1 April 25th, 2019

1.1 Review of Multiway Cut Problem

We are given an undirected graph $G = (V, E)$ with edge costs $c_e \geq 0$ for all $e \in E$. We also have k distinguished vertices $s_1, s_2, \ldots, s_k$.

Our goal is to partition the set of vertices into sets C_i such that $s_i \in C_i$ for all i and such that the cost of

$$F = \bigcup_i \delta(C_i)$$

is minimum. In other words, we are trying to minimize the cost of the edges that cross the cuts.

This can be formulated as an ILP:

- For each vertex $u \in V$, introduce variable x_u^i defined by:

$$x_u^i = \begin{cases} 1 & \text{if } u \text{ is assigned to } C_i \\ 0 & \text{otherwise} \end{cases}.$$

- For each edge $e \in E$, introduce variables z_e^i which is defined by:

$$z_e^i = \begin{cases} 1 & \text{if } e \in \delta(C_i) \\ 0 & \text{otherwise} \end{cases}.$$

- ILP:

$$\min_e c_e \left(\frac{1}{2} \sum_i z_e^i \right)$$

  Subject to:

$$\sum_i x_u^i = 1$$
$$z_e^i \geq x_u^i - x_v^i$$
$$z_e^i \geq x_v^i - x_u^i$$

  for all edges $e = (u, v)$, with connectivity and integer constraints:

$$x_{s_i}^i = 1$$

$$x_u^i \in \{0, 1\}.$$

- For the LP relaxation, we have:

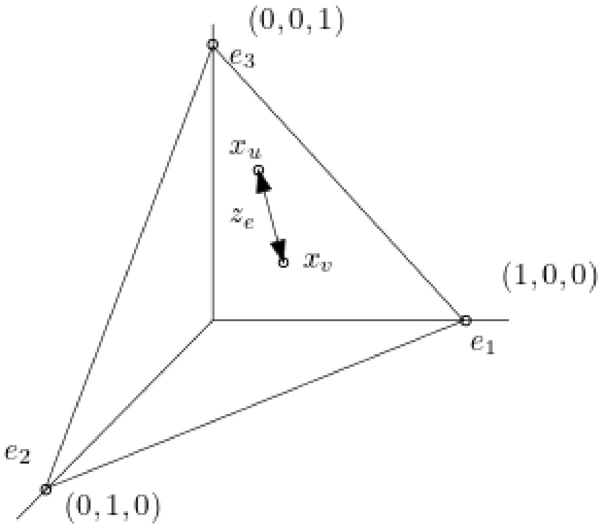
$$x_u^i \geq 0.$$

1.2 LP Relaxation for the Multiway Cut Problem

Think of x_u as a point in k dimensions:

$$x_u = (x_u^1, x_u^2, \ldots, x_u^k).$$

For $k = 3$ we can think as the following diagram:



Remark 1.1 — Note that $x_{s_1} = (1,0,0), x_{s_2} = (0,1,0)$, i.e.

$$x_{s_i} = e_i.$$

This means that x_u is on the hyperplane going through e_i. In addition, note that the x_u for the ILP lie on e_i.

Define the simplex in k dimension as:

$$\Delta_k = \{x \in R^k : \sum_i x^i = 1\}.$$

Note that:

$$\sum_i z_e^i = sum_i |x_u^i - x_v^i|$$

$$\|x_u - x_v\|_1,$$

which is the L_1 distance between x_u and x_v.

We can rewrite the relaxed LP as:

$$\min \frac{1}{2} \sum_{e=(u,v)} c_e \|x_u - x_v\|$$

subject to:

$$x_u \in \Delta_k$$

$$x_{s_i} = e_i.$$

In other words, we have to map the vertices the interior of this triangle, with cost being the weighted sum of the endpoints. This is called the **fractional multiway cut**

Remark 1.2 — Although this does not look like an LP, since it is the same as the LP, we can solve this in polynomial time.

Now we will convert this back into a feasible solution through randomized rounding.

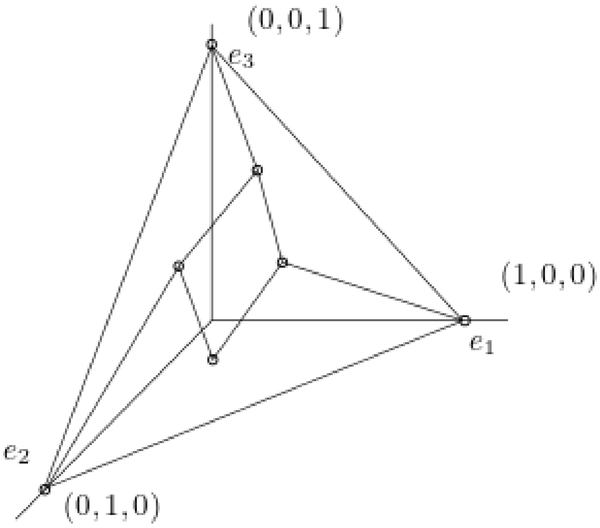


Figure 1: Fractional Mutiway Cut

1.3 Randomized Rounding

The deterministic approach would be to take each vertex and map it to its nearest corner. However, there isn't much analysis to prove its factor. As such we will take a randomized approach.

If we call the "ball distance" as half the L_1 distance, the maximum ball distance between any two vertices is 1 (if they are at the corners). Define $B(i_i, r)$ be the set of vertices u such that:

$$\frac{1}{2}\|e_i - x_u\| \leq r.$$

Note that all vertices are within ball distance 1 from s_i, that is $B(e_i, 1) = V$. (note that we use e_i and s_i interchangeably.

Algorithm 1.3
- Solve LP optimally
- Select $r \in (0, 1)$ uniformly at random
- Assign vertices within $B(s_i, r)$ to C_i

Remark 1.4 — The above algorithm has the following issues:

1. Some vertices are not within ball-distance r from any corner.
2. Some vertices for which there is a conflict, where it is being assigned to multiple corners.

Algorithm 1.5
- Select $r \in (0,1)$ uniformly at random.
- Select a random permutation Π of $\{1, 2, \dots, k\}$
- Examine the vertices in the order given by the permutation Π.
- For index $\Pi(i)$, assign all vertices not assigned so far in $B(s_{\Pi(i)}, r)$ to $C_{\Pi(i)}$
- At the end of the order (i.e. when considering $S_{\Pi(i)}$), assign all vertices not yet assigned to $C_{\Pi(k)}$.

Remark 1.6 — Note that both issues with the previous algorithms are solved, as each vertex is given a chance to grab vertices, and all vertices are assigned.

1.4 Analysis of Randomized Rounding Algorithm

To analyze this algorithm, we need to find the probability that edge $e = (u, v)$ makes a contribution to the cost, i.e. u and v are assigned to different corner (if they are assigned to the same, then they would not make a contribution).

First assume that the probability that we separate u and v is $\frac{1}{2}\|x_u - x_v\|$. If we can show this, then the expected cost of the multiway cut will be as good as the fractional multiway cut. (which is not possible). As such, to have a approximation factor of 1.5, we would like to show that the probability is at most $\frac{3}{4}\|x_u - x_v\|$, as:

$$\text{Expected Cost} = \sum_{e=(u,v)} \frac{3}{4} c_e \|x_u - x_v\| = \frac{3}{2} sum_{e=(u,v)} \frac{1}{2} \|x_u - x_v\|.$$

This is where we are headed.

(reworded) Recall that:

$$OPT \geq \frac{1}{2} \sum_{e=(u,v)} c_e \|x_u - x_v\|$$

where x_u and x_v correspond to vertices u, v respectively in the optimal LP solution. Suppose we can show that the probability that u and v are assigned to different C_i's is $\leq \frac{3}{4}\|x_u - x_v\|$, then the expected solution found has cost:

$$\leq \sum_{e=(u,v)} \frac{3}{4} c_e \|x_u - x_v\| = \frac{3}{2} \sum_{e=(u,v)} \frac{1}{2} \|x_u - x_v\| \leq \frac{3}{2} OPT.$$

We will show that this is the case.