ENM251 - Analytical Methods in Engineering

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These are the notes that I typed during the lectures/recitations. There's probably a lot of typo/mistakes since I haven't really gone through them after class, so keep an eye out for anything that doesn't make sense.

Contents

1 January 22nd, 2020

1.1 Separable Differential Equation

A general first-order ODE for a dependent variable y in the independent variable x can be written as:

$$\frac{dy}{dx} = F(x, y) \tag{1}$$

where F is some specified function of x and y. When F has the form

$$F(x, y) = f(x)g(y), \tag{2}$$

then 1 is said to be *separable* and such equation can always be solved by:

$$\frac{dy}{g(y)}f(x)dx \implies \int \frac{dy}{g(y)} + C_1 = \int f(x)dx + C_2 \implies \int \frac{dx}{g(y)} = \int f(x)dx + C.$$

as one form for the solution of 1.

1.1.1 Ideal Fluid Flow

We are concerned with a container that has a fluid with cross sectional area A with density ρ with a hole at the bottom of the container which causes it to flow out. We are concerned with the heigh x of the container. We also have a pipe that pumps in fluid with constant rate R.

This leads to following equation:

$$\frac{dx}{dt} = \alpha - \beta\sqrt{x}.$$

where

$$\alpha = \frac{R}{A} \quad \beta = \sqrt{\frac{2ga^2}{A^2 - a^2}} \quad g = 9.81\text{m s}^{-2}.$$

Note that this is a separable differential equation:

$$\frac{dx}{\alpha - \beta\sqrt{x}} = dt.$$

If we have α, β, we can solve, e.g. $\alpha = 60$ $\beta = 6$, we have:

$$\frac{dx}{dt} = 60 - 6\sqrt{x} \implies \frac{dx}{10 - \sqrt{x}} = 6dt.$$

Integrating on both sides, we have:

$$\int \frac{dx}{10 - \sqrt{x}} = \int 6dt = 6t + C.$$

Solving this, we get:

$$20\tan^{-1}\left(\frac{\sqrt{x}}{10}\right) - 10\ln(100 - x) - 2\sqrt{x} = 6t + C.$$

If we have initial conditions, e.g. at $t = 0, x = 0$, we would have:

$$0 - 10\ln(100) = C$$

allowing us to solve for C. This would allow us to solve for a time t for certain values of x.

1.2 Homogeneous Differential Equation

Again remember that the general form a differential equation of one a dependent variable y in the independent variable x is:

$$\frac{dy}{dx} = F(x, y).$$

If $F(x, y) = f(x)g(x)$ then this is separable. Remember that the goal is that we want to find $G(x, y) = C$, in other words, we want to get rid of the derivative and find the relationship between the two.

Definition 1.1. A function of form $F(x, y)$ is called **homogeneous** of order N if $F(tx, ty) = t^N F(x, y)$ for any scalar t.

Example 1.2

$$\begin{aligned} F(x,y) = x^3 + x^2y + 4xy^2 \implies F(tx, ty) &= (tx)^3 + (tx)^2(ty) + 4(tx)(ty)^2 \\ &= t^3\left(x^3 + x^2y + 4xy^2\right) = t^3F(x, y). \end{aligned}$$

Thus $F(x, y)$ is homogeneous to the order 3.

Example 1.3

$F(x, y) = x^3 + xy$ is not homogeneous.

Example 1.4

$$F(x, y) = \frac{xy}{x^2 + y^2}$$

$$F(tx, ty) = \frac{t^2xy}{t^2x^2 + t^2y^2} = t^2\left(\frac{xy}{x^2 + y^2}\right) = t^0F(x, y)$$

meaning that $F(x, y)$ is homogeneous to order 0.

Remark 1.5 — Typically if we say that a function is homogeneous but don't specify the order, it is assumed to be of order 0.

If a function in homogeneous to order 0, then it only depends on the ratio of $\frac{y}{x}$. In other words, rewrite $F(x, y) = f(\frac{y}{x})$.

Theorem 1.6

A function $F(x, y)$ is homogeneous of order 0 if and only if it can be expressed as $f(\frac{y}{x})$.

If we have a homogeneous function of order 0, we will be able to introduce a new variable $z = \frac{y}{x} \implies y = sz$, giving us:

$$\frac{d(xz)}{dx} = F(x, xz) = F(x(1), x(z)) = F(1, z).$$

Using the product rule, we have:

$$\frac{d(xz)}{dx} = \frac{dx}{dx}z + x\frac{dz}{dx} = F(1, z).$$

$$z + x\frac{dz}{dx} = F(1, z) \implies \frac{dz}{F(1, z) - z} = \frac{dx}{x},$$

which is a separable differential equation.

Remark 1.7 — The point is whenever you have a homogeneous equation, then introducing $z = \frac{y}{x}$ will allow us to convert it to a separable equation. Note that this only works for order 0 homogeneous equations.

1.2.1 Building an Radar Antenna

TL;DR the equation is:

$$\left(\frac{dy}{dx}\right)^2 - 2\frac{y - F}{x}\left(\frac{dy}{dx}\right) - 1 = 0.$$

If we use the quadratic formula, we get:

$$\frac{dy}{dx} = \frac{y - F}{x} + \sqrt{\left(\frac{y - F}{x}\right)^2 + 1}.$$

If we do the substitution, $z = \frac{y-F}{x}$, we get:

$$\frac{d(xz + F)}{dx} = z + \sqrt{z^2 + 1} \implies x\frac{dz}{dx} + z = z + \sqrt{z^2 + 1} \implies \frac{dz}{\sqrt{z^2 + 1}} = \frac{dx}{x}.$$

$$\int \frac{dz}{\sqrt{z^2 + 1}} = \ln x + C \implies \ln\left(z + \sqrt{z^2 + 1}\right) = \ln x + C.$$

$$\implies A^2x^2 - 2Axz = 1 \implies \frac{1}{2}Ax^2 + \left(F - \frac{1}{2A}\right),$$

which is the equation of a parabola. Thus the optimal shape of a radar dish is a parabola.

2 January 24th, 2020

2.1 Recitation 1

2.1.1 Homogeneous ODE

Recall that a homogeneous equation is

$$\frac{dy}{dx} = F(x, y), \quad \text{with } F(ax, ay) = a^n F(x, y).$$

What this typically means is that we won't have a constant.

Example 2.1

$F(x, y) = xy$ is homogeneous, as $F(ax, ay) = a^2xy$, while $F(x, y) = ax + 5$ is not homogeneous, as $F(ax, ay) = a^2xy + 5 \neq a^n F(x, y)$.

For 1st order homogeneous ODE, we have $n = 0$, with this we can introduce $z = \frac{y}{x}$ and convert this ODE into a separable differential equation.

2.1.2 Problem 1

Example 2.2

Let's consider

$$F(x, y) = \frac{dy}{dx} = \frac{2y^2 - x^2}{3xy}.$$

$$F(ax, ay) = \frac{2a^2y^2 - a^2x^2}{3a^2xy} = F(x, y),$$

meaning that it is a first order homogeneous equation.

With this, we have:

$$\begin{aligned} \frac{d(zx)}{dx} &= \frac{2(zx)^2 - x^2}{3x(zx)} \\ \implies z + x\frac{dz}{dx} &= \frac{2x^2z^2 - x^2}{3x^2z} = \frac{2z^2 - 1}{3z} \\ \implies x\frac{dz}{dx} &= \frac{2z^2 - 1 - 3z^2}{3z} = -\frac{z^2 + 1}{3z}. \end{aligned}$$

Now we can separate, giving us:

$$\frac{z}{z^2 + 1}dz = -\frac{1}{3x}dx \implies \int \frac{z}{z^2 + 1}dz = \int -\frac{1}{3x}dx$$

$$\implies \frac{1}{2}\ln(z^2 + 1) = -\frac{1}{3}\ln(x) + C_1$$

Solving for C_1, we get:

$$3\ln(z^2 + 1) = -2\ln(x) + 6C \implies C = 3\ln(z^2 + 1) + 2\ln(x) = 6C_1$$

$$\implies \ln\left(x^2(z^2 + 1)^3\right) = 6C_1 \implies x^2(z^2 + 1)^3 = e^{6C_1}.$$

Remembering that $z = \frac{y}{x}$, we have:

$$x^2\left(\frac{y^2}{x^2} + 1\right)^3 = e^{6C_1} \implies \frac{(y^2 + x^2)^3}{x^4} = e^{6C_1} \implies \frac{y^2 + x^2}{x^{\frac{4}{3}}} = e^{2C_1} = C.$$

$$y = \pm x^{\frac{2}{3}}\sqrt{C - x^{\frac{3}{2}}}.$$

2.1.3 Bernoulli Equation

Definition 2.3. A **Bernoulli Equation** is an equation of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

If $n = 0$ or $n = 1$, we separate this equation. If $n \neq 0, 1$, defining $y = z^\lambda$, we have:

$$\frac{dy}{dx} = \frac{d(z^\lambda)}{dx} = \frac{dz}{d\lambda}\frac{dz}{dx} = \lambda z^{\lambda-1}\frac{dz}{dx}$$

Substituting this back, we have:

$$\lambda z^{\lambda-1}\frac{dz}{dx} + P(x)z^\lambda = Q(x)(z^\lambda)^n.$$

Dividing both sides by $\lambda z^{\lambda-1}$, we have:

$$\frac{dz}{dx} + \frac{1}{\lambda}P(x)z = \frac{1}{\lambda}Q(x)z^{\lambda n-\lambda+1}.$$

Setting λ such that $\lambda n - \lambda + 1 = 0$, i.e. $\lambda = \frac{1}{1-n}$, the equation becomes:

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x).$$

Which is a linear equation, which we can solve:

$$z(x) = \frac{1}{\mu_n}\left(\int \mu_n(1-n)Q(x)dx + C\right), \quad \mu_n = \exp\left\{(1-n)P(x)dx\right\}.$$

And substituting back into the original equation, we have:

$$y = z^\lambda = z^{\frac{1}{1-n}} = \left(\frac{1}{\mu_n}\left(\int \mu_n(1-n)Q(x)dx + C\right)\right)^{\frac{1}{1-n}}.$$

2.1.4 Problem 2

Consider

$$vx\frac{dv}{dx} + v^2 + xg = \frac{FL}{m}.$$

Rearranging the equation, we get:

$$\frac{dv}{dx} + \frac{v}{x} + \frac{g}{v} = \frac{FL}{xvm} \implies \frac{dv}{dx} + \left(\frac{1}{x}\right)v = \left(\frac{FL}{mx} - g\right)v^{-1}.$$

which is the form of a Bernoulli equation. As such, we can just plug into the formula, and we get:

$$\mu = \exp\{\int (1-(-1))\frac{1}{x}dx\} = e^{\int \frac{2}{x}dx} = x^{2\ln(x)} = x^2.$$

$$V(x) = \left(\frac{1}{\mu}\left(\int (1-(-1))\mu Q(x)dx + C\right)\right)\frac{1}{(1-(-1))}$$

$$= \left(\frac{1}{x^2}\left(\int 2x^2\left(\frac{FL}{mx}-g\right)dx+C\right)\right)^{\frac{1}{2}}$$
$$= \left(\frac{1}{x^2}\left(\frac{FLx^2}{m}-\frac{2}{3}gx^3\right)+C\right)^{\frac{1}{2}} = \left(\frac{FL}{m}-\frac{2}{3}gx+\frac{C}{x^2}\right)^{\frac{1}{2}}.$$

If we have an constraint where V is finite with $x = 0$, we need $C = 0$, as otherwise $x = 0$ will be infinite. Thus:

$$V = \sqrt{\frac{FL}{m}-\frac{2}{3}gx}.$$

2.1.5 Problem 3 Hints from Homework 1

In the first homework, we have:

$$\frac{dx}{dt} = K\left(\alpha - mx\right)^2\left(\beta - nx\right),$$

for some positive constants α, β, m, n. Here we want to determine:

$$\lim_{t\to\infty} x(t).$$

when $\frac{\alpha}{m} < \frac{\beta}{n}$, $\frac{\alpha}{m} = \frac{\beta}{n}$, $\frac{\alpha}{m} > \frac{\beta}{n}$.

If we plug into the equation, we have:

$$\frac{dx}{dt} = Km^2n\left(\frac{\alpha}{m}-x\right)^2\left(\frac{\beta}{n}-x\right).$$

Note that these are all positive except for the last factor. Thus, for the first case, we have:

1. For $x < \frac{\alpha}{m}$, $\frac{dx}{dt} > 0$
2. For $x = \frac{\alpha}{m}$, $\frac{dx}{dt} = 0$
3. For $x > \frac{\alpha}{m}$ and $x < \frac{\beta}{m}$, $\frac{dx}{dt} > 0$
4. For $x = \frac{\beta}{n}$, $\frac{dx}{dt} = 0$
5. For $x > \frac{\beta}{n}$, $\frac{dx}{dt} < 0$

From 1 and 2, we have: if $x_0 \leq \frac{\alpha}{m}$, $\lim_{t\to\infty} x = \frac{\alpha}{m}$, while from 3,4,5, we have: if $x_0 > \frac{\alpha}{m} \lim_{t\to\infty} x = \frac{\beta}{n}$.

3 January 27th, 2020

3.1 Linear ODE

Definition 3.1. The basic form of first-order linear equation is:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x),$$

where $a_1(x) \neq 0$. The goal is given $a_1(x), a_0(x)$ and $b(x)$, solve for $y(x)$.

Example 3.2

$$x^2y'(x) + 2y(x) = x$$

is a first order linear ODE, where $a_1(x) = x^2,\ a_0(x) = 2,\ b(x) = x$.

To solve it, we first divide by $a_1(x)$, giving us:

$$\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = \frac{b(x)}{a_1(x)}.$$

which is of the standard form:

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Example 3.3

From the previous example, we'd have:

$$y'(x) + \frac{2}{x^2}y(x) = \frac{1}{x},$$

where $P(x) = \frac{2}{x^2}$ and $Q(x) = \frac{1}{x}$.

To solve this, we then multiply by $e^{\int P(x)dx}$, giving us:

$$e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx} = Q(x)e^{\int P(x)dx}.$$

Note that the second term is $\frac{d}{dx}\left(e^{\int P(x)dx}\right)$, thus by the product rule, this becomes:

$$\frac{d}{dx}\left(e^{\int P(x)dx}\right) = Q(x)e^{\int P(x)dx}.$$

If we call $\mu(x) = e^{\int P(x)dx}$ the **integrating factor** for the ODE, we can express this as:

$$\frac{d(\mu y)}{dx} = \mu Q \implies \mu y = \int \mu Q dx + C \implies y = \frac{1}{\mu}\left(\int \mu Q dx + C\right).$$

3.1.1 Steps for Solving $a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$

1. Change to standard form: $P(x) = \frac{a_0(x)}{a_1(x)}$, $Q(x) = \frac{b(x)}{a_1(x)}$.
2. Compute the integrating factor: $\mu(x) = e^{\int P(x)dx}$.
3. Plug into formula: $y(x) = \frac{1}{\mu(x)}\left(\int \mu(x)Q(x)dx + C\right)$.

Example 3.4

Returning to the previous example, considering $x^2y'(x) + 2y(x) = x$, we have:

- $P(x) = \frac{a_0(x)}{a_1(x)} = \frac{2}{x^2}$
- $Q(x) = \frac{b(x)}{a_1(x)} = \frac{1}{x}$

We now calculate the integral factor:

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{2}{x^2}dx} = e^{-\frac{2}{x}}.$$

Plugging into the formula, we get:

$$y(x) = \frac{1}{e^{-\frac{2}{x}}}\left(\int e^{-\frac{2}{x}}\frac{1}{x}\,dx + C_1\right).$$

Example 3.5

Now consider $x^2y'(x) + 2y(x) = 1$, following the same steps, we get:

$$y(x) = \frac{1}{e^{-\frac{2}{x}}}\left(\int e^{-\frac{2}{x}}\frac{1}{x^2}\,dx + C_1\right) = \frac{1}{e^{-\frac{2}{x}}}\left(\frac{1}{2}e^{-\frac{2}{x}} + C_1\right).$$

Example 3.6

$$\frac{dT}{dt} = -h(T - T_R) \implies \frac{dT}{dt} + hT = hT_R,$$

which can solved with the linear method. $P(t) = h$, $Q(t) = hT_R$, giving us:

$$\mu(t) = e^{\int hdt} = e^{ht} \implies T(t) = \frac{1}{e^{ht}}\left(\int e^{ht}hT_R\,dt + C_1\right)$$

$$T(t) = e^{-ht}\left(T_Re^{ht} + C_1\right) = T_R + C_1e^{-ht}.$$

Remark 3.7 — How to determine which method to use. Bring everything to one side:

$$\frac{dy}{dx} = F(x, y).$$

- If $F(x, y) = f(x)g(y)$, we can use the separable method.
- If $F(tx, ty) = F(x, y)$, we can use the homogeneous method.
- If $F(x, y) = -P(x)y + Q(x)$, then we can use the linear method.
- If $F(x, y) = -P(x)y + Q(x)y^m$, we can use the Bernoulli method.

3.1.2 Bernoulli Equation

Definition 3.8. A Bernoulli Equation is an equation of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^m,$$

for some number m.

Example 3.9

Giving initial condition $v(0) = 0$, solve v where:

$$\frac{dv}{dx} + \frac{1}{x}v = gv^{-1},$$

which is of the form of a Bernoulli Equation.

To solve the Bernoulli equation, we set $y = z^\lambda$ and choose λ so that the ODE for z is easier to solve than the ODE for y. This is because we'd get:

$$\frac{dy}{dx} + P(x)y = Q(x)y^m$$

$$\implies \frac{dz^\lambda}{dx} + P(x)z^\lambda = Q(x)(z^\lambda)^m$$

$$\implies \lambda z^{\lambda-1}\frac{dz}{dx} + P(x)z^\lambda = Q(x)z^{m\lambda}.$$

Dividing by λz^λ:

$$\implies \frac{dz}{dx} + \frac{1}{\lambda}P(x)z = \frac{1}{\lambda}Q(x)z^{m\lambda+1-\lambda}.$$

Thus we want to choose λ so that $m\lambda + 1 - \lambda = 0 \implies \lambda = \frac{1}{1-m}$ where $m \neq 1$.

If $m = 1$, then it is a separable equation, meaning that we have:

$$\frac{dy}{dx} = (Q(x) - P(x))\, y.$$

$$\frac{dy}{y} = (Q(x) - P(x))\, dx \implies y(x) = Ae^{\int (Q(x)-P(x))\ dx}.$$

3.1.3 Summary for Solving Bernoulli Equation

Consider

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)y^m.$$

1. First change to standard form with: $P(x) = \frac{a_0(x)}{a_1(x)}$, $Q(x) = \frac{b(x)}{a_1(x)}$
2. If $m = 1$, then, for some constant A, we have:

$$y(x) = Ae^{\int (Q(x)-P(x))dx}.$$

3. Otherwise, compute the integrating factor:

$$\mu(x) = e^{\int (1-m)p(x)dx}.$$

4. Giving us the equation:

$$y(x) = \left(\frac{1}{\mu(x)}\left(\int (1-m)\mu(x)Q(x)\ dx\right) + C\right)^{\frac{1}{1-m}}.$$

Remark 3.10 — Note that the linear case is when $m = 0$, which gives us the equation what we have before.

Example 3.11

Returning to our example earlier where we were considering $\frac{dv}{dx} = \frac{1}{x}v = gv^{-1}$, we have $P(x) = \frac{1}{x}$, $Q(x) = g$. Thus the integrating factor is:

$$\mu(x) = e^{\int (1-(-1))\frac{1}{x}\ dx} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2.$$

Thus we have:

$$\begin{aligned} v(x) &= \left(\frac{1}{x^2}\left(\int (1-(-1))x^2 g\ dx + C_1\right)\right)^{\frac{1}{1-(-1)}} \\ &= \left(\frac{1}{x^2}\left(\frac{2}{3}gx^3 + C_1\right)\right)^{\frac{1}{2}} \\ &= \sqrt{\frac{2gx}{3} + \frac{C_1}{x^2}}. \end{aligned}$$

Since $v(x) = 0 \implies C_1 = 0$, thus:

$$v(x) = \sqrt{\frac{2gx}{3}}.$$

4 January 29th, 2020

4.1 Phase Plot

Let us consider ODE's of the form:

$$\frac{dx}{dt} = f(x) = \dot{x}.$$

If we graph x vs $\dot{x}$ we can get a phase plot, for example:

Definition 4.1. A point where $f(x) = 0$ is called an **equilibrium point**. These equilibrium points can be unstable (empty circle), stable (filled circle), or left/right stable (half filled circle).

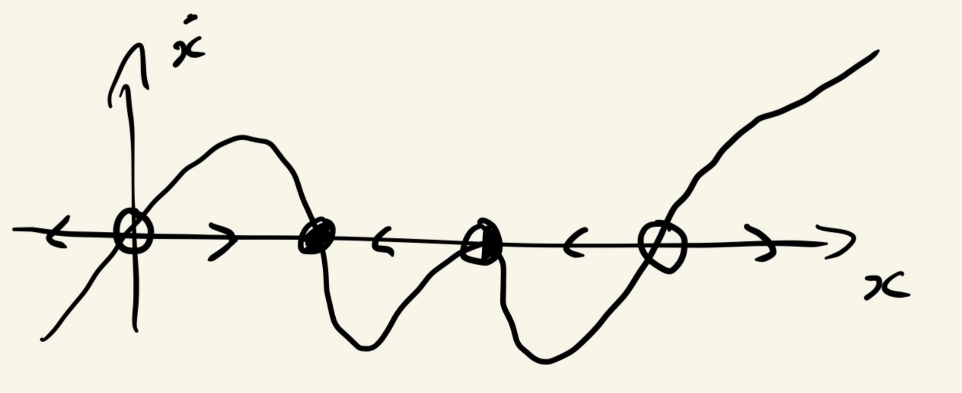

Figure 1: Phase plot of $\dot{x} = x(x-1)(x-2)^2(x-3)^3$

4.2 Computing Times

Since $\dot{x} = f(x)$, is separable, since $dt = \frac{dx}{f(x)}$, we have:

$$\int_{t_1}^{t_2} dt = \int_{x_1}^{x_2} \frac{dx}{f(x)} \implies t_2 - t_1 = \int_{x_1}^{x_2} \frac{dx}{f(x)}.$$

Which is the time interval between when $x = x_1$ and $x = x_2$.

Example 4.2

Let us try to compute the period of an object with mass m to travel from one end of a bowl to the other with radius R. TL;DR we get:

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{R}\cos(\theta)}.$$

Rearranging gives us:

$$dt = \sqrt{\frac{R}{2g\cos\theta}}d\theta \implies \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\cos(\theta)}} \approx \sqrt{\frac{R}{2g}}5.244.$$

4.3 Exact Equations

Whenever you have a function of form $\frac{dy}{dx} = F(x,y)$, you can always rewrite it in the form:

$$M(x,y)dx + N(x,y)dy = 0.$$

This might look familiar, as if we have $f(x,y) = C$, we have:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0.$$

As such, we'd like to ask when can $M(x,y)dx + N(x,y)dy = 0$ be written as $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$. It would be great if $M = \frac{\partial f}{\partial x}$ and $N = \frac{\partial f}{\partial y}$, so it's helpful to know when we can do this.

Consider

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

As such, if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

then $Mdx + ndy = 0$ is called exact.

Example 4.3

$2xydx + (x^2 - y^2)dy = 0$ is exact.

Example 4.4

$2x^2ydx + (x^3 - y^2)dy = 0$ is not exact.

Note that the two examples differ by a factor x, meaning that we have a further condition to determine whether something is exact.

5 January 31st, 2020

5.1 Problem 1

Find period of motion for the equation:

$$\dot{\theta} = \sqrt{\frac{g}{L}(3 + 2\cos\theta)} \quad 0 \leq \theta \leq 2\pi.$$

Since the RHS only has θ, this is separable, thus:

$$\int dt = \sqrt{\frac{L}{g}} \int \frac{d\theta}{\sqrt{3 + 2\cos(\theta)}}$$

Note that the RHS gives us an elliptical equation. Since we want the period, we have:

$$T = \sqrt{\frac{L}{g}} \int_0^{2\pi} \frac{d\theta}{3 + 2\cos\theta} + C.$$

We can consider C to be the start time, and just set it to 0. This is as far as you can go analytically, so plug it into a calculator.

5.1.1 How to use in MATLAB

```
T = integral(@(theta)1./sqrCos(1,theta),2,2*pi)
tspan = [0 2.5];
y0 = 0;
data = ode45(@sqrCos,tspan,y0);

function res = sqrCos(t,theta)
  L = 2,4;
  g = 9,8;
  res = sqrt(g/L*(3+2*cos(theta)));
end(function)
```

5.2 Problem 3

Consider the equation

$$v\frac{dv}{dx} + \frac{v^2}{x+\frac{m}{\rho}} = g.$$

With the initial condition: $v_0 = v(x_0) = v(0) = 0$. To solve for $v(x)$, note that this is a Bernoulli equation:

$$\frac{dv}{dx} + \frac{1}{x+\frac{m}{\rho}}v = g^{v-1}.$$

with:

$$p(x) = \frac{1}{x+\frac{m}{\rho}} \quad Q(x) = g \quad n = -1.$$

Plugging into the formula, we have:

$$V(x) = \left(\frac{1}{\mu(x)}\left(\int (1-n)\mu(x)Q(x)dx + C\right)\right)^{\frac{1}{1-n}}.$$

Calculating the integrating factor, we have:

$$\mu(x) = e^{\int(1-n)P(x)dx} = e^{2\ln(x+\frac{m}{\rho})} = \left(x+\frac{m}{\rho}\right)^2.$$

Thus we have:

$$V(x) = \left(\frac{1}{\left(x+\frac{m}{\rho}\right)^2}\left(2\int\left(x+\frac{m}{\rho}\right)^2 g\ dx + C\right)\right)^{\frac{1}{2}}$$

$$V(x) = \left(\frac{1}{\left(x+\frac{m}{\rho}\right)^2}\left(\frac{2}{3}\left(x+\frac{m}{\rho}\right)^3 + C\right)\right)^{\frac{1}{2}} = \frac{1}{x+\frac{m}{\rho}}\sqrt{\frac{2}{3}\left(x+\frac{m}{\rho}\right)^3 g + C}.$$

Plugging in the initial condition, we get: $C = -\frac{2}{3}\frac{m^3}{\rho^3}g$, giving us:

$$v(x) = \frac{1}{x+\frac{m}{\rho}}\sqrt{\frac{2}{3}\left(x+\frac{m}{\rho}\right)^3 g - \frac{2}{3}\left(\frac{m}{\rho}\right)^3 g}.$$

The acceleration is:

$$g - \frac{v^2}{x+\frac{m}{\rho}}.$$

6 February 3rd, 2020

6.1 Exact Equations

Remember that an exact equation is one where:

$$Mdx + Ndy = 0.$$

Where:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Consider the exact equation:

$$(y^2 - x^2)dx + 2xydy = 0.$$

To solve this exact ODE, we set:

$$\frac{\partial f}{\partial x} = M = y^2 - x^2 \implies \int_x (y^2 - x^2)dx + c_1(y) \implies f(x,y) = y^2 x - \frac{x^3}{3} + c_1(y).$$

Now if we take the partial with respect to y, we get:

$$\frac{\partial f}{\partial y} = 2yx + c_1'(y) = N = 2xy \implies c_1'(y) = 0 \implies c_1(y) = c_2.$$

This tells:

$$f(x,y) = y^2 x - \frac{1}{3}x^3 + c_2$$

satisfies both equations meaning that the solution to our ODE is of the form:

$$f(x,y) = xy^2 - \frac{1}{3}x^3 = C.$$

If we have an initial condition, then this will give us a unique solution.

Example 6.1

Consider the equation: $2xy^2dx + (2x^2y - y^3)dy = 0$. To solve this, we do the following:

$$\int_x 2xy^2 \; dx = x^2y^2 + c_1(y) \implies 2x^2y + c_1'(y) = 2x^2y - y^3 \implies c_1 = -\frac{y^4}{4}$$

Thus we have:

$$f(x,y) = 2x^2y^2 - \frac{1}{4}y^4 + C.$$

6.2 Inexact Equations

If $Mdx + Ndy = 0$ is not exact, then we try to introduce an integrating factor $\mu(x,y)$ to turn make $\mu Ndx + \mu Ndy = 0$. Thus we want:

$$\frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x}.$$

However this is usually as difficult to solve as the original equation. There are some special cases though:

- $\mu(x,y) = \mu(x)$. If this is the case, we have:

$$\frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x} \implies \mu \frac{\partial M}{\partial y} = \frac{d\mu}{dx} N + \mu \frac{\partial N}{\partial x}$$

$$\implies \mu\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\mu'(x)N \implies \frac{\mu'(x)}{\mu(x)}=\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}$$

and if the RHS is a function of only x, we can integrate, giving us:

$$\mu(x)=\exp\left\{\int\frac{\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)}{N}dx\right\}.$$

With this, we will be able to solve the differential equation with $\frac{\partial f}{\partial x}=\mu M$ and $\frac{\partial f}{\partial y}=\mu N$. This is true if:

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=k(x).$$

i.e. it's a function of only x

- $\mu(x,y)=\mu(y)$. Same thing but with y instead of x. We check if: $\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}$ is a function of only y. We will have:

$$\mu(y)=\exp\left\{\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}\right\}.$$

Example 6.2

Consider the equation $2xydx+(2x^2-y^2)dy=0$. Note that this is not exact. As such, we check:

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{2x-4x}{2x^2-y^2}=\frac{2x}{2x^2-y^2}\neq \text{ a function of only } x\ .$$

$$\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=\frac{4x-2x}{2xy}=\frac{1}{y}.$$

Thus we have:

$$\mu(y)=e^{\int\frac{1}{y}dy}=e^{\ln y}=y.$$

Example 6.3

Consider $\frac{dy}{dx} = \frac{2x^2-y^2}{3xy}$, rearranging gives us:

$$(x^2 - 2y^2)dx + 3xydy = 0.$$

Note that $\frac{\partial M}{\partial y} = -4y$ and $\frac{\partial N}{\partial x} = 3y$, thus it is not exact. Now we try:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-4y - 3y}{3xy} = \frac{-7}{3x}.$$

Which is a function of only x. As such, we have:

$$\mu(x) = e^{\int -\frac{7}{3x}dx} = x^{-\frac{7}{3}}.$$

Multiplying this in gives us:

$$(x^{-\frac{1}{3}} - 2x^{-\frac{7}{3}}y^2)dx + 3x^{-\frac{4}{3}}ydy = 0,$$

which is exact since:

$$\frac{\partial M}{\partial y} = -4x^{-\frac{7}{3}}y \quad \frac{\partial N}{\partial x} = -4x^{-\frac{7}{3}}y.$$

Solving this gives us:

$$f(x,y) = \int_x x^{-\frac{1}{3}} - 2x^{-\frac{7}{3}}y^2dx = \frac{3}{2}x^{\frac{2}{3}} + \frac{3}{2}x^{-\frac{4}{3}}y + c_1(y).$$

$$\frac{3}{2}x^{-\frac{4}{3}}y + c_1'(y) = \frac{3}{2}x^{-\frac{4}{3}}y \implies c_1 = C.$$

Thus

$$f(x,y) = \frac{3}{2}x^{\frac{2}{3}} + \frac{3}{2}x^{-\frac{4}{3}}y^2 = C.$$

7 February 5th, 2020

7.1 Applications

Given the family of curves $u(x,y) = c_1$, the family of curves orthogonal to these are the solution to:

$$\frac{\partial u}{\partial x}dy = \frac{\partial u}{\partial y}dx.$$

7.1.1 2nd-Order ODE

Definition 7.1. The general form of a 2nd order differential equation is:

$$y'' = F(x, y, y').$$

Where x is the independent variable and y is the dependent variable.

We want to consider a few special cases. The first one is when the dependent variable is missing, $y'' = f(x, y')$, for example $y'' = x - y'$. In this case, you can set $v = y'$ $v' = y''$, giving us:

$$v' = f(x, v)$$

which is a first order equation. Thus we can solve the first order ODE and then integrate to get y.

Example 7.2

Consider the earlier equation $y'' = x - y'$, we have:

$$v' = x - v \implies \frac{dv}{dx} + v = x$$

$$v = e^{-x}((x-1)e^x + c_1) = x - 1 + c_1 e^{-x} = \frac{dy}{dx}.$$

$$y = \frac{1}{2}x^2 + x + c_2 e^{-x} + c_3.$$

for some constants c_2 and c_3.

Remark 7.3 — Note that for a first order ODE, there should be one arbitrary constant, but for second order, there should be 2.

The second case is where the independent variable is missing, meaning:

$$\frac{d^2y}{dx^2} = F(y, \frac{dy}{dx}) \implies \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = v\frac{dv}{dy} = F(y, v).$$

Where v is once again $\frac{dy}{dx}$. Using this, we can solve for v in terms of y and then integrate twice.

8 February 10th, 2020

8.1 Review from MATH 240

Consider the homogenous equation:

$$a_2(x)y''(x) + a_1 y'(x) + a_0(x)y(x) = 0.$$

Note that this is homogeneous since $y(x) = 0$ is a valid solution. A general solution to a 2nd order lienar homogeneous ODE can be expressed as

$$y(x) = c_1 y_1(x) + c_2 y_2(x),$$

where c_1 and c_2 are arbitrary constrants and $y_1(x)$ and $y_2(x)$ are any two solutions to the ODE for which:

$$\underbrace{y_1(x)y_2'(x) - y_1'(x)y_2(x)}_{\text{Wronskian of } y_1 \text{ and } y_2} \neq 0.$$

Note that the LHS can be experssed as a determinant:

$$\det\left(\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix}\right).$$

Which is known as the **Wronskian** of y_1 and y_2.

Example 8.1

Consider $y''(x) - 3y'(x) + 3y(x) = 0$, we have:

$$y_1(x) = e^x \quad y_2(x) = e^{2x}.$$

and

$$\det\left(\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\right) = \det\left(\begin{bmatrix} e^x & e^{2x} \\ e^x & e^{2x} \end{bmatrix}\right) = 2e^{3x} - e^{3x} = e^{3x} \neq 0.$$

Meaning that the solution is of the form:

$$y(x) = c_1 e^x + c_2 e^{3x}.$$

Remark 8.2 — Note that we only need the Wronskian to not be the 0 function, and that it's ok for certain values of x for the Wronkian to be 0.

Example 8.3

If we used $y_1(x) = e^x$ and $y_2(x) = 2e^x$, then we'd get a Wronskian equal to 0, which would not work.

8.2 Constant Coefficients

Consider:

$$ay''(x) + by'(x) + cy(x) = 0,$$

where a, b and c are constants.

Example 8.4

Example 8.1 is an example of a constant equation with $a = 1$, $b = -3$, and $c = 2$.

Let us create a table to help us solve this problem. First we contstruct the descriminant: $D = b^2 - 4ac$. Depending on what value D is, we have:

Table 1: Table to Compute $ay'' + by' + cy = 0$

D	$y_1(x)$	$y_2(x)$	
$D < 0$	$e^{\alpha x}\cos(\beta x)$	$e^{\alpha x}\sin(\beta x)$	$\alpha = -\frac{b}{2a}$ $\beta = \sqrt{-D}/2a$
$D = 0$	$e^{\alpha x}$	$xe^{\alpha x}$	$\alpha = -\frac{b}{2a}$
$D > 0$	$e^{\alpha x}\cosh(\gamma x)$	$e^{\alpha x}\sinh(\gamma x)$	$\alpha = -\frac{b}{2a}$ $\gamma = \sqrt{D}/2a$
	$e^{(\alpha-\gamma)x}$	$e^{(\alpha+\gamma)x}$	$\alpha = -\frac{b}{2a}$ $\gamma = \sqrt{D}/2a$

Example 8.5

Consider $4y'' + y' + y = 0$. The discrminant is $D = b^2 - 4ac = -15 < 0$. Using the first row of the previous table, we have:

$$\alpha = -\frac{1}{8} \quad \beta = \frac{\sqrt{15}}{8}.$$

This means that:

$$y(x) = c_1 e^{-\frac{1}{8}x}\cos\left(\frac{x\sqrt{15}}{8}\right) + c_2 e^{-\frac{1}{8}x}\sin\left(\frac{x\sqrt{15}}{8}\right).$$

Example 8.6

Consider $4y'' + 4y' + y = 0$. Note that $D = b^2 - 4ac = 16 - 16 = 0$, thus using the second row, we have:

$$\alpha = -\frac{b}{2a} = -\frac{4}{8} = -\frac{1}{2}.$$

Thus we have:

$$y(x) = c_1 e^{-\frac{1}{2}x} + c_2 x e^{-\frac{1}{2}x}.$$

Example 8.7

Consider $y'' - 3y' + 2y = 0$, note that $D > 0$. We can either use the third or 4th row of the table. First we have:

$$\alpha = \frac{3}{2} \quad \gamma = \frac{1}{2}.$$

Thus we can write this as either:

$$y(x) = c_1 e^{\frac{3}{2}x}\cosh(\frac{1}{2}x) + c_2 e^{\frac{3}{2}x}\sinh(\frac{1}{2}x)$$

or

$$y(x) = c_1 e^{(\frac{3}{2}-\frac{1}{2})x} + c_2 e^{(\frac{3}{2}+\frac{1}{2})x} = c_1 e^{x} + c_2 e^{2x}.$$

8.3 Cauchy-Euler/Equidimentional Equation

Definition 8.8. A equidimensional or Cauchy-Euler 2nd order linear homogenous ODE is a equation of the form:

$$ax^2 y''(x) + bxy'(x) + cy(x) = 0.$$

for some constant a, b, c.

Remark 8.9 — Note that the exponent of the x matches the derivative of y.

Again, we can just use a table to solve these equations by checking the value of

$$D = (b-a)^2 - 4ac.$$

Table 2: Table to solve Cauchy-Euler Equations

D	$y_1(x)$	$y_2(x)$	
$D < 0$	$\lvert x\rvert^\alpha \cos(\beta \ln \lvert x\rvert)$	$\lvert x\rvert^\alpha \sin(\beta \ln \lvert x\rvert)$	$\alpha = -\frac{b-a}{2a}$ $\beta = \sqrt{-D}/2a$
$D = 0$	$\lvert x\rvert^\alpha$	$\lvert x\rvert^\alpha \ln \lvert x\rvert$	$\alpha = -\frac{b-a}{2a}$
$D > 0$	$\lvert x\rvert^\alpha \cosh(\gamma \ln \lvert x\rvert)$	$\lvert x\rvert^\alpha \sinh(\gamma \ln \lvert x\rvert)$	$\alpha = -\frac{b-a}{2a}$ $\gamma = \sqrt{D}/2a$
	$\lvert x\rvert^{\alpha-\gamma}$	$\lvert x\rvert^{\alpha+\gamma}$	$\alpha = -\frac{b-a}{2a}$ $\gamma = \sqrt{D}/2a$

Example 8.10

Consider $3x^2y'' + 2xy' + 5y = 0$, where $a = 3, b = 2, c = 5$. Note that:

$$d = (b-a)^2 - 4ac = (2-3)^2 - 4(3)(5) = -59 < 0.$$

Thus we have:

$$\alpha = -\frac{b-a}{2a} = \frac{1}{6}, \quad \beta = \frac{\sqrt{59}}{6}.$$

Thus we have:

$$y(x) = c_1 x^{\frac{1}{6}} \cos(\frac{\sqrt{59}}{6} \ln x) + c_2 x^{\frac{1}{6}} \sin(\frac{\sqrt{59}}{6} \ln x).$$

for $x > 0$.

Example 8.11

Consider $x^2y'' + 2xy' - 2y = 0$, $x > 0$, i.e. $a = 1, b = 2, c = -2$. Note that $D = (b-a)^2 - 4ac = 9 > 0$, thus we have:

$$\alpha = \frac{-(2-1)}{2(1)} = -\frac{1}{2} \quad \gamma = \frac{\sqrt{9}}{2(1)} = \frac{3}{2}.$$

With the general solution being:

$$y(x) = c_1 x^{-\frac{1}{2}} \cosh(\frac{3}{2} \ln x) + c_2 x^{-\frac{1}{2}} \sinh(\frac{3}{2} \ln x).$$

or

$$y(x) = c_1 x^{-2} + c_2 x.$$

8.4 Other Stuff from Math 240

If we once again consider the equation $a_2(x)y''(x) + a_1(x)y'(x) + a_0y(x) = 0$. Note that as we mentioned earlier, we need two linearly independent non-zero solutions to get the general solution. If we only have one, say $y_1(x)$, a second linearly independent solution $y_2(x)$ can be constructed using **Abel's equation** :

$$y_2(x) = Ay_1(x)\int \frac{e^{-\int \frac{a_1(x)}{a_2(x)}\,dx}}{y_1^2(x)}dx.$$

For any non-zero constant A.

Remark 8.12 — Derivation is in the notes.

Example 8.13

Consider $xy'' + (1-x)y' - y = 0$. Suppose we're told that one solution is $y_1(x) = e^x$. A second solution would be:

$$y_2(x) = Ae^x\int \frac{e^{-\int \frac{1-x}{x}\,dx}}{(e^x)^2}\,dx.$$

$$= Ae^x\int \frac{e^{\int 1-\frac{1}{x}\,dx}}{e^{2x}}\,dx = Ae^x\int \frac{e^{x-\ln x}}{e^{2x}}\,dx = Ae^x\int \frac{e^{-x}}{x}\,dx.$$

Which doesn't have a nice answer (oops)

Remark 8.14 — Note that whenever $a_2(x) + a_1(x) + a_0(x) = 0$, one solution is always $y_1(x) = e^x$, since we'd have $y'' = y' = y = e^x$.

Example 8.15

Consider $(1-x)y'' + xy' - y = 0$. Since we have $y_1(x) = e^x$, we have:

$$y_2(x) = Ae^x\int \frac{e^{-\int \frac{x}{1-x}\,dx}}{(e^x)^2}\,dx.$$

$$y_2(x) = Ae^x\int \frac{e^{x+\ln(x-1)}}{e^{2x}}\,dx = Ae^x\int \frac{x-1}{e^x}\,dx = Ae^x(-xe^{-x}) = -Ax.$$

Picking $A = -1$, we have: $y_2(x) = x$, thus the general solution would be:

$$y(x) = c_1e^x + c_2x.$$

8.5 Non-Homogeneous Equations

Now let's consider non-homogeneous equations:

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0y(x) = b(x).$$

A general solution is given by:

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x).$$

Where c_1, c_2 are arbitrary constants, y_1, y_2 are two linearly independent solutions to the homogeneous equation (where $b(x) = 0$), and y_p is any **particular solution** to the non-homogeneous equation.

When $\frac{b(x)}{a_0(x)} =$ a constant, then $y_p(x) = \frac{b(x)}{a_0(x)}$ works, otherwise:

$$y_p(x) = \int^x G(t,x) \frac{b(t)}{a_2(t)}\, dt.$$

Where:

$$G(t,x) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{y_1(t)y_2'(y) - y_1'(t)y_2(t)}.$$

Remark 8.16 — $G(t,x)$ is known as the **Green's function** associated with the ODE.

Remark 8.17 — When solving the integral, treat all x's as constant, then afterwards, replace all t 's with x 's.

Example 8.18

Consider the equation solved in 8.15 but with $b(x) = (x-1)^2$, i.e.:

$$(1-x)y'' + xy' - y = (x-1)^2.$$

We have:

$$\begin{aligned} y_1(x) &= e^x \quad y_2(x) = x. \\ y_1'(x) &= e^x \quad y_2(x) = 1. \end{aligned}$$

Thus we have:

$$G(t,x) = \frac{e^t x - e^x t}{e^t(1) - e^t t} = \frac{x - te^{x-t}}{1-t}.$$

Thus we have:

$$y_p(x) = \int^x \frac{x - te^{x-t}}{1-t} \frac{(t-1)^2}{1-t}\, dt = \int^x x - te^{x-t}\, dt = xt - e^x(t-1)e^{-t}\Big|^x = x^2 - x + 1.$$

9 February 12th, 2020

9.1 Example Mass/Spring/Damper System

We have a mass $m > 0$ attached to a spring with spring coefficient $k > 0$ and a dampener with coefficient $b \geq 0$. If we assume no coefficient of friction, we get

$$-k - x - b\dot{x} = m\ddot{x}.$$

Which can be rearranged to:

$$m\ddot{x} + b\dot{x} + kx = 0.$$

Which is a 2nd-order linear homogeneous ODE with constant coefficients, which we can use the table from earlier to solve. If we include an external force acting on the mass, we would have:

$$m\ddot{x} + b\dot{x} + kx = F(t) \tag{3}$$

Which would make it non homogeneous. There is an analog circuit equivalent called the LCR circuit, which would have an equation:

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = \Delta V.$$

Which is of the same form as Equation 3.

Let us consider the case without a driving force $F(t)$:

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0.$$

First, we will denote $\omega = \sqrt{\frac{k}{m}}$ which represents the **angular frequency** of the system, with units rad per sec, and $\gamma = \frac{b}{2\sqrt{mk}}$ be a **dampening ratio** (which represents how much dampening is in the system), making the equation:

$$\ddot{x} + 2\gamma\omega\dot{x} + \omega^2 x = 0.$$

Note that discriminant of this equation is:

$$D = \frac{b^2}{m^2} - 4\frac{k}{m} = \frac{4k}{m}\left(\frac{b^2}{4\sqrt{mk}} - 1\right) = 4\omega^2(\gamma^2 - 1).$$

Now depending on what γ and ω are, we can analyze the behaviour of the system.

9.2 No Dampening ($\gamma = 0$)

In this case, we would have:

$$\ddot{x} + \omega^2 x = 0.$$

The discriminant is thus:

$$D = 0^2 - 4(1)(\omega^2) = -4\omega^2 < 0.$$

Using the table, we have:

$$\alpha = -\frac{0}{2(1)} = 0, \quad \beta = \frac{\sqrt{-(-4\omega^2)}}{2(1)} = \omega.$$

Thus the solution will just be:

$$x(t) = c_1\cos(\omega t) + c_2\sin(\omega t).$$

If we want to find the constants, note that $x(0) = x_0 = c_1$. Meanwhile, differentiating the equation, we have:

$$v(t) = -c_1\omega\sin(\omega t) + \omega c_2(\omega t).$$

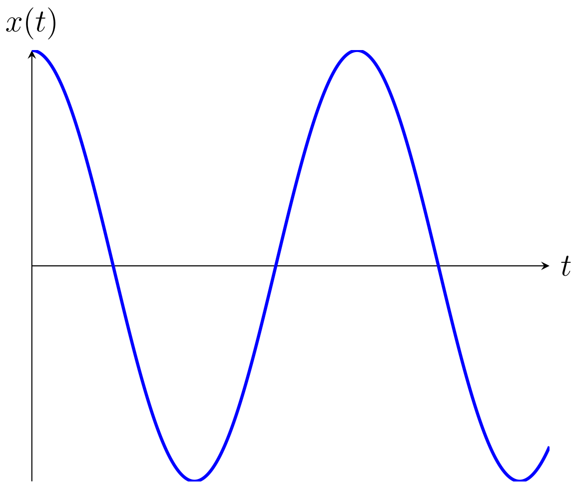

Figure 2: Example of Underdamped Motion

$$v(0) = v_0 = \omega c_2.$$

Thus the complete solution is:

$$x(t) = x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t).$$

This is just a sin curve with amplitude: $A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}$ and period: $T = \frac{2\pi}{\omega}$.

Remark 9.1 — Note that the period does not depend on x_0 or v_0, i.e. it doesn't depend on how it starts. This is different from SHM.

9.3 Under Damping ($0 < \gamma < 1$)

Returning to our equation, we have:

$$\ddot{x} + 2\gamma\omega\dot{x} + \omega^2 x = 0.$$

Thus the determinant is:

$$D = (2\gamma\omega)^2 - 4(1)(\omega^2) = 4w^2(\gamma^2 - 1).$$

If $0 < \gamma < 1$, we have $D < 0$, giving us:

$$\alpha = \frac{-(2\gamma\omega)}{2(1)} = -\gamma\omega, \quad \beta = \frac{\sqrt{-D}}{2(1)} = \omega\sqrt{1-\gamma^2}.$$

Plugging this into the equation, we get:

$$x(t) = c_1 e^{-\gamma\omega t} \cos(\omega t\sqrt{1-\gamma^2}) + c_2 e^{-\gamma\omega t} \sin(\omega t\sqrt{1-\gamma^2}).$$

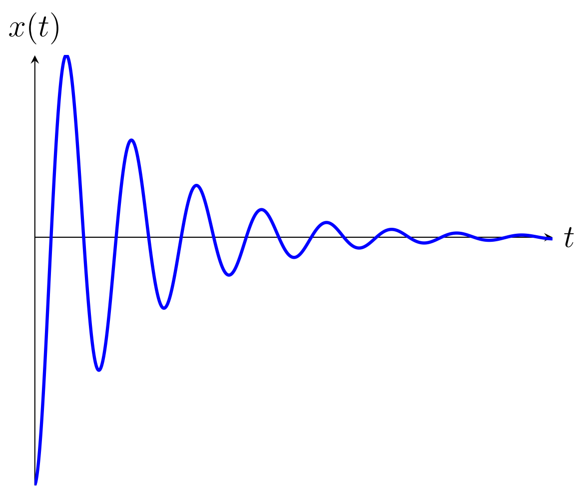

Figure 3: Example of Underdamped Motion

Remark 9.2 — Note that there will be infinite oscillations where the amplitude is decreasing to 0.

9.4 Critical Damping ($\gamma = 1$)

Notice in the case of $\gamma = 1$, we have $D = 0$, thus the solution is:

$$x(t) = c_1 e^{-\gamma\omega t} + c_2 t e^{-\gamma\omega t}.$$

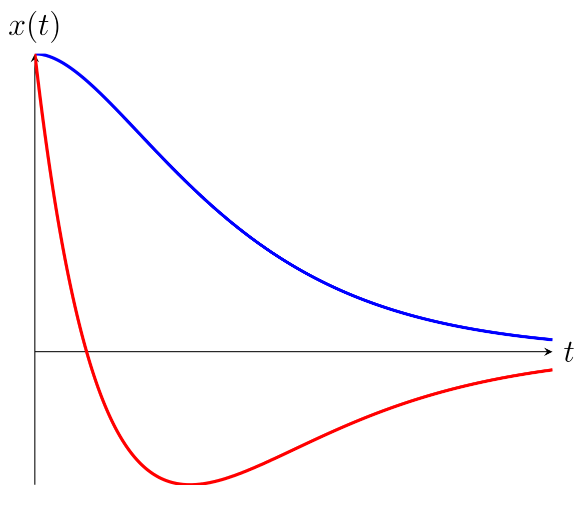

Figure 4: Example of Critical Damped / Over Damped Motion

Remark 9.3 — Note that in this case, there are no oscillations. There will never be two dips. This is because we have:

$$x(t) = c_1 e^{-\gamma\omega t} + c_2 t e^{-\gamma\omega t} = (c_1 + c_2 t)e^{-\gamma\omega t}.$$

Thus by looking at the sign of c_1 and c_2, it will either never cross the x axis (if same sign) or only cross it once (if signs are different). This can be shown by looking at the roots of the equation above.

9.5 Over Damping ($\gamma > 1$)

This yields $D > 0$, thus:

$$x(t) = c_1 e^{-\gamma\omega t}\cosh(\gamma t\sqrt{\gamma - 1}) + c_2 e^{-\gamma\omega t}\sinh(\omega t\sqrt{\gamma^2 - 1}).$$

Remark 9.4 — This is the case where we are taking away the energy a lot, which is useful in many cases. This will make it go to 0 a lot faster than critical damping. Thus for car suspension, we would rather it be critically damped than over damped.

Remark 9.5 — In circuits, this is analogous to using resistors to take away heat from the circuit.

9.6 Laplace Transforms

Laplace transforms are a special case of integral transforms. One way to think of an integral transform is that it's a function where the input is a function of t and output a function of s.

Definition 9.6. More specifically, a **integral transform** is of form:

$$\int_{\alpha(s)}^{\beta(s)} f(t)K(s,t)\,dt.$$

Where $K(s,t)$ is the **kernel** of the transform, and $\alpha(s)$ and $\beta(s)$ are the upper and lower limit.

Example 9.7

Consider the case where $\alpha(s) = s$, $\beta(s) = s^2$, $K(s,t) = st$, and an input $f(t) = t^3$. Then the output would be:

$$\int_s^{s^2} t^3(st)\,dt = \left.\frac{st^5}{5}\right|_{t=s}^{t=s^2} = \frac{1}{5}\left(s^{11} - s^6\right) = F(s).$$

Definition 9.8. Typically, we represent this integral transform as $T\{f(t)\} = F(s)$.

Definition 9.9. The **Laplace Transform** is a special case where:

$$\alpha(s) = 0 \quad \beta(s) = \infty \quad K(s,t) = e^{-st},$$

in other words:

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt = F(s).$$

Remark 9.10 — Note that st must be unitless, and if t represents time, then s represents frequency, thus making the Laplace transform a transformation from time space into frequency space.

Example 9.11
We have
$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \left.\frac{e^{-st}}{-s}\right|_0^\infty = \frac{1}{s}.$$
Note that $s > 0$

In order to go from s-space back to t-space, we take the inverse Laplace transform. This will be unique as long as we don't consider null functions.

Definition 9.12. A **null function** is a function that is zero except for finitely many points.

Example 9.13
An example of a null function is:
$$N(t) = \begin{cases} 1, & t = 0 \\ 2, & t = 1 \\ 0, & \text{otherwise} \end{cases}.$$

These null functions do not appear often for our situation, so we can have a Laplace transform table:

Table 3: Laplace Transform Table

1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$s > 0$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$s > 0$
	$\vdots$	

Remark 9.14 — Using the table, one example is: $\mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\}$

10 February 14th, 2020

10.1 Problem 1 - Solution 1

Consider
$$u''(r) + \frac{1}{r}u'(r) = -H.$$

With constraints:

$$u(a) = T_e \quad |u(0)| < \infty.$$

Let us take $v = u'(r)$, which gives us:

$$v'(r) + \frac{1}{r}v = -H.$$

which is a linear first order ODE, giving us:

$$v(r) = \frac{1}{\mu(r)}\left(\int \mu(r)(-H)dr + C_1\right), \quad \mu(r) = e^{\int \frac{1}{r}dr} = r.$$

Thus:

$$v(r) = \frac{1}{r}\left(\int -rH\ dr + C_1\right) = \frac{1}{r}\left(-\frac{1}{2}r^2H + C_1\right).$$

$$v(r) = -\frac{1}{2}rH + \frac{1}{r}C_1 = u'(r).$$

$$\implies u(r) = \int -\frac{1}{2}rH + \frac{1}{r}C_1\ dr = -\frac{1}{4}r^2H + \ln(r)C_1 + C_2.$$

To solve for constants, we apply initial conditions:

$$|u(0)| = |-\frac{1}{4}(0)^2H + \ln(0)C_1 + C_2| < \infty \implies C_1 = 0.$$

$$\implies u(a) = -\frac{1}{4}Ha^2 + C_2 = T_e \implies C_2 = T_e + \frac{1}{4}Ha^2.$$

Thus we have:

$$u(r) = T_e + \frac{1}{4}H(a^2 - r^2).$$

10.2 Problem 1 - Solution 2

We once again consider $u''(r) + \frac{1}{r}u'(r) = -H$. First we will solve the homogeneous equation:

$$u_h''(r) + \frac{1}{r}u_h'(r) = 0 \implies r^2u_h''(r) + ru_h'(r) = 0.$$

which is equidimensional. As such we just need to find the discriminant with $a = 1, b = 1, c = 0$:

$$D = (b-a)^2 - 4ac = (1-1)^2 - 0 = 0.$$

Using the table, we have:

$$u_1(r) = |r|^\alpha \ln(r) \quad u_2(r) = |r|^\alpha.$$

with

$$\alpha = -\frac{b-a}{2a} = 0.$$

Thus:

$$u_1(r) = \ln(r) \quad u_2(r) = 1.$$

Thus the overall homogeneous solution is:

$$u_h = C_1 \ln(r) + C_2.$$

Now we need to find the particular solution using Green's Function:

$$G(t,r) = \frac{u_1(t)u_2(r) - u_1(r)u_2(t)}{u_1(t)u_2'(t) - u_1'(t)u_2(t)} = \frac{\ln(t) - \ln(r)}{-\frac{1}{t}} = t\ln(r) - t\ln(t).$$

Using this, we have:

$$u_p(r) = \int^r G(t,r)g(t)dt = \int^r (t\ln(r) - t\ln(t))(-H)\ dt.$$

$$= -H\ln(r)\int^r t\ dt + H\int^r t\ln(t)\ dt = \frac{1}{2}r^2 H\ln(r).$$

Integrating by parts, with:

$$u = \ln(t) \quad dv = t\ dt$$

$$du = \frac{1}{t}\ dt \quad v = \frac{1}{2}t^2.$$

we have:

$$\int^r t\ln(t)\ dt = \frac{1}{2}t^2\ln(t) - \int \frac{1}{2}t\ dt = \frac{1}{2}t^2\ln(t) - \frac{1}{4}t^2\Big|_{t=r}.$$

Giving us $u_p(r) = -\frac{1}{4}r^2H$, thus giving us:

$$u_h + u_p = C_1\ln(r) + C_2 - \frac{1}{4}r^2 H.$$

Which is the same as the other solution before plugging in the initial conditions.

10.3 Problem 2

Consider the equation:

$$\ddot{x} + \omega^2 x = \ddot{x} + \frac{g}{L}x = g.$$

where $\omega = \sqrt{\frac{g}{L}}$ and initial conditions:

$$x(0) = 0 \quad \dot{x}(0) = 0.$$

This has constant coefficients, with $a = 1, b = 0, c = \omega^2$, thus the discriminant is:

$$D = b^2 - 4ac = -4\omega^2 < 0.$$

Thus we have:

$$x_1 = e^{\alpha t}\cos(\gamma t) \quad x_2 = e^{\alpha t}\sin(\gamma t).$$

with:

$$\gamma = \frac{\sqrt{-D}}{2a} = \omega \quad \alpha = -\frac{b}{2a} = 0.$$

Thus:

$$x_h = C_1\cos(\omega t) + C_2\sin(\omega t).$$

Now we need a particular solution. Looking back at the original equation, we can guess $x_p = L$. Since:

$$0 + \frac{g}{L}L = g.$$

Because of the existence-uniqueness theorem, this is the only solution that will work, meaning that overall solution before initial conditions is:

$$x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t) + L.$$

Applying initial conditions, we have:

$$x(0) = C_1(1) + C_2(0) + L = 0 \implies C_1 = -L.$$

$$\dot{x}(0) = -L\omega \sin(0) + C_2\omega \cos(0) = 0 \implies C_2 = 0.$$

Thus we have:

$$x(t) = -L\cos(\omega t) + L = L(1 - \cos(\omega t)).$$

With this we can solve for some stuff, for example:

$$x(t_{\frac{1}{2}}) = \frac{L}{2} \implies t_{\frac{1}{2}} = \frac{\pi}{3\omega}.$$

$$x(T) = L \implies T = \frac{\pi}{2\omega}.$$

11 February 17th, 2020

11.1 More Laplace Transform

Remember that the Laplace Transform for a function $f(t)$ is:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\ dt = F(s).$$

There is an associated inverse Laplace transform:

$$\mathcal{L}^{-1}\{F(s)\} = f(t).$$

Which maps frequency space back to time space. If we avoid null functions, this inverse Laplace transform is unique, giving us tables of these pairs such as:

Table 4: Example of $\mathcal{L}$ and $\mathcal{L}^{-1}$ Pair Table

$f(t)$	$F(s)$
$t^m e^{at}$	$\frac{m!}{(s-a)^{m+1}}, \quad s > a$
$\sin(\omega t)$	$\frac{\omega}{\omega^2+s^2}, \quad s > 0$
$\vdots$	$\vdots$

Theorem 11.1
The Laplace transform is linear, i.e.:

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\}.$$

Remark 11.2 — Proof in notes.

Example 11.3

$$\mathcal{L}\{t^3e^{-t} + 4\sin(8t)\} = \mathcal{L}\{t^3e^{-t}\} + 4\mathcal{L}\{\sin(8t)\}.$$
$$= \frac{3!}{(s-(-1))^{3+1}} + 4\left(\frac{8}{8^2+s^2}\right) = \frac{6}{(s+1)^4} + \frac{32}{64+s^2}.$$

Note that the first term has condition $s > -1$ and the second has $s > 0$, meaning that this domain is $s > 0$.

Remark 11.4 — When there are multiple conditions, we take the intersection of the domains.

11.1.1 Limit Theorems

Theorem 11.5 (Limit Theorem)
If $\mathcal{L}\{f(t)\} = F(s)$, we should find:

$$\lim_{s\to\infty} F(s) = 0.$$

with the exception of some impulse functions.

Example 11.6
We have $\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}$. Note that:

$$\lim_{s\to\infty}\left(\frac{s}{s^2+\omega^2}\right) = 0.$$

Remark 11.7 — This can be used as a check, as if you don't get $\lim\limits_{s\to\infty} F(s) = 0$, and you aren't dealing with impulse function, then you did something wrong.

Theorem 11.8 (Endpoint Theorem 1)

$$\lim_{s\to\infty}(sF(s)) = \underbrace{f(0^+)}_{\lim\limits_{t\to 0^+} f(t)}.$$

Example 11.9

Again consider $\mathcal{L}\{\cos(\omega t)\}$. We have:

$$\lim_{s\to\infty} s\left(\frac{s}{s^2+\omega^2}\right) = 1.$$

and

$$\cos(\omega \times t) = 1.$$

Theorem 11.10 (Endpoint Theorem 2)

$$\lim_{s\to 0}(sF(s)) = \underbrace{f(\infty)}_{\lim\limits_{t\to\infty} f(t)},$$

provided it exists.

Remark 11.11 — This allows us to the values of $f(t)$ without having to use the inverse Laplace transform.

Example 11.12

Suppose the Laplace transform of $f(t)$ is:

$$\mathcal{L}\{f(t)\} = \frac{1}{s\sqrt{s^2+1}}.$$

We would like to find out what $f(0)$ and $f(\infty)$ are. Using the endpoint theorem, we have:

$$f(0^+) = \lim_{s\to\infty} s\frac{1}{s\sqrt{s^2+1}} = \lim_{s\to\infty} \frac{1}{\sqrt{s^2+1}} = 0.$$

and

$$f(\infty) \lim_{s\to 0} s\frac{1}{s\sqrt{s^2+1}} = \lim_{s\to 0} \frac{1}{\sqrt{s^2+1}} = 1.$$

11.1.2 Existence of Laplace Transform of $f(t)$

Q: Can we take the integral of anything?
A: No, as the Laplace transform is an improper integral, which must converge.

Example 11.13

Note that

$$\mathcal{L}\{e^{t^2}\} = \int_0^\infty e^{-st}e^{t^2}\, dt = \infty.$$

Thus, $\mathcal{L}\{e^{t^2}\}$ does not have a Laplace transform.

For a function to have a Laplace transform, it must be of exponential order.

Definition 11.14 (Exponential Order)**.** For a function $f(t)$ to be of **exponential order**, there must be a constant α for which:

$$\lim_{t\to\infty} e^{-\alpha t} f(t) = 0.$$

The function is allowed to go to infinity, just not too fast.

11.1.3 Laplace Transforms for Derivatives

Consider the Laplace transform of $f'(t)$ and use integration by parts with:

$$\mathcal{L}\{f'(t)\} = \int_0^\infty \underbrace{e^{-st}}_{u} \underbrace{f'(t)\ dt}_{dv}.$$

$$\underbrace{e^{-st}}_{u} \underbrace{f(t)}_{v}\Big|_0^\infty - \int_0^\infty \underbrace{f(t)}_{v} \underbrace{\left(-se^{-st}\right)\ dt}_{du}$$

$$= \underbrace{e^{-\infty}}_{0} f(\infty) - \underbrace{e^{-0}}_{1} f(0^+) + s\int_0^\infty f(t)e^{-st}dt = s\mathcal{L}\{f(t)\} - f(0^+).$$

Theorem 11.15 (Laplace Transform for Derivatives)

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0^+).$$

Example 11.16

Consider the second derivative:

$$\mathcal{L}\{f''(t)\} = \mathcal{L}\{\frac{d}{dt}f'(t)\} = s\mathcal{L}\{f'(t)\} - f'(0^+) = s\left(s\mathcal{L}\{f(t)\} - f(0^+)\right) - f'(0^+).$$

Theorem 11.17

From the previous example:

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0^+) - f'(0^+).$$

Remark 11.18 — This can be generalized, and as such we have:

$$\mathcal{L}\{f'''(t)\} = s^3\mathcal{L}\{f(t)\} - s^2f(0^+) - sf'(0^+) - f''(0^+).$$

Note that for each of the negative terms, the power of s plus the order of the derivative of f will equal the order of the derivative being computed minus 1, with the s coefficient of $\mathcal{L}\{f(t)\}$ having the same power as the order.

Consider $ay''(t) + by'(t) + cy(t) = g(t)$ with initial conditions $y(0) = y_0$, $y'(0) = y'_0$ and with a, b, c being constant. Instead of solving by setting $g(t) = 0$, let us solve it using Laplace transform.
Let us begin by taking the Laplace transform of both sides:

$$\mathcal{L}\{ay''(t) + by'(t) + cy(t)\} = \mathcal{L}\{g(t)\}.$$

$$\implies a\mathcal{L}\{y''(t)\} + b\mathcal{L}\{y'(t)\} + c\mathcal{L}\{y(t)\} = \mathcal{L}\{g(t)\}.$$

$$\implies a\left(s^2\mathcal{L}\{y(t)\} - sy(0^+) - y'(0^+)\right) + b\left(s\mathcal{L}\{y(t)\} - y(0^+)\right) + c\mathcal{L}\{y(t)\} = \mathcal{L}\{g(t)\}.$$

Thus we have:

$$\mathcal{L}\{y(t)\} = \frac{(as+b)y_0 + ay'_0 + \mathcal{L}\{g(t)\}}{as^2 + bs + c}.$$

With this, we can get $y(t)$ by taking the inverse Laplace transform.

Example 11.19
Consider:

$$y''(t) + 2y'(t) + 3y(t) = t^3 \quad y(0) = 0 \quad y'(0) = 1.$$

With this we have: $a = 1, b = 2, c = 3, y_0 = 0, y'_0 = 1$, and:

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{t^3\} = \frac{3!}{s^{3+1}} = \frac{6}{s^4}.$$

Thus without solving the ODE, we can say that:

$$\mathcal{L}\{y(t)\} = \frac{(s+2)(0) + (1)(1) + \frac{6}{s^4}}{s^2 + 2s + 3} = \frac{s^4 + 6}{s^4(s^2 + 2s + 3)}.$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s^4 + 6}{s^4(s^2 + 2s + 3)}\right\}.$$

11.1.4 Other Properties of Laplace Transforms

Theorem 11.20 (First Shifting Theorem)
If $\mathcal{L}\{f(t)\} = F(s)$, then:

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a).$$

Remark 11.21 — The way to remember this, forget e^{at}, and then whoever we get an s, replace by $s - a$.

Theorem 11.22
If $\mathcal{L}\{f(t)\} = F(s)$, then:

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s).$$

$$\mathcal{L}\{t^m f(t)\} = (-1)^m \frac{d^m}{ds^m}F(s).$$

Remark 11.23 — The way to do this, forget the t, then afterward take the derivative w.r.t. s and negate it.

Example 11.24
We have:

$$\mathcal{L}\{e^{2t}\cos(4t)\} = \mathcal{L}\{\cos(4t)\}\Big|_{s\to s-2}.$$

$$= \frac{s}{s^2+4^2}\Big|_{s\to s-2} = \frac{s-2}{(s-2)^2+16}.$$

Example 11.25
We have:

$$\mathcal{L}\{t\cos(4t)\} = \frac{d}{ds}\mathcal{L}\{\cos(4t)\}.$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2+4^2}\right) = -\frac{d}{ds}\left(\frac{s}{s^2+16}\right).$$

$$= -\left(\frac{(s^2+16)-s(2s)}{(s^2+16)^2}\right) = \frac{s^2-16}{(s^2+16)^2}.$$

Example 11.26
We have:

$$\mathcal{L}\{te^{-t}\sin(t)\} = \mathcal{L}\{t\sin(t)\}\Big|_{s\to s-(-1)}$$

$$= -\frac{d}{ds}\mathcal{L}\{\sin(t)\}\Big|_{s\to s+1} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\Big|_{s\to s+1}$$

$$= \frac{2s}{(s^2+1)^2}\Big|_{s\to s+1} = \frac{2(s+1)}{((s+1)^2+1)^2} = \frac{2s+2}{(s^2+2s+2)^2}.$$

Remark 11.27 — Knowing these two properties, then we can compute Laplace transforms of functions with factors of $t^m e^{at}$.

11.1.5 Unit Step Function

Definition 11.28 (Unit Step Function)**.** The **unit step function** $u_a(t) = u(t-a)$ is defined as:

$$u_a(t) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

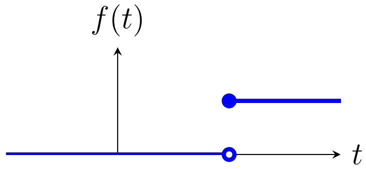

Figure 5: Example of a Unit Step Funciton

The Laplace transform for the unit step function is:

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty u(t-a)e^{-st}\,dt$$

$$= \int_0^a (0)e^{-st}\,dt + \int_a^\infty (1)e^{-st}\,dt = -\frac{e^{-st}}{s}\Big|_a^\infty = \frac{e^{-as}}{s}, \quad s > 0.$$

Remark 11.29 — We can use this for calculating the Laplace transforms for piecewise functions.

Example 11.30

Consider the piecewise function

$$f(t) = \begin{cases} 0, & t < 0 \\ 1, & 0 < t < 2 \\ t, & 2 \leq t \leq 3 \\ e^t, & 3 < t \end{cases}.$$

We can express this as:

$$1u(t) + (t-1)u(t-2) + (e^t - t)u(t-3).$$

Thus for any piecewise function, we can express it as:

$$f(t) = \begin{cases} 0, & t < 0 \\ f_1(t), & 0 < t < t_1 \\ f_2(t), & t_1 < t < t_2 \\ \vdots \\ f_{m+1}(t), & t_m < t \end{cases}$$

$$= f_1(t)u(t) + (f_2(t) - f_1(t))u(t-t_1) + (f_3(t) - f_2(t))u(t-t_2) + \ldots + (f_{m+1}(t) - f_m(t))u(t-t_m).$$

12 February 19th, 2020

12.1 Unit Step Function Continued

As a reminder, the unit step function is defined as:

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}.$$

Given a piecewise function, we can write it as a linear combination of step functions.

Example 12.1

Consider:

$$f(t) = \begin{cases} 7, & 0 < t < 2 \\ 6t, & 2 < t < 3 \\ t^2, & 3 < t < 7 \\ 0, & 7 < t \end{cases}.$$

We can rewrite this as:

$$f(t) = 7u(t) + (6t-7)u(t-2) + (t^2 - 6t)u(t-3) + (0 - t^2)u(t-7).$$

With this, we can take the Laplace transform of the function, but first, we need to consider the Laplace transform of $\mathcal{L}\{f(t)u(t-a)\}$. Looking at the definition, we have:

$$\mathcal{L}\{f(t)u(t-a)\} = \int_0^\infty f(t)u(t-a)e^{-st}\, dt.$$

Assuming $a > 0$, we have:

$$= \int_0^a f(t)\underbrace{u(t-a)}_{0}e^{-st}\, dt + \int_a^\infty f(t)\underbrace{u(t-a)}_{1}e^{-st}\, dt = \int_a^\infty f(t)e^{-st}\, dt.$$

If we set $z = t - a \implies dz = dt$,

$$= \int_0^\infty f(z+a)e^{-s(z+a)}\, dz = e^{-as}\mathcal{L}\{f(t+a)\}.$$

Theorem 12.2 (Shifting Theorem)

As shown above:

$$\mathcal{L}\{f(t)u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}.$$

Example 12.3

Considering $f(t)$ from Example 12.1, we have:

$$\mathcal{L}\{f(t)\} = \int_0^2 7e^{-st}\,dt + \int_2^3 6te^{-st}\,dt + \int_3^7 t^2e^{-st}\,dt + \int_7^\infty 0e^{-st}\,dt.$$

However, we can calculate this another way. From the table, we have $\mathcal{L}\{t^m\} = \frac{m!}{s^{m+1}}$, thus:

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{7u(t) + (6t-7)u(t-2) + (t^2-6t)u(t-3) + (0-t^2)u(t-7)\}$$
$$= \mathcal{L}\{7u(t)\} + \mathcal{L}\{(6t-7)u(t-2)\} + \mathcal{L}\{(t^2-6t)u(t-3)\} + \mathcal{L}\{-t^2u(t-7)\}$$
$$= e^{-0s}\mathcal{L}\{7\} + e^{-2s}\mathcal{L}\{6(t+2)-7\} + e^{-3s}\mathcal{L}\{(t+3)^2 - 6(t+3)\} - e^{-7s}\mathcal{L}\{(t+7)^2\}$$
$$= \frac{7}{s} + e^{-2s}\mathcal{L}\{6t-5\} + e^{-3s}\mathcal{L}\{t^2-9\} - e^{-7s}\mathcal{L}\{t^2+14t+49\}.$$

Thus:

$$\mathcal{L}\{f(t)\} = \frac{7}{s} + e^{-2s}\left(\frac{6}{s^2} + \frac{5}{s}\right) + e^{-3s}\left(\frac{2}{s^3} - \frac{9}{s}\right) - e^{-7s}\left(\frac{2}{s^3} + \frac{14}{s^2} + \frac{49}{s}\right), \quad s > 0.$$

Remark 12.4 — In the example earlier, we are using the Shifting Theorem and replacing t with $t+a$ in each of the functions that we are multiplying by the unit step function at a.

12.2 Examples of Where Unit Step Functions Occur

Example 12.5

Consider the equation:

$$L\frac{dI}{dt} + RI = \epsilon_1 u(t) + (\epsilon_2 - \epsilon_1)u(t-t_1).$$

Taking the Laplace transform of both sides, we get:

$$\mathcal{L}\left\{L\frac{dI}{dt} + RI\right\} = \mathcal{L}\{\epsilon_1 u(t) + (\epsilon_2-\epsilon_1)u(t-t_1)\}.$$

$$\implies L\mathcal{L}\{I'(t)\} + R\mathcal{L}\{I(t)\} = e^{-0s}\mathcal{L}\{\epsilon_1\} + e^{-t_1 s}\mathcal{L}\{\epsilon_2 - \epsilon_1\}$$
$$\implies L\left(s\mathcal{L}\{I\} - I(0)\right) + R\mathcal{L}\{I\} = \frac{\epsilon_1}{s} + e^{-t_1 s}\left(\frac{\epsilon_2 - \epsilon_1}{a}\right).$$
$$\implies \mathcal{L}\{I\} = \frac{LI_0 + \frac{\epsilon_1}{s} + e^{-t_1 s}\left(\frac{\epsilon_2-\epsilon_1}{s}\right)}{LS + R}.$$

There are many applications/cases where using a step function to describe a piecewise function might be useful. For example, if we have a spring with dampener with an external force $F(t)$, we might have $F(t)$ ramp up with t, and then stay constant after a certain amount of time.

Another example is consider a ball bouncing off the ground. The forces are:

$$F(t) = \begin{cases} -mg, & 0 < t < T_F \\ N(t) - mg, & T_F < t < T_F + T_C \\ -mg, T_F + T_C < t < T_F + T_C + T_R \end{cases}.$$

Where T_F is the time until hitting the ground, T_C is the contact duration, and T_R is the time to rebound back up, and $N(t)$ is the normal force. From this, we get figure out $N(t)$ and allow us to get the coefficient of restitution.

12.3 Impulse Function

Consider a function:

$$I_a(t) = \begin{cases} 0, & t < -\frac{a}{2} \\ \frac{1}{a}, & -\frac{a}{2} < t < \frac{a}{2} \\ 0, & \frac{a}{2} < t \end{cases}.$$

This can be expressed in terms of unit step functions as:

$$I_a(t) = \frac{1}{a}u(t + \frac{a}{2}) - \frac{1}{a}u(t - \frac{a}{2}).$$

Remark 12.6 — Note that the area under the curve is 1, as we choose the height to be inversely proportional to the width. This means that:

$$\int_{-\infty}^{\infty} I_a(t)\ dt = 1.$$

Definition 12.7 (Impulse Function)**.** An **impulse function** is:

$$\lim_{a \to 0} I_a(t) = I(t) = \begin{cases} 0, & t \neq 0 \\ +\infty, & t = 0 \end{cases}$$

with the property:

$$\int_{-\infty}^{\infty} I(t)\ dt = 1.$$

Or:

$$\int_R I(t)\ dt = \begin{cases} 0, & 0 \notin R \\ 1, & 0 \in R \end{cases}.$$

Remark 12.8 — With Laplace transform, if 0 is at the end of the domain, it is included, e.g.:

$$\int_0^7 I(t)\ dt = 1.$$

Remark 12.9 — Similar to the step function, we can shift the impulse function, i.e.:

$$I(t-a) = \begin{cases} 0, & t \neq a \\ +\infty, & t = a \end{cases},$$

with:

$$\int_R I(t-a)\ dt = \begin{cases} 0, & a \notin R \\ 1, & a \in R \end{cases}.$$

Example 12.10 (One Dimensional Crystal)
In a one dimensional crystal, we have atoms aligned in a line, and say they are separated by a. If we have an electron travelling along, the force it might see can be expressed as:

$$F(x) = a \sum_{k=-\infty}^{\infty} F_0 I(x - ka) = aF_0 \sum_{k=-\infty}^{\infty} I(x - ka).$$

Thus one way to model the force experienced by an electron is to use a bunch of impulse functions. This is called the **comb function**.

Remark 12.11 — If we had a continuous function $f(t)$, we'd have:

$$\int_R f(t)I(t-a)\ dt = \begin{cases} 0, & 0 \notin R \\ f(a), & a \in R \end{cases}.$$

Example 12.12
If we have:

$$\int_{-1}^{7} \frac{t^2}{\sqrt{3t^3+1}} e^t I(t-1)\ dt = \frac{1^2}{\sqrt{3(1)^3+1}} e^1 = \frac{1}{2}e.$$

Theorem 12.13
We have:

$$\mathcal{L}\{f(t)I(t-a)\}, a > 0 = \int_0^{\infty} f(t)I(t-a)e^{-st}\ dt = f(a)e^{-as}.$$

Example 12.14
$\mathcal{L}\{t^3 I(t-4)\} = 4^3 e^{-4s} = 64e^{-4s}$

Example 12.15

Consider $F(t) = aF\sum_{k=-\infty}^{\infty} I(t-ka)$. We have:

$$\mathcal{L}\{F(t)\} = Fa\sum_{k=-\infty}^{\infty} e^{-kas}.$$

13 February 21st, 2020

13.1 Midterm Notice

- Will be held on Wednesday, March 4th.
- Closed book and notes
- 1.86x11in page of notes and a calculator

13.2 Problem 1

Consider

$$xy''(x) - (1+2x)y'(x) + (1+x)y(x) = a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0.$$

Note that $a_0(x) + a_1(x) + a_2(x) = x - 1 + 2x + 1 + x = 0$, thus, $y(x) = e^x$ is one solution. Using Abel's formula, we have:

$$y_2(x) = y_1(x)\int \frac{1}{y_1(x)^2} e^{-\int(x)\ dx} dx.$$

Where:

$$-P(x) = -\frac{a_1(x)}{a_2(x)} = \frac{1+2x}{x} = 2 + \frac{1}{x}.$$

Thus:

$$y_2(x) = e^x \int \frac{xe^{2x}}{e^{2x}}\ dx = \frac{1}{2}x^2 e^x.$$

Thus:

$$y(x) = C_1 e^x + C_2 x^2 e^x.$$

13.3 Problem 2

For this problem, we have:

$$r^2\phi'' + 2r\phi' - n(n+1)\phi = 0.$$

Note that this is equidimensional. Here we can guess $\phi(r) = r^\alpha$.

13.4 Problem 3

Consider:

$$y'' - 2xy' + (x^2 - 1)y = 0.$$

Here we guess $y(x) = e^{ax^2}$ for some a (should be given by the problem). Thus we have:

$$y'(x) = 2axe^{ax^2} \quad y''(x) = (2a + 4a^2x^2)e^{ax^2}.$$

Plug into the equation, we get:

$$(2a + 4a^2x^2)e^{ax^2} - 2x(2ax)e^{ax^2} + (x^2 - 1)e^{ax^2} = 0, \quad \forall x.$$

$$\implies 2a + 4a^2x^2 - 4ax^2 + x^2 - 1 = 0$$

$$\implies (2a - 1) + (2a - 1)^2x^2 = 0 \implies a = \frac{1}{2}.$$

Giving us $y_1(x) = e^{\frac{1}{2}x^2}$. Thus:

$$P(x) = \frac{a_1(x)}{a_2(x)} = -2x \implies -\int P(x)\ dx = \int 2x\ dx = x^2.$$

$$\implies y_2(x) = e^{\frac{1}{2}x^2} \int \frac{e^{x^2}}{e^{x^2}} \implies y_2(x) = xe^{\frac{1}{2}x^2}.$$

Thus the complete solution is:

$$y(x) = C_1e^{\frac{1}{2}x^2} + C_2xe^{\frac{1}{2}x^2}.$$

13.5 Problem 4

Consider:

$$y'' - 2xy' + (x^2 - 1)y = x^2 - 1.$$

Note that y_h is the equation from before, so we need to find y_p. Note that $\frac{b(x)}{a_0(x)} = 1$ which is a constant, thus $y_p = 1$a is a solution. Thus, the complete solution is:

$$y(x) = C_1e^{\frac{1}{2}x^2} + C_2xe^{\frac{1}{2}x^2} + 1.$$

13.6 Problem 5

Consider the circuit with a battery, a resister and a coil. We have:

$$v(t) - RI(t) - L\frac{dI(t)}{dt} = 0.$$

Taking the Laplace of both sides, we get:

$$\mathcal{L}\{L\frac{dI}{dt} + RI\} = \mathcal{L}\{V\} \implies L\mathcal{L}\{\frac{dI}{dt}\} + R\mathcal{L}\{I\} = \mathcal{L}\{V\}$$

Since $\mathcal{L}\{y'(x)\} = s\mathcal{L}\{y(x)\} - y(0)$

$$\implies Ls\mathcal{L}\{I\} - LI_0 + R\mathcal{L}\{I\} = \mathcal{L}\{V\}$$

$$\implies \mathcal{L}\{I\} = \frac{LI_0 + \mathcal{L}\{V\}}{SL+R}$$

$$\implies I = \mathcal{L}^{-1}\left\{\frac{LI_0 + \mathcal{L}\{V\}}{SL+R}\right\} = I_0\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{R}{L}}\right\} + V.$$

Note that: $\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$. Thus:

$$I(t) = I_0 = e^{-\frac{R}{L}t} + V.$$

Let:

$$V(t) = \begin{cases} V_0, & 0 < t < a \\ 0, & a \leq t \end{cases}, \quad V_0 \in \mathbb{R} = V_0 - V_0 u(t - a_0.$$

Thus:

$$\mathcal{L}\{V(t)\} = \mathcal{L}\{V_0 - V_0 u(t-a\} = V_0\mathcal{L}\{1 - u(t-a)\}.$$

$$= V_0\mathcal{L}\{1\} - V_0\mathcal{L}\{u(t-a)\} = \frac{V_0}{s}(1 - e^{-as}).$$

Thus we have:

$$V(t) = \mathcal{L}^{-1}\left\{\frac{V_0(1-e^{-as})}{s(sL+R)}\right\}.$$

Using the fact that $\mathcal{L}^{-1}\{F(s)e^{-as}\} = u(t-a)\mathcal{L}\{F(x)\}$, we have:

$$= V_0\mathcal{L}^{-1}\left\{\frac{V_0}{s(sL+R)}\right\} - u(t-a)\mathcal{L}\left\{\frac{V_0}{s(sL+r)}\right\}\Bigg|_{t\to t-a}.$$

Using partial fraction, we'd get:

$$= V_0\mathcal{L}^{-1}\left\{\frac{1}{RS} - \frac{L}{R(SL+R)}\right\} = \frac{V_0}{R}\mathcal{L}\left\{\frac{1}{s} - \frac{1}{s+\frac{R}{L}}\right\} = \frac{V_0}{R}(1 - e^{-\frac{R}{L}t}).$$

Thus in the end, we get:

$$I(t) = I_0 e^{-\frac{R}{L}t} + \frac{V_0}{R}\left(1 - e^{-\frac{R}{L}t} - u(t-a)\frac{V_0}{R}\left(1 - e^{-\frac{R}{L}(t-a)}\right)\right).$$

14 February 24th, 2020

14.1 Computing Inverse Laplace Transform

Recall that:

1. $\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\}$
2. $\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}_{s\to s-a}$
3. $\mathcal{L}\{tf(t)\} = -\frac{d}{ds}\mathcal{L}\{f(t)\}$
4. $\mathcal{L}\{t^m f(t)\} = (-1)^m \frac{d^m}{ds^m}\mathcal{L}\{f(t)\}$

5. $\mathcal{L}\{u(t-a)f(t)\} = e^{-as}\mathcal{L}\{f(t+a)\}$

6. $\mathcal{L}\{I(t-a)f(t)\} = e^{-as}f(a)$

Now let's consider how to do compute inverse Laplace Transforms. Consider (5) from the above list, if we replace $f(t)$ by $f(t-a)$, we get:

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}\mathcal{L}\underbrace{\{f(t)\}}_{F(s)}.$$

Taking the inverse on both sides, we get:

Theorem 14.1 (First Shifting Theorem for Inverse Laplace Transforms)

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)\mathcal{L}^{-1}\{F(s)\}\big|_{t\to t-a}.$$

Example 14.2

Consider $\mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{s^2+1}\right\}$. Using the above, we have:

$$\mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{s^2+1}\right\} = u(t-a)\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}_{t\to t-a}.$$

$$= u(t-2)\sin t\big|_{t\to t-2} = u(t-2)\sin(t-2).$$

If we consider (2) from the above, and take the inverse of both sides, we would get:

Theorem 14.3 (Second Shifting Theorem for Inverse Laplace Transforms)

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}\mathcal{L}^{-1}\{F(s)\}.$$

Example 14.4

Suppose we want $\mathcal{L}^{-1}\left\{\frac{1}{2s^2+s+8}\right\}$. We have:

$$\mathcal{L}^{-1}\left\{\frac{1}{2s^2+s+8}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{2(s^2+\frac{1}{2}s)+8}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{2\left(s^2+\frac{1}{2}s+\frac{1}{16}-\frac{1}{16}\right)+8}\right\}$$
$$=\mathcal{L}^{-1}\left\{\frac{1}{2(s+\frac{1}{4})^2+\frac{63}{8}}\right\}.$$

Note that the above is of the form $\mathcal{L}^{-1}\{F(s+\frac{1}{4})\}$, thus we have:

$$=e^{-\frac{1}{4}t}\mathcal{L}^{-1}\left\{\frac{1}{2s^2+\frac{63}{8}}\right\}=\frac{1}{2}e^{-\frac{1}{4}t}\mathcal{L}\left\{\frac{1}{s^2+\frac{63}{16}}\right\}.$$

Using the fact that $\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\}=\frac{1}{a}\sin(at)$, we get:

$$=\frac{1}{2}e^{-\frac{1}{4}t}\frac{1}{\sqrt{\frac{63}{16}}}\sin\left(t\sqrt{\frac{63}{16}}\right)=\frac{2}{\sqrt{63}}e^{-\frac{1}{4}t}\sin\left(\frac{t}{4}\sqrt{63}\right).$$

Remark 14.5 — Essentially above we are using the fact that:

$$as^2+bs+c=a\left(s+\frac{b}{2a}\right)^2+\left(c-\frac{b^2}{4a}\right).$$

Instead of completing the square, this can also be useful in combination with partial fractions.

Example 14.6

From partial fractions, we know that:

$$\frac{s^2+1}{s^3(s-1)^2(s-2)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{D}{s-1}+\frac{E}{(s-1)^2}+\frac{F}{s-2}.$$

In addition, we know that $\mathcal{L}^{-1}\{\frac{1}{(s-a)^m}\}=\frac{1}{(m-1)!}t^{m-1}e^{at}$, thus if we know the coefficients, we have:

$$\frac{s^2+1}{s^3(s-1)^2(s-2)}=A+Bt+\frac{1}{2}Ct^2+De^t+Ete^t+Fe^{2t}.$$

Theorem 14.7 (Heavyside Expansion Theorem)
There is a special case, where all the powers in the denominators $Q(s)$ are to the first power, and numerator $P(s)$, we have:

$$\mathcal{L}\left\{\frac{P(s)}{(s-\alpha_1)(s-\alpha_2)\dots(s-\alpha_n)}\right\} = \sum_{y=1}^{n}\left(\frac{P(\alpha_y)}{Q'(\alpha_y)}\right)e^{\alpha_y t}.$$

Example 14.8
Consider the following:

$$\mathcal{L}\left\{\frac{s^2+1}{s(s-1)(s-2)}\right\} = \frac{P(0)}{Q'(0)}e^{0t} + \frac{P(1)}{Q'(1)}e^{1t} + \frac{P(2)}{Q'(2)}e^{2t}.$$

Where:

$$P(s) = s^2+1 \quad Q(s) = s^3-3s^2+2s \quad Q'(s) = 3s^2-6s+2.$$

Thus:

$$\mathcal{L}\left\{\frac{s^2+1}{s(s-1)(s-2)}\right\} = \frac{1}{2} - 2e^t + \frac{5}{2}e^{2t}.$$

14.2 Convolution Product

Definition 14.9. Given two functions $f(t)$ and $g(t)$, their **convolution product** is:

$$(f*g)(t) = \int_0^t f(\beta)g(t-\beta)\ d\beta.$$

Example 14.10
Let us consider $f(t)=t^2$, $g(t)=t$, we have:

$$(f*g)(t) = \int_0^t \beta^2(t-\beta)\ d\beta = \int_0^t \left(\beta^2 t - \beta^3\right)\ d\beta$$

$$= \frac{1}{3}\beta^3 t - \frac{1}{4}\beta^4\bigg|_{\beta=0}^{\beta=t} = \frac{1}{3}t^4 - \frac{1}{4}t^4 = \frac{1}{12}t^4.$$

Example 14.11
If we have $f(t) = \sin(t)$ and $g(t) = \cos(t)$, we have:

$$(f * g)(t) = \int_b^t \sin\beta \cos(t - \beta)\ d\beta.$$

Using the trig identity:

$$\sin A \cos B = \frac{\sin(A + B) + \sin(A - B)}{2}.$$

Thus we have:

$$(f * g)(t) = \int_0^t \frac{\sin(t) + \sin(2\beta - t)}{2}\ d\beta = \frac{1}{2}\beta \sin t - \frac{1}{4}\cos(2\beta - t)\Big|_{\beta=0}^{\beta=t} = \frac{1}{2}t \sin t.$$

Theorem 14.12 (Properties of Convultion Product)
The convolution product is:

- Distributive $f * (g + h) = f * g + f * h$
- Commutative $f * g = g * f$
- Associative $f * (g * h) = (f * g) * h$

Theorem 14.13 (Laplace Transform of Convolution Product)

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}.$$

Example 14.14
We have:

$$\mathcal{L}\{\sin t * \cos t\} = \mathcal{L}\{\sin t\}\mathcal{L}\{\cos t\} = \frac{1}{s^2 + 1}\frac{s}{s^2 + 1} = \frac{s}{(s^2 + 1)^2}.$$

From the other example, we found it to be $(\sin t * \cos t) = \frac{1}{2}t \sin t$, here we can check as:

$$\mathcal{L}\{\frac{1}{2}t \sin t\} = -\frac{1}{2}\frac{d}{ds}\mathcal{L}\{\sin t\} = -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s^2 + 1}\right) = \frac{s}{(s^2 + 1)^2}.$$

We can use the convolution product to get inverse Laplace transform, as:

Theorem 14.15

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \mathcal{L}^{-1}\{F(s)\} * \mathcal{L}^{-1}\{G(s)\}.$$

Example 14.16

Consider:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}.$$

Remark 14.17 —

$$\mathcal{L}\{\int_0^t f(\beta)\ d\beta\} = \mathcal{L}\{f * 1\} = \frac{1}{s}\mathcal{L}\{f\}.$$

15 February 26th, 2020

15.1 More Methods to Solve ODEs

We are dealing with linear, second order, non-homogeneous ODEs, which are the form:

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = b(x), \quad \alpha < x < \beta.$$

Remember, if $y_1(x)$ is any non-zero solution to $a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0$, then a second non-zero solution $y_2(x)$ can be computed using Abel's equation:

$$y_2(x) = Ay_1(x)\int \frac{e^{-\int \frac{a_1(x)}{a_2(x)}\ dx}}{(y_1(x))^2}\ dx,$$

for any $a \neq 0$. Thus a general solution to the homogeneous equation is:

$$y_h(x) = c_1y_2(x) + c_2y_2(x),$$

for arbitrary constants c_1 and c_2. Furthermore, if $y_p(x)$ is any solution to the non-homoegenous equation, then the general solution is:

$$y(x) = y_h(x) + y_p(x) = c_1y_1(x) + c_2y_2(x) + y_p(x).$$

If we can't easily guess $y_p(x)$ (is a constant if $\frac{b(x)}{a_0(x)}$, we can always use the Green's function expression:

$$y_p(x) = \int^x G(t,x)\frac{b(t)}{a_2(t)}\ dt.$$

Where:

$$G(t,x) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}.$$

As such, everything comes down to finding $y_1(x)$, as if we can find it, we can find y_2, y_p, and y_h. We've already seen for constant coefficients and equidimensional ODE's, we can get y_1 and y_2 from a table. We also know when $a_2 + a_1 + a_0 = 0$, we have a solution $y_1(x) = e^x$.

Another method is to first divide by $a_2(x)$, giving us:

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0, P(x) = \frac{a_1(x)}{a_2(x)} \quad Q(x) = \frac{a_0(x)}{a_2(x)}.$$

Which is putting it into the standard form. With this, we should compute the following:

$$\gamma(x) = \frac{Q'(x) + 2P(x)Q(x)}{(Q(x))^{\frac{3}{2}}}.$$

If $\gamma(x)$ is a constant, then the transformation:

$$z = \int \sqrt{AQ(x)}\, dx, \quad A \neq 0$$

will convert the original equation into one of constant coefficients:

$$\Psi''(z) + \frac{\gamma}{2\sqrt{A}}\Psi(z) + \frac{1}{A}\Psi(z) = 0.$$

Then

$$y(x) = \Psi\left(\int \sqrt{AQ(x)}\, dx\right).$$

Example 15.1 (One of Chebyshev Equation)
Consider:

$$(1 - x^2)y''(x) - xy'(x) + m^2 y(x) = 0, \quad -1 < x < 1, \quad m \in \mathbb{Z}.$$

Notice that in the normal form, we have:

$$P(x) = -\frac{x}{1 - x^2}, \quad Q(x) = \frac{m^2}{1 - x^2}.$$

Thus:

$$\gamma(x) = \frac{Q'(x) + 2P(x)Q(x)}{(Q(x))^{\frac{3}{2}}} = \frac{\frac{2xm^2}{(1-x^2)^2} + 2\left(-\frac{x}{1-x^2}\right)\left(\frac{m^2}{1-x^2}\right)}{\left(\frac{m^2}{1-x^2}\right)^{\frac{3}{2}}} = 0.$$

Thus, the transformation:

$$z = \int \sqrt{\frac{Am^2}{1 - x^2}}\, dx = m\sqrt{A} \int \frac{dx}{\sqrt{1 - x^2}} = m\sqrt{A}\sin^{-1}(x)$$

will give us constant coefficient. Let $A = 1$, we have:

$$\Psi(z) + \Psi(z) = 0 \implies \Psi(z) = c_1 \cos(z) + c_2 \sin(z).$$

Thus the solution to the original equation is:

$$y(x) = c_1 \cos(m \sin^{-1}(x)) + c_2 \sin(m \sin^{-1}(x)).$$

Remark 15.2 — This test is fast, as we just have to compute $\gamma(x)$ to see if it will work.

Another method would be, starting with the standard form, set:

$$y(x) = u(x)e^{-\frac{1}{2}\int P(x)dx}.$$

And the equation will become of the form:

$$u''(x) + R(x)u(x) = 0, \quad R(x) = Q(x) - \frac{1}{2}P(x) - \frac{1}{4}(P(x))^2.$$

If $R(x)$ is a constant, then solving for $u(x)$ is easy, as we would have constant coefficients, allowing us to use the table.

Example 15.3 (Another one of Chebyshev's Equations)
Consider:

$$y''(x) - 2\tan(x)y'(x) + m^2y(x) = 0.$$

This is already in standard form, with:

$$P(x) = -2\tan(x) \quad Q(x) = m^2.$$

Thus, we have:

$$R(x) = m^2 - \frac{1}{2}\left(-\sec^2(x)\right) - \frac{1}{4}(2\tan(x))^2 = m^2 + \sec^2(x) - \tan^2(x) = m^2 + 1.$$

Thus, we have:

$$u''(x) + (m^2+1)u(x) = 0 \implies u(x) = c_1\cos\left(x\sqrt{m^2+1}\right) + c_2\sin\left(x\sqrt{m^2+1}\right).$$

Thus the solution to the original equation will be:

$$y(x) = \left(c_1\cos\left(x\sqrt{m^2+1}\right) + c_2\sin\left(x\sqrt{m^2+1}\right)\right)e^{-\frac{1}{2}\int -2\tan(x)dx}.$$

Note that:

$$e^{\int tan(x)dx} = e^{-\ln(\cos(x))} = \frac{1}{\cos(x)}.$$

Thus we have:

$$y(x) = \frac{c_1\cos\left(x\sqrt{m^2+1}\right) + c_2\sin\left(x\sqrt{m^2+1}\right)}{\cos(x)}.$$

15.2 Taylor's Method

Starting with the equation in standard form:

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0, \quad \alpha < x < \beta.$$

If

$$\gamma(x) = \frac{Q'(x) + 2P(x)Q(x)}{(Q(x))^{\frac{3}{2}}} \neq \text{constant}$$

and

$$R(x) = Q(x) - \frac{1}{2}P'(x) - \frac{1}{4}\left(P(x)\right)^2 \neq \text{constant}.$$

Then we can try Taylor's method. First let's define the two equations.

Definition 15.4. **Legendre's Equation** is defined as

$$(1-x^2)y''(x) - 2xy'(x) + m(m+1)y(x) = 0, \quad -1 < x < 1, m \in \mathbb{Z}$$

Remark 15.5 — The Legendre's equation appears in many places whenever we're dealing with spherical symmetry.

Definition 15.6. **Bessel's Equation** is defined as

$$x^2y''(x) + xy'(x) \pm (x^2 - m^2)y(x) = 0, \quad 0 < x, m \in \mathbb{Z}$$

Remark 15.7 — Bessel's equation appears in many places as well, such as cylindrical symmetry.

Taylor's method will help with Legendre's equation, and there's an extension to Taylor's method called to Frobenius's Method.

Definition 15.8. A point $x = x_0$ for the ODE

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0$$

is called an **ordinary point** for the ODE if:

$$\lim_{x \to x_0} P(x) \text{ and } \lim_{x \to x_0} Q(x)$$

both exist.

Remark 15.9 — For Legendre's ODE, we have:

$$P(x) = -\frac{x}{1-x^2}, \quad Q(x) = \frac{m(m+1)}{1-x^2}.$$

$x = 0$ is an ordinary point for this ODE, as:

$$\lim_{x \to 0} P(x) = 0 \quad \lim_{x \to 0} Q(x) = m(m+1).$$

Definition 15.10. A point that is not ordinary is called a **singular point**.

Definition 15.11. A singular point is called **regular** if:

$$\lim_{x \to x_0} (x - x_0)P(x) \text{ and } \lim_{x \to x_0} (x - x_0)^2 Q(x)$$

both exist.

Definition 15.12. If a singular point is not regular, it is called an **essential point**.

Remark 15.13 — For Bessels' ODE, $x = 0$ is a singular point, as:

$$\lim_{x \to 0} P(x) = \lim_{x \to 0} \frac{1}{x} = \text{DNE}.$$

Note that $x = 0$ is regular, as:

$$\lim_{x \to 0} xP(x) = 1, \quad \lim_{x \to 0} x^2 Q(x) = \pm(x^2 - m^2).$$

Theorem 15.14 (Taylor's Theorem for ODEs)
If $x = x_0$ is an ordinary point for $y''(x) + P(x)y'(x) + Q(x)y(x) = 0, \quad \alpha < x, x_0 < \beta$, then two linearly independent solutions can be constructed as:

$$y_1(x) = \sum_{nm=0}^{\infty} a_m (x - x_0)^m \text{ and } y_2(x) = \sum_{m=0}^{\infty} b_m (x - x_0)^m$$

and these will converge absolutely for all $|x - x_0| < R$, where R is the nearest distance between x_0 and a singular point (if any).

Definition 15.15. If a series $\sum_{n=1}^{\infty} a_n$ **converges absolutely**, then $\sum_{n=1}^{\infty} |a_n|$ converges.

Definition 15.16. If a series $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ does not, then it **converges conditionally**.

Remark 15.17 — Note that we should always pick x_0 to be in the middle of two singular points. For example, consider Legendre's ODE $(1 - x^2)y'' - 2xy' + m(m + 1)y = 0, \ -1 < x < 1$. If we choose $x_0 = \frac{1}{2}$ then our power series will only converge absolutely if $0 < x < 1$. If we pick $x_0 = 0$, then it will converge absolutely everywhere in $-1 < x < 1$.

16 March 2nd, 2020

16.1 Taylor's Method

Recall that Taylor's method is where we assume the solution:

$$y(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m.$$

Where x_0 is an ordinary point for the ODE. Note that this is a power series expansion about x_0 with a_m being constants.

Consider

$$y''(x) + \omega^2 y(x) = 0, \quad -\infty < x < +\infty.$$

We know already that the solution involves $\sin(x)$ and $\cos(x)$. Note that for this ODE, all points are ordinary points. We will use $x_0 = 0$, since this will simplify things a lot.

Thus we have:

$$y(x) = \sum_{m=0}^{\infty} a_m x^m.$$

Which converges absolutely everywhere. Differentiating this W.R.T. x, we get:

$$y'(x) = \sum_{m=0}^{\infty} m a_m x^{m-1} \text{ and } y''(x) = \sum_{m=0}^{\infty} m(m-1) a_m x^{m-2}.$$

Plugging this into the original equation, we have:

$$\sum_{m=0}^{\infty} m(m-1) a_m x^{m-2} + \omega^2 \sum_{m=0}^{\infty} a_m x^m = 0.$$

Note that when $m = 0$ and $m = 1$, the first power series term would be 0. Now using the fact that:

$$\sum_{m=a}^{b} f(m) = \sum_{m=a\pm c}^{b\pm c} f(m \mp c).$$

we get:

$$\sum_{m=2-2}^{\infty} (m+2)((m+2)-1) a_{m+2} x^{(m+2)-2} + \omega^2 \sum_{m=0}^{\infty} a_m x^m = 0.$$

$$\implies \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m + \omega^2 \sum_{m=0}^{\infty} a_m x^m = 0.$$

Collecting the terms, we get:

$$\sum_{m=0}^{\infty} \left((m+2)(m+1) a_{m+2} + \omega^2 a_m\right) x^m = 0.$$

The only way we can get the LHS to equal zero for all values of x is if the coefficient equals zero, thus we have:

$$(m+2)(m+1) a_{m+2} + \omega^2 a_m = 0, \quad m = 0, 1, 2, \ldots.$$

$$a_{m+2} = -\frac{\omega^2 a_m}{(m+2)(m+1)}, \quad m = 0, 1, 2, \ldots.$$

From this, we get:

- for $m = 0$, $a_2 = -\frac{\omega^2}{2} a_0$
- for $m = 1$, $a_3 = -\frac{\omega^2}{(3)(2)} a_1$
- for $m = 2$, $a_4 = -\frac{\omega^2}{(4)(3)} a_2 = \frac{(-1)^2 \omega^4}{4!} a_0$
- for $m = 3$, $a_5 = -\frac{\omega^2}{(5)(4)} a_2 = \frac{(-1)^2 \omega^4}{5!} a_1$

As we keep going, we would have:

$$a_{2k} = \frac{(-1)^k \omega^{2k}}{(2k)!} a_0, \quad a_{2k+1} = \frac{(-1)^k \omega^{2k}}{(2k+1)!} a_1, \quad k = 0, 1, 2, 3, \dots.$$

Thus for this particular example, we can solve for all a_m from a_0 and a_1.

Since for this series, they break pretty naturally into even and odd powers, we can split it into:

$$\begin{aligned} y(x) = \sum_{m=0}^{\infty} a_m x^m &= \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k \omega^{2k}}{(2k)!} q_0 x^{2k} + \sum_{k=0}^{\infty} \frac{(-1)^k \omega^{2k}}{(2k+1)!} a_1 x^{2k+1} \\ &= \alpha \sum_{k=0}^{\infty} \frac{(-1)^k (\omega x)^{2k}}{(2k)!} + \frac{a_1}{\omega} \frac{(-1)^k (\omega x)^{2k+1}}{(2k+1)!} \\ &= a_0 \cos(\omega x) + \frac{a_1}{\omega} \sin(\omega x) \end{aligned}$$

Note that a_0 and a_1 are arbitrary constants, thus giving us the solution we had previously.

Remark 16.1 — As a reminder, for the ODE $y''(x) + P(x)y'(x) + Q(x)y(x) = 0$, x_0 is an ordinary point of the ODE if

$$\lim_{x \to x_0} P(x) \text{ and } \lim_{x \to x_0} Q(x)$$

both exist. Otherwise, it is called a singular point.

Remark 16.2 — Usually, if you can pick $x_0 = 0$, you should pick it, as then you can take advantage of even and odd properties.

16.2 Some Power Series Expansions

$$e^z = \sum_{m=0}^{\infty} \frac{z^m}{m!}, \quad |z| < \infty$$

$$e^{-z} = \sum_{m=0}^{\infty} \frac{(-1)^m z^m}{m!}, \quad |z| < \infty$$

$$\cos(z) = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{(2m)!}, \quad |z| < \infty$$

$$\cosh(z) = \sum_{m=1}^{\infty} \frac{z^{2m}}{(2m)!}, \quad |z| < \infty$$

$$\sin(z) = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m+1}}{(2m+1)!}, \quad |z| < \infty$$

$$\sinh(z) = \sum_{m=0}^{\infty} \frac{z^{2m+1}}{(2m+1)!}, \quad |z| < \infty$$

$$\frac{1}{1-z} = \sum_{m=0}^{\infty} z^m, \quad |z| < 1$$

Consider the ODE:

$$(1-x^2)y''(x) + 8xy'(x) - 20y(x) = 0, \quad -1 < x < +1.$$

Using $x_0 = 0$, which is an ordinary point, since:

$$\lim_{x \to x_0} P(x) = \lim_{x \to x_0} \frac{x_0}{1-x^2} \text{ and } \lim_{x \to x_0} Q(x) = \lim_{x \to x_0} \frac{-20}{1-x^2}.$$

both exists, we have:

$$(1-x^2)\sum_{m=0}^{\infty} m(m-1)a_m x^{m-2} + 8x\sum_{m=0}^{\infty} m a_m x^{m-1} - 20\sum_{m=0}^{\infty} a_m x^m = 0.$$

$$\implies \sum_{m=0}^{\infty} a_m m(m-1)x^{m-2} - \sum_{m=0}^{\infty} a_m m(m-1)x^m + 8\sum_{m=0}^{\infty} a_m m x^m - 20\sum_{m=0}^{\infty} a_m x^m = 0.$$

$$\implies \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=0}^{\infty} \underbrace{(m(m-1) - 8m + 20)}_{(m-4)(m-5)} a_m x^m = 0.$$

$$\implies \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m - \sum_{m=0}^{\infty} (m-4)(m-5)a_m x^m = 0.$$

$$\implies \sum_{m=0}^{\infty} \underbrace{((m+2)(m+1)a_{m+2} - (m-4)(m-5)a_m)}_{=0} x^m = 0.$$

Thus we get the relation:

$$a_{m+2} = \frac{(m-4)(m-5)}{(m+2)(m+1)} a_m, \quad m = 0, 1, 2, \ldots.$$

Plugging in values for m, we have:

- $m = 0$, $a_2 = \frac{20}{2}a_0 = 10a_0$
- $m = 1$, $a_3 = \frac{(-3)(-4)}{(3)(2)}a_1 = 2a_1$
- $m = 2$, $a_4 = \frac{(-2)(-3)}{(4)(3)}a_2 = 5a_0$
- $m = 3$, $a_5 = \frac{2}{(5)(4)}a_3 = \frac{1}{5}a_1$

When we get to $m = 4$ and $m = 5$, we would get a_6 and a_7 both equaling 0, thus all coefficients after that would be zero as well. Thus, we would have:

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$$

$$y(x) = a_0 + a_1x + 10a_0x^2 + 2a_1x^3 + 5a_0x^4 + \frac{1}{5}a_1x^5$$

$$y(x) = a_0\left(1 + 10x^2 + 5x^4\right) + a_1\left(x + 2x^3 + \frac{1}{5}x^5\right).$$

Remark 16.3 — As long as $P(-x) = -P(x)$ and $Q(-x) = Q(x)$, and $x_0 = 0$, we will always be able to break the sum into an even and an odd part. In other words, P is an odd function and Q is an even function.

Let us try one which doesn't separate into even and odd terms. Consider:

$$(1-x^2)y''(x) + 2y'(x) + xy(x) = 0, \quad -1 < x < +1.$$

Notice this time that

$$P(x) = \frac{2}{1-x^2} \text{ is not an odd function}$$

and

$$Q(x) = \frac{x}{1-x^2} \text{ is not an even function.}$$

Also note that $x_0 = 0$ is an ordinary point, meaning that we have:

$$y(x) = \sum_{m=0}^{\infty} a_m x^m.$$

$$y'(x) = \sum_{m=0}^{\infty} m a_m x^{m-1}.$$

$$y''(x) = \sum_{m=0}^{\infty} m(m-1) a_m x^{m-2}.$$

Plugging into the ODE, we have:

$$(1-x^2)\sum_{m=0}^{\infty} a_m m(m-1)x^{m-2} + 2\sum_{m=0}^{\infty} a_m m x^{m-1} + x\sum_{m=0}^{\infty} a_m x^m = 0$$

$$\implies \sum_{m=0}^{\infty} a_m m(m-1)x^{m-2} + \sum_{m=0}^{\infty} a_m m(m-1)x^m + 2\sum_{m=0}^{\infty} a_m m x^{m-1} + \sum_{m=0}^{\infty} a_m x^{m+1} = 0.$$

Note that the leading power of x for each of the power series are 0, 2, 1, and 1. Shifting the bounds, we would get:

$$\sum_{m=0}^{\infty} a_{m+2}(m+2)(m+1)x^m + \sum_{m=0}^{\infty} a_m m(m-1)x^m + 2\sum_{m=0}^{\infty} a_{m+1}(m+1)x^m + \sum_{m=0}^{\infty} a_m x^{m+1} = 0.$$

$$\implies \sum_{m=0}^{\infty} \left((m+1)(m+2)a_{m+2} - m(m-1)a_m + 2(m+1)a_{m+1}\right)x^m + \sum_{m=0}^{\infty} a_m x^{m+1} = 0.$$

Note that the first power sum has an extra term $(m = 0)$, that the first one does not have, thus we can separate it and get:

$$(2a_2 + 2a_1) + \sum_{m=1}^{\infty} \left((m+1)(m+2)a_{m+2} - m(m-1)a_m + 2(m+1)a_{m+1}\right)x^m + \sum_{m=1}^{\infty} a_{m-1}x^m = 0.$$

Thus we have:

$$2(a_2 + a_1) + \sum_{m=1}^{\infty} \left((m+1)(m+2)a_{m+2} - m(m-1)a_m + 2(m+1)a_{m+1} + a_{m-1}\right) x^m = 0.$$

Thus meaning that:

$$a_2 + a_1 = 0.$$

and

$$(m+1)(m+2)a_{m+2} - m(m-1)a_m + 2(m+1)a_{m+1} + a_{m-1} = 0.$$

Giving us:

$$a_2 = -a_1 \quad a_{m+2} = \frac{-2(m+1)a_{m+1} + m(m-1)a_m - a_{m-1}}{(m+2)(m+1)}, \quad m = 1, 2, 3, \ldots.$$

17 March 6th, 2020

17.1 Problem 1

Recall that if $x = x_0$ is an ordinary point, we have:

$$y(x) = \sum_{m=0}^{\infty} a_n (x - x_0)^n.$$

Consider the equation:

$$(1 - x^2)y''(x) + 4xy'(x) + 14y(x) = 0.$$

$$\implies y'' + \frac{4x}{1 - x^2}y' + \frac{14}{1 - x^2}y = 0.$$

We know that an ordinary point is one where this equation doesn't blow up, i.e. any point where $x^2 \neq 1$. Let's choose $x_0 = 0$, meaning that the solution would be in the form:

$$y = \sum_{nk=0}^{\infty} a_k x^k.$$

Now what's left to do is to solve for a_k. Plugging into the original equation, we have:

$$(1 - x^2) \left[\sum_{k=0}^{\infty} k(k-1)a_k x^{k-2}\right] + 4x \left[\sum_{k=0}^{\infty} k a_k x^{k-1}\right] + 14 \left[\sum_{k=0}^{\infty} a_k x^k\right] = 0.$$

Now let's collect the terms:

$$\implies \sum_{k=0}^{\infty} k(k-1)a_k x^{k-2} - \sum_{k=0}^{\infty} k(k-1)a_k x^k + 4\sum_{k=0}^{\infty} k a_k x^k + 14\sum_{k=0}^{\infty} a_k x^k = 0.$$

$$\implies \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2} x^k - \sum_{k=0}^{\infty} k(k-1)a_k x^k + 4\sum_{k=0}^{\infty} k a_k x^k + 14\sum_{k=0}^{\infty} a_k x^k = 0.$$

$$\implies \sum_{k=0}^{\infty}\left[(k+2)(k+1)a_{k+2}+\underbrace{(-k(k-1)+4k+14)}_{-(k-7)(k+2)}a_k\right]x^k=0.$$

Thus:

$$a_{k+2}=\frac{k-7}{k+1}a_k.$$

Note that the odd series truncate when $k=7$, the numerator of the right hand side equals, thus meaning $a_9=a_{11}=\ldots=0$, meaning that we only need to consider a_3, a_5, a_7 for the series of the odd terms. We have:

$$a_3=\frac{1-7}{1+1}a_1=-3a_1.$$

$$a_5=\frac{3-7}{3+1}a_3=-a_3=3a_1.$$

$$a_7=\frac{5-7}{5+1}a_5=-a_1.$$

Thus the first solution is:

$$y_1(x)=a_1x+a_3x^3+a_5x^5+a_7x^7=a_1\left(x-3x^3+3x^5-x^7\right).$$

With this, we would find a_1 from some initial condition, and then we can use Abel's equation to solve for y_2.

17.2 Problem 2

Now consider the equation:

$$y''-\alpha xy'+\beta y=0,\quad \alpha,\beta\in\mathbb{R}>0.$$

To apply Taylor's method, we first want to find an ordinary point to expand on. Note that the equation is defined for all x, thus we can choose any point to expand on. For simplicity, we should choose $x_0=0$, meaning that we have:

$$\sum_{k=0}^{\infty}k(k-1)a_kk^{k-2}-\alpha x\sum_{k=0}^{\infty}ka_kx^{k-1}+\beta\sum_{k=0}^{\infty}a_kx^k=0.$$

Now we play with the indicies again to get:

$$\sum_{k=2}^{\infty}k(k-1)a_kx^{k-2}-\alpha\sum_{k=0}^{\infty}ka_kx^k+\beta\sum_{k=0}^{\infty}a_kx^k=0$$

$$\implies \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k-\alpha\sum_{k=0}^{\infty}ka_kx^k+\beta\sum_{k=0}^{\infty}a_kx^k=0$$

$$\implies \sum_{k=0}^{\infty}\left[(k+2)(k+1)a_{k+2}-(\alpha k-\beta)a_k\right]x^k=0.$$

$$\implies a_{k+2}=\frac{\alpha k-\beta}{(k+2)(k+1)}a_k.$$

Note that this is polynomial if $\alpha k = \beta$ for some:

$$k = \frac{\beta}{\alpha} = \mathbb{Z}^+.$$

For example:

$$y'' - 5xy' + 25y = 0, \quad a_{k+2} = \frac{5(k-5)}{(k+2)(k+1)}a_k.$$

meaning that $a_7 = a_9 = \ldots = 0$, giving us:

$$a_3 = \frac{5(1-5)}{3 \cdot 2}a_1 = -\frac{10}{3}a_1.$$

$$a_5 = \frac{5(3-5)}{5 \cdot 4}a_3 = -\frac{1}{2}a_3 = \frac{5}{3}a_1.$$

Thus, we have:

$$y_1(x) = a_1x + a_3x^3 + a_5x^5 = a_1\left(x - \frac{10}{3}x^3 + \frac{5}{3}x^5\right).$$

And we can use Abel's equation to solve for y_2.

18 March 23rd, 2020

18.1 Legendre ODE and Legendre Polynomials

Consider the equation:

$$(1-x^2)y''(x) - 2xy'(x) + m(m+1)y(x) = 0, \quad -1 < x < 1.$$

Where m is a parameter taking on non-negative integer values. Note that $x_0 = 0$ is an ordinary point. Converting this to standard form, we get:

$$y''(x) - \frac{2x}{1-x^2}y'(x) + \frac{m(m+1)}{1-x^2}y(x) = 0.$$

This is called the **Legendre Equation**. This will come up with spherical symmetry.

Using Taylor's method, we have $y(x) = \sum_{k=0}^{\infty} a_k x^k$. Taking the derivative, we have:

$$y'(x) = \sum_{k=0}^{\infty} ka_kx^{k-1}.$$

$$y''(x) = \sum_{k=0}^{\infty} k(k-1)a_kx^{k-2}.$$

Plugging this in and collecting the common terms, we get:

$$\sum_{k=0}^{\infty} \left((k+2)(k+1)a_{k+2} - (k-m)(k+m+1)a_k\right)x^k = 0.$$

This means that:

$$a_{k+2} = \frac{(k-m)(k+m+1)a_k}{(k+2)(k+1)}, \quad k = 0, 1, 2, \dots.$$

Suppose $m = 4$, we have:

$$a_{k+2} = \frac{(k-4)(k+5)}{(k+2)(K+1)}a_k.$$

After plugging it in, note that the even coefficients go to zero after $k = 4$, while the odd coefficients is still an infinite sum. This tells us that one solution is:

$$y(x) = a_0(1 - 10x^2 + \frac{35}{3}x^4).$$

Note that because of the $(k-m)$ factor, for any m, there will always be one solution to the Legendre ODE that is a polynomial. That polynomial is called $P_m(x)$ and is known as a **Legendre Polynomial**. This polynomial has the following properties:

- $P_m(x)$ will have degree m
- $P_m(x)$ will only contain powers of the same parity of m.
- Note that can change the factor of a_0 or a_1. As such, we usually normalizes it as $P_m(\pm 1) = (\pm 1)^m$.
- $P_m(x)$ will have exactly m distinct roots between $x = -1$ and $x = +1$ if normalized.

> **Remark 18.1** — Doing this normalization will put force all $P_m(x)$ to lie in the square centered at the origin with width $= 2$.

With this, we could put $P_m(x)$ in a table. These polynomials have a very nice property, in that:

$$P_{m+1}(x) = \frac{(2m+1)P_m(x) - mP_{m-1}(x)}{m+1}.$$

This allows us to generate the higher order Legendre polynomials, and is how computers calculate them.

For the other equation, we can use Abel's equation, giving us:

$$\begin{aligned} Q_m(x) &= P_m(x)\left[\int \frac{e^{-\int \frac{2x}{1-x^2}dx}}{(P_m(x))^2}dx + A\right]. \\ &= P_m(x)\left[\int \frac{dx}{(1-x^2)P_m^2(x)} + A\right]. \end{aligned}$$

For Q_0, we have:

$$1\left[\int \frac{dx}{(1-x^2)} + A\right] = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) + A.$$

This can be generalized, with:

$$Q_m(x) = \text{polynomial} \ln\left(\frac{1+x}{1-x}\right) + \text{ polynomial}.$$

This means that it is not finite at $x = \pm 1$. Thus the general solution is:

$$y(x) = AP_m(x) + BQ_m(x).$$

And if we require that the solution must be finite at $x = \pm 1$, then $B = 0$ and $y(x) = AP_m(x)$.

19 March 25th, 2020

19.1 Important Result Involving ODEs

Let us consider the Legendre ODE:

$$(1-x^2)y''(x) - 2xy'(x) + m(m+1)y(x) = 0, \quad -1 < x < 1.$$

Recall that one solution is $y(x) = P_m(x)$. We can generalize this to a class of problems that look like:

$$a_2(x)y_m''(x) + a_1(x)y_m(x) + a_0(x)y_m(x) + \lambda_m^2 b(x)y_m(x) = 0, \quad \alpha < x < \beta.$$

With λ_m being some parameter for different values of m.

Remark 19.1 — Note that for the Legendre ODE falls into this category with $a_2(x) = 1-x^2$, $a_1(x) = -2x$, $a_0(x) = 0$, $b(x) = 1$, $\lambda_m^2 = m(m+1)$

Theorem 19.2

If $y_M(x)$ is a solution to the above equation and if $y_N(x)$ is a solution to the same ODE with M replaced by M, then after much algebra, we find that:

$$\left(\lambda_M^2 - \lambda_N^2\right)\int_\alpha^\beta y_M(x)y_N(x)w(x)\ dx = \left(s(x)\left|\begin{bmatrix} y_M(x) & y_N(x) \\ y_M'(x) & y_N'(x)\end{bmatrix}\right|\right)\Bigg|_\alpha^\beta.$$

Where:

$$s(x) = e^{\int \frac{a_1(x)}{a_2(x)}dx}.$$

$$w(x) = \frac{b(x)}{a_2(x)}s(x).$$

Example 19.3

Consider $y_N''(x) + N^2 y_N(x) = 0, \quad 0 < x < \pi$. One solution is $y_N(x) = \sin(Nx)$. Note that this is of the form with $a_2(x) = 1$, $a_1(x) = 0$, $a_0(x) = 0$, $b(x) = 1$, $\lambda_N^2 = N^2$. We have

$$s(x) = e^{\int \frac{0}{1}dx} = 1.$$

$$w(x) = \frac{1}{1}(1) = 1.$$

Plugging into the equation, we have:

$$(M^2 - N^2)\int_0^\pi \sin(Mx)\sin(Nx)(1)\ dx = \left((1)\left|\begin{bmatrix} \sin(Mx) & \sin(Nx) \\ M\cos(Mx) & N\cos(Nx)\end{bmatrix}\right|\right)\Bigg|_0^\pi$$

$$= \left|\begin{bmatrix} 0 & 0 \\ M(-1)^M & N(-1)^N\end{bmatrix}\right| = 0.$$

If $M \neq N$, then:

$$\int_0^\pi \sin(Mx)\sin(Nx)\ dx = 0.$$

For the Legendre ODE, we have: $a_2(x) = 1 - x^2$, $a_1(x) = -2x$, $a_0 = 0$, $\alpha_N^2 = N(N+1)$, $b(x) = 1$. Then we have:

$$s(x) = e^{\int \frac{a_1(x)}{a2(x)} dx} = 1 - x^2.$$

$$w(x) = \frac{b(x)}{a_2(x)} s(x) = \frac{1}{1-x^2}(1-x^2) = 1.$$

Using the earlier general result, we have:

$$(M(M+1) - N(N+1)) \int^{-} 1_{-}1 P_M(x) P_N(x)(1) \, dx = \left((1-x^2)\ldots\right)\Big|_{-1}^{1} = 0.$$

Theorem 19.4

Thus if $M \neq N$, we have:

$$\int_{-1}^{1} P_M(x) P_N(x) \, dx = 0.$$

19.1.1 Legendre Series

Recall that the Taylor series is a power series expansion:

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad a_n = \frac{f^{(n)}(x_0)}{n!}.$$

For this, we are expanding a function in terms of polynomials, with the basis set being:

$$\{1, (x - x_0), (x - x_0)^2, \ldots\}.$$

The **Legendre series** is similar, where the basis set being the Legendre polynomials:

$$\{P_0(x), P_1(x), P_2(x) \ldots\}.$$

Meaning that:

$$f(x) = \sum_{n=1}^{\infty} a_n P_n(x), \quad -1 < x < 1.$$

Note that this is an equation with an infinite number of equations and an infinite number of unknowns. To get the a_n, we will make use of the result that:

$$\int_{-1}^{1} P_M(x) P_N(x) \, dx = 0, \quad M \neq N.$$

Suppose $f(x)$ is given and all the $P_n(x)$ are known. Suppose you want to compute a_3. We multiply both sides by P_3, giving us:

$$f(x) P_3(x) = \sum_{n=1}^{\infty} a_n P_n(x) P_3(x).$$

Integrating both sides, we have:

$$\int_{-1}^{1} f(x)P_3(x)\ dx = \int_{-1}^{1} \sum_{n=0}^{\infty} a_n P_n(x) P_3(x)\ dx.$$

In this situation, we can swap the integral and sum (not always the case, but in this case it turns out to be true), thus we have:

$$= \sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_n(x)P_3(x)\ dx = a_3 \int_{-1}^{1} P_3(x)P_3(x)\ dx.$$

Thus we have:

$$a_3 = \frac{\int_{-1}^{1} f(x)P_3(x)\ dx}{\int_{-1}^{1} P_3(x)P_3(x)\ dx}.$$

Thus we have:

$$f(x) = \sum_{n=0}^{\infty} a_n P(x).$$

$$a_n = \frac{\int_{-1}^{1} f(x)P_n(x)\ dx}{\int_{-1}^{1} P_n(x)P_n(x)\ dx}.$$

It can be shown that:

$$\int_{-1}^{1} P_n(x)P_n(x)\ dx = \frac{2}{2n+1}, \quad n = 0, 1, 2, \ldots.$$

To make this easier to remember, remember that if:

$$A = A_1 i + A_2 j + A_3 k.$$

We have:

$$A_1 = \frac{A \cdot i}{i \cdot i}, A_2 = \frac{A \cdot j}{j \cdot j}, A_3 = \frac{A \cdot k}{k \cdot k}.$$

If we think of $\int f(x)g(x)\ dx$ as a "dot product" between equations, we can think of $a_n = \frac{f \cdot P_n}{P_n \cdot P_n}$. Note that this satisfies some special properties that are the same as the dot product between vectors, namely:

- $f \cdot g = g \cdot f$
- $f \cdot (g + h) = f \cdot g + f \cdot h$
- $f \cdot \alpha g = \alpha(f \cdot g)$
- $f \cdot f \geq 0$
- $f \cdot f = 0 \iff f = 0$

20 March 30th, 2020

20.1 Bessel's ODE and Bessel Functions

We know that solutions to the ODE:

$$y''(x) + \lambda^2 y(x) = 0$$

are:

$$y_1(x) = \cos(\lambda x) \quad y_2(x) = \sin(\lambda x).$$

With:

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

Using the series expansion for sine and cosine, we have:

$$y_1(x) = \cos(\lambda x) = \sum_{k=0}^{\infty} \frac{(-1)^k (\lambda x)^{2k}}{(2k)!}.$$

$$y_2(x) = \sin(\lambda x) = \sum_{k=0}^{\infty} \frac{(-1)^k (\lambda x)^{2k+1}}{(2k+1)!}.$$

The Bessel's ODE is the equation:

$$x^2 y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0, x > 0.$$

Without going through the series method, one solution to this ODE is given by:

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+\nu)!} \left(\frac{x}{2}\right)^{2k+\nu}, \quad |x| < \infty.$$

We can have any values real of ν even fractions or irrational values (using the gamma function).

> **Remark 20.1** — Note that $J_\nu(-x) = (-1)^\nu(x)$ and when ν is an even integer, than $J_\nu(x)$ is an even function, while when v is an odd integer, then $J_\nu(x)$ is an odd funciton.

> **Remark 20.2** — If $|x|$ is very small, then $J_\nu(x) \approx \left(\frac{x}{2}\right)^\nu \frac{1}{\nu!}$, $\nu \geq 0$. As such, $J_\nu(x)$ is finite at $x = 0$.

Using Abel's equation, we can get second linearly independent solution to Bessel's ODE, which is:

$$y_2(x) = Y_\nu(x) = J_\nu(x) = \left(A \int \frac{1}{xJ_\nu^2(x)}\, dx + B\right).$$

Noteablly, it is not finite at $x = 0$.

Thus the general solution to Bessel's ODE is:

$$y(x) = c_1 J_\nu(x) + c_2 Y_\nu(x).$$

You can always use $Y_\nu(x)$ as the second solution, but if it turns out that ν is not an integer, we can instead use $J_{-\nu}(x)$ for the second solution. Regardless, $J_{-\nu}(x)$ is also not finite at $x = 0$. This means that if we require that $y(\pm 1)$ to be finite, then we must set $c_2 = 0$, giving us:

$$y(x) = c_1 J_\nu(x).$$

There is a **modified Bessel's ODE**:

$$x^2 y''(x) + xy'(x) - (x^2 + \nu^2)y(x) = 0.$$

Which has a general solution

$$y(x) = c_1 I_\nu(x) + c_2 K_\nu(x).$$

Where:

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{k!(k+\nu)!} \left(\frac{x}{2}\right)^{2k+\nu}.$$

$$K_\nu(x) = I_\nu(x) \left(A \int \frac{1}{x I_\nu^2(x)}\, dx + B \right).$$

Remark 20.3 — $I_\nu(x)$ is finite for $\nu \geq 0$, and $K_\nu(x)$ is not finite.

20.2 Properties of $J_\nu(x)$

$$J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x).$$

$$Y_{\nu+1}(x) = \frac{2\nu}{x} Y_\nu(x) - Y_{\nu-1}(x).$$

$$I_{\nu+1}(x) = -\frac{2\nu}{x} Y_\nu(x) + Y_{\nu-1}(x).$$

$$K_{\nu+1}(x) = \frac{2\nu}{x} Y_\nu(x) + Y_{\nu-1}(x).$$

Theorem 20.4

Given the ODE:

$$x^2y''(x) + (a + 2bx^R)xy'(x) + (c + dx^{2s} - b(1 - a - R)x^R + b^2x^{2R})y(x) = 0, x > 0.$$

A general solution is given by:

$$y(x) = x^{\frac{1-a}{2}}e^{\frac{-bx^R}{R}}\left(c_1J_p\left(\frac{\sqrt{d}}{s}x^s\right) + c_2Y_p\left(\frac{\sqrt{d}}{s}x^s\right)\right).$$

when $d > 0$, and:

$$y(x) = x^{\frac{1-a}{2}}e^{\frac{-bx^R}{R}}\left(c_1I_p\left(\frac{\sqrt{-d}}{s}x^s\right) + c_2K_p\left(\frac{\sqrt{-d}}{s}x^s\right)\right).$$

when $d < 0$, where:

$$p = \left|\frac{1}{s}\sqrt{\left(\frac{1-a}{2}\right)^2 - c}\right|.$$

Example 20.5

Consider:

$$xy''(x) + 2y'(x) + \lambda^2x^2y(x) = 0.$$

Multiplying by x, we have:

$$x^2y''(x_0 + 2xy'(x) + \lambda^2x^3y(x) = 0.$$

Which means that

$$a + 2bx^R = 2 \implies a = 2, b = 0.$$

We also have:

$$c + dx^{2s} - b(1 - a - R)x^R + b^2x^{2R} = \lambda^2x^3 => c + d^{2s} = \lambda x^3.$$

$$\to c = 0, d = \lambda^2 \geq 0, s = \frac{3}{2}.$$

Thus:

$$p = \left|\frac{1}{s}\sqrt{\left(\frac{1-2}{2}\right)^2 - 0}\right| = \frac{1}{3}.$$

Thus:

$$y(x) = x^{-\frac{1}{2}}\left(c_1J_{\frac{1}{3}}\left(\frac{2}{3}\lambda x^{\frac{3}{2}}\right) + c_2Y_{\frac{1}{3}}\left(\frac{2}{3}\lambda x^{\frac{3}{2}}\right)\right).$$

21 April 1st, 2020

21.1 Dot Products

Recall that we defined the generic dot product between functions to be:

$$f \cdot g = \int_{\alpha}^{\beta} f(x)g(x)w(x)\ dx.$$

With $w(x)$ being a weight function. Recall that:

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta, \quad \cos\theta = \frac{\vec{A} \cdot \vec{B}}{\sqrt{\vec{A} \cdot \vec{A}}\sqrt{\vec{B} \cdot \vec{B}}}.$$

For

$$f \cdot g = \int_{\alpha}^{\beta} f(x)g(x)w(x)\ dx.$$

we have:

$$|f| = \sqrt{f \cdot f}.$$

$$\cos\theta = \frac{f \cdot g}{\sqrt{f \cdot f}\sqrt{g \cdot g}} = \frac{f \cdot g}{|f||g|}.$$

Note that f is perpendicular to g is $f \cdot g = 0$.

21.2 Recall Bessel's Equation

Consider

$$x^2y''(x) + (a + 2b^R)xy'(x) + (c + dx^{2s} - b(1 - a - R)x^R + b^2x^{2R})y(x) = 0.$$

Example 21.1 (Example where Bessel's equation does not work)

$$y''(x) + y'(x) + x^3y(x) = 0.$$

Multiplying by x^2, we have:

$$x^2y''(x) + y'(x) + x^5(x) = 0.$$

This means that $a + 2bx^R = x$, meaning that:

$$a = 0, b = \frac{1}{2}, R = 1.$$

$$c + dx^{2s} - b(1 - a - R)x^R + b^2x^{2R} = x^5.$$

$$c + dx^{2s} + \frac{1}{4}x^2 = x^5.$$

$$c + dx^{2s} = x^{5 - \frac{1}{4}x^2}.$$

Which does not work because c and d are constants.

Example 21.2

$$y''(x) + x^5 y(x) = 0.$$

Multiplying by x^2, we have:

$$x^2 y''(x) + x^5 y(x) = 0.$$

$$\implies a + 2bx^R = 0 \implies a = 0, b = 0.$$

$$c + dx^{2s} - b(1 - a - R)x^R + b^2 x^{2R} = x^5 \implies c = 0, d = 1 > 0, s = \frac{5}{2}.$$

$$p = \left| \frac{1}{s} \sqrt{\left(\frac{1-a}{2}\right)^2 - c} \right| = \frac{1}{5} \neq \text{ integer}.$$

$$y(x) = x^{\frac{1-a}{2}} e^{-b\frac{x^R}{12}} \left(c_1 J_p \left(\frac{\sqrt{d}}{s} x^s \right) + c_2 J_p \left(\frac{\sqrt{d}}{s} x^5 \right) \right).$$

$$y(x) = x^{1/2} \left(c_1 J_{\frac{1}{5}})\tfrac{2}{5} x^{\frac{5}{2}} + c_2 J_{\frac{1}{5}} \left(\frac{2}{5} x^{\frac{5}{2}} \right) \right).$$

For small x, $J_\mu(x) \approx \frac{1}{\nu!} x^\nu \to x^\nu$. Thus for small x

$$y(x) = c_1 x + c_2.$$

Thus if:

$$y(x) = 0 \implies c_2 = 0.$$

Giving us:

$$y(x) = x^{1/2} c_1 J_{1/5} \left(\frac{2}{5} x^{5/2} \right).$$

Ging back to vectors:

$$\vec{A} = a\vec{i} + b\vec{j} + c\vec{k}.$$

$\{\vec{i}, \vec{j}, \vec{j}\}$ is called an orthogonal basis set. Similarly you can have an infinite dimensional basis set:

$$\{\phi_1(x), \phi_2(x), \phi_3(x), \ldots\}.$$

if $\phi_m \cdot \phi_n = 0$ for $m \neq n$ with respect to some inner product

22 April 6th, 2020

22.1 Sturm-Liouville

Definition 22.1. An ODE of the form:

$$\frac{d}{dx}\left(s(x)\phi'(x)\right) + q(x)\phi(x) + \lambda w(x)\phi(x) = 0, \alpha < x < \beta.$$

along with the boundary conditions:

$$c_1\phi(\alpha) + c_2\phi'(\alpha) = 0.$$

$$d_1\phi(\beta) + d_2\phi'(\beta) = 0.$$

is called a **regular Sturm Liouville Problem** (denoted RSLP) if the following conditions hold.

1. $s(x), s'(x), q(x), w(x)$ are all continuous functions in the open interval $\alpha < x < \beta$
2. $s(x) > 0$ and $w(x) > 0$ for all $\alpha < x < \beta$
3. $c_1^2 + c_2^2 > 0$ and $d_1^2 + d_2^2 > 0$, i.e. can't have both c_1 and c_2 equal zero, same for d_1, d_2
4. The λ occurs only in the ODE as indicated by $\lambda w(x)\phi(x)$.

Recall that any linear 2nd order homogeneous ODE of the form:

$$a_2(x)\phi''(x) + a_1(x)\phi'(x) + a_0(x)\phi(x) + \lambda b(x)\phi(x) = 0.$$

can be placed in the form:

$$\frac{d}{dx}\left(s(x)\phi'(x)\right) + q(x)\phi(x) + \lambda w(x)\phi(x) = 0, \alpha < x < \beta.$$

by setting

$$s(x) = e^{\int \frac{a_1(x)}{a_2(x)}\,dx}.$$

$$q(x) = \frac{a_0(x)s(x)}{a_2(x)}.$$

$$w(x) = \frac{b(x)s(x)}{a_2(x)}.$$

Example 22.2

Consider:

$$x\phi''(x) + 2x\phi(x) + \phi(x) +_\lambda x^2\phi(x) = 0, \quad 0 < x < 1.$$

we have:

$$s(x) = e^{\int \frac{2x}{x}\,dx} = e^{2x}.$$

$$q(x) = \frac{1}{x}e^{2x}.$$

$$w(x) = \frac{x^2}{x}e^{2x} = xe^{2x}.$$

Meaning that the equation can be written in the form of:

$$\frac{d}{dx}\left(e^{2x}\phi'(x)\right) + \frac{1}{x}e^{2x}\phi(x) + \lambda xe^{2x}\phi(x) = 0, 0 < x < 1.$$

Remark 22.3 — The form above is called the **self-adjoint form**.

22.2 Properties of Regular Sturm-Liouville Problems

- There exist an infinite number of λ 's that lead to non-zero solutions $\phi(x)$ to the ODE and boundary conditions. These λ 's which can be ordered from smallest to largest are called the **eigenvalues** of the RSLP. Moreover

  $$\lim_{n\to\infty} \lambda_n = +\infty.$$

  This means if you solve a RLSP and found that $\lambda_n = \frac{n}{n+1}$ then there is a problem, since $\lim\limits_{n\to\infty} \frac{n}{n+1} = 1 \neq +\infty$.

- If these λ 's are ordered and if the non-zero functions $\phi_n(x)$ satisfy:

  $$\frac{d}{dx}\left(s(x)\phi_n'(x)\right) + q(x)\phi_n(x) + \lambda_n \omega(x)\phi_n(x) = 0.$$

  and it satisfies the boundary conditions, then $\phi_n(x)$ is called the eigenfunction associated with the eigenvalue λ_n and $\phi_n(x)$ goes through zero exactly $n-1$ times in the open interval $\alpha < x < \beta$. As such $\phi_1(x)$ does not go though zero, $\phi_2(x)$ goes through zero once, and so on.

- $$\phi_m \cdot \phi_n = \begin{cases} 0, n \neq m \\ > 0, n = m \end{cases}.$$

  where

  $$f \cdot g = \int_\alpha^\beta f(x)g(x)w(x)\, dx.$$

  In other words there is a dot product that can be defined on the functions ϕ_n.

- The set of eigenfunctions

  $$\{\phi_1(x), \phi_2(x), \ldots\}.$$

  is called a complete set of basic functions so that if $f(x)$ is any piecewise continuous function in the interval $\alpha < x < \beta$, then we may expand $f(x)$ as:

  $$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x).$$

  with

  $$a_n = \frac{\phi_n \cdot f}{\phi_n \cdot \phi_n}.$$

  moreover, the sum converges to:

  $$\frac{f(x^+) + f(x^-)}{2}.$$

  Where:

  $$f(a^{+)}) = \lim_{x\to a^+} f(x) \quad f(a^-) = \lim_{x\to a^-} f(x).$$

  Note that piecewise continuous means that there can only be a finite number of hole or jump discontinuities but not essential discontinuities.

Remark 22.4 — This means that the regular Sturm Liouville series will fill in all of the hole discontinuities and where there is a jump discontinuity, it will converge to the midpoint of the jump.

Definition 22.5. A function $f(x)$ is called piecewise continuous in a finite interval $\alpha < x < \beta$ is it has at most a finite number of hole or jumps in $\alpha < x < \beta$.

Example 22.6

Consider:

$$\phi''(x) + \lambda\phi(x) = 0, 0 < x < 1.$$

$$\phi(0) = 1 \quad \phi(1) = 0.$$

Thus we have:

$$s(x) = e^{\int \frac{0}{1} dx} = 1.$$

$$q(x) = \frac{(0)(1)}{1} = 0.$$

$$w(x) = \frac{(1)(1)}{1} = 1.$$

Note that this is a RSLP. To solve this problem, notice that this is a problem with constant coefficients, giving us:

$$\phi(x) = \begin{cases} A\cosh(x\sqrt{-\lambda}) + B\sinh(x\sqrt{-\lambda}), \lambda < 0 \\ A + Bx, \lambda = 0 \\ A\cos(x\sqrt{\lambda}) + B\sin(x\sqrt{\lambda}), \lambda > 0 \end{cases}.$$

If we consider $\lambda < 0$, because of the boundary conditions, we get $A = 0$, and $B = 0$, meaning that $\lambda < 0$ gives us $\phi(x) = 0$, meaning that there are no negative eigenvalues.

For $\lambda = 0$, we have:

$$\phi(x) = A = 0 \implies \phi(x) = Bx.$$

$$\phi(1) = B = 0 \implies \phi(x) = 0.$$

meaning that zero is not an eigenvalue.

For $\lambda > 0$, we have:

$$\phi(0) = A = 0 \implies \phi(x) = B\sin(x\sqrt{\lambda}).$$

$$\phi(1) = B\sin(x\sqrt{\lambda}) \implies \sqrt{\lambda} = n\pi \implies \lambda = (n\pi)^2 = \lambda_n.$$

Note that we will always get a multiplicative constant when we try to calculate ϕ_n , thus we can set $B_n = 1$ giving us:

$$\phi_n(x) = \sin(n\pi x).$$

Remark 22.7 — Note that for the above example, $\phi_1(x) = \sin(\pi x)$ does not go through zero on the open interval between 0 and 1. Similarly $\phi_2(x) = \sin(2\pi x)$ goes through zero once at $x = \frac{1}{2}$, etc.

23 April 8th, 2020

23.1 Examples of RSLP

Let us try to determine all λ's that lead to non-zero solutions $\phi(x)$ if:

$$x\phi''(x) - \phi'(x) + \lambda x^3 \phi(x) = 0, \quad 0 < x < 1.$$

$$\phi(0) = 0 \quad \phi(1) = 0.$$

Note that the standard form would be:

$$\phi''(x) - \frac{1}{x}\phi'(x) + \lambda x^2 \phi(x) = 0 \implies \phi''(x) + P(x)\phi'(x) + Q(x) = 0.$$

From HW6, can solve this, with:

$$\gamma = \frac{Q'(x) + 2P(x)Q(x)}{(Q(x))^{\frac{3}{2}}} = \frac{2\lambda x + 2(-1/x)(\lambda x^2)}{(\lambda x^2)^{\frac{3}{2}}} = 0 = \text{ a constant.}$$

Meaning we can use the transformation:

$$z = \int \sqrt{\alpha Q(x)}\ dx = \int \sqrt{\alpha \lambda x^2}\ dx = \int x\ dx = \frac{1}{2}x^2.$$

By picking $\alpha = \frac{1}{\lambda}$. Thus we have:

$$\Psi''(z) + \frac{\gamma}{2\sqrt{\alpha}}\Psi'(z) + \frac{1}{\alpha}\Psi(z) = 0.$$

which reduces to:

$$\Psi''(z) + \lambda\Psi(z) = 0.$$

From the example from last lecture, we would have:

$$\Psi(z) = \begin{cases} A\cosh(z\sqrt{-\lambda}) + B\sinh(z\sqrt{-\lambda}), & \lambda < 0 \\ A + Bz, & \lambda = 0 \\ A\cos(z\sqrt{\lambda}) + B\sin(z\sqrt{\lambda}), & \lambda > 0 \end{cases}.$$

Giving us:

$$\phi(x) = \begin{cases} A\cosh(\frac{1}{2}x^2\sqrt{-\lambda}) + B\sinh(\frac{1}{2}x^2\sqrt{-\lambda}), & \lambda < 0 \\ A + B\frac{1}{2}x^2, & \lambda = 0 \\ A\cos(\frac{1}{2}x^2\sqrt{\lambda}) + B\sin(\frac{1}{2}x^2\sqrt{\lambda}), & \lambda > 0 \end{cases}.$$

Putting in the boundary conditions, consider $\lambda < 0$ we would have:

$$\phi(0) = A = 0 \quad \text{since } \sinh(0) = 0 \ \cosh(0) = 1.$$

Thus:

$$\phi(1) = B\sinh(\frac{1}{2}\sqrt{-\lambda}) = 0 \implies B = 0.$$

This tells us that there are no negative eigenvalues for this problem. As shown in the last lecture, we can test $\lambda = 0$ and see that there are no zero eigenvalues either. For $\lambda > 0$, we would have:

$$\lambda_n = (2n\pi)^2 \quad \phi_n(x) = B_n \sin(n\pi x^2).$$

Picking $B_n = 1$, we would have $\phi_n(x) = \sin(n\pi x^2)$.

With this, we can define the dot product:

$$f \cdot g = \int_0^1 f(x)g(x)w(x)\ dx.$$

$$w(x) = \frac{b(x)}{a_2(x)}e^{\int \frac{a_1(x)}{a_2(x)}dx} = \frac{x^3}{x}e^{\int -\frac{1}{x}\ dx} = x^2\frac{1}{x} = x.$$

This also means that:

$$\phi_m \cdot \phi_n = \int_0^1 \phi_m(x)\phi_n x)x\ dx = 0, \quad m \neq n.$$

and

$$\phi_n \cdot \phi_n = \int_0^1 \phi_n(x)\phi_n x)x\ dx = \int_0^1 \sin(n\pi z)\sin(n\pi z)\frac{1}{2}\ dz = \frac{1}{4} > 0.$$

With this, if a piecewise continuous function $f(x)$ is expressed as:

$$f(x) = \sum_{n=1}^{\infty} a_n\phi_n(x), 0 < x < 1.$$

Then:

$$a_n = \frac{\phi_n \cdot f}{\phi_n \cdot \phi_n} = 4(\phi_n \cdot f).$$

Example 23.1

Consider $f(x) = 1$, we have:

$$a_n = \frac{\int_1^0 \phi_n(x)x\ dx}{\frac{1}{4}} = 4\int_0^1 \sin(n\pi x^2)x\ dx.$$

Using the substitution $z = x^2$, we get:

$$a_n = 4\int_0^1 \sin(n\pi z)\frac{1}{2}\ dz = 2\frac{1-\cos(n\pi)}{n\pi} = \frac{2(1-(-1)^n)}{n\pi}.$$

Thus we have:

$$1 = \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n\pi}\sin(n\pi x^2).$$

Remark 23.2 — Unlike Taylor series which tend to converge monotonically, Sturm-Liouville series tend to converge alternatively.

Sometimes we might not get clean values for the eigenvalues. To see this, we can try:

$$\phi''(x) + \lambda\phi(x) = 0, 0 < x < 1.$$

but with the boundary conditions:

$$\phi(0) = 0 \quad \phi(1) + \phi'(1) = 0.$$

For $\lambda < 0$, we would find:

$$\phi(1) + \phi'(1) = B\left(\sinh(\sqrt{-\lambda}) + \sqrt{-\lambda}\cosh(\sqrt{-\lambda})\right).$$

Which would mean that $B = 0$ since $\left(\sinh(\sqrt{-\lambda}) + \sqrt{-\lambda}\cosh(\sqrt{-\lambda})\right) = 0 \iff \lambda = 0$.

For the case of $\lambda = 0$, we would have: $\phi(1) + \phi'(1) = 2B = 0 \implies B = 0$ meaning that there is only the trivial condition.

For the case of $\lambda > 0$, we would have "

$$\phi(1) + \phi'(1) = B\sin(\sqrt{\lambda}) + B\sqrt{\lambda}\cos(\sqrt{\lambda}) = 0.$$

To avoid $B = 0$, we would need:

$$\sin(z) + z\cos(z) = 0.$$

which has a lot of solutions. However, these solutions are not easy to solve for and thus are calculated numerically. Note that there are infinite solutions, as this is equivalent to $tan(z) = -z$.

Remark 23.3 — Note that it is possible to have negative or zero eigenvalues, for example if it was instead $\phi(1) - \phi'(1) = 0$.

24 April 13th, 2020

24.1 Partial Differential Equation

Definition 24.1 (Wave Equation in 1-Dimension)**.** If we have $y(x,t)$ be the position of the point on the string at x and at time t, we would have the boundary conditions:

$$\begin{cases} y(\alpha,t) = h_1, & 0 \leq t \\ y(\beta,t) = h_1, & 0 \leq t \end{cases}.$$

along the left ($x = \alpha$) and right ($x = \beta$) boundaries. The equation of motion (which is derived from Newton's 2nd law) is:

$$\begin{cases} \frac{\partial}{\partial x}\left(\frac{T(x,t)\frac{\partial y}{\partial x}}{\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}}\right) = \rho(x)\left(y + \frac{\partial^2 y}{\partial t^2}\right)\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} \\ \frac{\partial}{\partial x}\left(\frac{T(x,t)}{\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}}\right) = 0 \end{cases}.$$

Where $T(x,t)$ is the tension function. With initial conditions:

$$y(t,x)\Big|_{t=0} = y(x,0) = y(x) \quad \text{(starting position)}.$$

$$\frac{\partial y(x,t)}{\partial t}\Big|_{t=0} = v(x) \quad \text{(starting velocity)}.$$

Some simplifying assumptions are:

$$T(x,t) = T(x) \qquad \text{meaning that the tension only depends on } x.$$

$$\frac{\partial y}{\partial x} \ll 1 \qquad \text{meaning that the tension is so high, it's hardly moving.}$$

Under these assumptions, we have $\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2} = 1$, making the PDE:

$$\begin{cases} T_0 \frac{\partial^2 y}{\partial x^2} = \rho(x)\left(g + \frac{\partial^2 y}{\partial t^2}\right) \\ \frac{\partial}{\partial x}\left(T(x)\right) = 0 \implies T(x) = T_0 \text{ which is a constant} \end{cases}.$$

If we further assume that $\rho(x) = \text{constant}$, we would get:

$$T_0 \frac{\partial^2 y}{\partial x^2} = \rho_0 g + \rho_0 \frac{\partial^2 y}{\partial t^2}.$$

Thus this problem becomes given $T_0, \rho_0, h_1, h_2, y(x), v(x)$, solve for $y(x,t)$ if:

$$T_0 \frac{\partial^2 y}{\partial x^2} = \rho_0 g + \rho_0 \frac{\partial^2 x}{\partial t^2}, \quad \alpha < x < \beta, 0 < t.$$

$$y(\alpha,t) = h_1 \quad y(\beta,t) = h_2, \quad 0 \le t.$$

$$y(x,0) = y(x), \quad \alpha < x < \beta.$$

$$\frac{\partial y(x,t)}{\partial t}\Big|_{t=0} = v(x), \quad \alpha < x < \beta.$$

This is called a boundary-value, initial-value problem (BCIVP). We can shift it by setting $\alpha = 0$ and $\beta = L$.

Remark 24.2 — BVIVP = PDE + BC + IC

To solve we will do the following:

1. First ask the question are the PDE and boundary conditions Homogeneous? i.e. does $y(x,t) = 0$ satisfy the PDE and boundary conditions. If the answer is no, proceed to step 2, if yes proceed to step 6.

2. Construct the ODE and BCs satisfied by the time independent solution $y(x,t) = y_e(x)$ (equilibrium solution). To construct, in the PDE, replace $y(x,t)$ by $y_e(x)$. For the 1-D wave equation, we have:

$$T_0 \frac{\partial^2 y_e(x)}{\partial x^2} = \rho_0 g + \rho_0 \underbrace{\frac{\partial^2 y_e(x)}{\partial t^2}}_{=0} \implies T_0 y_e''(x) = \rho_0 g, \quad 0 < x < L.$$

For the BCs, we have:

$$\begin{cases} y_e(0) = h_1 \\ y_e(L) = h_2 \end{cases}.$$

3. Solve the ODE and BCs for $y_e(x)$. For the 1-D wave equation, we have:

$$y_e''(x) = \frac{\rho_0 g}{T_0} \implies y_e'(x) = \frac{\rho_0 g}{T_0}x + c_1 \implies y_e(x) = \frac{\rho_0 g x^2}{2T_0} + c_1 x + c_2.$$

$$y_e(0) = c_2 = h_1 \implies y_e(x) = \frac{\rho_0 g x^2}{2T_0} + c_1 x + h_1.$$

$$y_e(L) = \frac{\rho_0 g L^2}{2T_0} + c_1 L + h_1 = h_2 \implies c_1 = \frac{h_2 - h_1}{L} - \frac{\rho_0 g L}{2T_0}.$$

Giving us:

$$y_e(x) = \frac{\rho_0 g x^2}{2T_0} + \left(\frac{h_2 - h_1}{L} - \frac{\rho_0 g L}{2T_0}\right) x + h_1, \quad 0 \le x \le L.$$

This is known as the **time-independent (steady state, equilibrium) shape**.

4. Define

$$u(x,t) = y(x,t) - y_e(x).$$

This can represent the shifting of the string from the equilibrium shape.

5. Construct the BVIVP satisfied by $u(x,t)$. This can done by substituting:

$$y(x,t) = u(x,t) + y_e(x),$$

into the BVIVP for $y(x,t)$. For the 1-D wave equation, we would get:

$$T_0 \frac{\partial^2 (u + y_e)}{\partial x^2} = \rho_0 g + \rho_0 \frac{\partial^2 (u + y_e)}{\partial t^2}.$$

$$\implies T_0 \frac{\partial^2 u}{\partial x^2} + \underbrace{T_0 y_e''(x)}_{\rho_0 g} = \rho_0 g + \rho_0 \frac{\partial^2 u}{\partial t^2}.$$

$$\implies T_0 \frac{\partial^2 u}{\partial x^2} = \rho_0 \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < L, x < t.$$

For the BC, we have:

$$y(0,t) = h_1 \implies u(0,t) + \underbrace{y_e(0)}_{=h_1} = h_1 \implies u(0,t) = 0.$$

and

$$y(L,t) = u(L,t) + \underbrace{y_e(L)}_{=h_2} = h_2 \implies u(L,t) = 0.$$

For the IC, we have:

$$y(x,0) = y(x) \implies u(x,0) = y(x) - y_e(x).$$

$$\left.\frac{\partial (u + y_e)}{\partial t}\right|_{t=0} \implies \left.\frac{\partial u}{\partial t}\right|_{t=0} = v(x).$$

Overall this means that:

$$\text{PDE} \left\{ T_0 \frac{\partial^2 u(x,t)}{\partial x^2} = \rho_0 \frac{\partial^2 u(x,t)}{\partial t^2}, \quad 0 < x < L, 0 < t \right. .$$

$$\text{BCs}\begin{cases} u(0,t)=0 \\ u(L,t)=0 \end{cases}, \quad 0<t.$$

$$\text{ICs}\begin{cases} u(x,0)=y(x)-y_e(x) \\ \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0}=v(x) \end{cases}, \quad 0<x<L.$$

Note that now the PDE and BCs are now homogeneous. We will continue from here next lecture.

25 April 15th, 2020

25.1 Solving Homogeneous BVIVP through Separation of Variables

Continuing from last time, we were able to transform a non-homogeneous BVIVP into one that is homogeneous. We have:

$$\text{PDE}\left\{T_0\frac{\partial^2 u(x,t)}{\partial x^2}=\rho_0\frac{\partial^2 u(x,t)}{\partial t^2}, \quad 0<x<L, 0<t\right. .$$

$$\text{BCs}\begin{cases} u(0,t)=0 \\ u(L,t)=0 \end{cases}, \quad 0<t.$$

$$\text{ICs}\begin{cases} u(x,0)=y(x)-y_e(x) \\ \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0}=v(x) \end{cases}, \quad 0<x<L.$$

Note that if our original BVIVP was homogeneous, we would be able to skip to this point. Moving on, we would like to construct a general solution to the PDE and BCs using the principle of linear-superpositions.

Suppose $u_1(x,t)$ and $u_2(x,t)$ are solutions to the PDE and BCs. Then $c_1u_1+c_2u_2$ would also be solutions for any constants c_1 and c_2. This can be generalized for more u_n. Note that this is possible because the PDE and BCs are homogeneous. As such, we will use

$$\{u_1(x,t),u_2(x,t),\dots\}.$$

as the basis set to generate the general solution.

> **Remark 25.1** — Note that for 2nd order linear ODE, the solution space is 2-dimensional, however for PDE the solution space is infinite dimensional.

In constructing the basis set of solutions, we should try solutions of form:

$$u(x,t)=\phi(x)\gamma(t)\neq 0.$$

To do this, we will replace this into the PDE and BCs, giving us:

$$T_0\frac{\partial^2\phi(x)\gamma(t)}{\partial x^2}=\rho_0\frac{\partial^2\phi(x)\gamma(t)}{\partial t^2}\implies T_0\gamma(t)\frac{d^2\phi(x)}{dx^2}=\rho_0\phi(x)\frac{d^2\gamma(t)}{dt^2}.$$

$$\implies \underbrace{\frac{\phi''(x)}{\phi(x)}}_{\text{function of only } x} = \underbrace{\frac{\rho_0}{T_0}\frac{\gamma''(t)}{\gamma(t)}}_{\text{function of only } t}, \quad 0 < x < L,\ 0 < t.$$

Since x and t are independent, we can set them to be any thing we want, setting $t = 1$ and leaving x alone, we get:

$$\frac{\phi''(x)}{\phi(x)} = \underbrace{\frac{\rho_0}{T_0}\frac{\gamma''(1)}{\gamma(1)}}_{\text{constant}}, \quad 0 < x < L.$$

In a similar way, if we set $x = \frac{L}{2}$ and leaving t alone, we would get:

$$\underbrace{\frac{\phi''(\frac{L}{2})}{\phi(\frac{L}{2})}}_{\text{constant}} = \frac{\rho_0}{T_0}\frac{\gamma''(t)}{\gamma(t)}, \quad 0 < t.$$

Note that because of this, the constants must both be the same, meaning that we have separated the PDE into two ODE:

$$\frac{\phi''(x)}{\phi(x)} = C, \quad 0 < x < L.$$

$$\frac{\rho_0}{T_0}\frac{\gamma''(t)}{\gamma(t)} = C, \quad 0 < t.$$

> **Remark 25.2** — Note that we can skip the working out, since whenever we come across a function of one variable equal a function of another variable, and both variables are independent, then both functions must be constants and equal the same constant.

Rearranging the two equations before, we have:

$$\phi''(x) - C\phi(x) = 0, \quad 0 < x < L.$$

$$\gamma''(t) - \frac{fT_0}{\rho_0}\gamma(t) = 0, \quad 0 < t.$$

Also note that:

$$u(0,t) = 0 \implies \phi(0)\gamma(t) = 0, 0 < t.$$
$$\implies \phi(x) = 0.$$

> **Remark 25.3** — Note that we cannot have $\gamma(t) = 0$, then $u(x,t) = \phi(x)\gamma(t) = 0$

Similarly for the other boundary condition, we would have:

$$u(L,t) = 0 \implies \phi(L)\gamma(t) = 0 \implies \phi(L) = 0.$$

Collecting the ϕ, we would get:

$$\phi''(x) - C\phi(x) = 0, \quad 0 < x < L.$$

$$\phi(0) = 0.$$
$$\phi(L) = 0.$$

which is a regular Sturm-Liouville Problem. As you recall this has the solution:

$$\phi(x) = \begin{cases} A\cosh(x\sqrt{C}) + B\sinh(x\sqrt{C}), & C > 0 \\ A + Bx, & C = 0 \\ A\cos(x\sqrt{-C}) + B\sin(x\sqrt{-C}), & C < 0 \end{cases}.$$

Similarly to before, if we consider $C > 0$, we have:

$$\phi(x) = A\cosh(x\sqrt{C}) + B\sinh(x\sqrt{C}).$$
$$\phi(0) = A = 0 \implies \phi(x) = B\sinh(x\sqrt{C}).$$
$$\phi(L) = B\sinh(L\sqrt{C}) = 0 \implies B = 0.$$

Meaning that $\phi(x) = 0$, which will not give us anything. (if on exam, they will tell us to find non-zero or tell us which cases to consider).

For $C = 0$, we would get the same conclusion $\phi(x) = 0$, since the BC force A and B to both be 0.

For $C < 0$, we have:

$$\phi(x) = A\cos(x\sqrt{-C}) + B\sin(x\sqrt{-C}).$$
$$\phi(0) = A = 0 \implies \phi(x) = B\sin(x\sqrt{-C}).$$
$$\phi(L) = B\sin(L\sqrt{-C}) = 0 \implies L\sqrt{-C} = n\pi \implies C = -\left(\frac{n\pi}{L}\right)^2 = C_n.$$

Note that there are an infinite possible solutions:

$$\phi_n(x) = B_n\sin(x\sqrt{-C_n}) = B_n\sin\left(\frac{n\pi x}{L}\right).$$

Also note that:

$$C_{-n} = -\left(-\frac{n\pi}{L}\right)^2 = -\left(\frac{n\pi}{L}\right)^2 = C_n.$$

and

$$\phi_{-n}(x) = B_{-n}\sin\left(-\frac{n\pi}{L}x\right) = -B_{-n}\sin\left(\frac{n\pi}{L}x\right) = (-B_{-n})\sin\left(\frac{n\pi}{L}x\right) = \phi_n(x).$$

In addition, note that we can throw away $n = 0$, since that would give us $C_0 = 0$ but we need $C < 0$. As such we can throw away all the negative n's, giving us:

$$C_n = -\left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots.$$
$$\phi_n(x) = B_n\sin\left(\frac{n\pi}{L}x\right), \quad n = 1, 2, 3, \ldots.$$

Note that B_n is an arbitrary constant we cannot do anything about, so we usually take $B_n = 1$, giving us:

$$\phi_n(x) = \sin\left(\frac{n\pi}{L}x\right), \quad n = 1, 2, 3, \ldots.$$

Remark 25.4 — Note that this is the solution to the RSLP in which:

$$\lambda_n = -\left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, ...$$

$$\phi_n(x) = \sin\left(\frac{n\pi}{L}x\right), \quad n = 1, 2, 3, \ldots.$$

Moving to γ, we have:

$$\gamma''(t) - C\frac{T_0}{\rho_0}\gamma(t) \implies \gamma_n''(t) + \underbrace{\frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2}_{\omega_n^2}\gamma_n(t) = 0, \quad 0 < t.$$

As such, we have:

$$\gamma_n''(t) + \omega_n^2\gamma_n(t) = 0.$$

Which, as we know has solution:

$$\gamma_n(t) = D_n\cos(\omega_n t) + E_n\sin(\omega_n t), \quad n = 1, 2, 3, \ldots.$$

This means that our basis set of solutions would now be:

$$U_n(x,t) = (D_n\cos(\omega_n t) + E_n\sin(\omega_n t))\sin\left(\frac{n\pi}{L}x\right).$$

$$\omega_n = \frac{n\pi}{L}\sqrt{\frac{T_0}{\rho_0}}.$$

Remark 25.5 — Note that if we carried the constants B_n, they could be absorbed into D_n and E_n.

Now finally, the principle of superposition means that the general solution is:

$$u(x,t) = \sum_{n=1}^{\infty}(D_n\cos(\omega_n t) + E_n\sin(\omega_n t))\,\phi_n(x).$$

$$\phi_n(x) = \sin\left(\frac{n\pi}{L}x\right), \quad 0 \le x \le L, \quad 0 \le t.$$

Which is the general solution to the PDE and the BCs.

Now we consider the initial conditions:

$$u(x,0) = y(0) - y_e(x) \implies \sum_{n=1}^{\infty} D_n\phi_n(x) = y(x) - y_e(x), \quad 0 < x < L.$$

Note that since this is a Sturm-Liouville problem, there is an associated dot product:

$$\phi_p \cdot \phi_q = \int_0^L \phi_p(x)\phi_q(x)\ dx = \begin{cases} 0, & p \neq q \\ \frac{L}{2}, & p = q \end{cases}.$$

Meaning that:

$$D_m = \frac{\phi_n \cdot (y - y_e)}{\phi_n \cdot \phi_n} = \frac{2}{L}\int_0^L \phi_n(x)\,(y(x) - y(e))\ dx.$$

Remark 25.6 — Note that the weight function is equal to 1, since our equation is given by:

$$\phi'' + \lambda\phi = 0.$$

If we go back to the function, and compute the partial derivative with respect to time, we have:

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left(-\omega_n D_n \sin(\omega_n t) + \omega_n E_n \cos(\omega_n t)\right) \phi_n(x).$$

Evaluating at $t = 0$, we have:

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} \omega_n E_n \phi_n(x) = v(x), \quad 0 < x < L.$$

Meaning that:

$$w_n E_n = \frac{\phi_n \cdot v}{\phi_n \cdot \phi_n}.$$

$$\implies E_n = \frac{1}{\omega_n} \frac{2}{L} \int_0^L \phi_n(x) v(x)\, dx.$$

In summary the complete solution to the BVIBP satisfied by $y(x, t)$, is given by:

$$y(x, t) = y_e(x) + u(x, t).$$

Where:

$$y_e(x) = \frac{\rho_0 g}{2T_0} x^2 + \left(\frac{H_2 - H_1}{L} - \frac{\rho_0 g L}{2T_0}\right) x + H_1.$$

and

$$u(x, t) = \sum_{n=1}^{\infty} \left(D_n \cos(\omega_n t) + E_n \sin(\omega_n t)\right) \phi_n(x).$$

where:

$$\omega_n = \frac{n\pi}{L}\sqrt{\frac{T_0}{\rho_0}} \quad \phi_n(x) = \sin\left(\frac{n\pi}{L} x\right).$$

$$D_n = \frac{2}{L} \int_0^L \phi_n(x)(y(x) - y_e(x))\, dx.$$

$$E_n = \frac{2}{L\omega_n} \int_0^L \phi_n(x) v(x)\, dx.$$

26 April 20th, 2020

26.1 Heat Equation

If we have a one dimensional rod with ends $x = \alpha$ and $x = \beta$, with

- mass density: $\rho(x)$
- conductivity: $\kappa(x)$

- specific heat per unit mass: $c(x)$
- temperature at point x at time t: $u(x,t)$
- heat generation term within rod: $R(x,t)$ (for our purpose, assume $R(x,t) = 0$)

Using conservation of energy, we get:

$$\frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial u}{\partial x}\right) = \rho(x)c(x)\frac{\partial u}{\partial t}.$$

This is known as the **one dimensional heat equation**.

Remark 26.1 — If we compare this with the one dimensional wave equation from earlier:

$$\frac{\partial}{\partial x}\left(T(x)\frac{\partial u}{\partial x}\right) = \rho(x)\frac{\partial^2 u}{\partial t^2}.$$

The only difference is that the wave equation has a second order partial derivative, while the heat has a first.

This makes a difference, as the second order derivatives means that the wave equation has oscillations, while the heat equation has exponential decay.

Let us consider the special case where $\kappa(x) = \kappa_0$, $\rho(x) = \rho_0$, $c(x) = c_0$, which gives us:

$$\kappa_0\frac{\partial^2 u}{\partial x^2} = \rho_0 c_0\frac{\partial u}{\partial t} \implies \frac{\partial^2 u}{\partial x^2} = \frac{1}{\gamma}\frac{\partial u}{\partial t}, \quad \gamma = \frac{k_0}{\rho_0 c_0}.$$

Note that since we can shift this equation to $x = 0$ and $x = L$.

Remark 26.2 — γ_0 is often known as the diffusity constant.

Now for the boundary conditions, let us fix $x = 0$ to be a certain temperature T_1, while we expose $x = L$ to a room of temperature T_2. This gives us the boundary conditions:

$$u(0,t) = T_1$$

$$hu(L,t) + \kappa_0\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=L} = hT_2.$$

Remark 26.3 — Note that these boundary conditions are only examples, if we fixed the temperature at $x = L$, we would have $u(L,t) = T_2$.

Now for the initial condition, note that since the order of the derivative is only one, we only need one initial condition:

$$u(x,0) = f(x), 0 < x < L.$$

Thus the mathematical formulation of the problem is:

Given the constants, $\rho_0, c_0, \kappa_0, h, T_1, T_2, \gamma = \frac{\kappa_0}{\rho_0 c_0}$ and the function $f(x)$, solve for $u(x,t)$ in the region $x \leq x \leq L$, $0 \leq t$, if:

$$\text{PDE}\left\{\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{\gamma}\frac{\partial u(x,t)}{\partial t}, \quad 0 < x < L, \quad 0 < t \right. .$$

$$\text{BCs}\begin{cases} u(0,t) = T_1, \quad 0 < t \\ hu(L,t) + \kappa_0 \frac{\partial u(x,t)}{\partial x}\Big|_{x=L} = hT_0, \quad 0 < t \end{cases}.$$

$$\text{ICs}\begin{cases} u(x,0) = f(x), \quad 0 < x < L \end{cases}.$$

Just like the wave equation, we ask the following questions:

1. Is the PDE and BCs Homogeneous by setting $u(x,t) = 0$

   - PDE: yes
   - BC ($x = 0$): no
   - BC ($x = L$): no

   If at least one of these is no, we move to step 2, otherwise we can skip to step 5.

2. Define and construct the boundary value problem satisfied by some time independent solution $u_e(x)$:

$$\frac{\partial^2 u_e(x)}{\partial x^2} = \frac{1}{\gamma}\frac{\partial u_e(x)}{\partial t}.$$

$$\implies u_e''(x) = 0, \quad 0 < x < L.$$

$$u_e(0) = T_1 \quad hu_e(L) + \kappa_0 u_e'(L) = hT_2.$$

3. Solve this BVP for u_e. For this case:

$$u_e(x) = Ax + B.$$

$$u_e(0) = B = T_1 \implies u_e'(x) = A.$$

$$hu_e(L) + \kappa_0 u_e'(x) = hT_2 \implies h(AL + T_1) + \kappa_0 A = hT_2.$$

$$\implies A = \frac{h(T_2 - T_1)}{hL + \kappa_0}.$$

   Thus the time independent solution is:

$$u_e(x) = \frac{h(T_2 - T_1)}{hL + \kappa_0}x + T_1, \quad 0 \le x \le L.$$

4. Define:

$$v(x,t) = u(x,t) - u_e(x).$$

   and the construct the BVIVP satisfied by $v(x,t)$:

$$\frac{\partial^2 (v(x,t) + u_e(x))}{\partial x^2} = \frac{1}{\gamma}\frac{\partial (v(x,t) + u_e(x))}{\partial t}.$$

$$\implies \frac{\partial^2 v(x,t)}{\partial x^2} + \underbrace{u_e''(x)}_{=0} = \frac{1}{\gamma}\left(\frac{\partial v(x,t)}{\partial t} + 0\right).$$

$$\implies \frac{\partial^2 v(x,t)}{\partial x^2} = \frac{1}{\gamma}\frac{\partial v(x,t)}{\partial t}, \quad 0 < x < L, \quad 0 < t.$$

$$v(0,t) + u_e(0) = T_1 \implies v(0,t) = 0, \quad 0 < t.$$

$$hu(L,t) + \kappa_0 \frac{\partial u(x,t)}{\partial x}\bigg|_{x=L} = hT_2 \implies hv(L,t) + \kappa_0 \frac{\partial v(x,t)}{\partial x} = 0, \quad 0 < t.$$

$$v(x,0) = f(x) - u_e(x), \quad 0 < x < L.$$

Note that the PDE and BCs for this BVIVP are homogeneous.

5. Construct a general solution to the PDE and BCs for $v(x,t) = u(x,t) - u_e(x)$ by constructing a basis set of solutions of the form:

$$v(x,t) = \phi(x)\beta(t) \neq 0.$$

Plugging in the PDE, we get:

$$\frac{\partial^2 (\phi(x)\beta(t))}{\partial x^2} = \frac{1}{\gamma} \frac{\partial (\phi(x)\beta(t))}{\partial t}.$$

$$\implies \phi''(x)\beta(t) = \frac{1}{\gamma}\phi(x)\beta'(t).$$

$$\implies \frac{\phi''(x)}{\phi(x)} = \frac{1}{\gamma}\frac{\beta'(t)}{\beta(t)} = \text{ a constant } C.$$

This gives us:

$$\phi''(x) - V\phi(x) = 0, \quad 0 < x < L.$$

$$\beta'(t) - \gamma c\beta(t) = 0, \quad 0 < t.$$

with boundary conditions

$$v(0,t) = 0 \implies \phi(0) = 0.$$

$$h\phi(L) + \kappa_0\phi'(L) = 0.$$

Note that this is a RSLP for x.

Recall the this has solutions:

$$\phi(x) = \begin{cases} A\cosh(x\sqrt{C}) + B\sinh(x\sqrt{C}), & C > 0 \\ A + Bx, & C = 0 \\ A\cos(x\sqrt{-C}) + B\sin(x\sqrt{-C}), & C < 0 \end{cases}.$$

As with before, note that for the cases $C > 0$ and $C = 0$, we would get $\phi(x) = 0$, with the only case $C < 0$:

$$\phi(x) = A\cos(x\sqrt{-C}) + B\sin(x\sqrt{-C}).$$

$$\phi(0) = A = 0 \implies \phi(x) = B\sin(x\sqrt{-C}) \implies \phi'(x) = B\sqrt{-C}\cos(x\sqrt{-C}).$$

$$h\phi(L) + \kappa_0\phi'(L) = 0 \implies B\left(h\sin(L\sqrt{C}) + \kappa_0\sqrt{-C}\cos(L\sqrt{-C})\right) = 0.$$

$$\implies \tan\left(L\sqrt{-C}\right) = -\frac{\kappa_0}{h}\sqrt{-C}.$$

Once again, this will give us $\lambda_1, \lambda_2, \ldots$ which are the eigenvalues for this problem. We will need to solve this numerically.

Once we have these, we have:

$$\phi_n(x) = \sin(\lambda_n x), \quad n = 1, 2, 3, \ldots.$$

We will continue from this point next lecture.

27 April 22th, 2020

27.1 Continuation of the Heat Equation

From last time, we had:

$$\tan(L\sqrt{-V}) = -\frac{k_0}{h}\sqrt{-C} = -\frac{\kappa_0}{hL}(L\sqrt{-C}).$$

Thus if we let $L\sqrt{-C} = z \implies C = -\left(\frac{z}{L}\right)^2$, then we want to find the solutions to the equation:

$$\tan z = -\frac{\kappa_0}{hL}z.$$

With this, we would get:

$$\phi_n(x) = \sin(z_n\frac{x}{L}).$$

Since this is in the RSLP form, it has an associated dot product:

$$\phi_n \cdot \phi_m = \int_0^L \phi_n(x)\phi_m(x)\ dx = \begin{cases} 0, & p \neq q \\ \frac{L}{2}\left(1 + \frac{\sin(2z_n)}{2z_n}\right) \end{cases}.$$

Note that the weight function $w(x) = 1$ (verify by comparing coefficients).

Going to the time equation:

$$\beta_n'(t) - \gamma C_n \beta_n(t) = 0.$$

$$\implies \beta_n'(t) + \gamma\left(\frac{z_n}{L}\right)^2\beta_n(t) = 0.$$

$$\implies \beta_n(t) = e^{-\gamma\left(\frac{z_n}{L}\right)^2 t}.$$

This gives us:

$$v_n(x,t) = \phi_n(x)\beta_n(t) = e^{-\gamma\left(\frac{z_n}{L}\right)^2 t}\sin\left(\frac{z_n}{L}x\right).$$

Meaning that the general solution is:

$$v(x,t) = \sum_{n=1}^{\infty} a_n e^{-\gamma\left(\frac{z)n}{L}\right)^2 t}\sin\left(\frac{z_n}{L}x\right).$$

For the initial condition, we have:

$$v(x,0) = \sum_{n=1}^{\infty} a_n\phi_n(x) = f(x) - u_e(x).$$

$$\implies a_n = \frac{\phi_n(f-u_e)}{\phi_n \cdot \phi_n} = \frac{1}{\frac{L}{2}\left(1+\frac{\sin(2z_n)}{2z_n}\right)}\int_0^L (f(x)-u_e(x))\phi_n(x)\ dx, \quad n = 1,2,3,...$$

Thus the final solution is:

$$u(x,t) = u_e(x) + \sum_{n=1}^{\infty} a_n e^{-\gamma\left(\frac{z_n}{L}\right)^2 t}\phi_n(x).$$

Note that the only difference between this solution and the wave equation is that the eigenvalues are different.

Remark 27.1 — Note that since $e^{-(\text{constant})t}$ approaches 0 as t increases, we have:

$$\lim_{t\to\infty} u(x,t) = u_e(x).$$

Note that this is not the case for the wave equation (unless there is a dampening term).

Remark 27.2 — The wave equation with damping has the form:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t}.$$

This will have a limiting solution equal to the time independent solution $u_e(x)$.

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