1 February 3rd, 2020

1.1 Exact Equations

Remember that an exact equation is one where:

$$Mdx + Ndy = 0.$$

Where:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Consider the exact equation:

$$(y^2 - x^2)dx + 2xydy = 0.$$

To solve this exact ODE, we set:

$$\frac{\partial f}{\partial x} = M = y^2 - x^2 \implies \int_x (y^2 - x^2)dx + c_1(y) \implies f(x, y) = y^2x - \frac{x^3}{3} + c_1(y).$$

Now if we take the partial with respect to y, we get:

$$\frac{\partial f}{\partial y} = 2yx + c_1'(y) = N = 2xy \implies c_1'(y) = 0 \implies c_1(y) = c_2.$$

This tells:

$$f(x, y) = y^2x - \frac{1}{3}x^3 + c_2$$

satisfies both equations meaning that the solution to our ODE is of the form:

$$f(x, y) = xy^2 - \frac{1}{3}x^3 = C.$$

If we have an initial condition, then this will give us a unique solution.

Example 1.1

Consider the equation: $2xy^2dx+(2x^2y-y^3)dy = 0$. To solve this, we do the following:

$$\int_x 2xy^2\ dx = x^2y^2 + c_1(y) \implies 2x^2y + c_1'(y) = 2x^2y - y^3 \implies c_1 = -\frac{y^4}{4}$$

Thus we have:

$$f(x, y) = 2x^2y^2 - \frac{1}{4}y^4 + C.$$

1.2 Inexact Equations

If $Mdx + Ndy = 0$ is not exact, then we try to introduce an integrating factor $\mu(x, y)$ to turn make $\mu Ndx + \mu Ndy = 0$. Thus we want:

$$\frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x}.$$

However this is usually as difficult to solve as the original equation. There are some special cases though:

- $\mu(x,y) = \mu(x)$. If this is the case, we have:

$$\frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x} \implies \mu\frac{\partial M}{\partial y} = \frac{d\mu}{dx}N + \mu\frac{\partial N}{\partial x}$$

$$\implies \mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \mu'(x)N \implies \frac{\mu'(x)}{\mu(x)} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

and if the RHS is a function of only x, we can integrate, giving us:

$$\mu(x) = \exp\left\{\int \frac{\left(\frac{\partial m}{\partial y} - \frac{\partial N}{\partial x}\right)}{N}dx\right\}.$$

With this, we will be able to solve the differential equation with $\frac{\partial f}{\partial x} = \mu M$ and $\frac{\partial f}{\partial y} = \mu N$. This is true if:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = k(x).$$

i.e. it's a function of only x

- $\mu(x,y) = \mu(y)$. Same thing but with y instead of x. We check if: $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{y}$ is a function of only y. We will have:

$$\mu(y) = \exp\left\{\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{m}\right\}.$$

Example 1.2

Consider the equation $2xydx + (2x^2 - y^2)dy = 0$. Note that this is not exact. As such, we check:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2x - 4x}{2x^2 - y^2} = \frac{2x}{2x^2 - y^2} \neq \text{ a function of only } x \ .$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{4x - 2x}{2xy} = \frac{1}{y}.$$

Thus we have:

$$\mu(y) = e^{\int \frac{1}{y}dy} = e^{\ln y} = y.$$

Example 1.3

Consider $\frac{dy}{dx} = \frac{2x^2-y^2}{3xy}$, rearranging gives us:

$$(x^2 - 2y^2)dx + 3xydy = 0.$$

Note that $\frac{\partial M}{\partial y} = -4y$ and $\frac{\partial N}{\partial x} = 3y$, thus it is not exact. Now we try:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-4y - 3y}{3xy} = \frac{-7}{3x}.$$

Which is a function of only x. As such, we have:

$$\mu(x) = e^{\int -\frac{7}{3x}dx} = x^{-\frac{7}{3}}.$$

Multiplying this in gives us:

$$(x^{-\frac{1}{3}} - 2x^{-\frac{7}{3}}y^2)dx + 3x^{-\frac{4}{3}}ydy = 0,$$

which is exact since:

$$\frac{\partial M}{\partial y} = -4x^{-\frac{7}{3}}y \quad \frac{\partial N}{\partial x} = -4x^{-\frac{7}{3}}y.$$

Solving this gives us:

$$f(x,y) = \int_x x^{-\frac{1}{3}} - 2x^{-\frac{7}{3}}y^2dx = \frac{3}{2}x^{\frac{2}{3}} + \frac{3}{2}x^{-\frac{4}{3}}y + c_1(y).$$

$$\frac{3}{2}x^{-\frac{4}{3}}y + c_1'(y) = \frac{3}{2}x^{-\frac{4}{3}}y \implies c_1 = C.$$

Thus

$$f(x,y) = \frac{3}{2}x^{\frac{2}{3}} + \frac{3}{2}x^{-\frac{4}{3}}y^2 = C.$$