1 February 12th, 2020

1.1 Example Mass/Spring/Damper System

We have a mass $m > 0$ attached to a spring with spring coefficient $k > 0$ and a dampener with coefficient $b \geq 0$. If we assume no coefficient of friction, we get

$$-k - x - b\dot{x} = m\ddot{x}.$$

Which can be rearranged to:

$$m\ddot{x} + b\dot{x} + kx = 0.$$

Which is a 2nd-order linear homogeneous ODE with constant coefficients, which we can use the table from earlier to solve. If we include an external force acting on the mass, we would have:

$$m\ddot{x} + b\dot{x} + kx = F(t) \tag{1}$$

Which would make it non homogeneous. There is an analog circuit equivalent called the LCR circuit, which would have an equation:

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = \Delta V.$$

Which is of the same form as Equation 1.

Let us consider the case without a driving force $F(t)$:

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0.$$

First, we will denote $\omega = \sqrt{\frac{k}{m}}$ which represents the **angular frequency** of the system, with units rad per sec, and $\gamma = \frac{b}{2\sqrt{mk}}$ be a **dampening ratio** (which represents how much dampening is in the system), making the equation:

$$\ddot{x} + 2\gamma\omega\dot{x} + \omega^2 x = 0.$$

Note that discriminant of this equation is:

$$D = \frac{b^2}{m^2} - 4\frac{k}{m} = \frac{4k}{m}\left(\frac{b^2}{4\sqrt{mk}} - 1\right) = 4\omega^2(\gamma^2 - 1).$$

Now depending on what γ and ω are, we can analyze the behaviour of the system.

1.2 No Dampening ($\gamma = 0$)

In this case, we would have:

$$\ddot{x} + \omega^2 x = 0.$$

The discriminant is thus:

$$D = 0^2 - 4(1)(\omega^2) = -4\omega^2 < 0.$$

Using the table, we have:

$$\alpha = -\frac{0}{2(1)} = 0, \quad \beta = \frac{\sqrt{-(-4\omega^2)}}{2(1)} = \omega.$$

Thus the solution will just be:

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

If we want to find the constants, note that $x(0) = x_0 = c_1$. Meanwhile, differentiating the equation, we have:

$$v(t) = -c_1\omega \sin(\omega t) + \omega c_2(\omega t).$$
$$v(0) = v_0 = \omega c_2.$$

Thus the complete solution is:

$$x(t) = x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t).$$

This is just a sin curve with amplitude: $A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}$ and period: $T = \frac{2\pi}{\omega}$.

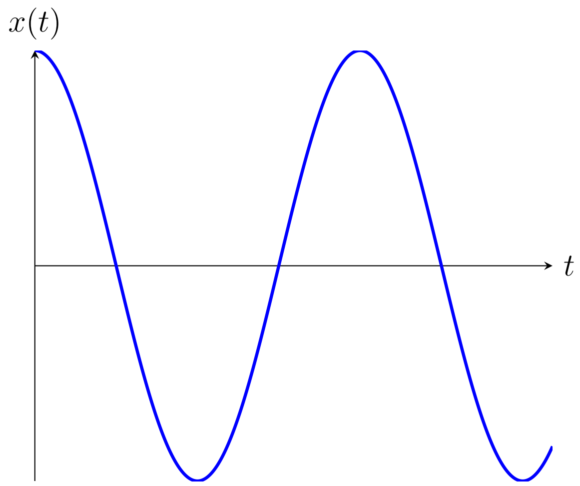

Figure 1: Example of Underdamped Motion

Remark 1.1 — Note that the period does not depend on x_0 or v_0, i.e. it doesn't depend on how it starts. This is different from SHM.

1.3 Under Damping ($0 < \gamma < 1$)

Returning to our equation, we have:

$$\ddot{x} + 2\gamma\omega\dot{x} + \omega^2 x = 0.$$

Thus the determinant is:

$$D = (2\gamma\omega)^2 - 4(1)(\omega^2) = 4`w^2(\gamma^2 - 1).$$

If $0 < \gamma < 1$, we have $D < 0$, giving us:

$$\alpha = \frac{-(2\gamma\omega)}{2(1)} = -\gamma\omega, \quad \beta = \frac{\sqrt{-D}}{2(1)} = \omega\sqrt{1-\gamma^2}.$$

Plugging this into the equation, we get:

$$x(t) = c_1 e^{-\gamma\omega t} \cos(\omega t\sqrt{10\gamma^2}) + c_2 e^{-\gamma\omega t} \sin(\omega t\sqrt{1-\gamma^2}).$$

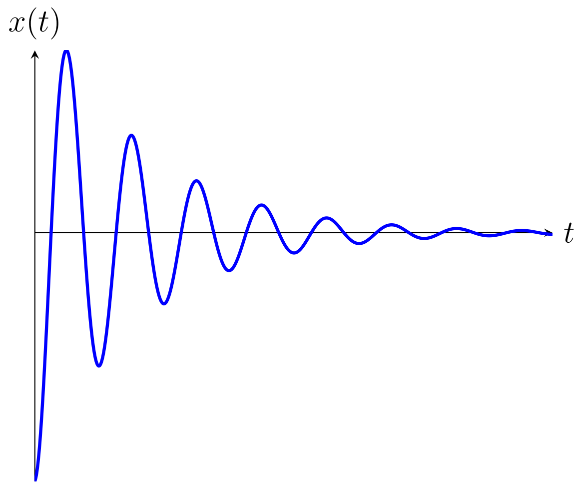

Figure 2: Example of Underdamped Motion

Remark 1.2 — Note that there will be infinite oscillations where the amplitude is decreasing to 0.

1.4 Critical Damping ($\gamma = 1$)

Notice in the case of $\gamma = 1$, we have $D = 0$, thus the solution is:

$$x(t) = c_1 e^{-\gamma\omega t} + c_2 t e^{-\gamma\omega t}.$$

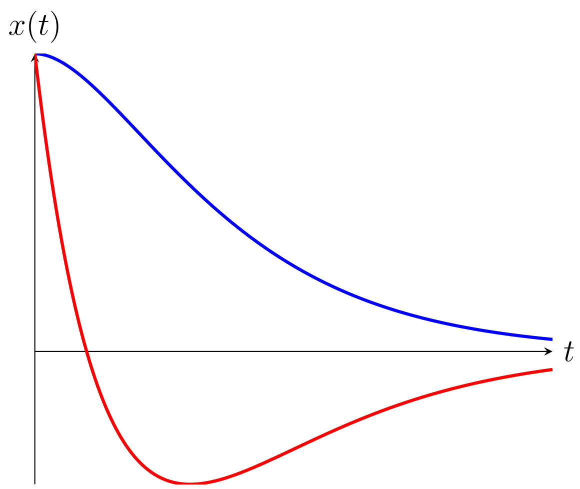

Figure 3: Example of Critical Damped / Over Damped Motion

Remark 1.3 — Note that in this case, there are no oscillations. There will never be two dips. This is because we have:

$$x(t) = c_1 e^{-\gamma\omega t} + c_2 t e^{-\gamma\omega t} = (c_1 + c_2 t) e^{-\gamma\omega t}.$$

Thus by looking at the sign of c_1 and c_2, it will either never cross the x axis (if same sign) or only cross it once (if signs are different). This can be shown by looking at the roots of the equation above.

1.5 Over Damping ($\gamma > 1$)

This yields $D > 0$, thus:

$$x(t) = c_1 e^{-\gamma\omega t}\cosh(\gamma t\sqrt{\gamma - 1}) + c_2 e^{-\gamma\omega t}\sinh(\omega t\sqrt{\gamma^2 - 1}).$$

Remark 1.4 — This is the case where we are taking away the energy a lot, which is useful in many cases. This will make it go to 0 a lot faster than critical damping. Thus for car suspension, we would rather it be critically damped than over damped.

Remark 1.5 — In circuits, this is analogous to using resistors to take away heat from the circuit.

1.6 Laplace Transforms

Laplace transforms are a special case of integral transforms. One way to think of an integral transform is that it's a function where the input is a function of t and output a function of s.

Definition 1.6. More specifically, a **integral transform** is of form:

$$\int_{\alpha(s)}^{\beta(s)} f(t)K(s,t)\ dt.$$

Where $K(s,t)$ is the **kernel** of the transform, and $\alpha(s)$ and $\beta(s)$ are the upper and lower limit.

Example 1.7

Consider the case where $\alpha(s) = s,\ \beta(s) = s^2,\ K(s,t) = st$, and an input $f(t) = t^3$. Then the output would be:

$$\int_s^{s^2} t^3(st)\ dt = \frac{st^5}{5}\Big|_{t=s}^{t=s^2} = \frac{1}{5}\left(s^{11} - s^6\right) = F(s).$$

Definition 1.8. Typically, we represent this integral transform as $T\{f(t)\} = F(s)$.

Definition 1.9. The **Laplace Transform** is a special case where:

$$\alpha(s) = 0 \quad \beta(s) = \infty \quad K(s,t) = e^{-st},$$

in other words:

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\ dt = F(s).$$

Remark 1.10 — Note that st must be unitless, and if t represents time, then s represents frequency, thus making the Laplace transform a transformation from time space into frequency space.

Example 1.11

We have
$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \frac{e^{-st}}{-s}\bigg|_0^\infty = \frac{1}{s}.$$
Note that $s > 0$

In order to go from s-space back to t-space, we take the inverse Laplace transform. This will be unique as long as we don't consider null functions.

Definition 1.12. A **null function** is a function that is zero except for finitely many points.

Example 1.13

An example of a null function is:
$$N(t) = \begin{cases} 1, & t = 0 \\ 2, & t = 1 \\ 0, & \text{otherwise} \end{cases}.$$

These null functions do not appear often for our situation, so we can have a Laplace transform table:

Table 1: Laplace Transform Table

1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$s > 0$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$s > 0$
	$\vdots$	

Remark 1.14 — Using the table, one example is: $\mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\}$