Game Theory Formula

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Preface

This is the third time that I use LaTeXto take the note. Thanks to the courses **'Linear Algebra with MATLAB'** in the **'2024 Summer School in Mathematical Economics & Financial Econometrics (UNNC)'**, which I learnt the LaTeXand tried my best to practice during my free time in UNUK.

The content in this notebook is mainly a complementary material suitable to **ECON 2001 Microeconomic Theory** in UNUK. The content taught in UNNC and UNUK is a little bit different and may be changed from time to time. However, the main knowledge always around what I had collected. During my study, I combined PPT, the suggested textbook [1], some knowledge I found in the previous UNNC Microeconomics Theory, and some knowledge in the PPT version of ECON 1049 (AUT) (I think this will always be provided in the moodle. If you do not find it, please go to ECON 3073, the content may be contained in the first part or the second part of the lecture note).

The course I learnt in UNUK also contains **Consumer Theory** and **Producer Theory** section, which I can also provide complementary notebooks.

Since this is my third time to write the LaTeX, I tried my best to make the knowledge in PPT and textbook flow. However, it seems the content I included is too much so it still looks like a bit messy and you may not want to read because of the number of pages (I can say that I had never imagined that there are so many things in Game Theory before I wrote it!). The only convenience that I can provide in this notebook is the outline or the structure of the microeconomics learning and collection of the formula (probably some detailed explanation that my not contained in the PPT but I think is important for learning it).

Moreover, I know that I may make mistake when I am writing this notebook. Therefore, if there is any error being detected, please do not hesitate to content me: `hmyhw8@nottingham.edu.cn` (If this not work, use my personal email).

Other notebooks can be found here.

Good luck with your Microeconomic Theory study!

1 Introduction to Game Theory

1.1 Basic Concepts

Key Components of a Game

1. A set of actions available to each player at each **decision point**.
2. **Information** available at each decision point.
3. Payoffs for each possible outcome.
4. At least two players.

Definition Complete Information
Each player knows the entire set of strategies and payoffs for all players in the game (and that each player knows all players have complete knowledge).

Math Knowledge Supplement

Cartesian Product

The **Cartesian product** is a mathematical operation that returns a set of all possible ordered pairs (or tuples) created by combining elements from two or more sets. Let's break it down:

Definition:

For two sets A and B, the Cartesian product $A \times B$ is defined as:

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

This means that each element a from set A is paired with each element b from set B to form a pair (a, b).

Example:

If $A = \{1, 2\}$ and $B = \{x, y\}$, then:

$$A \times B = \{(1, x), (1, y), (2, x), (2, y)\}$$

The result is a set of all ordered pairs where the first element comes from A, and the second comes from B.

Cartesian Product with More Sets:

The Cartesian product can be extended to more than two sets. For three sets A, B, and C, the Cartesian product $A \times B \times C$ consists of ordered triples:

$$A \times B \times C = \{(a, b, c) \mid a \in A, b \in B, c \in C\}$$

Example:

If $A = \{1\}$, $B = \{x, y\}$, and $C = \{p, q\}$, then:

$$A \times B \times C = \{(1, x, p), (1, x, q), (1, y, p), (1, y, q)\}$$

2 Normal Form Game

2.1 Simultaneous Games of Complete Information

Definition Normal Form Game
A normal form game (or static/strategic game)is a mathematical representation of a strategic interaction between players, where each player's possible strategies and their corresponding payoffs are explicitly defined.
An n-player *normal form game* is a $(2n+1)$-tuple

$$G = (N, S_1, \dots, S_n, u_1, \dots, u_n) \quad \text{or} \quad (N, (S_i)_{i\in N}, (u_i)_{i\in N})$$

where

- $N = \{1, \dots, n\}$ is the set of *players*;
- for every $i \in N$, S_i is the *strategy set* of player i;
- for every $i \in N$, $u_i : S \to \mathbb{R}$ is the *payoff function* of player i ($S = S_1 \times \cdots \times S_n = \prod_{i\in N} S_i$ is the *Cartesian product* of $S_1, \dots, S_n$); i.e., for every *strategy profile (or strategy combination/vector)* $s = (s_i)_{i\in N} = (s_1, \dots, s_n) \in S$, $u_i(s)$ is player i's payoff.

2.2 Pure Strategies

Definition Pure Strategies
A player choose one action or another with certainty. Its weight is 1.

NOTATION

1. **A Player** i $\quad (i \in I = \{1, ..., n\})$ In an n-player game we will have players $1, 2, ..., n$, with the generic player labeled i.
2. **Set of Players** $\quad I = \{1, 2, \dots, n\}$
3. **Strategy** s_i $\quad (i \in I = \{1, ..., n\})$ Each course of action open to a player during the game is called a strategy. In other words, A strategy is a complete plan of actions for playing a game; what to do for every possible situation.
Each player seeks to find a strategy that optimises their own objective function (do not care about the reduce the components' welfare or utility).
NOTE When there is no superscript $1, ..., m \in M$, then it represents a particular strategy that is chosen by the player i.
Example $\quad \mathbf{s_i} \in \mathbf{S_i}$ $\quad$ a particular strategy chosen by player i from the set of possibilities.
4. **The Set of Strategies** $\mathbf{S_i}$ $\quad (i \in I = \{1, ..., n\})$ The set of strategies open to player i.
Detaily, $\mathbf{S_i} = \{\mathbf{s_i^1}, \mathbf{s_i^2}, ..., \mathbf{s_i^m}\}$ $\quad (1, ..., m \in M)$ Each player i has a **set** of available **pure strategies**.
NOTE (1) When there is superscript $1, ..., m \in M$, then it represents all the available

strategies that the player i can choose.
(2) For different player, there are different numbers of elements in M.
Example player 1's set of strategies: $\mathbf{S_i} = \{\mathbf{s_i^1}, \mathbf{s_1^2}, ..., \mathbf{s_1^6}\}$
player 2's set of strategies: $\mathbf{S_2} = \{\mathbf{s_2^1}, \mathbf{s_2^2}, ..., \mathbf{s_2^{11}}\}$

5. **Strategy Profile** s_k A combination of the strategies chosen by all players.
Explanation A strategy profile specifies what every player in the game is doing simultaneously.
If there are n players in a game, and $S_1 = \{s_1^1, s_1^2, ..., s_1^m\}, S_2 = \{s_2^1, s_2^2, ..., s_2^q\}, \ldots,$ $S_n = \{s_n^1, s_n^2, ..., s_n^x\}$ are the sets of strategies available to players $1, 2, \ldots, n$, a **strategy profile** is an **ordered tuple**:
$$s_k = (s_1, s_2, \ldots, s_n)$$
where $s_i \in S_i$ (i.e., s_i is the strategy chosen by player i).
Example (1) **Profile** $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_n)$ The strategies of all players other than i.
(2) **Strategy Profile of Nash Equilibrium** $s^\star = (s_1^*, s_2^*, \ldots, s_n^*)$
(3) **Profile of Equilibrium Strategies** $\vec{s}^*_{-i}$
In this example we can also notice that the k in s_k is just a notation rather than a particular number, k can be replaced by any notation that can be used to describe this strategy profile.

6. **A set of Strategy Profiles** The set of strategy profiles is the collection of all possible strategy profiles for all players in the game. Formally, this is represented as the Cartesian product of all players' strategy sets:
$$S = S_1 \times S_2 \times ... \times S_n = \Pi_{i=1}^n S_i$$
Here:
   - S_i is the strategy set for player i.
   - Π represents the Cartesian product, which means the set S contains all possible combinations of strategies that the players can choose.

7. **Payoffs** $\mathbf{u_i(s_i, s_{-i})}$ The final returns to the players at the conclusion of a game.
   - Maps every combination of pure strategies to a player i's payoff.
$$u_i : \prod_{i \in I} S_i \to \mathbb{R}$$

Explanation The formula:
$$u_i : \prod_{i \in I} S_i \to \mathbb{R}$$
- u_i**:** This represents the *payoff function* for player i. It defines how much utility (or payoff) the player i receives based on the chosen strategies of all players.
- $\prod_{i \in I} S_i$**:**
  * The symbol $\prod$ denotes the Cartesian product.
  * S_i represents the set of *pure strategies* available to player i.
  * I is the set of all players in the game.
  * $\prod_{i \in I} S_i$ means the set of all possible combinations of strategies chosen

by all players in the game. For example, if there are two players with strategy sets S_1 and S_2, the Cartesian product $S_1 \times S_2$ represents all possible pairs of strategies (one from each player).

- **$\to \mathbb{R}$:** This indicates that the payoff function u_i maps from the Cartesian product of strategy sets (i.e., $\prod_{i \in I} S_i$) to the set of real numbers $\mathbb{R}$. In simpler terms, u_i takes as input a combination of strategies chosen by all players and outputs a real number, representing the payoff for player i.

- Measured in levels of utility obtained by the players.

  Example $U_1(s_1^x, s_2^y)$ is the player's payoffs expressed as utility in the payoff matrix.

- Functions of a profile of strategies chosen by all players. Depends on:
  (1) player i's own strategy s_i
  (2) the profile $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_n)$ of the strategies of all players other than i.

2.2.1 Best Response

In an n-player game, strategy s_i is a best response to rivals' strategies s_{-i} if player i cannot obtain a strictly higher payoff with any other possible strategy $s_i^{'} \in S_i$ given that rivals are playing s_{-i} .

Definition Best Response
s_i is a best response for player i to rivals' strategies s_{-i}, denoted $s_i \in BR_i(s_{-i})$, if

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \quad \text{for all } s_i' \in S_i. \tag{2.1}$$

ppt version Best response $BR_i(\vec{s}_{-i})$ is a strategy s_i' of player i which gives the most favourable outcome for a player, taking other players' strategies given:

$$BR_i(\vec{s}_{-i}) = \{s_i' \in S_i : u_i(s_i', \vec{s}_{-i}) \geq u_i(s_i, \vec{s}_{-i})\}$$

for any $s_i \in S_i$.

A technicality embedded in the definition is that there may be a set of best responses rather than a unique one; that is why we used the set inclusion notation $s_i \in BR_i(s_{-i})$. There may be a tie for the best response, in which case the set $BR_i(s_{-i})$ will contain more than one element. If there isn't a tie, then there will be a single best response s_i, and we can simply write $s_i = BR_i(s_{-i})$.

2.2.2 Dominate Strategy

Definition Dominate Strategy
A strategy that is a **best response** to any strategy the other players might choose is called a dominant strategy.

Property

1. Complicated strategic considerations do not matter when a player has a dominant strategy because what is best for that player is independent of what others are doing.
2. A dominant strategy must be a **best response** not just to the Nash equilibrium strategies of other players but to **all the strategies of those players**.
3. **Dominant Strategy Equilibrium** If all players in a game have a dominant strategy, then we say the game has a dominant strategy equilibrium.
4. **Relationship between Nash equilibrium and strictly dominant strategy equilibrium** In any game with a dominant strategy equilibrium, **the dominant strategy equilibrium is a Nash equilibrium**.
   When a dominant strategy exists, it is the *unique* Nash equilibrium.

NOTATION
A dominant strategy is a strategy s_i^* for player i that is a best response to all strategy profiles of other players ($\mathbf{s_{-i}}$). That is, $s_i^* \in BR_i(s_{-i})$ for all s_{-i}.

ppt version
Strictly Dominated Strategy A strategy $\hat{s}_i$ **strictly dominates** other strategies available in the set of strategies of layer i, s_i ($s_i \in S_i$) if it produces a higher payoff regardless of the strategies chosen by all other players (s_{-i}):

$$u_i(\hat{s}_i, s_{-i}) > u_i(s_i, s_{-i}) \quad \forall s_{-i} \in S_{-i} \text{ (where } S_{-i} \text{ is the set of strategy profiles).}$$

Rationality A rational player will never play a trictly dominated strategy.

Common knowledge of rationality Every player is rational and every player knows that every player is rational and so on.
Therefore, **iterated elimination of strictly dominated strategies**.
Players iteratedly eliminate strictly dominated strategies until there are none of them anymore.

2.2.3 Strictly and Weakly Dominated Strategy

Theorem Strict Dominance
Consider two strategies of player i, $s_i, s_i' \in S_i$.
We say s_i *strictly dominates* s_i' if s_i always gives player i a higher payoff than s_i' irrespectively of other players' strategies; i.e.,

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \quad \text{for all } s_{-i} \in S_{-i}$$

where

$$S_{-i} = S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n = \prod_{j \in N, j \neq i} S_j.$$

Thus, s_i' is called a *strictly dominated strategy* because there is some strategy s_i that strictly dominates s_i'.
If every player has a strictly dominant strategy, we say the game has a *strictly dominant strategy equilibrium*.

Example Iterated Elimination of Strictly Dominate Strategies for Common Knowledge of Rationality
Players iteratedly eliminate strictly dominated strategies until there are none of them anymore.
Background Setting
Two competing firms can set a high, medium or low price.

	p_H	p_M	p_L
p_H	$(100, 100)$	$(30, 150)$	$(-40, 90)$
p_M	$(150, 30)$	$(50, 50)$	$(5, 60)$
p_L	$(90, -40)$	$(60, 5)$	$(10, 10)$

We can firstly focus on player 1. No matter what is the player 2's strategy, for player 1, P_M always have higher payoff than P_H, i.e.P_M strictly dominates P_H for player 1. Since everyone is rational, the strictly dominated strategy will be eliminated by player 1, and the payoff matrix will be like:

	p_M	p_L
p_M	$(50, 50)$	$(5, 60)$
p_L	$(60, 5)$	$(10, 10)$

Symmetrically, player 2 will eliminate its strictly dominated strategyP_M, since whatever the strategy player 1 choose, for player 2, P_M is always strictly dominated by P_L.
So we will have:

	p_L
p_L	$(10, 10)$

2.2.4 Pareto Efficient

Definition Pareto Efficient
In a normal form game, an outcome (strategy profile) is Pareto efficient (or Pareto optimal) if there is no other outcome that makes at least one player strictly better off without making any other player worse off.
In other words:
A strategy profile s^* is Pareto efficient if there is no s such that $u_i(s) \geq u_i(s^*)$ for all players i, and $u_j(s) \geq u_j(s^*)$ for at least one player j.

For the **Prisoners' Dilemma**

Player 1 \ Player 2	Fink	Silent
Fink	$1, 1$	$3, 0$
Silent	$0, 3$	$2, 2$

Checking Pareto Efficiency:
To determine if the strictly dominant strategy equilibrium, (Fink, Fink), is Pareto efficient, compare it with the other outcomes:

1. (Fink, Silent) : Payoffs are $(3, 0)$. Here, Player 1 is strictly better off, but Player 2 is strictly worse off. Not a Pareto improvement.
2. (Silent, Fink) : Payoffs are $(0, 3)$. Here, Player 2 is strictly better off, but Player 1 is strictly worse off. Not a Pareto improvement.
3. (Silent, Silent) : Payoffs are $(2, 2)$. Both players are strictly better off compared to (Fink, Fink), making (Silent, Silent) a Pareto improvement.

Conclusion

- (Fink, Fink) is **not Pareto efficient**, because (Silent, Silent) provides higher payoffs to both players without making anyone worse off.
- The Pareto efficient outcome in this game is (Silent, Silent).

Theorem Weak Dominance
$s_i \in S_i$ *weakly dominates* $s_i' \in S_i$ if

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \quad \text{for all } s_{-i} \in S_{-i}$$

and

$$u_i(s_i, \tilde{s}_{-i}) > u_i(s_i', \tilde{s}_{-i}) \quad \text{for some } \tilde{s}_{-i} \in S_{-i}.$$

Weak domination is necessary but not sufficient for strict domination.

1. Sufficiency
   Weak domination is **not sufficient** for strict domination.
   Weak domination allows the payoffs to be equal $(u_i(s_i, s_{-i}) = u_i(s_i', s_{-i}))$ for some strategy profiles, which is not allowed in strict domination. In strict domination,

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \quad \text{must hold for all } s_{-i}.$$

2. Necessity
   Weak domination is a **necessary condition** for strict domination.
   If s_i strictly dominates s_i', then $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for all s_{-i}, which is part of the definition of weak domination. Moreover, strict domination inherently satisfies the condition of weak domination because strict inequality implies weak inequality.

Example Prisoner Dilemma

Player 1 \ Player 2	Fink	Silent
Fink	$1, 1$	$3, 0$
Silent	$0, 3$	$2, 2$
Run	$4, 0$	$4, 0$

The relationship between Nash equilibrium and weakly dominant strategy equilibrium

1. Every weakly dominant strategy equilibrium is a Nash equilibrium:
   If all players have weakly dominant strategies and they play them, no player has an incentive to deviate, as their weakly dominant strategy guarantees at least as good a payoff as any other strategy. Thus, the outcome is a Nash equilibrium.

2. Not every Nash equilibrium is a weakly dominant strategy equilibrium:
   A Nash equilibrium only requires that no player has an incentive to deviate, given the strategies of others. However, it does not require that a player's strategy dominates all other strategies in all cases. A Nash equilibrium can involve strategies that are not weakly dominant or even weakly dominated in some scenarios.

2.3 Nash Equilibrium

A Nash equilibrium is stable in that, even if all players revealed their strategies to each other, no player would have an incentive to deviate from his or her equilibrium strategy and choose something else.

Definition Nash equilibrium.
A Nash equilibrium is a strategy for each player that is the **best choice** for each player given the others' **equilibrium strategies**.

More formally, a Nash equilibrium is a strategy profile $s^\star = (s_1^*, s_2^*, \dots, s_n^*)$ such that, for each player $i = 1, 2, \dots, n \in N$, s_i^* is a best response to the other players' equilibrium strategies s_{-i}^*. That is,

$$s_i^* \in BR_i(s_{-i}^*) \text{ for every } i \in N$$

where

$$s_{-i}^* = (s_1^*, \dots, s_{i-1}^*, s_{i+1}^*, \dots, s_n^*).$$

A Nash equilibrium is an intersection point of graphs of $BR_i(\cdot)$'s.

ppt version For the n-player situation, the notation is like:
Nash Equilibrium is a profile of strategies s^* such that

$$s^* \iff s_i^* \in BR_i(s_{-i}^*) \quad \text{for any } i \in I.$$

and in $BR_i(s_{-i}^*)$ (or we can write it as $BR_i(\vec{s}_{-i}^*)$, these two are actually the same though the notation seems a bit different),

$$u_i(s_i^*, \vec{s}_{-i}^*) \geq u_i(s_i', \vec{s}_{-i}^*) \quad \text{for any } s_i' \in S_i.$$

($s_i^{'}$ is the strategies other than $s_i^{'}$ that is available to player i in its own set of strategy

$$S_i = \{s_i^1, s_i^2, ..., s_i^m\})$$

That means, for player i, a strategy s_i^* is a best response to the **profile of equilibrium strategies** $\vec{s}_{-i}^*$ of all other players.

Example Two-player Game
In a two-player game, (s_1, s_2) is a Nash equilibrium if

$$s_1 \in BR_1(s_2) \quad \text{and} \quad s_2 \in BR_2(s_1).$$

In more detial, in a two-player game, (s_1^*, s_2^*) is a Nash equilibrium if s_1^* and s_2^* are mutual best responses against each other:

$$u_1(s_1^*, s_2^*) \geq u_1(s_1, s_2^*) \quad \text{for all available strategies of player 1 } s_1 \in S_1$$

$$\text{where } S_1 = \{s_1^1, s_1^2, ..., s_1^m\} \text{ is the set of strategies available to player 1,}$$

and

$$u_2(s_2^*, s_1^*) \geq u_2(s_2, s_1^*) \quad \text{for all available strategies of player 2 } s_2 \in S_2$$

$$\text{where } S_2 = \{s_2^1, s_2^2, ..., s_2^m\} \text{ is the set of strategies available to player 2}$$

NOTATION

NE^P A Nash Equilibrium in pure strategies.

NE^M A Nash Equilibrium in mixed strategies.

There may be multiple Nash equilibria, making it hard to come up with a unique prediction.

Each player chooses a strategy given a set of available actions, the possible actions of others, and payoffs.

Example Find Nash Equilibrium in Prisoner Dilemma

Player 1 \ Player 2	Fink	Silent
Fink	$1, 1$	$3, 0$
Silent	$0, 3$	$2, 2$

Steps:

1. Underline the payoffs corresponding to player 1's best responses.
   - If player 2 finks, we are in the first column of the matrix.
     Player 1's best response is to fink if player 2 finks, so we underline $u_1 = 1$
   - Given that player 2 is silent, we focus our attention on the second column of the matrix.
     Player 1's best response is to fink if player 2 silent, so underline $u_1 = 3$
2. We move to underlining the payoffs corresponding to player 2's best responses.
   - Player 2's best response is to fink if player 1 finks, so we underline $u_2 = 1$.
   - Player 2's best response is to fink if player 1 silent, so we underline $u_2 = 3$

Example Find Nash Equilibrium in Pure Strategy

	L	S	R
U	$1, 3$	$4, 0$	$5, 2$
C	$5, 4$	$5, 1$	$-1, 2$
D	$2, 0$	$1, 3$	$6, 4$

- **Pure strategies**
  1. Pure strategies of Player 1 are the rows of the payoff matrix.
     A set of pure strategies of Player 1 is $S_1 = \{U, C, D\}$
  2. Pure strategies of Player 2 are the columns of the matrix.
     A set of pure strategies of Player 1 is $S_2 = \{L, S, R\}$
- **Payoff**
  Each cell contains players' payoffs obtained in a certain outcome determined by a combination of players' strategies:
  (1) The first element is the payoff of Player 1
  (2) The second element is the payoff of Player 2
- There are 9 possible strategy profiles: (U,L),(U,S), (U,R), (C,L), . . . , (D,R).

- **Dominate Strategy**
  Players iteratedly eliminate strictly dominated strategies until there are none of them anymore.

  1. For Player 2, compare strategies S and R, R strictly dominates S, as it gives higher payoffs for any strategy played by Player 1.
     Manipulation for solution *We can ignore S and remove it from the matrix, as it will never be an optimal strategy of Player 2.*
  2. For Player 1, compare strategies U and D, D strictly dominates U, as it gives higher payoffs for any strategy played by Player 2.
     Manipulation for solution *We can remove U from the payoff matrix.*

  Now there is the reduced-form matrix:

	L	R
C	$5^*, 4$	$-1, 2^*$
D	$2, 0$	$6^*, 4$

- **Best Response**
  Best Responses of Player 1:

  - C is a best response of P1 to L: $C = BR_1(L)$
  - D is a best response of P1 to R: $D = BR_1(R)$

  Best Responses of Player 2:

  - L is a best response of P2 to C: $L = BR_2(C)$
  - R is a best response of P2 to D: $R = BR_2(D)$

- Nash Equilibrium
  An intersection of best responses constitutes Nash Equilibrium.

$$NE^P \ni \{(C, L), (D, R)\} \rightarrow \{(5, 4), (6, 4)\}$$

2.4 Mixed Strategies

Definition Mixed Strategies
A mixed strategy is a general category that includes the special case of a pure strategy.
NOTE A pure strategy is the special case in which only **one action** is played with positive probability.

Definition Strictly Mixed Strategies Mixed strategies that involve **two or more actions** being played with positive probability are called strictly mixed strategies.
i.e. A mixed strategy for player i is strictly mixed if at least two elements in S_i are assigned with positive probability.
NOTE The key to guessing whether a game has a Nash equilibrium in strictly mixed strategies is the surprising result that almost all games have an **odd** number of Nash equilibria.

Principle of seeking Nash equilibria in strictly mixed strategies:

1. One player randomizes over only those actions among which he or she is indifferent, given other players' equilibrium mixed strategies.
2. One player's indifference condition pins down the other player's mixed strategy.

NOTATION

1. **Player i has a set of M possible actions** $A_i = \{a_i^1, \ldots, a_i^m, \ldots, a_i^M\}$, where the subscript refers to the player and the superscript to the different choices.
2. **Mixed Strategy** A probability distribution over the M actions, $s_i = (\sigma_i^1, \ldots, \sigma_i^m, \ldots, \sigma_i^M)$

   Explanation in a set of strategies $A_i = \{a_i^1, a_i^2, ..., a_i^m, ..., a_i^M\}$, suppose the probability of player i choosing a_i^1 is σ_i^1, then we can also denote that the probability of choosing a_i^2 is σ_i^2, ..., a_i^M is σ_i^M.
   More generally, we can say that

   $$\sigma_i(a_i^x) \text{ for x=1,...,m,...M}$$

   is the PDF of the set of strategies $A_i = \{a_i^1, a_i^2, ..., a_i^m, ..., a_i^M\}$.
   Or we can rewrite it as: given the particular strategy chosen by player i, a_i from its available strategies $\{a_i^1, a_i^2, ..., a_i^m, ..., a_i^M\}$ in the set of strategies A_i, we can get its mixed strategy by putting it into a probability distribution PDF σ_i, which can also be written as: For the random variable a_i, and the given possible value $a_i^1, a_i^2, ..., a_i^m, ..., a_i^M$,

   $$\Pr[a_i = a_i^x] = \sigma_i(a_i^x) \quad x = 1, 2, ..., M$$

   So a particular mixed strategy of player i can be written as $\sigma_i \in \Sigma_i$
   (Σ_i is the set of mixed strategies)

   $$\sigma_i = (\sigma_i(a_i^1), \sigma_i(a_i^2), ..., \sigma_i(a_i^M))$$

   and

   $$\Sigma_x \sigma_i(a_i^x) = 1$$

   When the PDF σ_i changes, the particular mixed strategy chosen by the player i changes. so the set of mixed strategies Σ_i contains different types of PDF σ_i.

Example
$A_1 = A_2 = \{ballet, boxing\}$, a mixed strategy as a pair of probabilities $(\sigma, 1-\sigma)$ (σ is the probability that the player chooses ballet).

(a) Mixed strategy (1/3, 2/3) the player plays ballet with probability 1/3 and boxing with probability 2/3;
(b) Mixed strategy (1/2, 1/2) the player is equally likely to play ballet or boxing;
(c) Mixed strategy (1, 0) the player chooses ballet with certainty;
(d) Mixed strategy (0, 1) the player chooses boxing with certainty.

3. **Probability of Player i Playing Action $\mathbf{a_i^m}$** $\sigma_i^m \in (0,1)$.
The probabilities in s_i must sum to unity:

$$\sigma_i^1 + \cdots + \sigma_i^m + \cdots + \sigma_i^M = 1. \tag{2.2}$$

4. **Payoff Function** $u_i(s_i, s_{-i})$, rather than being a certain payoff, must be reinterpreted as the **expected value of a random payoff**, with probabilities given by the strategies s_i and s_{-i}.
NOTE An expected value of a random variable equals the sum over all outcomes of the probability of the outcome multiplied by the value of the random variable in that outcome.

Example **Battle of the Sexes**
Player 1 is wife, Player 2 is husband.

	Ballet	**Boxing**
Ballet	$(2,1)$	$(0,0)$
Boxing	$(0,0)$	$(1,2)$

(a) Let's compute players' expected payoffs if the wife chooses the mixed strategy $(\frac{1}{9}, \frac{8}{9})$ and the husband $(\frac{4}{5}, \frac{1}{5})$ in the Battle of the Sexes. The wife's expected payoff is

$$\begin{aligned} U_1\left(\left(\frac{1}{9}, \frac{8}{9}\right), \left(\frac{4}{5}, \frac{1}{5}\right)\right) &= \left(\frac{1}{9}\right)\left(\frac{4}{5}\right) U_1(\text{ballet, ballet}) + \left(\frac{1}{9}\right)\left(\frac{1}{5}\right) U_1(\text{ballet, boxing}) \\ &+ \left(\frac{8}{9}\right)\left(\frac{4}{5}\right) U_1(\text{boxing, ballet}) + \left(\frac{8}{9}\right)\left(\frac{1}{5}\right) U_1(\text{boxing, boxing}) \\ &= \left(\frac{1}{9}\right)\left(\frac{4}{5}\right)(2) + \left(\frac{1}{9}\right)\left(\frac{1}{5}\right)(0) + \left(\frac{8}{9}\right)\left(\frac{4}{5}\right)(0) + \left(\frac{8}{9}\right)\left(\frac{1}{5}\right)(1) = \frac{16}{45}. \end{aligned}$$

(b) we compute the wife's expected payoff if she plays the pure strategy of going to ballet [the same as the mixed strategy (1, 0)] and the husband continues to play the mixed strategy $(4/5, 1/5)$. Now there are only two relevant outcomes, given by the two boxes in the row in which the wife plays ballet. The probabilities of the two outcomes are given by the probabilities in the husband's mixed strategy. Therefore,

$$\begin{aligned} U_1\left(\text{ballet}, \left(\frac{4}{5}, \frac{1}{5}\right)\right) &= \left(\frac{4}{5}\right) U_1(\text{ballet, ballet}) + \left(\frac{1}{5}\right) U_1(\text{ballet, boxing}) \\ &= \left(\frac{4}{5}\right)(2) + \left(\frac{1}{5}\right)(0) = \frac{8}{5}. \end{aligned}$$

Formula **The General Expression for expected payoff in two players scenario**
the Player 1's expected payoff when she plays mixed strategy $(w, 1-w)$ and the Player 2 plays $(h, 1-h)$: if the Player 1 plays ballet with probability w and the Player 2 with probability h, then:
$U_1(s_1^x, s_2^y)$ is the player's payoffs expressed as utility in the payoff matrix.

$$u_1\left((w, 1-w), (h, 1-h)\right) = (w)(h)U_1(\text{ballet, ballet}) + (w)(1-h)U_1(\text{ballet, boxing})$$

$$+(1-w)(h)U_1(\text{boxing, ballet}) + (1-w)(1-h)U_1(\text{boxing, boxing})$$

$$= (w)(h)(2) + (w)(1-h)(0) + (1-w)(h)(0) + (1-w)(1-h)(1) = 1 - h - w + 3hw.$$

So the player 1's **best response correspondence** is:

$$BR_1(h) : [0,1] \to [0,1] \text{ is}$$

$$BR_1(h) = \begin{cases} 0 & \text{if } h < \frac{1}{3}, \\ [0,1] & \text{if } h = \frac{1}{3}, \\ 1 & \text{if } h > \frac{1}{3}. \end{cases}$$

Nash equilibria We will compute values of w and h that make up Nash equilibria. Both players have a continuum of possible strategies between 0 and 1. And we will use graphical methods to solve for the Nash equilibria.
Continue to use the **Example of Battle of the Sexes**

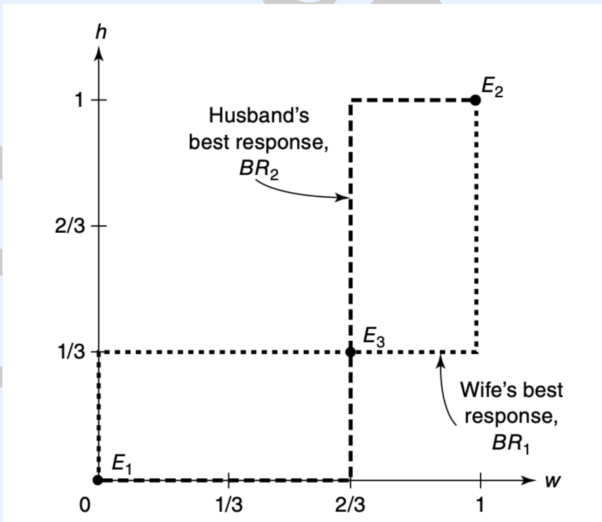

Figure 1: Nash Equilibria in Mixed Strategy

The Nash equilibria are given by the **intersection points** between the best responses. At these intersection points, both players are best responding to each other, which is what is required for the outcome to be a Nash equilibrium.

There are three Nash equilibria:

1. **pure-strategy Nash equilibria**
   - E_1 corresponding to the pure-strategy Nash equilibrium in which both play boxing.
   - E_2 to that in which both play ballet.
2. **mixed-strategy Nash equilibrium** Point E_3 is the strictly mixed-strategy Nash equilibrium, which can can be spelled out as "the wife plays ballet with probability 2/3 and boxing with probability 1/3 and the husband plays ballet with probability 1/3 and boxing with probability 2=3."
   More succinctly, having defined w and h, we may write the equilibruim as "$w^* = 2/3, h^* = 1/3$"

Manipulation for solution A shortcut to finding the Nash equilibrium in strictly mixed strategies is based on the insight that a player will be willing to randomize between two actions in **equilibrium** only if he or she gets **the same expected payoff** from playing either action or, in other words, is **indifferent** between the two actions in equilibrium.

Continue to use the **Example of Battle of the Sexes**
Suppose the husband is playing mixed strategy $(h, 1-h)$, that is, playing ballet with probability h and boxing with probability $1 - h$.
The wife's expected payoff from playing ballet is:

$$U_1(\text{ballet}, (h, 1-h)) = (h)(2) + (1-h)(0) = 2h.$$

Her expected payoff from playing boxing is

$$U_1(\text{boxing}, (h, 1-h)) = (h)(0) + (1-h)(1) = 1-h.$$

For the wife to be indifferent between ballet and boxing in equilibrium, these must be equal:

$$2h = 1 - h,$$

implying $h^* = \frac{1}{3}$.

Similar calculations based on the husband's indifference between playing ballet and boxing in equilibrium show that the wife's probability of playing ballet in the strictly mixed strategy Nash equilibrium is

$$w^* = \frac{2}{3}.$$

Recall the knowledge from statistics

Let X be a discrete random variable, and suppose that the possible values that it can assume are given by $x_1, x_2, x_3, \ldots$, arranged in some order. Suppose also that these values are assumed with probabilities given by

$$\Pr[X = x_k] = f(x_k) \quad k = 1, 2, \ldots$$

It is convenient to introduce the *probability density function (PDF)* or *probability mass function (PMF)*, given by

$$\Pr[X = x] = f(x)$$

For $x = x_k$, this becomes $f(x_k)$, while for other values of x, $f(x) = 0$. For a discrete random variable (RV) with sample space Ω, the PDF $f(x)$ is valid if:

1. $f(x) \geq 0$
2. $\sum_x f(x) = 1$

ppt version
A particular mixed strategy σ_i assigns a probability weight $\sigma_i(s_i) \geq 0$ to each pure strategy $s_i \in S_i$:

$$\sigma_1 = (\sigma_i(s_i^1), ..., \sigma_i(s_i^m))$$

In the ppt, this particular mixed strategy σ_i is denoted as

$$\sigma_i = \sum \sigma_i(s_i)s_i$$

To understand this expression, we can use the previous example:
Example Mixed strategy (1/3, 2/3) the player plays ballet with probability 1/3 and boxing with probability 2/3, so it mixed strategy in ppt can be denote as:

1/3 (bullet) + 2/3 (boxing)

- Let $S_i = \{s_i\}$ (here s_i represents all the available strategies that the player i can choose) be a set of pure strategies for Player i.
  Actually, $S_1 = \{s_1^1, s_1^2, ..., s_1^m\}$ is the set of strategies available to player 1
- Now let S_i be a discrete random variable, and suppose that the possible values that it can assume are given by $s_1^1, s_1^2, ..., s_1^m$, arranged in some order. Suppose also that these values are assumed with probabilities given by

$$\Pr[s_i = s_i^x] = \sigma_i(s_i^x) \quad x = 1, 2, \ldots, m$$

- If $\sigma_i(s_i) > 0$, we say that s_i is in the support of σ_i.
- $\sum \sigma_i(s_i) = \sum_{x=1}^m \sigma_i(s_i^x) = 1$
- **A Particular Mixed Strategy for Player i $\sigma_\mathbf{i}$** $\sigma_i(s_i)$ is a weight of pure strategy s_i (the probability that the player i will take this action x).

- If players play their mixed strategies, their **expected payoffs** can be calculated as a weighted sum of the corresponding payoffs:

$$u_i(\sigma_i, \sigma_{-i}) = u_i(\sigma) = \sum_{s \in S} \sigma(s) u_i(s)$$

    1. σ is the mixed strategy chosen by each player (mixed strategy profile)
    2. $\sigma(s)$ the general expression of PDF on each player's chosen strategies. It is different to different players.
    3. s the pure strategy profile including all the players' particular strategy.

Definition Nash Equilibrium in Mixed Strategies
Given that all other players play their equilibrium mixed strategies σ^*_{-i}, an equilibrium mixed strategy σ^*_i of player i gives an expected payoff not less than any other mixed strategy σ_i.

More formally, a strategy profile $\sigma^* = \{\sigma^*_1, \ldots, \sigma^*_n\}$ is a Nash Equilibrium in mixed strategies, if for each player i:

$$u_i(\sigma^*_i, \sigma^*_{-i}) \geq u_i(\sigma_i, \sigma^*_{-i}) \quad \text{for any } \sigma_i \in \Sigma_i.$$

Proposition 1
If two pure strategies constitute an equilibrium mixed strategy, then their expected payoffs are equal to $u_i(\sigma^*)$.
i.e. An equilibrium mixed strategy of player i must make the other players indifferent between playing their pure strategies.

Proposition 2
If some pure strategy s_i is not a component of an equilibrium mixed strategy, then its expected payoff is less than $u_i(\sigma^*)$.

Existence of Nash equilibrium
Allowing for mixed strategies, there must exist at least one Nash equilibrium in a wide class of games, particularly, all Önite games (games with a Önite number of players and a Önite number of actions).
This existence theorem does not apply to Nash equilibrium in pure strategies.

2.5 Simultaneous Games of Complete Information and Discrete Payoff Functions

Solve a Game for All Possible NE

1. Step 1. Represent a game in a normal form and define the sets of players, available strategies, and payoffs.
2. Step 2. Eliminate all strictly dominated strategies.
   - Start with those dominated by pure strategies.
   - Then eliminate those dominated by mixed strategies.
3. Step 3. Find all pure strategy NE.
4. Step 4. Find all mixed strategy NE
   - Use Proposition 1 to make each player indifferent between their pure strategies, constituting an equilibrium mixed strategy.
   - Compute equilibrium weights and write down equilibrium mixed strategies.
   - Compute equilibrium payoffs.

Example 1 Find the Mixed-Strategy Nash Equilibrium

probability of player's action	(q) L	$(1-q)$ R
(p) C	$5, 4$	$-1, 2$
$(1-p)$ D	$2, 0$	$6, 4$

Use the notation in ppt version for mixed strategy:

- Player 1 can mix C and D with some weights: $\sigma_1 = \sigma_1(C)C + \sigma_1(D)D$
- Player 2 can mix L and R with some weights: $\sigma_2 = \sigma_2(L)L + \sigma_1(R)R$

When Player 1 mixes C and D with weights p and $1-p$, Player 2 mixes L and R with weights q and $1-q$,
For Player 1:

$$Eu_1(C) = 5q + (-1)(1-q) = u_1(\sigma^*)$$
$$Eu_1(D) = 2q + 6(1-q) = u_1(\sigma^*)$$

Solve the equilibrium for player 1:

$$5q - 1 + q = 2q + 6 - 6q$$
$$10q = 7$$
$$q = 0.7$$

Recall **Principle of seeking Nash equilibria in strictly mixed strategies**:

1. One player randomizes over only those actions among which he or she is indifferent, given other players' equilibrium mixed strategies.
2. One player's indifference condition pins down the other player's mixed strategy.

This means:

In equilibrium, Player 2 plays a mixed strategy $\sigma_2^* = 0.7L + 0.3R$.

Player 1 obtains an expected payoff $u_1(\sigma^*) = 5 \cdot 0.7 - 0.3 = 3.2$.

Therefore, similarly, we can write solve the equilibrium for Player 2:

$$Eu_2(L) = 4p + 0(1-p) = u_2(\sigma^*)$$
$$Eu_2(R) = 2p + 4(1-p) = u_2(\sigma^*)$$
$$Eu_2(L) = Eu_2(R)$$
$$4p = 2p + 4 - 4p$$
$$6p = 4$$
$$p = \frac{2}{3}$$

In conclusion,

In equilibrium, Player 1 plays a mixed strategy $\sigma_1^* = \frac{2}{3}L + \frac{1}{3}R$.

Player 2 obtains an expected payoff $u_2(\sigma^*) = 4 \cdot \frac{2}{3} = \frac{8}{3} = 2\frac{2}{3}$.

So we can conclude that, for the game:

	L	S	R
U	$1,3$	$4,0$	$5,2$
C	$5,4$	$5,1$	$-1,2$
D	$2,0$	$1,3$	$6,4$

we will have two NE in pure strategies and one NE in mixed strategies:

$$NE^P \exists\{(C,L),(D,R)\} \to \{(5,4),(6,4)\}$$

$$NE^M \exists \left\{0.7\,C + 0.3\,D;\ \frac{2}{3}\,L + \frac{1}{3}\,R\right\} \to \{3.2;\ 2\frac{2}{3}\}$$

Example 2 Dominant Mixed Strategies
There are no strictly dominant pure strategies for both players in this game:

	L	S	R
U	$3,4$	$2,3$	$5,5^*$
D	$5,1$	$4,6^*$	$3,0$

For Player 2:

$$R = BR_2(U) \quad \text{and} \quad S = BR_2(D);$$

L is never a best response; it may be dominated by a mix of S and R.
Assume there is $\alpha \in (0,1)$ such that:

$$\alpha S + (1-\alpha)R \succ L.$$

A mix of S and R has a higher expected payoff than L:

$$\alpha \begin{bmatrix} 3 \\ 6 \end{bmatrix} + (1-\alpha)\begin{bmatrix} 5 \\ 0 \end{bmatrix} \gg \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

$$\begin{cases} 3\alpha + 5 - 5\alpha > 4 \\ 6\alpha > 1 \end{cases} \implies \begin{cases} \alpha < \frac{1}{2} \\ \alpha > \frac{1}{6} \end{cases}$$

A mix of S and R dominates L for any $\alpha \in \left(\frac{1}{6}, \frac{1}{2}\right)$.

2.6 Simultaneous Games of Complete Information and Continuous Payoff Functions

Not all games have discrete choices. When a set of available actions is infinite, we cannot form a payoff matrix and solve games in a standard manner. But we can still use the concepts of best responses and NE to solve games with **continuous choices**.

When choices are continuous, we can think of **the strategy sets as continuous variables**. The corresponding **players' payoffs are continuous functions** depending on the strategies of all players.

Steps to solve for the Nash equilibrium when the game involves a continuum of actions

1. Write down the payoff for each player as a function of all players' actions.
   - There will be one equation for each player.
   - With n players, the system of n equations for the n unknown equilibrium actions can be solved simultaneously.
2. Compute the first-order condition associated with each player's payoff maximum.
3. Rearrange the F.O.C function into the best response of each player as a function of all other players' actions.

Example Tragedy of the Commons

Basic setting:

- two players $\mathbf{i = 1, 2}$
- number of sheep that players graze on the common grass $\mathbf{q_i}$
- the per-sheep value of grazing on the commons

$$\mathbf{v(q_1, q_2) = 120 - (q_1 + q_2)}$$

this will give a listing of the players' payoff functions

$$\Rightarrow \begin{cases} u_1(q_1, q_2) = q_1 v(q_1, q_2) = q_1(120 - q_1 - q_2) \\ u_2(q_1, q_2) = q_2 v(q_1, q_2) = q_2(120 - q_1 - q_2) \end{cases}$$

To find the Nash equilibrium, we solve **player 1's** value-maximization problem **(satisfy F.O.C condition)**:

$$\max_{q_1} \{q_1(120 - q_1 - q_2)\}.$$

The first-order condition for a maximum is

$$120 - 2q_1 - q_2 = 0$$

or, rearranging,

$$q_1 = 60 - \frac{q_2}{2}.$$

Similar steps show that herder 2's best response is

$$q_2 = 60 - \frac{q_1}{2}.$$

The **Nash equilibrium** is given by the pair $(\mathbf{q_1^*}, \mathbf{q_2^*})$ that satisfies these equations simultaneously. Taking an algebraic approach to the simultaneous solution, Equation $q_2 = 60 - \frac{q_1}{2}$ can be substituted into the first equation, which yields

$$q_1 = 60 - \frac{1}{2}\left(60 - \frac{q_1}{2}\right);$$

Then we can get

$$q_1^* = q_2^* = 40$$

.

As we seek to maximise the payoff functions, the optimal values must give local maxima: The second-order partial derivatives **S.O.C** must be negative (remember to check after conducting the **F.O.C condition**). So we can finally find the Nash Equilibrium:

$$NE \ni \{q_1^* = 40; q_2^* = 40\} \rightarrow \{1600; 1600\}$$

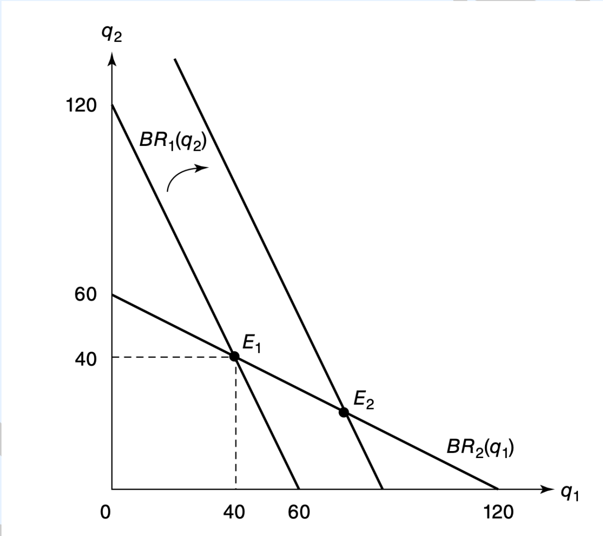

Figure 2: Best-Response Diagram for the Tragedy of the Commons

Graph Description

1. **Original Setting**
   - The two best responses on a graph with player 1's action on the horizontal axis and player 2's on the vertical axis.
   - These best responses are simply lines.
   - The two best responses intersect at the Nash equilibrium E_1.
2. **Change the Player 1's payoff function**
   - Shift the best response out for Player 1 while leaving 2's the same.
   - The new Nash Equilibrium, E_2, involves more sheep for 1 and fewer for 2.

However, the Nash equilibrium is **NOT** the best use of the commons.
In the scenario just now, we only maximize the Player 1's payoff, rather than consider the payoff of both players simultaneously to maximize the payoff. Therefore, we now focus on the **Joint Payoff Maximization** problem:

$$\max_{\mathbf{q_1}}\{(\mathbf{q_1}+\mathbf{q_2})\mathbf{v}(\mathbf{q_1},\mathbf{q_2})\} = \max_{\mathbf{q_1}}\{(\mathbf{q_1}+\mathbf{q_2})(\mathbf{120}-\mathbf{q_1}-\mathbf{q_2})\}$$

where

$$q_1^* = q_2^* = 30$$

The feature that Tragedy of the Commons shares with the Prisoners' Dilemma:
The Nash equilibrium is **less efficient** for all players than some other outcome.

- In the Prisoners' Dilemma, both fink in equilibrium when it would be more efficient for both to be silent.
- In the Tragedy of the Commons, the herders graze more sheep in equilibrium than is efficient.

3 Extensive Form Games

3.1 Sequential Games of Complete Information with Discrete Payoffs

Definition Sequential Games

Sequential games are interactions where players' actions take place at different points in time. A sequential game consists of **finite or infinite time periods**, in which players make decisions. The key differences between simultaneous and sequential games are dynamic interactions and the **order of moves**.

3.2 Extensive Form Game

Sequential games are usually represented in an **extensive form (a game tree).**
Actually, we have already seen the extensive form in the simultaneous game in the **Example Battle of the Sexes**:
Player 1 is wife, Player 2 is husband.

	Ballet	**Boxing**
Ballet	$(2, 1)$	$(0, 0)$
Boxing	$(0, 0)$	$(1, 2)$

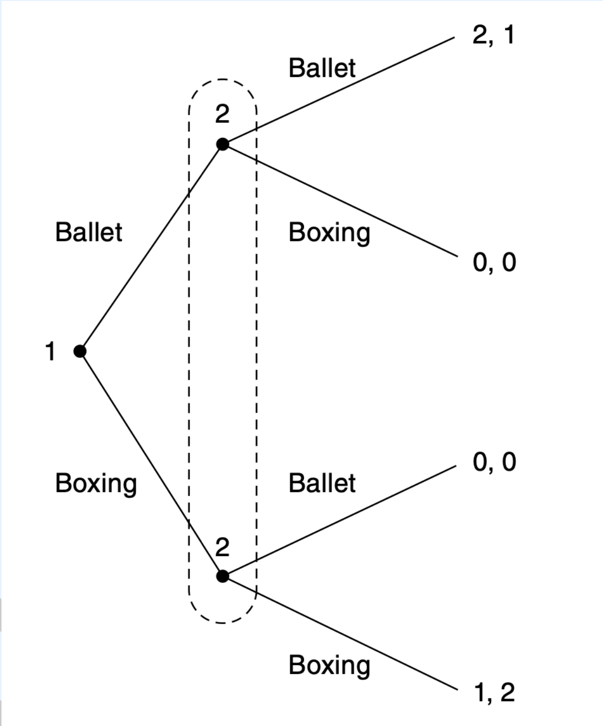

Figure 3: Battle of the Sexes in Simultaneous Game with Extensive Form

Graph Description

1. Players choose **simultaneously**, the player 2 does not know the player 1's choice when he moves, so his **decision nodes** are connected in the same **information set**.

2. There is only one decision node—the **topmost mode**—that is not connected to another in the same information set; hence there is only one **proper subgame**, the game itself.

3.2.1 Basic Settings of Extensive Form Game

Normally, the extensive form games specifies:

1. The players in the game
2. When each player has the move (the **order** of the moves) i.e. the timing of the game
3. What each player can do (what is each player's set of actions) at each of his or her chances to move i.e. A set of actions available to each player at every decision node
4. What each player knows at each of his or her chances to move i.e. Information available to each player at every decision node
5. The payoff received by each player for each combination of moves that could be chosen by the players. i.e. Payoffs at terminal points for each possible outcome (they are functions of action histories).
6. *Probability distribution of states in games with incomplete information

For fulling understanding the graph description, we need to introduce four definition, **Action**, **decision node**, **Perfect Information**, **Imperfect Information**, **information set**, and then describe this extensive form in simultaneous game and finally compare it to the sequential game.

3.2.2 Basic and Extended Concepts in Extensive Form Game

Definition Action
An action is a **move** (or a decision made) taken by a player at a particular **decision node**.

Extension History of Actions
A history of actions is a sequence of actions taken by players (from the beginning to a terminal point).

Definition Strategy A strategy is a complete plan of actions of a player (an ordered list of player's actions taken in every decision node).
A strategy for a player specifies **a feasible action** for the player at**every information set** for this player.

A **pure strategy** in an extensive game states what a player will choose at every decision node (from the beginning to a terminal node).

Definition Decision Node
A decision node is a situation where a player takes an **action**.
A decision node in game theory is a point in an extensive-form game (a game represented as a tree) where a player must choose among a set of possible actions. It represents a situation where a decision-maker (player) selects one action from several alternatives available at that point in the game.
Key Characteristics:

1. Player's Turn: Each decision node is associated with a specific player who has the choice to act.
2. Available Actions: The edges or branches emanating from the decision node represent the possible actions the player can take.
3. Outcome or Transition: The player's choice determines the next node in the game, which can either be another decision node, a chance node (random event), or a terminal node (end of the game).

Types of Nodes in a Game Tree:

1. Decision Node: Represents a player's decision.
2. Chance Node: Represents random events or probabilistic outcomes. (see this in the Bayesian Game)
3. Terminal Node: Represents the end of the game, where payoffs are determined.

Example

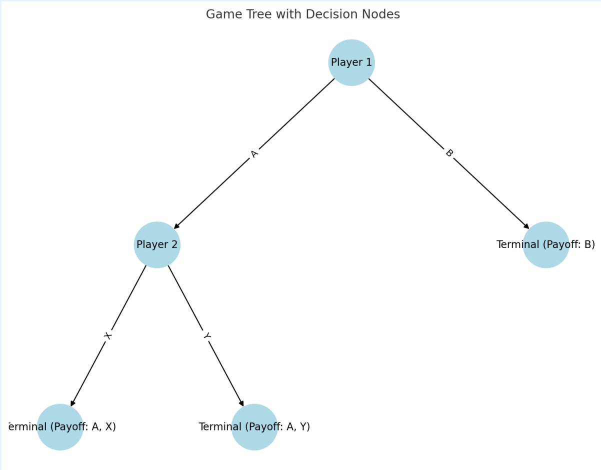

Figure 4: Decision Node

Explanation

Player 1′s Decision Node (1): The first decision node where Player 1 chooses between actions ″A″ and ″B.″
Player 2′s Decision Node (2): The second decision node (reached only if Player 1 chooses ″A″) where Player 2 chooses between actions ″X″ and ″Y.″

Definition Perfect Information
Perfect information in sequential games implies that all **information sets** (see this definition later) consist of **one decision node**.

Definition Imperfect Information
Imperfect information means that at least one information set contains more than two decision nodes.

Definition Information Set
Information set is information available to a player in a situation where the player makes a decision. It includes **all decision nodes that are not distinguishable.**
More formally, an information set for a player is a set of this player's decision nodes such that when the play of the game reaches a node in the information set, this player **does not know** which node in the information set has been **reached**, or a decision node of this player which is not **connected** with any other node by some information set described above.

Explanation
An information set in game theory represents the situation where a player does not have enough information to distinguish between two or more decision nodes. This uncertainty can arise when the player does not know exactly where they are in the game tree due to hidden information or simultaneous moves.

1. Connected Nodes
   An information set groups decision nodes together when the **player cannot distinguish between these nodes** based on the information available.
   When a game reaches any of the nodes in the information set, the player only knows they are in the set, not which specific node.

2. Single Node
   If a decision node is not connected to any other by an information set, it **simply forms an information set by itself**.

In conclusion,
Perfect information If a player **knows all moves** made by the other player(s) up to their turn, their information sets consist of only **single nodes**.
Imperfect information If the player **lacks information** about past moves or the game's state, their information set contains **multiple nodes**.

Theorem

The player must have the **same set of feasible actions** at each decision node in an information set.

Reasons

1. Consistency in Decision-Making: Imagine the player is at an information set where they do not know their exact position (which node they are at). If the possible actions differ between nodes, the player would effectively need to "know" where they are to pick an action, which contradicts the assumption of imperfect information. To avoid this contradiction, the set of feasible actions must be identical at all nodes within the same information set.

2. Logical Soundness:
   The concept of an information set is based on the idea that the player is uncertain about their location in the game. If the available actions differ between nodes, it would violate this uncertainty because the player could infer their position based on the differing options. Requiring the same feasible actions ensures that the player's uncertainty is preserved.

3. Strategy Definition:
   A player's strategy specifies a decision rule for each information set, not individual nodes. If the set of actions varies across nodes in the same information set, it would make it impossible to define a single, coherent strategy for that information set.

3.2.3 Formal Analysis of Sequential Game in Extensive Form

Now we can compare the extensive form game for the simultaneous game and the sequential game.

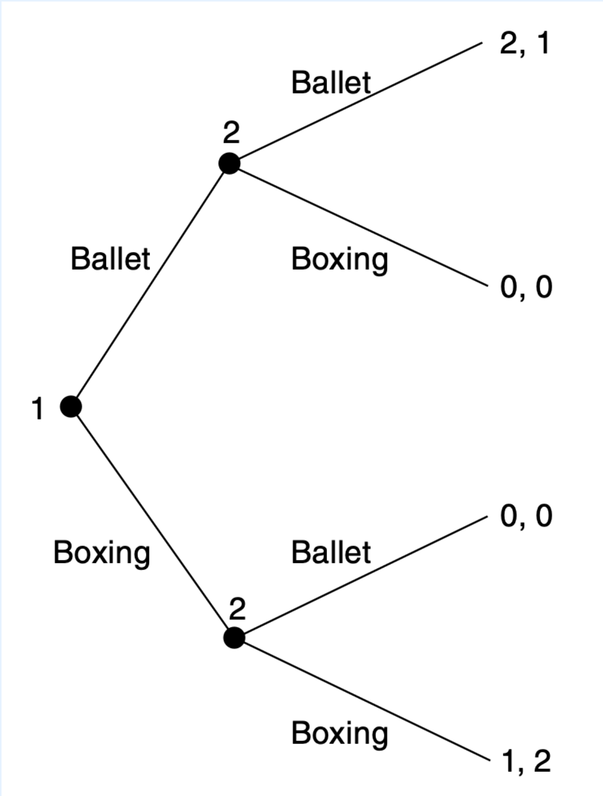

Figure 5: Battles of the Sexes of Sequential Games in the Extensive Form

Graph Description

1. In the sequential version of the Battle of the Sexes, the husband moves second after observing his wife's move. **The husband's decision nodes are not gathered in the same information set**.
   Wife also know what will the final payoff be and the possible movements of husband.
2. There are three **proper subgames**: the game itself and two lower subgames starting with decision nodes where the husband gets to move.

3.2.4 Formal Analysis of Sequential Game in Normal Form

Now we need to explore the sequential version of the Battle of the Sexes in detail.

Example The sequential version of the Battle of the Sexes
Basic Setting
Change the timing of moves: The wife moves first, choosing ballet or boxing;
The husband observes this choice (say, the wife calls him from her chosen location) and then the husband makes his choice.

Normal Form
The wife's possible strategies have not changed: she can choose the simple actions ballet or boxing (or perhaps a mixed strategy involving both actions, although this will not be a relevant consideration in the sequential game).
The husband's set of possible strategies has expanded. For each of the wife's two actions, he can choose one of two actions, so he has four possible strategies.

Let a = Ballet, o = Boxing The set of player 2's strategies is:

$$\{aa, ao, oa, oo\}$$

where:

aa = always ballet
ao = follow player 1
oa = do the opposite
oo = always boxing

Player 1 \2	aa	ao	oa	oo
a	$(2,1)$	$(2,1)$	$(0,0)$	$(0,0)$
o	$(0,0)$	$(1,2)$	$(0,0)$	$(1,2)$

Then we will have three pure-strategy Nash equilibria:

1. (a, aa)
   wife plays ballet, husband plays $(ballet|ballet, ballet|boxing)$
2. (a, ao), (o, ao)
   wife plays ballet, husband plays $(ballet|ballet, boxing|boxing)$
3. (o, oo)
   wife plays boxing, husband plays $(boxing|ballet, boxing|boxing)$

To analyze these three Nash equilibria, we need first to introduce one definition:

Definition Empty Threat
An empty threat is a strategic move or statement made by a player that they would not rationally follow through on because it would not be in their best interest to do so. In other words, the threat is not credible because carrying it out would harm the player making the threat more than it would benefit them.

The opposite definition of empty threat is **Credible Threat**.

Characteristics of an Empty Threat:

1. Non-credible
   The player making the threat lacks an incentive to actually carry it out.
2. Detrimental
   If the player were to carry out the threat, it would lead to worse outcomes for themselves.
3. Strategic Ineffectiveness
   Rational players in the game recognize that the threat is empty and thus do not change their behavior in response to it.

Applying the definition of empty threat, let's analyze the three equilibria in the sequential version of the Battle of the Sexes:

- For the third equilibrium: (o, oo)
  wife plays boxing, husband plays $(boxing|ballet, boxing|boxing)$ (i.e.whatever the wife chooses, husband always chooses boxing)
  The husband's strategy $(boxing|ballet, boxing|boxing)$ involves the implicit threat that he will choose boxing even if his wife chooses ballet.
  - This threat is sufficient to deter her from choosing ballet. Given that she chooses boxing in equilibrium, his strategy earns him 2, which is the best he can do in any outcome. So the outcome is a Nash equilibrium.
  - But the husband's threat is not credible— that is, it is an empty threat. If the wife really were to choose ballet first, then he would be giving up a payoff of 1 by choosing boxing rather than ballet.
  - It is clear why he would want to threaten to choose boxing, but it is not clear that such a threat should be believed.
- For in the first Nash equilibrium: (a, aa)
  The husband's strategy $(ballet|ballet, ballet|boxing)$ (i.e. whateve the wife will choose, husband always chooses ballet)
  - Also involves an empty threat.
- Therefore, the only Nash equilibrium with **credible threat** is the second Nash equilibrium:(a, ao), (o, ao)
  wife plays ballet, husband plays $(ballet|ballet, boxing|boxing)$ (i.e. follow the wife's movement).

From the above analysis, we can see that Nash equilibrium must be **refined** using sequential rationality, since sequential moves mean that players choose their actions, **anticipating best responses of players who will make their decisions afterwards**.
So, how to rule out the Nash equilibria with empty threat? we need to introduce the concept of **subgame perfect equilibrium** and the method of **Backward induction**.

3.3 Subgame

Nash equilibrium is sometimes too general for extensive games:

- Players may be able to anticipate other players' moves.
- Some equilibria may not be self-enforcing.

Game theory offers a formal way of selecting the reasonable Nash equilibria in sequential games using the concept of subgame-perfect equilibrium.
Subgame-perfect equilibrium is a refinement that rules out empty threats by requiring strategies to be rational even for contingencies that do not arise in equilibrium.

Definition Subgame
A subgame is a part of the extensive form beginning with a decision node and including everything that branches out to the right of it.
More mathemetically, Let G represents a sequential game as we defined it before. Then g is a subgame of G if it consists of a single decision node and all of its successors in G.

Properties

1. begins at a singleton information set
2. only includes all nodes following the game tree
   Explanation Once the subgame begins at a particular node, it must include every possible outcome and decision node that could follow logically from that starting point, based on the rules of the game.
3. does not cut any information sets
   Explanation A subgame cannot split an information set into pieces. If a decision node belongs to an information set, all other nodes in that information set must also be included in the subgame.
   $\iff$ If the player cannot distinguish between two nodes in the same information set, both nodes must be included in the subgame, or neither can be included.
   Importance This ensures that the uncertainty structure of the game is preserved. If you cut an information set, the player would suddenly gain "partial" information about their position in the game, which contradicts the original game design.
   Preserving entire information sets ensures that players' strategies remain consistent with their knowledge or lack of knowledge at that point in the game.

Definition Proper Subgame
A proper subgame is a subgame that starts at a decision node **not connected to another in an information set**. Conceptually, this means that the player who moves first in a proper subgame **knows the actions played by others that have led up to that point.**

Example Proper Subgame in the Simultaneous and Sequential Game of Battles of the Sexes in Extension Form

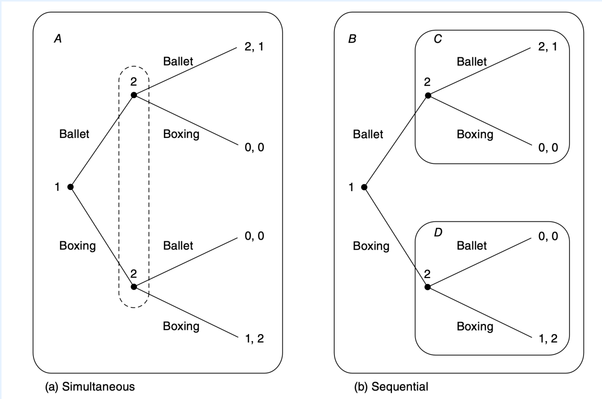

Figure 6: Proper Subgame in Battles of the Sexes in Extension Form

Definition Subgame-perfect equilibrium (SPE)
A subgame-perfect equilibrium is a strategy profile $(s_1^*, s_2^*, \ldots, s_n^*)$ that constitutes a **Nash equilibrium for every proper subgame.**
i.e. Nash equilibrium (NE) that survives after refinement is Subgame Perfect NE (SPNE).
i.e. The solution concept in sequential games of **complete information** is SPNE.

Property Subgame-perfect equilibrium
Subgame-perfect equilibrium **rules out** any sort of **empty threat** in a sequential game.

NOTE Subgame-perfect equilibrium is not a useful refinement for a simultaneous game. This is because a simultaneous game has no proper subgames besides the game itself and so subgame-perfect equilibrium would not reduce the set of Nash equilibria.

Example Subgame-perfect equilibrium in Battles of the Sexes

1. For subgame C
   Beginning with the husband's decision node following his wife's choosing ballet, he has a simple decision between ballet (which earns him a payoff of 1) and boxing (which earns him a payoff of 0).
   The Nash equilibrium in this simple decision subgame is for the husband to choose ballet.
2. For subgame, D
   Husband has a simple decision between ballet, which earns him 0, and boxing, which earns him 2.
   The Nash equilibrium in this simple decision subgame is for him to choose boxing.
3. The husband therefore has only one strategy that can be part of a subgame-perfect equilibrium: $(ballet|ballet, boxing|boxing)$.
   Any other strategy has him playing something that is not a Nash equilibrium for some proper subgame.

Theorem
A subgame perfect equilibrium is always a Nash equilibrium.

Explanation A strategy profile is a Nash equilibrium if no player can unilaterally deviate from their strategy and improve their payoff. This applies to the game as a whole, but it does not necessarily ensure optimality in every subgame of the extensive-form game.
An SPE is a Nash equilibrium that satisfies an additional condition: **the strategies must form a Nash equilibrium in every subgame of the game.** This ensures that the players' strategies are **credible** and **optimal** at every possible decision point, even off the **equilibrium path**.

3.4 Equilibrium Path

Definition Equilibrium Path
Equilibrium path is the connected path through the extensive form implied by equilibrium strategies.

Example Equilibrium Path in the Battle of the Sexes

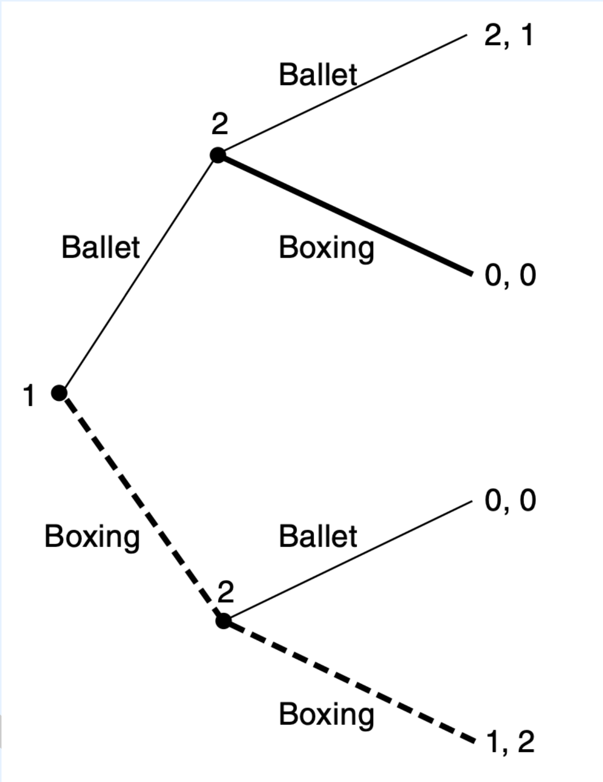

Figure 7: Equilibrium Path

3.5 Backward Induction

Definition Backward Induction
The process of solving for equilibrium by working backwards from the end of the game to the beginning.
Backward induction is particularly useful in games that feature multiple rounds of sequential play.

Method Steps

1. Identify all subgames at the bottom of the extensive form.
2. Find Nash equilibria for these subgames.
3. Replace these subgames with the actions and payo§s played in Nash equilibrium.
4. Move up to the next level of subgames and repeat (1) and (2).

Example Sequential Game with Normal Form, with Extensive Form, showing the Subgame, and Using Backward Induction to Find Subgame Perfect Nash Equilibrium

Background Setting

Timing of the game

1. Period 1.
   P1 chooses between A and B
2. Period 2.
   P2 observes the decision of P1 and decides what action to take: C or D after A; E or F after B.
   The game ends after C or D have been taken. Otherwise, Period 3 starts.
3. Period 3.
   P1 observes the decision of P2 and chooses between G and H.
   The game ends, and players obtain their **payoffs** (given in **terminal nodes**).

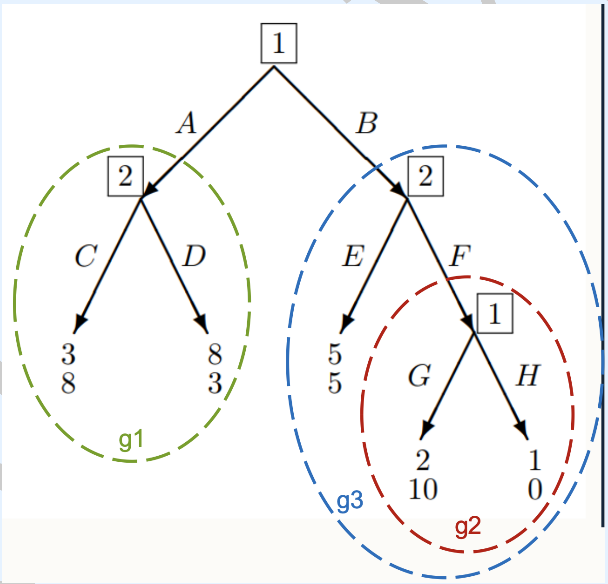

Figure 8: Sequential Game in Extensive Form

Pure Strategy Example

1. pure strategy BG
   P1 chooses action B in Period 1 and chooses action G in Period 3
2. A pure strategy AG
   P1 chooses action A in Period 1 but would have chosen action G if his second decision node had been reached.

Normal Form

	CE	CF	DE	DF
AG	$(3,8)$	$(3,8)$	$(8,3)$	$(8,3)$
AH	$(3,8)$	$(3,8)$	$(8,3)$	$(8,3)$
BG	$(5,5)$	$(2,10)$	$(5,5)$	$(2,10)$
BH	$(5,5)$	$(1,0)$	$(5,5)$	$(1,0)$

Backward Induction

1. Period 3 $\Longleftrightarrow$ subgame g2
   P1 chooses G
2. Period 2 $\Longleftrightarrow$ subgame g1, g3
   - In g1, P2 chooses C.
   - In g3, P2 chooses F.
3. Period 1 $\Longleftrightarrow$ game G
   P1 rationally anticipates what will happen in each subgame and chooses A.
4. Only one equilibrium:
$$SPNE \ni \{AG, CF\} \to (3,8)$$

3.6 Behavioural Strategies

Definition Behavioural Strategies
A behavioural strategy is a strategy that contains **mixed actions**.

Example

Timing of the game

1. Period 1.
   P1 chooses between A and B.
2. Period 2.
   P2 observes the decision of P1:
   (a) If P1 played A, P2 chooses between C and D
   The game ends.
   (b) If P1 played B, P1 and P2 play a **simultaneous game**, where P1 has two actions – E and F
   P2 chooses between actions G and H.

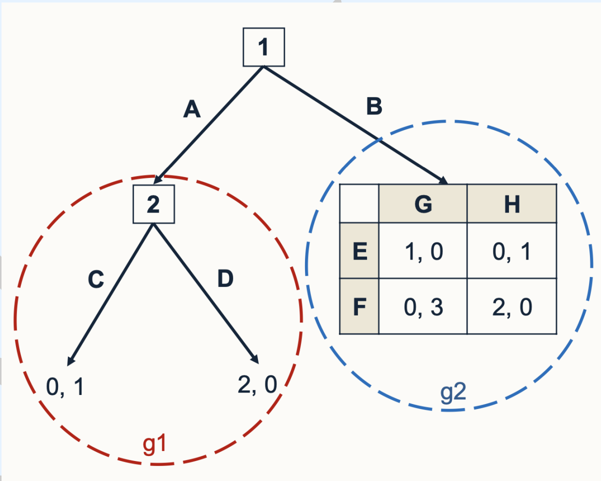

Figure 9: Behavioural Strategy: Combination of Sequential Game, Pure Strategy, Simultaneous Game, and Mixed Strategy

Subgame and Backward Induction to find SPNE

1. Period 2 $\iff$ Subgame g1, g2
   (a) For g1
   P2 chooses C.
   (b) For g2, the simultaneous game:
   - **no** Nash equilibrium in **pure strategies.**
   - one Nash equilibrium in **mixed strategies**:

$$Eu_1(E) = 1 \times q + 0 \times (1-q) = q$$
$$Eu_1(F) = 0 \times q + 2 \times (1-q) = 2 - 2q$$
$$q = 2 - 2q$$
$$3q = 3$$
$$q^* = \frac{2}{3}, \quad Eu_1^* = \frac{2}{3}$$
$$Eu_2(G) = 0 \times p + 3 \times (1-p) = 3 - 3p$$
$$Eu_2(H) = 1 \times p + 0 \times (1-p) = p$$
$$3 - 3p = p$$
$$4p = 3$$
$$p^* = \frac{3}{4}, \quad Eu_2^* = \frac{3}{4}$$

P1 choose mixed strategy $(2/3, 1/3)$
P2 choose mixed strategy $(1/4, 3/4)$

2. Period 1 $\Longleftrightarrow$ Game G
   P1 chooses action B.

3. Subgame Perfect Nash Equilibrium

$$SPNE \ni \{B|\frac{3}{4}E + \frac{1}{4}F; C|\frac{2}{3}G + \frac{1}{3}H\} \rightarrow (\frac{3}{4}, \frac{2}{3})$$

3.7 Imperfect competition – Sequential Games of Complete Information with Continuous Payoffs

Definition Oligopoly
A market with relatively few firms but more than one.
Oligopolies raise the possibility of **strategic interaction** among firms (game-theoretic analysis).

3.7.1 Bertrand Model

Basic Model Setting

1. Two identical firms, labeled 1 and 2
2. Producing identical products
   (a) Since firms' products are perfect substitutes, all sales go to the firm with the lowest price.
   (b) Sales are split evenly if $p_1 = p_2$.
3. A constant marginal cost (and constant average cost) c
4. The firms choose prices p_1 and p_2 simultaneously in a single period of competition.
5. $D(p)$ market demand
6. The game has a continuum of actions, but the profit functions are not differentiable here.
   Example

$$\pi_i(p_i, p_j) = \begin{cases} (p_i - c) \cdot D(p_i) & \text{if } p_i < p_j \, \forall j \neq i \\ 0 & \text{if } p_i > \min(p_j) \, \forall j \neq i \end{cases}$$

The only pure-strategy Nash equilibrium of the Bertrand game is $p_1^* = p_2^* = c$ (both firms charging marginal cost).
Therefore, there are two statements that need to be verified:

1. this outcome is a Nash equilibrium.
2. there is no other Nash equilibrium.

Proof
Statement 1: This outcome is a Nash equilibrium.
$\Leftrightarrow$ Show that both firms are playing a best response to each other.
$\Leftrightarrow$ Neither firm has an incentive to deviate to some other strategy.
In equilibrium, firms charge a price equal to marginal cost, which in turn is equal to average cost.
$\Rightarrow$ a price equal to average cost means firms earn zero profit in equilibrium.

- If it deviates to a higher price then it will make no sales and therefore no profit, not strictly more than in equilibrium.
- If it deviates to a lower price, then it will make sales but will be earning a negative margin on each unit sold, since price would be below marginal cost. So the firm would earn negative profit, less than in equilibrium.

Because there is no possible profitable deviation for the firm, we have succeeded in verifying that both firms' charging marginal cost is a Nash equilibrium.

Statement 2: There is no other Nash equilibrium.
Assume firm 1 is the low-price firm, $p_1 \leq p_2$ (The same conclusions would be reached taking 2 to be the low-price firm).

1. $c > p_1$
   Firm 1 earns a negative margin $p_1 - c$ on every unit it sells and, since it makes positive sales, it must earn negative profit. It could earn higher profit by deviating to a higher price.
   Example
   zero profit by deviating to $p_1 = c$.

2. $c < p_1$
   (a) $p_1 = p_2$ At best firm 2 gets only half of market demand
   (b) $p_1 < p_2$ At worst gets no demand. Firm 2 could capture all of market demand by undercutting firm 1's price by a tiny amount ε. This ε could be chosen small enough that market price and total market profit are hardly affected.
   If $p_1 = p_2$ prior to the deviation, the deviation would essentially double firm 2's profit.

3. $c = p_1$
   (a) $c = p_1 < p_2$
   Firm 1 earns zero profit here but could earn positive profit by deviating to a price slightly above c but still below p_2.
   (b) $c = p_1 = p_2$ Nash Equilibrium.

Generalized, the Nash equilibrium of the n-firm Bertrand game is:

$$p_1^* = p_2^* = ... = p_n^* = c$$

Definition Bertrand Paradox
The Nash equilibrium of the Bertrand model is the same as the perfectly competitive outcome. The Bertrand paradox is a general result in the sense that we did not specify the **marginal cost c** or the demand curve, so the result holds for any c and **any downward-sloping demand curve**.

Each of the next several sections will present a different model generated by changing a different one of the Bertrand assumptions.

- Assume that firms choose quantity rather than price, leading to what is called the Cournot game.
- If firms face capacity constraints rather than being able to produce an unlimited amount at cost c.
- If products are slightly differentiated rather than being perfect substitutes.
- If firms engage in repeated interaction rather than one round of competition.

3.7.2 Cournot Model

Firms are assumed to simultaneously choose quantities.
Price will be **above marginal** cost and firms will earn **positive profit** in the Nash equilibrium of the Cournot game. **Basic Model Setting**

1. n firms $i = 1, ..., n$
2. Each firm chooses its output q_i
3. An identical product
4. Simultaneous game
5. The outputs are combined into a total industry output:

   $$Q = q_1 + q_2 + ... + q_n$$

   resulting in:
6. market price $P(Q)$
7. Observe that $P(Q)$ is the inverse demand curve corresponding to the market demand curve $Q = D(p)$.
8. Assume market demand is **downward sloping**. i.e.$P'(Q) < 0$
9. Firm i's profit function is differentiable:

   $$\pi_i = P(Q)q_i - C_i(q_i)$$

   Total revenue $P(Q)q_i$.
   Total cost $C_i(q_i)$.
10. Find **each firm i's best response** by taking the **F.O.C** of the profit function with respect to q_i:

    $$\frac{\partial \pi_i}{\partial q_i} = \underbrace{P(Q) + P'(Q)q_i}_{MR} - \underbrace{C_i'(q_i)}_{MC} = 0.$$

    must hold for all $i = 1, ..., n$ in the Nash equilibrium.

Example

Cournot Model Nash Equilibrium

Following Cournot's nineteenth-century example of two natural springs, we assume that each spring owner has a large supply of (possibly healthful) water and faces the problem of how much to provide the market.
A firm's cost of pumping and bottling q_i liters is

$$C_i(q_i) = cq_i$$

, implying that **marginal costs are a constant c** per liter. Inverse demand for spring water is

$$P(Q) = a - Q$$

Total spring-water output

$$Q = q_1 + q_2$$

Profits for the two Cournot firms are

$$\pi_1 = P(Q)q_1 - cq_1 = (a - q_1 - q_2 - c)q_1,$$

$$\pi_2 = P(Q)q_2 - cq_2 = (a - q_1 - q_2 - c)q_2.$$

Using the first-order conditions to solve for the best-response functions, we obtain

$$q_1 = \frac{a - q_2 - c}{2}, \quad q_2 = \frac{a - q_1 - c}{2}.$$

Solving these equations simultaneously yields the Nash equilibrium

$$q_1^* = q_2^* = \frac{a - c}{3}.$$

Total output $Q^* = \frac{2}{3}(a - c)$.
Substituting total output into the inverse demand curve implies an equilibrium price of

$$P^* = \frac{a + 2c}{3}$$

. Substituting price and outputs into the profit functions implies

$$\pi_1^* = \pi_2^* = \frac{1}{9}(a - c)^2$$

Total market profit equals

$$\Pi^* = \pi_1^* + \pi_2^* = \frac{2}{9}(a - c)^2.$$

Best-Response Diagram for Cournot Duopoly
Graph the best-response functions
The **intersection** between the best responses is the **Nash equilibrium**.
Isoprofit curves for firm 1 increase until point M is reached, which is the monopoly outcome for firm 1.

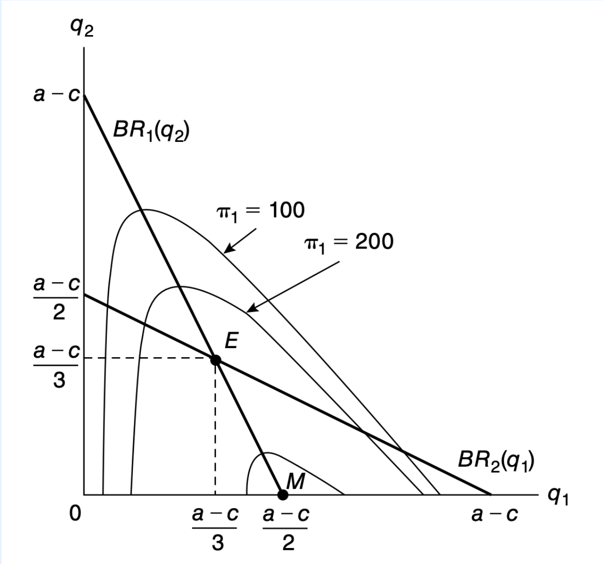

Figure 10: Best-Response for Cournot Duopoly

3.7.3 Product Differentiation

Bertrand competition with product differentiation.
With identical products, demanders were assumed to be indifferent about which firm's output they bought; hence they shop at the lowest-price firm, leading to the law of one price. The law of one price no longer holds if demanders strictly prefer one supplier to another at equal prices.

Basic Model Setting

1. n firms
2. Simultaneously choose
3. Prices $p_i(i = 1, ...n)$
4. Differentiated products
5. Product i has its own specific attributes a_i
   A product may be endowed with the attribute or the attribute may be the result of the firm's choice and spending level. The various attributes serve to differentiate the products.
   Example
   special options; quality; brand advertising; location
6. Firm i's demand:
   $$q_i(p_i, P_{-i}, a_i, A_{-i})$$
   - P_{-i} a list of all other firms' prices besides i's.
   - A_{-i} a list of all other firms' attributes besides i's.
7. Firm i's profit function is differentiable:
   $$\pi_i = p_i q_i - C_i(q_i, a_i)$$
8. Solving for best-response functions by taking each firm's **F.O.C**.
   $$\frac{\partial \pi_i}{\partial p_i} = \underbrace{q_i + p_i \frac{\partial q_i}{\partial p_i}}_{A} - \underbrace{\frac{\partial C_i}{\partial q_i} \cdot \frac{\partial q_i}{\partial p_i}}_{B} = 0.$$
   - A: marginal revenue from an increase in price
     The increase in price increases revenue on existing sales of q_i units, but we must also consider the negative effect of the reduction in sales ($\frac{\partial q_i}{\partial p_i}$ in $p_i \frac{\partial q_i}{\partial p_i}$).
   - B: the cost savings associated with the reduced sales that accompany an increased price.
9. The **Nash equilibrium** can be found by simultaneously solving the system of **F.O.C**. above for all $i = 1, ..., n$.
10. If the attributes a_i are also choice variables (rather than just endowments), there will be another set of **F.O.C** to consider.
    For firm i, the **F.O.C** with respect to a_i:
    $$\frac{\partial \pi_i}{\partial a_i} = p_i \frac{\partial q_i}{\partial a_i} - \frac{\partial C_i}{\partial a_i} - \frac{\partial C_i}{\partial q_i} \cdot \frac{\partial q_i}{\partial a_i} = 0.$$

Example

Differentiated Product

Suppose two firms produce toothpaste, one a green gel and the other a white paste.
Suppose for simplicity that production is **costless**.
Demand for product i is

$$q_i = a_i - p_i + \frac{p_j}{2}.$$

The positive coefficient on p_j, the other good's price, indicates that the goods are **gross substitutes**.
Firm i's demand is increasing in the attribute a_i, which we will take to be demanders' inherent preference for the variety in question; we will suppose that this is an **endowment** rather than a choice variable for the firm.

Firm i's profit is

$$\pi_i = p_i q_i - C_i(q_i) = p_i \left(a_i - p_i + \frac{p_j}{2} \right),$$

$C_i(q_i) = 0$ because i's production is costless.
F.O.C for profit maximization with respect to p_i is

$$\frac{\partial \pi_i}{\partial p_i} = a_i - 2p_i + \frac{p_j}{2} = 0.$$

Solving for p_i gives the following best-response functions for $i = 1, 2$:

$$p_1 = \frac{1}{2}\left(a_1 + \frac{p_2}{2} \right), \quad p_2 = \frac{1}{2}\left(a_2 + \frac{p_1}{2} \right).$$

Nash equilibrium prices:

$$p_i^* = \frac{8}{15} a_i + \frac{2}{15} a_j.$$

The associated profits are

$$\pi_i^* = \left(\frac{8}{15} a_i + \frac{2}{15} a_j \right)^2.$$

Interpretation

Firm i's equilibrium price is not only increasing in its own attribute, ai, but also in the other product's attribute, aj . An increase in aj causes firm j to increase its price, which increases firm i's demand and thus the price i charges.

Use graph to illustrate the Bertrand Model with Differentiated Products:

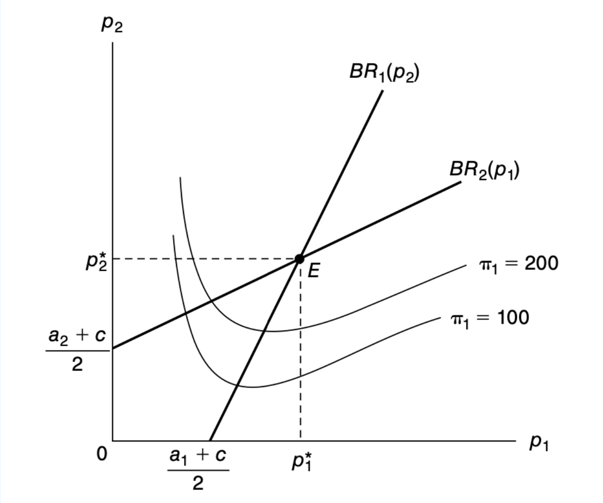

Figure 11: Best Responses for Bertrand Model with Differentiated Products

Definition Strategic Complements
Two strategies x_1 and x_2 are **strategic complements** if the best response function of one player is **increasing** in the other player's strategy. Mathematically, for player i's best response x_i as a function of the other player's strategy x_j:

$$\frac{\partial BR_i(x_j)}{\partial x_j} > 0.$$

- BR_i: Best response function of player i.
- A positive derivative means that when one player **increases** their strategy, the other player's optimal response is to also **increase** theirs.

Example
Prices are strategic complements: best response functions are upward-slopping.
Explanation
When one firm raises its price, the other firm optimally increases its price because demand shifts toward its product.

3.7.4 Stackelberg model

The model is similar to a duopoly version of the Cournot model
Firms move **sequentially**, with firm 1 (the leader) choosing its output first and then firm 2 (the follower) choosing after observing firm 1's output.
We use **backward induction** to solve for the subgame-perfect equilibrium of this sequential game.

Steps

1. Begin with the follower's output choice.
   Firm 2 chooses the output q_2 that maximizes its own profit, **taking firm 1's output** q_1 **as given.** $\Leftrightarrow$ firm 2 best-responds to firm 1's output.
   $\Rightarrow \quad BR_2(q_1)$ Best-response function for firm 2

2. Then the leader's output choice.
   Firm i's profit function is differentiable as in the Cournot Model:

$$\pi_i = P(Q)q_i - C_i(q_i)$$

   Firm 1 recognizes that it can influence the follower's action because the follower best-responds to 1's observed output, so $Q = q_1 + \mathbf{BR}_2(q_1)$

$$\pi_1 = P\left(q_1 + \mathbf{BR}_2(q_1)\right) q_1 - C_1(q_1).$$

   The **F.O.C** with respect to q_1:

$$\frac{\partial \pi_1}{\partial q_1} = P(Q) + P'(Q)q_1 + \underbrace{P'(Q)\mathbf{BR}_2'(q_1)q_1}_{S} - C_i'(q_i) = 0.$$

   - S: accounts for the **strategic effect** of firm 1's output on firm 2's.

   Explanation
   Recall in Cournot Model, **F.O.C**:

$$\frac{\partial \pi_i}{\partial q_i} = \underbrace{P(Q) + P'(Q)q_i}_{MR} - \underbrace{C_i'(q_i)}_{MC} = 0.$$

   The strategic effect S will lead firm 1 to produce more than it would have in a Cournot model.

   Due to the downward sloping demand curve, $P'(Q) < 0$
   Due to the definition of Strategic Substitutes (see below), $BR_2'(q_1) < 0$
   So The additional term $P'(Q)BR_2'(q_1)q_1$ (strategic effect S) **reduces the marginal cost** of increasing q_1 because it indirectly shrinks Firm 2's response.

   By **overproducing**, firm 1 leads firm 2 to reduce q_2 by the amount $\mathbf{BR}_2'(q_1)$;
   The fall in 2's output **increases market price**, thus **increasing the revenue that 1 earns** on its existing sales.
   q_2 falls with an increase in q_1 because best-response functions under quantity competition are generally downward sloping;

Definition Strategic Substitutes
Two strategies x_1 and x_2 are **strategic substitutes** if the best response function of one player is **decreasing** in the other player's strategy. Mathematically, for player i's best response x_i as a function of the other player's strategy x_j:

$$\frac{\partial BR_i(x_j)}{\partial x_j} < 0.$$

- BR_i: Best response function of player i.
- A negative derivative means that when one player **increases** their strategy, the other player's optimal response is to **decrease** theirs.

Example
Quantities are strategic substitutes: best response functions are downward-slopping.
Explanation
Firms compete by choosing output quantities. If Firm 1 increases its output q_1, the market price decreases, making it optimal for Firm 2 to reduce its output q_2.

Example
Stackelberg Model Nash Equilibrium
Recall all the basic setting in the Cournot Model:
We assume that each spring owner has a large supply of (possibly healthful) water and faces the problem of how much to provide the market.
A firm's cost of pumping and bottling q_i liters is

$$C_i(q_i) = cq_i$$

, implying that **marginal costs are a constant c** per liter. Inverse demand for spring water is

$$P(Q) = a - Q$$

Total spring-water output

$$Q = q_1 + q_2$$

Now, assume that they choose outputs sequentially as in the Stackelberg game, with firm 1 being the leader and firm 2 the follower.

Firm 2's output. We will solve for the subgame-perfect equilibrium using backward induction, starting with firm 2's output choice.
Profits for firm 2 is:

$$\pi_2 = P(Q)q_2 - cq_2 = (a - q_1 - q_2 - c)q_2.$$

By **F.O.C**, we obtain the same best-response function of firm 2:

$$q_2 = \frac{a - q_1 - c}{2}.$$

Firm 1's output. Now fold the game back to solve for firm 1's output choice. **Substituting 2's best response into 1's profit function**:

$$\pi_1 = \left[a - q_1 - \left(\frac{a - q_1 - c}{2}\right) - c\right] q_1 = \frac{1}{2}(a - q_1 - c)q_1.$$

Taking the first-order condition,

$$\frac{\partial \pi_1}{\partial q_1} = \frac{1}{2}(a - 2q_1 - c) = 0$$

Solving gives

$$q_1^* = \frac{(a-c)}{2}$$

. Substituting q_1^* back into firm 2's best-response function gives

$$q_2^* = \frac{(a-c)}{4}$$

. Profits are $\pi_1^* = \frac{1}{8}(a-c)^2$ and $\pi_2^* = \frac{1}{16}(a-c)^2$.

Draw the graph of Stackelberg Game:

The leader realizes that the follower will always best-respond, so the resulting **outcome will always be on the follower's best-response function**.
⇔ The leader effectively picks the point on the follower's best-response function that maximizes the leader's profit.
Point C is the Nash equilibrium of the Cournot game (involving simultaneous output choices).
The highest isoprofit is reached at the point S of tangency between firm 1's isoprofit and firm 2's best-response function. This is the **Stackelberg equilibrium, point S.**

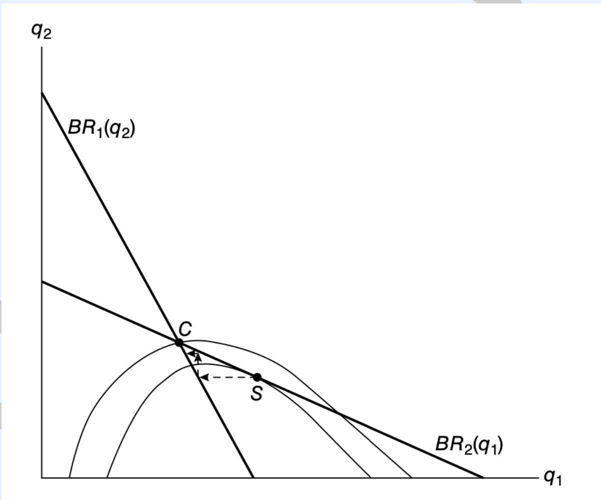

Figure 12: Stackelberg Game

Compared to the Cournot equilibrium at point C, the Stackelberg equilibrium involves higher output and profit for firm 1.
In other words, firm 1 has the first mover advantage.

Definition First-mover Advantage
The first-mover advantage arises when the leader firm can commit to an output level before the follower. By doing so, the leader anticipates the follower's best response and adjusts its output strategically to maximize its own profit.

Crucial to our analysis of *longer-run decisions* such as investment, entry, and exit is *how easy it is to reverse a decision once it has been made.*

1. It is better for a firm to be able to **easily reverse decisions**, since this would give the firm more flexibility in responding to changing circumstances.
2. Absent strategic considerations—and so for the case of a **monopolist**—a firm would always **value flexibility and reversibility.**
3. The strategic considerations that arise in an **oligopoly** setting may lead a firm to prefer its decision be **irreversible.**

Analysis for oligopoly case
What the firm loses in terms of flexibility may be offset by the value of being able to commit to the decision. If a firm can **commit to an action** before others move, the firm may gain a **first-mover advantage**. A firm may use its first-mover advantage to stake out a claim to a market by making a commitment to serve it and in the process limit the kinds of actions its rivals find profitable.
Commitment is essential for a first- mover advantage.

In the sequential version of the game, if a player were given the choice between being the first mover and having the ability to commit to attending an event or being the second mover and having the flexibility to be able to meet up with the first wherever he or she showed up, **a player would always choose the ability to commit.**
The first mover can guarantee his or her **preferred outcome as the unique subgame-perfect equilibrium** by **committing** to attend his or her favorite event.

3.7.5 Price Leadership

Before analyze this model, we need to introduce some definitions:

Definition Top Dog Strategy
The Top Dog Strategy is a concept in strategic firm behavior, particularly in models of oligopoly competition where firms **compete in quantities** simultaneously (e.g., Cournot) or sequentially (e.g., Stackelberg).
The Top Dog Strategy refers to a **predatory or aggressive strategy** where a firm **overproduces** (increasing its output) to **deliberately crowd the market**, thereby **reducing its rival's output and profits.**
This strategy is often employed by a firm with **first-mover power** (like a Stackelberg leader) that can commit to its output first to exploit its position and gain a competitive advantage.

Definition Second-mover Advantage
The second-mover advantage refers to the benefit a firm (the **follower**) gains by **moving second in a sequential game**, such as in the Stackelberg model. By observing the decision of the first mover (leader), **the second mover can optimize its own strategy** and respond in a way that maximizes its profit, given the leader's choice.

Definition Puppy Dog Strategy
The Puppy Dog Strategy is a concept in strategic competition where a firm **behaves non-aggressively** or appears weak and unthreatening to avoid provoking competitive responses from rivals. This strategy is often used to prevent rivals from adopting aggressive countermeasures, such as price wars or increased production.

1. Non-aggressive Behavior:
   The firm does not aggressively expand output, cut prices, or act like a market leader. It may:
   - Keep production or growth levels low.
   - Avoid signaling market dominance.
2. Avoiding Retaliation:
   By appearing weak or passive, the firm avoids provoking a strong competitive reaction, such as:
   - Price cuts (in price wars).
   - Overproduction or aggressive market entry by competitors.
3. **Long-term Strategy**:
   The firm sacrifices short-term profits or growth for a long-term payoff, often when it has a hidden strength or intends to expand quietly later.

Example

Price-Leadership Game

Recall in the Bertrand Model with Product Differentiation:
Suppose two firms produce toothpaste, one a green gel and the other a white paste.
Suppose for simplicity that production is **costless**.
Demand for product i is

$$q_i = a_i - p_i + \frac{p_j}{2}.$$

Firm i's profit is

$$\pi_i = p_i q_i - C_i(q_i) = p_i \left(a_i - p_i + \frac{p_j}{2} \right),$$

$C_i(q_i) = 0$ because i's production is costless.

Firm 1 is the price leader Now consider the game in which firm 1 chooses price before firm 2.

Solution We will solve for the subgame-perfect equilibrium using **backward induction**, starting with firm 2's move.

1. Firm 2's best response to its rival's choice p_1: recall the function **F.O.C** for profit maximization with respect to p_i is

$$\frac{\partial \pi_i}{\partial p_i} = a_i - 2p_i + \frac{p_j}{2} = 0.$$

   Firm 2's best-response function is:

$$p_2 = \frac{1}{2}\left(a_2 + \frac{p_1}{2} \right).$$

2. Fold the game back to firm 1's move. Substituting firm 2's best response into firm 1's profit function:

$$\pi_1 = p_1 q_1 - C_1(q_1) = p_1 \left(a_1 - p_1 + \frac{p_2}{2} \right) = p_1 \left(a_1 - p_1 + \frac{1}{2}\left(a_2 + \frac{p_1}{2} \right) \right)$$

   Taking the first-order condition and solving for the equilibrium price, Both firms' prices and profits are higher in this sequential game than in the 2 simultaneous one.

Now the **follower earns even more than the leader**.

Draw the graph for price leadership model.
Point B is the Nash equilibrium of the simultaneous game.
Point L is the subgame-perfect equilibrium of the sequential game in which firm 1 moves first. At L, 1's isoprofit is tangent to 2's best response.

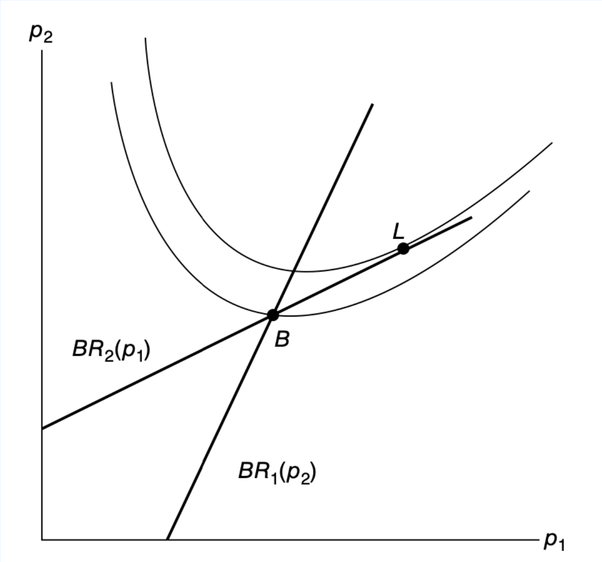

Figure 13: Price Leadership Game

The first mover is playing a **"puppy dog" strategy** because it **increases its price** relative to the simultaneous-move game; when translated into outputs, this means that the **first mover ends up producing less** than in the simultaneous-move game. It is as if the first mover strikes a less aggressive posture in the market and so **leads its rival to compete less aggressively.**

The crucial difference between the games that leads the first mover to play a **"top dog" strategy in the quantity game** and a **"puppy dog" strategy in the price game**:
the best-response functions have different slopes.
The goal is to induce the follower to compete less aggressively. The slopes of the best-response functions determine whether the leader can best do that by playing aggressively itself or by softening its strategy.

1. The first mover plays a "top dog" strategy in the sequential quantity game or, indeed, any game in which best responses slope down. When best responses slope down, playing more aggressively induces a rival to respond by competing less aggressively.

2. The first mover plays a "puppy dog" strategy in the price game or any game in which best responses slope up. When best responses slope up, playing less aggressively induces a rival to respond by competing less aggressively.

4 Repeated Game

Definition Repeated Game
A repeated game is a dynamic game in which the same static game (so called the "stage game") is played in every period.

Definition Stage Game
The simple constituent game that is played repeatedly is called the stage game.

Definition Trigger Strategies
Repeated play of the stage game opens up the possibility of **cooperation** in equilibrium. Players can adopt trigger strategies, whereby they continue to cooperate as long as all have cooperated up to that point but **revert to playing the Nash equilibrium if anyone deviates from cooperation.** And this deviation triggers some sort of **punishment**. In order for trigger strategies to form an equilibrium, the punishment must be **severe** enough to **deter deviation**.

Repeated games usually involve coordination or cooperation. If coordination or cooperation cannot be supported in a static base game, players can achieve a **socially efficient** equilibrium (the total surplus is the highest) in a repeated version of the base game. The outcome of a repeated game depends on the time horizon: finite or infinite T.

Definition Coordination Game
A coordination game is a type of game in game theory where players benefit by **making the same choices** or **aligning their strategies**. The goal is to achieve a **mutually beneficial equilibrium**.
Key Features:

1. **Multiple Equilibria**: These games often have multiple Nash equilibria, some of which may be more favorable than others (e.g., Pareto efficient).
2. **No Conflict of Interest**: All players want to **coordinate on the same strategy** to maximize their payoffs.
3. **Payoff Structure**: The **highest payoff** occurs when all players **choose the same or compatible strategies**.

Example Battle of the Sexes
Player 1 is wife, Player 2 is husband.

	Ballet	**Boxing**
Ballet	$(2, 1)$	$(0, 0)$
Boxing	$(0, 0)$	$(1, 2)$

Definition Coorperation Game
A cooperation game is a game where players can achieve higher collective payoffs by **collaborating** rather than acting independently or competitively.
Key Features:

1. **Potential for Conflict**: Players may face a **tension** between maximizing their **individual payoff** and the **collective payoff.**

2. **Enforcement Issues**: Cooperation often requires mechanisms to ensure compliance (e.g., trust, contracts, or repeated interactions).

Example Prisoner's Dilemma

	L_2	R_2
L_1	$1, 1$	$5, 0$
R_1	$0, 5$	$4, 4$

Now we will focus on **subgame-perfect equilibria** of the repeated games.
If the stage game has multiple Nash equilibria or the horizon is infinite, the repeated game has many **non-trivial subgame perfect equilibria**, such as using the trigger strategy.

4.1 Finitely Repeated Game

Theorem Selten's Theorem
If the stage game has a **unique Nash equilibrium**, then the **unique subgame-perfect equilibrium** of the finitely repeated game is to **play the Nash equilibrium every period.**
Every player plays the strategy in the unique Nash equilibrium for the stage game in every stage regardless of the outcome of preceding stages.

If the stage game has **multiple Nash equilibria**, it may be possible to **achieve some cooperation in a finitely repeated game**. Players can use trigger strategies, sustaining cooperation in early periods on an outcome that is not an equilibrium of the stage game, by threatening to play in later periods the Nash equilibrium that yields a worse outcome for the player who deviates from cooperation.

Game Setting
Given a stage game G

- **G (T)** The finitely repeated game in which G is played T times ($T < \infty$)
- The outcomes (**actions** taken and **payoffs** received) of all preceding plays observed before the next play begins (perfect monitoring).
- **Payoff for G (T)** the sum of the payoffs attained in all periods.
- A player's strategy specifies the action the player will take at each information set of this player (i.e., each possible history of play through the previous stages in each stage).
- A subgame beginning at stage t + 1 is the repeated game in which G is played T - t times (i.e., G (T - t)).
  There are many subgames that begin at stage t + 1, one for each possible history of play through stages 1 to t.
- **Subgame perfect (Nash) equilibrium** A strategy profile is a subgame perfect (Nash) equilibrium if the players' strategies constitute a Nash equilibrium in every subgame.

Example
Prisoners' Dilemma with unique Nash Equilibrium
Suppose the Prisoners' Dilemma were repeated for 2 periods.

	$L_2(cooperate)$	$R_2(defect)$
$L_1(cooperate)$	$1,1$	$5,0$
$R_1(defect)$	$0,5$	$4,4$

In a static base game, players never cooperate.
Now it repeated for **2 periods.**
Solve it using backward induction:

1. *Round 2* Both defect and the corresponding payoffs are (1,1)
2. *Round 1* Players rationally anticipate the outcome of the last round and **calculate the total payoffs** for the game.

Draw the game tree first, there are 5 information set for player 1, player 1's strategy in each information set is L_1, so for the whole 5 information set, player1's strategy is: $L_1L_1L_1L_1L_1$.
We can derive that in stage2, player 1 always play L_1.
After examination, we find that player 2 will always choose L_2
The unique subgame perfect equilibrium: play L_i in stage 1; play is L_i in stage 2 regardless of the outcome of stage 1.

Suppose the Prisoners' Dilemma were repeated for **T periods**.
Use backward induction to solve for the subgame-perfect equilibrium.

1. The lowest subgame is the Prisoners' Dilemma stage game played in period T :
   Regardless of what happened before, the Nash equilibrium on this subgame is for both to fink.

2. It is as if period T - 1 were the last, and the Nash equilibrium of this subgame is again for both to fink.
   Folding the game back to period T - 1, trigger strategies that condition period-T play on what happens in period T - 1 are ruled out. Although a player might like to promise to play cooperatively in period T and so reward the other for playing cooperatively in period T - 1, we have just seen that nothing that happens in period T - 1 affects what happens subsequently because players both fink in period T regardless.

3. Working backward in this way, we see that players will fink each period; that is, players will simply repeat the Nash equilibrium of the stage game T times.

Now we try to discuss the stage game has **multiple Nash equilibria**, which can **achieve some cooperation in a finitely repeated game**.
For cooperation to be sustained in a subgame-perfect equilibrium, the stage game must be repeated often enough that the **punishment** for deviation (repeatedly playing the less-preferred Nash equilibrium) is **severe enough to deter deviation**. The more repetitions of the stage game T , the more severe the possible punishment and thus the greater the level of cooperation and the higher the payoffs that can be sustained in a subgame-perfect equilibrium.

Theorem
If the normal-form base game G has **multiple NE**, one can find a **SPNE** where, at some round, players play a strategy that is **not part of a NE** of the base game G.

Firstly, we need to introduce a concept that we did not discussed in detail earlier, but we have met it (the opposite of empty threat).

Definition Credible Threat
A credible threat is a strategy or punishment that a player has a rational incentive to carry out if another player deviates. For a threat to be credible:

1. Rationality:
   - It must be in the **best interest of the threatening player** to follow through with the threat, given the actions of the other players.
   - The threatening player will **actually carry** it out because it **maximizes their utility** (payoff).

2. Subgame Perfection:
   - The threat must form part of a **subgame perfect Nash equilibrium (SPNE)**.
   - This means that the **threat remains optimal at every decision point (subgame)** of the game.

We will also need to recall the definition of pareto efficient later.

Example

Prisoners' Dilemma with two Nash Equilibria

The stage game is repeated twice.

	A	B	C
A	$(4,4)$	$(0,5)$	$(0,0)$
B	$(5,0)$	$(1,1)$	$(0,0)$
C	$(0,0)$	$(0,0)$	$(3,3)$

A more cooperative outcome can be sustained with the following strategy:
begin by playing A in the first period;
if no one deviates from A, play C in the second period;
if a player deviates from A, then play B in the second period.

Analysis

Since now this is not the one-shot time game, players need to use the payoff in stage two to help keep they gain the maximum utility together(cooperate) in the stage one.

"if no one deviates from A, play C in the second period"
Firstly, recall the definition of Pareto Efficient:
In a normal form game, an outcome (strategy profile) is Pareto efficient (or Pareto optimal) if there is no other outcome that makes at least one player strictly better off without making any other player worse off.
In other words, A strategy profile s^* is Pareto efficient if there is no s such that:

1. $u_i(s) \geq u_i(s^*)$ for all players i
2. $u_j(s) \geq u_j(s^*)$ for at least one player j

Examining all the payoff matrix, we can find that only (A,A) and (C,C) can satisfy the definition of pareto efficient.
In more detail, for example, when player 1 plays A, then for player 2, playing B or C will make A worse off. (A,A) and (C,C) satisfy 1 each.
Moreover, we can easily derive that the pure strategy Nash Equilibrium is (B,B) and (C,C). Since players need to cooperate in the stage 1 to gain a high payoff, they must have a credible threat in stage 2.
According to the definition of credible threat, the possible creadible threat is (B,B) and (C,C). However, for (B,B), when they play (B,B), player 1 can deviate to A to gain higher utility, symmetric for player2.
For, (C,C), if player 1 deviate, it will gain 0, symmetric for player2.
Therefore, the threat from (C,C) is greater than the threat from (B,B).
Therefore, to make two player cooperate (A,A) in the stage 1, they need to use (C,C) as threat in the stage 2.

Now, it remains to check whether the strategies form a Nash equilibrium on the game as a whole.
In equilibrium with these strategies, players earn 4 + 3 = 7 in total across the two periods.
By deviating to B in the first period, a player can increase his or her first-period payoff from 4 to 5, but this leads to both playing B in the second period, reducing the second-period payoff from 3 to 1. The total payoff across the two periods from this deviation is 5 + 1 = 6, less than the 7 earned in the proposed equilibrium.

Asymmetric equilibria
In one, player 1 begins by playing B and player 2 by playing A;
if no one deviates then both play the good stage-game Nash equilibrium (both play C), and if someone deviates then both play the bad equilibrium (both play B).
Player 2 does not want to deviate to playing B in the first period because he or she earns 1 from this deviation in the first period and 1 in the second when they play the bad equilibrium for a total of 1+1=2, whereas he or she earns more, 0+3=3; in equilibrium.

Definition Feasible Payoff

Convex Hull The convex hull of a set of points is the border and interior of the largest polygon that can be formed by connecting the points with line segments.

A feasible payoff is one that can be achieved by some **mixed-strategy profile** in the stage game. Graphically, the feasible payoff set appears as the convex hull of the pure-strategy stage-game payoffs. Therefore, we call the payoffs $(x_1, ..., x_n)$ feasible in the stage game G if they are a **convex combination** (i.e., a **weighted average**, where the weights are all nonnegative and sum to one) of the **pure-strategy payoffs** of G.

Example
From the above example,

	A	B	C
A	$(4,4)$	$(0,5)$	$(0,0)$
B	$(5,0)$	$(1,1)$	$(0,0)$
C	$(0,0)$	$(0,0)$	$(3,3)$

The feasible payoff set for the stage game, the *distinct pure-strategy payoffs* are (4, 4), (0, 5), (0, 0), (5, 0), (1, 1), and (3, 3). The *convex hull* is the polygon formed by line segments going from (0, 0) to (0, 5) to (4, 4) to (5, 0), and back to (0, 0).
Each point in the convex hull corresponds to the **expected payoffs** from some **combination of mixed strategies** for players 1 and 2 over actions A, B, and C.

Definition Minmax Value
The minmax value is the following payoff for player i:

$$\min_{s_{-i}} \left[\max_{s_i} u_i(s_i, s_{-i}) \right],$$

that is, the lowest payoff player i can be held to if **all other players work against him or her** but player i is allowed to choose a **best response** to them.
The folk theorem for finitely repeated games involves the **pure-strategy minmax** value—that is, the minmax value when players are restricted to using only pure strategies.

Example

	A	B	C
A	$(4,4)$	$(0,5)$	$(0,0)$
B	$(5,0)$	$(1,1)$	$(0,0)$
C	$(0,0)$	$(0,0)$	$(3,3)$

From the same example, the lowest that player 2 can hold 1 to is a payoff of 1:
player 2 does this by playing B and then 1 responds by playing B:

Theorem Folk Theorem for Finitely Repeated Games
Suppose that the stage game has **multiple Nash equilibria** and no player earns a constant payoff across all equilibria. Any **feasible payoff** in the stage game **greater than** the **player's pure-strategy minmax value** can be approached arbitrarily closely by the player's **per-period average payoff** in some subgame-perfect equilibrium of the finitely repeated game for large enough T.

For the above example,

	A	B	C
A	$(4,4)$	$(0,5)$	$(0,0)$
B	$(5,0)$	$(1,1)$	$(0,0)$
C	$(0,0)$	$(0,0)$	$(3,3)$

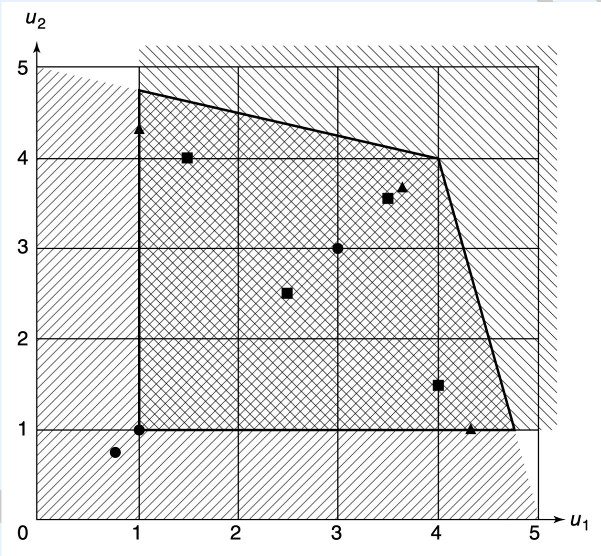

Figure 14: Folk Theorem for Finitely Repeated Games (Feasible Set and Minmax Value)

1. **feasible payoff set** the upward- hatched region
2. **pure-strategy minmax values** the downward-hatched region

4.2 Infinitely Repeated Game

Infinite Geometric Series
To derive the sum of an infinite geometric series, let the series be:

$$S = a + ar + ar^2 + ar^3 + \cdots$$

where S is the sum, a is the first term, and r is the common ratio ($|r| < 1$).
Now, multiply both sides by r:

$$rS = ar + ar^2 + ar^3 + ar^4 + \cdots$$

Subtract the second equation from the first:

$$S - rS = \left(a + ar + ar^2 + ar^3 + \cdots\right) - \left(ar + ar^2 + ar^3 + ar^4 + \cdots\right).$$

Notice that most terms cancel out, leaving:

$$S - rS = a.$$

Factor S on the left-hand side:

$$S(1 - r) = a.$$

Solve for S:

$$S = \frac{a}{1 - r}, \quad \text{where } |r| < 1.$$

In finitely repeated games, cooperation cannot be supported as a SPNE, because each player anticipates that the other will defect in the last round.
When a repeated game is infinite, cooperation can be supported as a SPNE if players value their future payoffs.

Definition Infinite Repeated Game
Given a stage game G , let $\mathbf{G}(\infty, \delta)$ denote the infinitely repeated game in which G is repeated forever and the players share the **discount factor** δ.
Each subgame beginning at stage t + 1 is identical to the original game $\mathbf{G}(\infty, \delta)$
There are as many subgames beginning at stage $t + 1$ as there are possible histories of play through stages 1 to t.

Discount Factor δ:

1. $\delta = \frac{1-p}{1+r}$
   - r = interest rate per stage
   - p = probability that the game ends at each stage
2. Measures how much a payoff unit is worth if received one period in the future rather than today. (i.e. how players value future payoffs)
   - $\delta \to 0$ players are extremely impatient, they don't value future payoffs.
   - $\delta \to 1$ players are extremely patient, they value future payoffs as much as present.
3. δ can also incorporate uncertainty about whether the game continues in future periods. The higher the probability that the game ends after the current period, the lower the expected return from stage games that might not actually be played.

For each t, the outcomes of the t - 1 preceding plays of the stage game are observed before stage t begins.
Each player's payoff is the **present value** of the player's payoffs from the infinite sequence of G.
$u_i(a_t)$ is a per-round payoff of player i if players play an action profile a_t in round t.
$u_i(a_t) = \pi_t$
The present value of the infinite sequence of payoff for player 1 in period 1, period 2, period 3, π_1, π_2, π_3, ..., is:

$$U_i^{\infty} = u_i(a_1) + \delta u_i(a_2) + \delta u_i(a_3) + \cdots = \pi_1 + \delta \pi_2 + \delta^2 \pi_3 + \cdots = \sum_{t=1}^{\infty} \delta^{t-1} u_i(a_t) = \sum_{t=1}^{\infty} \delta^{t-1} \pi_t.$$

Definition Average Payoff

Weight Normalization to Obtain the Average
The term δ^{t-1} represents the discount applied to the payoff π_t at time t. To find the **"average"** in the context of **discounting**, the **weights of the payoffs must sum to 1**. Since

$$\bar{\pi} = \frac{1}{\infty}\sum_{t=1}^{\infty}\pi_t$$

$$S = \sum_{t=1}^{\infty}\delta^{t-1}\bar{\pi} = \bar{\pi} + \bar{\pi}\delta + \bar{\pi}\delta^2 + \bar{\pi}\delta^3 + \cdots$$

$$\delta S = \delta\sum_{t=1}^{\infty}\delta^{t-1}\bar{\pi} = \delta(\bar{\pi} + \bar{\pi}\delta + \bar{\pi}\delta^2 + \bar{\pi}\delta^3 + \cdots)$$

$$S - \delta S = (1-\delta)S = (\bar{\pi} + \bar{\pi}\delta + \bar{\pi}\delta^2 + \bar{\pi}\delta^3 + \cdots) - \delta(\bar{\pi} + \bar{\pi}\delta + \bar{\pi}\delta^2 + \bar{\pi}\delta^3 + \cdots) = \bar{\pi}$$

$$S = \frac{1}{1-\delta}\bar{\pi}$$

which means series $\sum_{t=1}^{\infty}\delta^{t-1}\pi_t$ converges to $\frac{1}{1-\delta}\bar{\pi}$, we normalize the weights by multiplying by $(1-\delta)$.

$$(1-\delta)S = \bar{\pi}$$

Since $u_i(a_t) = \pi_t$ either constant or their fluctuations average out over time,
Given the discount factor δ, the average payoff of the infinite sequence of payoffs $u_i(a_1) = \pi_1, u_i(a_2) = \pi_2, u_i(a_3) = \pi_3, \cdots$ is:

$$\bar{u_i} = (1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}u_i(a_t) = (1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\pi_t$$

Theorem Folk Theorem for Infinitely Repeated Games
Any feasible payoff in the stage game greater than the player's minmax value can be obtained as the player's normalized payoff (normalized by multiplying by $(1-\delta)$ in some subgame-perfect equilibrium of the infinitely repeated game for δ close enough to 1.

Infinitely Repeated Game Model Setting

$P1/P2$	$cooperate$	$defect$
$cooperate$		
$defect$		

1. players cooperate during $\tau - 1$ rounds
   each obtains a **cooperative payoff** c_i in every round

What is the average payoff of the infinite sequence of payoffs $\pi, \pi, \pi, \cdots$?

Solution:

1. **Formula for Average Payoff**

The average payoff is defined as:

$$(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\pi_t$$

For the given sequence $\pi_t = \pi$ (constant at each time step), we can substitute this constant payoff:

$$(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\pi$$

2. **Simplify the Summation**

The summation $\sum_{t=1}^{\infty}\delta^{t-1}$ is an **infinite geometric series** with the first term 1 and common ratio δ. The formula for the sum of an infinite geometric series is:

$$\sum_{t=1}^{\infty}\delta^{t-1} = \frac{1}{1-\delta}, \quad \text{for } 0 < \delta < 1.$$

Substituting this into the formula:

$$(1-\delta)\cdot\frac{1}{1-\delta}\cdot\pi = \pi$$

Final Answer:

The average payoff of the infinite sequence of payoffs $\pi, \pi, \pi, \ldots$ is:

$$\pi$$

$$\bar{u}_i = c_i$$

2. In round τ, player i decides to defect and gets a **per-round gain** d_i:

$$d_i = \max u_i(a_i = \text{Defect}, a_{-i} = \text{Cooperate})$$

3. Then, the other player needs to punish the defector by switching to Defect in round $\tau + 1$ and playing Defect forevermore.
   Starting from round $\tau + 1$, the defector obtains the **minimum possible payoff** b_i in a base game G:

$$b_i = \min_{a_{-i}} \max_{a_i} u_i(a_i = \text{Defect}, a_{-i} = \text{Defect})$$

4. The defector's average per-round payoff across rounds $t \geq \tau$ is:

$$\bar{u}_i = (1-\delta)\left[d_i + \sum_{t=\tau+1}^{\infty} \delta^{t-\tau} b_i\right] = (1-\delta)d_i + (1-\delta)\frac{\delta}{1-\delta}b_i = (1-\delta)d_i + \delta b_i$$

   - $\sum_{t=\tau+1}^{\infty} \delta^{t-\tau} b_i = \delta b_i + \delta^2 b_i + \delta^3 b_i + \cdots = \delta(b_i + \delta b_i + \delta^2 b_i + \cdots) = \delta \times \frac{1}{1-\delta} \sum_{t=1}^{\infty} \delta^{t-1} b_i$

Definition Grim Strategy
The grim strategy is a concept in game theory, specifically in repeated games (like the iterated prisoner's dilemma). It is a strategy where a player cooperates initially but will punish the other player forever if they deviate (or "cheat") even once.

1. Initial Cooperation: At the start, the player chooses to cooperate.

2. Punishment for Defection: If the opponent defects (chooses a non-cooperative action) at any point, the player switches to defection forever as a form of punishment.

The strategy is called "grim" because once the punishment begins, it never stops — the player becomes unrelenting and refuses to cooperate again.

The promise of **indefinite future punishment** keeps both players **playing cooperate**. Therefore, Cooperation can be supported as a SPNE in an infinitely repeated game if players use Grim Strategies and their payoffs are such that:

$$c_i \geq (1-\delta)d_i + \delta b_i$$

Example
Prisoners' Dilemma in Infinite Repeated Games

Player 1 \ Player 2	Fink	Silent
Fink	$1,1$	$3,0$
Silent	$0,3$	$2,2$

Suppose both players use the following trigger strategy in the Prisoners' Dilemma: continue being silent if no one has deviated by playing fink; fink forever afterward if anyone has deviated to fink in the past.
To show that this trigger strategy **forms a subgame-perfect equilibrium**, we need to check that a player **cannot gain from deviating.**
No deviation:
Along the equilibrium path, both players are silent every period; this provides each with a payoff of 2 every period for a present discounted value of

$$\mathbf{V}^{\text{eq}} = 2 + 2\delta + 2\delta^2 + 2\delta^3 + \cdots = 2(1 + \delta + \delta^2 + \delta^3 + \cdots) = \frac{2}{1-\delta}.$$

Deviation:
A player who deviates by finking earns 3 in that period, but then both players fink every period from then on—each earning 1 per period for a total presented discounted payoff of

$$\mathbf{V}^{\text{dev}} = 3 + (1)(\delta) + (1)(\delta^2) + (1)(\delta^3) + \cdots = 3 + \delta(1 + \delta + \delta^2 + \cdots) = 3 + \frac{\delta}{1-\delta}.$$

The trigger strategies form a subgame-perfect equilibrium if:

$$\mathbf{V}^{\text{eq}} \geq \mathbf{V}^{\text{dev}}$$

In this example, it means

$$\frac{2}{1-\delta} \geq 3 + \frac{\delta}{1-\delta}$$

$$\delta \geq \frac{1}{2}$$

Players will find continued cooperative play desirable provided they do not discount future gains from such cooperation too highly.
If $\delta < \frac{1}{2}$, then no cooperation is possible in the infinitely repeated Prisoners' Dilemma; the only subgame-perfect equilibrium involves finking every period.

Since it involves the harshest punishment possible, the grim strategy elicits cooperation for the *largest range* of cases (the lowest value of δ) of any strategy. Harsh punishments work well because, if players succeed in cooperating, they never experience the losses from the punishment in equilibrium.

Tit-for- tat Strategy

Less harsh punishments which involves only one round of punishment for cheating.

5 Bayesian Games

We will focus on two-player games in which one of the players (player 1) has **private information** and the other (player 2) does not.
Firstly, model the private information.

Definition Type
Player 1 can be one of a number of possible such types, denoted t:

1. Player 1 knows his own type.
2. Player 2 is **uncertain** about t and must decide on her strategy based on beliefs about t.

Notation

1. **A set of possible types** $T = \{t_1, ..., t_k, ..., t_K\}$
2. **A particular value** t_k randomly drawn for player 1's type t from a set of possible types $T = \{t_1, ..., t_k, ..., t_K\}$
3. $Pr(t_k)$ the probability of drawing the particular type t_k.

Game Setting
The game begins at an initial node, called a **chance node**, (see definition of decision node in the earlier chapter) at which a particular value t_k is randomly drawn for player 1's type t from a set of possible types $T = \{t_1, ..., t_k, ..., t_K\}$
Let $Pr(t_k)$the probability of drawing the particular type t:

1. Player1 sees which type is drawn.
2. Player 2 does not see the draw and only knows the probabilities, using them to form her beliefs about player 1's type.
3. Thus the probability that player 2 places on player 1's being of type t_k is $Pr(t_k)$.

Since player 1 observes his type t before moving, his strategy can be conditioned on t.
Conditioning on this information may be a big benefit to a player.

Notation

1. $s_1(t)$ 1's strategy contingent on his type.
2. s_2 2's strategy is the unconditional one since 2 does not observe t.

Players' payoffs depend on strategies. In Bayesian games, payoffs may also depend on types.

Notation

1. $u_1(s_1(t), s_2, t)$ player 1's payoff
2. $u_2(s_2, s_1(t), t)$ player 2's payoff
   Player 1's type may have a direct effect on 2's payoffs. Player 1's type also has an indirect effect through its effect on 1's strategy $s_1(t)$, which in turn affects 2's payoffs.
   Since 2's payoffs depend on t in these two ways, her beliefs about t will be crucial in the calculation of her optimal strategy.

Example

Simultaneous Bayesian Games with discrete actions

Player 1 \ Player 2	L	R
U	$t, 2$	$0, 0$
D	$2, 0$	$2, 4$

t=6 with probability $\frac{1}{2}$and t=0 with probability $\frac{1}{2}$.

1. player 1 may play a different action for each of his types.
   $\Rightarrow$ Equilibrium requires that 1's strategy be a best response for each and every one of his types.
2. player 2 is uncertain about player 1's type.
   $\Rightarrow$ Equilibrium requires that 2's strategy maximize an **expected payoff**, where the expectation is taken with respect to **her beliefs about 1's type**.

Draw the game tree of a **Simultaneous Bayesian Game**

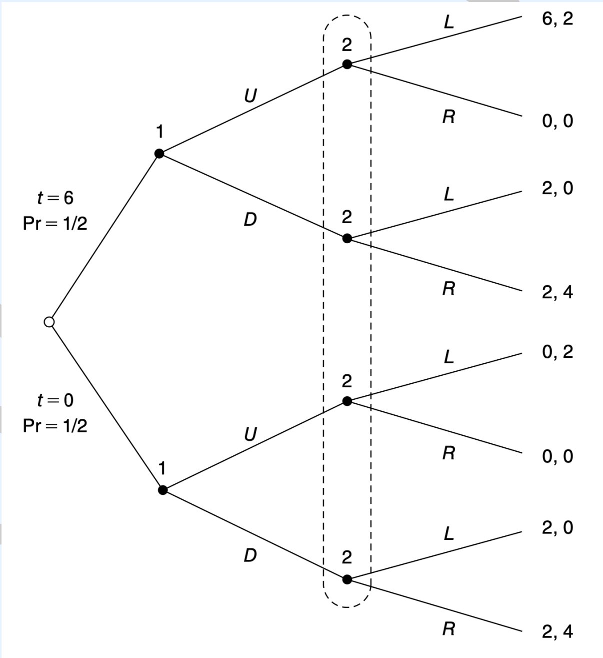

Figure 15: Simultaneous Bayesian Game with Incomplete Information Game Tree

NOTE

- The initial **chance node** is indicated by an open circle.
- Player 2's decision nodes are in the same information set because she does not observe 1's type or action prior to moving.

Definition Bayesian-Nash Equilibrium
In a two-player, simultaneous-move game in which player 1 has private information, a Bayesian-Nash equilibrium is a strategy profile $(s_1^*(t), s_2^*)$ such that $s_1^*(t)$ is a best response to s_2^* for each type $t \in T$ of player 1,

$$U_1(s_1^*(t), s_2^*, t) \geq U_1(s_1', s_2^*, t) \quad \text{for all} \quad s_1' \in S_1,$$

and such that s_2^* is a best response to $s_1^*(t)$ given player 2's beliefs $\Pr(t_k)$ about player 1's types:

$$\sum_{t_k \in T} \Pr(t_k) U_2(s_2^*, s_1^*(t_k), t_k) \geq \sum_{t_k \in T} \Pr(t_k) U_2(s_2', s_1^*(t_k), t_k) \quad \text{for all} \quad s_2' \in S_2.$$

Now continues the example above, try to calculate the Bayesian Nash Equilibrium in the game.
Firstly, we need to write the whole game with probability out:
$Pr(6) = \frac{1}{2}$

Player 1 \ Player 2	L	R
U	$6, 2$	$0, 0$
D	$2, 0$	$2, 4$

Ignoring the probability, the pure strategy nash equilibria are (6,2) and (2,4).
$Pr(0) = \frac{1}{2}$

Player 1 \ Player 2	L	R
U	$0, 2$	$0, 0$
D	$2, 0$	$2, 4$

Ignoring the probability, the pure strategy nash equilibrium is (2,4).
Categorize the conditional Nash equilibria candidates by plyaer 2 as it has unconditional strategy, we can gain:

1 plays $(U \mid t = 6, D \mid t = 0)$ **and 2 plays** L;
1 plays $(D \mid t = 6, D \mid t = 0)$ **and 2 plays** R.

Now examine whether they are Bayesian Nash Equilibrium one by one through calculating the 2's expected payoff. The first one:

$$U_1(s_1^*(t), s_2^*, t) = \frac{1}{2} \times 2 + \frac{1}{2} \times 0 = 1$$

Player 2 would gain by **deviating to R**, earning an expected payoff of 2.

$$U_1(s_1^*(t), s_2^*, t) = \frac{1}{2} \times 0 + \frac{1}{2} \times 4 = 2$$

So the first candidate cannot be an equilibrium.

The second candidate is a **Bayesian-Nash equilibrium**.

$$U_1(s_1^*(t), s_2^*, t) = \frac{1}{2} \times 4 + \frac{1}{2} \times 4 = 4$$

More complete way to illustrate this Bayesian-Nash equilibrium:
Given that 2 plays R, 1's best response is to play D, providing a payoff of 2 rather than 0 regardless of his type.
Given that both types of player 1 play D, player 2's best response is to play R, providing a payoff of 4 rather than 0.

6 Signaling Games

Now focus on the sequential games in which the informed player, 1, takes an action that is observable to 2 before 2 moves. Player 1's action provides information, a signal, that 2 can use to update her beliefs about 1's type, perhaps altering the way 2 would play in the absence of such information.

Basic setting of job market signaling game

1. Player 1 is a worker who can be one of two types.
   (a) high-skilled $(t = H)$
   A high-skilled worker generates revenue π:
   (b) low-skilled $(t = L)$
   A low-skilled worker is completely unproductive and generates no revenue for the firm.
2. Player 2 is a firm that considers hiring the applicant.
   If the applicant is hired, the firm must pay the worker w.
3. Assume $\pi > w > 0$: Therefore, the firm wishes to hire the applicant if and only if he or she is high-skilled.
4. But the firm cannot observe the applicant's skill;
   It can observe only the applicant's prior education.
5. Cost of obtaining an education:
   (a) high type c_H
   (b) low type c_L

   Assume $c_H < c_L$, i.e. education requires less effort for the high-skilled applicant than the low-skilled one.
6. Assume that education does not increase the worker's productivity directly. The applicant may still decide to obtain an education because of its value as a signal of ability to future employers.
7. Player 1 observes his or her type at the start;
   Player 2 observes only 1's action (education signal) before moving.
8. **Prior Beliefs** Let $Pr(H)$ and $Pr(L)$ be 2's beliefs prior to observing 1's education signal that 1 is high- or low-skilled, respectively.
9. **Posterior Beliefs** Observing 1's action will lead 2 to revise its beliefs.
   Player 2's posterior beliefs are used to **compute its best response** to 1's education decision.
   Example
   The probability that the worker is high-skilled is, conditional on the worker's having obtained an education, $Pr(H \mid E)$.
   The probability that the worker is high-skilled is, conditional on no education, $Pr(H \mid NE)$.

Suppose 2 sees 1 choose E
Since L and H are the only types, $Pr(L \mid E) = 1 - Pr(H \mid E)$
Player 2's payoff from playing NJ is 0.
Then to determine its best response to E, 2's expected payoff from playing J is

$$Pr(H \mid E)(\pi - w) + Pr(L \mid E)(-w) = Pr(H \mid E)\pi - w$$

So Player 2's best response is J if and only if

$$Pr(H \mid E) \geq \frac{w}{\pi}$$

The question remained is calculate the $Pr(H \mid E)$.

6.1 Bayes' rule

Conditional PDFs
The conditional probability of an event B given that A has occurred (written $P(B \mid A)$ is defined as

$$P(B \mid A) = \frac{P(A \quad \text{and} \quad B)}{P(A)}$$

DISCRETE CASE. If X and Y are discrete random variables and we have the events $(A : X = x), (B : Y = y)$, then

$$\Pr[Y = y \mid X = x] = \frac{f(x,y)}{f_X(x)}$$

where $f(x,y) = \Pr[X = x, Y = y]$ is the joint PDF and $f_X(x)$ is the marginal PDF for X. We define

$$f_{Y|X}(y \mid x) = \frac{f(x,y)}{f_X(x)}$$

and call it the conditional PDF of Y given X. Similarly, the conditional PDF of X given Y is

$$f_{X|Y}(x \mid y) = \frac{f(x,y)}{f_Y(y)}.$$

CONTINUOUS CASE. Similarly, for continuous RVs, the conditional PDFs of X (given Y) and Y (given X) are

$$f_{X|Y}(x \mid y) = \frac{f(x,y)}{f_Y(y)} \quad \& \quad f_{Y|X}(y \mid x) = \frac{f(x,y)}{f_X(x)}.$$

Note that $f_{X|Y}(x \mid y)$ and $f_{Y|X}(y \mid x)$ are not defined when $f_Y(y) = 0$ and $f_X(x) = 0$, respectively. They satisfy the properties of PDFs in their own right. The conditional PDF of X is used to assign probabilities to a range of values of X *given that* Y *takes the value* y, i.e.

$$\Pr[a \leq X \leq b \mid Y = y] = \int_a^b f_{X|Y}(x \mid y)\, dx = \frac{\int_a^b f(x,y)\, dx}{f_Y(y)}.$$

Similarly, conditional probabilities for Y given $X = x$ are,

$$\Pr[c \leq Y \leq d \mid X = x] = \int_c^d f_{Y|X}(y \mid x)\, dy = \frac{\int_c^d f(x,y)\, dy}{f_X(x)}.$$

By the definition of conditional probability, we can have:

$$\Pr(H \mid E) = \frac{\Pr(H \text{ and } E)}{\Pr(E)}.$$

Reversing the order of the two events in the conditional probability yields:

$$\Pr(E \mid H) = \frac{\Pr(H \text{ and } E)}{\Pr(H)}$$

Therefore,

$$\Pr(H \text{ and } E) = \Pr(E \mid H)\Pr(H).$$

Therefore, Bayes' rule gives the following formula for computing player 2's posterior belief $Pr(H \mid E)$:

$$\Pr(H \mid E) = \frac{\Pr(H \text{ and } E)}{\Pr(E)} = \frac{\Pr(E \mid H) \times \Pr(H)}{\Pr(E \mid H) \times \Pr(H) + \Pr(E \mid L) \times \Pr(L)}$$

Similarly, $\Pr(H \mid NE)$ is given by:

$$\Pr(H \mid NE) = \frac{\Pr(NE \mid H)\Pr(H)}{\Pr(NE \mid H)\Pr(H) + \Pr(NE \mid L)\Pr(L)}.$$

To solve the probability, $\Pr(H \mid E)$, we now need to find:

1. the prior beliefs $\Pr(H)$ and $\Pr(L)$ are given in the specification of the game by the probabilities of the different branches from the initial chance node.
2. The conditional probabilities such as $\Pr(E \mid H)$ and $\Pr(NE \mid L)$, which are given by player 1's equilibrium strategy.

For the second probability, we can consider several cases to illustrate how to derive it and the meaning of it:

1. When 1 plays a pure strategy, for example, player 1 obtains an education if and only if he or she is high-skilled,
   i.e. $\Pr(E \mid H) = 1$ and $\Pr(E \mid L) = 0$

   $$\mathbf{Pr}(H \mid E) = \frac{1 \cdot \mathbf{Pr}(H)}{1 \cdot \mathbf{Pr}(H) + 0 \cdot \mathbf{Pr}(L)} = 1.$$

2. Suppose player 1 obtains an education regardless of his or her type.
   i.e. $\Pr(E \mid H) = \Pr(E \mid L) = 1$

   $$\mathbf{Pr}(H \mid E) = \frac{1 \cdot \mathbf{Pr}(H)}{1 \cdot \mathbf{Pr}(H) + 1 \cdot \mathbf{Pr}(L)} = \mathbf{Pr}(H).$$

   Seeing 1 play E provides no information about 1's type, so 2's posterior belief is the same as its prior.

3. If 1 plays mixed strategy, $\Pr(E \mid H) = p$ and $\Pr(E \mid L) = q$, then

   $$\mathbf{Pr}(H \mid E) = \frac{p\,\mathbf{Pr}(H)}{p\,\mathbf{Pr}(H) + q\,\mathbf{Pr}(L)}.$$

Now we need to **rule out noncredible threats** in sequential games to move from Bayesian-Nash equilibrium to the refinement of perfect Bayesian equilibrium.

Definition Perfect Bayesian Equilibrium
A perfect Bayesian equilibrium consists of a **strategy profile** and **a set of beliefs** such that

1. At each information set, the strategy of the player moving there maximizes his or her expected payoff, where the expectation is taken with respect to his or her beliefs.

2. At each information set, where possible, the beliefs of the player moving there are formed using Bayes' rule (based on **prior beliefs** and other players' strategies).

   The requirement that players use Bayes' rule to update beliefs ensures that players incorporate the information from observing others' play in a rational way.

NOTE
Bayes' rule is **useless** following a **completely unexpected** event—in the context of a signaling model, an action that is not played in equilibrium by any type of player 1.
Example
If neither H nor L type chooses E in the job-market signaling game, then the denominators of $\Pr(H \mid E) = \frac{\Pr(H \text{ and } E)}{\Pr(E)} = \frac{\Pr(E|H)\times\Pr(H)}{\Pr(E|H)\times\Pr(H)+\Pr(E|L)\times\Pr(L)}$ equal zero and the fraction is undefined.
If Bayes' rule gives an **undefined answer**, then perfect Bayesian equilibrium **puts no restrictions on player 2's posterior beliefs** and so we can assume any beliefs we like.

Draw the game tree of the sequential Bayesian Game.

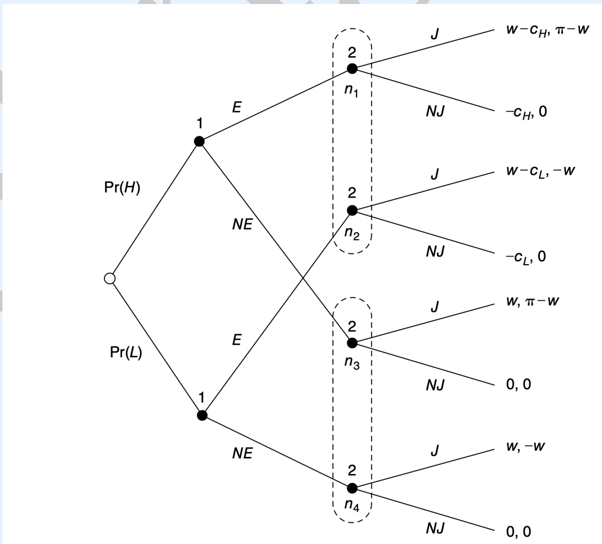

Figure 16: Sequential Bayesian Game

NOTE
The nodes in 2's information sets are labeled $n_1, \ldots, n_4$ for reference.

Describe the process:
Player 1 (worker) observes his or her own type.
Then 1 chooses to become educated (E) or not (NE).
After observing 1's action, player 2 (firm) decides to make him or her a job offer (J) or not (NJ).

The freedom to **specify any beliefs** when Bayes' rule gives an undefined answer may support **additional perfect Bayesian equilibria**.
There are three classes of equilibria: separating, pooling, and hybrid.

Definition Separating Equilibrium
Each type of player 1 chooses a **different action**.
Therefore, player 2 learns 1's type with **certainty** after observing 1's action.
The posterior beliefs that come from Bayes' rule are all **zeros and ones**.

Definition Pooling Equilibrium
Different types of player 1 choose the **same action**.
Observing 1's action provides 2 with **no information** about 1's type.
Pooling equilibria arise when one of player 1's types chooses an action that would otherwise be suboptimal in order to hide his or her private information.

Definition Hybrid Equilibrium
One type of player 1 plays a **strictly mixed strategy**; it is called a hybrid equilibrium because the mixed strategy sometimes results in the types being separated and sometimes pooled.
Player 2 learns a little about 1's type (Bayes' rule refines 2's beliefs a bit) but **doesn't learn 1's type with certainty**.
Player 2 may respond to the uncertainty by playing a **mixed strategy** itself.

Example
Separating Equilibrium in the Job-Market Signaling Game
The high-skilled worker signals his or her type by getting an education and the low-skilled worker does not.
Given these strategies, player 2's beliefs must be:
$\mathbf{Pr}(H \mid E) = \mathbf{Pr}(L \mid NE) = 1$ and $\mathbf{Pr}(H \mid NE) = \mathbf{Pr}(L \mid E) = 0$
Conditional on these beliefs, if player 2 observes that player 1 obtains an education then 2 knows it must be at node n_1 rather than n_2. Its best response is to offer a job (J), given the payoff of $\pi - w > 0$.
If player 2 observes that player 1 does not obtain an education then 2 knows it must be at node n_4 rather than n_3, and its best response is not to offer a job (NJ) because $0 > -w$.

Finally, go back and check that player 1 would not want to deviate from the separating strategy $(E \mid H, NE \mid L)$ given that 2 plays $(J \mid E, NJ \mid NE)$.

1. Type H of player 1 earns $w - c_H$ by obtaining an education in equilibrium. If type H deviates and does not obtain an education, then he or she earns 0 because player 2 believes that 1 is type L and does not offer a job. For type H not to prefer to deviate, it must be that $w - c_H \geq 0$.

2. Type L earns 0 by not obtaining an education in equilibrium. If type L deviates and obtains an education, then he or she earns $w - c_L$ because player 2 believes that 1 is type H and offers a job. For type L not to prefer to deviate, we must have $w - c_L \leq 0$.

Putting these conditions together, there is separating equilibrium in which the worker obtains an education if and only if he or she is high-skilled and in which the firm offers a job only to applicants with an education if and only if $c_H \leq w \leq c_L$.

Another possible separating equilibrium is for player 1 to obtain an education if and only if he or she is low-skilled. This is a bizarre outcome—since we expect education to be a signal of high rather than low skill—and fortunately we can rule it out as a perfect Bayesian equilibrium.

Player 2's best response would be to offer a job if and only if 1 did not obtain an education.
Type L would earn $-c_L$ from playing E and w from playing NE, so it would deviate to NE.

Example

Pooling Equilibria in the Job-Market Signaling Game

Pooling equilibrium: both types of player 1 choose E.

For player 1 not to deviate from choosing E, player 2's strategy must be to offer a job if and only if the worker is educated—that is, $(J \mid E, NJ \mid NE)$.

Analysis

1. When 2 choose to NJ after observing that players play E $(NJ \mid E)$, this means:
   If 2 doesn't offer jobs to educated workers, then 1 might as well **save the cost of obtaining an education** and choose NE.

2. When 2 choose to J after observing that players play NE $(J \mid NE)$, this means:
   If 2 offers jobs to uneducated workers, then 1 will again choose NE because he or she saves the cost of obtaining an education and still earns the wage from the job offer.

Next, we investigate when $(J \mid E, NJ \mid NE)$ is a best response for 2.
Player 2's posterior beliefs after seeing E are the same as its prior beliefs in this pooling equilibrium.
Player 2's expected payoff from choosing J is

$$\mathbf{Pr}(H \mid E)(\pi - w) + \mathbf{Pr}(L \mid E)(-w) = \mathbf{Pr}(H)(\pi - w) + \mathbf{Pr}(L)(-w) = \mathbf{Pr}(H)\pi - w.$$

For J to be a best response to E, the above expected payoff must exceed 2's zero payoff from choosing NJ, which upon rearranging implies that $\mathbf{Pr}(H) \geq \frac{w}{\pi}$.

Player 2's posterior beliefs at nodes n_3 and n_4 are not pinned down by Bayes' rule, because NE **is never played in equilibrium.** Equilibrium and so seeing 1 play NE is a completely **unexpected event**.
Perfect Bayesian equilibrium allows us to specify any probability distribution we like for the posterior beliefs $\mathbf{Pr}(H \mid NE)$ at node n_3 and $\mathbf{Pr}(L \mid NE)$ at node n_4.
Player 2's payoff from choosing NJ is 0. For NJ to be a best response to NE, 0 must exceed 2's expected payoff from playing J:

$$0 > \mathbf{Pr}(H \mid NE)(\pi - w) + \mathbf{Pr}(L \mid NE)(-w) = \mathbf{Pr}(H \mid NE)\pi - w,$$

where the right-hand side follows because $\mathbf{Pr}(H \mid NE) + \mathbf{Pr}(L \mid NE) = 1$. Rearranging yields $\mathbf{Pr}(H \mid NE) \leq \frac{w}{\pi}$.

In sum, in order for there to be a pooling equilibrium in which both types of player 1 obtain an education, we need $\mathbf{Pr}(H \mid NE) \leq \frac{w}{\pi} \leq \mathbf{Pr}(H)$.
The firm has to be:

- **optimistic** about **the proportion of skilled workers in the population—$\mathbf{Pr}(H)$** must be sufficiently **high**.
- pessimistic about the skill level of uneducated workers—$\mathbf{Pr}(H \mid NE)$ must be sufficiently low.

In this equilibrium, type L pools with type H in order to prevent player 2 from learning anything about the worker's skill from the education signal.

7 Asymmetric Information

Definition Market Failure

Market Failure is a situation when the market mechanism cannot effectively allocate goods and resources:

- Rational actions of economic agents fail to achieve the best social outcome.
- Market price is not equal to the value of used resources, and the resulting output is socially inefficient.

1. When qualities are discrete:
   - Only low-quality goods exist in the market.
   - High-quality goods are never supplied.
2. When qualities are continuous:
   - The market fails to sell anything.
   - Nobody buys anything.

Definition Asymmetric Information

Uncertainty need not lead to inefficiency when both sides of a transaction have the same limited knowledge concerning the future, but it can lead to **inefficiency** when **one side** has better information. The side with better information is said to have **private information** or, equivalently, **asymmetric information**.

Two models of asymmetric information are studied most often. Under the basic structure of **Agent-Principal Problem**, they are **moral hazard model (hidden-action model)** and **adverse selection model (hidden-type model)**. We will first introduce a basic definition, **contract**, and then the definition of these models and finally examine them one by one.

Definition Contract

A contract refers to a formal or informal **agreement** that:

1. Specifies the terms under which **actions, rewards**, or punishments are assigned to players (**agents**).
2. Is designed to **align the incentives of participants with the objectives of the principal** or the group. i.e. mitigate shirking by tying compensation to observable outcomes.

Contractual provisions can be added in order to **circumvent some of these inefficiencies.** Although contracts may help reduce the inefficiencies associated with asymmetric information, rarely do they eliminate the inefficiencies altogether.

7.1 Principal-Agent Problem

Definition Principal-Agent Model
Principal The party who **proposes** the contract. The principal usually cannot thoroughly monitor the agent and observe the agent's effort.

Agent The party who **decides whether or not to accept the contract** and then performs under the terms of the contract (if accepted). The **agent** is typically the party with the **asymmetric information.** The agent may have an incentive to act inappropriately as the interests of the agent and the principal are not aligned. The agent pursues their own goals.

Instead of contracting on the unobservable effort itself, the principal may be able to contract on an observable variable correlated to the agent's effort: final output or the success of the project managed by the agent.
The higher the agent's effort, the higher the principal's expected gain.
Paying for success (or punishing for failures) indirectly rewards effort - not deterministically, but **stochastically.**

- In the principal-agent model, the term "stochastically" refers to the **randomness** or **uncertainty** involved in the outcomes associated with the agent's effort.
  Unlike deterministic outcomes, where a specific level of effort guarantees a particular result, stochastic outcomes mean that the **agent's effort influences the probability of success but does not guarantee it.**
- The outcome may depend in part on random factors beyond the agent's control. So, tying the agent's compensation to outcomes exposes the agent to risk. A **risk-averse agent** will then require a **risk premium** to be paid before he will accept the contract.

We can first examine some examples.

Table 1: Principal-Agent Table with Hidden Information

Principal	**Agent**	**Hidden type**	**Hidden action**
Shareholders	Manager	Managerial skill	Effort, executive decisions
Manager	Employee	Job skill	Effort
Homeowner	Appliance repairer	Skill, severity of appliance malfunction	Effort, unnecessary repairs
Monopoly	Customer	Value for good	Care to avoid breakage
Health insurer	Insurance purchaser	Preexisting condition	Risky activity

Then, we need to introduce some basic definition for the later setting.

7.1.1 First Best

In a **full-information** environment, the principal could propose a contract to the agent that maximizes their joint surplus and captures all of this surplus for herself, leaving the agent with just enough surplus to make him indifferent between signing the contract or not. This outcome is called the first best.
The contract implementing this outcome is called the first-best contract.

7.1.2 Second Best

The outcome that maximizes the principal's surplus *subject to the constraint that the principal is less well informed* than the agent is called the second best.
The contract that implements this outcome is called the **second-best contract.**.

7.2 Moral Hazard Model & Hidden-Action Model

Definition Moral Hazard
The effect of insurance coverage on an individual's precautions, which may change the likelihood or size of losses.
Moral hazard is any situation in which one person makes the decision about **how much risk to take**, while someone else bears the cost if things go badly. It arises when individuals engage in risk sharing under conditions such that their privately taken actions affect the **probability distribution of the outcome.**

Definition Moral Hazard Model & Hidden-Action Model
The **agent's actions** taken during the term of the contract affect the principal, but the **principal** does **not observe these actions** directly. The **principal may observe outcomes that are correlated with the agent's actions** but **not the actions themselves**.

The classical Moral Hazard Model has two classical application.
First, **OWNER-MANAGER RELATIONSHIP**. Employment contracts signed between a firm's owners and a manager who runs the firm on behalf of the owners.
Job shirking: An employer cannot observe an employee's effort, who has a rational incentive to shirk responsibilities and do the least amount of work.
Second, **MORAL HAZARD IN INSURANCE**. Contracts offered by an insurance company to insure an individual against accident risk.
Insurance: An insured person might take less care to avoid damage to their insured property, as insurance companies bear the risk and potential costs.

We will examine the first model thoroughly to illustrate how this model work.

7.2.1 Owner-Manager Relationship

A normally distributed risk.
The **constant risk aversion utility function** can be combined with the assumption that a person faces a random threat to his or her wealth that follows a normal distribution to arrive at a particularly simple result.
Specifically, if a person's risky wealth follows a normal distribution with mean μ_W and variance σ_W^2, then the probability density function for wealth is given by

$$f(W) = \frac{1}{\sqrt{2\pi}} e^{z^2/2}, \quad \text{where } z = \frac{(W - \mu_W)}{\sigma_W}.$$

Recall the expected value or mean of a RV X, $E[X]$ (or μ), is defined as

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx \quad \text{(CONTINUOUS)}.$$

And the property of expected value:
If $g(X)$ is a continuous function of X then we have

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx \quad \text{or} \quad E[g(X)] = \sum_{x=-\infty}^{\infty} g(x) f(x)$$

$U(W)$is a von Neumann–Morgenstern utility index that reflects how he or she feels about various levels of wealth.
One kind of utility function for risk averse people is the exponential utility function:

$$U(W) = -e^{-AW} = -\exp(-AW)$$

(where A is a positive constant) exhibits constant absolute risk aversion over all ranges of wealth, because now

$$r(W) = -\frac{U''(W)}{U'(W)} = \frac{A^2 e^{-AW}}{Ae^{-AW}} = A.$$

Therefore, if this person has a utility function for wealth given by $U(W) = -e^{-AW}$, then **expected utility** from his or her risky wealth is given by

$$\mathbb{E}[U(W)] = \int_{-\infty}^{\infty} U(W) f(W)\, dW = \frac{1}{\sqrt{2\pi}} \int -e^{-AW} e^{-[(W-\mu_W)/\sigma_W]^2/2}\, dW.$$

Performing this integration and taking a variety of monotonic transformations of the resulting expression yields the final result that

$$\mathbb{E}[U(W)] \cong \mu_W - \frac{A}{2} \cdot \sigma_W^2.$$

General Introduction to the **Owner-Manager Relationship**

The owner would like the manager whom she hires to show up during business hours and work diligently.
The agent's actions may be unobservable to the principal. Observing the action may require the principal to monitor the agent at all times, and such **monitoring may be prohibitively expensive.** If the agent's action is unobservable then he will prefer to shirk, choosing an action to suit himself rather than the principal.
Although contracts cannot prevent shirking directly by tying the agent's compensation to his action—because his action is unobservable—**contracts can mitigate shirking by tying compensation to observable outcomes.** In the owner-manager application, the relevant observable outcome might be the **firm's profit**. The owner may be able to induce the manager to work hard by tying the manager's pay to the firm's profit, which depends on the **manager's effort.**

Model Basic Setting

1. **Principal** one representative owner
   plays the role of the **principal** in the model, **offers a contract to the manager**.
   individual shareholders who each own a small share of the firm as part of a diversified portfolio
   risk neutral

2. **Agent** one manager
   risk averse
   plays the role of the **agent**.
   The manager decides whether to:

   (a) accept the employment contract

   (b) if accept the employment contract, what action $e \geq 0$ to take.

   - e is the effort and time the manager puts in on the job.
   - An increase in e increases the firm's gross profit (not including payments to the manager) but is **personally costly** to the **manager**.
   - The manager's effort is not observable to the owner and not perfectly deducible from the value of profits.

3. Assume π is observable profit. It can take any values from $[\underline{\pi}, \overline{\pi}]$ with some probability determined by the conditional density function $f(\pi|e)$:

$$\text{if } e_l < e_k \text{ then } F(\pi|e_k) \leq F(\pi|e_l)$$

e affects the probability distribution of profits: a higher effort improves the likelihood of higher profit and reduces the likelihood of losses.

   (a) **Observable Profit (π)**: This refers to the profits that are measurable and observable by both the principal and the agent. The profits can vary within a range $[\underline{\pi}, \overline{\pi}]$.
   (b) **Conditional Density Function ($f(\pi|e)$)**: The probability distribution of profits (π) depends on the agent's effort level (e).
   (c) **Cumulative Distribution Function ($F(\pi|e)$)**: The probability that the profit is less than or equal to π, given effort e. If $e_k > e_l$, the distribution shifts such that higher effort e_k leads to a higher probability of higher profits.
   (d) **Key Statement**: If $e_l < e_k$, then $F(\pi|e_k) \leq F(\pi|e_l)$. This means that for any level of profit π, higher effort (e_k) results in a lower cumulative probability of earning less than π, implying higher probabilities for larger profits.

$\Rightarrow$ Gross profit is increasing in the manager's effort e.

Example
Firm's gross profit π_g:

$$\pi_g = e + \varepsilon$$

we use this distribution in the later specific calculation.

- Depends on a random variable ε (demand, cost, etc.).
- ε is normally distributed with mean 0 and variance σ^2

4. The manager's personal disutility (or cost) of undertaking effort $c(e)$:

- increasing $c'(e) > 0$
- convex $c''(e) > 0$

5. salary s

- depend on effort and/or gross profit, depending on what the owner can observe
- offered as part of the contract between the owner and manager.

6. net profit (gross profit minus payments to the manager) π_n

$$\pi_n = \pi_g - s$$

7. **Principal's Optimization Problem**:
   The optimal contract $[e^*, s^*(\pi)]$ must induce the effort that gives the highest expected net profit:

$$\max_{\pi} \quad E(\pi) = \int_{\underline{\pi}}^{\overline{\pi}} [\pi - s(\pi)]\, f(\pi|e)\, d\pi$$

   Example
   Risk-neutral owner wants to **maximize the expected value** of her **net profit**

$$E(\pi_n) = E(e + \varepsilon - s) = e - E(s)$$

8. The **payoff function** of the **manager** is

$$U(s, e) = v(s) - c(e)$$

   - $v(s)$ represents the utility or benefit that the agent (in this case, the manager) derives from their monetary remuneration s.
   - $v'(s) > 0$
   - The outside option of the manager gives a reservation utility $\bar{U}$ (minimal required payoff to accept the job contract).

   Example
   the **manager** has a **utility function** with respect to **salary** whose constant absolute risk aversion parameter is $A > 0$
   Expected utility function of manager:

$$E(U) = E(s) - \frac{A}{2} Var(s) - c(e)$$

We will examine the optimal salary contract that induces the manager to take appropriate effort e under different informational assumptions. We will study the first-best contract, when the owner can observe e perfectly, and then the second-best contract when there is asymmetric information about e.

First best (full-information case)
With full information, the owner can pay the manager a fixed salary s^* if he exerts the first-best level of effort e^* and nothing otherwise.
The manager's expected value: $E(s^*) = s^*$
The manager's variance: $Var(s^*) = 0$
Assume manager obtains 0 from his next-best job offer. (i.e. $\bar{U} = 0$)

Definition **Participation Constraint** i.e. **Individual Rationality Constraint**
For the manager to accept the contract, this **expected utility must exceed** what he would obtain from his **next-best job offer**.
The participation constraint becomes more severe and limits how much benefit the principal can extract from the contractual relationship.
Mathematical Expression

$$\underbrace{\int_{\underline{\pi}}^{\overline{\pi}} v(s(\pi))f(\pi|e^*)\,d\pi - c(e^*)}_{\text{Expected gain net of effort costs}} \geq \overline{U}$$

In the first best case, we have the Participation Constraint:

$$E(U) = s^* - c(e*) \geq 0$$

The owner optimally pays the lowest salary $s^* = c(e*)$
The owner's net profit then is $E(\pi_n) = e^* - E(s^*) = e^* - s^* = e^* - c(e^*)$

Second best (hidden-action case)
If the principal cannot observe effort, the contract cannot be conditioned on e.
However, the principal can still induce the manager to exert some effort if the manager's **salary depends on the firm's gross profit.**
The manager is given performance pay: the more the firm earns, the more the manager is paid.

Suppose the owner offers the manager one that is linear in gross profit:

$$s(\pi_g) = a + b\pi_g$$

- a The fixed component of salary (i.e. the manager's base salary).
- b power Measures the slope of the incentive scheme (e.g. the incentive pay in the form of stocks, stock options, and performance bonuses).
  - If b=0 then the salary is constant, providing no effort incentives.
  - As b $\to$ 1, the incentive scheme provides increasingly powerful incentives.

The owner-manager relationship can be viewed as a **three-stage game**.

1. In the first stage, the owner sets the salary, which amounts to choosing a and b.
2. In the second stage, the manager decides whether or not to accept the contract.
3. In the third stage, the manager decides how much effort to exert conditional on accepting the contract.

Solution
We will solve for the subgame-perfect equilibrium of this game by using backward induction.

1. Starting with the manager's choice of e in the last stage and taking as given that the manager was offered salary scheme $a + b\pi_g$ and accepted it.
Recall the expected utility function $E(U) = E(s) - \frac{A}{2}Var(s) - c(e)$ The manager's expected utility from the linear salary:

$$E(U) = E(a + b\pi_g) - \frac{A}{2}Var(a + b\pi_g) - c(e)$$

Simplify the above function:

(a) the expected value of a linear function of the random variable ε is a linear function of the expected value of ε
$\mathbb{E}(s) = \mathbb{E}(a + b\pi_g) = \mathbb{E}(a + be + b\varepsilon) = a + b\mathbb{E}(e) = a + be$

(b) the formula for the variance of a linear function of a random variable
$\mathrm{Var}(a + b\pi_g) = \mathrm{Var}(a + be + b\varepsilon) = b^2\,\mathrm{Var}(\varepsilon) = b^2\sigma^2$,

Manager's Expected Utility

$$E(U) = a + be - \frac{Ab^2\sigma^2}{2} - c(e).$$

Analysis

Definition Incentive Compatibility Constraint (ICC)
The **agent maximizes their utility** (or payoff) by choosing the **desired action**, effort, or reporting truthfully given the contract, rules, or incentives provided.
For Contracts/Principal-Agent Models:
Aligning agent's effort **with** the principal's goals. i.e. The ICC ensures that the agent chooses the optimal level of effort or action specified by the principal.
The agent exerts effort e because the expected utility from doing so is maximized:

$$U(e) \geq U(e') \quad \forall e' \neq e.$$

i.e. the optimal contract must induce the desired effort (the contract must incentivise the manager to choose e^*:

$$e^* = \arg\max \int_{\underline{\pi}}^{\overline{\pi}} v(s(\pi)) f(\pi|e)\, d\pi - c(e)$$

The first-order condition for the choice of e that **maximizes the manager's expected utility**:

$$c'(e) = b.$$

Because $c(e)$ is convex, the marginal cost of effort $c'(e)$ is increasing in e. Hence, , the higher is the power b of the incentive scheme, the more effort e the manager exerts. The manager's effort depends only on the slope, b, and not on the fixed part, a, of his incentive scheme.

2. Now fold the game back to the manager's second-stage choice of whether to accept the contract.
The manager accepts the contract if his expected utility is nonnegative.
Participation Constraint

$$E(U) = a + be - \frac{Ab^2\sigma^2}{2} - c(e) \geq 0$$
$$\Leftrightarrow$$
$$a \geq c(e) + \frac{Ab^2\sigma^2}{2} - be.$$

Analysis

   (a) **For the manager**, the fixed part of the salary, a, must be high enough for the manager to accept the contract.
   (b) **For the principal**, it is clear that the **owner will keep lowering** a **until the condition holds with equality**, since a does not affect the manager's effort and since the owner does not want to pay the manager any more than necessary to induce him to accept the contract.

3. Fold the game back to the owner's first-stage choice of the parameters a and b of the salary scheme.
The **owner's** objective is to **maximize her expected surplus**.

$$E(\pi_n) = E(\pi_g - s) = E(e + \varepsilon - s) = e - E(s) == e - (a + be) = e(1 - b) - a$$

subject to two constraints:

   (a) **participation constraint**
   (b) **incentive compatibility constraint**

Therefore, write this formula formally, it is:

$$\max \quad E(\pi_n) = E(\pi_g - s) = E(e + \varepsilon - s) = e - E(s) = e - (a + be) = e(1 - b) - a$$

$$\textbf{s.t}\begin{cases}\text{(incentive compatibility constraint)} & c'(e) = b\\ \text{(participation constraint)} & a = c(e) + \frac{Ab^2\sigma^2}{2} - be\end{cases}$$

It means we express the owner's surplus as a function **only of the manager's effort**.

So we can get:

$$\max \quad E(\pi_n) = e - c(e) - \frac{Ac'(e)^2\sigma^2}{2}$$

The second-best effort e^{**} satisfies the first-order condition:

$$\frac{\partial E(\pi_n)}{\partial e} = 1 - c'(e) - \frac{A}{2}\sigma^2 \times 2c'(e) \times c''(e) = 0$$

$$c'(e^{**}) = \frac{1}{1 + A\sigma^2 c''(e^{**})}.$$

7.3 Adverse Selection Model & Hidden Type Model

Definition Hidden Types
In the hidden-type model the agent has private information about an innate characteristic he **cannot choose.**

Definition Adverse Selection
Adverse selection occurs when the uninformed party knows that the informed party has more information. It is a form of market failure: products of different qualities are sold at the same price. Low-quality products drive out high-quality products.
For example, the problem facing insurers that risky types are both more likely to accept an insurance policy and more expensive to serve.

Definition Adverse Selection Model & Hidden Type Model
The **agent** has **private information** about the **state** of the world **before signing the contract** with the principal (In the hidden-action model, agent takes action after signing the contract). The agent's private information is called his **type**, consistent with our terminology from games of private information studied in Bayesian Games.

General Introduction to **Adverse Selection Model & Hidden Type Model**

- In the **hidden-action model**
  The principal shares **symmetric information** with the agent at the **contracting stage** so the principal can design a contract that extracts all of the agent's surplus.
- In the **hidden-type model**
  the **agent's private information** at the time of **contracting** puts him in **a better position**. Now there is no way for the principal to extract all the surplus from all types of agents.

Hidden Type Model

1. A contract that extracts all the surplus from the "high" types (those who benefit more from a given contract) would provide the "low" types with negative surplus, and they would refuse to sign it.
2. To extract as much surplus as possible from each type while ensuring that low types are not "scared off," the principal will offer a contract in the form of a cleverly **designed menu** that **include options targeted to each agent type.**
3. The menu of options will be more profitable for the principal than a contract with a single option, but the principal will still not be able to extract all the surplus from all agent types.
4. Since the agent's type is hidden, he cannot be forced to select the option targeted at his type but is free to select any of the options, and this ability will ensure that the **high types always end up with positive surplus.**

There are two classical applications of the hidden-type model that are important in economics. First is the optimal nonlinear pricing problem, second is private information in insurance. We will specifically study **nonlinear pricing problem** in the next section.
Moreover, since both parties suffer from adverse selection, both the informed and the uninformed try to devise tools to reveal the hidden information credibly. The most important two tools are:

1. **Screening**: the uninformed party designs a mechanism that makes the informed party reveal the hidden information.
   Example
   menu of contracts (menu pricing) (we will discuss later).

2. **Signaling**: the informed party sends a costly signal that can credibly reveal the hidden information.
   Example
   education (we will discuss it later), money burning, reputation, warranty, money back guarantee.

Actually, screening applies non-linear price, and signaling is the combination of job market signaling and non-linear price. We will see it later.

7.3.1 Nonlinear Pricing & Screening

Definition Nonlinear Price Schedule
The nonlinear price schedule is a menu of **different-sized bundles** at **different prices**, from which the consumer makes his selection. In such schedules, the larger bundle generally sells for a higher total price but a lower per-unit price than a smaller bundle.

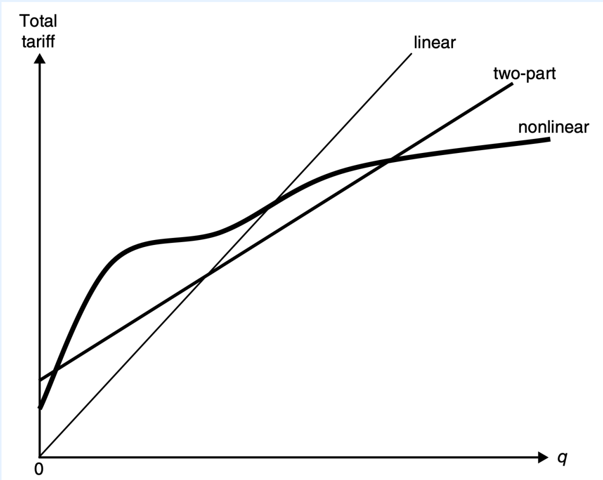

Figure 17: Different Kinds of Price Schedules

1. The linear pricing schedule is graphed as a straight line that intersects the origin (because nothing needs to be paid if no units are purchased).
2. The two-part tariff is also a straight line, but its intercept—reflecting the fixed fee—is above the origin.
3. The darkest curve is a general nonlinear pricing schedule.

Basic Model Setting

1. A **monopolist** (the **principal**)
2. **consumer** (the **agent**) with **private information** about his **own valuation for the good**.
3. the monopolist offers the consumer a **nonlinear price schedule**.
4. A single consumer obtains surplus

$$U = \theta v(q) - T$$

   (a) consuming a bundle of q units of a good
   (b) pays a total tariff of T (subtracted from his benefit to compute his net surplus.)
   (c) $\theta v(q)$ consumer's utility function
   [Diminishing return] The consumer prefers more of the good to less but that the marginal benefit of more units is declining:
   $v'(q) > 0$
   $v''(q) < 0$
   (d) θ The consumer's type:
      i. high θ_H with probability β
      ii. low θ_L with probability $1 - \beta$
   The high type enjoys consuming the good more than the low type: $0 < \theta_L < \theta_H$
   (e) A pair of $[q, T]$ is a **contract**: B pays T to purchase a bundle of q units.
5. There is **a single consumer** in the market.
   - The analysis would likewise apply to markets with many consumers, a proportion β of which are high types and $1 - \beta$ of which are low types.
   - **Feasible** assume that consumers cannot divide bundles into smaller packages for resale among themselves.
6. The monopolist has a **constant marginal and average cost** c of producing a unit of the good.
7. The monopolist's **profit** from selling a bundle of q units for a total tariff of T:

$$\Pi = T - cq$$

First-best Nonlinear Pricing
In the first-best case (**complete information**), the **monopolist can observe** the consumer's **type** θ **before** offering him a contract. The monopolist:
chooses the contract terms q and T to maximize her profit

1. subject to $\Pi = T - cq$
2. subject to a **participation constraint** that the consumer accepts the contract.

Setting the consumer's utility to 0 if he rejects the contract.

Participation Constraint

$$\theta v(q) - T \geq 0.$$

If Seller (monopolist) can **perfectly identify Buyer's (consumer) type**, Seller charges T equal to the Buyer's total surplus.
The monopolist will choose the highest value of T satisfying the participation constraint:

$$T_i = \theta_i v(q_i)$$

. Substituting this value of T into the monopolist's profit function yields

$$\Pi = T - cq = \theta v(q) - cq.$$

$$\max_{q_i} \theta_i v(q_i) - cq_i$$

Taking the first-order condition and rearranging provides a condition for the first-best quantity:

$$\frac{\partial \Pi}{\partial q_i} = \theta_i v'(q_i) - c = 0.$$

$$\theta_i v'(q_i) = c$$

$$MU = MC$$

Analysis
In the first best:

the marginal social benefit of increased quantity [the consumer's marginal private benefit, $\theta v'(q)$]
= the marginal social cost [the monopolist's marginal cost, c]

This is the socially **efficient** outcome because it ensures that resources are allocated such that the benefit of producing one more unit equals its cost.

The first-best quantity q:

- offered to the high type (q_H^*) satisfies the equation for $\theta = \theta_H$
- offered to the low type (q_L^*) satisfies the equation for $\theta = \theta_L$.

The tariffs are set so as to extract all the type's surplus. The first best for the monopolist is identical to what we termed first-degree price discrimination.

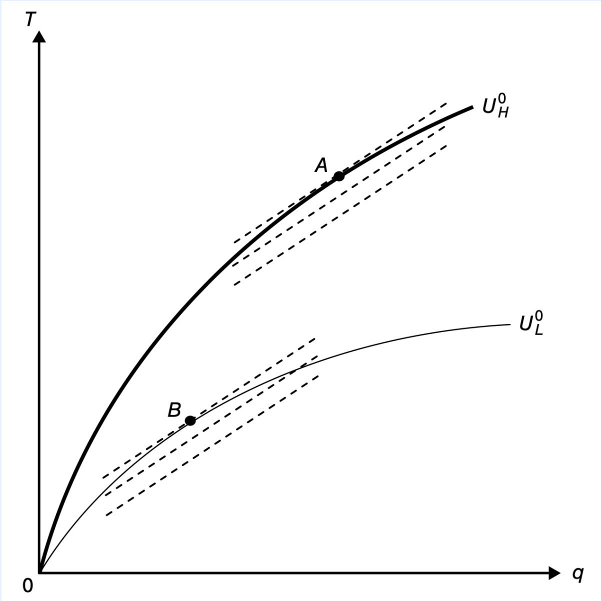

Figure 18: First-best Nonlinear Pricing

Interpretation
The contract (q, T) can be thought of as a bundle of two different "goods" over which the monopolist has preferences.
The **monopolist** $\Pi = T - cq$ regards:

1. T as a good (more money is better than less)
2. q as a bad (higher quantity requires higher production costs).

The slope of the monopolist's indifference curve is her marginal rate of substitution:

$$\textbf{MRS} = -\frac{\partial \Pi / \partial q}{\partial \Pi / \partial T} = -\frac{(-c)}{1} = c$$

Her indifference curve (actually an **isoprofit curve, T** $= \bar{\pi} +$ **cq**) over (q, T) combinations is a straight line with slope c.

Direction
Because q is a bad for the monopolist, her indifference curves are higher as one moves toward the upper left.

Indifference curves for the two **consumer** $U = \theta v(q) - T$ types:

1. Indifference curves are upward-slopping and concave

$$T = \theta v(q) - \bar{u}$$

2. the high type's (labeled U_H^0)
intersects the origin, implying that the high type gets the **same surplus** as if he **didn't sign the contract at all**.

3. the low type's (labeled U_L^0)

Direction
Because T is a bad for consumers, higher indifference curves for both types of consumer are reached as one moves toward the lower right. The slope of type θ's indifference curve is the marginal rate of substitution:

$$\mathbf{MRS} = -\frac{\partial U/\partial q}{\partial U/\partial T} = -\frac{\theta v'(q)}{-1} = \theta v'(q)$$

Therefore, indifference curve of **H-type** is **steeper**.
Combine the two indifference together to find the optimal quantity for high type (q_H^*) and low type (q_L^*):

- Point A is the first-best contract option offered to the high type
- Point B is the first-best contract option offered to the low type.

To be more specific, the graph should be:

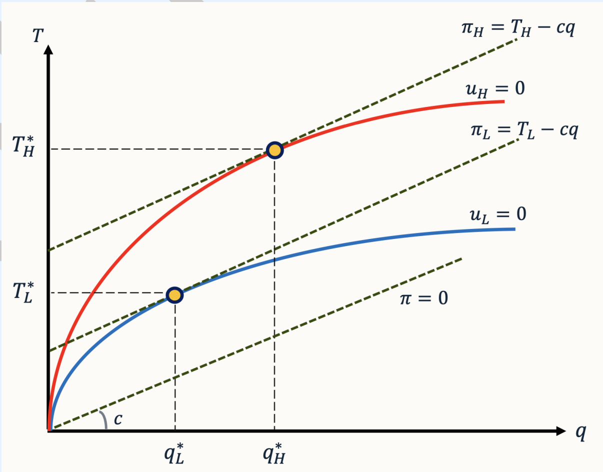

Figure 19: Detailed Nonlinear Price in First-best Case

Second-best Nonlinear Pricing

The monopolist **does not observe** the consumer's type when offering him a contract, but knows only the distribution:

- $\theta = \theta_H$ with probability β
- $\theta = \theta_L$ with probability $1-\beta$

The high type obtains more utility (moving from the indifference curve labeled U_H^0 to the one labeled U_H^2) by **choosing the bundle targeted to the low type (**B**)** rather than the bundle targeted to him (A).
In other words, choosing A is no longer incentive compatible for the high type.

Monopolist's reaction

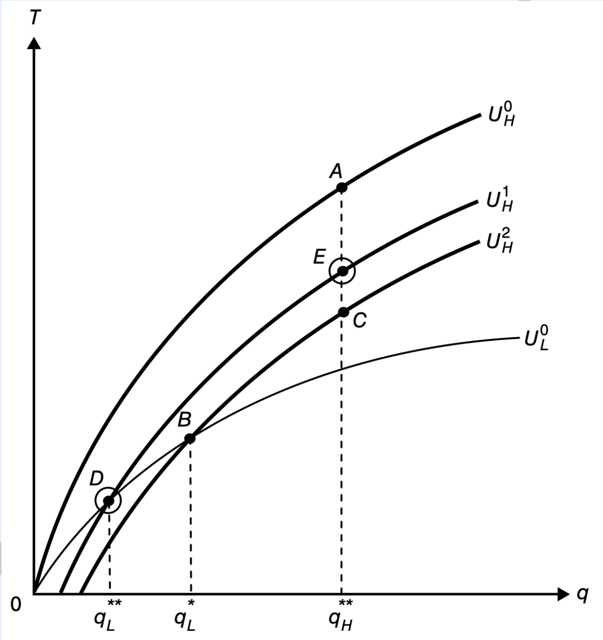

Figure 20: Second-best Nonlinear Pricing

1. In order to keep the high type from choosing B, the monopolist must reduce the high type's tariff, offering C instead of A.
2. **Better** Distort the low type's bundle in order to make it less attractive to the high type.
   (a) The monopolist **reduces the quantity** in the low type's bundle while **reducing the tariff** so that the low type stays on his U_L^0 indifference curve and thus continues to accept the contract, offering bundle D rather than B.
   (b) The high type obtains less utility from D than B, as D reaches only his U_H^1 indifference curve.
   (c) To keep the high type from choosing D, the monopolist needs only **lower the high type's tariff** from **A to E** rather than all the way down to C.

Question
How can we be sure that this reduction in the low type's tariff doesn't more than offset any increase in the high type's tariff?
Recall:

$$\textbf{MRS} = -\frac{\partial U/\partial q}{\partial U/\partial T} = -\frac{\theta v'(q)}{-1} = \theta v'(q)$$

A reduction in quantity harms the high type more than it does the low type. Since the high type values quantity more than does the low type, the high type would pay more to avoid the decrease in quantity in moving from B to D than would the low type.

3. For the low type, the second-best quantity is lower than the first best,i.e. the low type's quantity is distorted downward in the second best to extract surplus from the high type.

More Detailed Graphical Expression and Mathematical Expression

Definition Screening
Screening implies that the uninformed party offers the informed party a deal– a contract – which the informed party must either **accept or reject**.
By **accepting or rejecting**, the informed party **discloses their hidden information**.
Screening is a mechanism of self-selection: the informed party reveals its hidden information by choosing an **option** from the menu of available contracts.

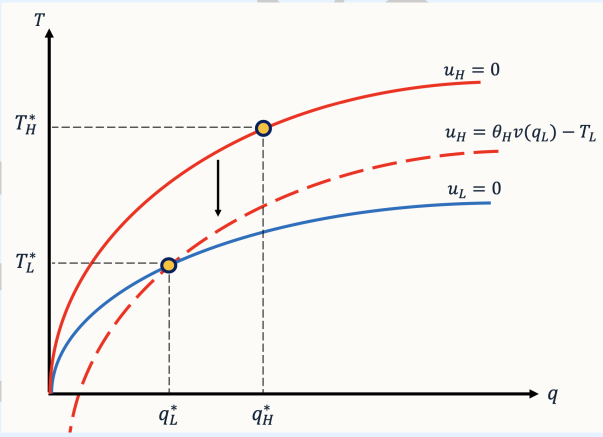

Figure 21: Detailed Nonlinear Price in Second-best Case

If Seller (monopolist) cannot observe Buyer's (consumer) type, H-type is willing to pretend to be L-type by choosing q_L^*. H-type increases his surplus by choosing a bundle $[q_L^*, T_L^*]$:

$$u_H = \theta_H v(q_H) - T_L > 0$$

Here, H-type enjoys an information rent: additional gain the informed party obtains as it knows more information.
To improve the situation, Seller must devise a **menu of contracts targeted to different types of Buyers**. More specifically, Seller must offer different bundles $[q_L, T_L]$ and $[q_H, T_H]$ such that both types purchase but no one could do better by choosing a bundle intended for another type.
In other words, the contracts must incentivise different types to choose different bundles. Therefore, the optimal contracts (bundles) must also satisfy the Participation Constraints (i.e. Individual Rationality Constraints (IR)) and the Incentive Compatibility Constraints (ICC).

Subsequently, the second-best contract is a menu that targets bundle (q_H, T_H) at the high type and (q_L, T_L) at the low type.

The contract maximizes the monopolist's expected profit

$$E(\Pi) = \beta(T_H - cq_H) + (1-\beta)(T_L - cq_L),$$

subject to four constraints:

1. **participation constraints**
   Ensure that both low type and high type consumer accept the contract rather than forgoing the monopolist's good.
   i.e. *Individual Rationality Constraints (IR)* (q_L, T_L) must be affordable for L-type, and (q_H, T_H) must be affordable for H-type.

   $$u_i = \theta_i v(q_i) - T_i \geq 0$$

   (a) *Keep the low type on his U_L^0 indifference curve to prevent him from rejecting the contract:

   $$\textbf{IR(L):} \quad \theta_L v(q_L) - T_L \geq 0,$$

   (b) The high type's second-best bundle E puts him on a higher indifference curve (U_H^1) than if he rejects the contract (U_H^0).
   i.e. H-type enjoys information rent. His surplus remains positive
   $\Rightarrow$ IR(H) holds as strict inequality.
   So the high type's participation constraint can be safely **ignored**:

   $$\textbf{IR(H):} \quad \theta_H v(q_H) - T_H \geq 0,$$

2. **incentive compatibility constraints**
   Ensuring that each type chooses the bundle targeted to him rather than the other type's bundle.

   $$q_i = \arg\max_{q_i} u_i$$

   (a) The low type would be on a lower indifference curve if he chose the high type's bundle (E) rather than his own (D), in 20.
   i.e. If L-type chooses (q_H, T_H) in 21, his surplus becomes negative
   $\Rightarrow$ ICC(L) holds as strict inequality.
   so the low type's incentive compatibility constraint can also be safely **ignored.**

   $$\textbf{ICC(L):} \quad \underbrace{\theta_L v(q_L) - T_L}_{\text{L-type buys } [q_L, T_L]} \geq \underbrace{\theta_L v(q_H) - T_H}_{\text{L-type buys } [q_H, T_H]}$$

   (b) *Keep the high type from choosing the low type's bundle:

   $$\textbf{ICC(H):} \quad \underbrace{\theta_H v(q_H) - T_H}_{\text{H-type buys } [q_H, T_H]} \geq \underbrace{\theta_H v(q_L) - T_L}_{\text{H-type buys } [q_L, T_L]}$$

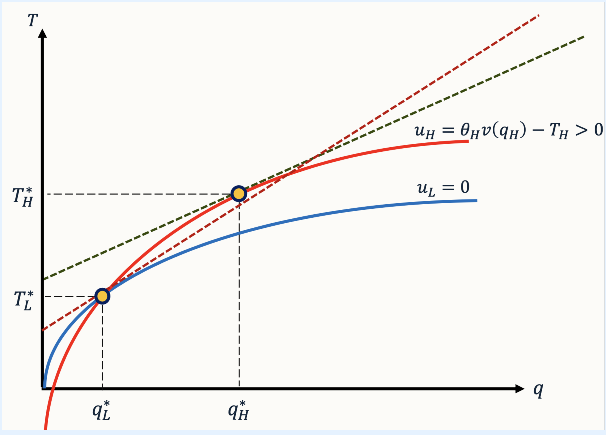

Figure 22: Optimal Nonlinear Pricing

Now the figure will be like in 22.

Description

These description will be examined through math below.

1. $[q_H, T_H]$ is *socially efficient*, as $MU(q_H) = \theta v'(q_H) = c$.
   The slope of the H-type's indifference curve is equal to the slope of the isoprofit line (dashed dark green line).

2. $[q_L, T_L]$ is socially inefficient, as $MU(q_L) = \theta v'(q_L) > c$.
   The slope of the L-type's indifference curve (dashed dark red line) is not equal to the slope of the isoprofit line.

3. The seller intentionally worsens the L-type's bundle to make it unattractive to H-type.

Seller offers (q_L, T_L) that turns the **entire surplus of L-type into 0** $u_L = 0$.
$\Rightarrow$ IR(L) holds as equality.

$$\textbf{IR(L):} \quad u_L = \theta_L v(q_L) - T_L = 0$$

Seller must offer (q_H, T_H) such that H-type becomes indifferent between (q_H, T_H) and (q_L, T_L).
$\Rightarrow$ ICC(H) holds as equality.

$$\textbf{ICC(H):} \quad \theta_H v(q_H) - T_H \geq \theta_H v(q_L) - T_L$$

Treating Equations 1(a) and 2(b) as equalities and using them to solve for T_L and T_H yields

$$T_L = \theta_L v(q_L)$$

and

$$T_H = \theta_H [v(q_H) - v(q_L)] + T_L = \theta_H [v(q_H) - v(q_L)] + \theta_L v(q_L).$$

By substituting these expressions for T_L and T_H into the monopolist's objective function $\beta(T_H - cq_H) + (1-\beta)(T_L - cq_L)$, we convert a complicated maximization problem with four inequality constraints into the simpler **unconstrained problem**:

$$\max_{q_H, q_L} \quad E(\Pi) = \beta\{\theta_H[v(q_H) - v(q_L)] + \theta_L v(q_L) - cq_H\} + (1-\beta)[\theta_L v(q_L) - cq_L].$$

The low type's quantity satisfies the first-order condition with respect to q_L,

$$\frac{\partial E[\Pi]}{\partial q_L} = \beta\left[-\theta_H v'(q_L) + \theta_L v'(q_L)\right] + (1-\beta)\left[\theta_L v'(q_L) - c\right] = 0$$

which (upon considerable rearranging) yields

$$\theta_L v'(q_L^*) = c + \frac{\beta(\theta_H - \theta_L)v'(q_L^*)}{1-\beta}.$$

L-type consumes inadequate quantity (worse quality) since now $\theta_L v'(q_L^*) > c$ i.e.$MU > MC$ (monopoly gains more than before, so consumer suffers).
(*NOTE: recall the consumer's utility function is $\theta v(q)$.)

The high type's quantity satisfies the first-order condition from the maximization of $\beta(T_H - cq_H) + (1-\beta)(T_L - cq_L)$ with respect to q_H

$$\frac{\partial E[\Pi]}{\partial q_H} = \beta\left[\theta_H v'(q_H) - c\right] = 0$$

yields:

$$\theta_H v'(q_H^{**}) = c$$

This condition is identical to the first best, H-type consumes a socially efficient bundle, implying that **there is no distortion of the high type's quantity in the second best.**

Re-examine the monopolist's expected profit

$$E(\Pi) = \beta(T_H - cq_H) + (1-\beta)(T_L - cq_L),$$

How much the monopolist distorts this quantity downward depends on the probabilities of the two consumer types.

1. If there are many low types (β is low) then the monopolist would **not be willing to distort** the low type's quantity very much, because the loss from this distortion would be substantial and there would be few high types from whom additional surplus could be extracted.
2. The more high types (the higher is β), the more the monopolist is willing to distort the low type's quantity downward.

7.3.2 Market Signaling: Market for Lemon

In all the models in asymmetric information part studied so far, the uninformed principal (e.g. monopoly) moved first, making a contract offer to the agent (e.g. consumer), who had private information.
Now in the lemon market, the uninformed principal is buyer, making a contract offer to the agent (e.g. seller), who had private information.

Example 1
Used-car Market with Uniform distribution of quality
The car's quality is not observable to the buyer, the buyer can only have some prior beliefs about the potential qualities (i.e. probability distribution of qualities available in the market), while the seller knows the car's condition.
This is a take-it-or-leave-it offer: Buyer (principal) offers a price, and Seller (agent) accepts or rejects.

Markets for used goods raise an interesting possibility for signaling.
Yet even the mere act of offering the car for sale can be taken as a signal of car quality by the market. **The signal is not positive: the quality of the good must be below the threshold that would have induced the seller to keep it.**

Recall the definition of adverse selection that a form of market failure: products of different qualities are sold at the same price. As a result, Low-quality products drive out high-quality products.
The market may unravel in equilibrium so that only the lowest-quality goods, the "lemons," are sold.

1. Suppose there is a continuum of qualities from low-quality lemons to high-quality gems and that only **the owner of a car knows its type**.
2. Because **buyers cannot differentiate** between lemons and gems, all used cars will sell for the **same price**, which is a function of the average car quality.
3. A car's owner will choose to:
   (a) **keep** it if the car is at the **upper end of the quality spectrum** (since a good car is worth more than the prevailing market price)
   (b) **sell** the car if it is at the **low end** (since these are worth less than the market price).
4. This reduction in average quality of cars offered for sale will reduce market price, leading would-be sellers of the **highest-quality remaining cars to withdraw from the market.**
5. The market continues to unravel until only the **worst-quality** lemons are **offered for sale.** In the extreme, the market can simply break down with nothing (or perhaps just a few of the worst items) being sold.

The lemons problem can be mitigated by:

1. trustworthy used-car dealers
2. development of car-buying expertise by the general public
3. sellers providing proof that their cars are trouble-free
4. sellers offering money-back guarantees

Two reasons why markets can still operate with asymmetric information:

1. **Gains from trade**: for the uninformed party, the overall gain from the transaction exceeds the potential risk of overpaying.
2. **Pledge**: ratings and feedback, certification, standardisation, reputation, warranties.

X has *Uniform Distribution* if

$$f(x) = \frac{1}{b-a}; \quad \Omega = [a, b].$$

We write $X \sim U[a, b]$.
The expected value of uniform distribution:

$$\mathbb{E}(x) = \frac{a+b}{2}$$

[Continue the above example]
Suppose the quality q of used cars is uniformly distributed between 0 and 20,000. Sellers value their cars at q.

$$q \sim U[0, 20,000].$$

- The probability density function (PDF) for this uniform distribution is:

$$f(q) = \begin{cases} \frac{1}{20{,}000}, & \text{if } q \in [0, 20,000], \\ 0, & \text{otherwise.} \end{cases}$$

Buyers (equal in number to the sellers) **place a higher value on cars, $\mathbf{q+b}$**, so there are *gains to be made from trade* in the used-car market.
When sellers have private information about quality and **buyers know only the distribution**.
Let p be the market price.
Sellers offer their cars for sale if and only if $q \leq p$.
The quality of a car offered for sale is thus uniformly distributed between 0 and p

$$q \sim U[0, p].$$

- The probability density function (PDF) for this distribution is:

$$f(q) = \begin{cases} \frac{1}{p}, & \text{if } q \in [0, p], \\ 0, & \text{otherwise.} \end{cases}$$

implying that expected quality is

$$\mathbb{E}(q) = \frac{p}{2}$$

Hence, a **buyer's expected net surplus** is

$$\mathbb{E}(q+b) - p = \frac{p}{2} + b - p = b - \frac{p}{2}$$

There may be multiple equilibria, but the one with the most sales involves the highest value of p:

$$b - \frac{p}{2} = 0$$
$$p^* = 2b$$

Only a fraction $\frac{2b}{20000}$ of the cars are sold.
As b falls, the market for used cars dries up.

Example 2

Used-car Market with Two Types of the Car
V represents the *value of the car* as perceived by the seller.
Two potential qualities with **equal probabilities**:

1. **Low-quality car**: $V = 5000$ (value to the seller is £5000).

2. **High-quality car**: $V = 10,000$ (value to the seller is £10,000).

$$P(V = 5000) = \frac{1}{2}, \quad P(V = 10,000) = \frac{1}{2}.$$

The seller only offers the car for sale if the buyer's price p meets or exceeds their valuation ($p \geq V$).

When buyers do not observe the quality but only know the distribution of V, they calculate the *expected value* $E[V]$:

$$E[V] = \frac{1}{2} \cdot 5000 + \frac{1}{2} \cdot 10,000 = 7500.$$

If the buyer offers a price p, the actual value V is conditioned on whether the seller accepts ($p \geq V$). The real value is then $E[V \mid V \leq p]$, which depends on p:
($E(V) = E[V \mid V \leq p]$ occurs when p is large enough to include all possible values of V. If $p \geq 10,000$, all values of V satisfy $V \leq q$. Therefore, conditioning on $V \leq q$ has no effect, and the conditional expected value is the same as the unconditional expected value.)

- **If** $p \geq 10,000$**:** All cars are sold, and

$$E[V \mid V \leq p] = 7500.$$

- **If** $5000 \leq p < 10,000$**:** Only low-quality cars are sold, and

$$E[V \mid V \leq p] = 5000.$$

- **If** $p < 5000$**:** No cars are sold.

In summary, V is the seller's internal valuation of the car, and the buyer uses it probabilistically to estimate whether the car is worth purchasing.

7.3.3 Signaling Game & Nonlinear Price

If the information structure is reversed and the informed player moves first, then the analysis becomes much more complicated, putting us in the world of signaling games.

Definition Signaling
Signalling is a mechanism which allows the **informed** party to credibly **disclose information** to the uninformed party by **sending a costly signal** in order to distinguish between the sender's types.

Difference between Screening and Signaling
Both signalling and screening are market mechanisms that resolve the problem of adverse selection. However, they differ in which party takes the action first:

1. **Screening**: the uninformed party designs conditions (incentives) which make the informed party reveal its hidden information.
2. **Signaling**: the informed party sends a costly signal to the uninformed.

As we now know that adverse selection is socially undesirable because it eliminates higher-quality goods from the market. However, both buyers and high-quality sellers would like to trade. And from previous example we conclude that there are some ways to make the signal positive for maintaining the market in a good condition.

So now, High-quality sellers are ready to give up some surplus to assure buyers that they sell a quality good. To achieve this, they can send a **costly signal** that credibly distinguishes the **types of sellers**.

In the job market, the workers (sellers) know their types, whether he or she is H-types, high productivity (high-quality) or L-types, low productivity (low-quality), but the firm (buyer) does not know.
Adverse selection eliminates H-types; in equilibrium, only L-types are hired.
Both the firm and the H-type would benefit if the H-type could credibly disclose his high productivity. So now, the **H-type** is ready to **give up some surplus** and incur a **signaling cost** to credibly **distinguish himself from the L-type**.
Let's examine signaling game in more detail combining it with nonlinear price.

Basic setting of more advanced job market signaling game combining nonlinear price

1. Player 1 is a worker who can be one of two types. The worker knows his type.
   (a) high-skilled: high productivity θ_H
   (b) low-skilled: low productivity θ_L
   (c) $\theta_H > \theta_L > 0$
2. Player 2 is a firm that considers hiring the applicant.
3. The firm cannot observe the worker's type, but it knows the probability distribution of types:
   (a) λ a probability of meeting H-type
   (b) $1 - \lambda$ a probability to meet L-type

   It can observe only the applicant's prior education.
4. The firm offers a wage w equal to the expected productivity of the worker:
$$\mathbb{E}(w) = \lambda\theta_H + (1 - \lambda)\theta_L$$
   (*NOTE: Adverse selection eliminates H-types; in equilibrium, only L-types are hired at $w = \theta_L$)
5. Before entering the job market, the worker can get some observable level of education. Cost of obtaining an education
$$c(e, \theta)$$
   - e: money, opportunity cost of time or effort.
     (a) $c'_e(e, \theta) > 0$ the cost increases as effort e increases.
     (b) $c''_{ee}(e, \theta) > 0$ the curve would appear to bend upwards, showing convexity.
     (c) $c'_\theta(e, \theta) < 0$ the cost decreases as the productivity of the worker is higher.
     (d) $c''_{e\theta}(e, \theta) < 0$ the interaction between e and θ results in a diminishing marginal effect of e on $c(e, \theta)$ as θ increases.
     the slopes of the curves become flatter for larger θ, indicating decreasing sensitivity to changes in e as θ grows.
   - θ_i the type of the worker
6. The worker's utility function
$$u(w, e \mid \theta) = w - c(e, \theta)$$
   - The worker's outside option is to be unemployed $\quad u(0) = 0$
7. Timing of the game:
   (a) Stage 1: The worker learns his type θ.
   (b) Stage 2: The worker chooses an education level e conditional on his type.
   (c) Stage 3: The firm observes e and makes a wage offer.

8. **Perfect Bayesian Equilibrium**
   The worker's strategy is optimal given the firm's strategy. The belief function $\mu(e)$ is derived from the worker's strategy using the Bayes' rule where possible.

   - If higher education e is costlier for low-ability workers but less costly for high-ability workers, then e might signal high ability to the firm, leading to a higher $\mu(e)$.
   - if e doesn't differentiate types, $\mu(e)$ might remain unchanged.

   The firm's wage offer following each choice of the signal constitutes a NE in which the probability of meeting H-type is $\mu(e)$.

Separating Equilibrium

As mentioned in the signaling game, different types choose different signals.
i.e. **Supporting beliefs** $\mu(\hat{e})$ is a probability of meeting H-type if the observed education level is $\hat{e}$.

- $\mu(\hat{e})$=1
  If the firm observes the education level $\hat{e}$, it believes with certainty $\mu = 1$ that the worker is of high type (H-type).
- $\mu(e \neq \hat{e}) = 0$
  If the firm observes any education level other than $\hat{e}$, it believes with certainty $\mu = 0$ that the worker is not of high type (i.e., the worker is of low type).

We will examine this supporting beliefs later.

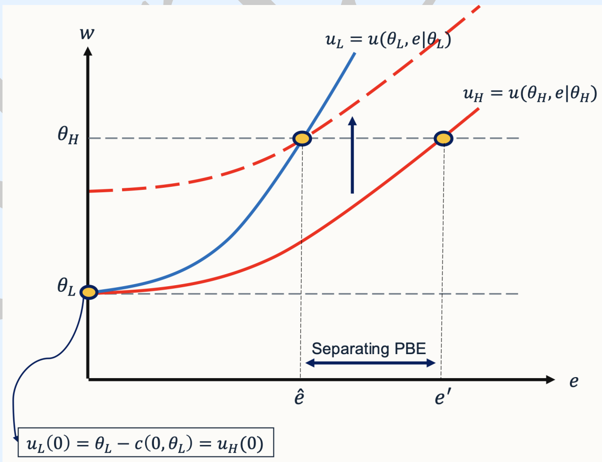

Figure 23: Separating Equilibrium in Job Market Signaling

Description

Given the background setting of the game, the constitution of the graph:

Indifference curves are shown by $\bar{u} = w - c(e, \theta)$.

- Indifference curves are increasing and convex.
- $IC_L(e)$ is deeper than $IC_H(e)$.
- Indifference curve shifts upward as utility increases since w is a good and e is a bad that is costly to workers.

Lemma 1
In a separating equilibrium, each worker's type receives a wage offer equal to the actual productivity as the **firm can distinguish** the worker's **types** by observing **different education levels**, it makes the **wage offer accordingly**:

1. $w_H = \theta_H$
2. $w_L = \theta_L$

Therefore, each type:

$$\max_e u_i(e, \theta_i) = \theta_i - c(e, \theta_i)$$

Lemma 2
In a separating equilibrium, L-type chooses no education since H-type chooses an education level not affordable for L-type. Thus, L-type has no incentive to choose any $e > 0$.

Still looking back to graph 23.

1. At $e = 0$, the worker is perceived as L-type: $w = \theta_L$.
2. At $\hat{e}$ L-type is indifferent between being perceived as L-type and being perceived as H-type:
$$\theta_L - c(0, \theta) = \theta_H - c(\hat{e}, \theta_L)$$
So L-type never chooses $e > \hat{e}$. Or to say, any education level $e > \hat{e}$ is not affordable for L-type.
3. At e' H-type is indifferent between being perceived as L-type and being perceived as H-type:
$$\theta_L - c(0, \theta_H) = \theta_H - c(e', \theta_H)$$
However, H-type chooses an education level $\hat{e}$ which is not affordable for L-type but that gives a payoff greater than $u(0, \theta_H)$:
$$u(0, \theta_L) \geq u(\hat{e}, \theta_L)$$
$$u(\hat{e}, \theta_H) \geq u(0, \theta_H)$$
H-type can increase his utility and shift his indifference curve upward by choosing $e \in [\hat{e}, e']$, as this education level credibly distinguishes H-type from L-type. And H-type never chooses $e > e'$.

Pooling Equilibrium

As mentioned in the signaling game, both types choose to send the same signal. Both types choose the same education level, and both types can afford this education level. Therefore, the firm still cannot distinguish the worker's types according to their education level.

Supporting Belief

In a pooling equilibrium, workers of different types (e.g., high type θ_H and low type θ_L) choose the same education level, denoted as e^P. The firm's belief that e^P corresponds to a mix of high and low types reflects the inability to separate types in this equilibrium.

1. $\mu(e^P) = \lambda$: When the firm observes the education level e^P, it assigns a probability λ that the worker is of high type (θ_H).
   - λ represents the proportion of high-type workers in the population or the firm's prior belief about the probability of encountering a high type.
2. $\mu(e \neq e^P) = 0$: If the firm observes any education level other than e^P, it believes with certainty ($\mu = 0$) that the worker is not of high type. This means that only e^P is consistent with the equilibrium behavior.

- The inequality $e^P \leq \tilde{e}$ implies that e^P, the chosen education level in the pooling equilibrium, is feasible and acceptable to both worker types. It is not excessively costly for the low-type worker, ensuring that they will also choose e^P.
- This belief structure ensures that both types of workers are incentivized to pool at the education level e^P. The firm uses its prior probability λ to make wage decisions, as it cannot update its belief based on the signal.

Subsequently, The firm offers w equal to the expected productivity:

$$w^* = E[\theta] = \lambda\theta_H + (1-\lambda)\theta_L$$

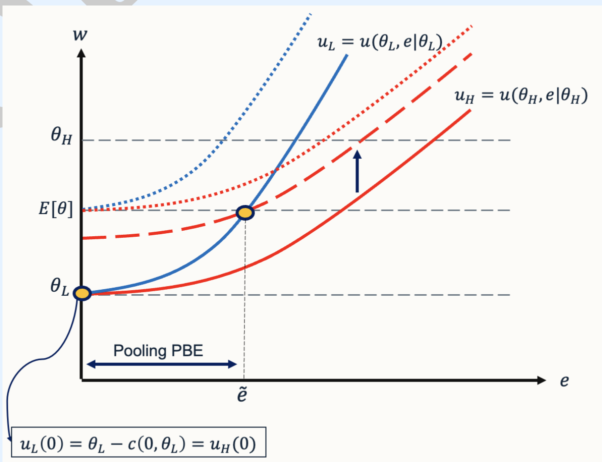

Figure 24: Pooling Equilibrium in Job Market Signaling

Definition Pareto Dominate
An outcome X Pareto dominates another outcome Y if:

1. At least one individual is **strictly better off** in X compared to Y, and
2. No individual is **worse off** in X compared to Y.

Mathematical Representation:
Let $u_i(X)$ and $u_i(Y)$ denote the utility of individual i in outcomes X and Y, respectively. Then X Pareto dominates Y if:

$$\forall i,\ u_i(X) \geq u_i(Y) \quad \text{and} \quad \exists j \text{ such that } u_j(X) > u_j(Y)$$

This means:

- Every agent is at least as well off in X as in Y.
- At least one agent is strictly better off in X.

Description

1. At $\tilde{e}$ L-type is indifferent between earning $w^* = E[\theta]$ and earning $w = \theta_L$:
$$E[\theta] - c(\tilde{e}, \theta_L) = \theta_L - c(0, \theta_L)$$
Thus, the highest possible education level is $\tilde{e}$ since
$$u(\tilde{e}, \theta_L) = u(0, \theta_L)$$
2. Any $e \in [0, \tilde{e}]$ can support a pooling equilibrium.
3. **No signal** $(e = 0)$ Pareto dominates any other pooling equilibria with positive e.
   (a) **Pooling Equilibrium with No Signal ($e = 0$):**
   - In this scenario, workers do not send any signal to the firm (i.e., they do not invest in education as a signal, and $e = 0$).
   - The firm treats all workers the same, as it cannot distinguish between high and low types due to the absence of a signal.

   (b) **Pareto Dominance:** Here, $e = 0$ Pareto dominates equilibria with $e > 0$, meaning that in the absence of a signal:
   - Workers save on the cost of signaling (education level $e > 0$ is costly for workers).
   - The firm does not gain additional information from $e > 0$ in a pooling equilibrium, so the firm's belief structure remains the same, as it relies on the prior distribution (λ) to make decisions in all pooling equilibria, its decision-making process remains unaffected.
   - Since no one is worse off and workers are better off, $e = 0$ is Pareto superior.
   - Therefore, no signal is preferred. As represented in dash line, Education cost is 0, and both types earn w^*.

References

[1] W. Nicholson. *Microeconomic theory: basic principles and extensions*. South Western Educational Publishing, 2005.