# 1 November 24th, 2020

## 1.1 Spectral Methods

Spectral methods are based on Fourier series and have high accuracy. If we are the domain $[0, 2\pi]$, we can define the periodic functions:

$$\phi_k(x) = e^{ikx} = \cos kx + i \sin kx.$$

For any function $u(x) \in L^2[0, 2\pi$, we can express it in terms of its Fourier series:

$$u(x) = \sum_{k=-\infty}^{+\infty} u_k e^{ikx}.$$

Where:

$$u_k = \frac{1}{2\pi} \int_0^{2\pi} u(x) e^{-ikx} dx.$$

$u_k$ are called the **fourier coefficients**.

> **Remark 1.1** — $L^2$ function means it is square integrable.

$\phi_k(x)$ are orthogonal as:

$$\int_0^{2\pi} \phi_k \overline{\phi_\ell} dx = \int_0^{2\pi} e^{ikx} e^{-i\ell x} dx = \int_0^{2\pi} e^{i(k-\ell)x} dx = \begin{cases} 2\pi & k = \ell \\ 0 & k \neq \ell \end{cases}.$$

$\phi_k(x)$ are complete, meaning for any $u \in L^2[0, 2\pi]$ it can be approximated by:

$$u(x) \sim \sum_{k=-N}^{N} u_k \phi_k(x.$$

in the sense that:

$$\lim_{N\to\infty} \|u(x) - \sum_{k=-N}^{N} u_k \phi_k(x)\| = 0.$$

As such $\{\phi_k\}$ forms a basis of $L^2[0, 2\pi]$.

**Definition 1.2** (Spectral Projection)**.** Given a function $u(x) \in L^2[0, 2\pi]$, we define:

$$P_N u(x) = \sum_{k=-N}^{N} u_k e^{ikx}$$

to be the **spectral projection** of $u$.

> **Remark 1.3** — In other words, the spectral projection is the finite truncation of the Fourier series.

This spectral projection has a few properties:

- $P_N$ is a projection, i.e. $P_N^2 = P_N$.

  *Proof.* Let $\tilde{u}(x) = P_N u(x)$ We have:

  $$P_N^2 u = P_N \tilde{u}.$$

  with .

  $$\begin{aligned}\tilde{u}_\ell(x) &= \frac{1}{2\pi}\int_0^{2\pi} \tilde{u}(x)e^{-i\ell x}dx \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{k=-N}^{N} u_k e^{ikx}\right)e^{-i\ell x}dx \\ &= \frac{1}{2\pi}\sum_{k=-N}^{N} u_k \int_0^{2\pi} e^{ikx}e^{-i\ell x}dx \\ &= u_\ell.\end{aligned}$$

  by using the orthogonality of $\phi_k$. □

- $\|P_N u(x)\|_{L^2}^2 = 2\pi \sum |u_k|^2$.

  *Proof.* We have:

  $$\begin{aligned}\|P_N u(x)\|_{L^2}^2 &= \int_0^{2\pi} |P_N u|^2 dx \\ &= \int_0^{2\pi}\left(\sum_{k=-N}^{N} u_k e^{ikx}\right)\left(\sum_{\ell=-N}^{N} \overline{u_\ell} e^{i\ell x}\right)dx \\ &= \int_0^{2\pi}\sum_{k=-N}^{N}\sum_{\ell=-N}^{N} u_k\overline{u_\ell}e^{i(k-\ell)x}dx \\ &= \sum_k\sum_\ell u_k\overline{u_\ell}\int_0^{2\pi} e^{i(k-\ell)x}dx \\ &= 2\pi\sum_k |u_k|^2.\end{aligned}$$

  again by using the orthogonality of $\phi_k$. □

## 1.2 Decay Rate of Spectral Methods

Let us consider the decay rates of the coefficients:

$$u_k = \frac{1}{2\pi}\int_0^{2\pi} u(x)e^{-ikx}dx.$$

If $u$ is differentiable, then do integration by parts giving us:

$$u_k = -\frac{1}{2\pi}\frac{1}{ik}u(x)e^{-ikx}\Big|_0^{2\pi} + \frac{1}{ik}\frac{1}{2\pi}\int_0^{2\pi} u'(x)e^{-ikx}dx.$$

Because $u$ is periodic, the first term is 0, meaning that:

$$u_k = \frac{1}{ik}\frac{1}{2\pi}\int_0^{2\pi} u'(x)e^{-ikx}dx.$$

Note that this is $\frac{1}{ik}(u')_k$. This can be continued, giving us:

$$u_k = \frac{1}{(ik)^r}\frac{1}{2\pi}\int_0^{2\pi} u^{(r)}e^{-ikx}dx.$$

Taking the absolute value, we have:

$$|u_k| \leq \frac{1}{|k|^r}\frac{1}{2\pi}\int_0^{2\pi} |u^{(r)}|dx \leq \frac{C}{|k|^r}\|u^{(r)}\|_{L^2}.$$

Note that this goes to zero, as $k$ goes to infinity. Note that since $r$ is arbitrary, this goes to zero faster than any power, as we can just increase $r$. As such, it goes to zero exponentially fast. Of course, this requires that the function is differentiable to any order.

We will now prove this exponential convergence of Fourier series for smooth functions.

*Proof.* Let $u(x)$ be smooth, i.e. differentiable to any order, then:

$$u(x) = \sum_{k=-\infty}^{+\infty} u_k e^{ikx}, \quad u_k = \frac{1}{2\pi}\int_0^{2\pi} u(x)e^{-ikx}dx.$$

Note that $u(x)$ is absolutely convergent, as $u_k$ decays faster than any power. As such, we have:

$$|u(x)| \leq \sum_k |u_k|.$$

If $u_k \leq \frac{1}{|k|^2}$, then the right hand side is convergent. Now let's consider:

$$\|u(x) - \sum_{k=-N}^{N} u_k e^{ikx}\|_\infty.$$

which converges to 0 as $N \to +\infty$. We have:

$$\begin{aligned}
\|u(x) - \sum_{k=-N}^{N} u_k e^{ikx}\|_\infty &= \|\sum_{|k|>N} u_k e^{ikx}\|_\infty \\
&\leq \sum_{|K|>N} |u_k||e^{ikx}| \\
&= \sum_{|k|>N} |u_k| \\
&\leq \sum_{|k|>N} \frac{1}{|k|^r}\|u^{(r)}\| \\
&\leq C\sum_{|k|>N} \frac{1}{|k|^r} \\
&= \frac{C}{(N+1)^r}\left(\sum_{j=1}\left(\frac{N+1}{N+j}\right)^r\right) \\
&\leq \frac{C}{(N+1)^r}\cdot C_1.
\end{aligned}$$

This means that:

$$\|u(x) - P_N u(x)\| \leq \frac{M}{(N+1)^r} \to 0.$$

for some constant $M$, which converges faster than any power. As such, it has exponential convergence. □

**Remark 1.4** — How fast the function decays in the frequency space depends on how smooth it is in the physical space.

**Definition 1.5** (Total Variation)**.** Let us define $u(x)$ on $[0, 2\pi]$ with a mesh $0 = x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq x_{n+1} = 2\pi$. We define the **total variation** as:

$$\gamma(u) = \sup_n \sup_{\{x_i\}} \sum_i |u(x_i) - u(x_{i-1})|.$$

**Example 1.6**

If $u(x)$ is a monotone increasing function on $[0, 2\pi]$, then we have $\gamma(u) = u(2\pi) - u(0)$.

**Remark 1.7** — If $u(x)$ is not monotone $[0, 2\pi]$, then if we can split the region into monotone sections, we can calculate the total variation as the absolute value of all the difference in the monotone sections.

**Example 1.8**

Consider $u(x) = \sin x$. We can split it into 3 sections, with the total variation being $\gamma(u) = 4$.

**Example 1.9**

If $u(x) = \sin \frac{1}{x}$, then $u(x)$ has no bounded variation.

Bounded variation is a very important property to determine the convergence of its Fourier series. It has the following properites:

- If $u$ is continuous, periodic, and has bounded variation, then we have: $P_N u(x) \to u(x)$ uniformly.
- If $u(x)$ has bounded variation, then $P_N u(x) \to \frac{1}{2}(u^+(x) + u^-(x))$, which is the average of its left and right limits (note that it does not need to be continuous), this is pointwise convergence.
- If $u \in L^2[0, 2\pi]$, then:

$$\|P_N u(x) - u(x)\|_{L^2} = \int_0^{2\pi} |P_N u(x) - u(x)|^2 dx \to 0.$$

This means it converges in the average sense.

- If $u(x)$ is continuous and periodic, but not of bounded variation, then $P_N u(x)$ does not necessarily converge for all $x$, e.g. $u(x) = x \sin \frac{1}{x}, x \in [-\frac{1}{\pi}, \frac{1}{\pi}]$. At $x = 0$, $P_N u(x)$ is not convergent.

**Remark 1.10 —** Uniform convergence means that:

$$\lim_{N \to \infty} \max_{x \in [0, 2\pi]} |u(x) - P_N u(x)| = 0.$$

Pointwise convergences means that for any $x \in [0, 2\pi]$:

$$\lim_{N \to \infty} |u(x) - P_N u(x)| = 0.$$