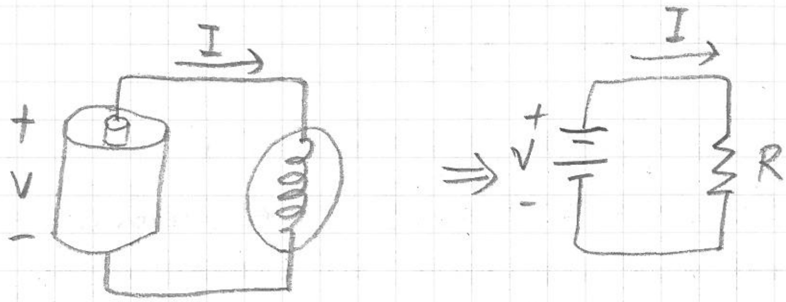
- The lumped circuit abstraction



for now, let's just consider elements having two terminals, e.g.,

battery,
lightbulb

We may ignore the internal structure of an element and consider it as a "lump", where we may completely describe relevant properties (such as voltage and current) by only observing its terminals.

For example, if a resistor with resistance R obeys Ohm's law, then we may compute the current flowing through the resistor by the voltage across its terminals : $I = V/R$.

The lumped circuit abstraction works only under certain constraints, which we call the lumped matter discipline (LMD).

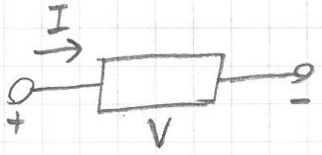
From Physics to Electronic Circuit:

☆ The lumped matter discipline (LMD) → a set of rules that control an activity or situation.

→ simplify the analysis of electronic circuit

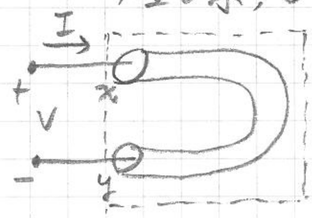
→ modularize a complex circuit into analyzable elements

Derivation of LMD:



$\boxed{\text{goal\#1: able to ascribe a unique voltage across the terminals } x \text{ and } y}$

example element
燈絲, U型, 長度為 l



definition of voltage:

$$V_{yx} = -\int_x^y \mathbb{E} \cdot dl$$

where $\mathbb{E}$ is the electrical field (a vector)

dl is a tiny portion of l

and Faraday's law of induction:

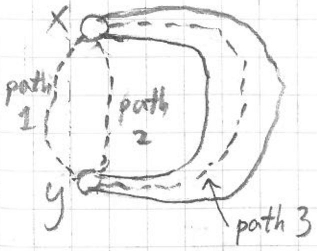
$$\oint \mathbb{E} \cdot dl = -\frac{\partial \phi_B}{\partial t}$$

where ϕ_B is the magnetic flux.

$\oint$ represents a closed path integral

Then we see that if there's no time-varying magnetic flux,

$$\oint \mathbb{E} \cdot dl = 0 \Rightarrow \underbrace{\int_x^y \mathbb{E} \cdot dl}_{\text{along path 1}} + \underbrace{\int_y^x \mathbb{E} \cdot dl}_{\text{path 2}} = 0$$



$$\Rightarrow \int_{x \atop \text{path 1}}^y \mathbb{E} \cdot dl = \int_{x \atop \text{path 2}}^y \mathbb{E} \cdot dl$$

也就是說 $\int_x^y \mathbb{E} \cdot dl$ 的值
和路徑無關! $\left(= \int_{x \atop \text{path 3}}^y \mathbb{E} \cdot dl\right)$

Therefore, the constraint for goal # 1 is

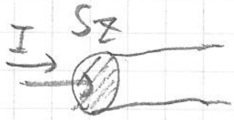
$$\frac{\partial \phi_B}{\partial t} = 0 ,$$

and we assumed that holds for all time.
(To make sure this constraint holds, we may need to revise the model and introduce an element called "inductor".)

$$\boxed{\begin{array}{l}\text{goal \# 2 :}\\ \text{able to define a unique current}\\ \text{through the terminals } x \text{ and } y\end{array}}$$

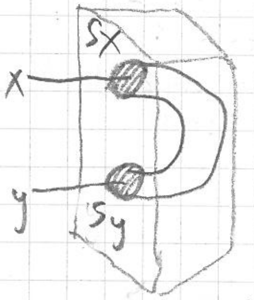
First of all, the definition of current :

$$I = \int_{S_z} J \cdot dS$$



where J is the current density at a given point within the filament and S_z is the cross-sectional surface of the filament at point z.
Due to the conservation of charge, we have

$$\underbrace{\oint J \cdot dS}_{\text{流出的电量}} = \underbrace{-\frac{\partial q}{\partial t}}_{\text{減少的电量}} \quad \text{for a closed surface}$$



假設 S_x 為唯一入口
S_y 為唯一出口.

Thus, if there's no time-varying charge within the closed surface, we have

$$\oint J \cdot ds = 0 \Rightarrow -\int_{S_x} J \cdot ds + \int_{S_y} J \cdot ds = 0$$

$$\Rightarrow \int_{S_y} J \cdot ds = \int_{S_x} J \cdot ds$$

$$I_{in} = I_{out}$$

Therefore, the constraint for goal # 2 is

$$\frac{\partial q}{\partial t} = 0,$$

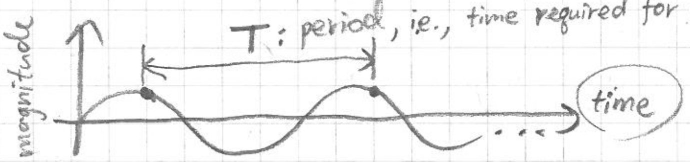
and we assume that holds for all time.
(To make sure this constraint holds, we may need to revise the model and introduce an element called "capacitor".)

Besides goals #1 and #2, we also need to assume that the signal timescale (→ rate of change, i.e., frequency) must be much larger than the propagation delay of electromagnetic waves across the lumped elements. (↳ ...0110100... for example) (Otherwise, ... see textbook P11)

⇒ the size of our lumped elements must be much smaller than the wavelength associated with the V and I signals, and such a condition may be challenging to hold as we reduce the element size and increase the operating frequency (e.g., a 2 GHz CPU).

→ definition of wavelength (λ): the distance between two adjacent points in the wave having the same phase.

T : period, i.e., time required for a wave to travel for the distance of one wavelength.



$$\lambda = v \cdot T = v \cdot \frac{1}{f} \Rightarrow f \uparrow \text{ then } \lambda \downarrow$$

wavelength (λ), wave speed (v), period (T), frequency (f)

for example, electromagnetic waves travel at about 15×10^4 km/s, or 15×10^9 cm/s, within a microprocessor (to be specific, through silicon dioxide). Now, suppose that the microprocessor operates at a clock rate of 2 GHz. This translates to the wavelength equal to

$$\lambda = 15\times10^9 / 2\times10^9 = 7.5 \text{ cm},$$

which means that LMD may not hold if the microprocessor chip is larger than 7.5 cm on a side.

Think about it : what if ① clock rate ↑ ?
② wave speed ↑ ?

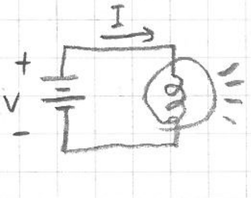
In general, in computer engineering, people are often working to meet various constraints (such as this) so that they may apply a previously established model (such as LMD) and make use of known results/properties that depend on the given model. This is like "Standing on the shoulders of giants."

- Basic lumped element 1: batteries

Two key properties $\begin{cases} \text{energy 能量 (unit: joule or ampere-hours or watt-hours)} \\ \text{power 功率 (unit: watt)} \end{cases}$

功率 → 能量轉換或使用的速率

$$\boxed{\begin{array}{c} P = V\cdot I \\ 1 \text{ watt} = 1 \text{ volt}\cdot 1 \text{ ampere} \end{array}}$$



let $\mathcal{E}$ be the amount of <u>energy</u> supplied to an element over an interval T, then we have $\mathcal{E} = p \cdot T$. In general, the amount of energy supplied is the time integral of the power.

1 joule = 1 watt - second

Example: Suppose a Raspberry Pi consumes 2W of power and its energy is supplied by a 3.7 V, 2600 mA-h battery. For how long can the battery power the Raspberry Pi ?

$$p = V \cdot I = 3.7 \times 2600 \times 10^{-3} \ \text{W-h}$$
$$= 9.62 \ \text{W-h}$$

$$\frac{9.62 \ \text{W-h}}{2 \ \text{W}} = 4.81 \ \text{hours} \ \divideontimes$$

- Basic lumped element 2: resistors (linear)

Ohm's law: the voltage measured across the terminals of a resistor is linearly proportional to the current flowing through the resistor.

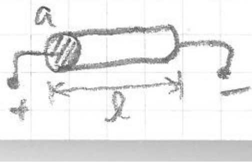
That is, $\boxed{v = i \cdot R}$

we call it the <u>resistance</u> of a resistor.

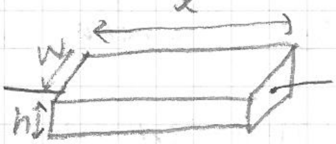
Further, $\boxed{R = \rho \frac{l}{a}}$

resistivity

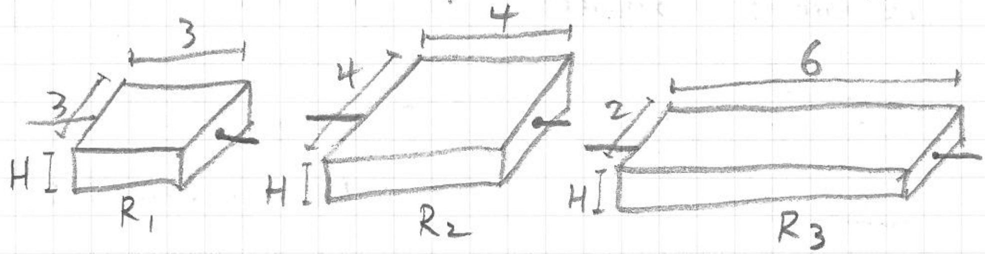
(see Appendix A.3 in the textbook)



also, $\boxed{R = \rho \frac{l}{wh}}$



Example: Consider three planar resistors as follows



Let $R_0 = \rho_0 \frac{1}{1 \cdot H} = 2\,k\Omega$ and assume $\rho_0 = \rho_1 = \rho_2 = \rho_3$

Then $R_1 = \rho_1 \frac{3}{3 \cdot H} = R_0 = 2\,k\Omega$

$$\frac{R_1}{R_2} = \frac{\rho_1 \frac{3}{3H}}{\rho_2 \frac{4}{4H}} = 1 \quad \text{and } R_2 = R_1 = 2\,k\Omega$$

$$\frac{R_2}{R_3} = \frac{\rho_2 \frac{4}{4H}}{\rho_3 \frac{6}{2H}} = \frac{1}{3} \Rightarrow R_3 = 6\,k\Omega$$

exercise: you can verify that $\frac{R_1}{R_3} = \frac{R_2}{R_3}$.

⇒ 等比例縮小長及寬，則相對電阻值 ____ ?

A. ↓ B. ↑ C. 不變

⇒ 縮小晶片的大小不會改變相對電阻值

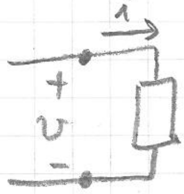
⇒ Often, signal values are derived as a function of resistance ratios. Therefore, by such a process shrink the chip may continue to function as before!

example: a voltage divider, which we will study later this semester.

P8

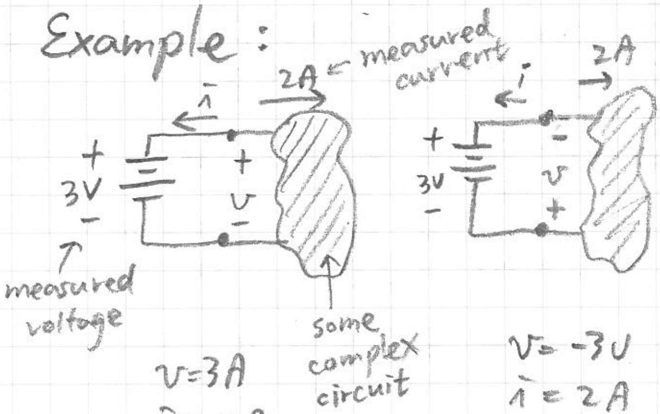
– Associated Variable Convention : (約定俗成)

For a two-terminal lumped element, define current to flow in at the element terminal assigned to be positive in voltage.



v and i are called "terminal variables."

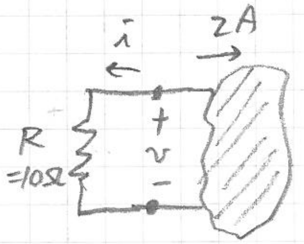
Example :



$v = 3A$
$i = -2A$
$P = v \cdot i = -6W$

$v = -3V$
$i = 2A$
$P = v \cdot i = -6W$

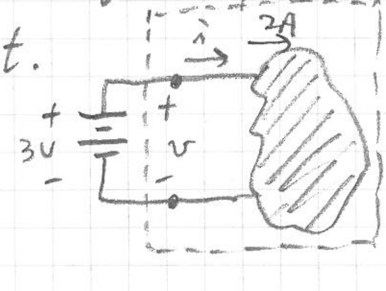
⇒ power supplied "from" battery
by



$i = -2A$
$v = iR = -20V$
$P = v \cdot i = 40W$

⇒ power supplied "to" resistor
(and then is dissipated in the form of heat)

Note that the above is from the viewpoint of the battery ; from the viewpoint of that complex circuit, you may try and see that now power is supplied "to" it.



☆ Signals, Systems, and Computing

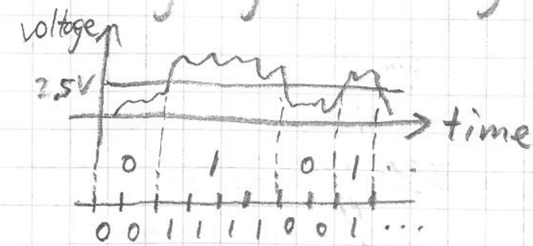
- In computing systems, information (and energy) is stored and transferred in terms of <u>signals</u>, which are <u>currents</u> or <u>voltages</u> as a function of time.

A circuit (or a system of circuits), as we study in this course, is used to
- ① carry signals
- ② transform signals from one to another

Computing is also a transformation of signals; though the signals are digital are the transformation is described at a higher layer of the abstraction (Figure 1.1 in the textbook).

Example: We use a smart phone to record this lecture (voice signal → currents and voltages), and the recording is stored in the phone, transferred via USB to your laptop, uploaded to a cloud drive, downloaded by your friend who cannot make it to the class, and finally played by your friend's speakers/headphone (currents and voltages → voice signals).

- Analog signal to digital signal via discretization:



P10 Section 1.8 in the textbook mentioned sinusoidal signals and the <u>root mean square</u> value. For your interest, the $\sqrt{2}$ ratio between the amplitude of a sinusoidal signal and its rms value comes from 三角函數2倍角轉換.

Example: let signal $i(t) = I_m \cos(\omega t)$

$$i_{rms} = \sqrt{\frac{1}{T}\int_0^T i^2(t) \cdot dt}$$

$$= \sqrt{\frac{1}{T}\int_0^T (I_m^2 \cos^2 \omega t)\, dt}$$

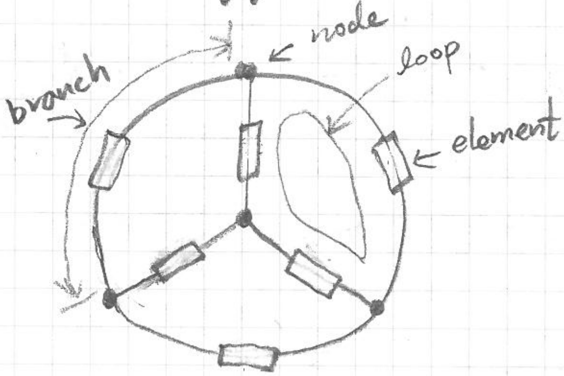
$$= \sqrt{\frac{I_m^2}{2T}\int_0^T (1+\cos 2\omega t)\, dt}$$

$$= \frac{I_m}{\sqrt{2}}\sqrt{\frac{1}{T}\int_0^T (1+\cos 2\omega t)\, dt}$$

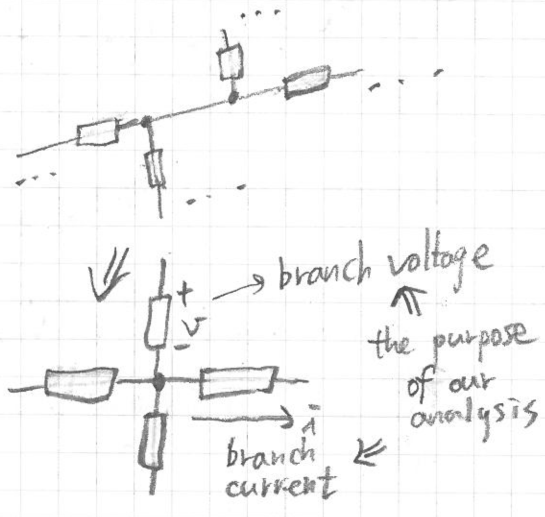
$$= \frac{I_m}{\sqrt{2}} \quad ※$$

☆ Resistive Networks and How to Analyze Them

- Terminology



ideal wire : no resistance



- Kirchhoff's Laws $\begin{cases} \underline{KCL} \\ \underline{KVL} \end{cases}$

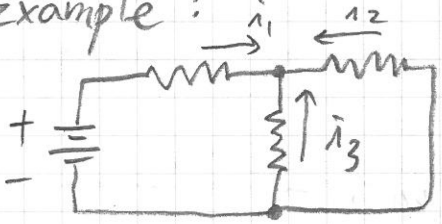
following LMD, Kirchhoff's laws are simplifications of Maxwell's Equations (see Appendix A.2 in the textbook)

$\longrightarrow$ KCL and KVL are extremely useful tools to help us analyze a circuit!

KCL : Kirchhoff's current law

$\boxed{\text{The algebraic sum of all branch currents flowing into any node must be zero.}}$

Example :



$$\sum_{n=1}^{3} i_n = 0$$

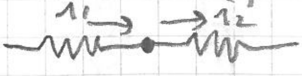
$$\sum_{n=1}^{3} (-i_n) = 0$$

In general, for integers N, M, we have

$$\sum_{n=1}^{N} i_n = 0 \Rightarrow \sum_{n=1}^{M} i_n + \sum_{n=M+1}^{N} i_n = 0$$

$$\Rightarrow \sum_{n=1}^{M} i_n = \sum_{n=M+1}^{N} (-i_n)$$

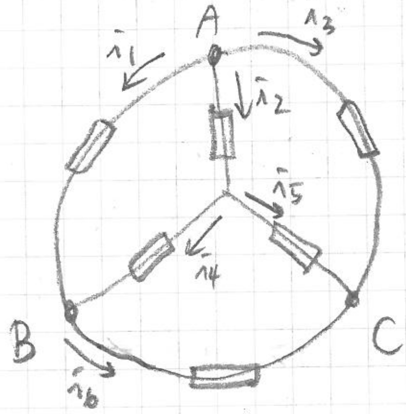
$\Rightarrow$ Sum of total current flowing into a node = sum of total current flowing out from a node.



$i_1 = i_2$ ideal wire

P12

Example:



Using KCL

node A: $0 = -i_1 - i_2 - i_3$

B: $0 = i_1 + i_4 - i_6$

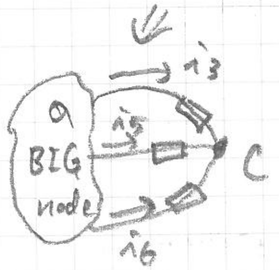
C: $0 = i_2 - i_4 - i_5$

D: $0 = i_3 + i_5 + i_6$

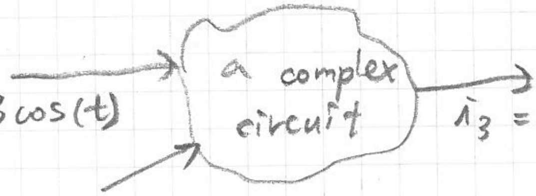
Suppose $i_1 = 1$, $i_3 = 3 \Rightarrow i_2 = -4$

In addition, suppose $i_5 = -2$

$\Rightarrow i_4 = -2$, $i_6 = -1$ #



In general, for N KCL statements, only N-1 of them are independent. Therefore, we need N-1 known values to solve the circuit.



$i_1 = 3\cos(t)$

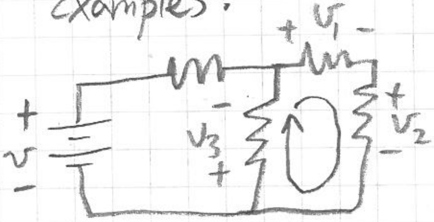
$i_2 = 6\cos(t)$

$i_3 = i_1 + i_2 = 9\cos(t)$

KVL: Kirchhoff's voltage law

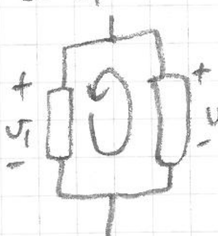
The algebraic sum of the branch voltages around any closed path in a network must be zero.

Examples:



$$\sum_{n=1}^{3} v_n = 0$$

並聯



$0 = v_1 - v_2$

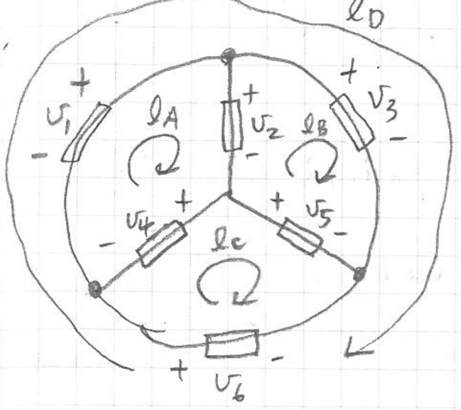
$\Rightarrow v_1 = v_2$

串聯

$0 = -v + v_1 + v_2$

$\Rightarrow v = v_1 + v_2$

Example:



Using KVL

loop l_A: $0 = -v_1 + v_2 + v_4$

l_B: $0 = -v_2 + v_3 - v_5$

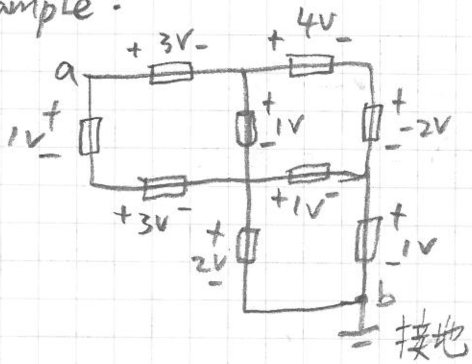
l_C: $0 = -v_4 + v_5 - v_6$

l_D: $0 = -v_1 + v_3 - v_6$

Suppose that $v_1 = 1$, $v_3 = 3$, $v_2 = 2$

then $v_4 = -1$, $v_5 = 1$, $v_6 = 2$ ☆

Example:



v_{ab} can be determined using KVL

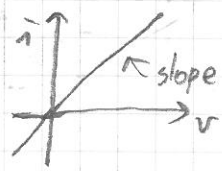
(which loop(s) would you pick?)

$v_{ab} = 6\ V.$

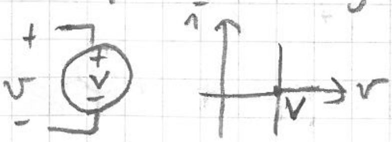
> In general, we model individual elements based on LMD assumptions and we analyze a circuit of elements using KCL and KVL.

☆ Two more basic elements and their i-v characteristics

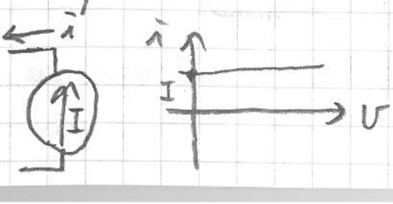
for linear resistor it obeys Ohm's law $R = \frac{v}{i}$



① Independent voltage source



② Independent current source



P14

☆ Basic Method to Analyze A Circuit

four steps

see Page 8

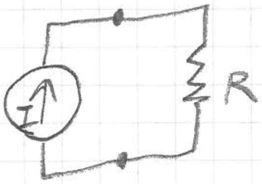
1° define branch current and voltage <u>consistently</u>

2° apply element laws for each elements (e.g., Ohm's law for linear resistors)

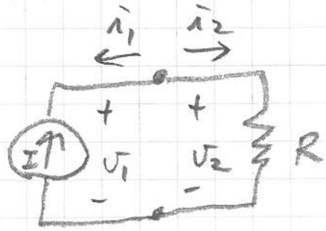
3° apply KCL and KVL

4° jointly solve the equations obtained from 2° and 3°.

Example :



1°



2° $i_1 = -I$, $v_2 = i_2 \cdot R$

3° KCL $\Rightarrow i_1 + i_2 = 0$

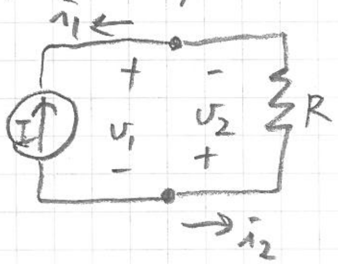
KVL $\Rightarrow -v_1 + v_2 = 0$

4° $i_2 = -i_1 = I$

$v_2 = i_2 R = IR$

$v_1 = v_2 = IR$ ※

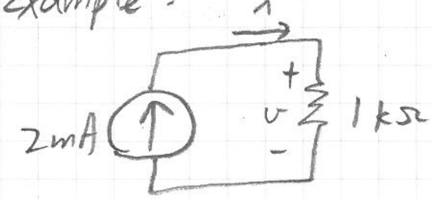
if at step 1° we define v_2 inversely, we must



also define i_2 inversely, to be consistent. Applying the basic method you will find $i_2 = -I$ and $v_2 = -IR$. This may seem to be strange. But if we recall that both i_2 and v_2 have a reversed direction, then it makes sense.

Alternatively, we may solve a circuit by considering "Energy conservation".

Example :



power out from the source :

$P_{out} = 2mA \times v$

power into the resistor :

$P_{in} = i \times v = \frac{v^2}{R}$

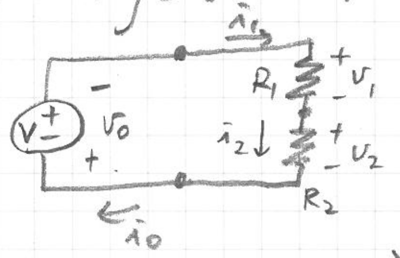
$= \frac{v^2}{1k\Omega}$

$P_{out} = P_{in} \Rightarrow 2\times10^{-3} \cdot v = \frac{v^2}{1\times10^3}$

$\Rightarrow v = 2\ V$ ※

(Example 2.14 in the textbook has a typo saying $v = 0.5\ V$)

☆ Voltage Divider



we may analyze it using the basic method :

$$\begin{cases} v_0 = V \\ v_1 = R_1 i_1 \\ v_2 = R_2 i_2 \end{cases} \quad \begin{cases} i_0 = i_1 \\ i_1 = i_2 \end{cases} \text{KCL} \quad v_0 + v_1 + v_2 = 0 \quad \text{KVL}$$

$$\Rightarrow \boxed{v_2 = \frac{R_2}{R_1 + R_2} V}$$

Further, from $i_2 = \frac{v_2}{R_2}$ we see $i = \frac{1}{R_1 + R_2} V$

in other word, $V = i \times (R_1 + R_2)$.

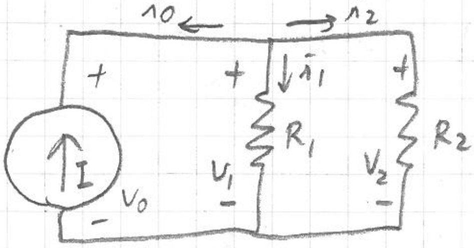
$\Rightarrow$ we may replace R_1 and R_2 by an equivalent resistor $R' = R_1 + R_2$

and the circuit is equivalent as

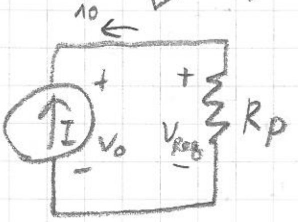
This lead to a general planar linear resistor analysis such as that in Example 2.21 in the textbook.

P16

✱ Current Divider



equivalent



Using the "basic method"

$$\begin{cases} i_0 = -I \\ v_1 = R_1 i_1 \\ v_2 = R_2 i_2 \end{cases} \quad \begin{cases} i_0 + i_1 + i_2 = 0 \\ v_0 = v_1 = v_2 \end{cases}$$

$$\Rightarrow i_1 + i_2 = I$$

$$i_1 + i_2 = \frac{v_1}{R_1} + \frac{v_2}{R_2} = \frac{v_0 R_2 + v_0 R_1}{R_1 R_2}$$

$$= \frac{R_1 + R_2}{R_1 R_2} v_0$$

$$\Rightarrow v_0 = \frac{R_1 R_2}{R_1 + R_2} \cdot I$$

$$\Rightarrow \boxed{\begin{aligned} i_1 &= \frac{v_1}{R_1} = \frac{R_2}{R_1 + R_2} I \\ i_2 &= \frac{v_2}{R_2} = \frac{R_1}{R_1 + R_2} I \end{aligned}}$$

$$v_1 = R_1 i_1 = \frac{R_1 R_2}{R_1 + R_2} I$$

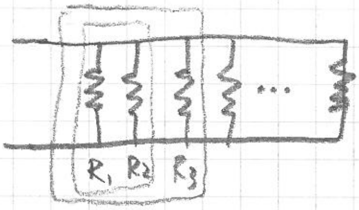
$$\parallel$$

$$v_0 = R_{eq} \cdot I$$

$$\Rightarrow \boxed{R_p = \frac{R_1 R_2}{R_1 + R_2}}$$

$$\Rightarrow \frac{1}{R_p} = \frac{1}{R_1} + \frac{1}{P_2}$$

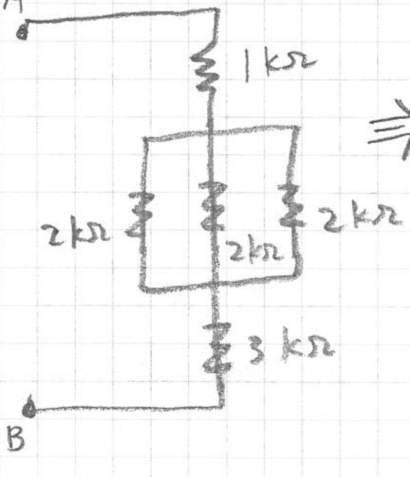
In general, for N resistors connected in parallel,



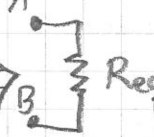
then

$$\frac{1}{R_p} = \sum_{n=1}^{N} \frac{1}{R_n}$$

this can be proved by induction.



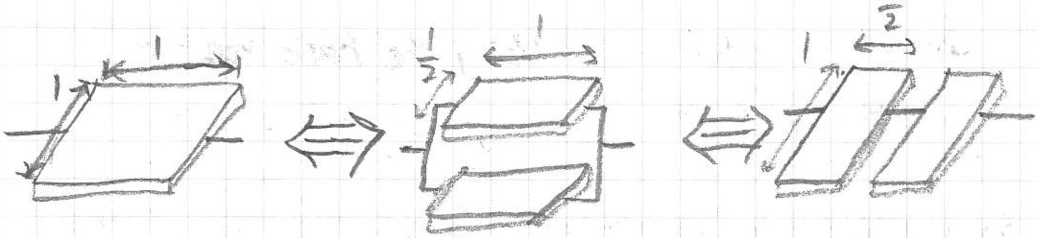
$\Rightarrow$



$$R_{eq} = 1 + \frac{1}{\frac{1}{2} + \frac{1}{2} + \frac{1}{2}} + 3$$

$$= \frac{14}{3} \Omega$$

※

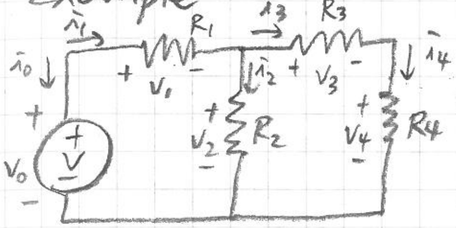


let $R_1 = R_\square$

$$R_2 = \frac{2R_\square \cdot 2R_\square}{2R_\square + 2R_\square} = R_\square = R_1$$

$$R_3 = \frac{1}{2}R_\square + \frac{1}{2}R_\square = R_\square = R_1$$

Example:



Determine v_1, v_2, v_3, v_4 and i_1, i_2, i_3, i_4.

Way ①: we may use the basic method

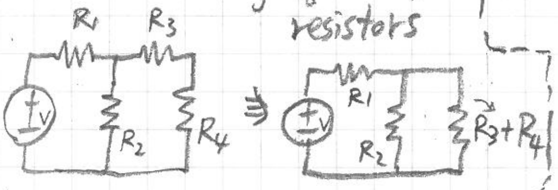
2° element law ...

3° KCL & KVL ...

4° 解聯立方程式

give it a try yourself here:

Way ②: transformation using equivalent resistors



$$\frac{R_2(R_3+R_4)}{R_2+R_3+R_4}$$

$$R_1 + \frac{R_2(R_3+R_4)}{R_2+R_3+R_4}$$

$i_1 = V \Big/ \left(R_1 + \dfrac{R_2(R_3+R_4)}{R_2+R_3+R_4}\right)$

$v_1 = i_1 R_1 =$ ___

$v_2 =$ ___ (voltage divider)

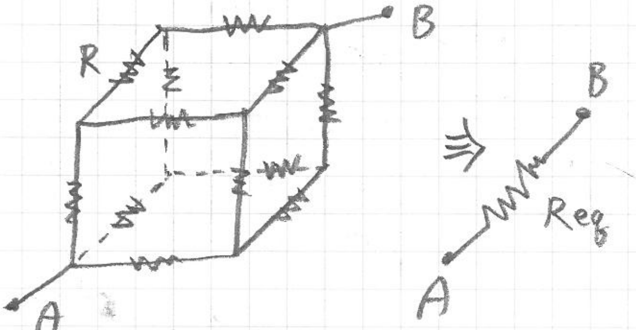
$i_2 = v_2/R_2$, $i_3 = v_2/(R_3+R_4)$

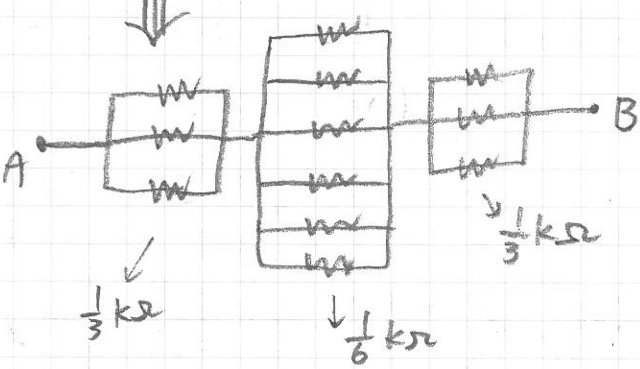
$v_3 = i_3 R_3$, $v_4 = i_3 R_4$

P18 Sometime we may leverage symmetry to greatly reduce the complexity of analysis:

Example: assuming all resistors are the same on a cube, with resistance $R = 1\,k\Omega$, determine R_{eq}

(see example 2.24 in the textbook)

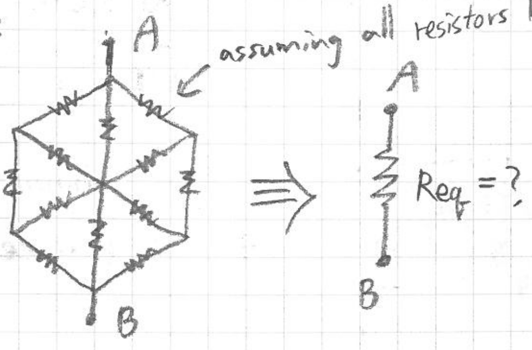




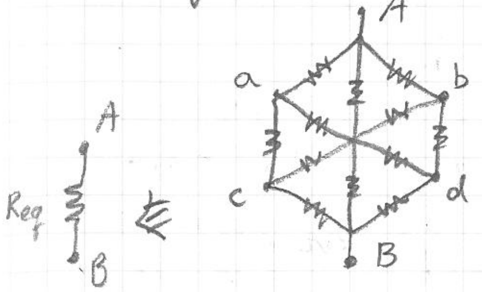
$$R_{eq} = \frac{1}{3} + \frac{1}{6} + \frac{1}{3} = \frac{5}{6}\,k\Omega \;\ast$$

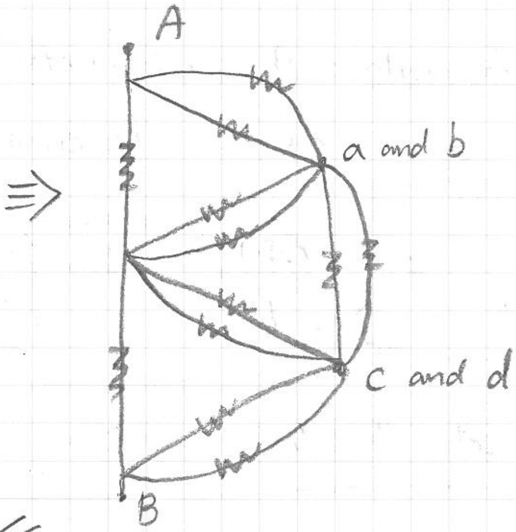
Often, we still need to apply the basic method after reducing a circuit by using equivalent resistors and symmetry!!

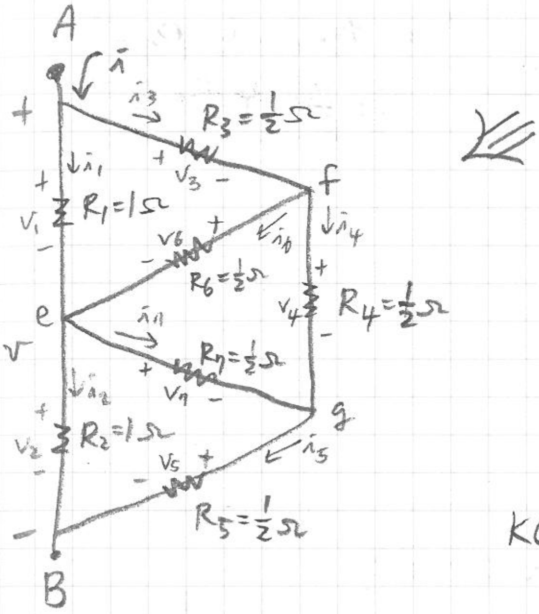
Example: assuming all resistors have 1Ω resistance



following P18,







element laws:

$$\begin{cases} v_1 = i_1,\ v_2 = i_2,\ v_3 = \frac{1}{2} i_3 \\ v_4 = \frac{1}{2} i_4,\ v_5 = \frac{1}{2} i_5,\ v_6 = \frac{1}{2} i_6 \\ v_7 = \frac{1}{2} i_7 \end{cases}$$

KCL:

① $i = i_1 + i_3 = i_2 + i_5$ (node A & B)

② $i_1 + i_6 = i_2 + i_7$ (node e)

③ $i_3 = i_6 + i_4$ (node f)

④ $i_5 = i_7 + i_4$ (node g)

KVL:

⑤ $v_1 + v_2 = v_3 + v_4 + v_5$

⑥ $v_1 = v_3 + v_6$

⑦ $v_2 = v_7 + v_5$

$$Req = \frac{v}{i} = \frac{v_1 + v_2}{i_1 + i_3} = \frac{i_1 + i_2}{i_1 + i_3}$$

therefore, 我們可將所有電壓值代換為電流值，去解電流的聯立方程式!!

$$\begin{array}{c} ① \\ ② \\ ③ \\ ④ \\ ⑤ \\ ⑥ \\ ⑦ \end{array} \begin{bmatrix} 1 & -1 & 1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & -1 \\ 2 & 2 & -1 & -1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

note: equation ④ is dependent.

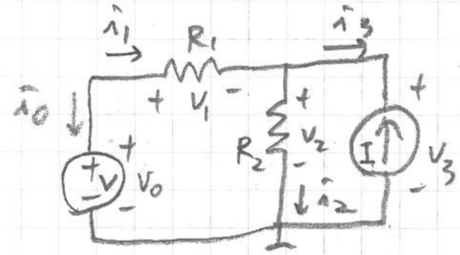
使用高斯消去法 可得 $\begin{cases} i_1 = 2 i_7 \quad i_3 = 3 i_7 \\ i_2 = 2 i_7 \end{cases}$

or www.mathtools.com Matrix Calculator

$$\therefore Req = \frac{i_1 + i_2}{i_1 + i_3} = \frac{2+2}{2+3} = \frac{4}{5}\Omega \ ※$$

Calculation is a necessary part in engineering!

P_{20} Example : A circuit with two independent sources



Determine i_2.

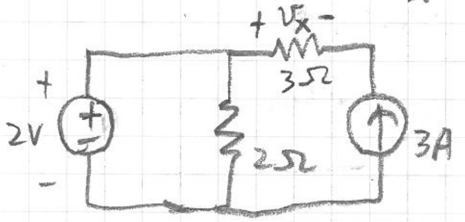
element law $\begin{cases} V_0 = V \\ V_1 = R_1 i_1 \\ V_2 = R_2 i_2 \\ i_3 = -I \end{cases}$ KCL $\begin{cases} i_0 = -i_1 \\ i_1 = i_2 + i_3 \end{cases}$

KVL $\begin{cases} V_0 = V_1 + V_2 \\ V_2 = V_3 \end{cases}$

→ then solve these linear equations (see textbook P95).

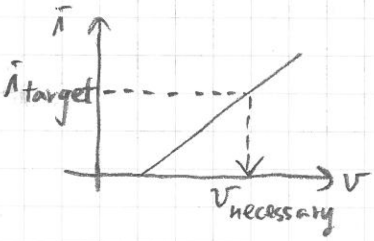
$$i_2 = \frac{R_1}{R_1+R_2} I + \frac{1}{R_1+R_2} V \quad ※$$

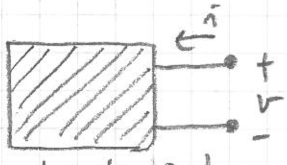
Exercise: $V_x = ?$



(ans : $V_x = -9V$)

☆☆ The I-V characteristic of a circuit

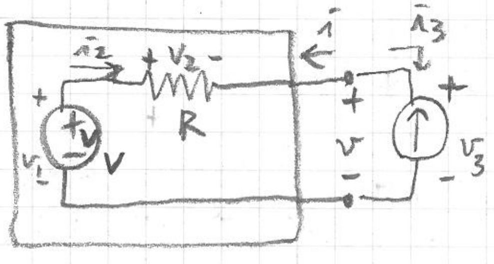




a two-terminal device

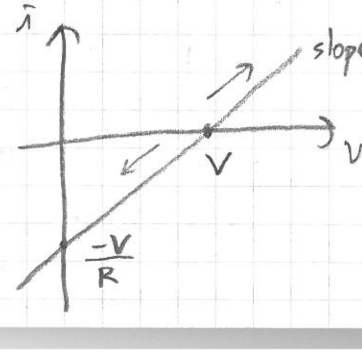
我們可藉由給定 v 量測 i (or vice versa) 來繪製 I-V relation.

If we know the device's internals, we may also determine its I-V relation:



using the basic method, we have

$$v = V + iR \Rightarrow i = \frac{1}{R} v - \frac{V}{R}$$



example usage :

預測电流流向

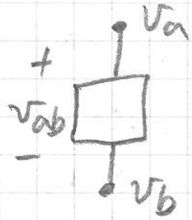
$\begin{cases} v \geq V \rightarrow i \geq 0 \\ v < V \rightarrow i < 0 \end{cases}$ ⇐

☆ Circuit Analysis using the Node Method

Motivation:

Often, the number of nodes in a circuit is much smaller than that of branches. Node method thus involves fewer number of variables, which means it is often easier to solve.

- definition of node voltage:



$v_{ab} = \underbrace{v_a}_{} - \underbrace{v_b}_{}$ node voltage

Procedure of the node method:

1° select a reference node (接地, $v=0$) $\perp$

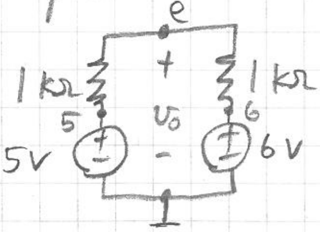
2° assign node variables

3° apply KCL

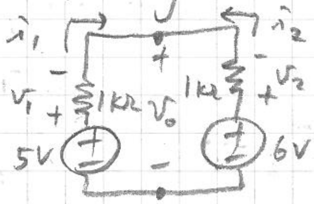
4° solve equations

5° back-solve the needed branch voltage/current

Example: find $v_0 = ?$



if using the basic method,



$i_1 = \frac{v_1}{1k\Omega}$

$i_2 = \frac{v_2}{1k\Omega}$

using the node method,

$$\frac{5-e}{1k\Omega} + \frac{6-e}{1k\Omega} = 0$$

$e = 5.5V$ $\quad v_0 = e - 0 = 5.5V$ ※

$i_1 + i_2 = 0$

$\Rightarrow v_1 + v_2 = 0$

using KVL,

$5 - v_1 + v_2 - 6 = 0$

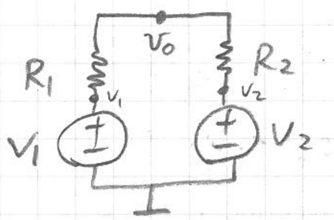
$\Rightarrow v_1 - v_2 = -1$

$\begin{cases} v_1 = -0.5 \\ v_2 = 0.5 \end{cases}$

$v_0 = 6 - v_2$ (KVL)

$= 5.5\,V$ ※

P22 Symbolic Computation can give us some insights:



KCL: $\frac{v_1 - v_0}{R_1} + \frac{v_2 - v_0}{R_2} = 0$

$\Rightarrow R_2(v_1 - v_0) + R_1(v_2 - v_0) = 0$

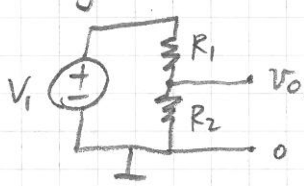
$\Rightarrow v_0 = \frac{1}{R_1 + R_2}(R_2 v_1 + R_1 v_2)$

$= \frac{R_2}{R_1 + R_2} v_1 + \frac{R_1}{R_1 + R_2} v_2$

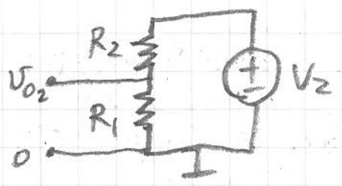
Insights ① v_0 is a linear combination of v_1 and v_2.
The circuit acts as an adder that gives a weighted sum of v_1 and v_2.

② if set $v_2 = 0$, then $v_{01} = \frac{R_2}{R_1 + R_2} v_1$,
which is equivalent to the result of a voltage divider:

i.e., having the same i-v characteristic.

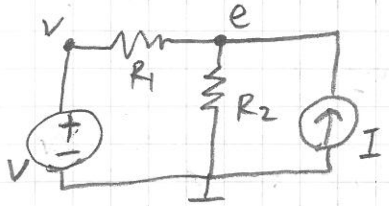


③ similarly, if set $v_1 = 0$, then
$v_{02} = \frac{R_1}{R_1 + R_2} v_2$, equivalent to the result of a voltage divider:



$\Rightarrow$ from ② and ③, the original circuit can be thought of as a superposition of two voltage dividers, with $v_0 = v_{01} + v_{02}$.

Another example:



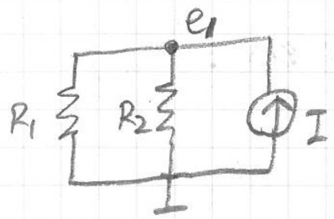
$$\text{KCL}: \frac{V-e}{R_1} + \frac{0-e}{R_2} + I = 0$$

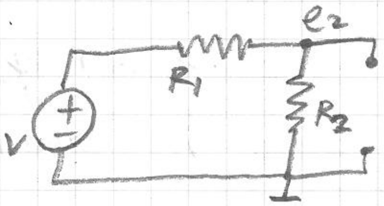
$$\Rightarrow R_2(V-e) + R_1(-e) + IR_1R_2 = 0$$

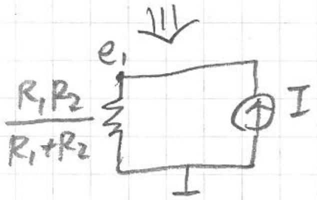
$$\Rightarrow e = \frac{1}{R_1+R_2}(R_2 V + I R_1 R_2)$$

$$= \frac{R_2}{R_1+R_2} V + \frac{R_1 R_2}{R_1+R_2} I$$

Study the case of $V=0$ and $I=0$, respectively, we see that the original circuit can be thought of as a superposistion of one current divider and one voltage divider, where $e = e_1 + e_2$:







$$e_1 = I \cdot \left(\frac{R_1 R_2}{R_1+R_2}\right)$$

$$e_2 = \frac{R_2}{R_1+R_2} V.$$

(Note: set $V=0$ 相當於將 (±) 短路 (short circuit)
set $I=0$ 相當於將 (↑) 斷路 (open circuit))

In general, for a linear circuit, we can use the concept of superposition to simply our analysis, by first considering one independent source at a time and then adding up the result.

Why does the concept of "superposition" make sense in circuit analysis?

→ Because ① each independent source contributes to the response of circuit "individually" and the contribution is independent from the contribution of any other independent source, and

② independent sources are assumed to have no resistance (see P13 of this note).

Why does the concept of "equivalence" make sense in circuit analysis?

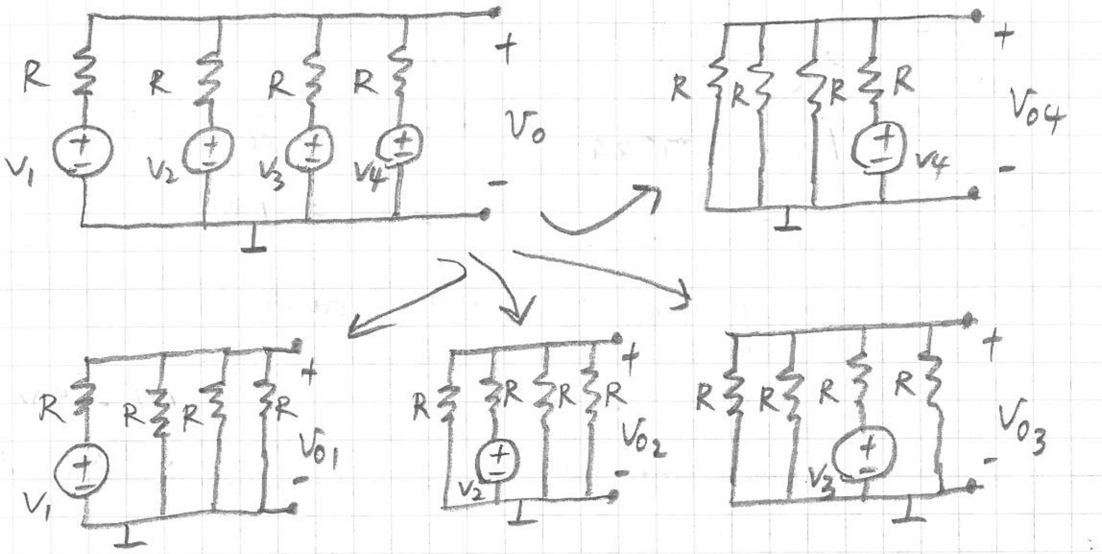
→ Because as long as the i-v characteristic are identical, from input/output viewpoint of a system, what's inside doesn't matter.

Therefore, we may replace some part of a circuit by its equivalence, solely for the purpose of simplifying our analysis. It is an extremely useful trick in engineering!

For example, we may use 訊號產生器 to feed an equivalent input to a system, emulating some physical input circuit.

Example of the use of superposition:

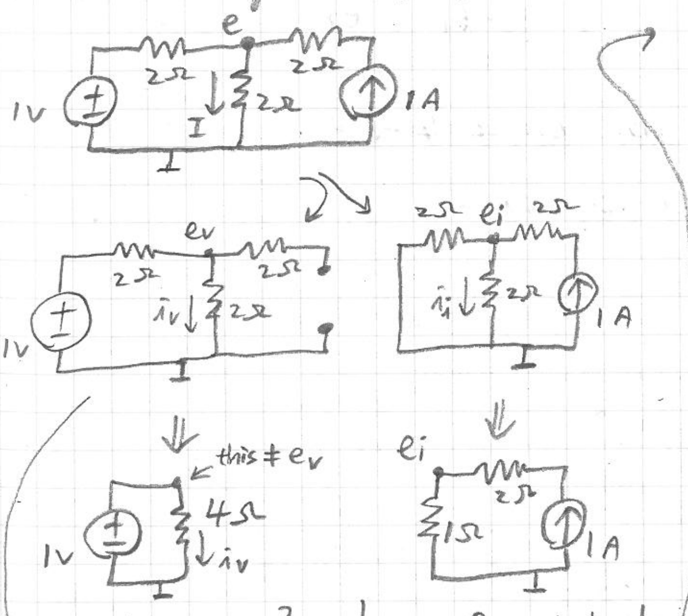
find $V_o = ?$



$$V_{o1} = \frac{\frac{1}{3}}{1+\frac{1}{3}} V_1 = \frac{1}{4} V_1, \quad V_{o2} = \frac{\frac{1}{3}}{1+\frac{1}{3}} V_2 = \frac{1}{4} V_2, \quad V_{o3} = \frac{\frac{1}{3}}{1+\frac{1}{3}} V_3 = \frac{1}{4} V_3, \quad V_{o4} = \frac{\frac{1}{3}}{1+\frac{1}{3}} V_4 = \frac{1}{4} V_4$$

$$\Rightarrow V_o = V_{o1} + V_{o2} + V_{o3} + V_{o4} = \frac{1}{4}(V_1 + V_2 + V_3 + V_4) \quad ※$$

Another example: find $I = ?$



$e_v = 1 \times \frac{2}{2+2} = \frac{1}{2}$ $\quad$ $e_i = 1 \times 1 = 1$

$e = e_v + e_i = \frac{3}{2}$

$I = \frac{e - 0}{2\Omega} = 0.75\ A$ ※

Alternatively, we may compute I directly:

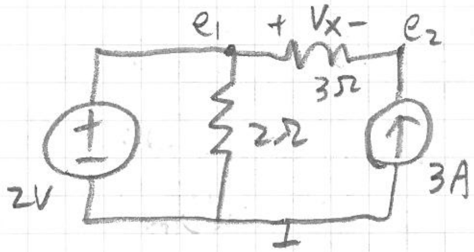
$i_v = \frac{1}{4}$ $\quad$ $i_i = \underbrace{\frac{2}{2+2}}_{\text{current divider}} \times 1 = \frac{1}{2}$

$\Rightarrow I = i_v + i_i$

$= \frac{1}{4} + \frac{1}{2} = 0.75\ A$ ※

P26

Some further example of the use of node method:



find $V_x = ?$

$e_1 = 2V$

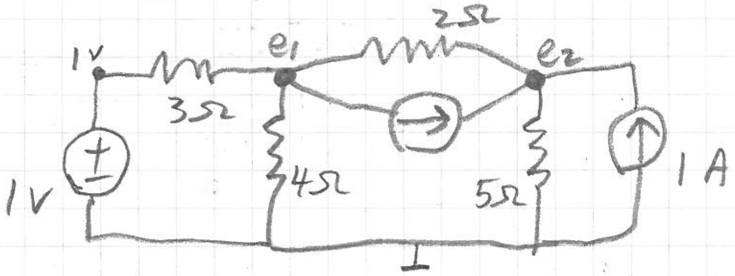
KCL: $\frac{e_1 - e_2}{3\Omega} + 3 = 0$

$\Rightarrow e_2 = 11\,V$

$\Rightarrow V_x = e_1 - e_2 = -9\,V$ ※

Compare this with the use of basic method as we did on P20 of this note! ($V_x = -9V$ there)

Another example: find e_1 and e_2



KCL on e_1: $\frac{e_1 - 1}{3} + \frac{e_1}{4} + \frac{e_1 - e_2}{2} + 2 = 0$

KCL on e_2: $-2 + \frac{e_2 - e_1}{2} + \frac{e_2}{5} - 1 = 0$

$$\Rightarrow \begin{cases} 4e_1 - 4 + 3e_1 + 6e_1 - 6e_2 + 24 = 0 \\ -20 + 5e_2 - 5e_1 + 2e_2 - 10 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 13e_1 - 6e_2 = -20 \\ -5e_1 + 7e_2 = 30 \end{cases}$$

$\Rightarrow 13e_1 - 6\left(\frac{1}{7}(30 + 5e_1)\right) = -20$

$\Rightarrow 91e_1 - 180 - 30e_1 = -140$

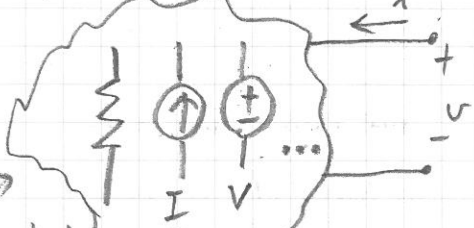
$\Rightarrow 61e_1 = 40 \Rightarrow e_1 \approx 0.655$, $e_2 = \frac{1}{6}(13e_1 + 20)$

$= \frac{1}{6}(13 \times 0.655 + 20) = 4.75$ ※

☆ Thévenin's Theorem

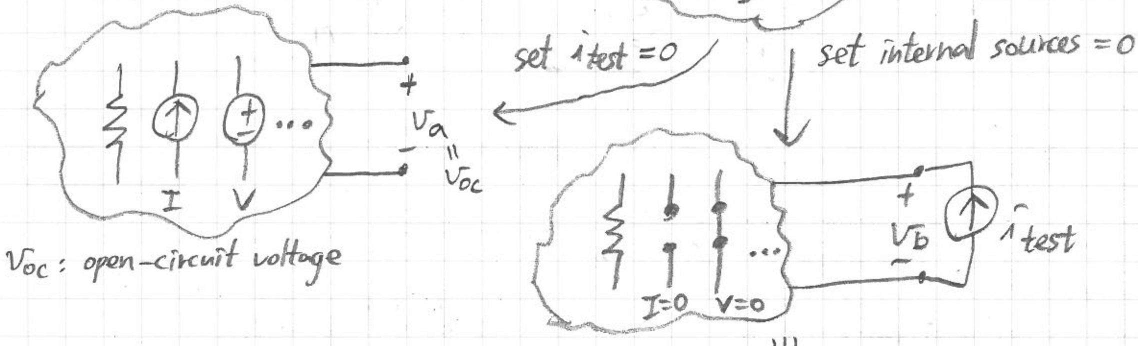
goal: Given an arbitrary linear circuit, we would like to know how it would respond to external excitation; in other word, we'd like to know its i-v characteristic.

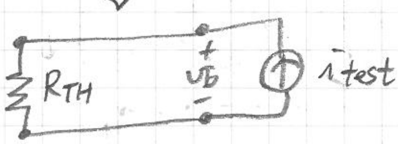
Approach: leverage the concept of superposition!



an arbitrary circuit

We append a testing current source, with the original circuit together they form a new circuit.



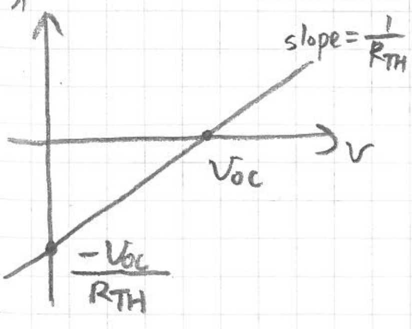


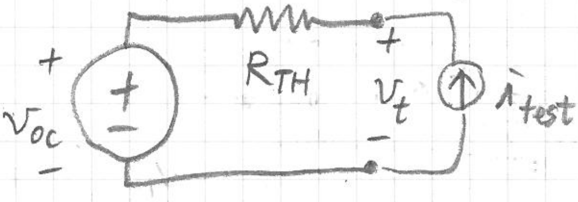
By superposition,

$$v_t = v_a + v_b$$

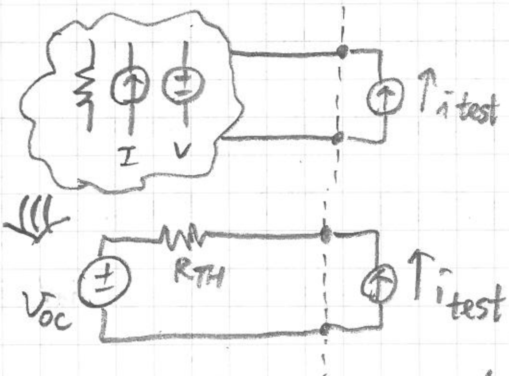
$$\Rightarrow \boxed{v_t = v_{oc} + i_{test} R_{TH}}$$

$\Rightarrow$ equivalently, this relation describes the following circuit:





P28 Therefore, we have the following equivalence:



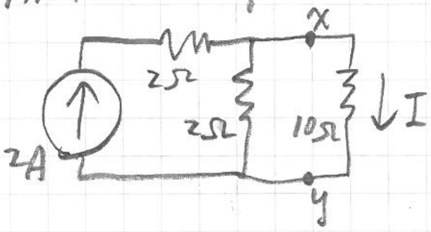
In other word, we may reduce an ~~arbitrary~~ linear circuit to an equivalent circuit of the form:

we name $V_{TH} = V_{oc}$ in honor of Thévenin.

Example:

$V_{TH} = v = 3 \times \frac{2}{1+2} = 2V$, $R_{TH} = \frac{1 \cdot 2}{1+2} = \frac{2}{3}\Omega$ from

Another example: find $I = ?$



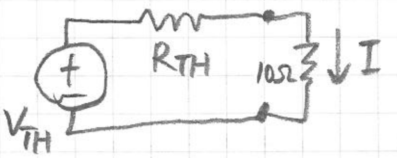
calculating V_{TH}:

$V_{TH} = 2A \times 2\Omega = 4V$

We may replace the left side of x-y by an equivalent circuit:

calculating R_{TH}:

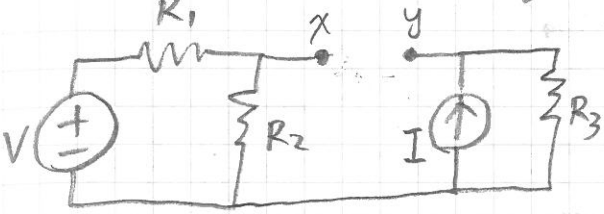
$\leftarrow R_{TH} = 2\Omega$



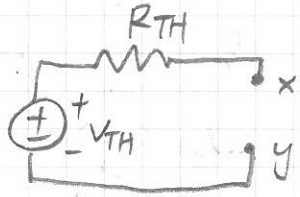
$\Rightarrow I = \frac{4}{2+10} = \frac{1}{3} A$ ※

Then $I = \frac{V_{TH}}{R_{TH} + 10\Omega}$

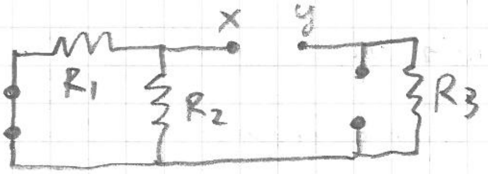
Exercise: create the Thévenin Equivalent Circuit for the following circuit:



$\Rightarrow$



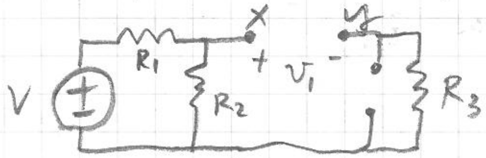
first of all, for R_{TH}:

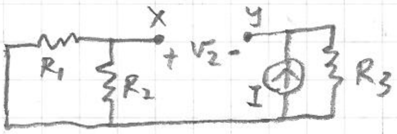


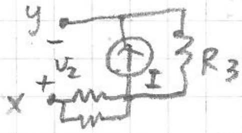
$$\Rightarrow R_{TH} = (R_1 /\!/ R_2) + R_3 = \frac{R_1 R_2}{R_1 + R_2} + R_3$$

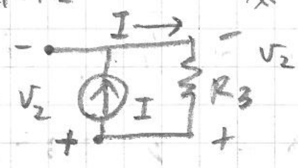
then, for V_{TH}:

approach ①, using superposition



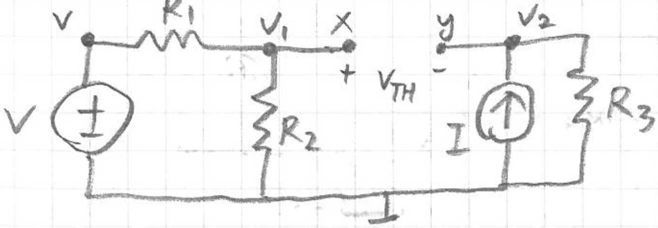






$$V_{TH} = V_1 + V_2 = \frac{R_1}{R_1+R_2}V + (-IR_3) = \frac{R_1}{R_1+R_2}V - IR_3 \quad ※$$

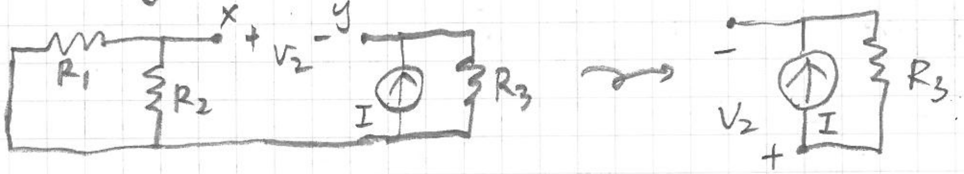
approach ②, using the node method (Page 21)



$$V_{TH} = V_1 - V_2 = \frac{R_2}{R_1+R_2}V - IR_3 \quad ※$$

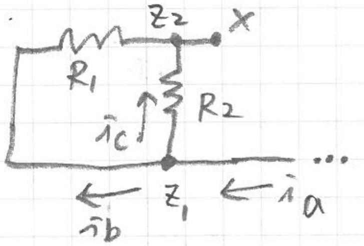
KCL:
$$\begin{cases} \dfrac{V - V_1}{R_1} + \dfrac{0 - V_1}{R_2} = 0 \\ I + \dfrac{0 - V_2}{R_3} = 0 \end{cases} \Rightarrow \begin{cases} V_1 = \dfrac{R_2}{R_1+R_2}V \\ V_2 = IR_3 \end{cases}$$

In the exercise on the previous page, you might wonder why we can do the following transformation:



Could it be possible that there are some current flowing through R_1 and/or R_2?

We can use KCL to figure it out:



for node z_1, we have

$$i_a - i_b - i_c = 0$$

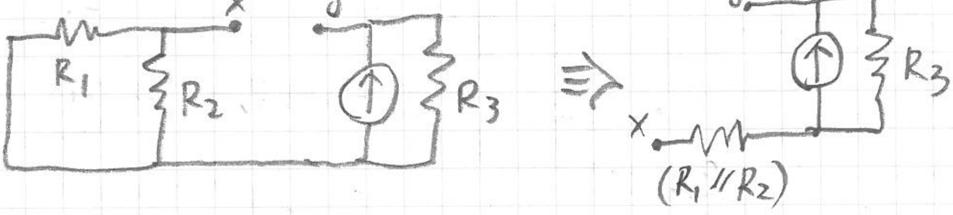
for node z_2, we have

$$i_b + i_c = 0$$

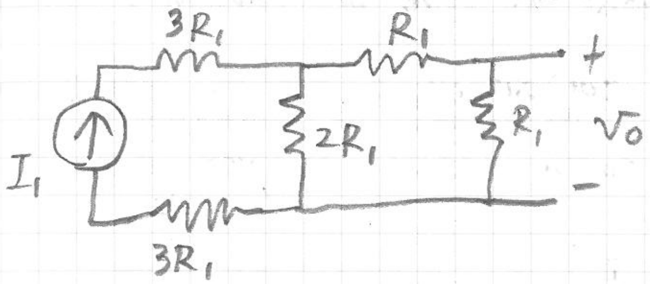
compare

Therefore, we see that $i_a = 0$.

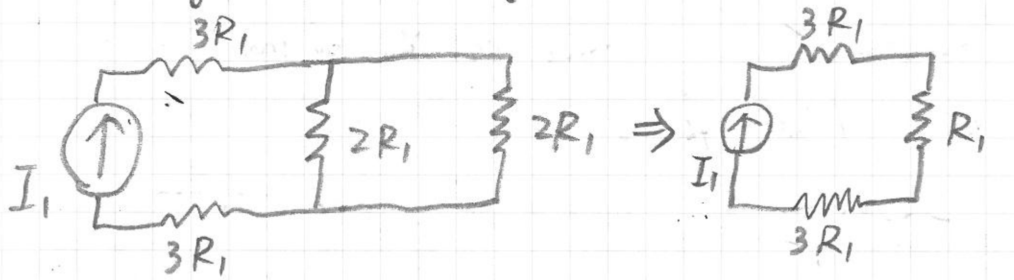
In practice, it is often helpful to think in terms of equivalent resistance, which may make the situation much more obvious:



Exercise: In the following circuit, determine voltage v_o:



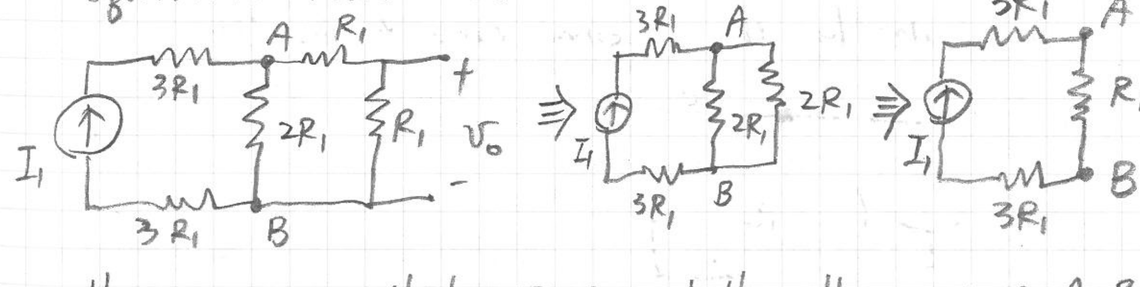
We may do the following transformation:



then we might conclude that $v_o = I_1 \cdot R_1$,

but it is wrong.

To see this, it could be helpful to <u>clearly label</u> the nodes between which you calculate the equivalent resistance:



thus we see that v_o is not the voltage across A, B,

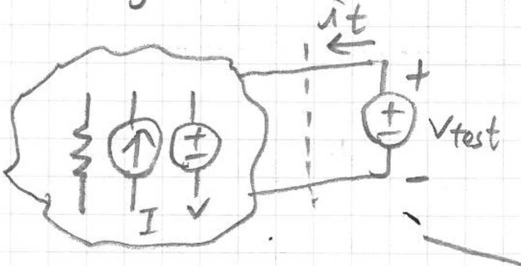
and $v_o = (\text{voltage across } A, B) \times \dfrac{R_1}{R_1 + R_1}$

$= (I_1 \cdot R_1) \times \dfrac{R_1}{R_1 + R_1} = \dfrac{1}{2} I_1 R_1$ is the correct answer.

#

P32 ☆ Norton's Theorem

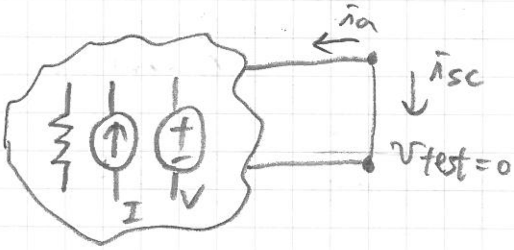
For the same goal as that of Thévenin's Theorem, here we choose to attach a testing independent voltage source to a possibly complex circuit:

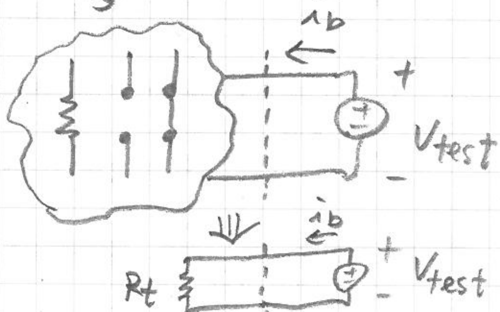


and leverage superposition

set internal sources = 0

set $V_{test} = 0$

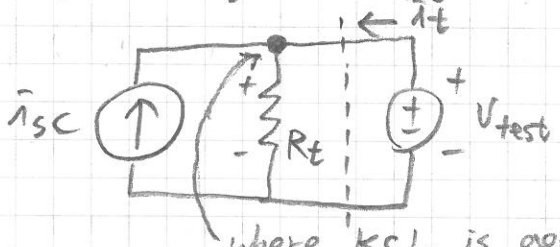




then $i_t = i_a + i_b = -i_{sc} + \dfrac{V_{test}}{R_t}$

$$\Rightarrow i_t + i_{sc} - \frac{V_{test}}{R_t} = 0$$

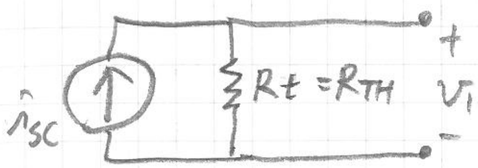
Think of above in terms of KCL, then, equivalently the original circuit is like

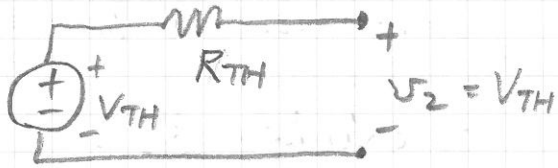


where KCL is applied.

And $R_t = R_{TH}$, since we applied same procedure of decomposition as we did in Thévenin's Theorem.

Relation between Norton's Equivalent Circuit and Thévenin's Equivalent Circuit:





since $v_1 = v_2$, we have

$$\hat{\imath}_{sc} \cdot R_t = v_2 = V_{TH}$$

$$\Rightarrow R_t = \frac{V_{TH}}{\hat{\imath}_{sc}} = \frac{v_{oc}}{\hat{\imath}_{sc}}$$

(recall that $V_{TH} = v_{oc}$ page 28)

in other word,

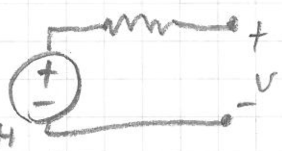
$$等效電阻 = \frac{開路電壓}{短路電流}$$

Example: find $v = ?$

approach ①: voltage divider

$$v = 3 \times \frac{2}{1+2} = \underline{2\ V}$$ ※

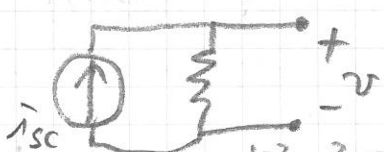
approach ②: Thévenin's Theorem



$V_{TH} = 2\ V$

$v = V_{TH} = \underline{2\ V}$ ※

approach ③: Norton's Theorem



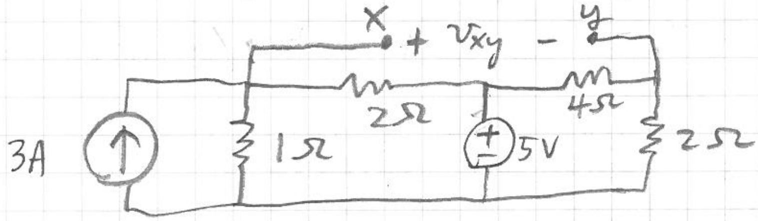
$\hat{\imath}_{sc} = 3A$

$R_t = \frac{1 \times 2}{1+2} = \frac{2}{3}\ \Omega$

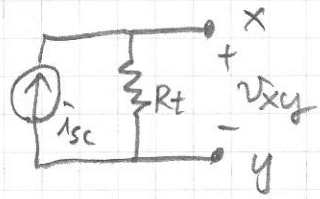
$v = 3 \times \frac{2}{3} = \underline{2\ V}$ ※

⇒ these three agree in one.

P34 Example: find $v_{xy} = ?$



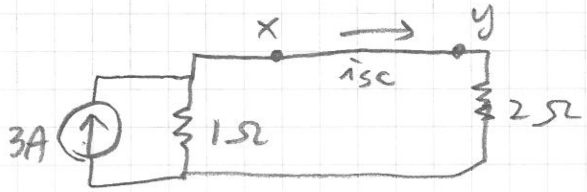
Using Norton's Theorem, we have



to calculate R_t:

$$R_t = (1\Omega // 2\Omega) + (4\Omega // 2\Omega)$$
$$= 2\ \Omega$$

to calculate i_{sc}:

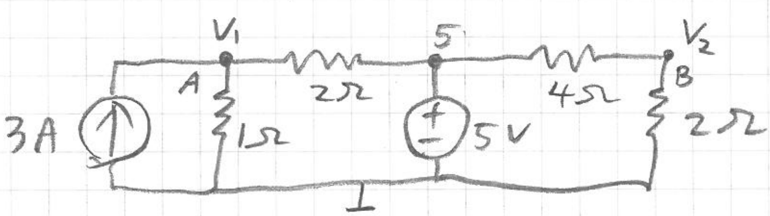


$$i_{sc} = 3 \times \frac{1}{1+2} = 1\ A$$

(current divider)

Therefore $V_{xy} = i_{sc} \times R_t = 2\ V$ #

Alternatively, we may use the node method:



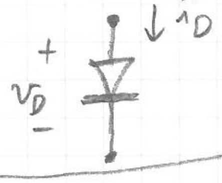
KCL at node A : $3 + \frac{0-V_1}{1} + \frac{5-V_1}{2} = 0 \Rightarrow V_1 = \frac{11}{3}$

KCL at node B : $\frac{5-V_2}{4} + \frac{0-V_2}{2} = 0 \Rightarrow V_2 = \frac{5}{3}$

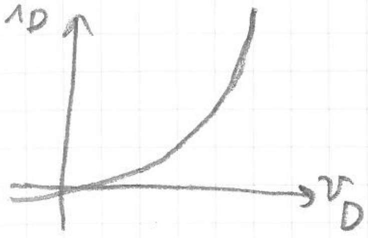
$$V_{xy} = V_1 - V_2 = \frac{11}{3} - \frac{5}{3} = 2\ V$$ #

☆ Nonlinear Devices

Silicon Diode



$$i_D = I_S \left(e^{v_D / v_{THE}} - 1 \right)$$



I_S : saturation current $\approx 10^{-12}$ A

v_{THE} : thermal voltage

$v_{THE} = \dfrac{kT}{q}$ T: temperature in kelvins

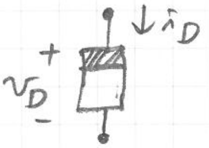
$°K = °C + 273.15$

k : Boltmann's constant $= 1.38 \times 10^{-23}$ J/°K

q : charge of an electron $= 1.602 \times 10^{-19}$ C

Example : CPU 溫度計, see example 16.1 in the textbook.

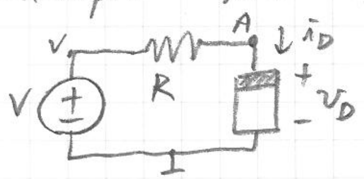
Hypothetical Nonlinear Device



$$i_D = \begin{cases} K \cdot v_D^2 & \text{for } v_D > 0 \\ 0 & \text{otherwise} \end{cases}$$

where K is a positive constant

Example : find v_D and i_D :



KCL at A : $\dfrac{V - v_D}{R} + (-i_D) = 0$

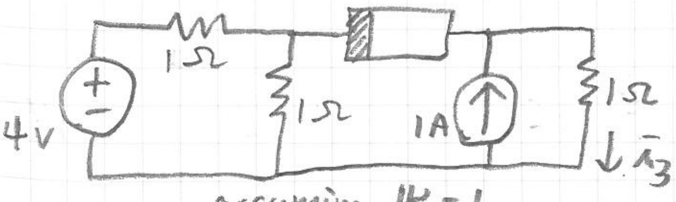
Plug in the i-v relation, we have

$$\begin{cases} \dfrac{V - v_D}{R} - K v_D^2 = 0 & \text{for } v_D > 0 \\ \dfrac{V - v_D}{R} = 0 & \text{otherwise} \end{cases}$$

$\Rightarrow v_D = \dfrac{-1 + \sqrt{1 + 4RKV}}{2RK}$ and $i_D = K \left(\dfrac{-1 + \sqrt{1+4RKV}}{2RK} \right)^2$ for $v_D > 0$ ※

Exercise : find i_3 = ?

(hint : Page 29 and the above example)



assuming K = 1

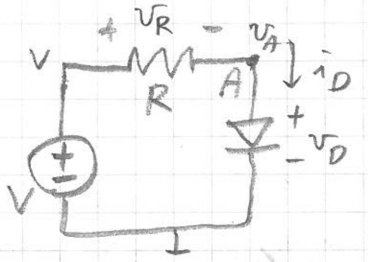
You can do it!

Ans: $i_3 = \left(\dfrac{\sqrt{7} - 1}{3} \right)^2 + 1$

※

Motivation for graphical analyses:

Consider the following example



KCL at node A:

$$\frac{v_A - V}{R} + I_s\left(e^{v_D/V_{THE}} - 1\right) = 0$$

$$\Rightarrow \frac{v_D - V}{R} + I_s\left(e^{v_D/V_{THE}} - 1\right) = 0$$

Solving for v_D is like solving for

x for $ax + be^x + c = 0$ for some constants a, b, and c.

→ May be solved by trail-and-error

→ little insight, however.

What if we want to know the impact of increasing/decreasing V to the value of v_D? And, how would v_D change with the change of R? These are critical questions to ask when designing a electronic circuit. Now, graphical analyses can be very helpful in this aspect!

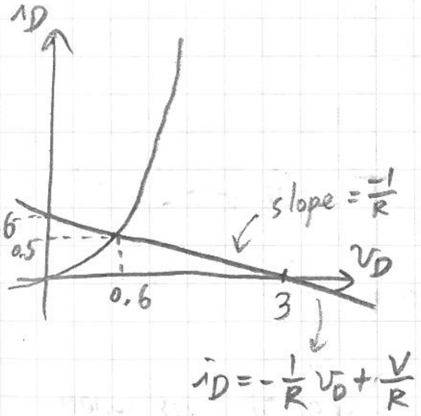
For concreteness, suppose $\begin{cases} V = 3\,\text{V} \\ R = 500\,\Omega \end{cases}$

in the example. $V_{THE} = 0.025\,\text{V}$ in room temperature

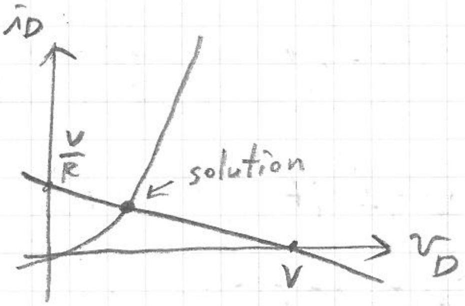
We can rearrange the KCL equation to give two equations for i_D:

$$\begin{cases} i_D = I_s\left(e^{v_D/v_{THE}} - 1\right) = 10^{-12}\left(e^{v_D/0.025} - 1\right) \\ i_D = \dfrac{V - v_D}{R} = -\dfrac{1}{R} v_D + 0.006 \qquad (\text{unit}: A) \end{cases}$$

then graphically speaking, the solution of i_D and v_D lies at the intersection point of the two curves on a v_D-i_D plot:

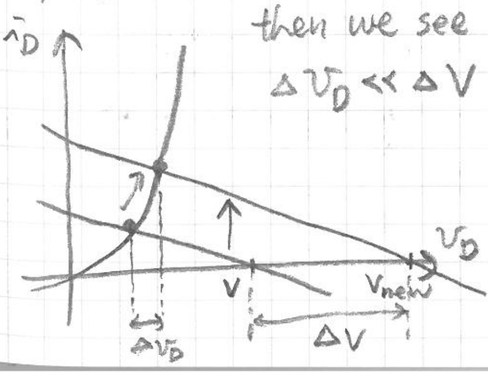


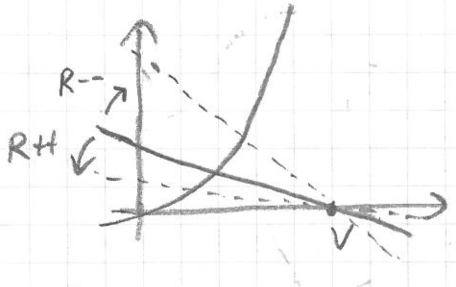
$\Rightarrow$



if R changes

if V increases
then we see
$\Delta v_D \ll \Delta V$

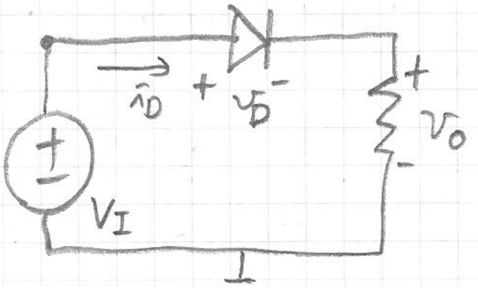


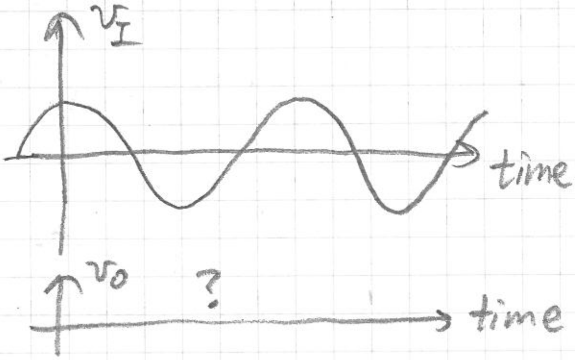


which give much insight in how the circuit behaves!

P38 Another example (half-wave voltage rectifier)

In the following circuit,
given a time-varying voltage source v_I,
what will be the output voltage v_O across a resistor ?



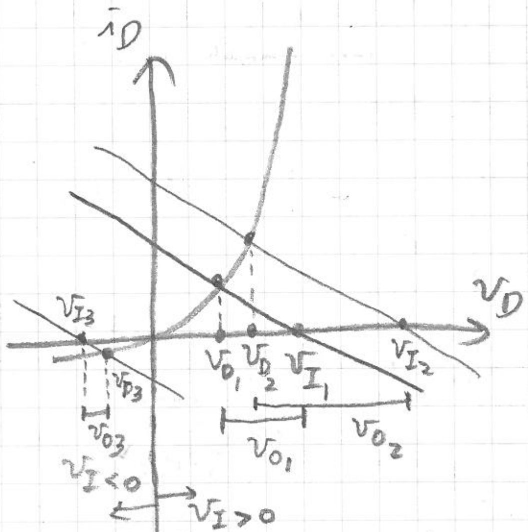


from KVL, $v_I - v_D - v_O = 0$

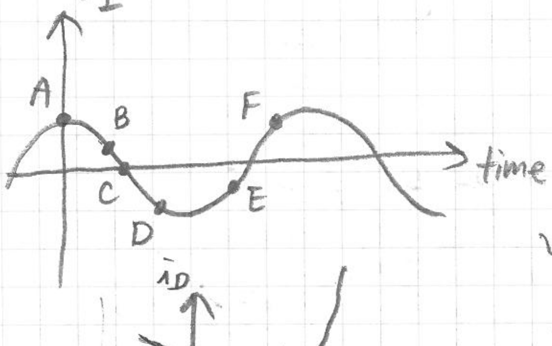
$$\Rightarrow v_O = v_I - v_D$$

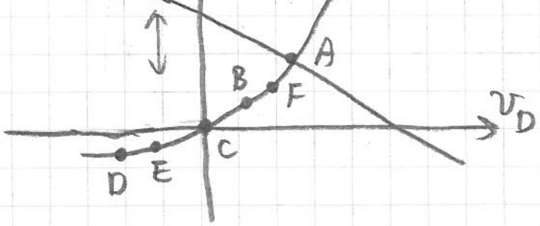
from KCL, element law, and the i-v characteristic of the diode:

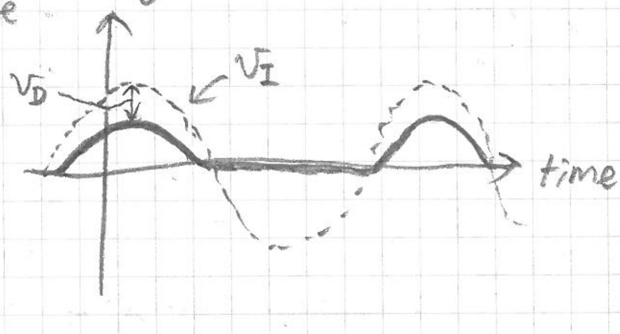
$$\begin{cases} i_D = I_S\left(e^{v_D/V_{THE}} - 1\right) \\ i_D = \dfrac{v_I - v_D}{R} \end{cases}$$



we see $v_{O_3} \approx 0$ when $v_D < 0$

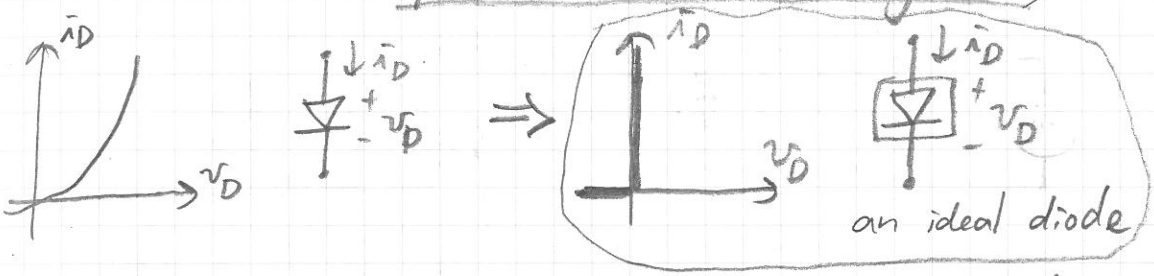






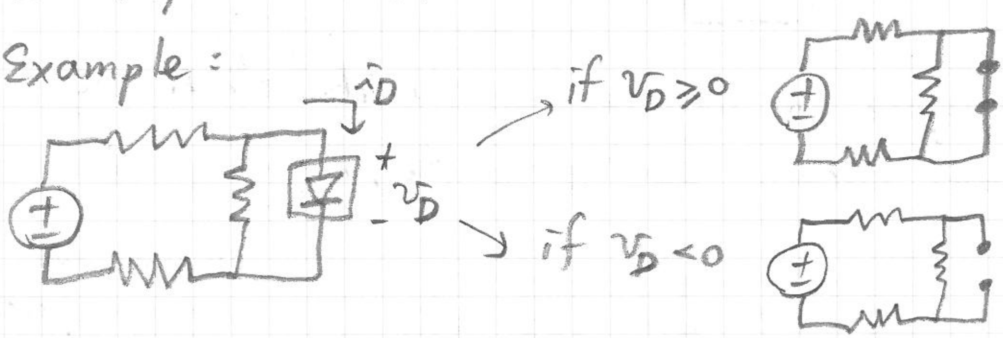
Besides the graphical analysis,
in some occasion we may simplify our analysis of a nonlinear circuit by considering an approximated version of the i-v characteristic of a given nonlinear element.

This is called the <u>piecewise linear analysis</u>.



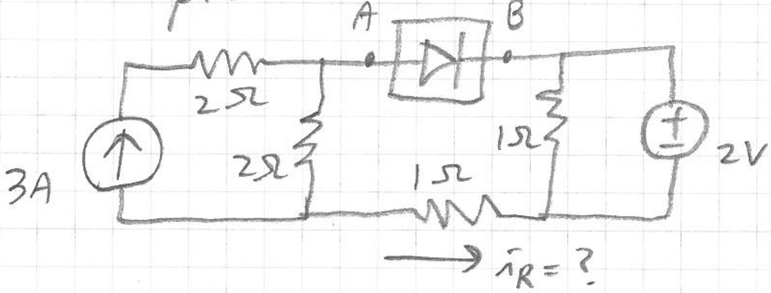
For an ideal diode, depending on the actual voltage (or the actual current direction), we may replace the diode by either a short circuit or an open circuit.

Example:

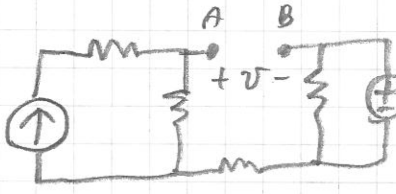


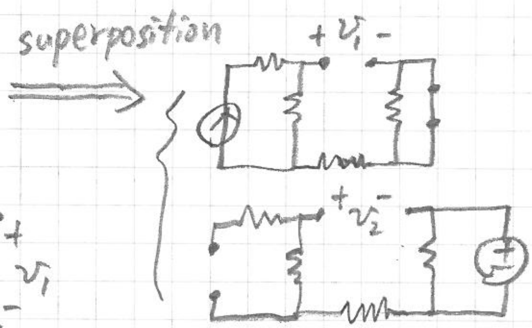
(Note: in class I conditioned on <u>the direction of the current flowing through the diode</u>, but perhaps it makes more sense to condition on v_D. Either way, we follow the associated variable convention (see page 8 of this note).)

P40 Example :

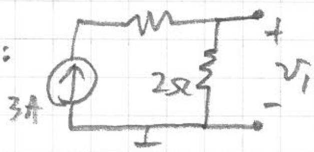


We first find the voltage across A and B.



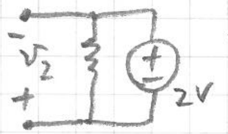


for v_1 :



$v_1 = 3A \times 2\Omega = 6\ V$

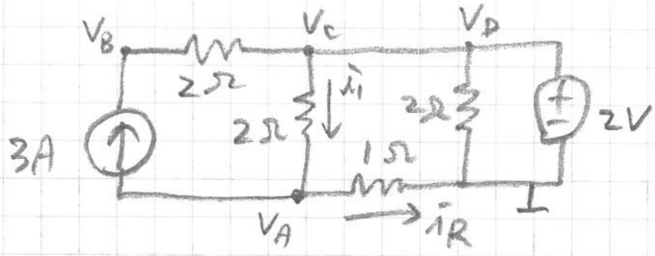
for v_2 :



$v_2 = -2V$

$\rightarrow v = v_1 + v_2 = 6 - 2 = 4\ V$

therefore we may replace the ideal diode by a short circuit, leading to the following equivalence :



in class we used superposition; here, let's try using the node method!

$V_C = V_D = 2$

$i_1 = 3 + i_R$

$$\Rightarrow \frac{2 - V_A}{2} = 3 + \frac{V_A - 0}{1}$$

$\rightarrow 2 - V_A = 6 + 2V_A$

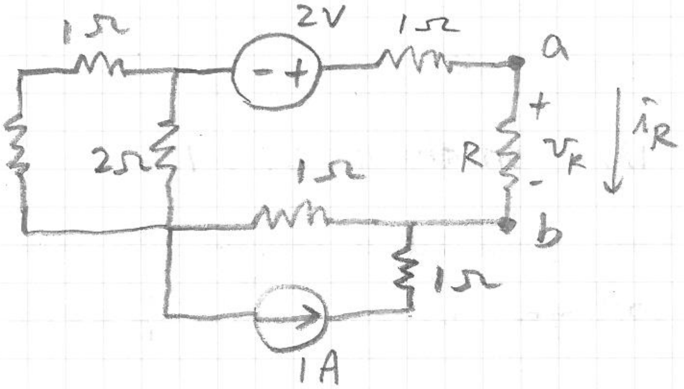
$\Rightarrow V_A = \frac{-4}{3}$

$$\Rightarrow i_R = \frac{V_A - 0}{1} = \frac{-4}{3}\ A$$

※

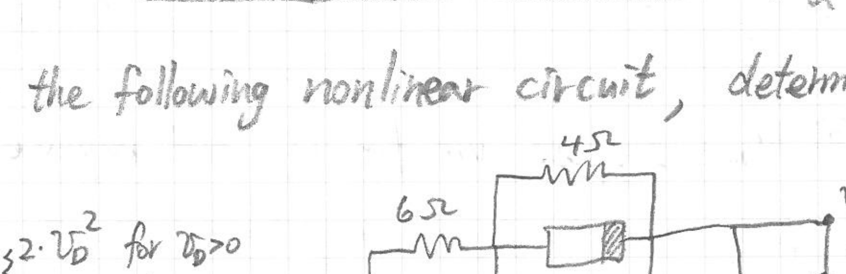
Some exercise problems since P_{32}:

A. Use Norton's Theorem to find i_R and v_R for ① $R=2$ and ② $R=4$

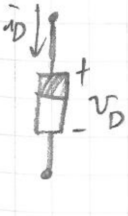


B. Following A, rather than using Norton's Theorem, directly apply superposition to find i_R for $R=2$.

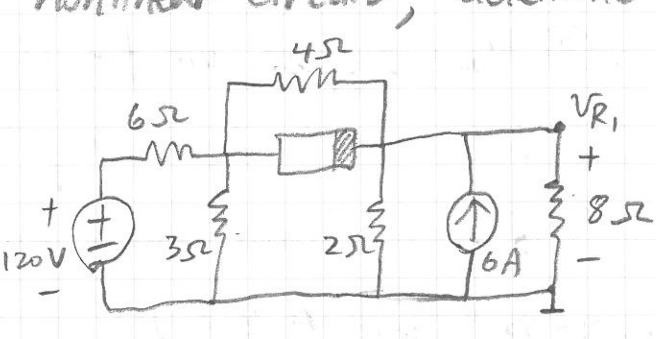
C. Determine the Norton equivalent at terminals a, b and at terminals c, d, for the following circuit:



D. For the following nonlinear circuit, determine v_{R_1}.



$$i_D = \begin{cases} 2 \cdot v_D^2 & \text{for } v_D > 0 \\ 0 & \text{otherwise} \end{cases}$$

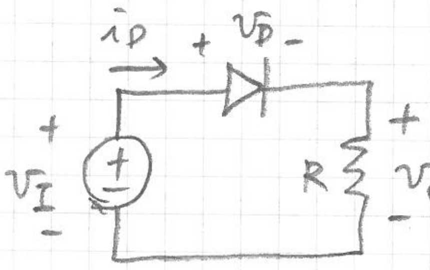


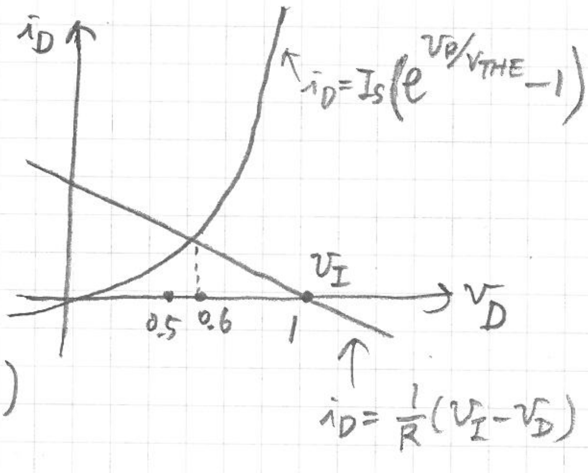
(hint: compare this with that in C.)

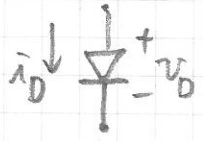
P42 E. For the following nonlinear circuit, with some initial analysis of its i_D-v_D relation at hand, try to answer three questions:

① if for some need we changed the input voltage v_I, so that $v_I > 1$, would that lead to a change to the output voltage $v_O > 0.4$ or $v_O < 0.4$?

② Now, suppose we operate the circuit at region $v_I \gg 1$. What can we say about v_O?







$i_D = I_s(e^{v_D/V_{THE}} - 1)$

③ Now, suppose we fix v_I but replace the linear resistor by a very heavy load, such that $R \gg 1$. What would i_D become?

Answers to Problems A, B, C, D:

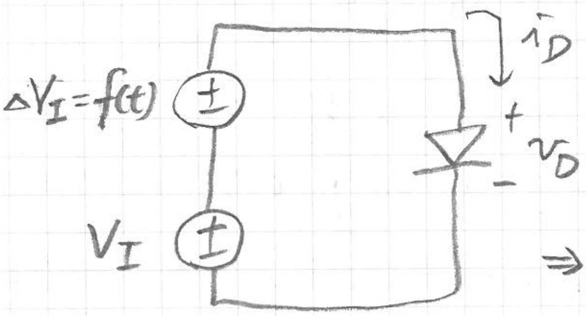
A. ① $i_R = \frac{1}{5}A$, $V_R = \frac{2}{5}V$ ② $i_R = \frac{1}{7}A$, $V_R = \frac{4}{7}V$ B. same as A.

C. $R_{tab} = 2\Omega$, $i_{scab} = 7\,A$
$R_{tcd} = \frac{3}{2}\Omega$, $i_{sccd} = \frac{38}{3}A$

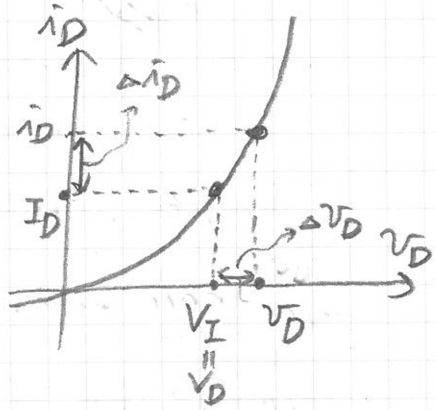
D. 16 V ※

☆ Small-Signal Analysis for Nonlinear Devices

- In many sensor applications and most audio amplifiers, the input voltage/current to a circuit often consists of two parts:
  - ① a time-invariant source (large signal)
  - ② a time-varying source (small signal)



$\Rightarrow$



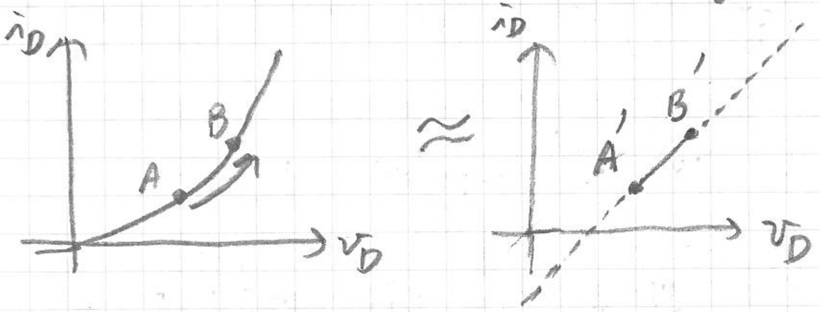
The large signal is used to determine the region of operation (i.e., which part of the i_D-v_D curve), and the small signal is the "real" input (e.g., those induced by human voice, as in the case of a microphone).

$$\underbrace{v_D}_{} = \underset{\text{large}}{V_I} + \underset{\text{small signal}}{\Delta v_D}$$

$$i_D = I_D + \Delta i_D$$

P44

as we will see, moving along a small distance on "the $i_D - v_D$ curve" can be approximated as moving along a small distance on "a straight line"

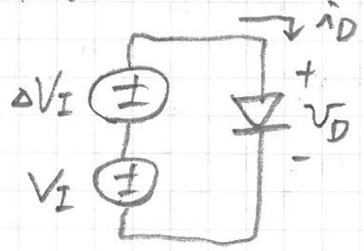


Therefore, we may simplify our analysis of small signal by considering the signal's response on a nonlinear device as if it is the response on a linear device (resistor).

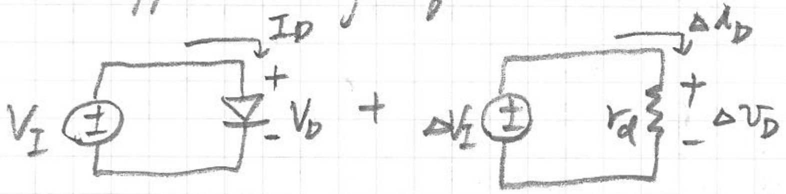
Review P13, where we've shown that the i-v characteristic of a resistor on the i-v plot is a straight line; further, the slope of the line is equal to the reciprocal of the resistance (R) of the resistor.

Now, a question is: how do we determine the resistance of that linear device?

Let r_d be the resistance of the linear device. Using small-signal analysis, we essentially transform the original circuit



into an approximately equivalence:

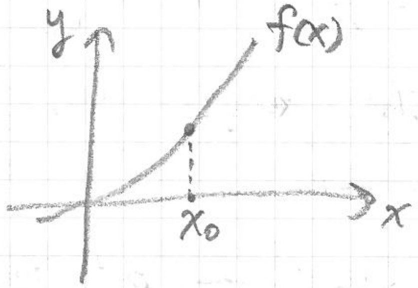


where $i_D = I_D + \Delta i_D$

$= \frac{v_{THE}}{I_D}$

Now, let's see how to determine r_d!

We use Taylor's Theorem, which provides a way to approximate a curve near a certain point $x = x_0$:



$$\underset{x \text{ near } x_0}{y} = \underset{x \text{ near } x_0}{f(x)} = f(x)\Big|_{x=x_0} + f'(x)\Big|_{x=x_0}(x-x_0)$$

$$+ \frac{1}{2!}f''(x)\Big|_{x=x_0}(x-x_0)^2 + \frac{1}{3!}f'''(x)\Big|_{x=x_0}(x-x_0)^3 + \cdots$$

P46 in our case of a nonlinear diode,

recall that

$$i_D = I_s\left(e^{v_D/v_{THE}} - 1\right) \underset{\uparrow}{=} f(v_D)$$

we define it

$$\Rightarrow \underset{v_D \text{ near } V_D}{i_D} = f(v_D)\Big|_{v_D=V_D} + f'(v_D)\Big|_{v_D=V_D}(v_D - V_D)$$

$$+ \frac{1}{2!} f''(v_D)\Big|_{v_D=V_D}(v_D - V_D)^2$$

$$+ \frac{1}{3!} f'''(v_D)\Big|_{v_D=V_D}(v_D - V_D)^3 + \cdots$$

$$= f(v_D)\Big|_{v_D=V_D} + (v_D - V_D)\Big(f'(v_D)\Big|_{v_D=V_D}$$

$$\left.\begin{aligned} &+ \frac{1}{2!} f''(v_D)\Big|_{v_D=V_D}(v_D - V_D) \\ &+ \frac{1}{3!} f''(v_D)\Big|_{v_D=V_D}(v_D - V_D)^2 \\ &+ \cdots \Big) \end{aligned}\right.$$

we choose to ignore these terms

using the chain rule, we get $f'(v_D) = \frac{I_s}{v_{THE}} e^{v_D/v_{THE}}$

$$\Rightarrow \underset{v_D \text{ near } V_D}{i_D} \approx f(v_D)\Big|_{v_D=V_D} + (v_D - V_D) \cdot \frac{I_s}{v_{THE}} \cdot e^{v_D/v_{THE}}\Big|_{v_D=V_D}$$

$$= I_s\left(e^{V_D/v_{THE}} - 1\right) + \Delta v_D \cdot \frac{I_s}{v_{THE}} \cdot e^{V_D/v_{THE}}$$

since we know that

$$e^{V_D/v_{THE}} \gg 1$$

so we can think of $e^{V_D/v_{THE}} \approx e^{V_D/v_{THE}} - 1$

With that, we may rewrite the equation as

$$\underset{v_D \text{ near } V_D}{i_D} = I_s\left(e^{V_D/v_{THE}} - 1\right) + \Delta v_D \frac{I_s}{v_{THE}}\left(e^{V_D/v_{THE}} - 1\right)$$

Now, by observation we see $I_s\left(e^{V_D/v_{THE}} - 1\right) = I_D$

Thus, $\underset{v_D \text{ near } V_D}{i_D} = I_D + \frac{\Delta v_D}{v_{THE}} I_D$

Compare to $i_D = I_D + \Delta i_D$, we have $\Delta i_D = \frac{\Delta v_D}{v_{THE}} I_D$

Think in terms of relation of Δi_D and Δv_D

and we may choose to define

$$\boxed{r_d = \frac{v_{THE}}{I_D}} \; ※$$

In the hindsight, we may generalize our result by saying that for an arbitrary nonlinear element, we have

$$\boxed{r_d = \frac{1}{f'(v_D)\big|_{v_D = V_D}} = \frac{1}{\left.\frac{d f(v_D)}{d v_D}\right|_{v_D = V_D}}}$$

※ (finally!)

$-V_D)$

$V_D)^2$

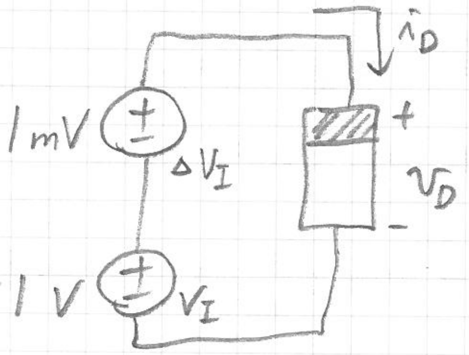
D/v_{THE}

$= V_D$

P48

Example :

find $i_D = ?$



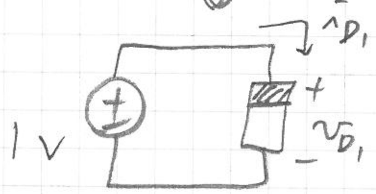
recall that for  (P35)

$i_D = K \cdot v_D^2$ for $v_D > 0$

and here we suppose

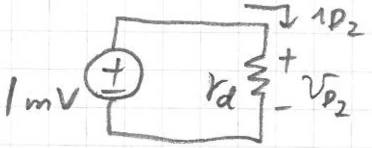
$K = 1\,mA/V^2$

Ans :



$v_{D_1} = 1 \quad (KVL)$

$\Rightarrow i_{D_1} = K \cdot v_{D_1}^2 = 1\,mA$



$v_{D_2} = 1\,mV \quad (KVL)$

$$r_d = \frac{1}{f'(v_D)\big|_{v_D = V_D}} = \frac{1}{2 \cdot K \cdot v_D\big|_{v_D = 1V}}$$

$$= 500\,\Omega$$

$$\Rightarrow i_{D_2} = \frac{v_{D_2}}{r_d} = 2\,\mu A$$

$$\Rightarrow i_D = i_{D_1} + i_{D_2} = 1.002\,mA$$

※

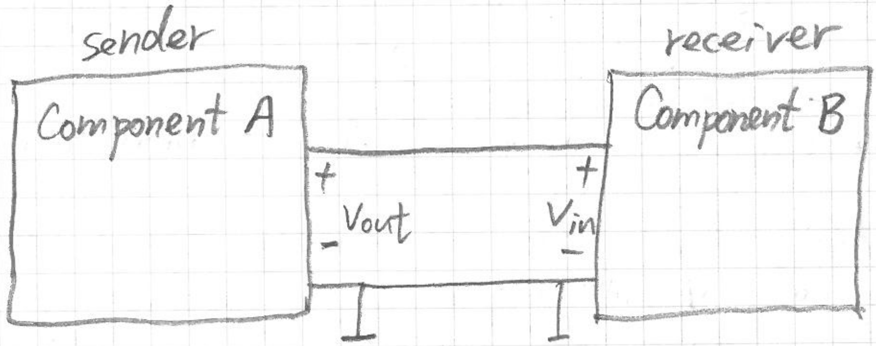
☆ The digital abstraction

So far, we've been studying analog signals like voltage and current, and we focus on how these signals, being transformed by a circuit, will impact the behavior of a certain element in a circuit. The impact manifests itself in terms of analog signals on the element, and we also called it the "response" of the element with respect to the analog signals from a voltage source or a current source .

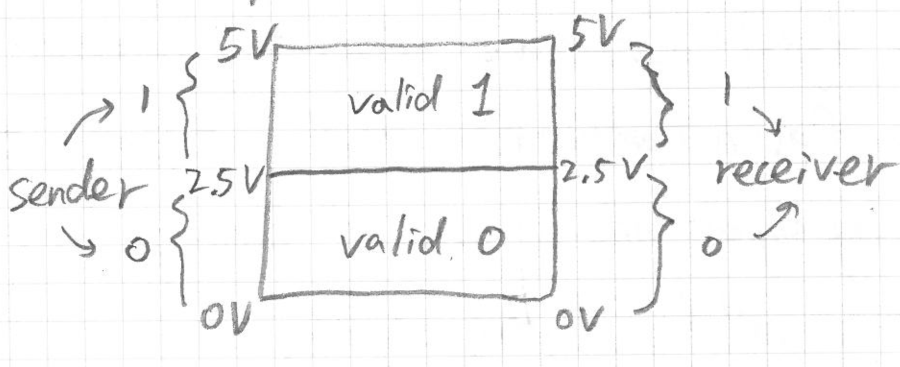
Now, how does an analog signal relates to the "digital world", the world built using some meaningful combinations of 0s and 1s? The digital abstraction serves this purpose. It specifies a transformation that interprets the analog signals into a series of binary digits, the so-called "digital signals."

Review Section 1.8 in the textbook.

- Digital signals are "interpretations" of analog signals, so that they can be understood and be used by a digital system (such as a computer).
- Between physical components in a system, it is still analog signals that are transferred.
- How to transform digital signals into analog ones?

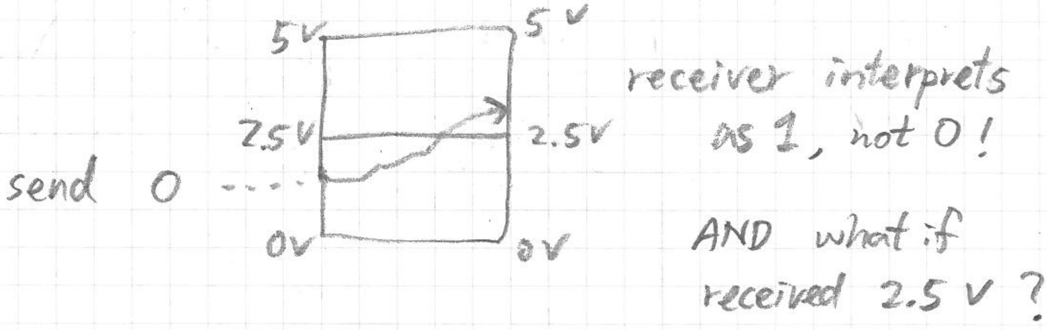


Our first attempt:

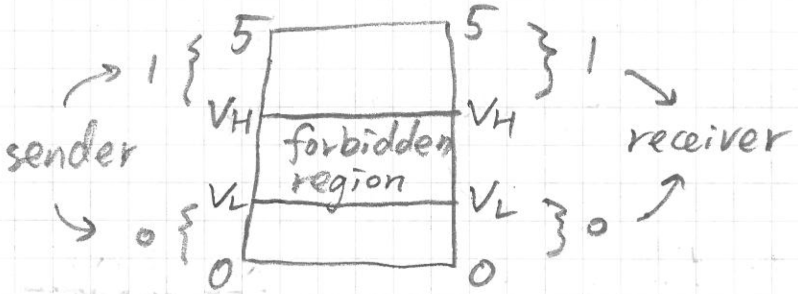


A serious problem:

there could be noise/interference during signal transmissions



An improved design:



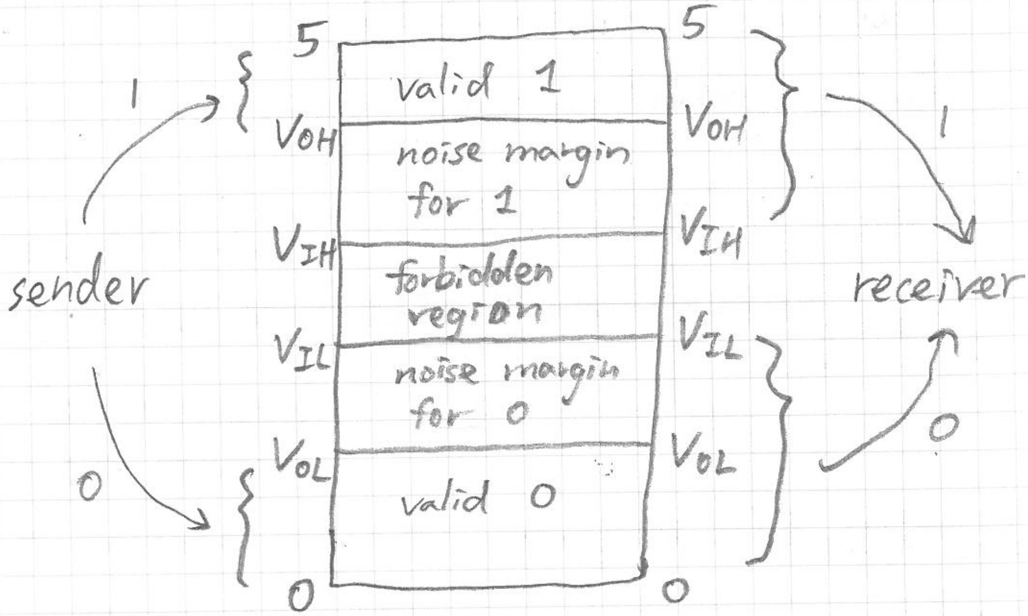
V_H and V_L are high/low voltage thresholds.

A further question:

how to quantify the immunity to noise?

and if we can do that, this can serve as a contract, and accordingly it would ensure device manufacturers to meet consumers' need, and components can be connected to form a system.

A better design :



The static discipline (principle)

A device must interpret correctly voltage inputs falling within the V_{IL} or V_{IH} threshold; with a valid input, the device must produce a valid voltage output that falls within the V_{OL} or V_{OH} threshold.

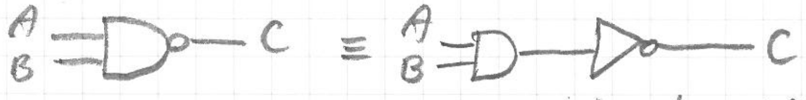
A specification of digital devices.

Sections 5.2, 5.3, 5.6 in the textbook are very good learning materials.
Be sure to study them yourselves.
We will briefly cover some of them at some appropriate opportunity in this course.

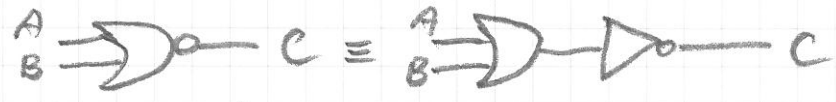
Logic gates (i.e., combinational gates):

AND gate OR gate NOT gate

A	C
1 →	0
0 →	1



NAND gate ≡ AND gate plus NOT gate



NOR gate ≡ OR gata plus NOT gate

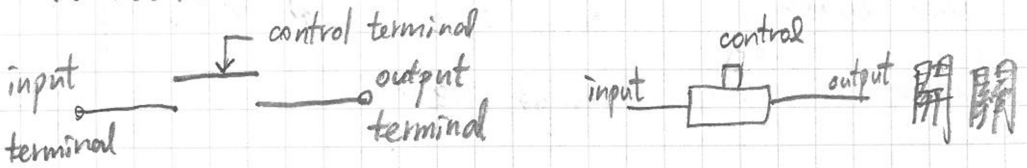
Truth table

inputs		output (C)			
A	B	AND	OR	NAND	NOR
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	1	0
1	1	1	1	0	0

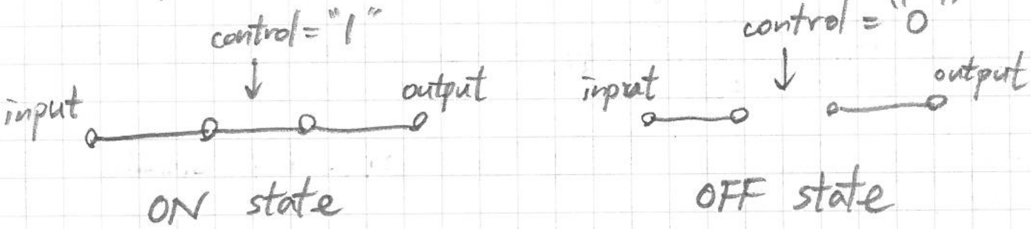
P54 ★ The MOSFET switch

(Metal Oxide Semiconductor Field-Effect Transistor)

A switch is a three-terminal element in a circuit.



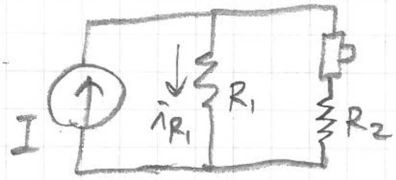
Ideally, input to the control terminal determines whether or not there is a short circuit between the input terminal and the output terminal:



(compare this with an ideal diode, P39)

Such a simple idea of a switch enables many ways to control the response of certain elements in a circuit. First, let's look at two examples showing how a switch may impact the response of another element in a circuit:

Example 1.

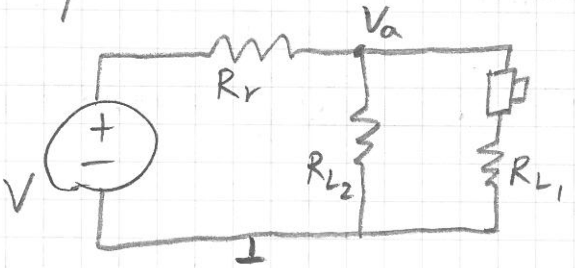


A switch connected in series with resistor R_2 will impact the current flowing through resistor R_1.

ON state : $i_{R_1} = I \times \frac{R_2}{R_1+R_2}$

OFF state : $i_{R_1} = I$ ※

Example 2.



ON state :
$$V_a = V \times \frac{R_{L_2} /\!/ R_{L_1}}{R_r + R_{L_2} /\!/ R_{L_1}}$$

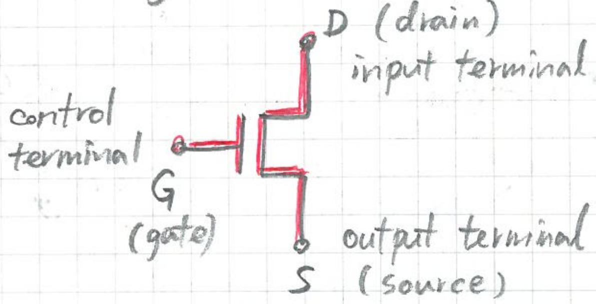
OFF state :
$$V_a = V \times \frac{R_{L_2}}{R_r + R_{L_2}}$$

The voltage across the resistor R_{L_2} will drop if the switch is in the ON state, because of the existance of R_r. ※

Conceptually, by connecting two switches in series we may implement the AND logic ; by connecting two switches in parallel, we may implement the OR logic. In both cases, the control terminals take the input logical values.

P56

The symbol of a MOSFET

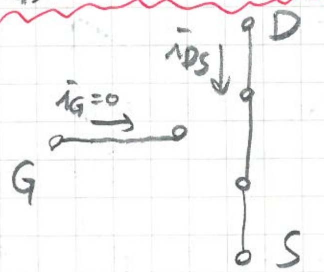


Let i_{DS} be the current flowing from the drain to the source, V_{GS} be the voltage across the gate and the source, and V_T be a threshold voltage.

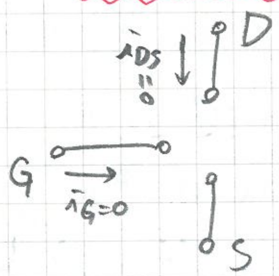
In its simplest model (S model), the MOSFET behaves as follows:

switch

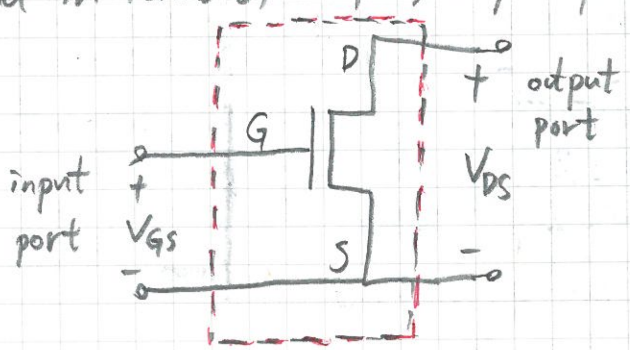
$V_{GS} \geq V_T \Rightarrow$ ON state



$V_{GS} < V_T \Rightarrow$ OFF state



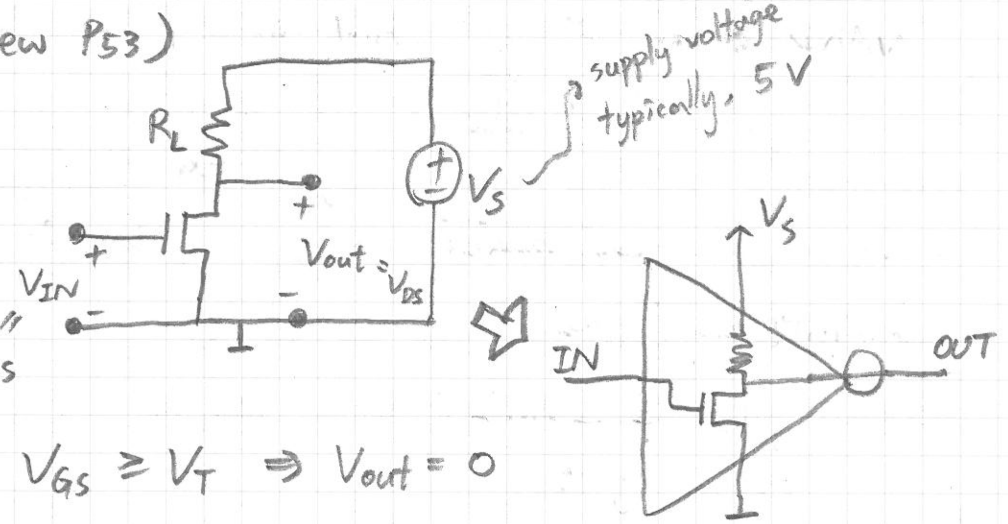
And in terms of input/output ports:



Later, we will study some more realistic models of MOSFET (for example, explicitly consider its internal resistance). For now, let's focus on the S model and its switching behavior.

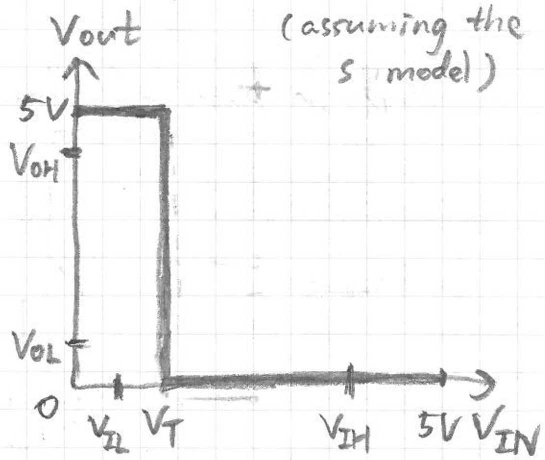
We may use a MOSFET to construct a logical NOT gate , so-called an "inverter":

(review P53)



$$\begin{cases} V_{GS} \geq V_T \Rightarrow V_{out} = 0 \\ \text{(logic 1)} \qquad \text{(logic 0)} \\ V_{GS} < V_T \Rightarrow V_{out} = V_S \\ \text{(logic 0)} \qquad \text{(logic 1)} \end{cases}$$

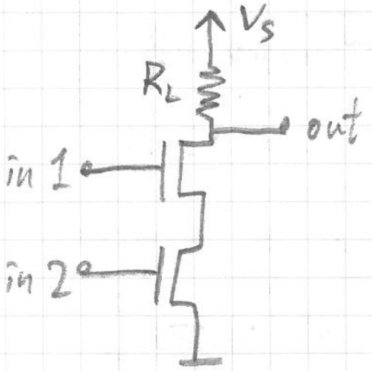
transfer characteristic of an inverter (assuming the S model)



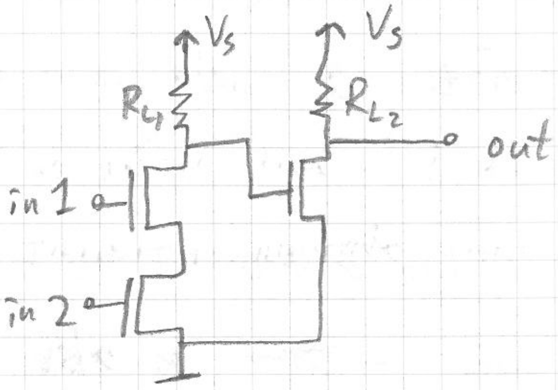
A valid mapping of voltage levels specified in the static discipline →

(review P52)

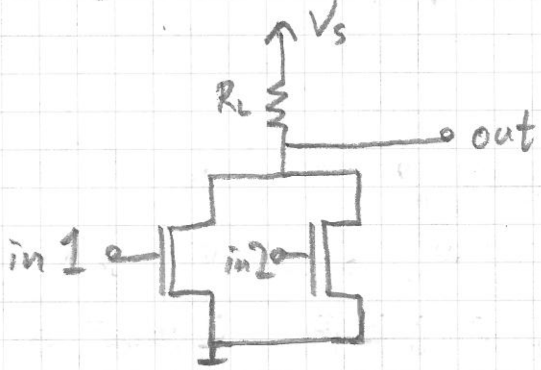
Similarly, we may use MOSFETs to construct other logic gates:



NAND gate



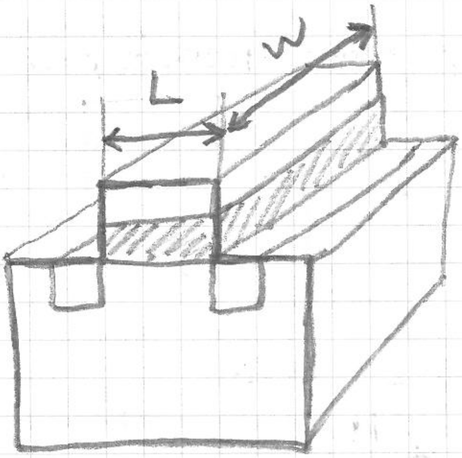
AND gate (NAND + NOT)



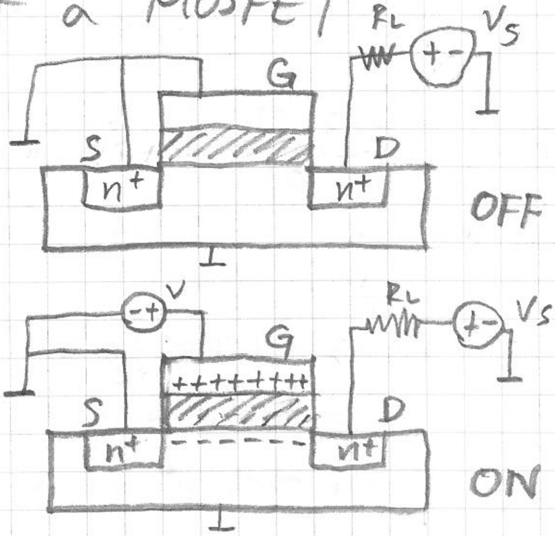
NOR gate

Exercise:
OR gate

☆ Physical structure of a MOSFET



3D view



2D view

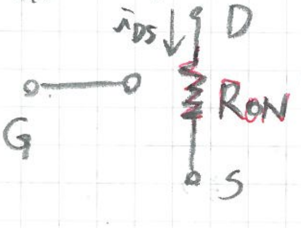
When a positive voltage is applied at the control terminal G, a conducting channel will be built up between the two n^+ regions, and therefore some nontrivial current may flow from D to S. Read Section 6.7 in the textbook for a more detailed account.

The physical structure also implies that in the ON state there really will exist some resistance between D and S, and the value of the resistance is characterised by the dimensions of that conducting channel. Let R_n be the resistance per square of the channel, and L be the channel length and W the channel width. We have

$$\underbrace{R_{ON}}_{\text{the resistance of the channel}} = R_N \cdot \frac{L}{W} \qquad (\text{review } P_6)$$

The S model neglects R_{ON}. A more realistic model takes R_{ON} into account, and we call it the SR model :

$V_{GS} \geq V_T \Rightarrow$ ON state



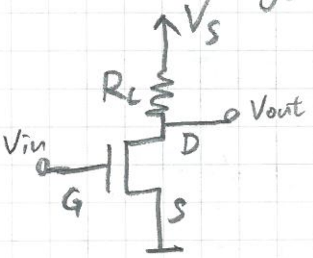
$$i_{DS} = \frac{V_{DS}}{R_{ON}}$$

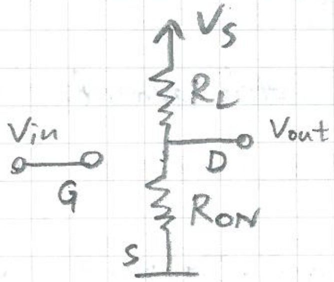
(the OFF state is same as that in the S model)

★ Analyzing a MOSFET circuit using the SR model

Taking into account the impact from R_{ON}, the analysis may seem complicated, but it is still based on what we've learned so far.

Let's analyze an inverter to illustrate this :



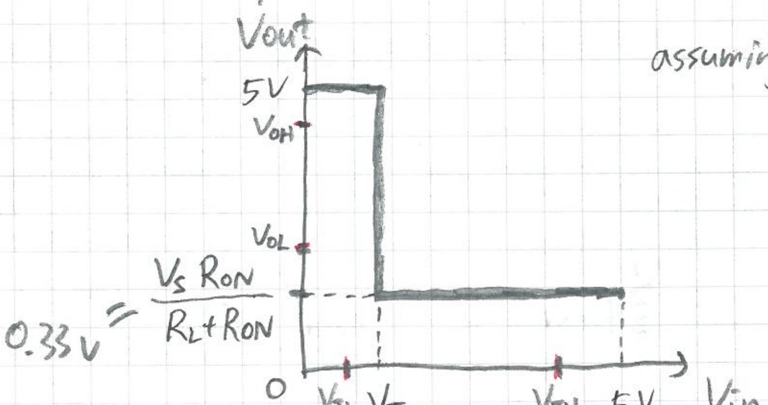


Because now we consider R_{ON}, the output voltage V_{out} will not be zero when $V_{GS} \geq V_T$; instead,

$$\boxed{V_{out} = V_S \times \frac{R_{ON}}{R_L + R_{ON}}}$$

following the voltage-divider relationship.

And thus the transfer characteristic will become as follows (compare to one on P57):



assuming $V_S = 5\,V$

$V_T = 1\,V$

$R_{ON} = 1\,k\Omega$

$R_L = 14\,k\Omega$

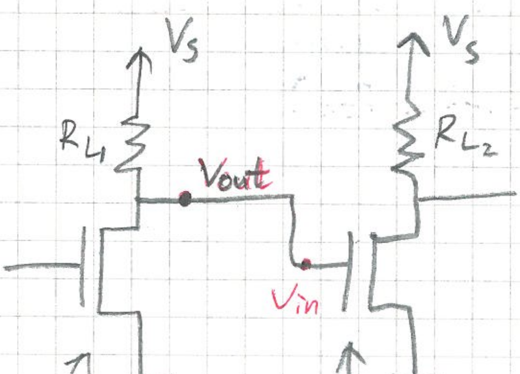
This change of Vout value has at least two impacts:

① To meet the static discipline, Vout (in ON state) needs to be lower than V_{OL}; otherwise, we may observe that even though the magnitude of noise is within the specified margin, the value of V_{in} still exceeds V_{IL}.

→ solution A: redesign to reduce Vout.
(from the manufacturer's viewpoint)

→ solution B: replace by a compatible device.
(from the consumer's viewpoint)

② To drive another MOSFET, Vout in ON state needs to be lower than V_T, since Vout in OFF state equals $V_S \geq V_T$ (usually).



the driving MOSFET

the MOSFET whose ON/OFF state is driven by the preceding MOSFET.

(As another example, see the AND gate on P_{58})

study Examples 6.5 and 6.6 in the textbook.

In general, since $\underset{\text{ON state}}{V_{out}} = V_s \times \frac{R_{ON}}{R_L + R_{ON}}$,

to change V_{out}, we may
① change V_s
② change R_L
③ change R_{ON}

① ⇒ as a side-effect, it will also change V_{out} in OFF state!

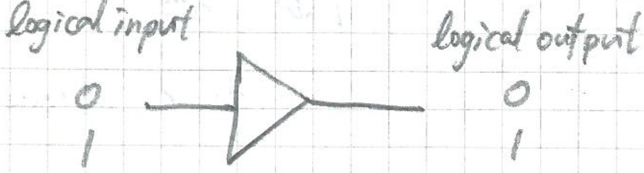
② ⇒ larger resistance is hard to achieve in VLSI;
larger resistance would cause nontrivial voltage drop in the presence of leakage current;

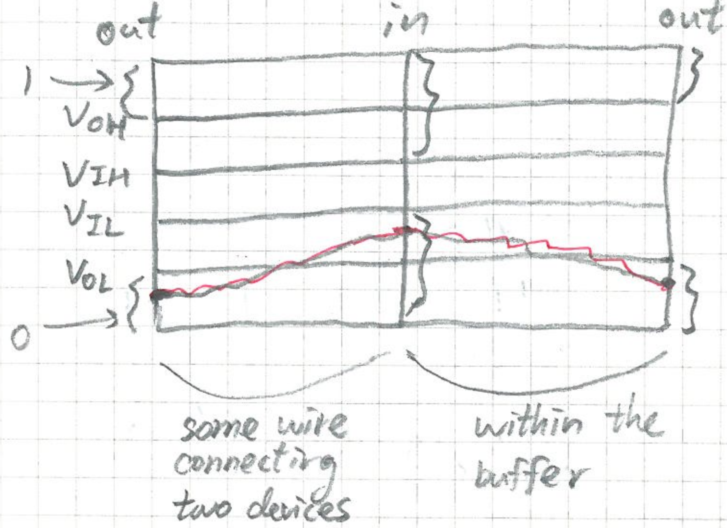
③ ⇒ may be achieved by changing the W/L ratio of the MOSFET.

⇨ An engineer's job often involves in finding the best solution among "multiple" options!!

☆ Signal restoration, gain, and nonlinearity
(Section 6.9 in the textbook)

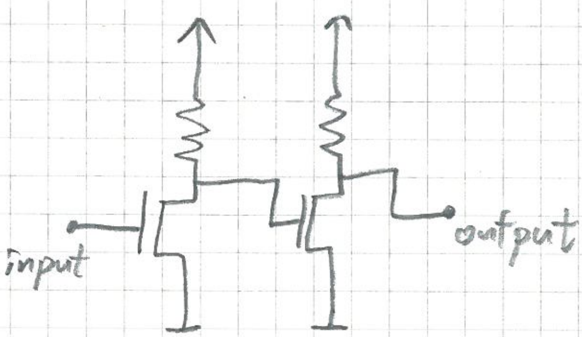
We may use a device called "buffer" to restore a distorted signal (distorted by noise, for example):





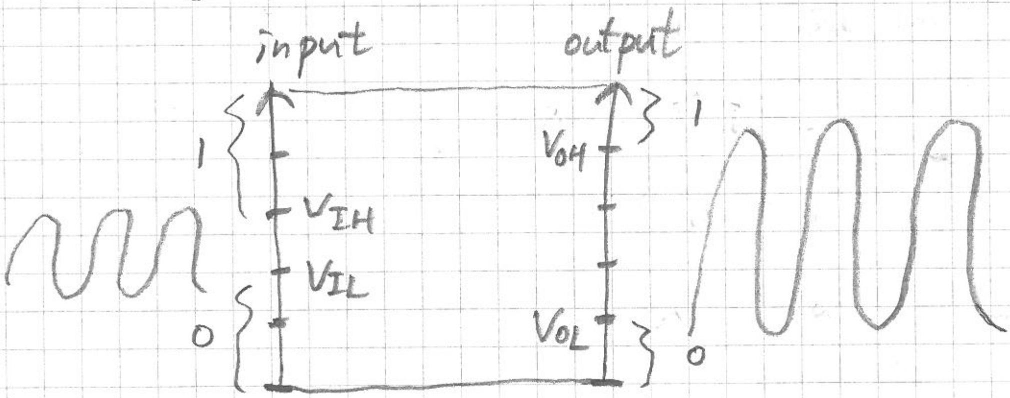
As long as the system follows the static descipline, the output of a buffer will be closer to the output of the device that precedes it.

A rough implementation of a buffer:



P64

In order for an electronic device to satisfies the system's specification of the static discipline, it turns out that such a device must be capable of <u>amplifying</u> an input signal. Why? Because a signal may fluctuate between logic 0 and logic 1 :



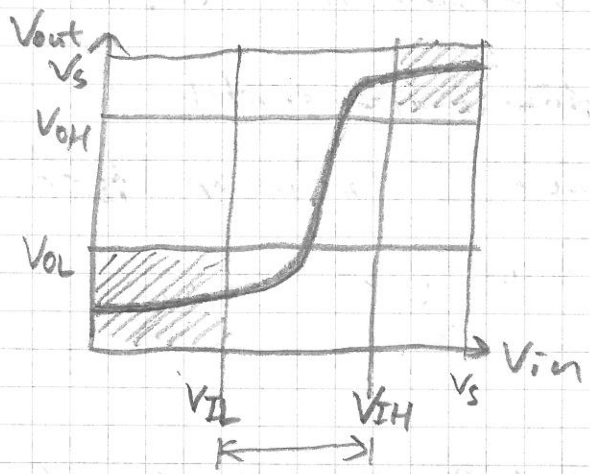
Definition :

$$\text{Gain} = \frac{V_{OH} - V_{OL}}{V_{IH} - V_{IL}} \quad \text{for } V_{IL} \to V_{IH} \text{ transition}$$

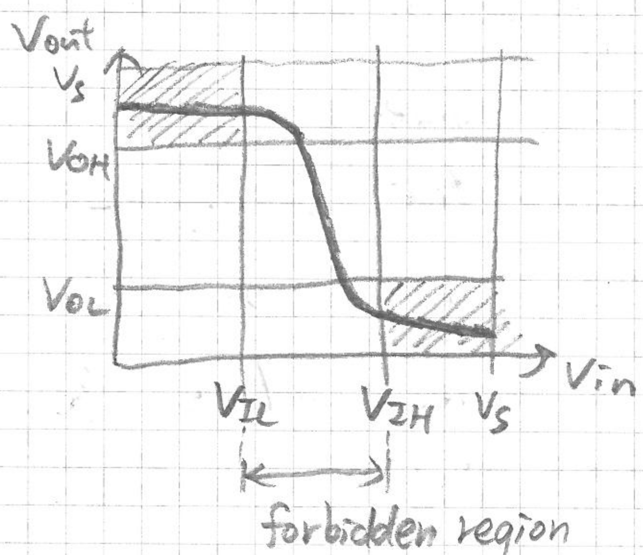
for example, if $V_{OH}=4$, $V_{IH}=3$, $V_{IL}=2$, $V_{OL}=1$

then the gain is 3.

The transfer characteristic of a buffer:



The transfer characteristic of an inverter:

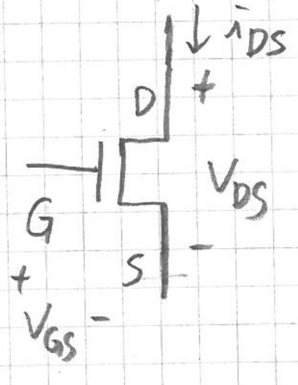


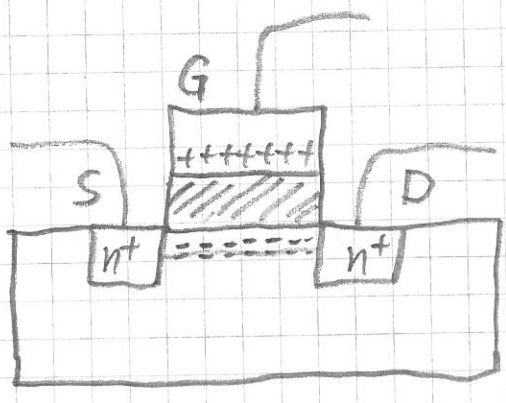
In both cases, the shaded region represents the valid region for the transfer curve.

Note that since $V_{OL} < V_{IL}$ and $V_S - V_{OH} < V_S - V_{IH}$, the magnitude of the slope of the curve in the valid region is smaller than 1.

So far, our discussion of MOSFET include its behavior as a switch (S model) as well as its linear v_{DS} - i_{DS} relation in its ON state (SR model, with resistor R_{ON}).

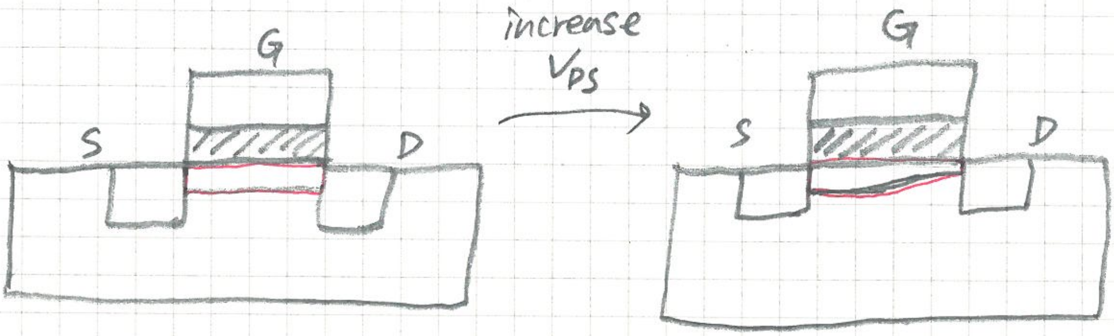
In science and engineering, we are often curious about how a system would behave should we increase/decrease the value of a certain parameter. We have seen that a MOSFET will go from OFF state to ON state as we increase voltage V_{GS}. Now, let's consider what will happen if we gradually increase V_{DS}.



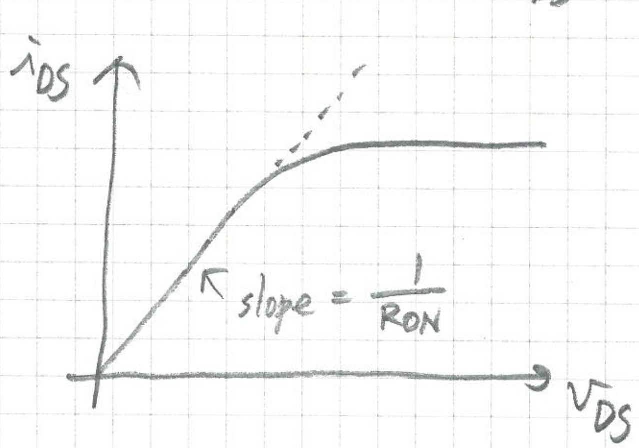


As we increase V_{DS}, the difference in electrical potentials between G and D decreases.
電位差

This in turn will reduce the amount of free electrons near D, esentially shrinking the thickness of the conductible channel near D:



Therefore we will see a bending of the curve on the i_{DS}-V_{DS} plot:



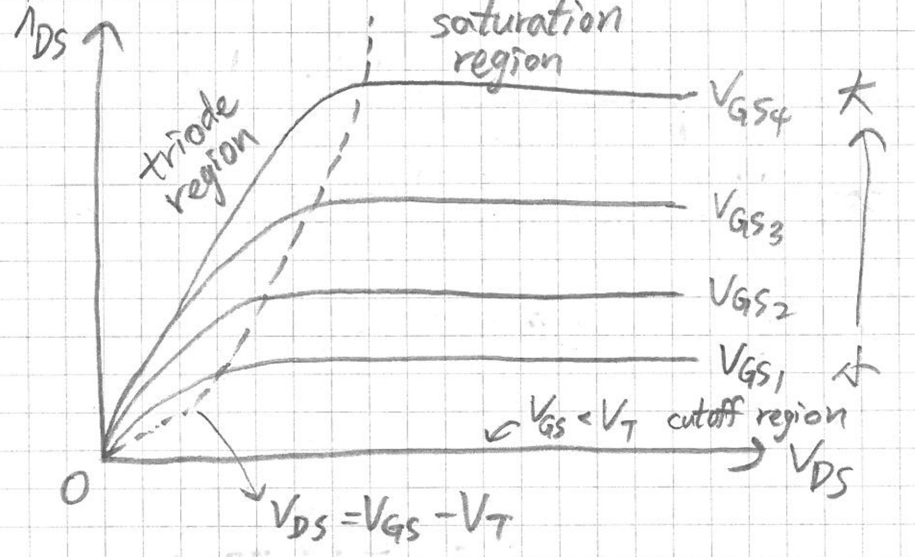
電阻截面積↓
⇒ 電阻值↑
⇒ slope↓

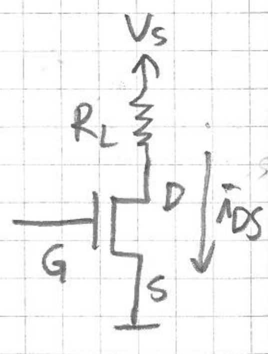
P68 ☆ The SCS model of a MOSFET

switch-current source

Compared to the S model and the SR model, the SCS model is a more accurate MOSFET model (closer to the real physical characteristic).

In the SCS model, a MOSFET can operate with three very different behaviors, and we say it has three "operational regions".

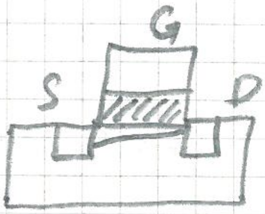




Conditions for each region:

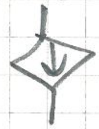
- cutoff: $V_{GS} < V_T$, i.e., $V_{GS} - V_T < 0$
- saturation: $0 \le V_{GS} - V_T \le V_{DS}$
- triode: $V_{DS} < V_{GS} - V_T$

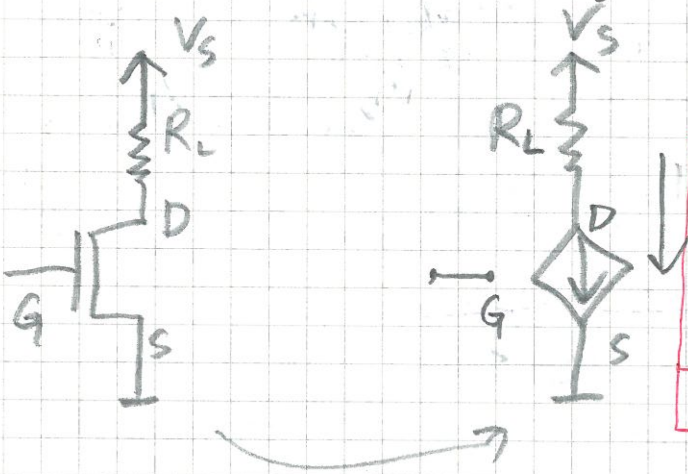
In the saturation region, current i_{DS} would stay the same as we keep increasing V_{DS}, because the channel between source and drain has become stable :



(review P_{58}, P_{66-67})

Therefore, we may consider such a behavior as if there is a "current source" (P_{13})
But i_{DS} would depend on voltage V_{GS}, and thus we say it is like a "voltage-controlled current source". symbol :





$$\boxed{i_{DS} = f(V_{GS}) = \frac{K(V_{GS}-V_T)^2}{2}}$$

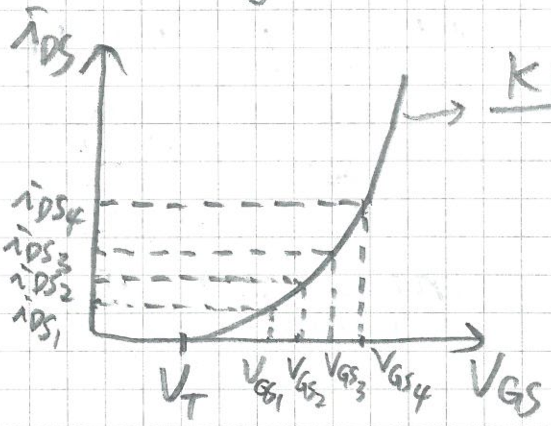
(according to physics)

Note that K is a coefficient ;
unit : mA/V^2

if $0 \le V_{GS}-V_T \le V_{DS}$
→ in saturation region

P70

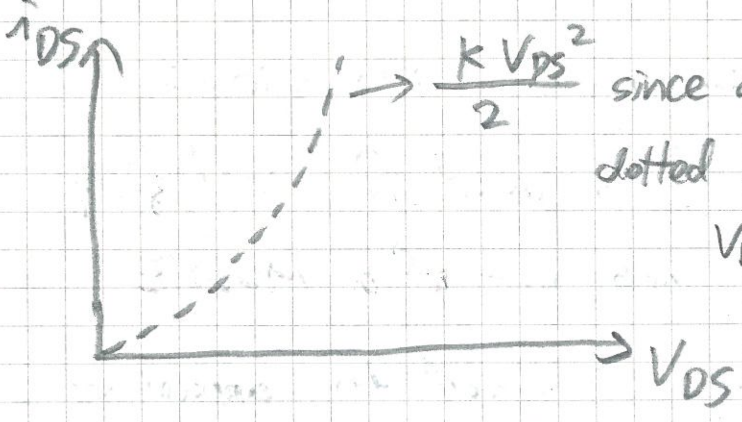
Plotting the $i_{DS}-V_{GS}$ relation, we see:



Therefore, if $|V_{GS_1}-V_{GS_2}| = |V_{GS_2}-V_{GS_3}|$
$= |V_{GS_3}-V_{GS_4}|$

we will have

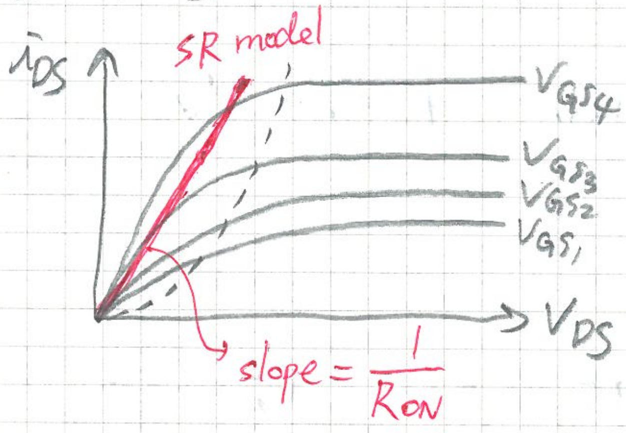
$$|i_{DS_1}-i_{DS_2}| < |i_{DS_2}-i_{DS_3}| < |i_{DS_3}-i_{DS_4}|$$



$\frac{K V_{DS}^2}{2}$ since along this dotted curve we have $V_{DS} = V_{GS}-V_T$

$$\Rightarrow \boxed{i_{DS} = \begin{cases} 0 & \text{for } V_{GS} < V_T \\ \frac{K(V_{GS}-V_T)^2}{2} & \text{for } 0 \le V_{GS}-V_T \le V_{DS} \end{cases}}$$

From the aspect of the SCS model, the R_{ON} in the SR model can be thought of as a piecewise approximation of the $i_{DS}-V_{DS}$ relation in the triode region in the SCS model: (P39) (Section 4.4 in textbook)



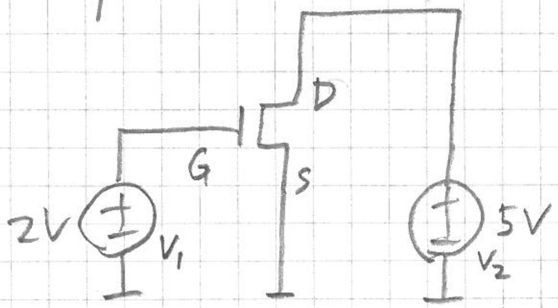
We may summarize in this way:

The S model is good for analyzing the ON/OFF behavior of a MOSFET.

The SR model is good for approximating the MOSFET's behavior in the triode region.

P72 Let's work on two examples to get ourselves familiar with the SCS model and its analysis:

Example 1.



Assuming
$V_T = 1\,V$
$K = 1\,mA/V^2$

① $i_{DS} = ?$ since $V_{GS} = 2V > V_T$
and $V_{DS} = 5V > V_{GS} - V_T = 1$,
we see the MOSFET is operating in the saturation region $\Rightarrow i_{DS} = \frac{K(V_{GS}-V_T)^2}{2} = 0.5\,mA$

② If we keep V_1 and decrease V_2, at what condition of V_2 will the MOSFET enter the triode region?

→ As long as $V_{DS} \geq V_{GS} - V_T$ the MOSFET will stay in the saturation region

→ if $V_2 = V_{DS} < V_{GS} - V_T = 1v$, the MOSFET will enter the triode region.

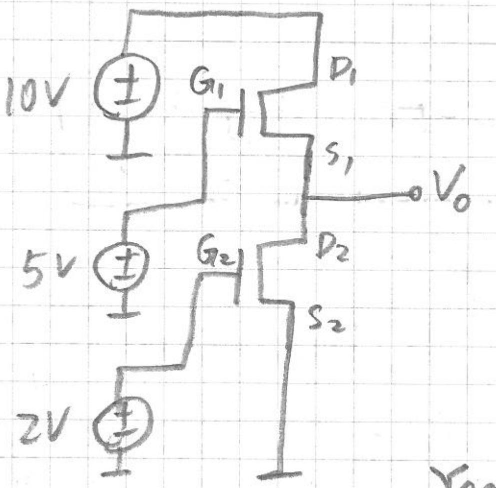
③ If we keep V_2, what would be the range of V_1 for the MOSFET to stay in the saturation region?

$\rightarrow \quad V_{DS} \geq V_{GS} - V_T$

$\rightarrow \quad 5 \geq V_{GS} - 1 \rightarrow V_{GS} \leq 6$

$\rightarrow \quad V_1 = V_{GS} \leq 6$ ※

Example 2.



Assuming $V_T = 1$ V

$k = 4$ mA/V^2

If $V_o = 3$, are both MOSFETs operating in the saturation region?

Yes, since $V_{G_1S_1} - V_T$

$= (5-3) - 1 = 1 > 0$

and $\underbrace{V_{G_1S_1} - V_T}_{=1} \leq \underbrace{V_{D_1S_1} = 10 - 3 = 7}$

And $V_{G_2S_2} - V_T = 2 - 1 = 1 > 0$

and $\underbrace{V_{G_2S_2} - V_T}_{= 2-1} \leq \underbrace{V_{D_2S_2}}_{=3}$

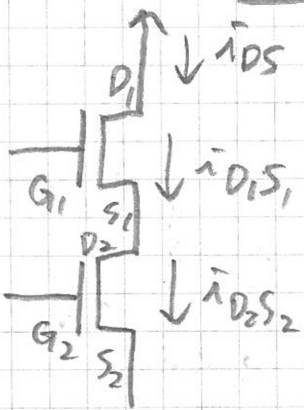
And $V_{G_1S_1} = V_{G_2S_2}$ correctly implies $i_{D_1S_1} = i_{D_2S_2}$ ※

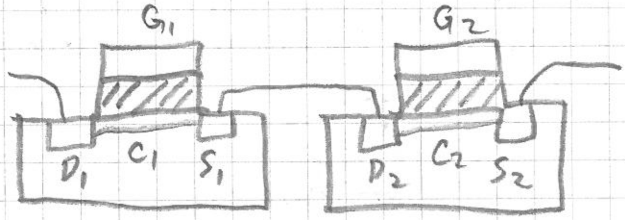
(Note on 2020/6/1 3PM:

I apologize for the confusion regarding whether $V_0 = 2$ in Example 2 makes sense or not.

After some more thoughts, I think what I said this morning is wrong.

It is <u>wrong</u> that $i_{DS} = i_{D_1S_1} + i_{D_2S_2}$





In physics, $i_{D_1S_1}$ is constrained by channel C_1, and $i_{D_2S_2}$ is constrained by channel C_2. It should be that

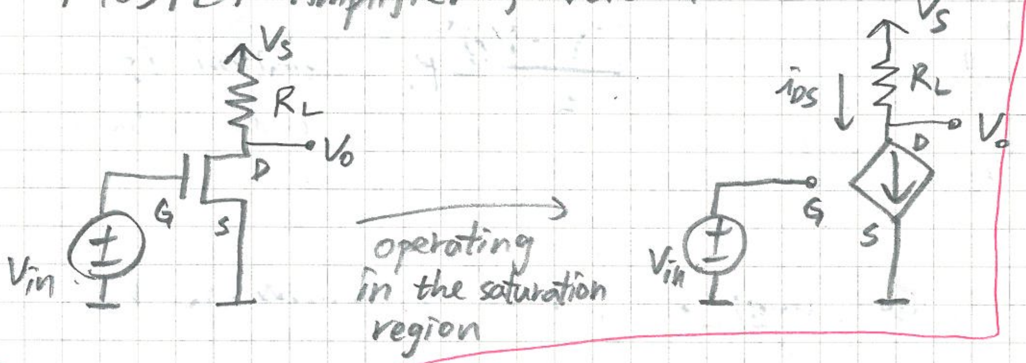
$$i_{DS} = i_{D_1S_1} = i_{D_2S_2}.$$

And therefore $V_0 = 2$ does NOT make sense.)

Now let's see how we may leverage a MOSFET operating in the saturation region for some useful purposes.

It turns out that in the saturation region a MOSFET may be used to amplify a signal. Amplifiers are used in many real-world applications and systems. Headphones and speakers are two examples.

- MOSFET Amplifier, Version 1



The voltage-controlled current source  gives

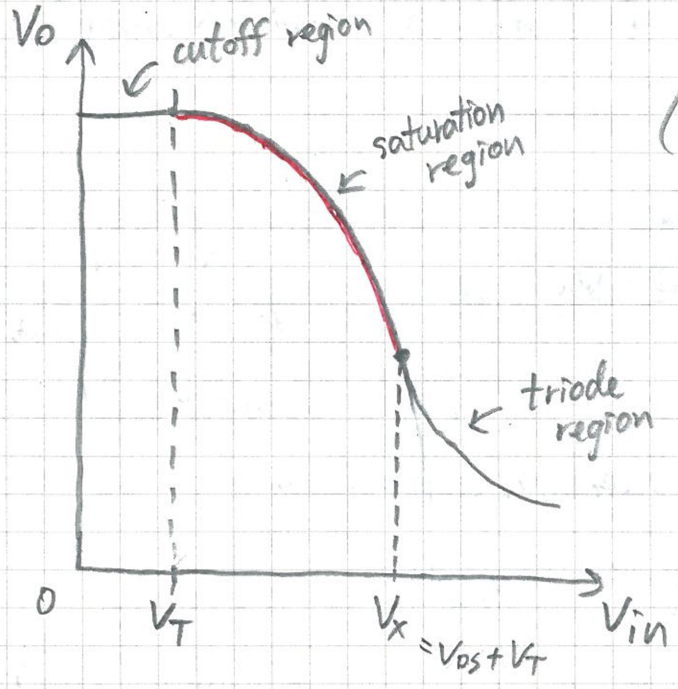
$$i_{DS} = \frac{K(V_{GS}-V_T)^2}{2} \quad (\text{see } P69)$$

and in addition, by Ohm's law $i_{DS} = \dfrac{V_S - V_o}{R_L}$

$$\Rightarrow \frac{K(V_{GS}-V_T)^2}{2} = \frac{V_S - V_o}{R_L} \Rightarrow \boxed{V_o = V_S - K \cdot \frac{(V_{in}-V_T)^2}{2} \cdot R_L}$$

and that $V_{GS} = V_{in}$

P76 following the equation on P75, we have



(compare this to the plots on P57 and P65)

The $V_o = V_s - \underbrace{\frac{K(V_{in}-V_T)^2}{2} R_L}_{\text{a parabola}}$ relation is only valid when $\underline{V_T \le V_{in} \le V_x}$.

Later we will study how to determine V_x. (P81)

The slope $\frac{V_o}{V_{in}}$ is also the ratio between V_{in} and V_o. We see that in some part of the saturation region the magnitude of the slope is greater than one. Therefore we may amplify V_{in} by $\frac{V_o}{V_{in}}$ times and output the result as V_o.

The "gain" of the amplifier is defined to be $\frac{V_o}{V_{in}}$.

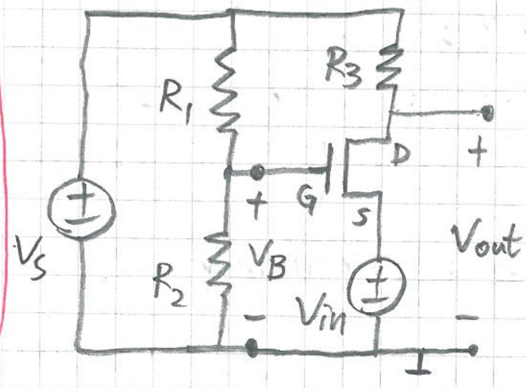
Using this MOSFET amplifier, however, the output will be inverted, which might not be what we want:

$$\begin{cases} V_{in} \downarrow \Rightarrow V_o \uparrow \\ V_{in} \uparrow \Rightarrow V_o \downarrow \end{cases}$$

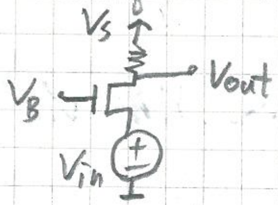
(can be verified by the sign of the slope in the $V_o - V_{in}$ plot.)

Let's study an alternative design ↓

- MOSFET Amplifier, Version 2



In this configuration, we use one voltage source to provide voltages to the gate and the drain. This is equivalent to:



$$V_{GS} = V_B - V_{in}$$
$$= \left(V_S \times \frac{R_2}{R_1+R_2}\right) - V_{in}$$

In saturation region, we have

$$i_{DS} = \frac{K(V_{GS}-V_T)^2}{2}$$

and by Ohm's law

$$i_{DS} = \frac{V_S - V_{out}}{R_3}$$

Therefore, we have

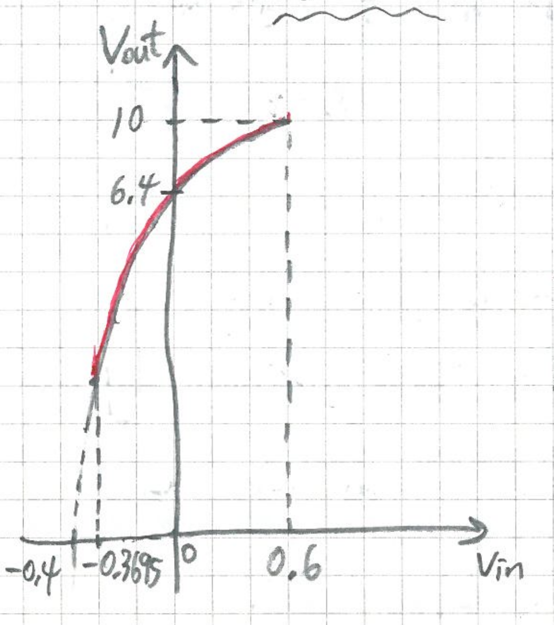
$$\frac{K(V_{GS}-V_T)^2}{2} = \frac{V_S - V_{out}}{R_3}$$

$$\Rightarrow \frac{K\left(V_S \times \frac{R_2}{R_1+R_2} - V_{in} - V_T\right)^2}{2} = \frac{V_S - V_{out}}{R_3}$$

P78 following the equation on P77, suppose that

$$\begin{cases} V_S = 10\ V \\ R_1 = 84\ k\Omega,\ R_2 = 16\ k\Omega,\ R_3 = 20\ k\Omega \\ V_T = 1\ V,\ K = 1\ mA/V^2 \end{cases}$$

then $V_B = 1.6$ and $V_{out} = 10 - 10(0.6 - V_{in})^2$



To operate in the saturation region, both of the following conditions must be satisified:

$$\begin{cases} V_{GS} \geq V_T & \text{—①} \\ V_{GS} - V_T \leq V_{DS} & \text{—②} \end{cases}$$

from ①, $1.6 - V_{in} \geq 1 \Rightarrow V_{in} \leq 0.6\ V$

from ②, $(1.6 - V_{in}) - 1 \leq V_{out} - V_{in}$

$\Rightarrow V_{out} \geq 0.6\ V$

$\Rightarrow 10 - 10(0.6 - V_{in})^2 \geq 0.6$

$\Rightarrow -0.3695\ V \leq V_{in} \leq 1.5695\ V$

$-0.3695\ V \leq V_{in} \leq 0.6\ V$

Thus we see that it is an amplifier ($\frac{V_{out}}{V_{in}} > 0$)

and $\begin{cases} V_{in} \uparrow \Rightarrow V_{out} \uparrow \\ V_{in} \downarrow \Rightarrow V_{out} \downarrow \end{cases}$ ※

For the purpose of signal amplification, and for the SCS model in general, we would want to make sure that the MOSFET operates in the saturation region. Given the circuit's parameters, in order to determine whether the MOSFET will operate in the saturation region (or to determine the valid range of values of parameters), we may use the <u>conditions listed on P68</u> or use <u>graphical analysis</u>.

For example, for the MOSFET amplifier on P75, suppose $V_S = 5V$, $R_L = 1\,k\Omega$, $V_T = 0.8\,V$, $V_{in} = 2.5\,V$ is the MOSFET in the saturation region?

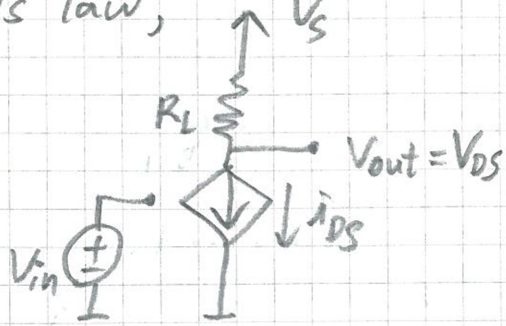
Answer: $V_{GS} = V_{in} = 2.5\,V$

$$V_{DS} = V_o = V_S - K\frac{(V_{in}-V_T)^2}{2}R_L = 4.28\,V$$

and we see that both $\begin{cases} V_{GS} \geq V_T \\ V_{DS} \geq V_{GS} - V_T \end{cases}$

Therefore, it is indeed in the saturation region. #

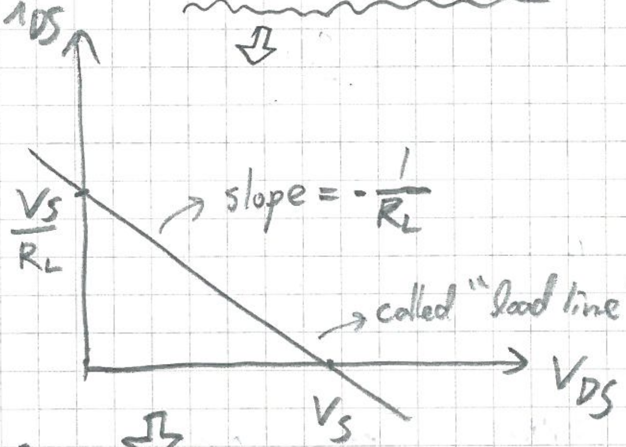
P80 Using graphically analysis, we first observe that, by Ohm's law,



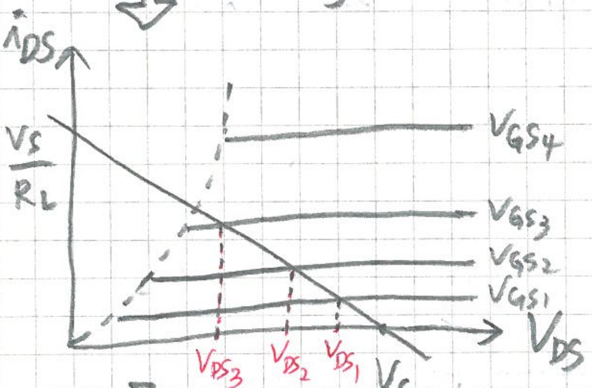
$$i_{DS} = \frac{V_S - V_{DS}}{R_L}$$

$$\Rightarrow i_{DS} = \frac{V_S}{R_L} - \frac{1}{R_L} V_{DS}$$

⇩



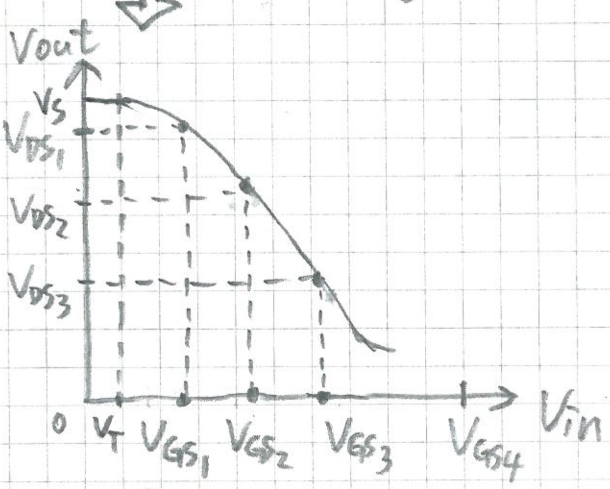
⇩



(see the plot on P68)

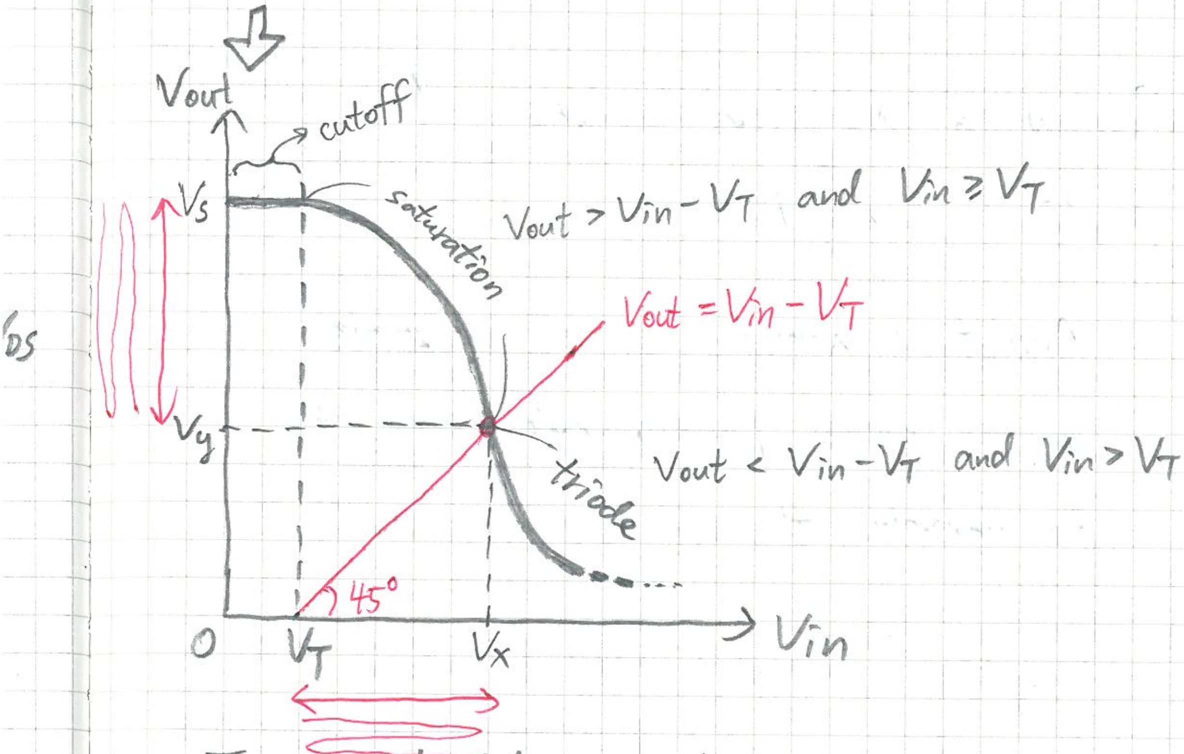
$$\begin{cases} V_{in} = V_{GS_x} & x = 1 \sim 4 \\ V_{out} = V_{DS_x} & x = 1 \sim 3 \end{cases}$$

⇩



(see the plot on P76)

⇩



To compute V_x, we have

$$\begin{cases} V_{out} = V_{in} - V_T \\ V_{out} = V_s - K\dfrac{(V_{in}-V_T)^2}{2}R_L \quad (P75) \end{cases}$$

$$\Rightarrow V_{in} - V_T = V_s - K\frac{(V_{in}-V_T)^2}{2}R_L$$

$$\Rightarrow \frac{KR_L}{2}(V_{in}-V_T)^2 + (V_{in}-V_T) - V_s = 0$$

$$\Rightarrow V_{in} - V_T = \frac{-1+\sqrt{1+2V_sR_LK}}{KR_L}$$

$$\Rightarrow \boxed{\begin{aligned} V_x &= \frac{-1+\sqrt{1+2V_sKR_L}}{KR_L} + V_T \\ V_y &= V_x - V_T \end{aligned}}$$

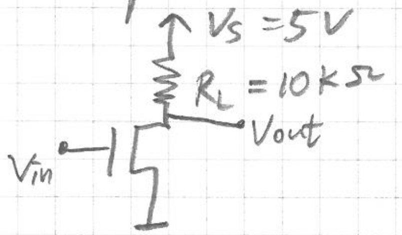
P82

Comparing analytical analysis and graphical analysis:

- Analytical analysis is as accurate as the model (and formula) can be. For example, the use of $i_{DS} = \frac{K(V_{GS}-V_t)^2}{2}$

- Graphical analysis may be more accurate, provided that the manufacturer of the electronic device often gives "data sheet", which includes the actual measured physical values of, for example, i_{DS} - V_{DS} characteristics.
- Graphical analysis also provides more insights (for example, see P37, 38).

Example:

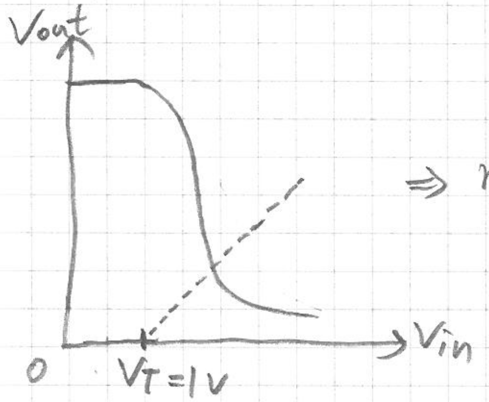


Assuming $V_T = 1V$

$K = 1\, mA/V^2$

What would be the maximum V_{in} that the MOSFET can still stay in the saturation region?

If the V_{out} - V_{in} plot is accurate, we may directly estimate the maximum V_{in}:



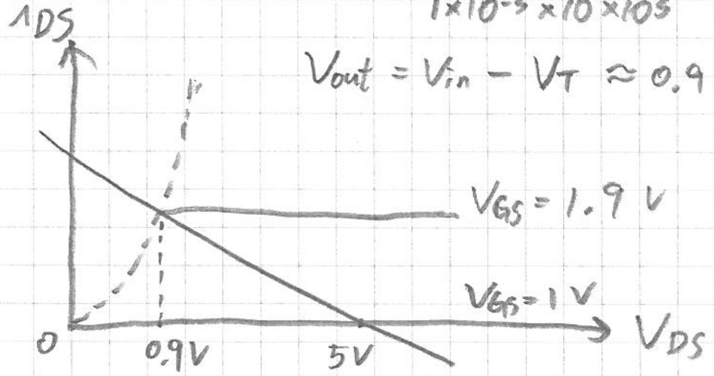
$\Rightarrow$ maximum V_{in} is around $2V$

Applying analytical analysis, we have

$$V_{in} = \frac{-1+\sqrt{1+2V_S K R_L}}{K R_L} + V_T$$

$$= \frac{-1+\sqrt{1+2\times5\times10^{-3}\times10\times10^{3}}}{1\times10^{-3}\times10\times10^{3}} + 1 \approx 1.9\,V$$

$$V_{out} = V_{in} - V_T \approx 0.9\,V$$



P84 ☆ The Small-Signal Model

We have learned some small-signal analysis when we were studying diode circuits early this semester (P43 - P48; Section 4.5 in the textbook).

Small-signal model and its analysis play a critical role in both design and usage of MOSFET amplifiers because

① many real-world input signals are small signals;

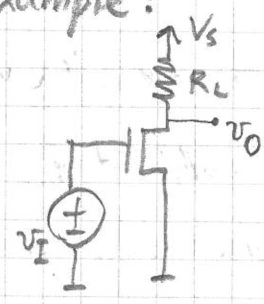
② the small-signal gain $\frac{v_o}{v_i}$ is "linear", which implies less signal distortion caused by the amplifier;

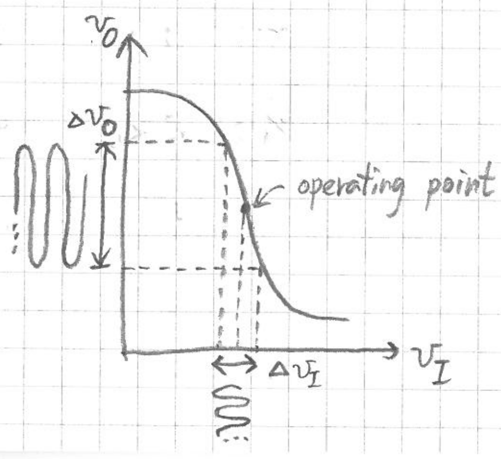
③ the circuit under the small-signal model is "linear", which means we may apply existing linear circuit analysis techniques, for example Thévenin's Theorem, to help us understand the circuit's behavior and its response to input signal.

It is important to remember that small-signal model is an "approximation" of the original non-linear model. In practice, it may be needed to estimate the qudity of such an approximation. That part is beyond the scope of this course, but you may study P218 in the textbook to get some idea. It is interesting to recall that the original non-linear model is itself an approximated description of how a circuit behaves in the real world.

In essence, the small-signal model describes how a circuit responds to a small, time-varying signal. The large, time-invariant signal is used to determine the operating point around which the small signal oscillates. Take our familiar MOSFET amplifier, for example:

$$\underbrace{v_I}_{\text{total input}} = \underbrace{V_I}_{\text{large signal}} + \underbrace{v_i}_{\text{small signal}}$$





P86

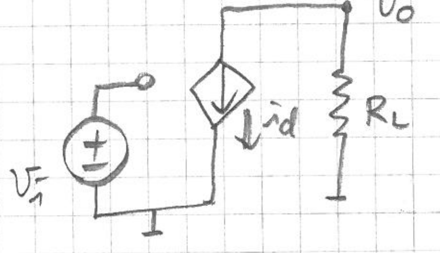
The large, time-invariant signal in this case is call the DC bias, or the DC offset.

直流偏壓

In the small-signal model, we consider that all of the independent current sources and independent voltage sources are shut off, because their impact to the circuit does not change along with the small signal; in other word, their impact to the circuit is time-invariant.

In this sense, the voltage-controlled current source is not considered shut off for our MOSFET amplifier, because the current depends on the small-signal v_i (and depends on the large-signal V_I, too).

The small-signal model of our MOSFET amplifier:



where $i_d = g_m \cdot v_i$,

and $g_m = K(V_I - V_T)$

is called the incremental transconductance.

To obtain the small-signal gain $\frac{v_o}{v_i}$, we may apply linear circuit analysis:

$$\frac{v_o - 0}{R_L} = -i_d$$

$$\Rightarrow v_o = -i_d \cdot R_L = -(g_m \cdot v_i) \cdot R_L$$

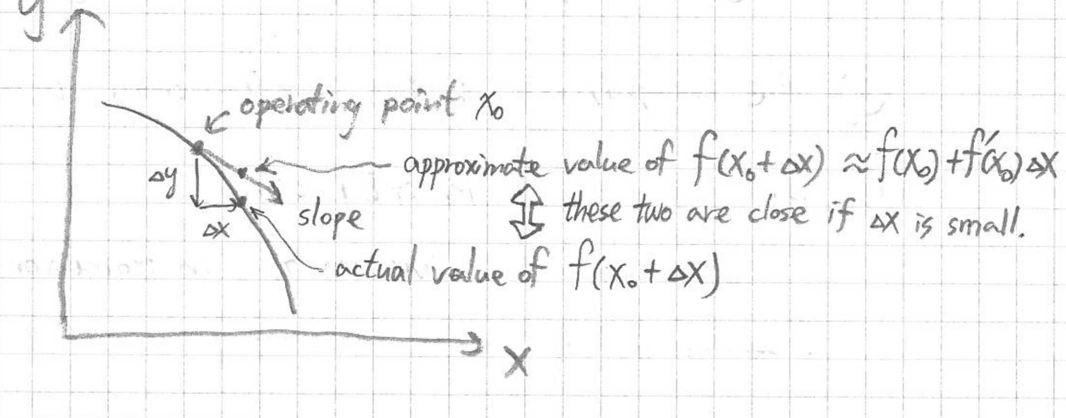
$$\Rightarrow \frac{v_o}{v_i} = -g_m \cdot R_L$$

and the magnitude of the gain is

$$\left|\frac{v_o}{v_i}\right| = g_m R_L .$$

In class, we've used the Taylor series expansion to derive that relation $i_d = K(V_I - V_T)v_i$.

The take-home message of such a derivation is that we may, in general, compute <u>the first derivative (一次微分)</u> of a function at the operating point to get a reasonable small-signal relation:



Study textbook Chapter 8 and Homework 6 for some examples of the small-signal analysis and the model. Finally, from a practical viewpoint, we may summarize four criteria regarding how to select an appropriate operating point, or in other word, how to set a proper DC bias:

1. determine the range of input signal v_I such that the MOSFET would operate in the saturation region; (P356 in textbook)
2. if the large-signal is also time-varying, we may want to maximize the peak-to-peak swing of the input signal by setting the operating point at the middle of the valid range of v_I for operation in the saturation region; (review P351 and P369 in the textbook)
3. the magnitude of the small-signal gain;
4. driving another MOSFET circuit; (P61 in this note; Questions 4, 5 in Homework 4) (Section 8.2.3 in the textbook)

We will discuss more on the <u>difference amplifier</u> after the final exam. It will not be included in the exam.