1 April 30th, 2019

1.1 Randomized Multiway Cut Problem

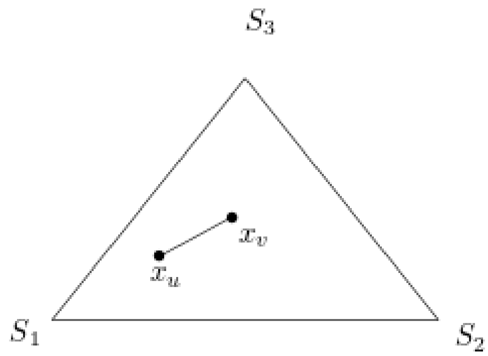

Figure 1: For $k = 3$

- Select $r \in (0, 1)$ uniformly at random
- Consider "corners" of simplex in random order
- Each corner considered "grabs" all vertices within "ball-distance" r from it that have not yet been grabbed. i.e.
$$u \in B(S, r).$$
- The last corner considered grabs all remaining vertices.
- To achieve an approximation factor of 1.5, we need to show that the probability that u and v are assigned to different corners is $\leq \frac{3}{4\|x_u - x_v\|}$.

Lemma 1.1
$u \in B(S_i, r)$ if and only if $1 - x_u^i \leq r$.

Proof. Ball distance of x_u from $S_i = \frac{1}{2}\|x_u - S_i\|$

$$= \frac{1}{2}\left[(1 - x_u^i) + \sum_{j \neq i} x_u^j\right] = \frac{1}{2}\left[(1 - x_u^i) + (1 - x_u^i\right] = 1 - x_u^i.$$

This is because the j-th component S_i is 0 for all $j \neq i$ and 1 for $j = i$, and $\sum x_u^j = 1$. $\square$

Definition 1.2. We say that an index i **cuts** (u, v) if exactly one of x_u and x_v is contained in ball $B(S_i, r)$.

For x_u and x_v to be assigned in different corner, it if necessary that there is some index that cuts (u, v). This is not sufficient since u and v could both be grabbed by a vertex, but another vertex later on might cut the edge (since they are selected in a random order).

When does index i cut (u, v)? In fact, what is the probability that index i cuts (u, v). From the lemma above, we only need to look at that i-th coordinate of x_u^i.

WLOG, assume that x_u is closer to S_i. Then

$$(u, v) \text{ is cut by index } i \text{ iff } r \in (1 - x_u^i, 1 - x_v^i).$$

$$\implies \Pr(\text{index } i \text{ cuts } (u, v)) = \frac{(1 - x_i^v) - (1 - x_i^u)}{1} = |x_u^i - x_v^i|.$$

Now we have:

$$\Pr((u, v) \in \text{multicut}) = \Pr(u \ \& \ v \text{ are cut by index } i)$$

$$\leq \sum_i \Pr(u \ \& \ v \text{ are cut by index } i\ ) = \sum_i |x_u^i - x_v^i| = \|x_u - x_v\|.$$

Now we would like to tighten the restriction to achieve an upper bound of $\frac{3}{4}\|x_u - x_v\|$. Note that we have not yet used the fact that "corners" are considered in random order.

Definition 1.3. We day that an index i settles (u, v) if i is the first index in the random permutation such that at least one of u or v belongs to $B(S_i, r)$.

Intuitively, i decides whether (u, v) will be in the multiway cut or not. Let us define the following events:

- S_i : Event that index i settles (u, v).
- X_i : Event that index i settles (u, v).

Then:

$$\Pr\left[(u, v) \in \text{ multicut}\right] \leq \sum_i \Pr(S_i \wedge X_i).$$

This is because (u, v) is in the multiway cut only if there is some index i that both settles and cuts (u, v), i.e. it is the first index that separates u and v. This is clearly necessary, but it is not sufficient, since it could be the last index.

Let l be the index such that S_i is closest to one of the two endpoints of (u, v). That is, $i = l$ minimizes the quantity $\min\left(\|S_i - u\|, \|S_i - v\|\right)$

Claim 1.4. Index $i \neq l$ cannot settle edge (u, v) if l is ordered before i in the random permutation.

Proof. If index i were to settle (u, v), then it would be l. □

We need to compute $\Pr(S_i \wedge X_i)$ for index i. There are 2 cases: $i \neq l$ and $i = l$.

1. For $i \neq l$, we have:

$$\Pr\left[S_i \wedge X_i\right]$$

$$= \Pr\left[S_i \wedge X_i \mid l \text{ occurs after } i \text{ in permutation}\right] \times \Pr[l \text{ occurs after } i \text{ in permutation}]$$

$$+ \Pr\left[S_i \wedge X_i \mid l \text{ occurs before } i \text{ in permutation}\right] \times \Pr\left[l \text{ occurs before } i \text{ in permutation}\right]$$

$$= \frac{1}{2}\Pr[S_i \wedge X_i \mid l \text{ occurs after } i \text{ in permutation}] \leq \frac{1}{2}\Pr[X_i] = \frac{1}{2}|x_u^i - x_v^i|.$$

2. For $i = l$ we have:
$$\Pr[S_l \wedge X_l] \leq \Pr[X_l] = |x_u^l - x_v^l|.$$

This means that:

$$\Pr\left[(u,v) \in \text{ multicut}\right] \leq \sum_i \Pr(S_i \wedge X_i) \leq \frac{1}{2}\sum_{i \neq l} |x_u^i - x_v^i| + |x_u^l - x_v^l|.$$

Lemma 1.5
For any index l and any 2 vertices u and v,
$$|x_u^l - x_v^l| \leq \frac{1}{2}\|x_u - x_v\|.$$

Proof. $|x_u^l - x_v^l| = \left|\left(1 - \sum_{j \neq l} x_u^j\right) - \left(1 - \sum_{j \neq l} x_v^l\right)\right| = \left|\sum_{j \neq l}(x_v^j - x_u^j)\right| \leq \sum_{i \neq j} |x_v^j - x_u^j| + \|x_v^l - x_u^l\|$ ∎

$$\Pr\left[(u,v) \in \text{ multicut}\right] \leq \frac{1}{2}\sum_i |x_u^i - x_v^i| + \frac{1}{2}|x_u^l - x_v^l| \leq \frac{1}{2}\|x_u - x_v\| + \frac{1}{2} \times \frac{1}{2}\|x_u - x_v\| = \frac{3}{4}\|x_u - x_v\|.$$