ISHIKA

Problem 1

Given

1D synthesis Function

$$\varphi_{int}(x) := \begin{cases} \frac{1}{2} + \frac{1}{2}\cos(\pi x) & \text{for } |x| < 1 \\ 0 & \text{else} \end{cases}$$

means φ_{int} is in interval $(-1, 1)$. it zero otherwise

a) To Prove :- It satisfy partition of unity

$$\sum_{k \in \mathbb{Z}} \varphi_{int}(x-k) = 1 \quad \text{for all } x \in \mathbb{R}$$

Proof

$$\sum_{k \in \mathbb{Z}} \varphi_{int}(x-k) = \sum_{k \in \mathbb{Z}} \left[\frac{1}{2} + \frac{1}{2}\cos(\pi(x-k))\right] \quad -(1)$$

when $|x-k| < 1$

since $\cos(\pi(x-k)) = \cos(\pi x - \pi k)$

By Formula $\cos(A-B) = \cos A \cos B + \sin A \sin B$

$$\cos(\pi x - \pi k) = \cos \pi x \cos \pi k + \sin \pi x \sin \pi k$$

$$\cos \pi x \cdot (-1)^k + \sin \pi x \cdot 0 \qquad \left[\cos(\pi k) = (-1)^k, \sin \pi k = 0\right]$$

$$= (-1)^k \cos \pi x$$

Put it equation 1

$$\Rightarrow \sum_{k \in \mathbb{Z}} \left[\frac{1}{2} + \frac{1}{2} (-1)^k \cos \pi x \right]$$

Because φ_{int} lies in $[-1, 1]$, only two integers nearest to x will satisfy $|x-k| < 1$

Let two integers be $k = \lfloor x \rfloor$ and $k = \lfloor x \rfloor + 1$

$$\Rightarrow \left[\frac{1}{2} + \frac{1}{2} (-1)^k \cos \pi x \right] + \left[\frac{1}{2} + \frac{1}{2} (-1)^{k+1} \cos \pi x \right]$$

$$\Rightarrow \left[\frac{1}{2} + \frac{1}{2} (-1)^k \cos \pi x \right] + \left[\frac{1}{2} - \frac{1}{2} (-1)^k \cos \pi x \right]$$

$$\left(\because (-1)^{k+1} = (-1)^k \cdot (-1)^1 \right)$$

$$\Rightarrow \frac{1}{2} + \frac{1}{2} + \cancel{\frac{1}{2} (-1)^k \cos \pi x} - \cancel{\frac{1}{2} (-1)^k \cos \pi x}$$

$$\Rightarrow \frac{2}{2} = 1$$

$$\sum_{x \in \mathbb{Z}} \varphi_{int} (x-k) = 1 \quad \text{for all } x \in \mathbb{R}$$

b) regularity order of given interpolant for regularity we need to check how many continuous derivatives are there

$$\psi_{int} \in C^r$$

Step 1 :- check if ψ_{int} is continuous

for $x = \pm 1$

$$\psi_{int} = \frac{1}{2} + \frac{1}{2} \cos(\pi x)$$

$x \to -1$

$$\psi_{int} = \frac{1}{2} + \frac{1}{2} \cos(-\pi)$$

$$\Rightarrow \frac{1}{2} + \frac{1}{2}(-1) = 0$$

for $x \geq 1$ it given that

$$\psi_{int} = 0$$

so its continuous at $x = \pm 1$

Step 2 first derivative

given function $\psi_{int} = \frac{1}{2} + \frac{1}{2} \cos(\pi x)$

differentiate

$$\Rightarrow 0 + \frac{1}{2} \cdot (-\sin(\pi x)) \cdot \pi$$

$\Rightarrow -\frac{\pi}{2} \sin \pi x$

for $x = -1$

$\Rightarrow -\frac{\pi}{2} \sin(-\pi)$

$\Rightarrow 0$ $[\sin \pi = 0]$

and for $x \geq 1$ its 0

So it is continuous on first derivative

step 3 second derivative

first derivative $= -\frac{\pi}{2} \sin \pi x$

differentiate

$\Rightarrow -\frac{\pi}{2} \cos \pi x \cdot \pi$

$\Rightarrow -\frac{\pi^2}{2} \cos \pi x$

for $x = -1$

$\Rightarrow -\frac{\pi^2}{2} (-1)$

$\Rightarrow \frac{\pi^2}{2}$

Now this function becomes Discontinuous

so

$$\varphi_{int} \in C^1$$

regularity order is 1

c) show truncated sinc function

$$\varphi_{int}(x) = \begin{cases} \frac{\sin \pi x}{\pi x} & \text{for } |x| \leq 2 \\ 0 & \text{else} \end{cases}$$

violates the partition of unity

here it given for function

$(x-k) \leq 2$ means here the two values possible $[k-2, k+2]$

we have to show

$$\sum_{k \in \mathbb{Z}} \varphi_{int}(x-k) \neq 1$$

Proof by example

$x = 0.5$

$$u(0.5) = \sum_{k \in \mathbb{Z}} \varphi_{int}(0.5 - k)$$

But we know by given φ_{int}

$$\varphi_{int}(0.5-k) \neq 0 \quad \text{only when} \quad |0.5-k| \leq 2$$

for values of k

$$|0.5-k| \leq 2$$

$$-2 \leq 0.5-k \leq 2$$
$$-2.5 \leq -k \leq 1.5$$
$$-1.5 \leq k \leq 2.5$$

So

$$k \in \{-1, 0, 1, 2\}$$

Now computing each term

when $k = -1$

$$x - k = 1.5$$

$$\sin \pi (x-k) = \sin(1.5\pi) = -1$$

$$\frac{\sin \pi(x-k)}{\pi(x-k)} = \frac{-1}{\pi \times 1.5} = -\frac{2}{3\pi}$$

when $k = 0$

$$x - k = 0.5$$

$\sin\pi(x-k) = \sin(0.5\pi) = 1$

$$\frac{\sin\pi(x-k)}{\pi(x-k)} = \frac{1}{0.5\pi} = \frac{2}{\pi}$$

when $k=1$

$x-k = -0.5$

$\sin\pi(-0.5) = -1$

$$\Rightarrow \frac{-1}{-0.5\pi} \Rightarrow \frac{2}{\pi}$$

when $x=2$

$x-k = -1.5$

$\sin(\pi(-1.5)) = +1$

$$\Rightarrow \frac{+1}{-1.5\pi} = -\frac{2}{3\pi}$$

Add all the term

$$-\frac{2}{3\pi} + \frac{2}{\pi} + \frac{2}{\pi} + \left(-\frac{2}{3\pi}\right)$$

$$-\frac{4}{3\pi} + \frac{4}{\pi} \Rightarrow \frac{1}{\pi}\left(\frac{-4+12}{3}\right)$$

$$= \frac{8}{3\pi} \approx 0.85 \neq 1$$

doesn't satisfy partition of unity.

Problem 2

given

$f_1 = 86$ at $x_1 = 0.8$

$f_2 = 60$ $x_2 = 1$

$f_3 = 42$ $x_3 = 1.2$

given synthesis function

$$\beta_2(x) := \begin{cases} \frac{3}{4} - x^2 & \text{for } |x| < \frac{1}{2} \\ \frac{1}{2}\left(\frac{3}{2} - |x|\right)^2 & \text{for } \frac{1}{2} \leq |x| < 3/2 \\ 0 & \text{else} \end{cases}$$

a) for coeff c_0, c_1, c_2

$$f(x_j) = \sum_i c_i \cdot \beta_2(x_j - i)$$

for linear system

coeff c_0, c_1, c_2 are centered around integer lattice points for our x values $0.8, 1.0, 1.2$ they lie near grid node $0, 1, 2$

so we will take $i = 0, 1, 2$

At $x = 0.8$

- $x - 0 = 0.8 \rightarrow |0.8| > 0.5$, use second case

$$\beta_2(0.8) = \frac{1}{2}(1.5 - 0.8)^2 = \frac{1}{2}(0.7)^2 = 0.245$$

- $x - 1 = -0.2 \rightarrow |-0.2| < 0.5$

$$\beta_2(-0.2) = \frac{3}{4} - (-0.2)^2 = 0.75 - 0.04 = 0.71$$

- $x - 2 = -1.2 \rightarrow |-1.2| > 0.5$ so

$$\beta_2(-1.2) = \frac{1}{2}(1.5 - 1.2)^2 = \frac{1}{2}(0.3)^2 = 0.045$$

so first row

$$[\beta_2(0.8), \beta_2(-0.2), \beta_2(-1.2)] = [0.245, 0.71, 0.045]$$

At $x = 1.0$

- $x - 0 = |1.0| > 0.5$

$$\beta_2(1.0) = \frac{1}{2}(0.5)^2 = 0.125$$

- $x - 1 = 0.0 \rightarrow |0.0| < |0.5$

$\beta_2(0) = 0.75$

- $x - 2 = |-1| > 0.5$

$\beta_2(-1) = \frac{1}{2}(0.5)^2 = 0.125$

Second row = [0.125, 0.75, 0.125]

At $x = 1.2$

- $x - 0 = |1.2| > 0.5$

$\beta_2(1.2) = \frac{1}{2}(0.3)^2 = 0.045$

- $x - 1 = |0.2| < 0.5$

$\beta_2(0.2) = 0.75 - 0.04 = 0.71$

- $x - \cancel{3} = |0.8| > 0.5$

$= \frac{1}{2}(0.7)^2 = 0.245$

Third row = [0.045, 0.71, 0.245]

Linear System

$$\begin{bmatrix} 0.245 & 0.71 & 0.045 \\ 0.125 & 0.75 & 0.125 \\ 0.045 & 0.71 & 0.245 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 86 \\ 60 \\ 42 \end{bmatrix}$$

Verifying

$c(c_0, c_1, c_2) = (245, 35, 25)$

row 1st $= 0.245 \times 245 + 0.71 \times 35 + 0.045 \times 25$
$= 86$

row 2nd $= 0.125 \times 245 + 0.75 \times 35 + 0.125 \times 25$
$= 60$

row 3rd $= 0.045 \times 245 + 0.71 \times 35 + 0.245 \times 25$
$= 42$

b) Analytic expression

$$u(x) = \sum_{i=0}^{2} c_i \cdot \beta_2(x-i)$$

$$= 245 \cdot \beta_2(x) + 35 \cdot \beta_2(x-1) + 25 \cdot \beta_2(x-2)$$

given locations

$x = 0.5$

- $\beta_2(0.5) = \frac{1}{2}(1)^2 = 0.5$

$\beta_2(-0.5) = 0.5$

$\beta_2(-1.5) = 0$

$u(0.5) = 245 \times 0.5 + 35 \times 0.5 + 0$

$\Rightarrow 140$

- $x = 1$

from part a $= 60$

- $x = 1.5$

$\beta_2(1.5) = 0$

$\beta_2(0.5) = 0.5$

$\beta_2(-0.5) = 0.5$

$u(1.5) = 35 \times 0.5 + 25 \times 0.5$

$= 17.5 + 12.5 = 30$

c) $f(x) = 100x^2 - 310x + 270$

locations $= \frac{1}{2}, 1, \frac{3}{2}$

$$F\left(\frac{1}{2}\right) = 100 \times \left(\frac{1}{2}\right)^2 - 310 \times \frac{1}{2} + 270$$

$$\Rightarrow 25 - 155 + 270 = 140$$

$$F(1) = 100 - 310 + 270 = 60$$

$$F(3/2) = 100 \times \left(\frac{3}{2}\right)^2 - 310 \times \frac{3}{2} + 270$$

$$\Rightarrow 25 \times 9 - 155 \times 3 + 270$$

$$225 - 465 + 270 = 30$$

Interpolation is same

d)

→ Quadratic B-splines have lower smoothness

→ They are less accurate for function with curve. Cubic model curve better.

→ visual quality is poor.

→ Cubic B-spline create twice continuously differentiable (C^2) interpolant whereas quadratic has only C^1