1 September 24th, 2020

1.1 Parabolic Equation in Two and Three Dimensions

Now let's consider the heat equation in 2 dimensions:

$$\begin{cases} u_t = \sigma(u_{xx} + u_{yy}) = \sigma \Delta u \\ u|_{\partial \Omega} = f_0 & \text{(boundary condition)} \\ u(x,0) = u_0(x) & \text{(initial temperature distribution)} \end{cases}.$$

> **Remark 1.1** — Δu is the Laplacian of u, and is $u_{xx} + u_{yy}$

To solve it numerically, we once again discretize it:

$$x_i = i\Delta x, \quad y_j = j\Delta y$$

for $0 \le i \le J_x$ and $0 \le j \le J_y$, with the step sizes Δx and Δy being the size of the corresponding domain divided by the number of grid points J_x and J_y. We once again have finite difference:

$$\frac{U_{i,j}^{n+1} - U_{i,j}^n}{\Delta t} \approx u_t(x_i, u_j, t_n).$$

$$\frac{U_{i+1,j}^n + U_{i-1,j}^n - 2U_{i,j}^n}{(\Delta x)^2} \approx u_{xx}(x_i, y_j, t_n).$$

$$\frac{U_{i,j+1}^n + U_{i,j-1}^n - 2U_{i,j}^n}{(\Delta y)^2} \approx u_{yy}(x_i, y_j, t_n).$$

Plugging into the heat equation, we have explicit scheme:

$$\frac{U_{i,j}^{n+1} - U_{i,j}^n}{\Delta t} = \sigma \left(\frac{U_{i+1,j}^n + U_{i-1,j}^n - 2U_{i,j}^n}{(\Delta x)^2} + \frac{U_{i,j+1}^n + U_{i,j-1}^n - 2U_{i,j}^n}{(\Delta y)^2} \right).$$

$$\implies U_{i,j}^{n+1} = \sigma \frac{\Delta t}{(\Delta x)^2} \left(U_{i+1,j}^n + U_{i-1,j}^n - 2U_{i,j}^n \right) + \sigma \frac{\Delta t}{(\Delta y)^2} \left(U_{i,j+1}^n + U_{i,j-1}^n - 2U_{i,j}^n \right) + U_{i,j}^n.$$

Since this is explicit, we expect some condition on

$$\nu_x = \frac{\Delta t}{(\Delta x)^2}, \text{ and } \nu_y = \frac{\Delta t}{(\Delta y)^2}.$$

Note that this is also a consistent scheme.

Now calculating the Truncation error by Taylor expansion, we have:

$$T(x,t) = \frac{1}{2}\Delta t u_{xx} - \frac{1}{12}\sigma \left[(\Delta x)^2 u_{xxxx} + (\Delta y)^2 u_{yyyy} \right] + \dots.$$

Note that $T_{ij}^n \to 0$ as $\Delta t \to 0$ and $(\Delta x, \Delta y) \to (0,0)$. In addition, it is first order in time, and second order in space.

> **Theorem 1.2**
>
> The explicit scheme converges under the condition $\nu_x + \nu_y \le \frac{1}{2}$.

Proof. Homework. $\qquad \square$

> **Remark 1.3 —** Note that if we let $\Delta x = \Delta y$, the above condition would be:
>
> $$\frac{\Delta t}{(\Delta x)^2} + \frac{\Delta t}{(\Delta y)^2} \leq \frac{1}{2} \implies 2\frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{2} \implies \Delta t \leq \frac{1}{4}(\Delta x)^2.$$
>
> This means that in 2D, the condition for convergence is even more restricted than the 1D case. Because of this, in higher dimensions, we don't want to use explicit schemes.

Let's now consider this scheme using Von Neumann stability analysis, for $\sigma = 1$, we have:

$$\implies U_{i,j}^{n+1} = U_{ij}^n + \frac{\Delta t}{(\Delta x)^2}\left(U_{i+1,j}^n + U_{i-1,j}^n - 2U_{i,j}^n\right) + \frac{\Delta t}{(\Delta y)^2}\left(U_{i,j+1}^n + U_{i,j-1}^n - 2U_{i,j}^n\right).$$

We have:

$$U_{ij}^n \sim \lambda^n e^{ik_x x_i + jk_j y_j}.$$

Substituting, we have:

$$\lambda^{n+1} e^{ik_x x_i + jk_y y_j} = \lambda^n e^{i(k_x x_i + k_y y_j)} + \nu_x \lambda^n \left(e^{ik_x x_{i+1}} - e^{ik_x x_i} + e^{ik_x x_{i-1}}\right) e^{ik_y y_j}$$
$$+ \nu_y \lambda^n \left(e^{ik_y y_{j+1}} - 2e^{ik_y y_j} + e^{ik_y y_{j-1}}\right) e^{ik_x x_i}.$$

$$= \lambda^n e^{ik_x x_i + jk_y y_j} + \nu_x \lambda^n \left(e^{ik_x \Delta x} - 2 + e^{-ik_x \Delta x}\right) e^{ik_x x_i + ik_y y_j} + \nu_y \lambda^n \left(e^{ik_y \Delta y} - 2 + e^{-ik_y \Delta y}\right) e^{ik_x x_i + jk_y y_j}.$$

$$\implies \lambda = 1 + \nu_x \left(-4\sin^2 \frac{k_x \Delta x}{2}\right) + \nu_y \left(-4\sin^2 \frac{k_y \Delta y}{2}\right).$$

Let us consider the case where $\Delta x = \Delta y$, meaning that $\nu_x = \nu_y$ and $k_x = k_y$, thus we have:

$$\lambda = 1 + 2\nu_x \left(-4\sin^2 \frac{k_x \Delta x}{2}\right) = 1 - 8\nu_x \sin^2 \frac{k_x \Delta x}{2} \leq 1.$$

Since we want $|\lambda| \leq 1$, we need:

$$1 - 8\nu_x \sin^2 \frac{k_x \Delta x}{2} \geq -1 \implies \nu_x \sin^2 \frac{k_x \Delta x}{2} \leq \frac{1}{4}.$$

If $\Delta x \neq \Delta y$, we can still do it, but just a bit more complicated.

1.2   Implicit Scheme

The implicit scheme of the 2D heat equation is of the form:

$$\implies U_{i,j}^{n+1} = U_{ij}^n + \frac{\Delta t}{(\Delta x)^2}\left(U_{i+1,j}^{n+1} + U_{i-1,j}^{n+1} - 2U_{i,j}^{n+1}\right) + \frac{\Delta t}{(\Delta y)^2}\left(U_{i,j+1}^{n+1} + U_{i,j-1}^{n+1} - 2U_{i,j}^{n+1}\right).$$

To solve this we need to solve a linear system. Since we discretize $i = 1, 2, \ldots, J_x - 1$ and $j = 1, 2, \ldots, J_y - 1$, we have a total of $(J_x - 1) \times (J_y - 1) = M$ equations, with the same number of unknowns. If we write this linear system in a matrix form, we have:

$$Ax = b$$

Where

$$A = \begin{bmatrix} D & L & & \\ L & D & \ddots & \\ & \ddots & \ddots & L \\ & & L & D \end{bmatrix}_{M \times M}$$

With

$$A = \begin{bmatrix} 1 + 2\nu_x + 2\nu_y & -\nu_x & & \\ -\nu_x & 1 + 2\nu_x + 2\nu_y & \ddots & \\ & \ddots & \ddots & -\nu_x \\ & & -\nu_x & 1 + 2\nu_x + 2\nu_y \end{bmatrix}$$

and

$$L = \begin{bmatrix} -\nu_y & & & \\ & \ddots & & \\ & & \ddots & \\ & & & -\nu_y \end{bmatrix}$$

Remark 1.4 — A is a symmetric-positive definite matrix.

Remark 1.5 — The form of matrix A, D, and L will depend on how you order x.

Rearrange the implicit scheme, we have:

$$-\nu_y U_{i,j-1}^{n+1} - \nu_x U_{i-1,j}^{n+1} + (1 - 2\nu_x - 2\nu_y)U_{i,j}^{n+1} - \nu_x U_{i+1,j}^{n+1} - \nu_y U_{i,j+1}^{n+1} = U_{i,j}^n.$$

To order x, we will first order them by y and then x, i.e.:

$$x = \begin{bmatrix} U_{11} \\ U_{21} \\ U_{31} \\ \vdots \\ U_{12} \\ U_{22} \\ \vdots \end{bmatrix}.$$

Note that x is a vector of lengths $(J_x - 1) \times (J_y - 1)$, as we only consider the interior points. Similar to the 1D case, the benefit of this implicit scheme is stability.

Remark 1.6 — If we use gaussian elimination to solve this linear system, the computation cost would be $O((M \times M)^2)$, which is unfeasible, especially for large M.