# 1 September 17th, 2020

## 1.1 Implicit Scheme for $u_t = u_{xx}$

Recall that the implicit scheme is:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{(\Delta x)^2}.$$

Note that when compared to the explicit scheme, the implicit scheme involves 3 unknown values of $U$ on the new level $n+1$. This is in contrast to the explicit scheme, for which the values of $U_j^{n+1}$ only depend on $U^n$. Thus there are $N-1$ unknowns: $U_1^{n+1}, U_2^{n+1}, \dots, U_{N-1}^{n+1}$, and $N-1$ equations:

$$(1+2\gamma)U_j^{n+1} - \gamma U_{j-1}^{n+1} - \gamma U_{j-1}^{n+1} = U_j^n.$$

This can be expressed as a linear system $AU = b$, with $A$ being tridiagonal.

The simplest way to solve this linear system is Gaussian elimination, which for a tridiagonal matrix is similar to Thomas algorithm which solves the equation:

$$-a_j U_{j-1} + b_j U_j - c_j U_{j+1} = d_j, \quad j = 1, \dots, N-1.$$

While assuming diagonally dominance:

$$a_j > 0, b_j > 0, c_j > 0, \quad b_j > a_j + c_j.$$

**Remark 1.1** — This diagonal dominance is to ensure there is a solution (not singular).

## 1.2 Stability Analysis for Implicit Scheme

Recall we are considering the equation:

$$\begin{cases} u_t = u_{xx} \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = u_0(x) \end{cases}.$$

Assuming we can do separation of variables, we have:

$$u(x,t) = Z(x) \cdot T(t).$$

Taking the Fourier series of the original equation, we have

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t) \sin k\pi x$$

$$\implies \sum_{k=1}^{\infty} a_k(t) \sin k\pi x = -\sum_{k=1}^{\infty} a_k(t)(k\pi)^2 \sin k\pi x.$$

Since $\sin k\pi x$ forms a basis, the coefficients must match, giving us:

$$a_k'(t) = -(k\pi)^2 a_k(t)$$
$$\implies a_k(t) = a_0 e^{-(k\pi)^2 t}.$$

Note that the evolution of $a_k$ is independent of other values of $k$. Thus in order to study how amplitude evolves with time, we don't need to look at the whole series, only how the amplitude decays with $k$. Thus for an exact solution of $u_t = u_{xx}$, we know that the amplitude decays exponentially fast.

For the discretized case, we want to see how the numeric scheme propagates the Fourier mode. Thus we let:

$$U_j^n = \lambda^n e^{ik(j\Delta x)}.$$

Plugging into the numerical implicit scheme, we have:

$$(1+2\nu)\lambda^{n+1}e^{ik(j\Delta x)} - \nu\lambda^{n+1}e^{ik(k+1)\Delta x} - \nu\lambda^{n+1}e^{ik(j-1)\Delta x} = \lambda^n e^{ikj\Delta x}.$$

$$\implies \lambda\left[(1+2\nu) - \nu e^{ik\Delta x} - \nu e^{-ik\Delta x}\right] = 1.$$
$$\implies \lambda\left(1+2\nu - 2\nu\cos k\Delta x\right) = 1.$$
$$\implies \lambda\left(1 + 4\nu\sin^2\frac{k\Delta x}{2}\right) = 1.$$
$$\implies \lambda = \frac{1}{1+4\nu\sin^2\frac{k\Delta x}{2}} < 1.$$

Thus this implicit scheme unconditionally stable, meaning there is no condition on $\nu$. Remember that for the explicit scheme, we needed the condition $\nu \leq \frac{1}{2}$.

## 1.3 The $\theta$-Method

Recall we have learned two schemes:

- Explicit Scheme:
$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{(\Delta x)^2}.$$

- Implicit Scheme:
$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{(\Delta x)^2}.$$

Both schemes have first order error in time $t$ and second order in space. This can be seen the truncation error $T_j^n$ using taylor expansion.

**Definition 1.2.** The ***$\theta$*-method** is a weighted average of explicit and implicit scheme. For the heat equation this is:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = (1-\theta)\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{(\Delta x)^2} + \theta\frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{(\Delta x)^2}, \quad 0 \leq \theta \leq 1.$$

> **Remark 1.3 —** If $\theta = 0$, we have a explicit scheme, and if $\theta = 1$ we have the implicit scheme, both with 1st order in time and 2nd order in space.

However, if use $\theta = \frac{1}{2}$, we have 2nd order in time and space. This is because there is some cancellation when $\theta = \frac{1}{2}$. For any other values of $\theta$, this will not be true. To calculate the truncation error for the $\theta$ method, we expand terms at $(x_j, t_{n+\frac{1}{2}})$:

$$
\begin{aligned}
u(x_j, t_n) &= u(x_j, t_{n+\frac{1}{2}}) - u_t(\frac{1}{2}\Delta t) + \frac{1}{2}u_{tt}\left(-\frac{1}{2}\Delta t\right)^2 + \frac{1}{6}u_{ttt}\left(-\frac{1}{2}\Delta t\right)^3 + \dots \\
u(x_j, t_{n+1}) &= u(x_j, t_{n+\frac{1}{2}}) + u_t(\frac{1}{2}\Delta t) + \frac{1}{2}u_{tt}\left(-\frac{1}{2}\Delta t\right)^2 + \frac{1}{6}u_{ttt}\left(-\frac{1}{2}\Delta t\right)^3 + \dots \\
u(x_{j\pm1}, t_n) &= u(x_j, t_{n+\frac{1}{2}}) - u_t(\frac{1}{2}\Delta t) \pm u_x\Delta x + \frac{1}{2}u_{tt}\left(-\frac{1}{2}\Delta t\right)^2 + \frac{1}{2}u_{xx}\left(\Delta x\right)^2 \mp \frac{1}{2}u_{xt}\Delta x\Delta t + \dots \\
u(x_{j\pm1}, t_{n+1}) &= u(x_j, t_{n+\frac{1}{2}}) + u_t(\frac{1}{2}\Delta t) \pm u_x\Delta x + \frac{1}{2}u_{tt}\left(-\frac{1}{2}\Delta t\right)^2 + \frac{1}{2}u_{xx}\left(\Delta x\right)^2 \pm \frac{1}{2}u_{xt}\Delta x\Delta t + \dots
\end{aligned}
$$

This gives truncation error:

$$
\begin{aligned}
T_j^{n+\frac{1}{2}} = \underbrace{(u_t - u_{xx})}_{=0} + \left[(\frac{1}{2} - \theta)\Delta t u_{xxt} - \frac{1}{12}\left(\Delta x\right)^2 u_{xxxx}\right] + \frac{1}{4!}\left(\frac{1}{2} - \theta\right)\Delta t u_{xxxxt}(\Delta x)^2 \\
+ O(\Delta t)^2 + O((\Delta x)^2)
\end{aligned}
$$

Note that when $\theta = \frac{1}{2}$, the truncation error is second order in both time and space. This is called the Czzrank-Nicolson scheme. Now the natural question is what is the stability of the this $\theta$-method. We have:

- $0 \leq \theta \leq \frac{1}{2}$: stable $\iff$ $\nu < \frac{1}{2}(1 - 2\theta)^{-1}$
- $\frac{1}{2} \leq \theta \leq 1$ : stable for all $\nu$

Thus the Crank-Nicolson scheme is unconditionally stable.