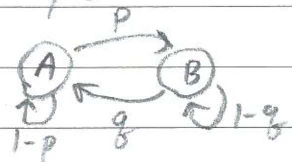
(Updated 2020/12/17)

Lecture note for analyzing the slotted Aloha protocol:

- From P279 in the textbook, we may apply what we've learned about the DTMC analysis.

As a quick reivew, there are two ways to analyze a DTMC to obtain the steady-state probabilities:

Example DTMC



① $\begin{cases} P_A = P_A(1-p) + P_B \cdot q \\ P_B = P_A \cdot p + P_B(1-q) \\ P_A + P_B = 1 \end{cases}$

(from transition probability matrix and total probability)

② $\begin{cases} P_A \cdot p = P_B \cdot q \\ P_A + P_B = 1 \end{cases}$

(from the balance equation and the total probability)

Notice that from equations in ① we can also derive the balance equation in ②.

Problem 4.1 in the textbook use approach ①; in the following we use approach ②:

And for simplicity we suppose $m = 4$:

$$\begin{cases} P_0(P_{02}+P_{03}+P_{04}) = P_1(P_{10}) \\ P_1(P_{12}+P_{13}+P_{14}+P_{10}) = P_2(P_{21}) \\ P_2(P_{23}+P_{24}+P_{21}) = P_3(P_{32}) + P_0(P_{02}) + P_1(P_{12}) \\ P_3(P_{34}+P_{32}) = P_4(P_{43}) + P_0(P_{03}) + P_1(P_{13}) + P_2(P_{23}) \\ P_0+P_1+P_2+P_3+P_4 = 1 \quad \text{i.e., } \sum_{i=0}^{4} P_i = 1 \\ P_4(P_{43}) = P_0(P_{04}) + P_1(P_{14}) + P_2(P_{24}) + P_3(P_{34}) \end{cases}$$

∴ Then the key observation is that we may plug $P_4(P_{43})$ into the equation for $P_3(P_{34}+P_{32})$ to simplify, and similarly plug the result into the equation for $P_2(\cdots)$ ...

$$\Rightarrow \begin{cases} P_1(P_{10}) = P_0(P_{02}+P_{03}+P_{04}) \\ P_2(P_{21}) = P_0(P_{02}+P_{03}+P_{04}) + P_1(P_{12}+P_{13}+P_{14}) \\ P_3(P_{32}) = P_0(P_{03}+P_{04}) + P_1(P_{13}+P_{14}) + P_2(P_{23}+P_{24}) \\ P_4(P_{43}) = P_0(P_{04}) + P_1(P_{14}) + P_2(P_{24}) + P_3(P_{34}) \\ P_0+P_1+P_2+P_3+P_4 = 1 \end{cases}$$

plugging in each of the transition probability we may obtain the steady-state probabilities, and from there we may derive the average latency for each packet

examples →

assuming that $m=4$

from the textbook :

$$\begin{cases} Q_a(i,n) = \binom{m-n}{i}(1-q_a)^{m-n-i} q_a^{\,i} \\ Q_r(i,n) = \binom{n}{i}(1-q_r)^{n-i} q_r^{\,i} \end{cases}$$

$P_{02} = Q_a(2,0) = \binom{4}{2}(1-q_a)^2 q_a^2 = 6\cdot(1-q_a)^2\cdot q_a^2$

$P_{03} = Q_a(3,0) = \binom{4}{3}(1-q_a)^1 q_a^3 = 4\cdot(1-q_a)\cdot q_a^3$

$P_{04} = Q_a(4,0) = \binom{4}{4}(1-q_a)^0 q_a^4 = q_a^4$

$P_{10} = Q_a(0,1)\cdot Q_r(1,1) = \binom{3}{0}(1-q_a)^3 q_a^0 \cdot \binom{1}{1}(1-q_r)^0 q_r^1 = (1-q_a)^3 q_r^1$

$P_{12} = Q_a(1,1)\cdot(1-Q_r(0,1)) = \binom{3}{1}(1-q_a)^2 q_a^1 (1-\binom{1}{0}(1-q_r)^1 q_r^0) = 3(1-q_a)^2 q_a q_r$

$P_{13} = Q_a(2,1) = \binom{3}{2}(1-q_a)^1 q_a^2 = 3\cdot(1-q_a)\cdot q_a^2$

$P_{14} = Q_a(3,1) = \binom{3}{3}(1-q_a)^0 q_a^3 = q_a^3$

$P_{21} = Q_a(0,2)\cdot Q_r(1,2) = \binom{2}{0}(1-q_a)^2 q_a^0 \cdot \binom{2}{1}(1-q_r)^1 q_r^1 = 2(1-q_a)^2(1-q_r)q_r$

$P_{23} = Q_a(1,2)\cdot(1-Q_r(0,2)) = \binom{2}{1}(1-q_a)^1 q_a^1\cdot(1-\binom{2}{0}(1-q_r)^2 q_r^0) = 2(1-q_a)q_a(1-(1-q_r)^2 q_r)$

$P_{24} = Q_a(2,2) = \binom{2}{2}(1-q_a)^0 q_a^2 = q_a^2$

$P_{32} = Q_a(0,3)\cdot Q_r(1,3) = \binom{1}{0}(1-q_a)^1 q_a^0\cdot\binom{3}{1}(1-q_r)^2 q_r = 3(1-q_a)^2(1-q_r)^2 q_r$

$P_{34} = Q_a(1,3)\cdot(1-Q_r(0,3)) = \binom{1}{1}(1-q_a)^0 q_a^1\cdot(1-\binom{3}{0}(1-q_r)^3 q_r^0) = q_a(1-(1-q_r)^3)$

$P_{43} = Q_a(0,4)\cdot Q_r(1,4) = \binom{0}{0}(1-q_a)^0 q_a^0\cdot\binom{4}{1}(1-q_r)^3 q_r = 4\cdot(1-q_r)^3 q_r$

Example ①: $\lambda = 0.4$ pkts/sec , $q_r = 0.5$

$\Rightarrow q_a = 1-e^{-\lambda/4} \approx 0.1$

$$\Rightarrow \begin{cases} P_{02} = 0.0486,\ P_{03} = 0.0036,\ P_{04} = 0.0001 \\ P_{10} = 0.3645,\ P_{12} = 0.1215,\ P_{13} = 0.027,\ P_{14} = 0.001,\ P_{43} = 0.25 \\ P_{21} = 0.405,\ P_{23} = 0.135,\ P_{24} = 0.01,\ P_{32} = 0.30375,\ P_{34} = 0.0875 \end{cases}$$

$$\Rightarrow \begin{cases} P_1 \times 0.3645 = P_0(0.0486+0.0036+0.0001) \\ P_2 \times 0.405 = P_0(0.0486+0.0036+0.0001) + P_1(0.1215+0.027+0.001) \\ P_3 \times 0.30375 = P_0(0.0036+0.0001) + P_1(0.027+0.001) + P_2(0.135+0.01) \\ P_4 \times 0.25 = P_0 \times 0.0001 + P_1 \times 0.001 + P_2 \times 0.01 + P_3 \times 0.0875 \\ P_0 + P_1 + P_2 + P_3 + P_4 = 1 \end{cases}$$

$\Rightarrow P_0 = 0.6933,\ P_1 = 0.0965,\ P_2 = 1225,\ P_3 = 0555,\ P_4 = 0.0319$

$\Rightarrow N = \sum_{k=1}^{4} k\cdot P_k = 0.6956$

$\Rightarrow T = N/\lambda = 1.739$ seconds

Example ② $\lambda = 4$ pkts/sec , $q_r = 0.5$

$\Rightarrow q_a \approx 0.6321$

following the same procedure, we have

$P_0 \simeq 0.000115,\ P_1 \approx 0.0038,\ P_2 \approx 0.0496,\ P_3 \simeq 0.2669$

$P_4 \approx 0.6939$

$\Rightarrow N = \sum_{k=1}^{4} k\cdot P_k = 3.6335$

$\Rightarrow T = N/\lambda \approx 0.9083$ seconds