# ENM251 - Analytical Methods in Engineering

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These are the notes that I typed during the lectures/recitations. There's probably a lot of typo/mistakes since I haven't really gone through them after class, so keep an eye out for anything that doesn't make sense.

## Contents

# 1 January 22nd, 2020

There are 5 types:

- Separable Differential Equation
- Homogeneous Differential Equation
- Linear Differential Equations
- Bernoulli Differential Equation
- ???

## 1.1 Separable Differential Equation

A general first-order ODE for a dependent variable $y$ in the independent variable $x$ can be written as:

$$\frac{dy}{dx} = F(x, y) \tag{1}$$

where $F$ is some specified function of $x$ and $y$. When $F$ has the form

$$F(x, y) = f(x)g(y), \tag{2}$$

then 1 is said to be *separable* and such equation can always be solved by:

$$\frac{dy}{g(y)} f(x)dx \implies \int \frac{dy}{g(y)} + C_1 = \int f(x)dx + C_2 \implies \int \frac{dx}{g(y)} = \int f(x)dx + C.$$

as one form for the solution of 1.

### 1.1.1 Ideal Fluid Flow

We are concerned with a container that has a fluid with cross sectional area $A$ with density $\rho$ with a hole at the bottom of the container which causes it to flow out. We are concerned with the heigh $x$ of the container. We also have a pipe that pumps in fluid with constant rate $R$.

This leads to following equation:

$$\frac{dx}{dt} = \alpha - \beta\sqrt{x}.$$

where

$$\alpha = \frac{R}{A} \quad \beta = \sqrt{\frac{2ga^2}{A^2 - a^2}} \quad g = 9.81\text{m s}^{-2}.$$

Note that this is a separable differential equation:

$$\frac{dx}{\alpha - \beta\sqrt{x}} = dt.$$

If we have $\alpha$, $\beta$, we can solve, e.g. $\alpha = 60$ $\beta = 6$, we have:

$$\frac{dx}{dt} = 60 - 6\sqrt{x} \implies \frac{dx}{10 - \sqrt{x}} = 6dt.$$

Integrating on both sides, we have:

$$\int \frac{dx}{10-\sqrt{x}} = \int 6dt = 6t + C.$$

Solving this, we get:

$$20\tan^{-1}\left(\frac{\sqrt{x}}{10}\right) - 10\ln(100-x) - 2\sqrt{x} = 6t + C.$$

If we have initial conditions, e.g. at $t = 0, x = 0$, we would have:

$$0 - 10\ln(100) = C$$

allowing us to solve for $C$. This would allow us to solve for a time $t$ for certain values of $x$.

## 1.2 Homogeneous Differential Equation

Again remember that the general form a differential equation of one a dependent variable $y$ in the independent variable $x$ is:

$$\frac{dy}{dx} = F(x, y).$$

If $F(x, y) = f(x)g(x)$ then this is separable. Remember that the goal is that we want to find $G(x, y) = C$, in other words, we want to get rid of the derivative and find the relationship between the two.

**Definition 1.1.** A function of form $F(x, y)$ is called **homogeneous** of order $N$ if $F(tx, ty) = t^N F(x, y)$ for any scalar $t$.

**Example 1.2**

$$\begin{aligned} F(x,y) = x^3 + x^2y + 4xy^2 \implies F(tx, ty) &= (tx)^3 + (tx)^2(ty) + 4(tx)(ty)^2 \\ &= t^3\left(x^3 + x^2y + 4xy^2\right) = t^3F(x, y). \end{aligned}$$

Thus $F(x, y)$ is homogeneous to the order 3.

**Example 1.3**

$F(x, y) = x^3 + xy$ is not homogeneous.

**Example 1.4**

$$F(x, y) = \frac{xy}{x^2 + y^2}$$

$$F(tx, ty) = \frac{t^2xy}{t^2x^2 + t^2y^2} = t^2\left(\frac{xy}{x^2 + y^2}\right) = t^0F(x, y)$$

meaning that $F(x, y)$ is homogeneous to order 0.

**Remark 1.5** — Typically if we say that a function is homogeneous but don't specify the order, it is assumed to be of order 0.

If a function in homogeneous to order 0, then it only depends on the ratio of $\frac{y}{x}$. In other words, rewrite $F(x,y) = f(\frac{y}{x})$.

**Theorem 1.6**

A function $F(x,y)$ is homogeneous of order 0 if and only if it can be expressed as $f(\frac{y}{x})$.

If we have a homogeneous function of order 0, we will be able to introduce a new variable $z = \frac{y}{x} \implies y = sz$, giving us:

$$\frac{d(xz)}{dx} = F(x, xz) = F(x(1), x(z)) = F(1, z).$$

Using the product rule, we have:

$$\frac{d(xz)}{dx} = \frac{dx}{dx}z + x\frac{dz}{dx} = F(1,z).$$

$$z + x\frac{dz}{dx} = F(1,z) \implies \frac{dz}{F(1,z) - z} = \frac{dx}{x},$$

which is a separable differential equation.

**Remark 1.7** — The point is whenever you have a homogeneous equation, then introducing $z = \frac{y}{x}$ will allow us to convert it to a separable equation. Note that this only works for order 0 homogeneous equations.

### 1.2.1 Building an Radar Antenna

TL;DR the equation is:

$$\left(\frac{dy}{dx}\right)^2 - 2\frac{y-F}{x}\left(\frac{dy}{dx}\right) - 1 = 0.$$

If we use the quadratic formula, we get:

$$\frac{dy}{dx} = \frac{y-F}{x} + \sqrt{\left(\frac{y-F}{x}\right)^2 + 1}.$$

If we do the substitution, $z = \frac{y-F}{x}$, we get:

$$\frac{d(xz+F)}{dx} = z + \sqrt{z^2+1} \implies x\frac{dz}{dx} + z = z + \sqrt{z^2+1} \implies \frac{dz}{\sqrt{z^2+1}} = \frac{dx}{x}.$$

$$\int \frac{dz}{\sqrt{z^2+1}} = \ln x + C \implies \ln\left(z + \sqrt{z^2+1}\right) = \ln x + C.$$

$$\implies A^2x^2 - 2Axz = 1 \implies \frac{1}{2}Ax^2 + \left(F - \frac{1}{2A}\right),$$

which is the equation of a parabola. Thus the optimal shape of a radar dish is a parabola.

# 2 January 24th, 2020

## 2.1 Recitation 1

### 2.1.1 Homogeneous ODE

Recall that a homogeneous equation is

$$\frac{dy}{dx} = F(x,y), \quad \text{with } F(ax,ay) = a^n F(x,y).$$

What this typically means is that we won't have a constant.

**Example 2.1**

$F(x,y) = xy$ is homogeneous, as $F(ax,ay) = a^2xy$, while $F(x,y) = ax+5$ is not homogeneous, as $F(ax,ay) = a^2xy + 5 \neq a^n F(x,y)$.

For 1st order homogeneous ODE, we have $n = 0$, with this we can introduce $z = \frac{y}{x}$ and convert this ODE into a separable differential equation.

### 2.1.2 Problem 1

**Example 2.2**

Let's consider

$$F(x,y) = \frac{dy}{dx} = \frac{2y^2 - x^2}{3xy}.$$

$$F(ax,ay) = \frac{2a^2y^2 - a^2x^2}{3a^2xy} = F(x,y),$$

meaning that it is a first order homogeneous equation.

With this, we have:

$$\begin{aligned}
\frac{d(zx)}{dx} &= \frac{2(zx)^2 - x^2}{3x(zx)} \\
\implies z + x\frac{dz}{dx} &= \frac{2x^2z^2 - x^2}{3x^2z} = \frac{2z^2-1}{3z} \\
\implies x\frac{dz}{dx} &= \frac{2z^2 - 1 - 3z^2}{3z} = -\frac{z^2+1}{3z}.
\end{aligned}$$

Now we can separate, giving us:

$$\frac{z}{z^2+1}dz = -\frac{1}{3x}dx \implies \int \frac{z}{z^2+1}dz = \int -\frac{1}{3x}dx$$

$$\implies \frac{1}{2}\ln(z^2+1) = -\frac{1}{3}\ln(x) + C_1$$

Solving for $C_1$, we get:

$$3\ln(z^2+1) = -2\ln(x) + 6C \implies C = 3\ln(z^2+1) + 2\ln(x) = 6C_1$$

$$\implies \ln\left(x^2(z^2+1)^3\right) = 6C_1 \implies x^2(z^2+1)^3 = e^{6C_1}.$$

Remembering that $z = \frac{y}{x}$, we have:

$$x^2\left(\frac{y^2}{x^2}+1\right)^3 = e^{6C_1} \implies \frac{(y^2+x^2)^3}{x^4} = e^{6C_1} \implies \frac{y^2+x^2}{x^{\frac{4}{3}}} = e^{2C_1} = C.$$

$$y = \pm x^{\frac{2}{3}}\sqrt{C - x^{\frac{3}{2}}}.$$

### 2.1.3 Bernoulli Equation

**Definition 2.3.** A **Bernoulli Equation** is an equation of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

If $n = 0$ or $n = 1$, we separate this equation. If $n \neq 0, 1$, defining $y = z^\lambda$, we have:

$$\frac{dy}{dx} = \frac{d(z^\lambda)}{dx} = \frac{dz}{d\lambda}\frac{dz}{dx} = \lambda z^{\lambda-1}\frac{dz}{dx}$$

Substituting this back, we have:

$$\lambda z^{\lambda-1}\frac{dz}{dx} + P(x)z^\lambda = Q(x)(z^\lambda)^n.$$

Dividing both sides by $\lambda z^{\lambda-1}$, we have:

$$\frac{dz}{dx} + \frac{1}{\lambda}P(x)z = \frac{1}{\lambda}Q(x)z^{\lambda n-\lambda+1}.$$

Setting $\lambda$ such that $\lambda n - \lambda + 1 = 0$, i.e. $\lambda = \frac{1}{1-n}$, the equation becomes:

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x).$$

Which is a linear equation, which we can solve:

$$z(x) = \frac{1}{\mu_n}\left(\int \mu_n(1-n)Q(x)dx + C\right), \quad \mu_n = \exp\left\{(1-n)P(x)dx\right\}.$$

And substituting back into the original equation, we have:

$$y = z^\lambda = z^{\frac{1}{1-n}} = \left(\frac{1}{\mu_n}\left(\int \mu_n(1-n)Q(x)dx + C\right)\right)^{\frac{1}{1-n}}.$$

### 2.1.4 Problem 2

Consider

$$vx\frac{dv}{dx} + v^2 + xg = \frac{FL}{m}.$$

Rearranging the equation, we get:

$$\frac{dv}{dx} + \frac{v}{x} + \frac{g}{v} = \frac{FL}{xvm} \implies \frac{dv}{dx} + \left(\frac{1}{x}\right)v = \left(\frac{FL}{mx} - g\right)v^{-1}.$$

which is the form of a Bernoulli equation. As such, we can just plug into the formula, and we get:

$$\mu = \exp\{\int (1-(-1))\frac{1}{x}dx\} = e^{\int \frac{2}{x}dx} = x^{2\ln(x)} = x^2.$$

$$V(x) = \left(\frac{1}{\mu}\left(\int (1-(-1))\mu Q(x)dx + C\right)\right)\frac{1}{(1-(-1))}$$

$$= \left(\frac{1}{x^2}\left(\int 2x^2\left(\frac{FL}{mx} - g\right)dx + C\right)\right)^{\frac{1}{2}}$$

$$= \left(\frac{1}{x^2}\left(\frac{FLx^2}{m} - \frac{2}{3}gx^3\right) + C\right)^{\frac{1}{2}} = \left(\frac{FL}{m} - \frac{2}{3}gx + \frac{C}{x^2}\right)^{\frac{1}{2}}.$$

If we have an constraint where $V$ is finite with $x = 0$, we need $C = 0$, as otherwise $x = 0$ will be infinite. Thus:

$$V = \sqrt{\frac{FL}{m} - \frac{2}{3}gx}.$$

### 2.1.5 Problem 3 Hints from Homework 1

In the first homework, we have:

$$\frac{dx}{dt} = K(\alpha - mx)^2(\beta - nx),$$

for some positive constants $\alpha, \beta, m, n$. Here we want to determine:

$$\lim_{t\to\infty} x(t).$$

when $\frac{\alpha}{m} < \frac{\beta}{n}$, $\frac{\alpha}{m} = \frac{\beta}{n}$, $\frac{\alpha}{m} > \frac{\beta}{n}$.

If we plug into the equation, we have:

$$\frac{dx}{dt} = Km^2n\left(\frac{\alpha}{m} - x\right)^2\left(\frac{\beta}{n} - x\right).$$

Note that these are all positive except for the last factor. Thus, for the first case, we have:

1. For $x < \frac{\alpha}{m}$, $\frac{dx}{dt} > 0$
2. For $x = \frac{\alpha}{m}$, $\frac{dx}{dt} = 0$
3. For $x > \frac{\alpha}{m}$ and $x < \frac{\beta}{m}$, $\frac{dx}{dt} > 0$
4. For $x = \frac{\beta}{n}$, $\frac{dx}{dt} = 0$
5. For $x > \frac{\beta}{n}$, $\frac{dx}{dt} < 0$

From 1 and 2, we have: if $x_0 \leq \frac{\alpha}{m}$, $\lim_{t\to\infty} x = \frac{\alpha}{m}$, while from 3,4,5, we have: if $x_0 > \frac{\alpha}{m} \lim_{t\to\infty} x = \frac{\beta}{n}$.

# 3 January 27th, 2020

## 3.1 Linear ODE

**Definition 3.1.** The basic form of first-order linear equation is:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x),$$

where $a_1(x) \neq 0$. The goal is given $a_1(x), a_0(x)$ and $b(x)$, solve for $y(x)$.

**Example 3.2**

$$x^2y'(x) + 2y(x) = x$$

is a first order linear ODE, where $a_1(x) = x^2,\ a_0(x) = 2,\ b(x) = x$.

To solve it, we first divide by $a_1(x)$, giving us:

$$\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = \frac{b(x)}{a_1(x)}.$$

which is of the standard form:

$$\frac{dy}{dx} + P(x)y = Q(x).$$

**Example 3.3**

From the previous example, we'd have:

$$y'(x) + \frac{2}{x^2}y(x) = \frac{1}{x},$$

where $P(x) = \frac{2}{x^2}$ and $Q(x) = \frac{1}{x}$.

To solve this, we then multiply by $e^{\int P(x)dx}$, giving us:

$$e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx} = Q(x)e^{\int P(x)dx}.$$

Note that the second term is $\frac{d}{dx}\left(e^{\int P(x)dx}\right)$, thus by the product rule, this becomes:

$$\frac{d}{dx}\left(e^{\int P(x)dx}\right) = Q(x)e^{\int P(x)dx}.$$

If we call $\mu(x) = e^{\int P(x)dx}$ the **integrating factor** for the ODE, we can express this as:

$$\frac{d(\mu y)}{dx} = \mu Q \implies \mu y = \int \mu Q dx + C \implies y = \frac{1}{\mu}\left(\int \mu Q dx + C\right).$$

### 3.1.1 Steps for Solving $a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$

1. Change to standard form: $P(x) = \frac{a_0(x)}{a_1(x)}$, $Q(x) = \frac{b(x)}{a_1(x)}$.
2. Compute the integrating factor: $\mu(x) = e^{\int P(x)dx}$.
3. Plug into formula: $y(x) = \frac{1}{\mu(x)}\left(\int \mu(x)Q(x)dx + C\right)$.

**Example 3.4**

Returning to the previous example, considering $x^2y'(x) + 2y(x) = x$, we have:

- $P(x) = \frac{a_0(x)}{a_1(x)} = \frac{2}{x^2}$
- $Q(x) = \frac{b(x)}{a_1(x)} = \frac{1}{x}$

We now calculate the integral factor:

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{2}{x^2}dx} = e^{-\frac{2}{x}}.$$

Plugging into the formula, we get:

$$y(x) = \frac{1}{e^{-\frac{2}{x}}}\left(\int e^{-\frac{2}{x}}\frac{1}{x}\, dx + C_1\right).$$

**Example 3.5**

Now consider $x^2y'(x) + 2y(x) = 1$, following the same steps, we get:

$$y(x) = \frac{1}{e^{-\frac{2}{x}}}\left(\int e^{-\frac{2}{x}}\frac{1}{x^2}\, dx + C_1\right) = \frac{1}{e^{-\frac{2}{x}}}\left(\frac{1}{2}e^{-\frac{2}{x}} + C_1\right).$$

**Example 3.6**

$$\frac{dT}{dt} = -h(T - T_R) \implies \frac{dT}{dt} + hT = hT_R,$$

which can solved with the linear method. $P(t) = h$, $Q(t) = hT_R$, giving us:

$$\mu(t) = e^{\int h dt} = e^{ht} \implies T(t) = \frac{1}{e^{ht}}\left(\int e^{ht}hT_R\, dt + C_1\right)$$

$$T(t) = e^{-ht}\left(T_Re^{ht} + C_1\right) = T_R + C_1e^{-ht}.$$

**Remark 3.7** — How to determine which method to use. Bring everything to one side:

$$\frac{dy}{dx} = F(x, y).$$

- If $F(x,y) = f(x)g(y)$, we can use the separable method.
- If $F(tx,ty) = F(x,y)$, we can use the homogeneous method.
- If $F(x,y) = -P(x)y + Q(x)$, then we can use the linear method.
- If $F(x,y) = -P(x)y + Q(x)y^m$, we can use the Bernoulli method.

### 3.1.2 Bernoulli Equation

**Definition 3.8.** A Bernoulli Equation is an equation of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^m,$$

for some number $m$.

**Example 3.9**

Giving initial condition $v(0) = 0$, solve $v$ where:

$$\frac{dv}{dx} + \frac{1}{x}v = gv^{-1},$$

which is of the form of a Bernoulli Equation.

To solve the Bernoulli equation, we set $y = z^\lambda$ and choose $\lambda$ so that the ODE for $z$ is easier to solve than the ODE for $y$. This is because we'd get:

$$\frac{dy}{dx} + P(x)y = Q(x)y^m$$

$$\implies \frac{dz^\lambda}{dx} + P(x)z^\lambda = Q(x)(z^\lambda)^m$$

$$\implies \lambda z^{\lambda-1}\frac{dz}{dx} + P(x)z^\lambda = Q(x)z^{m\lambda}.$$

Dividing by $\lambda z^\lambda$:

$$\implies \frac{dz}{dx} + \frac{1}{\lambda}P(x)z = \frac{1}{\lambda}Q(x)z^{m\lambda+1-\lambda}.$$

Thus we want to choose $\lambda$ so that $m\lambda + 1 - \lambda = 0 \implies \lambda = \frac{1}{1-m}$ where $m \neq 1$.

If $m = 1$, then it is a separable equation, meaning that we have:

$$\frac{dy}{dx} = (Q(x) - P(x))\,y.$$

$$\frac{dy}{y} = (Q(x) - P(x))\,dx \implies y(x) = Ae^{\int (Q(x)-P(x))\;dx}.$$

### 3.1.3 Summary for Solving Bernoulli Equation

Consider

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)y^m.$$

1. First change to standard form with: $P(x) = \frac{a_0(x)}{a_1(x)}$, $Q(x) = \frac{b(x)}{a_1(1)}$
2. If $m = 1$, then, for some constant $A$, we have:

$$y(x) = Ae^{\int (Q(x)-P(x))dx}.$$

3. Otherwise, compute the integrating factor:

$$\mu(x) = e^{\int (1-m)p(x)dx}.$$

4. Giving us the equation:

$$y(x) = \left(\frac{1}{\mu(x)}\left(\left(\int (1-m)\mu(x)Q(x)\ dx\right) + C\right)\right)^{\frac{1}{1-m}}.$$

**Remark 3.10** — Note that the linear case is when $m = 0$, which gives us the equation what we have before.

**Example 3.11**

Returning to our example earlier where we were considering $\frac{dv}{dx} = \frac{1}{x}v = gv^{-1}$, we have $P(x) = \frac{1}{x}$, $Q(x) = g$. Thus the integrating factor is:

$$\mu(x) = e^{\int (1-(-1))\frac{1}{x}\ dx} = e^{\int \frac{2}{x} dx} = e^{2\ln x} = x^2.$$

Thus we have:

$$\begin{aligned} v(x) &= \left(\frac{1}{x^2}\left(\int (1-(-1))x^2 g\ dx + C_1\right)\right)^{\frac{1}{1-(-1)}} \\ &= \left(\frac{1}{x^2}\left(\frac{2}{3}gx^3 + C_1\right)\right)^{\frac{1}{2}} \\ &= \sqrt{\frac{2gx}{3} + \frac{C_1}{x^2}}. \end{aligned}$$

Since $v(x) = 0 \implies C_1 = 0$, thus:

$$v(x) = \sqrt{\frac{2gx}{3}}.$$

# 4 January 29th, 2020

## 4.1 Phase Plot

Let us consider ODE's of the form:

$$\frac{dx}{dt} = f(x) = \dot{x}.$$

If we graph $x$ vs $\dot{x}$ we can get a phase plot, for example:

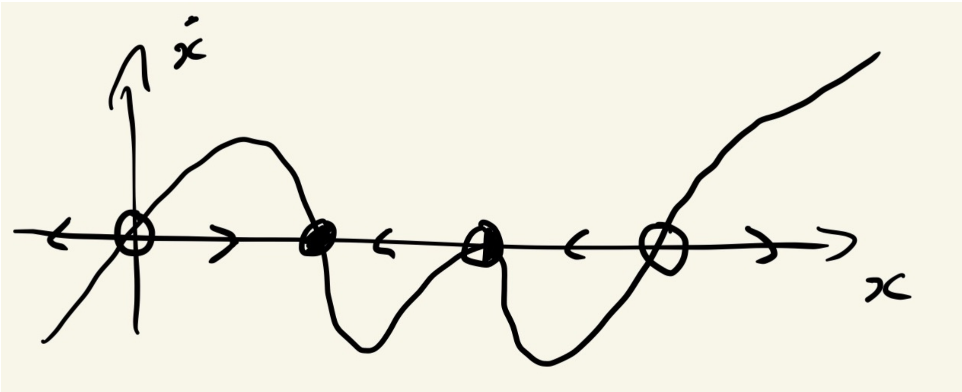


Figure 1: Phase plot of $\dot{x} = x(x-1)(x-2)^2(x-3)^3$

**Definition 4.1.** A point where $f(x) = 0$ is called an **equilibrium point**. These equilibrium points can be unstable (empty circle), stable (filled circle), or left/right stable (half filled circle).

## 4.2 Computing Times

Since $\dot{x} = f(x)$, is separable, since $dt = \frac{dx}{f(x)}$, we have:

$$\int_{t_1}^{t_2} dt = \int_{x_1}^{x_2} \frac{dx}{f(x)} \implies t_2 - t_1 = \int_{x_1}^{x_2} \frac{dx}{f(x)}.$$

Which is the time interval between when $x = x_1$ and $x = x_2$.

**Example 4.2**

Let us try to compute the period of an object with mass $m$ to travel from one end of a bowl to the other with radius $R$. TL;DR we get:

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{R}\cos(\theta)}.$$

Rearranging gives us:

$$dt = \sqrt{\frac{R}{2g\cos\theta}}d\theta \implies \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\cos(\theta)}} \approx \sqrt{\frac{R}{2g}}5.244.$$

## 4.3 Exact Equations

Whenever you have a function of form $\frac{dy}{dx} = F(x, y)$, you can always rewrite it in the form:

$$M(x, y)dx + N(x, y)dy = 0.$$

This might look familiar, as if we have $f(x, y) = C$, we have:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0.$$

As such, we'd like to ask when can $M(x, y)dx+N(x, y)dy = 0$ be written as $\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy = 0$. It would be great if $M = \frac{\partial f}{\partial x}$ and $N = \frac{\partial f}{\partial y}$, so it's helpful to know when we can do this.

Consider

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

As such, if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

then $Mdx + ndy = 0$ is called exact.

**Example 4.3**

$2xydx + (x^2 - y^2)dy = 0$ is exact.

**Example 4.4**

$2x^2ydx + (x^3 - y^2)dy = 0$ is not exact.

Note that the two examples differ by a factor $x$, meaning that we have a further condition to determine whether something is exact.

# 5 January 31st, 2020

## 5.1 Problem 1

Find period of motion for the equation:

$$\dot{\theta} = \sqrt{\frac{g}{L}(3 + 2\cos\theta)} \quad 0 \le \theta \le 2\pi.$$

Since the RHS only has $\theta$, this is separable, thus:

$$\int dt = \sqrt{\frac{L}{g}} \int \frac{d\theta}{\sqrt{3 + 2\cos(\theta)}}$$

Note that the RHS gives us an elliptical equation. Since we want the period, we have:

$$T = \sqrt{\frac{L}{g}} \int_0^{2\pi} \frac{d\theta}{3 + 2\cos\theta} + C.$$

We can consider $C$ to be the start time, and just set it to 0. This is as far as you can go analytically, so plug it into a calculator.

### 5.1.1 How to use in MATLAB

```
T = integral(@(theta)1./sqrCos(1,theta),2,2*pi)
tspan = [0 2.5];
y0 = 0;
data = ode45(@sqrCos,tspan,y0);

function res = sqrCos(t,theta)
  L = 2,4;
  g = 9,8;
  res = sqrt(g/L*(3+2*cos(theta)));
end(function)
```

## 5.2 Problem 3

Consider the equation

$$v\frac{dv}{dx} + \frac{v^2}{x + \frac{m}{\rho}} = g.$$

With the initial condition: $v_0 = v(x_0) = v(0) = 0$. To solve for $v(x)$, note that this is a Bernoulli equation:

$$\frac{dv}{dx} + \frac{1}{x + \frac{m}{\rho}} v = g^{v-1}.$$

with:

$$p(x) = \frac{1}{x + \frac{m}{\rho}} \quad Q(x) = g \quad n = -1.$$

Plugging into the formula, we have:

$$V(x) = \left(\frac{1}{\mu(x)}\left(\int (1-n)\mu(x)Q(x)dx + C\right)\right)^{\frac{1}{1-n}}.$$

Calculating the integrating factor, we have:

$$\mu(x) = e^{\int (1-n)P(x)dx} = e^{2\ln(x+\frac{m}{\rho})} = \left(x + \frac{m}{\rho}\right)^2.$$

Thus we have:

$$V(x) = \left(\frac{1}{\left(x + \frac{m}{\rho}\right)^2}\left(2\int \left(x + \frac{m}{\rho}\right)^2 g\ dx + C\right)\right)^{\frac{1}{2}}$$

$$V(x) = \left(\frac{1}{\left(x + \frac{m}{\rho}\right)^2}\left(\frac{2}{3}\left(x + \frac{m}{\rho}\right)^3 + C\right)\right)^{\frac{1}{2}} = \frac{1}{x + \frac{m}{\rho}}\sqrt{\frac{2}{3}\left(x + \frac{m}{\rho}\right)^3 g + C}.$$

Plugging in the initial condition, we get: $C = -\frac{2}{3}\frac{m^3}{\rho^3}g$, giving us:

$$v(x) = \frac{1}{x + \frac{m}{\rho}}\sqrt{\frac{2}{3}\left(x + \frac{m}{\rho}\right)^3 g - \frac{2}{3}\left(\frac{m}{\rho}\right)^3 g}.$$

The acceleration is:

$$g - \frac{v^2}{x + \frac{m}{\rho}}.$$

# 6 February 3rd, 2020

## 6.1 Exact Equations

Remember that an exact equation is one where:

$$Mdx + Ndy = 0.$$

Where:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Consider the exact equation:

$$(y^2 - x^2)dx + 2xydy = 0.$$

To solve this exact ODE, we set:

$$\frac{\partial f}{\partial x} = M = y^2 - x^2 \implies \int_x (y^2 - x^2)dx + c_1(y) \implies f(x, y) = y^2 x - \frac{x^3}{3} + c_1(y).$$

Now if we take the partial with respect to $y$, we get:

$$\frac{\partial f}{\partial y} = 2yx + c_1'(y) = N = 2xy \implies c_1'(y) = 0 \implies c_1(y) = c_2.$$

This tells:

$$f(x, y) = y^2 x - \frac{1}{3}x^3 + c_2$$

satisfies both equations meaning that the solution to our ODE is of the form:

$$f(x, y) = xy^2 - \frac{1}{3}x^3 = C.$$

If we have an initial condition, then this will give us a unique solution.

**Example 6.1**

Consider the equation: $2xy^2dx + (2x^2y - y^3)dy = 0$. To solve this, we do the following:

$$\int_x 2xy^2 \, dx = x^2y^2 + c_1(y) \implies 2x^2y + c_1'(y) = 2x^2y - y^3 \implies c_1 = -\frac{y^4}{4}$$

Thus we have:

$$f(x, y) = 2x^2y^2 - \frac{1}{4}y^4 + C.$$

## 6.2 Inexact Equations

If $Mdx + Ndy = 0$ is not exact, then we try to introduce an integrating factor $\mu(x, y)$ to turn make $\mu Ndx + \mu Ndy = 0$. Thus we want:

$$\frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x}.$$

However this is usually as difficult to solve as the original equation. There are some special cases though:

- $\mu(x, y) = \mu(x)$. If this is the case, we have:

$$\frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x} \implies \mu\frac{\partial M}{\partial y} = \frac{d\mu}{dx}N + \mu\frac{\partial N}{\partial x}$$

$$\implies \mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \mu'(x)N \implies \frac{\mu'(x)}{\mu(x)} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

  and if the RHS is a function of only x, we can integrate, giving us:

$$\mu(x) = \exp\left\{\int \frac{\left(\frac{\partial m}{\partial y} - \frac{\partial N}{\partial x}\right)}{N}dx\right\}.$$

  With this, we will be able to solve the differential equation with $\frac{\partial f}{\partial x} = \mu M$ and $\frac{\partial f}{\partial y} = \mu N$. This is true if:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = k(x).$$

  i.e. it's a function of only $x$

- $\mu(x, y) = \mu(y)$. Same thing but with $y$ instead of $x$. We check if: $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{y}$ is a function of only $y$. We will have:

$$\mu(y) = \exp\left\{\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{m}\right\}.$$

**Example 6.2**

Consider the equation $2xydx + (2x^2 - y^2)dy = 0$. Note that this is not exact. As such, we check:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2x - 4x}{2x^2 - y^2} = \frac{2x}{2x^2 - y^2} \neq \text{ a function of only } x\ .$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{4x - 2x}{2xy} = \frac{1}{y}.$$

Thus we have:

$$\mu(y) = e^{\int \frac{1}{y}dy} = e^{\ln y} = y.$$

**Example 6.3**
Consider $\frac{dy}{dx} = \frac{2x^2-y^2}{3xy}$, rearranging gives us:

$$(x^2 - 2y^2)dx + 3xydy = 0.$$

Note that $\frac{\partial M}{\partial y} = -4y$ and $\frac{\partial N}{\partial x} = 3y$, thus it is not exact. Now we try:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-4y - 3y}{3xy} = \frac{-7}{3x}.$$

Which is a function of only $x$. As such, we have:

$$\mu(x) = e^{\int -\frac{7}{3x}dx} = x^{-\frac{7}{3}}.$$

Multiplying this in gives us:

$$(x^{-\frac{1}{3}} - 2x^{-\frac{7}{3}}y^2)dx + 3x^{-\frac{4}{3}}ydy = 0,$$

which is exact since:

$$\frac{\partial M}{\partial y} = -4x^{-\frac{7}{3}}y \quad \frac{\partial N}{\partial x} = -4x^{-\frac{7}{3}}y.$$

Solving this gives us:

$$f(x,y) = \int_x x^{-\frac{1}{3}} - 2x^{-\frac{7}{3}}y^2dx = \frac{3}{2}x^{\frac{2}{3}} + \frac{3}{2}x^{-\frac{4}{3}}y + c_1(y).$$

$$\frac{3}{2}x^{-\frac{4}{3}}y + c_1'(y) = \frac{3}{2}x^{-\frac{4}{3}}y \implies c_1 = C.$$

Thus

$$f(x,y) = \frac{3}{2}x^{\frac{2}{3}} + \frac{3}{2}x^{-\frac{4}{3}}y^2 = C.$$

# 7 February 5th, 2020

## 7.1 Applications

Given the family of curves $u(x,y) = c_1$, the family of curves orthogonal to these are the solution to:

$$\frac{\partial u}{\partial x}dy = \frac{\partial u}{\partial y}dx.$$

### 7.1.1 2nd-Order ODE

**Definition 7.1.** The general form of a 2nd order differential equation is:

$$y'' = F(x,y,y').$$

Where $x$ is the independent variable and $y$ is the dependent variable.

We want to consider a few special cases. The first one is when the dependent variable is missing, $y'' = f(x, y')$, for example $y'' = x - y'$. In this case, you can set $v = y'$ $v' = y''$, giving us:

$$v' = f(x, v)$$

which is a first order equation. Thus we can solve the first order ODE and then integrate to get $y$.

**Example 7.2**

Consider the earlier equation $y'' = x - y'$, we have:

$$v' = x - v \implies \frac{dv}{dx} + v = x$$

$$v = e^{-x}((x-1)e^x + c_1) = x - 1 + c_1 e^{-x} = \frac{dy}{dx}.$$

$$y = \frac{1}{2}x^2 + x + c_2 e^{-x} + c_3.$$

for some constants $c_2$ and $c_3$.

**Remark 7.3** — Note that for a first order ODE, there should be one arbitrary constant, but for second order, there should be 2.

The second case is where the independent variable is missing, meaning:

$$\frac{d^2y}{dx^2} = F(y, \frac{dy}{dx}) \implies v\frac{dv}{dx} = F(y, v).$$

Where $v$ is once again $\frac{dy}{dx}$.

# 8 February 10th, 2020

## 8.1 Review from MATH 240

Consider the homogenous equation:

$$a_2(x)y''(x) + a_1 y'(x) + a_0(x)y(x) = 0.$$

Note that this is homogeneous since $y(x) = 0$ is a valid solution. A general solution to a 2nd order lienar homogeneous ODE can be expressed as

$$y(x) = c_1 y_1(x) + c_2 y_2(x),$$

where $c_1$ and $c_2$ are arbitrary constrants and $y_1(x)$ and $y_2(x)$ are any two solutions to the ODE for which:

$$\underbrace{y_1(x)y_2'(x) - y_1'(x)y_2(x)}_{\text{Wronskian of } y_1 \text{ and } y_2} \neq 0.$$

Note that the LHS can be expersed as a determinant:

$$\det\left(\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix}\right).$$

Which is known as the **Wronskian** of $y_1$ and $y_2$.

**Example 8.1**
Consider $y''(x) - 3y'(x) + 3y(x) = 0$, we have:

$$y_1(x) = e^x \quad y_2(x) = e^{2x}.$$

and

$$\det\left(\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\right) = \det\left(\begin{bmatrix} e^x & e^{2x} \\ e^x & e^{2x} \end{bmatrix}\right) = 2e^{3x} - e^{3x} = e^{3x} \neq 0.$$

Meaning that the solution is of the form:

$$y(x) = c_1 e^x + c_2 e^{3x}.$$

**Remark 8.2** — Note that we only the Wronskian to not be the 0 function, and that it's ok for certain values of $x$ for the Wronkian to be 0.

**Example 8.3**
If we used $y_1(x) = e^x$ and $y_2(x) = 2e^x$, then we'd get a Wronskian equal to 0, which would not work.

## 8.2 Constant Coefficients

Consider:

$$ay''(x) + by'(x) + cy(x) = 0,$$

where $a, b$ and $c$ are constants.

**Example 8.4**
Example 8.1 is an example of a constant equation with $a = 1$, $b = -3$, and $c = 2$.

Let us create a table to help us solve this problem. First we contstruct the descriminant: $D = b^2 - 4ac$. Depending on what value $D$ is, we have:

Table 1: Table to Compute $ay'' + by' + cy = 0$

| $D$ | $y_1(x)$ | $y_2(x)$ | |
|---|---|---|---|
| $D < 0$ | $e^{\alpha x}\cos(\beta x)$ | $e^{\alpha x}\sin(\beta x)$ | $\alpha = -\frac{b}{2a}$ $\beta = \sqrt{-D}/2a$ |
| $D = 0$ | $e^{\alpha x}$ | $xe^{\alpha x}$ | $\alpha = -\frac{b}{2a}$ |
| $D > 0$ | $e^{\alpha x}\cosh(\gamma x)$<br>$e^{(\alpha-\gamma)x}$ | $e^{\alpha x}\sin(\gamma x)$<br>$e^{(\alpha+\gamma)x}$ | $\alpha = -\frac{b}{2a}$ $\beta = \sqrt{D}/2a$<br>$\alpha = -\frac{b}{2a}$ $\beta = \sqrt{D}/2a$ |

**Example 8.5**

Consider $4y'' + y' + y = 0$. The discrminant is $D = b^2 - 4ac = -15 < 0$. Using the first row of the previous table, we have:

$$\alpha = -\frac{1}{8} \quad \beta = \frac{\sqrt{15}}{8}.$$

This means that:

$$y(x) = c_1 e^{-\frac{1}{8}x} \cos\left(\frac{x\sqrt{15}}{8}\right) + c_2 e^{-\frac{1}{8}x} \sin\left(\frac{x\sqrt{15}}{8}\right).$$

**Example 8.6**

Consider $4y'' + 4y' + y = 0$. Note that $D = b^2 - 4ac = 16 - 16 = 0$, thus using the second row, we have:

$$\alpha = -\frac{b}{2a} = -\frac{4}{8} = -\frac{1}{2}.$$

Thus we have:

$$y(x) = c_1 e^{-\frac{1}{2}x} + c_2 x e^{-\frac{1}{2}x}.$$

**Example 8.7**

Consider $y'' - 3y' + 2y = 0$, note that $D > 0$. We can either use the third or 4th row of the table. First we have:

$$\alpha = \frac{3}{2} \quad \gamma = \frac{1}{2}.$$

Thus we can write this as either:

$$y(x) = c_1 e^{\frac{3}{2}x} \cosh(\frac{1}{2}x) + c_2 e^{\frac{3}{2}x} \sinh(\frac{1}{2}x)$$

or

$$y(x) = c_1 e^{(\frac{3}{2} - \frac{1}{2})x} + c_2 e^{(\frac{3}{2} + \frac{1}{2})x} = c_1 e^{x} + c_2 e^{2x}.$$

## 8.3 Cauchy-Euler/Equidimentional Equation

**Definition 8.8.** A **equidimensional** or **Cauchy-Euler** 2nd order linear homogenous ODE is a equation of the form:

$$ax^2 y''(x) + bxy'(x) + cy(x) = 0.$$

for some constant $a, b, c$.

**Remark 8.9** — Note that the exponent of the $x$ matches the derivative of $y$.

Again, we can just use a table to solve these equations by checking the value of

$$D = (b - a)^2 - 4ac.$$

Table 2: Table to solve Cauchy-Euler Equations

| $D$ | $y_1(x)$ | $y_2(x)$ | |
|---|---|---|---|
| $D<0$ | $\lvert x\rvert^\alpha \cos(\beta \ln \lvert x\rvert)$ | $\lvert x\rvert^\alpha \sin(\beta \ln \lvert x\rvert)$ | $\alpha = -\frac{b-a}{2a}$ $\beta = \sqrt{-D}/2a$ |
| $D=0$ | $\lvert x\rvert^\alpha$ | $\lvert x\rvert^\alpha \ln \lvert x\rvert$ | $\alpha = -\frac{b-a}{2a}$ |
| $D>0$ | $\lvert x\rvert^\alpha \cosh(\gamma \ln \lvert x\rvert)$ | $\lvert x\rvert^\alpha \sinh(\gamma \ln \lvert x\rvert)$ | $\alpha = -\frac{b-a}{2a}$ $\gamma = \sqrt{D}/2a$ |
| | $\lvert x\rvert^{\alpha-\gamma}$ | $\lvert x\rvert^{\alpha+\gamma}$ | $\alpha = -\frac{b-a}{2a}$ $\gamma = \sqrt{D}/2a$ |

**Example 8.10**

Consider $3x^2y'' + 2xy' + 5y = 0$, where $a = 3, b = 2, c = 5$. Note that:

$$d = (b-a)^2 - 4ac = (2-3)^2 - 4(3)(5) = -59 < 0.$$

Thus we have:

$$\alpha = -\frac{b-a}{2a} = \frac{1}{6}, \quad \beta = \frac{\sqrt{59}}{6}.$$

Thus we have:

$$y(x) = c_1 x^{\frac{1}{6}} \cos(\frac{\sqrt{59}}{6} \ln x) + c_2 x^{\frac{1}{6}} \sin(\frac{\sqrt{59}}{6} \ln x).$$

for $x > 0$.

**Example 8.11**

Consider $x^2y'' + 2xy' - 2y = 0$, $x > 0$, i.e. $a = 1, b_=2, c = -2$. Note that $D = (b-a)^2 - 4ac = 9 > 0$, thus we have:

$$\alpha = \frac{-(2-1)}{2(1)} = -\frac{1}{2} \quad \gamma = \frac{\sqrt{9}}{2(1)} = \frac{3}{2}.$$

With the general solution being:

$$y(x) = c_1 x^{-\frac{1}{2}} \cosh(\frac{3}{2} \ln x) + c_2 x^{-\frac{1}{2}} \sinh(\frac{3}{2} \ln x).$$

or

$$y(x) = c_1 x^{-2} + c_2 x.$$

## 8.4 Other Stuff from Math 240

If we once again consider the equation $a_2(x)y''(x) + a_1(x)y'(x) + a_0 y(x) = 0$. Note that as we mentioned earlier, we need two linearly independent non-zero solutions to get the general solution. If we only have one, say $y_1(x)$, a second linearly independent solution $y_2(x)$ can be constructed using **Abel's equation** :

$$y_2(x) = A y_1(x) \int \frac{e^{-\int \frac{a_1(x)}{a_2(x)}\, dx}}{y_1^2(x)} dx.$$

For any non-zero constant $A$.

**Remark 8.12** — Derivation is in the notes.

**Example 8.13**
Consider $xy'' + (1-x)y' - y = 0$. Suppose we're told that one solution is $y_1(x) = e^x$. A second solution would be:

$$y_2(x) = Ae^x \int \frac{e^{-\int \frac{1-x}{x}\,dx}}{(e^x)^2}\,dx.$$

$$= Ae^x \int \frac{e^{\int 1-\frac{1}{x}\,dx}}{e^{2x}}\,dx = Ae^x \int \frac{e^{x-\ln x}}{e^{2x}}\,dx = Ae^x \int \frac{e^{-x}}{x}\,dx.$$

Which doesn't have a nice answer (oops)

**Remark 8.14** — Note that whenever $a_2(x) + a_1(x) + a_0(x) = 0$, one solution is always $y_1(x) = e^x$, since we'd have $y'' = y' = y = e^x$.

**Example 8.15**
Consider $(1-x)y'' + xy' - y = 0$. Since we have $y_1(x) = e^x$, we have:

$$y_2(x) = Ae^x \int \frac{e^{-\int \frac{x}{1-x}\,dx}}{(e^x)^2}\,dx.$$

$$y_2(x) = Ae^x \int \frac{e^{x+\ln(x-1)}}{e^{2x}}\,dx = Ae^x \int \frac{x-1}{e^x}\,dx = Ae^x(-xe^{-x}) = -Ax.$$

Picking $A = -1$, we have: $y_2(x) = x$, thus the general solution would be:

$$y(x) = c_1 e^x + c_2 x.$$

## 8.5 Non-Homogeneous Equations

Now let's consider non-homogeneous equations:

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0 y(x) = b(x).$$

A general solution is given by:

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x).$$

Where $c_1, c_2$ are arbitrary constants, $y_1, y_2$ are two linearly independent solutions to the homogeneous equation (where $b(x) = 0$), and $y_p$ is any **particular solution** to the non-homogeneous equation.

When $\frac{b(x)}{a_0(x)} =$ a constant, then $y_p(x) = \frac{b(x)}{a(x)}$ works, otherwise:

$$y_p(x) = \int^x G(t,x)\frac{b(t)}{a_2(t)}\,dt.$$

Where:

$$G(t,x) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{y_1(t)y_2'(y) - y_1'(t)y_2(t)}.$$

**Remark 8.16** — $G(t,x)$ is known as the **Green's function** associated with the ODE.

**Remark 8.17** — When solving the integral, treat all $x$'s as constant, then afterwards, replace all $t$ 's with $x$ 's.

**Example 8.18**

Consider the equation solved in 8.15. We have:

$$y_1(x) = e^x \quad y_2(x) = x.$$

$$y_1'(x) = e^x \quad y_2(x) = 1.$$

Thus we have:

$$G(t,x) = \frac{e^t x - e^x t}{e^t(1) - e^t t} = \frac{x - te^{x-t}}{1-t}.$$

Thus we have:

$$y_p(x) = \int^x \frac{x - te^{x-t}}{1-t}\frac{(t-1)^2}{1-t}\,dt = \int^x x - te^{x-t}\,dt = xt - e^x(t-1)e^{-t}\Big|^x = x^2 - x + 1.$$

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