1 April 15th, 2020

1.1 Solving Homogeneous BVIVP through Separation of Variables

Continuing from last time, we were able to transform a non-homogeneous BVIVP into one that is homogeneous. We have:

$$\text{PDE}\left\{T_0\frac{\partial^2 u(x,t)}{\partial x^2} = \rho_0\frac{\partial^2 u(x,t)}{\partial t^2}, \quad 0<x<L, 0<t\right. .$$

$$\text{BCs}\begin{cases} u(0,t)=0 \\ u(L,t)=0 \end{cases}, \quad 0<t.$$

$$\text{ICs}\begin{cases} u(x,0)=y(x)-y_e(x) \\ \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = v(x) \end{cases}, \quad 0<x<L.$$

Note that if our original BVIVP was homogeneous, we would be able to skip to this point. Moving on, we would like to construct a general solution to the PDE and BCs using the principle of linear-superpositions.

Suppose $u_1(x,t)$ and $u_2(x,t)$ are solutions to the PDE and BCs. Then $c_1u_1+c_2u_2$ would also be solutions for any constants c_1 and c_2. This can be generalized for more u_n. Note that this is possible because the PDE and BCs are homogeneous. As such, we will use

$$\{u_1(x,t), u_2(x,t), \dots\}.$$

as the basis set to generate the general solution.

> **Remark 1.1** — Note that for 2nd order linear ODE, the solution space is 2-dimensional, however for PDE the solution space is infinite dimensional.

In constructing the basis set of solutions, we should try solutions of form:

$$u(x,t)=\phi(x)\gamma(t)\neq 0.$$

To do this, we will replace this into the PDE and BCs, giving us:

$$T_0\frac{\partial^2\phi(x)\gamma(t)}{\partial x^2} = \rho_0\frac{\partial^2\phi(x)\gamma(t)}{\partial t^2} \implies T_0\gamma(t)\frac{d^2\phi(x)}{dx^2} = \rho_0\phi(x)\frac{d^2\gamma(t)}{dt^2}.$$

$$\implies \underbrace{\frac{\phi''(x)}{\phi(x)}}_{\text{function of only } x} = \underbrace{\frac{\rho_0}{T_0}\frac{\gamma''(t)}{\gamma(t)}}_{\text{function of only } t}, \quad 0<x<L,\ 0<t.$$

Since x and t are independent, we can set them to be any thing we want, setting $t=1$ and leaving x alone, we get:

$$\frac{\phi''(x)}{\phi(x)} = \underbrace{\frac{\rho_0}{T_0}\frac{\gamma''(1)}{\gamma(1)}}_{\text{constant}}, \quad 0<x<L.$$

In a similar way, if we set $x = \frac{L}{2}$ and leaving t alone, we would get:

$$\underbrace{\frac{\phi''(\frac{L}{2})}{\phi(\frac{L}{2})}}_{\text{constant}} = \frac{\rho_0}{T_0}\frac{\gamma''(t)}{\gamma(t)}, \quad 0 < t.$$

Note that because of this, the constants must both be the same, meaning that we have separated the PDE into two ODE:

$$\frac{\phi''(x)}{\phi(x)} = C, \quad 0 < x < L.$$

$$\frac{\rho_0}{T_0}\frac{\gamma''(t)}{\gamma(t)} = C, \quad 0 < t.$$

Remark 1.2 — Note that we can skip the working out, since whenever we come across a function of one variable equal a function of another variable, and both variables are independent, then both functions must be constants and equal the same constant.

Rearranging the two equations before, we have:

$$\phi''(x) - C\phi(x) = 0, \quad 0 < x < L.$$

$$\gamma''(t) - \frac{fT_0}{\rho_0}\gamma(t) = 0, \quad 0 < t.$$

Also note that:

$$u(0,t) = 0 \implies \phi(0)\gamma(t) = 0, 0 < t.$$
$$\implies \phi(x) = 0.$$

Remark 1.3 — Note that we cannot have $\gamma(t) = 0$, then $u(x,t) = \phi(x)\gamma(t) = 0$

Similarly for the other boundary condition, we would have:

$$u(L,t) = 0 \implies \phi(L)\gamma(t) = 0 \implies \phi(L) = 0.$$

Collecting the ϕ, we would get:

$$\phi''(x) - C\phi(x) = 0, \quad 0 < x < L.$$

$$\phi(0) = 0.$$

$$\phi(L) = 0.$$

which is a regular Sturm-Liouville Problem. As you recall this has the solution:

$$\phi(x) = \begin{cases} A\cosh(x\sqrt{C}) + B\sinh(x\sqrt{C}), & C > 0 \\ A + Bx, & C = 0 \\ A\cos(x\sqrt{-C}) + B\sin(x\sqrt{-C}), & C < 0 \end{cases}.$$

Similarly to before, if we consider $C > 0$, we have:

$$\phi(x) = A\cosh(x\sqrt{C}) + B\sinh(x\sqrt{C}).$$

$$\phi(0) = A = 0 \implies \phi(x) = B\sinh(x\sqrt{C}).$$

$$\phi(L) = B\sinh(L\sqrt{C}) = 0 \implies B = 0.$$

Meaning that $\phi(x) = 0$, which will not give us anything. (if on exam, they will tell us to find non-zero or tell us which cases to consider).

For $C = 0$, we would get the same conclusion $\phi(x) = 0$, since the BC force A and B to both be 0.

For $C < 0$, we have:

$$\phi(x) = A\cos(x\sqrt{-C}) + B\sin(x\sqrt{-C}).$$

$$\phi(0) = A = 0 \implies \phi(x) = B\sin(x\sqrt{-C}).$$

$$\phi(L) = B\sin(L\sqrt{-C}) = 0 \implies L\sqrt{-C} = n\pi \implies C = -\left(\frac{n\pi}{L}\right)^2 = C_n.$$

Note that there are an infinite possible solutions:

$$\phi_n(x) = B_n\sin(x\sqrt{-C_n}) = B_n\sin\left(\frac{n\pi x}{L}\right).$$

Also note that:

$$C_{-n} = -\left(-\frac{n\pi}{L}\right)^2 = -\left(\frac{n\pi}{L}\right)^2 = C_n.$$

and

$$\phi_{-n}(x) = B_{-n}\sin\left(-\frac{n\pi}{L}x\right) = -B_{-n}\sin\left(\frac{n\pi}{L}x\right) = (-B_{-n})\sin\left(\frac{n\pi}{L}x\right) = \phi_n(x).$$

In addition, note that we can throw away $n = 0$, since that would give us $C_0 = 0$ but we need $C < 0$. As such we can throw away all the negative n's, giving us:

$$C_n = -\left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots.$$

$$\phi_n(x) = B_n\sin\left(\frac{n\pi}{L}x\right), \quad n = 1, 2, 3, \ldots.$$

Note that B_n is an arbitrary constant we cannot do anything about, so we usually take $B_n = 1$, giving us:

$$\phi_n(x) = \sin\left(\frac{n\pi}{L}x\right), \quad n = 1, 2, 3, \ldots.$$

Remark 1.4 — Note that this is the solution to the RSLP in which:

$$\lambda_n = -\left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, ...$$

$$\phi_n(x) = \sin\left(\frac{n\pi}{L}x\right), \quad n = 1, 2, 3, \ldots.$$

Moving to γ, we have:

$$\gamma''(t) - C\frac{T_0}{\rho_0}\gamma(t) \implies \gamma_n''(t) + \underbrace{\frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2}_{\omega_n^2}\gamma_n(t) = 0, \quad 0 < t.$$

As such, we have:

$$\gamma_n''(t) + \omega_n^2\gamma_n(t) = 0.$$

Which, as we know has solution:

$$\gamma_n(t) = D_n\cos(\omega_n t) + E_n\sin(\omega_n t), \quad n = 1, 2, 3, \ldots.$$

This means that our basis set of solutions would now be:

$$U_n(x,t) = (D_n\cos(\omega_n t) + E_n\sin(\omega_n t))\sin\left(\frac{n\pi}{L}x\right).$$

$$\omega_n = \frac{n\pi}{L}\sqrt{\frac{T_0}{\rho_0}}.$$

Remark 1.5 — Note that if we carried the constants B_n, they could be absorbed into D_n and E_n.

Now finally, the principle of superposition means that the general solution is:

$$u(x,t) = \sum_{n=1}^{\infty}(D_n\cos(\omega_n t) + E_n\sin(\omega_n t))\,\phi_n(x).$$

$$\phi_n(x) = \sin\left(\frac{n\pi}{L}x\right), \quad 0 \leq x \leq L, \quad 0 \leq t.$$

Which is the general solution to the PDE and the BCs.

Now we consider the initial conditions:

$$u(x,0) = y(0) - y_e(x) \implies \sum_{n=1}^{\infty} D_n\phi_n(x) = y(x) - y_e(x), \quad 0 < x < L.$$

Note that since this is a Sturm-Liouville problem, there is an associated dot product:

$$\phi_p \cdot \phi_q = \int_0^L \phi_p(x)\phi_q(x)\;dx = \begin{cases} 0, & p \neq q \\ \frac{L}{2}, & p = q \end{cases}.$$

Meaning that:

$$D_m = \frac{\phi_n \cdot (y - y_e)}{\phi_n \cdot \phi_n} = \frac{2}{L}\int_0^L \phi_n(x)\,(y(x) - y(e))\;dx.$$

Remark 1.6 — Note that the weight function is equal to 1, since our equation is given by:

$$\phi'' + \lambda\phi = 0.$$

If we go back to the function, and compute the partial derivative with respect to time, we have:

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left(-\omega_n D_n \sin(\omega_n t) + \omega_n E_n \cos(\omega_n t)\right) \phi_n(x).$$

Evaluating at $t = 0$, we have:

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} \omega_n E_n \phi_n(x) = v(x), \quad 0 < x < L.$$

Meaning that:

$$w_n E_n = \frac{\phi_n \cdot v}{\phi_n \cdot \phi_n}.$$

$$\implies E_n = \frac{1}{\omega_n}\frac{2}{L}\int_0^L \phi_n(x) v(x)\ dx.$$

In summary the complete solution to the BVIBP satisfied by $y(x, t)$, is given by:

$$y(x,t) = y_e(x) + u(x,t).$$

Where:

$$y_e(x) = \frac{\rho_0 g}{2T_0}x^2 + \left(\frac{H_2 - H_1}{L} - \frac{\rho_0 g L}{2T_0}\right) x + H_1.$$

and

$$u(x,t) = \sum_{n=1}^{\infty} \left(D_n \cos(\omega_n t) + E_n \sin(\omega_n t)\right) \phi_n(x).$$

where:

$$\omega_n = \frac{n\pi}{L}\sqrt{\frac{T_0}{\rho_0}} \quad \phi_n(x) = \sin\left(\frac{n\pi}{L}x\right).$$

$$D_n = \frac{2}{L}\int_0^L \phi_n(x)(y(x) - y_e(x))\ dx.$$

$$E_n = \frac{2}{L\omega_n}\int_0^L \phi_n(x) v(x)\ dx.$$