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1.1 Error Analysis of Finite Element Methods

For finite different methods, we can get the error analysis using the Taylor expansion. However, for FEM, it is a bit more complicated. Recall that the truncation error is when we truncate the infinitely dimensional solution space $H_0^1(\Omega)$ into a finite-dimensional subspace $S_N \subset H_0^1(\Omega)$. Recall that for the problem:

$$\begin{cases} -\Delta u = f \\ u\big|_{\partial\Omega} = 0 \end{cases}.$$

we have an equivalent weak formulation of finding $u \in H_0^1(\Omega)$ such that:

$$\int_\Omega \nabla u \nabla v \; dx = \int_\Omega fv, \quad \forall v \in H_0^1(\Omega).$$

Let us define a **bi-linear form**:

$$a(u,v) = \int_\Omega \nabla u \nabla v \; dx.$$

Remark 1.1 — Bi linear form means that $a(u,v)$ is linear in both u and v.

Let us also define a norm:

$$\|v\|_E = \sqrt{a(v,v)} = \left(\int_\Omega (\nabla v)^2 \; dx \right)^{\frac{1}{2}}.$$

Let us consider a finite dimensional subspace $S_N \subset H_0^1(\Omega)$, meaning that the finite element solution u_N is such function that satisfies:

$$a(u_N, v) = (f, v), \quad \forall v \in S_N.$$

where $(u,v) = \int_\Omega uv$ is the inner product. For the exact solution u, we would have:

$$a(u,v) = (f,v) \quad \forall v \in S_N.$$

$$\implies a(u - u_N, v) = 0 \quad \forall v \in S_N \subset H_0^1(\Omega).$$

Let us consider the error, i.e. the difference between u and u_N:

$$\begin{aligned} \|u - u_N\|_E^2 &= a(u - u_N, u - u_N) \\ &= a(u - u_N, u - v + v - u_N), \quad \forall v \in S_N \\ &= a(u - u_N, u - v) + a(u - u_N, v - u_N) \\ &= a(u - u_N, u - v) \qquad (\text{since } v - u_N \in S_N) \\ &\leq \|u - u_N\|_E \|u - v\|_E, \quad \forall v \in S_N \quad (\text{Cauchy Schwartz inequality}). \end{aligned}$$

As such, we have:

$$\|u - u_N\|_E \leq \|u - v\|_E \quad \forall v \in S_N.$$

This means that out of all solutions in S_N, u_N is the best approximation to the exact solution.

To estimate the error, we will use a duality argument. Let w be a solution of the dual problem:

$$\begin{cases} -\Delta w = u - u_N \\ w(0) = w(1) = 0 \end{cases}.$$

Let us consider the L^2 norm:

$$\|v`\|_{L^2} = \left(\int_\Omega |v|^2 \, dx \right)^{\frac{1}{2}}.$$

we have:

$$\begin{aligned} \|u - u_N\|_{L^2}^2 &= \int |u - u_N|^2 \, dx \\ &= (u - u_N, u - u_N) \\ &= (u - u_N, -w'') \\ &= ((u - u_N)', w') \\ &= a(u - u_N, w) \\ &= a(u - u_N, w - v + v), \quad \forall v \in S_N \\ &= (u - u_N, w - v) + a(u - u_N, v) \\ &\le \|u - u_N\|_E \|w - v\|_E. \end{aligned}$$

As such, we have:

$$\|u - u_N\|_2 \le \frac{\|u - u_N\|_E \|w - v\|_E}{\|u - u_N\|_2} = \frac{\|u - u_N\|_E \|w - v\|_E}{\|w\|_2}, \quad \forall v \in S_N.$$

If we have the approximation assumption:

$$\int_{w \in S_N} \|w - v\|_E \le \epsilon \|w''\|_2 \implies \|u - u_N\|_2 \le \epsilon \|u - u_N\|_E.$$

Remark 1.2 — This approximation assumption is that any second order differentiable function can be approximated by a function in S_N.

Applying the approximation assumption again, we have:

$$\|u - u_N\|_E \le \epsilon \|u - v\|_E \le \epsilon \|u''\|_2.$$

$$\implies \|u - u_N\|_2 \le \epsilon^2 \|u''\|_2 = \epsilon^2 \|f\|_2.$$

As such, it is second order accurate in L^2 norm.

Consider $0 = x_0 < x_1 < x_2 < \ldots < x_N = 1$ a partition of $[0, 1]$. Let S_N be the continuous piecewise linear function space with hat basis functions $\phi_1, \ldots, \phi_{N-1}$. Take any function $u \in H_0^1(\Omega)$. Let $u_I(x) = \sum_{i=1}^{N-1} u(x_i)\phi_i(x)$.

Theorem 1.3

$$h = \max_i (x_{i+1} - x_i) \implies \|u - u_I\|_E \leq ch\|u''\|_2.$$

Proof.

$$\|u - u_I\|_E^2 = \int_0^1 (u' - u_I')^2 \, dx = \sum_{j=1}^N \int_{x_{j-1}}^{x_j} (u' - u_I')^2 \, dx.$$

Let $e(x) = u(x) - u_I(0)$, with $e(x_j) = 0$. For $[x_{j-1}, x_j]$:

$$\exists \xi \in (x_{j-1}, x_j), \text{ such that } e'(\xi) = 0, e'(y) = \int_\xi^y e''(x) \, dx.$$

from the fundamental theorem of calculus. As such, we have:

$$|e'(y)| \leq \int_\xi^y |e''(x)| \, dx \leq \left(\int_\xi^y \, dx\right)^{\frac{1}{2}} \left(\int_\xi^y |e''(x)|^2 \, dx\right)^{\frac{1}{2}}.$$

$$= (y - \xi)^{\frac{1}{2}} \left(\int_\xi^y |e''(x)|^2 \, dx\right)^{\frac{1}{2}} \leq (y - \xi)^{\frac{1}{2}} \left(\int_{x_{j-1}}^{x_j} |e''(x)|^2 \, dx\right)^{\frac{1}{2}}.$$

Thus, we have:

$$\begin{aligned}
\int_{x_{j-1}}^{x_j} |e'(y)| \, dy &\leq \int_{x_{j-1}}^{x_j} (y - \xi)) \left(\int_{x_{j-1}}^{x_j} |e''(x)|^2 \, dx\right) \, dy \\
&= \frac{1}{2}(y - \xi)^2 \Big|_{x_{j-1}}^{x_j} \int_{x_{j-1}}^{x_j} |e''(x)|^2 \, dx \\
&\leq c(x_j - x_{j-1})^2 \int_{x_{j-1}}^{x_j} |e''(y)|^2 \, dx \quad \text{for some constant } c \\
&\leq ch^2 \int_{x_{j-1}}^{x_j} (u'')^2 \, dx \quad \text{since } e'' = u''.
\end{aligned}$$

Thus, we have:

$$\left(\int_0^1 (u' - u_I')^2 \, dy\right)^{\frac{1}{2}} \leq ch \left(\int_0^1 (u'')^2 \, dx\right)^{\frac{1}{2}}.$$

$$\implies \|u - u_I\|_E \leq ch\|u''\|_2.$$

If we choose $v = w_I \in S_N$ to be the interpolation function of w, we would have:

$$\|w - v\|_E = \|w - w_I\|_E \leq ch\|w''\|_2.$$

□

As such, from before, we would have:

$$\|u - u_N\|_2 \leq ch\|u - u_N\|_E \leq ch^2\|u''\|_2 = ch^2\|f\|_2.$$

As such, this finite element scheme is second order accurate.

Remark 1.4 — We can increase this accuracy by choosing different basis functions, e.g. piecewise quadratic functions.