A Review of Poisson Distribution (from textbook [5]) P_1

- Bernoulli Trials, Definition:

  Repeated independent trials that have only two possible outcomes for each trial and their probabilities remain the same throughout trials.

  Example: tosses of a coin.

- Binomial Distribution, Definition:

  Let $b(k;n,p)$ be the probability that n Bernoulli trials with probability p for success and $q=1-p$ for failure result in k successes and $n-k$ failures.
  Then
  $$P\{S_n=k\}=b(k;n,p)=\binom{n}{k}p^k q^{n-k}$$
  is called the binomial distribution of S_n.
  Note that $\binom{n}{k}=\frac{n!}{k!(n-k)!}$ is called the binomial coefficient (coefficient of x^k in $(1+x)^n$)

- Poisson distribution is an approximation of the binomial distribution:
  In many real-world applications n is large and p is small, whereas the product $\lambda=np$ is of moderate magnitude.
  In this case,
  $$b(k;n,p)\approx\frac{\lambda^k}{k!}e^{-\lambda}$$
  let $P(k;\lambda)=\frac{\lambda^k}{k!}e^{-\lambda}$ and call it the Poisson distribution.

Proof of $b(k;n,p)\approx\frac{\lambda^k}{k!}e^{-\lambda}$:

$$b(0;n,p)=\binom{n}{0}p^0q^n=(1-p)^n=\left(1-\frac{\lambda}{n}\right)^n$$

$$\ln b(0;n,p)=n\ln\left(1-\frac{\lambda}{n}\right)=-\lambda-\frac{\lambda^2}{2n}-\cdots$$

(using Taylor expansion $\ln(1+t)=t-\frac{1}{2}t^2+\frac{1}{3}t^3\ldots$ and let $t=-\frac{\lambda}{n}$)

$\Rightarrow$ for large n, we have

$$b(0;n,p)\approx e^{-\lambda} \text{ since } \ln e^{-\lambda}=-\lambda$$

P_2

from the definition of $b(k;n,p)=\binom{n}{k}p^kq^{n-k}$

$$\frac{b(k;n,p)}{b(k-1;n,p)}=\frac{\lambda-(k-1)p}{kq}=\frac{\lambda-(k-1)p}{k(1-p)}\approx\frac{\lambda}{k}$$

for small p.

$$\Rightarrow b(1;n,p)\approx\frac{\lambda}{1}\cdot b(0;n,p)\approx\lambda e^{-\lambda}$$

$$b(2;n,p)\approx\frac{\lambda}{2}\cdot b(1;n,p)\approx\frac{\lambda^2}{2!}e^{-\lambda}$$

$$\cdots$$

$$\Rightarrow b(k;n,p)=\frac{\lambda^k}{k!}e^{-\lambda}$$

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