

Car Bid Analysis

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1 Solution

I address this problem under the assumption that all the players are smart and are trying to max out their reward, also the fact that all the players are smart and are trying to max out their reward, is a common knowledge among all the players. In this note I ignore the possibility that the player exploits the suboptimality of their opponents' decisions.

Firstly it's obvious that deterministic strategies will diverge. Hence we consider the case where each player is with a mixed strategy, which is represented by a distribution over the set of positive integers \mathbb{N}^+ . It's sufficient to find all the Nash equilibriums and pick the one with the best payoff. Let's assume that the players' strategies are the same i.e. symmetric Nash, otherwise the player can not determine which strategy she should pick out of the three. Denote the strategy as $p_1, p_2, \dots, p_k, \dots$ s.t. $\sum_k p_k = 1$, where p_k is the probability a player chooses k .

To get a Nash, $\{p_k\}$ has to be the maxima over all feasible strategies, given that the other 2 players do not modify their strategies. That induces that for any $p'_1, p'_2, \dots, p'_k, \dots$ s.t. $\sum_k p'_k = 1$, the expected payoff can only be lower than that of $\{p_k\}$, that is,

$$\sum_k p'_k \left(\sum_{i=1}^{k-1} p_i^2 + \left(\sum_{i=k+1}^{\infty} p_i \right)^2 \right) \leq \sum_k p_k \left(\sum_{i=1}^{k-1} p_i^2 + \left(\sum_{i=k+1}^{\infty} p_i \right)^2 \right)$$

To achieve this, it's necessary that the KKT condition is satisfied, which induces $\exists C, \forall k$ either

$$\sum_{i=1}^{k-1} p_i^2 + \left(\sum_{i=k+1}^{\infty} p_i \right)^2 = C \quad (1)$$

or

$$p_k = 0.$$

is satisfied. Suppose $p_k > 0, \forall k$ (proof it later), we have Eq. (1) satisfied $\forall k$. I first observe

$$p_k = 1 - \sum_{i=1}^{k-1} p_i - \left(C - \sum_{i=1}^{k-1} p_i^2 \right)^{1/2}.$$

It shows that each term p_k is deterministic given the prefix p_1, \dots, p_{k-1} , which indicates that given p_1 the strategy $\{p_k\}$ satisfying Eq. (1), if exists, is unique. Also if we can find a satisfied strategy $\{p_k\}$ with $\sum_k p_k = 1$, $\{\alpha p_k\}$ could satisfy

Eq. (1) with arbitrary αp_1 . Hence, the strategy satisfying both Eq. (1) and $\sum_k p_k = 1$, if exists, is unique.

Armed with the uniqueness I would try to find a solution and if I do, the problem is solved. It's natural to assume an exponential decay of p_k over k given the problem description. Let $p_k = s^{k-1}(1-s)$ and subsequently $C = s^2$, Eq. (1) indicates

$$\sum_{i=1}^{k-1} s^{2i-2}(1-s)^2 + \left(\sum_{i=k+1} s^{k-1}(1-s) \right)^2 = s^2.$$

That is,

$$(1 - s - s^2 - s^3)(1 - s^{2k-2}) = 0,$$

with solution

$$s = \frac{1}{3}((17 + 3 \cdot 33^{\frac{1}{2}})^{\frac{1}{3}} - 2 \cdot (17 + 3 \cdot 33^{\frac{1}{2}})^{-\frac{1}{3}}).$$

Since the solution must be unique, this is the only achievable Nash. I conclude that in order to achieve a Nash, each player should conduct a mixed strategy $\{p_k\}$, where $p_k = s^{k-1}(1-s)$ with the s value stated above.

I find the result instresting. With 3 players the mixed strategy comes with an exponential decay, where the decay rate is the solution of $(1 - s - s^2 - s^3) = 0$ which is slightly larger than $1/2$. With 4 players, could it be the solution of $(1 - s - s^2 - s^3 - s^4) = 0$, which is a number between $1/2$ and the rate corresponding to 3 players? What makes a lot of sense is that, with infinity many players, the rate should be the solution of $(1 - s - s^2 - \dots) = 0$, which is $1/2$.

2 Appendix

I append 2 things to make my note complete and technically correct.

- I show that the solution I'm giving is sufficient condition of a Nash, while all previous reasoning comes with the necessity. In fact with $p_k = s^{k-1}(1-s)$, Eq. (1) holds for all k . The expected payoff of a certain player

$$\sum_k p'_k \left(\sum_{i=1}^{k-1} p_i^2 + \left(\sum_{i=k+1} p_i \right)^2 \right) = s^2$$

no matter what p'_k value she picks. She can not improve her payoff by modifying her strategy.

- I show that $p_k = 0$ is not possible. By contrary if $p_k = 0$, we have both $\sum_{i=1}^{k-1} p_i^2 + (\sum_{i=k+1} p_i)^2 \geq \sum_{i=1}^k p_i^2 + (\sum_{i=k+2} p_i)^2$ and $\sum_{i=1}^{k-1} p_i^2 + (\sum_{i=k+1} p_i)^2 \geq \sum_{i=1}^{k-2} p_i^2 + (\sum_{i=k} p_i)^2$ satisfied. If either p_{k-1} or p_{k+1} is greater than 0, the player can swap that value with p_k and achieves a better expected payoff. So both of them must be 0, and by induction we have $p_1 \dots = p_k = p_{k+1} = \dots = 0$, which contradicts with the fact that $\sum_k p_k = 1$.