1 February 14th, 2020

1.1 Problem 1 - Solution 1

Consider

$$u''(r) + \frac{1}{r}u'(r) = -H.$$

With constraints:

$$u(a) = T_e \quad |u(0)| < \infty.$$

Let us take $v = u'(r)$, which gives us:

$$v'(r) + \frac{1}{r}v = -H.$$

which is a linear first order ODE, giving us:

$$v(r) = \frac{1}{\mu(r)}\left(\int \mu(r)(-H)dr + C_1\right), \quad \mu(r) = e^{\int \frac{1}{r}dr} = r.$$

Thus:

$$v(r) = \frac{1}{r}\left(\int -rH \ dr + C_1\right) = \frac{1}{r}\left(-\frac{1}{2}r^2H + C_1\right).$$

$$v(r) = -\frac{1}{2}rH + \frac{1}{r}C_1 = u'(r).$$

$$\implies u(r) = \int -\frac{1}{2}rH + \frac{1}{r}C_1 \ dr = -\frac{1}{4}r^2H + \ln(r)C_1 + C_2.$$

To solve for constants, we apply initial conditions:

$$|u(0)| = |-\frac{1}{4}(0)^2H + \ln(0)C_1 + C_2| < \infty \implies C_1 = 0.$$

$$\implies u(a) = -\frac{1}{4}Ha^2 + C_2 = T_e \implies C_2 = T_e + \frac{1}{4}Ha^2.$$

Thus we have:

$$u(r) = T_e + \frac{1}{4}H(a^2 - r^2).$$

1.2 Problem 1 - Solution 2

We once again consider $u''(r) + \frac{1}{r}u'(r) = -H$. First we will solve the homogeneous equation:

$$u_h''(r) + \frac{1}{r}u_h'(r) = 0 \implies r^2u_h''(r) + ru_h'(r) = 0.$$

which is equidimensional. As such we just need to find the discriminant with $a = 1, b = 1, c = 0$:

$$D = (b-a)^2 - 4ac = (1-1)^2 - 0 = 0.$$

Using the table, we have:

$$u_1(r) = |r|^\alpha \ln(r) \quad u_2(r) = |r|^\alpha.$$

with

$$\alpha = -\frac{b-a}{2a} = 0.$$

Thus:

$$u_1(r) = \ln(r) \quad u_2(r) = 1.$$

Thus the overall homogeneous solution is:

$$u_h = C_1 \ln(r) + C_2.$$

Now we need to find the particular solution using Green's Function:

$$G(t,r) = \frac{u_1(t)u_2(r) - u_1(r)u_2(t)}{u_1(t)u_2'(t) - u_1'(t)u_2(t)} = \frac{\ln(t) - \ln(r)}{-\frac{1}{t}} = t\ln(r) - t\ln(t).$$

Using this, we have:

$$u_p(r) = \int^r G(t,r)g(t)dt = \int^r (t\ln(r) - t\ln(t))(-H)\ dt.$$

$$= -H\ln(r)\int^r t\ dt + H\int^r t\ln(t)\ dt = \frac{1}{2}r^2 H\ln(r).$$

Integrating by parts, with:

$$u = \ln(t) \quad dv = t\ dt$$

$$du = \frac{1}{t}\ dt \quad v = \frac{1}{2}t^2.$$

we have:

$$\int^r t\ln(t)\ dt = \frac{1}{2}t^2\ln(t) - \int \frac{1}{2}t\ dt = \frac{1}{2}t^2\ln(t) - \frac{1}{4}t^2\bigg|_{t=r}.$$

Giving us $u_p(r) = -\frac{1}{4}r^2 H$, thus giving us:

$$u_h + u_p = C_1\ln(r) + C_2 - \frac{1}{4}r^2 H.$$

Which is the same as the other solution before plugging in the initial conditions.

1.3 Problem 2

Consider the equation:

$$\ddot{x} + \omega^2 x = \ddot{x} + \frac{g}{L}x = g.$$

where $\omega = \sqrt{\frac{g}{L}}$ and initial conditions:

$$x(0) = 0 \quad \dot{x}(0) = 0.$$

This has constant coefficients, with $a = 1, b = 0, c = \omega^2$, thus the discriminant is:

$$D = b^2 - 4ac = -4\omega^2 < 0.$$

Thus we have:

$$x_1 = e^{\alpha t}\cos(\gamma t) \quad x_2 = e^{\alpha t}\sin(\gamma t).$$

with:
$$\gamma = \frac{\sqrt{-D}}{2a} = \omega \quad \alpha = -\frac{b}{2a} = 0.$$
Thus:
$$x_h = C_1 \cos(\omega t) + C_2 \sin(\omega t).$$
Now we need a particular solution. Looking back at the original equation, we can guess $x_p = L$. Since:
$$0 + \frac{g}{L} L = g.$$
Because of the existence-uniqueness theorem, this is the only solution that will work, meaning that overall solution before initial conditions is:
$$x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t) + L.$$
Applying initial conditions, we have:
$$x(0) = C_1(1) + C_2(0) + L = 0 \implies C_1 = -L.$$
$$\dot{x}(0) = -L\omega \sin(0) + C_2 \omega \cos(0) = 0 \implies C_2 = 0.$$
Thus we have:
$$x(t) = -L\cos(\omega t) + L = L(1 - \cos(\omega t)).$$
With this we can solve for some stuff, for example:
$$x(t_{\frac{1}{2}}) = \frac{L}{2} \implies t_{\frac{1}{2}} = \frac{\pi}{3\omega}.$$
$$x(T) = L \implies T = \frac{\pi}{2\omega}.$$