④

$\boxed{1}$ $f * g = g * f$

$$(f * g)(x) := \int_{-\infty}^{\infty} f(x-x')\, g(x')\, dx'$$

$u = x - x'$

$x' = x - u$

$dx' = -du$

$$\int_{\infty}^{-\infty} f(u)\, g(x-u)\, (-du)$$

$$= \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du = (g * f)(x)$$

$\boxed{2}$ $(f * g) * h = f * (g * h)$

$$((f * g) * h)(x) = \int_{-\infty}^{\infty} (f*g)(u)\, h(x-u)\, du$$

$$(f * g)(u) = \int_{-\infty}^{\infty} f(s)\, g(u-s)\, ds$$

$$((f * g) * h)x = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(s) g(u-s)\, ds \right) h(x-u)\, du$$

$$= \int_{-\infty}^{\infty} f(s) \left(\int_{-\infty}^{\infty} h(x-u)\, g(u-s)\, du \right) ds$$

$$\left[\begin{matrix} w = u-s & \dfrac{dw}{du} = 1 & dw = du \\ u = s+w & & \end{matrix} \right]$$

$$= \int_{-\infty}^{\infty} f(s) \left(\int_{-\infty}^{\infty} g(w)\, h(x-(s+w))\, dw \right) ds$$

$$= \int_{-\infty}^{\infty} f(s) \underbrace{\left(\int_{-\infty}^{\infty} g(w)\, h((x-s)-w)\, dw \right)}_{(g*h)(x-s)} ds$$

$$= \int_{-\infty}^{\infty} f(s)\, (g*h)(x-s)\, ds = (f*(g*h))(x)$$

$\boxed{3}$ $(f+g)*h = f*h + g*h$

$$((f+g)*h)(x) = \int_{-\infty}^{\infty} (f(u)+g(u))\,h(x-u)\,du$$

$$= \underbrace{\int_{-\infty}^{\infty} f(u)\,h(x-u)\,du}_{(f*h)(x)} + \underbrace{\int_{-\infty}^{\infty} g(u)\,h(x-u)\,du}_{(g*h)(x)}$$

$$= (f*h)(x) + (g*h)(x)$$

Due to commutativity, $(f+g)*h = h*(f+g)$

Therefore, replacing functions

$$(g+h)*f = f*(g+h) = f*g + f*h$$

$\boxed{4}$ $(f * g)' = f * g'$

$$(f * g)(x) = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$$

$$\frac{d}{dx}(f * g)(x) = \frac{d}{dx} \int_{-\infty}^{\infty} \underline{f(u)}\, g(x-u)\, du$$

$$= \int_{-\infty}^{\infty} f(u) \frac{d}{dx} g(x-u)\, du$$

$$= \int_{-\infty}^{\infty} f(u)\, g'(x-u)\, du$$

$$= (f * g')(x)$$

(assuming f and g are integrable)

$f \in C^0(\mathbb{R})$
$g \in C^n(\mathbb{R})$

$$(f * g)(x) = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$$

$$\frac{d^k}{dx^k} (f * g)(x) = \int_{-\infty}^{\infty} f(u) \frac{d^k}{dx^k} g(x-u)\, du$$

$$= \int_{-\infty}^{\infty} f(u)\, g^k(x-u)\, du$$

for $k \leq n$.

$f \rightarrow$ continuous

$g \rightarrow$ has derivatives up to n

$f * g \rightarrow$ has derivatives up to n

$\boxed{5}$ $(\alpha f + \beta g) * h = \alpha (f * h) + \beta (g * h)$

$$((\alpha f + \beta g) * h)(x) = \int_{-\infty}^{\infty} (\alpha f(u) + \beta g(u)) h(x-u) du$$

$$\int_{-\infty}^{\infty} (f(x) + g(x)) dx = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i) + g(x_i)] \Delta x_i$$

$$= \lim_{n \to \infty} \left(\sum_{i=1}^{n} f(x_i) \Delta x_i + \sum_{i=1}^{n} g(x_i) \Delta x_i \right)$$

$$= \underbrace{\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x_i}_{} + \underbrace{\lim_{n \to \infty} \sum_{i=1}^{n} g(x_i) \Delta x_i}_{}$$

$$= \int_{-\infty}^{\infty} f(x) dx + \int_{-\infty}^{\infty} g(x) dx$$

(assuming that functions are integrable)

$$\int_{-\infty}^{\infty} (\alpha f(u) + \beta g(u))\, h(x-u)\, du$$

$$= \int_{-\infty}^{\infty} \alpha f(u)\, h(x-u)\, du + \int_{-\infty}^{\infty} \beta g(u)\, h(x-u)\, du$$

$$= \underbrace{\alpha \int_{-\infty}^{\infty} f(u)\, h(x-u)\, du}_{f \star h} + \underbrace{\beta \int_{-\infty}^{\infty} g(u)\, h(x-u)\, du}_{g \star h}$$

$$= \alpha (f \star h) + \beta (g \star h)$$

$\boxed{6}$ $(T_b f) * g = T_b (f * g)$, $(T_b f)(x) := f(x-b)$

$$((T_b f) * g)(x) = \int_{-\infty}^{\infty} (T_b f)(u)\, g(x-u)\, du$$

$$= \int_{-\infty}^{\infty} f(u-b)\, g(x-u)\, du$$

$$\boxed{\begin{array}{l} w = u - b \\ u = w + b \\ dw = du \end{array}}$$

$$= \int_{-\infty}^{\infty} f(w)\, g(x-(w+b))\, dw$$

$$= \underbrace{\int_{-\infty}^{\infty} f(w)\, g((x-b)-w)\, dw}_{(f * g)(x-b)}$$

$$T_b(f \star g)(x) = (f \star g)(x-b)$$

$$= ((T_b f) \star g)(x)$$