1 September 10th, 2020

1.1 More Linear Algebra Review

Definition 1.1. If $T : \mathbb{R}^m \to \mathbb{R}^n$, then the **image** of T is:

$$\text{image}T = \{y \in \mathbb{R}^n : \exists x \in \mathbb{R}^m \text{ s.t. } y = T(x)\}.$$

The image

Definition 1.2. Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. The **rank** of T is the dimension of the image of T.

Lemma 1.3

If $T : \mathbb{R}^m \to \mathbb{R}^n$ is linear, then:

1. $\text{rank } T \leq n$
2. $\text{rank } T \leq m$

Proof. The proof of 1) is trivial. For 2), we use the theorem below. □

Theorem 1.4

$$\text{rank } T + \dim \ker T = m.$$

Definition 1.5. A linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ has **full rank** if:

$$\text{rank } T = \min\{n, m\}.$$

Lemma 1.6

If $T : \mathbb{R}^m \to \mathbb{R}^n$ has full rank, and:

1. if $m \geq n$, then the matrix of T relative to certain basis is $\begin{bmatrix} I_n & 0 \end{bmatrix}$
2. if $m \leq n$, then the matrix of T relative to a certain basis is $\begin{bmatrix} I_m \\ 0 \end{bmatrix}$

Example 1.7

Fix $e \in \mathbb{R}^n$, define $T : \mathbb{R}^n \to \mathbb{R}$ by $T(x) = \langle x, e\rangle$, $\forall x \in \mathbb{R}^n$, then:

- its rank is 1 if $e \neq 0$
- its rank is 0 if $e = 0$.

Definition 1.8. In the case we choose $e = e_j$, which is the j-th standard basis of $\mathbb{R}^n$, then the map T above is called the **projection to the j-th coordinate**, symbolically written as Π_j.

Definition 1.9. For a given $c \in \mathbb{R}$, then the set $\{x \in \mathbb{R}^n : T(x) = c\}$ forms a **hyperplane**

Definition 1.10. Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation whose matrix relative to the standard basis is

$$\begin{bmatrix} t_{11} & t_{12} & \dots & t_m \\ \vdots & \ddots & & \vdots \\ t_{m_1} & t_{m_2} & \dots & t_{mn} \end{bmatrix}$$

then the **norm** of T is

$$\|T\| = \sqrt{\sum_{i,j} t_{ij}^2} = \sqrt{\operatorname{tr}(TT^\top)}.$$

Theorem 1.11

Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be linear. Then for every $x \in \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \in \mathbb{R}^m$, then:

$$\|T(x)\| \leq \|T\|\|x\|.$$

Proof. Denote the matrix T relatie to the standard basis as above. If $x \in \mathbb{R}^m$, we have:

$$\begin{aligned}
\|T(x)\|^2 &= \left\| T\left(\sum_{i=1}^m x_i e_i\right) \right\|^2 \\
&= \left\| \sum_{i=1}^m x_i T(e_i) \right\|^2 \\
&= \left\| \sum_{i=1}^m \sum_{j=1}^n x_i t_{ij} e_j \right\|^2 \\
&= \sum_{j=1}^n \left(\sum_{i=1}^m x_i t_{ij} \right)^2 \\
&\leq \sum_{j=1}^n \left(\left(\sum_{i=1}^m x_i^2 \right) \left(\sum_{i=1}^m t_{ij}^2 \right) \right) && \text{Cauchy-Schwarz} \\
&= \sum_{j=1}^n \sum_{i=1}^m t_{ij}^2 \|x\| \\
&= \|T\|\|x\|
\end{aligned}$$

□

Lemma 1.12
1. If $\|T\| = 0$, then $T = 0$
2. If $T, S : \mathbb{R}^m \to \mathbb{R}^n$ are linear, then: $\|T + S\| \leq \|T\| + \|S\|$
3. If $T : \mathbb{R}^m \to \mathbb{R}^n$, $S : \mathbb{R}^\ell \to \mathbb{R}^m$ are linear, then $\|TS\| \leq \|T\|\|S\|$

Definition 1.13. For a given sequence $\{a_n\}$ of real numbers, $\{a_n\}$ converges to $a \in \mathbb{R}$ if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$, such that $n > N$ implies that $|a_n - a| < \epsilon$

Definition 1.14. For a given sequence $\{v_n\}$ of vectors in $\mathbb{R}^n$, $\{v_n\}$ converges to $v \in \mathbb{R}^n$ if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$, such that $n > N$ implies that $|v_n - v| < \epsilon$

Definition 1.15. For a given sequence $\{T_n\}$ of linear transformations $\mathbb{R}^\ell \to \mathbb{R}^n$, $\{T_n\}$ converges to T if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$, such that $n > N$ implies that $\|T_n - T\| < \epsilon$

Lemma 1.16
Let $\{v_n\}$ be a sequence in $\mathbb{R}^m$ which converges to $v \in \mathbb{R}^m$. Then: $\forall j \in \{1, \ldots, m\}$, $\{\Pi_j(v_n)\}$ converges to $\Pi_j(v)$

Proof. Since $\{v_n\}$ converges, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. if $n > N$, then $\|v_n - v\| < \epsilon >$, Thus if $n > N$, then $|\Pi_j(v_n) - \Pi_j(v) = |\Pi_j(v_n - v)$ For every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n > N$, then if $n > N$, then $|\Pi_j(v_n) - \Pi_j(v)| \leq \|v_n - v\| < \epsilon$ □

Lemma 1.17
Let $\{v_n\}$ be a sequence in $\mathbb{R}^m$ so that $\forall j \in \{1, .., m\}$, $\{\Pi_j(v_n)\}$ converges. Then $\{v_n\}$ is a convergent sequence of vectors.

Proof. Suppose that $\{\Pi_j(v_n)\}$ converges to a_j and let $v = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$. Thus if $n > N_j$, then $|\Pi_j(v_n - v)| < \frac{\epsilon}{m}$ if $n > N = \max\{N_1, N_2, N_m\}$, we have:

$$\|v_n - v\| < \sum_j |\Pi_j(v_n - v)| < \frac{\epsilon}{m} \times m = \epsilon.$$

Since □