1 February 8th, 2022

1.1 Overview of the Course

Considering $S_n = \sum_{i=1}^{n} X$, for regular CLT we assume that we have:

- Second moment condition
- In $\mathbb{R}$
- Independence

In this course, we will extend CLT to remove these three conditions.

1.1.1 Stable Law

First, we will remove the second moment condition, leading to the **stable law, infinitely divisible distribution**.

If the second moment exists, then there is a normalization that allows the limiting distribution go to a gaussian distribution. However, if we don't have the second moment, it depends on the tail of the distribution, giving us a class of distributions. We will be able to show that this No matter if it is a triangular array or a sequence of random variable.

This part will take 3-4 lectures.

1.1.2 Functional Limiting Theorem

In the previous case, we only concerned variables in $\mathbb{R}$. Now we will extend them to get the weak convergence of random functions.

This can be useful for looking at empirical distributions in statistics. For example if we have:

$$X \sim F, \text{ with } F \text{ unknown}$$

If $X_1, \ldots, X_n$ F i.i.d., we can use the empirical distribution:

$$F_n(t) = \frac{1}{n} \sum_{i=1}^{n} \not\Vdash(X_i \leq t)$$

We want to compare this distribution with the original distribution, which we can use the Kologoromov Statistics

$$\sqrt{n} \sup_t |F_n(t) - F(t)|$$

Note that $\sqrt{n}(F_n(t) - F(t))$ is a random function with $f \in \mathbb{R}$.

To do this, we need to prove the weak convergence of the whole stochastic process. With that, we would have the weak convergence of the random function. Later we will show that $\sqrt{n}(F_n(t) - F(t))$ converges to the Brownian bridge.

Reference:

- Convergence of Probability Measure $\rightarrow$ Billingsley Chapter 2

This part will also be quite short.

1.1.3 Martingale and it's Limiting Theorem

Roughly speaking, a martingale can be thought of the sum of a random variable. We do not need independence. Here we will introduce martingale differences, and this part will take up the majority of the course.

Reference:

- Durrett Chapter 5
- Hall and Heyde → Martingale Limit Theory and its Application

1.1.4 Concentration (if time permits)

Reference:

- R. Vershynin → High-dimensional probability

1.2 Heavy Tail Limiting (Poisson) Convergence

let $N(s,t)$ be the number of arrivals at a bank during $[s,t]$. Suppose:

(i) The number in disjoint intervals are independent

(ii) The distribution of $N(s,t)$ only depends on $t-s$

(iii) $\Pr(N(0,h)=1)=\lambda h+o(h)$, and

(iv) $\Pr(N(0,h)\geq 2)=o(h)$

Theorem 1.1
If (i) - (iv) hold, then $N(0,t)$ has a poisson distribution with mean λt.

Definition 1.2 (Poisson process with rate λ)**.** A family of random variables $N_t, t\geq 0$, satisfying:

1. If $0 =$

1.3 Stable Law

We have:

$$X_1, X_2, \ldots X_n \text{ i.i.d.} \quad S_n=\sum_{i=1}^{n} X_i$$

If $\mathbf{E}X_i=\mu$ and $\mathbf{Var}X_i=\sigma^2$, we have:

$$\frac{S_n-n\mu}{\sqrt{n}\sigma} \implies N(0,1)$$

Now, if $\mathbf{E}X_i^2=\infty$, do we have a_n, b_n, Y s.t.:

$$\frac{S_n-b_n}{a_n} \implies Y \quad (Y \text{ nondegenerate})$$

Let us start with a simple case where everything about X_i is known.

Example 1.3

Consider $X_1, X_2, \ldots$ i.i.d.

$$\Pr(X_1 > x) = \Pr(X_1 < -x) = \frac{x^{-\alpha}}{2},\ \text{for } x \geq_1, 0 < \alpha < 2$$

Density $f(x) = \alpha\frac{|x|^{-\alpha-1}}{2}$, $|x| > 1$ Note that this is:

- symmetric (indicates $b_n = 0$)
- $\mathbf{E}X_1^2 = 2\int\limits_1^\infty x\Pr(|x_1| > x)dx = \int\limits_1^\infty x^{-\alpha+1}dx = \infty$

The solution is:

$$\mathbf{E}[e^{isS_n}] = \left[\underbrace{\mathbf{E}e^{isX_1}}_{\phi(s)}\right]^n = [1 - (1 - \phi(s))]^n$$

$$1 - \phi(s) = \int\limits_1^\infty (1 - e^{ist})\frac{\alpha}{2|x|^{\alpha+1}}dx + \int\limits_{-\infty}^{-1}(1 - e^{isx})\frac{\alpha}{2|x|^{\alpha+1}}dx$$