ENM251 - Analytical Methods in Engineering

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These are the notes that I typed during the lectures/recitations. There's probably a lot of typo/mistakes since I haven't really gone through them after class, so keep an eye out for anything that doesn't make sense.

Contents

1 January 22nd, 2020

1.1 Separable Differential Equation

A general first-order ODE for a dependent variable y in the independent variable x can be written as:

$$\frac{dy}{dx} = F(x, y) \tag{1}$$

where F is some specified function of x and y. When F has the form

$$F(x, y) = f(x)g(y), \tag{2}$$

then 1 is said to be *separable* and such equation can always be solved by:

$$\frac{dy}{g(y)} f(x)dx \implies \int \frac{dy}{g(y)} + C_1 = \int f(x)dx + C_2 \implies \int \frac{dx}{g(y)} = \int f(x)dx + C.$$

as one form for the solution of 1.

1.1.1 Ideal Fluid Flow

We are concerned with a container that has a fluid with cross sectional area A with density ρ with a hole at the bottom of the container which causes it to flow out. We are concerned with the heigh x of the container. We also have a pipe that pumps in fluid with constant rate R.

This leads to following equation:

$$\frac{dx}{dt} = \alpha - \beta\sqrt{x}.$$

where

$$\alpha = \frac{R}{A} \quad \beta = \sqrt{\frac{2ga^2}{A^2 - a^2}} \quad g = 9.81\text{m s}^{-2}.$$

Note that this is a separable differential equation:

$$\frac{dx}{\alpha - \beta\sqrt{x}} = dt.$$

If we have α, β, we can solve, e.g. $\alpha = 60$ $\beta = 6$, we have:

$$\frac{dx}{dt} = 60 - 6\sqrt{x} \implies \frac{dx}{10 - \sqrt{x}} = 6dt.$$

Integrating on both sides, we have:

$$\int \frac{dx}{10 - \sqrt{x}} = \int 6dt = 6t + C.$$

Solving this, we get:

$$20\tan^{-1}\left(\frac{\sqrt{x}}{10}\right) - 10\ln(100 - x) - 2\sqrt{x} = 6t + C.$$

If we have initial conditions, e.g. at $t = 0, x = 0$, we would have:

$$0 - 10\ln(100) = C$$

allowing us to solve for C. This would allow us to solve for a time t for certain values of x.

1.2 Homogeneous Differential Equation

Again remember that the general form a differential equation of one a dependent variable y in the independent variable x is:

$$\frac{dy}{dx} = F(x, y).$$

If $F(x, y) = f(x)g(x)$ then this is separable. Remember that the goal is that we want to find $G(x, y) = C$, in other words, we want to get rid of the derivative and find the relationship between the two.

Definition 1.1. A function of form $F(x, y)$ is called **homogeneous** of order N if $F(tx, ty) = t^N F(x, y)$ for any scalar t.

Example 1.2

$$\begin{aligned} F(x, y) = x^3 + x^2y + 4xy^2 \implies F(tx, ty) &= (tx)^3 + (tx)^2(ty) + 4(tx)(ty)^2 \\ &= t^3\left(x^3 + x^2y + 4xy^2\right) = t^3F(x, y). \end{aligned}$$

Thus $F(x, y)$ is homogeneous to the order 3.

Example 1.3

$F(x, y) = x^3 + xy$ is not homogeneous.

Example 1.4

$$F(x, y) = \frac{xy}{x^2 + y^2}$$

$$F(tx, ty) = \frac{t^2xy}{t^2x^2 + t^2y^2} = t^2\left(\frac{xy}{x^2 + y^2}\right) = t^0F(x, y)$$

meaning that $F(x, y)$ is homogeneous to order 0.

Remark 1.5 — Typically if we say that a function is homogeneous but don't specify the order, it is assumed to be of order 0.

If a function in homogeneous to order 0, then it only depends on the ratio of $\frac{y}{x}$. In other words, rewrite $F(x, y) = f(\frac{y}{x})$.

Theorem 1.6

A function $F(x, y)$ is homogeneous of order 0 if and only if it can be expressed as $f(\frac{y}{x})$.

If we have a homogeneous function of order 0, we will be able to introduce a new variable $z = \frac{y}{x} \implies y = sz$, giving us:

$$\frac{d(xz)}{dx} = F(x, xz) = F(x(1), x(z)) = F(1, z).$$

Using the product rule, we have:

$$\frac{d(xz)}{dx} = \frac{dx}{dx}z + x\frac{dz}{dx} = F(1, z).$$

$$z + x\frac{dz}{dx} = F(1, z) \implies \frac{dz}{F(1, z) - z} = \frac{dx}{x},$$

which is a separable differential equation.

Remark 1.7 — The point is whenever you have a homogeneous equation, then introducing $z = \frac{y}{x}$ will allow us to convert it to a separable equation. Note that this only works for order 0 homogeneous equations.

1.2.1 Building an Radar Antenna

TL;DR the equation is:

$$\left(\frac{dy}{dx}\right)^2 - 2\frac{y - F}{x}\left(\frac{dy}{dx}\right) - 1 = 0.$$

If we use the quadratic formula, we get:

$$\frac{dy}{dx} = \frac{y - F}{x} + \sqrt{\left(\frac{y - F}{x}\right)^2 + 1}.$$

If we do the substitution, $z = \frac{y-F}{x}$, we get:

$$\frac{d(xz + F)}{dx} = z + \sqrt{z^2 + 1} \implies x\frac{dz}{dx} + z = z + \sqrt{z^2 + 1} \implies \frac{dz}{\sqrt{z^2 + 1}} = \frac{dx}{x}.$$

$$\int \frac{dz}{\sqrt{z^2 + 1}} = \ln x + C \implies \ln\left(z + \sqrt{z^2 + 1}\right) = \ln x + C.$$

$$\implies A^2x^2 - 2Axz = 1 \implies \frac{1}{2}Ax^2 + \left(F - \frac{1}{2A}\right),$$

which is the equation of a parabola. Thus the optimal shape of a radar dish is a parabola.

2 January 24th, 2020

2.1 Recitation 1

2.1.1 Homogeneous ODE

Recall that a homogeneous equation is

$$\frac{dy}{dx} = F(x, y), \quad \text{with } F(ax, ay) = a^n F(x, y).$$

What this typically means is that we won't have a constant.

Example 2.1

$F(x,y) = xy$ is homogeneous, as $F(ax,ay) = a^2xy$, while $F(x,y) = ax + 5$ is not homogeneous, as $F(ax,ay) = a^2xy + 5 \neq a^n F(x,y)$.

For 1st order homogeneous ODE, we have $n = 0$, with this we can introduce $z = \frac{y}{x}$ and convert this ODE into a separable differential equation.

2.1.2 Problem 1

Example 2.2

Let's consider

$$F(x,y) = \frac{dy}{dx} = \frac{2y^2 - x^2}{3xy}.$$

$$F(ax,ay) = \frac{2a^2y^2 - a^2x^2}{3a^2xy} = F(x,y),$$

meaning that it is a first order homogeneous equation.

With this, we have:

$$\begin{aligned} \frac{d(zx)}{dx} &= \frac{2(zx)^2 - x^2}{3x(zx)} \\ \implies z + x\frac{dz}{dx} &= \frac{2x^2z^2 - x^2}{3x^2z} = \frac{2z^2 - 1}{3z} \\ \implies x\frac{dz}{dx} &= \frac{2z^2 - 1 - 3z^2}{3z} = -\frac{z^2 + 1}{3z}. \end{aligned}$$

Now we can separate, giving us:

$$\frac{z}{z^2+1}dz = -\frac{1}{3x}dx \implies \int \frac{z}{z^2+1}dz = \int -\frac{1}{3x}dx$$

$$\implies \frac{1}{2}\ln(z^2+1) = -\frac{1}{3}\ln(x) + C_1$$

Solving for C_1, we get:

$$3\ln(z^2+1) = -2\ln(x) + 6C \implies C = 3\ln(z^2+1) + 2\ln(x) = 6C_1$$

$$\implies \ln\left(x^2(z^2+1)^3\right) = 6C_1 \implies x^2(z^2+1)^3 = e^{6C_1}.$$

Remembering that $z = \frac{y}{x}$, we have:

$$x^2\left(\frac{y^2}{x^2} + 1\right)^3 = e^{6C_1} \implies \frac{(y^2+x^2)^3}{x^4} = e^{6C_1} \implies \frac{y^2+x^2}{x^{\frac{4}{3}}} = e^{2C_1} = C.$$

$$y = \pm x^{\frac{2}{3}}\sqrt{C - x^{\frac{3}{2}}}.$$

2.1.3 Bernoulli Equation

Definition 2.3. A Bernoulli Equation is an equation of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

If $n = 0$ or $n = 1$, we separate this equation. If $n \neq 0, 1$, defining $y = z^\lambda$, we have:

$$\frac{dy}{dx} = \frac{d(z^\lambda)}{dx} = \frac{dz}{d\lambda}\frac{dz}{dx} = \lambda z^{\lambda-1}\frac{dz}{dx}$$

Substituting this back, we have:

$$\lambda z^{\lambda-1}\frac{dz}{dx} + P(x)z^\lambda = Q(x)(z^\lambda)^n.$$

Dividing both sides by $\lambda z^{\lambda-1}$, we have:

$$\frac{dz}{dx} + \frac{1}{\lambda}P(x)z = \frac{1}{\lambda}Q(x)z^{\lambda n-\lambda+1}.$$

Setting λ such that $\lambda n - \lambda + 1 = 0$, i.e. $\lambda = \frac{1}{1-n}$, the equation becomes:

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x).$$

Which is a linear equation, which we can solve:

$$z(x) = \frac{1}{\mu_n}\left(\int \mu_n(1-n)Q(x)dx + C\right), \quad \mu_n = \exp\left\{(1-n)P(x)dx\right\}.$$

And substituting back into the original equation, we have:

$$y = z^\lambda = z^{\frac{1}{1-n}} = \left(\frac{1}{\mu_n}\left(\int \mu_n(1-n)Q(x)dx + C\right)\right)^{\frac{1}{1-n}}.$$

2.1.4 Problem 2

Consider

$$vx\frac{dv}{dx} + v^2 + xg = \frac{FL}{m}.$$

Rearranging the equation, we get:

$$\frac{dv}{dx} + \frac{v}{x} + \frac{g}{v} = \frac{FL}{xvm} \implies \frac{dv}{dx} + \left(\frac{1}{x}\right)v = \left(\frac{FL}{mx} - g\right)v^{-1}.$$

which is the form of a Bernoulli equation. As such, we can just plug into the formula, and we get:

$$\mu = \exp\{\int (1-(-1))\frac{1}{x}dx\} = e^{\int \frac{2}{x}dx} = x^{2\ln(x)} = x^2.$$

$$V(x) = \left(\frac{1}{\mu}\left(\int (1-(-1))\mu Q(x)dx + C\right)\right)\frac{1}{(1-(-1))}$$

$$= \left(\frac{1}{x^2}\left(\int 2x^2\left(\frac{FL}{mx} - g\right) dx + C\right)\right)^{\frac{1}{2}}$$

$$= \left(\frac{1}{x^2}\left(\frac{FLx^2}{m} - \frac{2}{3}gx^3\right) + C\right)^{\frac{1}{2}} = \left(\frac{FL}{m} - \frac{2}{3}gx + \frac{C}{x^2}\right)^{\frac{1}{2}}.$$

If we have an constraint where V is finite with $x = 0$, we need $C = 0$, as otherwise $x = 0$ will be infinite. Thus:

$$V = \sqrt{\frac{FL}{m} - \frac{2}{3}gx}.$$

2.1.5 Problem 3 Hints from Homework 1

In the first homework, we have:

$$\frac{dx}{dt} = K(\alpha - mx)^2(\beta - nx),$$

for some positive constants α, β, m, n. Here we want to determine:

$$\lim_{t\to\infty} x(t).$$

when $\frac{\alpha}{m} < \frac{\beta}{n}$, $\frac{\alpha}{m} = \frac{\beta}{n}$, $\frac{\alpha}{m} > \frac{\beta}{n}$.

If we plug into the equation, we have:

$$\frac{dx}{dt} = Km^2n\left(\frac{\alpha}{m} - x\right)^2\left(\frac{\beta}{n} - x\right).$$

Note that these are all positive except for the last factor. Thus, for the first case, we have:

1. For $x < \frac{\alpha}{m}$, $\frac{dx}{dt} > 0$
2. For $x = \frac{\alpha}{m}$, $\frac{dx}{dt} = 0$
3. For $x > \frac{\alpha}{m}$ and $x < \frac{\beta}{m}$, $\frac{dx}{dt} > 0$
4. For $x = \frac{\beta}{n}$, $\frac{dx}{dt} = 0$
5. For $x > \frac{\beta}{n}$, $\frac{dx}{dt} < 0$

From 1 and 2, we have: if $x_0 \leq \frac{\alpha}{m}$, $\lim_{t\to\infty} x = \frac{\alpha}{m}$, while from 3,4,5, we have: if $x_0 > \frac{\alpha}{m} \lim_{t\to\infty} x = \frac{\beta}{n}$.

3 January 27th, 2020

3.1 Linear ODE

Definition 3.1. The basic form of first-order linear equation is:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x),$$

where $a_1(x) \neq 0$. The goal is given $a_1(x), a_0(x)$ and $b(x)$, solve for $y(x)$.

Example 3.2

$$x^2y'(x) + 2y(x) = x$$

is a first order linear ODE, where $a_1(x) = x^2,\ a_0(x) = 2,\ b(x) = x$.

To solve it, we first divide by $a_1(x)$, giving us:

$$\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = \frac{b(x)}{a_1(x)}.$$

which is of the standard form:

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Example 3.3

From the previous example, we'd have:

$$y'(x) + \frac{2}{x^2}y(x) = \frac{1}{x},$$

where $P(x) = \frac{2}{x^2}$ and $Q(x) = \frac{1}{x}$.

To solve this, we then multiply by $e^{\int P(x)dx}$, giving us:

$$e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx} = Q(x)e^{\int P(x)dx}.$$

Note that the second term is $\frac{d}{dx}\left(e^{\int P(x)dx}\right)$, thus by the product rule, this becomes:

$$\frac{d}{dx}\left(e^{\int P(x)dx}\right) = Q(x)e^{\int P(x)dx}.$$

If we call $\mu(x) = e^{\int P(x)dx}$ the **integrating factor** for the ODE, we can express this as:

$$\frac{d(\mu y)}{dx} = \mu Q \implies \mu y = \int \mu Q dx + C \implies y = \frac{1}{\mu}\left(\int \mu Q dx + C\right).$$

3.1.1 Steps for Solving $a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$

1. Change to standard form: $P(x) = \frac{a_0(x)}{a_1(x)}$, $Q(x) = \frac{b(x)}{a_1(x)}$.
2. Compute the integrating factor: $\mu(x) = e^{\int P(x)dx}$.
3. Plug into formula: $y(x) = \frac{1}{\mu(x)}\left(\int \mu(x)Q(x)dx + C\right)$.

Example 3.4

Returning to the previous example, considering $x^2y'(x) + 2y(x) = x$, we have:

- $P(x) = \frac{a_0(x)}{a_1(x)} = \frac{2}{x^2}$
- $Q(x) = \frac{b(x)}{a_1(x)} = \frac{1}{x}$

We now calculate the integral factor:

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{2}{x^2}dx} = e^{-\frac{2}{x}}.$$

Plugging into the formula, we get:

$$y(x) = \frac{1}{e^{-\frac{2}{x}}}\left(\int e^{-\frac{2}{x}}\frac{1}{x}\,dx + C_1\right).$$

Example 3.5

Now consider $x^2y'(x) + 2y(x) = 1$, following the same steps, we get:

$$y(x) = \frac{1}{e^{-\frac{2}{x}}}\left(\int e^{-\frac{2}{x}}\frac{1}{x^2}\,dx + C_1\right) = \frac{1}{e^{-\frac{2}{x}}}\left(\frac{1}{2}e^{-\frac{2}{x}} + C_1\right).$$

Example 3.6

$$\frac{dT}{dt} = -h(T - T_R) \implies \frac{dT}{dt} + hT = hT_R,$$

which can solved with the linear method. $P(t) = h$, $Q(t) = hT_R$, giving us:

$$\mu(t) = e^{\int h dt} = e^{ht} \implies T(t) = \frac{1}{e^{ht}}\left(\int e^{ht}hT_R\,dt + C_1\right)$$

$$T(t) = e^{-ht}\left(T_Re^{ht} + C_1\right) = T_R + C_1e^{-ht}.$$

Remark 3.7 — How to determine which method to use. Bring everything to one side:

$$\frac{dy}{dx} = F(x, y).$$

- If $F(x, y) = f(x)g(y)$, we can use the separable method.
- If $F(tx, ty) = F(x, y)$, we can use the homogeneous method.
- If $F(x, y) = -P(x)y + Q(x)$, then we can use the linear method.
- If $F(x, y) = -P(x)y + Q(x)y^m$, we can use the Bernoulli method.

3.1.2 Bernoulli Equation

Definition 3.8. A Bernoulli Equation is an equation of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^m,$$

for some number m.

Example 3.9

Giving initial condition $v(0) = 0$, solve v where:

$$\frac{dv}{dx} + \frac{1}{x}v = gv^{-1},$$

which is of the form of a Bernoulli Equation.

To solve the Bernoulli equation, we set $y = z^\lambda$ and choose λ so that the ODE for z is easier to solve than the ODE for y. This is because we'd get:

$$\frac{dy}{dx} + P(x)y = Q(x)y^m$$

$$\implies \frac{dz^\lambda}{dx} + P(x)z^\lambda = Q(x)(z^\lambda)^m$$

$$\implies \lambda z^{\lambda-1}\frac{dz}{dx} + P(x)z^\lambda = Q(x)z^{m\lambda}.$$

Dividing by λz^λ:

$$\implies \frac{dz}{dx} + \frac{1}{\lambda}P(x)z = \frac{1}{\lambda}Q(x)z^{m\lambda+1-\lambda}.$$

Thus we want to choose λ so that $m\lambda + 1 - \lambda = 0 \implies \lambda = \frac{1}{1-m}$ where $m \neq 1$.

If $m = 1$, then it is a separable equation, meaning that we have:

$$\frac{dy}{dx} = (Q(x) - P(x))\,y.$$

$$\frac{dy}{y} = (Q(x) - P(x))\,dx \implies y(x) = Ae^{\int (Q(x)-P(x))\ dx}.$$

3.1.3 Summary for Solving Bernoulli Equation

Consider

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)y^m.$$

1. First change to standard form with: $P(x) = \frac{a_0(x)}{a_1(x)}$, $Q(x) = \frac{b(x)}{a_1(1)}$
2. If $m = 1$, then, for some constant A, we have:

$$y(x) = Ae^{\int (Q(x)-P(x))dx}.$$

3. Otherwise, compute the integrating factor:

$$\mu(x) = e^{\int (1-m)p(x)dx}.$$

4. Giving us the equation:

$$y(x) = \left(\frac{1}{\mu(x)}\left(\int (1-m)\mu(x)Q(x)\ dx\right) + C\right)^{\frac{1}{1-m}}.$$

Remark 3.10 — Note that the linear case is when $m = 0$, which gives us the equation what we have before.

Example 3.11

Returning to our example earlier where we were considering $\frac{dv}{dx} = \frac{1}{x}v = gv^{-1}$, we have $P(x) = \frac{1}{x}$, $Q(x) = g$. Thus the integrating factor is:

$$\mu(x) = e^{\int (1-(-1))\frac{1}{x}\ dx} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2.$$

Thus we have:

$$\begin{aligned} v(x) &= \left(\frac{1}{x^2}\left(\int (1-(-1))x^2 g\ dx + C_1\right)\right)^{\frac{1}{1-(-1)}} \\ &= \left(\frac{1}{x^2}\left(\frac{2}{3}gx^3 + C_1\right)\right)^{\frac{1}{2}} \\ &= \sqrt{\frac{2gx}{3} + \frac{C_1}{x^2}}. \end{aligned}$$

Since $v(x) = 0 \implies C_1 = 0$, thus:

$$v(x) = \sqrt{\frac{2gx}{3}}.$$

4 January 29th, 2020

4.1 Phase Plot

Let us consider ODE's of the form:

$$\frac{dx}{dt} = f(x) = \dot{x}.$$

If we graph x vs $\dot{x}$ we can get a phase plot, for example:

Definition 4.1. A point where $f(x) = 0$ is called an **equilibrium point**. These equilibrium points can be unstable (empty circle), stable (filled circle), or left/right stable (half filled circle).

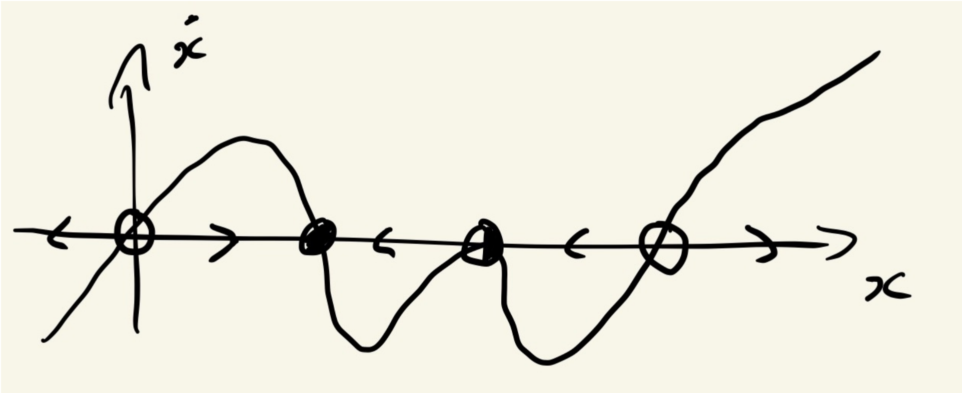

Figure 1: Phase plot of $\dot{x} = x(x-1)(x-2)^2(x-3)^3$

4.2 Computing Times

Since $\dot{x} = f(x)$, is separable, since $dt = \frac{dx}{f(x)}$, we have:

$$\int_{t_1}^{t_2} dt = \int_{x_1}^{x_2} \frac{dx}{f(x)} \implies t_2 - t_1 = \int_{x_1}^{x_2} \frac{dx}{f(x)}.$$

Which is the time interval between when $x = x_1$ and $x = x_2$.

Example 4.2

Let us try to compute the period of an object with mass m to travel from one end of a bowl to the other with radius R. TL;DR we get:

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{R}\cos(\theta)}.$$

Rearranging gives us:

$$dt = \sqrt{\frac{R}{2g\cos\theta}} d\theta \implies \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\cos(\theta)}} \approx \sqrt{\frac{R}{2g}} 5.244.$$

4.3 Exact Equations

Whenever you have a function of form $\frac{dy}{dx} = F(x,y)$, you can always rewrite it in the form:

$$M(x,y)dx + N(x,y)dy = 0.$$

This might look familiar, as if we have $f(x,y) = C$, we have:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0.$$

As such, we'd like to ask when can $M(x,y)dx + N(x,y)dy = 0$ be written as $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$. It would be great if $M = \frac{\partial f}{\partial x}$ and $N = \frac{\partial f}{\partial y}$, so it's helpful to know when we can do this.

Consider

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

As such, if
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$
then $Mdx + ndy = 0$ is called exact.

Example 4.3
$2xydx + (x^2 - y^2)dy = 0$ is exact.

Example 4.4
$2x^2ydx + (x^3 - y^2)dy = 0$ is not exact.

Note that the two examples differ by a factor x, meaning that we have a further condition to determine whether something is exact.

5 January 31st, 2020

5.1 Problem 1

Find period of motion for the equation:
$$\dot{\theta} = \sqrt{\frac{g}{L}(3 + 2\cos\theta)} \quad 0 \leq \theta \leq 2\pi.$$
Since the RHS only has θ, this is separable, thus:
$$\int dt = \sqrt{\frac{L}{g}} \int \frac{d\theta}{\sqrt{3 + 2\cos(\theta)}}$$
Note that the RHS gives us an elliptical equation. Since we want the period, we have:
$$T = \sqrt{\frac{L}{g}} \int_0^{2\pi} \frac{d\theta}{3 + 2\cos\theta} + C.$$
We can consider C to be the start time, and just set it to 0. This is as far as you can go analytically, so plug it into a calculator.

5.1.1 How to use in MATLAB

```
T = integral(@(theta)1./sqrCos(1,theta),2,2*pi)
tspan = [0 2.5];
y0 = 0;
data = ode45(@sqrCos,tspan,y0);

function res = sqrCos(t,theta)
  L = 2,4;
  g = 9,8;
  res = sqrt(g/L*(3+2*cos(theta)));
end(function)
```

5.2 Problem 3

Consider the equation

$$v\frac{dv}{dx} + \frac{v^2}{x + \frac{m}{\rho}} = g.$$

With the initial condition: $v_0 = v(x_0) = v(0) = 0$. To solve for $v(x)$, note that this is a Bernoulli equation:

$$\frac{dv}{dx} + \frac{1}{x + \frac{m}{\rho}}v = g^{v-1}.$$

with:

$$p(x) = \frac{1}{x + \frac{m}{\rho}} \quad Q(x) = g \quad n = -1.$$

Plugging into the formula, we have:

$$V(x) = \left(\frac{1}{\mu(x)}\left(\int (1-n)\mu(x)Q(x)dx + C\right)\right)^{\frac{1}{1-n}}.$$

Calculating the integrating factor, we have:

$$\mu(x) = e^{\int (1-n)P(x)dx} = e^{2\ln(x+\frac{m}{\rho})} = \left(x + \frac{m}{\rho}\right)^2.$$

Thus we have:

$$V(x) = \left(\frac{1}{\left(x+\frac{m}{\rho}\right)^2}\left(2\int \left(x + \frac{m}{\rho}\right)^2 g\ dx + C\right)\right)^{\frac{1}{2}}$$

$$V(x) = \left(\frac{1}{\left(x+\frac{m}{\rho}\right)^2}\left(\frac{2}{3}\left(x + \frac{m}{\rho}\right)^3 + C\right)\right)^{\frac{1}{2}} = \frac{1}{x + \frac{m}{\rho}}\sqrt{\frac{2}{3}\left(x + \frac{m}{\rho}\right)^3 g + C}.$$

Plugging in the initial condition, we get: $C = -\frac{2}{3}\frac{m^3}{\rho^3}g$, giving us:

$$v(x) = \frac{1}{x + \frac{m}{\rho}}\sqrt{\frac{2}{3}\left(x + \frac{m}{\rho}\right)^3 g - \frac{2}{3}\left(\frac{m}{\rho}\right)^3 g}.$$

The acceleration is:

$$g - \frac{v^2}{x + \frac{m}{\rho}}.$$

6 February 3rd, 2020

6.1 Exact Equations

Remember that an exact equation is one where:

$$Mdx + Ndy = 0.$$

Where:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Consider the exact equation:

$$(y^2 - x^2)dx + 2xydy = 0.$$

To solve this exact ODE, we set:

$$\frac{\partial f}{\partial x} = M = y^2 - x^2 \implies \int_x (y^2 - x^2)dx + c_1(y) \implies f(x,y) = y^2 x - \frac{x^3}{3} + c_1(y).$$

Now if we take the partial with respect to y, we get:

$$\frac{\partial f}{\partial y} = 2yx + c_1'(y) = N = 2xy \implies c_1'(y) = 0 \implies c_1(y) = c_2.$$

This tells:

$$f(x,y) = y^2 x - \frac{1}{3}x^3 + c_2$$

satisfies both equations meaning that the solution to our ODE is of the form:

$$f(x,y) = xy^2 - \frac{1}{3}x^3 = C.$$

If we have an initial condition, then this will give us a unique solution.

Example 6.1

Consider the equation: $2xy^2dx + (2x^2y - y^3)dy = 0$. To solve this, we do the following:

$$\int_x 2xy^2 \, dx = x^2y^2 + c_1(y) \implies 2x^2y + c_1'(y) = 2x^2y - y^3 \implies c_1 = -\frac{y^4}{4}$$

Thus we have:

$$f(x,y) = 2x^2y^2 - \frac{1}{4}y^4 + C.$$

6.2 Inexact Equations

If $Mdx + Ndy = 0$ is not exact, then we try to introduce an integrating factor $\mu(x,y)$ to turn make $\mu N dx + \mu N dy = 0$. Thus we want:

$$\frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x}.$$

However this is usually as difficult to solve as the original equation. There are some special cases though:

- $\mu(x,y) = \mu(x)$. If this is the case, we have:

$$\frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x} \implies \mu\frac{\partial M}{\partial y} = \frac{d\mu}{dx}N + \mu\frac{\partial N}{\partial x}$$

$$\implies \mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \mu'(x)N \implies \frac{\mu'(x)}{\mu(x)} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

and if the RHS is a function of only x, we can integrate, giving us:

$$\mu(x) = \exp\left\{\int \frac{\left(\frac{\partial m}{\partial y} - \frac{\partial N}{\partial x}\right)}{N} dx\right\}.$$

With this, we will be able to solve the differential equation with $\frac{\partial f}{\partial x} = \mu M$ and $\frac{\partial f}{\partial y} = \mu N$. This is true if:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = k(x).$$

i.e. it's a function of only x

- $\mu(x, y) = \mu(y)$. Same thing but with y instead of x. We check if: $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{y}$ is a function of only y. We will have:

$$\mu(y) = \exp\left\{\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{m}\right\}.$$

Example 6.2

Consider the equation $2xydx + (2x^2 - y^2)dy = 0$. Note that this is not exact. As such, we check:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2x - 4x}{2x^2 - y^2} = \frac{2x}{2x^2 - y^2} \neq \text{ a function of only } x\ .$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{4x - 2x}{2xy} = \frac{1}{y}.$$

Thus we have:

$$\mu(y) = e^{\int \frac{1}{y} dy} = e^{\ln y} = y.$$

Example 6.3

Consider $\frac{dy}{dx} = \frac{2x^2-y^2}{3xy}$, rearranging gives us:

$$(x^2 - 2y^2)dx + 3xydy = 0.$$

Note that $\frac{\partial M}{\partial y} = -4y$ and $\frac{\partial N}{\partial x} = 3y$, thus it is not exact. Now we try:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-4y - 3y}{3xy} = \frac{-7}{3x}.$$

Which is a function of only x. As such, we have:

$$\mu(x) = e^{\int -\frac{7}{3x}dx} = x^{-\frac{7}{3}}.$$

Multiplying this in gives us:

$$(x^{-\frac{1}{3}} - 2x^{-\frac{7}{3}}y^2)dx + 3x^{-\frac{4}{3}}ydy = 0,$$

which is exact since:

$$\frac{\partial M}{\partial y} = -4x^{-\frac{7}{3}}y \quad \frac{\partial N}{\partial x} = -4x^{-\frac{7}{3}}y.$$

Solving this gives us:

$$f(x,y) = \int_x x^{-\frac{1}{3}} - 2x^{-\frac{7}{3}}y^2 dx = \frac{3}{2}x^{\frac{2}{3}} + \frac{3}{2}x^{-\frac{4}{3}}y + c_1(y).$$

$$\frac{3}{2}x^{-\frac{4}{3}}y + c_1'(y) = \frac{3}{2}x^{-\frac{4}{3}}y \implies c_1 = C.$$

Thus

$$f(x,y) = \frac{3}{2}x^{\frac{2}{3}} + \frac{3}{2}x^{-\frac{4}{3}}y^2 = C.$$

7 February 5th, 2020

7.1 Applications

Given the family of curves $u(x,y) = c_1$, the family of curves orthogonal to these are the solution to:

$$\frac{\partial u}{\partial x}dy = \frac{\partial u}{\partial y}dx.$$

7.1.1 2nd-Order ODE

Definition 7.1. The general form of a 2nd order differential equation is:

$$y'' = F(x, y, y').$$

Where x is the independent variable and y is the dependent variable.

We want to consider a few special cases. The first one is when the dependent variable is missing, $y'' = f(x, y')$, for example $y'' = x - y'$. In this case, you can set $v = y'$ $v' = y''$, giving us:

$$v' = f(x, v)$$

which is a first order equation. Thus we can solve the first order ODE and then integrate to get y.

Example 7.2

Consider the earlier equation $y'' = x - y'$, we have:

$$v' = x - v \implies \frac{dv}{dx} + v = x$$

$$v = e^{-x}((x-1)e^x + c_1) = x - 1 + c_1 e^{-x} = \frac{dy}{dx}.$$

$$y = \frac{1}{2}x^2 + x + c_2 e^{-x} + c_3.$$

for some constants c_2 and c_3.

Remark 7.3 — Note that for a first order ODE, there should be one arbitrary constant, but for second order, there should be 2.

The second case is where the independent variable is missing, meaning:

$$\frac{d^2y}{dx^2} = F(y, \frac{dy}{dx}) \implies \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = v\frac{dv}{dx} = F(y, v).$$

Where v is once again $\frac{dy}{dx}$. Using this, we can solve for v in terms of y and then integrate twice.

8 February 10th, 2020

8.1 Review from MATH 240

Consider the homogenous equation:

$$a_2(x)y''(x) + a_1 y'(x) + a_0(x)y(x) = 0.$$

Note that this is homogeneous since $y(x) = 0$ is a valid solution. A general solution to a 2nd order lienar homogeneous ODE can be expressed as

$$y(x) = c_1 y_1(x) + c_2 y_2(x),$$

where c_1 and c_2 are arbitrary constrants and $y_1(x)$ and $y_2(x)$ are any two solutions to the ODE for which:

$$\underbrace{y_1(x)y_2'(x) - y_1'(x)y_2(x)}_{\text{Wronskian of } y_1 \text{ and } y_2} \neq 0.$$

Note that the LHS can be experssed as a determinant:

$$\det\left(\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix}\right).$$

Which is known as the **Wronskian** of y_1 and y_2.

Example 8.1

Consider $y''(x) - 3y'(x) + 3y(x) = 0$, we have:

$$y_1(x) = e^x \quad y_2(x) = e^{2x}.$$

and

$$\det\left(\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\right) = \det\left(\begin{bmatrix} e^x & e^{2x} \\ e^x & e^{2x} \end{bmatrix}\right) = 2e^{3x} - e^{3x} = e^{3x} \neq 0.$$

Meaning that the solution is of the form:

$$y(x) = c_1 e^x + c_2 e^{3x}.$$

Remark 8.2 — Note that we only need the Wronskian to not be the 0 function, and that it's ok for certain values of x for the Wronkian to be 0.

Example 8.3

If we used $y_1(x) = e^x$ and $y_2(x) = 2e^x$, then we'd get a Wronskian equal to 0, which would not work.

8.2 Constant Coefficients

Consider:

$$ay''(x) + by'(x) + cy(x) = 0,$$

where a, b and c are constants.

Example 8.4

Example 8.1 is an example of a constant equation with $a = 1$, $b = -3$, and $c = 2$.

Let us create a table to help us solve this problem. First we contstruct the descriminant: $D = b^2 - 4ac$. Depending on what value D is, we have:

Table 1: Table to Compute $ay'' + by' + cy = 0$

D	$y_1(x)$	$y_2(x)$	
$D < 0$	$e^{\alpha x}\cos(\beta x)$	$e^{\alpha x}\sin(\beta x)$	$\alpha = -\frac{b}{2a}$ $\beta = \sqrt{-D}/2a$
$D = 0$	$e^{\alpha x}$	$xe^{\alpha x}$	$\alpha = -\frac{b}{2a}$
$D > 0$	$e^{\alpha x}\cosh(\gamma x)$	$e^{\alpha x}\sinh(\gamma x)$	$\alpha = -\frac{b}{2a}$ $\gamma = \sqrt{D}/2a$
	$e^{(\alpha-\gamma)x}$	$e^{(\alpha+\gamma)x}$	$\alpha = -\frac{b}{2a}$ $\gamma = \sqrt{D}/2a$

Example 8.5

Consider $4y'' + y' + y = 0$. The discrminant is $D = b^2 - 4ac = -15 < 0$. Using the first row of the previous table, we have:

$$\alpha = -\frac{1}{8} \quad \beta = \frac{\sqrt{15}}{8}.$$

This means that:

$$y(x) = c_1 e^{-\frac{1}{8}x}\cos\left(\frac{x\sqrt{15}}{8}\right) + c_2 e^{-\frac{1}{8}x}\sin\left(\frac{x\sqrt{15}}{8}\right).$$

Example 8.6

Consider $4y'' + 4y' + y = 0$. Note that $D = b^2 - 4ac = 16 - 16 = 0$, thus using the second row, we have:

$$\alpha = -\frac{b}{2a} = -\frac{4}{8} = -\frac{1}{2}.$$

Thus we have:

$$y(x) = c_1 e^{-\frac{1}{2}x} + c_2 x e^{-\frac{1}{2}x}.$$

Example 8.7

Consider $y'' - 3y' + 2y = 0$, note that $D > 0$. We can either use the third or 4th row of the table. First we have:

$$\alpha = \frac{3}{2} \quad \gamma = \frac{1}{2}.$$

Thus we can write this as either:

$$y(x) = c_1 e^{\frac{3}{2}x}\cosh(\frac{1}{2}x) + c_2 e^{\frac{3}{2}x}\sinh(\frac{1}{2}x)$$

or

$$y(x) = c_1 e^{(\frac{3}{2}-\frac{1}{2})x} + c_2 e^{(\frac{3}{2}+\frac{1}{2})x} = c_1 e^{x} + c_2 e^{2x}.$$

8.3 Cauchy-Euler/Equidimentional Equation

Definition 8.8. A equidimensional or Cauchy-Euler 2nd order linear homogenous ODE is a equation of the form:

$$ax^2 y''(x) + bxy'(x) + cy(x) = 0.$$

for some constant a, b, c.

Remark 8.9 — Note that the exponent of the x matches the derivative of y.

Again, we can just use a table to solve these equations by checking the value of

$$D = (b-a)^2 - 4ac.$$

Table 2: Table to solve Cauchy-Euler Equations

D	$y_1(x)$	$y_2(x)$	
$D < 0$	$\|x\|^\alpha \cos(\beta \ln \|x\|)$	$\|x\|^\alpha \sin(\beta \ln \|x\|)$	$\alpha = -\frac{b-a}{2a}$ $\beta = \sqrt{-D}/2a$
$D = 0$	$\|x\|^\alpha$	$\|x\|^\alpha \ln \|x\|$	$\alpha = -\frac{b-a}{2a}$
$D > 0$	$\|x\|^\alpha \cosh(\gamma \ln \|x\|)$	$\|x\|^\alpha \sinh(\gamma \ln \|x\|)$	$\alpha = -\frac{b-a}{2a}$ $\gamma = \sqrt{D}/2a$
	$\|x\|^{\alpha-\gamma}$	$\|x\|^{\alpha+\gamma}$	$\alpha = -\frac{b-a}{2a}$ $\gamma = \sqrt{D}/2a$

Example 8.10

Consider $3x^2y'' + 2xy' + 5y = 0$, where $a = 3, b = 2, c = 5$. Note that:

$$d = (b-a)^2 - 4ac = (2-3)^2 - 4(3)(5) = -59 < 0.$$

Thus we have:

$$\alpha = -\frac{b-a}{2a} = \frac{1}{6}, \quad \beta = \frac{\sqrt{59}}{6}.$$

Thus we have:

$$y(x) = c_1 x^{\frac{1}{6}} \cos(\frac{\sqrt{59}}{6} \ln x) + c_2 x^{\frac{1}{6}} \sin(\frac{\sqrt{59}}{6} \ln x).$$

for $x > 0$.

Example 8.11

Consider $x^2y'' + 2xy' - 2y = 0$, $x > 0$, i.e. $a = 1, b_= 2, c = -2$. Note that $D = (b-a)^2 - 4ac = 9 > 0$, thus we have:

$$\alpha = \frac{-(2-1)}{2(1)} = -\frac{1}{2} \quad \gamma = \frac{\sqrt{9}}{2(1)} = \frac{3}{2}.$$

With the general solution being:

$$y(x) = c_1 x^{-\frac{1}{2}} \cosh(\frac{3}{2} \ln x) + c_2 x^{-\frac{1}{2}} \sinh(\frac{3}{2} \ln x).$$

or

$$y(x) = c_1 x^{-2} + c_2 x.$$

8.4 Other Stuff from Math 240

If we once again consider the equation $a_2(x)y''(x) + a_1(x)y'(x) + a_0y(x) = 0$. Note that as we mentioned earlier, we need two linearly independent non-zero solutions to get the general solution. If we only have one, say $y_1(x)$, a second linearly independent solution $y_2(x)$ can be constructed using **Abel's equation** :

$$y_2(x) = Ay_1(x) \int \frac{e^{-\int \frac{a_1(x)}{a_2(x)}\,dx}}{y_1^2(x)} dx.$$

For any non-zero constant A.

Remark 8.12 — Derivation is in the notes.

Example 8.13

Consider $xy'' + (1-x)y' - y = 0$. Suppose we're told that one solution is $y_1(x) = e^x$. A second solution would be:

$$y_2(x) = Ae^x \int \frac{e^{-\int \frac{1-x}{x}\,dx}}{(e^x)^2}\, dx.$$

$$= Ae^x \int \frac{e^{\int 1-\frac{1}{x}\,dx}}{e^{2x}}\, dx = Ae^x \int \frac{e^{x-\ln x}}{e^{2x}}\, dx = Ae^x \int \frac{e^{-x}}{x}\, dx.$$

Which doesn't have a nice answer (oops)

Remark 8.14 — Note that whenever $a_2(x) + a_1(x) + a_0(x) = 0$, one solution is always $y_1(x) = e^x$, since we'd have $y'' = y' = y = e^x$.

Example 8.15

Consider $(1-x)y'' + xy' - y = 0$. Since we have $y_1(x) = e^x$, we have:

$$y_2(x) = Ae^x \int \frac{e^{-\int \frac{x}{1-x}\,dx}}{(e^x)^2}\, dx.$$

$$y_2(x) = Ae^x \int \frac{e^{x+\ln(x-1)}}{e^{2x}}\, dx = Ae^x \int \frac{x-1}{e^x}\, dx = Ae^x(-xe^{-x}) = -Ax.$$

Picking $A = -1$, we have: $y_2(x) = x$, thus the general solution would be:

$$y(x) = c_1e^x + c_2x.$$

8.5 Non-Homogeneous Equations

Now let's consider non-homogeneous equations:

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0y(x) = b(x).$$

A general solution is given by:

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x).$$

Where c_1, c_2 are arbitrary constants, y_1, y_2 are two linearly independent solutions to the homogeneous equation (where $b(x) = 0$), and y_p is any **particular solution** to the non-homogeneous equation.

When $\frac{b(x)}{a_0(x)} =$ a constant, then $y_p(x) = \frac{b(x)}{a_0(x)}$ works, otherwise:

$$y_p(x) = \int^x G(t, x) \frac{b(t)}{a_2(t)} \, dt.$$

Where:

$$G(t, x) = \frac{y_1(t) y_2(x) - y_1(x) y_2(t)}{y_1(t) y_2'(y) - y_1'(t) y_2(t)}.$$

Remark 8.16 — $G(t, x)$ is known as the **Green's function** associated with the ODE.

Remark 8.17 — When solving the integral, treat all x's as constant, then afterwards, replace all t 's with x 's.

Example 8.18

Consider the equation solved in 8.15 but with $b(x) = (x-1)^2$, i.e.:

$$(1-x)y'' + xy' - y = (x-1)^2.$$

We have:

$$y_1(x) = e^x \quad y_2(x) = x.$$
$$y_1'(x) = e^x \quad y_2(x) = 1.$$

Thus we have:

$$G(t, x) = \frac{e^t x - e^x t}{e^t(1) - e^t t} = \frac{x - te^{x-t}}{1-t}.$$

Thus we have:

$$y_p(x) = \int^x \frac{x - te^{x-t}}{1-t} \frac{(t-1)^2}{1-t} \, dt = \int^x x - te^{x-t} \, dt = xt - e^x(t-1)e^{-t} \Big|^x = x^2 - x + 1.$$

9 February 12th, 2020

9.1 Example Mass/Spring/Damper System

We have a mass $m > 0$ attached to a spring with spring coefficient $k > 0$ and a dampener with coefficient $b \geq 0$. If we assume no coefficient of friction, we get

$$-k - x - b\dot{x} = m\ddot{x}.$$

Which can be rearranged to:

$$m\ddot{x} + b\dot{x} + kx = 0.$$

Which is a 2nd-order linear homogeneous ODE with constant coefficients, which we can use the table from earlier to solve. If we include an external force acting on the mass, we would have:

$$m\ddot{x} + b\dot{x} + kx = F(t) \tag{3}$$

Which would make it non homogeneous. There is an analog circuit equivalent called the LCR circuit, which would have an equation:

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = \Delta V.$$

Which is of the same form as Equation 3.

Let us consider the case without a driving force $F(t)$:

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0.$$

First, we will denote $\omega = \sqrt{\frac{k}{m}}$ which represents the **angular frequency** of the system, with units rad per sec, and $\gamma = \frac{b}{2\sqrt{mk}}$ be a **dampening ratio** (which represents how much dampening is in the system), making the equation:

$$\ddot{x} + 2\gamma\omega\dot{x} + \omega^2 x = 0.$$

Note that discriminant of this equation is:

$$D = \frac{b^2}{m^2} - 4\frac{k}{m} = \frac{4k}{m}\left(\frac{b^2}{4\sqrt{mk}} - 1\right) = 4\omega^2(\gamma^2 - 1).$$

Now depending on what γ and ω are, we can analyze the behaviour of the system.

9.2 No Dampening ($\gamma = 0$)

In this case, we would have:

$$\ddot{x} + \omega^2 x = 0.$$

The discriminant is thus:

$$D = 0^2 - 4(1)(\omega^2) = -4\omega^2 < 0.$$

Using the table, we have:

$$\alpha = -\frac{0}{2(1)} = 0, \quad \beta = \frac{\sqrt{-(-4\omega^2)}}{2(1)} = \omega.$$

Thus the solution will just be:

$$x(t) = c_1\cos(\omega t) + c_2\sin(\omega t).$$

If we want to find the constants, note that $x(0) = x_0 = c_1$. Meanwhile, differentiating the equation, we have:

$$v(t) = -c_1\omega\sin(\omega t) + \omega c_2(\omega t).$$

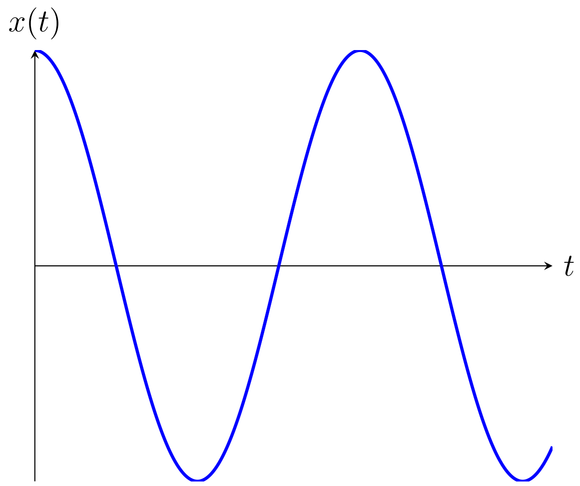

Figure 2: Example of Underdamped Motion

$$v(0) = v_0 = \omega c_2.$$

Thus the complete solution is:

$$x(t) = x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t).$$

This is just a sin curve with amplitude: $A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}$ and period: $T = \frac{2\pi}{\omega}$.

Remark 9.1 — Note that the period does not depend on x_0 or v_0, i.e. it doesn't depend on how it starts. This is different from SHM.

9.3 Under Damping ($0 < \gamma < 1$)

Returning to our equation, we have:

$$\ddot{x} + 2\gamma\omega\dot{x} + \omega^2 x = 0.$$

Thus the determinant is:

$$D = (2\gamma\omega)^2 - 4(1)(\omega^2) = 4w^2(\gamma^2 - 1).$$

If $0 < \gamma < 1$, we have $D < 0$, giving us:

$$\alpha = \frac{-(2\gamma\omega)}{2(1)} = -\gamma\omega, \quad \beta = \frac{\sqrt{-D}}{2(1)} = \omega\sqrt{1-\gamma^2}.$$

Plugging this into the equation, we get:

$$x(t) = c_1 e^{-\gamma\omega t} \cos(\omega t\sqrt{1-\gamma^2}) + c_2 e^{-\gamma\omega t} \sin(\omega t\sqrt{1-\gamma^2}).$$

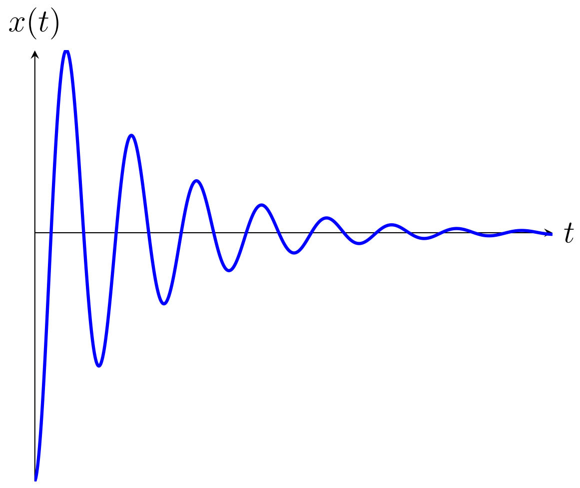

Figure 3: Example of Underdamped Motion

Remark 9.2 — Note that there will be infinite oscillations where the amplitude is decreasing to 0.

9.4 Critical Damping ($\gamma = 1$)

Notice in the case of $\gamma = 1$, we have $D = 0$, thus the solution is:

$$x(t) = c_1 e^{-\gamma\omega t} + c_2 t e^{-\gamma\omega t}.$$

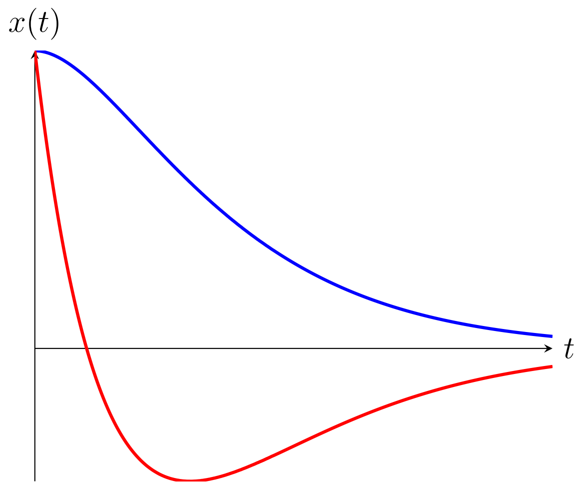

Figure 4: Example of Critical Damped / Over Damped Motion

Remark 9.3 — Note that in this case, there are no oscillations. There will never be two dips. This is because we have:

$$x(t) = c_1 e^{-\gamma\omega t} + c_2 t e^{-\gamma\omega t} = (c_1 + c_2 t) e^{-\gamma\omega t}.$$

Thus by looking at the sign of c_1 and c_2, it will either never cross the x axis (if same sign) or only cross it once (if signs are different). This can be shown by looking at the roots of the equation above.

9.5 Over Damping ($\gamma > 1$)

This yields $D > 0$, thus:

$$x(t) = c_1 e^{-\gamma\omega t}\cosh(\gamma t\sqrt{\gamma - 1}) + c_2 e^{-\gamma\omega t}\sinh(\omega t\sqrt{\gamma^2 - 1}).$$

Remark 9.4 — This is the case where we are taking away the energy a lot, which is useful in many cases. This will make it go to 0 a lot faster than critical damping. Thus for car suspension, we would rather it be critically damped than over damped.

Remark 9.5 — In circuits, this is analogous to using resistors to take away heat from the circuit.

9.6 Laplace Transforms

Laplace transforms are a special case of integral transforms. One way to think of an integral transform is that it's a function where the input is a function of t and output a function of s.

Definition 9.6. More specifically, a **integral transform** is of form:

$$\int_{\alpha(s)}^{\beta(s)} f(t)K(s,t)\ dt.$$

Where $K(s,t)$ is the **kernel** of the transform, and $\alpha(s)$ and $\beta(s)$ are the upper and lower limit.

Example 9.7

Consider the case where $\alpha(s) = s,\ \beta(s) = s^2,\ K(s,t) = st$, and an input $f(t) = t^3$. Then the output would be:

$$\int_s^{s^2} t^3(st)\ dt = \left.\frac{st^5}{5}\right|_{t=s}^{t=s^2} = \frac{1}{5}\left(s^{11} - s^6\right) = F(s).$$

Definition 9.8. Typically, we represent this integral transform as $T\{f(t)\} = F(s)$.

Definition 9.9. The **Laplace Transform** is a special case where:

$$\alpha(s) = 0 \quad \beta(s) = \infty \quad K(s,t) = e^{-st},$$

in other words:

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\ dt = F(s).$$

Remark 9.10 — Note that st must be unitless, and if t represents time, then s represents frequency, thus making the Laplace transform a transformation from time space into frequency space.

Example 9.11

We have
$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\, dt = \frac{e^{-st}}{-s}\bigg|_0^\infty = \frac{1}{s}.$$
Note that $s > 0$

In order to go from s-space back to t-space, we take the inverse Laplace transform. This will be unique as long as we don't consider null functions.

Definition 9.12. A **null function** is a function that is zero except for finitely many points.

Example 9.13

An example of a null function is:
$$N(t) = \begin{cases} 1, & t = 0 \\ 2, & t = 1 \\ 0, & \text{otherwise} \end{cases}.$$

These null functions do not appear often for our situation, so we can have a Laplace transform table:

Table 3: Laplace Transform Table

1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$s > 0$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$s > 0$
	$\vdots$	

Remark 9.14 — Using the table, one example is: $\mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\}$

10 February 14th, 2020

10.1 Problem 1 - Solution 1

Consider
$$u''(r) + \frac{1}{r}u'(r) = -H.$$

With constraints:

$$u(a) = T_e \quad |u(0)| < \infty.$$

Let us take $v = u'(r)$, which gives us:

$$v'(r) + \frac{1}{r}v = -H.$$

which is a linear first order ODE, giving us:

$$v(r) = \frac{1}{\mu(r)}\left(\int \mu(r)(-H)dr + C_1\right), \quad \mu(r) = e^{\int \frac{1}{r}dr} = r.$$

Thus:

$$v(r) = \frac{1}{r}\left(\int -rH\ dr + C_1\right) = \frac{1}{r}\left(-\frac{1}{2}r^2H + C_1\right).$$

$$v(r) = -\frac{1}{2}rH + \frac{1}{r}C_1 = u'(r).$$

$$\implies u(r) = \int -\frac{1}{2}rH + \frac{1}{r}C_1\ dr = -\frac{1}{4}r^2H + \ln(r)C_1 + C_2.$$

To solve for constants, we apply initial conditions:

$$|u(0)| = |-\frac{1}{4}(0)^2H + \ln(0)C_1 + C_2| < \infty \implies C_1 = 0.$$

$$\implies u(a) = -\frac{1}{4}Ha^2 + C_2 = T_e \implies C_2 = T_e + \frac{1}{4}Ha^2.$$

Thus we have:

$$u(r) = T_e + \frac{1}{4}H(a^2 - r^2).$$

10.2 Problem 1 - Solution 2

We once again consider $u''(r) + \frac{1}{r}u'(r) = -H$. First we will solve the homogeneous equation:

$$u_h''(r) + \frac{1}{r}u_h'(r) = 0 \implies r^2u_h''(r) + ru_h'(r) = 0.$$

which is equidimensional. As such we just need to find the discriminant with $a = 1, b = 1, c = 0$:

$$D = (b-a)^2 - 4ac = (1-1)^2 - 0 = 0.$$

Using the table, we have:

$$u_1(r) = |r|^\alpha \ln(r) \quad u_2(r) = |r|^\alpha.$$

with

$$\alpha = -\frac{b-a}{2a} = 0.$$

Thus:

$$u_1(r) = \ln(r) \quad u_2(r) = 1.$$

Thus the overall homogeneous solution is:

$$u_h = C_1\ln(r) + C_2.$$

Now we need to find the particular solution using Green's Function:

$$G(t,r) = \frac{u_1(t)u_2(r) - u_1(r)u_2(t)}{u_1(t)u_2'(t) - u_1'(t)u_2(t)} = \frac{\ln(t) - \ln(r)}{-\frac{1}{t}} = t\ln(r) - t\ln(t).$$

Using this, we have:

$$u_p(r) = \int^r G(t,r)g(t)dt = \int^r (t\ln(r) - t\ln(t))(-H)\ dt.$$

$$= -H\ln(r)\int^r t\ dt + H\int^r t\ln(t)\ dt = \frac{1}{2}r^2 H\ln(r).$$

Integrating by parts, with:

$$u = \ln(t) \quad dv = t\ dt$$

$$du = \frac{1}{t}\ dt \quad v = \frac{1}{2}t^2.$$

we have:

$$\int^r t\ln(t)\ dt = \frac{1}{2}t^2\ln(t) - \int \frac{1}{2}t\ dt = \frac{1}{2}t^2\ln(t) - \frac{1}{4}t^2\Big|_{t=r}.$$

Giving us $u_p(r) = -\frac{1}{4}r^2 H$, thus giving us:

$$u_h + u_p = C_1\ln(r) + C_2 - \frac{1}{4}r^2 H.$$

Which is the same as the other solution before plugging in the initial conditions.

10.3 Problem 2

Consider the equation:

$$\ddot{x} + \omega^2 x = \ddot{x} + \frac{g}{L}x = g.$$

where $\omega = \sqrt{\frac{g}{L}}$ and initial conditions:

$$x(0) = 0 \quad \dot{x}(0) = 0.$$

This has constant coefficients, with $a = 1, b = 0, c = \omega^2$, thus the discriminant is:

$$D = b^2 - 4ac = -4\omega^2 < 0.$$

Thus we have:

$$x_1 = e^{\alpha t}\cos(\gamma t) \quad x_2 = e^{\alpha t}\sin(\gamma t).$$

with:

$$\gamma = \frac{\sqrt{-D}}{2a} = \omega \quad \alpha = -\frac{b}{2a} = 0.$$

Thus:

$$x_h = C_1\cos(\omega t) + C_2\sin(\omega t).$$

Now we need a particular solution. Looking back at the original equation, we can guess $x_p = L$. Since:

$$0 + \frac{g}{L}L = g.$$

Because of the existence-uniqueness theorem, this is the only solution that will work, meaning that overall solution before initial conditions is:

$$x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t) + L.$$

Applying initial conditions, we have:

$$x(0) = C_1(1) + C_2(0) + L = 0 \implies C_1 = -L.$$

$$\dot{x}(0) = -L\omega \sin(0) + C_2\omega \cos(0) = 0 \implies C_2 = 0.$$

Thus we have:

$$x(t) = -L\cos(\omega t) + L = L(1 - \cos(\omega t)).$$

With this we can solve for some stuff, for example:

$$x(t_{\frac{1}{2}}) = \frac{L}{2} \implies t_{\frac{1}{2}} = \frac{\pi}{3\omega}.$$

$$x(T) = L \implies T = \frac{\pi}{2\omega}.$$

11 February 17th, 2020

11.1 More Laplace Transform

Remember that the Laplace Transform for a function $f(t)$ is:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\ dt = F(s).$$

There is an associated inverse Laplace transform:

$$\mathcal{L}^{-1}\{F(s)\} = f(t).$$

Which maps frequency space back to time space. If we avoid null functions, this inverse Laplace transform is unique, giving us tables of these pairs such as:

Table 4: Example of $\mathcal{L}$ and $\mathcal{L}^{-1}$ Pair Table

$f(t)$	$F(s)$
$t^m e^{at}$	$\frac{m!}{(s-a)^{m+1}}, \quad s > a$
$\sin(\omega t)$	$\frac{\omega}{\omega^2+s^2}, \quad s > 0$
$\vdots$	$\vdots$

Theorem 11.1

The Laplace transform is linear, i.e.:

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\}.$$

Remark 11.2 — Proof in notes.

Example 11.3

$$\mathcal{L}\{t^3e^{-t} + 4\sin(8t)\} = \mathcal{L}\{t^3e^{-t}\} + 4\mathcal{L}\{\sin(8t)\}.$$
$$= \frac{3!}{(s-(-1))^{3+1}} + 4\left(\frac{8}{8^2+s^2}\right) = \frac{6}{(s+1)^4} + \frac{32}{64+s^2}.$$

Note that the first term has condition $s > -1$ and the second has $s > 0$, meaning that this domain is $s > 0$.

Remark 11.4 — When there are multiple conditions, we take the intersection of the domains.

11.1.1 Limit Theorems

Theorem 11.5 (Limit Theorem)
If $\mathcal{L}\{f(t)\} = F(s)\}$, we should find:

$$\lim_{s\to\infty} F(s) = 0.$$

with the exception of some impulse functions.

Example 11.6
We have $\mathcal{L}\{\cos(\omega t) = \frac{s}{s^2+\omega^2}$. Note that:

$$\lim_{s\to\infty}\left(\frac{s}{s^2+\omega^2}\right) = 0.$$

Remark 11.7 — This can be used as a check, as if you don't get $\lim\limits_{s\to\infty} F(s) = 0$, and you aren't dealing with impulse function, then you did something wrong.

Theorem 11.8 (Endpoint Theorem 1)

$$\lim_{s\to\infty}(sF(s)) = \underbrace{f(0^+)}_{\lim\limits_{t\to0^+} f(t)}.$$

Example 11.9

Again consider $\mathcal{L}\{\cos(\omega t)\}$. We have:

$$\lim_{s\to\infty} s\left(\frac{s}{s^2+\omega^2}\right) = 1.$$

and

$$\cos(\omega \times t) = 1.$$

Theorem 11.10 (Endpoint Theorem 2)

$$\lim_{s\to\infty}(sF(s)) = \underbrace{f(\infty)}_{\lim_{t\to\infty} f(t)},$$

provided it exists.

Remark 11.11 — This allows us to the values of $f(t)$ without having to use the inverse Laplace transform.

Example 11.12

Suppose the Laplace transform of $f(t)$ is:

$$\mathcal{L}\{f(t)\} = \frac{1}{s\sqrt{s^2+1}}.$$

We would like to find out what $f(0)$ and $f(\infty)$ are. Using the endpoint theorem, we have:

$$f(0^+) = \lim_{s\to\infty} s\frac{1}{s\sqrt{s^2+1}} = \lim_{s\to\infty}\frac{1}{\sqrt{s^2+1}} = 0.$$

and

$$f(\infty)\lim_{s\to 0} s\frac{1}{s\sqrt{s^2+1}} = \lim_{s\to 0}\frac{1}{\sqrt{s^2+1}} = 1.$$

11.1.2 Existence of Laplace Transform of $f(t)$

Q: Can we take the integral of anything?
A: No, as the Laplace transform is an improper integral, which must converge.

Example 11.13

Note that

$$\mathcal{L}\{e^{t^2}\} = \int_0^\infty e^{-st}e^{t^2}\,dt = \infty.$$

Thus, $\mathcal{L}\{e^{t^2}\}$ does not have a Laplace transform.

For a function to have a Laplace transform, it must be of exponential order.

Definition 11.14 (Exponential Order)**.** For a function $f(t)$ to be of **exponential order**, there must be a constant α for which:

$$\lim_{t\to\infty} e^{-\alpha t} f(t) = 0.$$

The function is allowed to go to infinity, just not too fast.

11.1.3 Laplace Transforms for Derivatives

Consider the Laplace transform of $f'(t)$ and use integration by parts with:

$$\mathcal{L}\{f'(t)\} = \int_0^\infty \underbrace{e^{-st}}_{u} \underbrace{f'(t)\ dt}_{dv}.$$

$$\underbrace{e^{-st}}_{u} \underbrace{f(t)}_{v} \Big|_0^\infty - \int_0^\infty \underbrace{f(t)}_{v} \underbrace{\left(-se^{-st}\right) dt}_{du}$$

$$= \underbrace{e^{-\infty}}_{0} f(\infty) - \underbrace{e^{-0}}_{1} f(0^+) + s\int_0^\infty f(t)e^{-st}dt = s\mathcal{L}\{f(t)\} - f(0^+).$$

Theorem 11.15 (Laplace Transform for Derivatives)

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0^+).$$

Example 11.16

Consider the second derivative:

$$\mathcal{L}\{f''(t)\} = \mathcal{L}\{\frac{d}{dt}f'(t)\} = s\mathcal{L}\{f'(t)\} - f'(0^+) = s\left(s\mathcal{L}\{f(t)\} - f(0^+)\right) - f'(0^+).$$

Theorem 11.17

From the previous example:

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0^+) - f'(0^+).$$

Remark 11.18 — This can be generalized, and as such we have:

$$\mathcal{L}\{f'''(t)\} = s^3\mathcal{L}\{f(t)\} - s^2 f(0^+) - sf'(0^+) - f''(0^+).$$

Note that for each of the negative terms, the power of s plus the order of the derivative of f will equal the order of the derivative being computed minus 1, with the s coefficient of $\mathcal{L}\{f(t)\}$ having the same power as the order.

Consider $ay''(t) + by'(t) + cy(t) = g(t)$ with initial conditions $y(0) = y_0$, $y'(0) = y_0'$ and with a, b, c being constant. Instead of solving by setting $g(t) = 0$, let us solve it using Laplace transform.
Let us begin by taking the Laplace transform of both sides:

$$\mathcal{L}\{ay''(t) + by'(t) + cy(t)\} = \mathcal{L}\{g(t)\}.$$

$$\implies a\mathcal{L}\{y''(t)\} + b\mathcal{L}\{y'(t)\} + c\mathcal{L}\{y(t)\} = \mathcal{L}\{g(t)\}.$$

$$\implies a\left(s^2\mathcal{L}\{y(t)\} - sy(0^+) - y'(0^+)\right) + b\left(s\mathcal{L}\{y(t)\} - y(0^+)\right) + c\mathcal{L}\{y(t)\} = \mathcal{L}\{g(t)\}.$$

Thus we have:

$$\mathcal{L}\{y(t)\} = \frac{(as + b)y_0 + ay_0' + \mathcal{L}\{g(t)\}}{as^2 + bs + c}.$$

With this, we can get $y(t)$ by taking the inverse Laplace transform.

Example 11.19

Consider:

$$y''(t) + 2y'(t) + 3y(t) = t^3 \quad y(0) = 0 \quad y'(0) = 1.$$

With this we have: $a = 1, b = 2, c = 3, y_0 = 0, y_0' = 1$, and:

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{t^3\} = \frac{3!}{s^{3+1}} = \frac{6}{s^4}.$$

Thus without solving the ODE, we can say that:

$$\mathcal{L}\{y(t)\} = \frac{(s + 2)(0) + (1)(1) + \frac{6}{s^4}}{s^2 + 2s + 3} = \frac{s^4 + 6}{s^4(s^2 + 2s + 3)}.$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s^4 + 6}{s^4(s^2 + 2s + 3)}\right\}.$$

11.1.4 Other Properties of Laplace Transforms

Theorem 11.20 (First Shifting Theorem)

If $\mathcal{L}\{f(t)\} = F(s)$, then:

$$\mathcal{L}\{e^{at}f(t)\} = F(s - a).$$

Remark 11.21 — The way to remember this, forget e^{at}, and then whoever we get an s, replace by $s - a$.

Theorem 11.22

If $\mathcal{L}\{f(t)\} = F(s)$, then:

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s).$$

$$\mathcal{L}\{t^m f(t)\} = (-1)^m \frac{d^m}{ds^m}F(s).$$

Remark 11.23 — The way to do this, forget the t, then afterward take the derivative w.r.t. s and negate it.

Example 11.24

We have:

$$\mathcal{L}\{e^{2t}\cos(4t)\} = \mathcal{L}\{\cos(4t)\}\Big|_{s\to s-2}.$$

$$= \frac{s}{s^2+4^2}\Big|_{s\to s-2} = \frac{s-2}{(s-2)^2+16}.$$

Example 11.25

We have:

$$\mathcal{L}\{t\cos(4t)\} = \frac{d}{ds}\mathcal{L}\{\cos 4(t)\}.$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2+4^2}\right) = -\frac{d}{ds}\left(\frac{s}{s^2+16}\right).$$

$$= -\left(\frac{(s^2+16)-s(2s)}{(s^2+16)^2}\right) = \frac{s^2-16}{(s^2+16)^2}.$$

Example 11.26

We have:

$$\mathcal{L}\{te^{-t}\sin(t)\} = \mathcal{L}\{t\sin(t)\}\Big|_{s\to s-(-1)}$$

$$= -\frac{d}{ds}\mathcal{L}\{\sin(t)\}\Big|_{s\to s+1} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\Big|_{s\to s+1}$$

$$= \frac{2s}{(s^2+1)^2}\Big|_{s\to s+1} = \frac{2(s+1)}{((s+1)^2+1)^2} = \frac{2s+2}{(s^2+2s+2)^2}.$$

Remark 11.27 — Knowing these two properties, then we can compute Laplace transforms of functions with factors of $t^m e^{at}$.

11.1.5 Unit Step Function

Definition 11.28 (Unit Step Function)**.** The **unit step function** $u_a(t) = u(t-a)$ is defined as:

$$u_a(t) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

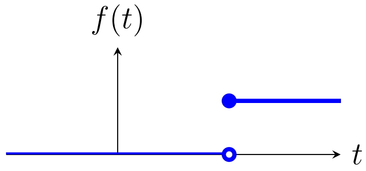

Figure 5: Example of a Unit Step Funciton

The Laplace transform for the unit step function is:

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty u(t-a)e^{-st}\, dt$$

$$= \int_0^a (0)e^{-st}\, dt + \int_a^\infty (1)e^{-st}\, dt = -\frac{e^{-st}}{s}\bigg|_a^\infty = \frac{e^{-as}}{s}, \quad s > 0.$$

Remark 11.29 — We can use this for calculating the Laplace transforms for piecewise functions.

Example 11.30

Consider the piecewise function

$$f(t) = \begin{cases} 0, & t < 0 \\ 1, & 0 < t < 2 \\ t, & 2 \leq t \leq 3 \\ e^t, & 3 < t \end{cases}.$$

We can express this as:

$$1u(t) + (t-1)u(t-2) + (e^t - t)u(t-3).$$

Thus for any piecewise function, we can express it as:

$$f(t) = \begin{cases} 0, & t < 0 \\ f_1(t), & 0 < t < t_1 \\ f_2(t), & t_1 < t < t_2 \\ \vdots \\ f_{m+1}(t), & t_m < t \end{cases}$$

$$= f_1(t)u(t) + (f_2(t) - f_1(t))u(t-t_1) + (f_3(t) - f_2(t))u(t-t_2) + \ldots + (f_{m+1}(t) - f_m(t))u(t-t_m).$$

12 February 19th, 2020

12.1 Unit Step Function Continued

As a reminder, the unit step function is defined as:

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}.$$

Given a piecewise function, we can write it as a linear combination of step functions.

Example 12.1
Consider:

$$f(t) = \begin{cases} 7, & 0 < t < 2 \\ 6t, & 2 < t < 3 \\ t^2, & 3 < t < 7 \\ 0, & 7 < t \end{cases}.$$

We can rewrite this as:

$$f(t) = 7u(t) + (6t-7)u(t-2) + (t^2-6t)u(t-3) + (0-t^2)u(t-7).$$

With this, we can take the Laplace transform of the function, but first, we need to consider the Laplace transform of $\mathcal{L}\{f(t)u(t-a)\}$. Looking at the definition, we have:

$$\mathcal{L}\{f(t)u(t-a)\} = \int_0^\infty f(t)u(t-a)e^{-st}\,dt.$$

Assuming $a > 0$, we have:

$$= \int_0^a f(t)\underbrace{u(t-a)}_{0}e^{-st}\,dt + \int_a^\infty f(t)\underbrace{u(t-a)}_{1}e^{-st}\,dt = \int_a^\infty f(t)e^{-st}\,dt.$$

If we set $z = t - a \implies dz = dt$,

$$= \int_0^\infty f(z+a)e^{-s(z+a)}\,dz = e^{-as}\mathcal{L}\{f(t+a)\}.$$

Theorem 12.2 (Shifting Theorem)
As shown above:

$$\mathcal{L}\{f(t)u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}.$$

Example 12.3
Considering $f(t)$ from Example 12.1, we have:

$$\mathcal{L}\{f(t)\} = \int_0^2 7e^{-st}\,dt + \int_2^3 6te^{-st}\,dt + \int_3^7 t^2e^{-st}\,dt + \int_7^\infty 0e^{-st}\,dt.$$

However, we can calculate this another way. From the table, we have $\mathcal{L}\{t^m\} = \frac{m!}{s^{m+1}}$, thus:

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{7u(t) + (6t-7)u(t-2) + (t^2-6t)u(t-3) + (0-t^2)u(t-7)\}$$
$$= \mathcal{L}\{7u(t)\} + \mathcal{L}\{(6t-7)u(t-2)\} + \mathcal{L}\{(t^2-6t)u(t-3)\} + \mathcal{L}\{-t^2u(t-7)\}$$
$$= e^{-0s}\mathcal{L}\{7\} + e^{-2s}\mathcal{L}\{6(t+2)-7\} + e^{-3s}\mathcal{L}\{(t+3)^2 - 6(t+3)\} - e^{-7s}\mathcal{L}\{(t+7)^2\}$$
$$= \frac{7}{s} + e^{-2s}\mathcal{L}\{6t-5\} + e^{-3s}\mathcal{L}\{t^2-9\} - e^{-7s}\mathcal{L}\{t^2+14t+49\}.$$

Thus:

$$\mathcal{L}\{f(t)\} = \frac{7}{s} + e^{-2s}\left(\frac{6}{s^2} + \frac{5}{s}\right) + e^{-3s}\left(\frac{2}{s^3} - \frac{9}{s}\right) - e^{-7s}\left(\frac{2}{s^3} + \frac{14}{s^2} + \frac{49}{s}\right), \quad s > 0.$$

Remark 12.4 — In the example earlier, we are using the Shifting Theorem and replacing t with $t+a$ in each of the functions that we are multiplying by the unit step function at a.

12.2 Examples of Where Unit Step Functions Occur

Example 12.5
Consider the equation:

$$L\frac{dI}{dt} + RI = \epsilon_1 u(t) + (\epsilon_2 - \epsilon_1)u(t-t_1).$$

Taking the Laplace transform of both sides, we get:

$$\mathcal{L}\left\{L\frac{dI}{dt} + RI\right\} = \mathcal{L}\{\epsilon_1 u(t) + (\epsilon_2-\epsilon_1)u(t-t_1)\}.$$

$$\implies L\mathcal{L}\{I'(t)\} + R\mathcal{L}\{I(t)\} = e^{-0s}\mathcal{L}\{\epsilon_1\} + e^{-t_1 s}\mathcal{L}\{\epsilon_2 - \epsilon_1\}$$
$$\implies L\left(s\mathcal{L}\{I\} - I(0)\right) + R\mathcal{L}\{I\} = \frac{\epsilon_1}{s} + e^{-t_1 s}\left(\frac{\epsilon_2 - \epsilon_1}{a}\right).$$
$$\implies \mathcal{L}\{I\} = \frac{LI_0 + \frac{\epsilon_1}{s} + e^{-t_1 s}\left(\frac{\epsilon_2-\epsilon_1}{s}\right)}{LS + R}.$$

There are many applications/cases where using a step function to describe a piecewise function might be useful. For example, if we have a spring with dampener with an external force $F(t)$, we might have $F(t)$ ramp up with t, and then stay constant after a certain amount of time.

Another example is consider a ball bouncing off the ground. The forces are:

$$F(t) = \begin{cases} -mg, & 0 < t < T_F \\ N(t) - mg, & T_F < t < T_F + T_C \\ -mg, T_F + T_C < t < T_F + T_C + T_R \end{cases}.$$

Where T_F is the time until hitting the ground, T_C is the contact duration, and T_R is the time to rebound back up, and $N(t)$ is the normal force. From this, we get figure out $N(t)$ and allow us to get the coefficient of restitution.

12.3 Impulse Function

Consider a function:

$$I_a(t) = \begin{cases} 0, & t < -\frac{a}{2} \\ \frac{1}{a}, & -\frac{a}{2} < t < \frac{a}{2} \\ 0, & \frac{a}{2} < t \end{cases}.$$

This can be expressed in terms of unit step functions as:

$$I_a(t) = \frac{1}{a}u(t + \frac{a}{2}) - \frac{1}{a}u(t - \frac{a}{2}).$$

Remark 12.6 — Note that the area under the curve is 1, as we choose the height to be inversely proportional to the width. This means that:

$$\int_{-\infty}^{\infty} I_a(t)\ dt = 1.$$

Definition 12.7 (Impulse Function)**.** An **impulse function** is:

$$\lim_{a \to 0} I_a(t) = I(t) = \begin{cases} 0, & t \neq 0 \\ +\infty, & t = 0 \end{cases}$$

with the property:

$$\int_{-\infty}^{\infty} I(t)\ dt = 1.$$

Or:

$$\int_R I(t)\ dt = \begin{cases} 0, & 0 \notin R \\ 1, & 0 \in R \end{cases}.$$

Remark 12.8 — With Laplace transform, if 0 is at the end of the domain, it is included, e.g.:

$$\int_0^7 I(t)\ dt = 1.$$

Remark 12.9 — Similar to the step function, we can shift the impulse function, i.e.:

$$I(t-a) = \begin{cases} 0, & t \neq a \\ +\infty, & t = a \end{cases},$$

with:

$$\int_R I(t-a)\ dt = \begin{cases} 0, & a \notin R \\ 1, & a \in R \end{cases}.$$

Example 12.10 (One Dimensional Crystal)

In a one dimensional crystal, we have atoms aligned in a line, and say they are separated by a. If we have an electron travelling along, the force it might see can be expressed as:

$$F(x) = a \sum_{k=-\infty}^{\infty} F_0 I(x-ka) = aF_0 \sum_{k=-\infty}^{\infty} I(x-ka).$$

Thus one way to model the force experienced by an electron is to use a bunch of impulse functions. This is called the **comb function**.

Remark 12.11 — If we had a continuous function $f(t)$, we'd have:

$$\int_R f(t)I(t-a)\ dt = \begin{cases} 0, & 0 \notin R \\ f(a), & a \in R \end{cases}.$$

Example 12.12

If we have:

$$\int_{-1}^{7} \frac{t^2}{\sqrt{3t^3+1}} e^t I(t-1)\ dt = \frac{1^2}{\sqrt{3(1)^3+1}} e^1 = \frac{1}{2}e.$$

Theorem 12.13

We have:

$$\mathcal{L}\{f(t)I(t-a)\}, a > 0 = \int_0^\infty f(t)I(t-a)e^{-st}\ dt = f(a)e^{-as}.$$

Example 12.14

$\mathcal{L}\{t^3 I(t-4)\} = 4^3 e^{-4s} = 64e^{-4s}$

Example 12.15

Consider $F(t) = aF\sum_{k=-\infty}^{\infty} I(t-ka)$. We have:

$$\mathcal{L}\{F(t)\} = Fa\sum_{k=-\infty}^{\infty} e^{-kas}.$$

Index