# 1 February 19th, 2020

## 1.1 Unit Step Function Continued

As a reminder, the unit step function is defined as:

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}.$$

Given a piecewise function, we can write it as a linear combination of step functions.

**Example 1.1**

Consider:

$$f(t) = \begin{cases} 7, & 0 < t < 2 \\ 6t, & 2 < t < 3 \\ t^2, & 3 < t < 7 \\ 0, & 7 < t \end{cases}.$$

We can rewrite this as:

$$f(t) = 7u(t) + (6t-7)u(t-2) + (t^2 - 6t)u(t-3) + (0 - t^2)u(t-7).$$

With this, we can take the Laplace transform of the function, but first, we need to consider the Laplace transform of $\mathcal{L}\{f(t)u(t-a)\}$. Looking at the definition, we have:

$$\mathcal{L}\{f(t)u(t-a)\} = \int_0^\infty f(t)u(t-a)e^{-st}\,dt.$$

Assuming $a > 0$, we have:

$$= \int_0^a f(t)\underbrace{u(t-a)}_{0}e^{-st}\,dt + \int_a^\infty f(t)\underbrace{u(t-a)}_{1}e^{-st}\,dt = \int_a^\infty f(t)e^{-st}\,dt.$$

If we set $z = t - a \implies dz = dt$,

$$= \int_0^\infty f(z+a)e^{-s(z+a)}\,dz = e^{-as}\mathcal{L}\{f(t+a)\}.$$

**Theorem 1.2** (Shifting Theorem)

As shown above:

$$\mathcal{L}\{f(t)u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}.$$

**Example 1.3**

Considering $f(t)$ from Example 1.1, we have:

$$\mathcal{L}\{f(t)\} = \int_0^2 7e^{-st}\,dt + \int_2^3 6te^{-st}\,dt + \int_3^7 t^2 e^{-st}\,dt + \int_7^\infty 0e^{-st}\,dt.$$

However, we can calculate this another way. From the table, we have $\mathcal{L}\{t^m\} = \frac{m!}{s^{m+1}}$, thus:

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{7u(t) + (6t-7)u(t-2) + (t^2-6t)u(t-3) + (0-t^2)u(t-7)\}$$

$$= \mathcal{L}\{7u(t)\} + \mathcal{L}\{(6t-7)u(t-2)\} + \mathcal{L}\{(t^2-6t)u(t-3)\} + \mathcal{L}\{-t^2u(t-7)\}$$

$$= e^{-0s}\mathcal{L}\{7\} + e^{-2s}\mathcal{L}\{6(t+2)-7\} + e^{-3s}\mathcal{L}\{(t+3)^2 - 6(t+3)\} - e^{-7s}\mathcal{L}\{(t+7)^2\}$$

$$= \frac{7}{s} + e^{-2s}\mathcal{L}\{6t-5\} + e^{-3s}\mathcal{L}\{t^2-9\} - e^{-7s}\mathcal{L}\{t^2+14t+49\}.$$

Thus:

$$\mathcal{L}\{f(t)\} = \frac{7}{s} + e^{-2s}\left(\frac{6}{s^2} + \frac{5}{s}\right) + e^{-3s}\left(\frac{2}{s^3} - \frac{9}{s}\right) - e^{-7s}\left(\frac{2}{s^3} + \frac{14}{s^2} + \frac{49}{s}\right), \quad s > 0.$$

**Remark 1.4** — In the example earlier, we are using the Shifting Theorem and replacing $t$ with $t+a$ in each of the functions that we are multiplying by the unit step function at $a$.

## 1.2 Examples of Where Unit Step Functions Occur

**Example 1.5**

Consider the equation:

$$L\frac{dI}{dt} + RI = \epsilon_1 u(t) + (\epsilon_2 - \epsilon_1)u(t-t_1).$$

Taking the Laplace transform of both sides, we get:

$$\mathcal{L}\left\{L\frac{dI}{dt} + RI\right\} = \mathcal{L}\{\epsilon_1 u(t) + (\epsilon_2-\epsilon_1)u(t-t_1)\}.$$

$$\implies L\mathcal{L}\{I'(t)\} + R\mathcal{L}\{I(t)\} = e^{-0s}\mathcal{L}\{\epsilon_1\} + e^{-t_1 s}\mathcal{L}\{\epsilon_2 - \epsilon_1\}$$

$$\implies L\left(s\mathcal{L}\{I\} - I(0)\right) + R\mathcal{L}\{I\} = \frac{\epsilon_1}{s} + e^{-t_1 s}\left(\frac{\epsilon_2-\epsilon_1}{a}\right).$$

$$\implies \mathcal{L}\{I\} = \frac{LI_0 + \frac{\epsilon_1}{s} + e^{-t_1 s}\left(\frac{\epsilon_2-\epsilon_1}{s}\right)}{LS+R}.$$

There are many applications/cases where using a step function to describe a piecewise function might be useful. For example, if we have a spring with dampener with an external force $F(t)$, we might have $F(t)$ ramp up with $t$, and then stay constant after a certain amount of time.

Another example is consider a ball bouncing off the ground. The forces are:

$$F(t) = \begin{cases} -mg, & 0 < t < T_F \\ N(t) - mg, & T_F < t < T_F + T_C \\ -mg, T_F + T_C < t < T_F + T_C + T_R \end{cases}.$$

Where $T_F$ is the time until hitting the ground, $T_C$ is the contact duration, and $T_R$ is the time to rebound back up, and $N(t)$ is the normal force. From this, we get figure out $N(t)$ and allow us to get the coefficient of restitution.

## 1.3 Impulse Function

Consider a function:

$$I_a(t) = \begin{cases} 0, & t < -\frac{a}{2} \\ \frac{1}{a}, & -\frac{a}{2} < t < \frac{a}{2} \\ 0, & \frac{a}{2} < t \end{cases}.$$

This can be expressed in terms of unit step functions as:

$$I_a(t) = \frac{1}{a}u(t + \frac{a}{2}) - \frac{1}{a}u(t - \frac{a}{2}).$$

**Remark 1.6** — Note that the area under the curve is 1, as we choose the height to be inversely proportional to the width. This means that:

$$\int_{-\infty}^{\infty} I_a(t)\ dt = 1.$$

**Definition 1.7** (Impulse Function)**.** An **impulse function** is:

$$\lim_{a \to 0} I_a(t) = I(t) = \begin{cases} 0, & t \neq 0 \\ +\infty, & t = 0 \end{cases}$$

with the property:

$$\int_{-\infty}^{\infty} I(t)\ dt = 1.$$

Or:

$$\int_R I(t)\ dt = \begin{cases} 0, & 0 \notin R \\ 1, & 0 \in R \end{cases}.$$

**Remark 1.8** — With Laplace transform, if 0 is at the end of the domain, it is included, e.g.:

$$\int_0^7 I(t)\ dt = 1.$$

**Remark 1.9** — Similar to the step function, we can shift the impulse function, i.e.:

$$I(t-a) = \begin{cases} 0, & t \neq a \\ +\infty, & t = a \end{cases},$$

with:

$$\int_R I(t-a)\ dt = \begin{cases} 0, & a \notin R \\ 1, & a \in R \end{cases}.$$

**Example 1.10** (One Dimensional Crystal)
In a one dimensional crystal, we have atoms aligned in a line, and say they are separated by $a$. If we have an electron travelling along, the force it might see can be expressed as:

$$F(x) = a \sum_{k=-\infty}^{\infty} F_0 I(x-ka) = aF_0 \sum_{k=-\infty}^{\infty} I(x-ka).$$

Thus one way to model the force experienced by an electron is to use a bunch of impulse functions. This is called the **comb function**.

**Remark 1.11** — If we had a continuous function $f(t)$, we'd have:

$$\int_R f(t)I(t-a)\ dt = \begin{cases} 0, & 0 \notin R \\ f(a), & a \in R \end{cases}.$$

**Example 1.12**
If we have:

$$\int_{-1}^{7} \frac{t^2}{\sqrt{3t^3+1}} e^t I(t-1)\ dt = \frac{1^2}{\sqrt{3(1)^3+1}} e^1 = \frac{1}{2}e.$$

**Theorem 1.13**
We have:

$$\mathcal{L}\{f(t)I(t-a)\}, a > 0 = \int_0^{\infty} f(t)I(t-a)e^{-st}\ dt = f(a)e^{-as}.$$

**Example 1.14**
$\mathcal{L}\{t^3 I(t-4)\} = 4^3 e^{-4s} = 64e^{-4s}$

**Example 1.15**

Consider $F(t) = aF\sum_{k=-\infty}^{\infty} I(t-ka)$. We have:

$$\mathcal{L}\{F(t)\} = Fa\sum_{k=-\infty}^{\infty} e^{-kas}.$$