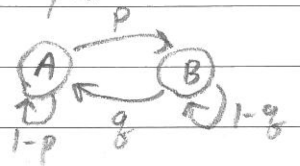
Lecture note for analyzing the slotted Aloha protocol:

- From P279 in the textbook, we may apply what we've learned about the DTMC analysis.

As a quick reivew, there are two ways to analyze a DTMC to obtain the steady-state probabilities:

Example DTMC



① $\begin{cases} P_A = P_A(1-p) + P_B \cdot q \\ P_B = P_A \cdot p + P_B(1-q) \\ P_A + P_B = 1 \end{cases}$

(from transition probability matrix and total probability)

② $\begin{cases} P_A \cdot p = P_B \cdot q \\ P_A + P_B = 1 \end{cases}$

(from the balance equation and the total probability)

Notice that from equations in ① we can also derive the balance equation in ②.

Problem 4.1 in the textbook use approach ①; in the following we use approach ②:

And for simplicity we suppose $m=4$:

$$\begin{cases} P_0(P_{02}+P_{03}+P_{04}) = P_1(P_{10}) \\ P_1(P_{12}+P_{13}+P_{14}) = P_2(P_{21}) \\ P_2(P_{23}+P_{24}) = P_3(P_{32}) \\ P_3(P_{34}) = P_4(P_{43}) \\ P_0+P_1+P_2+P_3+P_4 = 1 \quad \text{i.e.,} \ \sum_{i=0}^{4} P_i = 1 \end{cases}$$

$$\begin{aligned} \Rightarrow P_4 &= P_3\left(\frac{P_{34}}{P_{43}}\right) \\ &= P_2\left(\frac{P_{23}+P_{24}}{P_{32}}\right)\left(\frac{P_{34}}{P_{43}}\right) \\ &= P_1\left(\frac{P_{12}+P_{13}+P_{14}}{P_{21}}\right)\left(\frac{P_{23}+P_{24}}{P_{32}}\right)\left(\frac{P_{34}}{P_{43}}\right) \\ &= P_0\left(\frac{P_{02}+P_{03}+P_{04}}{P_{10}}\right)\left(\frac{P_{12}+P_{13}+P_{14}}{P_{21}}\right)\left(\frac{P_{23}+P_{24}}{P_{32}}\right)\left(\frac{P_{34}}{P_{43}}\right) \end{aligned}$$

$$P_3 = P_0\left(\frac{P_{02}+P_{03}+P_{04}}{P_{10}}\right)\left(\frac{P_{12}+P_{13}+P_{14}}{P_{21}}\right)\left(\frac{P_{23}+P_{24}}{P_{32}}\right)$$

$$P_2 = P_0\left(\frac{P_{02}+P_{03}+P_{04}}{P_{10}}\right)\left(\frac{P_{12}+P_{13}+P_{14}}{P_{21}}\right)$$

$$P_1 = P_0\left(\frac{P_{02}+P_{03}+P_{04}}{P_{10}}\right)$$

and from $1 = \sum_{i=0}^{4} P_i$

$$\Rightarrow 1 = P_0\left(1 + \frac{P_{02}+P_{03}+P_{04}}{P_{10}}\left(1 + \frac{P_{12}+P_{13}+P_{14}}{P_{21}}\left(1 + \frac{P_{23}+P_{24}}{P_{32}}\left(1 + \frac{P_{34}}{P_{43}}\right)\right)\right)\right)$$

plugging in each of the transition probability we may obtain the steady-state probabilities, and from there we may derive the average latency for each packet

Examples →

For example, consider two cases, both with $m=4$ and $q_r = 0.5$:

Case ①: $\lambda = 0.4$ packets/second

$$\Rightarrow q_a = 1 - e^{-\lambda/4} \approx 0.1$$

$$1 = P_0 \left(1 + \frac{6\times 0.81\times 0.01 + 4\times 0.9\times 0.001 + 1\times 0.0001}{3\times 0.3645}\right.$$

$$\left(1 + \frac{3\times 0.81\times 0.1 + 3\times 0.9\times 0.01 + 1\times 0.001}{2\times 0.81\times 2\times 0.25}\right.$$

$$\left(1 + \frac{2\times 0.9\times 0.1 + 1\times 0.01}{0.9\times 3\times 0.25\times 0.5}\right.$$

$$\left.\left.\left.\left(1 + \frac{0.1}{4\times 0.125\times 0.5}\right)\right)\right)\right)$$

$$\Rightarrow 1 = P_0 \left(1 + \frac{0.0523}{1.0935}\left(1 + \frac{0.291}{0.81}\left(1 + \frac{0.19}{0.3375}\left(1 + \frac{0.1}{0.25}\right)\right)\right)\right)$$

$$= P_0 (1.0764)$$

$$\Rightarrow P_0 \approx 0.929,\ P_1 \approx 0.044,\ P_2 \approx 0.014,\ P_3 \approx 0.008,\ P_4 \approx 0.003$$

$$N = \sum_{i=0}^{4} i \cdot P_i = 0.044 + 0.028 + 0.024 + 0.012 = 0.108$$

$$T = \frac{N}{\lambda} = \frac{0.108}{0.4} = 0.27 = \underline{270 \text{ ms}}$$ ※

Case ②: $\lambda = 4$ packets/second $\Rightarrow q_a \approx 0.6321$

$$1 = P_0 \left(1 + \frac{0.8557}{1.0935}\left(1 + \frac{0.7474}{0.81}\left(1 + \frac{0.8646}{0.3375}\left(1 + \frac{0.6321}{0.25}\right)\right)\right)\right)$$

$$= P_0 (9.031)$$

$$\Rightarrow P_0 \approx 0.1107,\ P_1 \approx 0.0866,\ P_2 \approx 0.0799,\ P_3 \approx 0.2048,\ P_4 \approx 0.5178$$

$$N = \sum_{i=0}^{4} i \cdot P_i = 2.932$$

$$T = \frac{N}{\lambda} = \frac{2.932}{4} = \underline{733 \text{ ms}}.$$ ※

Question to consider: could T be larger than 1s if $\lambda \uparrow$?