# 1 October 6th, 2020

## 1.1 Explicit Scheme for Hyperbolic Equation

Applying the explicit scheme method, we have:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a\frac{U_{j+1}^n - U_j^n}{\Delta x} = 0.$$

If we set $\nu = \frac{a\Delta t}{\Delta x}$, we have:

$$U_j^{n+1} = U_j^n - \nu(U_{j+1}^n - U_j^n) = (1+\nu)U_j^n - \nu U_{j+1}^n.$$

**Definition 1.1** (**FLC Condition**)**.** For a convergent scheme, the domain of dependence of the PDE must lie within the domain of dependence of the numerical scheme.

**Example 1.2**

The scheme above cannot converge for $a > 0$. This is because if $a > 0$, $U_j^{n+1}$ depends on $U_j^n$, $U_{j+1}^n$, $U_{j+1}^{n-1}$, $U_{j+2}^{n-1}$, etc. However, the characteristic line $x = at + x_0$ does not lie in this domain, thus the scheme does not hold.

**Example 1.3**

If we instead use a backward difference:

$$U_j^{n+1} = U_j^n - \nu(U_j^n - U_{j-1}^n) = (1+\nu)U_j^n - \nu U_j^n.$$

Then the characteristic line falls within the domain.

**Example 1.4**

For $a < 0$, we must have $a\frac{\Delta t}{\Delta x} \leq 1$, in order to have the characteristic line to lie within the domain.

If we attempt to use central difference, we can eliminate the sign condition by using a symmetric scheme in space:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} = 0.$$

Note that for this scheme, CFL condition holds for $|a\frac{\Delta t}{\Delta x}| \leq 1$. However, if we look at this from stability analysis, we have:

$$\frac{\lambda - 1}{\Delta t}\lambda^n e^{ik(j\Delta x)} + \frac{a}{2\Delta x}\lambda^n e^{ik(j\Delta x)}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right) = 0.$$

$$\implies \lambda = 1 - a\frac{\Delta t}{\Delta x} i \sin k\Delta x.$$

which is complex. Thus we have:

$$|\lambda| = \sqrt{1 + a^2 \left(\frac{\Delta t}{\Delta x}\right)^2 \sin^2 k\Delta x} > 1.$$

meaning it does not converge. Thus we cannot central difference for an explicit scheme for hyperbolic equation. Instead, we have to use forward or backwards depending on the value of $a$:

$$U_j^{n+1} = \begin{cases} U_j^n - \nu(U_{j+1}^n - U_j^n) & a < 0 \\ U_j^n - \nu(U_j^n - U_{j-1}^n) & a > 0 \end{cases}.$$

which is called the **upwind scheme**.

**Example 1.5**
In laymen terms, if the wave is going forward ($a > 0$) we use backward difference, if the wave is going backward we use forward difference.

Analyzing the stability, for $a > 0$, (backward difference) we have:

$$\begin{aligned} \lambda &= 1 - \nu(1 - e^{-ik\Delta x}) \\ &= 1 - \nu(1 - (\cos k\Delta x - i \sin k\Delta x)) \\ &= 1 - \nu(1 - \cos k\Delta x) + i\nu \sin k\Delta x \\ \implies |\lambda|^2 &= (1 - \nu(1 - \cos k\Delta x))^2 + \nu^2 \sin^2 k\Delta x \\ &= 1 - 2\nu(1 - \nu)(1 - \cos k\Delta x) \\ &= 1 - 4\nu(1 - \nu) \sin^2 \left(\frac{1}{2} k\Delta x\right) < 1 \quad \text{if } \nu < 1. \end{aligned}$$

This matches the CFL condition.

**Remark 1.6** — Since this only uses first order derivatives, this scheme is only first order in time and space.

Previously to get second order accuracy, we used central difference. However, as we showed above, this is not a stable scheme. To achieve this accuracy, we will use quadratic interpolation.

# 2 Lax-Wendroff Scheme

Suppose we want to find the value at $P$ in Figure 1. Using the characteristic line, we have that $U(P) = U(Q)$, and as such, we want to find the value at $Q$. To do this, we can try linear interpolation with points $B = U_{j-1}^n$, $A = U_j^n$, and $C = U_{j+1}^n$ that lie on the lattice:

$$\begin{aligned} U(Q) &= \nu U(B) + (1 - \nu) U(A) \\ \implies U_j^{n+1} &= \nu U_{j-1}^n + (1 - \nu) U_j^n. \end{aligned}$$

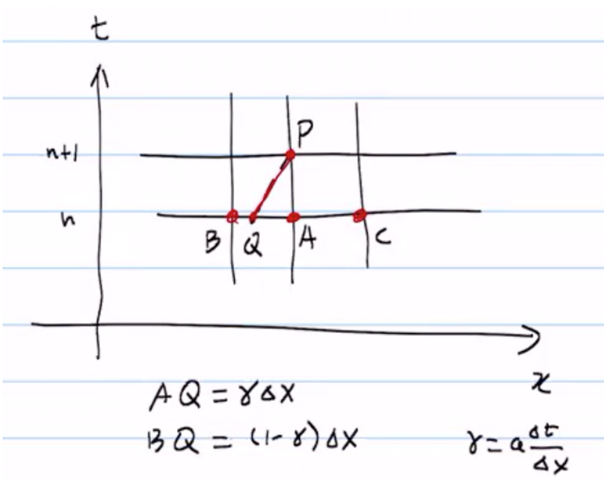


Figure 1

which is exactly the upwind scheme. If we try quadratic interpolation, we would have: In other words

$$P(x) = U_{j-1}\frac{(x-\Delta x)}{(0-\Delta x)}\cdot\frac{(x-2\Delta x)}{(0-2\Delta x)} + U_j\frac{(x-0)}{(\Delta x-0)}\cdot\frac{(x-2\Delta x)}{(\Delta x-2\Delta x)} + U_{j+1}\frac{(x-0)}{(2\Delta x-0)}\cdot\frac{(x-\Delta x)}{(2\Delta x-\Delta x)}.$$

Plugging in $Q = (1-\nu)\Delta x$, we have:

$$\begin{aligned} P(Q) &= \frac{\nu}{2}(1+\nu)U_{j-1} + (1-\nu)(1+\nu)U_j - \frac{1}{2}\nu(1-\nu)U_{j+1} \\ \implies U_j^{n+1} &= U_j^n + \frac{\nu}{2}(U_{j-1}^n U_{j+1}^n) + \frac{\nu^2}{2}\left(U_{j-1}^n - 2U_j^n + U_{j+1}^n\right). \end{aligned}$$

Remember we need the characteristic line to lie within the scheme to satisfy the CFL condition, meaning we need $|\nu| \le 1$. This scheme is called the **Lax-Wendroff method**.

**Remark 2.1** — Note that this scheme is similar to the central difference but with a higher order correction term.

**Remark 2.2** — The above scheme is second order in space. This can be verified by calculating the truncation error.

Checking the stability analysis, we would have:

$$\begin{aligned} \lambda &= (1 - 2\nu^2\sin^2\frac{k\Delta x}{2})^2 + \nu^2\sin^2 k\Delta x \\ |\lambda|^2 &= (1 - 2\nu^2\sin^2\frac{k\Delta x}{2})^2 + \nu^2\sin^2 k\Delta x \\ &= 1 - 4\nu^2\sin^2\frac{k\Delta x}{2} + 4\nu^4\sin^4\frac{k\Delta x}{2} + 4\nu^2\sin^2\frac{k\Delta x}{2}\cos^2\frac{k\Delta x}{2} \\ &= 1 - 4\nu^2(1-\nu^2)\sin^4\left(\frac{1}{2}k\Delta x\right) \le 1 \quad \text{if } \nu \le 1 \end{aligned}$$

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This means the scheme is stable for $|\nu| \leq 1$ (CFL condition).

To summarize, the design of numerical methods for hyperbolic equations is very different that for parabolic equations, as the issues we have to consider are very different.