1 March 2nd, 2020

1.1 Taylor's Method

Recall that Taylor's method is where we assume the solution:

$$y(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m.$$

Where x_0 is an ordinary point for the ODE. Note that this is a power series expansion about x_0 with a_m being constants.

Consider

$$y''(x) + \omega^2 y(x) = 0, \quad -\infty < x < +\infty.$$

We know already that the solution involves $\sin(x)$ and $\cos(x)$. Note that for this ODE, all points are ordinary points. We will use $x_0 = 0$, since this will simplify things a lot. Thus we have:

$$y(x) = \sum_{m=0}^{\infty} a_m x^m.$$

Which converges absolutely everywhere. Differentiating this W.R.T. x, we get:

$$y'(x) = \sum_{m=0}^{\infty} m a_m x^{m-1} \text{ and } y''(x) = \sum_{m=0}^{\infty} m(m-1) a_m x^{m-2}.$$

Plugging this into the original equation, we have:

$$\sum_{m=0}^{\infty} m(m-1) a_m x^{m-2} + \omega^2 \sum_{m=0}^{\infty} a_m x^m = 0.$$

Note that when $m = 0$ and $m = 1$, the first power series term would be 0. Now using the fact that:

$$\sum_{m=a}^{b} f(m) = \sum_{m=a\pm c}^{b\pm c} f(m \mp c).$$

we get:

$$\sum_{m=2-2}^{\infty} (m+2)((m+2)-1) a_{m+2} x^{(m+2)-2} + \omega^2 \sum_{m=0}^{\infty} a_m x^m = 0.$$

$$\implies \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m + \omega^2 \sum_{m=0}^{\infty} a_m x^m = 0.$$

Collecting the terms, we get:

$$\sum_{m=0}^{\infty} \left((m+2)(m+1) a_{m+2} + \omega^2 a_m \right) x^m = 0.$$

The only way we can get the LHS to equal zero for all values of x is if the coefficient equals zero, thus we have:

$$(m+2)(m+1) a_{m+2} + \omega^2 a_m = 0, \quad m = 0, 1, 2, \ldots.$$

$$a_{m+2} = -\frac{\omega^2 a_m}{(m+2)(m+1)}, \quad m = 0, 1, 2, \ldots.$$

From this, we get:

- for $m=0$, $a_2 = -\frac{\omega^2}{2}a_0$
- for $m=1$, $a_3 = -\frac{\omega^2}{(3)(2)}a_1$
- for $m=2$, $a_4 = -\frac{\omega^2}{(4)(3)}a_2 = \frac{(-1)^2\omega^4}{4!}a_0$
- for $m=3$, $a_5 = -\frac{\omega^2}{(5)(4)}a_2 = \frac{(-1)^2\omega^4}{5!}a_1$

As we keep going, we would have:

$$a_{2k} = \frac{(-1)^k\omega^{2k}}{(2k)!}a_0, \quad a_{2k+1} = \frac{(-1)^k\omega^{2k}}{(2k+1)!}a_1, \quad k = 0,1,2,3,\ldots.$$

Thus for this particular example, we can solve for all a_m from a_0 and a_1.

Since for this series, they break pretty naturally into even and odd powers, we can split it into:

$$\begin{aligned}
y(x) = \sum_{m=0}^{\infty} a_m x^m &= \sum_{k=0}^{\infty} a_{2k}x^{2k} + \sum_{k=0}^{\infty} a_{2k+1}x^{2k+1} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k\omega^{2k}}{(2k)!}q_0 x^{2k} + \sum_{k=0}^{\infty} \frac{(-1)^k\omega^{2k}}{(2k+1)!}a_1 x^{2k+1} \\
&= \alpha \sum_{k=0}^{\infty} \frac{(-1)^k(\omega x)^{2k}}{(2k)!} + \frac{a_1}{\omega}\frac{(-1)^k(\omega x)^{2k+1}}{(2k+1)!} \\
&= a_0\cos(\omega x) + \frac{a_1}{\omega}\sin(\omega x)
\end{aligned}$$

Note that a_0 and a_1 are arbitrary constants, thus giving us the solution we had previously.

Remark 1.1 — As a reminder, for the ODE $y''(x) + P(x)y'(x) + Q(x)y(x) = 0$, x_0 is an ordinary point of the ODE if

$$\lim_{x\to x_0} P(x) \text{ and } \lim_{x\to x_0} Q(x)$$

both exist. Otherwise, it is called a singular point.

Remark 1.2 — Usually, if you can pick $x_0 = 0$, you should pick it, as then you can take advantage of even and odd properties.

1.2 Some Power Series Expansions

$$e^z = \sum_{m=0}^{\infty} \frac{z^m}{m!}, \quad |z| < \infty$$

$$e^{-z} = \sum_{m=0}^{\infty} \frac{(-1)^m z^m}{m!}, \quad |z| < \infty$$

$$\cos(z) = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{(2m)!}, \quad |z| < \infty$$

$$\cosh(z) = \sum_{m=1}^{\infty} \frac{z^{2m}}{(2m)!}, \quad |z| < \infty$$

$$\sin(z) = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m+1}}{(2m+1)!}, \quad |z| < \infty$$

$$\sinh(z) = \sum_{m=0}^{\infty} \frac{z^{2m+1}}{(2m+1)!}, \quad |z| < \infty$$

$$\frac{1}{1-z} = \sum_{m=0}^{\infty} z^m, \quad |z| < 1$$

Consider the ODE:

$$(1-x^2)y''(x) + 8xy'(x) - 20y(x) = 0, \quad -1 < x < +1.$$

Using $x_0 = 0$, which is an ordinary point, since:

$$\lim_{x\to x_0} P(x) = \lim_{x\to x_0} \frac{x_0}{1-x^2} \text{ and } \lim_{x\to x_0} Q(x) = \lim_{x\to x_0} \frac{-20}{1-x^2}.$$

both exists, we have:

$$(1-x^2)\sum_{m=0}^{\infty} m(m-1)a_m x^{m-2} + 8x\sum_{m=0}^{\infty} m a_m x^{m-1} - 20\sum_{m=0}^{\infty} a_m x^m = 0.$$

$$\implies \sum_{m=0}^{\infty} a_m m(m-1)x^{m-2} - \sum_{m=0}^{\infty} a_m m(m-1)x^m + 8\sum_{m=0}^{\infty} a_m m x^m - 20\sum_{m=0}^{\infty} a_m x^m = 0.$$

$$\implies \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=0}^{\infty} \underbrace{(m(m-1) - 8m + 20)}_{(m-4)(m-5)} a_m x^m = 0.$$

$$\implies \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m - \sum_{m=0}^{\infty}(m-4)(m-5)a_m x^m = 0.$$

$$\implies \sum_{m=0}^{\infty} \underbrace{((m+2)(m+1)a_{m+2} - (m-4)(m-5)a_m)}_{=0} x^m = 0.$$

Thus we get the relation:

$$a_{m+2} = \frac{(m-4)(m-5)}{(m+2)(m+1)}a_m, \quad m = 0, 1, 2, \ldots.$$

Plugging in values for m, we have:

- $m = 0$, $a_2 = \frac{20}{2}a_0 = 10a_0$
- $m = 1$, $a_3 = \frac{(-3)(-4)}{(3)(2)}a_1 = 2a_1$

- $m = 2$, $a_4 = \frac{(-2)(-3)}{(4)(3)} a_2 = 5a_0$
- $m = 3$, $a_5 = \frac{2}{(5)(4)} a_3 = \frac{1}{5} a_1$

When we get to $m = 4$ and $m = 5$, we would get a_6 and a_7 both equaling 0, thus all coefficients after that would be zero as well. Thus, we would have:

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

$$y(x) = a_0 + a_1 x + 10a_0 x^2 + 2a_1 x^3 + 5a_0 x^4 + \frac{1}{5} a_1 x^5$$

$$y(x) = a_0 \left(1 + 10x^2 + 5x^4\right) + a_1 \left(x + 2x^3 + \frac{1}{5} x^5\right).$$

Remark 1.3 — As long as $P(-x) = -P(x)$ and $Q(-x) = Q(x)$, and $x_0 = 0$, we will always be able to break the sum into an even and an odd part. In other words, P is an odd function and Q is an even function.

Let us try one which doesn't separate into even and odd terms. Consider:

$$(1 - x^2) y''(x) + 2y'(x) + xy(x) = 0, \quad -1 < x < +1.$$

Notice this time that

$$P(x) = \frac{2}{1 - x^2} \text{ is not an odd function}$$

and

$$Q(x) = \frac{x}{1 - x^2} \text{ is not an even function.}$$

Also note that $x_0 = 0$ is an ordinary point, meaning that we have:

$$y(x) = \sum_{m=0}^{\infty} a_m x^m.$$

$$y'(x) = \sum_{m=0}^{\infty} m a_m x^{m-1}.$$

$$y''(x) = \sum_{m=0}^{\infty} m(m-1) a_m x^{m-2}.$$

Plugging into the ODE, we have:

$$(1 - x^2) \sum_{m=0}^{\infty} a_m m(m-1) x^{m-2} + 2 \sum_{m=0}^{\infty} a_m m x^{m-1} + x \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\implies \sum_{m=0}^{\infty} a_m m(m-1) x^{m-2} + \sum_{m=0}^{\infty} a_m m(m-1) x^m + 2 \sum_{m=0}^{\infty} a_m m x^{m-1} + \sum_{m=0}^{\infty} a_m x^{m+1} = 0.$$

Note that the leading power of x for each of the power series are 0, 2, 1, and 1. Shifting the bounds, we would get:

$$\sum_{m=0}^{\infty} a_{m+2}(m+2)(m+1)x^m + \sum_{m=0}^{\infty} a_m m(m-1)x^m + 2\sum_{m=0}^{\infty} a_{m+1}(m+1)x^m + \sum_{m=0}^{\infty} a_m x^{m+1} = 0.$$

$$\implies \sum_{m=0}^{\infty} \left((m+1)(m+2)a_{m+2} - m(m-1)a_m + 2(m+1)a_{m+1}\right) x^m + \sum_{m=0}^{\infty} a_m x^{m+1} = 0.$$

Note that the first power sum has an extra term ($m = 0$), that the first one does not have, thus we can separate it and get:

$$(2a_2+2a_1)+\sum_{m=1}^{\infty} \left((m+1)(m+2)a_{m+2} - m(m-1)a_m + 2(m+1)a_{m+1}\right) x^m + \sum_{m=1}^{\infty} a_{m-1}x^m = 0.$$

Thus we have:

$$2(a_2+a_1)+\sum_{m=1}^{\infty} \left((m+1)(m+2)a_{m+2} - m(m-1)a_m + 2(m+1)a_{m+1} + a_{m-1}\right) x^m = 0.$$

Thus meaning that:

$$a_2 + a_1 = 0.$$

and

$$(m+1)(m+2)a_{m+2} - m(m-1)a_m + 2(m+1)a_{m+1} + a_{m-1} = 0.$$

Giving us:

$$a_2 = -a_1 \quad a_{m+2} = \frac{-2(m+1)a_{m+1} + m(m-1)a_m - a_{m-1}}{(m+2)(m+1)}, \quad m = 1, 2, 3, \ldots.$$