

# Probabilistic Robotics Course

## Least Squares Introduction ICP Optimization

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# Maximum Likelihood Estimation

Using

$$\mathbf{x}^* = \underset{\mathbf{x}}{\operatorname{argmax}} p(\mathbf{x}|\mathbf{z})$$

- Bayes' Rule

$$p(\mathbf{x}|\mathbf{z}) = \frac{p(\mathbf{z}|\mathbf{x})p(\mathbf{x})}{p(\mathbf{z})}$$

$$\propto p(\mathbf{z}|\mathbf{x})$$

- Independence,

$$= \prod_i p(\mathbf{z}^{[i]}|\mathbf{x})$$

We can further  
simplify the task

# Gaussian Assumption

Measurements affected by Gaussian noise

$$\begin{aligned} p(\mathbf{z}_i | \mathbf{x}) &= \mathcal{N}(\mathbf{z}^{[i]}; \mathbf{h}^{[i]}(\mathbf{x}), \Sigma^{[i]}) \\ &\propto \exp \left( - \underbrace{(\mathbf{h}^{[i]}(\mathbf{x}) - \mathbf{z}^{[i]})^T}_{\mathbf{e}^{[i]}(\mathbf{x})} \underbrace{\Sigma^{[i]-1}}_{\Omega^{[i]}} (\mathbf{h}^{[i]}(\mathbf{x}) - \mathbf{z}^{[i]}) \right) \end{aligned}$$

# Gaussian Assumption

Measurements affected by Gaussian noise

The diagram illustrates the Gaussian Assumption equation for the probability of a measurement  $\mathbf{z}_i$  given a state  $\mathbf{x}$ . The equation is presented with color-coded terms and labels:

- prediction**: A label pointing to the predicted measurement  $\mathbf{h}^{[i]}(\mathbf{x})$  (red box).
- measurement**: A label pointing to the observed measurement  $\mathbf{z}^{[i]}$  (blue box).
- state**: A label pointing to the state  $\mathbf{x}$  (red box).
- error function**: A label pointing to the error term  $\mathbf{e}^{[i]}(\mathbf{x})$  (red box).

$$p(\mathbf{z}_i|\mathbf{x}) = \mathcal{N}(\mathbf{z}^{[i]}; \mathbf{h}^{[i]}(\mathbf{x}), \Sigma^{[i]})$$
$$\propto \exp \left( - \underbrace{(\mathbf{h}^{[i]}(\mathbf{x}) - \mathbf{z}^{[i]})^T}_{\mathbf{e}^{[i]}(\mathbf{x})} \underbrace{\Sigma^{[i]-1}}_{\Omega^{[i]}} (\mathbf{h}^{[i]}(\mathbf{x}) - \mathbf{z}^{[i]}) \right)$$

# Gaussian Assumption

Through Gaussian assumption

- Maximization becomes minimization
- Product turns into sum

$$\begin{aligned}\mathbf{x}^* &= \operatorname{argmax}_x \prod_i p(\mathbf{z}^{[i]} | \mathbf{x}) \\ &= \operatorname{argmax}_x \prod_i \exp(-\mathbf{e}^{[i]}(\mathbf{x})^T \boldsymbol{\Omega}^{[i]} \mathbf{e}^{[i]}(\mathbf{x})) \\ &= \operatorname{argmin}_x \sum_i \mathbf{e}^{[i]}(\mathbf{x})^T \boldsymbol{\Omega}^{[i]} \mathbf{e}^{[i]}(\mathbf{x})\end{aligned}$$

# Gauss Method Overview

Iterative minimization of

$$F(\mathbf{x}) = \sum_i \mathbf{e}^{[i]}(\mathbf{x})^T \boldsymbol{\Omega}^{[i]} \mathbf{e}^{[i]}(\mathbf{x})$$

Each iteration refines the current estimate by applying a perturbation

$$\mathbf{x} \leftarrow \mathbf{x} + \Delta \mathbf{x}$$

Perturbation obtained by minimizing a quadratic approximation of the problem in  $\Delta \mathbf{x}$

$$F(\mathbf{x} + \Delta \mathbf{x}) \simeq \Delta \mathbf{x}^T \mathbf{H} \Delta \mathbf{x} + 2\mathbf{b}^T \Delta \mathbf{x} + c$$

# Linearization

The quadratic approximation is obtained by linearizing the error functions around  $\mathbf{x}^*$

$$\mathbf{e}^{[i]}(\mathbf{x}^* + \Delta \mathbf{x}) \simeq \underbrace{\mathbf{e}^{[i]}(\mathbf{x}^*)}_{\mathbf{e}} + \underbrace{\left. \frac{\partial \mathbf{e}^{[i]}(\mathbf{x})}{\partial (\mathbf{x})} \right|_{\mathbf{x}=\mathbf{x}^*}}_{\mathbf{J}^{[i]}} \Delta \mathbf{x}$$

...expanding the products

$$\mathbf{e}^{[i]}(\mathbf{x}^* + \Delta \mathbf{x})^T \boldsymbol{\Omega}^{[i]} \mathbf{e}^{[i]}(\mathbf{x}^* + \Delta \mathbf{x}) \simeq \underbrace{\Delta \mathbf{x}^T \mathbf{J}^{[i]T} \boldsymbol{\Omega}^{[i]} \mathbf{J}^{[i]} \Delta \mathbf{x}}_{\mathbf{H}^{[i]}} + 2 \underbrace{\mathbf{J}^{[i]T} \boldsymbol{\Omega}^{[i]} \mathbf{e}^{[i]}}_{\mathbf{b}^{[i]T}} \Delta \mathbf{x} + \underbrace{\mathbf{e}^{[i]T} \boldsymbol{\Omega}^{[i]} \mathbf{e}^{[i]}}_{c^{[i]}}$$

...and grouping the terms

$$\mathbf{H} = \sum_i \mathbf{H}^{[i]} \quad \mathbf{b} = \sum_i \mathbf{b}^{[i]} \quad c = \sum_i c^{[i]}$$

# Quadratic form

Find the  $\Delta \mathbf{x}$  that minimizes the quadratic approximation of the objective function

$$\begin{aligned}\Delta \mathbf{x}^* &= \underset{\Delta \mathbf{x}}{\operatorname{argmin}} F(\mathbf{x}^* + \Delta \mathbf{x}) \\ &\simeq \underset{\Delta \mathbf{x}}{\operatorname{argmin}} \Delta \mathbf{x}^T \mathbf{H} \Delta \mathbf{x} + 2\mathbf{b}^T \Delta \mathbf{x} + c\end{aligned}$$

Find  $\Delta \mathbf{x}$  that nulls the derivative of quadratic form

$$\begin{aligned}0 &= \frac{\partial \left[ \Delta \mathbf{x}^T \mathbf{H} \Delta \mathbf{x} + 2\mathbf{b}^T \Delta \mathbf{x} + c \right]}{\partial \Delta \mathbf{x}} \\ -\mathbf{b} &= \mathbf{H} \Delta \mathbf{x}\end{aligned}$$



# Algorithm (one Iteration)

Clear **H** and **b**

$$\mathbf{H} \leftarrow 0 \quad \mathbf{b} \leftarrow 0$$

For each measurement, update h and b

$$\mathbf{e}^{[i]} \leftarrow \mathbf{h}^{[i]}(\mathbf{x}^*) - \mathbf{z}^{[i]}$$

$$\mathbf{J}^{[i]} \leftarrow \left. \frac{\partial \mathbf{e}^{[i]}(\mathbf{x})}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}^*}$$

$$\mathbf{H} \leftarrow \mathbf{H} + \mathbf{J}^{[i]T} \boldsymbol{\Omega}^{[i]} \mathbf{J}^{[i]}$$

$$\mathbf{b} \leftarrow \mathbf{b} + \mathbf{J}^{[i]T} \boldsymbol{\Omega}^{[i]} \mathbf{e}^{[i]}$$

Update the estimate with the perturbation

$$\Delta \mathbf{x} \leftarrow \text{solve}(\mathbf{H} \Delta \mathbf{x} = -\mathbf{b})$$

$$\mathbf{x}^* \leftarrow \mathbf{x}^* + \Delta \mathbf{x}$$

# Methodology

Identify the state space **X**

- Qualify the domain
- Find a locally Euclidean parameterization

Identify the measurement space(s) **Z**

- Qualify the domain
- Find a locally Euclidean parameterization

Identify the prediction functions **h(x)**

# Gauss-Newton in SLAM

Typical problems where GN is used

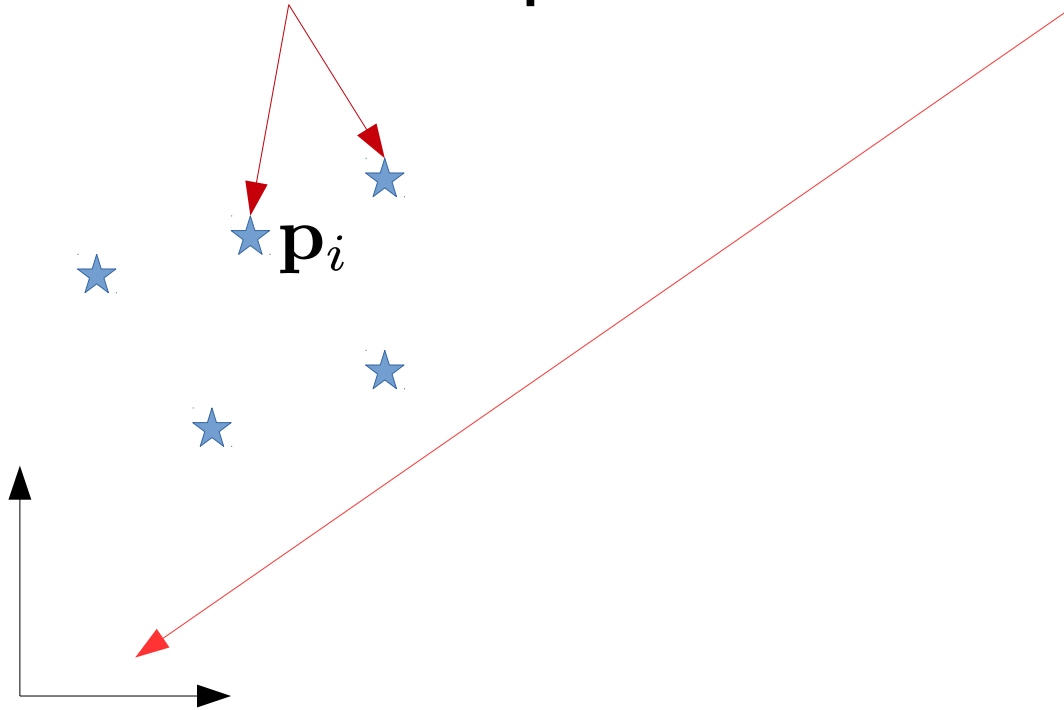
- Calibration
- Registration
  - Cloud to Cloud (ICP)
  - Image to Cloud (Posit)
- Global Optimization
  - Pose-SLAM
  - Bundle Adjustment

## Warning

- **Data association is assumed to be known known**
- **Gauss-Newton alone is not sufficient to solve a full problem**
- **One needs a strategy to compute data association**

# Example ICP Optimization in 2D

Given a set of points in the world frame



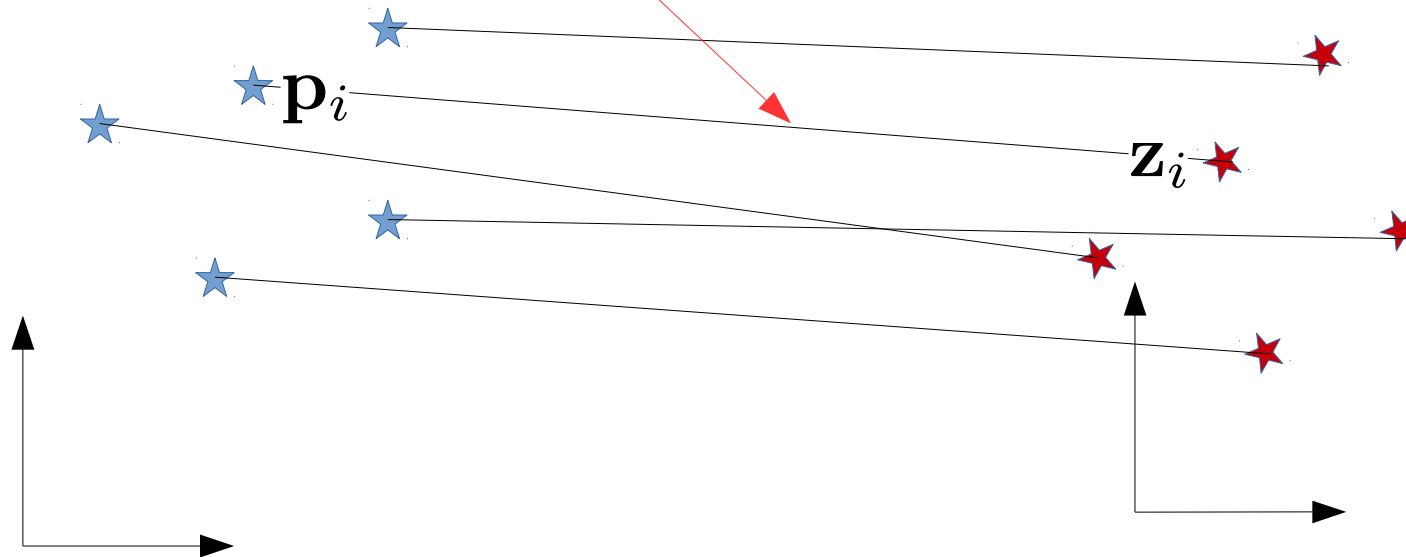
# Example ICP Optimization in 2D

A set of 3D measurements in the robot frame



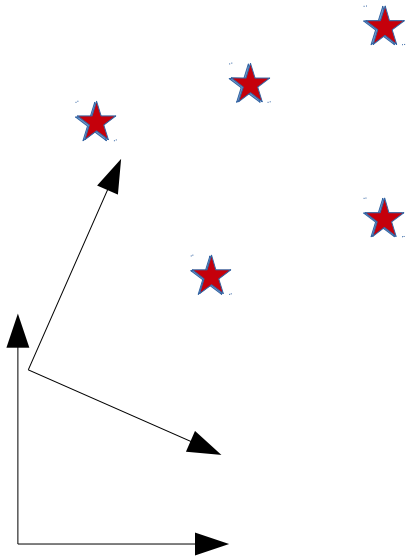
# Example ICP Optimization in 2D

Roughly known correspondences



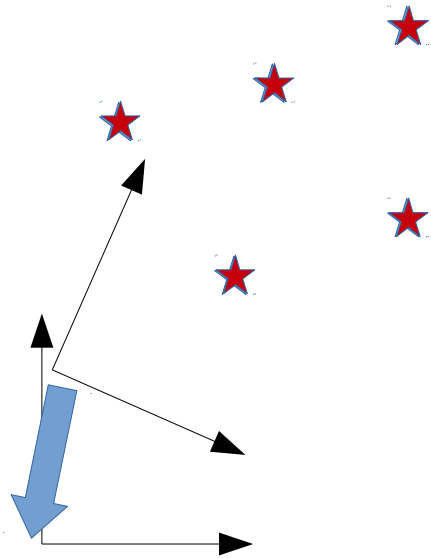
# Example ICP Optimization in 2D

We want to find a transform that minimizes distance between corresponding points



# Example ICP Optimization in 2D

Such a transform will be the pose of world w.r.t. robot



**Note:** we can also estimate robot w.r.t world, but it leads to longer calculations



# ICP: State and Measurements

State

$$\mathbf{x} \in SE(2)$$

$$\mathbf{x} = \underbrace{(x \ y \ \alpha)^T}_{\mathbf{t}}$$

Measurements

$$\mathbf{z} \in \mathbb{R}^2$$

$$\mathbf{h}_i(\mathbf{x}) = \mathbf{R}(\alpha)\mathbf{p}_i + \mathbf{t}$$

# On Rotation Matrices

Recall a rotation matrix in 2D and the corresponding derivative w.r.t. the rotation angle

$$\mathbf{R}(\alpha) = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \quad \mathbf{R}'(\alpha) = \begin{pmatrix} -s & -c \\ c & -s \end{pmatrix}$$

```
function
R=rotation2D(alpha)
    s=sin(alpha);
    c=cos(alpha);
    R=[c -s;
        s  c];
endfunction
```

```
function
Rp=rotation2Dgradient(alpha)
    s=sin(alpha);
    c=cos(alpha);
    Rp=[-s -c;
        c -s];
endfunction
```

# ICP: Jacobian

$$\mathbf{h}_i(\mathbf{x}) = \mathbf{R}(\alpha)\mathbf{p}_i + \mathbf{t}$$

measurement  
function

$$\frac{\partial \mathbf{h}_i(\mathbf{x})}{\partial \mathbf{x}} = \left( \frac{\partial \mathbf{h}_i(\mathbf{x})}{\partial \mathbf{t}} \quad \frac{\partial \mathbf{h}_i(\mathbf{x})}{\partial \alpha} \right)$$

measurement  
Jacobian

$$\frac{\partial \mathbf{h}_i(\mathbf{x})}{\partial \mathbf{t}} = \mathbf{I}$$

$$\frac{\partial \mathbf{h}_i(\mathbf{x})}{\partial \alpha} = \mathbf{R}'(\alpha)\mathbf{p}_i$$

# ICP: Octave Code

```
function [e,J]=errorAndJacobian(x,p,z)
    t=x(1:2);
    alpha=x(3);
    R=rotation2D(alpha);
    z_hat=R*p+t;
    e=z_hat-z;
    J=zeros(2,3);
    J(1:2,1:2)=eye(2);
    J(1:2,3)=rotation2Dgradient(alpha)*p;
endfunction;
```

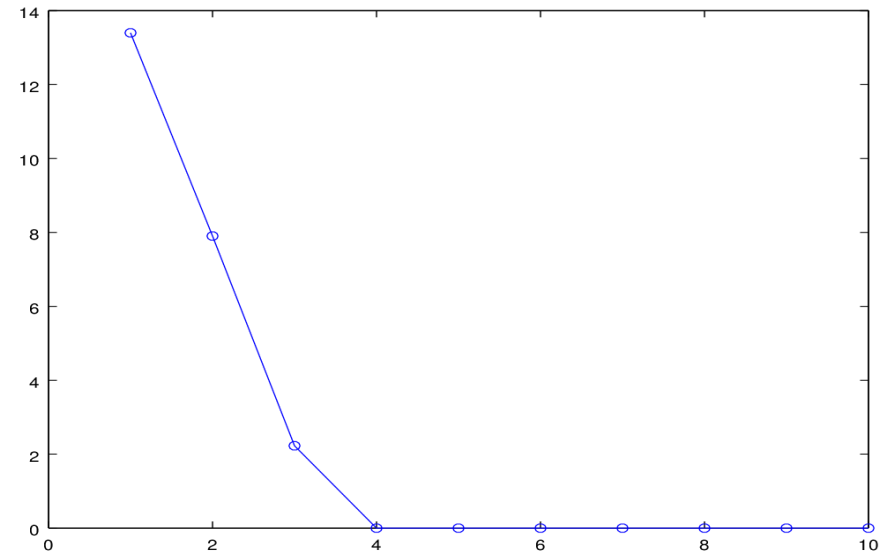
# ICP: Octave Code

```
function [chi, x_new]=icp2d(x,P,Z)
    chi=0; %cumulative chi2
    H=zeros(3,3); b=zeros(3,1); %accumulators for H and b
    for(i=1:size(P,2))
        p=P(:,i); z=Z(:,i); % fetch point and measurement
        [e,J]=errorAndJacobian(x,p,z); %compute e and J
        H+=J'*J; %assemble H and B
        b+=J'*e;
        chi+=e'*e; %update cumulative error
    endfor
    dx=-H\b; %solve the linear system
    x_new=x+dx; %apply update

    %normalize theta between -pi and pi
    alpha=x(3);
    s=sin(alpha); c=cos(alpha);
    x(3)=atan2(s,c);
endfunction
```

# Testing, good initial guess

- Spawn a set of random points in 2D
- Define a location of the robot
- Compute synthetic measurements from that location
- Set the a point close to the true location as initial guess
- Run ICP and plot the evolution of the error



- When started from a good guess, the system converges nicely

# Testing, bad initial guess

- Spawn a set of random points in 2D
  - Define a location of the robot
  - Compute synthetic measurements from that location
  - Set the origin as initial guess
  - Run ICP and plot the evolution of the error
- If the guess is poor, the system might take long to converge or **not** reach the minimum

