1 March 25th, 2020

1.1 Important Result Involving ODEs

Let us consider the Legendre ODE:

$$(1-x^2)y''(x) - 2xy'(x) + m(m+1)y(x) = 0, \quad -1 < x < 1.$$

Recall that one solution is $y(x) = P_m(x)$. We can generalize this to a class of problems that look like:

$$a_2(x)y_m''(x) + a_1(x)y_m(x) + a_0(x)y_m(x) + \lambda_m^2 b(x)y_m(x) = 0, \quad \alpha < x < \beta.$$

With λ_m being some parameter for different values of m.

Remark 1.1 — Note that for the Legendre ODE falls into this category with $a_2(x) = 1-x^2$, $a_1(x) = -2x$, $a_0(x) = 0$, $b(x) = 1$, $\lambda_m^2 = m(m+1)$

Theorem 1.2

If $y_M(x)$ is a solution to the above equation and if $y_N(x)$ is a solution to the same ODE with M replaced by M, then after much algebra, we find that:

$$\left(\lambda_M^2 - \lambda_N^2\right)\int_\alpha^\beta y_M(x)y_N(x)w(x)\ dx = \left(s(x)\left|\begin{bmatrix} y_M(x) & y_N(x) \\ y_M'(x) & y_N'(x)\end{bmatrix}\right|\right)\Bigg|_\alpha^\beta.$$

Where:

$$s(x) = e^{\int \frac{a_1(x)}{a_2(x)}dx}.$$

$$w(x) = \frac{b(x)}{a_2(x)}s(x).$$

Example 1.3

Consider $y_N''(x) + N^2 y_N(x) = 0, \quad 0 < x < \pi$. One solution is $y_N(x) = \sin(Nx)$. Note that this is of the form with $a_2(x) = 1$, $a_1(x) = 0$, $a_0(x) = 0$, $b(x) = 1$, $\lambda_N^2 = N^2$. We have

$$s(x) = e^{\int \frac{0}{1}dx} = 1.$$

$$w(x) = \frac{1}{1}(1) = 1.$$

Plugging into the equation, we have:

$$(M^2 - N^2)\int_0^\pi \sin(Mx)\sin(Nx)(1)\ dx = \left((1)\left|\begin{bmatrix} \sin(Mx) & \sin(Nx) \\ M\cos(Mx) & N\cos(Nx)\end{bmatrix}\right|\right)\Bigg|_0^\pi$$

$$= \left|\begin{bmatrix} 0 & 0 \\ M(-1)^M & N(-1)^N\end{bmatrix}\right| = 0.$$

If $M \neq N$, then:

$$\int_0^\pi \sin(Mx)\sin(Nx)\ dx = 0.$$

For the Legendre ODE, we have: $a_2(x) = 1 - x^2$, $a_1(x) = -2x$, $a_0 = 0$, $\alpha_N^2 = N(N+1)$, $b(x) = 1$. Then we have:

$$s(x) = e^{\int \frac{a_1(x)}{a2(x)} dx} = 1 - x^2.$$

$$w(x) = \frac{b(x)}{a_2(x)} s(x) = \frac{1}{1 - x^2}(1 - x^2) = 1.$$

Using the earlier general result, we have:

$$(M(M+1) - N(N+1)) \int^{-} 1_{-}1 P_M(x) P_N(x)(1) \, dx = \left((1 - x^2) \dots\right) \Big|_{-1}^{1} = 0.$$

Theorem 1.4

Thus if $M \neq N$, we have:

$$\int_{-1}^{1} P_M(x) P_N(x) \, dx = 0.$$

1.1.1 Legendre Series

Recall that the Taylor series is a power series expansion:

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad a_n = \frac{f^{(n)}(x_0)}{n!}.$$

For this, we are expanding a function in terms of polynomials, with the basis set being:

$$\{1, (x - x_0), (x - x_0)^2, \dots\}.$$

The **Legendre series** is similar, where the basis set being the Legendre polynomials:

$$\{P_0(x), P_1(x), P_2(x) \dots\}.$$

Meaning that:

$$f(x) = \sum_{n=1}^{\infty} a_n P_n(x), \quad -1 < x < 1.$$

Note that this is an equation with an infinite number of equations and an infinite number of unknowns. To get the a_n, we will make use of the result that:

$$\int_{-1}^{1} P_M(x) P_N(x) \, dx = 0, \quad M \neq N.$$

Suppose $f(x)$ is given and all the $P_n(x)$ are known. Suppose you want to compute a_3. We multiply both sides by P_3, giving us:

$$f(x) P_3(x) = \sum_{n=1}^{\infty} a_n P_n(x) P_3(x).$$

Integrating both sides, we have:

$$\int_{-1}^{1} f(x)P_3(x)\ dx = \int_{-1}^{1}\sum_{n=0}^{\infty} a_n P_n(x)P_3(x)\ dx.$$

In this situation, we can swap the integral and sum (not always the case, but in this case it turns out to be true), thus we have:

$$= \sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_n(x)P_3(x)\ dx = a_3 \int_{-1}^{1} P_3(x)P_3(x)\ dx.$$

Thus we have:

$$a_3 = \frac{\int_{-1}^{1} f(x)P_3(x)\ dx}{\int_{-1}^{1} P_3(x)P_3(x)\ dx}.$$

Thus we have:

$$f(x) = \sum_{n=0}^{\infty} a_n P(x).$$

$$a_n = \frac{\int_{-1}^{1} f(x)P_n(x)\ dx}{\int_{-1}^{1} P_n(x)P_n(x)\ dx}.$$

It can be shown that:

$$\int_{-1}^{1} P_n(x)P_n(x)\ dx = \frac{2}{2n+1}, \quad n = 0, 1, 2, \ldots.$$

To make this easier to remember, remember that if:

$$A = A_1 i + A_2 j + A_3 k.$$

We have:

$$A_1 = \frac{A \cdot i}{i \cdot i}, A_2 = \frac{A \cdot j}{j \cdot j}, A_3 = \frac{A \cdot k}{k \cdot k}.$$

If we think of $\int f(x)g(x)\ dx$ as a "dot product" between equations, we can think of $a_n = \frac{f \cdot P_n}{P_n \cdot P_n}$. Note that this satisfies some special properties that are the same as the dot product between vectors, namely:

- $f \cdot g = g \cdot f$
- $f \cdot (g + h) = f \cdot g + f \cdot h$
- $f \cdot \alpha g = \alpha(f \cdot g)$
- $f \cdot f \geq 0$
- $f \cdot f = 0 \iff f = 0$