1 September 8th, 2020

1.1 Introduction

1.1.1 Numerical Differentiation

Recall that the derivative is defined as:

$$u(x_0) = \lim_{h \to 0} \frac{u(x_0 + h) - u(x_0)}{h}.$$

Thus we can approximate the difference

Definition 1.1. Central Difference

$$u_x(x_i) = \frac{u_{i+1} - u_{i-1}}{2h}.$$

Theorem 1.2

The central difference method is second order accurate.

Proof. Using Taylor Expansion, we have:

$$u_{i+1} = u_i + u_x h + \frac{1}{2} u_{xx}.$$

□

Remark 1.3 — Note that if the derivative is

1.1.2 Numerical Integration

Numerical Integration can be approximation by the **Riemann Sum**

$$\int_a^b f(x)\ dx \approx \sum_{i=1}^{N} f(x_i) \Delta x_i.$$

We can also use the **Trapezoidal Rule**, as:

$$\int_{x_i}^{x_{i+1}} \approx \frac{h}{2} \left[f(x_i) + f(x_{i+1}) \right].$$

Remark 1.4 — The above equation is approximating the curve in interval $[x_i, x_{i+1}]$ with a straight line

Thus if we add up over all intevals x_i, we'd get:

$$\int_a^b f(x)\ dx \approx \sum_{i=0}^{N-1} \frac{h}{2} \left[f(x_i) + f(x_{i+1}) \right].$$

This is **second order accurate**.

1.1.3 Simpson's Rule

1.2 Solution of ODE's

Say we have an ODE:

$$\begin{cases} \dot{x} = f(x,y) \\ x(0) = x_0 \end{cases}.$$

One simple approximation is the **Euler method**:

$$\dot{x} \approx = \frac{x^{n+1} - x^n}{\Delta t} = f(x^n, t_n).$$

with time step Δt and:

$$x^{n+1} = x^n + \Delta t f(x^n, t_n).$$

This is a first order accurate scheme.

> **Remark 1.5** — There are some disadvantages to this method, as Δt must be chosen carefully to be stable.

The Euler method can be modified to make higher order methods, such as the **Modified Euler**

$$\begin{cases} x^* = x^n + \Delta t f(x^n, t_n) \\ x^{n+1} = x^n + \frac{\Delta t}{2} \left(f(x^n, t_n) \right) + f(x^*, t_{n+1}) \end{cases}.$$

Essentially, x^* is a predictor of the This is second order accurate.

> **Remark 1.6** — There is also a **Runge-Katta Method**, which is a 4th order method.

1.2.1 Solving Linear System

We have a linear system:

$$Ax = b.$$

We can solve this with a variety of methods, such as

- Guassian Eliminations
- LU factorization
- Iterative methods