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1.1 Unit Step Function Continued

As a reminder, the unit step function is defined as:

$$u(t - a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}.$$

Given a piecewise function, we can write it as a linear combination of step functions.

Example 1.1

Consider:

$$f(t) = \begin{cases} 7, & 0 < t < 2 \\ 6t, & 2 < t < 3 \\ t^2, & 3 < t < 7 \\ 0, & 7 < t \end{cases}.$$

We can rewrite this as:

$$f(t) = 7u(t) + (6t - 7)u(t - 2) + (t^2 - 6t)u(t - 3) + (0 - t^2)u(t - 7).$$

With this, we can take the Laplace transform of the function, but first, we need to consider the Laplace transform of $\mathcal{L}\{f(t)u(t - a)\}$. Looking at the definition, we have:

$$\mathcal{L}\{f(t)u(t - a)\} = \int_0^\infty f(t)u(t - a)e^{-st} dt.$$

Assuming $a > 0$, we have:

$$= \int_0^a f(t) \underbrace{u(t - a)}_0 e^{-st} dt + \int_a^\infty f(t) \underbrace{u(t - a)}_1 e^{-st} dt = \int_a^\infty f(t) e^{-st} dt.$$

If we set $z = t - a \implies dz = dt$,

$$= \int_0^\infty f(z + a) e^{-s(z+a)} dz = e^{-as} \mathcal{L}\{f(t + a)\}.$$

Theorem 1.2 (Shifting Theorem)

As shown above:

$$\mathcal{L}\{f(t)u(t - a)\} = e^{-as} \mathcal{L}\{f(t + a)\}.$$

Example 1.3

Considering $f(t)$ from Example 1.1, we have:

$$\mathcal{L}\{f(t)\} = \int_0^2 7e^{-st} dt + \int_2^3 6te^{-st} dt + \int_3^7 t^2 e^{-st} dt + \int_7^\infty 0e^{-st} dt.$$

However, we can calculate this another way. From the table, we have $\mathcal{L}\{t^m\} = \frac{m!}{s^{m+1}}$, thus:

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{7u(t) + (6t - 7)u(t - 2) + (t^2 - 6t)u(t - 3) + (0 - t^2)u(t - 7)\} \\ &= \mathcal{L}\{7u(t)\} + \mathcal{L}\{(6t - 7)u(t - 2)\} + \mathcal{L}\{(t^2 - 6t)u(t - 3)\} + \mathcal{L}\{-t^2 u(t - 7)\} \\ &= e^{-0s} \mathcal{L}\{7\} + e^{-2s} \mathcal{L}\{6(t + 2) - 7\} + e^{-3s} \mathcal{L}\{(t + 3)^2 - 6(t + 3)\} - e^{-7s} \mathcal{L}\{(t + 7)^2\} \\ &= \frac{7}{s} + e^{-2s} \mathcal{L}\{6t - 5\} + e^{-3s} \mathcal{L}\{t^2 - 9\} - e^{-7s} \mathcal{L}\{t^2 + 14t + 49\}. \end{aligned}$$

Thus:

$$\mathcal{L}\{f(t)\} = \frac{7}{s} + e^{-2s} \left(\frac{6}{s^2} + \frac{5}{s} \right) + e^{-3s} \left(\frac{2}{s^3} - \frac{9}{s} \right) - e^{-7s} \left(\frac{2}{s^3} + \frac{14}{s^2} + \frac{49}{s} \right), \quad s > 0.$$

Remark 1.4 — In the example earlier, we are using the Shifting Theorem and replacing t with $t + a$ in each of the functions that we are multiplying by the unit step function at a .

1.2 Examples of Where Unit Step Functions Occur

Example 1.5

Consider the equation:

$$L \frac{dI}{dt} + RI = \epsilon_1 u(t) + (\epsilon_2 - \epsilon_1) u(t - t_1).$$

Taking the Laplace transform of both sides, we get:

$$\begin{aligned} \mathcal{L} \left\{ L \frac{dI}{dt} + RI \right\} &= \mathcal{L}\{\epsilon_1 u(t) + (\epsilon_2 - \epsilon_1) u(t - t_1)\}. \\ \implies L \mathcal{L}\{I'(t)\} + R \mathcal{L}\{I(t)\} &= e^{-0s} \mathcal{L}\{\epsilon_1\} + e^{-t_1 s} \mathcal{L}\{\epsilon_2 - \epsilon_1\} \\ \implies L(s \mathcal{L}\{I\} - I(0)) + R \mathcal{L}\{I\} &= \frac{\epsilon_1}{s} + e^{-t_1 s} \left(\frac{\epsilon_2 - \epsilon_1}{a} \right). \\ \implies \mathcal{L}\{I\} &= \frac{LI_0 + \frac{\epsilon_1}{s} + e^{-t_1 s} \left(\frac{\epsilon_2 - \epsilon_1}{s} \right)}{LS + R}. \end{aligned}$$

There are many applications/cases where using a step function to describe a piecewise function might be useful. For example, if we have a spring with dampener with an

external force $F(t)$, we might have $F(t)$ ramp up with t , and then stay constant after a certain amount of time.

Another example is consider a ball bouncing off the ground. The forces are:

$$F(t) = \begin{cases} -mg, & 0 < t < T_F \\ N(t) - mg, & T_F < t < T_F + T_C \\ -mg, & T_F + T_C < t < T_F + T_C + T_R \end{cases}.$$

Where T_F is the time until hitting the ground, T_C is the contact duration, and T_R is the time to rebound back up, and $N(t)$ is the normal force. From this, we get figure out $N(t)$ and allow us to get the coefficient of restitution.

1.3 Impulse Function

Consider a function:

$$I_a(t) = \begin{cases} 0, & t < -\frac{a}{2} \\ \frac{1}{a}, & -\frac{a}{2} < t < \frac{a}{2} \\ 0, & \frac{a}{2} < t \end{cases}.$$

This can be expressed in terms of unit step functions as:

$$I_a(t) = \frac{1}{a}u\left(t + \frac{a}{2}\right) - \frac{1}{a}u\left(t - \frac{a}{2}\right).$$

Remark 1.6 — Note that the area under the curve is 1, as we choose the height to be inversely proportional to the width. This means that:

$$\int_{-\infty}^{\infty} I_a(t) dt = 1.$$

Definition 1.7 (Impulse Function). An **impulse function** is:

$$\lim_{a \rightarrow 0} I_a(t) = I(t) = \begin{cases} 0, & t \neq 0 \\ +\infty, & t = 0 \end{cases}$$

with the property:

$$\int_{-\infty}^{\infty} I(t) dt = 1.$$

Or:

$$\int_R I(t) dt = \begin{cases} 0, & 0 \notin R \\ 1, & 0 \in R \end{cases}.$$

Remark 1.8 — With Laplace transform, if 0 is at the end of the domain, it is included, e.g.:

$$\int_0^7 I(t) dt = 1.$$

Remark 1.9 — Similar to the step function, we can shift the impulse function, i.e.:

$$I(t - a) = \begin{cases} 0, & t \neq a \\ +\infty, & t = a \end{cases},$$

with:

$$\int_R I(t - a) dt = \begin{cases} 0, & a \notin R \\ 1, & a \in R \end{cases}.$$

Example 1.10 (One Dimensional Crystal)

In a one dimensional crystal, we have atoms aligned in a line, and say they are separated by a . If we have an electron travelling along, the force it might see can be expressed as:

$$F(x) = a \sum_{k=-\infty}^{\infty} F_0 I(x - ka) = aF_0 \sum_{k=-\infty}^{\infty} I(x - ka).$$

Thus one way to model the force experienced by an electron is to use a bunch of impulse functions. This is called the **comb function**.

Remark 1.11 — If we had a continuous function $f(t)$, we'd have:

$$\int_R f(t) I(t - a) dt = \begin{cases} 0, & 0 \notin R \\ f(a), & a \in R \end{cases}.$$

Example 1.12

If we have:

$$\int_{-1}^7 \frac{t^2}{\sqrt{3t^3 + 1}} e^t I(t - 1) dt = \frac{1^2}{\sqrt{3(1)^3 + 1}} e^1 = \frac{1}{2}e.$$

Theorem 1.13

We have:

$$\mathcal{L}\{f(t)I(t - a)\}, a > 0 = \int_0^\infty f(t)I(t - a)e^{-st} dt = f(a)e^{-as}.$$

Example 1.14

$$\mathcal{L}\{t^3 I(t - 4)\} = 4^3 e^{-4s} = 64e^{-4s}$$

Example 1.15

Consider $F(t) = aF \sum_{k=-\infty}^{\infty} I(t - ka)$. We have:

$$\mathcal{L}\{F(t)\} = Fa \sum_{k=-\infty}^{\infty} e^{-kas}.$$