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1.1 Heat Equation

If we have a one dimensional rod with ends $x = \alpha$ and $x = \beta$, with

- mass density: $\rho(x)$
- conductivity: $\kappa(x)$
- specific heat per unit mass: $c(x)$
- temperature at point x at time t: $u(x,t)$
- heat generation term within rod: $R(x,t)$ (for our purpose, assume $R(x,t) = 0$)

Using conservation of energy, we get:

$$\frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial u}{\partial x}\right) = \rho(x)c(x)\frac{\partial u}{\partial t}.$$

This is known as the **one dimensional heat equation**.

Remark 1.1 — If we compare this with the one dimensional wave equation from earlier:

$$\frac{\partial}{\partial x}\left(T(x)\frac{\partial u}{\partial x}\right) = \rho(x)\frac{\partial^2 u}{\partial t^2}.$$

The only difference is that the wave equation has a second order partial derivative, while the heat has a first.

This makes a difference, as the second order derivatives means that the wave equation has oscillations, while the heat equation has exponential decay.

Let us consider the special case where $\kappa(x) = \kappa_0$, $\rho(x) = \rho_0$, $c(x) = c_0$, which gives us:

$$\kappa_0\frac{\partial^2 u}{\partial x^2} = \rho_0 c_0\frac{\partial u}{\partial t} \implies \frac{\partial^2 u}{\partial x^2} = \frac{1}{\gamma}\frac{\partial u}{\partial t}, \quad \gamma = \frac{k_0}{\rho_0 c_0}.$$

Note that since we can shift this equation to $x = 0$ and $x = L$.

Remark 1.2 — γ_0 is often known as the diffusity constant.

Now for the boundary conditions, let us fix $x = 0$ to be a certain temperature T_1, while we expose $x = L$ to a room of temperature T_2. This gives us the boundary conditions:

$$u(0,t) = T_1$$

$$hu(L,t) + \kappa_0\frac{\partial u(x,t)}{\partial x}\bigg|_{x=L} = hT_2.$$

> **Remark 1.3** — Note that these boundary conditions are only examples, if we fixed the temperature at $x = L$, we would have $u(L,t) = T_2$.

Now for the initial condition, note that since the order of the derivative is only one, we only need one initial condition:

$$u(x,0) = f(x), 0 < x < L.$$

Thus the mathematical formulation of the problem is:

Given the constants, $\rho_0, c_0, \kappa_0, h, T_1, T_2, \gamma = \frac{\kappa_0}{\rho_0 c_0}$ and the function $f(x)$, solve for $u(x,t)$ in the region $x \le x \le L$, $0 \le t$, if:

$$\text{PDE} \begin{cases} \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{\gamma}\frac{\partial u(x,t)}{\partial t}, \quad 0 < x < L, \quad 0 < t \end{cases}.$$

$$\text{BCs} \begin{cases} u(0,t) = T_1, \quad 0 < t \\ hu(L,t) + \kappa_0 \frac{\partial u(x,t)}{\partial x}\Big|_{x=L} = hT_0, \quad 0 < t \end{cases}.$$

$$\text{ICs} \begin{cases} u(x,0) = f(x), \quad 0 < x < L \end{cases}.$$

Just like the wave equation, we ask the following questions:

1. Is the PDE and BCs Homogeneous by setting $u(x,t) = 0$

   - PDE: yes
   - BC $(x = 0)$: no
   - BC $(x = L)$: no

   If at least one of these is no, we move to step 2, otherwise we can skip to step 5.

2. Define and construct the boundary value problem satisfied by some time independent solution $u_e(x)$:

$$\frac{\partial^2 u_e(x)}{\partial x^2} = \frac{1}{\gamma}\frac{\partial u_e(x)}{\partial t}.$$

$$\implies u_e''(x) = 0, \quad 0 < x < L.$$

$$u_e(0) = T_1 \quad hu_e(L) + \kappa_0 u_e'(L) = hT_2.$$

3. Solve this BVP for u_e. For this case:

$$u_e(x) = Ax + B.$$

$$u_e(0) = B = T_1 \implies u_e'(x) = A.$$

$$hu_e(L) + \kappa_0 u_e'(x) = hT_2 \implies h(AL + T_1) + \kappa_0 A = hT_2.$$

$$\implies A = \frac{h(T_2 - T_1)}{hL + \kappa_0}.$$

   Thus the time independent solution is:

$$u_e(x) = \frac{h(T_2 - T_1)}{hL + \kappa_0} x + T_1, \quad 0 \le x \le L.$$

4. Define:
$$v(x,t) = u(x,t) - u_e(x).$$
and the construct the BVIVP satisfied by $v(x,t)$:
$$\frac{\partial^2 (v(x,t) + u_e(x)}{\partial x^2} = \frac{1}{\gamma}\frac{\partial (v(x,t) + u_e(x)}{\partial t}.$$
$$\implies \frac{\partial^2 v(x,t)}{\partial x^2} + \underbrace{u_e''(x)}_{=0} = \frac{1}{\gamma}\left(\frac{\partial v(x,t)}{\partial t} + 0\right).$$
$$\implies \frac{\partial^2 v(x,t)}{\partial x^2} = \frac{1}{\gamma}\frac{\partial v(x,t)}{\partial t}, \quad 0 < x < L, \quad 0 < t.$$
$$v(0,t) + u_e(0) = T_1 \implies v(0,t) = 0, \quad 0 < t.$$
$$hu(L,t) + \kappa_0 \frac{\partial u(x,t)}{\partial x}\bigg|_{x=L} = hT_2 \implies hv(L,t) + \kappa_0 \frac{\partial v(x,t)}{\partial x} = 0, \quad 0 < t.$$
$$v(x,0) = f(x) - u_e(x), \quad 0 < x < L.$$
Note that the PDE and BCs for this BVIVP are homogeneous.

5. Construct a general solution to the PDE and BCs for $v(x,t) = u(x,t) - u_e(x)$ by constructing a basis set of solutions of the form:
$$v(x,t) = \phi(x)\beta(t) \neq 0.$$
Plugging in the PDE, we get:
$$\frac{\partial^2 (\phi(x)\beta(t))}{\partial x^2} = \frac{1}{\gamma}\frac{\partial (\phi(x)\beta(t))}{\partial t}.$$
$$\implies \phi''(x)\beta(t) = \frac{1}{\gamma}\phi(x)\beta'(t).$$
$$\implies \frac{\phi''(x)}{\phi(x)} = \frac{1}{\gamma}\frac{\beta'(t)}{\beta(t)} = \text{ a constant } C.$$
This gives us:
$$\phi''(x) - V\phi(x) = 0, \quad 0 < x < L.$$
$$\beta'(t) - \gamma c\beta(t) = 0, \quad 0 < t.$$
with boundary conditions
$$v(0,t) = 0 \implies \phi(0) = 0.$$
$$h\phi(L) + \kappa_0\phi'(L) = 0.$$
Note that this is a RSLP for x.

Recall the this has solutions:
$$\phi(x) = \begin{cases} A\cosh(x\sqrt{C}) + B\sinh(x\sqrt{C}), & C > 0 \\ A + Bx, & C = 0 \\ A\cos(x\sqrt{-C}) + B\sin(x\sqrt{-C}), & C < 0 \end{cases}.$$

As with before, note that for the cases $C > 0$ and $C = 0$, we would get $\phi(x) = 0$, with the only case $C < 0$:

$$\phi(x) = A\cos(x\sqrt{-C}) + B\sin(x\sqrt{-C}).$$

$$\phi(0) = A = 0 \implies \phi(x) = B\sin(x\sqrt{-C}) \implies \phi'(x) = B\sqrt{-C}\cos(x\sqrt{-C}).$$

$$h\phi(L) + \kappa_0\phi'(L) = 0 \implies B\left(h\sin(L\sqrt{C}) + \kappa_0\sqrt{-C}\cos(L\sqrt{-C})\right) = 0.$$

$$\implies \tan\left(L\sqrt{-C}\right) = -\frac{\kappa_0}{h}\sqrt{-C}.$$

Once again, this will give us $\lambda_1, \lambda_2, \ldots$ which are the eigenvalues for this problem. We will need to solve this numerically.

Once we have these, we have:

$$\phi_n(x) = \sin(\lambda_n x), \quad n = 1, 2, 3, \ldots.$$

We will continue from this point next lecture.