参考解答 for problem 1.16 :

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let $g(n) = A(n)\alpha + B(n)\beta_0 + C(n)\beta_1 + D(n)\cdot r$ ⓪

① consider

$g(n)=1 \Rightarrow \begin{cases} \alpha = 1 \\ 1 = 3 + rn + \beta_0 \\ 1 = 3 + rn + \beta_1 \end{cases}$ since $\begin{cases} g(1) = \alpha \\ g(2n) = 3g(n) + rn + \beta_0 \\ g(2n+1) = 3g(n) + rn + \beta_1 \end{cases}$

$\Rightarrow (\alpha, \beta_0, \beta_1, r) = (1, -2, -2, 0)$

$\Rightarrow A(n) - 2B(n) - 2C(n) = 1$

② consider

$g(n) = n \Rightarrow \begin{cases} \alpha = 1 & \text{since } g(1) = \alpha \\ 2n = 3n + rn + \beta_0 & \text{since } g(2n) = 3g(n) + rn + \beta_0 \\ 2n+1 = 3n + rn + \beta_1 & \text{since } g(2n+1) = 3g(n) + rn + \beta_1 \end{cases}$

$\Rightarrow (\alpha, \beta_0, \beta_1, r) = (1, 0, 1, -1) \Rightarrow A(n) + C(n) - D(n) = n$

③ consider $(\alpha, \beta_0, \beta_1, r) = (1, 0, 0, 0)$

because in this way we may simplify the original recurrence relation:

$\Rightarrow \begin{cases} g(1) = \alpha \\ g(2n) = 3g(n) \\ g(2n+1) = 3g(n) \end{cases}$ which means $\begin{aligned} &g(2) = 3g(1) \\ &g(3) = 3g(1) \\ &g(4) = 3g(2) = 3^2 g(1) \\ &g(5) = 3g(2) \\ &g(6) = 3g(3) = 3^2 g(1) \\ &\vdots \end{aligned}$

$\Rightarrow g(n) = 3^m$ where $n = 2^m + l$ and $0 \le l < 2^m$

$\Rightarrow A(n) = g(n) = 3^m$

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④ consider $(\alpha, \beta_0, \beta_1, r) = (1, 0, 0, 1)$

because that also helps simplifies the recurrence relation:

$\Rightarrow \begin{cases} g(1) = 1 \\ g(2n) = 3g(n) + n \\ g(2n+1) = 3g(n) + n = g(2n) \end{cases}$ which means

$g(2) = 3g(1) + 1$

$g(3) = 3g(1) + 1$

$g(4) = 3g(2) + 2 = 3(3g(1)+1) + 2$

$g(5) = 3g(2) + 2$

$g(6) = 3g(3) + 3 = 3(3g(1)+1) + 3$

$g(7) = g(6)$

$g(8) = 3g(4) + 4 = 3(3(3g(1)+1)+2) + 4$

$g(9) = g(8)$

$g(10) = 3g(5) + 5 = 3(3g(2)+2) + 5 = 3(3(3g(1)+1)+2) + 5$

$\vdots$

$\Rightarrow g(n) = 3^m g(1) + 3^{m-1}\cdot 1 + 3^{m-2}\cdot 2 + 3^{m-3}\cdot 3 + \cdots + 3^1\cdot(m-1) + \lfloor n/2 \rfloor$

$= 3^m g(1) + \sum_{k=1}^{m-1} 3^{m-k}\cdot k + \lfloor \frac{n}{2} \rfloor$

$= A(n) + D(n)$

$\Rightarrow D(n) = \sum_{k=1}^{m-1} 3^{m-k}\cdot k + \lfloor \frac{n}{2} \rfloor$

Finally, jointly solve ①, ②, ③, and ④ we can get $B(n)$ and $C(n)$. Plugging $A(n)$, $B(n)$, $C(n)$, $D(n)$ into ⓪ and we are done.

(Note that we might consider $(\alpha, \beta_0, \beta_1, r) = (1, 1, 1, 0)$ which will give $A(n) + B(n) + C(n) = \frac{3}{2}\cdot 3^m - \frac{1}{2}$ but unfortunately this is dependent to ① and ③.)

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