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1.1 Inverse Power Iteration

If $\lambda_i,\ i \in 1,\dots,n$ with $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n|$ are eigenvalues of A, then $\frac{1}{\lambda_i}$ are eigenvalues of A^{-1} and

$$\frac{1}{|\lambda_1|} \leq \frac{1}{|\lambda_2|} \leq \dots \leq \frac{1}{|\lambda_n|}.$$

Therefore, we can apply power iteration to A^{-1} to get λ_n and hence x_n. This is called the inverse power iteration.

Algorithm 1.1
1. Choose $y^{(0)} \in \mathbb{R}^n$ s.t. $\|y^{(0)}\|_2 = 1$
2. for $k = 1, 2, \dots$

$$z^{(k)} = A^{-1} y^{(k-1)}$$
$$y^{(k)} = \frac{z^{(k)}}{\|z^{(k)}\|_2}$$
$$\mu^{(k)} = \left(y^{(k)}\right)^T A y^{(k)}.$$

Remark 1.2 —
1. From the convergence of power iteration, if:
   - $\langle y^{(0)}, x_n \rangle \neq 0$
   - $\frac{1}{|\lambda_n|} > \frac{1}{|\lambda_{n-1}|}$ (i.e. $|\lambda_n| < |\lambda_{n-1}|$)
   - A^{-1} is symmetric (always true because A is symmetric.

   then the limit of the iteration is:

$$y^{(k)} \to \pm x_n, \quad \mu^{(k)} \to \lambda_n,$$

   with a rate $\left(\frac{|\lambda_n|}{|\lambda_{n-1}|}\right)^{\frac{k}{2}}$
2. We need to solve $Az^{(k)} = y^{(k-1)}$ in each iteration, which can be done by Gaussian Elimination. But we only need to compute $A = LU$ before the iteration and then, in each iteration, we obtain:

$$z^{(k)} = U^{-1} L^{-1} y^{(k_-1)},$$

   which is just a forward and backward substitution.
   - Thus the total computational cost is:

$$O(n^3) + O\left(n^2 \cdot \log\left(\frac{1}{\epsilon}\right)\right).$$