1 November 12th, 2020

1.1 Iterative Methods to Solve Linear Systems

Last time, we introduced three methods, Jacobi, Gauss-Seidel, and SOR. Let us consider the Jacobi iteration. We want to show that it is irreducibly diagonally dominant matrix.

Proof. Assume A is irreducibly diagonally dominant. Recall that the Jacobi iteration is of the form

$$G = D^{-1}(L+U).$$

We want to show that $\rho(G) < 1$. Assume λ is an eigenvalue of G. We need to now show that $|\lambda| < 1$. We have:

$$\begin{aligned} Gv &= \lambda v \\ \implies D^{-1}(L+U)v &= \lambda v \\ \implies (L+U)v &= \lambda Dv \\ \implies (\lambda D - L - U)v &= 0. \end{aligned}$$

Writing down each row explicitly, we have:

$$\lambda A_{kk} v_k + \sum_{\ell \neq k} A_{k\ell} v_\ell = 0 \quad \forall k.$$

$$\lambda A_{kk} v_k = -\sum_{\ell \neq k} A_{k\ell} v_\ell \quad \forall k.$$

Taking absolute value on both sides, we have:

$$|\lambda||A_{kk}||v_k| = |\sum_{\ell \neq k} A_{k\ell} v_\ell| \leq \sum_{\ell \neq k} |A_{k\ell}||v_\ell|.$$

If we choose $|v_k| = \max_i |v_i| \neq 0$, we would have:

$$|\lambda| \leq \sum_{\ell \neq k} \frac{|A_{k\ell}|}{|A_{kk}|} \frac{|v_\ell|}{|v_k|} \leq \frac{1}{|A_{kk}|} \sum_{\ell \neq k} |A_{k\ell}| \leq 1.$$

The last inequality is because we assume that A is diagonally dominant.

If $|\lambda| = 1$, we would have:

$$1 \leq \sum_{\ell \neq k} \frac{|A_{k\ell}|}{|A_{kk}|} \frac{|v_\ell|}{|v_k|} \leq 1 \implies |v_\ell| = |v_k|, \forall \ell.$$

Now for all k, we have:

$$\sum_{\ell \neq k} |A_{k\ell}| = |A_{kk}|.$$

which is a contradiction to the irreducibility of A. As such $|\lambda| < 1$ for all eigenvalues of A, meaning that the Jacobi iteration converges. ∎

As shown last time, if we consider the central difference for $-u'' = f$, then the matrix is irreducibly diagonally dominant. We can compute G, with:

$$G = D^{-1}(L+U) = \frac{1}{2}\begin{bmatrix} 0 & 1 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{bmatrix}_{(N-1)\times(N-1)}.$$

In addition, we can directly compute the eigenvalues of the original differential operator, as we have:

$$-u'' = \lambda u, \quad u(0) = u(1) = 0.$$

Since the general solution to $-u'' = \lambda u$ is:

$$u = \begin{cases} A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x) & \lambda > 0 \\ Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x} & \lambda < 0 \end{cases}.$$

Note that the second case does not satisfy the boundary condition, as:

$$u(0) = A + B = 0 \quad u(1) = Ae^{\sqrt{-\lambda}} + Be^{-\sqrt{-\lambda}} = 0.$$

$$\implies Ae^{\sqrt{-\lambda}}(1 - e^{-2\sqrt{-\lambda}}) = 0.$$

which is untrue, since $e^{-2\sqrt{-\lambda}} < 1$. As such, $\lambda > 0$. Considering the boundary conditions, we have:

$$u(0) = B = 0.$$

$$u(1) = A\sin(\sqrt{\lambda}) = 0 \implies \lambda_n = (n\pi)^2.$$

since $A \neq 0$, otherwise we would get the trivial solution. The eigenvectors are thus:

$$u_n(x) = \sin(n\pi x).$$

Now consider the eigenvalues and eigenvectors of G, which are discretized. We predict that the eigenvectors are of form:

$$u_n = \sin(n\pi x_j) = \begin{bmatrix} \sin(n\pi x_1) \\ \sin(n\pi x_2) \\ \vdots \\ \sin(n\pi x_{N-1}) \end{bmatrix}.$$

where $x_j = jh = \frac{j}{N}$. We want to show that:

$$Gu_n = \lambda_n u_n.$$

Consider the j-th row of Gu_n, we have:

$$\frac{1}{2}\begin{bmatrix} 0 & 0 & \dots & 1 & 0 & 1 & 0\dots \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_j \\ u_{j+1} \\ \vdots \\ u_{N-1} \end{bmatrix} = \frac{1}{2}(u_{j-1} + u_{j+1}) = \lambda_n u_j.$$

Thus, we have:

$$\frac{1}{2}(\sin(n\pi x_{j-1}) + \sin(n\pi x_{j+1})) = \lambda_n \sin(n\pi x_j).$$

$$\implies \frac{1}{2}(\sin(n\pi(j-1)h) + \sin(n\pi(j+1)h)) = \lambda_n \sin(n\pi jh).$$

Recalling that $\sin(x+y) = \sin x \cos y + \cos x \sin y$, giving us:

$$(\sin(n\pi jh)\cos(n\pi h) - \cos(n\pi jh)\sin(n\pi h)) + (\sin(n\pi jh)\cos(n\pi h) + \cos(n\pi jh)\sin(n\pi h)).$$

$$\implies \cos(n\pi h)\sin(n\pi jh) = \lambda_n \sin(n\pi jh).$$

This means that:

$$\lambda_n = \cos(n\pi h).$$

with eigen vectors:

$$u_n(x) = \sin(n\pi jh).$$

Thus, we have:

$$|\lambda_n| = |\cos(n\pi h)| < 1, \quad n = 1, 2, \ldots, N-1.$$

We now want to investigate the rate of iteration, which indicates the number of iterations required to obtain a required accuracy.

Definition 1.1. The **rate of convergence** is

$$R = \ln(\rho).$$

Example 1.2

To reduce the error by a factor of 10^p, the number of iterations required is $\frac{P}{R}\ln(10)$.

Let's assume that $G = S\Lambda S^{-1}$, i.e. G is diagonalizable. Recall, we have:

$$x_{n+1} = Gx_n + C.$$

$$x_* = Gx_* + C.$$

We have:

$$\begin{aligned} x_{n+1} - x_* &= G(x_n - x_*) \\ \implies e_{n+1} &= Ge_n = G^2 e_{n-1} = \ldots = G^{n+1} e_0. \end{aligned}$$

This gives us:

$$\|e_n\| \le \|G^n\|\|e_0\| \le \|S\Lambda^n S^{-1}\|\|e_0\| \le \|S\|\|S^{-1}\|\|\Lambda^n\|\|e_0\|.$$

As such:

$$\frac{\|e_n\|}{\|e_0\|} \le \|S\|\|S^{-1}\|\|\Lambda\|^n \le \|S\|\|S^{-1}\|\rho^n.$$

Say we want to reduce the error by a factor of 10^p, we have:

$$\begin{aligned}
&\|S\|\|S^{-1}\|\rho^n \leq 10^{-p} \\
\implies &(\|S\|\|S^{-1}\|)^{\frac{1}{n}}\rho \leq 10^{-\frac{p}{n}} \\
\implies &\frac{1}{n}\ln(\|S\|\|S^{-1}\|) + \ln\rho \leq -\frac{p}{n}\ln 10 \\
\implies &\frac{\ln(\|S\|\|S^{-1}\|)}{\ln\rho} + n \geq -\frac{p}{\ln\rho}\ln 10 \\
\implies &n \geq \frac{P}{R}\ln 10 - C.
\end{aligned}$$

where $R = -\ln\rho$ for some constant C.

Remark 1.3 — If ρ is close to 1, then $n \sim \infty$.

Remark 1.4 — If N is very large, then the first eigenvalue will be close to 1, meaning that ρ is close to 1. This is not good, since for accuracy reason we want N to be large, but convergence would be slow.