1 March 27th, 2019

1.1 Computation of Eigenvalue Decomposition

For simplicity, we assume that $A \in \mathbb{R}^{n\times n}$ is symmetric, so that all eigenvalues/eigenvectors are real. Let λ_i $i = 1, 2 \ldots, n$ be the eigenvalues of A, which are sorted in magnitude, i.e.

$$|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|.$$

The corresponding eigenvectors are denoted by q_i. We have

$$Q = \begin{bmatrix} q_1 & q_2 & \ldots & q_n \end{bmatrix} \in \mathbb{R}^{n\times n}$$

satisfying $Q^T Q = Q^T = I$.

Definition 1.1. Let $A \in \mathbb{R}^{n\times n}$ be symmetric. For a given vector $x \in \mathbb{R}^n$, the **Rayleigh Quotient** is defined by

$$r(x) = \frac{x^T Ax}{x^T x}.$$

If x is an eigenvector,

$$r(x) = \frac{x^T Ax}{x^T x} = \frac{\lambda x^T}{x^T x} = \lambda,$$

i.e. $r(x)$ is an eigenvalue.

The eigenvalues are critical points of $r(x)$, with $\nabla r(x) = 0$. It can be proven that

$$\min_i \lambda_i = \min_{x\neq 0} r(x).$$

Remark 1.2 — This can be extended to non-symmetric matrices/ matrices or eigenvalues that are complex.

1.2 Power Iteration

Purpose: Find λ_1 and its associated eigenvector x_1, with $\|x_1\|_2 = 1$.

Algorithm 1.3 1. Choose $y^{(0)} \in R^n$ s.t. $\|y^{(0)}\|_2 = 1$.
2. for $k = 1, 2, \ldots, n$

$$z^{(k)} = Ay^{(k-1)}$$

$$y^{(k)} = \frac{z^{(k)}}{\|z^{(k)}\|_2}$$

$$\mu^{(k)} = \frac{\left(y^{(k)}\right)^T Ay^{(k)}}{\left(y^{(k)}\right)^T y^{(k)}} = \left(y^{(k)}\right)^T Ay^{(k)}.$$

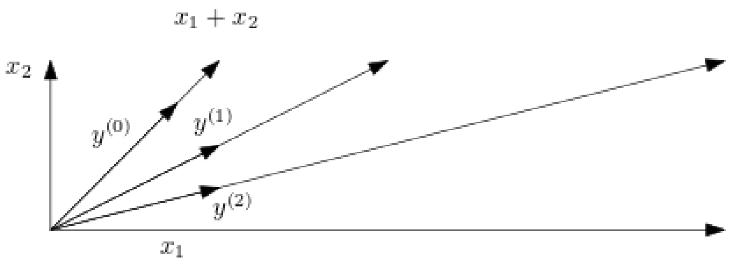

Figure 1

Remark 1.4 — $y^{(k)}$ is an approximation to $\pm x_1$, $\mu^{(k)}$ is an approximation to λ_1.

- Assume $(2, x_1)$, $(1, x_2)$ are two eigenpairs of $A \in \mathbb{R}^{2\times 2}$ (so that $x_1 \perp x_2$).
- Assume $y^{(0)} = \frac{1}{\sqrt{2}}(x_1 + x_2)$
- $k = 1$:

$$z^{(1)} = Ay^{(0)} = A\left(\frac{1}{\sqrt{2}}(x_1 + x_2)\right) = \frac{1}{\sqrt{2}}(Ax_1 + Ax_2) = \frac{1}{\sqrt{2}}(2x_1 + x_2).$$

$$y^{(1)} = \frac{1}{\sqrt{5}}(2x_1 + x_2).$$

Note that $y^{(k)}$ approaches x_1 more than x_2.

$\vdots$

- $k + 1$:

$$z^{(k+1)} = Ay^{(k)} = A\left(\frac{1}{\sqrt{2^{2k}+1}}\left(2^k x_1 + x_2\right)\right) = \frac{1}{\sqrt{2^{2k}+1}}\left(2^{k+1}x_1 + x_2\right).$$

If the component of x_1 is non-zero, then it will converge to x_1, i.e. as long as $y^{(0)}$ is not a multiple of x_2, it will converge to x_1.

Claim 1.5. Power iteration may not be convergent:

Example 1.6
Assume $(1, x_1)$, $(-1, x_2)$ are two eigenpairs of $A \in \mathbb{R}^{2\times 2}$. Assume $y^{(0)} = \frac{1}{\sqrt{2}}(x_1 + x_2)$.

$$k = 1 : z^{(1)} = Ay^{(0)} = \frac{1}{\sqrt{2}}(x_1 - x_2).$$

$$y^{(1)} = \frac{1}{\sqrt{2}}(x_1 - x_2).$$

$$k = 2 : z^{(2)} = \frac{1}{\sqrt{2}}(x_1 + x_2)$$

$$y^{(2)} = \frac{1}{\sqrt{2}}(x_1 + x_2).$$

which just repeats itself.

Remark 1.7 — Try with $(-2, x_1)$, $(1, x_2)$. Does not converge, but we can get the direction of x_1 since both x_1 and $-x_1$ are eigenvectors.

Remark 1.8 — Power iteration may not converge to (λ_1, x_1), e.g. $y^{(0)} = x_2$. This is because there is no x_1 component.

1.3 Analysis of Power Iteration

We will show $|\langle y^{(k)}, x\rangle| \to 1$. It is the same as $1 - \langle y^{(k)}, x_1\rangle^2 \to 0,\ k \to \infty$

Theorem 1.9

Assume $A \in \mathbb{R}^{n\times n}$ is symmetric and $|\lambda_1| > |\lambda_2|$ (otherwise they might be amplified at the same rate).

If $\langle y^{(0)}, x_1\rangle \neq 0$, then $\exists C_0 > 0$ depending on $y^{(0)}$ only such that

$$\left(1 - \langle y^{(k)}, x_1\rangle^2\right)^{\frac{1}{2}} \leq C_0 \left|\frac{\lambda_2}{\lambda_1}\right|^k \to 0,\ \text{as } k \to \infty.$$

Consequently,

- $\min\{\|y^{(k)} - x_1\|_2, \|y^{(k)} + x_1\|_2\} \leq \sqrt{2}C_o \left|\frac{\lambda_2}{\lambda_1}\right|^k$, i.e. $y^{(k)} \to \pm x_1$
- $|\mu^{(k)} - \lambda_1| \leq 2\sqrt{2}C_o \left|\frac{\lambda_2}{\lambda_1}\right|^k \to 0$

Proof. Note that

$$y^{(k)} = \frac{A^k y^{(0)}}{\|A^k y^{(0)}\|_2}.$$

Let $A = X\Lambda X^T$ be the eigenvalue decomposition of A. Then

$$A^k = X\Lambda X^T X\Lambda X^T \dots X\lambda X^T = X\Lambda^k X^T.$$

So

$$A^k y^{(0)} = X\Lambda^k X^T y^{(0)} = X\Lambda^k v$$

$$A^k y^{(0)} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1^k v_1 \\ \vdots \\ \lambda_n^k v_n \end{bmatrix} = \sum_{i=1}^n \lambda_i^k v_i x_i,\ v_i \in \mathbb{R},\ x_i \in \mathbb{R}^n.$$

Because x_i are othronormal,

$$\|A^k y^{(0)}\|_2^2 = \sum_{i=1}^n \lambda_i^{2k} v_1^2 = \sum_{i=1}^n |\lambda_i|^{2k} |v_i|^2 = |\lambda_1|^{2k}|v_1|^2 (1 + \dots) \geq \left(|\lambda_1|^k |v_1|\right)^2.$$

and

$$\langle y^{(k)}, x_1\rangle^2 = \frac{1}{\|A^k y^{(0)}\|_2^2}\langle A^k y^{(0)}, x_1\rangle^2 = \frac{1}{\|A^k y^{(0)}\|_2^2}\langle \sum_{i=1}^n \lambda_i^k v_i x_i, x_1\rangle^2 = \frac{1}{\|A^k y^{(0)}\|_2^2}\left(\lambda_1^k v_1\right)^2.$$

$\square$