1 January 24th, 2020

1.1 Recitation 1

1.1.1 Homogeneous ODE

Recall that a homogeneous equation is

$$\frac{dy}{dx} = F(x, y), \quad \text{with } F(ax, ay) = a^n F(x, y).$$

What this typically means is that we won't have a constant.

Example 1.1

$F(x, y) = xy$ is homogeneous, as $F(ax, ay) = a^2xy$, while $F(x, y) = ax + 5$ is not homogeneous, as $F(ax, ay) = a^2xy + 5 \neq a^n F(x, y)$.

For 1st order homogeneous ODE, we have $n = 0$, with this we can introduce $z = \frac{y}{x}$ and convert this ODE into a separable differential equation.

1.1.2 Problem 1

Example 1.2

Let's consider

$$F(x, y) = \frac{dy}{dx} = \frac{2y^2 - x^2}{3xy}.$$

$$F(ax, ay) = \frac{2a^2y^2 - a^2x^2}{3a^2xy} = F(x, y),$$

meaning that it is a first order homogeneous equation.

With this, we have:

$$\begin{aligned}
\frac{d(zx)}{dx} &= \frac{2(zx)^2 - x^2}{3x(zx)} \\
\implies z + x\frac{dz}{dx} &= \frac{2x^2z^2 - x^2}{3x^2z} = \frac{2z^2 - 1}{3z} \\
\implies x\frac{dz}{dx} &= \frac{2z^2 - 1 - 3z^2}{3z} = -\frac{z^2 + 1}{3z}.
\end{aligned}$$

Now we can separate, giving us:

$$\frac{z}{z^2 + 1}dz = -\frac{1}{3x}dx \implies \int \frac{z}{z^2 + 1}dz = \int -\frac{1}{3x}dx$$

$$\implies \frac{1}{2}\ln(z^2 + 1) = -\frac{1}{3}\ln(x) + C_1$$

Solving for C_1, we get:

$$3\ln(z^2 + 1) = -2\ln(x) + 6C \implies C = 3\ln(z^2 + 1) + 2\ln(x) = 6C_1$$

$$\implies \ln\left(x^2(z^2 + 1)^3\right) = 6C_1 \implies x^2(z^2 + 1)^3 = e^{6C_1}.$$

Remembering that $z = \frac{y}{x}$, we have:

$$x^2\left(\frac{y^2}{x^2}+1\right)^3 = e^{6C_1} \implies \frac{(y^2+x^2)^3}{x^4} = e^{6C_1} \implies \frac{y^2+x^2}{x^{\frac{4}{3}}} = e^{2C_1} = C.$$

$$y = \pm x^{\frac{2}{3}}\sqrt{C - x^{\frac{3}{2}}}.$$

1.1.3 Bernoulli Equation

Definition 1.3. A **Bernoulli Equation** is an equation of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

If $n = 0$ or $n = 1$, we separate this equation. If $n \neq 0, 1$, defining $y = z^\lambda$, we have:

$$\frac{dy}{dx} = \frac{d(z^\lambda)}{dx} = \frac{dz}{d\lambda}\frac{dz}{dx} = \lambda z^{\lambda-1}\frac{dz}{dx}$$

Substituting this back, we have:

$$\lambda z^{\lambda-1}\frac{dz}{dx} + P(x)z^\lambda = Q(x)(z^\lambda)^n.$$

Dividing both sides by $\lambda z^{\lambda-1}$, we have:

$$\frac{dz}{dx} + \frac{1}{\lambda}P(x)z = \frac{1}{\lambda}Q(x)z^{\lambda n-\lambda+1}.$$

Setting λ such that $\lambda n - \lambda + 1 = 0$, i.e. $\lambda = \frac{1}{1-n}$, the equation becomes:

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x).$$

Which is a linear equation, which we can solve:

$$z(x) = \frac{1}{\mu_n}\left(\int \mu_n(1-n)Q(x)dx + C\right), \quad \mu_n = \exp\left\{(1-n)P(x)dx\right\}.$$

And substituting back into the original equation, we have:

$$y = z^\lambda = z^{\frac{1}{1-n}} = \left(\frac{1}{\mu_n}\left(\int \mu_n(1-n)Q(x)dx + C\right)\right)^{\frac{1}{1-n}}.$$

1.1.4 Problem 2

Consider

$$vx\frac{dv}{dx} + v^2 + xg = \frac{FL}{m}.$$

Rearranging the equation, we get:

$$\frac{dv}{dx} + \frac{v}{x} + \frac{g}{v} = \frac{FL}{xvm} \implies \frac{dv}{dx} + \left(\frac{1}{x}\right)v = \left(\frac{FL}{mx} - g\right)v^{-1}.$$

which is the form of a Bernoulli equation. As such, we can just plug into the formula, and we get:

$$\mu = \exp\{\int (1-(-1))\frac{1}{x}dx\} = e^{\int \frac{2}{x}dx} = x^{2\ln(x)} = x^2.$$

$$V(x) = \left(\frac{1}{\mu}\left(\int (1-(-1))\mu Q(x)dx + C\right)\right)\frac{1}{(1-(-1))}$$

$$= \left(\frac{1}{x^2}\left(\int 2x^2\left(\frac{FL}{mx} - g\right)dx + C\right)\right)^{\frac{1}{2}}$$

$$= \left(\frac{1}{x^2}\left(\frac{FLx^2}{m} - \frac{2}{3}gx^3\right) + C\right)^{\frac{1}{2}} = \left(\frac{FL}{m} - \frac{2}{3}gx + \frac{C}{x^2}\right)^{\frac{1}{2}}.$$

If we have an constraint where V is finite with $x = 0$, we need $C = 0$, as otherwise $x = 0$ will be infinite. Thus:

$$V = \sqrt{\frac{FL}{m} - \frac{2}{3}gx}.$$

1.1.5 Problem 3 Hints from Homework 1

In the first homework, we have:

$$\frac{dx}{dt} = K(\alpha - mx)^2(\beta - nx),$$

for some positive constants α, β, m, n. Here we want to determine:

$$\lim_{t\to\infty} x(t).$$

when $\frac{\alpha}{m} < \frac{\beta}{n}$, $\frac{\alpha}{m} = \frac{\beta}{n}$, $\frac{\alpha}{m} > \frac{\beta}{n}$.

If we plug into the equation, we have:

$$\frac{dx}{dt} = Km^2n\left(\frac{\alpha}{m} - x\right)^2\left(\frac{\beta}{n} - x\right).$$

Note that these are all positive except for the last factor. Thus, for the first case, we have:

1. For $x < \frac{\alpha}{m}$, $\frac{dx}{dt} > 0$
2. For $x = \frac{\alpha}{m}$, $\frac{dx}{dt} = 0$
3. For $x > \frac{\alpha}{m}$ and $x < \frac{\beta}{m}$, $\frac{dx}{dt} > 0$
4. For $x = \frac{\beta}{n}$, $\frac{dx}{dt} = 0$
5. For $x > \frac{\beta}{n}$, $\frac{dx}{dt} < 0$

From 1 and 2, we have: if $x_0 \leq \frac{\alpha}{m}$, $\lim_{t\to\infty} x = \frac{\alpha}{m}$, while from 3,4,5, we have: if $x_0 > \frac{\alpha}{m} \lim_{t\to\infty} x = \frac{\beta}{n}$.