1 February 10th, 2022

1.1 Stable Law Continued

Recall from the last time, we concluded that most of the contribution for S_n is from the large points of scale $O(n^{1/\alpha})$ and that this is of constant order. Let us define an index set of large points:

$$I_n(\epsilon) = \{m \leq n : |X_m| > \epsilon n^{1/\alpha}\}$$

and define the sums:

$$\hat{S}_n(\epsilon) = \sum_{m \in I_n(\epsilon)} X_m = \sum_{m=1}^{n} X_m \mathbb{1}(|x_m| > \epsilon n^{1/\alpha})$$

$$\overline{S}_n(\epsilon) = S_n - \hat{S}_n(\epsilon) = \sum_{m=1}^{n} X_m \mathbb{1}(|X_m| \leq \epsilon^{1/\alpha})$$

Intuitively speaking $\hat{S}_n(\epsilon)$ represents the sum of large points and $\overline{S}_n(\epsilon)$ represents the sum of small points.

Remark 1.1 — Later on, ϵ will be chosen to be as small as possible. Later we will it to go to zero along with n, e.g. $1/\log n$, since we might exclude relevant points. For now we will consider it fixed.

Now we have two task, to show

1. Show $\frac{\overline{S}_n(\epsilon)}{n^{1/\alpha}}$ is small if ϵ is small.
2. Find the limit of $\frac{\hat{S}_n(\epsilon)}{n^{1/\alpha}}$.

Proof. of 1.

$$\mathbf{E}\left[\left(\frac{\overline{S}_n(\epsilon)}{n^{1/\alpha}}\right)^2\right] = n^{-\frac{2}{\alpha}} \cdot n \cdot \mathbf{E}\left[\left(\overline{X_1}(\epsilon)\right)^2\right], \quad \overline{X_i}(\epsilon) = X_i \mathbb{1}(|X_i| \leq \epsilon n^{1/\alpha})$$

$$\begin{aligned}
\mathbf{E}\left[\left(\overline{X_1}(\epsilon)\right)^2\right] &= \int_0^\infty 2y \Pr(|\overline{X}_1(\epsilon)| > y)\, dy \\
&\leq \int_0^{\epsilon n^{1/\alpha}} 2y \Pr(|X_1| > y) dy \\
&= \int_0^1 2y\, dy + \int_1^{\epsilon n^{1/\alpha}} \cdot 2y y^{-\alpha} dy \leq \frac{2\epsilon^{2-\alpha}}{2-\alpha} n^{\frac{2}{\alpha}-1}
\end{aligned}$$

This gives us:

$$\mathbf{E}\left[\left(\frac{\overline{S}_n(\epsilon)}{n^{1/\alpha}}\right)^2\right] \leq \frac{2\epsilon^{2-\alpha}}{2-\alpha}, \quad 0 < \alpha < 2$$

Later we choose $\epsilon = \epsilon_n \downarrow 0$ as $n \to \infty$. ☐

Proof. of 2.

Note that $\hat{S}_n(\epsilon)$ is a sum of a random number of r.v. We will find the characteristic function using the total law of expectation:

$$\mathbf{E}\left[\exp\left(it\frac{\hat{S}_n(\epsilon)}{n^{1/\alpha}}\right)\right] = \sum_{m=0}^{n}\mathbf{E}\left[\exp\left(it\frac{\hat{S}_n(\epsilon)}{n^{1/\alpha}}\right)\Bigg||I_n(\epsilon)| = m\right]\Pr(|I_n(\epsilon)| = m)$$

Now, we need to find these two terms. We will start with finding $\Pr(|I_n(\epsilon)| = m)$. We will use two facts:

1. $|I_n(\epsilon)| = \sum_{m=1}^{n} \mathbb{1}(|X_m| > \epsilon n^{1/\alpha})$ is $\text{Bin}\left(n, \frac{\epsilon^{-\alpha}}{n}\right) \sim \text{Poisson}(\epsilon^{-\alpha})$, giving us $\Pr(|X_n| > \epsilon n^{\frac{1}{\alpha}}) = \epsilon^{-\alpha}\frac{1}{n}$.

2. The conditional distribution of $\hat{S}_n(\epsilon)\Big||I_n(\epsilon)| = m$ equals the distribution of the sum of m i.i.d. r.v. with c.d.f. F_ϵ defined as:

$$1 - F_\epsilon(x) = \Pr\left(\frac{X_1}{n^{1/\alpha}} > x\Bigg|\frac{|X_1|}{n^{1/\alpha}} > \epsilon\right).$$

i.e. F_ϵ is the conditional distribution of $\frac{X_1}{n^{1/\alpha}}$ given $\frac{|X_1|}{n^{1/\alpha}} > \epsilon$.

Proof.

$$\begin{aligned}\Pr(\hat{S}_n(\epsilon) \in B\big||I_n(\epsilon)| = m) &= \frac{\Pr(\hat{S}_n(\epsilon) \in B, |I_n(\epsilon)| = m)}{\Pr(|I_n(\epsilon)| = m)}\\ &= \frac{\binom{n}{m}\Pr\left(\sum_{i=1}^{m} X_i \in B, |X_1| > \epsilon n^{1/\alpha}, \dots, |X_m| > \epsilon n^{1/\alpha}\right)}{\binom{n}{m}\Pr\left(|X_1| > \epsilon n^{1/\alpha}, \dots, |X_m| > \epsilon n^{1/\alpha}\right)}\end{aligned}$$

□

For our distribution, we have:

$$1 - F_\epsilon(x) = \frac{x^{-\alpha}}{2\epsilon^{-\alpha}}, \quad x \geq \epsilon.$$

i.e. F_ϵ is the c.d.f. of ϵX_1, meaning that the characteristic function of F_ϵ is $\varphi(\epsilon t)$. Consequently:

$$\begin{aligned}\mathbf{E}\left[\exp\left\{it\hat{S}_n(n^{1/\alpha})\right\}\right] &= \sum_{m=0}^{n}\binom{n}{m}\left(\frac{\epsilon^{-\alpha}}{n}\right)^m\left(1 - \frac{\epsilon^{-\alpha}}{n}\right)^{n-m}[\varphi(\epsilon t)]^m\\ &\to \sum_{m=0}^{\infty}\exp(-\epsilon^{-\alpha})\cdot(-\epsilon^{-\alpha})^m\frac{[\varphi(\epsilon t)]^m}{m!} = \exp\left\{-\epsilon^{-\alpha}(1 - \varphi(\epsilon t))\right\}\end{aligned}$$

using Poisson approximation for binomial and DCT.

Recall earlier that we have an approximation for $1-\varphi(\epsilon t) = C_\alpha \epsilon^\alpha |t|^\alpha$ if $\epsilon \to 0$, giving us:

$$\mathbf{E}\left[\exp\left\{it\hat{S}_n(n^{1/\alpha})\right\}\right] = \exp(-C_\alpha |t|^\alpha),$$

which is the same as Solution 1. Note that we need to choose $\epsilon = \epsilon_n \downarrow 0$. For more details see Lemma 3.7.1 of [Dur19]. $\square$

From this solution, we can see that only the tail part matters. Now, we will try to generalize this solution.

Definition 1.2 (slowly varying function). $L : \mathbb{R} \to \mathbb{R}$ is a slowly varying function if it satisfies:

$$\lim_{x \to +\infty} \frac{L(tx)}{L(x)} = 1,$$

for any fixed $t > 0$.

Example 1.3

$\log x$, $\log\log x$, $\log\sqrt{x}$ are slowly varying functions, but any power function x^t is not.

Theorem 1.4 (stable law)

Suppose $X_1, X_2 \ldots$ are i.i.d. with distribution satisfying:

(i) $\lim_{x\to+\infty} \Pr(X_1 > x)/\Pr(|X_1| > x) = \theta \in [0,1]$ (tails may not be significant)

(ii) $\Pr(|X_1| > x) = x^{-\alpha} L(x)$, $\alpha < 2$, and L is slowly varying (general total tail)

Let $S_n = \sum_{i=1}^n X_i$, $a_n = \inf\{x : \Pr(|X_1| > x) \le \frac{1}{n}\}$, $b_n = n\mathbf{E}\left[X_1 \mathbb{1}(|X_1| \le a_n)\right]$, then as $n \to \infty$:

$$\frac{S_n - b_n}{a_n} \implies Y,$$

for a non-degenerate r.v. Y.

Remark 1.5 — θ in Theorem 1.4 indicates the relative heaviness between the right and left tail. If θ close to 1, the right tail is dominant, if $\theta \approx \frac{1}{2}$ then both tails are roughly equal.

We want to choose a_n s.t. $\Pr\left(\frac{X_1}{a_n} \in (\alpha,\beta)\right) \sim \frac{1}{n}$ since $\frac{S_n}{a_n} = \sum_{i=1}^n \frac{X_1}{a_n}$, and we want the number of large points to be a constant order random variable. A natural choice is $\Pr(|X_1| \ge a_n) \sim \frac{1}{n}$, which gives us the quantile of $\frac{1}{n}$, i.e. $a_n = \inf\{x : \Pr(|X_1| > x) \le \frac{1}{n}\}$.

Remark 1.6 — We could have used ca_n for any constant c. In this case, we just choose $c = 1$.

For choosing b_n, we can choose $b_n = n\mathbf{E}\left[X_1 \mathbb{1}(|X_1| \le ca_n)\right]$ for any constant as well. This is because $b_n = n\mathbf{E}[\underbrace{X_1}_{a_n} \underbrace{\mathbb{1}(|X_1| \le ca_n)}_{\Pr(\cdot)\sim 1/n}] \sim a_n$, meaning that the limit would differ by a constant factor.

> **Remark 1.7** — The reason why we can truncate to of order a_n instead of something much larger say a_n^2 is because with high probability there are no such points.

1.2 Proof of Stable Law

Claim 1.8.

$$n \Pr(|X_1| > \alpha_n) \to 1, \quad n \to \infty$$

Proof. omitted. □

For the tail behavior, we get:

$$\begin{aligned} &n \Pr(|X_1| > x\alpha_n) \to \theta x^{-\alpha}, \quad n \to \infty, x > 0 \\ \sim &n \Pr(|X_1| > \alpha_n) \cdot \theta = n(xa_n)^\alpha L(xa_n) \cdot \theta \\ \sim &n(xa_n)^\alpha L(a_n) \cdot \theta = nx^{-\alpha} \Pr(|X_1| > a_n) \cdot \theta \sim x^{-\alpha} \cdot \theta \end{aligned}$$

meaning that a constant in front of a_n does not affect the convergence.

This also tells us that if we use compute the counting measure:

$$N_n((x, \infty)) = \sum_{m=1}^{n} \mathbb{1}\left(\frac{X_m}{a_n} > x\right) \implies \text{Poisson}(\theta x^{-\alpha}).$$

More generally $N_n(A)$ converges to a Poisson point process $N(A)$ with mean measure

$$\mathbf{E}N(A) = \mu(A) = \int_{A\cap(0,\infty)} \theta\alpha|x|^{-(\alpha+1)}dx + \int_{A\cap(-\infty,0)} (1-\theta)\alpha|x|^{-(\alpha+1)}dx.$$

Now we will decompose the points into large and small parts. Let us define index set:

$$I_n(\epsilon) = \{m \le n : |X_m| > \epsilon a_n\}$$

and define the following:

$$\begin{aligned} \hat{S}_n(\epsilon) &= \sum_{m \in I_n(\epsilon)} X_m \quad \text{(sum of large points)} \\ \overline{\mu}(\epsilon) &= \mathbf{E}[X_m \mathbb{1}(|X_n| \le \epsilon a_n)] = \mathbf{E}\overline{X_m}(\epsilon) \\ \hat{\mu}(\epsilon) &= \mathbf{E}[X_m \mathbb{1}(\epsilon a_n < |X_n| \le a_n)] \\ \overline{S}_n(\epsilon) &= (S_n - b_n) - (\hat{S}_n(\epsilon) - n\hat{\mu}(\epsilon)) \\ &= \sum_{m=1}^{n} \left(\overline{X_m}(\epsilon) - \overline{\mu}(\epsilon)\right) \text{ (centered sum of small points)} \end{aligned}$$

Remark 1.9 — Unlike the special case, we need to subtract by b_n, since it is no long symmetric.

Remark 1.10 — Note from the definition of b_n, we truncate $\hat{\mu}(\epsilon)$ instead of going to infinity.

Now we have once again have two tasks:

1. Show $\frac{\overline{S}_n(\epsilon)}{a_n}$ is small if ϵ is small.
2. Find the limit of $\frac{\hat{S}_n(\epsilon)-n\hat{\mu}(\epsilon)}{a_n}$.

Proof. of 1.

$$\begin{aligned}\mathbf{E}\left[\left(\frac{\overline{S}_n(\epsilon)}{a_n}\right)^2\right] &\leq n\mathbf{E}\left[\left(\frac{\overline{X}_1(\epsilon)}{a_n}\right)^2\right] \\ &\leq \int_0^\epsilon 2y\Pr(|\overline{X}_1(\epsilon)| > ya_n)\,dy \\ &= \underbrace{n\Pr(|X_1| > a_n)}_{\to 1}\int_0^\epsilon 2y\underbrace{\frac{\Pr(|X_1| \geq ya_n)}{\Pr(|X_1| > a_n)}}_{y^{-\alpha}}\,dy \\ &\to \int_0^\epsilon 2yy^{-\alpha}dy = \frac{2}{2-\alpha}\epsilon^{2-\alpha} \to 0 \text{ if } \epsilon \to 0\end{aligned}$$

□

Proof. of 2.
Let us first consider trying to compute the characteristic function of $\frac{S_n}{a_n}$, since we can add the constant part later. We have the following:

(i) $|I_n(\epsilon)| \to \text{Poisson}(\epsilon^{-\alpha})$

(ii) Given $|I_n(\epsilon)| = m$, $\frac{\hat{S_n}(\epsilon)}{a_n}$ has the same distribution as the sum of m i.i.d. r.v. with c.d.f. F_ϵ, which is again the conditional distribution of $\frac{X_1}{a_n}$ given $\frac{|X_1|}{a_n} \geq \epsilon$. This time, we need to distinguish the left and right tails:

$$\begin{aligned}1 - F_n^\epsilon(x) &= \Pr\left(\frac{X_1}{a_n} > x \,\middle|\, \frac{|X_1|}{a_n} > \epsilon\right) \to \theta\frac{x^{-\alpha}}{\epsilon^{-\alpha}} \\ F_n^\epsilon(-x) &= \Pr\left(\frac{X_1}{a_n} < -x \,\middle|\, \frac{|X_1|}{a_n} > \epsilon\right) \to (1-\theta)\frac{|x|^{-\alpha}}{\epsilon^{-\alpha}}\end{aligned}$$

Let $\Psi_n^\epsilon(t) \to \Psi^\epsilon(t)$ be the c.f. of F_n^ϵ:

$$\Psi_n^\epsilon(t) \to \Psi^\epsilon(t) = \int_\epsilon^\infty e^{itx}\theta\epsilon^\alpha\alpha x^{-(\alpha+1)}dx + \int_{-\infty}^{-\epsilon} e^{itx}(1-\theta)\epsilon^\alpha|x|^{-(\theta+1)}dx$$

Note that these tails only hold when $|x| > \epsilon$, as the density would be zero otherwise. This will be continued next lecture.

□