1 February 18th, 2021

1.1 Spectral Radius Cont.

Corollary 1.1

Let $A \in \mathbb{R}^{n\times n}$. Then:

$$\lim_{k\to\infty} A^k = 0 \iff \rho(A) < 1$$

Proof.

- " $\implies$ " Assume $\lim_{k\to\infty} A^k = 0$. Let λ be the eigenvalue of A s.t. $\rho(A) = |\lambda|$. For any k,, then A^k is an eigenvalue of A^k. We have:

$$(\rho(A))^k = |\lambda|^k = |\lambda^k| \leq \rho(A^k) \leq \|A^k\|$$

for any operator norm. Thus:

$$\lim_{k\to\infty}(\rho(A))^k \leq \lim_{k\to\infty}\|A^k\| = 0 \implies \rho(A) < 1$$

- " $\impliedby$ " Assume $\rho(A) < 1$. Choose $\epsilon = \frac{1}{2}(1-\rho(A)) > 0$. Thus there exists $\|\cdot\|_\epsilon$ s.t.:

$$\|A\|_\epsilon \leq \rho(A) + \epsilon = \rho(A) + \frac{1}{2}(1-\rho(A)) = \frac{1}{2} + \frac{1}{2}\rho(A) < 1$$

Then:

$$\|A^k\|_\epsilon \leq (\|A\|_\epsilon)^k \to 0 \text{ as } k \to \infty \implies \lim_{kn\to\infty}\|A^k\|_\epsilon = 0$$

Since norms are continuous functions for finite dimension, we have:

$$\|\lim_{k\to\infty} A^k\| = 0 \to \lim_{k\to\infty} A^k = 0$$

□

1.2 Convergence of Jacobi Iteration

Recall that the Jacobi iteration can be written in the stationary iteration form:

$$x_{k+1} = Gx_k + f$$

where $G = I - D^{-1}A$, $f = D^{-1}b$. Let x_* be the solution of $Ax = b$. i.e. ($Ax_* = b$). Then:

$$\begin{aligned} Dx_* - b &= (D-A)x_* \\ Dx_* &= (D-A)x_* + b \\ x_* &= D^{-1}(D-A)x_* + D^{-1}b \\ x_* &= Gx_* + f. \end{aligned}$$

Taking the difference, we have:

$$(x_{k+1} - x_*) = G(x_k - x_*)$$

Now, taking the norms on both sides, we have:

$$\begin{aligned}\|x_{k+1} - x_*\| &= \|G(x_k - x_*)\| \\ &\leq \|G\|\|x_k - x_*\|.\end{aligned}$$

If $\rho(G) < 1$, then we can choose $\epsilon = \frac{1}{2}(1 - \rho(G))$ and construct the norm $\|\cdot\|$ (depending on G) s.t.

$$\|G\|_\epsilon \leq \rho(G) + \epsilon = \frac{1}{2} + \frac{1}{2}\rho(G) < 1$$

Then:

$$\|x_{k+1} - x_*\|_\epsilon \leq \|G\|_\epsilon \|x_k - x_*\|_\epsilon = \rho\|x_k - x_*\|_\epsilon, \quad \forall k.$$

As a result, we have:

$$\|x_k - x_*\|_\epsilon \leq \rho^k \|x_0 - x_*\|_\epsilon \to 0 \text{ as } k \to +\infty,$$

i.e. $x_k \to x_*$.

In addition, the convergence rate is “linear”, because:

$$\frac{\|x_k - x_*\|}{\|x_{k-1} - x_*\|} \leq \rho < 1$$

In order to obtain a $\tilde{\epsilon}$-precision solution, i.e.:

$$\begin{aligned}&\|x_k - x_*\|_\epsilon \leq \rho^k \|x_0 - x_*\|_\epsilon < \tilde{\epsilon} \\ \iff\ & \rho^k \leq \frac{\tilde{\epsilon}}{\|x_0 - x_*\|_\epsilon} \\ \iff\ & k \geq \frac{\log\left(\frac{\|x_0 - x_*\|_\epsilon}{\tilde{\epsilon}}\right)}{\log(1/\rho)} \sim O(1/\log \rho^{-1}).\end{aligned}$$

Remark 1.2 — Note that ρ can be arbitrarily close to $\rho(G)$. Thus, ρ is called the **convergence factor**.

Remark 1.3 — If $\rho \approx \rho(G) = 1 - O(1/n^\alpha)$, where $\alpha > 0$, then:

$$\log \rho^{-1} \sim O(n^\alpha)$$

meaning we require $k \sim O(n^\alpha \cdot \log \tilde{\epsilon}^{-1})$. Usually $\tilde{\epsilon}^{-1}$ can be treated as a constant.

Corollary 1.4

Jacobi converges to x_* for any x_0 if and only if $\rho(G) < 1$.

1.3 Computation Cost of Jacobi Iteration

Note that the Jacobi iteration only uses matrix-vector product (and $O(n)$ operations for calculating D^{-1}). Thus:

$$\text{computational cost per step:} \begin{cases} O(n^2) & \text{for general } A \\ O(m+n) & \text{for sparse } A \text{ with } m \text{ non-zero entries} \end{cases}$$

Thus the total computational cost is:

$$O(m+n) \times O(n^\alpha \cdot \log \tilde{\epsilon}^{-1}) = O((m+n)n^\alpha \cdot \log \tilde{\epsilon}^{-1})$$

1.4 Jacobi Iteration for 1D Discrete Laplacian

Recall that the Laplacian equation in 1D is:

$$\begin{cases} -u_{xx} = f & x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

Using central difference, we have $Ax = b$, where:

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}$$

We have $Au = \lambda u$, i.e.

$$\begin{cases} -u_{j-1} + 2u_j - u_{j+1} = \lambda u_j & j = 1, \ldots, n \\ u_0 = u_{n+1} = 0 \end{cases}$$

Recall that this is a discrete difference eq. whose solutions are given by:

$$u_j = c_1 \alpha_1^j + c_2 \alpha_2^j$$

where c_1, c_2 are constants, α_1, α_2 are roots of

$$-1 + 2\alpha - \alpha^2 = \lambda \alpha.$$

i.e. $\alpha_1 + \alpha_2 = 2 - \lambda$ and $\alpha_1 \alpha_2 = 1$. Because $u_0 = u_{n+1} = 0$, we have:

$$\begin{cases} c_1 + c_2 = u_0 = 0 \\ c_1 \alpha_1^{n+1} + c_2 \alpha_2^{n+1} = u_{n+1} = 0 \end{cases}.$$

$$\begin{aligned} \det\left(\begin{bmatrix} 1 & 1 \\ \alpha_1^{n+1} & \alpha_2^{n+1} \end{bmatrix}\right) = 0 &\iff \alpha_1^{n+1} = \alpha_2^{n+1} \\ &\iff \left(\frac{\alpha_1}{\alpha_2}\right)^{n+1} = 1 \\ &\iff \frac{\alpha_1}{\alpha_2} = e^{i \cdot \frac{2\pi}{n+1} \cdot k}, \ k = 0, 1, \ldots, n. \end{aligned}$$

Since, $\alpha_1\alpha_2 = 1$, we have:

$$\implies 1 = \alpha_2^2 \frac{\alpha_1}{\alpha_2} = \alpha_2^2 e^{i\frac{2\pi}{n+1}k}$$
$$\implies \begin{cases} \alpha_2 &= e^{-i\frac{\pi}{n+1}k} \\ \alpha_1 &= e^{i\frac{\pi}{n+1}k} \end{cases}, k = 0, 1, .., n.$$

Thus:

$$\alpha_1 + \alpha_2 = 2 - \lambda \implies \lambda = 2 - (\alpha_1 + \alpha_2) = 2 - 2\text{Re}(e^{i\frac{\pi}{n+1}k}) = 2(1 - \cos\left(\frac{\pi}{n+1}k\right))$$

However, there are $n+1$ values of k, but there are only n eigenvalues of A. When $k = 0$, $\alpha_1 = \alpha_2 = 1$. However, in this case, we have:

$$u_j = c_1\alpha_1^j + c_2\alpha_2^j \implies c_1 + c_2 = 0$$

which is a contradiction since $c_1 + c_2 = 0 \implies u = 0$. Thus when $k = 1, 2, \ldots, n$, we can find the corresponding u (left as homework).

Thus, the eigenvalues of A are:

$$\lambda_k = 2(1 - \cos\left(\frac{\pi}{n+1}k\right)), \quad k = 1, 2, \ldots, n$$

Consequently:

$$\rho(G) = \max_{k=1,2,\ldots,n} |1 - \frac{1}{2}\lambda_k| = \max_{k=1,2,\ldots,n} |\cos\left(\frac{\pi}{n+1}k\right)| = \cos\left(\frac{\pi}{n+1}\right) < 1$$

Thus the Jacobi converges, and:

$$\|x_k - x_*\|_2 \le \|G\|_g \|x_{k-1} - x_*\|_2 = \rho(G)\|x_{k-1} - x_*\|_2$$

Thus,

$$\rho = \rho(G) = \cos\frac{\pi}{n+1} = 1 - 2\sin^2\frac{\pi}{2(n+1)} = 1 - O(\frac{1}{n^2})$$

This gives us $\alpha = 2$.

As such, the number of iteration for $\|x_k - x_*\|_2 \le \tilde{\epsilon}$ is:

$$k \sim O(n^2 \cdot \log\tilde{\epsilon}^{-1})$$

Since we only need matrix product, we only need $O(n)$ FLOPs per iteration, meaning that the total FLOPs needed is:

$$O(n^3 \cdot \log\tilde{\epsilon}^{-1})$$

As a comparison, Gaussian Elimination needs $O(n)$. Thus, the Jacobi iteration (in this version), is not efficient.