MATH3033 - Real Analysis

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1.1 Course Logistics

1.1.1 Grading

- Assignments: 20%

- Midterm (Early November): 30%

- Final: 50%

1.1.2 Content

There will be 4 chapters in this course:

1. Point Set Topology

2. Functions in Several Variables

3. Sequences of Functions

4. Lebesgue Integral

1.2 Point Set Topology

To motivate this chapter, consider the following question:

> "Given a function $f : \mathbb{Z} \to \mathbb{R}$, could it be continuous or differentiable? What about other domains, e.g. $\mathbb{Q}, \mathbb{R}^n, \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$?"

Clearly for different domains, the notion of continuity and/or differentiability are different, thus we want to formulate the geometric properties of these sets, which leads to the idea of **Point Set Topology**.

> **Remark 1.1** — For the purpose of this course, we are only considering point set topology in $\mathbb{R}^n$, not in general.

1.3 Reviews/Definitions

Definition 1.2. $\mathbb{R}^n$ consists of n-tuples of real numbers, with the following operations defined:

- Addition: $(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$

- Scalar Multiplication: $k(x_1, \ldots, x_n) = (kx_1, \ldots, kx_n)$

for all $k, x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$

Lemma 1.3

$\mathbb{R}^n$ with the operations above form a real vector space.

Definition 1.4. An **inner product** in $\mathbb{R}^n$ is a function $\langle x, y \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, such that:

1. $\langle \ , \ \rangle$ is bilinear, i.e. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

2. $\langle \ , \ \rangle$ is symmetric, i.e. $\langle x, y \rangle = \langle y, x \rangle$

3. $\langle x, x \rangle \geq 0, \forall x \in \mathbb{R}^n$ and $\langle x, x \rangle = 0$ iff $x = 0$

Definition 1.5. The **standard inner product/dot product** is:

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n.$$

Definition 1.6. If $x \in \mathbb{R}^n$, its norm $\|x\| = \sqrt{\langle x, x \rangle}$

Theorem 1.7

For every $x, y, z \in \mathbb{R}^n$, we have:

1. $\|x - y\| = \|y - x\|$

2. $\|x - y\| \geq 0$

3. $\|x - y\| = 0$ only when $x = y$

4. $\|x - z\| \leq \|x - y\| + \|y - z\|$ (**triangular inequality**), proof below

Theorem 1.8

Cauchy-Schwarz inequality For every $x, y \in \mathbb{R}^n$, we have:

$$|\langle x, y \rangle| \leq \|x\| \|y\|.$$

Proof. We have:

$$0 \leq \left\langle x - \frac{\langle x, y \rangle}{\|y\|^2} y, x - \frac{\langle x, y \rangle}{\|y\|^2} y \right\rangle$$

$$\implies x \leq \langle x, x \rangle + \left\langle \frac{\langle x, y \rangle}{\|y\|^2} y, \frac{\langle x, y \rangle}{\|y\|^2} y \right\rangle - 2 \left\langle x, \frac{\langle x, y \rangle}{\|y\|^2} y \right\rangle$$

$$\implies x \leq \langle x, x \rangle + \frac{\langle x, y \rangle^2}{\|y\|^{42}} \langle y, y \rangle - 2 \frac{\langle x, y \rangle}{\|y\|^2} \langle x, y \rangle$$

$$\implies 0 \leq \|x\|^2 - \frac{\langle x, y \rangle^2}{\|y\|^2} \implies \langle x, y \rangle^2 \leq \|x\|^2 \|y\|^2$$

$\square$

Returning to the proof of the triangle inequality, we have:

Proof.

$$
\begin{aligned}
\|x - z\|^2 = \|x - y + y - z\|^2 &= \langle x - y + y - z, x - y + y - z \rangle \\
&= \langle (x - y) + (y - z), (x - y) + (y - z) \rangle \\
&= \langle x - y, x - y \rangle + \langle y - z, y - z \rangle + 2 \langle x - y, y - z \rangle \\
&= \|x - y\|^2 + \|y - z\|^2 + 2 \langle x - y, y - z \rangle \\
&\leq \|x - y\|^2 + \|y - z\|^2 + 2\|x - y\|\|y - z\| = (\|x - y\| + \|y - z\|)^2
\end{aligned}
$$

Taking square roots on both sides, we have:

$$
\|x - z\| \leq \|x - y\| + \|y - z\|.
$$

$\square$

Definition 1.9. A function $T : \mathbb{R}^m \to \mathbb{R}^n$ is a **linear transformation** if:

$$
T(ax + by) = aT(x) + bT(y), \quad \forall x, y, \in \mathbb{R}^m, \quad a, b \in \mathbb{R}.
$$

Definition 1.10. A **basis** of $\mathbb{R}^n$ is a linearly independent set whose span is $\mathbb{R}^n$, i.e. every vector in $\mathbb{R}^n$ is a unique linear combination of the vectors in the basis.

Definition 1.11. Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation and let $A = (a_1, \ldots, a_m)$ and $B = (b_1, \ldots, b_n)$ be basis of $\mathbb{R}^m$ and $\mathbb{R}^n$ Then write

$$
T(a_1) = t_{11}b_1 + t_{12}b_2 + \ldots + t_{1n}b_n
$$

$$
T(a_2) = t_{12}b_1 + t_{13}b_2 + \ldots + t_{2n}b_n
$$

$$
\ldots
$$

$$
T(a_m) = t_{m1}b_1 + t_{m2}b_2 + \ldots + t_{mn}b_n
$$

, we call:

$$
\begin{bmatrix}
t_{11} & t_{12} & \ldots & t_{1n} \\
 & & \ldots & \\
t_{m_1} & t_{m_2} & \ldots & t_{mn}
\end{bmatrix}^T
$$

the matrix of T relative to the basis A and B.

Example 1.12

Let vectors in $\mathbb{R}^n$ be presented as column vectors and:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

A map $T : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T(x) = Ax$ for all $x \in \mathbb{R}^3$. The matrix of T relative to the basis:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \text{ is } \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

The matrix of T relative to the basis:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\} \text{ is } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

This is because:

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$