MATH5412 - Advanced Probability Theory II

Aaron Wang

aswang@connect.ust.hk

Spring 2022

Abstract

These are notes for MATH5412 at HKUST, the second course in a two part graduate-level course taught by Bao Zhigang in Spring 2022. The main focus is as a continuation of MATH5411.

Contents

1 February 8th, 2022

1.1 Overview of the Course

Considering $S_n = \sum\limits_{i=1}^{n} X$, for regular CLT we assume that we have:

- Second moment condition

- In $\mathbb{R}$

- Independence

In this course, we will extend CLT to remove these three conditions.

1.1.1 Stable Law

First, we will remove the second moment condition, leading to the .

If the second moment exists, then there is a normalization that allows the limiting distribution go to a gaussian distribution. However, if we don't have the second moment, it depends on the tail of the distribution, giving us a class of distributions. We will be able to show that this No matter if it is a triangular array or a sequence of random variable.

This part will take 3-4 lectures.

1.1.2 Functional Limiting Theorem

In the previous case, we only concerned variables in $\mathbb{R}$. Now we will extend them to get the weak convergence of random functions.

This can be useful for looking at empirical distributions in statistics. For example if we have:

$$X \sim F, \text{ with } F \text{ unknown}$$

If $X_1, \ldots, X_n$ F i.i.d., we can use the empirical distribution:

$$F_n(t) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(X_i \leq t)$$

We want to compare this distribution with the original distribution, which we can use the Kologoromov Statistics

$$\sqrt{n} \sup_t |F_n(t) - F(t)|$$

Note that $\sqrt{n}(F_n(t) - F(t))$ is a random function with $f \in \mathbb{R}$.

To do this, we need to prove the weak convergence of the whole stochastic process. With that, we would have the weak convergence of the random function. Later we will show that $\sqrt{n}(F_n(t) - F(t))$ converges to the Brownian bridge.

Reference:

- Convergence of Probability Measure $\rightarrow$ Billingsley Chapter 2

This part will also be quite short.

1.1.3 Martingale and it's Limiting Theorem

Roughly speaking, a martingale can be thought of the sum of a random variable. We do not need independence. Here we will introduce martingale differences, and this part will take up the majority of the course.

  Reference:

- Durrett Chapter 5 [Dur19]

- Hall and Heyde $\rightarrow$ Martingale Limit Theory and its Application [HH80]

1.1.4 Concentration (if time permits)

Reference:

- R. Vershynin $\rightarrow$ High-dimensional probability

1.2   Heavy Tail Limiting (Poisson) Convergence

let $N(s,t)$ be the number of arrivals at a bank during $[s,t]$. Suppose:

 (i) The number in disjoint intervals are independent

 (ii) The distribution of $N(s,t)$ only depends on $t-s$

 (iii) $\Pr(N(0,h)=1) = \lambda h + o(h)$, and

 (iv) $\Pr(N(0,h) \geq 2) = o(h)$

> **Theorem 1.1**
>
> If (i) - (iv) hold, then $N(0,t)$ has a poisson distribution with mean λt.

Definition 1.2 (Poisson process with rate λ)**.** A family of random variables $N_t, t \geq 0$, satisfying:

 1. If $0 =$

1.3   Stable Law

We have:
$$X_1, X_2, \ldots X_n \text{ i.i.d.} \quad S_n = \sum_{i=1}^{n} X_i$$

If $\mathbf{E}X_i = \mu$ and $\mathbf{Var}X_i = \sigma^2$, we have:
$$\frac{S_n - n\mu}{\sqrt{n}\sigma} \implies N(0,1)$$

Now, if $\mathbf{E}X_i^2 = \infty$, do we have a_n, b_n, Y s.t.:
$$\frac{S_n - b_n}{a_n} \implies Y \quad (Y \text{ nondegenerate})$$

Let us start with a simple case where everything about X_i is known.

Example 1.3

Consider $X_1, X_2, \ldots$ i.i.d.

$$\Pr(X_1 > x) = \Pr(X_1 < -x) = \frac{x^{-\alpha}}{2}, \text{ for } x \geq 1, 0 < \alpha < 2$$

Density $f(x) = \alpha \frac{|x|^{-\alpha-1}}{2}$, $|x| > 1$ Note that this is:

- symmetric (indicates $b_n = 0$)

- $\mathbf{E}X_1^2 = 2 \int\limits_1^\infty x \Pr(|x_1| > x) dx = \int\limits_1^\infty x^{-\alpha+1} dx = \infty$

The solution is:

$$\mathbf{E}[e^{isS_n}] = \left[\underbrace{\mathbf{E}e^{isX_1}}_{\phi(s)} \right]^n = [1 - (1 - \phi(s))]^n$$

$$1 - \phi(s) = \int\limits_1^\infty (1 - e^{ist}) \frac{\alpha}{2|x|^{\alpha+1}} dx + \int\limits_{-\infty}^{-1} (1 - e^{isx}) \frac{\alpha}{2|x|^{\alpha+1}} dx$$

2 February 10th, 2022

2.1 Cont.

Recall from the last time, we found that most of the contribution is from the large points of scale $O(n^{1/\alpha})$.

Let us define an index set of large points:

$$I_n(\epsilon) = \{m \leq n : |X_m| > \epsilon n^{1/\alpha}\}$$

and define the sums:

$$\hat{S}_n(\epsilon) = \sum_{m \in I_n(\epsilon)} X_m = \sum_{m=1}^n X_m \mathbb{1}(|x_m| > \epsilon n^{1/\alpha})$$

$$\overline{S}_n(\epsilon) = S_n - \hat{S}_n(\epsilon) = \sum_{m=1}^n X_m \mathbb{1}(|X_m| \leq \epsilon \leq \epsilon^{1/\alpha})$$

Now we have two task:

- Show $\frac{\overline{S}_n(\epsilon)}{n^{1/\alpha}}$ is small if ϵ is small

- Find the limit of $\frac{\hat{S}_n(\epsilon)}{n^{1/\alpha}}$

Proof.

$$\mathbf{E}\left[\frac{\overline{S}_n(\epsilon)}{n^{1/\alpha}}\right]^2 = n^{-\frac{2}{\alpha}} \cdot n \cdot \mathbf{E}\left[\overline{X_1}(\epsilon)\right]^2, \quad \overline{X_i}(\epsilon) = X_i \mathbb{1}(|X_i| \leq \epsilon n^{1/\alpha})$$

$$\mathbf{E}[\overline{X_1}(\epsilon)]^2 = \int\limits_0^\infty 2y \Pr(|\overline{X}_1(\epsilon)| \geq y \ dy) \leq \int\limits_0^{\epsilon n^{1/\alpha}} 2y$$

Later we choose $\epsilon = \epsilon \to 0$ as $n \to \infty$.   □

Proof. Proof of (2).

$$\mathbf{E}\exp\left(it\frac{\hat{S}_n(\epsilon)}{n^{1/\alpha}}\right) = \sum_{m=0}^n \mathbf{E}\left[\exp\left(it\frac{\hat{S}_n(\epsilon)}{n^{1/\alpha}}\right)\Big||I_n(\epsilon)| = m\right]\Pr(|I_n(\epsilon)| = m)$$

We will use two facts:

1. $|I_n(\epsilon)|$ is $\mathrm{Bin}\left(n, \frac{\epsilon^{-\alpha}}{n}\right) \sim \mathrm{Poisson}(\epsilon^{-\alpha})$. $\Pr(|X_n| > \epsilon n^{\frac{1}{\alpha}}) = \epsilon^{-\alpha}\frac{1}{n}$.

                                                                                                □

References

[HH80]   P. Hall and C.C. Heyde. *Martingale Limit Theory and Its Application.* Probability and mathematical statistics. Academic Press, 1980. ISBN: 9781483240244. URL: http://www.stat.yale.edu/~mjk56/MartingaleLimitTheoryAndItsApplication.pdf.

[Dur19] Rick Durrett. *Probability: Theory and Examples.* 2019. URL: https://services.math.duke.edu/~rtd/PTE/PTE5_011119.pdf.

Index