- Translating a recurrence into sum by a summation factor s_n:

Consider $a_n T_n = b_n T_{n-1} + c_n$

$$s_n a_n T_n = s_n b_n T_{n-1} + s_n c_n \quad \text{— ①}$$

if $s_n b_n = s_{n-1} a_{n-1}$

then ① becomes $s_n a_n T_n = s_{n-1} a_{n-1} T_{n-1} + s_n c_n$

let $S_n = s_n a_n T_n$

then we have $S_n = S_{n-1} + s_n c_n$

which gives

$$S_n = S_0 + \sum_{k=1}^{n} s_k c_k = s_0 a_0 T_0 + \sum_{k=1}^{n} s_k c_k = s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k$$

and $T_n = \frac{1}{s_n a_n}\left(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k\right)$ ※

$$s_n = \frac{a_{n-1}}{b_n} s_{n-1} = \cdots = \frac{a_{n-1} a_{n-2} a_{n-3} \cdots a_1}{b_n b_{n-1} b_{n-2} \cdots b_2} \cdot s_1$$

for $n \ge 1$

as long as $s_1 \ne 0$ it will be cancelled out after we plug s_n into the original equation.

Example: Analyzing the average # of comparisons in quicksort.

$$\begin{cases} C_0 = C_1 = 0 \\ C_n = n+1 + \frac{2}{n} \cdot \sum_{k=0}^{n-1} C_k, \text{ for } n > 1. \end{cases}$$

① — $n C_n = n^2 + n + 2 \sum_{k=0}^{n-1} C_k$, for $n > 1$

② — $(n-1) C_{n-1} = (n-1)^2 + (n-1) + 2 \sum_{k=0}^{n-1-1} C_k$, for $n-1 > 1$ i.e. $n > 2$

$= n^2 - n + 2 \sum_{k=0}^{n-2} C_k$

①-②: $n C_n - (n-1) C_{n-1} = 2n + 2 C_{n-1}$, for $n > 2$

$$\Rightarrow \begin{cases} C_0 = C_1 = 0;\ C_2 = 3 \\ n C_n = (n+1) C_{n-1} + 2n, \text{ for } n > 2 \end{cases}$$

$$\Rightarrow \underbrace{n}_{a_n} C_n = \underbrace{(n+1)}_{b_n} C_{n-1} + \underbrace{2n - 2[n=1] + 2[n=2]}_{c_n}, \text{ for } n \ge 1$$

therefore set $s_n = \frac{a_{n-1} a_{n-2} \cdots a_1}{b_n b_{n-1} \cdots b_2} = \frac{(n-1)(n-2)\cdots 4 \cdot 3 \cdot 2 \cdot 1}{(n+1)(n)(n-1)\cdots 4 \cdot 3} = \frac{2}{n(n+1)}$

$$C_n = \frac{1}{s_n a_n}\left(s_1 b_1 C_0 + \sum_{k=1}^{n} s_k c_k\right)$$

$$= \frac{n+1}{2}\left(\sum_{k=1}^{n} \frac{2}{k(k+1)} \cdot (2k - 2[k=1] + 2[k=2])\right) \quad \text{for } n \ge 1$$

$$= 2(n+1) \sum_{k=1}^{n} \frac{1}{k+1} + \frac{n+1}{2} \cdot \frac{2(-2)}{1(1+1)} + \frac{n+1}{2} \cdot \frac{2 \cdot 2}{2(2+1)} \quad \text{for } n \ge 2$$

$$= 2(n+1) \sum_{k=1}^{n} \frac{1}{k+1} - \frac{2}{3}(n+1), \text{ for } n \ge 2$$

$$C_n = 2(n+1)\sum_{k=1}^{n}\frac{1}{k+1} - \frac{2}{3}(n+1)$$

$$= 2(n+1)\sum_{1\le k\le n}\frac{1}{k+1} - \frac{2}{3}(n+1)$$

$$= 2(n+1)\sum_{1\le k-1\le n}\frac{1}{(k-1)+1} - \frac{2}{3}(n+1)$$

$$= 2(n+1)\sum_{2\le k\le n+1}\frac{1}{k} - \frac{2}{3}(n+1) = 2(n+1)\left(\sum_{1\le k\le n}\frac{1}{k} - \frac{1}{1} + \frac{1}{n+1}\right) - \frac{2}{3}(n+1)$$

$$= 2(n+1)\sum_{1\le k\le n}\frac{1}{k} - 2n - \frac{2}{3}n - \frac{2}{3}$$

$$= 2(n+1)\cdot H_n - \frac{8}{3}n - \frac{2}{3} \quad \text{for } n>1$$

where $H_n = \sum_{k=1}^{n}\frac{1}{k}$ is called ✱ harmonic number.

§ Section 2.3

Fundamental rules of Σ manipulations:

① $\sum_{k\in K} c\cdot a_k = c\sum_{k\in K} a_k$ (distributive)

② $\sum_{k\in K}(a_k+b_k) = \sum_{k\in K} a_k + \sum_{k\in K} b_k$ (associative)

☆ ③ $\sum_{k\in K} a_k = \sum_{p(k)\in K} a_{p(k)}$ (communitative)

p(k) is any permutation of the set of all integers.
K is an index set.

for example, if $p(k) = -k$

$$\sum_{k=-2}^{2} a_k = a_{-2}+a_{-1}+a_0+a_1+a_2$$

$$\sum_{p(k)\text{ for } k\in[-2,2]} a_k = a_2+a_1+a_0+a_{-1}+a_{-2}$$

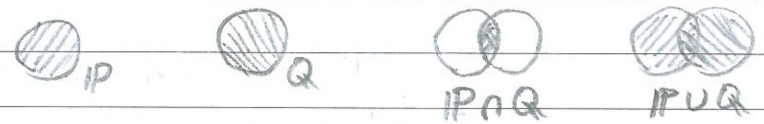
Rule ③ permits powerful ways to manipulate Σ
In general, we only need to require that for every $n\in K$, we have $p(k)=n$ for exactly one k:

Example 1:

$$\sum_{\substack{k\in K\\ k \text{ even}}} a_k = \sum_{\substack{l\in K\\ l \text{ even}}} a_l = \sum_{\substack{2k\in K\\ 2k \text{ even}}} a_{2k} = \sum_{2k\in K} a_{2k}$$

Example 2: let P and Q be two sets of integers

$$\sum_{k\in P} a_k + \sum_{k\in Q} a_k = \sum_{k\in P\cap Q} a_k + \sum_{k\in P\cup Q} a_k$$



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Example: For a general sum of an arithmetic progression $S=\sum_{0\le k\le n}(a+bk)$ we want to get a closed form of it.

Solution 1:

$$S=\sum_{0\le k\le n} a + b\sum_{0\le k\le n} k$$

by reciting the formula

$$= a(n+1)+b\cdot\frac{n(n+1)}{2}$$

$$= \frac{1}{2}(a+(a+bn))\cdot(n+1) \quad ※$$

Solution 2:

$$S=\sum_{0\le k\le n}(a+bk) \quad —①$$

replace k by n-k

$$=\sum_{0\le n-k\le n}(a+b(n-k))$$

$0\le n-k\le n$

$\Rightarrow \cancel{-n\le k-n\le 0}$

$\Rightarrow 0\le k\le n$

$$=\sum_{0\le k\le n}(a+bn-bk) \quad —②$$

$$①+②:\ 2S=\sum_{0\le k\le n}(2a+bn)$$

$$=(2a+bn)\sum_{0\le k\le n}1$$

$$=(2a+bn)(n+1)$$

$$\Rightarrow S=\frac{1}{2}(a+(a+bn))\cdot(n+1) \quad ※$$

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- The Perturbation Method:

To get a closed form of $S_n=\sum_{k=0}^{n}a_k$, we may express S_{n+1} in two ways and simplify our calculation:

$$S_{n+1}=S_n+a_{n+1} \quad \text{留尾}$$

$$=\sum_{0\le k\le n+1}a_k = a_0+\sum_{1\le k\le n+1}a_k$$

$$=a_0+\sum_{1\le k+1\le n+1}a_{k+1}$$

$$=a_0+\sum_{0\le k\le n}a_{k+1} \quad \text{想辦法轉成 } f(S_n)$$

留頭

$$\Rightarrow S_n - f(S_n) = a_0 - a_{n+1}$$

and we can solve for S_n :)

Note: from $\sum_{1\le k+1\le n+1}a_{k+1}$ to $\sum_{0\le k\le n}a_{k+1}$

it makes sense because $1\le k+1\le n+1$ and $0\le k\le n$ are equivalent statements about $k+1$.

Review Eq (2.4) on page 23 of the textbook.

Example: Find a closed form of a general geometric progression:

$$S_n = \sum_{0 \le k \le n} a x^k$$

Solution:

$$S_n + a x^{n+1} = a x^0 + \sum_{0 \le k \le n} a x^{k+1}$$

$$= a + x \sum_{0 \le k \le n} a x^k$$

$$= a + x \cdot S_n$$

$$\Rightarrow S_n - x \cdot S_n = a - a x^{n+1}$$

$$S_n = \frac{a(1-x^{n+1})}{1-x}$$

※

Example: An IoT sensor needs to send its data to a base station. Suppose that for each transmission of a packet of data it will success with probability p. What would be the expected number of transmissions to successfully send a packet of data?

Solution:

$$S_\infty = 1 \cdot p + 2 \cdot (1-p) \cdot p + 3 \cdot (1-p)^2 \cdot p + \cdots$$

$$= \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} \cdot p$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} k \cdot (1-p)^{k-1} \cdot p$$

$$= \lim_{n \to \infty} \left(\frac{p}{1-p} \sum_{k=1}^{n} k \cdot (1-p)^k \right)$$

let $1-p = x$ and we face $\sum_{k=0}^{n} k \cdot x^k = T_n$.

Using the perturbation method, we have

$$T_n + (n+1) x^{n+1} = T_0 + \sum_{k=0}^{n} (k+1) \cdot x^{k+1}$$

$$= \sum_{k=0}^{n} k \cdot x^{k+1} + \sum_{k=0}^{n} x^{k+1}$$

$$= x \cdot \sum_{k=0}^{n} k \cdot x^k + x \cdot \sum_{k=0}^{n} x^k$$

$$= x \cdot T_n + x \cdot \frac{1-x^{n+1}}{1-x} \Rightarrow T_n = \frac{x \cdot \frac{1-x^{n+1}}{1-x} - (n+1) x^{n+1}}{1-x}$$

$$= \frac{x - x^{n+2} - (1-x)(n+1) x^{n+1}}{(1-x)^2}$$

$$\Rightarrow S_\infty = \lim_{n \to \infty} \left(\frac{p}{1-p} \cdot T_n \right)$$

$$= \lim_{n \to \infty} \left(\frac{p}{1-p} \cdot \frac{(1-p) + (1-p)^{n+2} - p(n+1)(1-p)^{n+1}}{p^2} \right)$$

$$= p^{-1} \quad \text{since } 0 \le p < 1$$

※

(note that $\lim_{n \to \infty} n \cdot x^n = 0$ for $0 \le x < 1$.)

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(Proof of $\lim_{n\to\infty} n\cdot x^n = 0$ for $0 \le x < 1$:

let $x = \frac{1}{1+t}$ for some $t > 0$

By the binomial theorem, we have

$$(1+t)^n = 1 + nt + \frac{n(n-1)}{2}t^2 + \cdots$$
$$\ge \frac{n(n-1)}{2}t^2$$

$$\Rightarrow n\cdot x^n = \frac{n}{(1+t)^n} \le \frac{2}{(n-1)t^2}$$

and since $0 \le n\cdot x^n$

$$\Rightarrow 0 \le n\cdot x^n \le \frac{2}{(n-1)t^2}$$

and Q.E.D. by the squeeze theorem. ※)

§ Section 2.4 Multiple Sums

$$\sum_{\substack{1\le j\le 3\\ 1\le k\le 3}} a_j b_k \equiv \sum_{1\le j,k\le 3} a_j b_k \equiv a_1b_1 + a_1b_2 + a_1b_3 + a_2b_1 + a_2b_2 + a_2b_3 + a_3b_1 + a_3b_2 + a_3b_3$$

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Since we may sum those nine terms in any order (the commutative rule), in general we have

$$\sum_{P(j,k)} a_{j,k} \equiv \sum_{j,k} a_{j,k}[P(j,k)]$$
$$\equiv \sum_j \sum_k a_{j,k}[P(j,k)]$$
$$\equiv \sum_k \sum_j a_{j,k}[P(j,k)]$$

where $P(j,k)$ is some property of j and k.

Example: $\sum_{1\le j,k\le 3} a_j b_k = \sum_{j,k} a_j b_k [1\le j\le 3][1\le k\le 3]$

$$= \sum_j \sum_k a_j b_k [1\le j\le 3][1\le k\le 3]$$
$$= \sum_j a_j [1\le j\le 3] \sum_k b_k [1\le k\le 3]$$
$$= \left(\sum_{j=1}^{3} a_j\right)\left(\sum_{k=1}^{3} b_k\right)$$
$$= \left(\sum_{k=1}^{3} b_k\right)\left(\sum_{j=1}^{3} a_j\right)$$ ※

In practice, the range of the inner sum may depend on the index variable of the outer sum:

$$\sum_{j\in\mathbb{J}}\sum_{k\in\mathbb{K}(j)} a_{j,k} = \sum_{k\in\mathbb{K}'}\sum_{j\in\mathbb{J}'(k)} a_{j,k}$$

where sets $\mathbb{J}$, $\mathbb{K}(j)$, $\mathbb{K}'$, and $\mathbb{J}'(k)$ are related in the following way

$$\underbrace{[j\in\mathbb{J}][k\in\mathbb{K}(j)]}_{①} = \underbrace{[k\in\mathbb{K}'][j\in\mathbb{J}'(k)]}_{②}$$

An important specific case:

$$\underbrace{[1\le j\le n][j\le k\le n]}_{①} = [1\le j\le k\le n]$$

$$= \underbrace{[1\le k\le n][1\le j\le k]}_{②}$$

Example: (j, k; ①, ②)

$(1,1),(1,2),\cdots,(1,n)$
$(2,2),\cdots,(2,n)$
$\vdots$
(n,n)

which means

$$\sum_{j=1}^{n}\sum_{k=j}^{n} a_{j,k} = \sum_{1\le j\le k\le n} a_{j,k} = \sum_{k=1}^{n}\sum_{j=1}^{k} a_{j,k}$$

(Exercise: Problem 2.14 in the textbook)

Example: Consider the array of n^2 products $a_i a_j$:

$$\begin{bmatrix} a_1a_1 & a_1a_2 & \cdots & a_1a_n \\ a_2a_1 & a_2a_2 & \cdots & a_2a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_na_1 & a_na_2 & \cdots & a_na_n \end{bmatrix}$$

double sum

Simplify $S_{\triangledown} = \sum_{1\le j\le k\le n} a_j a_k$.

Solution:

$$S_{\triangledown} = \sum_{1\le j\le k\le n} a_j a_k = \sum_{1\le k\le j\le n} a_k a_j$$

$$= \sum_{1\le k\le j\le n} a_j a_k = S_{\triangle}$$

$$\Rightarrow 2S_{\triangledown} = \underbrace{\sum_{1\le j,k\le n} a_j a_k}_{\text{整個 array}} + \underbrace{\sum_{1\le j=k\le n} a_j a_k}_{\text{對角線}}$$

$$= \left(\sum_{j=1}^{n} a_j\right)^2 + \sum_{j=1}^{n} a_j^2$$

$$\Rightarrow S_{\triangledown} = \frac{1}{2}\left(\left(\sum_{j=1}^{n} a_j\right)^2 + \sum_{j=1}^{n} a_j^2\right) \quad ※$$

single sums

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Example: simplify $S_n = \sum_{1\le j<k\le n} \frac{1}{k-j}$

$$S_n = \sum_{1\le k\le n}\sum_{1\le j<k}\frac{1}{k-j}$$

$$= \sum_{1\le k\le n}\sum_{1\le k-j<k}\frac{1}{k-(k-j)} \qquad \begin{matrix} -k<-k+j\le -1 \\ 0<j\le k-1 \end{matrix}$$

$$= \sum_{1\le k\le n}\sum_{0<j\le k-1}\frac{1}{j} = \sum_{1\le k\le n} H_{k-1}$$

$$= \sum_{1\le k+1\le n} H_k = \sum_{0\le k<n} H_k$$

Alternatively, we can replace $k-j$ first:

$$S_n = \sum_{1\le k-j<k\le n}\frac{1}{j}$$

$$= \sum_{1\le k\le n}\sum_{1\le k-j<k}\frac{1}{j} \qquad \begin{matrix} -k<j-k\le -1 \\ 0<j\le k-1 \end{matrix}$$

$$= \sum_{1\le k\le n} H_{k-1} \;\cdots \text{ same as above}$$

Still, we can replace $k-j$ another way around:

$$S_n = \sum_{1\le j<k+j\le n}\frac{1}{k} \qquad \left(\begin{matrix} \text{from } 1\le j<k+j \\ \Rightarrow 1-j\le 0<k \Rightarrow 1\le k \\ \text{from } k+j\le n \\ \Rightarrow k\le n-j\le n \\ \Rightarrow 1\le k\le n \end{matrix}\right)$$

$$= \sum_{1\le k\le n}\sum_{1\le j\le n-k}\frac{1}{k}$$

from $k+j\le n$

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$$= \sum_{1\le k\le n}\frac{n-k}{k}$$

(note $\sum_{1\le k\le n-j}\sum_{1\le j\le n-k} = \sum_{1\le k\le n}\sum_{1\le j\le n-k}$ because for $k>n-j$ it is discounted by the inner sum.)

$$= \sum_{1\le k\le n}\frac{n}{k} - \sum_{1\le k\le n} 1$$

$$= nH_n - n$$

$$\Rightarrow S_n = nH_n - n$$

Note that we've also shown that

$$\boxed{\sum_{0\le k<n} H_k = nH_n - n}$$

by observing the result of S_n in our previous attempt.

A good lesson we've learned here is that if the term is $f(k)$ then we rearrange the index of the inner sum to be j (i.e., independent of k); in this way we may simply replace the inner sum by a coefficient which equals the size of the index range :)