1 March 30th, 2020

1.1 Bessel's ODE and Bessel Functions

We know that solutions to the ODE:

$$y''(x) + \lambda^2 y(x) = 0$$

are:

$$y_1(x) = \cos(\lambda x) \quad y_2(x) = \sin(\lambda x).$$

With:

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

Using the series expansion for sine and cosine, we have:

$$y_1(x) = \cos(\lambda x) = \sum_{k=0}^{\infty} \frac{(-1)^k (\lambda x)^{2k}}{(2k)!}.$$

$$y_2(x) = \sin(\lambda x) = \sum_{k=0}^{\infty} \frac{(-1)^k (\lambda x)^{2k+1}}{(2k+1)!}.$$

The Bessel's ODE is the equation:

$$x^2 y''(x) + xy'(x) + (x^2 - \nu^2) y(x) = 0, x > 0.$$

Without going through the series method, one solution to this ODE is given by:

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+\nu)!} \left(\frac{x}{2}\right)^{2k+\nu}, \quad |x| < \infty.$$

We can have any values real of ν even fractions or irrational values (using the gamma function).

Remark 1.1 — Note that $J_\nu(-x) = (-1)^\nu(x)$ and when ν is an even integer, than $J_\nu(x)$ is an even function, while when v is an odd integer, then $J_\nu(x)$ is an odd funciton.

Remark 1.2 — If $|x|$ is very small, then $J_\nu(x) \approx \left(\frac{x}{2}\right)^\nu \frac{1}{\nu!}$, $\nu \geq 0$. As such, $J_\nu(x)$ is finite at $x = 0$.

Using Abel's equation, we can get second linearly independent solution to Bessel's ODE, which is:

$$y_2(x) = Y_\nu(x) = J_\nu(x) = \left(A \int \frac{1}{x J_\nu^2(x)}\, dx + B\right).$$

Noteablly, it is not finite at $x = 0$.

Thus the general solution to Bessel's ODE is:

$$y(x) = c_1 J_\nu(x) + c_2 Y_\nu(x).$$

You can always use $Y_\nu(x)$ as the second solution, but if it turns out that ν is not an integer, we can instead use $J_{-\nu}(x)$ for the second solution. Regardless, $J_{-\nu}(x)$ is also not finite at $x = 0$. This means that if we require that $y(\pm 1)$ to be finite, then we must set $c_2 = 0$, giving us:

$$y(x) = c_1 J_\nu(x).$$

There is a **modified Bessel's ODE**:

$$x^2 y''(x) + xy'(x) - (x^2 + \nu^2)y(x) = 0.$$

Which has a general solution

$$y(x) = c_1 I_\nu(x) + c_2 K_\nu(x).$$

Where:

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{k!(k+\nu)!} \left(\frac{x}{2}\right)^{2k+\nu}.$$

$$K_\nu(x) = I_\nu(x) \left(A \int \frac{1}{x I_\nu^2(x)}\, dx + B \right).$$

Remark 1.3 — $I_\nu(x)$ is finite for $\nu \geq 0$, and $K_\nu(x)$ is not finite.

1.2 Properties of $J_\nu(x)$

$$J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x).$$

$$Y_{\nu+1}(x) = \frac{2\nu}{x} Y_\nu(x) - Y_{\nu-1}(x).$$

$$I_{\nu+1}(x) = -\frac{2\nu}{x} Y_\nu(x) + Y_{\nu-1}(x).$$

$$K_{\nu+1}(x) = \frac{2\nu}{x} Y_\nu(x) + Y_{\nu-1}(x).$$

Theorem 1.4

Given the ODE:

$$x^2y''(x) + (a + 2bx^R)xy'(x) + (c + dx^{2s} - b(1 - a - R)x^R + b^2x^{2R})y(x) = 0, x > 0.$$

A general solution is given by:

$$y(x) = x^{\frac{1-a}{2}} e^{\frac{-bx^R}{R}} \left(c_1 J_p \left(\frac{\sqrt{d}}{s} x^s \right) + c_2 Y_p \left(\frac{\sqrt{d}}{s} x^s \right) \right).$$

when $d > 0$, and:

$$y(x) = x^{\frac{1-a}{2}} e^{\frac{-bx^R}{R}} \left(c_1 J_p \left(\frac{\sqrt{-d}}{s} x^s \right) + c_2 Y_p \left(\frac{\sqrt{-d}}{s} x^s \right) \right).$$

when $d < 0$, where:

$$p = \left| \frac{1}{s} \sqrt{\left(\frac{1-a}{2} \right)^2 - c} \right|.$$

Example 1.5

Consider:

$$xy''(x) + 2y'(x) + \lambda^2 x^2 y(x) = 0.$$

Multiplying by x, we have:

$$x^2y''(x_0 + 2xy'(x) + \lambda^2 x^3 y(x) = 0.$$

Which means that

$$a + 2bx^R = 2 \implies a = 2, b = 0.$$

We also have:

$$c + dx^{2s} - b(1 - a - R)x^R + b^2x^{2R} = \lambda^2 x^3 => c + d^{2s} = \lambda x^3.$$

$$\to c = 0, d = \lambda^2 \geq 0, s = \frac{3}{2}.$$

Thus:

$$p = \left| \frac{1}{s} \sqrt{\left(\frac{1-2}{2} \right)^2 - 0} \right| = \frac{1}{3}.$$

Thus:

$$y(x) = x^{-\frac{1}{2}} \left(c_1 J_{\frac{1}{3}} \left(\frac{2}{3} \lambda x^{\frac{3}{2}} \right) + c_2 Y_{\frac{1}{3}} \left(\frac{2}{3} \lambda x^{\frac{3}{2}} \right) \right).$$