EI-MOR: A Hybrid Exponential Integrator and Model Order Reduction Approach for Transient Power/Ground Network Analysis

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Outline

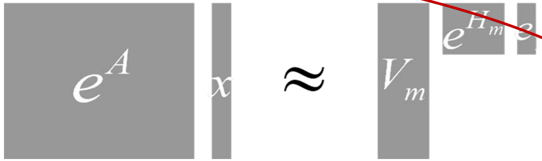
- Background
- Relations between EI and MOR
- Hybrid EI-MOR method
- Result
- Conclusion

Background

- **Circuits are bigger**
  - Millions of unknowns for post-layout sim
  - Even larger for power/ground (P/G) network simulation
- **SPICE is slow**
  - Huge sparse matrix LU, many time points
  - Weeks of verification time
- **Acceleration techniques exist**

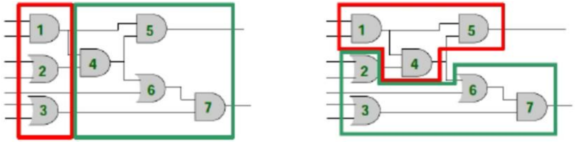
Mainstream circuit simulation acceleration methods

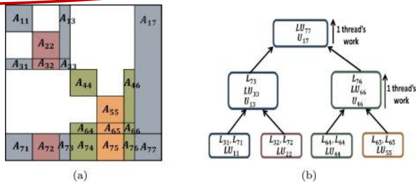
Mode Order Reduction(MOR)
- frequency-domain
- moment-matching
- maintain the I/O relationship



Exponential Integrator(EI)
- time-domain
- accuracy, large step sizes，high parallelism
- General circuit simulation scenarios

Focus of this work





Parallel computing
- graph theory
- circuit/matrix division
- parallel hardware (multi-threaded/multi-core/GPU) acceleration

Krylov Subspace based Model Order Reduction

Time-domain

$$\mathbf{C}\dot{x}(t) + \mathbf{G}x(t) = \mathbf{B}u(t)$$

Build reduced order system

$$\mathbf{C}_r = \mathbf{V}^T C\mathbf{V}, \quad \mathbf{G}_r = \mathbf{V}^T \mathbf{G}\mathbf{V}$$

$$\mathbf{B}_r = \mathbf{V}^T \mathbf{B}, \quad x(t) = \mathbf{V}z(t)$$

Time-domain solution of the reduced order system

Frequency-domain

$$s\mathbf{C}x(s) + \mathbf{G}x(s) = \mathbf{B}u(s)$$

Build Kylov subspace orthonormal basis with block Arnoldi algorithm

$$span\{\mathbf{V}\} = \mathcal{K}_m(\mathbf{M}, \mathbf{R})$$

$$= span(\mathbf{R}, \mathbf{MR}, \mathbf{M}^2\mathbf{R}, ..., \mathbf{M}^{m-1}\mathbf{R})$$

$$\mathbf{M} = -\mathbf{G}^{-1}\mathbf{C} \quad \mathbf{R} = \mathbf{G}^{-1}\mathbf{B}$$

PRIMA [Odabasioglu' 98]

Exponential Integrator based on Rational Krylov Subspace

$$\mathbf{C}\dot{x}(t) + \mathbf{G}x(t) = \mathbf{B}u(t)$$

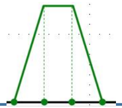
Convert to ODE

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{C}^{-1}\mathbf{B}u(t)$$

Analytical solution

$$x_{n+1} = e^{\mathbf{A}h}x_n + \boxed{\int_0^h e^{\mathbf{A}(h-\tau)}\mathbf{C}^{-1}\mathbf{B}u(\tau)\,d\tau}$$

Input piece-wise-linear (pwl) approximation



MEXP

$$x_{n+1} = \exp\left(\begin{bmatrix}\mathbf{A} & \mathbf{C}^{-1}\mathbf{B}u_n \\ 0 & 0\end{bmatrix} h\right)\begin{bmatrix}x_n \\ 1\end{bmatrix}$$

Get $\mathbf{V_m}$, $\mathbf{H_m}$ with Arnoldi algorithm

$$span\{\mathbf{V}_m\} = \mathcal{K}_m\left(\left(I - \gamma\mathbf{A}\right)^{-1}, \mathbf{R}\right)$$

$$\mathbf{H_m} = \mathbf{V_m^T}\mathbf{A}\mathbf{V_m} \qquad \gamma: shift\ parameter$$

$$e^{\mathbf{A}h}v \approx \beta\mathbf{V}_m e^{\left(I - \mathbf{H}_m^{-1}\right)h/\gamma} e_1$$

Posteriori residual estimate

$$res = \beta h_{m+1,m}(\mathbf{C}/\gamma + \mathbf{G})v_{m+1}e_m^T\mathbf{H_m}^{-1}e^{\mathcal{H}_m h}e_1$$

All in time-domain

Outline

Why EI and MOR

- **Same goal**
  - Accelerate transient circuit simulation
- **Similar approximation method**
  - Krylov subspace approximation
- **Separate research topics**
  - MOR: work mostly in the frequency-domain, classical
  - EI: pure time-domain method, younger (since 2010)

Contribution 1: The relation between MOR and EI

[C. WANG ICCAD’22]

Equivalence between EI and MOR

MOR

$$s\mathbf{C}x(s) + \mathbf{G}x(s) = \mathbf{B}u(s)$$

Generate V_M matching *m* moments around s_0

Obtain the reduced system

$$\mathbf{C}_r\dot{z}(t) + \mathbf{G}_r z(t) = \mathbf{B}_r u(t)$$

Advancing one step using the **EI method**

$$z_1 = e^{\mathbf{A}_r h} z_0 + h\varphi_1(\mathbf{A}_r h)\mathbf{C}^{-1}\mathbf{B}u_0$$

Recover the original solution

$$x_1^M = \mathbf{V}_M z_1$$

=

EI

$$\mathbf{C}\dot{x}(t) + \mathbf{G}x(t) = \mathbf{B}u(t)$$

Analytical solution with PWL approximation

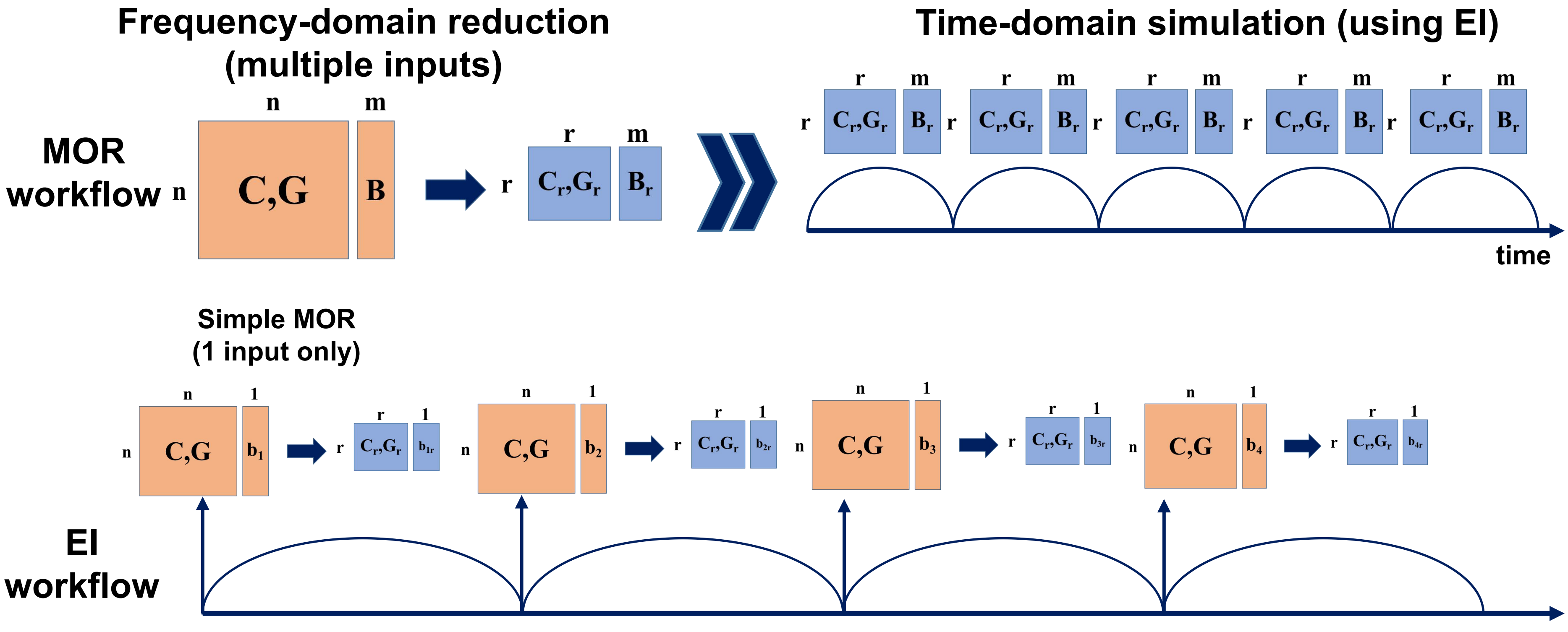
$$x_1 = e^{\mathbf{A}h} x_0 + h\varphi_1(\mathbf{A}h)\mathbf{C}^{-1}\mathbf{B}u_0$$

Rational Krylov subspace with shift parameter $\gamma = 1/s_0$ to get V_E and H_E

$$x_1^E = e^{\mathbf{A}h}\mathbf{v} \approx \|\mathbf{v}\|_2 \mathbf{V}_E e^{\mathbf{H}_E h} e_1$$

Zero initial condition x_0, a single-column input

Illustration of EI and MOR Equivalence



Direct time-domain simulation (one MOR per time step)

Equivalence: Corollary

- Main Theorem also holds for inhomogeneous initial condition $_0 \neq 0$
  - Decompose original system into two sub-systems to MOR process
- EI using (+ 1)-dimensional V matches the first moment of the exact solution
  - Another perspective explaining the superiority of EI

Comparisons of EI and MOR in circuit simulation

MOR: "**once-and-for-all**"	EI: "**on-the-fly**"
• Reduction once in frequency-domain, use ROM repeatedly in time-domain	• One MOR per time step, pure time-domain
• No knowledge of actual inputs, so require accuracy for any input during whole span	• Full knowledge of actual inputs, require accuracy for one input within one step
• Cost is high especially for many-input and wide-spectrum systems	• Independent of the number of inputs
• Difficult to choose expansion points and moments	• A single expansion point usually suffices. Easy to determine moments
• Difficult to handle nonlinear circuits	• Easy to deal with nonlinear circuits
• ROMs are re-usable	• ROMs not re-usable

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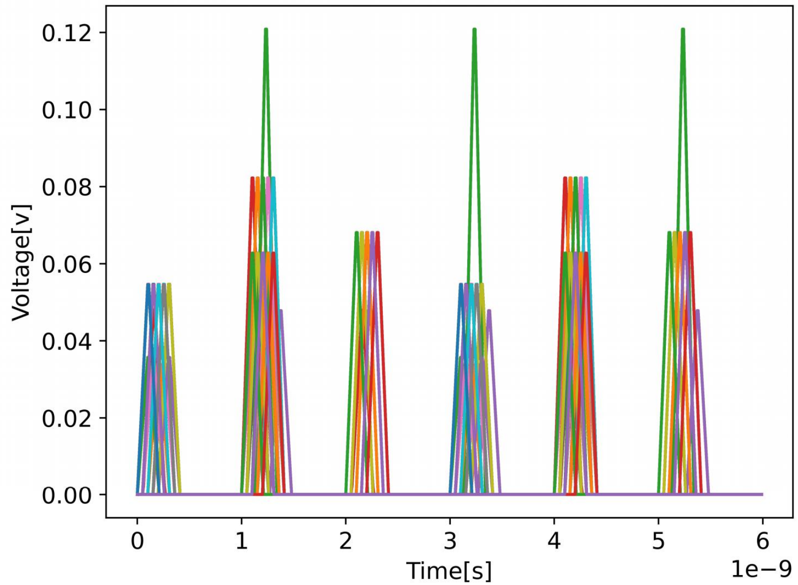
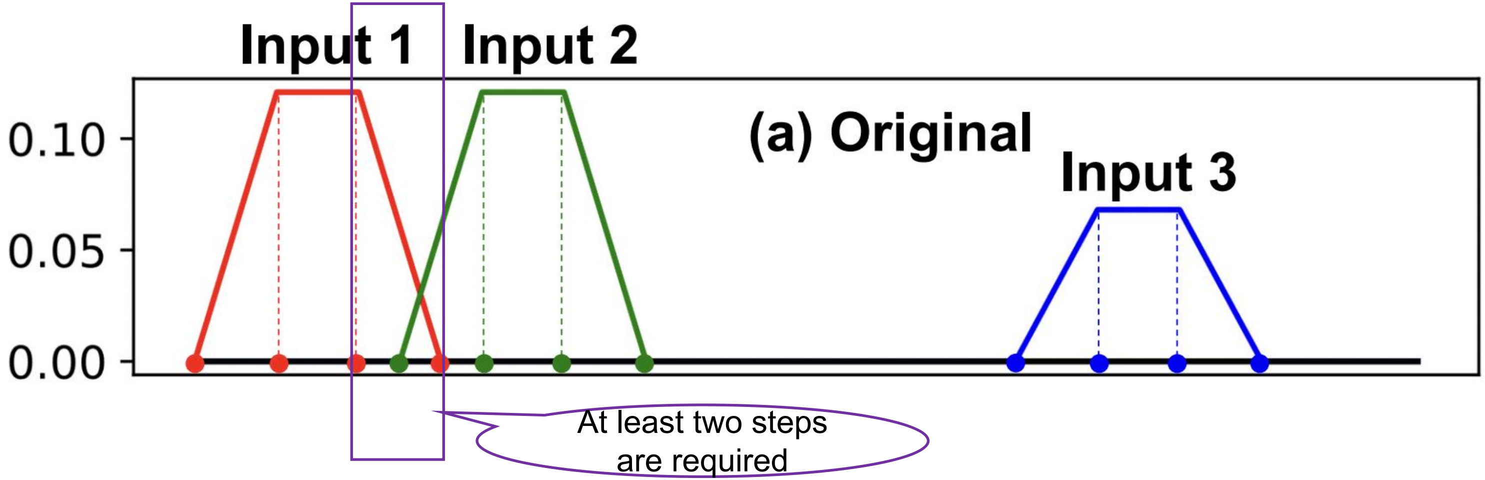
Hybrid EI-MOR method for P/G Analysis

- In P/G networks nonlinear devices are modeled as a large number of independent current sources
- Typically pulse or PWL waveforms



Input Misalignment Problem for EI



- PWL assumption in EI → every input linear within the same step
- Input misalignment will induce many breakpoints
- Usually not an issue for traditional SPICE, but serious for EI

Contribution 2: Inspired by EI-MOR equivalence, develop a hybrid EI-MOR method to address the misaligned input issue

Motivations

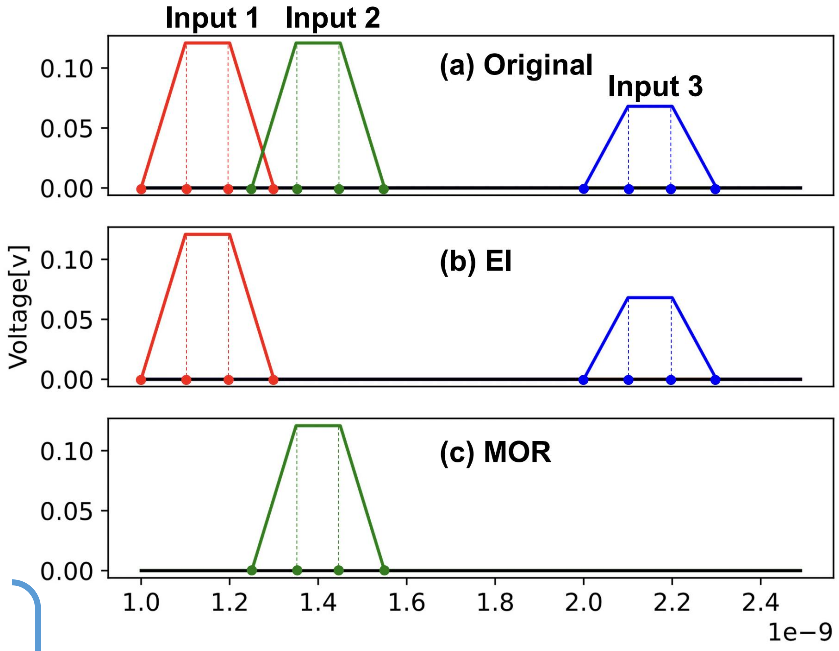
	Input #	Time step #
EI cost	Independent	Sensitive
MOR cost	Sensitive	Not sensitive

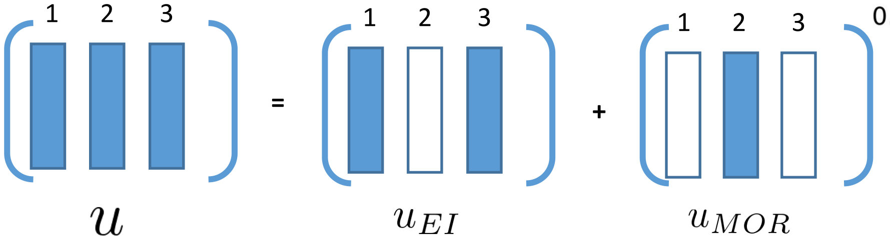
Our idea: a hybrid EI-MOR method

- **EI handles well-aligned inputs (majority)**
- **MOR deals with misaligned inputs(minority)**

Divide the original inputs

- Divide inputs into two groups
    - A majority group of well-aligned inputs for EI
    - A minority group of misaligned inputs for MOR
- EI inputs handled as usual with all breakpoints
- MOR inputs used in PRIMA to generate a reduced model for later time domain simulation
- The EI solution and MOR solution superimposed to obtain the final solution





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Result: Verification of EI-MOR Equivalence

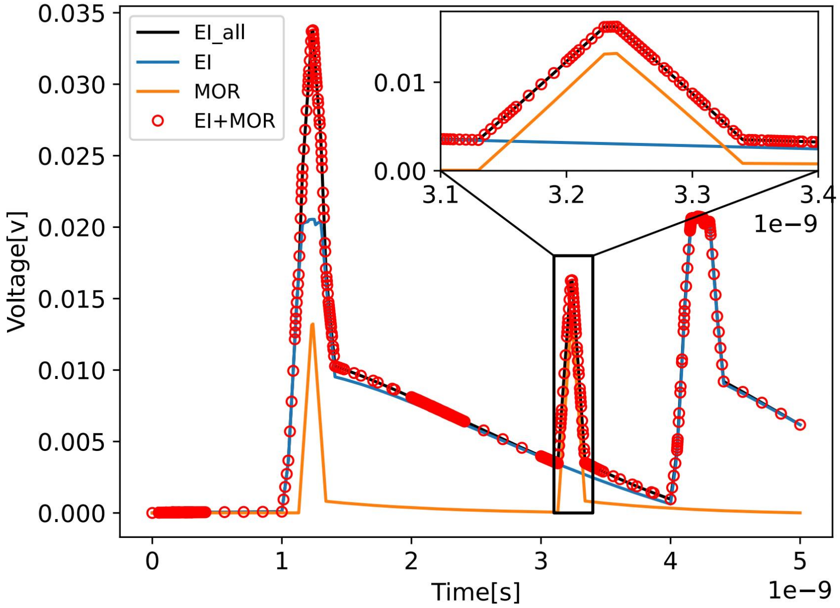
- Data: ibmpg1t
- Nonzero initial condition x_0
- a single-column input
- 3 different expansion points and 3 different moments matched of each expansion point

EI solution and MOR solution are numerically equal

Expansion Point	Moments	$\frac{\|x^{EI}-x^{MOR}\|}{\|x^{EI}\|}$
1e7	4	2.5418×10^{-14}
	6	8.1615×10^{-14}
	8	1.2139×10^{-13}
1e8	4	2.3310×10^{-14}
	6	5.9894×10^{-14}
	8	1.0884×10^{-13}
1e9	4	4.6395×10^{-14}
	6	7.7582×10^{-13}
	8	6.3915×10^{-12}

Result: Validity of "divide-and-superposition"

- Test instance: ibmpg1t（5883 current source，3 for MOR，5880 for EI）
- MOR based on PRIMA（s0=1e8, m=15)
- Both EI and MOR inputs have non-trivial contribution to the transient responses
- Produce the correct total solution when superimposed



Result: Performance of EI-MOR

Case	Node	VSource	ISource
ibmpg1t	39681	14308	5883
ibmpg2t	164238	330	18535
ibmpg3t	1041535	955	114191

$$i_{MOR} = \arg\max_{i} (k_0 - k_i) .$$

MOR inputs picked based on ability to reduce breakpoints

Case	Method	Isource	bp_{All}	bp_{EI}	$Arnoldi_{EI}$	$t_{EI}(s)$	m_{MOR}	ROM Size	$t_{MOR}^{Reduce}(s)$	$t_{MOR}^{Sim}(s)$	$t_{total}(s)$
ibmpg1t	EI	5883	517	517	16006	225.39	-	-	-	-	225.39
	EI-MOR	5880(EI)3(MOR)	517	342	12570	177.14	15	45	1.08	17.31	195.52
	EI-MOR	5877(EI)6(MOR)	517	277	8496	122.41	15	90	3.48	28.60	154.48
ibmpg2t	EI	18535	533	533	15840	1173.51	-	-	-	-	1173.51
	EI-MOR	18532(EI)3(MOR)	533	389	11509	865.06	15	45	5.32	86.77	957.16
	EI-MOR	18529(EI)6(MOR)	533	277	7700	634.85	15	90	17.12	84.71	736.68
ibmpg3t	EI	114191	509	509	13882	8420.87	-	-	-	-	8420.87
	EI-MOR	114188(EI)3(MOR)	509	375	10067	6306.15	20	60	93.33	471.59	6871.07
	EI-MOR	114185(EI)6(MOR)	509	277	7318	4733.61	20	120	194.13	672.05	5599.79

Note: EI or EI-MOR does not exclude other techniques like circuit partition, parallel LU and spectral graph sparsifier

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Conclusion

- **Equivalence between EI and MOR**
  - EI equivalent to one MOR per time step
- **A hybrid EI-MOR method for P/G networks**
  - Combine EI and MOR to address the misaligned inputs in EI
  - Good performance gain
- **New insights and opportunities for this classic EDA topic**.

Thank you!
Q & A

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Equivalence: Lemma

Lemma1: With zero initial condition $x_0 = 0$, a single-column $\mathbf{b}$ matrix and a real expansion point $s_0 = 1/\gamma$, let $\mathbf{V}_M \in \mathcal{R}^{N\times m}$ be an dimensional orthonormal basis with

MOR

$$colspan\{\mathbf{V}_M\} = \mathcal{K}_m(\mathbf{M},\ \mathbf{R})$$
$$= \mathcal{K}_m\left(-(s_0\mathbf{C}+\mathbf{G})^{-1}\mathbf{C},\ \ (s_0\mathbf{C}+\mathbf{G})^{-1}\mathbf{b}\right)$$

and $\mathbf{V}_E \in \mathcal{R}^{N\times(m+1)}$ be an $+1$ dimensional orthonormal basis with

$$colspan\{\mathbf{V}_E\} = \mathcal{K}_m\left(\left(\mathbf{I}-\gamma\tilde{\mathbf{A}}\right)^{-1},\ \tilde{x}_n\right)$$

EI

So, we can get the **relationship of the orthonormal basis**

$$\mathbf{V}_E = \begin{bmatrix} \mathbf{o}\mathbf{V}_M \\ 1\ \mathbf{o}^T \end{bmatrix}$$

$\mathbf{o} \in \mathcal{R}^{N\times 1}$

Equivalence: Main Theorem

With the assumptions stated in Lemma 3.1, EI, when solved with a (+ 1)-dimensional rational Krylov subspace with , is equivalent to applying a Krylov subspace based MOR, with the expansion point $s_0 = 1/\gamma$ and the subspace dimension , to the original system, and advancing the reduced system one step in the time domain using again the EI approach.

$$x_1^{MOR} = \mathbf{V_M} z_1 = -h \mathbf{V_M} \varphi_1 \left(\left(s_0 \mathbf{I_M} + \mathbf{H_M}^{-1} \right) h \right) \mathbf{H_M}^{-1} e_1 u_0$$

$$\mathbf{x}_1^{MOR} = \mathbf{x}_1^{EI}$$

$$\begin{aligned}
x_1^{EI} &\approx h \mathbf{V_M} \varphi_1 \left(\left(s_0 \mathbf{I_M} + \mathbf{H_M}^{-1} \right) h \right) \mathbf{V_M}^T \mathbf{C}^{-1} \mathbf{b} u_0 \\
&= h \mathbf{V_M} \varphi_1 \left(\left(s_0 \mathbf{I_M} + \mathbf{H_M}^{-1} \right) h \right) \mathbf{V_M}^T \mathbf{C}^{-1} \mathbf{K} \mathbf{K}^{-1} \mathbf{b} u_0 \\
&= h \mathbf{V_M} \varphi_1 \left(\left(s_0 \mathbf{I_M} + \mathbf{H_M}^{-1} \right) h \right) \mathbf{V_M}^T \left(\mathbf{K} \mathbf{C}^{-1} \right)^{-1} \mathbf{V_M} e_1 u_0 \\
&= -h \mathbf{V_M} \varphi_1 \left(\left(s_0 \mathbf{I_M} + \mathbf{H_M}^{-1} \right) h \right) \mathbf{H_M}^{-1} e_1 u_0
\end{aligned}$$