1 February 10th, 2022

1.1 Cont.

Recall from the last time, we found that most of the contribution is from the large points of scale $O(n^{1/\alpha})$.

Let us define an index set of large points:

$$I_n(\epsilon) = \{m \leq n : |X_m| > \epsilon n^{1/\alpha}\}$$

and define the sums:

$$\hat{S}_n(\epsilon) = \sum_{m \in I_n(\epsilon)} X_m = \sum_{m=1}^{n} X_m \mathbb{1}(|x_m| > \epsilon n^{1/\alpha})$$

$$\overline{S}_n(\epsilon) = S_n - \hat{S}_n(\epsilon) = \sum_{m=1}^{n} X_m \mathbb{1}(|X_m| \leq \epsilon \leq \epsilon^{1/\alpha})$$

Now we have two task:

- Show $\frac{\overline{S}_n(\epsilon)}{n^{1/\alpha}}$ is small if ϵ is small
- Find the limit of $\frac{\hat{S}_n(\epsilon)}{n^{1/\alpha}}$

Proof.

$$\mathbf{E}\left[\frac{\overline{S}_n(\epsilon)}{n^{1/\alpha}}\right]^2 = n^{-\frac{2}{\alpha}} \cdot n \cdot \mathbf{E}\left[\overline{X_1}(\epsilon)\right]^2, \quad \overline{X_i}(\epsilon) = X_i \mathbb{1}(|X_i| \leq \epsilon n^{1/\alpha})$$

$$\mathbf{E}[\overline{X_1}(\epsilon)]^2 = \int_0^\infty 2y \Pr(|\overline{X}_1(\epsilon)| \geq y \; dy) \leq \int_0^{\epsilon n^{1/\alpha}} 2y$$

Later we choose $\epsilon = \epsilon \to 0$ as $n \to \infty$. □

Proof. Proof of (2).

$$\mathbf{E} \exp\left(it \frac{\hat{S}_n(\epsilon)}{n^{1/\alpha}}\right) = \sum_{m=0}^{n} \mathbf{E}\left[\exp\left(it \frac{\hat{S}_n(\epsilon)}{n^{1/\alpha}}\right) \Big| |I_n(\epsilon)| = m\right] \Pr(|I_n(\epsilon)| = m)$$

We will use two facts:

1. $|I_n(\epsilon)|$ is Bin $\left(n, \frac{\epsilon^{-\alpha}}{n}\right) \sim$ Poisson$(\epsilon^{-\alpha})$. $\Pr(|X_n| > \epsilon n^{\frac{1}{\alpha}}) = \epsilon^{-\alpha} \frac{1}{n}$.

□