Consumer Theory Formula

Haoling Wang

Contents

Preface

This is the first time that I use LaTeXto take the note. Thanks to the courses **'Linear Algebra with MATLAB'** in the **'2024 Summer School in Mathematical Economics & Financial Econometrics (UNNC)'**, which I learnt the LaTeXand tried my best to practice during my free time in UNUK.

The content in this notebook is mainly a complementary material suitable to **ECON 2001 Microeconomic Theory** in UNUK. The content taught in UNNC and UNUK is a little bit different and may be changed from time to time. However, the main knowledge always around what I had collected. During my study, I combined both PPT and the suggested textbook [1]

The course I learnt in UNUK also contains **Producer Theory** and **Game Theory** section, which I can also provide complementary notebooks.

Since this is my first time to write the LaTeX, there are some content such as proofs that are not contained in this pdf. The only convenience that I can provide in this notebook is the outline or the structure of the microeconomics learning and collection of the formula. You can find proofs or the details by yourself. In the later materials, I will write all the things with more detail.

Moreover, I know that I may make mistake when I am writing this notebook. Therefore, if there is any error being detected, please do not hesitate to content me: `hmyhw8@nottingham.edu.cn` (If this not work, use my personal email).

Other notebooks can be found here.

Good luck with your Microeconomic Theory study!

1 Preference and Utility

1.1 Marginal Rate of Substitution

$$MRS = -\frac{dy}{dx}\bigg|_{U(x,y)=k} = \frac{U_x}{U_y} \tag{1.1}$$

Proof
Hint: Total differentiation

Example
For Cobb-Douglas function

$$U(x,y) = x^{\alpha}y^{\beta}$$

or its log form

$$lnU(x,y) = \alpha lnx + \beta lny$$

$$MRS = -\frac{dy}{dx}\bigg|_{U(x,y)=k} = \frac{U_x}{U_y} = \frac{\alpha x^{\alpha-1}y^{\beta}}{\beta x^{\alpha}y^{\beta-1}} = \frac{\alpha}{\beta}\cdot\frac{y}{x} \tag{1.2}$$

1.2 CES utility

The more general **constant elasticity** of substitution function (CES) can contain the most common utility functions (perfect substitution, perfect complementary, Cobb-Douglas) that we always see.
The marginal rate of substitution (MRS) for CES utility functions depends only on **the ratio of the amounts of the two goods**, not on the total quantities of the goods. So they are **homothetic**.

Definition Homothetic
Monotonic transformations preserve the order of the relationship between the arguments of a function and the value of that function.
A **homothetic function** is one that is formed by taking a monotonic transformation of a homogeneous function.

Properties Homothetic
The implicit trade-offs among the variables in a function depend only on **the ratios of those variables**, not on their absolute values.

$$U(x,y) = \frac{x^\delta}{\delta} + \frac{y^\delta}{\delta} \quad \text{when } \delta \leq 1,\ \delta \neq 0 \tag{1.3}$$

$$U(x,y) = \ln x + \ln y \quad \text{when } \delta = 0 \tag{1.4}$$

Table 1: Different Type of Utilities Generated from CES Utility Function

Perfect Substitutes	Cobb-Douglas	Perfect Complements
$\delta = 1$	$\delta = 0$	$\delta = -\infty$

Example

Table 2: Real Different Type of Utility Function

Perfect Substitutes	Cobb-Douglas	Perfect Complements
$U(x,y) = \alpha x + \beta y$	$U(x,y) = x^\alpha y^\beta$	$U(x,y) = min(\alpha x, \alpha y)$

Notation

The elasticity of substitution σ

$$\sigma = \frac{1}{1-\sigma} \tag{1.5}$$

Example

Perfect substitutes $\sigma = \infty$
Perfect complements $\sigma = 0$

2 Optimal Choice

2.1 Maximise Utility Subject to Constraint

General Form

$$\max_{(x_1,x_2,...,x_n)} U(x_1,x_2,...,x_n) \quad \text{s.t.} \quad g(x_1,x_2,...,x_n) = I - p_1x_1 - P_2x_2 - ... - P_nx_n = 0 \tag{2.1}$$

Largange Form

$$\mathcal{L}(\mathbf{P},\mathbf{x},I) = U(x_1,x_2,\ldots,x_n) + \lambda\left(I - p_1x_1 - p_2x_2 - \cdots - p_nx_n\right) \tag{2.2}$$

F.O.C for finding max

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial U}{\partial x_1} - \lambda p_1 = 0 \tag{2.3}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{\partial U}{\partial x_2} - \lambda p_2 = 0 \tag{2.4}$$

$$\vdots$$

$$\frac{\partial \mathcal{L}}{\partial x_n} = \frac{\partial U}{\partial x_n} - \lambda p_n = 0 \tag{2.5}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = I - p_1x_1 - p_2x_2 - \cdots - p_nx_n = 0 \tag{2.6}$$

For any two goods, x_i and x_j, derive

$$MRS_{(x_i \text{ for } x_j)} = \frac{\partial U/\partial x_i}{\partial U/\partial x_j} = \frac{P_i}{P_j} \tag{2.7}$$

We can also derive:
Lagrange Multiplier

$$\lambda = \frac{\partial U/\partial x_1}{p_1} = \frac{\partial U/\partial x_2}{p_2} = \cdots = \frac{\partial U/\partial x_n}{p_n} \tag{2.8}$$

Price

$$p_i = \frac{\partial U/\partial x_i}{\lambda}, \quad \text{for every } i \tag{2.9}$$

- Regarding to **interior solution**, we need to check the S.O.C:

$$\frac{\partial^2 \mathcal{L}}{\partial x_1^2} = \frac{\partial^2 U}{\partial x_1^2} < 0 \tag{2.10}$$

- Otherwise, we need to check the **corner solution**:

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial U}{\partial x_1} - \lambda p_1 < 0 \quad \text{then} \quad x_i = 0 \tag{2.11}$$

Economically, this means any good whose price exceeds its marginal value to the consumer will not be purchased

$$p_i > \frac{\partial U / \partial x_i}{\lambda} = \frac{MU_{x_i}}{\lambda} \tag{2.12}$$

2.2 Demand and Indirect Utility Functions

Marshallian Demand Function

$$x_i^*(p_1, p_2, \ldots, p_n, I) \tag{2.13}$$

Indirect Utility Functions

Because of the individual's desire to maximize utility given a budget constraint, the optimal level of utility obtainable will depend indirectly on the prices of the goods being bought and the individual's income.

$$V(p_1, p_2, \ldots, p_n, I) = U[x_1^*(p_1, p_2, \ldots, p_n, I), x_2^*(p_1, p_2, \ldots, p_n, I), \ldots, x_n^*(p_1, p_2, \ldots, p_n, I)] \tag{2.14}$$

Proof

Hint: Envelope Theorem

2.3 Dual Approach to Optimal Choice - Expenditure Minimisation

General Form

$$\min_{(x_1,x_2,...,x_n)} E(\mathbf{P},x_1,x_2,...,x_n)=P_1x_1+P_2x_2+...+P_nx_n \quad \text{s.t.} \quad g(x_1,x_2,...,x_n)=U(x_1,x_2,...,x_n)-\bar{U}=0 \tag{2.15}$$

Lagrange Form

$$\mathcal{L}(\mathbf{P},\mathbf{x},\bar{U})=E(\mathbf{P},\mathbf{x})+\lambda\left(U(x_1,x_2,...,x_n)-\bar{U}\right) \tag{2.16}$$

To find the minimization, satisfies **F.O.C.**:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1}&=p_1+\lambda\frac{\partial U}{\partial x_1}=0\\
\frac{\partial \mathcal{L}}{\partial x_2}&=p_2+\lambda\frac{\partial U}{\partial x_2}=0\\
&\vdots\\
\frac{\partial \mathcal{L}}{\partial x_n}&=p_n+\lambda\frac{\partial U}{\partial x_n}=0\\
\frac{\partial \mathcal{L}}{\partial \lambda}&=U(x_1,x_2,...,x_n)-\bar{U}=0
\end{aligned} \tag{2.17}$$

We can still derive:

$$\text{MRS}=\frac{\frac{\partial U}{\partial x_i}}{\frac{\partial U}{\partial x_j}}=\frac{p_i}{p_j} \tag{2.18}$$

And then use the above equation to derive:

$$\begin{gathered}
x_1^c(p_1,...,p_n,\bar{U})\\
x_2^c(p_1,...,p_n,\bar{U})\\
\vdots\\
x_n^c(p_1,...,p_n,\bar{U})
\end{gathered} \tag{2.19}$$

These are **Compensated (Hicksian) Demand Functions**.

3 Income and Substitution Effects

3.1 Marshallian Demand Function

$$x_n^* = D_n(p_1, p_2, \ldots, p_n, I) \tag{3.1}$$

Property

Marshallian Demand Function is **homogeneous** of degree zero in prices and income.

$$x_i^* = x_i(p_1, p_2, \ldots, p_n, I) = x_i(tp_1, tp_2, \ldots, tp_n, tI)$$

3.1.1 Marshallian Demand Elasticities

- Price elasticity of demand

$$e_{x,p_x} = \frac{\Delta x/x}{\Delta p_x/p_x} = \frac{\partial x}{\partial p_x} \cdot \frac{p_x}{x} \tag{3.2}$$

- Income elasticity of demand

$$e_{x,I} = \frac{\Delta x/x}{\Delta I/I} = \frac{\partial x}{\partial I} \cdot \frac{I}{x} \tag{3.3}$$

- Cross-price elasticity of demand

$$e_{x,p_y} = \frac{\Delta x/x}{\Delta p_y/p_y} = \frac{\partial x}{\partial p_y} \cdot \frac{p_y}{x} \tag{3.4}$$

- Total Spending and Price Elasticity Total Spending on x is $P_x \times x$

$$\frac{\partial (p_x x)}{\partial p_x} = p_x \cdot \frac{\partial x}{\partial p_x} + x = x\left[e_{x,p_x} + 1\right] \tag{3.5}$$

3.2 Compensated (Hicksian) demand functions

Theorem

Shephard's Lemma

$$\frac{\partial \mathcal{L}(\mathbf{P}, \mathbf{x}, \bar{U})}{\partial p_i} = \frac{\partial E(\mathbf{P}, x_1, x_2, ..., x_n)}{\partial p_i} = x^c(\mathbf{P}, \bar{U}) \tag{3.6}$$

Proof

Hint: Envelope Theorem

Definition

Compensated Demand Function

Hold real income (or utility) constant while examining reactions to changes in p_x.

Formula

$$\begin{aligned} & x^*(P_x, P_y, I) = x^*(P_x, P_y, E(P_x, P_y, \bar{U})) \\ & = x^c(p_x, p_y, \bar{U}) = \frac{\partial \mathcal{L}(p_x, p_y, U)}{\partial p_x} \overset{(3.6)}{=} \frac{\partial E(p_x, p_y, U)}{\partial p_x} \end{aligned} \tag{3.7}$$

where$E(P_x, P_y, \bar{U})$is just the minimized income$I$given the fixed utility$\bar{U}$.

In $x^*(P_x, P_y, I) = x^*(P_x, P_y, E(P_x, P_y, \bar{U})) = x^c(p_x, p_y, \bar{U})$ we can see that $I = E(P_x, P_y, \bar{U})$. This means that the minimum cost in the expenditure minimization problem is just the budget constraint (income) in the maximization problem. Here we have the expenditure minimization function, we can also invert it into profit miximization problem, that is $V(P_x, P_y, I) = U$, where U is the given parameter ($\bar{U}$ in the expenditure minimization problem). The above is also a way to explain the **Indirect Utility Function**.

3.2.1 Compensated Price Elasticities

- Compensated Price Elasticity of Demand for Good x^c with Respect to Price p_x

$$e_{x^c, p_x} = \frac{\Delta x^c / x^c}{\Delta p_x / p_x} = \frac{\Delta x^c}{\Delta p_x} \cdot \frac{p_x}{x^c} = \frac{\partial x^c(p_x, p_y, U)}{\partial p_x} \cdot \frac{p_x}{x^c} \tag{3.8}$$

- Compensated Price Elasticity of Demand for Good x^c with Respect to Cross Price p_y

$$e_{x^c, p_y} = \frac{\Delta x^c / x^c}{\Delta p_y / p_y} = \frac{\Delta x^c}{\Delta p_y} \cdot \frac{p_y}{x^c} = \frac{\partial x^c(p_x, p_y, U)}{\partial p_y} \cdot \frac{p_y}{x^c} \tag{3.9}$$

3.3 Relations among Demand Elasticities

- The relationship between Marshallian demand elasticity and Compensated demand elasticity

$$\text{recall:} \quad x = x^c$$

and we will learn this equation later

$$\text{Slutsky Equation}: \quad \frac{\partial x}{\partial p_x} = \frac{\partial x^c}{\partial p_x} - \frac{\partial x}{\partial E} \cdot \frac{\partial E}{\partial p_x}$$

and we already know

$$\text{Marshallian demand elasticity of own price:} \quad e_{x,p_x} = \frac{\Delta x / x}{\Delta p_x / p_x} = \frac{\partial x}{\partial p_x} \cdot \frac{p_x}{x}$$

so we can derive:

$$e_{x,p_x} = \frac{p_x}{x} \cdot \frac{\partial x}{\partial p_x} = \frac{p_x}{x^c} \cdot \frac{\partial x^c}{\partial p_x} - \frac{p_x}{x} \cdot x \cdot \frac{\partial x}{\partial I} = e^c_{x,p_x} - \frac{P_x x}{I} e_{x,I}$$

which equals to

$$e_{x,p_x} = e^c_{x,p_x} - \frac{P_x x}{I} e_{x,I} \tag{3.10}$$

This means:
The compensated and uncompensated price elasticities will be **similar** if:
(1) The share of income devoted to x is small
(2) The income elasticity of x is small

- relationships among Marshallian demand elasticities

Theorem

Euler's Theorem
For a function $f(x_1, x_2, \ldots, x_n)$ that is homogeneous of degree k, Euler's Theorem states that:

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = k f(x_1, x_2, \ldots, x_n) \tag{3.11}$$

This means that for a homogeneous function of degree k, if we multiply all the input variables by a constant, the function itself will be scaled by the constant raised to the power of k.

- If $k = 1$, this function exhibits **constant returns to scale**.
- If $k > 1$, it shows **increasing returns to scale**.
- If $k < 1$, it shows **decreasing returns to scale**.

Since Marshallian Demand Functions are **homogeneous of degree zero** in all prices and income,

$$0 = p_x \cdot \frac{\partial x}{\partial p_x} + p_y \cdot \frac{\partial x}{\partial p_y} + I \cdot \frac{\partial x}{\partial I}$$

divide by x:

$$0 = e_{x,p_x} + e_{x,p_y} + e_{x,I} \tag{3.12}$$

3.3.1 Engel Aggregation

Differentiate the budget constraint with respect to income

$$\text{Budget constraint: } I = P_x x + P_y y$$

$$1 = p_x \cdot \frac{\partial x}{\partial I} + p_y \cdot \frac{\partial y}{\partial I}$$

$$1 = p_x \cdot \frac{\partial x}{\partial I} \cdot \frac{xI}{xI} + p_y \cdot \frac{\partial y}{\partial I} \cdot \frac{yI}{yI} = \frac{P_x x}{I} e_{x,I} + \frac{P_y y}{I} e_{y,I}$$

so

$$\frac{P_x x}{I} e_{x,I} + \frac{P_y y}{I} e_{y,I} = 1 \tag{3.13}$$

Engel's law:
(1) Income elasticity of demand for food items is < 1
(2) Income elasticity of demand for all nonfood items must be > 1

3.3.2 Cournot Aggregation

Differentiate the budget constraint with respect to P_x

$$\frac{\partial I}{\partial p_x} = 0 = p_x \cdot \frac{\partial x}{\partial p_x} + x + p_y \cdot \frac{\partial y}{\partial p_x}$$

$$0 = p_x \cdot \frac{\partial x}{\partial p_x} \cdot \frac{p_x}{I} \cdot \frac{x}{x} + x \cdot \frac{p_x}{I} + p_y \cdot \frac{\partial y}{\partial p_x} \cdot \frac{p_x}{I} \cdot \frac{y}{y}$$

$$\frac{p_x}{I} e_{x,p_x} + \frac{p_x}{I} + \frac{p_y}{I} e_{y,p_x} = 0$$

$$\frac{p_x x}{I} e_{x,p_x} + \frac{p_y y}{I} e_{y,p_x} = -\frac{p_x x}{I} \tag{3.14}$$

Cross-price effects are restricted because of the budget constraint.

3.4 The Slutsky Equation

Since the Duality, we know that the quantity demanded generated by Marshallian Demand Function and Compensated Demand Function is the same, we can differentiate P_x on the equation:

$$x^c(p_x, p_y, U) = x^*(p_x, p_y, E(p_x, p_y, U)) \tag{3.15}$$

where budget constraint

$$I = E(p_x, p_y, U)$$

to derive

$$\frac{\partial x^c}{\partial p_x} = \frac{\partial x}{\partial p_x} + \frac{\partial x}{\partial E} \cdot \frac{\partial E}{\partial p_x}$$

This is because x^* has P_x inside, and the expenditure function $E(p_x, p_y, U)$ also has P_x.

So now we have the **Slutsky Equation**:

$$\frac{\partial x}{\partial p_x} = \frac{\partial x^c}{\partial p_x} - \frac{\partial x}{\partial E} \cdot \frac{\partial E}{\partial p_x} \overset{(3.7)}{=} \frac{\partial x^c}{\partial p_x} - \frac{\partial x}{\partial E} \cdot x^c \tag{3.16}$$

where

$$\textbf{Substitution Effect} = \frac{\partial x^c}{\partial p_x} \tag{3.17}$$

$$\textbf{Income Effect} = -\frac{\partial x}{\partial I} \cdot x^c \quad (\text{since} \quad I = E(p_x, p_y, U)) \tag{3.18}$$

Assumption: Income has been all spent and there is no saving so E = I

3.5 Compensating Variation

Definition Compensating Variation
Compensating Variation is defined as the amount of **money that would need to be given** to or taken from the consumer after a price change, to allow them to reach the **same level of utility** they enjoyed **before the price change**.

- **When prices increase** The consumer needs to be compensated by receiving an amount of money (the compensating variation) to maintain their previous utility level.
- **When prices decrease** The compensating variation would reflect how much the consumer would be willing to pay to maintain their previous utility level.

General Formula

$$CV = E(p_x^1 1, p_y, U_0) - E(p_x^0, p_y, U_0) \tag{3.19}$$

Where to reach U_0:
Expenditure at $p_x^0 : E(p_x^0, p_y, U_0)$
Expenditure at $p_x^1 : E(p_x^1, p_y, U_0)$

This integral is the area to the left of the compensated demand curve between p_x^0 and p_x^1
Integration Formula

$$CV = \int_{p_x^0}^{p_x^1} dE = \int_{p_x^0}^{p_x^1} x^c(p_x, p_y, U_0)dp_x \tag{3.20}$$

4 Choice under Uncertainty

4.1 Expected Utility Theory

Assumption The marginal utility of wealth declines as wealth increases.

$$E(U) = P_1 U(X_1) + P_2 U(X_2) + ... + P_n U(X_n) \tag{4.1}$$

Principle Expected Utility Maximization
If individuals obey the von Neumann–Morgenstern axioms (see it in the textbook detailly) of behavior in uncertain situations, they will act as if they choose the option that **maximizes the expected value** of their von Neumann–Morgenstern utility index.

4.2 Measures of Risk Aversion

Utility Function is U(W), a concave function of W to reflect the assumption of a diminishing marginal utility.

Definition Risk Aversion
An individual who always refuses fair bets is said to be risk averse. If individuals exhibit a diminishing marginal utility of wealth, they will be risk averse. As a consequence, they will be willing to pay something to avoid taking fair bets.

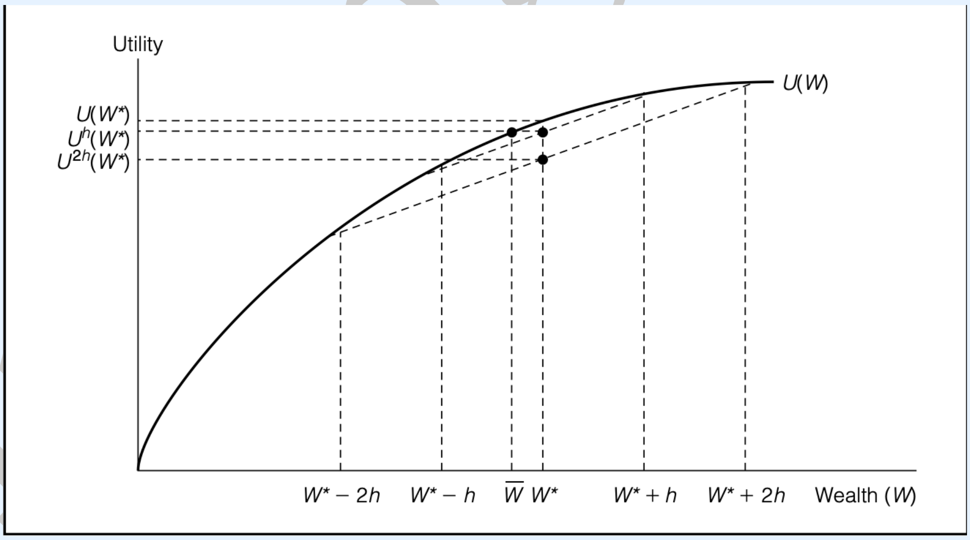

Figure 1: Gamble, Risk Aversion and Insurance

A certain wealth of $\bar{W}$ provides the same utility as does participating in gamble h. This person would be willing to pay up to $W^\star - \bar{W}$ in order to avoid participating in the gamble. This explains why people buy insurance.

Coefficient of Absolute Risk Aversion

$$r(W) = -\frac{U''(W)}{U'(W)} \tag{4.2}$$

See the proof in the textbook.
However, whether risk aversion increases or decreases with wealth is indeterminate. It all depends on the precise shape of the utility function.

Coefficient of Relative Risk Aversion

Assumption The willingness to pay to avoid a given gamble is inversely proportional to wealth.

$$rr(W) = W \times r(W) = -W\frac{U''(W)}{U'(W)} \tag{4.3}$$

References

[1] W. Nicholson. *Microeconomic theory: basic principles and extensions*. South Western Educational Publishing, 2005.