1 April 22th, 2020

1.1 Continuation of the Heat Equation

From last time, we had:

$$\tan(L\sqrt{-V}) = -\frac{k_0}{h}\sqrt{-C} = -\frac{\kappa_0}{hL}(L\sqrt{-C}).$$

Thus if we let $L\sqrt{-C} = z \implies C = -\left(\frac{z}{L}\right)^2$, then we want to find the solutions to the equation:

$$\tan z = -\frac{\kappa_0}{hL}z.$$

With this, we would get:

$$\phi_n(x) = \sin(z_n\frac{x}{L}).$$

Since this is in the RSLP form, it has an associated dot product:

$$\phi_n \cdot \phi_m = \int_0^L \phi_n(x)\phi_m(x)\ dx = \begin{cases} 0, & p \neq q \\ \frac{L}{2}\left(1 + \frac{\sin(2z_n)}{2z_n}\right) \end{cases}.$$

Note that the weight function $w(x) = 1$ (verify by comparing coefficients).

Going to the time equation:

$$\beta_n'(t) - \gamma C_n \beta_n(t) = 0.$$

$$\implies \beta_n'(t) + \gamma\left(\frac{z_n}{L}\right)^2\beta_n(t) = 0.$$

$$\implies \beta_n(t) = e^{-\gamma\left(\frac{z_n}{L}\right)^2 t}.$$

This gives us:

$$v_n(x,t) = \phi_n(x)\beta_n(t) = e^{-\gamma\left(\frac{z_n}{L}\right)^2 t}\sin\left(\frac{z_n}{L}x\right).$$

Meaning that the general solution is:

$$v(x,t) = \sum_{n=1}^{\infty} a_n e^{-\gamma\left(\frac{z)n}{L}\right)^2 t}\sin\left(\frac{z_n}{L}x\right).$$

For the initial condition, we have:

$$v(x,0) = \sum_{n=1}^{\infty} a_n\phi_n(x) = f(x) - u_e(x).$$

$$\implies a_n = \frac{\phi_n(f - u_e)}{\phi_n \cdot \phi_n} = \frac{1}{\frac{L}{2}\left(1 + \frac{\sin(2z_n)}{2z_n}\right)}\int_0^L (f(x) - u_e(x))\phi_n(x)\ dx, \quad n = 1, 2, 3, ...$$

Thus the final solution is:

$$u(x,t) = u_e(x) + \sum_{n=1}^{\infty} a_n e^{-\gamma\left(\frac{z_n}{L}\right)^2 t}\phi_n(x).$$

Note that the only difference between this solution and the wave equation is that the eigenvalues are different.

Remark 1.1 — Note that since $e^{-(\text{constant})t}$ approaches 0 as t increases, we have:

$$\lim_{t\to\infty} u(x,t) = u_e(x).$$

Note that this is not the case for the wave equation (unless there is a dampening term).

Remark 1.2 — The wave equation with damping has the form:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t}.$$

This will have a limiting solution equal to the time independent solution $u_e(x)$.