1 January 22nd, 2020

1.1 Separable Differential Equation

A general first-order ODE for a dependent variable y in the independent variable x can be written as:

$$\frac{dy}{dx} = F(x, y) \tag{1}$$

where F is some specified function of x and y. When F has the form

$$F(x, y) = f(x)g(y), \tag{2}$$

then 1 is said to be *separable* and such equation can always be solved by:

$$\frac{dy}{g(y)} f(x)dx \implies \int \frac{dy}{g(y)} + C_1 = \int f(x)dx + C_2 \implies \int \frac{dx}{g(y)} = \int f(x)dx + C.$$

as one form for the solution of 1.

1.1.1 Ideal Fluid Flow

We are concerned with a container that has a fluid with cross sectional area A with density ρ with a hole at the bottom of the container which causes it to flow out. We are concerned with the heigh x of the container. We also have a pipe that pumps in fluid with constant rate R.

This leads to following equation:

$$\frac{dx}{dt} = \alpha - \beta\sqrt{x}.$$

where

$$\alpha = \frac{R}{A} \quad \beta = \sqrt{\frac{2ga^2}{A^2 - a^2}} \quad g = 9.81\text{m s}^{-2}.$$

Note that this is a separable differential equation:

$$\frac{dx}{\alpha - \beta\sqrt{x}} = dt.$$

If we have α, β, we can solve, e.g. $\alpha = 60$ $\beta = 6$, we have:

$$\frac{dx}{dt} = 60 - 6\sqrt{x} \implies \frac{dx}{10 - \sqrt{x}} = 6dt.$$

Integrating on both sides, we have:

$$\int \frac{dx}{10 - \sqrt{x}} = \int 6dt = 6t + C.$$

Solving this, we get:

$$20\tan^{-1}\left(\frac{\sqrt{x}}{10}\right) - 10\ln(100 - x) - 2\sqrt{x} = 6t + C.$$

If we have initial conditions, e.g. at $t = 0, x = 0$, we would have:

$$0 - 10\ln(100) = C$$

allowing us to solve for C. This would allow us to solve for a time t for certain values of x.

1.2 Homogeneous Differential Equation

Again remember that the general form a differential equation of one a dependent variable y in the independent variable x is:

$$\frac{dy}{dx} = F(x, y).$$

If $F(x, y) = f(x)g(x)$ then this is separable. Remember that the goal is that we want to find $G(x, y) = C$, in other words, we want to get rid of the derivative and find the relationship between the two.

Definition 1.1. A function of form $F(x, y)$ is called **homogeneous** of order N if $F(tx, ty) = t^N F(x, y)$ for any scalar t.

Example 1.2

$$\begin{aligned} F(x, y) = x^3 + x^2y + 4xy^2 \implies F(tx, ty) &= (tx)^3 + (tx)^2(ty) + 4(tx)(ty)^2 \\ &= t^3\left(x^3 + x^2y + 4xy^2\right) = t^3 F(x, y). \end{aligned}$$

Thus $F(x, y)$ is homogeneous to the order 3.

Example 1.3

$F(x, y) = x^3 + xy$ is not homogeneous.

Example 1.4

$$F(x, y) = \frac{xy}{x^2 + y^2}$$

$$F(tx, ty) = \frac{t^2xy}{t^2x^2 + t^2y^2} = t^2\left(\frac{xy}{x^2 + y^2}\right) = t^0 F(x, y)$$

meaning that $F(x, y)$ is homogeneous to order 0.

Remark 1.5 — Typically if we say that a function is homogeneous but don't specify the order, it is assumed to be of order 0.

If a function in homogeneous to order 0, then it only depends on the ratio of $\frac{y}{x}$. In other words, rewrite $F(x, y) = f(\frac{y}{x})$.

Theorem 1.6

A function $F(x, y)$ is homogeneous of order 0 if and only if it can be expressed as $f(\frac{y}{x})$.

If we have a homogeneous function of order 0, we will be able to introduce a new variable $z = \frac{y}{x} \implies y = sz$, giving us:

$$\frac{d(xz)}{dx} = F(x, xz) = F(x(1), x(z)) = F(1, z).$$

Using the product rule, we have:

$$\frac{d(xz)}{dx} = \frac{dx}{dx}z + x\frac{dz}{dx} = F(1, z).$$

$$z + x\frac{dz}{dx} = F(1, z) \implies \frac{dz}{F(1, z) - z} = \frac{dx}{x},$$

which is a separable differential equation.

Remark 1.7 — The point is whenever you have a homogeneous equation, then introducing $z = \frac{y}{x}$ will allow us to convert it to a separable equation. Note that this only works for order 0 homogeneous equations.

1.2.1 Building an Radar Antenna

TL;DR the equation is:

$$\left(\frac{dy}{dx}\right)^2 - 2\frac{y - F}{x}\left(\frac{dy}{dx}\right) - 1 = 0.$$

If we use the quadratic formula, we get:

$$\frac{dy}{dx} = \frac{y - F}{x} + \sqrt{\left(\frac{y - F}{x}\right)^2 + 1}.$$

If we do the substitution, $z = \frac{y-F}{x}$, we get:

$$\frac{d(xz + F)}{dx} = z + \sqrt{z^2 + 1} \implies x\frac{dz}{dx} + z = z + \sqrt{z^2 + 1} \implies \frac{dz}{\sqrt{z^2 + 1}} = \frac{dx}{x}.$$

$$\int \frac{dz}{\sqrt{z^2 + 1}} = \ln x + C \implies \ln\left(z + \sqrt{z^2 + 1}\right) = \ln x + C.$$

$$\implies A^2x^2 - 2Axz = 1 \implies \frac{1}{2}Ax^2 + \left(F - \frac{1}{2A}\right),$$

which is the equation of a parabola. Thus the optimal shape of a radar dish is a parabola.