

# 1 January 22nd, 2020

There are 5 types:

- Separable Differential Equation
- Homogeneous Differential Equation
- Linear Differential Equations
- Bernoulli Differential Equation
- ???

## 1.1 Separable Differential Equation

A general first-order ODE for a dependent variable  $y$  in the independent variable  $x$  can be written as:

$$\frac{dy}{dx} = F(x, y) \quad (1)$$

where  $F$  is some specified function of  $x$  and  $y$ . When  $F$  has the form

$$F(x, y) = f(x)g(y), \quad (2)$$

then 1 is said to be *separable* and such equation can always be solved by:

$$\frac{dy}{g(y)} f(x) dx \implies \int \frac{dy}{g(y)} + C_1 = \int f(x) dx + C_2 \implies \int \frac{dx}{g(y)} = \int f(x) dx + C.$$

as one form for the solution of 1.

### 1.1.1 Ideal Fluid Flow

We are concerned with a container that has a fluid with cross sectional area  $A$  with density  $\rho$  with a hole at the bottom of the container which causes it to flow out. We are concerned with the height  $x$  of the container. We also have a pipe that pumps in fluid with constant rate  $R$ .

This leads to following equation:

$$\frac{dx}{dt} = \alpha - \beta\sqrt{x}.$$

where

$$\alpha = \frac{R}{A} \quad \beta = \sqrt{\frac{2ga^2}{A^2 - a^2}} \quad g = 9.81 \text{ m s}^{-2}.$$

Note that this is a separable differential equation:

$$\frac{dx}{\alpha - \beta\sqrt{x}} = dt.$$

If we have  $\alpha, \beta$ , we can solve, e.g.  $\alpha = 60 \quad \beta = 6$ , we have:

$$\frac{dx}{dt} = 60 - 6\sqrt{x} \implies \frac{dx}{10 - \sqrt{x}} = 6dt.$$

Integrating on both sides, we have:

$$\int \frac{dx}{10 - \sqrt{x}} = \int 6dt = 6t + C.$$

Solving this, we get:

$$20 \tan^{-1} \left( \frac{\sqrt{x}}{10} \right) - 10 \ln(100 - x) - 2\sqrt{x} = 6t + C.$$

If we have initial conditions, e.g. at  $t = 0, x = 0$ , we would have:

$$0 - 10 \ln(100) = C$$

allowing us to solve for  $C$ . This would allow us to solve for a time  $t$  for certain values of  $x$ .

## 1.2 Homogeneous Differential Equation

Again remember that the general form a differential equation of one a dependent variable  $y$  in the independent variable  $x$  is:

$$\frac{dy}{dx} = F(x, y).$$

If  $F(x, y) = f(x)g(y)$  then this is separable. Remember that the goal is that we want to find  $G(x, y) = C$ , in other words, we want to get rid of the derivative and find the relationship between the two.

**Definition 1.1.** A function of form  $F(x, y)$  is called **homogeneous** of order  $N$  if  $F(tx, ty) = t^N F(x, y)$  for any scalar  $t$ .

### Example 1.2

$$\begin{aligned} F(x, y) = x^3 + x^2y + 4xy^2 &\implies F(tx, ty) = (tx)^3 + (tx)^2(ty) + 4(tx)(ty)^2 \\ &= t^3 (x^3 + x^2y + 4xy^2) = t^3 F(x, y). \end{aligned}$$

Thus  $F(x, y)$  is homogeneous to the order 3.

### Example 1.3

$F(x, y) = x^3 + xy$  is not homogeneous.

### Example 1.4

$$\begin{aligned} F(x, y) &= \frac{xy}{x^2 + y^2} \\ F(tx, ty) &= \frac{t^2xy}{t^2x^2 + t^2y^2} = t^2 \left( \frac{xy}{x^2 + y^2} \right) = t^0 F(x, y) \end{aligned}$$

meaning that  $F(x, y)$  is homogeneous to order 0.

**Remark 1.5** — Typically if we say that a function is homogeneous but don't specify the order, it is assumed to be of order 0.

If a function is homogeneous to order 0, then it only depends on the ratio of  $\frac{y}{x}$ . In other words, rewrite  $F(x, y) = f(\frac{y}{x})$ .

**Theorem 1.6**

A function  $F(x, y)$  is homogeneous of order 0 if and only if it can be expressed as  $f(\frac{y}{x})$ .

If we have a homogeneous function of order 0, we will be able to introduce a new variable  $z = \frac{y}{x} \implies y = xz$ , giving us:

$$\frac{d(xz)}{dx} = F(x, xz) = F(x(1), x(z)) = F(1, z).$$

Using the product rule, we have:

$$\begin{aligned} \frac{d(xz)}{dx} &= \frac{dx}{dx}z + x\frac{dz}{dx} = F(1, z). \\ z + x\frac{dz}{dx} &= F(1, z) \implies \frac{dz}{F(1, z) - z} = \frac{dx}{x}, \end{aligned}$$

which is a separable differential equation.

**Remark 1.7** — The point is whenever you have a homogeneous equation, then introducing  $z = \frac{y}{x}$  will allow us to convert it to a separable equation. Note that this only works for order 0 homogeneous equations.

### 1.2.1 Building an Radar Antenna

TL;DR the equation is:

$$\left(\frac{dy}{dx}\right)^2 - 2\frac{y-F}{x}\left(\frac{dy}{dx}\right) - 1 = 0.$$

If we use the quadratic formula, we get:

$$\frac{dy}{dx} = \frac{y-F}{x} + \sqrt{\left(\frac{y-F}{x}\right)^2 + 1}.$$

If we do the substitution,  $z = \frac{y-F}{x}$ , we get:

$$\begin{aligned} \frac{d(xz + F)}{dx} &= z + \sqrt{z^2 + 1} \implies x\frac{dz}{dx} + z = z + \sqrt{z^2 + 1} \implies \frac{dz}{\sqrt{z^2 + 1}} = \frac{dx}{x}. \\ \int \frac{dz}{\sqrt{z^2 + 1}} &= \ln x + C \implies \ln(z + \sqrt{z^2 + 1}) = \ln x + C. \\ \implies A^2x^2 - 2Axz &= 1 \implies \frac{1}{2}Ax^2 + \left(F - \frac{1}{2A}\right), \end{aligned}$$

which is the equation of a parabola. Thus the optimal shape of a radar dish is a parabola.