1 February 21st, 2020

1.1 Midterm Notice

- Will be held on Wednesday, March 4th.
- Closed book and notes
- 1.86x11in page of notes and a calculator

1.2 Problem 1

Consider

$$xy''(x) - (1+2x)y'(x) + (1+x)y(x) = a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0.$$

Note that $a_0(x) + a_1(x) + a_2(x) = x - 1 + 2x + 1 + x = 0$, thus, $y(x) = e^x$ is one solution. Using Abel's formula, we have:

$$y_2(x) = y_1(x) \int \frac{1}{y_1(x)^2} e^{-\int (x)\ dx} dx.$$

Where:

$$-P(x) = -\frac{a_1(x)}{a_2(x)} = \frac{1+2x}{x} = 2 + \frac{1}{x}.$$

Thus:

$$y_2(x) = e^x \int \frac{xe^{2x}}{e^{2x}}\ dx = \frac{1}{2}x^2e^x.$$

Thus:

$$y(x) = C_1e^x + C_2x^2e^x.$$

1.3 Problem 2

For this problem, we have:

$$r^2\phi'' + 2r\phi' - n(n+1)\phi = 0.$$

Note that this is equidimensional. Here we can guess $\phi(r) = r^\alpha$.

1.4 Problem 3

Consider:

$$y'' - 2xy' + (x^2 - 1)y = 0.$$

Here we guess $y(x) = e^{ax^2}$ for some a (should be given by the problem). Thus we have:

$$y'(x) = 2axe^{ax^2} \quad y''(x) = (2a + 4a^2x^2)e^{ax^2}.$$

Plug into the equation, we get:

$$(2a + 4a^2x^2)e^{ax^2} - 2x(2ax)e^{ax^2} + (x^2 - 1)e^{ax^2} = 0, \quad \forall x.$$

$$\implies 2a + 4a^2x^2 - 4ax^2 + x^2 - 1 = 0$$

$$\implies (2a-1) + (2a-1)^2x^2 = 0 \implies a = \frac{1}{2}.$$

Giving us $y_1(x) = e^{\frac{1}{2}x^2}$. Thus:

$$P(x) = \frac{a_1(x)}{a_2(x)} = -2x \implies -\int P(x)\ dx = \int 2x\ dx = x^2.$$

$$\implies y_2(x) = e^{\frac{1}{2}x^2} \int \frac{e^{x^2}}{e^{x^2}} \implies y_2(x) = xe^{\frac{1}{2}x^2}.$$

Thus the complete solution is:

$$y(x) = C_1 e^{\frac{1}{2}x^2} + C_2 x e^{\frac{1}{2}x^2}.$$

1.5 Problem 4

Consider:

$$y'' - 2xy' + (x^2-1)y = x^2 - 1.$$

Note that y_h is the equation from before, so we need to find y_p. Note that $\frac{b(x)}{a_0(x)} = 1$ which is a constant, thus $y_p = 1$a is a solution. Thus, the complete solution is:

$$y(x) = C_1 e^{\frac{1}{2}x^2} + C_2 x e^{\frac{1}{2}x^2} + 1.$$

1.6 Problem 5

Consider the circuit with a battery, a resister and a coil. We have:

$$v(t) - RI(t) - L\frac{dI(t)}{dt} = 0.$$

Taking the Laplace of both sides, we get:

$$\mathcal{L}\{L\frac{dI}{dt} + RI\} = \mathcal{L}\{V\} \implies L\mathcal{L}\{\frac{dI}{dt}\} + R\mathcal{L}\{I\} = \mathcal{L}\{V\}$$

Since $\mathcal{L}\{y'(x)\} = s\mathcal{L}\{y(x)\} - y(0)$

$$\implies Ls\mathcal{L}\{I\} - LI_0 + R\mathcal{L}\{I\} = \mathcal{L}\{V\}$$

$$\implies \mathcal{L}\{I\} = \frac{LI_0 + \mathcal{L}\{V\}}{SL + R}$$

$$\implies I = \mathcal{L}^{-1}\left\{\frac{LI_0 + \mathcal{L}\{V\}}{SL+R}\right\} = I_0\mathcal{L}^{-1}\left\{\frac{1}{s + \frac{R}{L}}\right\} + V.$$

Note that: $\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$. Thus:

$$I(t) = I_0 = e^{-\frac{R}{L}t} + V.$$

Let:

$$V(t) = \begin{cases} V_0, & 0 < t < a \\ 0, & a \le t \end{cases}, \quad V_0 \in \mathbb{R} = V_0 - V_0 u(t - a_0.$$

Thus:

$$\mathcal{L}\{V(t)\} = \mathcal{L}\{V_0 - V_0 u(t-a\} = V_0\mathcal{L}\{1 - u(t-a)\}.$$
$$= V_0\mathcal{L}\{1\} - V_0\mathcal{L}\{u(t-a)\} = \frac{V_0}{s}(1-e^{-as}).$$

Thus we have:

$$V(t) = \mathcal{L}^{-1}\left\{\frac{V_0(1-e^{-as})}{s(sL+R)}\right\}.$$

Using the fact that $\mathcal{L}^{-1}\{F(s)e^{-as}\} = u(t-a)\mathcal{L}\{F(x)\}$, we have:

$$= V_0\mathcal{L}^{-1}\left\{\frac{V_0}{s(sL+R)}\right\} - u(t-a)\mathcal{L}\left\{\frac{V_0}{s(sL+r)}\right\}\Bigg|_{t\to t-a}.$$

Using partial fraction, we'd get:

$$= V_0\mathcal{L}^{-1}\left\{\frac{1}{RS} - \frac{L}{R(SL+R)}\right\} = \frac{V_0}{R}\mathcal{L}\left\{\frac{1}{s} - \frac{1}{s+\frac{R}{L}}\right\} = \frac{V_0}{R}(1-e^{-\frac{R}{L}t}).$$

Thus in the end, we get:

$$I(t) = I_0 e^{-\frac{R}{L}t} + \frac{V_0}{R}\left(1 - e^{-\frac{R}{L}t} - u(t-a)\frac{V_0}{R}\left(1 - e^{-\frac{R}{L}(t-a)}\right)\right).$$