1 March 8th, 2022

1.1 Properties of Conditional Expectation Cont.

Definition 1.1 (tower property)**.** For sub σ-field $\mathcal{H}_1 \subset \mathcal{H}_2 \subset \mathscr{F}$, we have:

$$\mathbf{E}[\mathbf{E}[X|\mathcal{H}_2]|\mathcal{H}_1] = \mathbf{E}[\mathbf{E}[X|\mathcal{H}_1]|\mathcal{H}_2] = \mathbf{E}[X|\mathcal{H}_1]$$

Proof. Note that $\mathbf{E}[\mathbf{E}[X|\mathcal{H}_1]|\mathcal{H}_2] = \mathbf{E}[X|\mathcal{H}_1]$ is trivial, since $\mathbf{E}[X|\mathcal{H}_1]$ is $\mathcal{H}_1$-measurable so is $\mathcal{H}_2$.

Second, denote $Y = \mathbf{E}[X|\mathcal{H}_\in]$. We have that:

$$\mathbf{E}[\mathbf{E}[X|\mathcal{H}_2]|\mathcal{H}_1] = \mathbf{E}[X|\mathcal{H}_1] \iff \mathbf{E}[Y|\mathcal{H}_1] = E[X|\mathcal{H}_1].$$

By the definition of $\mathbf{E}[Y|\mathcal{H}_1]$, it shall be $\mathcal{H}_1$-measurable and $\mathbf{E}[\mathbf{E}[Y|\mathcal{H}_1]\mathbb{1}_A] = \mathbf{E}Y\mathbb{1}_A$, $A \in \mathcal{H}_1$. Thus, we need to show that:

1. $\mathbf{E}[X|\mathcal{H}_1]$ is $\mathcal{H}_1$-measurable (trivial)
2. $\mathbf{E}[\mathbf{E}[X|\mathcal{H}_1]\mathbb{1}_A] = \mathbf{E}Y\mathbb{1}_A$ for all $A \in \mathcal{H}_1$. We claim that both sides are equal to $\mathbf{E}X\mathbb{1}_A$. The LHS is true for any $A \in \mathcal{H}_1$ by definition, and the RHS is true for any $A \in \mathcal{H}_2$ by definition of $\mathcal{H}[X|\mathcal{H}_2]$.

□

Example 1.2

The **law of total expectation** is a special case, as we have:

$$\mathbf{E}\left[\mathbf{E}[Z|\mathcal{H}]\right] = \mathbf{E}\left[\mathbf{E}[Z|\mathcal{H}]|\{\emptyset, \Omega\}\right] = \mathbf{E}[Z|\{\emptyset, \Omega\}] = \mathbf{E}Z.$$

Note that $\mathbf{E}[Z|\{\emptyset, \Omega\}] = \mathbf{E}Z$.

Corollary 1.3

Here are some corollaries of the tower property:

- Law of total expectation
- If Y is $\mathcal{H}$-measurable, then $\mathbf{E}[\mathbf{E}[X|\mathcal{H}]|Y] = \mathbf{E}[X|Y]$.
- $\mathbf{E}[\mathbf{E}[X|Y]|f(Y)] = \mathbf{E}[X|f(Y)]$
- $\mathbf{E}[\mathbf{E}[X|Y, Z]|Y] = \mathbf{E}[X|Y]$

1.1.1 Additional Properties of Conditional Expectation

Linearity: $\mathbf{E}[aX_1 + bX_2|\mathcal{H}] = a\mathbf{E}[X_1|\mathcal{H}] + B\mathbf{E}[X_2|\mathcal{H}]$

Monotonicity: $\mathbf{E}[X_1|\mathcal{H}] \leq \mathbf{E}[X_2|\mathcal{H}]$ if $X_1 \leq X_2$

Jensen: If $\varphi : \mathbb{R} \to \Omega$ is convex, then $\varphi(\mathbf{E}[X|\mathcal{H}]) \leq \mathbf{E}[\varphi(X)|\mathcal{H}]$

1.2 Conditional Variance

Definition 1.4 (**conditional variance**)**.** The conditional variance is defined by:

$$\mathbf{Var}[X|\mathcal{H}] = \mathbf{E}[(X - \mathbf{E}[X|\mathcal{H}])^2|\mathcal{H}] = \mathbf{E}[X^2|\mathcal{H}] - (\mathbf{E}[X|\mathcal{H}])^2$$

Theorem 1.5 (law of total variance)

$$\mathbf{Var}(X) = \underbrace{\mathbf{E}(\mathbf{Var}[X|\mathcal{H}])}_{\mathbf{E}(X-\mathbf{E}[X|\mathcal{H}])^2} + \underbrace{\mathbf{Var}(\mathbf{E}[X|\mathcal{H}])}_{\mathbf{E}(\mathbf{E}[X|\mathcal{H}]-\mathbf{E}X)^2}$$

1.3 Introduction to Martingales

Consider a discrete stochastic process $\{X(t), t \in T\}$, with T being discrete.

Definition 1.6 (**filtration**)**.** Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. A filtration $\{\mathscr{F}_n\}_{n\geq 1}$ is a sequence of σ-fields, s.t. $\mathscr{F}_i \subseteq \mathscr{F}_{i+1} \subseteq \mathscr{F}$ for all i.

Let $X_1, X_2, \dots X_n$ be a random process on $(\Omega, \mathscr{F}, \mathbb{P})$. Let $\mathscr{F} = \sigma(X_1, \dots, X_i)$

Definition 1.7 (**adapted sequence**)**.** A random process $\{X_n\}$ is adapted to $\{\mathscr{F}_n\}$ if X_n is $\mathscr{F}_n$-measurable for all n.

Example 1.8

$\mathscr{F}_n$: a collection of info of stock market up to day n. X_n: stock price of day n.

Definition 1.9 (**martingale**)**.** If X_n is a sequence of r.v. and $\{\mathscr{F}_n\}$ is a filtration, with:

- $\mathbf{E}|X_n| < \infty$
- $\{X_n\}$ is adapted to $\{\mathscr{F}_n\}$
- $\mathbf{E}[X_{n+1}|\mathscr{F}_n] = X_n$ for all n

then we call $\{X_n\}$ to be a martingale w.r.t $\{\mathscr{F}_n\}$.

Definition 1.10 (**submartingale** and **supermartingale**)**.** If the "=" is replaced with "$\geq$", then it is called a submartingale. If it's replaced with a "$\leq$" it is called a supermartingale.

Remark 1.11 — Submartingale means that the expectation is greater than previous, meaning that you're playing a favorable game. However, most casinos are supermartingales.

Example 1.12

Let us consider successive tosses of a fair coin:

$$\xi_n = \begin{cases} 1 & \text{if } n\text{-th coin is H} \\ -1 & \text{if } n\text{-th coin is T} \end{cases}, \quad \mathbf{E}\xi_n = 0.$$

Let $X_n = \sum_{i=1}^{n} \xi_i$. Then $\{X_n\}$ is a martingale w.r.t. $\{\mathscr{F}_n\}$ with $\mathscr{F}_n = \sigma(X_1, \ldots, X_n)$.

Example 1.13

(**Polya's urn model**) Consider a urn with r red and g green balls. At each time we draw a ball and replace it with $c+1$ balls of the same color drawn. Let X_n be the fraction of green balls after the n-th draw. We claim that X_n is a martingale w.r.t. $\mathscr{F}_n = \sigma(X_1, ..., X_n)$.

Proof. We have:

$$\mathbf{E}[X_{n+1}|\mathscr{F}_n] = \mathbf{E}[X_{n+1}|X_n].$$

Suppose at step n, there are i red and j green. Then:

$$X_{n+1} = \begin{cases} \frac{j+c}{i+j+c}, & \text{with probability } \frac{j}{j+i} \\ \frac{j}{i+j+c}, & \text{with probability } \frac{i}{j+i}, \end{cases}$$

Thus:

$$\mathbf{E}\left[X_{n+1}|X_n = \frac{j}{i+j}\right] = \frac{j+c}{i+j+c}\frac{j}{i+j} + \frac{j}{i+j+c}\frac{i}{i+j} = \frac{j}{i+j}.$$

□

Example 1.14

(**Galton-Watson process**) Let ξ_i^n, $i, n \geq 1$ be i.i.d. nonnegative integer-valued. Let z_n $n > 0$ be defined as:

$$Z_0 = 1 \quad Z_{n+1} = \begin{cases} \xi_i^{n+1} + \ldots + \xi_{Z_n}^{n+1} & \text{if } Z_n \neq 0 \\ 0 & \text{if } Z_n = 0 \end{cases}$$

Here ξ_i^{n+1} is the number of offspring of i-th individual in n-th generation. Let $\mathscr{F}_n = \sigma\{\xi_i^m, i >= 1, 1 \leq m \leq n\}$ and $\mu : \mathbf{E}\xi_i^m \in (0, \infty)$. Then:

$$\frac{Z_n}{\mu^n} \text{ is a martingale w.r.t. } \mathscr{F}_n.$$

Proof. We claim that $\mathbf{E}Z_n = \mu^n$, since:

$$\begin{aligned}
\mathbf{E}Z_{n+1} &= \sum_{k=0}^{\infty} \mathbf{E}\left(\sum_{i=1}^{k} \xi_i^{n+1} | Z_n = k\right) \mathbb{P}(Z_n = k) \\
&= \sum_{k=0}^{\infty} \left(\mathbf{E}\sum_{i=1}^{k} \xi_i^{n+1}\right) \cdot \mathbb{P}(Z_n = k) \\
&= \sum_{k=0}^{\infty} k\mathbb{P}(Z_n = k) \cdot \mu = \mu \mathbf{E}Z_n.
\end{aligned}$$

To show that it's a martingale, we have:

$$\begin{aligned}
\mathbf{E}\left(Z_{n+1}|\mathscr{F}_n\right) &= \sum_{k=1}^{\infty} \mathbf{E}\left(Z_{n+1}\mathbb{1}(Z_n = k)|\mathscr{F}_n\right) \\
&= \sum_{k=1}^{\infty} \mathbf{E}\left(\sum_{i=0}^{k} \xi_i^{n+1}\mathbb{1}(Z_n = k)|\mathscr{F}_n\right) \\
&= \sum_{k=1}^{\infty} k\mathbb{1}(Z_n = k)\mu = \mu Z_n
\end{aligned}$$

$\square$

Example 1.15

(**de Moivre's martingale**) Consider an unfair coin with probability p of H:

$$\xi_n = \begin{cases} 1 & n - \text{th flipping is H} \\ 0 & \text{otherwise} \end{cases}$$

Let $X_n = \sum_{i=1}^{n} \xi_i$ and $\mathscr{F}_n = \sigma(X_1, \ldots, X_n)$. Let $Y_n = \left(\frac{1-p}{p}\right)^{X_n}$, then Y_n is a martingale w.r.t. $\mathscr{F}_n$.

Proof.

$$\mathbf{E}[Y_{n+1}|\mathscr{F}_n] = p \cdot \left(\frac{1-p}{p}\right)^{X_n+1} + (1-p)\left(\frac{p}{p}\right)^{X_n-1} = \left(\frac{1-p}{p}\right)^{X_n} = Y_n$$

$\square$