1 January 29th, 2020

1.1 Phase Plot

Let us consider ODE's of the form:

$$\frac{dx}{dt} = f(x) = \dot{x}.$$

If we graph x vs $\dot{x}$ we can get a phase plot, for example:
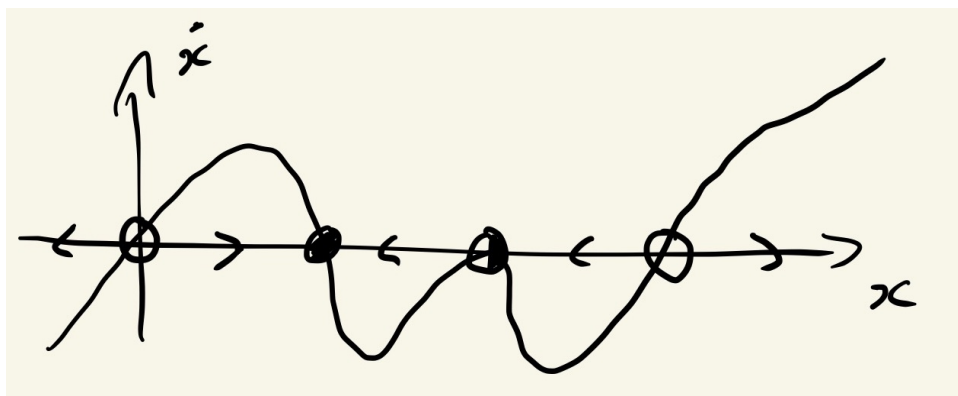

Figure 1: Phase plot of $\dot{x} = x(x-1)(x-2)^2(x-3)^3$

Definition 1.1. A point where $f(x) = 0$ is called an **equilibrium point**. These equilibrium points can be unstable (empty circle), stable (filled circle), or left/right stable (half filled circle).

1.2 Computing Times

Since $\dot{x} = f(x)$, is separable, since $dt = \frac{dx}{f(x)}$, we have:

$$\int_{t_1}^{t_2} dt = \int_{x_1}^{x_2} \frac{dx}{f(x)} \implies t_2 - t_1 = \int_{x_1}^{x_2} \frac{dx}{f(x)}.$$

Which is the time interval between when $x = x_1$ and $x = x_2$.

Example 1.2

Let us try to compute the period of an object with mass m to travel from one end of a bowl to the other with radius R. TL;DR we get:

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{R}} \cos(\theta).$$

Rearranging gives us:

$$dt = \sqrt{\frac{R}{2g \cos \theta}} d\theta \implies \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\cos(\theta)}} \approx \sqrt{\frac{R}{2g}} 5.244.$$

1.3   Exact Equations

Whenever you have a function of form $\frac{dy}{dx} = F(x, y)$, you can always rewrite it in the form:
$$M(x, y)dx + N(x, y)dy = 0.$$

This might look familiar, as if we have $f(x, y) = C$, we have:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0.$$

As such, we'd like to ask when can $M(x, y)dx + N(x, y)dy = 0$ be written as $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$. It would be great if $M = \frac{\partial f}{\partial x}$ and $N = \frac{\partial f}{\partial y}$, so it's helpful to know when we can do this.

Consider

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

As such, if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

then $Mdx + ndy = 0$ is called exact.

Example 1.3

$2xydx + (x^2 - y^2)dy = 0$ is exact.

Example 1.4

$2x^2ydx + (x^3 - y^2)dy = 0$ is not exact.

Note that the two examples differ by a factor x, meaning that we have a further condition to determine whether something is exact.