Producer Theory Formula

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Preface

This is the second time that I use LaTeXto take the note. Thanks to the courses **'Linear Algebra with MATLAB'** in the **'2024 Summer School in Mathematical Economics & Financial Econometrics (UNNC)'**, which I learnt the LaTeXand tried my best to practice during my free time in UNUK.

The content in this notebook is mainly a complementary material suitable to **ECON 2001 Microeconomic Theory** in UNUK. The content taught in UNNC and UNUK is a little bit different and may be changed from time to time. However, the main knowledge always around what I had collected. During my study, I combined PPT and the suggested textbook [1].

The course I learnt in UNUK also contains **Consumer Theory** and **Game Theory** section, which I can also provide complementary notebooks.

Since this is my second time to write the LaTeX, The General Equilibrium part is not included in this notebook since the arrays and tables always lead to 'fail to compiling'. The only convenience that I can provide in this notebook is the outline or the structure of the microeconomics learning and collection of the formula.

Moreover, I know that I may make mistake when I am writing this notebook. Therefore, if there is any error being detected, please do not hesitate to content me: `hmyhw8@nottingham.edu.cn` (If this not work, use my personal email).

Other notebooks can be found here.

Good luck with your Microeconomic Theory study!

1 Production Function

1.1 Production Function

Generally, the inputs are $(x_1, x_2, ..., x_n)$, the out put is Y, so the production function is:

$$Y = f(x_1, x_2, ..., x_n)$$

More specifically, the inputs are Labor (L) and Capital (K).

$$Y = f(L, K) \tag{1.1}$$

Example

For Cobb-Douglas function

$$f(x_1, x_2) = Ax_1^{\alpha}x_2^{\beta} \quad (A = 1, \alpha + \beta = 1)$$

1. **A**: the scale of production
2. α, β:how output reacts to an increase in inputs

Properties Production Function

1. **Monotonicity**
2. **Convexity**
3. **Homogeneity**: $f(\lambda x_1, \lambda x_2) = \lambda^r f(x_1, x_2)$ represent **return to scale**
   $r > 1$ Increasing Return to Scale
   $r < 1$ Dncreasing Return to Scale
   $r = 1$ Constant Return to Scale

1.2 Isoquant

Definition Isoquant
The Y_1 isoquant is a curve that represents all the combinations of L and K that give a level of output equal to Y_1. i.e.

$$f(L,K) = Y_1 \tag{1.2}$$

1.3 Marginal Product

$$MP_L = \frac{\partial f(L,K,...)}{\partial L} \tag{1.3}$$

$$MP_K = \frac{\partial f(L,K,...)}{\partial K} \tag{1.4}$$

Properties Marginal Product

- **Positive but Diminishing Marginal Rate of Return**
- the marginal productivity functions derived from a **constant** returns-to-scale production function are **homogeneous of degree 0**.
- **Homothetic** The marginal productivity of any input depends only on the ratio of capital to labor input, not on the absolute levels of these inputs.

$$\begin{aligned} MP_k &= \frac{\partial f(k,l)}{\partial k} = \frac{\partial f(tk,tl)}{\partial k}(t>0) \xrightarrow{t=\frac{1}{l}} MP_k = \frac{\partial f\left(\frac{k}{l},1\right)}{\partial k} \\ MP_l &= \frac{\partial f(k,l)}{\partial l} = \frac{\partial f(tk,tl)}{\partial l}(t>0) \xrightarrow{t=\frac{1}{l}} MP_l = \frac{\partial f\left(\frac{k}{l},1\right)}{\partial l} \end{aligned} \tag{1.5}$$

1.4 Marginal Rate of Technical Substitution

Definition. Marginal Rate of Technical Substitution
$\mathbf{MRTS_{L,K}}$ is the rate at which a factor of production (L) can be substituted for another (K) to keep the quantity produced constant.

Formula

$$MRTS_{L,K} = \frac{\Delta K}{\Delta L} = -\frac{MP_L}{MP_K} \tag{1.6}$$

Proof
Total differentiation

$$\Delta y = \frac{\partial f(L,K)}{\partial L}\Delta L + \frac{\partial f(L,K)}{\partial K}\Delta K = 0$$

Properties Marginal Rate of Technical Substitution

- **Diminishing**
- **Homothetic**

2 Cost Function

2.1 Cost Minimization

Definition Cost Minimization
Choose inputs to minimize costs subject to producing a given level of output.

2.1.1 Interior Solution

Two goods senario
First, we assume that there are only two inputs: homogeneous labor (l, measured in labor-hours) and homoge- neous capital (k, measured in machine-hours).
Second, we assume that inputs are hired in perfectly competitive markets. Firms can buy (or sell) all the labor or capital services they want at the prevailing rental rates (w and v).
Now the problem is: To minimize the cost of producing a given level of output, a firm should choose that point on the q_0 isoquant at which the rate of technical substitution of l for k is equal to the ratio $\frac{w}{v}$.
Mathematically, it is a **cost minimization problem**:

$$\min_{k,l} \quad c(v,w,k,l) = vk + wl \quad \text{s.t.} \quad f(k,l) = q_0 \tag{2.1}$$

set the **Lagrange Form**:

$$\mathcal{L}(v,w,q_0,k,l) = vk + wl + \lambda[q_0 - f(k,l)] \tag{2.2}$$

To find the minimum, satisfies **F.O.C**:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial l} &= w - \lambda \frac{\partial f}{\partial l} = 0 \\ \frac{\partial \mathcal{L}}{\partial k} &= v - \lambda \frac{\partial f}{\partial k} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= q_0 - f(k,l) = 0 \end{aligned} \tag{2.3}$$

Dividing the first two equations we can get: the cost-minimizing firm should equate the RTS for the two inputs to the ratio of their prices.

$$\frac{w}{v} = \frac{\partial f/\partial l}{\partial f/\partial k} = RTS\ (l \text{ for } k). \tag{2.4}$$

This means: **Cost minimization Principle** In order to minimize the cost of any given level of input (q_0), the firm should produce at that point on the q_0 isoquant for which the RTS (of l for k) is equal to the ratio of the inputs' rental prices ($\frac{w}{v}$).

We can also rewrite the above equation as:

$$\frac{MPK}{v} = \frac{MPL}{w} \tag{2.5}$$

This means: for costs to be minimized, the marginal productivity per dollar spent should be the same for all inputs.

Lagrangian Multiplier

The Lagrangian multiplier shows how much in extra costs would be incurred by increasing the output constraint slightly.

$$\lambda = \frac{w}{MPL} = \frac{v}{MPK} \tag{2.6}$$

This is the extra cost of obtaining an extra unit of output by hiring either added labor or added capital input.

Notice that if we use q rather than q_0 meaning that the given q is also a parameter that can be changed, we can derive:

$$\frac{\partial \mathcal{L}}{\partial q} = \lambda \quad \text{i.e.} \quad \frac{\partial c(w, v, q)}{\partial q} = \text{Marginal Cost} \tag{2.7}$$

Here $c(w, v, q)$ is actually the cost function that we will introduce later.

General Form

$$\min_{x_1\geq 0,\dots,x_n\geq 0} \quad c(\mathbf{w},\mathbf{x}) = w_1x_1 + \dots + w_nx_n \quad \text{s.t.} \quad f(\mathbf{x}) = f(x_1,\dots,x_n) = y_0 \tag{2.8}$$

Lagrange Form

$$\mathcal{L}(\omega, y_0, \mathbf{x}) = c(\mathbf{w},\mathbf{x}) - \lambda[f(\mathbf{x}) - y_0] = w_1x_1 + \dots + w_nx_n - \lambda[f(x_1,\dots,x_n) - y_0] \tag{2.9}$$

Suppose thee are only two inputs, so the equations are:

$$\begin{aligned}&\min_{x_1\geq 0,x_2\geq 0} \quad c(\mathbf{w},\mathbf{x}) = w_1x_1 + w_2x_2 \quad \text{s.t.} \quad f(\mathbf{x}) = f(x_1,x_2) = y_0\\ &\text{Transfer to:}\\ &\mathcal{L}(w_1,w_2,y_0,x_1,x_2) = c(w_1,w_2,x_1,x_2) - \lambda[f(x_1,x_2) - y_0]\\ &= w_1x_1 + w_2x_2 - \lambda[f(x_1,x_2) - y_0]\end{aligned} \tag{2.10}$$

F.O.C for finding max

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x_1} &: w_1 - \lambda\frac{\partial f(x_1,x_2)}{\partial x_1} = 0\\ \frac{\partial \mathcal{L}}{\partial x_2} &: w_2 - \lambda\frac{\partial f(x_1,x_2)}{\partial x_2} = 0\\ \frac{\partial \mathcal{L}}{\partial \lambda} &: f(x_1,x_2) - y_0 = 0\end{aligned} \tag{2.11}$$

Lagrange Multiplier

$$\begin{aligned}&\lambda = \frac{w_1}{MP_1} = \frac{w_2}{MP_2}\\ &\overset{Generalise}{\Rightarrow}\\ &\lambda = \frac{w_1}{MP_1} = \frac{w_2}{MP_2} = \dots = \frac{w_n}{MP_n}\end{aligned} \tag{2.12}$$

Price for the input (i.e.) cost for each input

$$\begin{aligned}&w_1 = \lambda\frac{\partial f(x_1,x_2)}{\partial x_1} = \lambda MP_1\\ &w_2 = \lambda\frac{\partial f(x_1,x_2)}{\partial x_2} = \lambda MP_2\\ &\overset{Generalise}{\Rightarrow}\\ &w_i = \lambda\frac{\partial f(x_1,\dots x_n)}{\partial x_i} = \lambda MP_i\end{aligned} \tag{2.13}$$

Now we focus on two goods solution. **To solve for the solution:**

$$\frac{MP_1}{MP_2} = \frac{w_1}{w_2} \tag{2.14}$$

and then substitute it into:

$$\frac{\partial \mathcal{L}}{\partial \lambda} : f(x_1,x_2) - y_0 = 0$$

we can derive:

$$x_1^\star(\omega_1,\omega_2,y_0) \quad x_2^\star(\omega_1,\omega_2,y_0) \tag{2.15}$$

we can directly denote it as:

$$x_1^c(\omega_1, \omega_2, y_0) \quad x_2^c(\omega_1, \omega_2, y_0) \tag{2.16}$$

This solution is called: **Conditional Factor Demand Functions**.
For only two inputs scenario, the Conditional Factor Demand Functions are:

$$x_1^c(\omega_1, \omega_2, y_0) \text{ and } x_2^c(\omega_1, \omega_2, y_0)$$

We also need to check **S.O.C.**:

$$\frac{\partial^2 \mathcal{L}}{\partial x_1^2} = -\lambda \frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} \geq 0$$
$$\frac{\partial^2 \mathcal{L}}{\partial x_2^2} = -\lambda \frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} \geq 0$$
$$\frac{\partial^2 \mathcal{L}}{\partial x_1^2} \frac{\partial^2 \mathcal{L}}{\partial x_2^2} - \left(\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} \right)^2 \geq 0$$

Substituting the Conditional Factor Demand Functions into the expression for the cost (i.e. using the **Envelope Theorem**), we obtain:
Cost Function

$$\begin{aligned} c(w_1, w_2, y) &\overset{\text{Envelope Theorem}}{=} c(w_1, w_2, y, x_1^c(\omega_1, \omega_2, y_0), x_2^c(\omega_1, \omega_2, y_0)) \\ c(w_1, w_2, y) &= w_1 x_1^c(w1, w2, y) + w_2 x_2^c(w1, w2, y) \end{aligned} \tag{2.17}$$

Notion: Since y_0 can be given at any level, I use y here to represent that the output level can be any given level.

Properties Cost Function

1. $\mathbf{c(\omega_1, \omega_2, 0) = 0}$
2. **Increasing in y**
3. **Increasing in** $\boldsymbol{\omega_1, \omega_2}$
4. **Homogeneous of degree 1 in** $\boldsymbol{\omega_1, \omega_2}$
5. **Concave in** $\boldsymbol{\omega_1, \omega_2}$
6. **Shepherd's Lemma:** $\frac{\partial \mathcal{L}}{\partial w_i} = \frac{\partial c(\omega_1, \omega_2, y)}{\partial \omega_i} = x_i^c(\omega_1, \omega_2, y)$ for i = 1,2
   is just the **Conditional Factor Demand Functions**

2.1.2 Corner Solution

$x_1 = 0$ or $x_2 = 0$, or both

$$\begin{aligned} &①\frac{\partial \mathcal{L}}{\partial x_1} \geq 0 \\ &\text{if } x_1 > 0 \text{ then } \frac{\partial \mathcal{L}}{\partial x_1} = 0 \\ &\text{if } x_1 = 0 \text{ then } \frac{\partial \mathcal{L}}{\partial x_1} > 0 \end{aligned} \tag{2.18}$$

$$\begin{aligned} &②\frac{\partial \mathcal{L}}{\partial x_2} \geq 0 \\ &\text{if } x_2 > 0 \text{ then } \frac{\partial \mathcal{L}}{\partial x_2} = 0 \\ &\text{if } x_2 = 0 \text{ then } \frac{\partial \mathcal{L}}{\partial x_2} > 0 \end{aligned} \tag{2.19}$$

2.2 Isocosts

Definition Isocosts
Combinations of x_1, x_2 that have some given cost C_0.
i.e.

$$w_1x_1 + w_2x_2 = C_0 \tag{2.20}$$

$$\text{The negative slope of Isoquant} = \frac{w_1}{w_2} = \frac{MP_1}{MP_2} \tag{2.21}$$

To minimize cost, the slope of the isoquant needs to be equal to the slope the isocost (tangency condition).

3 Cost curve

Recall **Cost Function**:

$$c(w_1, w_2, y) = w_1 x_1^c(w1, w2, y) + w_2 x_2^c(w1, w2, y) \tag{3.1}$$

Now we need to transfer it into a function that contain only one parameter,y.
i.e., The **cost function** become $c(y)$.
Since now we only focus on two inputs: labor(L) and capital(K), and suppose the wage for labor is w, the rent for capital is r, then the cost function become more specifically:

$$c(w, r, y) = wL(w, r, y) + rK(w, r, y) \tag{3.2}$$

3.1 Short Run

It is important to understand that total short-run costs are not the minimal costs for producing the various output levels. Because we are holding capital fixed in the short run, the firm does not have the flexibility of input choice that we assumed. Rather, to vary its output level in the short run, the firm will be forced to use "nonoptimal" input combinations: The MRTS will not be equal to the ratio of the input prices.

3.1.1 Formula

In the short run, the capital is fixed, which means $K = \bar{K}$. So the production function is $y = f(L, \bar{K})$.
Therefore, "inversing" the function we can have $L = g(y, \bar{K})$. So now we use $L(y, \bar{K})$ to represent labor(L).

1. **Total Cost**

$$\begin{aligned} c(y) &= F + c_v(y) \\ c_s(y, \bar{K}) &= r\bar{K} + wL(y, \bar{K}) \end{aligned} \tag{3.3}$$

2. **Fixed Cost**

$$FC_s = r\bar{K} \tag{3.4}$$

3. **Variable Cost**

$$c_{v,s}(y, \bar{K}) = wL(y, \bar{K}) \tag{3.5}$$

4. **Average Total Cost**

$$\begin{aligned} AC(y) &= \frac{c(y)}{y} = \frac{F}{y} + \frac{c_v(y)}{y} \\ AC_s(y, \bar{K}) &= \frac{wL(y, \bar{K}) + r\bar{K}}{y} \end{aligned} \tag{3.6}$$

5. **Average Fixed Cost**

$$\begin{aligned} AFC(y) &= \frac{F}{y} \\ AFC_s(\bar{K}) &= \frac{r\bar{K}}{y} \end{aligned} \tag{3.7}$$

6. **Average Variable Cost**

$$\begin{aligned} AVC(y) &= \frac{c_v(y)}{y} \\ AVC_s(y,\bar{K}) &= \frac{wL(y,\bar{K})}{y} \end{aligned} \tag{3.8}$$

7. **Marginal Cost**

$$\begin{aligned} MC(y) &= \frac{\partial c(y)}{\partial y} \\ MC_s(y,\bar{K}) &= \frac{\partial c_s(y,\bar{K})}{\partial y} = w\frac{\partial L(y,\bar{K})}{\partial y} \end{aligned} \tag{3.9}$$

3.1.2 Relationship between Formula

Derivative Total Cost function (TC), Average Cost function (AC) and Average Variable Cost function (AVC) $w.r.t.y$, we can have:

$$\begin{aligned}
\frac{\partial TC}{\partial y} &= \frac{\partial c(y)}{\partial y} = \frac{\partial (F + c_v(y))}{\partial y} = \frac{\partial c_v(y)}{\partial y} = \frac{\partial VC}{\partial y} = MC(y) \\
\frac{\partial AC(y)}{\partial y} &= \frac{\partial \left(\frac{c(y)}{y}\right)}{\partial y} = \frac{y\frac{\partial c(y)}{\partial y} - c(y)}{y^2} = \frac{MC(y) - AC(y)}{y} \\
\frac{\partial AVC(y)}{\partial y} &= \frac{\partial \left(\frac{c_v(y)}{y}\right)}{\partial y} = \frac{y\frac{\partial c_v(y)}{\partial y} - c_v(y)}{y^2} = \frac{MC(y) - AVC(y)}{y}
\end{aligned} \tag{3.10}$$

To satisfy **F.O.C.** of the above equation, we need to make MC intersect with $\min AC$ and $\min AVC$, because in **F.O.C.** ,

$$\begin{aligned} MC(y) &= AC(y) \\ MC(y) &= AVC(y) \end{aligned}$$

3.2 Long Run

In the long run, the capital is not fixed, so K is also a variable. Therefore, the firm will be able to change its level of capital input and will adjust its input usage to the cost-minimizing combinations.
Now we can use $y = f(L, K)$ as the production function. And $c(w, r, y) = wL(w, r, y) + rK(w, r, y)$ as the **Total Cost** function. And there is **no fixed cost**. i.e.

Variable Cost (VC)= Total Cost(TC)
Average Total Cost (AC) = Average Variable Cost(AVC)

4 Profit Function

4.1 Profit Maximization Problem

Definition Profit Maximization
Given the output level, choose quantity of inputs that maximizes profit, the difference between revenue and costs.

General Form

$$\max_{x_1 \geq 0, \ldots, x_n \geq 0} \quad \pi = py - c(\mathbf{w}, \mathbf{x}) = py - w_1 x_1 + \ldots + w_n x_n \quad \text{s.t.} \quad y = f(\mathbf{x}) = f(x_1, \ldots x_n) \tag{4.1}$$

$(y = y_0$ is the chosen output level)

Lagrange Form

$$\begin{aligned} \max_{x_1 \geq 0, \ldots, x_n \geq 0} \quad \mathcal{L}(\omega, \mathbf{x}) &= pf(\mathbf{x}) - w_1 x_1 - \ldots - w_n x_n \\ &= pf(x_1, \ldots x_n) - w_1 x_1 - \ldots - w_n x_n \end{aligned} \tag{4.2}$$

To find the maximization, satisfy the **F.O.C.**:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &: p\frac{\partial f(x_1, \ldots, x_n)}{\partial x_1} - w_1 = 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} &: p\frac{\partial f(x_1, \ldots, x_n)}{\partial x_2} - w_2 = 0 \\ &\vdots \\ \frac{\partial \mathcal{L}}{\partial x_n} &: p\frac{\partial f(x_1, \ldots, x_n)}{\partial x_n} - w_n = 0 \end{aligned} \tag{4.3}$$

This means **Marginal Cost of input (MC)** ω_i**= Marginal Revenue(MR)** $p\frac{\partial f(x_1,\ldots,x_n)}{\partial x_i}$ as:

$$\begin{aligned} \underbrace{p\frac{\partial f(x_1, \ldots, x_n)}{\partial x_1}}_{\text{Marginal Revenue}} &= \underbrace{w_1}_{\text{Marginal Cost of input 1}} \\ \underbrace{p\frac{\partial f(x_1, \ldots, x_n)}{\partial x_2}}_{\text{Marginal Revenue}} &= \underbrace{w_2}_{\text{Marginal Cost of input 2}} \\ &\vdots \end{aligned} \tag{4.4}$$

Recall we have:

$$\frac{w}{v} = \frac{\partial f/\partial l}{\partial f/\partial k} = RTS \ (l \text{ for } k). \tag{4.5}$$

in the cost minimization problem for two input senario (L and K), and now we can also have:

$$\frac{w}{v} = \frac{\partial f/\partial l}{\partial f/\partial k} \tag{4.6}$$

in this problem, so the profit maximization problem also imply cost minimization.

The solution to this **F.O.C.** is **Factor Demand Functions**

$$\begin{gathered} x_1^\star(p, w_1, w_2, ..., w_n) \\ x_2^\star(p, w_1, w_2, ..., w_n) \\ \vdots \end{gathered} \tag{4.7}$$

These demand functions are "unconditional".

By substituting the **Factor Demand Functions** into the **Production Function** we obtain the **Output Supply Function**, which gives us the amount of output that maximizes profit:

$$y^\star(p, w_1, ..., w_n) = f(x_1^\star(p, w_1, ..., w_n), x_2^\star(p, w_1, ..., w_n)) \tag{4.8}$$

(here y is the variable rather than a given parameter. The given parameters are $p, w_1, ..., w_n$)

Now we can derive the **Profit Function** by substituting the **Factor Demand Functions** and the **Output Supply Function** into the expression for profit maximization problem:

$$\pi(p, w_1, ..., w_n) = y^\star(p, w_1, ..., w_n) - w_1 x_1^\star(p, \mathbf{w}) - w_2 x_2^\star(p, \mathbf{w}) - ... - x_n^\star(p, \mathbf{w}) \tag{4.9}$$

Properties Profit Function

1. **Increasing in p**
2. **Dncreasing in** $\omega_{\mathbf{1}}, \omega_{\mathbf{2}}, ..., \omega_{\mathbf{n}}$
3. **Homogeneous of degree 1 in** $\mathbf{p}, \omega_{\mathbf{1}}, \omega_{\mathbf{2}}, ..., \omega_{\mathbf{n}}$
4. **Convex in p**
5. **Hotelling's Lemma:** (homogeneous of degree 0)

   As the price of the final output increases, then the increase in profit is equal to the quantity of output we are selling.

   $$\frac{\partial \pi(p, w_1, ..., w_n)}{\partial p} = y^\star(p, w_1, ..., w_n) \quad \textbf{Output Supply Function} \tag{4.10}$$

   (Although in the expression of Profit Function, $x_i^\star(p, \mathbf{w})$ has price(p) as parameter, when looking back to the **F.O.C.** condition, we can see that when deriving the Factor Demand Function, the p has been cancelled out. Therefore, when derivate profit function $w.r.t$ p, p in $x_i^\star(p, \mathbf{w})$ does not take into account.)

   As the price of an input increases, the decrease in profit is equal to the quantity of that input we are using for production.

   $$\frac{\partial \pi(p, w_1, ..., w_n)}{\partial w_i} = -x_i^\star(p, w_1, ..., w_n) \quad \textbf{Factor Demand Function} \tag{4.11}$$

   (In the profit function, changing w_i affects the cost term associated with $-w_i x_1^\star$directly. Although the term $y^\star$ is implicitly influenced by all prices, including w_i, But in the partial differentiation $w.r.t.w_i$, the immediate impact comes directly from $-w_i x_1^\star$.)

4.2 Profit Maximization Link to Monopoly

Recall in equation 4.4 we have:
Marginal Cost of input (MC) ωi = Marginal Revenue(MR) $p\frac{\partial f(x_1,\ldots,x_n)}{\partial x_i}$.

Since we know that in the monopoly, the single supplier in the market may choose to produce at any point on the market demand curve. So the price is affected by the quantity Q that this supplier want to produce. So we can have the revenue function:

$$\textbf{Total Revenue} = p(Q) \times Q \quad \text{where} \quad Q = f(x_1, \ldots x_n) \tag{4.12}$$

We can derive **Marginal Revenue** as:

$$\text{Marginal Revenue} = MR(Q) = \frac{d}{dQ}[p(Q) \times Q] = \frac{d}{dQ}[p(Q)] \times Q + p \tag{4.13}$$

Now we have another definition relate to elasticity:

Definition Elasticity of Demand($e_{q,p}$)
The percentage change in quantity demanded (q) that results from a 1 percent change in price (p).

$$e_{q,p} = \frac{\frac{dq}{q}}{\frac{dp}{p}} = \frac{\frac{dq}{dp}}{\frac{p}{q}} = \frac{dq}{dp} \cdot \frac{p}{q} \tag{4.14}$$

So rearrange the Marginal Revenue we can have:

$$MR = \frac{d}{dQ}[p(Q)] \times Q + p = p\left(1 + \frac{Q}{p} \cdot \frac{dp}{dQ}\right) = p\left(1 + \frac{1}{e_{Q,p}}\right).$$

Table 1: Relationship between Elasticity and Marginal Revenue

Elastic	$e_{Q,p} < -1$	MR > 0
Unit Elastic	$e_{Q,p} = -1$	MR $= 0$
Inelastic	$e_{Q,p} > -1$	MR < 0

So we can conclude that a **monopoly** will choose to operate only in regions in which the market demand curve is **elastic** ($e_{Q,p} < -1$).

Since MR=MC for profit maximization, we can have:

$$MC = \frac{d}{dQ}[p(Q)] \times Q + p = p\left(1 + \frac{Q}{p} \cdot \frac{dp}{dQ}\right) = p\left(1 + \frac{1}{e_{Q,p}}\right).$$

Rearrange it, we can have:

Formula The Inverse Elasticity Rule

$$\frac{P - MC}{P} = -\frac{1}{e_{Q,p}} \tag{4.15}$$

Index Classification of Elasticity

1. **Elastic Demand/Supply ($|e_{q,p}| > 1$):**
   - When the percentage change in quantity is greater than the percentage change in price.
   - Formula representation:
   $$|e_{q,p}| > 1$$
2. **Inelastic Demand/Supply ($|e_{q,p}| < 1$):**
   - When the percentage change in quantity is less than the percentage change in price.
   - Formula representation:
   $$|e_{q,p}| < 1$$
3. **Unitary Elastic Demand/Supply ($|e_{q,p}| = 1$):**
   - When the percentage change in quantity is equal to the percentage change in price.
   - Formula representation:
   $$|e_{q,p}| = 1$$

4.2.1 Price Discrimination in Monopoly

Definition Price Discrimination
A monopoly engages in price discrimination if it is able to sell otherwise **identical units** of output at **different prices**.

Feasible Price discrimination strategy is feasible depends crucially on the inability of buyers of the good to practice **arbitrage**. i.e. When resale is costly or can be pre- vented entirely, then price discrimination becomes possible.

First-degree (perfect) price discrimination

If each buyer can be **separately identified** by a monopolist, then it may be possible to charge each the **maximum price he or she would willingly pay for the good.**
This strategy of perfect (or first-degree) price discrimination would then extract **all available consumer surplus**, leaving demanders as a group indifferent between buying the monopolist's good or doing without it.
The monopolist will proceed in this way up to the point at which the marginal buyer is no longer willing to pay the good's marginal cost.

Third-degree Price Discrimination through Market Separation

The monopoly can separate its buyers into relatively **few identifiable markets** (such as "rural-urban," "domestic-foreign," or "prime-time–off-prime") and **pursue a separate monopoly pricing** policy in each market.
Assume that **marginal cost** is the **same** in **all markets**.
Recall the price elasticities of demand:

$$MC = \frac{d}{dQ}[p(Q)] \times Q + p = p\left(1 + \frac{Q}{p} \cdot \frac{dp}{dQ}\right) = p\left(1 + \frac{1}{e_{Q,p}}\right).$$

We can have:

$$P_i(1 + \frac{1}{e_i}) = P_j(1 + \frac{1}{e_j})$$

- P_i and P_j: the prices charged in markets i and j.
- price elasticities of demand given by e_i and e_j.
- profit-maximizing **price** will be **higher** in markets in which **demand is less elastic**.
- The two-price discriminatory policy is clearly more profitable for the monopoly than a single-price policy would be, because the firm can always opt for the latter policy should market conditions warrant.
- The **welfare** consequences of third-degree price discrimination are, in principle, **ambiguous**.
  The multiple-price policy will be allocationally superior to a single- price policy only in situations in which **total output is increased through discrimination.**

Second-degree Price Discrimination through Price Schedules

The monopoly to choose a (possibly rather complex) price schedule that provides incentives for demanders to **separate themselves** depending on how much they wish to buy.

- quantity discounts
- minimum purchase requirements or "cover" charges
- tie-in sales
  a seller conditions the sale of one product (the "tying product") on the purchase of another product (the "tied product").

Feasible Second-degree price discrimination is feasible only when there are **no arbitrage** possibilities.

One form of pricing schedule that has been extensively studied is a **linear two-part tariff.**

Two-part Tariffs
Demanders must pay:

1. a fixed fee for the right to consume a good
2. a uniform price for each unit consumed

The tariff any demander must pay to purchase q units of a good:

$$T(q) = a + pq$$

- a the fixed fee
- p the marginal price to be paid

The average price paid by any demander:

$$\bar{p} = \frac{T(q)}{q} = \frac{a}{q} + p$$

Feasible When those who pay low average prices (those for whom q is large) cannot resell the good to those who must pay high average prices (those for whom q is small).

The monopolist's goal Choose a and p to maximize profits, given the demand for this product.

1. firm set $p = MC$ to extract the maximum consumer surplus from a given set of buyers.
2. a could be set equal to the surplus enjoyed by the least eager buyer. He or she would then be indifferent about buying the good, but all other buyers would experience net gains from the purchase.

This feasible tariff might not be the most profitable, however. Consider the effects on profits of a small increase in p above MC.

1. This would result in no net change in the profits earned from the least willing buyer.
2. Quantity demanded would drop slightly at the margin where $p = MC$, and some of what had previously been consumer surplus (and therefore part of the fixed fee, a) would be converted into variable profits because now $p > MC$.
3. For all other demanders, profits would be increased by the price rise.

Although each will pay a bit less in fixed charges, profits per unit bought will rise to a greater extent.

5 Duality for Cost Minimization and Profit Maximization Problem

Like in the consumer theory, we can also have the **Slutsky-style equation** here for substitution effect and output effect. Let's first look at a specific input case:
Two concepts of demand for any input (say, labor):
(1) the conditional demand for labor, denoted by $l^c(v, w, q)$;
(2) the unconditional demand for labor, which is denoted by $l(P, v, w)$.
At the profit-maximizing choice for labor input, these two concepts agree about the amount of labor hired. The two concepts also agree on the level of output produced (which is a function of all the prices):

$$\begin{aligned}&l(P, v, w) = l^c(v, w, q) = l^c(v, w, q(P, v, w))\\&\text{(since output supply function} \quad q^\star(P, w, v) = f(k^\star(P, w, v), l^\star(P, w, v))\\&\text{by Envelope Theorem)}\end{aligned} \tag{5.1}$$

Differentiation of this expression with respect to the wage (and holding the other prices constant) yields:

$$\frac{\partial l(P, v, w)}{\partial w} = \frac{\partial l^c(v, w, q)}{\partial w} + \frac{\partial l^c(v, w, q)}{\partial q} \cdot \frac{\partial q(P, v, w)}{\partial w}. \tag{5.2}$$

Principle **Substitution and output effects in input demand**
When the price of an input falls, two effects cause the quantity demanded of that input to rise:

1. **Substitution Effect** $\frac{\partial l(P,v,w)}{\partial w} < 0$ causes any given output level to be produced using more of the input.
2. **Output Effect** $\frac{\partial l^c(v,w,q)}{\partial q} \cdot \frac{\partial q(P,v,w)}{\partial w} < 0$ the fall in costs causes more of the good to be sold, thereby creating an additional output effect that increases demand for the input.
3. For a rise in input price, both substitution and output effects cause the quantity of an input demanded to decline.

Conditional Factor Demand Functions and **Factor Demand Functions** are equal to each other.
i.e.

$$\begin{aligned}&\frac{\partial c(w_1, ..., w_n)}{\partial w_i} = x_i^c(w_1, ..., w_n, y) \quad (y = y_0 \text{ is a given output level}) \\ &= x_i^\star(p, w_1, w_2, ..., w_n)\end{aligned} \tag{5.3}$$

Proof
The **Output Supply Function** is:

$$y^\star(p, w_1, ..., w_n) = f(x_1^\star(p, w_1, ..., w_n), x_2^\star(p, w_1, ..., w_n))$$

Since the parameter is $p, w_1, ..., w_n$ and the variable is y, we can now suppose that the p is a variable and "invert" the function to use y to express p:

$$p = g(w_1, ..., w_n, y)$$

Therefore, the **Factor Demand Functions** $x_i^\star(p, w_1, w_2, ..., w_n)$ now can be expressed as:

$$x_i^\star(g(w_1, ..., w_n, y), w_1, w_2, ..., w_n) = x_i^c(w_1, ..., w_n, y)$$

since $w_1, w_2, ..., w_n$ are all parameter and only y and p has inverted the expression as they are now regarded as variable.

6 Partial Equilibrium

6.1 Short Run Decision Supply

In **Perfect Competitive Market**, with the **F.O.C** that we have already derived, we derive:

$$p = MR = MC(y) \tag{6.1}$$

Actually, in the shot run , the capital (K) is fixed, so the **Supply Curve** is:

$$p = MC(y) = MC(y, \bar{K}) \,.$$

The **Shutdown Condition** is:

$$AVC(y^{\star}) > p$$

Proof
The **Shutdown Condition** comes from when firm is better off not producing, i.e. $y^{\star} = 0$, this means:

$$\begin{aligned}
&\pi(y = y^{\star}) < \pi(y = 0) \\
&py^{\star} - c_v(y^{\star}) - F < -F \\
&\frac{c_v(y^{\star})}{y} > p
\end{aligned}$$

This means if the revenues do not even cover the variable cost, then the firm should shut down.

6.2 Long Run Decision Supply

In the long run, all the inputs are variable, the **Cost Function** become

$$c(w, r, y) = wL(w, r, y) + rK(w, r, y)$$

and the **Marginal Cost Function** i.e. **Long Run Supply Curve**

$$p = MC(y) = MC(y, K(y)) \tag{6.2}$$

Fixed Cost (FC) is zero, so the **Shutdown Condition** is:

$$AC(y^\star) \geq p \tag{6.3}$$

Proof
The **Shutdown Condition** comes from when firm is better off not producing, i.e. $y^\star = 0$, this means:

$$\begin{aligned}
&\text{now } c(y) = c_v(y) \text{ since no fixed cost} \\
&\pi(y = y^\star) < \pi(y = 0) \\
&py^\star - c_v(y^\star) = py^\star - c(y^\star) < 0 \\
&\frac{c(y^\star)}{y} > p
\end{aligned}$$

This means if the revenues do not even cover the average cost, then the firm should shut down. However, in the long run, we have the assumption that we firms will earn **zero profit** in the **market equilibrium**. And since no fixed cost here, $AC(y) = AVC(y)$, they represent the same curve. To maximize the profit, we need $p = MC(y)$ and we know that $MC(y) = AC(y) = AVC(y)$.Therefore, it is possible to have $p = AC(y^\star)$ for individual firm to stay in the market.

6.3 The Relationship Between Short Run and Long Run Supply Curve

Since in the short run, the fixed capital $\bar{K}$ is possible to have the number just equal to the long-run optimal capital $K(y^\star) = K^\star$. If $\bar{K} = K^\star$, then for this output level, $y^\star$,

$$MC(y^\star, \bar{K}) = MC(y^\star, K^\star) = MC(y^\star, K(y^\star)) \tag{6.4}$$

Then $(y^\star, K^\star)$ is the only intersection between Short Run and Long Run Supply Curve.
Since in the short-run some factors are fixed, then the **short-run supply curve** is **less elastic** to a change in price (steeper).

6.4 Short Run Industry Supply

$$S(p) = \sum_{i=1}^{n} S_i(p) \tag{6.5}$$

6.5 Short Run Industry Equilibrium

Once we determine the **industry supply curve**, we can **intersect this with the demand curve** to determine the **equilibrium price** $p^\star$.
Recall that for an **individual firm**, it will choose whether to stay or leave this market through its condition: $AVC(y^\star) > p \quad$ [stay].

6.6 Long Run Industry Supply and Equilibrium

In the long run, the individual firm will choose to stay in the business if $AC(y^\star) \geq p$.
For the identical firms with same production function $y = f(L, K)$ and cost function $c(y)$, The long run industry supply curve will be flat at the price:

$$p^\star = \min AC$$

As more and more firms will enter the market for positive profit until the price decrease to $p^\star$ because of the flatter and flatter industry supply curve intersect with fixed demand curve.
Then all the firms are keeping for the zero profit, there is no incentive for entering or existing.

References

[1] W. Nicholson. *Microeconomic theory: basic principles and extensions*. South Western Educational Publishing, 2005.