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1.1 Legendre ODE and Legendre Polynomials

Consider the equation:

$$(1-x^2)y''(x) - 2xy'(x) + m(m+1)y(x) = 0, \quad -1 < x < 1.$$

Where m is a parameter taking on non-negative integer values. Note that $x_0 = 0$ is an ordinary point. Converting this to standard form, we get:

$$y''(x) - \frac{2x}{1-x^2}y'(x) + \frac{m(m+1)}{1-x^2}y(x) = 0.$$

This is called the **Legendre Equation**. This will come up with spherical symmetry.

Using Taylor's method, we have $y(x) = \sum_{k=0}^{\infty} a_k x^k$. Taking the derivative, we have:

$$y'(x) = \sum_{k=0}^{\infty} k a_k x^{k-1}.$$

$$y''(x) = \sum_{k=0}^{\infty} k(k-1) a_k x^{k-2}.$$

Plugging this in and collecting the common terms, we get:

$$\sum_{k=0}^{\infty} \left((k+2)(k+1)a_{k+2} - (k-m)(k+m+1)a_k\right) x^k = 0.$$

This means that:

$$a_{k+2} = \frac{(k-m)(k+m+1)a_k}{(k+2)(k+1)}, \quad k = 0, 1, 2, \ldots.$$

Suppose $m = 4$, we have:

$$a_{k+2} = \frac{(k-4)(k+5)}{(k+2)(K+1)}a_k.$$

After plugging it in, note that the even coefficients go to zero after $k = 4$, while the odd coefficients is still an infinite sum. This tells us that one solution is:

$$y(x) = a_0(1 - 10x^2 + \frac{35}{3}x^4).$$

Note that because of the $(k-m)$ factor, for any m, there will always be one solution to the Legendre ODE that is a polynomial. That polynomial is called $P_m(x)$ and is known as a **Legendre Polynomial**. This polynomial has the following properties:

- $P_m(x)$ will have degree m
- $P_m(x)$ will only contain powers of the same parity of m.
- Note that can change the factor of a_0 or a_1. As such, we usually normalizes it as $P_m(\pm 1) = (\pm 1)^m$.
- $P_m(x)$ will have exactly m distinct roots between $x = -1$ and $x = +1$ if normalized.

Remark 1.1 — Doing this normalization will put force all $P_m(x)$ to lie in the square centered at the origin with width $= 2$.

With this, we could put $P_m(x)$ in a table. These polynomials have a very nice property, in that:

$$P_{m+1}(x) = \frac{(2m+1)P_m(x) - mP_{m-1}(x)}{m+1}.$$

This allows us to generate the higher order Legendre polynomials, and is how computers calculate them.

For the other equation, we can use Abel's equation, giving us:

$$\begin{aligned} Q_m(x) &= P_m(x)\left[\int \frac{e^{-\int \frac{2x}{1-x^2}dx}}{(P_m(x))^2}dx + A\right]. \\ &= P_m(x)\left[\int \frac{dx}{(1-x^2)P_m^2(x)} + A\right]. \end{aligned}$$

For Q_0, we have:

$$1\left[\int \frac{dx}{(1-x^2)} + A\right] = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) + A.$$

This can be generalized, with:

$$Q_m(x) = \text{polynomial}\ln\left(\frac{1+x}{1-x}\right) + \text{ polynomial}.$$

This means that it is not finite at $x = \pm 1$. Thus the general solution is:

$$y(x) = AP_m(x) + BQ_m(x).$$

And if we require that the solution must be finite at $x = \pm 1$, then $B = 0$ and $y(x) = AP_m(x)$.