# 1 April 4th, 2019

## 1.1 Fractional Min-cut

**Review** - LP relaxation of integer LP for min cut

$$
\begin{aligned}
\text{minimize:} \quad & \textstyle\sum_{(u,v)\in e} c(u,v)y(u,v) \\
\text{subject to:} \quad & \textstyle\sum_{(u,v)\in p} y(u,v) \geq 1, \quad \forall p \in p \\
& y(u,v) \geq 0, \quad \forall (u,v) \in e
\end{aligned}
$$

$y(u,v)$ can be thought of the length of an edge, with the constraints being that the length of $t$ from $s$ is at least 1.

To prove that the integrality gap to be 1, we need to show that there exists a integer cut that is just as good as the fractional min-cut

**Lemma 1.1**
Given any feasible solution $y(u,v)$ for $(u,v) \in E$, it is possible to find a $s - t$ cut $A$ such that

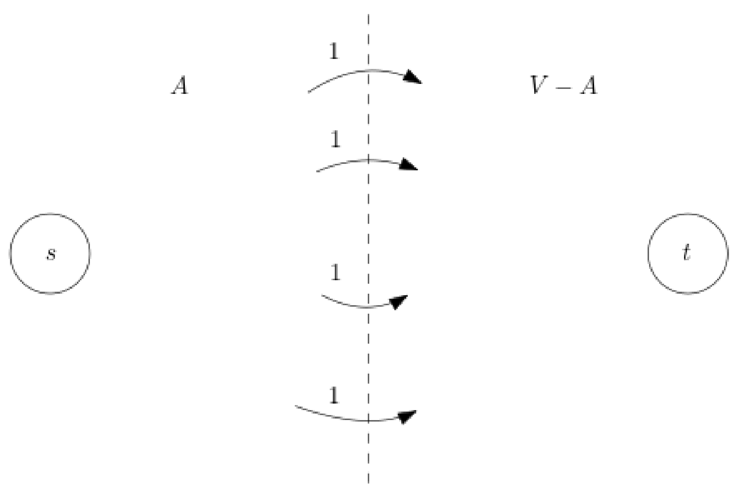
$$c(A) \leq \sum c(u,v)y(u,v).$$

Constraints of LP imply $d(t) \geq 1$.

Pick $T$ (threshold) uniformly at random in interval $[0,1)$. Define $A$ to be the set $A = \{v \ : d(v) \leq T\}$, if 1 was included in the interval, then $t$ could be in $A$. Now, we will show that

$$E[c(A)] \leq \sum c(u,v)y(u,v).$$

If this is true, then we have proven the above lemma, as there cannot be a cut $A$ which has less capacity then the fractional min-cut. As it's less than or equal to the fractional min-cut, then they must be equal. For this to be consistent, then all edges crossing $A$ are 1, and everything else is 0.
There must be a $s - t$ cut $A$ s.t. this is true, (since it is true on average).



*Proof.* Define a random variable $X(u,v)$ to be 1 if $u \in A$ and $v \notin A$. Otherwise it is 0.

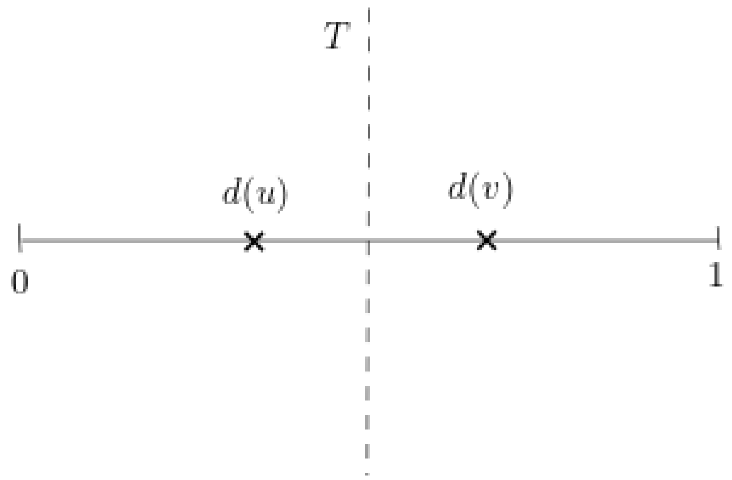
$$c(A) = \sum_{(u,v)\in E} X(u,v)c(u,v)$$

$$E[c(A)] = E[\sum X(u,v)c(u,v).$$

By the linearity of expectation:

$$= \sum c(u,v)E[X(u,v)],$$

$$E[X(u,v)] = \Pr[X(u,v) = 1] = \Pr[d(u) \leq T \text{ and } d(v) > T] \leq d(v) - d(u)$$




$$d(v) \leq d(u) + y(u,v)$$

$$E[X(u,v)] \leq d(v) - d(u) \leq y(u,v).$$

□

## 1.2 Steiner Forest Problem

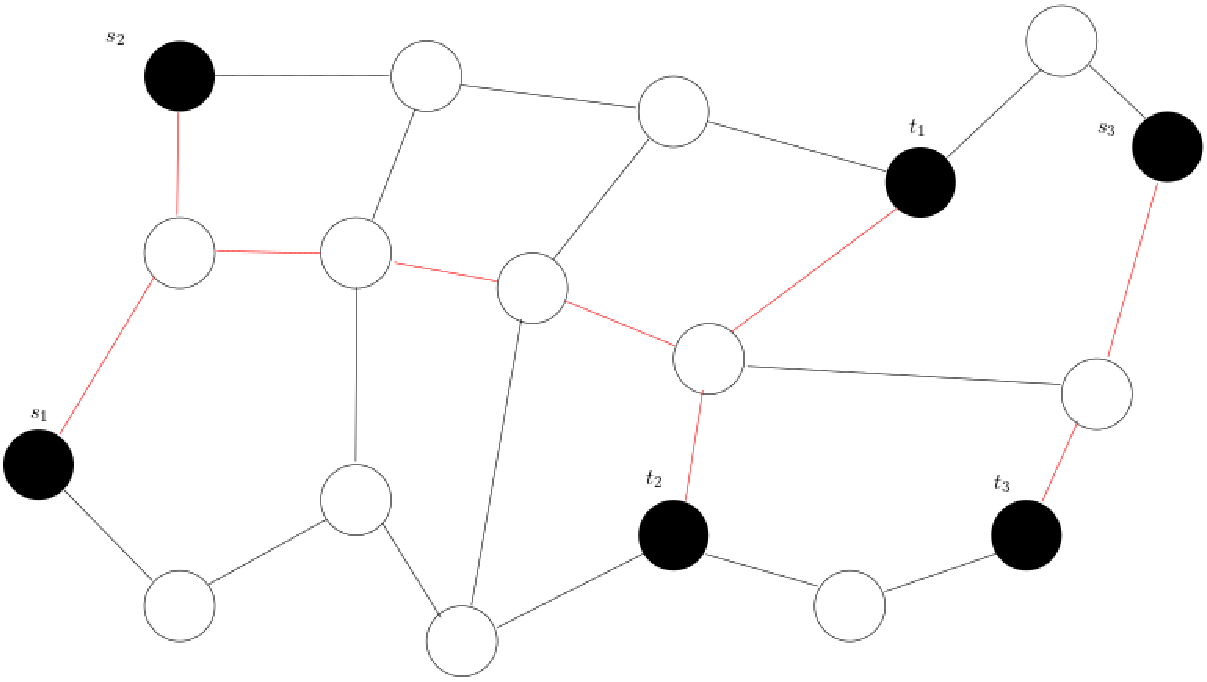
Given a graph $G = (V, E)$, edge cost $C : E \to R^+$. In the Steiner tree problem, we have a set of required vertices that must be connected. In the Steiner Forest problem, we have sets $S_i \subseteq V$, find a minimum cost subgraph $F$ (for forest) such that each pair of vertices belonging to the same set $S_i$ is connected.

**Problem Restatement** Define a connectivity requirement function $r$ (for requirement) that maps unordered pairs of vertices to $\{0, 1\}$:

$$r(u,v) = \begin{cases} 1, & \text{if } u \text{ and } \text{ belong to some set } S_i \\ 0, & \text{otherwise} \end{cases}$$

Note that there cannot be cycles (can remove an edge to reduce the cost)

$$\text{The ILP is:} \quad \begin{array}{ll} \text{minimize:} & \sum_{e\in E} c_e x_e \\ \text{subject to:} & \ldots \qquad \text{but what are the constraints} \\ & x_e \in \{0,1\} \end{array}$$




$r(s_1, t_1) = 1$

$r(s_2, t_2) = 1$

$r(s_3, t_3) = 1$

Note that if $r(u, v) = 1$, for every $u - v$ cut, there must be an edge crossing such cut. This is a necessary condition, but is it sufficient? It is, as if $u$, $v$ were not connected, there would have been a cut that separates them.

We let $\delta(S)$ denote the set of edges with exactly one endpoint in $S$. Let $\overline{S} = V - S$. Consider any cut $(S, \overline{S})$ in $G$ that separates a pair $(u, v)$ that should be connected. Then we **must** pick at least one edge $e \in \delta(S)$. Clearly this is necessary, but it is also sufficient.

Let $S^\star$ be the collection of all sets $S$ such that $(S, \overline{S})$ separate a pair $(u, v)$ for which $r(u, v) = 1$. Introduce a 0/1 variable $x_e$ for each edge $e \in E$:

Integer LP for Steiner Forest:

$$\begin{array}{rl} \text{minimize:} & \sum_{e \in E} c_e x_e \\ \text{subject to:} & \sum_{e: e \in \delta(S)} x_e \geq 1 \quad \forall S \in S^\star \\ & x_e \in \{0, 1\} \end{array}$$

This is because each $S \in S^\star$ is a cut that must be crossed.