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Problem 1

According to sampling theorem
The highest frequency that can be accurately represented is half the sampling rate

Given

Sampling rate :- 384 kHz = f

$$f = 2W$$

$$\frac{384}{2} = W$$

$$\boxed{192\ \text{kHz} = W}$$

reason

In order to sample a band-limited signal without aliasing artefacts, one must sample the highest frequency more than twice per cycle

$$h < h_{max} := \frac{1}{2W}$$

Problem 2

Given

$$f := (12, 16, 12, 16, 4, 8, 4, 8)^T$$

a) Fourier Coefficients

for $m = 8$

Fourier basis vectors

$$b_0 = \frac{1}{\sqrt{8}}(1, 1, 1, 1, 1, 1, 1, 1)^T$$

$$b_1 = \frac{1}{\sqrt{8}}\left(1, e^{-i\pi/4}, e^{-\pi/2}, e^{-3\pi/4}, e^{-i\pi}, e^{-i5\pi/4}, e^{-3\pi/2}, e^{-i7\pi/4}\right)^T,$$

$$b_2 = \frac{1}{\sqrt{8}}\left(1, e^{-\pi/2}, e^{-i\pi}, e^{i3\pi/2}, e^{i2\pi}, e^{-5\pi/2}, e^{-i3\pi}, e^{-i7\pi/2}\right)^T$$

$$b_3 = \frac{1}{\sqrt{8}}\left(1, e^{-i3\pi/4}, e^{-i3\pi/2}, e^{-i9\pi/4}, e^{-i3\pi}, e^{-i15\pi/4}, e^{-i9\pi/2}, e^{-i21\pi/4}\right)^T$$

$$b_4 = \frac{1}{\sqrt{8}}\left(1, -1, 1, -1, 1, -1, 1, -1\right)^T$$

Now using Inner Product, The Fourier Coefficient

Formula $\Rightarrow$ $F_k = \langle f, b_k \rangle = f^T \overline{b_k}$

f_0

$$F_0 = \frac{1}{\sqrt{8}}(12\cdot1 + 16\cdot1 + 12\cdot1 + 16\cdot1 + 4\cdot1 + 8\cdot1 + 4\cdot1 + 8\cdot1) = \frac{80}{\sqrt{8}} = 20\sqrt{2}$$

F_1

$$F_1 = \frac{1}{\sqrt{8}}\left(12\cdot 1 + 16e^{i\pi/4} + 12e^{i\pi/2} + 16e^{i3\pi/4} + 4e^{i\pi} + 8e^{i5\pi/4} + 4e^{i3\pi/2} + 8e^{i7\pi/4}\right)$$

using Euler formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$F_1 = \frac{1}{\sqrt{8}}\left[12 + 16\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) + 12i + 16\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) - 4 + 8\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)\right.$$

$$\left. - 4i + 8\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)\right]$$

$$\Rightarrow \frac{1}{\sqrt{8}}\left(8 + (4 - 8\sqrt{2})i\right) = \frac{8}{\sqrt{8}} + \frac{(4-8\sqrt{2})}{\sqrt{8}}i$$

F_2

$$F_2 = \frac{1}{\sqrt{8}}\left(12\cdot 1 + 16e^{i\pi/2} + 12e^{i\pi} + 16e^{i3\pi/2} + 4e^{i2\pi} + 8e^{i5\pi/2} + 4e^{i3\pi} + 8e^{i7\pi/2}\right)$$

simplifying using Euler formula.

$$= \frac{1}{\sqrt{8}}\left(12 + 16i - 12 - 16i + 4 + 8i - 4 - 8i\right) = 0$$

F_3

This is similar to F_1 but with $3\pi/4$ phase

so by simplifing

$$F_3 = \frac{1}{\sqrt{8}}\left(8 + (4 + 8\sqrt{2})i\right)$$

F_4

$$F_4 = \frac{1}{\sqrt{8}}(12-16+12-16+4-8+4-8) = \frac{-16}{\sqrt{8}} = -4\sqrt{2}$$

b)

F_5

By symmetry: $F_5 = \overline{F_3}$

we know $F_3 = \dfrac{8+(4+8\sqrt{2})i}{\sqrt{8}}$

Complex Conjugate of F_3

$$F_5 = \frac{8-(4+8\sqrt{2})i}{\sqrt{8}}$$

F_6

By Symmetry: $F_6 = \overline{F_2}$

$F_2 = 0$

so $F_6 = 0$

F_7

By Symmetry: $F_7 = \overline{F_1}$

$$F_1 = \frac{8+(4-8\sqrt{2})i}{\sqrt{8}}$$

Complex Conjugate of F_7

$$F_7 = \frac{8 - (4 - 8\sqrt{2})i}{\sqrt{8}}$$

c)

Highest Frequency coefficient

$$F_4 = -4\sqrt{2}$$

d)

by removing F_4 and computing Inverse DFT

$$f_m = \frac{1}{\sqrt{m}} \sum_{p=0}^{m-1} \hat{F_p} \exp\left(\frac{i 2\pi p m}{m}\right)$$

$$(m = 0, \cdots, m-1)$$

$$f_m = \frac{1}{\sqrt{8}} \left(\sum_{p=0}^{3} F_p e^{\frac{i 2\pi p n}{8}} + \sum_{p=5}^{7} F_p e^{\frac{i 2\pi p n}{8}} \right)$$

for real signal

$$F_5 = \overline{F_3},\ F_6 = \overline{F_2} \text{ and } F_7 = \overline{F_1}$$

$$f_m = \frac{1}{\sqrt{8}} \left(F_0 + 2\mathrm{Re}\left(F_1 e^{i\frac{\pi n}{4}} \right) + 2\mathrm{Re}\left(F_3 e^{\frac{i 3\pi n}{4}} \right) \right)$$

Putting values

$$F_m = \frac{1}{\sqrt{8}}\left(20\sqrt{2} + 2\mathrm{Re}\left(\frac{8 + (4-8\sqrt{2})i}{\sqrt{8}}\, e^{\frac{i\pi n}{4}}\right)\right.$$

$$\left. + 2\mathrm{Re}\left(\frac{8 + (4+8\sqrt{2})i}{\sqrt{8}}\, e^{\frac{i3\pi n}{4}}\right)\right)$$

Now

$$\mathrm{Re}\left(\frac{8+(4-8\sqrt{2})i}{\sqrt{8}}\, e^{\frac{i\pi n}{4}}\right) = \mathrm{Re}\left(\left(\frac{8+(4-8\sqrt{2})i}{\sqrt{8}}\right)\left(\cos\left(\frac{\pi n}{4}\right) + i\sin\left(\frac{\pi n}{4}\right)\right)\right)$$

$$\mathrm{Re}\left(\frac{8+(4+8\sqrt{2})i}{\sqrt{8}}\, e^{\frac{i3\pi n}{4}}\right) = \mathrm{Re}\left(\frac{8+(4+8\sqrt{2})i}{\sqrt{8}}\left(\cos\left(\frac{3\pi n}{4}\right) + i\sin\left(\frac{3\pi n}{4}\right)\right)\right)$$

simplifying

$$\mathrm{Re}\left(F_1 e^{\frac{i\pi n}{4}}\right) = \frac{8\cos\left(\frac{\pi n}{4}\right) - (4-8\sqrt{2})\sin\left(\frac{\pi n}{4}\right)}{\sqrt{8}}$$

$$\mathrm{Re}\left(F_3 e^{\frac{i3\pi n}{4}}\right) = \frac{8\cos\left(\frac{3\pi n}{4}\right) - (4+8\sqrt{2})\sin\left(\frac{3\pi n}{4}\right)}{\sqrt{8}}$$

substituting back in F_m

$$= \frac{1}{\sqrt{8}}\left(20\sqrt{2} + \frac{16\cos\left(\frac{\pi n}{4}\right) - 2(4-8\sqrt{2})\sin\left(\frac{\pi n}{4}\right)}{\sqrt{8}} + \right.$$

$$\frac{16\cos\left(\frac{3\pi n}{4}\right) - 2(4+8\sqrt{2})\sin\left(\frac{3\pi n}{4}\right)\Big)}{\sqrt{8}}$$

Now putting for $m = 0$ to 7 and solving

	F_m
$m=0$	10
$m=1$	14
$m=2$	10
$m=3$	14
$m=4$	6
$m=5$	10
$m=6$	6
$m=7$	10

So new signal

$$F = \{10, 14, 10, 14, 6, 10, 6, 10\}^T$$

e) effects.

→ It becomes smoother

→ Blurring because of high frequency oscillations