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1.1 Eigenvalue Decomposition

Definition 1.1. Let $A \in \mathbb{R}^{n\times n}$ be a square matrix. A non-zero vector x is an **eigenvector** of A with $\lambda \in \mathbb{C}$ being the corresponding **eigevalue** if:

$$Ax = \lambda x.$$

- Even if A is a real matrix, its eigenvalue and eigenvectors can be complex
- The set of eigenvalues of A is called the spectrum of A. The spectral radius $\rho(A)$ is the maximum value $|\lambda|$ over all eigenvalues of A.
- If (λ, x) is an eigenpair of A, then:

$$\begin{aligned} \left(\lambda^2, x\right) &\text{ is a eigenpair of } A^2 \\ (\lambda - \sigma, x) &\text{ is a eigenpair of } A - \sigma I \\ \left(\frac{1}{\lambda - \sigma}, x\right) &\text{ is a eigenpair of } (A - \sigma I)^{-1}. \end{aligned}$$

Proof. Since (λ, x) is an eigenpair of A, $Ax = \lambda x$ Multiplying both sides by A from the left:

$$\begin{aligned} A \cdot A = \lambda Ax &\implies A^2 x = \lambda Ax = \lambda \cdot \lambda x = \lambda^2 x. \\ Ax - \sigma x = \lambda x - \sigma x &\implies (A - \sigma I)\, x = (\lambda - \sigma)\, x \\ \implies x = (\lambda - \sigma)\,(A - \sigma I)^{-1}\, x &\implies (A - \sigma I)^{-1}\, x. \end{aligned}$$

□

Definition 1.2. Two matrices A and B are **similar** with each other if there exists a nonsingular matrix T such that

$$B = TAT^{-1}.$$

Theorem 1. If A and B are similar, then A and B have the same eigenvalues.

Proof. Since A, B are similar, $B = TAT^{-1}$, which implies $A = T^{-1}BT$. If (λ, x) is an eigenpair of A, then $Ax = \lambda x$, so that

$$T^{-1}BTx = \lambda x \implies B\,(Tx) = \lambda\,(Tx)\,.$$

Thus, (λ, Tx) is an eigenpair of B. i.e. any eigenvalue of A is an eigenvalue of B. The reverse is similar. □

Definition 1.3. An eigenvalue decomposition of a square matrix $A \in \mathbb{R}^{n\times n}$ is a factorization

$$A = X\Lambda X^{-1},$$

where $X \in \mathbb{C}^{n\times n}$ is non-singular and $\Lambda \in \mathbb{C}^{n\times n}$ is diagonal.

- If $A \in \mathbb{R}^{n \times n}$ admits an eigenvalue decomposition, then
$$AX = X\Lambda.$$
If we rewrite $X = [x_1 x_2 \dots x_n]$ with $x_i \in \mathbb{C}^n$ the i-th column of x, and $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2 \dots, \lambda_n) = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ with $\lambda_i \in \mathbb{C}$ being the i-th diagonal of Λ, then
$$A[x_1 x_2 \dots x_n] = [x_1 x_2 \dots x_n] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$
$$\implies [Ax_1 Ax_2 \dots Ax_n] = [\lambda_1 x_1 \lambda_2 x_2 \dots \lambda_n x_n].$$
$$\implies Ax_i = \lambda_i x_i, \quad i = 1, 2 \dots, n.$$
In other words $(\lambda_i, x_i)\,, i = 1, 2, \dots, n$ are eigenpairs of A.
- Since X is nonsingular, x_i are linearly independent. So, x_i are n independent eigenvectors, which span $\mathbb{C}^n$.
- Eigenvalue decomposition implies $X^{-1}\Lambda X - \Lambda$, so that we also say A is diagonalizable.
- Eigenvalue decomposition does not always exist, as a square matrix $A \in \mathbb{R}^{n \times n}$ does not always have n independent eigenvectors.
- Though $A \in \mathbb{R}^{n \times n}$ is real, the eigenvalue decomposition may be complex.

1.2 Characteristic Polynomial

Definition 1.4. The **characteristic polynomial** of $A \in \mathbb{R}^{n \times n}$ denoted P_A is a degree n polynomial defined by
$$P_A(z) = \det(zI - A), \quad \text{where } z \in \mathbb{C}.$$

Let (λ_1, x) be an eigenpair of A. Then $Ax = \lambda x$, which is equivalent to:
$$(\lambda I - A)\, x = 0.$$
Since x is non-zero, $\lambda I - A$ has a non-zero solution. Therefore, $\lambda I - A$ is singular. That is $\det(\lambda I - A) = P_A(\lambda) = 0$. Thus, λ is an eigenvalue of A iff $P_A(\lambda) = 0$, and the corresponding eigenvector x are non-zero solutions of $(\lambda I - A)\, x = 0$.

Example 1. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. The characteristic polynomial is:
$$P_A(z) = \det\left(zI - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \det\left(\begin{bmatrix} z & -1 \\ 0 & z \end{bmatrix}\right) = z^2.$$
Therefore, $P_A(\lambda) = \lambda^2 = 0 \implies \lambda_1 = \lambda_2 = 0$ are the eigenvalues of A.

For eigenvectors, solve $(0I - A) = 0$, i.e.
$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} x = 0 \implies x = \begin{bmatrix} q \\ 0 \end{bmatrix}.$$
As there is only one independent eigenvector, A is not diagonalizable (i.e. no eigenvalue decomposition.

Example 2. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The characteristic polynomial is:

$$P_A(z) = \det\left(\begin{bmatrix} z & 1 \\ -1 & z \end{bmatrix}\right) = z^2 + 1.$$

Therefore, $P_A(\lambda) = \lambda^2 + 1 = 0 \implies \lambda_1 = i, \quad \lambda_2 = -i$ are the eigenvalues.
For eigenvector of $\lambda_1 = i$, solve $(iI - A =)\, x = 0$, i.e.

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} x = 0 \implies x = \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \quad \alpha \in \mathbb{C}.$$

Therefore, a corresponding eigenvector is $x_i = \begin{bmatrix} i \\ 1 \end{bmatrix}$.
For eigenvector of $\lambda_2 = -i$:

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} x = 0 \implies x = \beta \begin{bmatrix} i \\ -1 \end{bmatrix} \quad \beta \in \mathbb{C}.$$

The corresponding eigenvector is $x_2 = \begin{bmatrix} i \\ -1 \end{bmatrix}$.
As such, define $X = \begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} i & i \\ 1 & -1 \end{bmatrix}, \Lambda = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} = \begin{bmatrix} i & \\ & -i \end{bmatrix}, X^{-1} = \begin{bmatrix} -\frac{1}{2}, & \frac{1}{2} \\ -\frac{1}{2}i & -\frac{1}{2} \end{bmatrix}$,
Therefore $A = X\Lambda X^{-1}$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} i & i \\ 1 & -1 \end{bmatrix} \begin{bmatrix} i & \\ & -i \end{bmatrix} \begin{bmatrix} -\frac{1}{2}i & \frac{1}{2} \\ -\frac{1}{2}i & -\frac{1}{2} \end{bmatrix}.$$

This shows that a real matrix may have a complex eigenvalue decomposition.

Remark. However, we don't usually solve the characteristic equation, as polynomial root-finding is not numerically stable in general.

1.3 Special Case: Symmetric Matrix and SPD Matrix

Assume $A \in \mathbb{R}^{n\times n}$ is symmetric. Then

1. The eigenvalues of A are real.

   Proof. Let (λ, x) be an eigenpair of A. Then, $Ax = \lambda x$.
   Multiply both sides by $x^* \equiv \overline{x^*}$ (conjugate transpose) from the left:

   $$x^*Ax = \lambda x^*x \implies \lambda = \frac{x^*Ax}{x^*x}.$$

   - x^*Ax is real because $\overline{x^*Ax} = \overline{(x^*Ax)^T} = \overline{x^T A^T \overline{x}} = x^*Ax$
   - x^*x is also real, because $\overline{x^*x} = \overline{(x^*x)^T} = \overline{x^T\overline{x}} = x^*x$.
   - As such, $\lambda = \frac{x^*Ax}{x^*x}$ is real.

   $\square$