1 February 17th, 2022

1.1 Proof of Stable Law Continued

Let $\Psi_n^\epsilon(t)$ be the c.f. of F_n^ϵ. We have:

$$\Psi_n^\epsilon(t) \to \Psi^\epsilon(t) = \int_\epsilon^\infty e^{itx}\theta\epsilon^\alpha x^{-(\alpha+1)}dx + \int_{-\infty}^{-\epsilon} e^{itx}(1-\theta)\epsilon^\alpha |x|^{-(\alpha+1)}dx$$

Now, we have:

$$\begin{aligned}
\mathbf{E}\exp\left(it\frac{\hat{s_n}(\epsilon)}{a_n}\right) &= \sum_m \mathbf{E}\exp\left(it\frac{\hat{s_n}(\epsilon)}{a_n}\middle| |I_n(\epsilon)| = m\right)\cdot \Pr\left(|I_n(\epsilon) = m\right) \\
&\sim \sum_{m=0}^\infty [\Psi^\epsilon(t)]^m \frac{(\epsilon^{-\alpha})^m e^{-\epsilon^{-\alpha}}}{m!} \\
&= \exp\left(-\epsilon^{-\alpha}(1-\Psi^\epsilon(t))\right) \\
&= \exp\left[\int_\epsilon^\infty (e^{itx}-1)\theta\alpha x^{-(\alpha+1)}dx + \int_{-\infty}^{-\epsilon}(e^{itx}-1)(1-\theta)\alpha|x|^{-(\alpha+1)}dx\right]
\end{aligned}$$

> **Remark 1.1** — The approximation should be justified by DCT, since we need to justify the convergence of the total sum.

Note that for the above case, ϵ is fixed. In the general case, we need to send $\epsilon \downarrow 0$. However, when $x \to 0$, $e^{itx} - 1 \sim itx$, and $x \cdot x^{-(\alpha+1)} = x^{-\alpha}$ is not integrable around 0 if $\alpha \geq 1$.

> **Remark 1.2** — When $\theta \neq \frac{1}{2}$, this singularity appears, which does not happen when we consider the special case.

As such, we need to consider the centered sum $\exp\left(-it\frac{n\hat{\mu(\epsilon)}}{a_n}\right)$, with:

$$\hat{\mu(\epsilon)} = \mathbf{E}X_1\mathbb{1}(\epsilon a_n < |X_1| \leq a_n).$$

As seen previously, from the assumption of the tail behavior and slowly varying function, we have:

$$\Pr\left(x < \frac{X_1}{a_n} \le y\right) = \frac{1}{n}\theta(x^{-\alpha} - y^{-\alpha})$$

$$\implies n\hat{\mu(\epsilon)}a_n \to \int_{\epsilon}^{1} x\theta\alpha x^{-(\alpha+1)}dx + \int_{-1}^{-\epsilon} x(1-\theta)\alpha|x|^{-(\alpha+1)}dx$$

$$\implies \mathbf{E}\exp\left(it\frac{\hat{S_n}(\epsilon) - n\hat{\mu}(\epsilon)}{a_n}\right) \to \exp\Bigg[\int_{1}^{\infty}(e^{itx}-1)\theta\alpha x^{-(\alpha+1)}dx$$
$$+ \int_{\epsilon}^{1}(e^{-tx} - 1 - itx)\theta\alpha x^{-(\alpha+1)}dx$$
$$+ \int_{-1}^{-\epsilon}(e^{itx} - 1 - itx)(1-\theta)\alpha|x|^{-(\alpha+1)}dx$$
$$+ \int_{-\infty}^{-1}(e^{itx} - 1)(1-\theta)\alpha|x|^{-(\alpha+1)}dx\Bigg]$$

which is integrable.

Simplifying, and sending $\epsilon \downarrow 0$, we get:

$$\exp\left[itc + \int_{0}^{\infty}(e^{itx} - 1 - \frac{itx}{1+x^2})\theta\alpha x^{-(\alpha+1)}dx + \int_{-\infty}^{0}(e^{itx} - 1 - \frac{itx}{1+x^2})(1-\theta)\alpha|x|^{-(\alpha+1)}dx\right]. \tag{1}$$

Definition 1.3 (**stable law**)**.** Distribution with characteristic function of the form 1.

Remark 1.4 (Alternative representation) —

$$\exp\left[itc - b|t|^{\alpha}(1 + i\kappa\text{sgn}(t)w_alpha(t))\right]$$

with:

$$k = 2\theta - 1 \in [-1, 1], \quad w_alpha(t) = \begin{cases} \tan(\frac{\pi\alpha}{2}), & \alpha \neq 1 \\ \frac{2}{\pi}\log|t| & \alpha = 1 \end{cases}$$

for $0 < \alpha \le 2$. See (Brenman. 1968, page 204-206)

Example 1.5

If $\alpha = 2$, the stable law becomes Gaussian.

Example 1.6

If $\alpha = 1$, $c = 0$, $\kappa = 0$, we get the Cauchy distribution.

Example 1.7
If $\alpha = \frac{1}{2}$, $c = 0$, $\kappa = 1$, $b = 1$, we get density function:

$$(2\pi y^3)^{-1/2} \exp(-1/2y), \quad y \geq 0.$$

Remark 1.8 — The density function are not known except for the above 3 cases.

Theorem 1.9

Y is stable law $\iff$ Y is the weak limit of $\frac{\sum_{i=1}^{n} X_i - b_n}{a_n}$ for a given sequence of i.i.d. X_i's.

Example 1.10
Let $X_1, X_2, \ldots$ be i.i.d. with a density function that is symmetric about 0 and continuous and positive at 0. We claim:

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{X_i} \implies \text{a Cauchy distribution } (\alpha = 1, \kappa = 0).$$

Proof. Consider when $x \to \infty$:

$$\Pr\left(\frac{1}{X_1} > x\right) = \Pr(0 \leq X_1 < \boldsymbol{x}^{-1}) = \int_0^{\boldsymbol{x}^{-1}} f(y)dy = \frac{f(0)}{x}$$

Similarly, for the left tail:

$$\Pr\left(\frac{1}{X_1} < -x\right) = \frac{f(0)}{x}.$$

In addition, we have $\theta = \frac{1}{2}$ by assumption (of symmetry), giving us $b_n = 0$. Thus:

$$\Pr\left(\left|\frac{1}{X_1}\right| > a_n\right) = \frac{2f(0)}{a_n} = \frac{1}{n} \implies a_n = 2f(0)\cdot n$$

Thus:

$$\frac{1}{n}\sum_{i=1}^{n} X_i \implies \text{Cauchy}.$$

□

Remark 1.11 — Whenever we prove with stable law, we check the tail behavior.

Note that the centralization constant is not necessary if $\alpha < 1$.
Consider $X_1, X_2, \ldots$ i.i.d. with exact distribution:

$$\Pr(X_1 > x) = \theta x^{-\alpha} \quad \Pr(X_1 < -x) = (1-\theta)x^{-\alpha}, \quad 0 < \alpha < 2, |x| \geq 1.$$

In this case, we know that $a_n = n^{1/\alpha}$. Meanwhile, we have:

$$\begin{aligned} b_n &= n\mathbf{E}X_1 \mathbb{1}(|X_n| < a_n) \\ &= n \int_1^{n^{1/\alpha}} (2\theta - 1)\alpha x^{-\alpha} dx \sim \begin{cases} cn & \alpha > 1 \\ cn \log n & \alpha = 1 \\ cn^{1/\alpha} & \alpha < 1 \end{cases}. \end{aligned}$$

Note that if $\alpha < 1$, we don't need to subtract by b_n to have convergence, but we will have a different limit if we do/don't.

Remark 1.12 — If $\alpha > 1$, the constant cn.

1.2 Infinitely Divisible Distribution

As we mentioned previously, the stable law is the limit of $\frac{\sum_{i=1}^{n} X_i - b_n}{a_n}$ for a given sequence of i.i.d. X_i's.

On the other hand, the **infinitely divisible distribution** is the limit of $\frac{\sum_{i=1}^{n} X_{n,i} - b_n}{a_n}$ for triangular array with i.i.d. $X_{n,i}$'s for each n.

Example 1.13

Gaussian $\in$ stable law, Poisson $\in$ infinitely divisible law

Here we won't derive the infinitely divisible distributions, but we will state some results. If interested, consult the textbook.

Example 1.14 (Poisoon as an infinitely divisible distribution)

Poisson is the limit of triangular array of Bernoulli r.v. $X_{n,1}, \dots, X_{n,n}$ with:

$$\Pr(X_{n,i} = 1) = 1 - \Pr(X_{n,i} = 0) = \frac{\lambda}{n}$$

Note that the c.f. of Poisson(λ) is $\exp(\lambda(e^{it} - 1))$ which is not a stable law.

Theorem 1.15 (**Levy-Khinchin Theorem**)

Z has an infinitely divisible distribution $\iff$ its c.f. is of the form:

$$\varphi(t) = \exp\left[ict - \frac{\sigma^2 t^2}{2} + \int \left(e^{itx} - 1 - \frac{itx}{1 + x^2}\right) \mu(dx)\right]$$

where μ is a measure (not necessarily probability measure) with:

$$\mu(\{0\}) = 0, \qquad \int \frac{x^2}{1 + x^2} \mu(dx) < \infty.$$

Example 1.16 (Examples of infinitely divisible distributions)
If we consider:

1. Gaussian, $\mu = 0$ measure.
2. Poisson, we have:

$$c = \int \frac{x}{1+x^2}\mu(dx), \quad \sigma^2 = 0, \quad \mu(\{1\}) = \lambda \text{ (single point mass)}$$

3. all stable law: $\sigma^2 = 0$.
4. Compound Poisson:
Let $\xi_1, \xi_2, \ldots$ be i.i.d. and $N(\lambda)$ be an independent Poisson(λ) with c.f.:

$$\varphi(t) = \mathbf{E}\exp(it\xi_1) = \int \exp(itx)\mu_\xi(dx).$$

Let $Z = \xi_1 + \ldots + \xi_{N(\lambda)}$ is infinitely divisible:

$$\mathbf{E}\exp(itZ) = \exp(-\lambda(1-\varphi(t))) = \exp\left[\lambda \int (e^{itx} - 1)\right]$$

This is the end of this chapter about stable law.

1.3 Functional Limit Theorems

Our aim for this chapter is to study the weak convergence in the space $C[0,1]$, which is the space of all continuous functions supported on $[0,1]$.

Remark 1.17 — The choice of considering on $[0,1]$ is just for convenience. We can do on other set as long as they are compact.

The weak convergence of a function means that as $n \to \infty$, $X_n(t) \to X(t), t \in [0,1]$ in distribution. First, we will consider the weak convergence on a much simpler space, namely $\mathbb{R}^k$.

We denote a **random vector** as:

$$\vec{X} = (X^1, \ldots, X^k) : (\Omega, \mathscr{F}) \to (\mathbb{R}^K, \mathscr{B})$$

so that $X^{-1}(B) \in \mathscr{F}, \forall B \in \mathscr{B}(\mathbb{R}^k)$.

Consider a random vector sequence $X_n = (X_n^1, \ldots, X_n^k)$, $n = 1, \ldots$, with c.d.f. $F_n : \mathbb{R}^k \to [0,1]$:

$$F_n(\vec{x}) = \Pr(X_n^1 \le x_1, \ldots, X_n^k \le x_n).$$

Definition 1.18 (Convergence of a random vector sequence)**.** We say that F_n converges to F weakly if $F_n(x) \to F(x)$ at all continuity point of F, denoted by $F_n \implies F$ Further we say X_n converges to X weakly (in distribution) if $F_n \implies F$, denoted by $X_n \implies X$.

Definition 1.19 (Alternative definition of $X_n \implies X$)**.** We say $X_n \implies X$ if for any bounded continuous function: $f : \mathbb{R}^k \to \mathbb{R}$, $\mathbf{E}f(X_n) \to \mathbf{E}f(X)$.

Definition 1.20 (tightness)**.** We say a sequence of probability measure μ_n on $(\mathbb{R}^k, \mathscr{B}(\mathbb{R}^k))$ is **tight** if for any $\epsilon > 0, \exists M = M_\epsilon > 0$, s.t.:

$$\mu_n([-M, M]^k) \geq 1 - \epsilon.$$

Definition 1.21 (**characteristic function of random vector**)**.** The c.f. of $X = (X^1, \ldots, X^k)$ is defined as:

$$\varphi_X(t) = \mathbf{E}\exp(it \cdot X) = \mathbf{E}\exp\left(i\sum_{a=1}^{k} t_a X_a\right), \quad t = (t_1, \ldots, t_k) \in \mathbb{R}^k$$

Theorem 1.22 (inversion formula)

If $A = [a_1, b_1] \times \ldots \times [a_k, b_k]$ with $\mu(\partial A) = 0$:

$$\mu(A) = \lim_{T\to\infty} (2\pi)^k \int_{[-T,T]^k} \prod_{j=1}^{k} \left(\frac{e^{-is_j a_j} - e^{is_j b_j}}{is_j}\right) \varphi(s) ds$$

Theorem 1.23 (continuity theorem)

Let $X_n, 1 \leq n \leq \infty$ be a random vectors with c.f. φ_n, then:

$$X_n \implies X_\infty \iff \varphi_n(t) \to \varphi_\infty(t)$$

for any given $t \in \mathbb{R}$.