Algorithm

Input:

P data matrices $\mathbf{X^1},\mathbf{X^2},\ldots,\mathbf{X^P}$, parameters β,ω

Output:

P basis matrices $\mathbf{U^1},\mathbf{U^2},\ldots,\mathbf{U^P}$,$P$ relation matrices $\mathbf{W^1},\mathbf{W^2},\ldots,\mathbf{W^P}$, factor matrices $\mathbf{V}$, weight vector $\Pi(\pi^1,\ldots,\pi^p)$

Begin

Initialize $\mathbf{U^1},\mathbf{U^2},\ldots,\mathbf{U^P},\mathbf{W^1},\mathbf{W^2},\ldots,\mathbf{W^P},\mathbf{V}$

Initialize $(\pi^1,\ldots,\pi^p) = \left(\frac{1}{P},\frac{1}{P},\ldots,\frac{1}{P}\right)$

loop

  for p=1 to P **do**

    Fix $\mathbf{V}$, update $\mathbf{U^P},\mathbf{W^P}$

  end for

  Fix $\mathbf{U^1},\mathbf{U^2},\ldots,\mathbf{U^P}$, update $\mathbf{V_l}$

  Fix $\mathbf{U^1},\mathbf{U^2},\ldots,\mathbf{U^P}$, update $\mathbf{V_{ul}}$

  for p=1 to P **do**

    Fix $\mathbf{U},\mathbf{W},\mathbf{V}$, compute $c^p = \|X^p - U^p W^p V\|_F^2$

  end for

  Update Π

break loop if convergence

End

$$\min_{\{U^{(p)}\}_{p=1}^{P},\{W^{(p)}\}_{p=1}^{P},V} \sum_{p=1}^{P} \pi^{(p)} \left\|X^{(p)} - U^{(p)}W^{(p)}V\right\|_F^2 + \beta\left\{tr\left[V_l L^a (V_l)^T\right] - tr\left[V_l L^p (V_l)^T\right]\right\} + \omega\|\Pi\|^2$$

$$s.t.\ \forall p, U_{ij}^{(p)} \ge 0, V_{ij} \ge 0, \pi^{(p)} \ge 0, \sum_{p=1}^{P} \pi^{(p)} = 1$$

We define:

$$O(\mathrm{U},V,W,\Pi) = \sum_{p=1}^{P} \pi^{(p)} \left\|X^{(p)} - U^{(p)}W^{(p)}V\right\|_F^2 + \beta\left\{tr\left[V_l L^a (V_l)^T\right] - tr\left[V_l L^p (V_l)^T\right]\right\} + \omega\|\Pi\|^2$$

1. Fix V,W, update U

   Lagrange function:

   $$L\left(U^{(P)}\right) = \mathrm{O}\left(U^{(P)}\right) + tr\left(\Psi^T U^{(P)}\right)$$

   $$\frac{\partial L\left(U^{(P)}\right)}{\partial U^{(P)}} = -2X^{(p)}V^T\left(W^{(p)}\right)^T + 2U^{(P)}W^{(p)}VV^T\left(W^{(p)}\right)^T + \Psi$$

   Using KKT condition, we get:

$$U^{(P)} \leftarrow U^{(P)} \circ \frac{X^{(P)}V^T\left(W^{(p)}\right)^T}{U^{(P)}W^{(p)}VV^T\left(W^{(p)}\right)^T}$$

2. Fix U,V, update W

   Lagrange function:

$$L\left(W^{(P)}\right) = \mathrm{O}\left(W^{(P)}\right) + tr\left(\Phi^T W^{(P)}\right)$$

$$\frac{\partial L\left(W^{(P)}\right)}{\partial W^{(P)}} = -2\left(U^{(p)}\right)^T\left(X - U^{(p)}W^{(P)}V\right)V^T + \Phi$$

   Using KKT condition, we get:

$$W^{(P)} \leftarrow W^{(P)} \circ \frac{\left(U^{(p)}\right)^T X^{(p)}V^T}{\left(U^{(p)}\right)^T U^{(P)}W^{(p)}VV^T}$$

3. Fix U,W, update V_l

$$L(V) = \mathrm{O}(V) + tr\left(\Gamma^T V\right)$$

$$\frac{\partial L}{\partial V_l} = \sum_{p=1}^{P}\pi^{(p)}\left(-2\left(W^{(p)}\right)^T\left(U^{(p)}\right)^T X_l^{(p)} + 2\left(W^{(p)}\right)^T\left(U^{(p)}\right)^T U^{(p)}W^{(p)}V_l\right) + \beta\left(2V_lL^a - 2V_lL^p\right)_l + \Gamma_l$$

$$L^a = D^a - W^a, L^p = D^p - W^p$$

   Using KKT condition,

$$V_l \leftarrow V_l \circ \frac{\sum_{p=1}^{P}\pi^{(p)}\left(\left(W^{(p)}\right)^T\left(U^{(p)}\right)^T X_l^{(p)}\right) + \beta V_l\left(D^p + W^a\right)}{\sum_{p=1}^{P}\pi^{(p)}\left(W^{(p)}\right)^T\left(U^{(p)}\right)^T U^{(p)}W^{(p)}V_l + \beta V_l\left(D^a + W^p\right)}$$

   update V_{ul}

$$\frac{\partial L}{\partial V_{ul}} = \sum_{p=1}^{P}\pi^{(p)}\left(-2\left(W^{(p)}\right)^T\left(U^{(p)}\right)^T X_l^{(p)} + 2\left(W^{(p)}\right)^T\left(U^{(p)}\right)^T U^{(p)}W^{(p)}V_l\right)_l + \Gamma_{ul}$$

   Using KKT condition,

$$V_{ul} \leftarrow V_{ul} \circ \frac{\sum_{p=1}^{P}\pi^{(p)}\left(\left(W^{(p)}\right)^T\left(U^{(p)}\right)^T X_{ul}^{(p)}\right)}{\sum_{p=1}^{P}\pi^{(p)}\left(W^{(p)}\right)^T\left(U^{(p)}\right)^T U^{(p)}W^{(p)}V_{ul}}$$

4. Update Π ,using optimization tools

   When U,V,W are fixed,minimization of $\mathrm{O}(\Pi)$ is a convex optimization

$$\min \sum_{p=1}^{P} \pi^{(p)} \left\| X^{(p)} - U^{(p)} W^{(p)} V \right\|_F^2 + \lambda \left\| \Pi \right\|^2$$

$$s.t.\ \ \pi^{(p)} > 0, \sum_{p=1}^{P} \pi^{(p)} = 1$$

$\rightarrow$

$$\min \sum_{p=1}^{P} \pi^{(p)} c^{(p)} + \omega \left\| \Pi \right\|^2$$

$$s.t\ .\ \pi^{(p)} > \ \ 0 \sum_{p=1}^{P} \pi^{(p)} =$$

Graph Laplacian :

$$\frac{1}{2} \sum_i \sum_j W_{ij} \left\| v_i - v_j \right\|_2^2 = tr\left[VLV^T \right]$$

Prove:

$$\begin{aligned}
left &= \frac{1}{2} \sum_i \sum_j W_{ij} \sum_k \left[v_{i(k)} - v_{j(k)} \right]^2 \\
&= \frac{1}{2} \sum_i \sum_j W_{ij} \sum_k \left[{v_{i(k)}}^2 + {v_{j(k)}}^2 - 2 v_{i(k)} v_{j(k)} \right] \\
&= \frac{1}{2} \sum_i \sum_j W_{ij} \left(v_i^T v_i - 2 v_i^T v_j + v_j^T v_j \right) \\
&= \frac{1}{2} \times 2 \left(\sum_i v_i^T v_i D_{ii} - \sum_{i,j} v_i^T v_j W_{ij} \right) \\
&= \sum_i v_i^T v_i D_{ii} - \sum_{i,j} v_i^T v_j W_{ij} \\
&= tr\left(VDV^T \right) - tr\left(VWV^T \right) \\
&= tr\left(VLV^T \right) = right
\end{aligned}$$