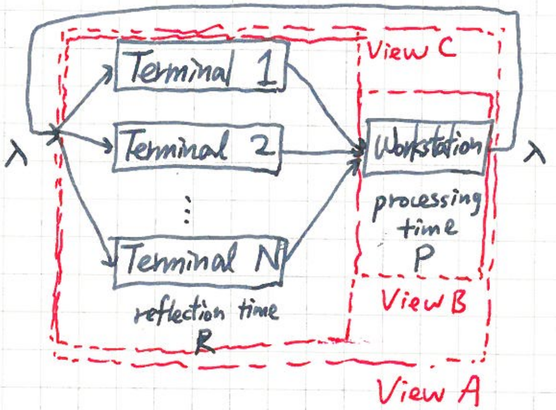
Example 4 (Exp 3.7) Bounding the attainable system throughput λ



View A

$\Rightarrow \lambda = \frac{N}{T}$, $T = R + D$

where

$P \le D \le NP$

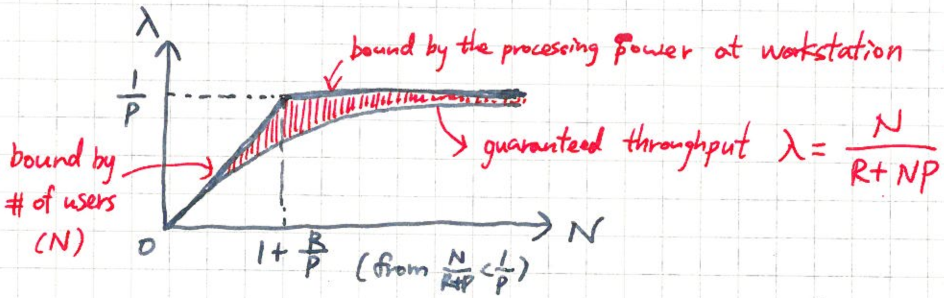
$$\Rightarrow \frac{N}{R+NP} \le \lambda \le \frac{N}{R+P}$$

View B $\Rightarrow 1 \ge \lambda P \Rightarrow \lambda \le \frac{1}{P}$

(View C $\Rightarrow N' = \lambda R \Rightarrow N' \le \frac{R}{P}$)

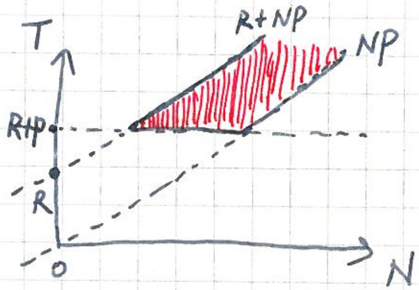
From Views A and B, we have

$$\frac{N}{R+NP} \le \lambda \le \min\left\{\frac{1}{P}, \frac{N}{R+P}\right\}$$



Further, by $T = \frac{N}{\lambda}$

$$\Rightarrow \max\{NP, R+P\} \le T \le R+NP$$



~~Note for lecture 07~~

Little's Theorem provides us a very useful way to estimate a system's performance regarding N, λ, and T:

$$N = \lambda \cdot T \quad \text{at steady state}$$

But N and T cannot be specified further without knowledge of some statistics of the system. A way to proceed is to try to compute

$$N = \sum_{n=0}^{\infty} n \cdot p_n .$$

where p_n stands for the stedy-state probabilities of n customers in the system. Then we can get T using Little's Theorem.

The theory of Markov chains provides us a powerful tool to obtain p_n. In the following, we first introduce the theory of Markov chains, followed by some statistics of the system where the theory is applicable:

☆ Markov chain theory (materials borrowed from textbook 2 by Prof. Mor Harchol Balter)

Definition: A discrete-time Markov chain (DTMC) is a stochastic process $\{X_n, n = 0, 1, 2, \ldots\}$ where X_n denotes the state at time step n and such that, $\forall n \ge 0$, $\forall i, j$, and $\forall i_0, \ldots, i_{n-1}$,

a sequence of random variables

$$P\{X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\}$$
$$= P\{X_{n+1} = j \mid X_n = i\} = P_{ij} \text{ (at steady state)}$$

P_{ij} ← called "transition probability" is independent of time step and of past history.

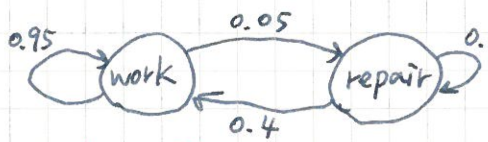
Note: a "stochastic process" is a term for a sequence of random variables.

Definition: The Markovian Property

The conditional distribution of any future state X_{n+1} depends only on the present state X_n.

Definition: The transition probability matrix associated with any DTMC is a matrix $\mathbb{P}$ whose (i,j)-th entry P_{ij} represents the probability of moving to state j on the next transition, given that the current state is i.

Example: Consider a machine is either working or in repair, with the following Markov chain diagram:



○ : state

↷ : transition and its probability

$$\mathbb{P} = \begin{array}{c c} & \begin{array}{cc} W & R \end{array} \\ \begin{array}{c} W \\ R \end{array} & \begin{bmatrix} 0.95 & 0.05 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

$(\mathbb{P}^2)_{RR} = 0.38$

$$\mathbb{P}^2 = \begin{bmatrix} 0.9225 & 0.0775 \\ 0.62 & 0.38 \end{bmatrix}$$

$$\mathbb{P}^3 = \begin{bmatrix} 0.9073 & 0.0926 \\ 0.741 & 0.259 \end{bmatrix}$$

$$\vdots$$

$$\mathbb{P}^{10} = \begin{bmatrix} \sim 0.89 & \sim 0.11 \\ \sim 0.89 & \sim 0.11 \end{bmatrix}$$

$$\mathbb{P}^{11} = \begin{bmatrix} \sim 0.89 & \sim 0.11 \\ \sim 0.89 & \sim 0.11 \end{bmatrix}$$

$$\vdots$$

$$P_{ij}^2 = \sum_{k=0}^{M-1} \boxed{P_{ik}} \cdot P_{kj}$$

$$(\mathbb{P}^2)_{ij} = i\,[\quad]\,[\quad]^{\,j}$$

$$\Rightarrow P_{ij}^n = \sum_{k=0}^{M-1} \boxed{P_{ik}^{n-1}} \cdot P_{kj}$$

(from i, go to j in n steps)

in general, let $\mathbb{P} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$ $\quad 0 < a < 1$, $\quad 0 < b < 1$

$$\lim_{n\to\infty} \mathbb{P}^n = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$$

← all rows are the same, implying that the starting state does not matter!

let $\pi_j = \lim_{n\to\infty} P_{ij}^n = (\lim_{n\to\infty} P^n)_{ij}$

↑ the limiting probability

$$\begin{bmatrix} \pi_0 & \pi_1 & \cdots & \pi_M \\ \pi_0 & \pi_1 & \cdots & \pi_M \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

for an M-state DTMC,

$\vec{\pi} = (\pi_0, \pi_1, \pi_2, \ldots, \pi_M)$ with $\sum_{i=0}^{M-1} \pi_i = 1$

represents the limiting distribution in each state.

(for proofs of existence, etc., see textbook2 Chapters 8 and 9)

It turns out that the above limiting distribution is equal to an unique stationary distribution where $\vec{\pi} \cdot \mathbb{P} = \vec{\pi}$

$\Rightarrow$ in general, we can compute π_j using $\vec{\pi} \cdot \mathbb{P} = \vec{\pi}$ and $\sum_{i=0}^{M-1} \pi_i = 1$

↖ called "stationary equations"

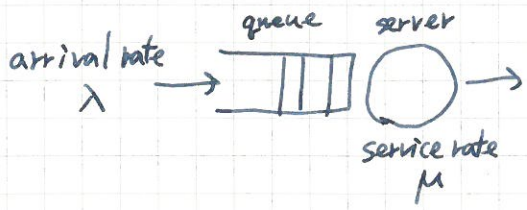
in the previous example, for example,

$$\begin{cases} \pi_W = \pi_W \times 0.95 + \pi_R \times 0.4 \\ \pi_R = \pi_W \times 0.05 + \pi_R \times 0.6 \\ \pi_W + \pi_R = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \pi_W = \frac{8}{9} \approx 0.89 \\ \pi_R = \frac{1}{9} \approx 0.11 \end{cases}$$

Back to our analysis of queueing systems, we may obtain p_n by computing π_j if the system has the Markovian Property! See next page ⟶

★ Arrival statistics and service statistics of queueing system



The Kendall notation to classify the queueing system

$$X / Y / Z$$

- X: $\boxed{\text{the distribution of interarrival times for the arrival process}}$
  M: memoryless
- Y: $\boxed{\text{the distribution of the service time}}$
  M: memoryless
  G: general
  D: deterministic
- Z: # of servers

We start with the M/M/1 queue.

Exponential distribution, Definition:

a random variable X with the probability density function

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

Its cumulative distribution function is

$$F_X(s) = P\{X \leq s\} = \int_0^s f(y)dy = 1 - e^{-\lambda s}$$

The mean $E[X] = \int_0^\infty x \cdot f(x) dx = \frac{1}{\lambda}$, and we call λ "rate"

Exponential distribution is memoryless in the sense that

$$P\{X > s+t \mid X > s\} = \frac{P\{X > s+t\}}{P\{X > s\}} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t}$$

$$= P\{X > t\}$$

We model the arrival process as the Poisson process, the most widely used model for its good approximation in describing the aggregate behavior of a large number of low-rate arrivals (e.g., an IoT network with many low-rate data streams).

Poisson process, Definition:

A stochastic process $\{A(t) \mid t \geq 0\}$ is a Poisson process with rate λ if

① $A(t)$ is a counting process that represents the total # of arrivals in $[0, t]$

② # of arrivals occurring in disjoint time intervals are independent

③ # of arrivals in any intervals of length τ is Poisson distributed with parameter $\lambda\tau$. That is, for all $t, \tau > 0$,

See later page for a review of Poisson distribution.

$$\cancel{P\{A(t)} \; P\{A(t+\tau) - A(t) = n\} = e^{-\lambda\tau} \frac{(\lambda\tau)^n}{n!}$$

$n = 0, 1, \ldots$

The Poisson process has an important property that interarrival times are independent and exponentially distributed with rate λ.

↳ thus memoryless

Therefore, we may use the Markov chain theory to analyze the M/M/1 queue!

Poisson process with rate λ

service times have an exponential distribution with rate μ

In M/M/1, let $N(t)$ denote the # of customers in the system at time t. The times at which customers will arrive or complete service in the future are independent of ① the arrival times of customers presently in the system, and of ② how much service the customer currently in service has already received.

We now first focus on discrete times

$$0,\ \delta,\ 2\delta,\ 3\delta,\ \ldots \quad \text{and then let } \delta \to 0$$

Let N_k denote the # of customers in the system at $k\delta$

The transition probabilities of DTMC in this case is

$$P_{ij} = P\{N_{k+1} = j \mid N_k = i\}$$

We can obtain these probabilities by considering # of arrivals and departures in a δ interval.

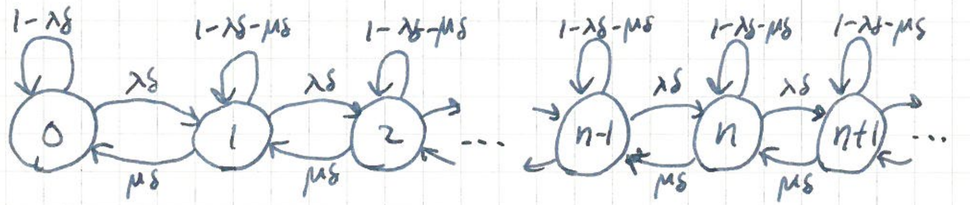
For example, $P_{ii} = (e^{-\lambda\delta})(e^{-\mu\delta})$

$$= e^{-\lambda\delta-\mu\delta}$$

using Taylor expansion $e^{-a} = 1 - a + a^2/2 - \cdots$

$$\Rightarrow P_{ii} = 1 - \lambda\delta - \mu\delta + o(\delta) \quad \text{where } \lim_{\delta\to 0}\frac{o(\delta)}{\delta} = 0$$

The DTMC for the M/M/1 queue is therefore:



where state n corresponds to n customers in the system, and the shown transition probabilities are correct up to an $o(\delta)$ term.

At long term, the frequency of transition from n to $n+1$ is equal to that from $n+1$ to n. This can be derived by considering that the probability of leaving a state is equal to that of entering the state.

$$p_n \lambda\delta + o(\delta) = p_{n+1}\mu\delta + o(\delta)$$

$$\Rightarrow p_n \lambda = p_{n+1}\mu \quad \text{as } \delta \to 0$$

let $\rho = \frac{\lambda}{\mu}$, then $p_{n+1} = \rho\, p_n$ and $p_{n+1} = \rho^{n+1} p_0$

$$1 = \sum_{n=0}^{\infty} p_n = \sum_{n=0}^{\infty} \rho^n p_0 = \frac{p_0}{1-\rho} \Rightarrow p_n = \rho^n(1-\rho)$$

Finally, $\boxed{N} = \sum_{n=0}^{\infty} n \cdot p_n = \sum_{n=0}^{\infty} n \cdot \rho^n (1-\rho)$

$$= (1-\rho)\sum_{n=0}^{\infty} n\cdot\rho^n = (1-\rho)\cdot\frac{\rho}{(1-\rho)^2} = \boxed{\frac{\rho}{1-\rho}}$$

and from Little's Theorem $\boxed{T = \frac{N}{\lambda} = \frac{1}{\mu-\lambda}}$

A Review of Poisson Distribution (from textbook [illegible]) P_1

- Bernoulli Trials, Definition:

  Repeated independent trials that have only two possible outcomes for each trial and their probabilities remain the same throughout trials.

  Example: tosses of a coin.

- Binomial Distribution, Definition:

  Let $b(k;n,p)$ be the probability that n Bernoulli trials with probability p for success and $q=1-p$ for failure result in k successes and $n-k$ failures.
  Then
  $$P\{S_n=k\}=b(k;n,p)=\binom{n}{k}p^kq^{n-k}$$
  is called the binomial distribution of S_n.
  Note that $\binom{n}{k}=\frac{n!}{k!(n-k)!}$ is called the binomial coefficient (coefficient of x^k in $(1+x)^n$)

- Poisson distribution is an approximation of the binomial distribution:
  In many real-world applications n is large and p is small, whereas the product $\lambda=np$ is of moderate magnitude.
  In this case,
  $$b(k;n,p)\approx\frac{\lambda^k}{k!}e^{-\lambda}$$
  let $p(k;\lambda)=\frac{\lambda^k}{k!}e^{-\lambda}$ and call it the Poisson distribution.

Proof of $b(k;n,p)\approx\frac{\lambda^k}{k!}e^{-\lambda}$:

$$b(0;n,p)=\binom{n}{0}p^0q^n=(1-p)^n=\left(1-\frac{\lambda}{n}\right)^n$$

$$\ln b(0;n,p)=n\ln\left(1-\frac{\lambda}{n}\right)$$
$$=-\lambda-\frac{\lambda^2}{2n}-\cdots$$

(using Taylor expansion $\ln(1+t)=t-\frac{1}{2}t^2+\frac{1}{3}t^3\ldots$ and let $t=-\frac{\lambda}{n}$

$\Rightarrow$ for large n, we have
$$b(0;n,p)\approx e^{-\lambda} \text{ since } \ln e^{-\lambda}=-\lambda$$

P_2 from the definition of $b(k;n,p)=\binom{n}{k}p^kq^{n-k}$

$$\frac{b(k;n,p)}{b(k-1;n,p)}=\frac{\lambda-(k-1)p}{kq}=\frac{\lambda-(k-1)p}{k(1-p)}\approx\frac{\lambda}{k}$$
for small p.

$$\Rightarrow b(1;n,p)\approx\frac{\lambda}{1}\cdot b(0;n,p)\approx\lambda e^{-\lambda}$$
$$b(2;n,p)\approx\frac{\lambda}{2}\cdot b(1;n,p)\approx\frac{\lambda^2}{2!}e^{-\lambda}$$
$$\cdots$$
$$\Rightarrow b(k;n,p)=\frac{\lambda^k}{k!}e^{-\lambda}$$
※

Derivation of the mean of S_n

$$E[S_n]=\sum_{k=0}^{\infty}k\cdot p(k;\lambda)$$
$$=\sum_{k=0}^{\infty}k\cdot\frac{\lambda^k}{k!}e^{-\lambda}$$
$$=\sum_{k=1}^{\infty}k\cdot\frac{\lambda^k}{k!}e^{\lambda}$$
$$=\lambda e^{-\lambda}\cdot\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!}$$
$$=\lambda e^{-\lambda}\cdot e^{\lambda}$$
$$=\lambda$$
※