1 October 15th, 2020

1.1 Box Scheme

From last time, we looked at the modified equations for the upwind scheme and the Lax-Wendroff schemes and saw that they showed different properties because of the leading order error term. For upwind, this was u_{xx}, giving it a dispersive effect, while for Lax-Wendroff this was u_{xxx}, giving us an oscillatory effect.

Let us now consider the **Box Scheme**:

$$\frac{1}{2}\left(\frac{U_j^{n+1}-U_j^n}{\Delta t}+\frac{U_{j+1}^{n+1}-U_{j+1}^n}{\Delta t}\right)+\frac{a}{2}\left(\frac{U_{j+1}^n-U_j^n}{\Delta x}+\frac{U_{j+1}^{n+1}-U_j^{n+1}}{\Delta x}\right)=0.$$

Remark 1.1 — This is essentially taking the average of the time and space derivatives.

If we let $\nu=\frac{\Delta t}{\Delta x}$, we have:

$$(1-a\nu)U_j^{n+1}+(1+a\nu)U_{j+1}^{n+1}=(1+a\nu)U_j^n+(1-a\beta)U_{j+1}^n.$$

Assume for $j=0$, U_j^{n+1} is given for the boundary condition. If this is the case, then we have:

$$(1+a\nu)U_{j+1}^{n+1}=-(1-a\nu)U_j^{n+1}+(1+a\nu)U_j^n+(1-a\beta)U_{j+1}^n.$$

allowing us to solve for U_{j+1}^{n+1} recursively. If the boundary condition is given on the right boundary, i.e. U_N^{n+1} is given, then we can solve recursively from the right, as:

$$(1-a\nu)U_j^{n+1}=-(1+a\nu)U_{j+1}^{n+1}+(1+a\nu)U_j^n+(1-a\beta)U_{j+1}^n.$$

Remark 1.2 — Note that for a transport equation to be solved, we need to specify the left boundary condition $a>0$ or the right boundary condition if $a<0$.

For the stability, let us consider:

$$U_j^n\sim\lambda^n e^{ikj\Delta x}.$$

Plugging this into the equation, we have:

$$(1-a\nu)\lambda^{n+1}e^{ikj\Delta x}=-(1+a\nu)\lambda^{n+1}e^{ik(j+1)\Delta x}+(1+a\nu)\lambda^n e^{ikj\Delta x}+(1-a\nu)\lambda^n e^{ik(j+1)\Delta x}$$

$$\implies (1-a\nu)\lambda-(1+a\nu)\lambda e^{ik\Delta x}+(1+a\nu)+(1-a\nu)e^{ik\Delta x}$$

$$\implies \lambda=\frac{(1+a\nu)+(1-a\nu)e^{ik\Delta x}}{(1+a\nu)e^{ik\Delta x}+(1-a\nu)}.$$

If we let $a_1=1+a\nu$ and $a_2=1-a\nu$, we have:

$$\lambda=\frac{a_1+a_2e^{ik\Delta x}}{a_1e^{ik\Delta x}+a_2}.$$

$$\implies |\lambda|=|\frac{a_1+a_2\cos x+ia_2\sin x}{a_1\cos x+ia\sin x+a_2}=\frac{(a_1+a_2\cos x)^2+a_2^2\sin^2 x}{(a_1\cos x+a_2)^2+a_1^2\sin^2 x}.$$

$$=\frac{a_1^2+2a_1a_2\cos x+a_2^2}{a_1^2+2a_1a_2\cos x+a_2^2}=1.$$

and thus the box scheme is unconditionally stable.

Remark 1.3 — The box scheme is also second order in time and space.

Remark 1.4 — The box scheme has a phase advance error if $|a\nu| \leq 1$ and a phase lag error if $|a\nu| > 1$.

1.2 Leap Frog Scheme

The leap-frog scheme takes the central difference in both time and space. The equation we want to solve is:

$$u_t + f(a)_x = 0.$$

or

$$u_t + au_x = 0.$$

$$\frac{U_j^{n+1} - U_j^{n-1}}{2\Delta t} + \frac{f(U_{j+1}^n) - f(U_{j-1}^n)}{2\Delta x} = 0.$$

This has a CFL condition of $|\nu| < 1$ where $\nu = a\frac{\Delta t}{\Delta x}$.

Remark 1.5 — The leap frog scheme is second order in time and space, as it is using the central difference.

For the stability, we have:

$$\frac{1}{2\Delta t}(\lambda^{n+1} - \lambda^{n-1})e^{ikj\Delta x} + a\frac{1}{\Delta x}(e^{ik\Delta x} - e^{-ik\Delta x})\lambda^n e^{ikj\Delta x} = 0$$

$$\implies \left(\lambda - \frac{1}{\lambda}\right) + a\frac{\Delta t}{\Delta x}(2i \sin k\Delta x) = 0$$

$$\implies \lambda^2 + (2\nu i \sin k\Delta x)\lambda - 1 = 0$$

$$\implies \lambda = \frac{-2\nu i \sin k\Delta x \pm \sqrt{-4\nu^2 \sin^2 k\Delta x + 4}}{2} = -\nu i \sin k\Delta x \pm \sqrt{1 - \nu^2 \sin^2 k\Delta x}$$

$$\implies |\lambda| = \nu^2 \sin^2 k\Delta x + 1 - \nu^2 \sin^2 k\Delta x = 1.$$

Thus the leap frog is stable if $|\nu| \leq 1$.

1.3 Leap-Frog Scheme for Wave Equation

Now let's consider the leap frog scheme for:

$$u_{tt} - a^2 u_{xx} = 0.$$

Remark 1.6 — This second order equation can be converted into a system of first order equation. Let $u_t = -av_x$. We have:

$$u_{tt} = -av_{tx} = a^2 u_{xx}.$$

Thus we can rewrite it as

$$\begin{cases} u_t + av_x = 0 \\ v_t + au_x = 0 \end{cases}.$$

To solve this, let U_j^n be defined on the grid points and $V_{j+\frac{1}{2}}^{n+\frac{1}{2}}$ be defined on the center of the grid squares. With this, we can apply the leap frog scheme:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a\frac{V_{j+\frac{1}{2}}^{n+\frac{1}{2}} - V_{j-\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta x} = 0.$$

$$\frac{V_{j+\frac{1}{2}}^{n+\frac{3}{2}} - V_{j+\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta t} + a\frac{U_{j+1}^{n+1} - U_j^{n+1}}{\Delta x} = 0.$$

Remark 1.7 — This also has second order time and space accuracy, since it is using central difference.

For the stability, let us construct a Fourier solution:

$$(U^n, V^{n+\frac{1}{2}}) = \lambda^n e^{ik\Delta x}(\hat{u}, \hat{v}).$$

where $\hat{u}$, $\hat{v}$ are constants. This gives us:

$$\frac{\hat{u}(\lambda^{n+1}e^{ikj\Delta x} - \lambda^n e^{ikj\Delta x})}{\Delta t} + a\frac{\hat{v}(\lambda^n e^{ik(j+\frac{1}{2})\Delta x} - \lambda^n e^{ik(j-\frac{1}{2})\Delta x})}{\Delta x} = 0.$$

$$\frac{\hat{v}(\lambda^{n+1}e^{ik(j+\frac{1}{2})\Delta x} - \lambda^n e^{ik(j+\frac{1}{2})\Delta x})}{\Delta t} + a\frac{\hat{u}(\lambda^{n+1}e^{ik(j+1)\Delta x} - \lambda^{n+1}e^{ikj\Delta x})}{\Delta x} = 0.$$

$$\implies \begin{cases} \hat{u}(\lambda - 1) + \nu\hat{v}(2i\sin k\frac{\Delta x}{2}) = 0 \\ \hat{v}(\lambda - 1)e^{ik\frac{\Delta x}{2}} + \nu\hat{u}\lambda(e^{ik\Delta x} - 1) = 0 \end{cases}.$$

$$\implies \begin{bmatrix} \lambda - 1 & 2i\sin\frac{k\Delta x}{2} \\ 2i\lambda\nu\sin\frac{k\Delta x}{2} & \lambda - 1 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} = 0.$$

for nontrivial solutions, this requires

$$(\lambda - 1)^2 + \lambda 4\nu^2 \sin^2\frac{k\Delta x}{2} = 0.$$

$$\implies \lambda^2 - 2(1 - 2\nu\sin^2\frac{k\Delta x}{2})\lambda + 1 = 0.$$

solving this would give us $|\lambda| = 1$ which is stable.

Remark 1.8 — We can also do direct discretization of the scheme which is second order in time and space and is stable.