1 April 24th, 2019

1.1 Case Studies 3

Let

- $W = [w_{ij}]$
- $D = \text{diag}(W_1)$

Define $L = D - W$. Computer the eigenvector x_{n-1} corresponding to λ_{n-1}, the 2nd smallest eigenvalue of L. Define:

$$S = \{i \mid x_{n-1}(i) > 0\}$$

and

$$S^C = \{i \mid x_{n-1}(i) \leq 0\}.$$

However, this method doesn't work well, as it tends to produce unbalanced groups, with one being huge and the other tiny. A better cut would be a "normalized cut".

To make the two groups have similar sizes, define the size of a group:

$$\text{Vol}(A) = \sum_{\substack{i \in A \\ j \in V}} w_{ij}.$$

Definition 1.1. A normalized cut of A and B is:

$$N_{cut}(A, B) = \frac{\text{cut}(A, B)}{\text{Vol}(A)} + \frac{\text{cut}(A, B)}{\text{Vol}(B)}.$$

This is minimized if and only if $\text{cut}(A, B)$ is minimized and $\text{Vol}(A)$ and $\text{Vol}(B)$ are similar. However, this is not solvable, so we just solve:

$$\min_{S \subseteq V} N_{vut}(S, \overline{S}), \quad \text{where } \overline{S} = V \setminus S.$$

This again cannot be efficiently solved, so it can be solved approximately by an eigenvector/eigenvalue solver:

Define $z \in \mathbb{R}^m$ by:

$$z_i = \begin{cases} \frac{1}{\text{Vol}(S)} & \text{if } i \in S \\ -\frac{1}{\text{Vol}(\overline{S})} & \text{if } i \notin S \end{cases}.$$

Then

$$z^T L z = Z^T (D - W) z.$$

1.2 Singular Value Decomposition (SVD)

If $A \in \mathbb{R}^{n\times n}$ is symmetric, then

$$A = U\Lambda U^T,$$

where $U \in \mathbb{R}^{n\times n}$ satisfies $U^TU = UU^T = I$ and $\Lambda \in \mathbb{R}^{n\times n}$ is diagonal. This allows A to be it numerically stable.

However, this only only applies if A is a square and symmetric matrix. As such, we would like to extend the decomposition to any matrix, regardless of if they are square or symmetric. This is SVD.

Let $A \in R^{m\times n}$ with $m \geq n$ be an arbitrary real matrix (we can generalize this to $m < n$ and complex matrix). We will make use of the following facts:

- Since $A^TA \in R^{n\times n}$ is square, symmetric, and SPSD, there exists an eigenvector decomposition of A^TA.

  Let $(\lambda_i, v_i), i = 1, 2, \ldots, n$ be the eigenpairs of A^{TA}, with

  $$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0.$$

  So

  $$A^TA = V\Lambda V^T, V = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}, \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ \lambda_n & & \end{bmatrix}.$$

- $A^T \in R^{m\times m}$ is also square, symmetric, and SPSD.

Theorem 1.2

Let $A \in R^{m\times n}$ and $B \in \mathbb{R}^{m\times m}$. Let $\lambda_1, \ldots, \lambda_n$ be eigenvalues of BA. Then the eigenvalues of AB are

$$\lambda_1, \lambda_2, \ldots, \lambda_n, \underbrace{0, \ldots, 0}_{\text{m-n 0's}}.$$

By this theorem, the eigenvalues of