Problem 2

(a) from 2D Taylor expansion we know that: (let $h_1 = h_2 = h$).

$$f_{i+1,j+1} = f_{ij} + h\partial_x f_{ij} + h\partial_y f_{ij} + \frac{1}{2}h^2\partial_{xx} f_{ij} + h^2\partial_{xy} f_{ij} + \frac{1}{2}h^2\partial_{yy} f_{ij} + O((\sqrt{2}h)^3)$$

$$f_{i-1,j-1} = f_{ij} - h\partial_x f_{ij} - h\partial_y f_{ij} + \frac{1}{2}h^2\partial_{xx} f_{ij} + h^2\partial_{xy} f_{ij} + \frac{1}{2}h^2\partial_{yy} f_{ij} + O((\sqrt{2}h)^3)$$

$$f_{i-1,j+1} = f_{ij} - h\partial_x f_{ij} + h\partial_y f_{ij} + \frac{1}{2}h^2\partial_{xx} f_{ij} - h^2\partial_{xy} f_{ij} + \frac{1}{2}h^2\partial_{yy} f_{ij} + O((\sqrt{2}h)^3)$$

$$f_{i+1,j-1} = f_{ij} + h\partial_x f_{ij} - h\partial_y f_{ij} + \frac{1}{2}h^2\partial_{xx} f_{ij} - h^2\partial_{xy} f_{ij} + \frac{1}{2}h^2\partial_{yy} f_{ij} + O((\sqrt{2}h)^3)$$

$$\frac{1}{4h^2}\left[(f_{i-1,j+1} + f_{i+1,j-1}) - (f_{i-1,j-1} + f_{i+1,j+1})\right]$$

$$= \frac{1}{4h^2}\left[2f_{ij} + h^2(\partial_{xx} f_{ij} - 2\partial_{xy} f_{ij} + \partial_{yy} f_{ij}) - 2f_{ij} - h^2(\partial_{xx} f_{ij} + 2\partial_{xy} f_{ij} + \partial_{yy} f_{ij})\right] + O(h)$$

$$= -\partial_{xy} f_{ij} + O(h).$$

(b) from (a) we know that it is an approximation with consistency oder 1

(c). ~~$\mathcal{F}[g](u,v) = \mathcal{F}[\frac{1}{4h^2}[f(u-1,v+1) + f(u+1,$~~

$$\mathcal{F}[g](u,v) = \mathcal{F}\left[\frac{1}{4h^2}\left[f(x+h,y-h) + f(x-h,y+h) - (f(x+h,y+h) + f(x-h,y-h))\right]\right]$$

$$= \frac{1}{4h^2}\left[e^{i2\pi h(u-v)} + e^{i2\pi h(v-u)} - (e^{i2\pi h(u+v)} + e^{i2\pi h(-u-v)})\right]\mathcal{F}[f](u,v).$$

$$= \frac{1}{4h^2}\cdot 2\left[\cos(u-v)h\cdot 2\pi - \cos(u+v)h\cdot 2\pi\right]\mathcal{F}[f](u,v),\quad w(u,v) = \frac{1}{2h^2}\left[\cos(u-v)2\pi h - \cos(u+v)2\pi h\right].$$

(d). $\cos x = 1 - \frac{x^2}{2} + O(x^4)$

$$\frac{1}{2h^2}\left[1 - \frac{4\pi^2h^2(u-v)^2}{2} - \left(1 - \frac{4\pi^2h^2(u+v)^2}{2}\right)\right]\mathcal{F}[f](u,v) + O(h^2(u^2+v^2)^2).$$

$$= 4\pi^2 uv\,\mathcal{F}[f](u,v) + O(h^2(u^2+v^2)^2),\ \cancel{w(u,v)}$$

~~thus $\mathcal{F}[g](u$~~

$$\mathcal{F}[Df](u,v) = -(i2\pi u)(i2\pi v)\mathcal{F}[f](u,v) = 4\pi^2 uv\,\mathcal{F}[f](u,v).$$

thus: $\mathcal{F}[g](u,v) - \mathcal{F}[Df](u,v) = O(h^2(u^2+v^2)^2)$.

(e): For ~~a~~ fixed frequency (u,v), the approximation error is $O(h^2)$. For a fixed grid size h, the approximation error is $O((u^2+v^2)^2)$.

$O(h^2)$ is of course $O(h)$. But analysis in the frequency domain gives more accurate error approximation.