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1.1 Applications of Functional CLT

To show an application, we ask what is the distribution of $\max\limits_{0<=i\leq n} S_i = M_n$. To be more precise, we are asking this for a general ξ_i's. Notice that

$$\frac{M_n}{\sqrt{n}} = \sup_{t\in[0,1]} X_n(t) \implies \sup_{t\in[0,1]} X(t)$$

by the continuous mapping theorem. Since this is a general result, we only need to find the limiting distribution of a special case, in this case the random walk. Thus, we want to know what's the maximum point the random walk has reached. Let ξ_i's i.i.d $\pm 1\text{Ber}(\frac{1}{2})$.

Claim 1.1.

$$\mathbb{P}(M_n \geq a) = 2\mathbb{P}(S_n > a) + \mathbb{P}(S_n = a), \quad \forall a \geq 0$$

Proof. Note we don't need to consider for $a < 0$, which is trivial, since $S_0 = 0$. If $a = 0$, we have:

$$\mathbb{P}(M_n > 0) = 1 \quad \text{ since } S_0 = 0.$$

By law of total probability, we have:

$$2\mathbb{P}(S_n > 0) + \mathbb{P}(S_n = 0) = \mathbb{P}(S_n > 0) + \mathbb{P}(S_n < 0) + \mathbb{P}(S_n = 0) = 1.$$

If $a > 0$, we have:

$$\mathbb{P}(M_n \geq a) = \underbrace{\mathbb{P}(M_n \geq a, S_n > a)}_{\mathbb{P}(S_n > a)} + \mathbb{P}(M_n \geq a, S_n < a) + \underbrace{\mathbb{P}(M_n \geq a, S_n = a)}_{\mathbb{P}(S_n = a)}$$

What remains is to show that $\mathbb{P}(M_n \geq a, S_n < a) = \mathbb{P}(M_n \geq a, S_n > a)$. This is true, since there is a 1-to-1 correspondence between the paths of both sides by the reflection principle. This is simply by reflecting after the first time the trajectory reaches level a. □

Now, for any $\alpha \geq 0$, we set $a_n = \lceil \alpha n^{1/2} \rceil$. We have:

$$\begin{aligned}
\mathbb{P}\left(\frac{M_n}{\sqrt{n}} \geq \alpha\right) = \mathbb{P}(M_n \geq a_n) &= 2\mathbb{P}(S_n > a_n) + \mathbb{P}(S_n = a_n) \\
&= 2\mathbb{P}(S_n/\sqrt{n} > a_n/\sqrt{n}) + \mathbb{P}(S_n/\sqrt{n} = a_n/\sqrt{n}) \\
&= 2\mathbb{P}(N(0,1) > \alpha).
\end{aligned}$$

This means that the distribution function goes to:

$$\mathbb{P}\left(\frac{M_n}{\sqrt{n}} \leq \alpha\right) \to 1 - 2\mathbb{P}(N(0,1) > \alpha) = \frac{2}{\sqrt{2\pi}} \int_0^\alpha e^{-\frac{u^2}{2}} du, \quad \alpha \geq 0.$$

Remark 1.2 — In probability theory, the general framework is to find the limiting distribution of a specific case and also prove the universal result.

Now we will consider the limiting distribution of the empirical process. This time, it converges to a Brownian bridge.

Definition 1.3 (**Brownian Bridge**)**.** Let $X(t), t \in [0,1]$ be Brownian Motion. We say:

$$\mathring{X}(t) = X(t) - tX(1), \quad t \in [0,1],$$

is a Brownian Bridge.

Remark 1.4 — We call it a Brownian Bridge since $\mathring{X}(0) = \mathring{X}(1) = 0$.

Definition 1.5 (alternate defintion of Brownian Bridge)**.** Brownian Bridge $\mathring{X}(t)$ is a Gaussian process with:

$$\mathbf{E}(\mathring{X}(t)) = 0, \quad \mathbf{E}(\mathring{X}(s)\mathring{X}(t)) = s \wedge t - \text{ s.t. } (s(1-t) \text{ if } s \leq t)$$

Definition 1.6 (**empirical distribution**)**.** Given a r.v. ξ with c.d.f. F (often unknown in reality), we want to estimate F. Let ξ_i be i.i.d. samples of ξ. The empirical distribution is:

$$F_n(t) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(\xi_i \in [0,t])$$

The empirical distribution is often used to estimate the underlying distribution, since by the SLLN, for any fixed $t \in [0,1]$:

$$F_n(t) \overset{\text{a.s.}}{\to} \mathbf{E}\mathbb{1}(\xi_i \in [0,1]) = F(t)$$

meaning it approximates it pointwise. Using the CLT, for any fixed $t \in [0,1]$:

$$\sqrt{n}[F_n(t) - F(t)] \implies N(0, F(t)(1-F(t))).$$

What we really want to know is if it can approximate it as a whole, not just pointwise convergence. This makes use of the following theorem:

Theorem 1.7 (**Glivenko-Cantelli Theorem**)

$$\|F_n - F\|_\infty = \sup_{0 \leq t \leq 1} |F_n(t) - F(t)| \overset{\text{a.s.}}{\to} 0$$

Again, this distance is called the Kolmogorov-Smirnov statistic. We might now ask what is the limiting distribution of this distance, i.e. the limiting distribution of:

$$\sup_{0 \leq t \leq 1} |\sqrt{n}(F_n(t) - F(t))|.$$

Let us define this as a random function $X_n(t) = \sqrt{n}(F_n(t) - F(t)), t \in [0,1]$.

Remark 1.8 — Note that $X_n(t)$ is not in $C[0,1]$. Instead, $X_n(t) \in D[0,1]$, which is the space of functions which are right continuous with left limits. We will ignore this issue. For more rigor, check the textbook.

This is a Gaussian process, and when we choose $t = 0$, then $X_n(1) = 0$. At point $t = 1$, we have $X_n(1) = 0$. Thus, we might suspect that this is a Brownian Bridge.

Theorem 1.9

The empirical process $\sqrt{n}(F_n(t) - F(t))$ converges weakly to a Gaussian process $Y(t)$ with mean 0 and covariance:

$$\mathbf{E}(Y(S)Y(t)) = F(s \wedge t) - F(s)F(t)$$

Hence, $Y(t) \stackrel{d}{=} \mathring{X}(F(t))$. In particular, if $F(t) = t$, i.e. ξ_i are uniformly distributed, then $Y(t) = \mathring{X}(t)$. Further, by continuous mapping, we have:

$$\sqrt{n} \sup_{0 \le t \le 1} |\sqrt{n}(F_n(t) - F(t))| \implies \sup_{0 <= t \le 1} \mathring{X}(F(t)) = \sup_{0 \le t < 1} \mathring{X}(t)$$

if F is continuous.

This means that whenever F is continuous, the K-S statistic is non-parametric, as it does not depend on F.

Remark 1.10 — There are other statistics that can be used, such as the Cramer Von-Mikes statistics defined as:

$$\int |\sqrt{n}(F_n(t) - F(t))|^2 dF(t).$$

This is the end of this chapter on functional limiting theorems.