Further Simplified Abstract
And First Steps

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1 Most Simple Version

Solve the 1D Poisson's equation to steady-state for a stuck-on glacier

$$\frac{\partial^2 u}{\partial z^2} = -1; \qquad 0 \leq z \leq 1 \tag{1}$$

with the boundary conditions

$$u\,(0) = 0; \tag{2}$$

$$\frac{\partial u}{\partial z}(1) = 0. \tag{3}$$

We can discretize the domain in space (z) using the standard second-order centered difference approximation

$$\frac{\partial^2 u_j}{\partial z^2} = \frac{u_{j-1} - 2u_j + u_{j+1}}{(z_{j+1} - z_j)\,(z_j - z_{j-1})} = -1 \tag{4}$$

which, using a consistent distance between nodes, becomes

$$\frac{u_{j-1} - 2u_j + u_{j+1}}{\Delta z^2} = -1 \text{ for } 1 \leq j \leq N, \tag{5}$$

with the boundary conditions

$$u_1 = 0; \tag{6}$$

$$\frac{\partial u_N}{\partial z} = 0. \tag{7}$$

For the first equation, when $j = 2$, we have

$$\frac{u_1 - 2u_2 + u_3}{\Delta z^2} = -1 \implies \frac{-2u_2 + u_3}{\Delta z^2} = -1; \tag{8}$$

and for the final equation when $j = N$ we have

$$\frac{u_{N-1} - 2u_N + u_{N+1}}{\Delta z^2} = -1 \quad \text{with} \quad u_N = u_{N+1} \implies \frac{u_{N-1} - u_N}{\Delta z^2} = -1. \tag{9}$$

So the system of equations is

$$\begin{cases} \dfrac{-2u_2 + u_3}{\Delta z^2} = -1 \\ \dfrac{u_{j-1} - 2u_j + u_{j+1}}{\Delta z^2} = -1; \quad 3 \leq j \leq N-1 \\ \dfrac{u_{N-1} - u_N}{\Delta z^2} = -1. \end{cases} \tag{10}$$

We can represent this system of equations as a matrix (A) of the form

$$A = \frac{1}{\Delta z^2} \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -1 \end{bmatrix} \tag{11}$$

with the u-column vector and b-solution vector as

$$u = \begin{bmatrix} u_2 \\ \vdots \\ u_N \end{bmatrix} \qquad b = \begin{bmatrix} -1 \\ \vdots \\ -1 \end{bmatrix} \tag{12}$$

such that $Au = b$.