1 October 27th, 2020

1.1 Weak Derivative

From last time, we have that the general approach to solving a PDE using finite element methods is:

1. Find the weak formulation of the equation
2. Define the solution space
3. Find the N-dimensional subspace
4. Reduce the weak formulation by a linear system

Recall that the weak formulation is to find $u \in S_N$ s.t.:

$$\int_0^1 u'v' = \int_0^1 fv \, dx \quad \forall v \in S_N.$$

In our previous example, we had S_N being continuous piecewise linear functions. However these piecewise linear functions are not differentiable at the corners, meaning we cannot interpret derivatives in the point wise sense. To do this, we will introduce the concept of a weak derivative.

Definition 1.1 (weak derivative). w is a **weak derivative** of u if w is integrable and:

$$\int_{-\infty}^{+\infty} uv' = -\int_{-\infty}^{+\infty} wv \, dx, \quad \forall v \in C_0^\infty(\mathbb{R}).$$

Remark 1.2 — If u, v are differentiable in the classical sense, assuming zero boundary conditions, then:

$$\int_{-\infty}^{+\infty} u'v \, dx = uv\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} uv' \, dx = -\int_{-\infty}^{+\infty} uv' \, dx.$$

This means that if u is differentiable, then $w = u'$.

Example 1.3 (Weak derivative of hat function)

Consider the hat function:

$$u(x) = \begin{cases} 0 & x < -1 \text{ or } x > 1 \\ 1+x & -1 \leq x < 1 \\ 1-x & 0 \leq x \leq 1 \end{cases}.$$

Plugging this in, we have:

$$\begin{aligned} \int_{-\infty}^{+\infty} uv' \, dx &= \int_{-1}^{0} (1+x)v' \, dx + \int_{0}^{1} (1-x)v' \, dx \\ &= (1+x)v\Big|_{-1}^{0} - \int_{-1}^{0} 1 \cdot v \, dx + (1-x)v\Big|_{0}^{1} + \int_{0}^{1} 1 \cdot v \, dx \\ &= v(0) - \int_{-1}^{0} v \, dx - v(0) + \int_{0}^{1} v \, dx \\ &= -\int_{-\infty}^{+\infty} wv \, dx. \end{aligned}$$

where:

$$w(x) = \begin{cases} 0 & x < -1 \text{ or } x > 1 \\ +1 & -1 \leq x < 1 \\ -1 & 0 \leq x \leq 1 \end{cases}.$$

As such, the hat function is weakly differentiable.

Example 1.4 (Piecewise smooth function that is not weakly differentiable)

Consider the step function:

$$u(x) = \begin{cases} 1 & x < 0 \\ -1 & x \geq 0 \end{cases}.$$

We have:

$$\int_{-\infty}^{+\infty} uv' \, dx = \int_{-\infty}^{0} v' \, dx - \int_{0}^{\infty} v' \, dx = v(0) - (-v(0)) = 2v(0) = \int_{-\infty}^{+\infty} wv \, dx.$$

In this situation, w is the delta function at 0. However, it is not integrable in the classical sense. As such, u is not weakly differentiable.

Theorem 1.5

If f is piecewise smooth and continuous, and f' is integrable in each Ω_i, then f is weakly differentiable.

Proof. Let f be smooth on Ω_i, $[a,b] = \bigcup_i \Omega_i$. We define g by $g\big|_{\Omega_i}$ which is the classical derivative of f on Ω_i. Now we will show that w is the weak derivative of f. By definition, we need to show:

$$\int_a^b gv\ dx = -\int_a^b fv'\ dx, \quad v \in C_0^\infty(a,b).$$

To do this, we have:

$$\begin{aligned}
\int_a^b gv\ dx &= \sum_i \int_{\Omega_i} gv\ dx = \sum_i \int_{\Omega_i} f'v\ dx = \sum_i \int_{x_i}^{x_{i+1}} f'v\ dx \\
&= \sum_i fv\Big|_{x_i}^{x_{i+1}} - \sum_i \int_{x_i}^{x_{i+1}} fv'\ dx \\
&= \underbrace{\sum_i (fv(x_{i+1}) - fv(x_i))}_{=0 \text{ since } v \text{ is continuous}} - \sum_i \int_{x_i}^{x_{i+1}} fv'\ dx \\
&= -\int_a^b fv'\ dx
\end{aligned}$$

.

as desired. Thus f is weakly differentiable with a weak derivative g. □

We can generalize this to higher space dimensions:

Definition 1.6 (weak partial derivative)**.** If ω_i is integrable over Ω and:

$$\int_\Omega \omega_i v\ dx = -\int_\Omega u\frac{\partial v}{\partial x_i}\ dx \quad \forall v \in C_0^\infty(\Omega).$$

then w_i is the **weak partial derivative** of u with respect to x_i.

We can also define higher order derivatives:

Definition 1.7. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ where α_j are non-negative integers. Let us define $|\alpha| = \sum_{j=1}^n \alpha_j$, then w_α, integrable in Ω is the multivariable weak derivative of u if:

$$\int_\Omega u\partial^\alpha v\ d\Omega = (-1)^{|\alpha|}\int_\Omega w_\alpha v\ d\Omega, \quad \forall v \in C_0^\infty(\Omega).$$

with:

$$w_\alpha = \frac{\partial^{|\alpha|}u}{\partial^{\alpha_1}x_1\partial^{\alpha_2}x_2\ldots\partial^{\alpha_n}x_n}.$$

Remark 1.8 — Consider the case for 2nd order derivative. Integration by parts gives us:

$$\int_{-\infty}^{+\infty} u''v\ dx = -\int_{-\infty}^{+\infty} u'v'\ dx = \int_{-\infty}^{+\infty} uv''\ dx.$$

Then if there exists a w that is integrable such that:

$$\int_{-\infty}^{+\infty} wv\ dx = \int_{-\infty}^{+\infty} uv''\ dx.$$

then u has a second order weak derivative equal to w.

Example 1.9

The hat function is not second order weakly differentiable, since it's first order weak derivative is a step function, which is not weakly differentiable.

Remark 1.10 — A piecewise quadratic function is not necessarily second order weakly differentiable, as you need the additional constraint that the first order derivative at the boundaries are the same.

Definition 1.11. Let us define

$$L^2(\Omega) = \{u : \text{ integrable}, \int_\Omega u^2 \, dx < +\infty\}, \quad \|u\|_{L^2} = \left(\int_\Omega u^2 \, dx\right)^{\frac{1}{2}}.$$

Then the **Soboler space** is:

$$H^1(\Omega) = \{u : u \in L^2, \quad u \text{ has weak first order derivate which are in } L^2(\Omega)\}.$$

$$\|u\|_{H^1(\Omega)} = \left(\int_\Omega u^2 + \sum_{i=1}^n (\partial x_i u)^2 \, dx\right).$$

Remark 1.12 — The solution space for FEM is a Soboler space.

Some of the properties of $H^1(\Omega)$ include:

- Functions in $H^1(a,b)$ are continuous. $(\Omega = (a,b))$
- f is piecewise smooth and f is continuous implies that $f \in H^1(\Omega)$
- Let $C^1(\Omega) = \{f \text{ which are continuously differentiable in } \Omega\}$, then $C^1(\Omega)$ is dense in $H^1(\Omega)$

1.1.1 Treatment of Boundary Conditions

For now we have only been considering zero boundary conditions. If instead we have a inhomogeneous Dirichlet boundary condition, e.d.

$$\begin{cases} -u'' = f \\ u(0) = a \\ u(1) = b \end{cases}.$$

Then we would have:

$$u(x) = u_H + u_I, \quad u_I = a + (b-a)x.$$

where u_H is zero at the boundary. With this, we have:

$$\begin{aligned} u_H &= u(x) - u_I \\ -u_H'' &= -u'' + u_I'' = -u'' = f \\ u_H(0) &= u(0) - u_I(0) = a - a = 0 \\ u_H(1) &= u(1) - u_I(1) = b - b = 0. \end{aligned}$$

Thus we would have:

$$\begin{cases} -u_H'' = f \\ u_H(0) = u_H(1) = 0 \end{cases}.$$