1 February 10th, 2020

1.1 Review from MATH 240

Consider the homogenous equation:

$$a_2(x)y''(x) + a_1y'(x) + a_0(x)y(x) = 0.$$

Note that this is homogeneous since $y(x) = 0$ is a valid solution. A general solution to a 2nd order lienar homogeneous ODE can be expressed as

$$y(x) = c_1y_1(x) + c_2y_2(x),$$

where c_1 and c_2 are arbitrary constrants and $y_1(x)$ and $y_2(x)$ are any two solutions to the ODE for which:

$$\underbrace{y_1(x)y_2'(x) - y_1'(x)y_2(x)}_{\text{Wronskian of } y_1 \text{ and } y_2} \neq 0.$$

Note that the LHS can be expersssed as a determinant:

$$\det\left(\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix}\right).$$

Which is known as the **Wronskian** of y_1 and y_2.

Example 1.1

Consider $y''(x) - 3y'(x) + 3y(x) = 0$, we have:

$$y_1(x) = e^x \quad y_2(x) = e^{2x}.$$

and

$$\det\left(\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\right) = \det\left(\begin{bmatrix} e^x & e^{2x} \\ e^x & e^{2x} \end{bmatrix}\right) = 2e^{3x} - e^{3x} = e^{3x} \neq 0.$$

Meaning that the solution is of the form:

$$y(x) = c_1e^x + c_2e^{3x}.$$

Remark 1.2 — Note that we only need the Wronskian to not be the 0 function, and that it's ok for certain values of x for the Wronkian to be 0.

Example 1.3

If we used $y_1(x) = e^x$ and $y_2(x) = 2e^x$, then we'd get a Wronskian equal to 0, which would not work.

1.2 Constant Coefficients

Consider:

$$ay''(x) + by'(x) + cy(x) = 0,$$

where a, b and c are constants.

Example 1.4

Example 1.1 is an example of a constant equation with $a = 1$, $b = -3$, and $c = 2$.

Let us create a table to help us solve this problem. First we contstruct the descriminant: $D = b^2 - 4ac$. Depending on what value D is, we have:

Table 1: Table to Compute $ay'' + by' + cy = 0$

D	$y_1(x)$	$y_2(x)$	
$D < 0$	$e^{\alpha x}\cos(\beta x)$	$e^{\alpha x}\sin(\beta x)$	$\alpha = -\frac{b}{2a}$ $\beta = \sqrt{-D}/2a$
$D = 0$	$e^{\alpha x}$	$xe^{\alpha x}$	$\alpha = -\frac{b}{2a}$
$D > 0$	$e^{\alpha x}\cosh(\gamma x)$	$e^{\alpha x}\sinh(\gamma x)$	$\alpha = -\frac{b}{2a}$ $\gamma = \sqrt{D}/2a$
	$e^{(\alpha-\gamma)x}$	$e^{(\alpha+\gamma)x}$	$\alpha = -\frac{b}{2a}$ $\gamma = \sqrt{D}/2a$

Example 1.5

Consider $4y'' + y' + y = 0$. The discrminant is $D = b^2 - 4ac = -15 < 0$. Using the first row of the previous table, we have:

$$\alpha = -\frac{1}{8} \quad \beta = \frac{\sqrt{15}}{8}.$$

This means that:

$$y(x) = c_1 e^{-\frac{1}{8}x}\cos\left(\frac{x\sqrt{15}}{8}\right) + c_2 e^{-\frac{1}{8}x}\sin\left(\frac{x\sqrt{15}}{8}\right).$$

Example 1.6

Consider $4y'' + 4y' + y = 0$. Note that $D = b^2 - 4ac = 16 - 16 = 0$, thus using the second row, we have:

$$\alpha = -\frac{b}{2a} = -\frac{4}{8} = -\frac{1}{2}.$$

Thus we have:

$$y(x) = c_1 e^{-\frac{1}{2}x} + c_2 x e^{-\frac{1}{2}x}.$$

Example 1.7

Consider $y'' - 3y' + 2y = 0$, note that $D > 0$. We can either use the third or 4th row of the table. First we have:

$$\alpha = \frac{3}{2} \quad \gamma = \frac{1}{2}.$$

Thus we can write this as either:

$$y(x) = c_1 e^{\frac{3}{2}x} \cosh(\frac{1}{2}x) + c_2 e^{\frac{3}{2}x} \sinh(\frac{1}{2}x)$$

or

$$y(x) = c_1 e^{(\frac{3}{2}-\frac{1}{2})x} + c_2 e^{(\frac{3}{2}+\frac{1}{2})x} = c_1 e^{x} + c_2 e^{2x}.$$

1.3 Cauchy-Euler/Equidimentional Equation

Definition 1.8. A **equidimensional** or **Cauchy-Euler** 2nd order linear homogenous ODE is a equation of the form:

$$ax^2 y''(x) + bxy'(x) + cy(x) = 0.$$

for some constant a, b, c.

Remark 1.9 — Note that the exponent of the x matches the derivative of y.

Again, we can just use a table to solve these equations by checking the value of

$$D = (b-a)^2 - 4ac.$$

Table 2: Table to solve Cauchy-Euler Equations

D	$y_1(x)$	$y_2(x)$	
$D < 0$	$\lvert x\rvert^{\alpha} \cos(\beta \ln \lvert x\rvert)$	$\lvert x\rvert^{\alpha} \sin(\beta \ln \lvert x\rvert)$	$\alpha = -\frac{b-a}{2a}$ $\beta = \sqrt{-D}/2a$
$D = 0$	$\lvert x\rvert^{\alpha}$	$\lvert x\rvert^{\alpha} \ln \lvert x\rvert$	$\alpha = -\frac{b-a}{2a}$
$D > 0$	$\lvert x\rvert^{\alpha} \cosh(\gamma \ln \lvert x\rvert)$	$\lvert x\rvert^{\alpha} \sinh(\gamma \ln \lvert x\rvert)$	$\alpha = -\frac{b-a}{2a}$ $\gamma = \sqrt{D}/2a$
	$\lvert x\rvert^{\alpha-\gamma}$	$\lvert x\rvert^{\alpha+\gamma}$	$\alpha = -\frac{b-a}{2a}$ $\gamma = \sqrt{D}/2a$

Example 1.10

Consider $3x^2y'' + 2xy' + 5y = 0$, where $a = 3, b = 2, c = 5$. Note that:

$$d = (b-a)^2 - 4ac = (2-3)^2 - 4(3)(5) = -59 < 0.$$

Thus we have:

$$\alpha = -\frac{b-a}{2a} = \frac{1}{6}, \quad \beta = \frac{\sqrt{59}}{6}.$$

Thus we have:

$$y(x) = c_1 x^{\frac{1}{6}} \cos(\frac{\sqrt{59}}{6} \ln x) + c_2 x^{\frac{1}{6}} \sin(\frac{\sqrt{59}}{6} \ln x).$$

for $x > 0$.

Example 1.11

Consider $x^2y'' + 2xy' - 2y = 0$, $x > 0$, i.e. $a = 1, b_= 2, c = -2$. Note that $D = (b-a)^2 - 4ac = 9 > 0$, thus we have:

$$\alpha = \frac{-(2-1)}{2(1)} = -\frac{1}{2} \quad \gamma = \frac{\sqrt{9}}{2(1)} = \frac{3}{2}.$$

With the general solution being:

$$y(x) = c_1 x^{-\frac{1}{2}} \cosh(\frac{3}{2} \ln x) + c_2 x^{-\frac{1}{2}} \sinh(\frac{3}{2} \ln x).$$

or

$$y(x) = c_1 x^{-2} + c_2 x.$$

1.4 Other Stuff from Math 240

If we once again consider the equation $a_2(x)y''(x) + a_1(x)y'(x) + a_0 y(x) = 0$. Note that as we mentioned earlier, we need two linearly independent non-zero solutions to get the general solution. If we only have one, say $y_1(x)$, a second linearly independent solution $y_2(x)$ can be constructed using **Abel's equation** :

$$y_2(x) = Ay_1(x) \int \frac{e^{-\int \frac{a_1(x)}{a_2(x)}\, dx}}{y_1^2(x)} dx.$$

For any non-zero constant A.

Remark 1.12 — Derivation is in the notes.

Example 1.13

Consider $xy'' + (1-x)y' - y = 0$. Suppose we're told that one solution is $y_1(x) = e^x$. A second solution would be:

$$y_2(x) = Ae^x \int \frac{e^{-\int \frac{1-x}{x}\,dx}}{(e^x)^2}\,dx.$$

$$= Ae^x \int \frac{e^{\int 1-\frac{1}{x}\,dx}}{e^{2x}}\,dx = Ae^x \int \frac{e^{x-\ln x}}{e^{2x}}\,dx = Ae^x \int \frac{e^{-x}}{x}\,dx.$$

Which doesn't have a nice answer (oops)

Remark 1.14 — Note that whenever $a_2(x) + a_1(x) + a_0(x) = 0$, one solution is always $y_1(x) = e^x$, since we'd have $y'' = y' = y = e^x$.

Example 1.15

Consider $(1-x)y'' + xy' - y = 0$. Since we have $y_1(x) = e^x$, we have:

$$y_2(x) = Ae^x \int \frac{e^{-\int \frac{x}{1-x}\,dx}}{(e^x)^2}\,dx.$$

$$y_2(x) = Ae^x \int \frac{e^{x+\ln(x-1)}}{e^{2x}}\,dx = Ae^x \int \frac{x-1}{e^x}\,dx = Ae^x(-xe^{-x}) = -Ax.$$

Picking $A = -1$, we have: $y_2(x) = x$, thus the general solution would be:

$$y(x) = c_1 e^x + c_2 x.$$

1.5 Non-Homogeneous Equations

Now let's consider non-homogeneous equations:

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0 y(x) = b(x).$$

A general solution is given by:

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x).$$

Where c_1, c_2 are arbitrary constants, y_1, y_2 are two linearly independent solutions to the homogeneous equation (where $b(x) = 0$), and y_p is any **particular solution** to the non-homogeneous equation.

When $\frac{b(x)}{a_0(x)} =$ a constant, then $y_p(x) = \frac{b(x)}{a_0(x)}$ works, otherwise:

$$y_p(x) = \int^x G(t,x)\frac{b(t)}{a_2(t)}\,dt.$$

Where:

$$G(t,x) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{y_1(t)y_2'(y) - y_1'(t)y_2(t)}.$$

Remark 1.16 — $G(t,x)$ is known as the **Green's function** associated with the ODE.

Remark 1.17 — When solving the integral, treat all x's as constant, then afterwards, replace all t 's with x 's.

Example 1.18

Consider the equation solved in 1.15 but with $b(x) = (x-1)^2$, i.e.:

$$(1-x)y'' + xy' - y = (x-1)^2.$$

We have:

$$y_1(x) = e^x \quad y_2(x) = x.$$
$$y_1'(x) = e^x \quad y_2(x) = 1.$$

Thus we have:

$$G(t,x) = \frac{e^t x - e^x t}{e^t(1) - e^t t} = \frac{x - te^{x-t}}{1-t}.$$

Thus we have:

$$y_p(x) = \int^x \frac{x - te^{x-t}}{1-t}\frac{(t-1)^2}{1-t}\,dt = \int^x x - te^{x-t}\,dt = xt - e^x(t-1)e^{-t}\Big|^x = x^2 - x + 1.$$