ISHIKA

Problem 1

a) given $N=8$

i) DFT

for matrix A

$$A_{k,n} = \frac{1}{\sqrt{N}} e^{\frac{-i2\pi pn}{N}} \quad , \quad k,n = 0,\dots\dots, m-1.$$

for $N=8$

$$A \; \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^2 & \omega^4 & \omega^6 & 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega & \omega^4 & \omega^7 & \omega^2 & \omega^5 \\ 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 \\ 1 & \omega^5 & \omega^2 & \omega^7 & \omega^4 & \omega & \omega^6 & \omega^3 \\ 1 & \omega^6 & \omega^4 & \omega^2 & 1 & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega^1 \end{bmatrix} , \; \omega = e^{-i2\pi/8}$$

Now B is conjugate transport of A

$$B = A^* \; , \; B_{k,n} = \frac{1}{\sqrt{8}} e^{i2\pi kn/8}$$

b) DCT

for matrix A

$$A_{k,n} = C_k \sqrt{\frac{2}{N}} \cos\left(\frac{\pi k (2n+1)}{2N}\right), \quad C_0 = \frac{1}{\sqrt{2}}, \quad C_k = 1 \text{ for } k > 0$$

$$A = \sqrt{\frac{2}{8}} \begin{pmatrix} \frac{1}{\sqrt{2}}\cos(0) & \frac{1}{\sqrt{2}}\cos(0) & \cdots & \frac{1}{\sqrt{2}}\cos(0) \\ \cos\left(\frac{\pi(2n+1)}{16}\right) & \cos\left(\frac{3\pi(2n+1)}{16}\right) & \cdots & \cos\left(\frac{15\pi(2n+1)}{16}\right) \\ \vdots & \vdots & & \vdots \end{pmatrix}$$

for matrix B, it is transpose of A

$B = A^T$

iii) DWT

for matrix A where $N = 8$

$$A = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}$$

Again B matrix is transpose of A

$B = A^T$

b)

for DFT B is conjugate transpose of A

$$B = A^*$$

for DCT and DWT : B is transpose of A

$$B = A^T$$

Problem 2

given signal

$$F = (3.25, 1.25, 3.25, 1.25, -2.75, -1.75, -0.75, -1.75)^T$$

from Problem 1

$$A = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & \sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}$$

$g = AF$

first coefficient

$$g_0 = \frac{3.25 + 1.25 + 3.25 + 1.25 - 2.75 - 1.75 - 0.75 - 1.75}{\sqrt{8}}$$

$$= \frac{2.0}{\sqrt{8}} \approx 0.707$$

Second Coefficient

$$g_1 = \frac{3.25 + 1.25 + 3.25 + 1.25 - (-2.75 - 1.75 - 0.75 - 1.75)}{\sqrt{8}}$$

$$= \frac{12.0}{\sqrt{8}} \approx 4.243$$

third Coefficient

$$g_2 = \frac{\sqrt{2}\,(3.25 + 1.25 - 3.25 - 1.25)}{\sqrt{8}} = 0$$

Fourth coefficient

$$g_3 = \frac{\sqrt{2}\,(-2.75 - 1.75 - (-0.75 - 1.75))}{\sqrt{8}} = \frac{\sqrt{2}\,(-2.0)}{\sqrt{8}}$$

$$= -1$$

Fifth coefficient

$$g_4 = \frac{2(3.25 - 1.25)}{\sqrt{8}} \simeq 1.414$$

Sixth coefficient

$$g_5 = \frac{2(3.25 - 1.25)}{\sqrt{8}} \simeq 1.414$$

Seventh coefficient

$$g_6 = \frac{2(-2.75 - (-1.75))}{\sqrt{8}} \simeq -0.707$$

Eighth coefficient

$$g_7 = \frac{2(-0.75 - (-1.75))}{\sqrt{8}} \simeq 0.707$$

Total

$$g = (0.707, 4.243, 0, -1, 1.414, 1.414, -0.707, 0.707)^T$$

b) Smallest among these eight coefficients are g_0, g_2, g_6, g_7

$g_0 = 0$, $g_2 = 0$, $g_6 = 0$, $g_7 = 0$

New g

$g_{new} = (0, 4.243, 0, -1, 1.414, 1.414, 0, 0)^T$

c) Backtransform the new g

$g_{new} = (0, 4.243, 0, -1, 1.414, 1.414, 0, 0)^T$

The inverse Haar transform

$$F = A^T g_{new}$$

A^T is transpose of Haar matrix A

But as A is orthonormal $A^T = A^{-1}$ so

$$A^T = A$$

Now multiply A^T to g_{new}

$$
F = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & \sqrt{2} & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & \sqrt{2} & 0 & -2 & 0 & 0 & 0 \\ 1 & 1 & -\sqrt{2} & 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & -\sqrt{2} & 0 & 0 & -2 & 0 & 0 \\ 1 & -1 & 0 & \sqrt{2} & 0 & 0 & 2 & 0 \\ 1 & -1 & 0 & \sqrt{2} & 0 & 0 & -2 & 0 \\ 1 & -1 & 0 & -\sqrt{2} & 0 & 0 & 0 & 2 \\ 1 & -1 & 0 & -\sqrt{2} & 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 4.243 \\ 0 \\ -1 \\ 1.414 \\ 1.414 \\ 0 \\ 0 \end{bmatrix}
$$

row 1

$$
F_0 = \frac{1}{\sqrt{8}} \left(1\times 0 + 1\times 4.243 + \sqrt{2}\times 0 + 0\times(-1) + 2\times 1.414 + 0\times 1.414 + 0\times 0 + 0\times 0 \right)
$$

$$
F_0 = \frac{1}{\sqrt{8}} \left(0 + 4.243 + 0 + 0 + 2.828 + 0 + 0 + 0 \right) = \frac{7.071}{\sqrt{8}}
$$

$$
= 2.5
$$

row 2

$$
F_1 = \frac{1}{\sqrt{8}} \left(1\times 0 + 1\times 4.243 + \sqrt{2}\times 0 + 0\times(-1) + (-2)\times 1.414 + 0\times 1.414 + 0\times 0 + 0\times 0 \right)
$$

$$
= \frac{1}{\sqrt{8}} \left(0 + 4.243 + 0 + 0 - 2.828 + 0 + 0 + 0 \right) = \frac{1.415}{\sqrt{8}}
$$

$$
\simeq 0.5
$$

row 3

$F_2 = \frac{1}{\sqrt{8}} (1 \times 0 + 1 \times 4.243 + (-\sqrt{2}) \times 0 + 0 \times (-1) + 0 \times 1.414 + 2 \times 1.414$
$+ 0 \times 0 + 0 \times 0)$

$= \frac{1}{\sqrt{8}} (0 + 4.243 + 0 + 0 + 0 + 2.828 + 0 + 0)$

$= \frac{7.071}{\sqrt{8}} \simeq 2.5$

row 4

$F_3 = \frac{1}{\sqrt{8}} (1 \times 0 + 1 \times 4.243 + (-\sqrt{2}) \times 0 + 0 \times (-1) + 0 \times 1.414 + (-2) \times 1.414$
$+ 0 \times 0 + 0 \times 0)$

$= \frac{1}{\sqrt{8}} (0 + 4.243 + 0 + 0 + 0 - 2.828 + 0 + 0) = \frac{1.415}{\sqrt{8}} \simeq 0.5$

row 5

$F_4 = \frac{1}{\sqrt{8}} (1 \times 0 + (-2) \times 4.243 + 0 \times 0 + \sqrt{2} \times (-1) + 0 \times 1.414 + 0 \times 1.414 + 2.0 + 0)$

$= \frac{1}{\sqrt{8}} (0 - 4.243 + 0 - 1.414 + 0 + 0 + 0 + 0) = \frac{-5.657}{\sqrt{8}}$

$\simeq -2.0$

row 6

$F_5 = \frac{1}{\sqrt{8}} (1 \times 0 + (-1) \times 4.243 + 0 \times 0 + \sqrt{2} \times (-1) + 0 \times 1.414 + 0 \times 1.414$
$+ (-2) \times 0 + 0 \times 0)$

$= \frac{1}{\sqrt{8}}(0 - 4.243 + 0 - 1.414 + 0 + 0 + 0 + 0) = \frac{-5.657}{\sqrt{8}}$

$\simeq -2.0$

row 7

$f_6 = \frac{1}{\sqrt{8}} (1 \times 0 + (-1) \times 4.243 + 0 \times 0 + (-\sqrt{2})(-1.0) + 0 \times 1.414 +$
$0 \times 1.414 + 0.0 + 2 \times 0)$

$= \frac{1}{\sqrt{8}} (0 - 4.243 + 0 + 1.414 + 0 + 0 + 0 + 0) = \frac{-2.829}{\sqrt{8}}$

$\simeq 1.0$

row 8

$f_7 = \frac{1}{\sqrt{8}} (1 \times 0 + (-1) \times 4.243 + 0 \times 0 + (-\sqrt{2}) \times (-1.0) + 0 \times 1.414$
$+ 0 \times 1.414 + 0 \times 0 + (-2) \times 0)$

$= \frac{1}{\sqrt{8}} (0 - 4.243 + 0 + 1.414 + 0 + 0 + 0 + 0)$

$= \frac{-2.829}{\sqrt{8}} \simeq -1$

Final

$f = (2.5, 0.5, 2.5, 0.5, -2, -2, -1, -1)^T$

d) given Fourier - denoised signal

$\tilde{f} \simeq (1.25, 2.77, 2.81, 0.65, -1.75, -2.27, -1.31, -0.15)^T$

→ Haar wavelet has sharp edges [step like]

→ Fourier shows smooth transitions like blurred edges