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ECE 661: Computer Vision
September 1, 2022

ECE661 Homework 1

Question 1. What are all the points in the representational space $\mathbb{R}^3$ that are the homogeneous coordinates of the origin in the physical space $\mathbb{R}^2$?

The origin represented in $\mathbb{R}^3$ is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. If we explore the 3D representational space $\mathbb{R}^3$ we note that for any multiple $k \in \mathbb{R}$, $k \neq 0$, $k \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ is the same physical point in $\mathbb{R}^2$ as that corresponding to $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$. Thus all the points in the representational space $\mathbb{R}^3$ that are the homogeneous coordinates of the origin in the physical space $\mathbb{R}^2$ are:

$$\left\{ k \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mid k \in \mathbb{R}, k \neq 0 \right\}$$

Question 2. Are all points at infinity in the physical plane $\mathbb{R}^2$ the same? Justify your answer.

Any point at infinity in the physical plane $\mathbb{R}^2$ is known as an ideal point and takes the following form:

$$\begin{bmatrix} u \\ v \\ 0 \end{bmatrix}$$

where u and v are both real numbers. The pair (u, v) determines the trajectory towards infinity, meaning two ideal points with differing (u, v) pairs indeed are not the same.

Question 3. Argue that the matrix rank of a degenerate conic can never exceed 2.

A degenerate conic is mathematically defined as:

$$C = \boldsymbol{l}\boldsymbol{m}^T + \boldsymbol{m}\boldsymbol{l}^T$$

where $\mathbf{l}$ and $\mathbf{m}$ are both HC representations of the two lines that form the conic. Both $\boldsymbol{l}\boldsymbol{m}^T$ and $\boldsymbol{m}\boldsymbol{l}^T$ are outer products meaning the rank of the resulting matrices will be one. Since the degenerate conic is the sum of two rank one matrices, the resulting matrix cannot have a rank that exceeds 2.

Question 4. A line in $\mathbb{R}^2$ is defined by two points. That raises the question - how many points define a conic in $\mathbb{R}^2$? Justify your answer

A general conic is defined by 5 points, as may be seen by counting the number of coefficients in the second-degree implicit algebraic form of a conic: $ax^2 + bxy + cy^2 + dx + ey + f = 0$

Question 5. Derive in just 3 steps the intersection of two lines $\boldsymbol{l_1}$ and $\boldsymbol{l_2}$ with $\boldsymbol{l_1}$ passing through the points $(0, 0)$ and $(1, 2)$, and with $\boldsymbol{l_2}$ passing through the points $(3, 4)$ and $(5, 6)$. How many steps would take you if the second line passed through $(7, -8)$ and $(-7, 8)$?

Part 1:

$$\boldsymbol{l_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \tag{1}$$

$$\boldsymbol{l_2} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \times \begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} \tag{2}$$

$$\bar{\boldsymbol{x}} = \boldsymbol{l_1} \times \boldsymbol{l_2} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ -2 \end{bmatrix} \tag{3}$$

The lines $\boldsymbol{l_1}$ and $\boldsymbol{l_2}$ intersect at $(1, 2)$

Part 2:

$$\boldsymbol{l_2} = \begin{bmatrix} 7 \\ -8 \\ 1 \end{bmatrix} \times \begin{bmatrix} -7 \\ 8 \\ 1 \end{bmatrix} = \begin{bmatrix} -16 \\ -14 \\ 0 \end{bmatrix} \tag{1}$$

The intersection between $\boldsymbol{l_1}$ and the new $\boldsymbol{l_2}$ would be the origin since both lines share a c value of 0 in their HC form. Understanding this principal allows you to skip the computation of the cross product between the two lines. Hence it would only take a minimum of two steps.

Question 6. Let $\boldsymbol{l_1}$ be the line passing through points $(-4, 0)$ and $(-2, 8)$ and $\boldsymbol{l_2}$ be the line passing through points $(0, -2)$ and $(4, 14)$. Find the intersection between these two lines. Comment on your answer.

$$\boldsymbol{l_1} = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} -2 \\ 8 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ 2 \\ -32 \end{bmatrix} \tag{1}$$

$$\boldsymbol{l_2} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 4 \\ 14 \\ 1 \end{bmatrix} = \begin{bmatrix} -16 \\ 4 \\ 8 \end{bmatrix} \tag{2}$$

$$\bar{\boldsymbol{x}} = \boldsymbol{l_1} \times \boldsymbol{l_2} = \begin{bmatrix} -8 \\ 2 \\ -32 \end{bmatrix} \times \begin{bmatrix} -16 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 144 \\ 576 \\ 0 \end{bmatrix} \tag{3}$$

$\boxed{\text{The intersection of lines } \boldsymbol{l_1} \text{ and } \boldsymbol{l_2} \text{ are an ideal point. This means the two lines are parallel.}}$

Question 7. Find the intersection of two lines whose equations are given by $x = 1$ and $y = -1$.

If we write $x = 1$ and $y = -1$ in the first order implicit algebraic form of a line, $ax + by + c = 0$ we get:

$$\boldsymbol{l_1} = 1x + 0y - 1 = 0 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\boldsymbol{l_2} = 0x + 1y + 1 = 0 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\bar{\boldsymbol{x}} = \boldsymbol{l_1} \times \boldsymbol{l_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\boxed{\text{Thus the two lines intersect at } (1, -1)}$

Question 8. As you known, when a point $\mathbf{p}$ is on a conic $\mathbf{C}$, the tangent to the conic at that point is given by $\boldsymbol{l} = \boldsymbol{Cp}$. That raises the question as to what $\boldsymbol{Cp}$ would correspond to when $\mathbf{p}$ was outside the conic. AS you'll see later in class, when $\mathbf{p}$ is outside the conic, $\boldsymbol{Cp}$ is the line that joins the two points of contact if you draw tangents to $\mathbf{C}$ from the point $\mathbf{p}$. This line is referred to as the polar line. Now let our conic $\mathbf{C}$ be an ellipse that is centered at the coordinates $(2, 3)$, with $a = \frac{1}{2}$ and $b = 1$, where a and b, respectively, are the lengths of semi-minor and semi-major axis. For simplicity, assume that the minor axis is parallel to the x-axis and the major axis is parallel to the y-axis. Let $\mathbf{p}$ be the origin of the $\mathbb{R}^2$ physical plane. Find the intersection points of the polar line with the x and y axis.

The ellipse in question takes the following:

$$\frac{(x-2)^2}{\frac{1}{4}} + \frac{(y-3)^3}{1} = 1$$

We then convert this ellipse form into the implicit form for a conic to identify the $\boldsymbol{C}$ matrix

$$4(x-2)^2 + (y-3)^2 = 1$$

$$4(x^2 - 4x + 4) + (y^2 - 6y + 9) = 1$$

$$4x^2 - 16x + 16 + y^2 - 6y + 9 = 1$$

$$4x^2 - 16x + y^2 - 6y + 24 = 0$$

$$a = 4, d = -16, c = 1, e = -6, f = 24$$

$$\therefore \boldsymbol{C} = \begin{bmatrix} 4 & 0 & -8 \\ 0 & 1 & -3 \\ -8 & -3 & 24 \end{bmatrix}$$

The polar line can then be calculated by:

$$\boldsymbol{l_p} = \boldsymbol{C}\boldsymbol{p} = \begin{bmatrix} 4 & 0 & -8 \\ 0 & 1 & -3 \\ -8 & -3 & 24 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ -3 \\ 24 \end{bmatrix}$$

The intersection of the polar line with the x-axis is:

$$\bar{\boldsymbol{x}}_{\boldsymbol{x-axis}} = \begin{bmatrix} -8 \\ -3 \\ 24 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -24 \\ 0 \\ -8 \end{bmatrix}$$

The intersection of the polar line with the y-axis is:

$$\bar{\boldsymbol{x}}_{\boldsymbol{y-axis}} = \begin{bmatrix} -8 \\ -3 \\ 24 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 24 \\ 3 \end{bmatrix}$$

$\boxed{\text{The polar line intersects the x-axis at } (3, 0) \text{ and the y-axis at } (0, 8)}$