

★ Signals, Systems, and Computing

- In computing systems, information (and energy) is stored and transferred in terms of signals, which are currents or voltages as a function of time.

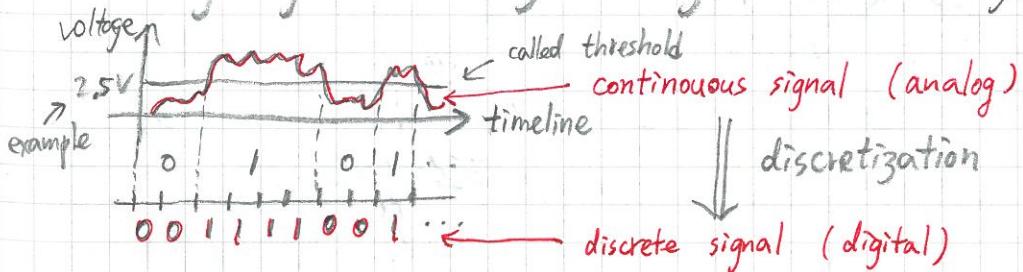
A circuit (or a system of circuits), as we will study in this course, is used to

- ① carry signals
- ② transform signals from one to another

Computing is also a transformation of signals; though the signals are digital, and the transformation is described at a higher layer of the abstraction (Figure 1.1 in the textbook).

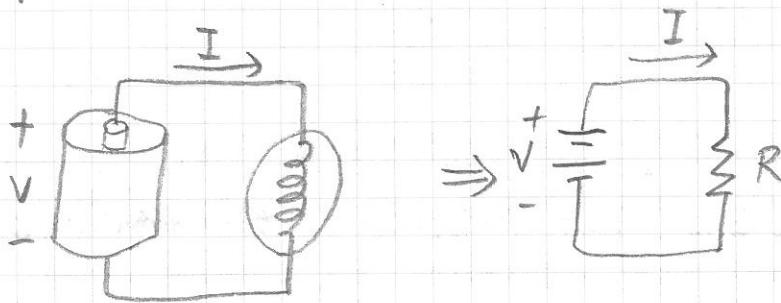
Example: We use a smart phone to record this lecture (voice signal \rightarrow currents and voltages), and the recording is stored in the phone, transferred via USB to your laptop, uploaded to a cloud drive, downloaded by your friend who cannot make it to the class, and finally played by your friend's speakers/headphone (currents and voltages \rightarrow voice signals).

- Analog signal to digital signal via discretization:

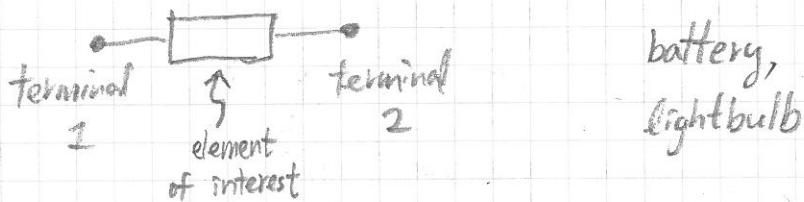


P₂

* The lumped circuit abstraction



for now, let's just consider elements having two terminals, e.g,



battery,
lightbulb

We may ignore the internal structure of the element and consider it as a "lump", where we may completely describe relevant properties (such as voltage and current) by only observing its terminals.

For example, if a resistor with resistance R obeys Ohm's law, then we may compute the current flowing through the resistor by the voltage across its terminals : $I = V/R$.

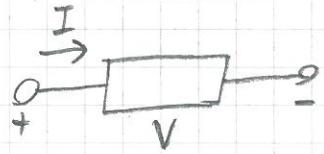
The lumped circuit abstraction works only under certain constraints, which we call the lumped matter discipline (LMD).

From Physics to Electronic Circuit:

★ The lumped matter discipline (LMD) → a set of rules that control an activity or situation.

objectives { simplify the analysis of electronic circuit;
modularize a complex circuit into analyzable elements.

Derivation of LMD:



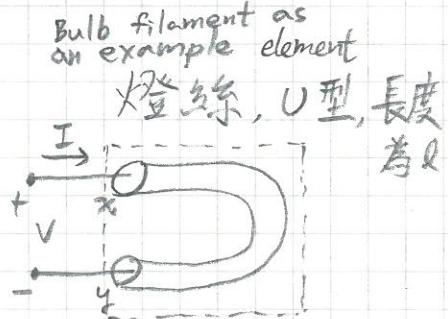
strategy #1:

to describe a unique voltage across the terminals X and Y

definition of voltage:

$$V_{yx} = - \int_x^y E \cdot dl$$

where E is the electrical field (a vector)
 dl is a tiny portion of l



and Faraday's law of induction:

$$\oint E \cdot dl = - \frac{\partial \Phi_B}{\partial t}$$

where Φ_B is the magnetic flux.

(\oint represents a closed path integral)

a sufficient condition
充分條件

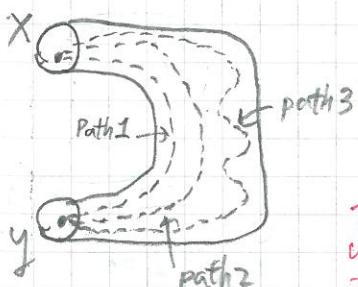
Thus, we see that if there's no time-varying magnetic flux,

$$\oint E \cdot dl = 0 \Rightarrow \int_X^Y E \cdot dl + \int_Y^X E \cdot dl = 0$$

along path 1 along path 2

$$\Rightarrow \int_X^Y E \cdot dl = \int_X^Y E \cdot dl$$

path 1 path 2



Think about it.
Why is this important?

也就是說 $\int_X^Y E \cdot dl$ 的值和路徑無關! ($= \int_X^Y E \cdot dl$)

P4 Therefore, the constraint for strategy #1 to work is

$$\frac{\partial \phi_B}{\partial t} = 0,$$

and we assume that holds for all time.
 (To make sure this constraint holds, we may need to revise the model and introduce an element called "inductor".)

strategy #2:
 to define a unique current through the terminals X and Y

First of all, the definition of current :

$$I = \int_{S_z} J \cdot dS$$

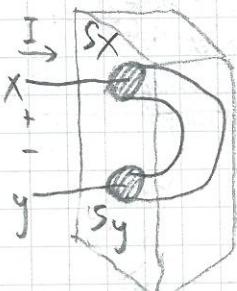


where J is the current density at a given point within a filament and S_z is the cross-sectional surface of the filament at point z .

Due to the conservation of charge, we have

total current out of a closed surface = $\oint J \cdot dS = -\frac{\partial Q}{\partial t}$ for a closed surface

流出的电量 減少的电量



Thus, if there's no time-varying charge within the closed surface, we have

$$\oint J \cdot dS = 0 \Rightarrow -\int_{S_x} J \cdot dS + \int_{S_y} J \cdot dS = 0$$

假設 S_x 為唯一入口
 S_y 為唯一出口.

$$\Rightarrow \int_{S_y} J \cdot dS = \int_{S_x} J \cdot dS$$



Therefore, the constraint for strategy #2 to work is

P5

$$\frac{\partial \mathcal{L}}{\partial t} = 0,$$

and we assume that holds for all time.

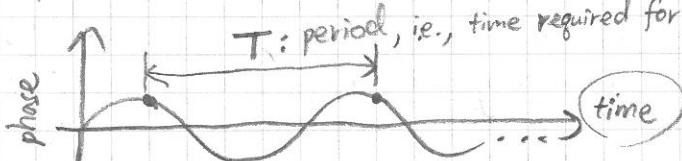
(To make sure this constraint holds, we may need to revise the model and introduce an element called "capacitor".)

Besides goals #1 and #2, we also need to assume that the signal ^{rate of change, i.e., frequency} timescale must be much larger $\rightarrow \dots 0110100\dots$ for example

than the propagation delay of electromagnetic waves across the lumped elements. (Otherwise, ... see textbook pages 11, 12)

\Rightarrow the size of our lumped elements must be much smaller than the wavelength associated with the V and I signals, and such a condition may be challenging to hold as we reduce the element size and increase the operating frequency (e.g., a 2 GHz CPU).

definition of wavelength: the ^(λ) distance between two adjacent points in the wave having the same phase.



$$\lambda = v \cdot T = v \cdot \frac{1}{f} \Rightarrow f \uparrow \text{then } \lambda \downarrow$$

wavelength wave speed period frequency

For example, electromagnetic waves travel at about 15×10^4 km/s, or 15×10^9 cm/s, within a microprocessor (to be specific, through silicon dioxide).

Now, suppose that the microprocessor operates at a clock rate of 2 GHz. This translates to the wavelength equal to $\lambda = 15 \times 10^9 / 2 \times 10^9 = 7.5$ cm,

which means that LMD may not hold if the microprocessor chip is larger than 7.5 cm on a side.

Think about it : what if
 ① clock rate \uparrow ?
 ② wave speed \uparrow ?

In general, in computer engineering, people are often working to meet various constraints (such as this) so that they may apply a previously established model (such as LMD) and make use of known results/ properties that depend on the given model. This is like "Standing on the shoulders of giants."

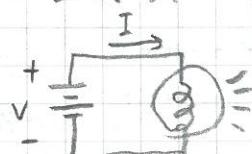
- Basic lumped element #1: batteries

Two key properties { energy 能量 (unit: joule or ampere-hours)
 power 功率 (unit: watt) or watt-hours)

能量轉換或使用的速率

$$P = V \cdot I$$

1 watt = 1 volt \cdot 1 ampere



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Let Σ be the amount of energy supplied to an element over an interval T , then we have $\Sigma = P \cdot T$. In general, the amount of energy supplied is the time integral of the power.

$$1 \text{ joule} = 1 \text{ watt-second}$$

Example: Suppose a Raspberry Pi consumes 2W of power and its energy is supplied by a 3.7V, 2600 mAh battery. For how long can the battery power the Raspberry Pi?

$$\begin{aligned} P = V \cdot I &= 3.7 \times 2600 \times 10^{-3} \text{ W-h} \\ &= 9.62 \text{ W-h} \end{aligned}$$

$$\frac{9.62 \text{ W-h}}{2 \text{ W}} = 4.81 \text{ hours} \quad \star$$

- Basic lumped element #2: resistors (linear)

Ohm's law: the voltage measured across the terminals of a resistor is linearly proportional to the current flowing through the resistor.

That is,

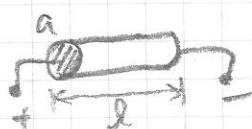
$$\boxed{\frac{V}{I} = R}$$

Further,

$$\boxed{R = \rho \frac{l}{a}}$$

(see
Appendix A.3
in the textbook)

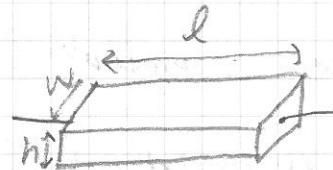
resistivity



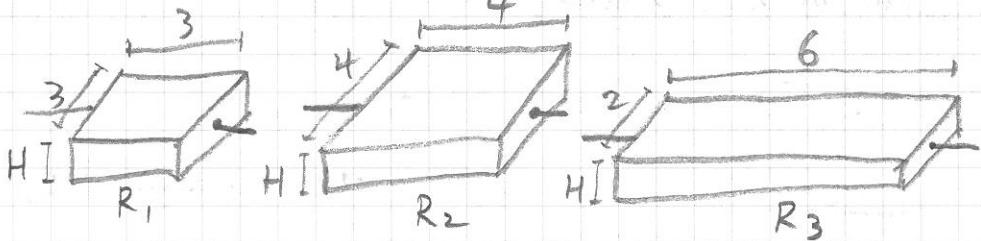
we call it the resistance of a resistor.

also, $R = \rho \frac{l}{wh}$

for a cube



Example : Consider three planar resistors as follows



Let $R_0 = \rho_0 \frac{1}{l \cdot H} = 2 k\Omega$ and assume $\rho_0 = \rho_1 = \rho_2 = \rho_3$

$$\text{Then } R_1 = \rho_1 \frac{3}{3 \cdot H} = R_0 = 2 k\Omega$$

$$\frac{R_1}{R_2} = \frac{\rho_1 \frac{3}{3H}}{\rho_2 \frac{4}{4H}} = 1 \quad \text{and } R_2 = R_1 = 2 k\Omega$$

$$\frac{R_2}{R_3} = \frac{\rho_2 \frac{4}{4H}}{\rho_3 \frac{6}{6H}} = \frac{1}{3} \Rightarrow R_3 = 6 k\Omega$$

exercise : you can verify that $\frac{R_1}{R_3} = \frac{R_2}{R_3}$.

\Rightarrow 等比例縮小長及寬，則相對電阻值 ?

\Rightarrow 缩小晶片的大小不會改變相對電阻值

A. \downarrow B. \uparrow C. 不變

\Rightarrow Often, signal values are derived as a function of resistance ratios. Therefore, by such a process shrink, the chip may continue to function as before!

\Rightarrow example : a "voltage divider," which we will study soon this semester.

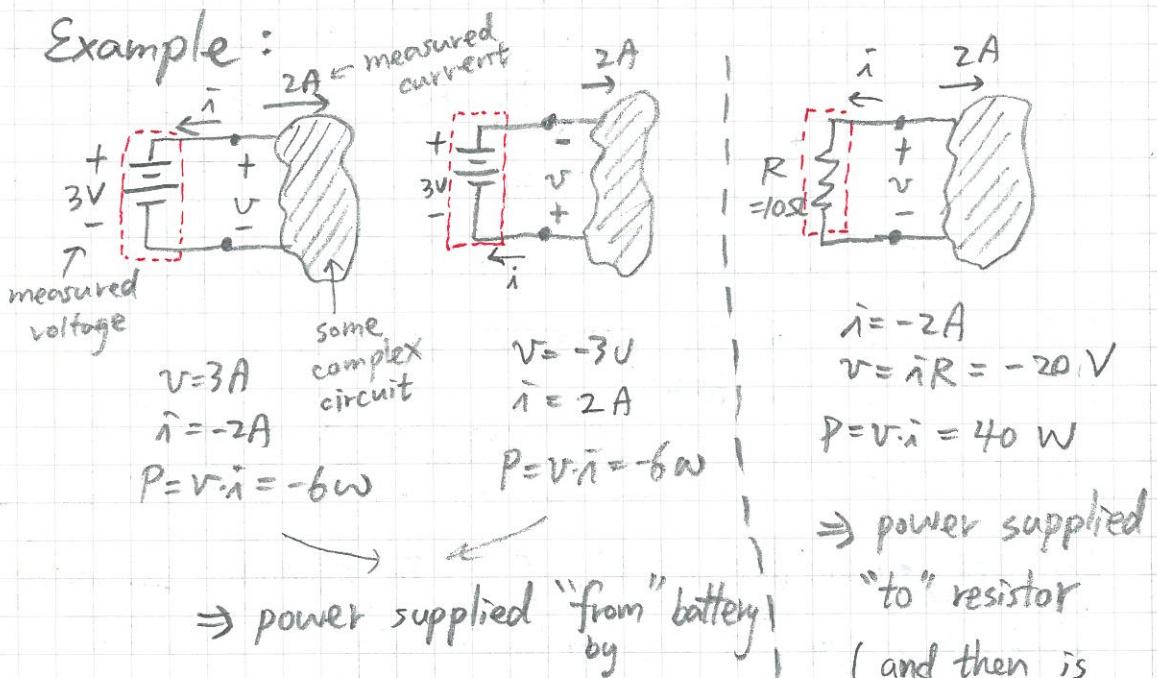
P9 - Associated Variable Convention : (約定値)

For a two-terminal lumped element, define current to flow in at the terminal that is assigned to be positive in voltage.

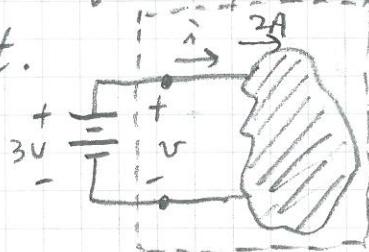


v and i are called "terminal variables."

Example :



Note that the above is from the viewpoint of the battery; from the viewpoint of that complex circuit, you may try and see that now power is supplied "to" it.



\Rightarrow power supplied "to" resistor
(and then is dissipated in the form of heat)

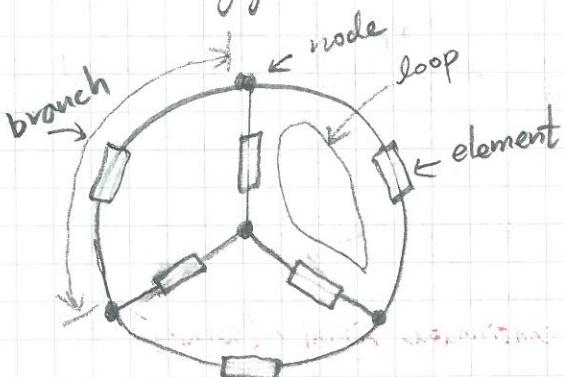
P10 Section 1.8 in the textbook mentioned sinusoidal Signals and the root mean square value. For your interest, the $\sqrt{2}$ ratio between the amplitude of a sinusoidal signal and its rms value comes from 三角函數 2 倍角轉換.

Example: Let signal $i(t) = I_m \cos(\omega t)$

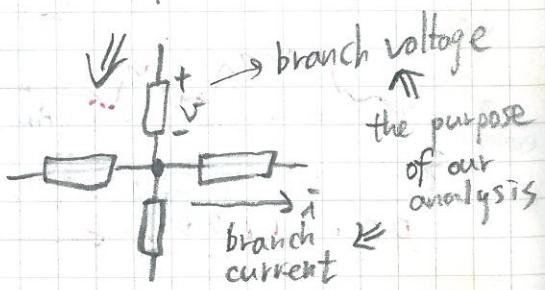
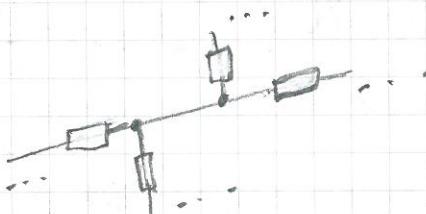
$$\begin{aligned} i_{rms} &= \sqrt{\frac{1}{T} \int_0^T i^2(t) \cdot dt} \\ &= \sqrt{\frac{1}{T} \int_0^T (I_m^2 \cos^2 \omega t) dt} \\ &= \sqrt{\frac{I_m^2}{2T} \int_0^T (1 + \cos 2\omega t) dt} \\ &= \frac{I_m}{\sqrt{2}} \sqrt{\frac{1}{T} \int_0^T (1 + \cos 2\omega t) dt} \\ &= \frac{I_m}{\sqrt{2}} \times \end{aligned}$$

★ Resistive Networks, and How to Analyze Them

- Terminology



ideal wire: no resistance



- Kirchhoff's Laws { KCL
KVL

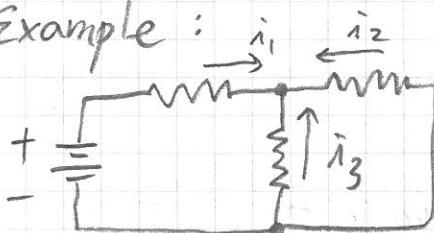
following LMD, Kirchhoff's laws are simplifications of Maxwell's Equations
(see Appendix A.2 in the textbook)

→ KCL and KVL are extremely useful tools to help us analyze a circuit!

KCL : Kirchhoff's current law —

The algebraic sum of all branch currents flowing into any node must be zero.

Example :



$$\sum_{n=1}^3 i_n = 0$$

$$\sum_{n=1}^3 (-i_n) = 0$$

stance

In general, for integers N, M, we have

$$\sum_{n=1}^N i_n = 0 \Rightarrow \sum_{n=1}^M i_n + \sum_{n=M+1}^N i_n = 0$$

$$\Rightarrow \sum_{n=1}^M i_n = \sum_{n=M+1}^N (-i_n)$$

⇒ Sum of total current flowing into a node

= sum of total current flowing out from a node.



pose
ur
ysis

P12

Example:

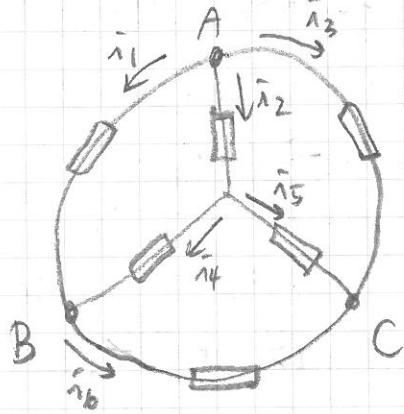
Using KCL

$$\text{node A: } 0 = -\bar{i}_1 - \bar{i}_2 - \bar{i}_3$$

$$B: 0 = \bar{i}_1 + \bar{i}_4 - \bar{i}_6$$

$$C: 0 = \bar{i}_2 - \bar{i}_4 - \bar{i}_5$$

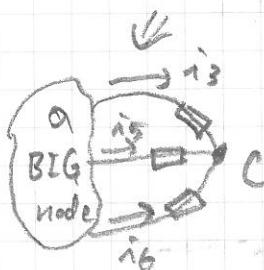
$$D: 0 = \bar{i}_3 + \bar{i}_5 + \bar{i}_6$$



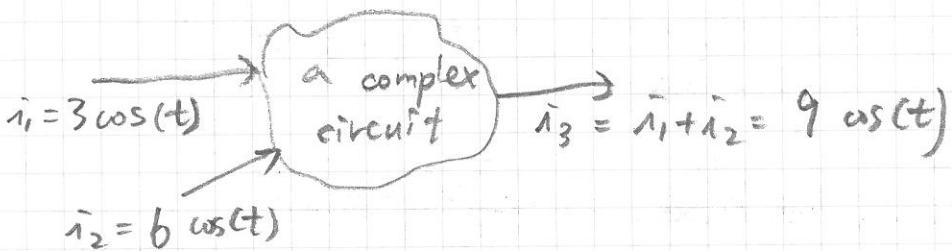
$$\text{Suppose } \bar{i}_1 = 1, \bar{i}_3 = 3 \Rightarrow \bar{i}_2 = -4$$

In addition, suppose $\bar{i}_5 = -2$

$$\Rightarrow \bar{i}_4 = -2, \bar{i}_6 = -1$$



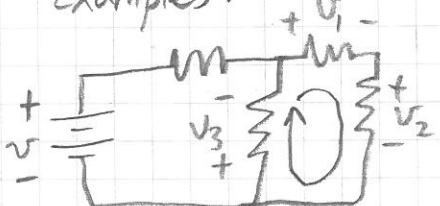
In general, for N KCL statements, only $N-1$ of them are independent. Therefore, we need $N-1$ known values to solve the circuit.



KVL: Kirchhoff's voltage law

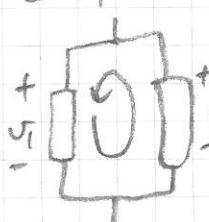
The algebraic sum of the branch voltages around any loop (i.e., closed path) in a network must be zero.

Examples:



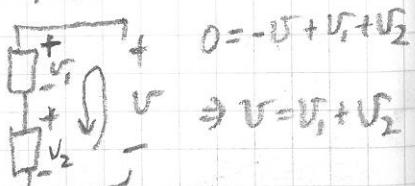
$$\sum_{n=1}^3 V_n = 0$$

並聯



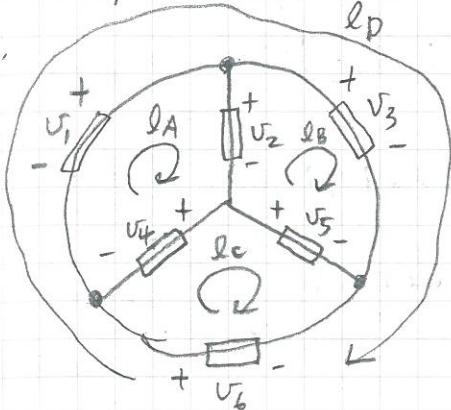
$$0 = V_1 - V_2$$

串聯



$$0 = -V + V_1 + V_2 \Rightarrow V = V_1 + V_2$$

Example:



Using KVL

$$\text{loop } l_A: 0 = -V_1 + V_2 + V_4$$

$$l_B: 0 = -V_2 + V_3 - V_5$$

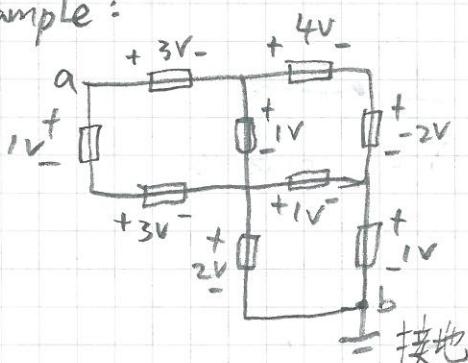
$$l_C: 0 = -V_4 + V_5 - V_6$$

$$l_D: 0 = -V_1 + V_3 - V_6$$

Suppose that $V_1 = 1$, $V_3 = 3$, $V_2 = 2$

then $V_4 = -1$, $V_5 = 1$, $V_6 = 2$

Example:



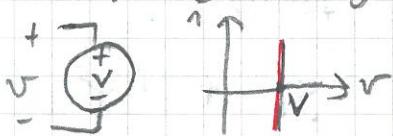
V_{ab} can be determined using KVL
(which loop(s) would you pick?)

$$V_{ab} = 6 \text{ V.}$$

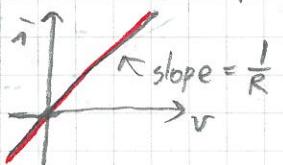
In general, we model individual elements based on LMD assumptions and we analyze a circuit of elements using KCL and KVL.

* Two more basic elements and their $i-v$ characteristics

① Independent voltage source



for linear resistor it obeys Ohm's law $R = \frac{V}{i}$



② Independent current source



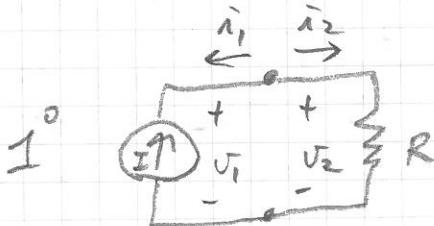
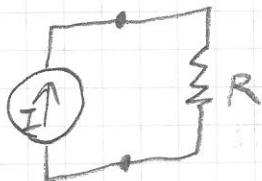
* Basic Method to Analyze A Circuit

four steps

see Page 9

- 1° define branch current and voltage consistently
- 2° apply element laws for each elements
(e.g., Ohm's law for linear resistors)
- 3° apply KCL and KVL
- 4° jointly solve the equations obtained from 2° and 3°.

Example :



$$2^{\circ} \quad i_1 = -I, \quad v_2 = i_2 \cdot R$$

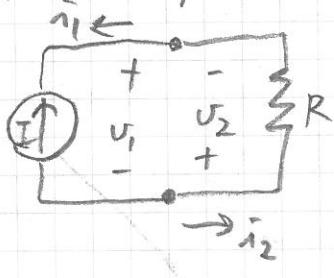
$$3^{\circ} \quad \text{KCL} \Rightarrow i_1 + i_2 = 0 \\ \text{KVL} \Rightarrow -v_1 + v_2 = 0$$

$$4^{\circ} \quad i_2 = -i_1 = I$$

$$v_2 = i_2 R = IR$$

$$v_1 = v_2 = IR \quad *$$

if at step 1° we define v_2 inversely, we must

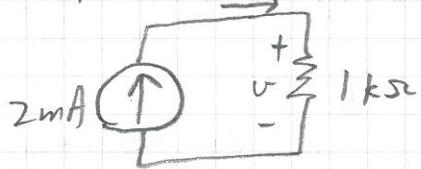


also define i_2 inversely, to be consistent. Applying the basic method you will find

$i_2 = -I$ and $v_2 = -IR$. This may seem to be strange. But if we recall that both i_2 and v_2 have a reversed direction, then it makes sense.

Alternatively, we may solve a circuit by considering "Energy conservation".

Example : power out from the source :



$$P_{out} = 2 \text{ mA} \times V$$

power into the resistor :

$$P_{in} = i \times V = \frac{V^2}{R}$$

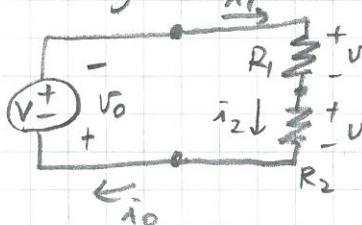
$$= \frac{V^2}{1 \text{ k}\Omega}$$

$$P_{out} = P_{in} \Rightarrow 2 \times 10^{-3} \cdot V = \frac{V^2}{1 \times 10^3}$$

$$\Rightarrow V = 2 \text{ V} *$$

(Example 2.14 in the textbook has a typo saying $V = 0.5 \text{ V}$)

* Voltage Divider



We may analyze it using the basic method :

$$\begin{cases} V_0 = V \\ V_1 = R_1 i_1 \\ V_2 = R_2 i_2 \end{cases} \quad \begin{cases} i_0 = i_1 \\ i_1 = i_2 \end{cases} \quad V_0 + V_1 + V_2 = 0 \quad \text{KVL}$$

$$\Rightarrow V_2 = \frac{R_2}{R_1 + R_2} V$$

$$\text{Further, from } i_2 = \frac{V_2}{R_2} \text{ we see } i = \frac{1}{R_1 + R_2} V$$

in other word, $V = i \times (R_1 + R_2)$.

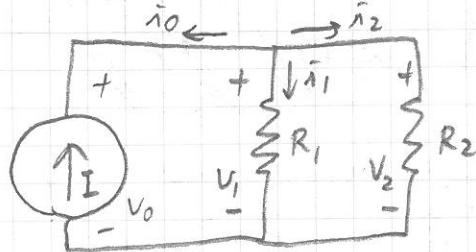
\Rightarrow we may replace \nwarrow an equivalent resistor $R' = R_1 + R_2$
by R_1 and R_2

and the circuit is equivalent as $\text{voltage source } V \parallel R'$

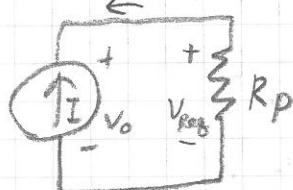
This lead to a general planar linear resistor analysis,
such as that in Example 2.21 in the textbook.

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★ Current Divider



↓ equivalent



$$V_1 = R_1 \bar{i}_1 = \frac{R_1 R_2}{R_1 + R_2} I$$

$$V_0 = R_{\text{eq}} \cdot I$$

$$\Rightarrow R_p = \frac{R_1 R_2}{R_1 + R_2}$$

$$\Rightarrow \frac{1}{R_p} = \frac{1}{R_1} + \frac{1}{R_2}$$

Using the "basic method"

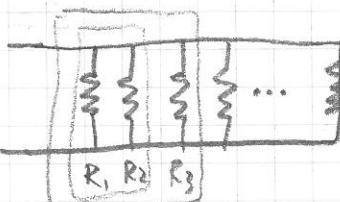
$$\begin{cases} \bar{i}_0 = -I \\ V_1 = R_1 \bar{i}_1 \\ V_2 = R_2 \bar{i}_2 \end{cases} \quad \begin{cases} \bar{i}_0 + \bar{i}_1 + \bar{i}_2 = 0 \\ V_0 = V_1 = V_2 \end{cases}$$

$$\Rightarrow \bar{i}_1 + \bar{i}_2 = I$$

$$\begin{aligned} \bar{i}_1 + \bar{i}_2 &= \frac{V_1}{R_1} + \frac{V_2}{R_2} = \frac{V_0 R_2 + V_0 R_1}{R_1 R_2} \\ &= \frac{R_1 + R_2}{R_1 R_2} V_0 \end{aligned}$$

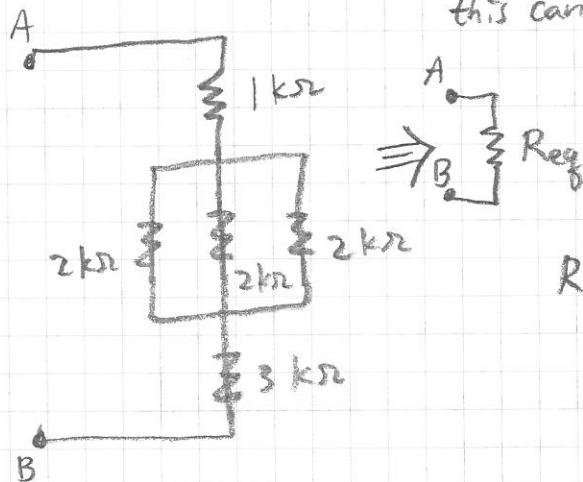
$$\Rightarrow V_0 = \frac{R_1 R_2}{R_1 + R_2} I$$

$$\boxed{\begin{aligned} \bar{i}_1 &= \frac{V_1}{R_1} = \frac{R_2}{R_1 + R_2} I \\ \bar{i}_2 &= \frac{V_2}{R_2} = \frac{R_1}{R_1 + R_2} I \end{aligned}}$$

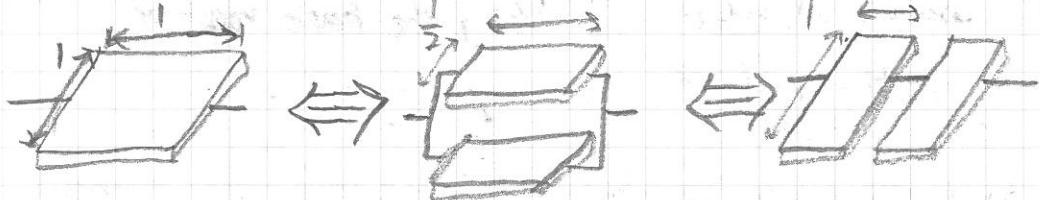
In general, for N resistors connected in parallel,

$$\frac{1}{R_p} = \sum_{n=1}^N \frac{1}{R_n}$$

this can be proved by induction.



$$\begin{aligned} R_p &= 1 + \frac{1}{\frac{1}{2} + \frac{1}{2} + \frac{1}{2}} + 3 \\ &= \frac{14}{3} \text{ k}\Omega \end{aligned}$$



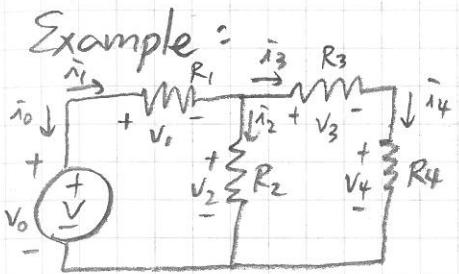
$$\text{let } R_1 = R_2$$

$$R_2 = \frac{2R_0 \cdot 2R_0}{2R_0 + 2R_0}$$

$$= R_0 = R_1$$

$$R_3 = \frac{1}{2}R_0 + \frac{1}{2}R_0$$

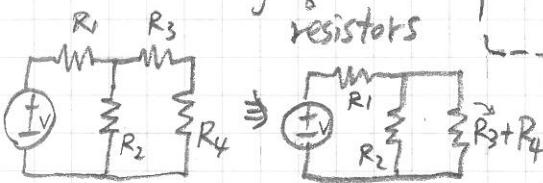
$$= R_0 = R_1$$



Determine V_1, V_2, V_3, V_4
and i_1, i_2, i_3, i_4 .

Way ① : we may use the basic method

Way ② : transformation using equivalent resistors



2° element law ...

3° KCL & KVL ...

4° 解聯立方程式

$$V_2 = \frac{R_2(R_3 + R_4)}{R_2 + R_3 + R_4}$$

$$i_1 = V / \left(R_1 + \frac{R_2(R_3 + R_4)}{R_2 + R_3 + R_4} \right)$$

$$V_1 = i_1 R_1 = \underline{\quad}$$

$$V_2 = \underline{\quad} \text{ (voltage divider)}$$

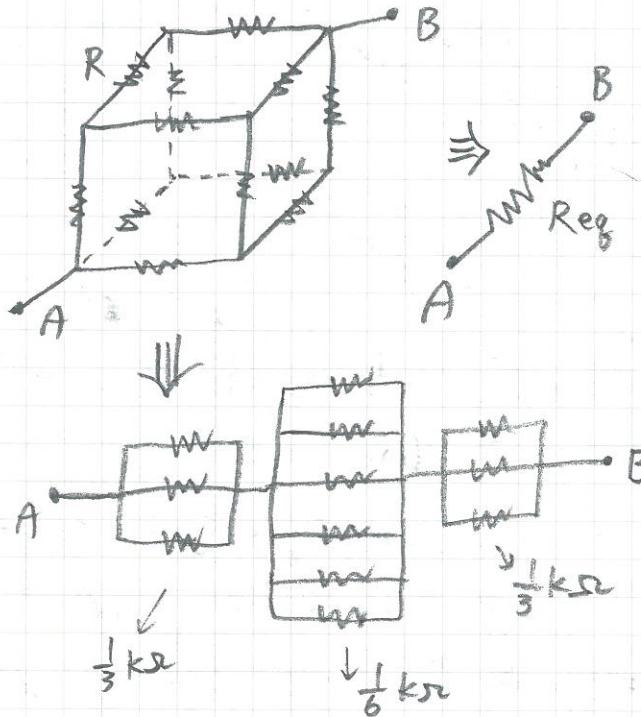
$$i_2 = V_2 / R_2, i_3 = V_2 / (R_3 + R_4),$$

$$V_3 = i_3 R_3, V_4 = i_3 R_4$$

give it a try yourself here :

P18 Sometimes, we may leverage symmetry to greatly reduce the complexity of analysis:

Example: assuming all resistors are the same on a cube, with resistance $R = 1 \text{ k}\Omega$, determine Req

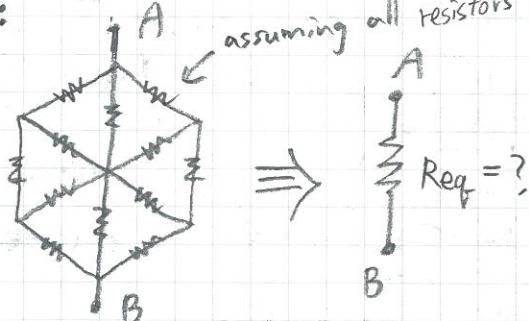


(See example 2.24
in the textbook)

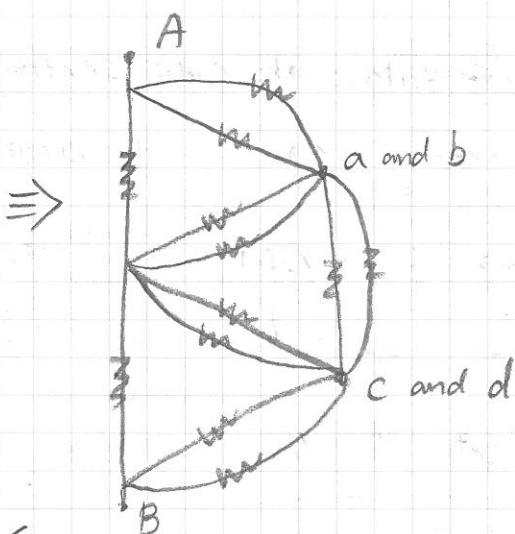
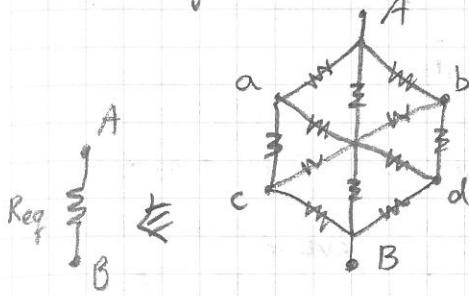
$$\begin{aligned}\text{Req} &= \frac{1}{\frac{1}{3}} + \frac{1}{\frac{1}{3}} + \frac{1}{\frac{1}{3}} \\ &= \frac{5}{6} \text{ k}\Omega\end{aligned}$$

Often, we still need to apply the basic method, after reducing a circuit by using equivalent resistors and symmetry !!

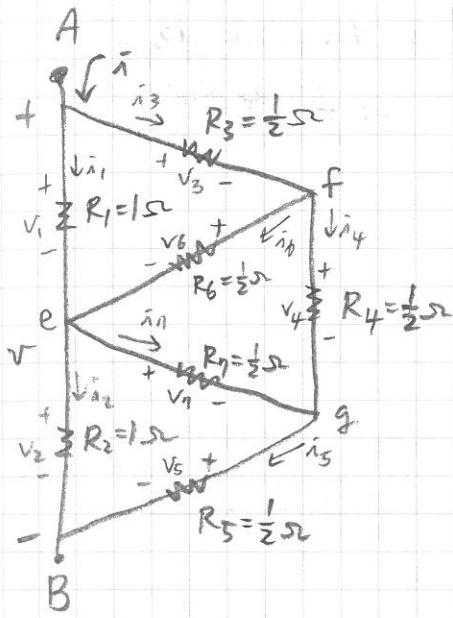
Example:



following P18,



24



$$\begin{aligned} \text{KVL: } & \left\{ \begin{array}{l} v_1 + v_2 = v_3 + v_4 + v_5 \\ v_1 = v_3 + v_6 \\ v_2 = v_7 + v_5 \end{array} \right. \\ & \left\{ \begin{array}{l} \text{⑤} \\ \text{⑥} \\ \text{⑦} \end{array} \right. \end{aligned}$$

$$\text{Req} = \frac{v}{i} = \frac{v_1 + v_2}{i_1 + i_3} = \frac{i_1 + i_2}{i_1 + i_3}$$

element laws:

$$\left\{ \begin{array}{l} v_1 = i_1, v_2 = i_2, v_3 = \frac{1}{2}i_3 \\ v_4 = \frac{1}{2}i_4, v_5 = \frac{1}{2}i_5, v_6 = \frac{1}{2}i_6 \\ v_7 = \frac{1}{2}i_7 \end{array} \right.$$

$$\text{KCL: } \left\{ \begin{array}{l} i = i_1 + i_3 = i_2 + i_5 \quad (\text{node A \& B}) \\ \text{②} \\ \text{③} \\ \text{④} \end{array} \right. \quad \begin{array}{l} i_1 + i_6 = i_2 + i_7 \quad (\text{node e}) \\ i_3 = i_6 + i_4 \quad (\text{node f}) \\ i_5 = i_7 + i_4 \quad (\text{node g}) \end{array}$$

$$\left\{ \begin{array}{l} i = i_1 + i_3 = i_2 + i_5 \\ i_1 + i_6 = i_2 + i_7 \\ i_3 = i_6 + i_4 \\ i_5 = i_7 + i_4 \end{array} \right. \quad \begin{array}{l} (\text{node A \& B}) \\ (\text{node e}) \\ (\text{node f}) \\ (\text{node g}) \end{array}$$

therefore, 我們可將所有電壓值代換為電流值，去解電流的聯立方程式!!)

$$\left\{ \begin{array}{l} \text{①} \\ \text{②} \\ \text{③} \\ \text{④} \\ \text{⑤} \\ \text{⑥} \\ \text{⑦} \end{array} \right. \left[\begin{array}{cccccc} 1 & -1 & 1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & -1 \\ 2 & 2 & -1 & -1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 & -1 & 0 & -1 \end{array} \right] \left[\begin{array}{c} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right]$$

note: equation ④ is dependent.

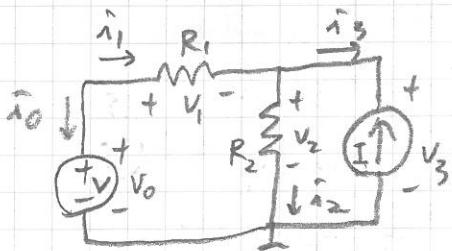
使用高斯消去法 可得 $\left\{ \begin{array}{l} i_1 = 2i_7 \\ i_2 = 2i_7 \\ i_3 = 3i_7 \end{array} \right.$

or www.mathstools.com Matrix Calculator

$$\therefore \text{Req} = \frac{i_1 + i_2}{i_1 + i_3} = \frac{2+2}{2+3} = \frac{4}{5} \Omega$$

Calculation is a necessary part in engineering!

P20 Example : A circuit with two independent sources

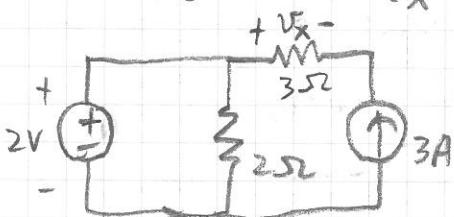


Determine i_2 .

$$\begin{array}{l} \text{element law} \\ \left\{ \begin{array}{l} V_0 = V \\ V_1 = R_1 i_1 \\ V_2 = R_2 i_2 \\ i_3 = -I \end{array} \right. \end{array} \quad \begin{array}{l} \text{KCL} \\ \left\{ \begin{array}{l} i_0 = -i_1 \\ i_1 = i_2 + i_3 \end{array} \right. \end{array} \quad \begin{array}{l} \text{KVL} \\ \left\{ \begin{array}{l} V_0 = V_1 + V_2 \\ V_2 = V_3 \end{array} \right. \end{array}$$

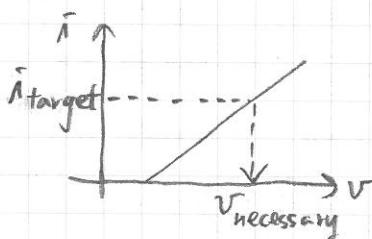
then solve these linear equations (see textbook Pg5).

Exercise: $V_x = ?$



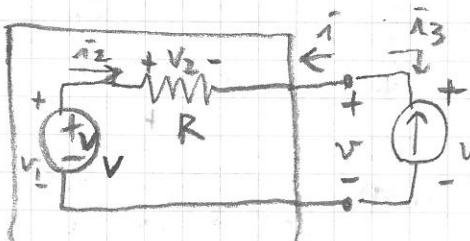
(ans: $V_x = -9V$)

★★ The I-V characteristic of a circuit



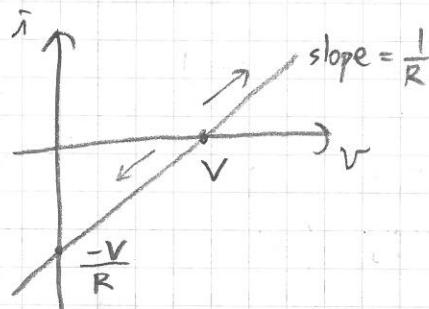
我們可藉由給定 v 量測 i
(or vice versa) 來繪製
I-V relation.

If we know the device's internals,
we may also determine its I-V relation:



using the basic method, we have

$$V = V + iR \Rightarrow i = \frac{1}{R}V - \frac{V}{R}$$



example usage:

預測電流流向

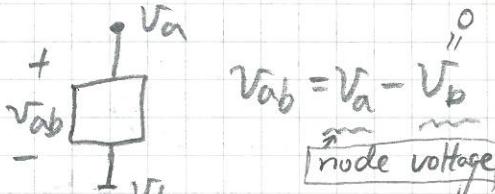
$$\left\{ \begin{array}{l} V \geq V \rightarrow i \geq 0 \\ V < V \rightarrow i < 0 \end{array} \right.$$

★ Circuit Analysis using the Node Method

Motivation:

Often, the number of nodes in a circuit is much smaller than that of branches. Node method thus involves fewer number of variables, which means it is often easier to solve.

- definition of node voltage:

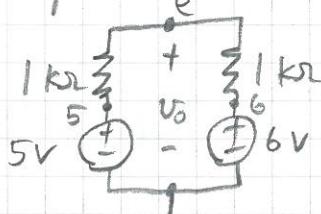


$$V_{ab} = V_a - V_b$$

Procedure of the node method:

- 1° select a reference node (接地, $v=0$)
- 2° assign node variables
- 3° apply KCL
- 4° solve equations
- 5° back-solve the needed branch voltage/current

Example: find $V_0 = ?$



using the node method,

$$\frac{5-e}{1\text{ k}\Omega} + \frac{6-e}{1\text{ k}\Omega} = 0$$

$$e = 5.5\text{ V} \quad V_0 = e - 0 = 5.5\text{ V}$$

if using the basic method,

$$\begin{aligned} i_1 &= \frac{V_1}{1\text{ k}\Omega} \\ i_2 &= \frac{V_2}{1\text{ k}\Omega} \end{aligned}$$

$$\begin{aligned} i_1 + i_2 &= 0 \\ \Rightarrow V_1 + V_2 &= 0 \end{aligned} \quad \left\{ \begin{array}{l} V_1 = -0.5 \\ V_2 = 0.5 \end{array} \right.$$

using KVL,

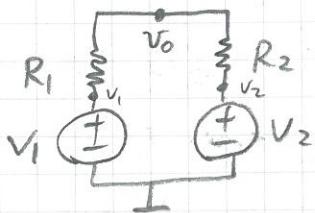
$$5 - V_1 + V_2 - 6 = 0$$

$$\Rightarrow V_1 - V_2 = -1$$

$$V_0 = 6 - V_2 \quad (\text{KVL})$$

$$= 5.5 \text{ V}$$

P₂₂ Symbolic Computation can give us some insights:



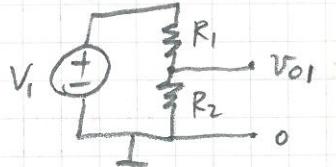
$$\begin{aligned} \text{KCL: } & \frac{V_1 - V_0}{R_1} + \frac{V_2 - V_0}{R_2} = 0 \\ \Rightarrow & R_2(V_1 - V_0) + R_1(V_2 - V_0) = 0 \\ \Rightarrow & V_0 = \frac{1}{R_1 + R_2}(R_2 V_1 + R_1 V_2) \\ = & \frac{R_2}{R_1 + R_2} V_1 + \frac{R_1}{R_1 + R_2} V_2 \end{aligned}$$

Insights ① V_0 is a linear combination of V_1 and V_2 .

The circuit acts as an adder that gives a weighted sum of V_1 and V_2 .

② if set $V_2 = 0$, then $V_{01} = \frac{R_2}{R_1 + R_2} V_1$, which is equivalent to the result

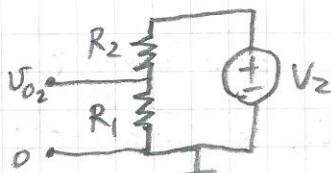
i.e., having the
some i-v
characteristic.
of a voltage divider :



③ similarly, if set $V_1 = 0$, then

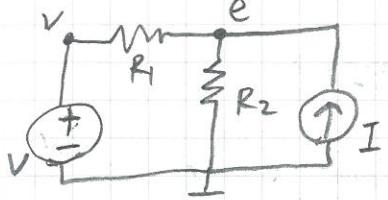
$$V_{02} = \frac{R_1}{R_1 + R_2} V_2, \text{ equivalent to the}$$

result of a voltage divider :



\Rightarrow from ② and ③, the original circuit can be thought of as a superposition of two voltage dividers, with $V_0 = V_{01} + V_{02}$. we will talk more about it soon!

Another example:



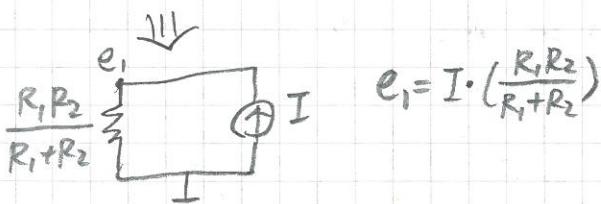
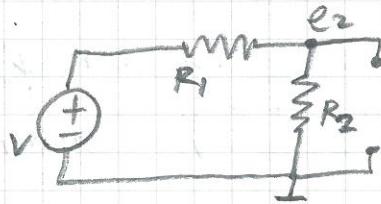
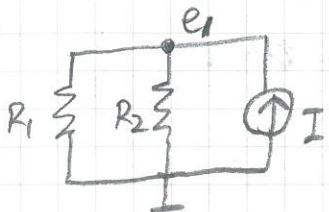
$$\text{KCL: } \frac{V-e}{R_1} + \frac{0-e}{R_2} + I = 0$$

$$\Rightarrow R_2(V-e) + R_1(-e) + I R_1 R_2 = 0$$

$$\Rightarrow e = \frac{1}{R_1 + R_2} (R_2 V + I R_1 R_2)$$

$$= \frac{R_2}{R_1 + R_2} V + \frac{R_1 R_2}{R_1 + R_2} I$$

Study the case of $V=0$ and $I=0$, respectively, we see that the original circuit can be thought of as a superposition of one current divider and one voltage divider, where $e = e_1 + e_2$:



$$e_2 = \frac{R_2}{R_1 + R_2} V.$$

(Note: set $V=0$ 相當於將 \oplus 短路 (short circuit))
 set $I=0$ 相當於將 \oplus 斷路 (open circuit))

In general, for a linear circuit, we can use the concept of superposition to simplify our analysis, by first considering one independent source at a time and then adding up the result.

P24

Why does the concept of "superposition" make sense in circuit analysis?

- Because ① each independent source contributes to the response of circuit "individually" and the contribution is independent from the contribution of any other independent source, and
② independent sources are assumed to have no resistance (see P13 of this note).

Why does the concept of "equivalence" make sense in circuit analysis?

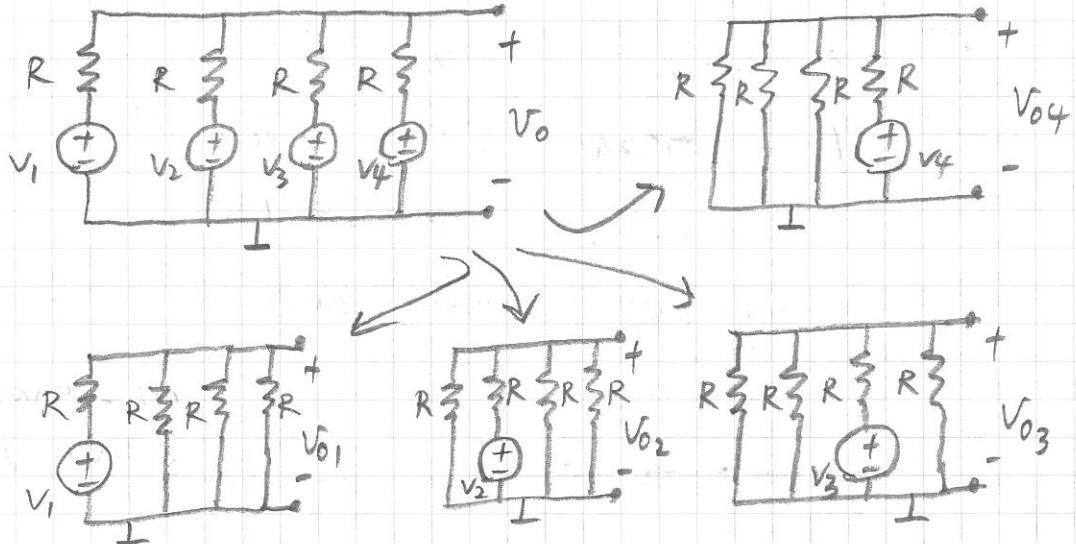
- Because as long as the i-v characteristics are identical, from input/output viewpoint of a system, what's inside doesn't matter.

Therefore, we may replace some part of a circuit by its equivalence, solely for the purpose of simplifying our analysis. It is an extremely useful trick in engineering!

For example, we may use signal generator to feed an equivalent input to a system, emulating some physical input circuit.

Example of the use of superposition:

find $V_o = ?$

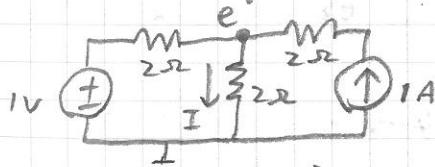


$$V_{o1} = \frac{\frac{1}{3}}{1 + \frac{1}{3}} V_1, \quad V_{o2} = \frac{\frac{1}{3}}{1 + \frac{1}{3}} V_2, \quad V_{o3} = \frac{\frac{1}{3}}{1 + \frac{1}{3}} V_3, \quad V_{o4} = \frac{\frac{1}{3}}{1 + \frac{1}{3}} V_4$$

$$= \frac{1}{4} V_1 \quad = \frac{1}{4} V_2 \quad = \frac{1}{4} V_3 \quad = \frac{1}{4} V_4$$

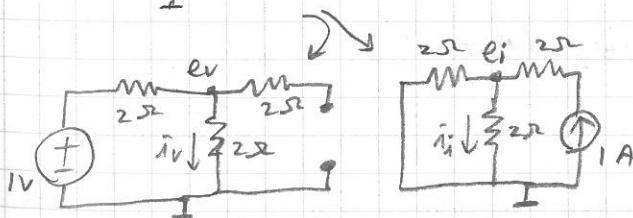
$$\Rightarrow V_o = V_{o1} + V_{o2} + V_{o3} + V_{o4} = \frac{1}{4}(V_1 + V_2 + V_3 + V_4) *$$

Another example: find $I = ?$



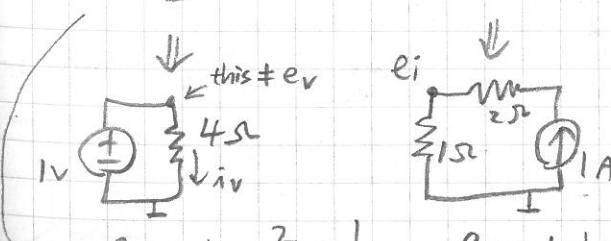
$$e = e_v + e_i = \frac{3}{2}$$

$$I = \frac{e - 0}{2\Omega} = 0.75 \text{ A} *$$



Alternatively, we may compute I directly:

$$i_v = \frac{1}{4}, \quad i_i = \underbrace{\frac{2}{2+2} \times 1}_{\text{current divider}} = \frac{1}{2}$$



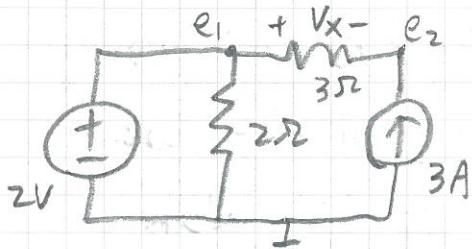
$$e_v = 1 \times \frac{2}{2+2} = \frac{1}{2}$$

$$e_i = 1 \times 1 = 1$$

$$\Rightarrow I = i_v + i_i$$

$$= \frac{1}{4} + \frac{1}{2} = 0.75 \text{ A} *$$

P26 Some further example of the use of node method:



find $V_x = ?$

$$e_1 = 2V$$

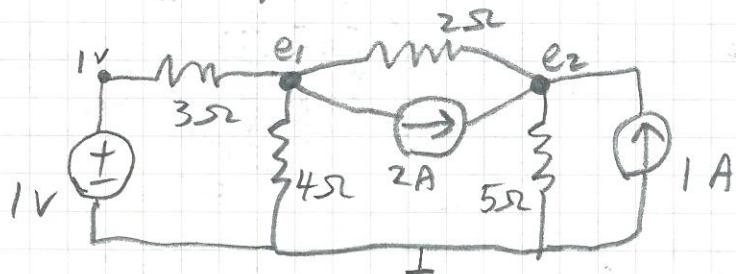
$$\text{KCL: } \frac{e_1 - e_2}{3\Omega} + 3 = 0$$

$$\Rightarrow e_2 = 11V$$

$$\Rightarrow V_x = e_1 - e_2 = -9V$$

Compare this with the use of basic method
as we did on P20 of this note! ($V_x = -9V$ there)

Another example: find e_1 and e_2



$$\text{KCL on } e_1: \frac{e_1 - 1}{3} + \frac{e_1}{4} + \frac{e_1 - e_2}{2} + 2 = 0$$

$$\text{KCL on } e_2: -2 + \frac{e_2 - e_1}{2} + \frac{e_2}{5} - 1 = 0$$

$$\Rightarrow \begin{cases} 4e_1 - 4 + 3e_1 + 6e_1 - 6e_2 + 24 = 0 \\ -20 + 5e_2 - 5e_1 + 2e_2 - 10 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 13e_1 - 6e_2 = -20 \\ -5e_1 + 7e_2 = 30 \end{cases}$$

$$\Rightarrow 13e_1 - 6\left(\frac{1}{7}(30 + 5e_1)\right) = -20$$

$$\Rightarrow 91e_1 - 180 - 30e_1 = -140$$

$$\Rightarrow 61e_1 = 40 \Rightarrow e_1 \approx 0.655, e_2 = \frac{1}{6}(13e_1 + 20)$$

$$= \frac{1}{6}(13 \times 0.655 + 20) = 4.75$$

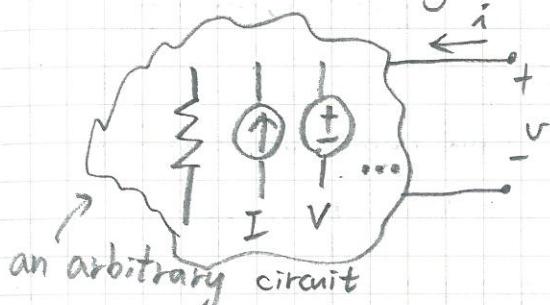
d:

Thevenin's Theorem

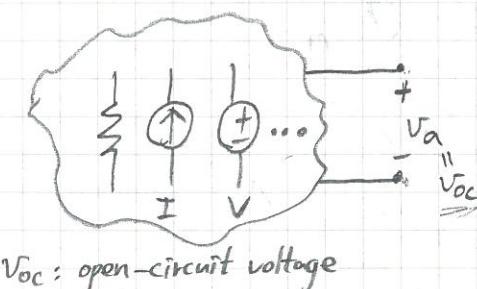
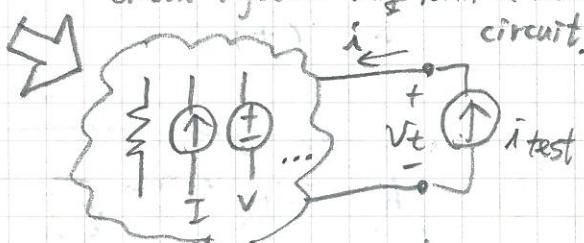
P27

Goal: Given an arbitrary linear circuit, we would like to know how it would respond to external excitation; in other word, we'd like to know its i-v characteristic.

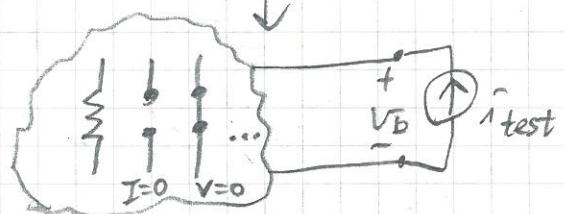
Approach: leverage the concept of superposition!



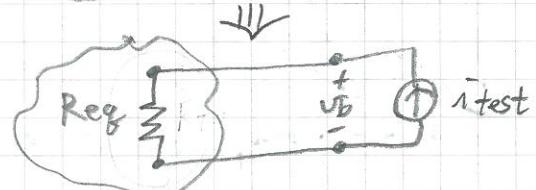
We append a testing current source, with the original circuit together they form a new circuit.



set $i_{\text{test}} = 0$



set internal sources = 0

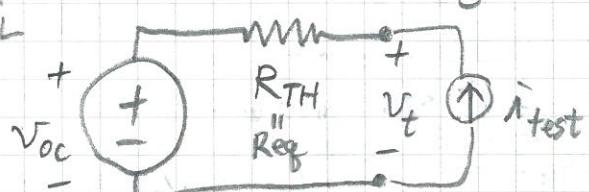
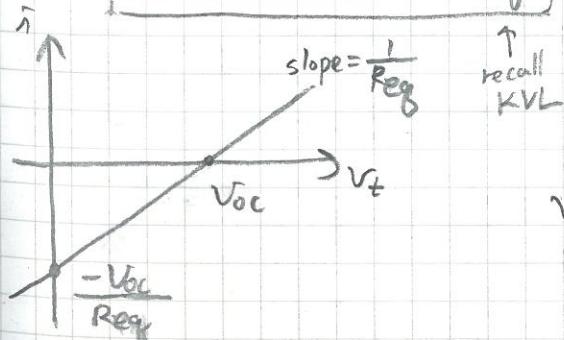


By superposition,

$$V_t = V_a + V_b$$

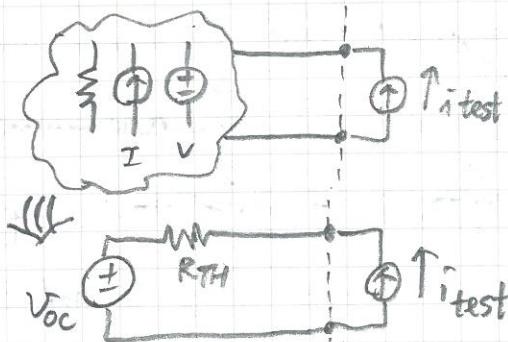
$$\Rightarrow V_t = V_{oc} + i_{\text{test}} R_{\text{eq}}$$

\Rightarrow equivalently, this relation describes the following circuit:



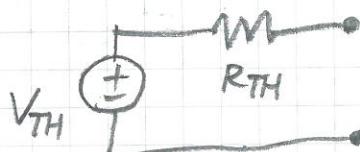
We rename R_{eq} to R_{TH} to honor Thevenin

P28 Therefore, we have the following equivalence:



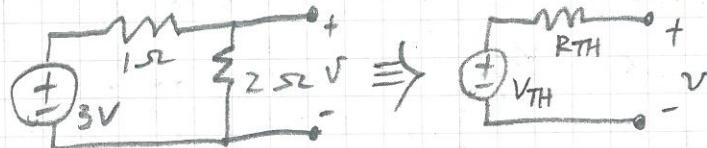
linear

In other word, we may reduce an arbitrary circuit to an equivalent circuit of the form:



We name $V_{TH} = V_{oc}$
in honor of Thévenin.

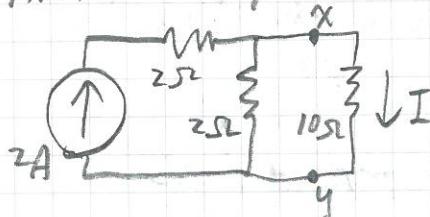
Example:



$$V_{TH} = V = 3 \times \frac{2}{1+2} = 2V, \quad R_{TH} = \frac{1 \cdot 2}{1+2} = \frac{2}{3}\Omega \text{ from } \boxed{\begin{array}{c} 1\Omega \\ \parallel \\ 2\Omega \end{array}}$$



Another example: find $I = ?$

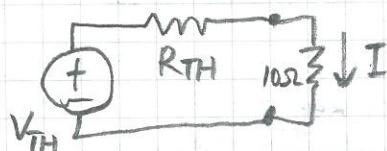


calculating V_{TH} :

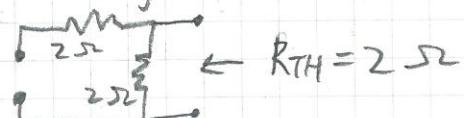


$$V_{TH} = 2A \times 2\Omega = 4V$$

We may replace the left side of X-Y by an equivalent circuit:



calculating R_{TH} :



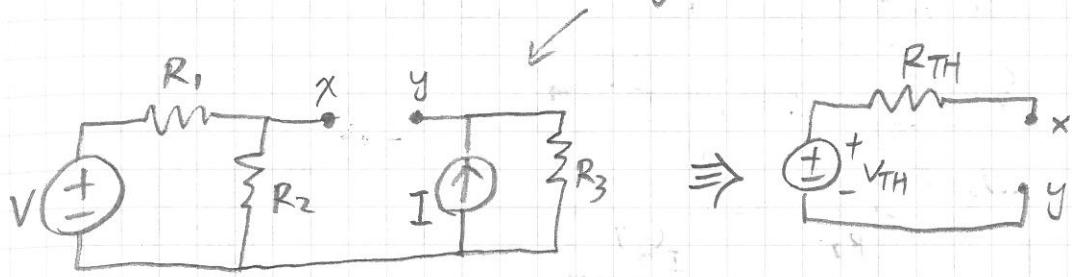
$$\Rightarrow I = \frac{4}{2+10} = \frac{1}{3}A$$

**

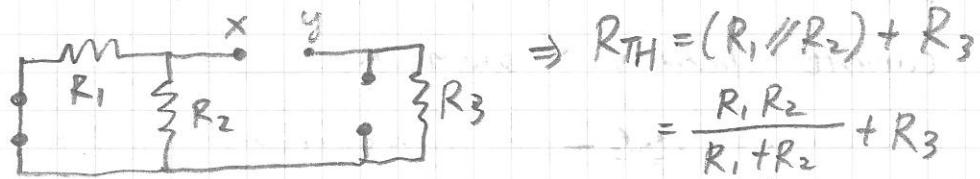
$$\text{Then } I = \frac{V_{TH}}{R_{TH} + 10\Omega}$$

Exercise : create the Thévenin Equivalent Circuit
for the following circuit :

P29

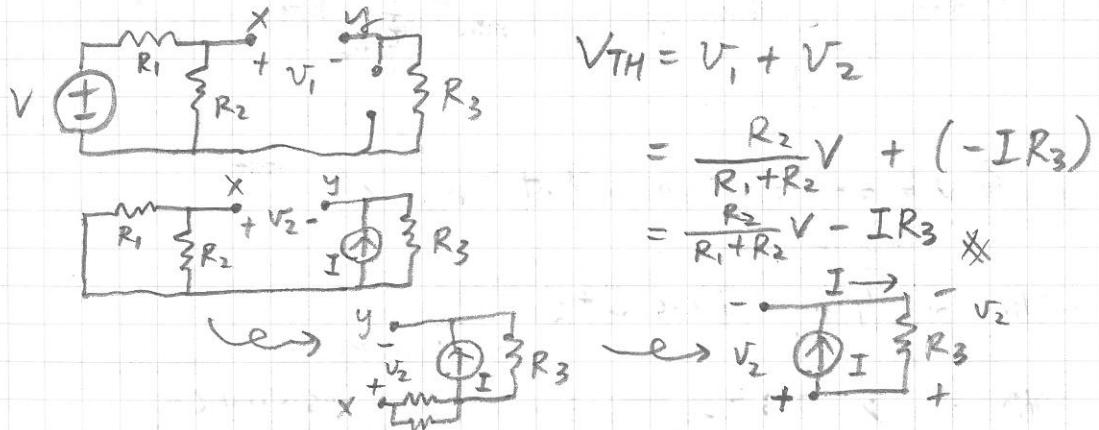


first of all, for R_{TH} :

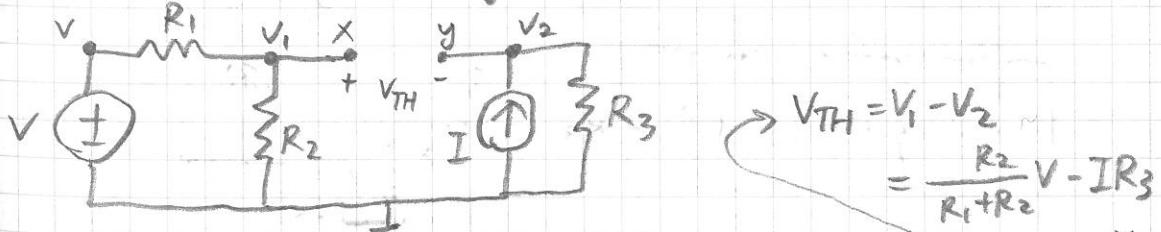


then, for V_{TH} :

approach ①, using superposition



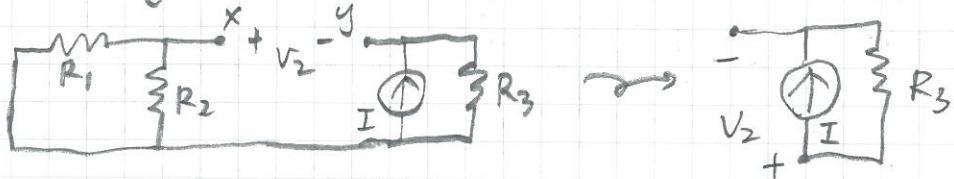
approach ②, using the node method (Page 21)



KCL:

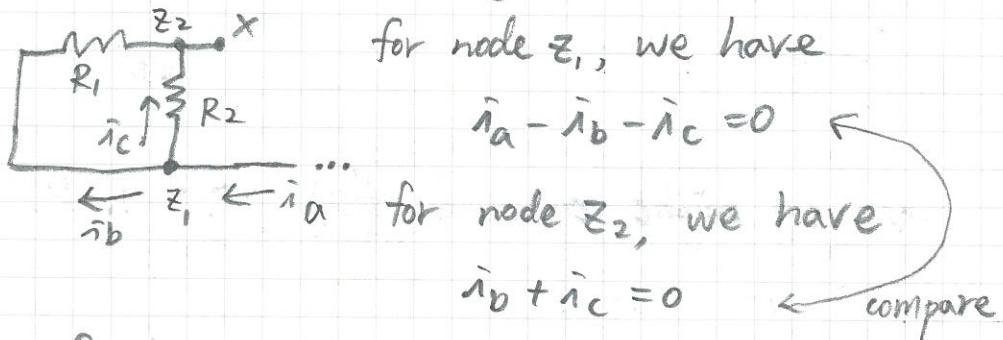
$$\left\{ \begin{array}{l} \frac{V - V_1}{R_1} + \frac{0 - V_1}{R_2} = 0 \\ I + \frac{0 - V_2}{R_3} = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} V_1 = \frac{R_2}{R_1 + R_2} V \\ V_2 = IR_3 \end{array} \right.$$

P30 In the exercise on the previous page, you might wonder why we can do the following transformation:



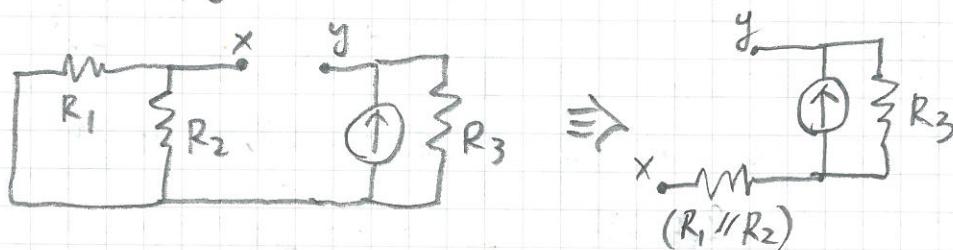
Could it be possible that there are some current flowing through R_1 and/or R_2 ?

We can use KCL to figure it out:

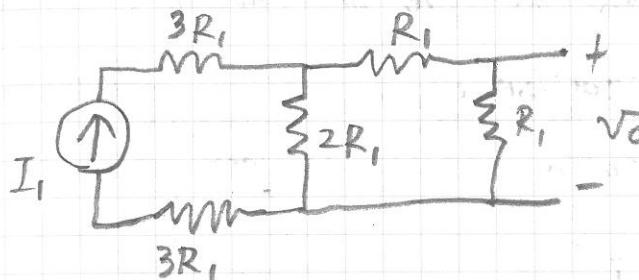


Therefore, we see that $i_a = 0$.

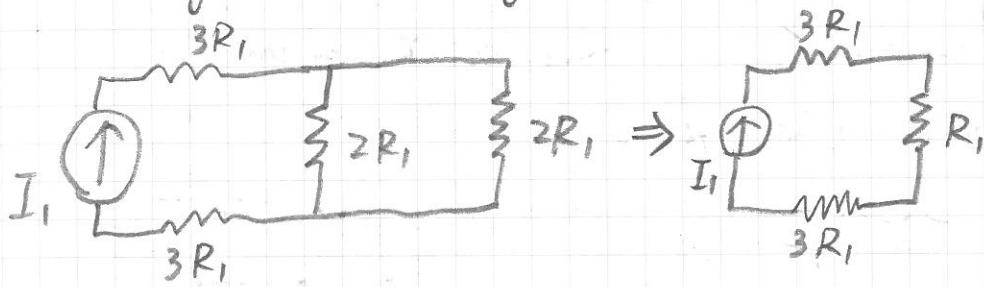
In practice, it is often helpful to think in terms of equivalent resistance, which may make the situation much more obvious:



Exercise : In the following circuit, determine voltage V_o :

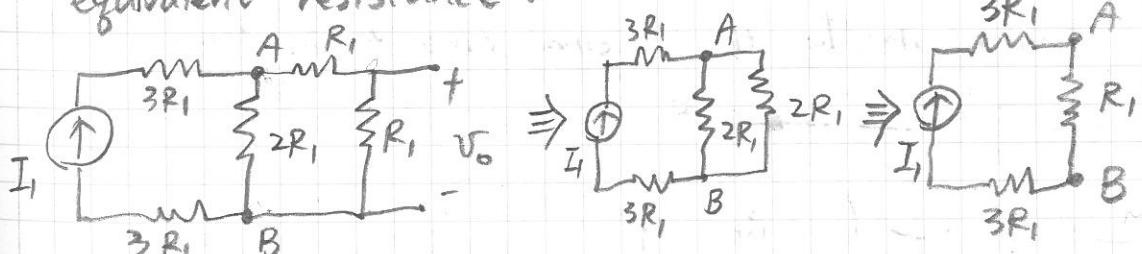


We may do the following transformation :



then we might conclude that $V_o = I_1 \cdot R_1$,
but it is wrong.

To see this, it could be helpful to clearly label the nodes between which you calculate the equivalent resistance :

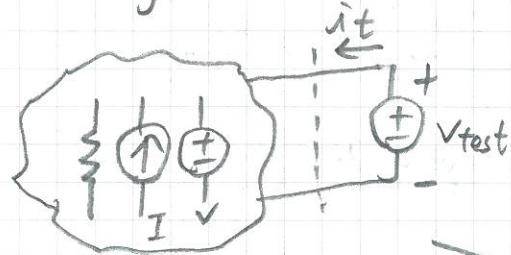


thus we see that V_o is not the voltage across A, B, and $V_o = (\text{voltage across } A, B) \times \frac{R_1}{R_1 + R_1}$

$$= (I_1 \cdot R_1) \times \frac{R_1}{R_1 + R_1} = \frac{1}{2} I_1 R_1 \text{ is the correct answer.}$$

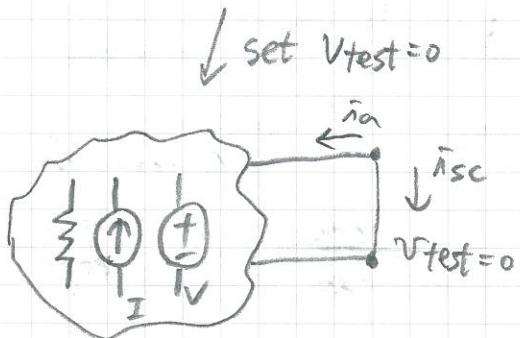
P32 ★ Norton's Theorem

For the same goal as that of Thévenin's Theorem, here we choose to attach a testing independent voltage source to a possibly complex circuit:

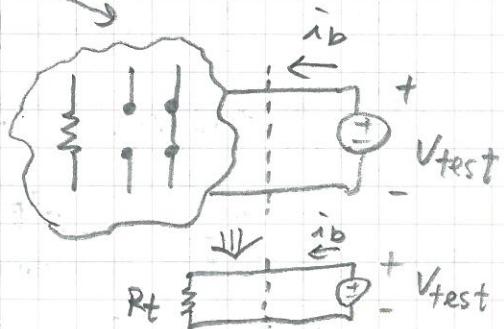


and leverage superposition

set internal sources = 0



set $V_{test} = 0$

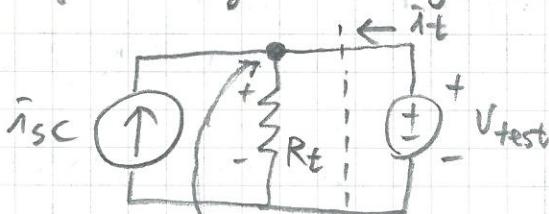


$$R_t \equiv \frac{V_{test}}{i_ssc}$$

$$\text{then } i_t = i_a + i_b = -i_ssc + \frac{V_{test}}{R_t}$$

$$\Rightarrow i_t + i_ssc - \frac{V_{test}}{R_t} = 0$$

Think of above in terms of KCL, then,
equivalently the original circuit is like



where KCL is applied.

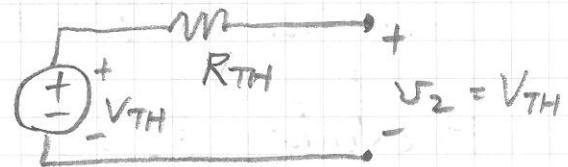
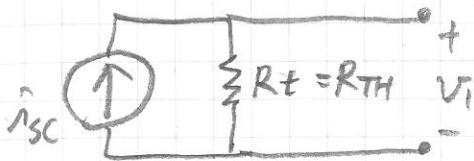
And $R_t = R_{TH}$, since we applied same procedure of decomposition as we did in Thévenin's Theorem.

Relation between

Norton's Equivalent Circuit

and

Thevenin's Equivalent Circuit:



since $V_1 = V_2$, we have

$$i_{sc} \cdot R_t = V_2 = V_{TH}$$

(recall that

$$\Rightarrow R_t = \frac{V_{TH}}{i_{sc}} = \frac{V_{oc}}{i_{sc}}$$

$V_{TH} = V_{oc}$
page 28)

in other word,

$$\text{等效電阻} = \frac{\text{開路電壓}}{\text{短路電流}}$$

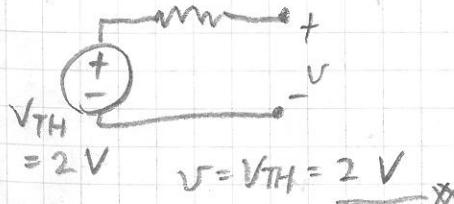
Example: find $V = ?$



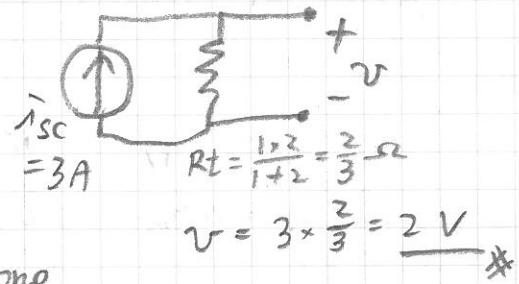
approach ①: voltage divider

$$V = 3 \times \frac{2}{1+2} = \frac{2}{3} V$$

approach ②: Thevenin's Theorem

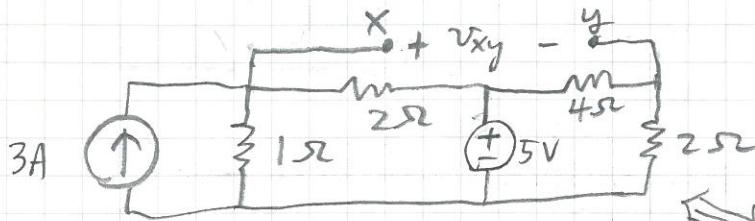


approach ③: Norton's Theorem

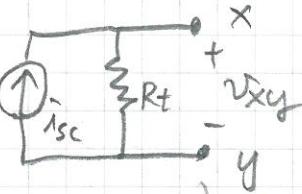


⇒ these three agree in one.

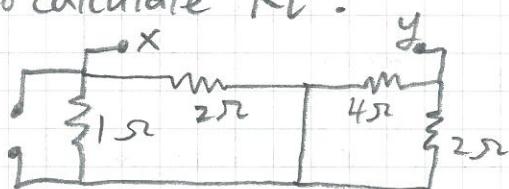
P34 Example : find $V_{xy} = ?$



Using Norton's Theorem, we have:



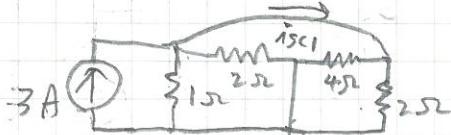
Now, to calculate R_t :



$$R_t = \left(\frac{1}{1\Omega} + \frac{1}{2\Omega} \right) + \left(\frac{1}{4\Omega} + \frac{1}{2\Omega} \right) = 2\Omega$$

To calculate i_{sc} , we may use superposition:

$$i_{sc} = i_{sc1} + i_{sc2}$$

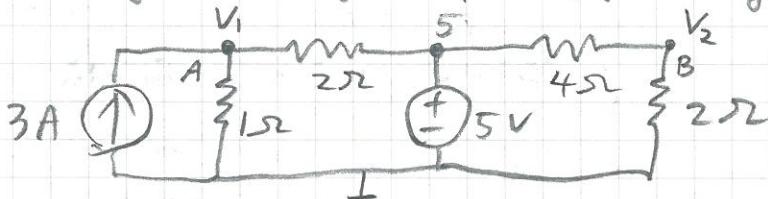


$$\text{therefore, } i_{sc} = 1A$$

$$V_{xy} = i_{sc} \times R_t \\ = 2V$$

from KCL, we may see that $i_{sc2}=0$

Alternatively, we may compute V_{xy} directly, using the node analysis method:



$$\text{KCL at node A : } 3 + \frac{0-V_1}{1} + \frac{5-V_1}{2} = 0 \Rightarrow V_1 = \frac{11}{3}$$

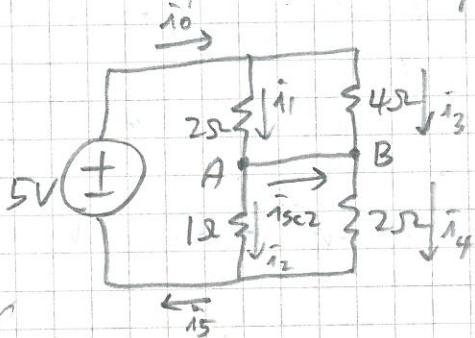
$$\text{KCL at node B : } \frac{5-V_2}{4} + \frac{0-V_2}{2} = 0 \Rightarrow V_2 = \frac{5}{3}$$

$$V_{xy} = V_1 - V_2 = \frac{11}{3} - \frac{5}{3} = 2V$$

From the above example, we see that in order to use Norton's Theorem correctly, we need to be very careful when determining the short-circuit current i_{sc} .

KCL is a great tool here to help us.

To be specific, when we were determining i_{sc2} in the previous example:



Applying KCL at nodes A and B, we have

$$\bar{i}_{sc2} = \bar{i}_2 - \bar{i}_1$$

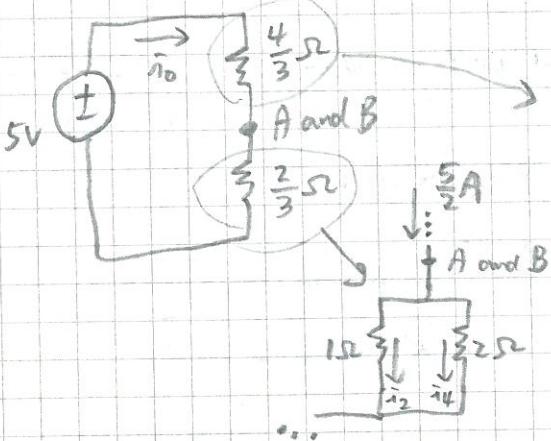
$$\bar{i}_{sc2} = \bar{i}_4 - \bar{i}_3$$

The above two equalities must hold, and we can use one of them to calculate \bar{i}_{sc2} and then use the other one to verify our answer.

equivalently

$$\text{Hence, } i_o = \frac{5}{\frac{4}{3} + \frac{2}{3}} = \frac{5}{2} \text{ A}$$

according to Ohm's law.



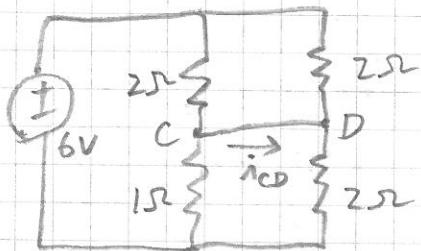
$$i_1 = i_o \times \frac{4}{2+4} = \frac{5}{3} \text{ A}$$

$$i_3 = i_o \times \frac{2}{2+4} = \frac{5}{6} \text{ A}$$

$$i_2 = \frac{5}{2} \times \frac{2}{1+2} = \frac{5}{3} \text{ A}$$

$$i_4 = \frac{5}{2} \times \frac{1}{1+2} = \frac{5}{6} \text{ A}$$

Interestingly, if we have the following circuit instead, the current flowing from node C to node D would be non-zero:



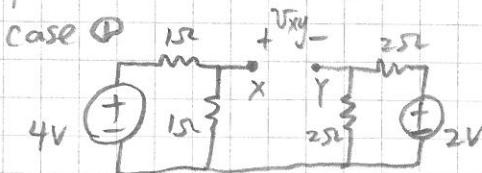
You can verify, as an exercise, that $i_{CD} = -\frac{3}{5} \text{ A}$

In essence, the condition to have no current flowing from node X to node Y is to have the node voltage at node X equal to that at node Y. Starting from here and using the symbolic computation, you might rediscover the "Wheatstone Bridge" circuit :)

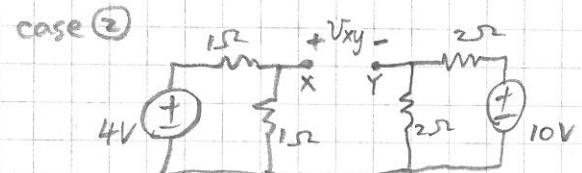
One final note on circuit transformation:

When we are using either Thévenin's Theorem or Norton's Theorem to transform a circuit into its equivalence, it is possible that we might define the V_{OC} or i_{SC} variable in the direction opposite to what's really happening in the circuit, since for a complex circuit we might not readily perceive the real direction of the voltage or current. But that's fine, because our computed result will have a negative sign if the direction was wrong.

Example: determine V_{XY} in these two cases:



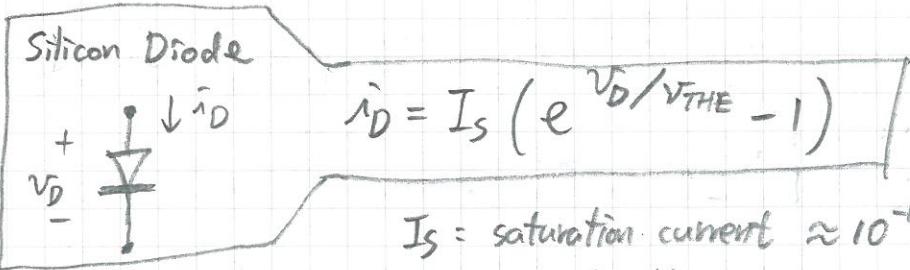
$$(\text{ans: } V_{XY} = 1 \text{ V})$$



$$(\text{ans: } V_{XY} = -3 \text{ V})$$

* Nonlinear Elements and Circuits

P37



I_s : saturation current $\approx 10^{-12} A$

V_{THE} : thermal voltage

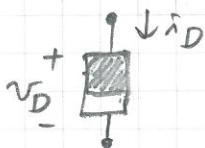
$$V_{THE} = \frac{kT}{q} \quad T: \text{temperature in kelvins}$$

$$^{\circ}\text{K} = ^{\circ}\text{C} + 273.15$$

$$k: \text{Boltzmann's constant} \\ = 1.38 \times 10^{-23} \text{ J/}^{\circ}\text{K}$$

$$q: \text{charge of an electron} \\ = 1.602 \times 10^{-19} \text{ C}$$

Ideal Nonlinear Device



$$i_D = \begin{cases} IK \cdot V_D^2 & \text{for } V_D > 0 \\ 0 & \text{otherwise} \end{cases}$$

where IK is a positive constant

Example: find V_D and i_D :



$$\text{KCL at A: } \frac{V - V_D}{R} + (-i_D) = 0$$

Plug in the i - v relation, we have

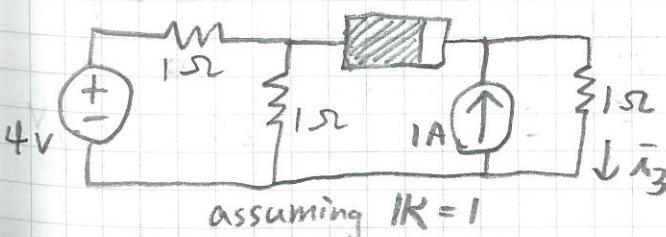
$$\begin{cases} \frac{V - V_D}{R} - IK V_D^2 = 0 & \text{for } V_D > 0 \\ \frac{V - V_D}{R} = 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow V_D = \frac{-1 + \sqrt{1 + 4RIKV}}{2RK}$$

$$\text{and } i_D = IK \left(\frac{-1 + \sqrt{1 + 4RIKV}}{2RK} \right)^2 \text{ for } V_D > 0$$

Exercise: find $i_3 = ?$

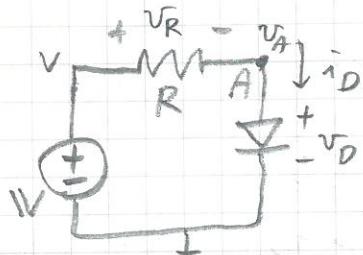
(hint: Page 29 and the above example)



You can do it!

$$\text{Ans: } i_3 = \left(\frac{\sqrt{7} - 1}{3} \right)^2 + 1$$

P38 Motivation for graphical analyses:
Consider the following example



KCL at node A:

$$\frac{v_A - V}{R} + I_s(e^{\frac{v_D}{V_{THE}}} - 1) = 0$$

$$\Rightarrow \frac{v_D - V}{R} + I_s(e^{\frac{v_D}{V_{THE}}} - 1) = 0$$

Solving for v_D is like solving for

x for $ax + be^x + c = 0$ for some constants a, b , and c .

→ May be solved by trial-and-error

→ little insight, however.

What if we want to know the impact of increasing/decreasing V to the value of v_D ?

And, how would v_D change with the change of R ? These are critical questions to ask when designing an electronic circuit.

Now, graphical analyses can be very helpful in this aspect!

For concreteness, suppose $V = 3 \text{ V}$

$$R = 500 \Omega$$

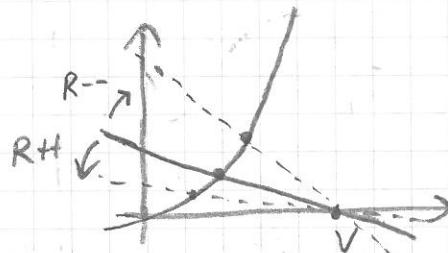
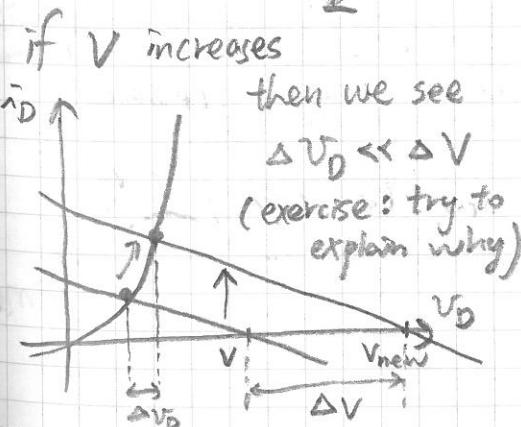
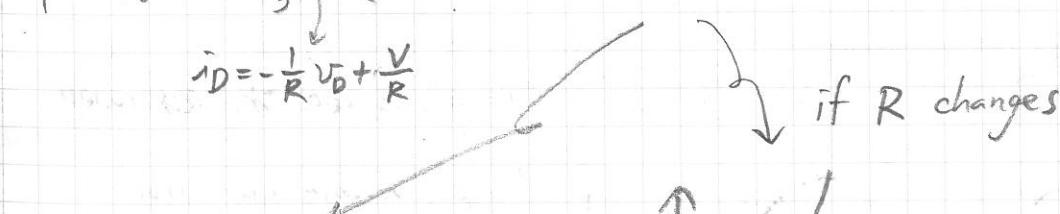
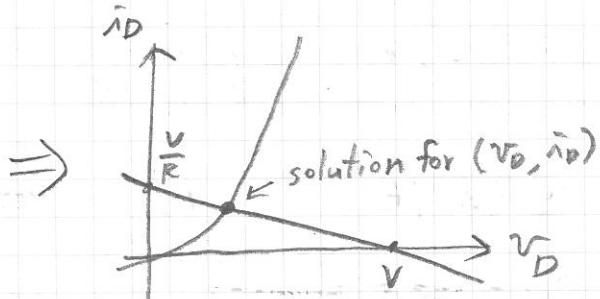
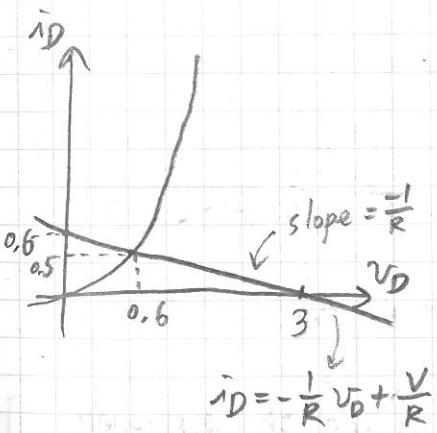
in the example on P28.

$$V_{THE} = 0.025 \text{ V} \text{ in room temperature}$$

We can leverage KCL and KVL
to state two equations for i_D :

$$\begin{cases} i_D = I_s (e^{v_D/V_{THE}} - 1) = 10^{-12} (e^{v_D/0.025} - 1) \\ i_D = \frac{V_R}{R} = \frac{V - v_D}{R} = -\frac{1}{R} v_D + 0.006 \quad (\text{unit: A}) \end{cases}$$

then graphically speaking, the solution
of i_D and v_D lies at the intersection point
of the two curves on the $i_D - v_D$ plane:



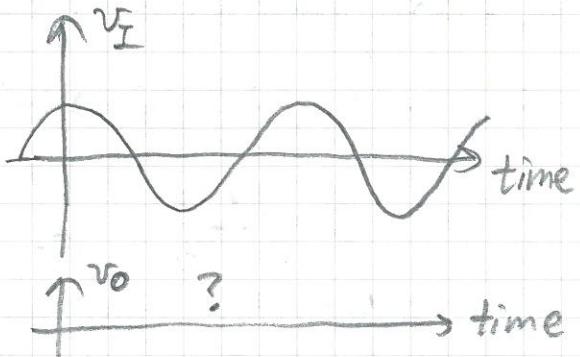
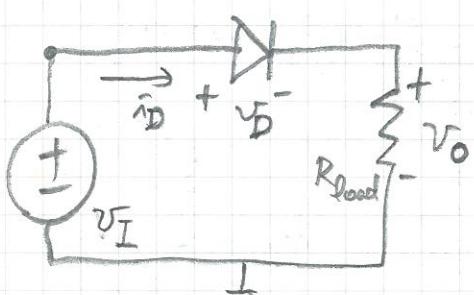
which give much insight
in how the circuit would behave!

P40 Another example (half-wave voltage rectifier)

In the following circuit,

given a time-varying voltage source V_I ,

what will be the output voltage V_O across a resistor?

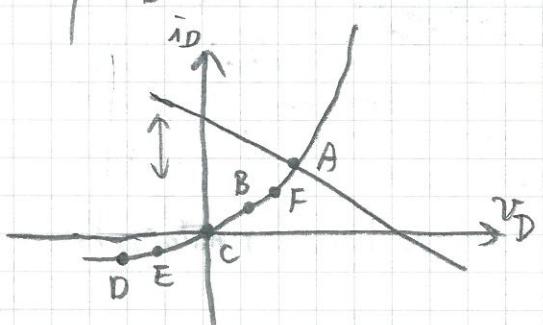
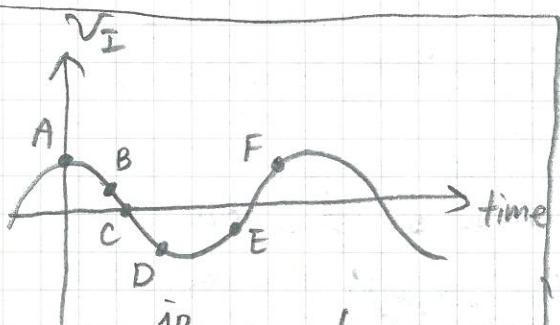
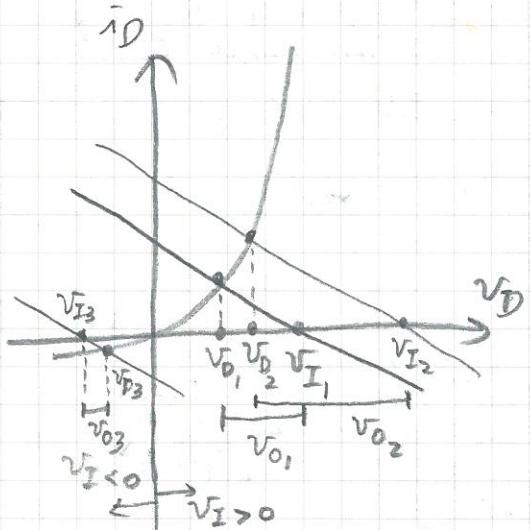


$$\text{From KVL, } V_I - V_D - V_O = 0$$

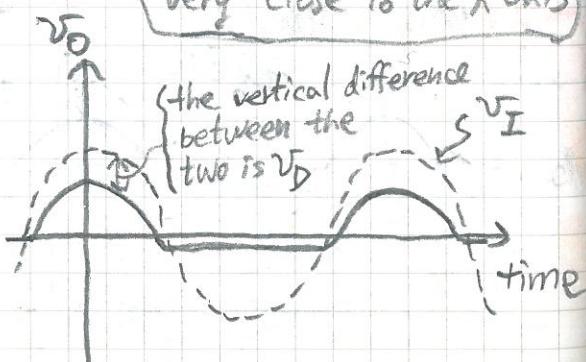
$$\Rightarrow V_O = V_I - V_D$$

From KCL, element law, and the $i-v$ characteristic of the diode:

$$\begin{cases} i_D = I_s (e^{V_D/V_{TH}} - 1) \\ i_D = \frac{V_O}{R} = \frac{V_I - V_D}{R} \end{cases}$$



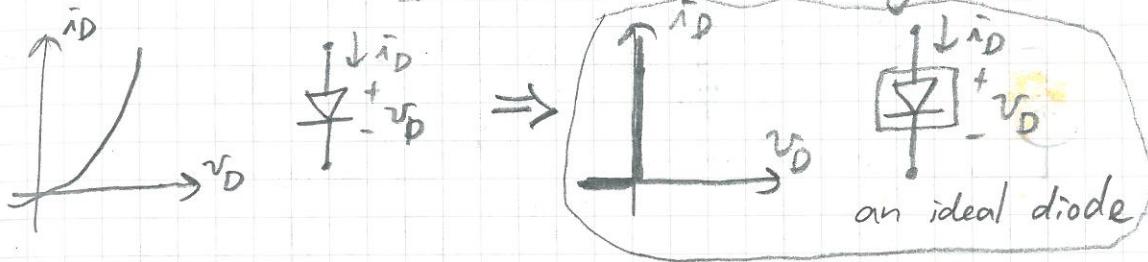
We see $V_{O_3} \approx 0$ when $V_D < 0$
because the curve is very close to the X axis



Besides the graphical analysis,

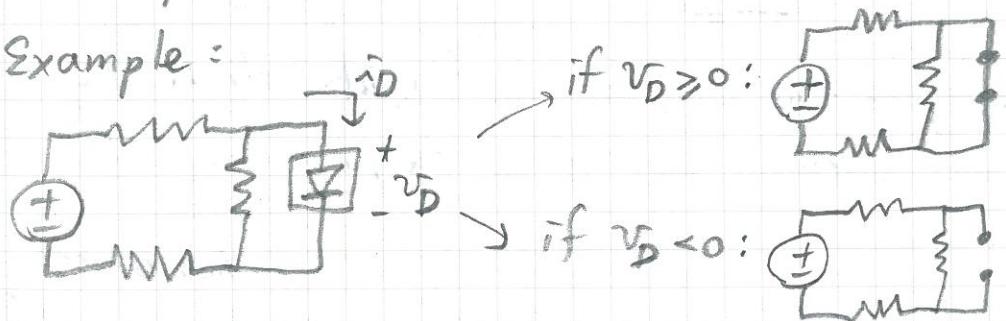
in some occasion we may simplify our analysis of a nonlinear circuit by considering an approximated version of the $i-v$ characteristic of a given nonlinear element.

This is called the piecewise linear analysis.



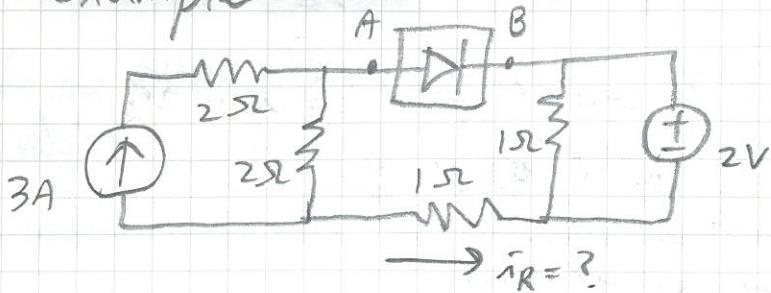
For an ideal diode, depending on the actual voltage (or the actual current direction), we may replace the diode by either a short circuit or an open circuit.

Example:

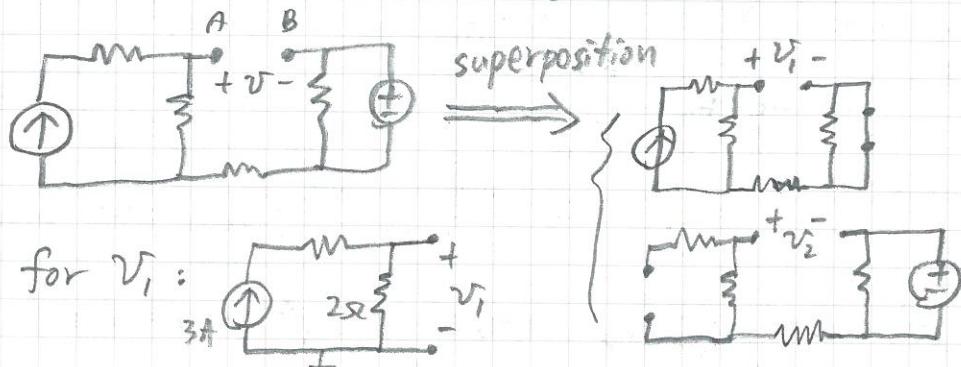


(Note: study textbook P206~209 for an alternative exposition of this subject; study Example 4.11 on P209 for an advanced example.)

P42 Example:



We first find the voltage across A and B.

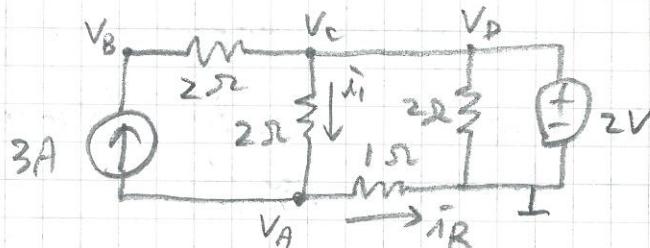


$$V_1 = 3A \times 2\Omega = 6V$$

$$\left. \begin{array}{l} \text{superposition} \\ \xrightarrow{\quad} \end{array} \right\} V = V_1 + V_2$$

$$\text{for } V_2: \quad \begin{array}{c} V_2 \\ + \\ 2V \\ - \end{array} \quad V_2 = -2V \quad \left. \begin{array}{l} \\ \Rightarrow V = 6 - 2 = 4V \end{array} \right)$$

therefore we may replace the ideal diode by a short circuit, leading to the following equivalence:



(exercise:
try to analyse
this using
superposition!)

$$V_C = V_D = 2$$

$$\bar{i}_1 = 3 + \bar{i}_R$$

$$\Rightarrow \frac{2 - V_A}{2} = 3 + \frac{V_A - 0}{1}$$

$$\Rightarrow 2 - V_A = 6 + 2V_A$$

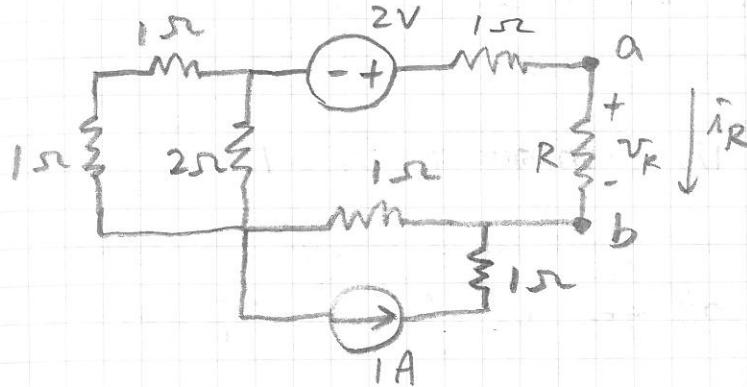
$$\Rightarrow V_A = -\frac{4}{3}$$

$$\Rightarrow \bar{i}_R = \frac{V_A - 0}{1} = \frac{-4}{3} A$$

Some exercise problems since P32:

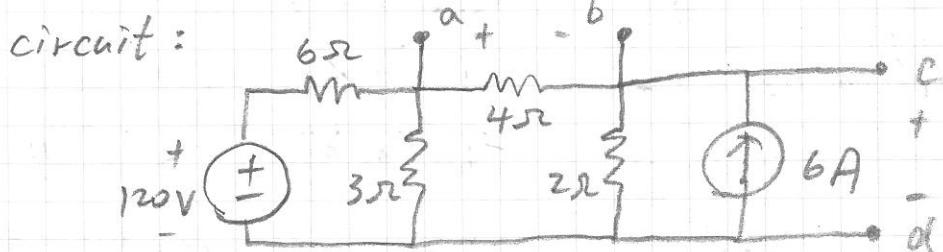
A. Use Norton's Theorem to find i_R and v_R

for ① $R=2$ and ② $R=4$



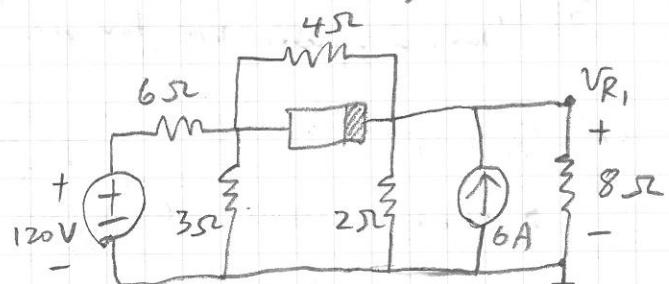
B. Following A, rather than using Norton's Theorem, directly apply superposition to find i_R for $R=2$.

C. Determine the Norton equivalent at terminals a,b and at terminals c,d, for the following circuit:



D. For the following nonlinear circuit, determine V_{R_1} .

$$i_D = \begin{cases} 2 \cdot V_D^2 & \text{for } V_D > 0 \\ 0 & \text{otherwise} \end{cases}$$

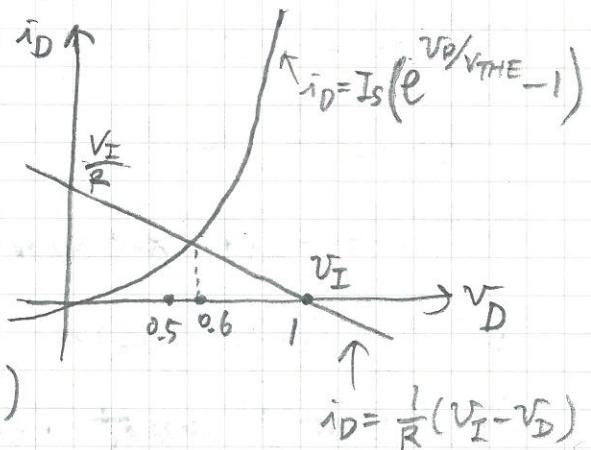
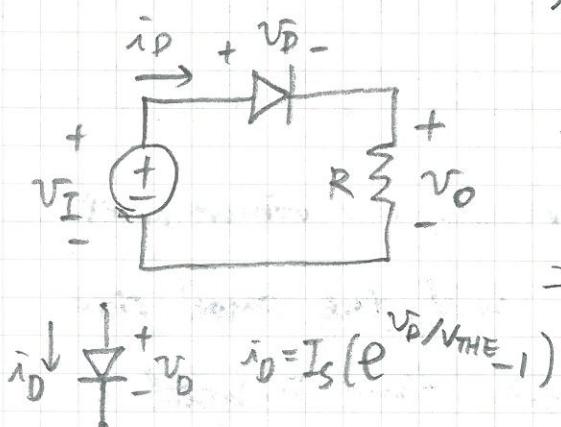


(hint: compare this with that in C.)

P44 E. For the following nonlinear circuit, with some initial analysis of its $i_D - v_D$ relation at hand, try to answer three questions:

① If for some need we changed the input voltage v_I , so that $v_I > 1$, would that lead to a change to the output voltage $v_O > 0.4$ or $v_O < 0.4$?

② Now, suppose we operate the circuit at region $v_I \gg 1$. What can we say about v_O ?



③ Now, suppose we fix v_I but replace the linear resistor by a very heavy load, such that $R \gg 1$. What would i_D become?

Answers to Problems A, B, C, D:

A. ① $i_R = \frac{1}{5}A$, $v_R = \frac{2}{5}V$ ② $i_R = \frac{1}{9}A$, $v_R = \frac{4}{9}V$ B. same as A. D. 16 V

C. $R_{tab} = 2\Omega$, $i_{scab} = 9A$
 $R_{tcd} = \frac{3}{2}\Omega$, $i_{scd} = \frac{38}{3}A$

Interlude : A note on an extremely useful tool, "the Taylor's Theorem" (A.K.A. Taylor Expansion)

Motivation : To approximate a complex function by a simpler one, given some input.

Idea : for function $f(x)$, we may say that function $g(x)$ is approximately the same as $f(x)$ at $x=x_0$ if $f(x)$ and $g(x)$ have similar trend around $x=x_0$.

Approach : Construct $g(x)$ such that

$$g(x_0) = f(x_0)$$

$g'(x_0) = f'(x_0)$ ← first derivative, i.e., the change rate of a function.

$g''(x_0) = f''(x_0)$ ← second derivative, i.e., the change rate of the change rate of the function.

:

$$\text{So we write } g(x) = f(x)\Big|_{x \text{ near } x_0} + f'(x)\Big|_{x=x_0}(x-x_0)$$

$$+ \frac{1}{2!} f''(x)\Big|_{x=x_0} (x-x_0)^2$$

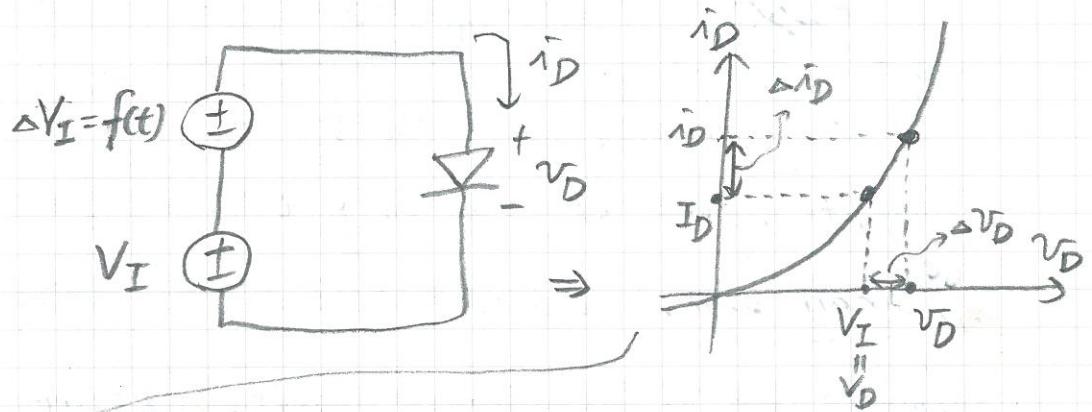
$$+ \frac{1}{3!} f'''(x)\Big|_{x=x_0} (x-x_0)^3 + \dots$$

① Try to compute and compare $\{g(x), f(x)\}$, $\{g'(x), f'(x)\}$ and $\{g''(x), f''(x)\}$ to see this really makes sense.

② Take a look at P218 in the textbook and Equation 4.66 to see how we may control the error.

★ Small-Signal Analysis for Nonlinear Devices P46

- In many sensor applications and most audio amplifiers, the input voltage/current to a circuit often consists of two parts:
 - ① a time-invariant source (large signal)
 - ② a time-varying source (small signal)

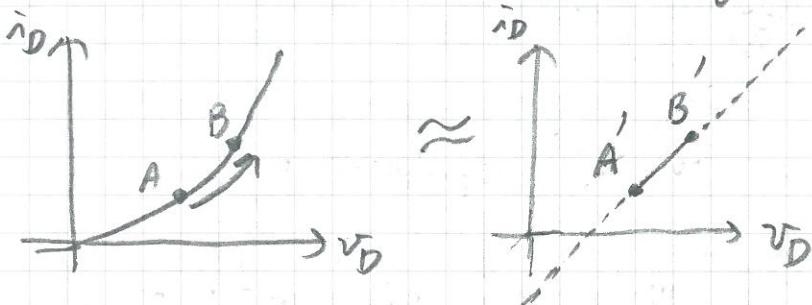


The large signal is used to determine the region of operation (i.e., which part of the $i_D - v_D$ curve), and the small signal is the "real" input (e.g., those induced by human voice, as in the case of a microphone).

$$\begin{aligned} v_D &= V_I + \Delta v_D \\ &\quad \begin{matrix} \uparrow & \uparrow \\ \text{large} & \text{small signal} \end{matrix} \\ i_D &= I_D + \Delta i_D \end{aligned}$$

P47

as we will see, moving along a small distance on "the $i_D - v_D$ curve" can be approximated as moving along a small distance on "a straight line."

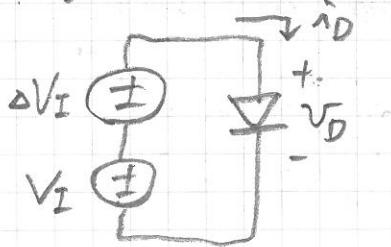


Therefore, we may simplify our analysis of small signal by considering the signal's response on a nonlinear device as if it is the response on a linear device (resistor).

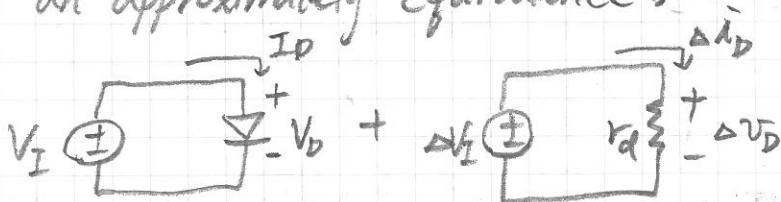
Review P13, where we've shown that the $i-v$ characteristic of a resistor on the $i-v$ plot is a straight line; further, the slope of the line is equal to the reciprocal of the resistance (R) of the resistor.

Now, a question is : how do we determine the resistance of that linear device ?

Let r_d be the resistance of the linear device. Using small-signal analysis, we essentially transform the original circuit



into an approximately equivalence:

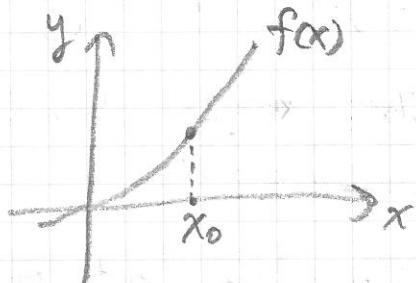


$$\text{where } i_D = I_D + \Delta i_D$$

$$\frac{V_{THE}}{I_D}$$

Now, let's see how to determine r_d !

We use Taylor's Theorem, which provides a way to approximate a curve near a certain point $x = x_0$:



$$\begin{aligned} y &= f(x) = f(x_0) \Big|_{x=x_0} + f'(x) \Big|_{x=x_0} (x - x_0) \\ &\quad + \frac{1}{2!} f''(x) \Big|_{x=x_0} (x - x_0)^2 + \frac{1}{3!} f'''(x) \Big|_{x=x_0} (x - x_0)^3 + \dots \end{aligned}$$

P49

in our case of a nonlinear diode,
recall that

$$i_D = I_s \left(e^{\frac{v_D}{V_{THE}}} - 1 \right) = f(v_D)$$

↑
we define it

$$\Rightarrow i_D \underset{v_D \text{ near } V_D}{=} f(v_D) \Big|_{v_D=V_D} + f'(v_D) \Big|_{v_D=V_D} (v_D - V_D)$$

$$+ \frac{1}{2!} f''(v_D) \Big|_{v_D=V_D} (v_D - V_D)^2$$

$$+ \frac{1}{3!} f'''(v_D) \Big|_{v_D=V_D} (v_D - V_D)^3 + \dots$$

$$= f(v_D) \Big|_{v_D=V_D} + (v_D - V_D) \left(f'(v_D) \Big|_{v_D=V_D} \right.$$

we choose to ignore these terms

$$\left. \begin{array}{l} + \frac{1}{2!} f''(v_D) \Big|_{v_D=V_D} (v_D - V_D) \\ + \frac{1}{3!} f'''(v_D) \Big|_{v_D=V_D} (v_D - V_D)^2 \\ + \dots \end{array} \right)$$

using the chain rule, we get $f'(v_D) = \frac{I_s}{V_{THE}} e^{\frac{v_D}{V_{THE}}}$

$$\Rightarrow i_D \underset{v_D \text{ near } V_D}{\approx} f(v_D) \Big|_{v_D=V_D} + \underbrace{(v_D - V_D) \cdot \frac{I_s}{V_{THE}} e^{\frac{v_D}{V_{THE}}}}_{\text{we define it}} \Big|_{v_D=V_D}$$

$$= \underbrace{I_s \left(e^{\frac{v_D}{V_{THE}}} - 1 \right)}_{\text{we define it}} + \underbrace{\Delta v_D \cdot \frac{I_s}{V_{THE}} \cdot e^{\frac{v_D}{V_{THE}}}}$$

since we know that

$$e^{\frac{V_D}{V_{THE}}} \gg 1$$

so we can think of $e^{\frac{V_D}{V_{THE}}} \approx e^{\frac{V_D}{V_{THE}}} - 1$

With that, we may rewrite the equation as

$$\frac{i_D}{v_0 \text{ near } V_D} = I_S (e^{\frac{V_D}{V_{THE}}} - 1) + \Delta v_D \frac{I_S}{V_{THE}} (e^{\frac{V_D}{V_{THE}}} - 1)$$

Now, by observation we see $I_S (e^{\frac{V_D}{V_{THE}}} - 1) = I_D$

$$\text{Thus, } \frac{i_D}{v_0 \text{ near } V_D} = I_D + \frac{\Delta v_D}{V_{THE}} I_D$$

Compare to $i_D = I_D + \Delta i_D$, we have $\Delta i_D = \underbrace{\frac{\Delta v_D}{V_{THE}} I_D}_{r_d}$

Think in terms of relation of Δi_D and Δv_D

and we may choose to define

$$r_d = \frac{V_{THE}}{I_D}$$

$$\frac{\Delta v_D}{\Delta i_D} = \frac{V_{THE}}{I_D} r_d$$

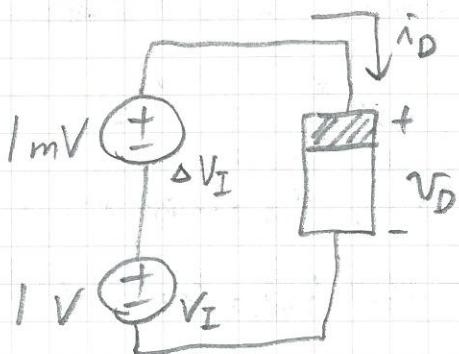
In the hindsight, we may generalize our result by saying that for an arbitrary nonlinear element, we have

$$r_d = \frac{1}{f'(v_D)} \Big|_{v_D=v_D} = \frac{1}{\frac{df(v_D)}{dv_D}} \Big|_{v_D=v_D}$$

(finally!)

P51

Example :

find $i_D = ?$ 

recall that for \square (P35)

$$\dot{i}_D = K \cdot V_D^2 \text{ for } V_D > 0$$

and here we suppose
 $K = 1 \text{ mA/V}^2$

Ans:

$$\left. \begin{array}{l} \downarrow \\ \dot{i}_{D_1} \end{array} \right\} \quad V_{D_1} = 1 \quad (\text{KVL})$$

$$1V \quad \left. \begin{array}{l} \downarrow \\ \dot{i}_{D_1} \end{array} \right\} \quad \Rightarrow \dot{i}_{D_1} = K \cdot V_{D_1}^2 = 1 \text{ mA}$$

$$V_{D_2} = 1 \text{ mV} \quad (\text{KVL})$$

$$r_d = \frac{1}{f(V_D)} \Big|_{V_D=V_0} = \frac{1}{2 \cdot K \cdot V_D} \Big|_{V_D=1} = 500 \Omega$$



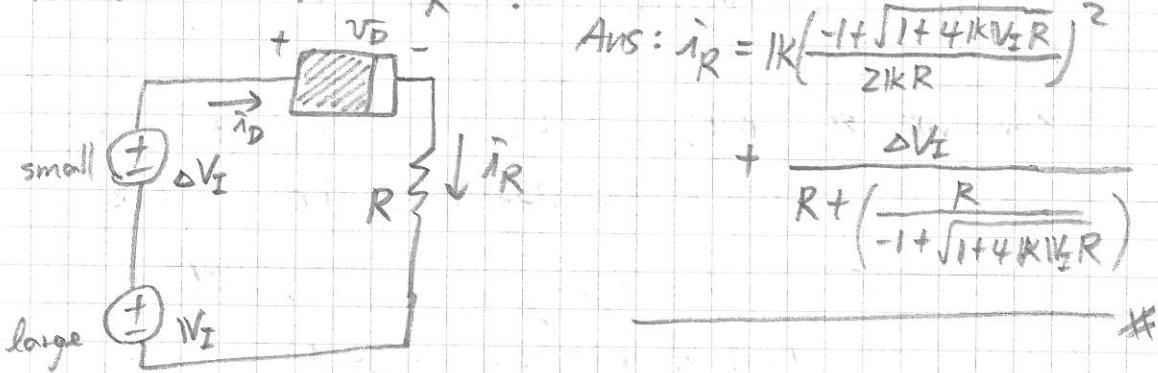
$$= 500 \Omega$$

$$\Rightarrow \dot{i}_{D_2} = \frac{V_{D_2}}{r_d} = 2 \mu A$$

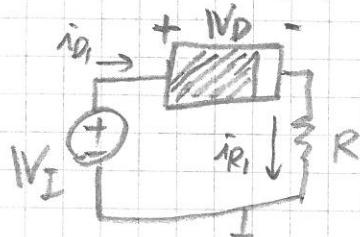
$$\Rightarrow \dot{i}_D = \dot{i}_{D_1} + \dot{i}_{D_2} = 1.002 \text{ mA}$$

Exercises for small-signal analysis:

Exercise ①: find $\bar{i}_R = ?$ (The diode is defined on page 37)



Solution: First of all, it is clear that $\bar{V}_D > 0$ in this case.

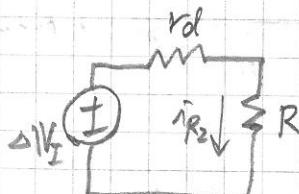


From analytical analysis, we have

$$\frac{V_I - V_D}{R} = 1K V_D^2$$

$$\Rightarrow V_D = \frac{-1 + \sqrt{1 + 4K V_I R}}{2K R}$$

$$\Rightarrow \bar{i}_{R_1} = \bar{i}_{D_1} = 1K V_D^2$$

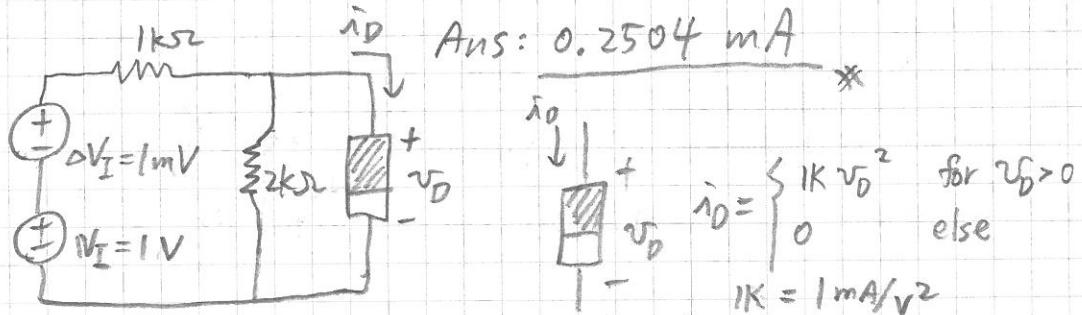


$$r_d = \frac{1}{f'(V_D)} \Big|_{V_D=V_D} = \frac{1}{2K V_D} \Big|_{V_D=V_D} = \frac{1}{-1 + \sqrt{1 + 4K V_I R}}$$

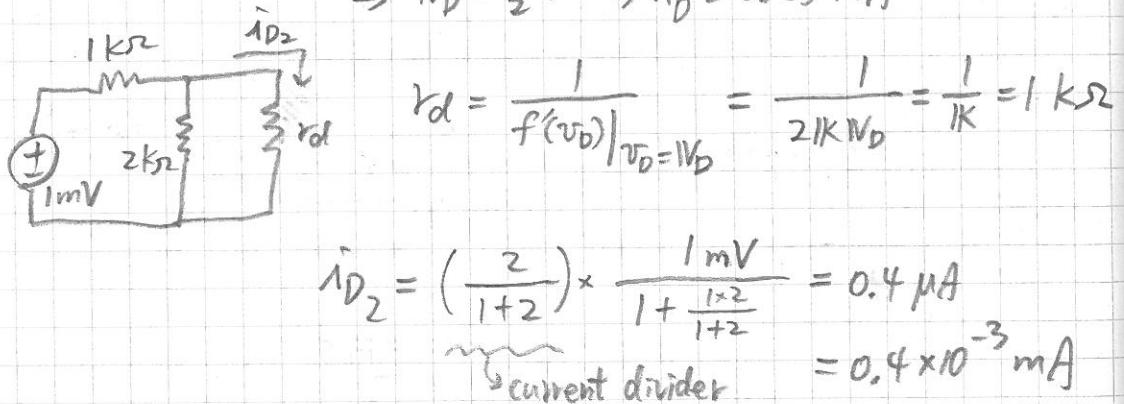
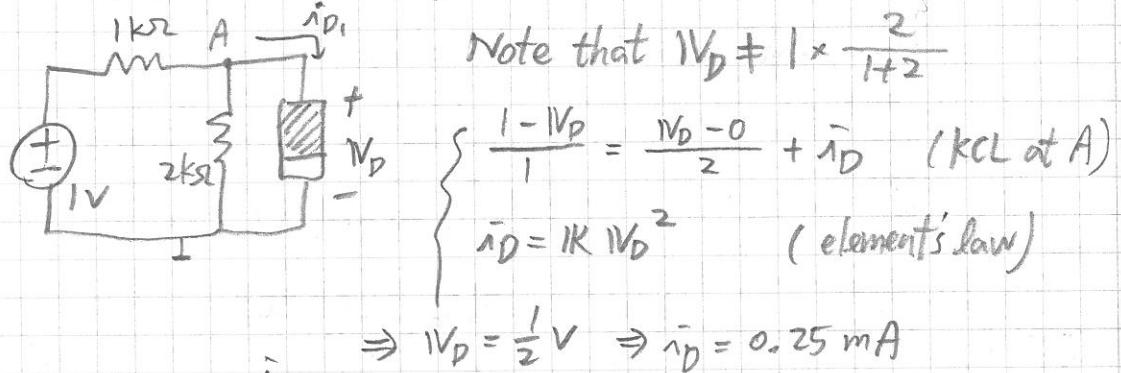
$$\bar{i}_{R_2} = \frac{\Delta V_I}{R + r_d} = \frac{\Delta V_I}{R + \left(\frac{1}{-1 + \sqrt{1 + 4K V_I R}} \right)}$$

finally, $\bar{i}_R = \bar{i}_{R_1} + \bar{i}_{R_2}$

Exercise ②: find $i_D = ?$



Solution: First of all, from voltage divider we know that $V_D > 0$



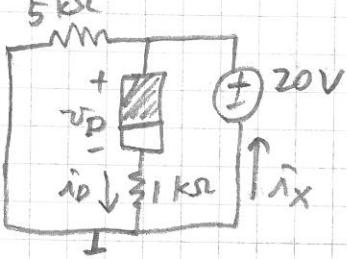
$$\text{finally, } \bar{i}_D = \bar{i}_{D1} + \bar{i}_{D2}$$

$$= 0.25\text{ mA} + 0.0004\text{ mA}$$

$$= 0.2504\text{ mA}$$

*

Exercise ③ : find $\bar{i}_x = ?$ Ans: 20 mA *

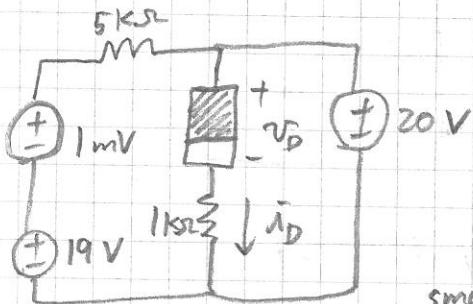


Solution: It's clear that $v_D > 0$

Apply the node analysis method, we have

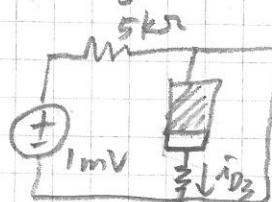
$$\begin{cases} \frac{20}{5\text{k}\Omega} + \bar{i}_D = \bar{i}_X \\ \bar{i}_D = \frac{20 - v_D}{1\text{k}\Omega} \\ \bar{i}_D = 1\text{k} \cdot v_D^{-2} \end{cases} \Rightarrow \begin{cases} v_D = 4\text{ V} \\ \bar{i}_D = 16 \text{ mA} \\ \bar{i}_X = 20 \text{ mA} \end{cases}$$

Exercise ④ find $\bar{i}_D = ?$ Ans: 16 mA *

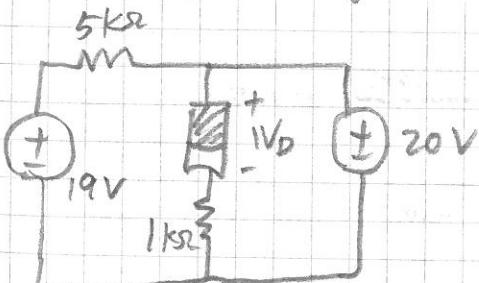


Solution: see the following decomposition

small signal



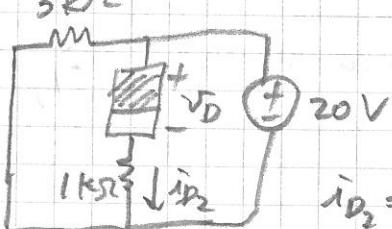
$$\therefore i_{D1} = 0$$



$$\bar{i}_D = \bar{i}_{D1} + \bar{i}_{D2} + \bar{i}_{D3} = 16 \text{ mA}$$

*

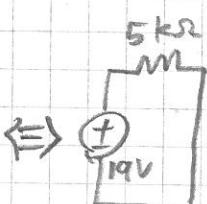
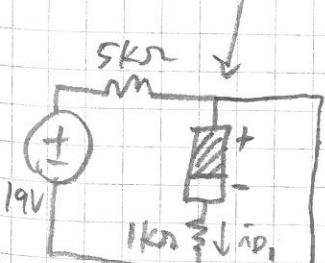
using superposition



$$i_{D2} = 16 \text{ mA}$$

which we solved
in exercise ③

*



$$\therefore i_{D1} = 0$$

The digital abstraction

P55

So far, we've been studying analog signals like voltage and current, and we focus on how these signals, being transformed by a circuit, will impact the behavior of a certain element in a circuit. The impact manifests itself in terms of analog signals on the element, and we also called it the "response" of the element with respect to the analog signals from a voltage source  or a current source .

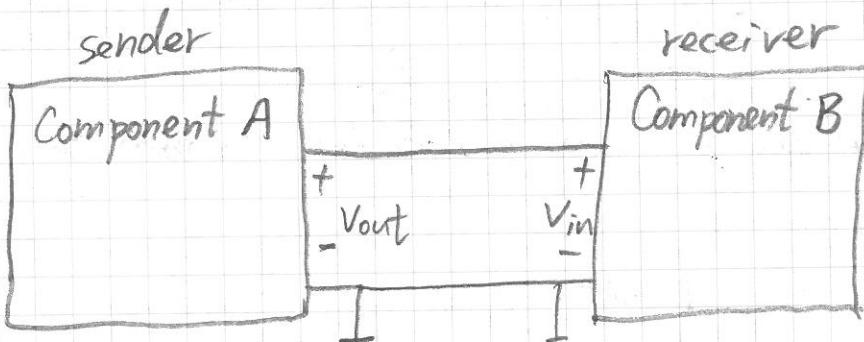
Now, how does an analog signal relates to the "digital world", the world built using some meaningful combinations of 0s and 1s?

The digital abstraction serves this purpose. It specifies a transformation that interprets the analog signals into a series of binary digits, the so-called "digital signals."

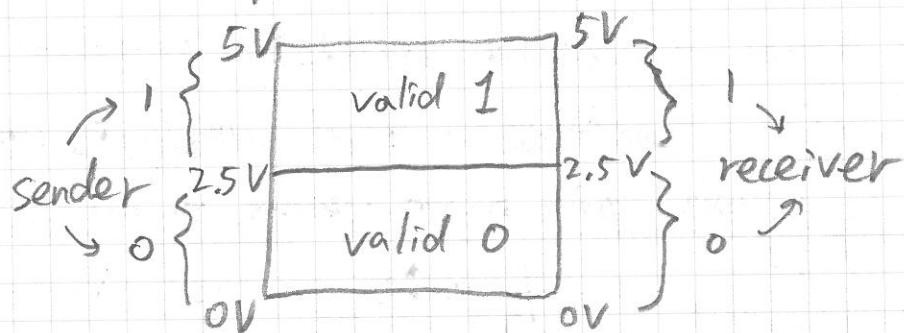
P56

Review Section 1.8 in the textbook.

- Digital signals are "interpretations" of analog signals, so that they can be understood and be used by a digital system (such as a computer).
- Between physical components in a system, it is still analog signals that are transferred.
- How to transform digital signals into analog ones?

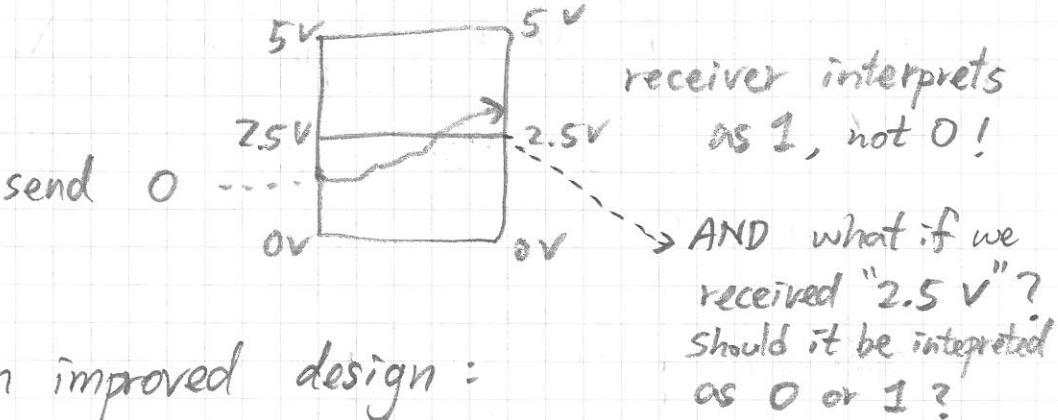


Our first attempt :

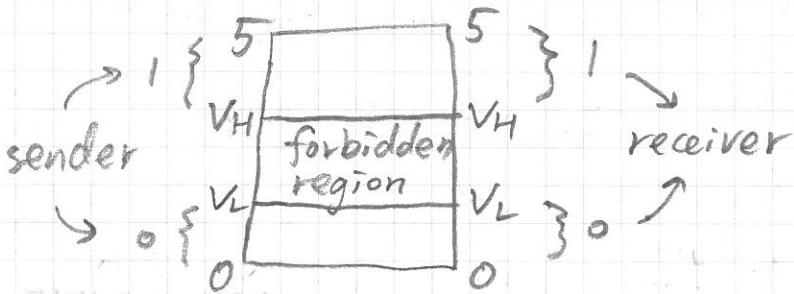


A serious problem :

there could be noise / interference
during signal transmissions



An improved design :



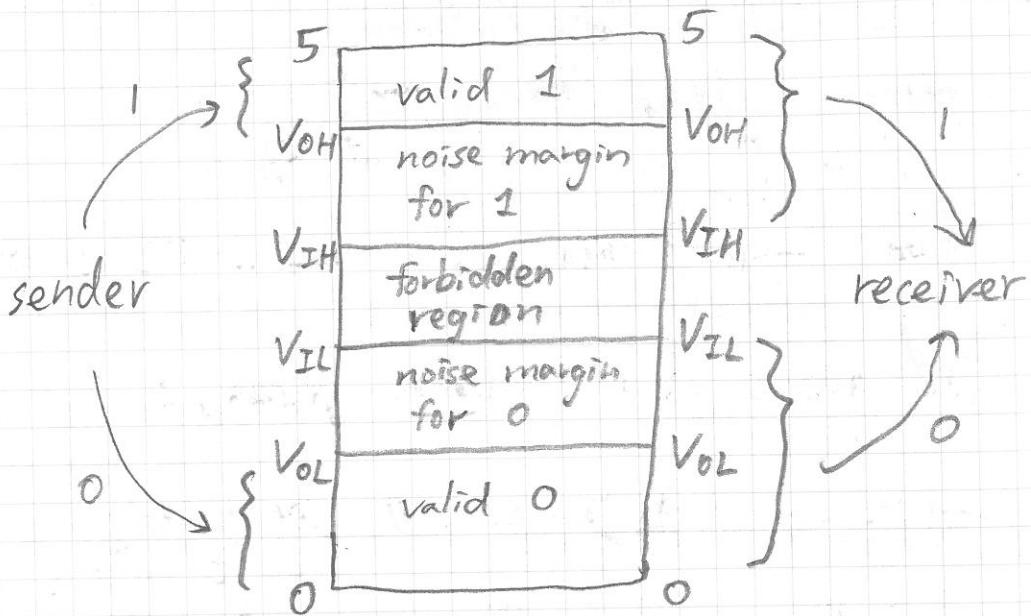
V_H and V_L are high / low voltage thresholds.

A further question :

how to quantify the resilience
to noise ?

and if we can do that, this can serve
as a "contract", and accordingly it would
ensure device manufacturers to meet consumers'
need, and components can be connected
to form a system.

A better design :



The static discipline (principle) —

A device must interpret correctly voltage inputs falling within the V_{IL} or V_{IH} threshold; with a valid input, the device must produce a valid voltage output that falls within the V_{OL} or V_{OH} threshold.

~ A specification of digital devices.

- Figure 5.9 in the textbook is a great illustration for us to remember the relative levels of V_{OH} , V_{IH} , V_{IL} , and V_{OL} .

Sections 5.2, 5.3, 5.6 in the textbook
are very good learning materials.

Be sure to study them yourselves.

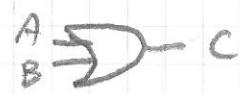
We will briefly cover some of them

at some appropriate opportunity in this course.

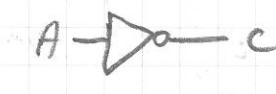
Logic gates (^{i.e.,} combinational gates):



AND gate

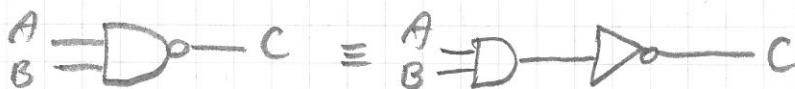


OR gate



NOT gate

$$\begin{array}{rcl} A & C \\ \hline 1 & 0 \\ 0 & 1 \end{array}$$



NAND gate \equiv AND gate plus NOT gate



NOR gate \equiv OR gate plus NOT gate

Truth table

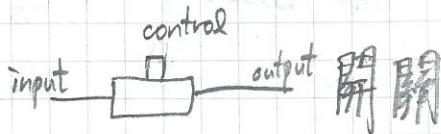
inputs		output (c)			
A	B	AND	OR	NAND	NOR
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	1	0
1	1	1	1	0	0

P₆₀

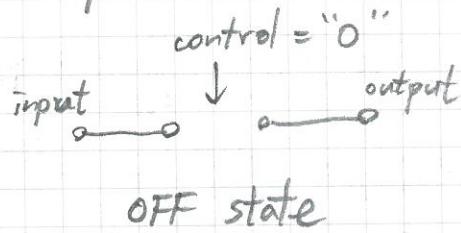
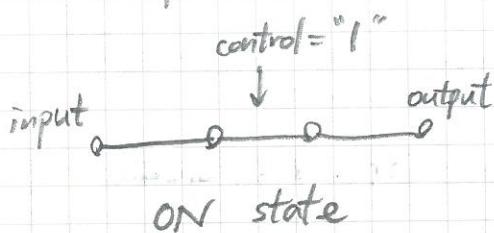
* The MOSFET switch

(Metal Oxide Semiconductor Field-Effect Transistor)

A switch is a three-terminal element in a circuit.



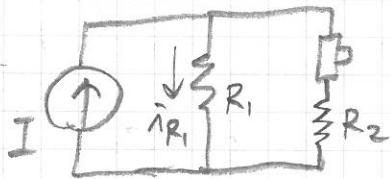
Ideally, input to the control terminal determines whether or not there is a short circuit between the input terminal and the output terminal:



(compare this with an ideal diode $\text{\textcircled{D}}$, P₃₉)

Such a simple idea of a switch enables many ways to control the response of certain elements in a circuit. First, let's look at two examples showing how a switch may impact the response of another element in a circuit:

Example 1.

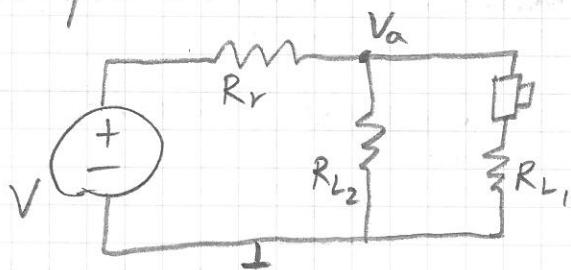


A switch connected in series with resistor R_2 will impact the current flowing through resistor R_1 .

$$\text{ON state: } i_{R_1} = I \times \frac{R_2}{R_1 + R_2}$$

$$\text{OFF state: } i_{R_1} = I$$

Example 2.



$$\text{ON state: } V_a = V \times \frac{R_{L2} // R_{L1}}{R_r + R_{L2} // R_{L1}}$$

$$\text{OFF state: } V_a = V \times \frac{R_{L2}}{R_r + R_{L2}}$$

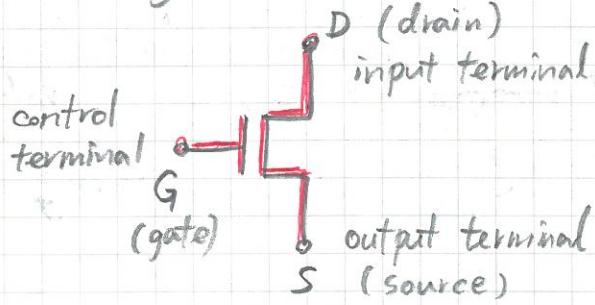
The voltage across the resistor R_{L2} will drop if the switch is in the ON state, because of the existence of R_r .

**

Conceptually, by connecting two switches in series we may implement the AND logic; by connecting two switches in parallel, we may implement the OR logic. In both cases, the control terminals take the input logical values.

P62

The symbol of a MOSFET (to be specific,

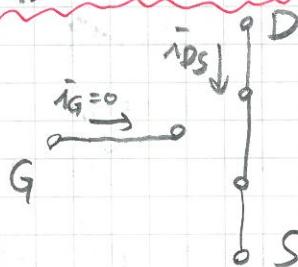


we study
n-channel
enhancement-type
MOSFETs,
also known as
NMOS FETs.
see Pg 9 for more)

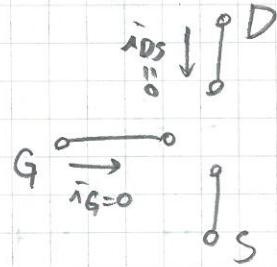
Let i_{DS} be the current flowing from the drain to the source, V_{GS} be the voltage across the gate and the source, and V_T be a threshold voltage.

In its simplest model (S model), the MOSFET behaves as follows :

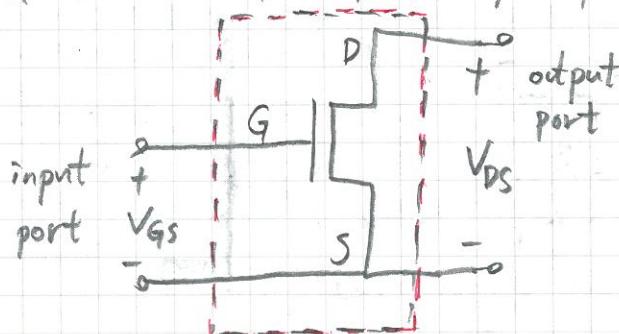
$$V_{GS} \geq V_T \Rightarrow \text{ON state}$$



$$V_{GS} < V_T \Rightarrow \text{OFF state}$$



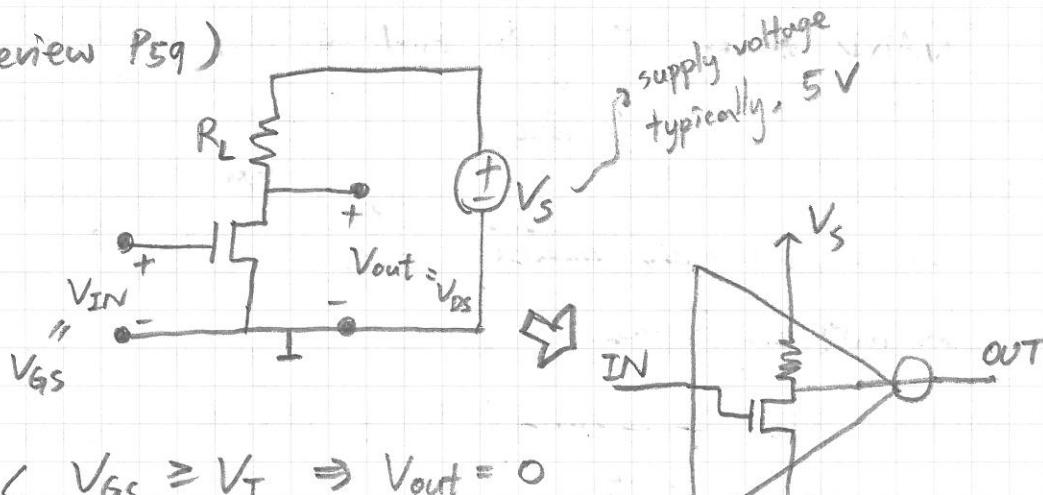
And in terms of input/output ports :



Later, we will study some more realistic models of MOSFET (for example, explicitly consider its internal resistance). For now, let's focus on the S model and its switching behavior. P63

We may use a MOSFET to construct a logical NOT gate $\rightarrow \neg$, so-called an "inverter":

(review P59)



$$\left. \begin{array}{l} V_{GS} \geq V_T \Rightarrow V_{out} = 0 \\ (\text{logic 1}) \quad (\text{logic 0}) \end{array} \right\}$$

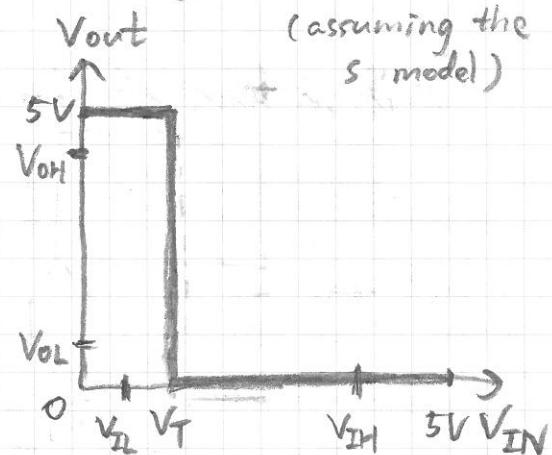
$$\left. \begin{array}{l} V_{GS} < V_T \Rightarrow V_{out} = V_S \\ (\text{logic 0}) \quad (\text{logic 1}) \end{array} \right\}$$

transfer characteristic of an inverter (assuming the S model)

A valid mapping of voltage levels specified

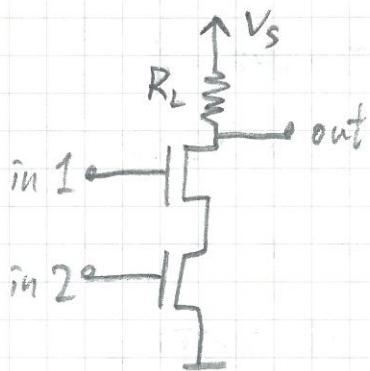
in the static discipline \rightarrow

(review P58)

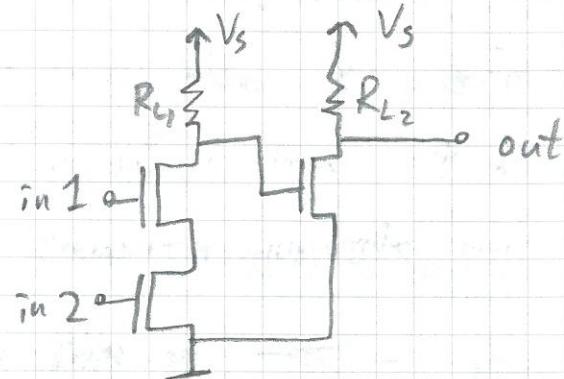


P64

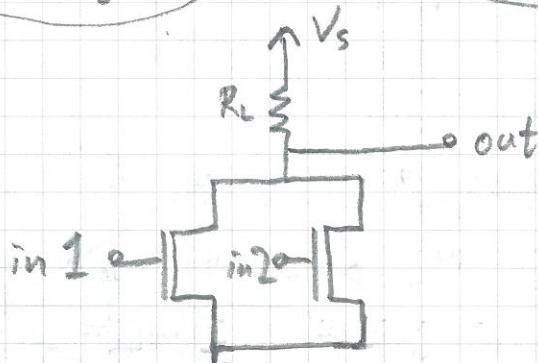
Similarly, we may use MOSFETs to construct other logic gates:



NAND gate



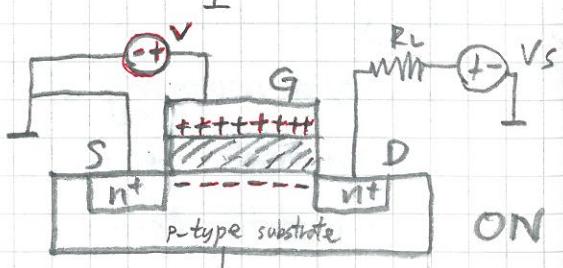
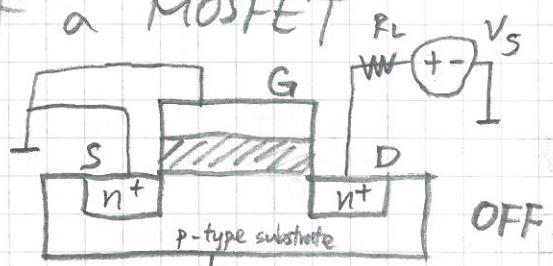
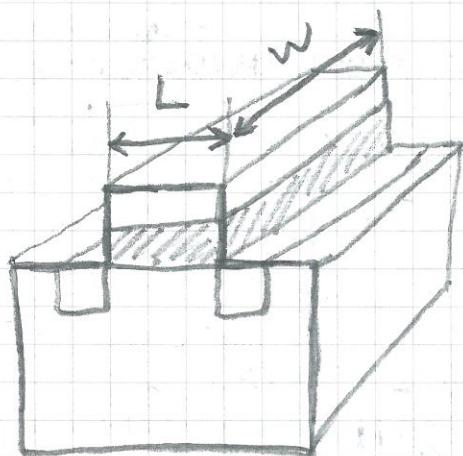
AND gate (NAND + NOT)



NOR gate

Exercise:
OR gate

* Physical structure of a MOSFET



3D view

2D view

When a positive voltage is applied at the control terminal G, a conducting channel will be built up between the two n^+ regions, and therefore some nontrivial current may flow from D to S. Read Section 6.7 in the textbook for a more detailed account.

The physical structure also implies that in the ON state there really will exist some resistance between D and S, and the value of the resistance is characterised by the dimensions of that conducting channel. Let R_N be the resistance per square of the channel, and L be the channel length and W the channel width.

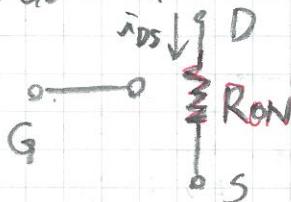
We have

$$R_{ON} = R_N \cdot \frac{L}{W} \quad (\text{review P8})$$

→ the resistance of the channel

The S model neglects R_{ON} . A more realistic model takes R_{ON} into account, and we call it the SR model:

$$V_{GS} \geq V_T \Rightarrow \text{ON state}$$



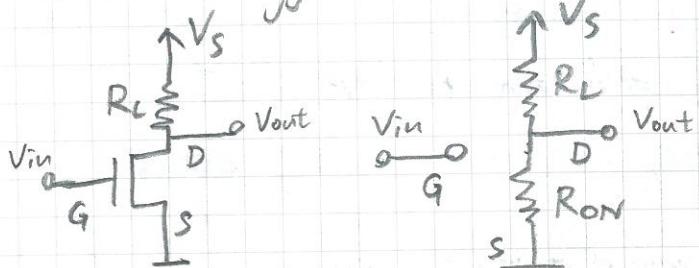
$$i_{DS} = \frac{V_{GS}}{R_{ON}}$$

(the OFF state is same as that in the S model)

P66 * Analyzing a MOSFET circuit using the SR model

Taking into account the impact from R_{ON} , the analysis may seem complicated, but it is still based on what we've learned so far.

Let's analyze an inverter to illustrate this:

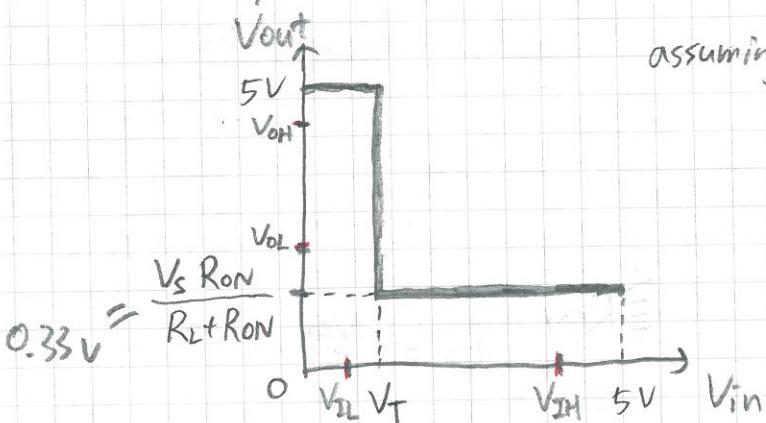


Because now we consider R_{ON} , the output voltage V_{out} will not be zero when $V_{GS} \geq V_T$; instead,

$$V_{out} = V_s \times \frac{R_{on}}{R_L + R_{on}}$$

following the voltage-divider relationship.

And thus the transfer characteristic will become as follows (compare to one on P57) :



assuming $V_s = 5V$

$V_T = 1V$

$R_{on} = 1k\Omega$

$R_L = 14k\Omega$

This change of V_{out} value has at least two impacts:

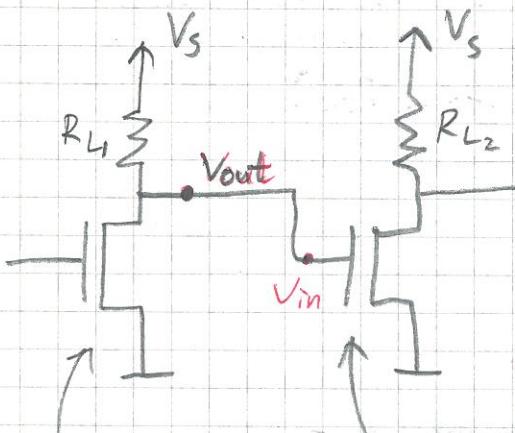
① To meet the static discipline, V_{out} needs to be lower than V_{OL} ; otherwise, we may observe that even though the magnitude of noise is within the specified margin, the value of V_{in} still exceeds V_{IL} .

→ solution A: redesign to reduce V_{out} .
(from the manufacturer's viewpoint)

→ solution B: replace by a compatible device.
(from the consumer's viewpoint)

② To drive another MOSFET, V_{out} in ON state needs to be lower than V_T , since V_{out} in OFF state equals $V_s \geq V_T$.

usually



the driving
MOSFET

(As another example,
see the AND gate)
on P4

the MOSFET whose ON/OFF state
is driven by the preceding MOSFET.

P68

study Examples 6.5 and 6.6 in the textbook.

In general, since $V_{out} = V_s \times \frac{R_{on}}{R_L + R_{on}}$,

to change V_{out} , we may

- { ① change V_s
- ② change R_L
- ③ change R_{on}

① \Rightarrow as a side-effect, it will also
change V_{out} in OFF state!

② \Rightarrow

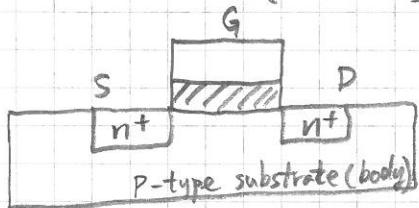
- { larger resistance is hard to achieve in VLSI;
- larger resistance would cause nontrivial
voltage drop in the presence of leakage current;

③ \Rightarrow may be achieved by changing the W/L ratio
of the MOSFET.

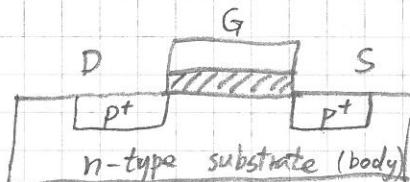
\Rightarrow An engineer's job often involves in finding
the best solution among "multiple" options !!

A note on different types of MOSFETs:

n channel (NMOS)

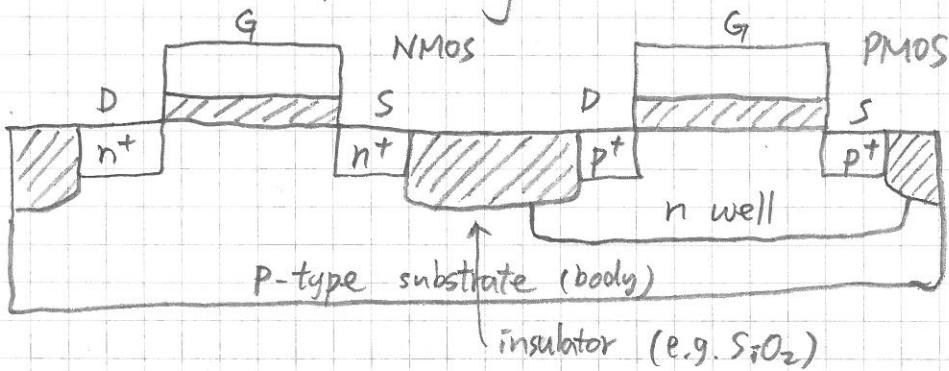


p channel (PMOS)



Generally speaking, x channel MOS means that when some appropriate voltage is applied between the gate and the source (i.e., V_{GS}), a "channel" of x -type will be formed between the source and the drain.

CMOS (Complementary MOS)



- enhancement-type vs. depletion-type;

Enhancement-type MOSs do not have a channel between S and D by default, and a channel may be induced by some V_{GS} ; depletion-type MOSs have a physically implanted channel, and the channel may be reduced by some V_{GS} .

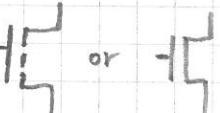
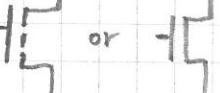
Viewing a MOS as a switch, we may say that an enhancement-type MOS is normally open (i.e., OFF) and that a depletion-type MOS is normally closed (i.e., ON).

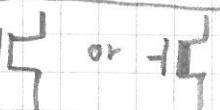
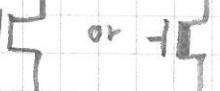
Here is a summary of MOSFET symbols:



In variation #1, the arrow indicates the polarity of the body and the channel ($+ \rightarrow -$); in variation #2, the arrow indicates the normal current direction (\vec{i}), and the arrow mark is placed at the source terminal.

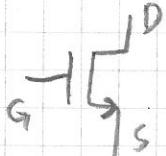
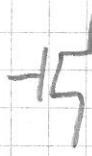
Another popular notation:

enhancement  or 

depletion  or 

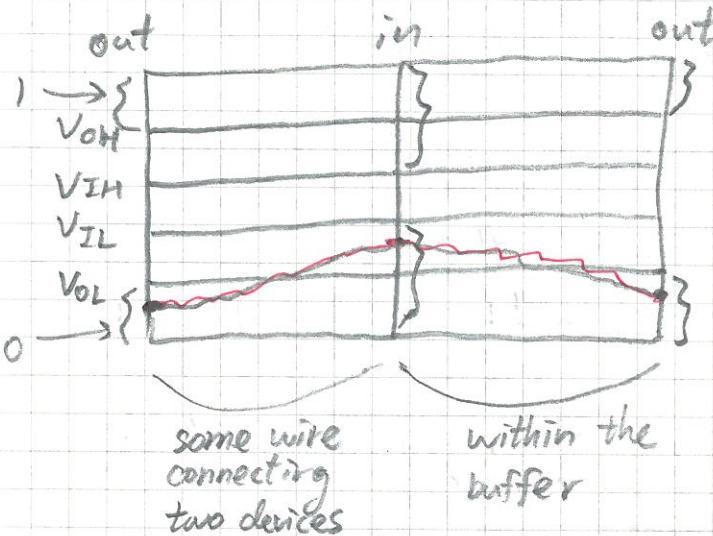
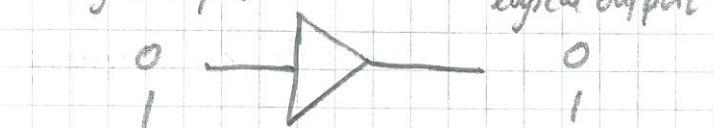
variation 1 2

* In this course, we focus on the n-channel enhancement-type MOSFETs, and we draw

 or simply 

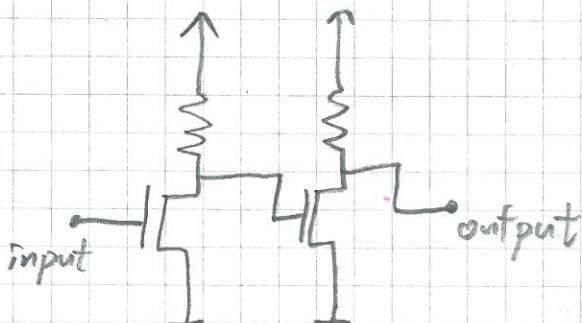
* Signal restoration, gain, and nonlinearity Pg 1
(Section 6.9 in the textbook)

We may use a device called "buffer" to restore a distorted signal (distorted by noise, logical input logical output for example):



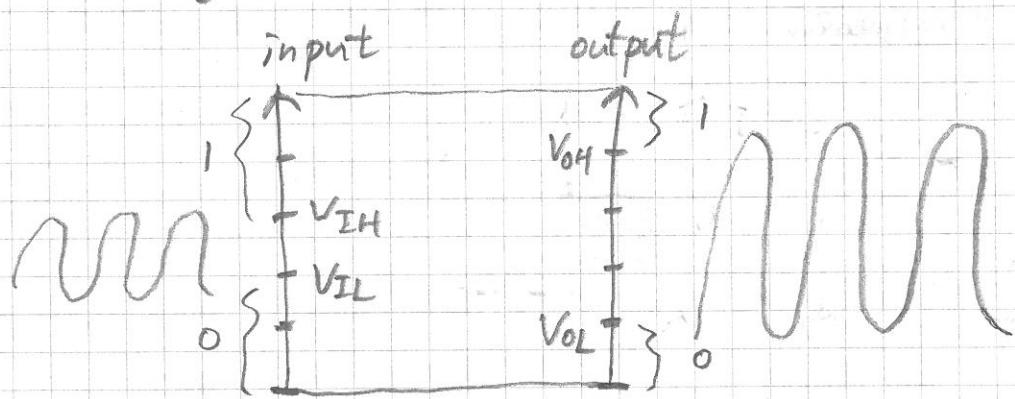
As long as the system follows the static discipline, the output of a buffer will be closer to the output of the device that precedes it.

A rough implementation
of a buffer:



P72

In order for an electronic device to satisfies the system's specification of the static discipline, it turns out that such a device must be capable of amplifying an input signal. Why? Because a signal may fluctuate between logic 0 and logic 1:



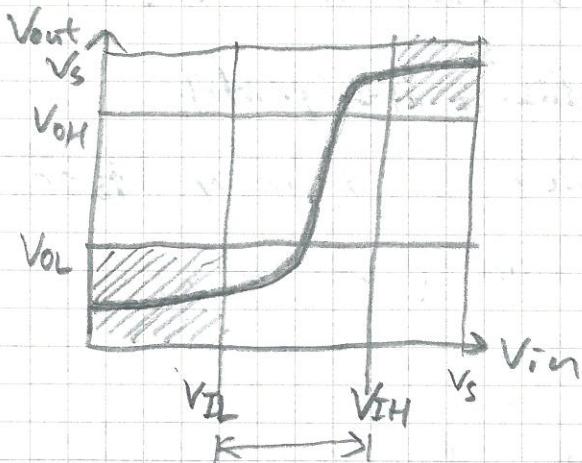
Definition :

$$\text{Gain} = \frac{V_{OH} - V_{OL}}{V_{IH} - V_{IL}} \quad \text{for } V_{IL} \rightarrow V_{IH} \text{ transition}$$

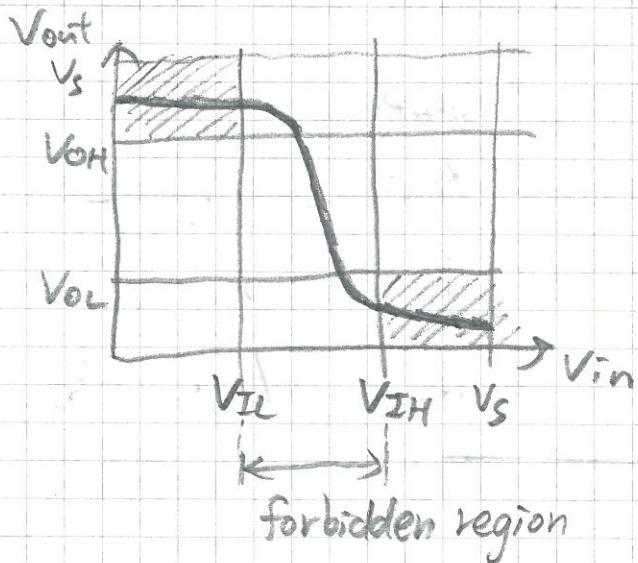
for example, if $V_{OH}=4$, $V_{IH}=3$, $V_{IL}=2$, $V_{OL}=1$

then the gain is 3.

The transfer characteristic of a buffer:



The transfer characteristic of an inverter:



In both cases, the shaded region represents the valid region for the transfer curve.

Note that since $V_{OL} < V_{IL}$ and $V_s - V_{OH} < V_s - V_{IH}$, the magnitude of the slope of the curve in the valid region is smaller than 1.

P74

So far, our discussion of MOSFET

include its behavior as a switch

(S model) as well as its linear $V_{DS} - I_{DS}$

relation in its ON state (SR model, with

resistor R_{ON}).

In science and engineering, we are often

curious about how a system would behave

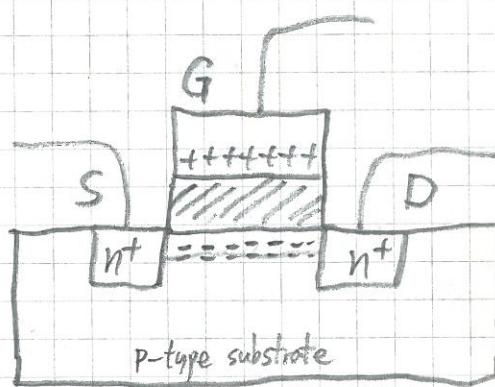
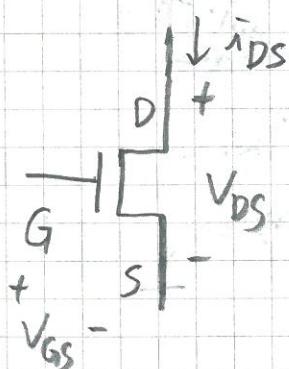
should we increase/decrease the value
of a certain parameter. We have

seen that a MOSFET will go from OFF state

to ON state as we increase voltage V_{GS} .

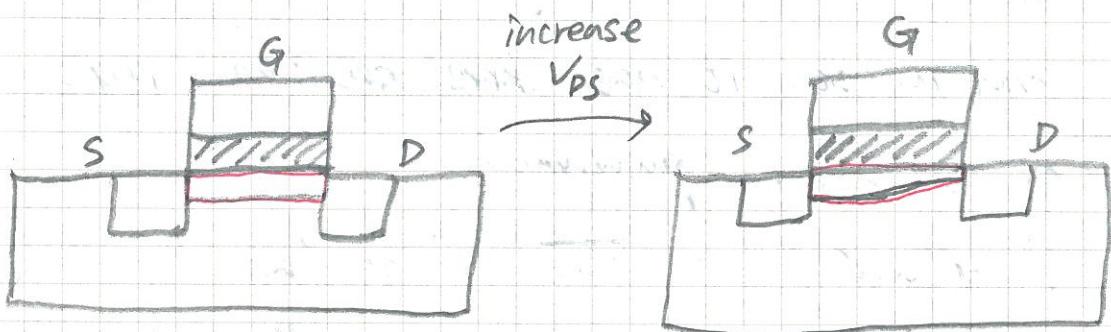
Now, let's consider what will happen

if we gradually increase V_{DS} .

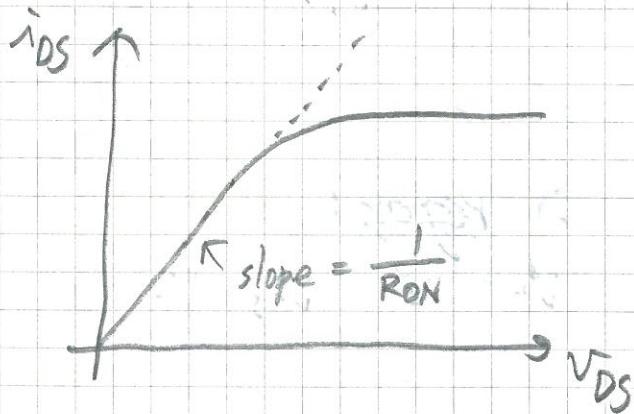


As we increase V_{DS} , the difference in electrical potentials between G and D decreases. → 電位差

This in turn will reduce the amount of free electrons near D, essentially shrinking the thickness of the conductible channel near D :



Therefore we will see a bending of the curve on the i_{DS} - V_{DS} plot :

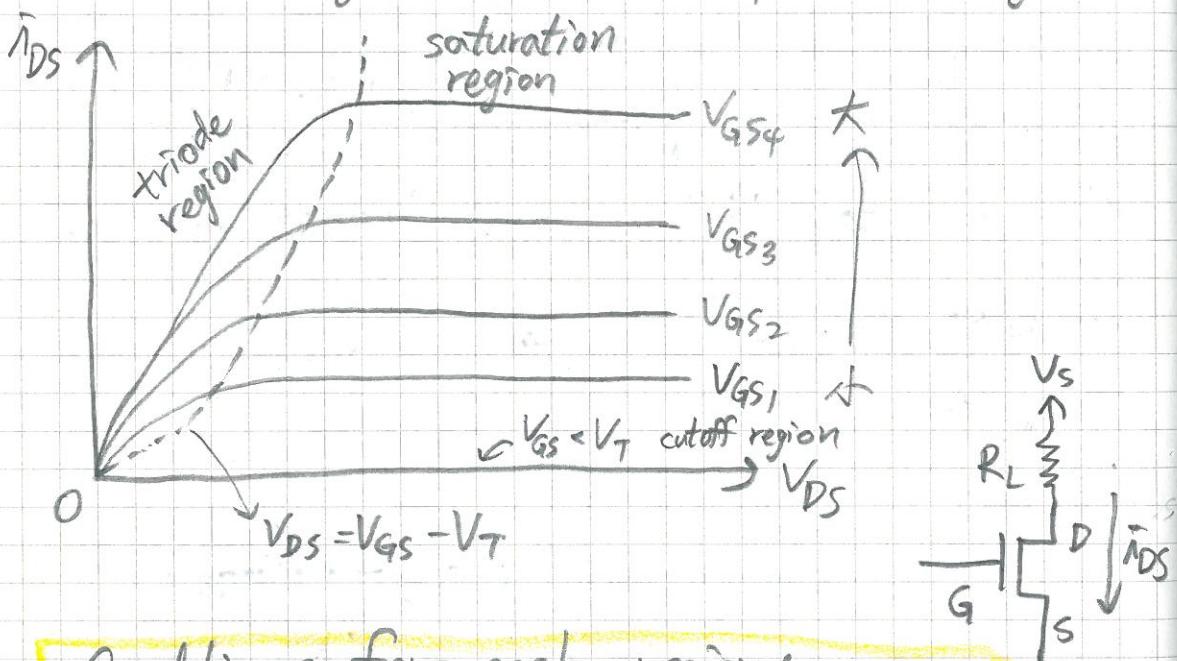


電阻截面積 ↓
 \Rightarrow 電阻值 ↑
 \Rightarrow slope ↓

P76 *The SCS model of a MOSFET switch-current source

Compared to the S model and the SR model, the SCS model is a more accurate MOSFET model (closer to the real physical characteristic).

In the SCS model, a MOSFET can operate with three very different behaviors, and we say it has three "operational regions".



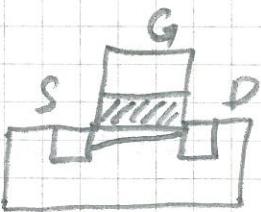
Conditions for each region:

{ cutoff : $V_{GS} < V_T$, i.e., $\underline{V_{GS} - V_T} < 0$

{ saturation : $0 \leq \underline{V_{GS} - V_T} \leq V_{DS}$

{ triode : $V_{DS} < \underline{V_{GS} - V_T}$

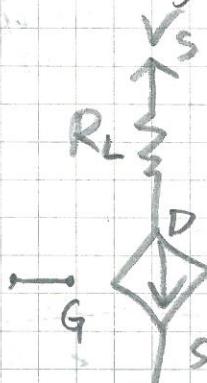
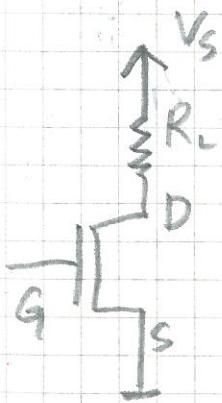
In the saturation region, current i_{DS} would stay the same as we keep increasing V_{DS} , because the channel between source and drain has become stable :



(review P58, P66-67)

Therefore, we may consider such a behavior as if there is a "current source" (P13)

But i_{DS} would depend on voltage V_{GS} , and thus we say it is like a "voltage-controlled current source". symbol:



$$i_{DS} = f(V_{GS})$$

$$= \frac{K(V_{GS} - V_T)^2}{2}$$

(according to physics)

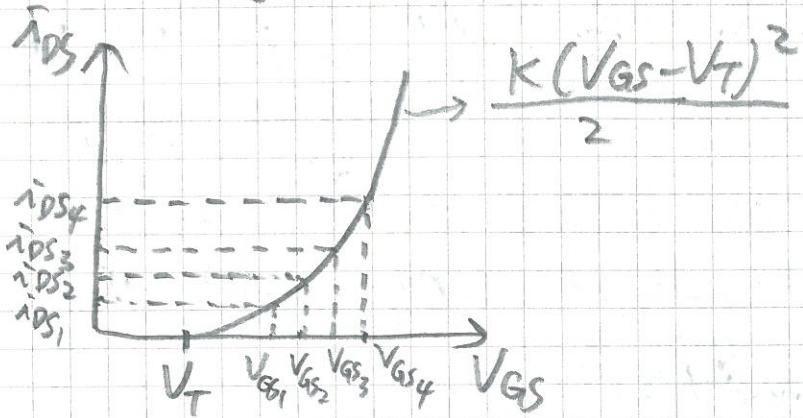
Note that K is a coefficient;

unit: mA/V^2

if $0 \leq V_{GS} - V_T \leq V_{DS}$
→ in saturation region

Pn8

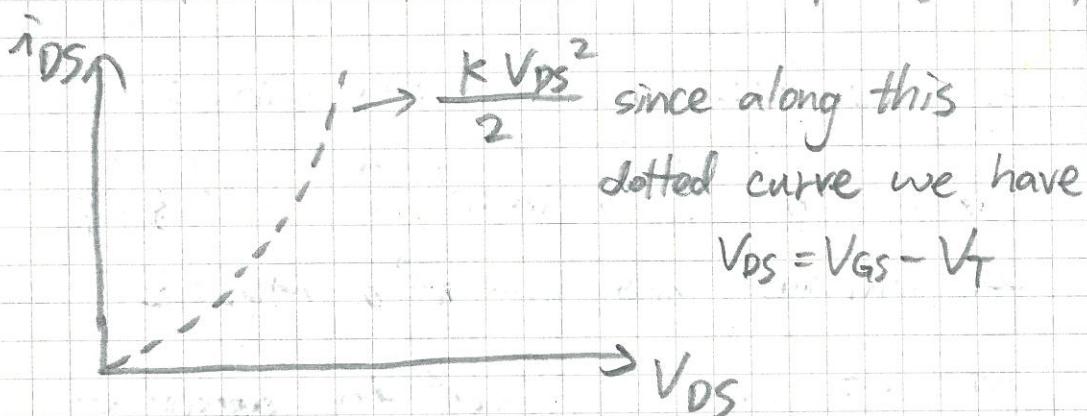
Plotting the i_{DS} - V_{GS} relation, we see:



$$\begin{aligned}\text{Therefore, if } |V_{GS_1} - V_{GS_2}| &= |V_{GS_2} - V_{GS_3}| \\ &= |V_{GS_3} - V_{GS_4}|\end{aligned}$$

we will have

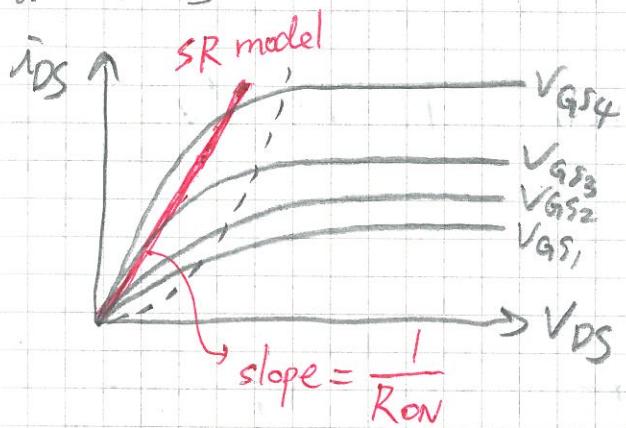
$$|i_{DS_1} - i_{DS_2}| < |i_{DS_2} - i_{DS_3}| < |i_{DS_3} - i_{DS_4}|$$



$$\Rightarrow i_{DS} = \begin{cases} 0 & \text{for } V_{GS} < V_T \\ \frac{K(V_{GS}-V_T)^2}{2} & \text{for } 0 \leq V_{GS}-V_T \leq V_{DS} \end{cases}$$

From the aspect of the SCS model,
 the R_{ON} in the SR model can be thought
 of as a piecewise approximation of the
 $i_{DS} - V_{DS}$ relation in the triode region in (P39) (Section 4.4
in textbook)

the SCS model :



We may summarize in this way :

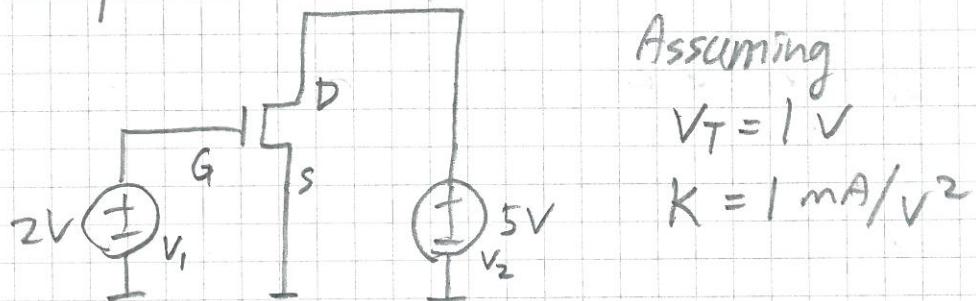
The S model is good for analyzing
 the ON/OFF behavior of a MOSFET.

The SR model is good for approximating
 the MOSFET's behavior in the triode
 region.

P80 Let's work on two examples to

get ourselves familiar with the SCS model
and its analysis:

Example 1.



Assuming

$$V_T = 1 \text{ V}$$

$$K = 1 \text{ mA/V}^2$$

① $i_{DS} = ?$ since $V_{GS} = 2 \text{ V} > V_T$
and $V_{DS} = 5 \text{ V} > V_{GS} - V_T = 1$,

we see the MOSFET is operating in the

saturation region $\Rightarrow i_{DS} = \frac{K(V_{GS}-V_T)^2}{2} = 0.5 \text{ mA}$

② If we keep V_1 and decrease V_2 , at what condition
of V_2 will the MOSFET enter the triode region?

\rightarrow As long as $V_{DS} \geq V_{GS} - V_T$ the MOSFET
will stay in the saturation region

\rightarrow if $V_2 = V_{DS} < V_{GS} - V_T = 1 \text{ V}$, the MOSFET will
enter the triode region.

③ If we keep V_2 , what would be the range of V_1 for the MOSFET to stay in the saturation region?

$$\rightarrow V_{DS} \geq V_{GS} - V_T$$

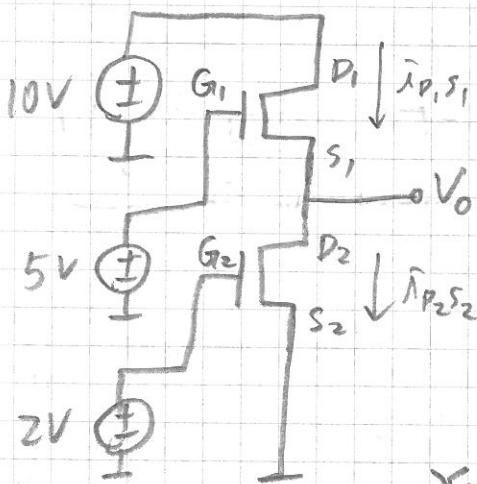
$$\rightarrow 5 \geq V_{GS} - 1 \rightarrow V_{GS} \leq 6$$

$$\rightarrow V_1 = V_{GS} \leq 6 *$$

Example 2.

Assuming $V_T = 1$ V

$$k = 4 \text{ mA/V}^2$$



If $V_0 = 3$, are both MOSFETs operating in the saturation region?

Yes, since $V_{G_1S_1} - V_T$

$$= (5-3) - 1 = 1 > 0$$

$$\text{and } \underbrace{V_{G_1S_1} - V_T}_{1} \leq \underbrace{V_{DS_1}}_{10-3=7}$$

$$\text{And } V_{G_2S_2} - V_T = 2 - 1 = 1 > 0$$

$$\text{and } \underbrace{V_{G_2S_2} - V_T}_{2-1} \leq \underbrace{V_{DS_2}}_{3}$$

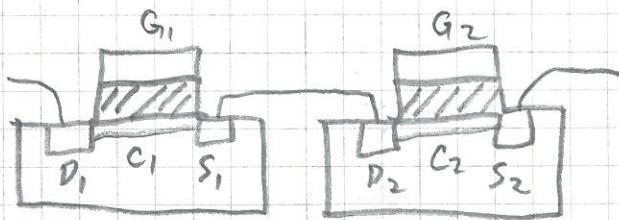
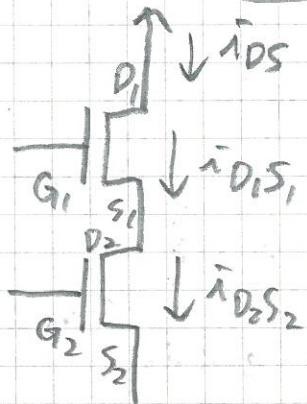
And $V_{G_1S_1} = V_{G_2S_2}$ correctly implies $i_{D_1S_1} = i_{D_2S_2}$ ✘

P82 (Note on 2020/6/1 3PM :

I apologize for the confusion regarding whether $V_0=2$ in Example 2 makes sense or not.

After some more thoughts, I think what I said this morning was wrong.

It is wrong that $\bar{I}_{DS} = \bar{I}_{D_1S_1} + \bar{I}_{D_2S_2}$



In physics, $\bar{I}_{D_1S_1}$ is constrained by channel C_1 , and $\bar{I}_{D_2S_2}$ is constrained by channel C_2 . It should be that

$$\bar{I}_{DS} = \bar{I}_{D_1S_1} = \bar{I}_{D_2S_2}.$$

And therefore $V_0=2$ does NOT make sense.)

Now let's see how we may leverage

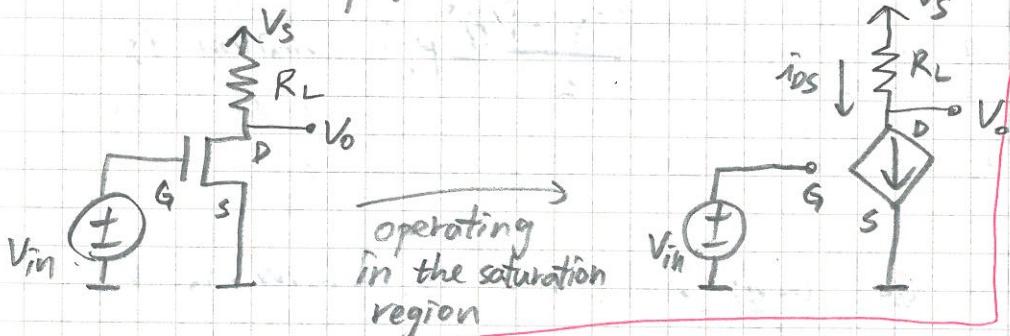
P83

a MOSFET operating in the saturation region for some useful purposes.

It turns out that in the saturation region a MOSFET may be used to amplify a signal. Amplifiers are used in many real-world applications and systems. Headphones and speakers are two examples.

Example 7.7 in the textbook

- MOSFET Amplifier, Version 1



The voltage-controlled current source gives

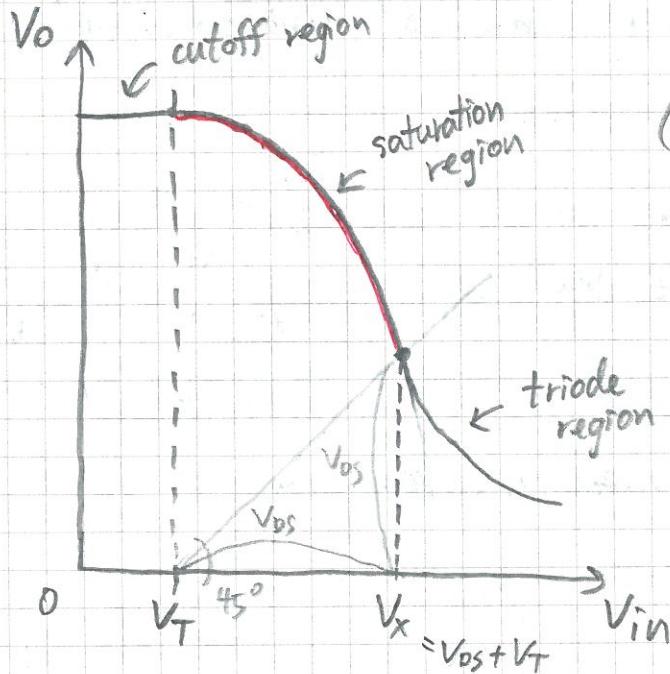
$$i_{DS} = \frac{K(V_{GS} - V_T)^2}{2} \quad (\text{see P69})$$

and in addition, by Ohm's law $i_{DS} = \frac{V_s - V_o}{R_L}$

$$\Rightarrow \left\{ \frac{K(V_{GS} - V_T)^2}{2} = \frac{V_s - V_o}{R_L} \right\} \Rightarrow V_o = V_s - K \cdot \frac{(V_{in} - V_T)^2}{2} \cdot R_L$$

(and that $V_{GS} = V_{in}$)

P84 following the equation on P75, we have



(compare this to the plots on P63 and P66)

The $V_o = V_s - \frac{K(V_{in} - V_T)^2}{2} R_L$ relation is a parabola only valid when $V_T \leq V_{in} \leq V_x$.

Later we will study how to determine V_x .
(P89)

The slope $\frac{V_o}{V_{in}}$ is also the ratio between V_{in} and V_o .

We see that in some part of the saturation region the magnitude of the slope is greater than one. Therefore we may amplify V_{in} by $\frac{V_o}{V_{in}}$ times and output the result as V_o .

The "gain" of the amplifier is defined to be $\frac{V_o}{V_{in}}$.

Using this MOSFET amplifier, however,

the output will be inverted, which might not be what we want:

$$\begin{cases} V_{in} \downarrow \Rightarrow V_o \uparrow \\ V_{in} \uparrow \Rightarrow V_o \downarrow \end{cases}$$

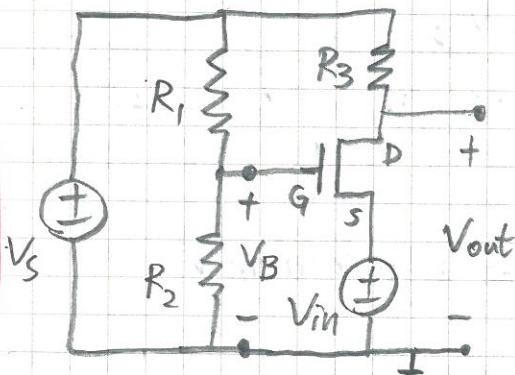
(can be verified by the sign of the slope in the $V_o - V_{in}$ plot.)

Let's study an alternative design →

example 7.12

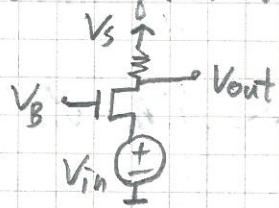
in the textbook

- MOSFET Amplifier, Version 2



In this configuration, we use one voltage source to provide voltages to the gate and the drain.

This is equivalent to :



$$V_{GS} = V_B - V_{in}$$

$$= \left(V_s \times \frac{R_2}{R_1 + R_2} \right) - V_{in}$$

Therefore, we have

$$\frac{K(V_{GS} - V_T)^2}{2} = \frac{V_s - V_{out}}{R_3}$$

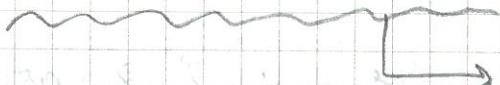
In saturation region, we have

$$I_{DS} = \frac{K(V_{GS} - V_T)^2}{2}$$

and by Ohm's law

$$I_{DS} = \frac{V_s - V_{out}}{R_3}$$

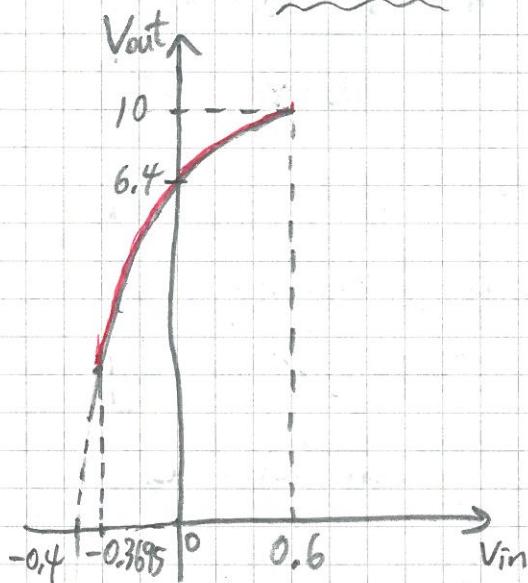
$$\frac{K\left(V_s \times \frac{R_2}{R_1 + R_2} - V_{in} - V_T\right)^2}{2} = \frac{V_s - V_{out}}{R_3}$$



P86 following the equation on P85, suppose that

$$\begin{cases} V_S = 10 \text{ V} \\ R_1 = 84 \text{ k}\Omega, R_2 = 16 \text{ k}\Omega, R_3 = 20 \text{ k}\Omega \\ V_T = 1 \text{ V}, K = 1 \text{ mA/V}^2 \end{cases}$$

then $V_B = 1.6$ and $V_{out} = 10 - 10(0.6 - V_{in})^2$



To operate in the saturation region, both of the following conditions must be satisfied:

$$\begin{cases} V_{GS} \geq V_T \quad \text{--- (1)} \\ V_{GS} - V_T \leq V_{DS} \quad \text{--- (2)} \end{cases}$$

from (1), $1.6 - V_{in} \geq 1 \Rightarrow V_{in} \leq 0.6 \text{ V}$

from (2), $(1.6 - V_{in}) - 1 \leq V_{out} - V_{in}$

$$\Rightarrow V_{out} \geq 0.6 \text{ V}$$

$$\Rightarrow 10 - 10(0.6 - V_{in})^2 \geq 0.6$$

$$\Rightarrow -0.3695 \text{ V} \leq V_{in} \leq 1.5695 \text{ V}$$

$-0.3695 \text{ V} \leq V_{in} \leq 0.6 \text{ V}$

Thus we see that

it is an amplifier ($\frac{V_{out}}{V_{in}} > 1$)

and $\begin{cases} V_{in} \uparrow \Rightarrow V_{out} \uparrow \\ V_{in} \downarrow \Rightarrow V_{out} \downarrow \end{cases}$

$\begin{cases} V_{in} \downarrow \Rightarrow V_{out} \downarrow \end{cases} *$

For the purpose of signal amplification, and for the SCS model in general, we would want to make sure that the MOSFET operates in the saturation region. Given the circuit's parameters, in order to determine whether the MOSFET will operate in the saturation region (or to determine the valid range of values of parameters), we may use the conditions listed on Pg6 or use graphical analysis.

For example, for the MOSFET amplifier on P83, suppose $V_S = 5V$, $R_L = 1k\Omega$, $V_T = 0.8V$, $V_{in} = 2.5V$
 $K = 0.5 \text{ mA/V}^2$
is the MOSFET in the saturation region?

Answer: $V_{GS} = V_{in} = 2.5V$

$$V_{DS} = V_o = V_S - K \frac{(V_{in} - V_T)^2}{2} R_L = 4.28V$$

and we see that both $\left\{ \begin{array}{l} V_{GS} \geq V_T \\ V_{DS} \geq V_{GS} - V_T \end{array} \right.$

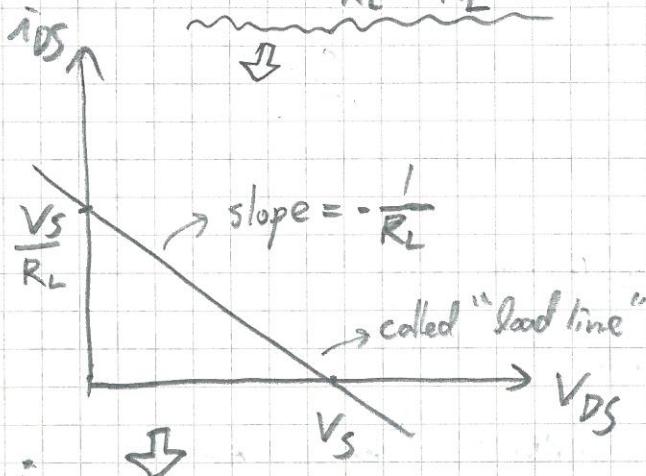
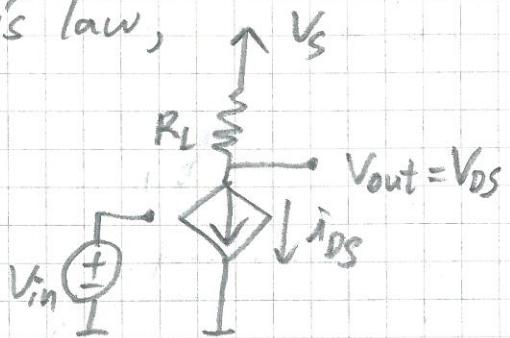
6V

Therefore, it is indeed in the saturation region. #

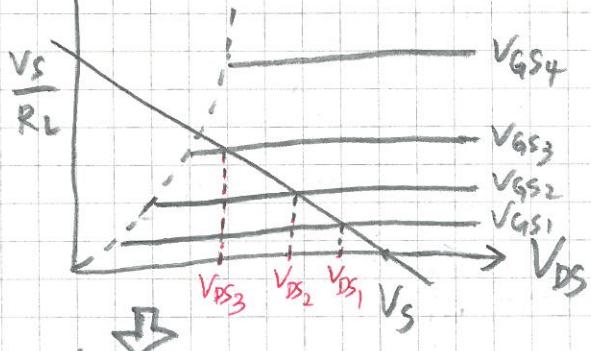
P88 Using graphically analysis, we first observe that, by Ohm's law,

$$i_{DS} = \frac{V_s - V_{DS}}{R_L}$$

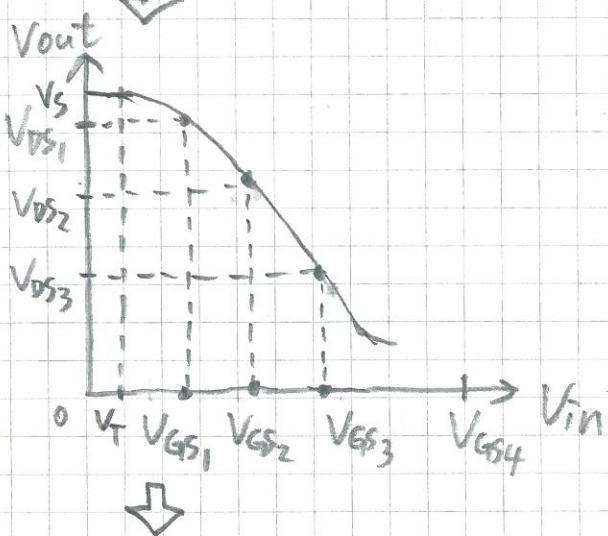
$$\Rightarrow i_{DS} = \frac{V_s}{R_L} - \frac{1}{R_L} V_{DS}$$



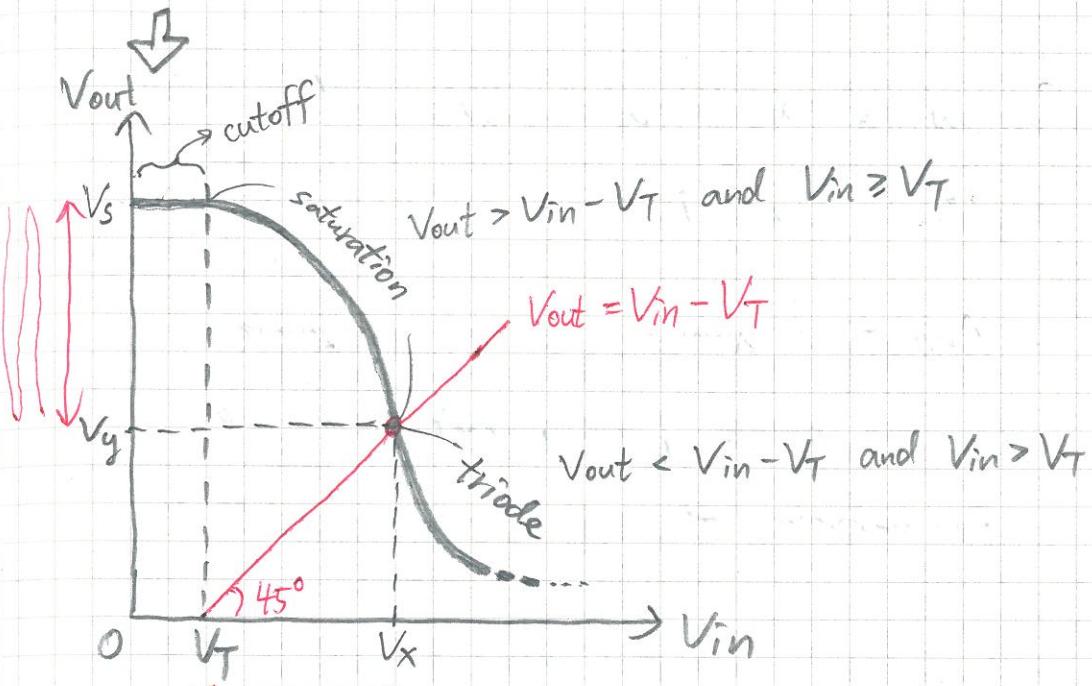
(see the plot on P76)



$$\begin{cases} V_{in} = V_{GSx} & x=1 \sim 4 \\ V_{out} = V_{DSx} & x=1 \sim 3 \end{cases}$$



(see the plot on P84)



To compute V_x , we have

$$\begin{cases} V_{out} = V_{in} - V_T \\ V_{out} = V_s - K \frac{(V_{in} - V_T)^2}{2} R_L \end{cases} \quad (\text{P83})$$

$$\Rightarrow V_{in} - V_T = V_s - K \frac{(V_{in} - V_T)^2}{2} R_L$$

$$\Rightarrow \frac{K R_L}{2} (V_{in} - V_T)^2 + (V_{in} - V_T) - V_s = 0$$

$$\Rightarrow V_{in} - V_T = \frac{-1 + \sqrt{1 + 2V_s R_L K}}{K R_L}$$

$$\Rightarrow V_x = \frac{-1 + \sqrt{1 + 2V_s K R_L}}{K R_L} + V_T$$

$$V_y = V_x - V_T$$

P90

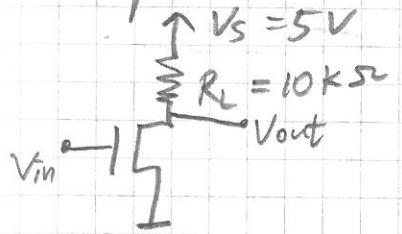
Comparing analytical analysis and graphical analysis:

- Analytical analysis is as accurate as the model (and formula) can be.

For example, the use of $i_{DS} = \frac{k(V_{GS}-V_T)^2}{2}$

- Graphical analysis may be more accurate, provided that the manufacturer of the electronic device often gives "data sheet", which includes the actual measured physical values of, for example, i_{DS} - V_{GS} characteristics.
- Graphical analysis also provides more insights (for example, see P39, 40).

Example:

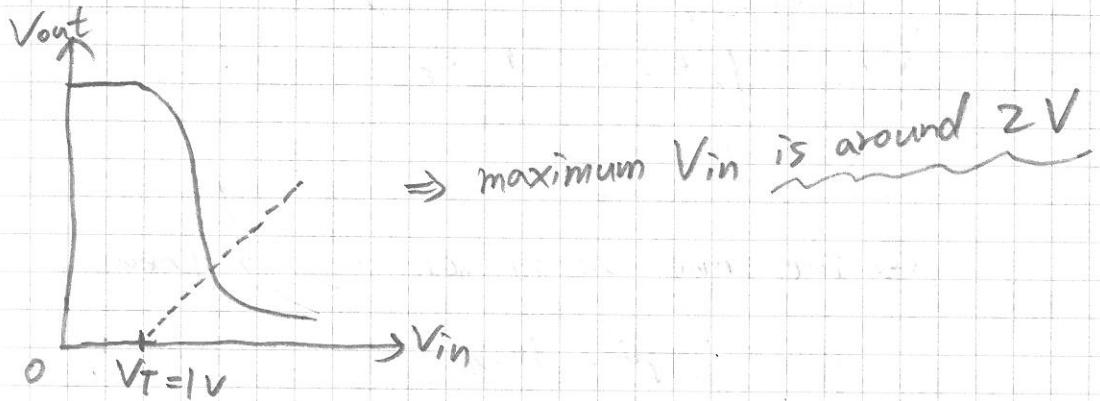


Assuming $V_T = 1\text{V}$

$$K = 1 \text{ mA/V}^2$$

What would be the maximum V_{in} that the MOSFET can still stay in the saturation region?

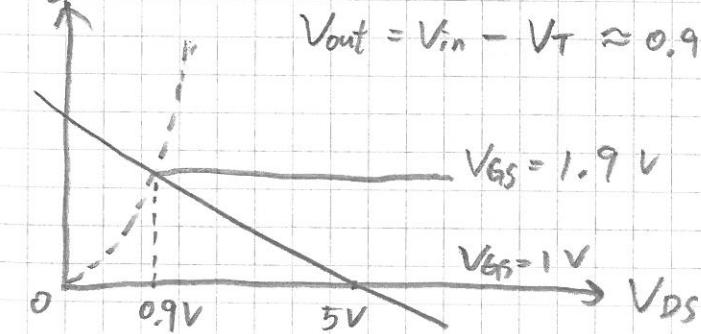
If the $V_{out} - V_{in}$ plot is accurate, we may directly estimate the maximum V_{in} :



Applying analytical analysis, we have

$$\begin{aligned} V_{in} &= \frac{-1 + \sqrt{1 + 2V_s K R_L}}{K R_L} + V_T \\ &= \frac{-1 + \sqrt{1 + 2 \times 5 \times 10^{-3} \times 10 \times 10^3}}{1 \times 10^{-3} \times 10 \times 10^3} + 1 \approx 1.9 \text{ V} \end{aligned}$$

$$V_{out} = V_{in} - V_T \approx 0.9 \text{ V}$$



P92 ★ The Small-Signal Model And Analysis

We have learned some small-signal analysis when we were studying diode circuits early this semester (P₄₅₋₅₄; Section 4.5 in the textbook).

Small-signal model and its analysis play a critical role in both design and usage of MOSFET amplifiers because

- ① many real-world input signals are small signals;
- ② the small-signal gain $\frac{V_o}{V_i}$ is "linear", which implies less signal distortion caused by the amplifier;
- ③ the circuit under the small-signal model is "linear", which means we may apply existing linear circuit analysis techniques, for example Thévenin's Theorem, to help us understand the circuit's behavior and its response to input signal.

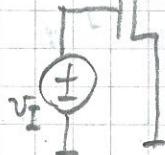
It is important to remember that small-signal model is an "approximation" of the original non-linear model. In practice, it may be needed to estimate the quality of such an approximation. That part is beyond the scope of this course, but you may study P218 in the textbook to get some idea. It is interesting to recall that the original non-linear model is itself an approximated description of how a circuit behaves in the real world.

In essence, the small-signal model describes how a circuit responds to a small, time-varying signal. The large, time-invariant signal is used to determine the operating point around which the small signal oscillates. Take our familiar MOSFET amplifier, for example:

notation convention!

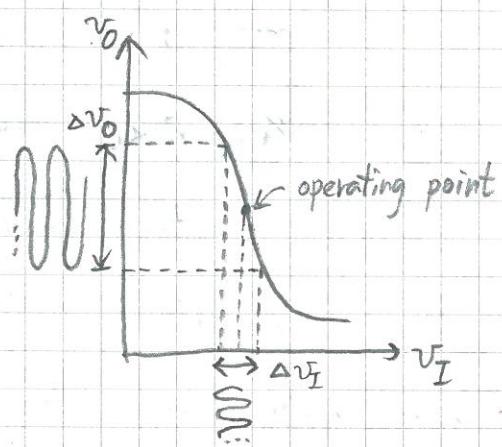
$$V_I = V_I^+ + V_I^-$$

total input
 ↓
 large signal
 ↓
 small signal



$$V_S$$

↓
 R_L
 ↓
 V_O



P94

The large, time-invariant signal in this case is called the DC bias, or the DC offset.

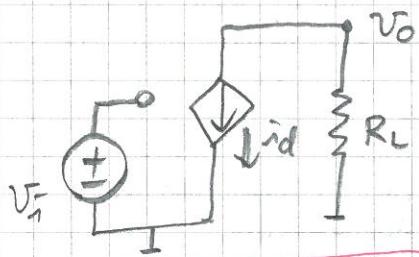
直流偏壓

In the small-signal model, we consider that all of the independent current sources \oplus and independent voltage sources \oplus are shut off, because their impact to the circuit does not change along with the small signal; in other word, their impact to the circuit is time-invariant.

In this sense, the voltage-controlled current source \downarrow is not considered shut off for our MOSFET amplifier, because the current depends on the small-signal $V_{\tilde{i}}$ (and depends on the large-signal V_i , too).

The small-signal model of our MOSFET amplifier:

on Page 93



trans : input to output
conductance : $\frac{\text{current}}{\text{voltage}}$

where $i_d = g_m \cdot V_{\tilde{i}}$,

and $g_m = k(V_i - V_T)$

is called the

incremental transconductance.
(or simply, transconductance)

* This result applies to other circuits as well, as long as the MOSFET operates in saturation.

To obtain the small-signal gain $\frac{V_o}{V_i}$, we may apply linear circuit analysis : P95

$$\frac{V_o - 0}{R_L} = -i_d$$

$$\Rightarrow V_o = -i_d \cdot R_L = -(g_m \cdot V_i) \cdot R_L$$

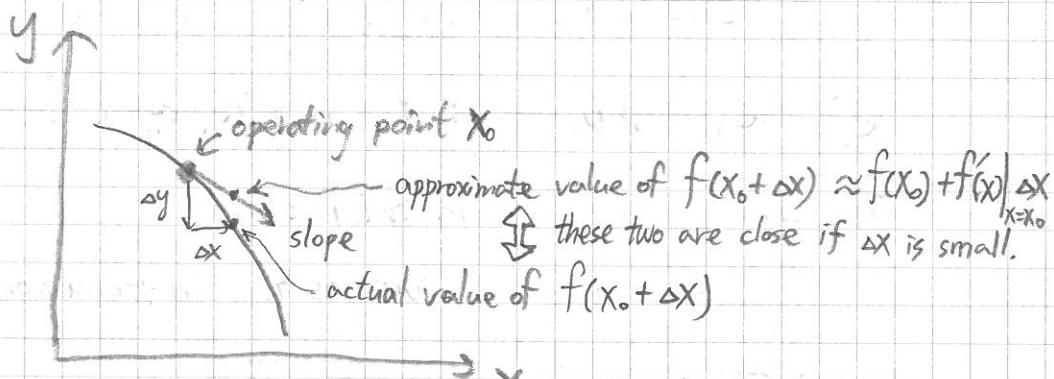
$$\Rightarrow \frac{V_o}{V_i} = -g_m \cdot R_L$$

and the magnitude of the gain is

$$\left| \frac{V_o}{V_i} \right| = g_m R_L$$

In class, we've used the Taylor series expansion to derive that relation $i_d = K(V_I - V_T)V_i$.

The take-home message of such a derivation is that we may, in general, compute the first derivative ($=$ 1st 微分) of a function at the operating point to get a reasonable small-signal relation :



P96

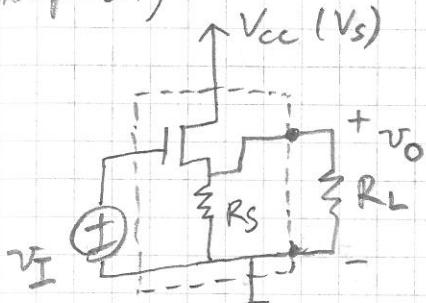
both

Study textbook Chapter 8 and Homework for some examples of the small-signal analysis and the model. Finally, from a practical viewpoint, we may summarize four criteria regarding how to select an appropriate operating point, or in other word, how to set a proper DC bias:

- ① determine the range of input signal V_I such that the MOSFET would operate in the saturation region; (P356 in textbook)
- ② if the large-signal is also time-varying, we may want to maximize the peak-to-peak swing of the input signal by setting the operating point at the middle of the valid range of V_I for operation in the saturation region; (review P351 and P369 in the textbook)
- ③ the magnitude of the small-signal gain;
- ④ driving another MOSFET circuit; (P61 in this note; Questions 4, 5 in Homework 4) (Section 8.2.3 in the textbook)

in the Spring 2020
semester

Example: Small-Signal Analysis of
(Textbook) A Source Follower Circuit
example 8.4)

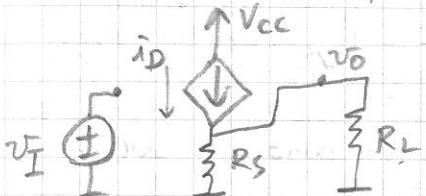


We've seen this circuit

before (Example 7.8)

← consider this resistor as a load attached at the output port.

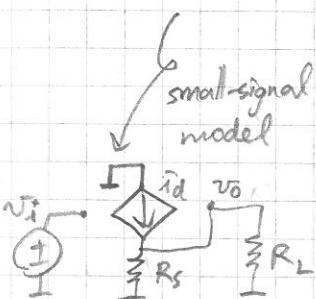
↙ MOSFET operates in saturation



$$\begin{aligned} \text{recall that } id &= \frac{1}{2} k(V_{GS} - V_T)^2 \\ &= I_D + id \\ &= \frac{1}{2} k(V_{GS} - V_T)^2 + k(V_{GS} - V_T)v_{GS} \end{aligned}$$

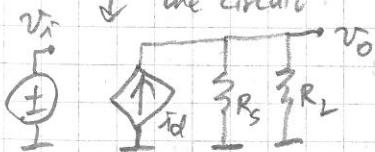
interpret the result as these

an approximation from Taylor expansion



$$id = k(V_{GS} - V_T) \cdot v_{GS} = g_m (v_i - v_o)$$

↙ rearrange the circuit $\Rightarrow v_o = id \cdot (R_s // R_L)$ ← node analysis



$$= g_m (v_i - v_o) (R_s // R_L)$$

$$\Rightarrow \text{small-signal voltage gain } \frac{v_o}{v_i} = \frac{R_s g_m}{R_s + R_L + R_s g_m}$$

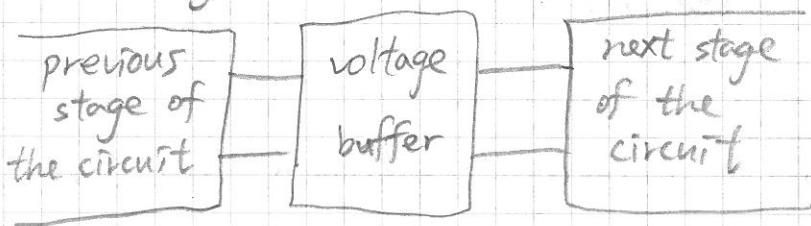
⇒ if g_m is large, we have

you may wonder $\frac{v_o}{v_i} \approx 1$ which is why we call
why we ever want this unity-gain circuit.
Read on to learn more.

this a source follower.

Furthermore, → next page

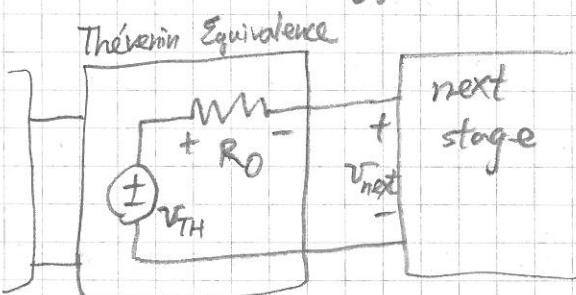
A source follower can be used as a "voltage buffer", ^① to provide a unity gain of voltage to the next stage of the complex circuit, ^② to provide a smaller internal resistance viewed from the next stage, and ^③ to provide a larger internal resistance viewed from the previous stage.



* Why is the view of internal resistance important?

Reason 1. From the view of the next stage, having a smaller internal resistance makes less energy waste:

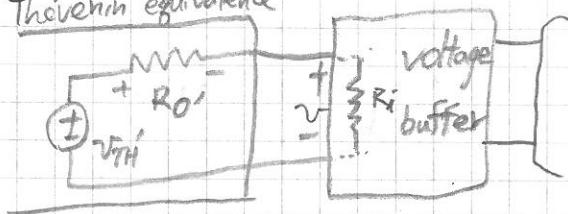
R_O : resistance viewed from the output



From the concept of voltage divider, we see that R_O & then v_{next} will be closer to V_{TH} .
(Think about internal resistance as Thevenin resistance)

Reason 2. From the view of the previous stage, having a larger internal resistance could also make less energy waste:

Thevenin equivalence



R_i : resistance viewed from the input.

Also, from the concept of voltage divider, we see that V will be closer to V_{TH} if $R_i \gg R_0'$

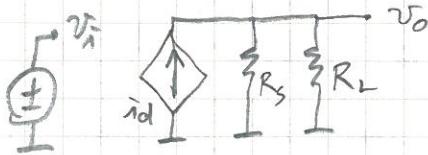
Now, let's see what is R_o and R_i of the source follower circuit!

Note that from now on, we'll call R_o the output resistance, r_o the small-signal output resistance, R_i the input resistance, and r_i the small-signal input resistance.

Since the need for this input/output resistance notion arises from the practical concern, we may attach a testing voltage V_{test} at the input/output port and get its current response, say i_{test} , and the resistance can be calculated by $R = \frac{V_{test}}{i_{test}}$

P100

The small-signal source follower circuit is



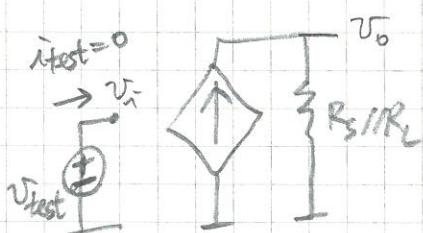
and now we're going to find r_i and r_o .

r_i and r_o are important parameters, not only because of energy efficiency, but also because small signals themselves are represented by voltage variations. With a high r_i and a low r_o we may better preserve the signal. In this regard, energy waste can be thought of as signal distortion.

physically

Now, to find r_i , we first note that there's an insulating layer between the gate and the rest of the MOSFET. Therefore, there's no current flowing through the gate insulation.

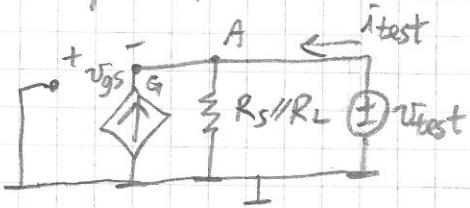
Thus, if we attach v_{test} to the gate-ground (i.e., the input port) we see $i_{test} = 0$.



This implies $r_i = \frac{v_{test}}{i_{test}} = \infty$

Similarly, $R_i = \infty$

Now, to find r_o , we attach v_{test} at the output port of the source follower:



Applying the node analysis at A, we have

$$(i_d + i_{test})(R_s \parallel R_L) = v_{test}$$

$$\Rightarrow (g_m v_{gs} + i_{test})(R_s \parallel R_L) = v_{test}$$

$$\Rightarrow (g_m \cdot (-v_{test}) + i_{test})(R_s \parallel R_L) = v_{test}$$

$$\Rightarrow i_{test}(R_s \parallel R_L) = v_{test}(1 + g_m(R_s \parallel R_L))$$

$$\Rightarrow r_o = \frac{v_{test}}{i_{test}} = \frac{R_s \parallel R_L}{1 + g_m(R_s \parallel R_L)} \approx \frac{1}{g_m}$$

If g_m is large, then r_o is small.

We can increase g_m by increasing the DC bias voltage and/or by widening the channel between the drain and the source, because $g_m = k'(V_{GS} - V_T)$

$$= k' \left(\frac{W}{L} \right) (V_{GS} - V_T)$$

Thus, we now see that

the source follower circuit is good for voltage buffering because it has large r_i and may have small r_o and have an unity voltage gain. (To be continued \downarrow)

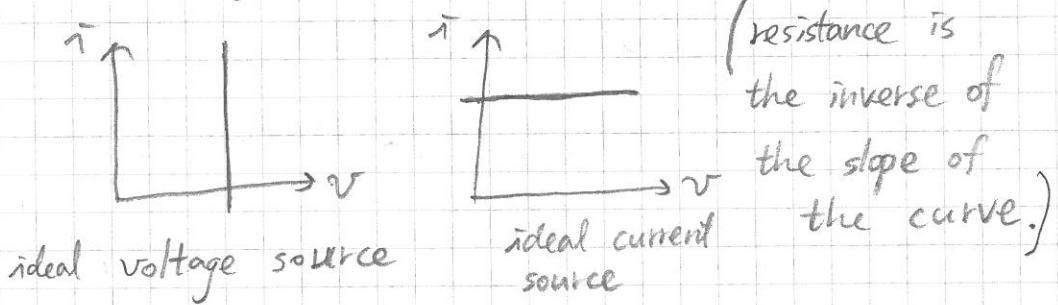
P1.02

When calculating the input/output resistance, it is important that we suppose all "independent" voltage sources and current sources are turned off:



are turned off:

Why? Because those voltage/current sources theoretically have $i-v$ characteristics that resemble zero/ ∞ resistance:

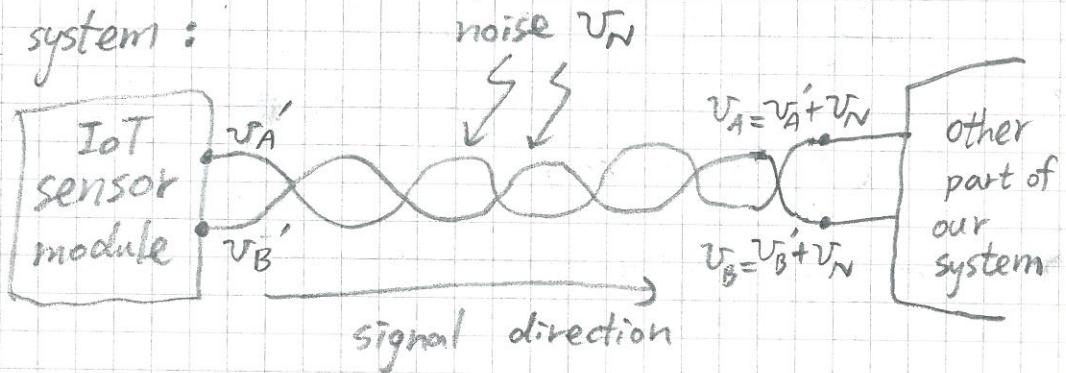


For "dependent" voltage sources and current sources, however, we do not always ^{suppose} they are turned off, because their values depend on others.

Finally, for small-signal r_i and R_o , we will still consider the corresponding DC biases because it is their existence that determines the structure of the small-signal circuit.

* Difference Amplifier

The physical environment in which an electronic system operates often introduces noises to the system :

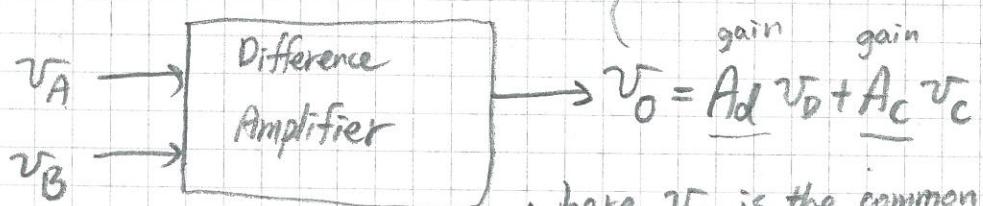


It would be good if we can ^①reduce the noise and ^②amplify the original signal.

Difference amplifiers are designed for this purpose.

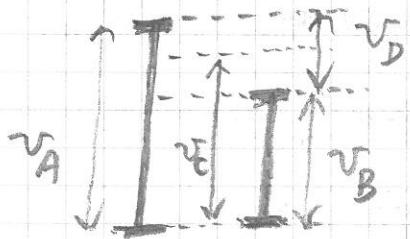
(Note that the two wires are twisted together so that the same amount of noise infects both signals, which then can be reduced by having the amplifier reduce ^{the magnitude of} the common component of the signals.)

think about it as
 $v_o = f(v_b, v_c)$



where v_C is the common component shared by v_A, v_B and v_D is the difference between v_A, v_B

We can visualize the setup this way:



$$V_D = V_A - V_B$$

is called the difference-mode component signal.

$$V_C = (V_A + V_B)/2 \text{ is called the common-mode component signal.}$$

Therefore, we may represent V_A and V_B in terms of V_C and V_D :

$$\begin{cases} V_A = V_C + \frac{1}{2}V_D \\ V_B = V_C - \frac{1}{2}V_D \end{cases}$$

A reason to choose $V_C = (V_A + V_B)/2$ is just for convenience; as long as the quantity $V_A - V_B$ is represented by some variable (V_D here), we may choose other linear combination of $\{V_A, V_B\}$ for V_C .

It is convenient if we choose V_C such that $V_A \leq V_C \leq V_B$. Otherwise, for example, letting

$V_C = V_B - \alpha V$, then we'll have

$$\begin{cases} V_A = V_C + \alpha V + V_D \\ V_B = V_C - \alpha V \end{cases} \text{ which is ugly.}$$

Going back to equation $V_o = A_D V_d + A_c V_c$, P105

A_D is called the difference-mode gain, and

A_C is called the common-mode gain.

Our goal is to design a circuit so that

A_D is large and A_C is small (relatively),
to reduce noise.

In particular, people use $\frac{A_D}{A_C}$ to quantify
the ability of the circuit to reject noise.

$\frac{A_D}{A_C}$ is called Common-Mode Rejection Ratio.
(CMRR)

We are in particular interested in the small-signal behavior of a difference amplifier. To be specific, represent the small-signal output

as $V_o = \underbrace{A_D V_d}_{\frac{V_o}{V_d}} + \underbrace{A_c V_c}_{\frac{V_o}{V_c}}$, and we want a large A_D and a small A_C.

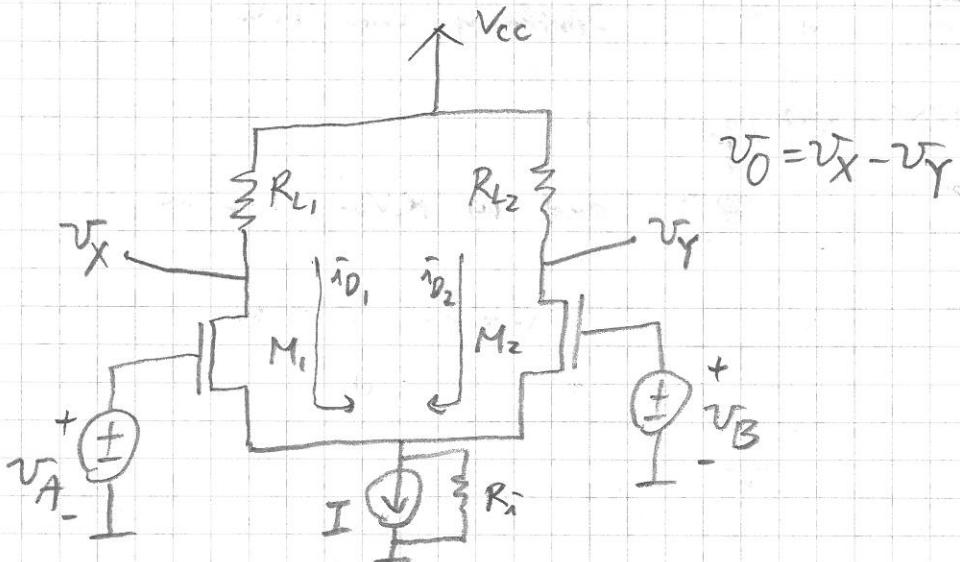
Represent the small-signal inputs as

$$\begin{cases} V_a = V_c + \frac{1}{2} V_d \\ V_b = V_c - \frac{1}{2} V_d \end{cases}$$

Conventionally, people use superposition to consider each mode separately. We'll do this in the following pages.

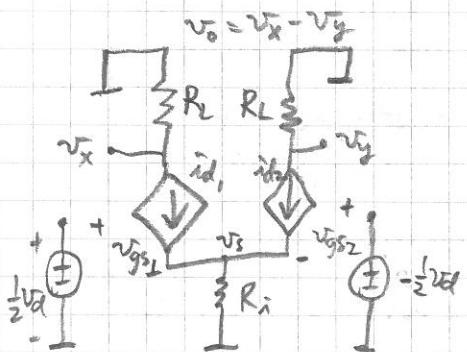
Prob

First of all, the overall structure:



Assuming symmetry, that is, $R_{L1} = R_{L2}$ and MOSFETs $M_1 = M_2$. And assume R_i is very large. Also, assume that both M_1 and M_2 operates under the saturation discipline.

Difference Mode (small signal):



$$\text{We have } id_1 = g_m \cdot v_{gs1},$$

$$id_2 = g_m \cdot v_{gs2}$$

$$\text{from KVL, we may replace } v_{gs1}, v_{gs2}$$

$$\Rightarrow id_1 = g_m \left(\frac{1}{2} v_d - v_s \right)$$

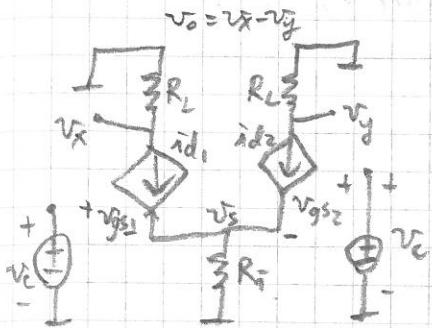
$$id_2 = g_m \left(-\frac{1}{2} v_d - v_s \right)$$

$$v_o = v_x - v_y = -id_1 R_L + id_2 R_L$$

$$= -g_m v_d \cdot R_L$$

$$\Rightarrow Ad = \frac{v_o}{v_d} = -g_m \cdot R_L$$

Common Mode (small signal):



Similarly, we have

$$\begin{cases} id_1 = g_m \cdot v_{gs1}, \\ id_2 = g_m \cdot v_{gs2} \end{cases}$$

and by KVL, we get

$$\begin{cases} id_1 = g_m (v_c - v_s) \\ id_2 = g_m (v_c - v_s) \end{cases}$$

$$v_o = v_x - v_y = -id_1 R_L - (-id_2 R_L) = 0$$

$$\Rightarrow A_c = \frac{v_o}{v_c} = 0$$

Therefore, with $A_d = -g_m R_L$ and $A_c = 0$, we see

$$v_o = A_d v_d + A_c v_c = -g_m R_L v_d$$

which means that, from small-signal viewpoint, the output of the difference amplifier is effectively the amplified version of the signal difference between input v_a and v_b , and with no component from the common component among v_a and v_b !

Next, let's analyze the ^{small-signal} output resistance of this difference amplifier (the input resistance is ∞ , since

To do so, we in the follow page use

Thévenin's Theorem to transform the circuit in each mode, and then use superposition to jointly calculate the output resistance.

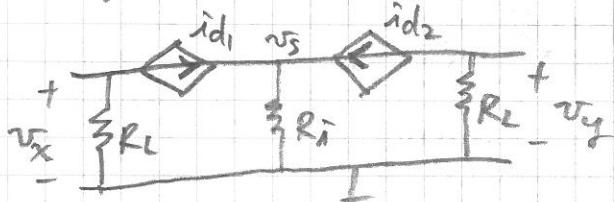
Note that this is a good strategy because \rightarrow

$\frac{V_o}{I_d}$
insulation

Prob

often Thévenin's Theorem simplifies the circuit, and, more importantly, Thévenin's Theorem will yield the same structure of the circuit, which makes it trivial to apply superposition.

Difference Mode (small signal):



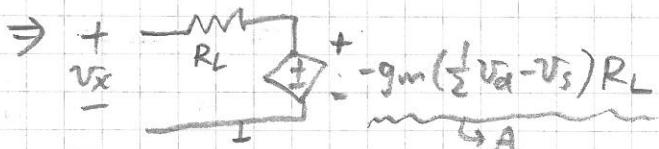
Seen from v_x \Rightarrow setting $v_d = 0$, we have

$$v_{gs1} = v_{gs2} = 0 - v_s = -v_s,$$

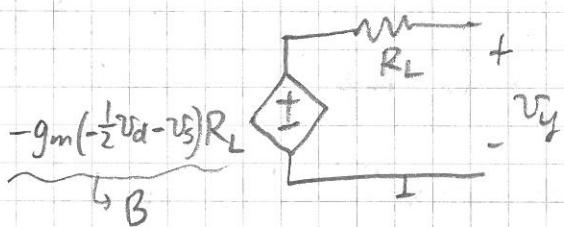
which implies $i_{d1} = i_{d2} = 0$

$$\Rightarrow R_{TH} = R_L$$

$$\Rightarrow V_{TH} = -i_{d1} \cdot R_L = -g_m \left(\frac{1}{2} v_d - v_s \right) R_L$$

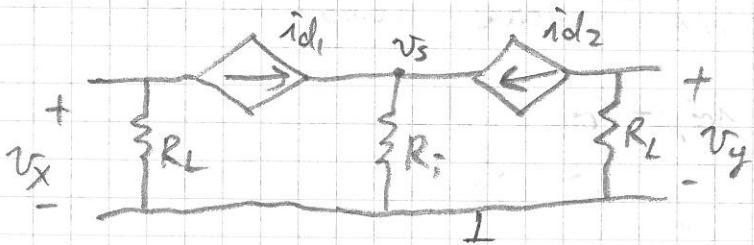


Seen from v_y , using similar derivation, we have



Common Mode (Small signal):

the following derivation will be very similar to what we did for the difference mode, since the only difference in the circuit structure is the different input voltage source:

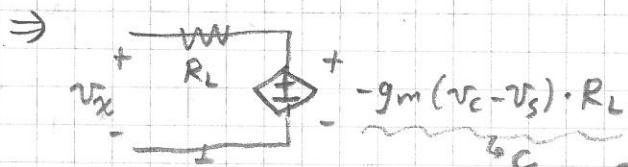


Seen from $v_x \Rightarrow$ setting $v_c = 0$, we have

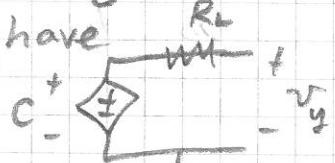
$v_{gs1} = v_{gs2} = -v_s$, which again implies $i_{d1} = i_{d2} = 0$

$$\Rightarrow R_{TH} = R_L$$

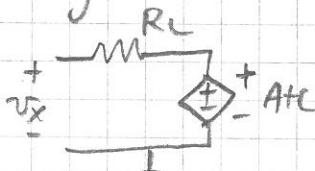
$$\Rightarrow V_{TH} = -i_{d1} \cdot R_L = -g_m(v_c - v_s) \cdot R_L$$



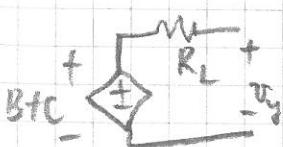
Seen from v_y , by symmetry, we have



Combining the two modes, we have



seen from v_x , and thus output resistance $r_o = R_L$



seen from v_y , and thus output resistance $r_o = R_L$

You might notice that in the previous[✓], the resulting voltage source we got is a bit different from that on P435 in the textbook.

Here is some explanation :

- In the common mode, applying KCL

we have $i_{d1} + i_{d2} = \frac{v_s}{R_i}$

$$\Rightarrow 2g_m(v_c - v_s) = \frac{v_s}{R_i} \text{ and since both inputs are } v_c$$

therefore, we can say $i_{d1} = i_{d2} = \frac{v_s}{2R_i}$.

Further, because of the biasing current source, I, we can say that $v_s \neq 0$.

- In the difference mode, applying KCL

we have $i_{d1} + i_{d2} = \frac{v_s}{R_i}$

$$\Rightarrow -2g_m \cdot v_s = \frac{v_s}{R_i} \text{ and the textbook said}$$

that since g_m and R_i are independent $\Rightarrow v_s = 0$

So, what we've got in the previous page is equivalent to that in the textbook.

Interestingly, in the difference mode v_s does not necessarily equal zero; it equals zero^{only} because the inputs happened to be $\frac{1}{2}v_d$ and $-\frac{1}{2}v_d$.

To see this, let's consider the two inputs are v_d and 0 instead :

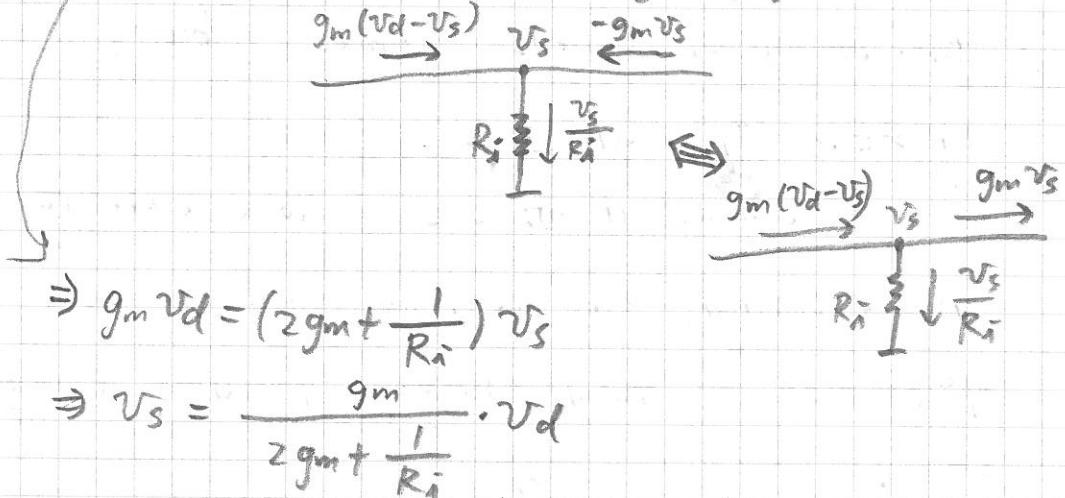
P111

in this case we will have, from KCL,

$$g_m(v_d - v_s) + g_m(0 - v_s) = \frac{v_s}{R_i}$$

$$\Rightarrow g_m v_d - 2 g_m v_s = \frac{v_s}{R_i}$$

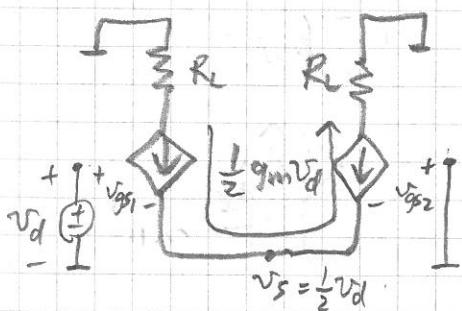
which does not necessarily imply $v_s = 0$



if $R_i \gg 1$, this implies $v_s = \frac{1}{2} v_d$ and thus

$$\frac{\frac{1}{2} g_m v_d}{R_i} \xrightarrow{R_i \gg 1} 0 \text{ because } R_i \gg 1$$

on the whole circuit it means :

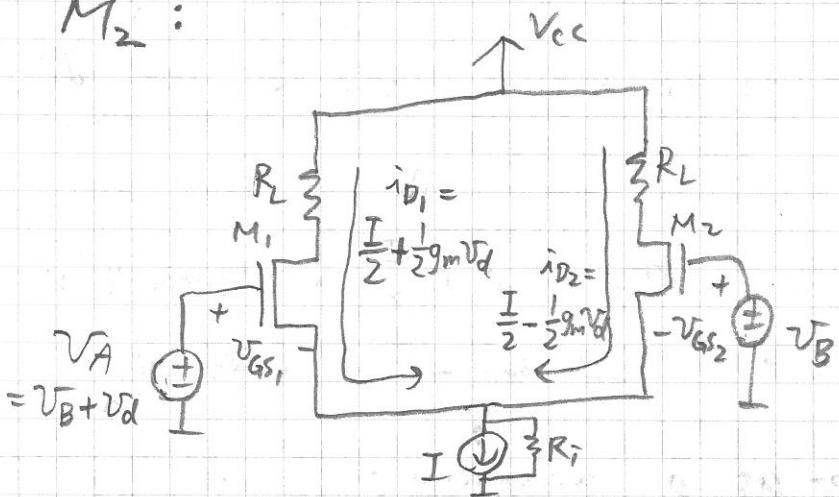


notice that now $v_{gs1} = \frac{1}{2} v_d$
and $v_{gs2} = -\frac{1}{2} v_d$

Now, it is very interesting to ponder on
this from the aspect of "total signal"]

P112 ↴ If we think about the case where V_A and V_B differs only by a tiny amount, say, ΔV . Then we have $V_D = V_A - V_B = \Delta V$ and $v_d = \Delta V$, $V_D = 0$. The fact that a tiny amount of current, $\frac{1}{2}g_m v_d$, flowing through MOSFETs M_1 and M_2 is equivalent as both a small amount of increase of total current flowing through M_1 and a small amount of decrease of total current flowing through M_2 .

M_2 :



which makes sense, because we see that v_{GS_1} being slightly larger than v_{GS_2} will cause i_{D_1} being slightly larger than i_{D_2} .

We may double check this by observing

that $v_{GS_1} = v_{GS_2} + v_d$, and thus

$$\bar{i}_{D_1} = \frac{1}{2}IK(v_{GS_2} + v_d - V_T)^2$$

$$\bar{i}_{D_2} = \frac{1}{2}IK(v_{GS_2} - V_T)^2$$

$$\bar{i}_{D_1} - \bar{i}_{D_2} = \frac{1}{2}IK((v_{GS_2} - V_T + v_d)^2 - (v_{GS_2} - V_T)^2)$$

$$= \frac{1}{2}IK(2v_{GS_2} - 2V_T + v_d) \cdot v_d \quad (\frac{a^2 - b^2}{(a+b)(a-b)})$$

$$\approx \frac{1}{2}IK(2v_{GS_2} - 2V_T) \cdot v_d \quad (\because v_d \ll v_{GS_2})$$

$$= IK(v_{GS_2} - V_T) \cdot v_d$$

$$= g_m \cdot v_d$$

which is equal to

$$\bar{i}_{D_1} - \bar{i}_{D_2} = \left(\frac{I}{2} + \frac{1}{2}g_m v_d\right) - \left(\frac{I}{2} - \frac{1}{2}g_m v_d\right) = g_m \cdot v_d$$

Amazing.

- Now, back to P98 and P101 we saw that

$A_d = -g_m R_L$ and the output resistance is R_L .

This gives us a dilemma:

If we want to have a larger voltage gain we will have a larger output resistance should we choose to increase R_L ; and if we want to have a smaller output resistance we will have to decrease the voltage gain.

Facing trade-offs is common to engineers. Often, a good system design involves finding a good balance that suits a particular application requirement.

Fortunately, in our case here we may have a way to increase the voltage gain while not sacrificing the output resistance.

In Homework 7, Question 2, we see that

$$A_{\text{d}} = -R_L \sqrt{I_R} \sqrt{I}$$

↖ in Fall 2020 semester

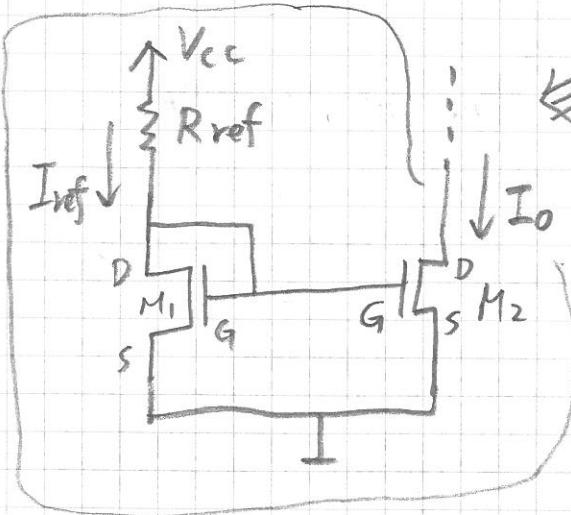
This means that we may increase the voltage gain by increasing I :)

(Perhaps that's also a reason why we use such a biasing current source)

To conclude our discussion of MOSFET, we look at two interesting applications :

① MOSFET as a small-signal resistor
(see Example 8.1 in the textbook)

② MOSFET as a current source (next page)



In this configuration,

we'll have $I_o = I_{ref}$

if both MOSFETs are

① identical, and

② operating under the

saturation discipline.

The left MOSFET is in saturation because

$$V_{GS} = V_{DS} \quad (\text{i.e., } V_{GS} - V_T < V_{DS})$$

$$\text{Then } I_{ref} = \frac{1}{2} K_1 (V_{GS_1} - V_T)^2$$

$$I_o = \frac{1}{2} K_2 (V_{GS_2} - V_T)^2$$

Since M_1, M_2 have both gates and sources short-circuited,

$$V_{GS_1} = V_{GS_2} \quad \text{and} \quad \boxed{\frac{I_o}{I_{ref}} = \frac{K_2}{K_1}}$$

$$\text{Considering that } \left\{ \begin{array}{l} K_2 = K' \sqrt{\frac{W_2}{L_2}} \\ K_1 = K' \sqrt{\frac{W_1}{L_1}} \end{array} \right.$$

$$\left(\begin{array}{l} K_2 = K' \sqrt{\frac{W_2}{L_2}} \\ K_1 = K' \sqrt{\frac{W_1}{L_1}} \end{array} \right)$$

\Rightarrow We may change the ratio between I_o and I_{ref} by changing the geometric structure of the MOSFET :)

Finally, observe that I_{ref} can be determined by

$$\left\{ \begin{array}{l} I_{ref} = \frac{1}{2} K_1 (V_{GS_1} - V_T)^2 \\ V_{DS_1} = I_{ref} \cdot R_L \end{array} \right.$$

$$V_{DS_1} = V_{GS_1}$$

