MATH3322 - Matrix Computation

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Contents

Eigenvalue Decomposition

Definition 1. Let $A \in \mathbb{R}^{n\times n}$ be a square matrix. A non-zero vector x is an eigenvector of A with $\lambda \in \mathbb{C}$ being the corresponding eigevalue if:

$$Ax = \lambda x.$$

- Even if A is a real matrix, its eigenvalue and eigenvectors can be complex
- The set of eigenvalues of A is called the spectrum of A. The spectral radius $\rho(A)$ is the maximum value $|\lambda|$ over all eigenvalues of A.
- If (λ, x) is an eigenpair of A, then:

$$\begin{aligned} \left(\lambda^2, x\right) &\text{ is a eigenpair of } A^2 \\ (\lambda - \sigma, x) &\text{ is a eigenpair of } A - \sigma I \\ \left(\frac{1}{\lambda - \sigma}, x\right) &\text{ is a eigenpair of } (A - \sigma I)^{-1}. \end{aligned}$$

  Proof. Since (λ, x) is an eigenpair of A, $Ax = \lambda x$ Multiplying both sides by A from the left:

$$\begin{aligned} A \cdot A = \lambda Ax \implies A^2 x = \lambda Ax = \lambda \cdot \lambda x = \lambda^2 x. \\ Ax - \sigma x = \lambda x - \sigma x \implies (A - \sigma I)\, x = (\lambda - \sigma)\, x \\ \implies x = (\lambda - \sigma)(A - \sigma I)^{-1} x \implies (A - \sigma I)^{-1} x. \end{aligned}$$

  □

Definition 2. Two matricies A and B are similar with each other if there exists a nonsingular matrix T such that

$$B = TAT^{-1}.$$

Theorem 1. If A and B are similar, then A and B have the same eigenvalues.

Proof. Since A, B are similar, $B = TAT^{-1}$, which implies $A = T^{-1}BT$. If (λ, x) is an eigenpair of A, then $Ax = \lambda x$, so that

$$T^{-1}BTx = \lambda x \implies B\,(Tx) = \lambda\,(Tx).$$

Thus, (λ, Tx) is an eigenpair of B. i.e. any eigenvalue of A is an eigenvalue of B. The reverse is similar. □

Definition 3. An eigenvalue decomposition of a square matrix $A \in \mathbb{R}^{n\times n}$ is a factorization

$$A = X\Lambda X^{-1},$$

where $X \in \mathbb{C}^{n\times n}$ is non-singular and $\Lambda \in \mathbb{C}^{n\times n}$ is diagonal.

- If $A \in \mathbb{R}^{n\times n}$ admits an eigenvalue decomposition, then

$$AX = X\Lambda.$$

  If we rewrite $X = [x_1 x_2 \dots x_n]$

Decomposition