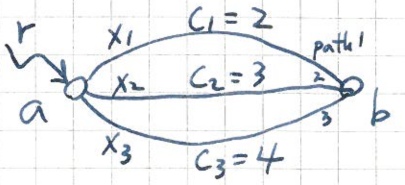
Note for lecture 13

(Page 452)

Supplementary materials for Section 5.5:

In class, we have studied the use of partial derivatives of a cost function to find a necessary condition for optimal routing. Here is an example to further illustrate the idea:

Consider the follow OD pair with 3 paths:



Suppose that the cost function is

$$D(x) = D_1(x_1) + D_2(x_2) + D_3(x_3)$$

where $D_i(x_i) = \dfrac{1}{C_i - x_i}$ $\begin{cases} D_1(x_1) = (2-x_1)^{-1} \\ D_2(x_2) = (3-x_2)^{-1} \\ D_3(x_3) = (4-x_3)^{-1} \end{cases}$

If $r=1$, then we will see the optimal routing is to use the third path exclusively and

$x^* = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Now, first consider the partial derivatives

(Next page →)

$D_\bullet = D$

$$\frac{\partial D_\bullet(x)}{\partial x_1} = (2-x_1)^{-2}$$

(the first derivative of $\frac{1}{c-x}$ is $\frac{1}{(c-x)^2}$)

$$\frac{\partial D_\bullet(x)}{\partial x_2} = (3-x_2)^{-2}$$

$$\frac{\partial D_\bullet(x)}{\partial x_3} = (4-x_3)^{-2}$$

which show the rate of change (the definition of derivative) of $D(x)$ with respect to x_1, x_2, x_3.

Now if we shift some tiny amount of flow from the third path to the second path, for example, then that means we essentially ~~deduct~~ subtract $\dfrac{\partial D(x)}{\partial x_3} \cdot \delta$ amount of cost from the total cost and

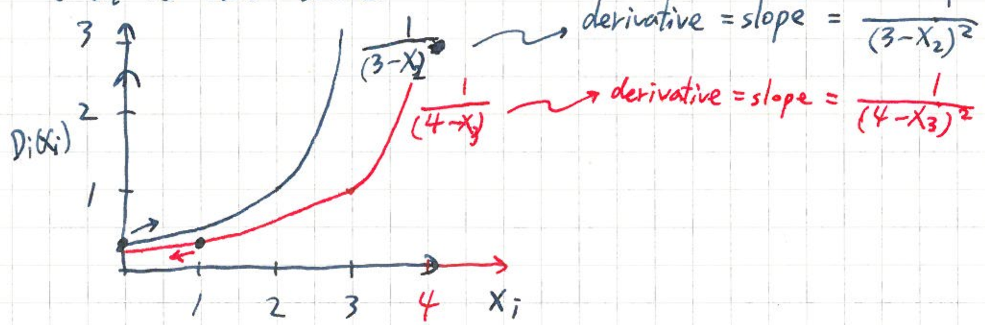
↑ that tiny amount of flow

add $\dfrac{\partial D(x)}{\partial x_2} \cdot \delta$ amount of cost ~~from~~ to the total cost.

In other words, we will make the following change in the total cost:

$$\frac{\partial D(x)}{\partial x_2} \cdot \delta - \frac{\partial D(x)}{\partial x_3} \cdot \delta$$

At $x^* = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\dfrac{\partial D(x)}{\partial x_2} = \dfrac{\partial D(x)}{\partial x_3}$ and it seems that the shift might not increase the total cost, but looking at the following plots we shall see that is not true:



although the slope of $D_2(x_2)$ at $x_2=0$ is equal to the slope of $D_3(x_3)$ at $x_3=1$, shifting flow in the way as mentioned means

① moving to the right of $(0, \frac{1}{3})$ along the curve $\frac{1}{3-x_2}$

and ② moving to the left of $(1, \frac{1}{3})$ along the curve $\frac{1}{4-x_3}$

Because both curves are monotonic, we shall see that such shift of flow will increase the total cost. This tells us that we should not attempt to shift flow (i.e., change route) if the values of the partial derivatives are equal. In fact, if we know the rates of change to the total cost are equal, there really is no point of changing the current setting of route. Now, let's see what happen if the routing is not

optimal. Consider $x^1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, for example.

Then $\frac{\partial D(x)}{\partial x_2}$ at $x_2 = 1$ would be $(3-1)^{-2} = \frac{1}{4}$

and $\frac{\partial D(x)}{\partial x_3}$ at $x_3 = 0$ would be $(4-0)^{-2} = \frac{1}{16}$

which lead to an observation that if we make the following flow shift

$$\begin{cases} x_2 = 1 \rightarrow x_2 = 1-\delta \\ x_3 = 0 \rightarrow x_3 = 0+\delta \end{cases}$$

then the change to the total cost would be

$$-\frac{\partial D(x)}{\partial x_2} \cdot \delta + \frac{\partial D(x)}{\partial x_3} \cdot \delta < 0$$

and we may reduce the total cost this way. I hope the above illustration could help you see why on Page 452 we made use of partial derivatives.

Finally, note that in Example 5.1, the slopes of x_1^* and x_2^* after r became larger than $c_1 - \sqrt{c_1 c_2}$ are not equal. I think I did not say it right in class.

In the illustration in this note, we see $x^* = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is optimal by the following step (same as that in Example 5.1): Suppose we use the third path exclusively. By condition (5.53) we have $\frac{1}{(4-r)^2} \le \frac{1}{9}$ and will arrive at a condition for r, i.e., $r \le 1$ ※