1 April 6th, 2020

1.1 Sturm-Liouville

Definition 1.1. An ODE of the form:

$$\frac{d}{dx}\left(s(x)\phi'(x)\right) + q(x)\phi(x) + \lambda w(x)\phi(x) = 0, \alpha < x < \beta.$$

along with the boundary conditions:

$$c_1\phi(\alpha) + c_2\phi'(\alpha) = 0.$$

$$d_1\phi(\beta) + d_2\phi'(\beta) = 0.$$

is called a **regular Sturm Liouville Problem** (denoted RSLP) if the following conditions hold.

1. $s(x), s'(x), q(x), w(x)$ are all continuous functions in the open interval $\alpha < x < \beta$
2. $s(x) > 0$ and $w(x) > 0$ for all $\alpha < x < \beta$
3. $c_1^2 + c_2^2 > 0$ and $d_1^2 + d_2^2 > 0$, i.e. can't have both c_1 and c_2 equal zero, same for d_1, d_2
4. The λ occurs only in the ODE as indicated by $\lambda w(x)\phi(x)$.

Recall that any linear 2nd order homogeneous ODE of the form:

$$a_2(x)\phi''(x) + a_1(x)\phi'(x) + a_0(x)\phi(x) + \lambda b(x)\phi(x) = 0.$$

can be placed in the form:

$$\frac{d}{dx}\left(s(x)\phi'(x)\right) + q(x)\phi(x) + \lambda w(x)\phi(x) = 0, \alpha < x < \beta.$$

by setting

$$s(x) = e^{\int \frac{a_1(x)}{a_2(x)}\, dx}.$$

$$q(x) = \frac{a_0(x)s(x)}{a_2(x)}.$$

$$w(x) = \frac{b(x)s(x)}{a_2(x)}.$$

Example 1.2

Consider:

$$x\phi''(x) + 2x\phi(x) + \phi(x) +_{\lambda} x^2\phi(x) = 0, \quad 0 < x < 1.$$

we have:

$$s(x) = e^{\int \frac{2x}{x}\ dx} = e^{2x}.$$

$$q(x) = \frac{1}{x}e^{2x}.$$

$$w(x) = \frac{x^2}{x}e^{2x} = xe^{2x}.$$

Meaning that the equation can be written in the form of:

$$\frac{d}{dx}\left(e^{2x}\phi'(x)\right) + \frac{1}{x}e^{2x}\phi(x) + \lambda xe^{2x}\phi(x) = 0, 0 < x < 1.$$

Remark 1.3 — The form above is called the **self-adjoint form**.

1.2 Properties of Regular Sturm-Liouville Problems

- There exist an infinite number of λ 's that lead to non-zero solutions $\phi(x)$ to the ODE and boundary conditions. These λ 's which can be ordered from smallest to largest are called the **eigenvalues** of the RSLP. Moreover

$$\lim_{n\to\infty} \lambda_n = +\infty.$$

  This means if you solve a RLSP and found that $\lambda_n = \frac{n}{n+1}$ then there is a problem, since $\lim\limits_{n\to\infty} \frac{n}{n+1} = 1 \neq +\infty$.

- If these λ 's are ordered and if the non-zero functions $\phi_n(x)$ satisfy:

$$\frac{d}{dx}\left(s(x)\phi_n'(x)\right) + q(x)\phi_n(x) + \lambda_n\omega(x)\phi_n(x) = 0.$$

  and it satisfies the boundary conditions, then $\phi_n(x)$ is called the eigenfunction associated with the eigenvalue λ_n and $\phi_n(x)$ goes through zero exactly $n-1$ times in the open interval $\alpha < x < \beta$. As such $\phi_1(x)$ does not go though zero, $\phi_2(x)$ goes through zero once, and so on.

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$$\phi_m \cdot \phi_n = \begin{cases} 0, n \neq m \\ > 0, n = m \end{cases}.$$

  where

$$f \cdot g = \int_{\alpha}^{\beta} f(x)g(x)w(x)\ dx.$$

  In other words there is a dot product that can be defined on the functions ϕ_n.

- The set of eigenfunctions
$$\{\phi_1(x), \phi_2(x), \ldots\}.$$
is called a complete set of basic functions so that if $f(x)$ is any piecewise continuous function in the interval $\alpha < x < \beta$, then we may expand $f(x)$ as:
$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x).$$
with
$$a_n = \frac{\phi_n \cdot f}{\phi_n \cdot \phi_n}.$$
moreover, the sum converges to:
$$\frac{f(x^+) + f(x^-)}{2}.$$
Where:
$$f(a^{+)}) = \lim_{x \to a^+} f(x) \quad f(a^-) = \lim_{x \to a^-} f(x).$$
Note that piecewise continuous means that there can only be a finite number of hole or jump discontinuities but not essential discontinuities.

Remark 1.4 — This means that the regular Sturm Liouville series will fill in all of the hole discontinuities and where there is a jump discontinuity, it will converge to the midpoint of the jump.

Definition 1.5. A function $f(x)$ is called piecewise continuous in a finite interval $\alpha < x < \beta$ is it has at most a finite number of hole or jumps in $\alpha < x < \beta$.

Example 1.6

Consider:

$$\phi''(x) + \lambda\phi(x) = 0, 0 < x < 1.$$

$$\phi(0) = 1 \quad \phi(1) = 0.$$

Thus we have:

$$s(x) = e^{\int \frac{0}{1} dx} = 1.$$

$$q(x) = \frac{(0)(1)}{1} = 0.$$

$$w(x) = \frac{(1)(1)}{1} = 1.$$

Note that this is a RSLP. To solve this problem, notice that this is a problem with constant coefficients, giving us:

$$\phi(x) = \begin{cases} A\cosh(x\sqrt{-\lambda}) + B\sinh(x\sqrt{-\lambda}), \lambda < 0 \\ A + Bx, \lambda = 0 \\ A\cos(x\sqrt{\lambda}) + B\sin(x\sqrt{\lambda}), \lambda > 0 \end{cases}.$$

If we consider $\lambda < 0$, because of the boundary conditions, we get $A = 0$, and $B = 0$, meaning that $\lambda < 0$ gives us $\phi(x) = 0$, meaning that there are no negative eigenvalues.

For $\lambda = 0$, we have:

$$\phi(x) = A = 0 \implies \phi(x) = Bx.$$

$$\phi(1) = B = 0 \implies \phi(x) = 0.$$

meaning that zero is not an eigenvalue.

For $\lambda > 0$, we have:

$$\phi(0) = A = 0 \implies \phi(x) = B\sin(x\sqrt{\lambda}).$$

$$\phi(1) = B\sin(x\sqrt{\lambda}) \implies \sqrt{\lambda} = n\pi \implies \lambda = (n\pi)^2 = \lambda_n.$$

Note that we will always get a multiplicative constant when we try to calculate ϕ_n , thus we can set $B_n = 1$ giving us:

$$\phi_n(x) = \sin(n\pi x).$$

Remark 1.7 — Note that for the above example, $\phi_1(x) = \sin(\pi x)$ does not go through zero on the open interval between 0 and 1. Similarly $\phi_2(x) = \sin(2\pi x)$ goes through zero once at $x = \frac{1}{2}$, etc.