Probabilistic Robotics Course

Sparse Optimization

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Large Problems

Many relevant estimation problems require estimating a large number of variables given a very large number of measurements.

Examples include

- Pose-graphs
- Pose-Landmark
- Calibration
- **<add your own>**
- **<combine the above problems>**

The larger the problem, the slower the solution

Large and Sparse Problems

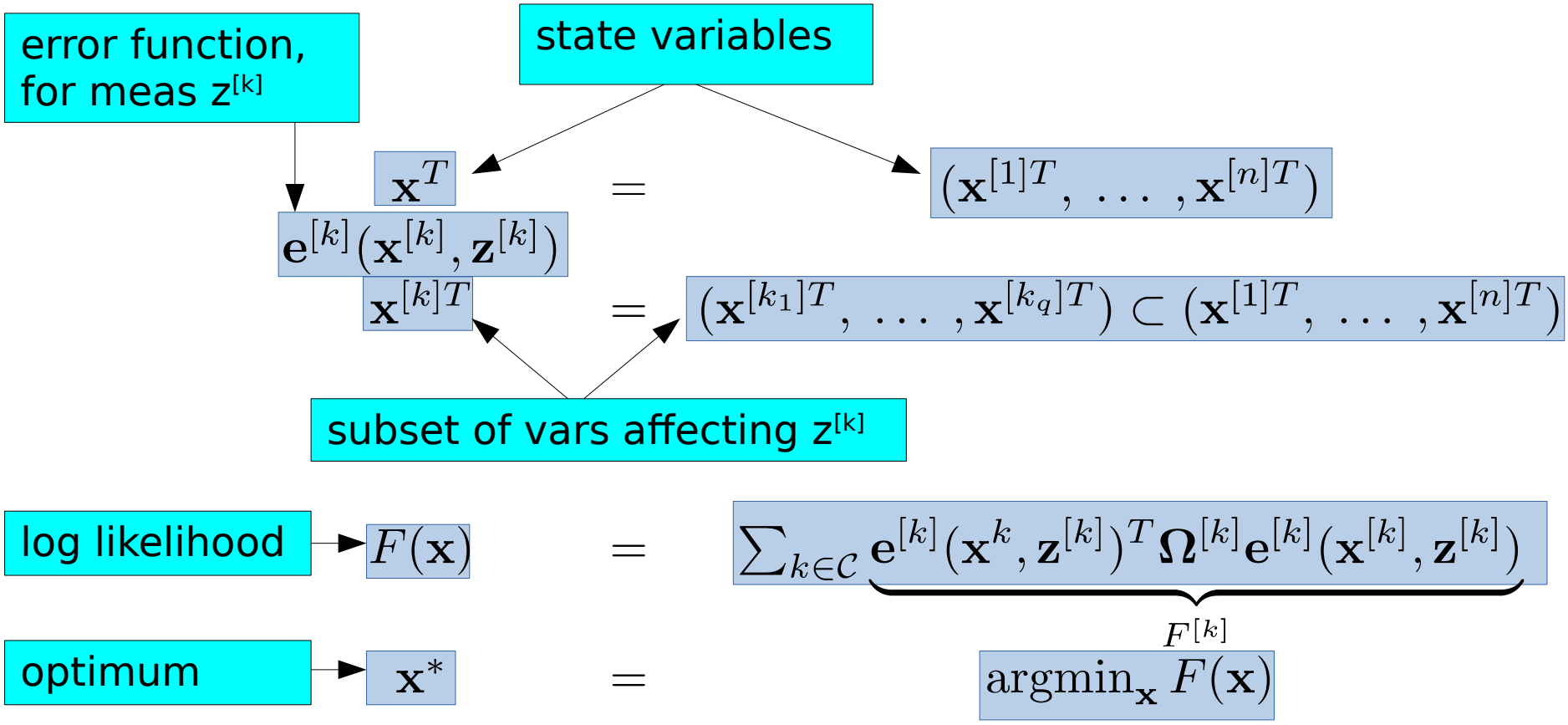
Despite the high number of variables, a single measurement is typically only affected by a subset of state variables

Examples:

- *Landmark measurement* determined by :
  - the observer position $X_r^{[n]}$
  - the position of the landmark $X_l^{[m]}$
- Pose-Pose measurement determined by
  - the observer position $X_r^{[i]}$
  - the observed position $X_r^{[j]}$

Graphical Representation

Formally we can highlight the dependency of a measurement from a subset of state variables by the following notation

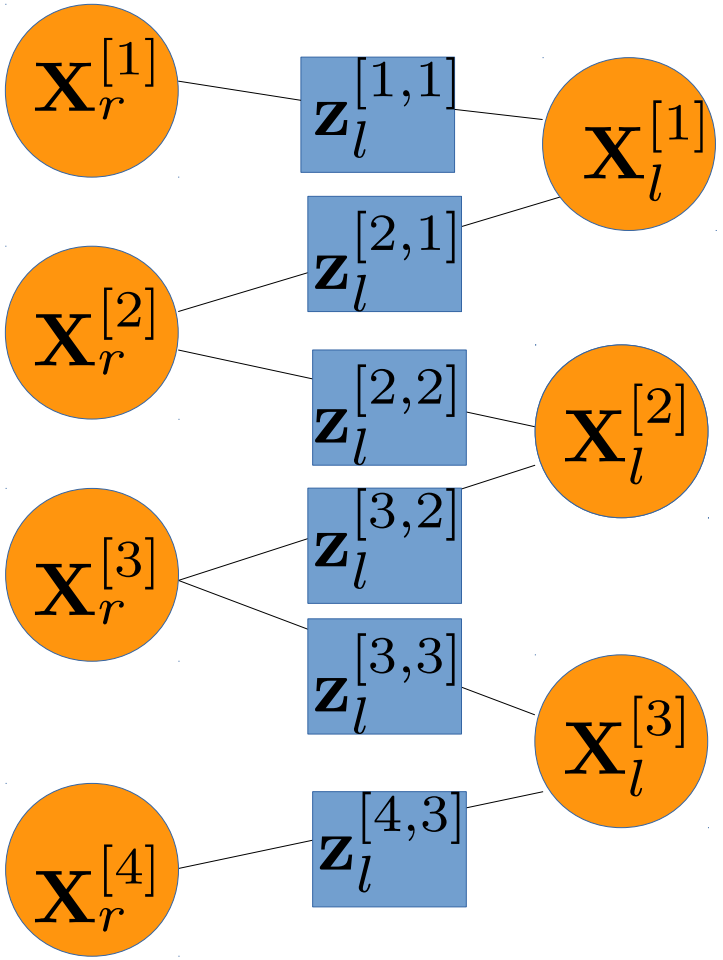


Graphical Representation

We can represent the problem with a graph

- A node for each state variable
- A node for each measurement
- An edge between a variable and a measurement if they are dependant

Example: pose-landmark SLAM



Graphical Representation

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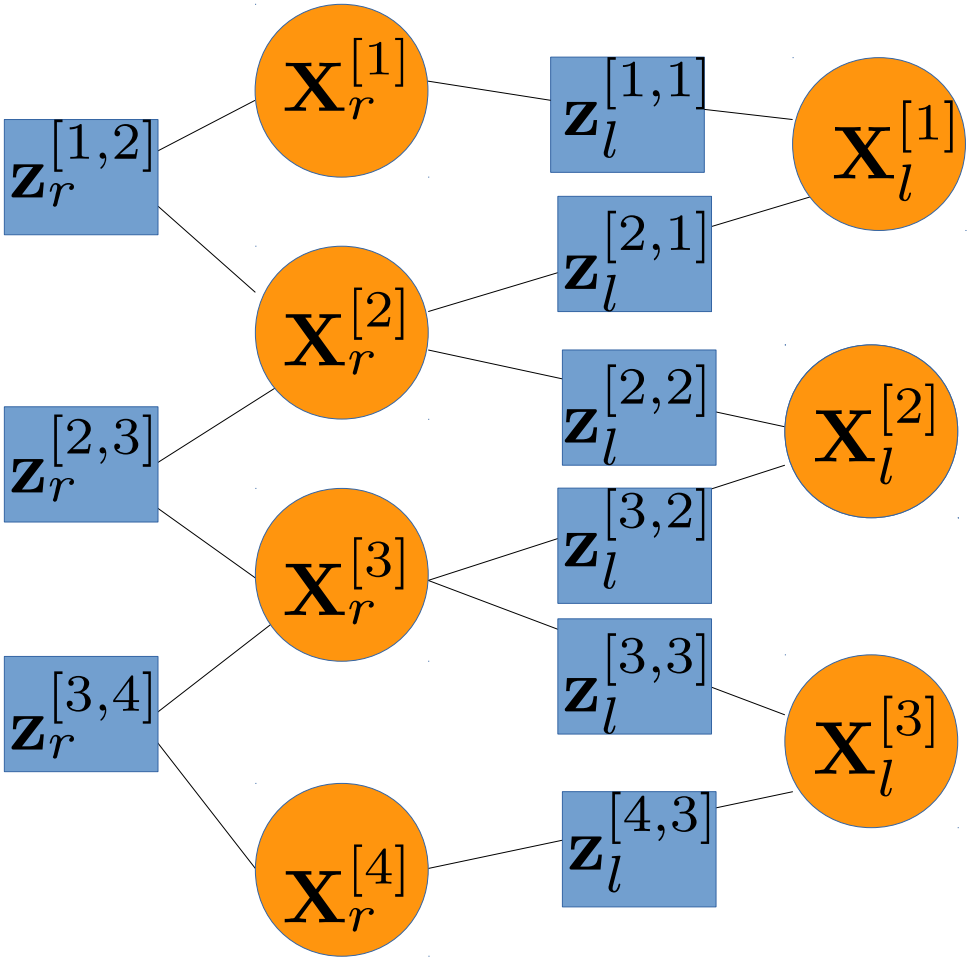
Example: landmark+ odometry



Graphical Representation

This representation is known as a factor graph.

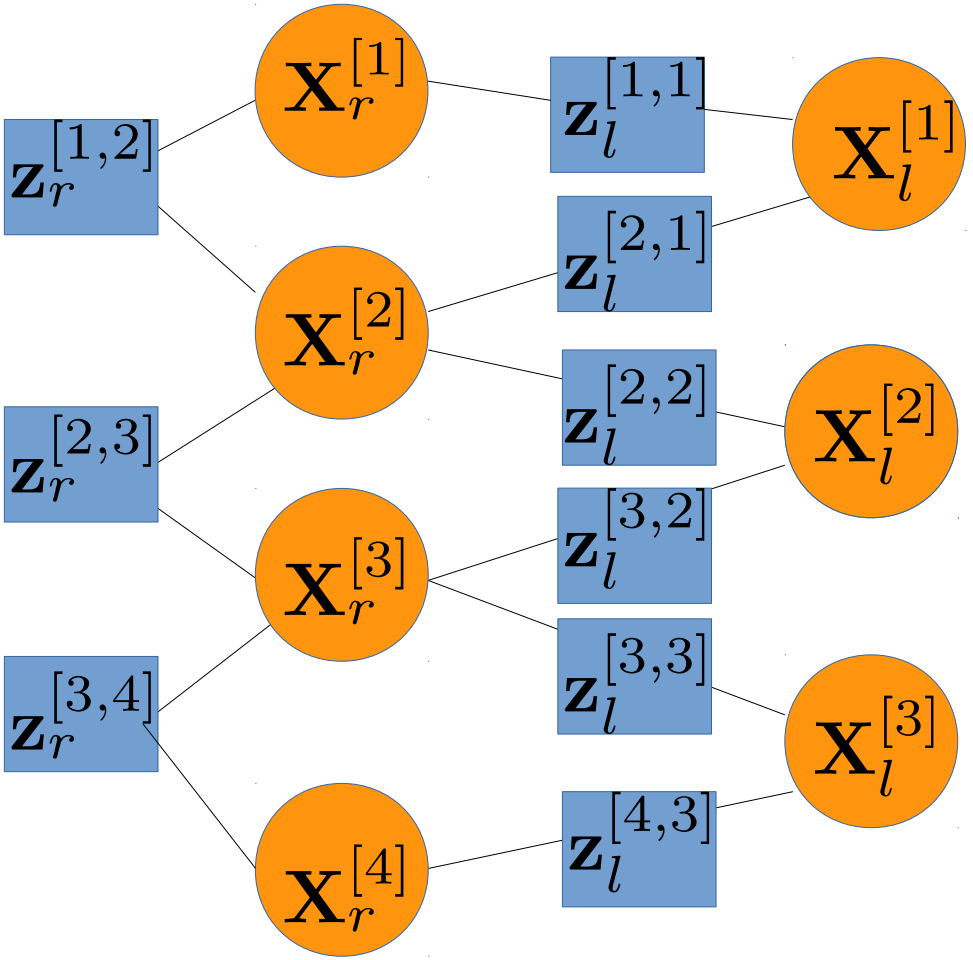
Can represent arbitrary PDF

In our case we restrict to Gaussians

It is bipartite:

- two variable nodes are connected only through a measurement
- factors can be seen as hyper-edges connecting multiple variables

Example: landmark+ odometry



Structure

The contribution to the K matrix for a measurement $z^{[k]}$ measurement will affect only the state components in $x^{[k]}$

$$\mathbf{J}^{[k]} = \begin{pmatrix} \mathbf{0} \cdots \mathbf{0} & \mathbf{J}^{[k_1]} & \cdots & \mathbf{J}^{[k_i]} & \cdots \mathbf{0} \cdots & \mathbf{J}^{[k_q]} \mathbf{0} \cdots \mathbf{0} \end{pmatrix}$$

$$\mathbf{H}^{[k]} = \begin{pmatrix} \ddots & & & & & \\ & \mathbf{J}^{[k_1]T}\mathbf{\Omega}^{[k]}\mathbf{J}^{[k_1]} & \cdots & \mathbf{J}^{[k_1]T}\mathbf{\Omega}^{[k]}\mathbf{J}^{[k_i]} & \cdots & \mathbf{J}^{[k_1]T}\mathbf{\Omega}^{[k]}\mathbf{J}^{[k_q]} \\ & \vdots & & \vdots & & \vdots \\ & \mathbf{J}^{[k_i]T}\mathbf{\Omega}^{[k]}\mathbf{J}^{[k_1]} & \cdots & \mathbf{J}^{[k_i]T}\mathbf{\Omega}^{[k]}\mathbf{J}^{[k_i]} & \cdots & \mathbf{J}^{[k_i]T}\mathbf{\Omega}^{[k]}\mathbf{J}^{[k_q]} \\ & \vdots & & \vdots & & \vdots \\ & \mathbf{J}^{[k_q]T}\mathbf{\Omega}^{[k]}\mathbf{J}^{[k_1]} & \cdots & \mathbf{J}^{[k_q]T}\mathbf{\Omega}^{[k]}\mathbf{J}^{[k_i]} & \cdots & \mathbf{J}^{[k_q]T}\mathbf{\Omega}^{[k]}\mathbf{J}^{[k_q]} \\ & & & & & \ddots \end{pmatrix}$$

$$\mathbf{b}^{[k]T} = \begin{pmatrix} \vdots \\ \mathbf{J}^{[k_1]T}\mathbf{\Omega}^{[k]}\mathbf{e}^{[k]} \\ \vdots \\ \mathbf{J}^{[k_i]T}\mathbf{\Omega}^{[k]}\mathbf{e}^{[k]} \\ \vdots \\ \mathbf{J}^{[k_q]T}\mathbf{\Omega}^{[k]}\mathbf{e}^{[k]} \\ \vdots \end{pmatrix}$$

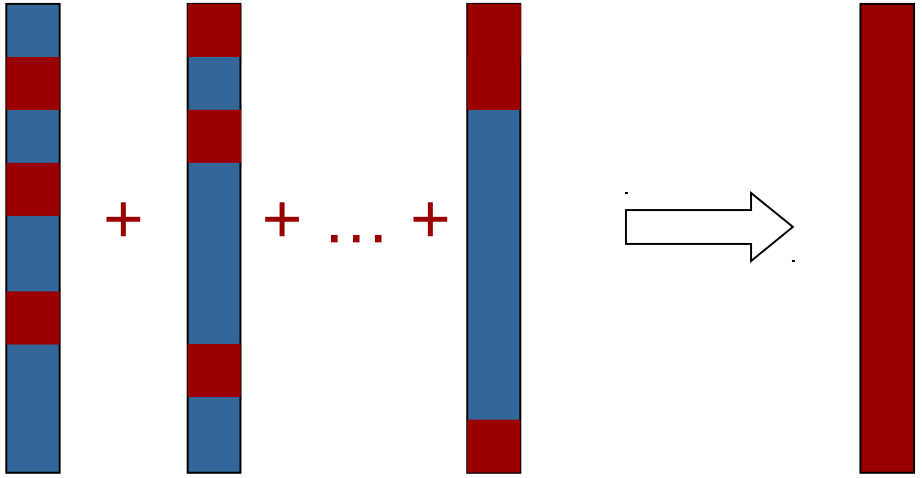
Structure

Structure of **H**: location of the non zero elements

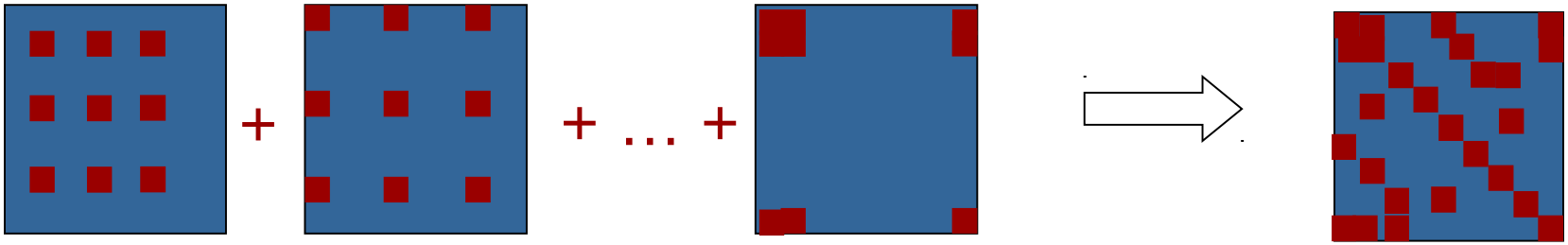
The structure of H depends only on the structure of the measurements

- preallocate before the iterations
- do not allocate memory for zero blocks

$$\mathbf{b} = \sum \mathbf{b}^{[k]}$$



$$\mathbf{H} = \sum \mathbf{H}^{[k]}$$



Full diagonal.
each measurement correlates a fixed number of variables with q^2 blocks

Structure and Efficiency

The number of non-zero blocks in H depends on the number of measurements

Bounded by:

- number of poses
- perception range
- landmark density

In typical SLAM-related problems the number of measurements grows linearly with the length of the trajectory

H has a *linear* number of non-zero blocks (it is mostly empty)

Use special techniques to solve sparse linear systems

Solving Sparse Systems

A common solution to solve a symmetric positive definite system of equation is through Cholesky factorization

$$\mathbf{H}\Delta\mathbf{x} = -\mathbf{b}$$

System we want to solve

$$\mathbf{H} = \mathbf{L}\mathbf{L}^T$$

Cholesky (L lower triangular)

$$\mathbf{L}\underbrace{\mathbf{L}^T\Delta\mathbf{x}}_{\mathbf{y}} = -\mathbf{b}$$

$$\mathbf{L}\mathbf{y} = -\mathbf{b}$$

Solve for y by backward substitution

$$\mathbf{L}^T\Delta\mathbf{x} = \mathbf{y}$$

Solve for x by forward substitution

Big Issue: H sparse does not mean L sparse!!
We lose the benefits of sparsity

Permutations and Cholesky

Permutation matrix

- Encodes a reordering of the variables
- The elements are either 0 or 1
- Exactly 1 non zero for each row
- Exactly 1 non zero for each column
- The inverse of a permutation is its transpose

If a matrix is sparse, there is a reordering of the variables that renders the Cholesky factor maximally sparse

Computing such an ordering is **NP complete**

Efficient heuristics solve the problem

Permutations and Cholesky

Solve the linear system through a permutation

$$\mathbf{H}\Delta\mathbf{x} = -\mathbf{b}$$

System we want to solve

$$\mathbf{PH}\underbrace{\mathbf{P}^T\mathbf{P}}_{\mathbf{I}}\Delta\mathbf{x} = -\mathbf{Pb}$$

Apply a permutation

$$\underbrace{\mathbf{PHP}^T}_{\mathbf{H}'}\underbrace{\mathbf{P}\Delta\mathbf{x}}_{\Delta\mathbf{x}'} = -\underbrace{\mathbf{Pb}}_{\mathbf{b}'}$$

$$\mathbf{H}'\Delta\mathbf{x}' = -\mathbf{b}'$$

Solve the system under permutation

$$\mathbf{H}' = \mathbf{L}'\mathbf{L}'^T$$

Cholesky decomposition of H' (sparse)

$$\mathbf{L}'\underbrace{\mathbf{L}'^T\Delta\mathbf{x}}_{\mathbf{y}} = -\mathbf{b}'$$

$$\mathbf{L}'\mathbf{y} = -\mathbf{b}$$

solve through forward/backward subst.

$$\mathbf{L}'^T\Delta\mathbf{x}' = \mathbf{y}$$

$$\Delta\mathbf{x} = \mathbf{P}^T\Delta\mathbf{x}'$$

Recover dx applying inverse permutation

Conclusions

We approached a complex multi-robot multi-landmark problem

- The measurement independence leads to a sparse structure of H
- The structure of H does not change during the iterations
- It can be efficiently solved by using sparse methods
- Cholesky is not the only possible way (also other approaches such as QR factorization will do)
- Sparse methods rely on finding an ordering that keeps the triangular system sparse

Tomorrow: Total Least Squares

Sparse Optimization with

- Pose-Pose
- Pose-Landmark (3d)
- Pose-Landmark (Projection)

Using a factor graph

Pose-Landmark Constraint

- Here we represent the state as the pose of the robot in the world, i.e. $\mathbf{X} \quad : \quad {}^{W}\mathbf{T}_{R}$
- Consequently, prediction and error functions become

$$\mathrm{h}^{[i,j]}(\mathbf{X}) = \mathbf{X}_r^{[i]-1}\mathbf{X}_l^{[j]} = \mathbf{R}_\theta^T\mathbf{X}_l^{[j]} + \mathbf{R}_\theta^T\mathbf{t}$$

$$\mathrm{e}^{[i,j]}(\mathbf{X}_r \boxplus \boldsymbol{\Delta}\mathbf{x}_r, \mathbf{X}_l + \boldsymbol{\Delta}\mathbf{x}_l) = [v2t(\boldsymbol{\Delta}\mathbf{x}_r)\mathbf{X}_r]^{-1}\left(\mathbf{X}_l + \boldsymbol{\Delta}\mathbf{x}_l\right) - \mathbf{z}_j$$

$$\mathbf{J}_R = \frac{\partial [v2t(\boldsymbol{\Delta}\mathbf{x}_r)\mathbf{X}_r]^{-1}(\mathbf{X}_l+\boldsymbol{\Delta}\mathbf{x}_l)-\mathbf{z}_j}{\partial \boldsymbol{\Delta}\mathbf{x}_r}\Big|_{\boldsymbol{\Delta}\mathbf{x}_r=0,\boldsymbol{\Delta}\mathbf{x}_l=0}$$

$$\mathbf{J}_L = \frac{\partial [v2t(\boldsymbol{\Delta}\mathbf{x}_r)\mathbf{X}_r]^{-1}(\mathbf{X}_l+\boldsymbol{\Delta}\mathbf{x}_l)-\mathbf{z}_j}{\partial \boldsymbol{\Delta}\mathbf{x}_l}\Big|_{\boldsymbol{\Delta}\mathbf{x}_r=0,\boldsymbol{\Delta}\mathbf{x}_l=0}$$

Pose-Pose Constraint

- Here we represent the state as the pose of the robot in the world, i.e. $\mathbf{X} \quad : \quad {}^{W}\mathbf{T}_{R}$

- Consequently, prediction and error functions become

$$\mathrm{h}^{[i,j]}(\mathbf{X}) = \hat{\mathbf{Z}}^{[i,j]} = \mathbf{X}_r^{[i]-1}\mathbf{X}_r^{[j]}$$

$$\mathrm{e}^{[i,j]}(\mathbf{X}) = t2v\left[\mathbf{Z}^{[i,j]-1}\mathbf{X}_r^{[i]-1}\mathbf{X}_r^{[j]}\right]$$

$$\mathrm{e}^{[i,j]}(\mathbf{X}_r \boxplus \boldsymbol{\Delta}\mathbf{x}_r) = t2v\left[\mathbf{Z}^{[i,j]-1}v2t(\boldsymbol{\Delta}\mathbf{x}_r^i)^{-1}\hat{\mathbf{Z}}^{[i,j]}v2t(\boldsymbol{\Delta}\mathbf{x}_r^j)\right]$$