1 February 17th, 2020

1.1 More Laplace Transform

Remember that the Laplace Transform for a function $f(t)$ is:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s).$$

There is an associated inverse Laplace transform:

$$\mathcal{L}^{-1}\{F(s)\} = f(t).$$

Which maps frequency space back to time space. If we avoid null functions, this inverse Laplace transform is unique, giving us tables of these pairs such as:

Table 1: Example of $\mathcal{L}$ and $\mathcal{L}^{-1}$ Pair Table

$f(t)$	$F(s)$
$t^m e^{at}$	$\frac{m!}{(s-a)^{m+1}}, \quad s > a$
$\sin(\omega t)$	$\frac{\omega}{\omega^2+s^2}, \quad s > 0$
$\vdots$	$\vdots$

Theorem 1.1

The Laplace transform is linear, i.e.:

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\}.$$

Remark 1.2 — Proof in notes.

Example 1.3

$$\mathcal{L}\{t^3e^{-t} + 4\sin(8t)\} = \mathcal{L}\{t^3e^{-t}\} + 4\mathcal{L}\{\sin(8t)\}.$$
$$= \frac{3!}{(s-(-1))^{3+1}} + 4\left(\frac{8}{8^2+s^2}\right) = \frac{6}{(s+1)^4} + \frac{32}{64+s^2}.$$

Note that the first term has condition $s > -1$ and the second has $s > 0$, meaning that this domain is $s > 0$.

Remark 1.4 — When there are multiple conditions, we take the intersection of the domains.

1.1.1 Limit Theorems

Theorem 1.5 (Limit Theorem)
If $\mathcal{L}\{f(t)\} = F(s)\}$, we should find:

$$\lim_{s\to\infty} F(s) = 0.$$

with the exception of some impulse functions.

Example 1.6
We have $\mathcal{L}\{\cos(\omega t) = \frac{s}{s^2+\omega^2}$. Note that:

$$\lim_{s\to\infty}\left(\frac{s}{s^2+\omega^2}\right) = 0.$$

Remark 1.7 — This can be used as a check, as if you don't get $\lim_{s\to\infty} F(s) = 0$, and you aren't dealing with impulse function, then you did something wrong.

Theorem 1.8 (Endpoint Theorem 1)

$$\lim_{s\to\infty}(sF(s)) = \underbrace{f(0^+)}_{\lim_{t\to 0^+} f(t)}.$$

Example 1.9
Again consider $\mathcal{L}\{\cos(\omega t)\}$. We have:

$$\lim_{s\to\infty} s\left(\frac{s}{s^2+\omega^2}\right) = 1.$$

and

$$\cos(\omega \times t) = 1.$$

Theorem 1.10 (Endpoint Theorem 2)

$$\lim_{s\to\infty}(sF(s)) = \underbrace{f(\infty)}_{\lim_{t\to\infty} f(t)},$$

provided it exists.

Remark 1.11 — This allows us to the values of $f(t)$ without having to use the inverse Laplace transform.

Example 1.12
Suppose the Laplace transform of $f(t)$ is:

$$\mathcal{L}\{f(t)\} = \frac{1}{s\sqrt{s^2+1}}.$$

We would like to find out what $f(0)$ and $f(\infty)$ are. Using the endpoint theorem, we have:

$$f(0^+) = \lim_{s\to\infty} s\frac{1}{s\sqrt{s^2+1}} = \lim_{s\to\infty} \frac{1}{\sqrt{s^2+1}} = 0.$$

and

$$f(\infty) \lim_{s\to 0} s\frac{1}{s\sqrt{s^2+1}} = \lim_{s\to 0} \frac{1}{\sqrt{s^2+1}} = 1.$$

1.1.2 Existence of Laplace Transform of $f(t)$

Q: Can we take the integral of anything?
A: No, as the Laplace transform is an improper integral, which must converge.

Example 1.13
Note that

$$\mathcal{L}\{e^{t^2}\} = \int_0^\infty e^{-st}e^{t^2}\,dt = \infty.$$

Thus, $\mathcal{L}\{e^{t^2}\}$ does not have a Laplace transform.

For a function to have a Laplace transform, it must be of exponential order.

Definition 1.14 (Exponential Order)**.** For a function $f(t)$ to be of **exponential order**, there must be a constant α for which:

$$\lim_{t\to\infty} e^{-\alpha t} f(t) = 0.$$

The function is allowed to go to infinity, just not too fast.

1.1.3 Laplace Transforms for Derivatives

Consider the Laplace transform of $f'(t)$ and use integration by parts with:

$$\mathcal{L}\{f'(t)\} = \int_0^\infty \underbrace{e^{-st}}_{u}\,\underbrace{f'(t)\,dt}_{dv}.$$

$$\underbrace{e^{-st}}_{u}\,\underbrace{f(t)}_{v}\Big|_0^\infty - \int_0^\infty \underbrace{f(t)}_{v}\,\underbrace{\left(-se^{-st}\right)\,dt}_{du}$$

$$= \underbrace{e^{-\infty}}_{0} f(\infty) - \underbrace{e^{-0}}_{1} f(0^+) + s\int_0^\infty f(t)e^{-st}dt = s\mathcal{L}\{f(t)\} - f(0^+).$$

Theorem 1.15 (Laplace Transform for Derivatives)

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0^+).$$

Example 1.16

Consider the second derivative:

$$\mathcal{L}\{f''(t)\} = \mathcal{L}\{\frac{d}{dt}f'(t)\} = s\mathcal{L}\{f'(t)\} - f'(0^+) = s\left(s\mathcal{L}\{f(t)\} - f(0^+)\right) - f'(0^+).$$

Theorem 1.17

From the previous example:

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0^+) - f'(0^+).$$

Remark 1.18 — This can be generalized, and as such we have:

$$\mathcal{L}\{f'''(t)\} = s^3\mathcal{L}\{f(t)\} - s^2 f(0^+) - sf'(0^+) - f''(0^+).$$

Note that for each of the negative terms, the power of s plus the order of the derivative of f will equal the order of the derivative being computed minus 1, with the s coefficient of $\mathcal{L}\{f(t)\}$ having the same power as the order.

Consider $ay''(t) + by'(t) + cy(t) = g(t)$ with initial conditions $y(0) = y_0$, $y'(0) = y_0'$ and with a, b, c being constant. Instead of solving by setting $g(t) = 0$, let us solve it using Laplace transform.

Let us begin by taking the Laplace transform of both sides:

$$\mathcal{L}\{ay''(t) + by'(t) + cy(t)\} = \mathcal{L}\{g(t)\}.$$

$$\implies a\mathcal{L}\{y''(t)\} + b\mathcal{L}\{y'(t)\} + c\mathcal{L}\{y(t)\} = \mathcal{L}\{g(t)\}.$$

$$\implies a\left(s^2\mathcal{L}\{y(t)\} - sy(0^+) - y'(0^+)\right) + b\left(s\mathcal{L}\{y(t)\} - y(0^+)\right) + c\mathcal{L}\{y(t)\} = \mathcal{L}\{g(t)\}.$$

Thus we have:

$$\mathcal{L}\{y(t)\} = \frac{(as+b)y_0 + ay_0' + \mathcal{L}\{g(t)\}}{as^2 + bs + c}.$$

With this, we can get $y(t)$ by taking the inverse Laplace transform.

Example 1.19
Consider:
$$y''(t) + 2y'(t) + 3y(t) = t^3 \quad y(0) = 0 \quad y'(0) = 1.$$
With this we have: $a = 1, b = 2, c = 3, y_0 = 0, y_0' = 1$, and:
$$\mathcal{L}\{g(t)\} = \mathcal{L}\{t^3\} = \frac{3!}{s^{3+1}} = \frac{6}{s^4}.$$
Thus without solving the ODE, we can say that:
$$\mathcal{L}\{y(t)\} = \frac{(s+2)(0) + (1)(1) + \frac{6}{s^4}}{s^2 + 2s + 3} = \frac{s^4 + 6}{s^4(s^2 + 2s + 3)}.$$
$$y(t) = \mathcal{L}^{-1}\left\{\frac{s^4 + 6}{s^4(s^2 + 2s + 3)}\right\}.$$

1.1.4 Other Properties of Laplace Transforms

Theorem 1.20 (First Shifting Theorem)
If $\mathcal{L}\{f(t)\} = F(s)$, then:
$$\mathcal{L}\{e^{at}f(t)\} = F(s-a).$$

Remark 1.21 — The way to remember this, forget e^{at}, and then whoever we get an s, replace by $s - a$.

Theorem 1.22
If $\mathcal{L}\{f(t)\} = F(s)$, then:
$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s).$$
$$\mathcal{L}\{t^m f(t)\} = (-1)^m \frac{d^m}{ds^m}F(s).$$

Remark 1.23 — The way to do this, forget the t, then afterward take the derivative w.r.t. s and negate it.

Example 1.24
We have:
$$\mathcal{L}\{e^{2t}\cos(4t)\} = \mathcal{L}\{\cos(4t)\}\Big|_{s \to s-2}.$$
$$= \frac{s}{s^2 + 4^2}\Big|_{s \to s-2} = \frac{s-2}{(s-2)^2 + 16}.$$

Example 1.25

We have:

$$\mathcal{L}\{t\cos(4t)\} = \frac{d}{ds}\mathcal{L}\{\cos 4(t)\}.$$
$$= -\frac{d}{ds}\left(\frac{s}{s^2+4^2}\right) = -\frac{d}{ds}\left(\frac{s}{s^2+16}\right).$$
$$= -\left(\frac{(s^2+16)-s(2s)}{(s^2+16)^2}\right) = \frac{s^2-16}{(s^2+16)^2}.$$

Example 1.26

We have:

$$\mathcal{L}\{te^{-t}\sin(t)\} = \mathcal{L}\{t\sin(t)\}\Big|_{s\to s-(-1)}$$
$$= -\frac{d}{ds}\mathcal{L}\{\sin(t)\}\Big|_{s\to s+1} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\Big|_{s\to s+1}$$
$$= \frac{2s}{(s^2+1)^2}\Big|_{s\to s+1} = \frac{2(s+1)}{((s+1)^2+1)^2} = \frac{2s+2}{(s^2+2s+2)^2}.$$

Remark 1.27 — Knowing these two properties, then we can compute Laplace transforms of functions with factors of $t^m e^{at}$.

1.1.5 Unit Step Function

Definition 1.28 (Unit Step Function)**.** The **unit step function** $u_a(t) = u(t-a)$ is defined as:

$$u_a(t) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

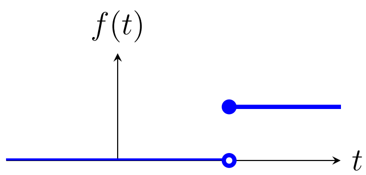

Figure 1: Example of a Unit Step Funciton

The Laplace transform for the unit step function is:

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty u(t-a)e^{-st}\,dt$$
$$= \int_0^a (0)e^{-st}\,dt + \int_a^\infty (1)e^{-st}\,dt = -\frac{e^{-st}}{s}\Big|_a^\infty = \frac{e^{-as}}{s}, \quad s > 0.$$

Remark 1.29 — We can use this for calculating the Laplace transforms for piecewise functions.

Example 1.30
Consider the piecewise function

$$f(t) = \begin{cases} 0, & t < 0 \\ 1, & 0 < t < 2 \\ t, & 2 \leq t \leq 3 \\ e^t, & 3 < t \end{cases} .$$

We can express this as:

$$1u(t) + (t-1)u(t-2) + (e^t - t)u(t-3).$$

Thus for any piecewise function, we can express it as:

$$f(t) = \begin{cases} 0, & t < 0 \\ f_1(t), & 0 < t < t_1 \\ f_2(t), & t_1 < t < t_2 \\ \vdots \\ f_{m+1}(t), & t_m < t \end{cases}$$

$$= f_1(t)u(t)+(f_2(t)-f_1(t))u(t-t_1)+(f_3(t)-f_2(t))u(t-t_2)+\ldots+(f_{m+1}(t)-f_m(t))u(t-t_m).$$