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Problem 1

Given

$f = (f_{i,j}) \Rightarrow$ noisy image

$g = (g_{i,j}) \Rightarrow$ original image

$n = (n_{i,j}) \Rightarrow$ additive noise with mean 0.

$f_{i,j} = g_{i,j} + n_{i,j}$

a) we have to Find the Case where $PSNR(F, g) = 0$

Formula

$$PSNR(f, g) := 10 \log_{10} \left(\frac{(255)^2}{MSE(f, g)} \right)$$

$\therefore$ here 255^2 is maximal MSE

we can write it as MAX^2

So

$$0 = 10 \log_{10} \frac{(MAX^2)}{MSE(f, g)}$$

Divide both side by 10

$$0 = \log_{10} \left(\frac{MAX^2}{MSE(F, g)} \right)$$

To get rid up log we can raise it to Power of 10

$$10^0 = \frac{(MAX)^2}{MSE(f, g)}$$

$$1 = \frac{(MAX)^2}{MSE(f,g)}$$

$$MSE(f,g) = (MAX)^2$$

So $PSNR(f,g)$ will be equal to zero when MSE is equal to maximum possible Intensity difference

It means noisy image f is as far as possible from original image means its completely noisy.

b) Now given a Filtered version u of f s.t.

$$PSNR(u,g) = PSNR(f,g) + 30dB \quad — (1)$$

By Formula

$$PSNR(u,g) = 10 \log_{10} \frac{(MAX^2)}{MSE(u,g)}$$

Put in eq — (1)

$$10 \log_{10} \frac{(MAX^2)}{MSE(u,g)} = 10 \log_{10} \frac{(MAX^2)}{MSE(f,g)} + 30$$

$$10 \log_{10} \left(\frac{MAX^2}{MSE(u,g)}\right) - 10 \log_{10} \frac{(MAX^2)}{MSE(f,g)} = 30$$

we know $\log A - \log B = \log(A/B)$

$$10 \log_{10} \left[\frac{\frac{\cancel{MAX^2}}{MSE(u,g)}}{\frac{\cancel{MAX^2}}{MSE(f,g)}} \right] = 30$$

$$10 \log_{10} \left[\frac{MSE(f,g)}{MSE(u,g)} \right] = 30$$

$$\log_{10} \left[\frac{MSE(f,g)}{MSE(u,g)} \right] = 3$$

Raising it to power of 10 on both side

$$\left[\frac{MSE(f,g)}{MSE(u,g)} \right] = 1000$$

It means MSE decreased by factor of 1000.

c) Assume $n \rightarrow 0$

$PSNR(f,g) = ?$

Given

$$f_{i,j} = g_{i,j} + n_{i,j}$$

if $n_{i,j} = 0$

$f_{i,j} = g_{i,j}$ means $MSE(f,g) = 0$

Now Putting in Formula

$$PSNR(F,g) = 10 \log_{10} \frac{(MAX^2)}{(MSE(F,g))}$$

$$PSNR(F,g) = 10 \log_{10} \frac{(MAX^2)}{0} = \infty$$

This means noisy image is same as original.

Problem 3

a) Given Continuous signal

$$f(x) = \begin{cases} 1/2 & (-1 \le x \le 1) \\ 0 & \text{else} \end{cases}$$

we know this is box function with area = 1

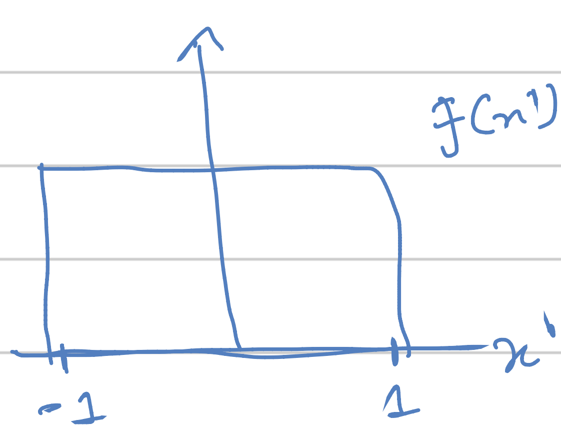
first Convolution $h_1 = f * f$

e)

$$(f * f)(x) = \int_{-\infty}^{\infty} f(x') \cdot f(x - x')\, dx'$$

Case 1

$x < -2$



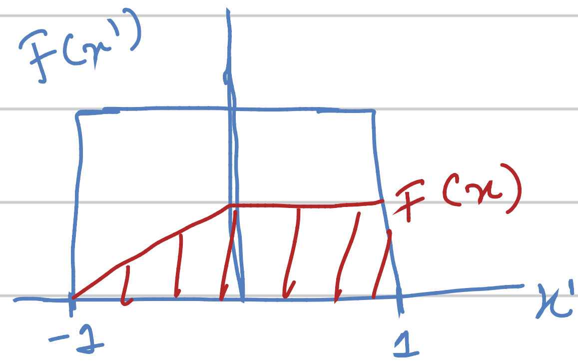
for this two boxes will not collide.

Convolution = 0 = $h_1(x)$

Case 2

$-2 \leq x < 0$

$\rightarrow f(x') \neq 0$ only for $-1 \leq x' \leq 1$



$f(x-x') \neq 0$ only when

$-1 \leq x-x' \leq 1$

$x-1 \leq -x' \leq x+1$

multiply by -1

$x+1 \geq x' \geq x-1$

Now compute using Formula

$$h_1(x) = \int_{-1}^{x+1} \frac{1}{2} \cdot \frac{1}{2} \, dx' = \frac{1}{4} \int_{-1}^{x+1} 1 \, dx' = \frac{1}{4} \left[x \right]_{-1}^{x+1}$$

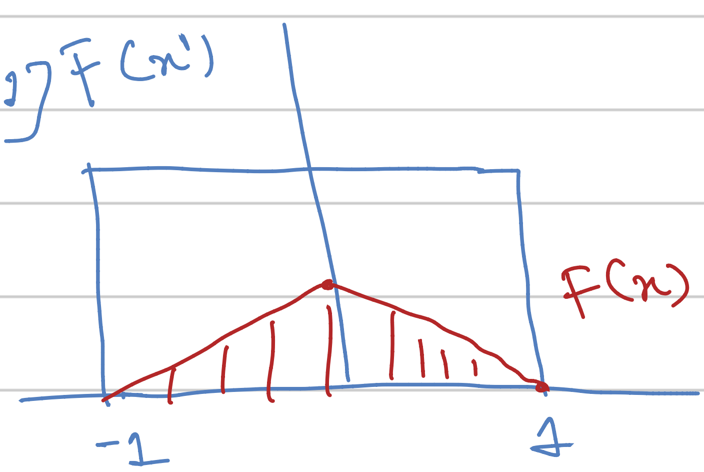
$$= \frac{1}{4} \left[x+1 - (-1) \right] = \frac{1}{4} \left[x+2 \right)$$

$$h_1(x) = \frac{1}{4} (x+2)$$

Case 3

$0 \leq x \leq 2$

$f(x') \neq 0$ only for $x' \in [-1, 1]$



$F(x-x') \neq 0$ only if

$-1 \leq x-x' \leq 1$

$x' \in [x-1, x+1]$

$$h_1(x) = \int_{x-1}^{1} \frac{1}{2} \cdot \frac{1}{2} \, dx' = \frac{1}{4} \left[x \right]_{x-1}^{1} = \frac{1}{4} \left[1 - x + 1 \right]$$

$$\Rightarrow \frac{1}{4} (2-x)$$

Case 4

$x > 2$

here again they don't overlap

Convolution $= 0$

ii) $h_2 = F * F * F$

$h_2(x) = h_1(x) * F$

$$h_2(x) = \int_{-1}^{1} h_1(x - x') F(x') \, dx'$$

Now we know

$$F(x) = \begin{cases} \frac{1}{2} & -1 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

$$h_1(x) = \begin{cases} \frac{1}{4}(2 - |x| & -2 \le x \le 2 \\ 0 & \text{else} \end{cases}$$

Case 1

$x < -3$

$h_2(-3) = 0$

they will not overlap

Case 2

$-3 \leq x \leq -2$

$$h_1(x) = \frac{1}{4}(2+x)$$

$$h_2(x) = \frac{1}{2}\int_{-2}^{x+1} \frac{1}{4}(2+x)\,dx$$

$$\frac{1}{8}\int_{-2}^{x+1} (2+x)\,dx = \frac{1}{8}\left[2x + \frac{1}{2}x^2\right]_{-2}^{x+1}$$

$$\Rightarrow \frac{1}{8}\left[2(x+1) + \frac{(x+1)^2}{2} - (-4+2)\right]$$

$$\frac{1}{8}\left[2x+2+\frac{(x+1)^2}{2} + 2\right]$$

$$\frac{1}{8}\left[2x+4+\frac{(x+1)^2}{2}\right]$$

Case 3

$-2 \leq x \leq 0$

$$h_2(x) = \frac{1}{2}\int_{x-1}^{x+1} \frac{1}{4}(2-|x|)\,dx$$

$$\Rightarrow \frac{1}{2}\times\frac{1}{4}\left[\int_{x-1}^{0}(2+x)\,dx + \int_{0}^{x+1}(2-x)\,dx\right]$$

$$\frac{1}{8}\left[\left(2x+\frac{x^2}{2}\right)_{x-1}^{0} + \left[2x-\frac{x^2}{2}\right]_{0}^{x+1}\right]$$

$\frac{1}{8}\left[-2(x-1) + \frac{(x-1)^2}{2} + 2(x+1) - \frac{(x+1)^2}{2} \right]$

$\frac{1}{8}\left[-\cancel{2x} + 2 - \frac{x^2}{2} - \frac{1}{2} + \cancel{\frac{2x}{2}} + \cancel{2x} + 2 - \frac{x^2}{2} - \frac{1}{2} - \cancel{\frac{2x}{2}} \right.$

$\frac{1}{8}\left[4 - \frac{2x^2}{2} - \frac{2}{2} \right] = \frac{1}{8}\left[3 - x^2 \right]$

Case 3

$0 \leq x \leq 2$

Same as Case 2 (Symmetry of Convolution)

$h_2(x) = \frac{1}{8}(3 - x^2)$

Case 4

$2 \leq x \leq 3$

$h_2(x) = \frac{1}{2} \int_{x-1}^{2} \frac{1}{4}(2-x)\, dx$

$\frac{1}{8}\left[2x - \frac{x^2}{2} \right]_{x-1}^{2}$

$\frac{1}{8}\left[4 - \frac{4}{2} - 2(x-1) + \frac{(x-1)^2}{2} \right]$

$$\frac{1}{8}\left[2 - 2n + 2 + \frac{n^2}{2} + \frac{1}{2} - \frac{2n}{2}\right]$$

$$\frac{1}{8}\left[4 - 2n + \frac{n^2}{2} + \frac{1}{2} - n\right]$$

$$\frac{1}{8}\left[4 - 3n + \frac{n^2}{2} + \frac{1}{2}\right]$$

$$\frac{1}{8} \times \frac{1}{2}\left[8 - 6n + n^2 + 1\right]$$

$$\frac{1}{8} \times \frac{1}{2}\left[9 - 6n + n^2\right]$$

$$\frac{1}{16}\left[3 - n\right)^2$$

Case 5

$n \in [-3, -2]$

same as 4 but reflected

$$h_2(n) = \frac{1}{16}(3+n)^2$$

So for $h_2(n)$

$$h_2(n) = \begin{cases} 0 & |n| > 3 \\ \frac{1}{16}(3+n)^2 & -3 \leq n \leq -2 \\ \frac{1}{8}(3-\cancel{x}) & -2 \leq n \leq 2 \\ \frac{1}{16}(3-n)^2 & 2 \leq n \leq 3 \end{cases}$$

(b)

for $h_1(n)$ at points $-2, -1, 0, 1, 2$

$h_1(-2) = 0$

$h_1(-1) = 1/2$

$h_1(0) = 1$

$h_1(1) = 1/2$

$h_1(2) = 0$

for $h_2(n)$ at points $-3, -2, -1, 0, 1, 2, 3$

$h_2(-3) = 0$

$h_2(-2) = 0$

$h_3(-1) = 1/4$

$h_3(0) = 1$

$h_3(1) = 1/4$

$h_2(2) = 0$

$h_2(3) = 0$

c)

Both sequence of Convolution Smooths the Signal but in Continuous Convolution Smooter curves are due to integration and for discrete Convolution there are Stepwise curves due to Summation