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1.1 Circulant Preconditioners

Definition 1.1. Let

$$S = \begin{bmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \in \mathbb{R}^{n\times n} \text{ be a shift matrix , i.e. } S\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_n \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

Then a matrix C is called a **circulant matrix** if:

$$C = \begin{bmatrix} c & Sc & S^2 & \dots & S^{n-1}c \end{bmatrix} = \begin{bmatrix} c_0 & c_{n-1} & \dots & c_1 \\ c_1 & c_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \dots & c_0 \end{bmatrix} \text{ for some } c = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}$$

Theorem 1.2

$C = F^*\Lambda F$, where $F = \frac{1}{\sqrt{n}}[\omega^{jk}]_{j=0,k=0}^{n-1,n-1}$ is the discrete Fourier transform, and:

$$\Lambda = \operatorname{diag}(Fc)$$

Then:

$$Cx = F^*\Lambda Fx = F^*(Fc \circ Fx)$$

Proof. Consider the k-th column of F^*, $k = 0, 1, \dots, n-1$:

$$f_k = \frac{1}{\sqrt{n}}[\overline{\omega}^{jk}]_{j=0}^{n-1} = \frac{1}{\sqrt{n}}[\omega^{-jk}]_{j=0}^{n-1}$$

Then:

$$\begin{aligned} [Cf_k]_\ell &= e_\ell^* C f_k \\ &= e_\ell^*\left(\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\omega^{-jk}S^j c\right) \\ &= \frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\omega^{-jk}(e_\ell^* S^j c) \\ &= \frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\omega^{-jk}C_{(\ell-1-j) \bmod n}. \end{aligned}$$

Let $\tilde{j} = (\ell - 1 - j)$ mod n, i.e. $j = (\ell - 1 - \tilde{j})$ mod n we have:

$$\begin{aligned}
[Cf_k]_\ell &= \frac{1}{\sqrt{n}} \sum_{\tilde{j}=0}^{n-1} C_{\tilde{j}} \omega^{(\tilde{j}+1-\ell)k} \\
&= \frac{1}{\sqrt{n}} \left(\sum_{\tilde{j}=0}^{n-1} C_{\tilde{j}} \omega^{\tilde{j}k} \right) \omega^{-(\ell-1)k} \\
&= (Fc)_k e_\ell^* f_k = (Fc)_k \cdot (f_k)_\ell.
\end{aligned}$$

Therefore:

$$Cf_k = (Fc)_k \cdot f_k$$

giving us:

$$CF^* = F^* \Lambda$$

Finally:

$$C = F^* \Lambda F$$

□

With this theorem, now solving:

$$Cd = r \iff F^* \Lambda F d = r \iff d = F^* \Lambda^{-1} F r \iff d = F^*(Fr/Fc)$$

Where Fr/Fc is entrywise division.

Thus, we:

- Compute Fr and Fc by FFT in $O(n \log n)$.
- Compute Fr/Fc in $O(n)$
- Compute $F^*(Fr/Fc)$ by Inverse FFT in $O(n \log n)$

Thus the total computational cost is $O(n \log n)$.

1.1.1 Circulant Preconditioner for $1 - D$ Discrete Laplacian

Consider:

$$A = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & -1 & \ddots & -1 \\ & & -1 & 2 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

This is almost circulant. Thus, let is consider:

$$\begin{bmatrix} 2 & -1 & & -1 \\ -1 & 2 & \ddots & \\ & -1 & \ddots & -1 \\ -1 & & -1 & 2 \end{bmatrix} + \alpha I$$

where α is small. Then P is SPD and circulant. We use it as the preconditioner for CG for $Ax = b$.

For the convergence, we need to get the eigenvalues of $P^{-1}A$. We expect that most of the eigenvalues are clustered around 1.

Since:

$$A = P - \alpha I + E$$

where $E = \begin{bmatrix} 0 & \cdots & \cdots & 1 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 1 & \cdots & \cdots & 0 \end{bmatrix}$, we have:

$$A^{-1}P = A^{-1}(A + \alpha I - E) = (I + \alpha A^{-1}) - A^{-1}E$$

whose eigenvalues are the same as:

$$\underbrace{(I + \alpha A^{-1})}_{B} - \underbrace{A^{-\frac{1}{2}} E A^{-\frac{1}{2}}}_{L}$$

- Let $\lambda_i(B)$ be eigenvalues of $B = I + \alpha A^{-1}$, then:

$$1 \le \lambda_1(B) \le \lambda_n(B) \le 3$$

Proof. By direct calculation, we have:

$$\lambda_j(A) = 2\left(1 - \cos\frac{j\pi}{n+1}\right), \quad j = 1, \dots, n$$

So the eigenvalues of B are

$$\lambda_j(B) = 1 + \alpha\left(1 - \cos\frac{j\pi}{n+1}\right)^{-1}, \quad j = 1, \dots, n$$

We have:

$$\begin{aligned} \lambda_1(B) &\ge 1 \\ \lambda_n(B) &\le 1 + \alpha\left(1 - \cos\frac{n\pi}{n+1}\right)^{-1} \\ &\le 1 + \alpha \cdot O(n^2) \\ &\le 3 \quad \text{by choosing } \alpha = O(n^{-2}). \end{aligned}$$

□

> **Remark 1.3** — Instead of 3, we can choose any number arbitrarily close to 1 by tuning alpha.

- Estimate eigenvalues of B. We have: $D = B - A$. Clearly

$$\operatorname{rank}(L) = 2$$

as $\operatorname{rank}(E) = 2$ and A is a full rank matrix.

Theorem 1.4 (Cauchy Interlacing Theorem)
Let $N \in \mathbb{R}^{n\times n}$, $M \in \mathbb{R}^{m\times m}$ with $m \leq n$ be two symmetric matrices satisfying:

$$M = P^T N P,$$

where $P \in \mathbb{R}^{n\times m}$ is a matrix satisfying $P^T P = I$, i.e. P is orthogonal. Then:

$$\lambda_j(N) \leq \lambda_j(M) \leq \lambda_{n-m+j}(N), \quad \forall j = 1, \dots, m$$

where $\lambda_j(\cdot)$ is the j-th smallest eigenvalue.

Proof. Will be covered in the eigenvalue decomposition chapter. □

Remark 1.5 — Essentially, the Cauchy Interlacing Theorem says that:

- the j-th smallest eigenvalue of N is smaller than that of M
- the j-th largest eigenvalue of N is larger than that of M (by letting $j = m - j$)

Choose P to be the orthogonal basis of $\ker(L)$. Thus:

$$P \in \mathbb{R}^{n\times(n-2)}$$

Consider $P^T BP = P^T DP$. Applying the Cauchy Interlacing Theorem, we have:

$$\begin{aligned}
&\lambda_1(B) \leq \lambda_1(P^T BP) = \lambda_1(P^T DP) \leq \lambda_3(B) \\
&\lambda_{n-2}(D) \leq \lambda_{n-2}(P^T DP) = \lambda_{n-2}(P^T BP) \leq \lambda_n(B) \\
\implies &1 \leq \lambda_1(B) \leq \lambda_3(D) \leq \lambda_{n-2}(D) \leq \lambda_n(B) \leq 3.
\end{aligned}$$

Since $\lambda_i(D) = \lambda_i(A^{-1}P)$, as D is similar to $A^{-1}P$, we have:

$$1 \leq \lambda_3(A^{-1}P) \leq \lambda_{n-2}(A^{-1}P) \leq 3$$

This means that most of the eigenvalues of $A^{-1}P$ are between 1 and 3.

- Since eigenvalues of $P^{-1}A$ are the inverse of the eigenvalue of $A^{-1}P$ (since they are inverse to each other) and all eigenvalues are positive (since they are the product of SPD matrices), we have $\lambda_i(P^{-1}A) = (\lambda_{n-i+1}(A^{-1}P))^{-1}$, giving us:

$$\frac{1}{3} \leq \lambda_3(P^{-1}A) \leq \lambda_{n-2}(P^{-1}A) \leq 1$$

In other words, except $\lambda_1, \lambda_2, \lambda_{n-1}, \lambda_n$, all eigenvalues of $P^{-1}A$ are in $[\frac{1}{3}, 1]$.

- It remains to estimate $\lambda_1(P^{-1}A)$. We have:

$$\lambda_n(A^{-1}P) = \|A^{-1}P\|_2 \leq \|D\|_2 = \|B - L\|_2 \leq \|B\|_2 + \|L\|_2$$

Because $\|B\|_2 \leq 3$ and $\|L\|_2 = \|A^{-1}E\|_2 \leq \|A^{-1}\|_2\|E\|_2$. Note that the smallest eigenvalue of $A^{-1} = O(n^{-2})$, meaning $\|A^{-1}\|_2 = O(n^2)$. In addition, $\|E\|_2 \leq \|E\|_F = 2$. Thus, $\|L\|_2 \leq Cn^2$. As such, we have:

$$
\begin{aligned}
&\lambda_n(A^{-1}P) \leq Cn^2 \text{ for some } C > 0 \\
\implies &\lambda_1(P^{-1}A) \sim O(n^{-2}).
\end{aligned}
$$

Which is the same order as $\lambda_1(A)$.

As such, in order to achieve an ϵ-solution (with ϵ being a constant independent of n), we have:

$$k \sim O(\log n)$$

The cost per iteration is $O(n \log n)$ (since we need to use FFT). All together, the computation cost of PCG is $O(n \log^2 n)$, which is optimal up to a log factor.

Remark 1.6 — PCG is the state of the art of iterative methods to solve linear systems for SPD A.