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1.1 Jacobi for 2D Discrete Laplacian

For the 2D Laplacian, we have:

$$\begin{cases} -u_{xx} - u_{yy} = f & (x, u) \in \Omega = (0,1)^2 \\ u(x,y) = 0 & \text{on } \partial\Omega \end{cases}$$

By central difference:

$$A_2 x = b, x \in \mathbb{R}^N \quad A_2 \in \mathbb{R}^{N \times N}$$

with $N = n^2$ and:

$$A_2 = A \otimes I + I \otimes A$$

, where A is the 1D discrete Laplacian.

Definition 1.1 (**Kronecker Product (Tensor Product)**).

$$B \otimes C = \begin{bmatrix} b_{11}C & b_{12}C & \dots & b_{1q}C \\ \vdots & & & \vdots \\ b_{p1}C & b_{p2}C & \dots & b_{pq}C \end{bmatrix}$$

Theorem 1.2

$$(A \otimes B)(C \otimes D) = (AC) \otimes (CD)$$

Lemma 1.3
The eigenvalues of A_2 are $\lambda_i + \lambda_j$, where λ_i, λ_j are eigenvalues of A and $1 \le i, j \le n$.

Proof. Let (λ_i, u_i) be eigenpairs of A. Then:

$$A_2(u_i \otimes u_j) = (A \otimes I)(u_i \otimes u_j) + (I \otimes A)(u_i \otimes u_j)$$

$$\begin{aligned} A_2(u_i \otimes u_j) &= (A \otimes I)(u_i \otimes u_j) + (I \otimes A)(u_i \otimes u_j) \\ &= Au_i \otimes u_j + u_j \otimes Au_j \\ &= \lambda_i u_i \otimes u_j + u_i \otimes (\lambda_j u_j) \\ &= (\lambda_i + \lambda_j)(u_i \otimes u_j). \end{aligned}$$

$\square$

Thus:

$$\begin{aligned} G_2 &= I - D_2^{-1} A_2 \\ &= I - \frac{1}{4} A_2. \end{aligned}$$

Meaning that the eigenvalues of G_2 are:

$$\begin{aligned}1-\frac{1}{2}(\lambda_i+\lambda_j) &= 1-\frac{1}{4}\left(4-2\cos\frac{i\pi}{n+1}-2\cos\frac{j\pi}{n+1}\right)\\ &= \frac{1}{2}\left(\cos\frac{i\pi}{n+1}+\cos\frac{j\pi}{n+1}\right).\end{aligned}$$

Thus:

$$\rho(G_2)=\max_{1\le i,j\le n}\left|\frac{1}{2}\left(\cos\frac{i\pi}{n+1}+\cos\frac{j\pi}{n+1}\right)\right| = \cos\frac{\pi}{n+1}<1$$

Because G_2 is symmetric, $\|G_2\|_2=\rho(G_2)$:

$$\|x_k-x_*\|_2\le\rho(G_2)\cdot\|x_{k-1}-x_*\|_2$$

Similar to before, we have:

$$1-O\left(\frac{1}{n^2}\right)=1-O\left(\frac{1}{N}\right)$$

This gives us $\alpha=1$, meaning that:

- number of iterations needed: $O(N\log\tilde{\epsilon}^{-1})$
- number of FLOPs needed per iterations: $O(N)$, which is the number of non-zero entries

Thus the total computation cost is $O(N^2\cdot\log\tilde{\epsilon}^{-1})$. This is the same order as Gaussian Elimination, since $\tilde{\epsilon}$ is usually a constant.

Remark 1.4 — More examples of Jacobi Iteration include the strictly/irreducibly diagonally dominant matrix, which have been covered in MAT5311.

1.2 Jacobi for SPD Matrices

Theorem 1.5

Let $A\in\mathbb{R}^{n\times n}$ be SPD. Then Jacobi converges to x_* for any x_0 if and only if $2D-A$ is SPD too.

Proof. Recall that Jacobi converges to x_* for any x_0 if and only if $\rho(G)<1$.

- Assume Jacobi converges, then:

  $$\rho(I-D^{-1}A)<1\iff\rho(I-D^{-\frac{1}{w}}AD^{-\frac{1}{2}})<1$$

  because $D^{\frac{1}{2}}(I-D^{-1}A)D^{-\frac{1}{2}}$ is similar, thus meaning they share the same eigenvalue and thus spectral radius. Let λ be an eigenvalue of $I-D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$. Then $|\lambda|<1$ and $1+\lambda$ is an eigenvalue of $2I-D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$.

  Since A is symmetric, λ is real, meaning that $1+\lambda$ is positive, thus meaning $2I-D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$ is SPD.

Consider:

$$D^{\frac{1}{2}}(2I - D^{-\frac{1}{2}}AD^{-\frac{1}{2}})D^{\frac{1}{2}} = 2D - A$$

which is also SPD, since they are similar.

- The reverse is very similar, just in reverse. Assume $2D - A$ is SPD. We have:

$$\begin{aligned} 2D - A \text{ is SPD} &\implies 2I - D^{-\frac{1}{2}}AD^{-\frac{1}{2}} \text{ is SPD} \\ &\implies 1 + \lambda, \text{ where } \lambda \text{ is an eigenvalue of } I - D^{-\frac{1}{2}}AD^{-\frac{1}{2}}.. \end{aligned}$$

We also have:

$$\begin{aligned} A \text{ is SPD} &\implies D^{-\frac{1}{2}}AD^{-\frac{1}{2}} = I - (I - D^{-\frac{1}{2}}AD^{-\frac{1}{2}}) \text{ is SPD} \\ &\implies 1 - \lambda > 0 \\ &\implies \lambda > -1. \end{aligned}$$

where λ is an eigenvalue of $I - D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$. As such, we have:

$$-1 < \lambda < 1 \implies |\lambda| < 1 \implies \rho(I - D^{-1}A) < 1$$

□

Example 1.6
Consider the 1D Laplacian:

$$\begin{cases} A \text{ is SPD} \\ 2D - A = 4I - A \text{ is SPD} \end{cases} \implies \text{Jacobi converges}$$

1.3 Lower Bound of Jacobi Convergence Rate

We have:

$$\frac{\|x_k - x_*\|}{\|x_{k-1} - x_*\|} \le \rho(G) + \epsilon$$

for an arbitrarily small ϵ, or $\epsilon = 0$ if G is symmetric. This is a worse-case, i.e. an upper bound. However, this factor is asymptotically optimal, meaning:

$$\lim_{k\to\infty} \frac{\|x_k - x_*\|}{\|x_{k-1} - x_*\|} \ge \rho(G)$$

As such, convergence factor ρ is **tight**.

Let us demonstrate this when G is symmetric first.

Remark 1.7 — For nonsymmetric matrices, this is also true, we will prove later.

$|\lambda_1| > |\lambda_2|$ where λ_1, λ_2 are the largest and 2nd largest eigenvalue of G in absolute value.

Let $G = U \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} U^T$ be the eigenvalue decomposition of G, where $U = \begin{bmatrix} u_1 & u_2 & \dots & u_2 \end{bmatrix}$ is unitary.

Let $x_k - x_* = z_k$. Then:

$$z_k = G^k z_0 = \left(U \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} U^T \right) z_0 = U \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix} U^T z_0.$$

Denote $U^T z_0 = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$. Thus we have:

$$\begin{aligned}
\|z_k\|_2 &= \| \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \| \\
&= \left(\sum_{i=1}^n \lambda_i^{2k} \alpha_i^2 \right)^{1/2} \\
&= |\lambda_1|^k \left(\sum_{i=1}^n \left(\frac{\lambda_i}{\lambda_1} \right)^{2k} \alpha_i^2 \right)^{1/2}.
\end{aligned}$$

Thus, we have:

$$\begin{aligned}
\lim_{k\to\infty} \frac{\|z_k\|_2}{\|z_{k-1}\|_2} &= |\lambda_1| \lim_{k\to\infty} \frac{\left(\sum_{i=1}^n \left(\frac{\lambda_i}{\lambda_1} \right)^{2k} \alpha_i^2 \right)^{1/2}}{\left(\sum_{i=1}^n \left(\frac{\lambda_i}{\lambda_1} \right)^{2(k-1)} \alpha_i^2 \right)^{1/2}} \\
&= |\lambda_1| \lim_{k\to\infty} \frac{\alpha_1^2 + \left(\frac{\lambda_2}{\lambda_1} \right)^{2k} \alpha_2^2 + \dots}{\alpha_1^2 + \left(\frac{\lambda_2}{\lambda_1} \right)^{2(k-1)} \alpha_2^2 + \dots} \\
&= |\lambda_1| \\
&= \rho(G).
\end{aligned}$$