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1.1 More Methods to Solve ODEs

We are dealing with linear, second order, non-homogeneous ODEs, which are the form:

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = b(x), \quad \alpha < x < \beta.$$

Remember, if $y_1(x)$ is any non-zero solution to $a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0$, then a second non-zero solution $y_2(x)$ can be computed using Abel's equation:

$$y_2(x) = Ay_1(x) \int \frac{e^{-\int \frac{a_1(x)}{a_2(x)}\, dx}}{(y_1(x))^2}\, dx,$$

for any $a \neq 0$. Thus a general solution to the homogeneous equation is:

$$y_h(x) = c_1 y_2(x) + c_2 y_2(x),$$

for arbitrary constants c_1 and c_2. Furthermore, if $y_p(x)$ is any solution to the non-homoegenous equation, then the general solution is:

$$y(x) = y_h(x) + y_p(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x).$$

If we can't easily guess $y_p(x)$ (is a constant if $\frac{b(x)}{a_0(x)}$, we can always use the Green's function expression:

$$y_p(x) = \int^x G(t, x) \frac{b(t)}{a_2(t)}\, dt.$$

Where:

$$G(t, x) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}.$$

As such, everything comes down to finding $y_1(x)$, as if we can find it, we can find y_2, y_p, and y_h. We've already seen for constant coefficients and equidimensional ODE's, we can get y_1 and y_2 from a table. We also know when $a_2 + a_1 + a_0 = 0$, we have a solution $y_1(x) = e^x$.

Another method is to first divide by $a_2(x)$, giving us:

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0, P(x) = \frac{a_1(x)}{a_2(x)} \quad Q(x) = \frac{a_0(x)}{a_2(x)}.$$

Which is putting it into the standard form. With this, we should compute the following:

$$\gamma(x) = \frac{Q'(x) + 2P(x)Q(x)}{(Q(x))^{\frac{3}{2}}}.$$

If $\gamma(x)$ is a constant, then the transformation:

$$z = \int \sqrt{AQ(x)}\, dx, \quad A \neq 0$$

will convert the original equation into one of constant coefficients:

$$\Psi''(z) + \frac{\gamma}{2\sqrt{A}}\Psi(z) + \frac{1}{A}\Psi(z) = 0.$$

Then

$$y(x) = \Psi\left(\int \sqrt{AQ(x)}\, dx\right).$$

Example 1.1 (One of Chebyshev Equation)
Consider:

$$(1-x^2)y''(x) - xy'(x) + m^2y(x) = 0, \quad -1 < x < 1, \quad m \in \mathbb{Z}.$$

Notice that in the normal form, we have:

$$P(x) = -\frac{x}{1-x^2}, \quad Q(x) = \frac{m^2}{1-x^2}.$$

Thus:

$$\gamma(x) = \frac{Q'(x) + 2P(x)Q(x)}{(Q(x))^{\frac{3}{2}}} = \frac{\frac{2xm^2}{(1-x^2)^2} + 2\left(-\frac{x}{1-x^2}\right)\left(\frac{m^2}{1-x^2}\right)}{\left(\frac{m^2}{1-x^2}\right)^{\frac{3}{2}}} = 0.$$

Thus, the transformation:

$$z = \int \sqrt{\frac{Am^2}{1-x^2}}\,dx = m\sqrt{A}\int \frac{dx}{\sqrt{1-x^2}} = m\sqrt{A}\sin^{-1}(x)$$

will give us constant coefficient. Let $A = 1$, we have:

$$\Psi(z) + \Psi(z) = 0 \implies \Psi(z) = c_1\cos(z) + c_2\sin(z).$$

Thus the solution to the original equation is:

$$y(x) = c_1\cos(m\sin^{-1}(x)) + c_2\sin(m\sin^{-1}(x)).$$

Remark 1.2 — This test is fast, as we just have to compute $\gamma(x)$ to see if it will work.

Another method would be, starting with the standard form, set:

$$y(x) = u(x)e^{-\frac{1}{2}\int P(x)dx}.$$

And the equation will become of the form:

$$u''(x) + R(x)u(x) = 0, \quad R(x) = Q(x) - \frac{1}{2}P(x) - \frac{1}{4}(P(x))^2.$$

If $R(x)$ is a constant, then solving for $u(x)$ is easy, as we would have constant coefficients, allowing us to use the table.

Example 1.3 (Another one of Chebyshev's Equations)
Consider:
$$y''(x) - 2\tan(x)y'(x) + m^2 y(x) = 0.$$
This is already in standard form, with:
$$P(x) = -2\tan(x) \quad Q(x) = m^2.$$
Thus, we have:
$$R(x) = m^2 - \frac{1}{2}\left(-\sec^2(x)\right) - \frac{1}{4}(2\tan(x))^2 = m^2 + \sec^2(x) - \tan^2(x) = m^2 + 1.$$
Thus, we have:
$$u''(x) + (m^2+1)u(x) = 0 \implies u(x) = c_1 \cos\left(x\sqrt{m^2+1}\right) + c_2 \sin\left(x\sqrt{m^2+1}\right).$$
Thus the solution to the original equation will be:
$$y(x) = \left(c_1 \cos\left(x\sqrt{m^2+1}\right) + c_2 \sin\left(x\sqrt{m^2+1}\right)\right) e^{-\frac{1}{2}\int -2\tan(x)dx}.$$
Note that:
$$e^{\int tan(x)dx} = e^{-\ln(\cos(x))} = \frac{1}{\cos(x)}.$$
Thus we have:
$$y(x) = \frac{c_1 \cos\left(x\sqrt{m^2+1}\right) + c_2 \sin\left(x\sqrt{m^2+1}\right)}{\cos(x)}.$$

1.2 Taylor's Method

Starting with the equation in standard form:
$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0, \quad \alpha < x < \beta.$$
If
$$\gamma(x) = \frac{Q'(x) + 2P(x)Q(x)}{(Q(x))^{\frac{3}{2}}} \neq \text{constant}$$
and
$$R(x) = Q(x) - \frac{1}{2}P'(x) - \frac{1}{4}(P(x))^2 \neq \text{constant}.$$
Then we can try Taylor's method. First let's define the two equations.

Definition 1.4. Legendre's Equation is defined as
$$(1-x^2)y''(x) - 2xy'(x) + m(m+1)y(x) = 0, \quad -1 < x < 1, m \in \mathbb{Z}$$

Remark 1.5 — The Legendre's equation appears in many places whenever we're dealing with spherical symmetry.

Definition 1.6. **Bessel's Equation** is defined as

$$x^2y''(x) + xy'(x) \pm (x^2 - m^2)y(x) = 0, \quad 0 < x, m \in \mathbb{Z}$$

Remark 1.7 — Bessel's equation appears in many places as well, such as cylindrical symmetry.

Taylor's method will help with Legendre's equation, and there's an extension to Taylor's method called to Frobenius's Method.

Definition 1.8. A point $x = x_0$ for the ODE

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0$$

is called an **ordinary point** for the ODE if:

$$\lim_{x \to x_0} P(x) \text{ and } \lim_{x \to x_0} Q(x)$$

both exist.

Remark 1.9 — For Legendre's ODE, we have:

$$P(x) = -\frac{x}{1-x^2}, \quad Q(x) = \frac{m(m+1)}{1-x^2}.$$

$x = 0$ is an ordinary point for this ODE, as:

$$\lim_{x \to 0} P(x) = 0 \quad \lim_{x \to 0} Q(x) = m(m+1).$$

Definition 1.10. A point that is not ordinary is called a **singular point**.

Definition 1.11. A singular point is called **regular** if:

$$\lim_{x \to x_0} (x - x_0)P(x) \text{ and } \lim_{x \to x_0} (x - x_0)^2 Q(x)$$

both exist.

Definition 1.12. If a singular point is not regular, it is called an **essential point**.

Remark 1.13 — For Bessels' ODE, $x = 0$ is a singular point, as:

$$\lim_{x \to 0} P(x) = \lim_{x \to 0} \frac{1}{x} = \text{DNE}.$$

Note that $x = 0$ is regular, as:

$$\lim_{x \to 0} xP(x) = 1, \quad \lim_{x \to 0} x^2 Q(x) = \pm(x^2 - m^2).$$

Theorem 1.14 (Taylor's Theorem for ODEs)
If $x = x_0$ is an ordinary point for $y''(x) + P(x)y'(x) + Q(x)y(x) = 0, \quad \alpha < x, x_0 < \beta$, then two linearly independent solutions can be constructed as:

$$y_1(x) = \sum_{nm=0}^{\infty} a_m(x - x_0)^m \quad \text{and} \quad y_2(x) = \sum_{m=0}^{\infty} b_m(x - x_0)^m$$

and these will converge absolutely for all $|x - x_0| < R$, where R is the nearest distance between x_0 and a singular point (if any).

Definition 1.15. If a series $\sum_{n=1}^{\infty} a_n$ **converges absolutely**, then $\sum_{n=1}^{\infty} |a_n|$ converges.

Definition 1.16. If a series $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ does not, then it **converges conditionally**.

Remark 1.17 — Note that we should always pick x_0 to be in the middle of two singular points. For example, consider Legendre's ODE $(1 - x^2)y'' - 2xy' + m(m + 1)y = 0, \ -1 < x < 1$. If we choose $x_0 = \frac{1}{2}$ then our power series will only converge absolutely if $0 < x < 1$. If we pick $x_0 = 0$, then it will converge absolutely everywhere in $-1 < x < 1$.