1 April 8th, 2020

1.1 Examples of RSLP

Let us try to determine all λ's that lead to non-zero solutions $\phi(x)$ if:

$$x\phi''(x) - \phi'(x) + \lambda x^3\phi(x) = 0, \quad 0 < x < 1.$$

$$\phi(0) = 0 \quad \phi(1) = 0.$$

Note that the standard form would be:

$$\phi''(x) - \frac{1}{x}\phi'(x) + \lambda x^2\phi(x) = 0 \implies \phi''(x) + P(x)\phi'(x) + Q(x) = 0.$$

From HW6, can solve this, with:

$$\gamma = \frac{Q'(x) + 2P(x)Q(x)}{(Q(x))^{\frac{3}{2}}} = \frac{2\lambda x + 2(-1/x)(\lambda x^2)}{(\lambda x^2)^{\frac{3}{2}}} = 0 = \text{ a constant.}$$

Meaning we can use the transformation:

$$z = \int \sqrt{\alpha Q(x)}\ dx = \int \sqrt{\alpha\lambda x^2}\ dx = \int x\ dx = \frac{1}{2}x^2.$$

By picking $\alpha = \frac{1}{\lambda}$. Thus we have:

$$\Psi''(z) + \frac{\gamma}{2\sqrt{\alpha}}\Psi'(z) + \frac{1}{\alpha}\Psi(z) = 0.$$

which reduces to:

$$\Psi''(z) + \lambda\Psi(z) = 0.$$

From the example from last lecture, we would have:

$$\Psi(z) = \begin{cases} A\cosh(z\sqrt{-\lambda}) + B\sinh(z\sqrt{-\lambda}), & \lambda < 0 \\ A + Bz, & \lambda = 0 \\ A\cos(z\sqrt{\lambda}) + B\sin(z\sqrt{\lambda}), & \lambda > 0 \end{cases}.$$

Giving us:

$$\phi(x) = \begin{cases} A\cosh(\frac{1}{2}x^2\sqrt{-\lambda}) + B\sinh(\frac{1}{2}x^2\sqrt{-\lambda}), & \lambda < 0 \\ A + B\frac{1}{2}x^2, & \lambda = 0 \\ A\cos(\frac{1}{2}x^2\sqrt{\lambda}) + B\sin(\frac{1}{2}x^2\sqrt{\lambda}), & \lambda > 0 \end{cases}.$$

Putting in the boundary conditions, consider $\lambda < 0$ we would have:

$$\phi(0) = A = 0 \quad \text{ since } \sinh(0) = 0\ \cosh(0) = 1.$$

Thus:

$$\phi(1) = B\sinh(\frac{1}{2}\sqrt{-\lambda}) = 0 \implies B = 0.$$

This tells us that there are no negative eigenvalues for this problem. As shown in the last lecture, we can test $\lambda = 0$ and see that there are no zero eigenvalues either. For $\lambda > 0$, we would have:

$$\lambda_n = (2n\pi)^2 \quad \phi_n(x) = B_n\sin(n\pi x^2).$$

Picking $B_n = 1$, we would have $\phi_n(x) = \sin(n\pi x^2)$.

With this, we can define the dot product:

$$f \cdot g = \int_0^1 f(x)g(x)w(x)\ dx.$$

$$w(x) = \frac{b(x)}{a_2(x)} e^{\int \frac{a_1(x)}{a_2(x)} dx} = \frac{x^3}{x} e^{\int -\frac{1}{x}\ dx} = x^2 \frac{1}{x} = x.$$

This also means that:

$$\phi_m \cdot \phi_n = \int_0^1 \phi_m(x)\phi_n x)x\ dx = 0, \quad m \neq n.$$

and

$$\phi_n \cdot \phi_n = \int_0^1 \phi_n(x)\phi_n x)x\ dx = \int_0^1 \sin(n\pi z)\sin(n\pi z)\frac{1}{2}\ dz = \frac{1}{4} > 0.$$

With this, if a piecewise continuous function $f(x)$ is expressed as:

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x), 0 < x < 1.$$

Then:

$$a_n = \frac{\phi_n \cdot f}{\phi_n \cdot \phi_n} = 4(\phi_n \cdot f).$$

Example 1.1

Consider $f(x) = 1$, we have:

$$a_n = \frac{\int_1^0 \phi_n(x)x\ dx}{\frac{1}{4}} = 4\int_0^1 \sin(n\pi x^2)x\ dx.$$

Using the substitution $z = x^2$, we get:

$$a_n = 4\int_0^1 \sin(n\pi z)\frac{1}{2}\ dz = 2\frac{1-\cos(n\pi)}{n\pi} = \frac{2(1-(-1)^n)}{n\pi}.$$

Thus we have:

$$1 = \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n\pi} \sin(n\pi x^2).$$

Remark 1.2 — Unlike Taylor series which tend to converge monotonically, Sturm-Liouville series tend to converge alternatively.

Sometimes we might not get clean values for the eigenvalues. To see this, we can try:

$$\phi''(x) + \lambda\phi(x) = 0, 0 < x < 1.$$

but with the boundary conditions:

$$\phi(0) = 0 \quad \phi(1) + \phi'(1) = 0.$$

For $\lambda < 0$, we would find:

$$\phi(1) + \phi'(1) = B\left(\sinh(\sqrt{-\lambda}) + \sqrt{-\lambda}\cosh(\sqrt{-\lambda})\right).$$

Which would mean that $B = 0$ since $\left(\sinh(\sqrt{-\lambda}) + \sqrt{-\lambda}\cosh(\sqrt{-\lambda})\right) = 0 \iff \lambda = 0.$

For the case of $\lambda = 0$, we would have: $\phi(1) + \phi'(1) = 2B = 0 \implies B = 0$ meaning that there is only the trivial condition.

For the case of $\lambda > 0$, we would have "

$$\phi(1) + \phi'(1) = B\sin(\sqrt{\lambda}) + B\sqrt{\lambda}\cos(\sqrt{\lambda}) = 0.$$

To avoid $B = 0$, we would need:

$$\sin(z) + z\cos(z) = 0.$$

which has a lot of solutions. However, these solutions are not easy to solve for and thus are calculated numerically. Note that there are infinite solutions, as this is equivalent to $tan(z) = -z$.

> **Remark 1.3 —** Note that it is possible to have negative or zero eigenvalues, for example if it was instead $\phi(1) - \phi'(1) = 0$.