Probabilistic Robotics Course

Least Squares on Manifolds

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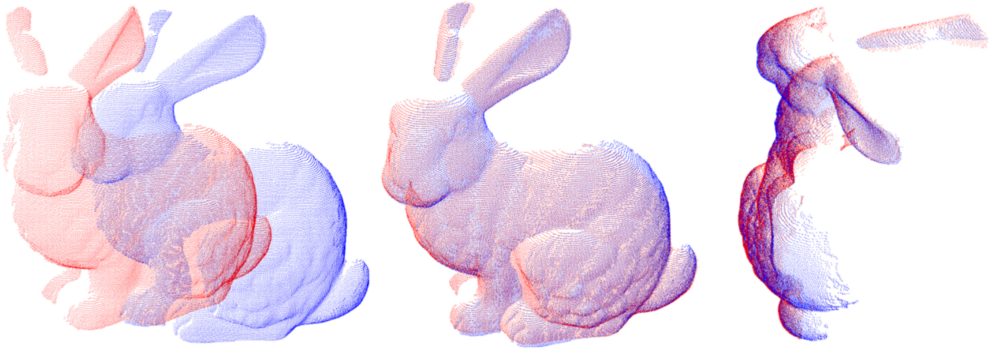
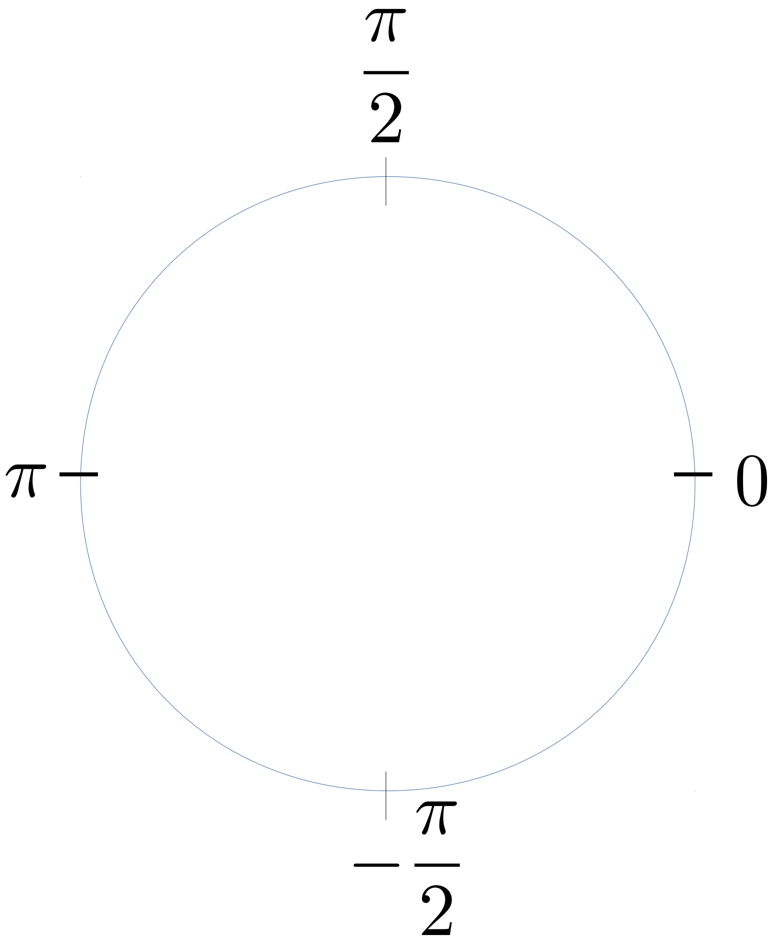


Courtesy of Li et. al

Non-Euclidean Spaces

In robotics we often encounter spaces that have a non-euclidean topology

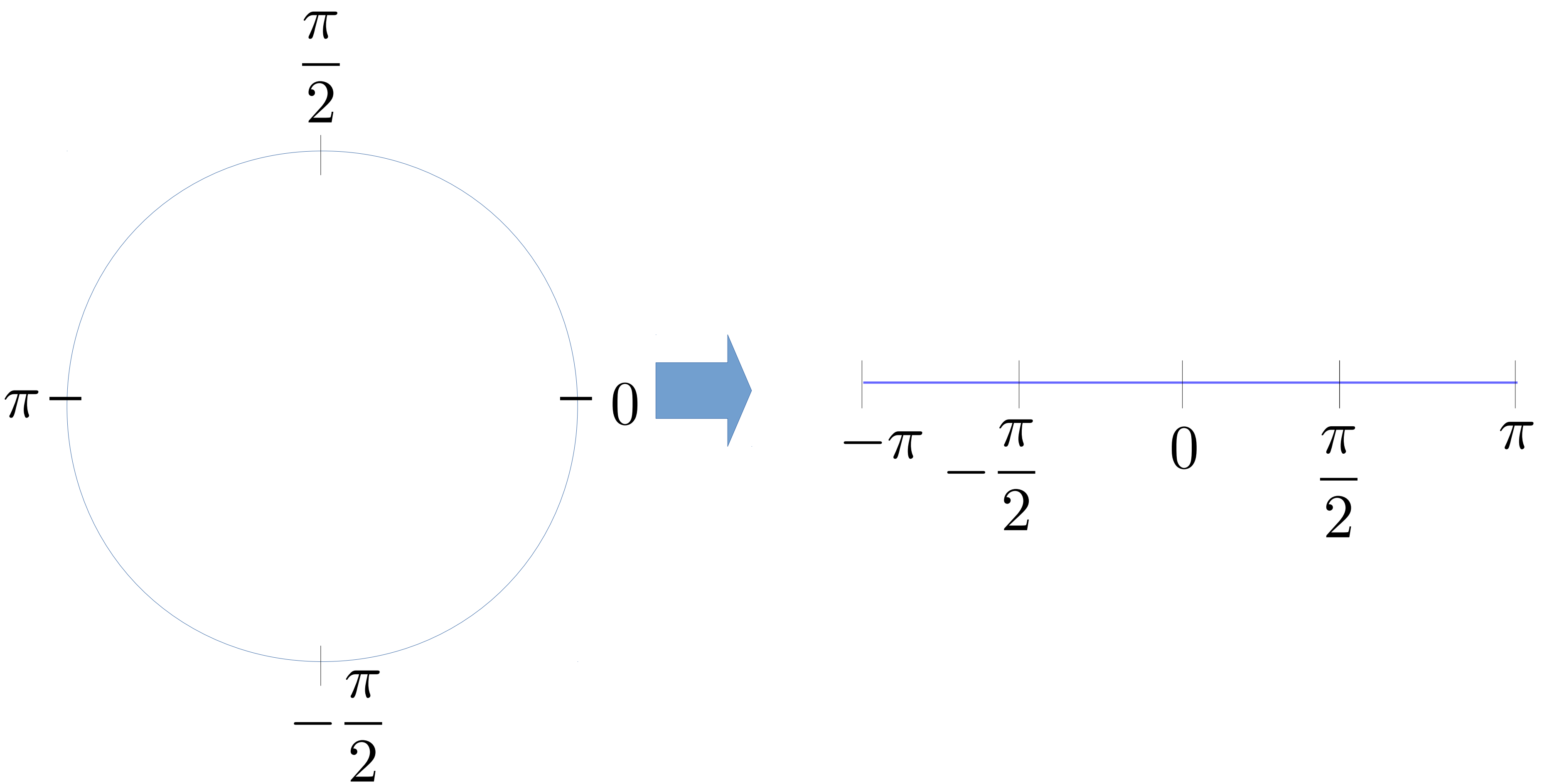
- E.g.: 2D angles



Non-Euclidean Spaces

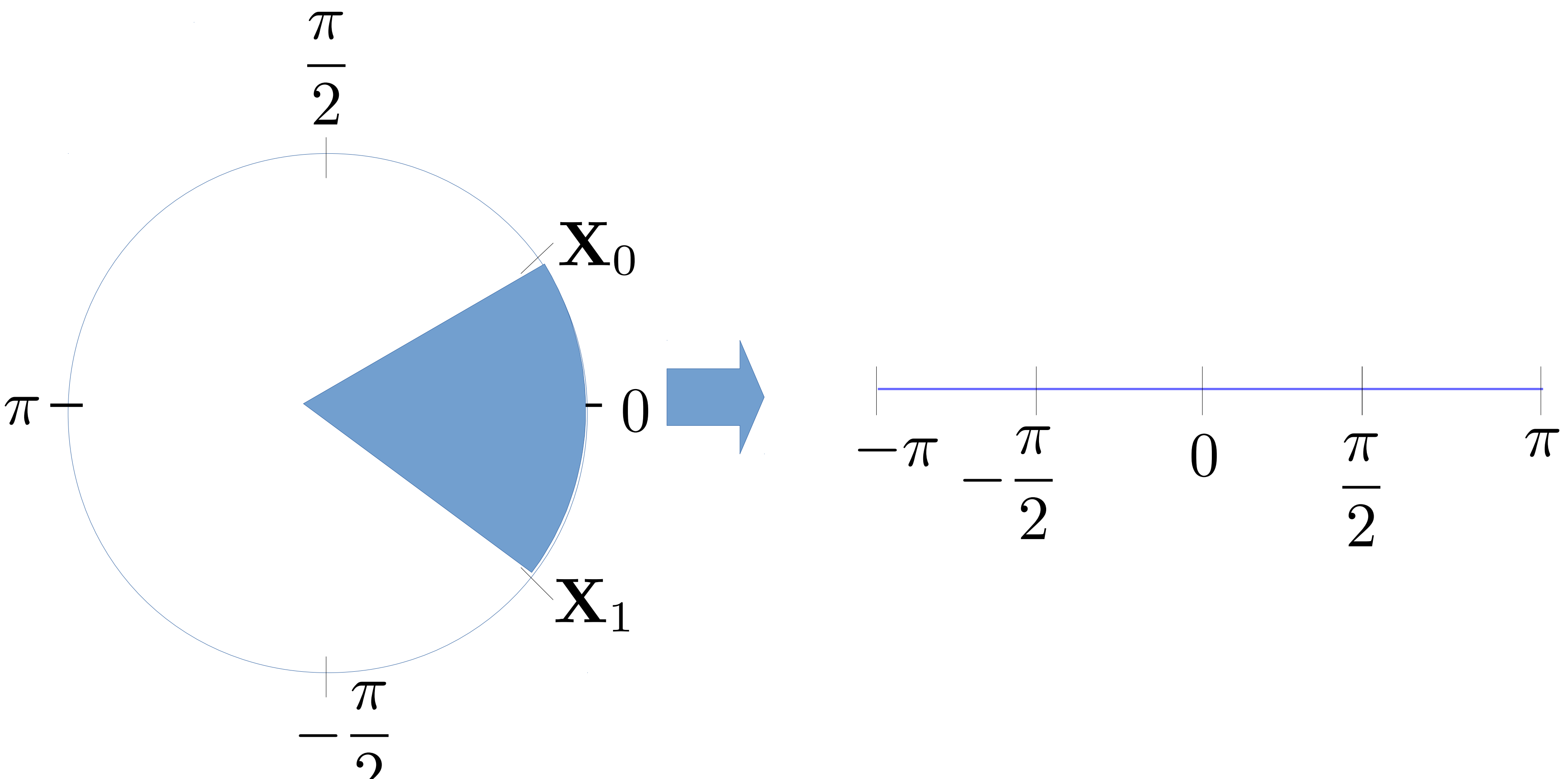
In such cases we commonly operate on a locally Euclidean parameterization

- E.g. we map the angles in the interval [-pi:pi]



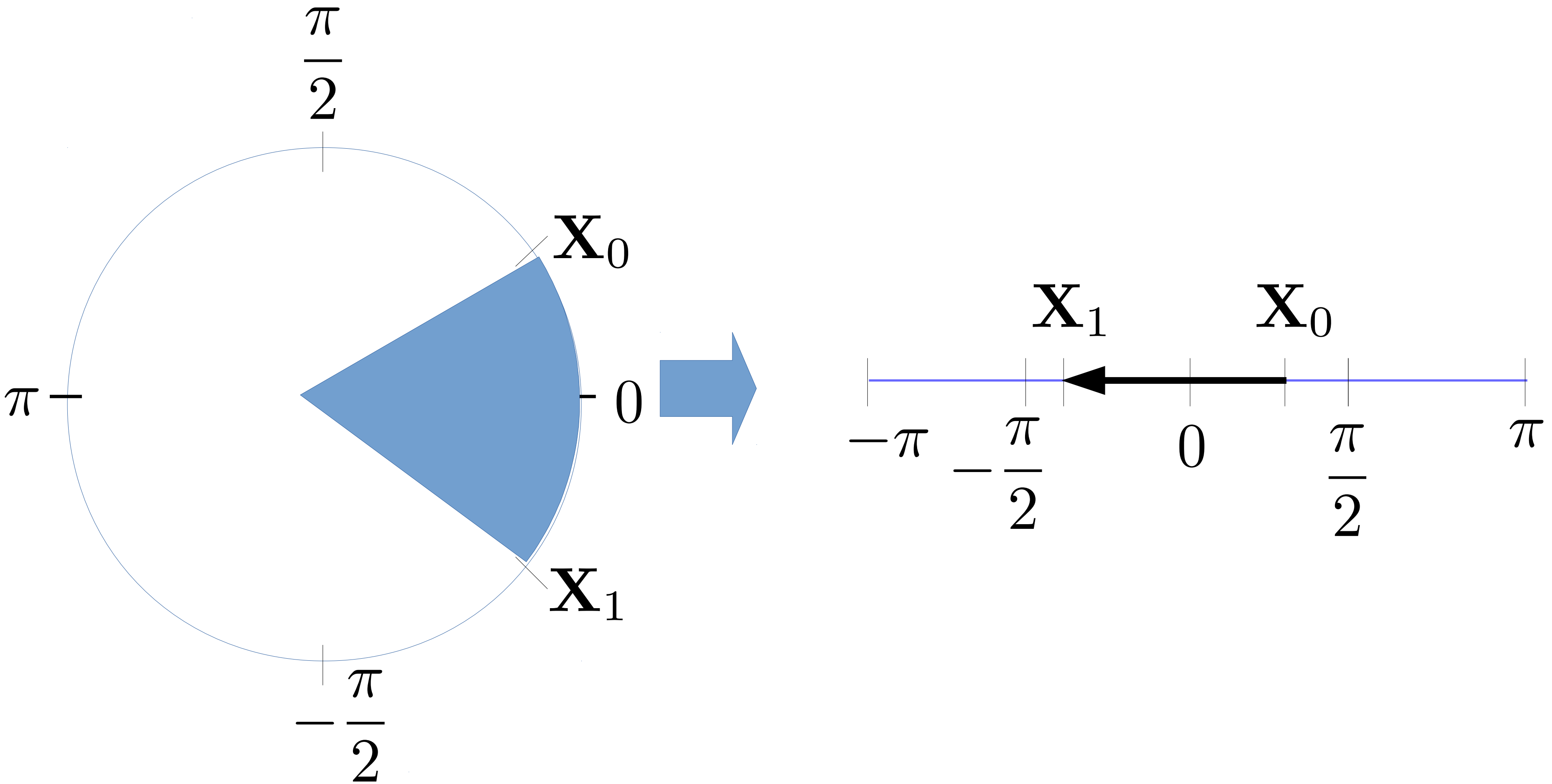
Non-Euclidean Spaces

We can then measure distances in the Euclidean mapping through a regular subtraction



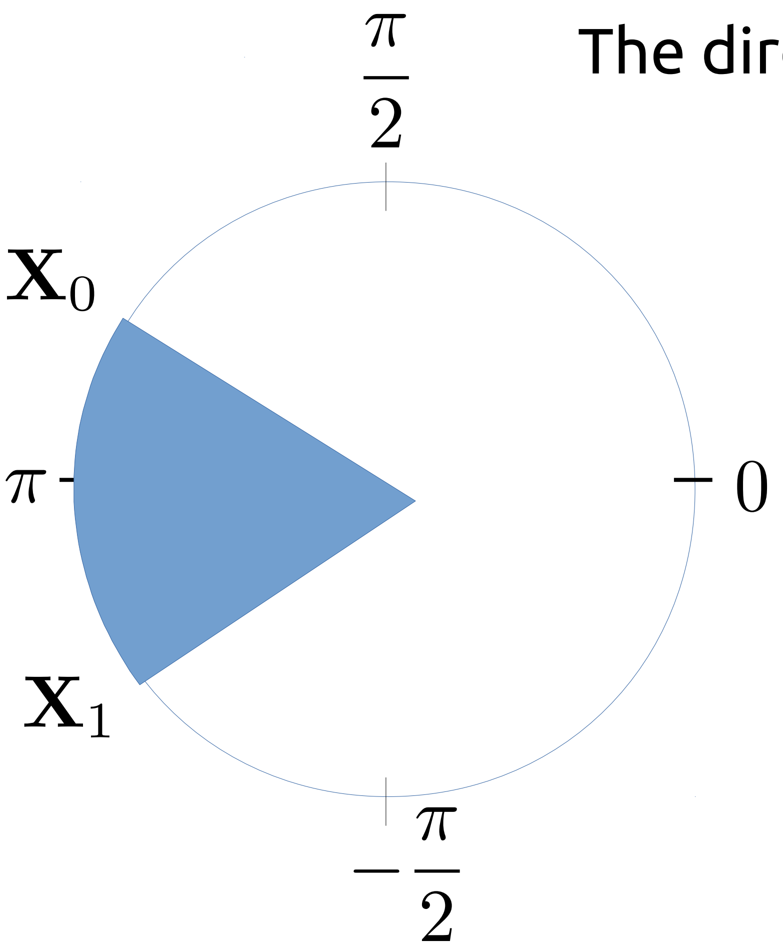
Non-Euclidean Spaces

We can then measure distances in the Euclidean mapping through a regular subtraction

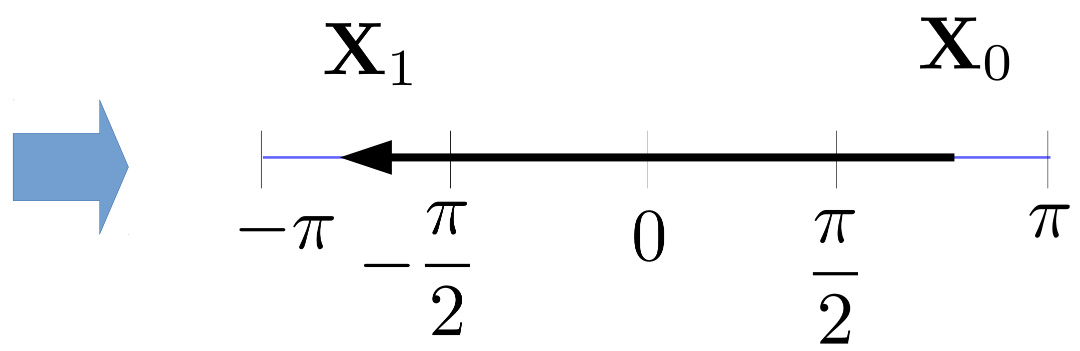


Non-Euclidean Spaces

We can then measure distances in the Euclidean mapping ~~through a regular subtraction~~



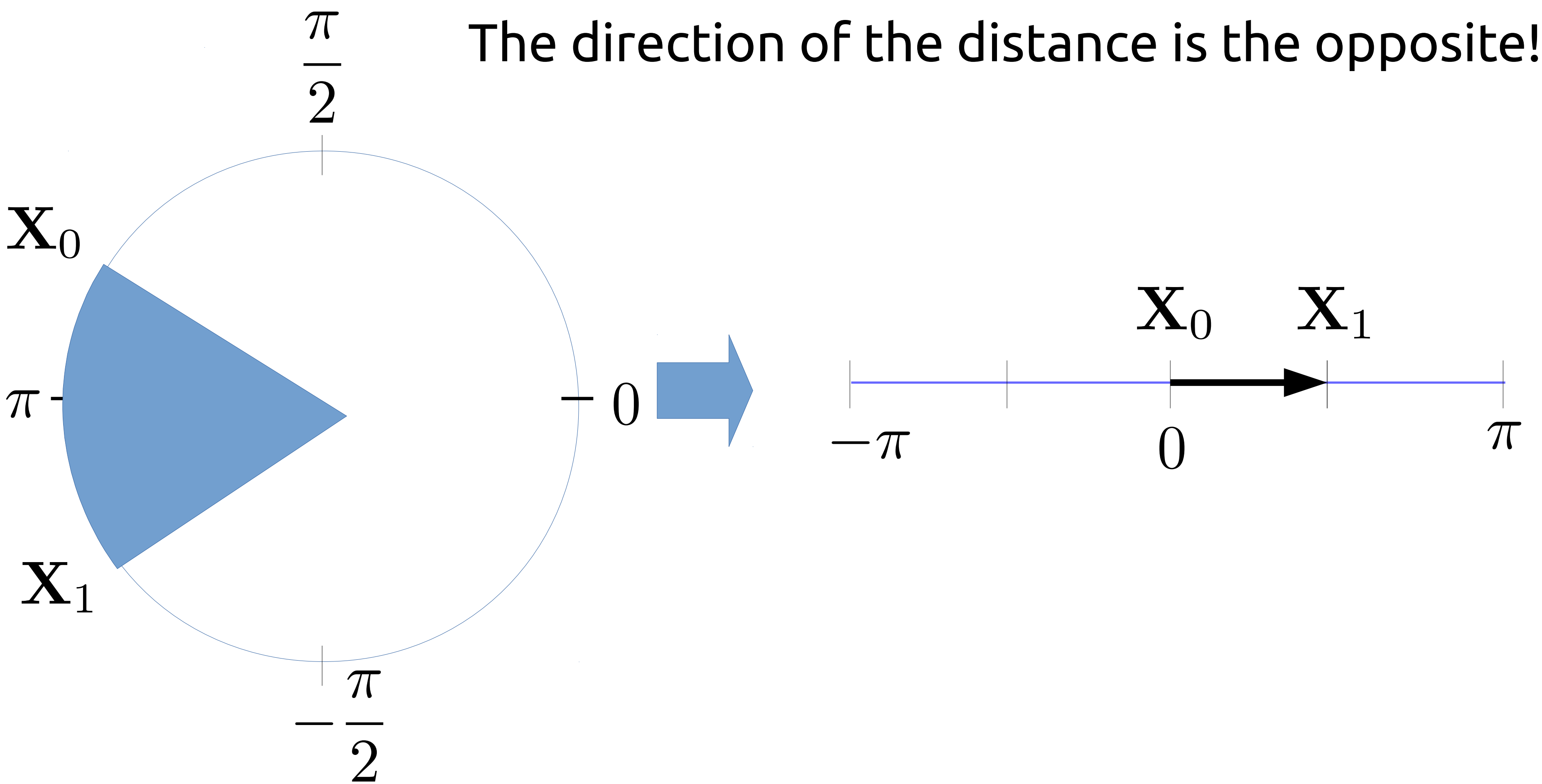
The direction of the distance is the opposite!!!



Non-Euclidean Spaces

Idea: when computing the distances, build the Euclidean mapping in the neighborhood of one of the points: the **chart around $\mathbf{x}_0$.**

The direction of the distance is the opposite!!!



Computing Differences

$\mathbf{x}_0$: start point, on manifold

$\mathbf{x}_1$: end point, on manifold

$\mathbf{\Delta x}$: difference, on chart

- Compute a chart around $\mathbf{x}_0$
- Compute the location of $\mathbf{x}_1$ on the chart
- Measure the difference between points in the chart
- Chart is Euclidean: $\mathbf{x}_0 = \mathbf{x}_1 \Rightarrow \mathbf{\Delta x} = 0$
- Use an operator $\mathbf{x}_1 \boxminus \mathbf{x}_0 = \mathbf{\Delta x}$

Hertzberg, Wagner, Frese and Schroeder 2011

Applying Differences

$\mathbf{X}_0$: start point, on manifold

$\mathbf{\Delta x}$: difference on chart

$\mathbf{X}_1$: end point, on manifold reachable from $\mathbf{X}_0$ by moving of $\mathbf{\Delta x}$ on the chart

- Compute a chart around $\mathbf{X}_0$
- Move of $\mathbf{\Delta x}$ in the chart and go back to the manifold
- Encapsulate the operation with an operator

$$\mathbf{X}_0 \boxplus \mathbf{\Delta x} = \mathbf{X}_1$$

Euclidean Least Squares

Clear **H** and **b**

$$\mathbf{H} \leftarrow 0 \qquad \mathbf{b} \leftarrow 0$$

For each measurement, update h and b

$$\begin{aligned}
\mathbf{e}^{[i]} &\leftarrow \mathbf{h}^{[i]}(\mathbf{x}^*) - \mathbf{z}^{[i]} \\
\mathbf{J}^{[i]} &\leftarrow \left.\frac{\partial \mathbf{e}^{[i]}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^*} \\
\mathbf{H} &\leftarrow \mathbf{H} + \mathbf{J}^{[i]T} \mathbf{\Omega}^{[i]} \mathbf{J}^{[i]} \\
\mathbf{b} &\leftarrow \mathbf{b} + \mathbf{J}^{[i]T} \mathbf{\Omega}^{[i]} \mathbf{e}^{[i]}
\end{aligned}$$

Update the estimate with the perturbation

$$\begin{aligned}
\mathbf{\Delta x} &\leftarrow \mathrm{solve}(\mathbf{H}\mathbf{\Delta x} = -\mathbf{b}) \\
\mathbf{x}^* &\leftarrow \mathbf{x}^* + \mathbf{\Delta x}
\end{aligned}$$

Gauss in Non Euclidean Spaces

Beware of the + and – operators

- Error function

$$\mathbf{e}^{[i]}(\mathbf{x}) \quad = \quad \mathbf{h}^{[i]}(\mathbf{x}) \boxminus \mathbf{z}_i$$

- Taylor expansion

$$\mathbf{e}^{[i]}(\mathbf{X} \boxplus \mathbf{\Delta x}) \quad = \quad \underbrace{\mathbf{e}^{[i]}(\mathbf{X})}_{\mathbf{e}^{[i]}} + \underbrace{\left.\frac{\partial \mathbf{e}^{[i]}(\mathbf{X} \boxplus \mathbf{\Delta x})}{\partial \mathbf{\Delta x}}\right|_{\mathbf{\Delta x}=\mathbf{0}}}_{\mathbf{J}^{[i]}} \mathbf{\Delta x}$$

- Increments

$$\mathbf{X} \leftarrow \mathbf{X} \boxplus \mathbf{\Delta x}$$

Manifold Least Squares

Clear **H** and **b**

$$\mathbf{H} \leftarrow 0 \qquad \mathbf{b} \leftarrow 0$$

For each measurement

$$\begin{aligned}
\mathbf{e}^{[i]} &\leftarrow \mathbf{h}^{[i]}(\mathbf{X}^*) \boxminus \mathbf{Z}^{[i]} \\
\mathbf{J}^{[i]} &\leftarrow \left.\frac{\partial \mathbf{e}(\mathbf{X}^* \boxplus \mathbf{\Delta x})}{\partial \mathbf{\Delta x}}\right|_{\mathbf{\Delta x}=\mathbf{0}} \\
\mathbf{H} &\mathrel{+}= \mathbf{J}^{[i]} \mathbf{\Omega}^{[i]} \mathbf{J}^{[i]} \\
\mathbf{b} &\mathrel{+}= \mathbf{J}^{[i]} \mathbf{\Omega}^{[i]} \mathbf{e}^{[i]}
\end{aligned}$$

Compute and apply the perturbation

$$\begin{aligned}
\mathbf{\Delta x} &\leftarrow \mathrm{solve}(\mathbf{H}\mathbf{\Delta x} = -\mathbf{b}) \\
\mathbf{X}^* &\leftarrow \mathbf{X}^* \boxplus \mathbf{\Delta x}
\end{aligned}$$

Methodology

State space **X**

- Qualify the Domain
- Define an Euclidean parameterization for the perturbation
- Define boxplus operator

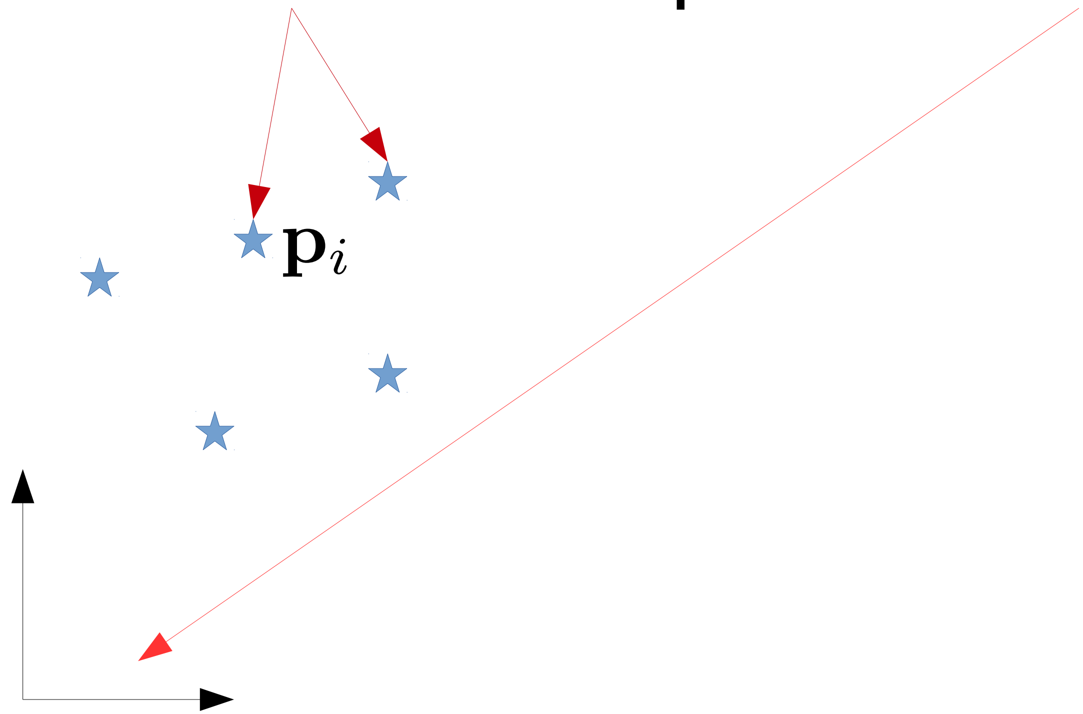
Measurement space(s) **Z**

- Qualify the Domain
- Define an Euclidean parameterization for the perturbation
- Define boxminus operator

Identify the prediction functions **h(X)**

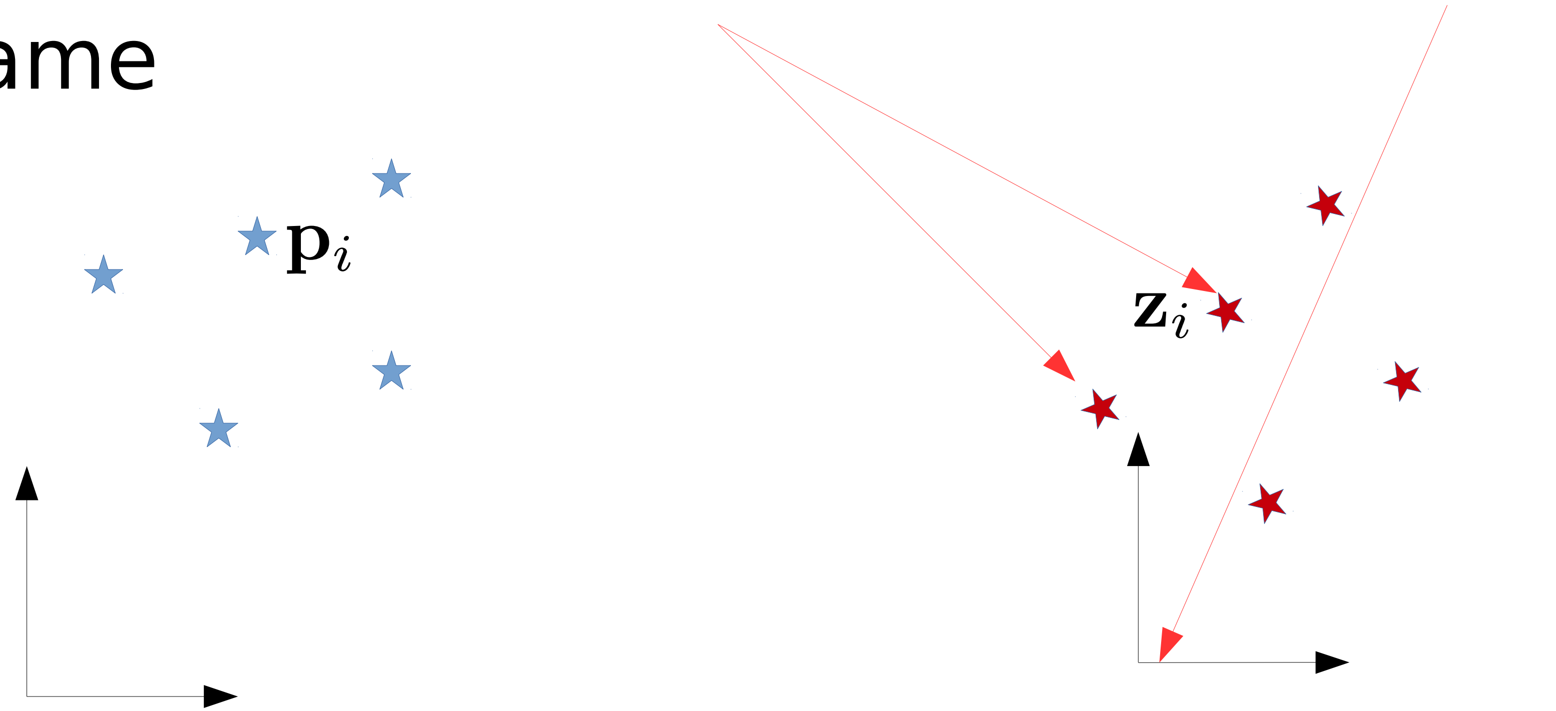
Example ICP Optimization 2D

Given a set of 2D points in the world frame



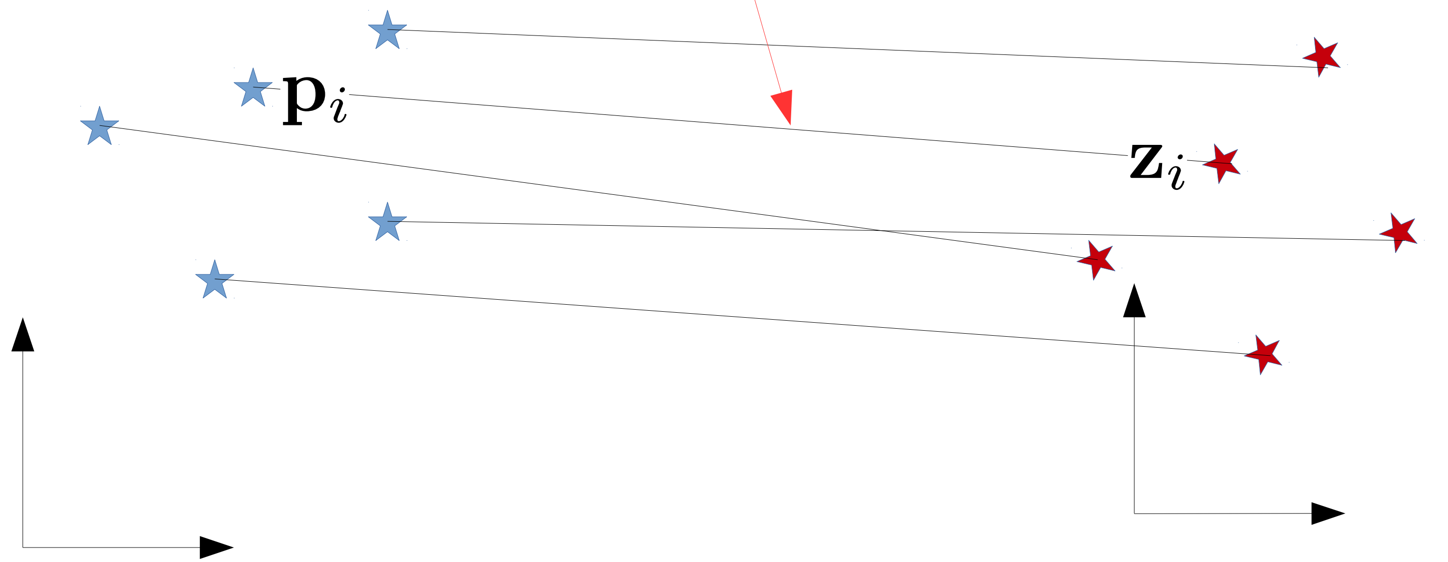
Example ICP Optimization 2D

A set of 2D measurements in the robot frame



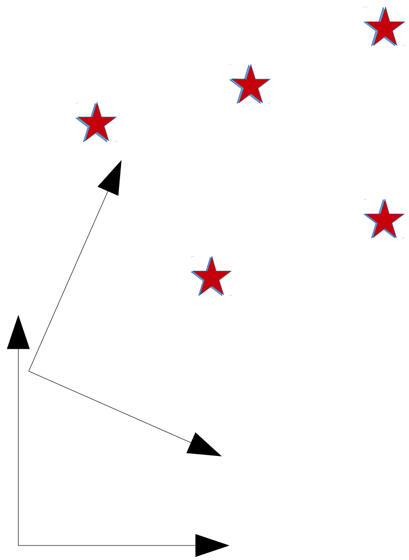
Example ICP Optimization 2D

Roughly known correspondences



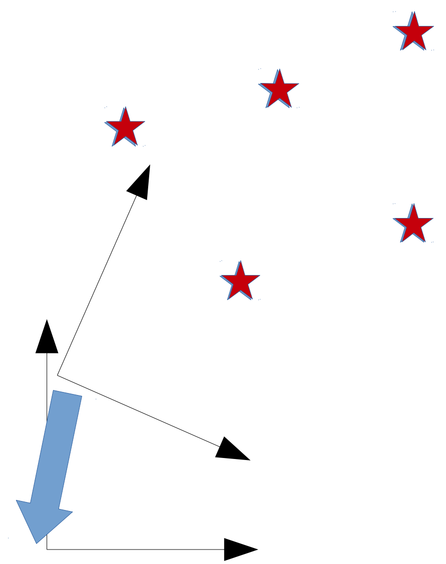
Example ICP Optimization 2D

We want to find a transform that minimizes distance between corresponding points



Example ICP Optimization 2D

Such a transform will be the pose of world w.r.t. robot



Note: we can also estimate robot w.r.t world, but it leads to longer calculations

ICP: State Space

State

$$\mathbf{X} \in SE(2), \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$$

Manifold representation as homogeneous transformation

$$\mathbf{\Delta x} = (\underbrace{\Delta x \ \Delta y}_{\mathbf{\Delta t}} \ \Delta\theta)^T$$

Euclidean parameterization for the chart

$$\mathbf{\Delta X} = v2t(\mathbf{\Delta x}) = \begin{bmatrix} \mathbf{\Delta R} & \mathbf{\Delta t} \\ \mathbf{0} & 1 \end{bmatrix}$$

Convenient function that converts a perturbation into a matrix

$$\begin{aligned} \mathbf{X} \boxplus \mathbf{\Delta x} &= v2t(\mathbf{\Delta x})\mathbf{X} \\ &= \mathbf{\Delta X} \cdot \mathbf{X} \end{aligned}$$

Definition of the boxplus operator

ICP: Measurements

$$
\begin{aligned}
\mathbf{z} &\in \Re^2 \\
\mathbf{h}^{[i]}(\mathbf{X}) &= \mathbf{R}\mathbf{p}^{[i]} + \mathbf{t} \\
&= \mathbf{X}\mathbf{p}^{[i]} \\
\mathbf{h}^{[i]}(\mathbf{X} \boxplus \mathbf{\Delta x}) &= (\mathbf{X} \boxplus \mathbf{\Delta x})\mathbf{p}^{[i]} \\
&= v2t(\mathbf{\Delta x})\,\underbrace{\mathbf{X}\mathbf{p}^{[i]}}_{\tilde{\mathbf{p}}^{[i]}} \\
&= \mathbf{R}(\Delta\theta)\tilde{\mathbf{p}}^{[i]} + \mathbf{\Delta t}
\end{aligned}
$$

ICP: Jacobian

$$
\begin{aligned}
\mathbf{h}^{[i]}(\mathbf{X} \boxplus \boldsymbol{\Delta}\mathbf{x}) &= \mathbf{R}(\boldsymbol{\Delta}\theta)\tilde{\mathbf{p}}^{[i]} + \boldsymbol{\Delta}\mathbf{t} \\
\left.\frac{\partial \mathbf{h}^{[i]}(\mathbf{X} \boxplus \boldsymbol{\Delta}\mathbf{x})}{\partial \boldsymbol{\Delta}\mathbf{x}}\right|_{\boldsymbol{\Delta}\mathbf{x}=0} &= \left.\begin{pmatrix} \frac{\partial \mathbf{h}^{[i]}(\cdot)}{\partial \boldsymbol{\Delta}\mathbf{t}} & \frac{\partial \mathbf{h}^{[i]}(\cdot)}{\partial \Delta\theta} \end{pmatrix}\right|_{\boldsymbol{\Delta}\mathbf{x}=0} \\
\left.\frac{\partial \mathbf{h}^{[i]}(\cdot)}{\partial \boldsymbol{\Delta}\mathbf{t}}\right|_{\boldsymbol{\Delta}\mathbf{x}=0} &= \mathbf{I} \\
\left.\frac{\partial \mathbf{h}^{[i]}(\cdot)}{\partial \Delta\theta}\right|_{\boldsymbol{\Delta}\mathbf{x}=0} &= \mathbf{R}'(0)\tilde{\mathbf{p}}_i \\
&= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tilde{\mathbf{p}}^{[i]} = \begin{pmatrix} -\tilde{p}^{[i]}.y \\ \tilde{p}^{[i]}.x \end{pmatrix}
\end{aligned}
$$

ICP: Octave Program

```
function [e,J]=errorAndJacobianManifold(X,p,z)
 t=X(1:2,3);
 R=X(1:2,1:2);
 z_hat=R*p+t;
 e=z_hat-z;

 J=zeros(2,3);
 J(1:2,1:2)=eye(2);
 J(1:2,3)=[-z_hat(2),
   z_hat(1)]';
endfunction;
```

ICP: Octave Program

```
function [chi,X]=icp2dManifold(X,P,Z)
  chi=0;          %cumulative chi2
  H=zeros(3,3); %accumulators for H and b
  b=zeros(3,1);
  for(i=1:size(P,2))
     p=P(:,i); z=Z(:,i);
    [e,J]=errorAndJacobianManifold(X,p,z);
    H+=J'*J;                %assemble H and B
     b+=J'*e;
     chi+=e'*e;               %update cumulative error
  endfor
  dx=-H\b;                    %solve the linear system
  X=v2t(dx)*X;                     %apply perturbation
endfunction
```

Uncertainty of the Solution

Optimizing on a Manifold generates a H matrix that is computed **on the chart**

$\mathbf{H}^{-1}$ represents a covariance of the solution around the origin of the chart

We can write that around the optimum

$$\mathbf{\Delta x} \sim \mathcal{N}(\mathbf{\Delta x}; \mathbf{0}, \mathbf{H}^{-1})$$

The Gaussian approximation of the distribution of solution around the optimum, is related to the chart through boxplus

$$\begin{aligned} \mathbf{X} &= \mathbf{X}^* \boxplus \mathbf{\Delta x} \\ &= g_{\mathbf{X}^*}(\mathbf{\Delta x}) \end{aligned}$$

Uncertainty (cont)

Using our manipulation skills on the Gaussian distribution, we can compute the approximation of a Gaussian in X, by either

- Linearization

$$\mathbf{J_X} = \left. \frac{\partial \mathbf{X}^* \boxplus \mathbf{\Delta x}}{\partial \mathbf{\Delta x}} \right|_{\mathbf{\Delta x} = \mathbf{0}}$$

$$\mathbf{X} \sim \mathcal{N}(\mathbf{X}; \mathbf{X}^*, \mathbf{J_X} \mathbf{H}^{-1} \mathbf{J_X}^T)$$

- Unscented Transform

$$\mathbf{X}^{(i)} = \mathbf{X}^* \boxplus \mathbf{\Delta x}^{(i)}$$

Where $\mathbf{\Delta x}^{(i)}$ are sigma points extracted from the Gaussian distribution in the chart.

Measurement Uncertainty

While computing the error function in the manifold case, we also replaced the vector difference between prediction and observation

- This has a more subtle effect, since the error is not anymore a linear function between the prediction and the measurement
- Accordingly, the information matrix **has to be recomputed at each iteration**, based on the current initial guess

$$\mathbf{Z}^{[i]} \sim \mathcal{N}(\mathbf{Z}^{[i]}, \mathbf{\Omega}^{[i]})$$

$$\mathbf{e}^{[i]}(\mathbf{X}) = \mathbf{h}^{[i]}(\mathbf{X}) \boxminus \mathbf{Z}^{[i]}$$

$$\mathbf{e}^{[i]}(\mathbf{X}) \sim \mathcal{N}\left(\mathbf{h}^{[i]}(\mathbf{X}) \boxminus \mathbf{Z}^{[i]}, \mathbf{\Omega}_{\mathbf{X}}^{[i]}\right)$$

Measurement Uncertainty

The remapped information matrix of a measurement can be computed as usual either by

- Linearization

$$\mathbf{Z}^{[i]} \sim \mathcal{N}(\mathbf{Z}^{[i]}, \mathbf{\Omega}^{[i]})$$

$$\mathbf{J}_{\mathbf{e}}^{[i]} = \left. \frac{\partial \mathbf{h}^{[i]}(\mathbf{X}) \boxminus \mathbf{Z}}{\partial \mathbf{Z}} \right|_{\mathbf{Z}=\mathbf{Z}^{[i]}}$$

$$\mathbf{e}^{[i]}(\mathbf{X}) \sim \mathcal{N}[\mathbf{h}^{[i]}(\mathbf{X}) \boxminus \mathbf{Z}^{[i]}, \overbrace{(\mathbf{J}_{\mathbf{e}}^{[i]} \mathbf{\Omega}^{[i]^{-1}} \mathbf{J}_{\mathbf{e}}^{[i]T})^{-1}}^{\mathbf{\Omega}_{\mathbf{X}}^{[i]}}]$$

- Unscented Transform

$$\mathbf{e}^{[i](j)}(\mathbf{X}) = \mathbf{h}^{[i]}(\mathbf{X}) \boxminus \mathbf{Z}^{[i](j)}$$

Manifold Least Squares

Clear **H** and **b**

$$\mathbf{H} \leftarrow 0 \qquad \mathbf{b} \leftarrow 0$$

For each measurement

$$\begin{aligned}
\mathbf{e}^{[i]} &\leftarrow \mathbf{h}^{[i]}(\mathbf{X}^*) \boxminus \mathbf{Z}^{[i]} \\
\mathbf{J}^{[i]} &\leftarrow \left.\frac{\partial \mathbf{e}(\mathbf{X}^* \boxplus \mathbf{\Delta x})}{\partial \mathbf{\Delta x}}\right|_{\mathbf{\Delta x}=\mathbf{0}} \\
\mathbf{J}_{\mathbf{e}} &= \left.\frac{\partial \mathbf{h}^{[i]}(\mathbf{X}^*) \boxminus \mathbf{Z}}{\partial \mathbf{Z}}\right|_{\mathbf{Z}=\mathbf{Z}^{[i]}} \\
\mathbf{\Omega}_{\mathbf{X}}^{[i]} &\leftarrow (\mathbf{J}_{\mathbf{e}} \mathbf{\Omega}^{[i]-1} \mathbf{J}_{\mathbf{e}}^T)^{-1} \\
\mathbf{H} &\mathrel{+}= \mathbf{J}^{T\,[i]} \mathbf{\Omega}_{\mathbf{X}}^{[i]} \mathbf{J}^{[i]} \\
\mathbf{b} &\mathrel{+}= \mathbf{J}^{T\,[i]} \mathbf{\Omega}_{\mathbf{X}}^{[i]} \mathbf{e}^{[i]}
\end{aligned}$$

Compute and apply the perturbation

$$\begin{aligned}
\mathbf{\Delta x} &\leftarrow \mathrm{solve}(\mathbf{H}\mathbf{\Delta x} = -\mathbf{b}) \\
\mathbf{X} &\leftarrow \mathbf{X} \boxplus \mathbf{\Delta x}
\end{aligned}$$

Conclusions

- Least squares on smooth manifolds offers a more robust formulation of non-linear least squares on non-Euclidean spaces
- Key idea: linearize the problem with respect to the current optimum, around the perturbations
- Using the boxplus and boxminus operators to encapsulate operations on the manifolds

  Download the octave code and compare plain vs manifold ICP
- Beware of the Uncertainties