1 February 24th, 2020

1.1 Computing Inverse Laplace Transform

Recall that:

1. $\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\}$
2. $\mathcal{L}\{e^{at} f(t)\} = \mathcal{L}\{f(t)\}_{s\to s-a}$
3. $\mathcal{L}\{tf(t)\} = -\frac{d}{ds}\mathcal{L}\{f(t)\}$
4. $\mathcal{L}\{t^m f(t)\} = (-1)^m \frac{d^m}{ds^m}\mathcal{L}\{f(t)\}$
5. $\mathcal{L}\{u(t-a)f(t)\} = e^{-as}\mathcal{L}\{f(t+a)\}$
6. $\mathcal{L}\{I(t-a)f(t)\} = e^{-as}f(a)$

Now let's consider how to do compute inverse Laplace Transforms. Consider (5) from the above list, if we replace $f(t)$ by $f(t-a)$, we get:

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}\mathcal{L}\underbrace{\{f(t)\}}_{F(s)}.$$

Taking the inverse on both sides, we get:

Theorem 1.1 (First Shifting Theorem for Inverse Laplace Transforms)

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-as)\mathcal{L}^{-1}\{F(s)\}\big|_{t\to t-a}.$$

Example 1.2

Consider $\mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{s^2+1}\right\}$. Using the above, we have:

$$\mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{s^2+1}\right\} = u(t-a)\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}_{t\to t-a}.$$

$$= u(t-2)\sin t\big|_{t\to t-2} = u(t-2)\sin(t-2).$$

If we consider (2) from the above, and take the inverse of both sides, we would get:

Theorem 1.3 (Second Shifting Theorem for Inverse Laplace Transforms)

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}\mathcal{L}^{-1}\{F(s)\}.$$

Example 1.4

Suppose we want $\mathcal{L}^{-1}\left\{\frac{1}{2s^2+s+8}\right\}$. We have:

$$\mathcal{L}^{-1}\left\{\frac{1}{2s^2+s+8}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2(s^2+\frac{1}{2}s)+8}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2\left(s^2+\frac{1}{2}s+\frac{1}{16}-\frac{1}{16}\right)+8}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{2(s+\frac{1}{4})^2+\frac{63}{8}}\right\}.$$

Note that the above is of the form $\mathcal{L}^{-1}\{F(s+\frac{1}{4})\}$, thus we have:

$$= e^{-\frac{1}{4}t}\mathcal{L}^{-1}\left\{\frac{1}{2a^2+\frac{63}{8}}\right\} = \frac{1}{2}e^{-\frac{1}{4}t}\mathcal{L}\left\{\frac{1}{s^2+\frac{63}{16}}\right\}.$$

Using the fact that $\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin(at)$, we get:

$$= \frac{1}{2}e^{-\frac{1}{4}t}\frac{1}{\sqrt{\frac{63}{16}}}\sin\left(t\sqrt{\frac{63}{16}}\right) = \frac{2}{\sqrt{63}}e^{-\frac{1}{4}t}\sin\left(\frac{t}{4}\sqrt{63}\right).$$

Remark 1.5 — Essentially above we are using the fact that:

$$as^2+bs+c = a\left(s+\frac{b}{2a}\right)^2 + \left(c-\frac{b^2}{4a}\right).$$

Instead of completing the square, this can also be useful in combination with partial fractions.

Example 1.6

From partial fractions, we know that:

$$\frac{s^2+1}{s^3(s-1)^2(s-2)} = \frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{D}{s-1}+\frac{E}{(s-1)^2}+F(s-2).$$

In addition, we know that $\mathcal{L}^{-1}\{\frac{1}{(s-a)^m}\} = \frac{1}{(m-1)!}t^{m-1}e^{at}$, thus if we know the coefficients, we have:

$$\frac{s^2+1}{s^3(s-1)^2(s-2)} = A+Bt+\frac{1}{2}Ct^2+De^t+Ete^t+Fe^{2t}.$$

Theorem 1.7 (Heavyside Expansion Theorem)
There is a special case, where all the powers in the denominators $Q(s)$ are to the first power, and numerator $P(s)$, we have:

$$\mathcal{L}\left\{\frac{P(s)}{(s-\alpha_1)(s-\alpha_2)\ldots(s-\alpha_n)}\right\} = \sum_{y=1}^{n}\left(\frac{P(\alpha_y)}{Q'(\alpha_y)}\right)e^{\alpha_y t}.$$

Example 1.8
Consider the following:

$$\mathcal{L}\left\{\frac{s^2+1}{s(s-1)(s-2)}\right\} = \frac{P(0)}{Q'(0)}e^{0t} + \frac{P(1)}{Q'(1)}e^{1t} + \frac{P(2)}{Q'(2)}e^{2t}.$$

Where:

$$P(s) = s^2+1 \quad Q(s) = s^3-3s^2+2s \quad Q'(s) = 3s^2-6s+2.$$

Thus:

$$\mathcal{L}\left\{\frac{s^2+1}{s(s-1)(s-2)}\right\} = \frac{1}{2} - 2e^t + \frac{5}{2}e^{2t}.$$

1.2 Convolution Product

Definition 1.9. Given two functions $f(t)$ and $g(t)$, their **convolution product** is:

$$(f*g)(t) = \int_0^t f(\beta)g(t-\beta)\ d\beta.$$

Example 1.10
Let us consider $f(t) = t^2$, $g(t) = t$, we have:

$$(f*g)(t) = \int_0^t \beta^2(t-\beta)\ d\beta = \int_0^t \left(\beta^2 t - \beta^3\right)\ d\beta$$

$$= \frac{1}{3}\beta^3 t - \frac{1}{4}\beta^4\Big|_{\beta=0}^{\beta=t} = \frac{1}{3}t^4 - \frac{1}{4}t^4 = \frac{1}{12}t^4.$$

Example 1.11

If we have $f(t) = \sin(t)$ and $g(t) = \cos(t)$, we have:

$$(f * g)(t) = \int_b^t \sin\beta\cos(t-\beta)\ d\beta.$$

Using the trig identity:

$$\sin A\cos B = \frac{\sin(A+B)+\sin(A-B)}{2}.$$

Thus we have:

$$(f * g)(t) = \int_0^t \frac{\sin(t)+\sin(2\beta-t)}{2}\ d\beta = \frac{1}{2}\beta\sin t - \frac{1}{4}\cos(2\beta-t)\Big|_{\beta=0}^{\beta=t} = \frac{1}{2}t\sin t.$$

Theorem 1.12 (Properties of Convultion Product)

The convolution product is:

- Distributive $f*(g+h) = f*g+f*h$
- Commutative $f*g = g*f$
- Associative $f*(g*h) = (r*g)\cdot h$

Theorem 1.13 (Laplace Transform of Convolution Product)

$$\mathcal{L}\{f*g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}.$$

Example 1.14

We have:

$$\mathcal{L}\{\sin t * \cos t\} = \mathcal{L}\{\sin t\}\mathcal{L}\{\cos t\} = \frac{1}{s^2+1}\frac{s}{s^2+1} = \frac{s}{(s^2+1)^2}.$$

From the other example, we found it to be $(\sin t * \cos t) = \frac{1}{2}t\sin t$, here we can check as:

$$\mathcal{L}\{\frac{1}{2}t\sin t\} = -\frac{1}{2}\frac{d}{ds}\mathcal{L}\{\sin t\} = -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{s}{(s^2+1)^2}.$$

We can use the convolution product to get inverse Laplace transform, as:

Theorem 1.15

$$\mathcal{L}\{F(s)G(s)\} = \mathcal{L}^{-1}\{F(s)\} * \mathcal{L}^{-1}\{G(s)\}.$$

Example 1.16

Consider:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}.$$

Remark 1.17 —

$$\mathcal{L}\{\int_0^t f(\beta)\, d\beta\} = \mathcal{L}\{f * 1\} = \frac{1}{s}\mathcal{L}\{f\}.$$