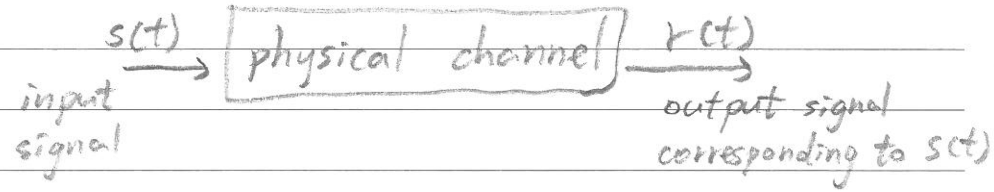
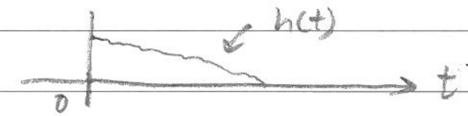
☆ A supplementary note for textbook pages 42-45



For any practical physical channel, its output $r(t)$ will be a bit different from its input $s(t)$, even if there's no noise. For example, the signal strength will declay as the signal travels through the channel. We also call $r(t)$ the "response" of the channel corresponding to input $s(t)$.

Now, let $\delta(t) = \begin{cases} 1 & \text{for } t=0 \\ 0 & \text{else} \end{cases}$ $\quad \delta(t-\tau) = \begin{cases} 1 & \text{for } t=\tau \\ 0 \end{cases}$

describes a unit impulse of input at $t=0$, and let $h(t)$ be its response:



The $r(t)$ is essentially the sum of all pulse responses at time t. Recall that the input signal $s(t)$ can be thought of as a sum of pulses $s(\tau)\delta(t-\tau)$ at each τ.

we can think in this way:

Therefore, for each pulse of the input signal happening in the past, its response may span up to the current time t.

So, we can compute $r(t)$ by considering all those responses:

$$r(t) = \underbrace{\int_{-\infty}^{\infty} s(\tau) h(t-\tau)\, d\tau}$$

this is also called <u>convolution</u> of $s(t)$ and $h(t)$.

Now, consider $s(t) = e^{j2\pi f t}$

$$r(t) = \int_{-\infty}^{\infty} s(\tau) h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} e^{j2\pi f\tau} h(t-\tau)\, d\tau$$

let $\tau' = t-\tau \Rightarrow \frac{d\tau'}{d\tau} = -1 \Rightarrow d\tau' = -d\tau \Rightarrow d\tau = -d\tau'$

so $r(t) = \int_{\infty}^{-\infty} e^{j2\pi f(t-\tau')} h(\tau') (-1)\, d\tau'$

$$= \int_{-\infty}^{\infty} e^{j2\pi f(t-\tau')} h(\tau')\, d\tau'$$

$$= e^{j2\pi f t} \underbrace{\int_{-\infty}^{\infty} e^{-j2\pi f\tau'} h(\tau')\, d\tau'}_{\text{let this be } \mathbb{H}(f)}$$

$\Rightarrow r(t) = s(t) \cdot \mathbb{H}(f)$

$$\mathbb{H}(f) = \int_{-\infty}^{\infty} h(\tau) \cdot e^{-j2\pi f\tau}\, d\tau$$

is called the Fourier transform of $h(\tau)$.

$$h(t) = \int_{-\infty}^{\infty} \mathbb{H}(f) \cdot e^{j2\pi f\tau}\, df$$

is called the inverse Fourier transform of $\mathbb{H}(f)$

We have shown that $r(t) = s(t) \cdot \mathbb{H}(f)$.

$r(t) = \mathbb{H}(f) \cdot s(t)$

$= \mathbb{H}(f) \cdot \int_{-\infty}^{\infty} \mathbb{S}(f) \cdot e^{j2\pi f t}\, df$ ← inverse Fourier transform of $\mathbb{S}(f)$

$= \int_{-\infty}^{\infty} \mathbb{H}(f) \cdot \mathbb{S}(f)\, e^{j2\pi f t}\, df$

and since also $r(t) = \int_{-\infty}^{\infty} R(f)\, e^{j2\pi f t}\, df$ ← inverse Fourier transform

$\Rightarrow R(f) = \mathbb{H}(f) \cdot \mathbb{S}(f)$

Finally, let $R'(f) = R(f) \cdot \mathbb{H}^{-1}(f)$ ← our customized filter for this channel; attached at the receiving side.

then we see

$$R'(f) = \mathbb{H}(f) \cdot \mathbb{S}(f) \cdot \mathbb{H}^{-1}(f)$$

$$= \mathbb{S}(f)$$

and we may get back the original signal by yet another inverse Fourier transform:

$$s(t) = \int_{-\infty}^{\infty} \mathbb{S}(f) \cdot e^{-j2\pi f\tau}\, df$$

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