1 September 8th, 2020

1.1 Course Logistics

1.1.1 Grading

- Assignments: 20%
- Midterm (Early November): 30%
- Final: 50%

1.1.2 Content

There will be 4 chapters in this course:

1. Point Set Topology
2. Functions in Several Variables
3. Sequences of Functions
4. Lebesgue Integral

1.2 Point Set Topology

To motivate this chapter, consider the following question:

> "Given a function $f : \mathbb{Z} \to \mathbb{R}$, could it be continuous or differentiable? What about other domains, e.g. $\mathbb{Q}, \mathbb{R}^n, \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$?"

Clearly for different domains, the notion of continuity and/or differentiability are different, thus we want to formulate the geometric properties of these sets, which leads to the idea of **Point Set Topology**.

> **Remark 1.1 —** For the purpose of this course, we are only considering point set topology in $\mathbb{R}^n$, not in general.

1.3 Reviews/Definitions

Definition 1.2. $\mathbb{R}^n$ consists of n-tuples of real numbers, with the following operations defined:

- Addition: $(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$
- Scalar Multiplication: $k(x_1, \ldots, x_n) = (kx_1, \ldots, kx_n)$

for all $k, x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$

Lemma 1.3

$\mathbb{R}^n$ with the operations above form a real vector space.

Definition 1.4. An **inner product** in $\mathbb{R}^n$ is a function $\langle x, y\rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, such that:

1. $\langle\ ,\ \rangle$ is bilinear, i.e. $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
2. $\langle\ ,\ \rangle$ is symmetric, i.e. $\langle x, y\rangle = \langle y, x\rangle$
3. $\langle x, x\rangle \geq 0, \forall x \in \mathbb{R}^n$ and $\langle x, x\rangle = 0$ iff $x = 0$

Definition 1.5. The **standard inner product/dot product** is:

$$\langle (x_1, \dots, x_n), (y_1, \dots, y_n)\rangle = x_1 y_1 + x_2 y_2 + \dots + x_n y_n.$$

Definition 1.6. If $x \in \mathbb{R}^n$, its norm $\|x\| = \sqrt{\langle x, x\rangle}$

Theorem 1.7

For every $x, y, z \in \mathbb{R}^n$, we have:

1. $\|x-y\| = \|y-x\|$
2. $\|x-y\| \geq 0$
3. $\|x-y\| = 0$ only when $x = y$
4. $\|x-z\| \leq \|x-y\| + \|y-z\|$ (**triangular inequality**), proof below

Theorem 1.8

Cauchy-Schwarz inequality For every $x, y \in \mathbb{R}^n$, we have:

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

Proof. We have:

$$0 \leq \left\langle x - \frac{\langle x, y\rangle}{\|y\|^2} y, x - \frac{\langle x, y\rangle}{\|y\|^2} y \right\rangle$$

$$\implies x \leq \langle x, x\rangle + \left\langle \frac{\langle x, y\rangle}{\|y\|^2} y, \frac{\langle x, y\rangle}{\|y\|^2} y \right\rangle - 2\left\langle x, \frac{\langle x, y\rangle}{\|y\|^2} y \right\rangle$$

$$\implies x \leq \langle x, x\rangle + \frac{\langle x, y\rangle^2}{\|y\|^{\cancel{4}2}} \cancel{\langle y, y\rangle} - 2\frac{\langle x, y\rangle}{\|y\|^2} \langle x, y\rangle$$

$$\implies 0 \leq \|x\|^2 - \frac{\langle x, y\rangle^2}{\|y\|^2} \implies \langle x, y\rangle^2 \leq \|x\|^2 \|y\|^2$$

□

Returning to the proof of the triangle inequality, we have:

Proof.

$$\begin{aligned}
\|x-z\|^2 = \|x-y+y-z\|^2 &= \langle x-y+y-z, x-y+y-z\rangle \\
&= \langle (x-y)+(y-z), (x-y)+(y-z)\rangle \\
&= \langle x-y, x-y\rangle + \langle y-z, y-z\rangle + 2\langle x-y, y-z\rangle \\
&= \|x-y\|^2 + \|y-z\|^2 + 2\langle x-y, y-z\rangle \\
&\le \|x-y\|^2 + \|y-z\|^2 + 2\|x-y\|\|y-z\| = (\|x-y\| + \|y-z\|)^2
\end{aligned}$$

Taking square roots on both sides, we have:

$$\|x-z\| \le \|x-y\| + \|y-z\|.$$

□

Definition 1.9. A function $T : \mathbb{R}^m \to \mathbb{R}^n$ is a **linear transformation** if:

$$T(ax+by) = aT(x) + bT(y), \quad \forall x, y, \in \mathbb{R}^m, \quad a, b \in \mathbb{R}.$$

Definition 1.10. A **basis** of $\mathbb{R}^n$ is a linearly independent set whose span is $\mathbb{R}^n$, i.e. every vector in $\mathbb{R}^n$ is a unique linear combination of the vectors in the basis.

Definition 1.11. Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation and let $A = (a_1, \dots, a_m)$ and $B = (b_1, \dots, b_n)$ be basis of $\mathbb{R}^m$ and $\mathbb{R}^n$ Then write

$$T(a_1) = t_{11}b_1 + t_{12}b_2 + \ldots + t_{1n}b_n$$

$$T(a_2) = t_{12}b_1 + t_{13}b_2 + \ldots + t_{2n}b_n$$

$$\ldots$$

$$T(a_m) = t_{m1}b_1 + t_{m2}b_2 + \ldots + t_{mn}b_n$$

, we call:

$$\begin{bmatrix} t_{11} & t_{12} & \ldots & t_{1n} \\ & & \ldots & \\ t_{m_1} & t_{m_2} & \ldots & t_{mn} \end{bmatrix}^T$$

the matrix of T relative to the basis A and B.

Example 1.12

Let vectors in $\mathbb{R}^n$ be presented as column vectors and:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

A map $T : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T(x) = Ax$ for all $x \in \mathbb{R}^3$. The matrix of T relative to the basis:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \text{ is } \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

The matrix of T relative to the basis:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\} \text{ is } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

This is because:

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$