1 March 6th, 2020

1.1 Problem 1

Recall that if $x = x_0$ is an ordinary point, we have:

$$y(x) = \sum_{m=0}^{\infty} a_n (x - x_0)^n.$$

Consider the equation:

$$(1 - x^2)y''(x) + 4xy'(x) + 14y(x) = 0.$$
$$\implies y'' + \frac{4x}{1 - x^2}y' + \frac{14}{1 - x^2}y = 0.$$

We know that an ordinary point is one where this equation doesn't blow up, i.e. any point where $x^2 \neq 1$. Let's choose $x_0 = 0$, meaning that the solution would be in the form:

$$y = \sum_{nk=0}^{\infty} a_k x^k.$$

Now what's left to do is to solve for a_k. Plugging into the original equation, we have:

$$(1 - x^2)\left[\sum_{k=0}^{\infty} k(k-1)a_k x^{k-2}\right] + 4x\left[\sum_{k=0}^{\infty} k a_k x^{k-1}\right] + 14\left[\sum_{k=0}^{\infty} a_k x^k\right] = 0.$$

Now let's collect the terms:

$$\implies \sum_{k=0}^{\infty} k(k-1)a_k x^{k-2} - \sum_{k=0}^{\infty} k(k-1)a_k x^k + 4\sum_{k=0}^{\infty} k a_k x^k + 14\sum_{k=0}^{\infty} a_k x^k = 0.$$
$$\implies \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k - \sum_{k=0}^{\infty} k(k-1)a_k x^k + 4\sum_{k=0}^{\infty} k a_k x^k + 14\sum_{k=0}^{\infty} a_k x^k = 0.$$
$$\implies \sum_{k=0}^{\infty}\left[(k+2)(k+1)a_{k+2} + \underbrace{(-k(k-1) + 4k + 14)}_{-(k-7)(k+2)} a_k\right] x^k = 0.$$

Thus:

$$a_{k+2} = \frac{k-7}{k+1}a_k.$$

Note that the odd series truncate when $k = 7$, the numerator of the right hand side equals, thus meaning $a_9 = a_{11} = \ldots = 0$, meaning that we only need to consider a_3, a_5, a_7 for the series of the odd terms. We have:

$$a_3 = \frac{1-7}{1+1}a_1 = -3a_1.$$
$$a_5 = \frac{3-7}{3+1}a_3 = -a_3 = 3a_1.$$
$$a_7 = \frac{5-7}{5+1}a_5 = -a_1.$$

Thus the first solution is:

$$y_1(x) = a_1 x + a_3 x^3 + a_5 x^5 + a_7 x^7 = a_1\left(x - 3x^3 + 3x^5 - x^7\right).$$

With this, we would find a_1 from some initial condition, and then we can use Abel's equation to solve for y_2.

1.2 Problem 2

Now consider the equation:

$$y'' - \alpha x y' + \beta y = 0, \quad \alpha, \beta \in \mathbb{R} > 0.$$

To apply Taylor's method, we first want to find an ordinary point to expand on. Note that the equation is defined for all x, thus we can choose any point to expand on. For simplicity, we should choose $x_0 = 0$, meaning that we have:

$$\sum_{k=0}^{\infty} k(k-1)a_k k^{k-2} - \alpha x \sum_{k=0}^{\infty} k a_k x^{k-1} + \beta \sum_{k=0}^{\infty} a_k x^k = 0.$$

Now we play with the indicies again to get:

$$\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} - \alpha \sum_{k=0}^{\infty} k a_k x^k + \beta \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\implies \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k - \alpha \sum_{k=0}^{\infty} k a_k x^k + \beta \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\implies \sum_{k=0}^{\infty} \left[(k+2)(k+1)a_{k+2} - (\alpha k - \beta)a_k\right] x^k = 0.$$

$$\implies a_{k+2} = \frac{\alpha k - \beta}{(k+2)(k+1)} a_k.$$

Note that this is polynomial if $\alpha k = \beta$ for some:

$$k = \frac{\beta}{\alpha} = \mathbb{Z}^+.$$

For example:

$$y'' - 5xy' + 25y = 0, \quad a_{k+2} = \frac{5(k-5)}{(k+2)(k+1)} a_k.$$

meaning that $a_7 = a_9 = \ldots = 0$, giving us:

$$a_3 = \frac{5(1-5)}{3 \cdot 2} a_1 = -\frac{10}{3} a_1.$$

$$a_5 = \frac{5(3-5)}{5 \cdot 4} a_3 = -\frac{1}{2} a_3 = \frac{5}{3} a_1.$$

Thus, we have:

$$y_1(x) = a_1 x + a_3 x^3 + a_5 x^5 = a_1 \left(x - \frac{10}{3} x^3 + \frac{5}{3} x^5\right).$$

And we can use Abel's equation to solve for y_2.