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Problem 1

Given

$F = (f_{i,j})$ ⇒ noisy image

$g = (g_{i,j})$ ⇒ original image

$n = (n_{i,j})$ ⇒ additive noise with mean 0.

$f_{i,j} = g_{i,j} + n_{i,j}$

a) we have to find the case where $PSNR(F,g) = 0$

Formula

$$PSNR(F,g) := 10 \log_{10} \left( \frac{(255)^2}{MSE(F,g)} \right)$$

∴ here $255^2$ is maximal MSE

we can write it as $MAX^2$

So

$$0 = 10 \log_{10} \frac{(MAX^2)}{MSE(F,g)}$$

Divide both side by 10

$$0 = \log_{10} \left( \frac{MAX^2}{MSE(F,g)} \right)$$

To get rid up log we can raise it to Power of 10

$$10^0 = \frac{(MAX)^2}{MSE(F,g)}$$

$$1 = \frac{(MAX)^2}{MSE(f,g)}$$

$$MSE(f,g) = (MAX)^2$$

So PSNR(f,g) will be equal to zero when MSE is equal to maximum possible Intensity difference

It means noisy image f is as far as possible from original image means its completely noisy.

b) Now given a Filtered version u of f s.t.

$$PSNR(u,g) = PSNR(f,g) + 30dB \quad \text{--- (1)}$$

By Formula

$$PSNR(u,g) = 10 \log_{10} \frac{(MAX^2)}{MSE(u,g)}$$

Put in eq-(1)

$$10 \log_{10} \frac{(MAX^2)}{MSE(u,g)} = 10 \log_{10} \frac{(MAX^2)}{MSE(f,g)} + 30$$

$$10 \log_{10} \left(\frac{MAX^2}{MSE(u,g)}\right) - 10 \log_{10} \frac{(MAX^2)}{MSE(f,g)} = 30$$

we know $\log A - \log B = \log(A/B)$

$$10 \log_{10}\left[\frac{\cancel{MAX^2}}{MSE(f,g)} \Big/ \frac{\cancel{MAX^2}}{MSE(f,g)}\right] = 30$$

$$10 \log_{10}\left[\frac{MSE(f,g)}{MSE(f,g)}\right] = 30$$

$$\log_{10}\left[\frac{MSE(f,g)}{MSE(f,g)}\right] = 3$$

Raising it to power of 10 on both side

$$\left[\frac{MSE(f,g)}{MSE(f,g)}\right] = 1000$$

It means MSE decreased by factor of 1000.

c) Assume $n \to 0$

$PSNR(f,g) = ?$

Given

$$f_{i,j} = g_{ij} + n_{ij}$$

if $n_{i,j} = 0$

$f_{i,j} = g_{i,j}$ means $MSE(f,g) = 0$

Now Putting in Formula

$$PSNR(F,g) = 10 \log_{10} \frac{(MAx^2)}{(MSE(F,g)}$$

$$PSNR(F,g) = 10 \log_{10} \frac{(MAx^2)}{0} = \infty$$

This means noisy image is same as original.

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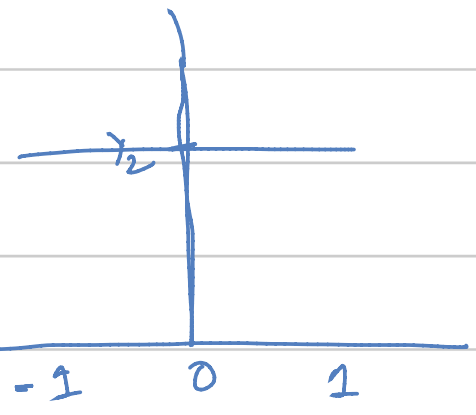
Problem 3




a) Given Continuous signal

$$f(x) = \begin{cases} 1/2 & (-1 \le x \le 1) \\ 0 & \text{else} \end{cases}$$

we know this is box function with area = 1

first convolution $h_1 = f * f$

b)

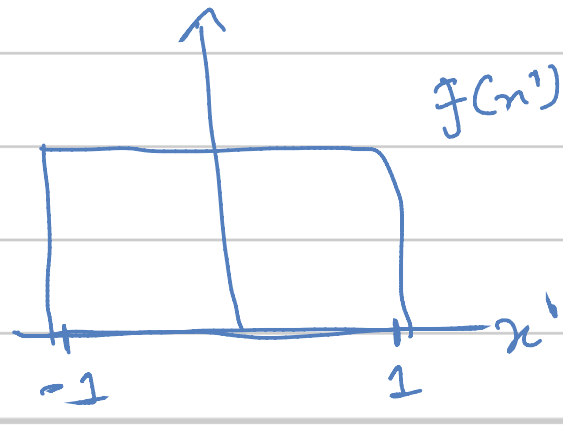
$$(f * f)(n) = \int_{-\infty}^{\infty} f(x') \cdot f(n-x')\, dx'$$

Case 1

$n < -2$

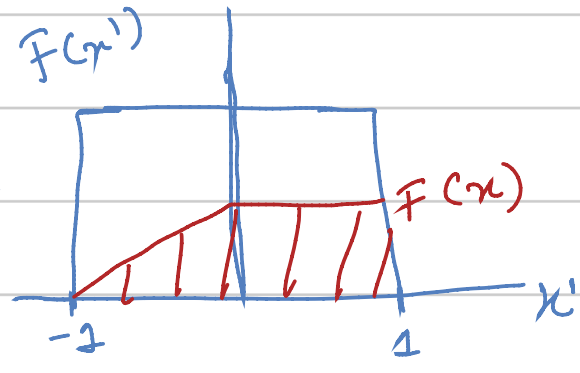
for this two boxes will not collide.

Convolution = 0 = $h_1(n)$



Case 2

$-2 \leq n < 0$




→ $F(x') \neq 0$ only for $-1 \leq x' \leq 1$

$F(n-x') \neq 0$ only when

$-1 \leq n-x' \leq 1$

$n-1 \leq -x' \leq n+1$

multiply by $-1$

$n+1 \geqslant x' \geqslant n-1$

Now compute using formula

$$h_1(n) = \int_{-1}^{n+1} \frac{1}{2} \cdot \frac{1}{2}\, dx' = \frac{1}{4} \int_{-1}^{n+1} 1\, dx' = \frac{1}{4} [x]_{-1}^{n+1}$$

$$= \frac{1}{4} [n+1-(-1)] = \frac{1}{4}[n+2]$$

$$h_1(n) = \frac{1}{4}(n+2)$$

Case 3

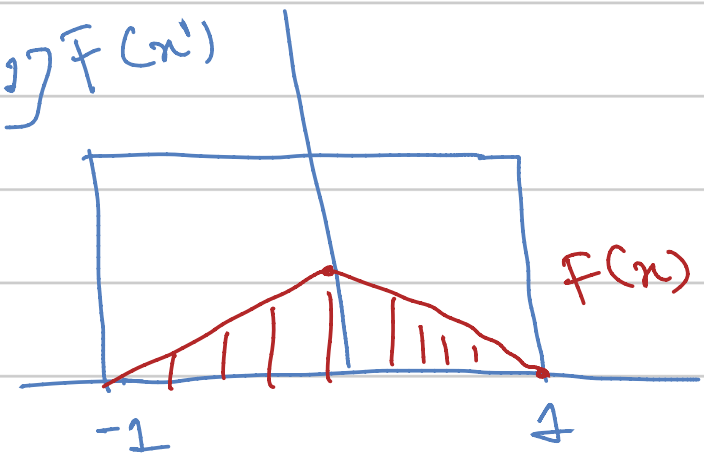
$0 \leq n \leq 2$

$F(x') \neq 0$ only for $x' \in [-1, 1]$

$F(n-x') \neq 0$ only if

$-1 \leq n-x' \leq 1$

$x' \in [n-1, n+1]$




$$h_1(n) = \int_{n-1}^{1} \frac{1}{2} \cdot \frac{1}{2}\, dx' = \frac{1}{4} [x]_{n-1}^{1} = \frac{1}{4}[1-n+1]$$

$$\Rightarrow \frac{1}{4}(2-n)$$

Case 4

$x > 2$

here again they don't overlap

Convolution = 0

ii) $h_2 = f * f * f$

$h_2(x) = h_1(x) * f$

$$h_2(x) = \int_{-1}^{1} h_1(x - x') f(x') dx'$$

Now we know

$$f(x) = \begin{cases} \frac{1}{2} & -1 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

$$h_1(x) = \begin{cases} \frac{1}{4}(2 - |x| & -2 \le x \le 2 \\ 0 & \text{else} \end{cases}$$

Case 1

$x < -3$ & $x > 3$

$h_2(-3) = 0$ & $h_2(3) = 0$

they will not overlap

Case 2

$-3 \le n \le -1$

$$h_1(n) = \frac{1}{4}(2+n)$$

here $n-t$ ranges from $n-1$ to $n+1$

for $t \in [-1, 1]$, $n-t \in [n-1, n+1]$

the nonzero part of $h_1(n-t)$ is $\frac{1}{4}(2+(n-t))$ but only 2 if $n-t \in [-2,0]$

Now

$$h_2(n) = \frac{1}{2}\int_{-1}^{n+2} \frac{1}{4}(n-t+2)\,dt$$

$$\frac{1}{8}\left[nt - \frac{t^2}{2} + 2t\right]_{-1}^{n+2} = \frac{1}{8}\left[(n+2)t - \frac{t^2}{2}\right]_{-1}^{n+2}$$

$$= \frac{1}{8}\left[(n+2)(n+2) - \frac{(n+2)^2}{2} - \left((n+2)(-1) - \frac{1}{2}\right)\right]$$

$$\frac{1}{8}\left[(n+2)^2 - \frac{(n+2)^2}{2} + n + 2 + \frac{1}{2}\right]$$

$$\frac{1}{8}\times\frac{1}{2}\left[2(n+2)^2 - (n+2)^2 + 2n + 4 + 1\right]$$

$$\frac{1}{16}\left[(n+2)^2 + 2n + 5\right]$$

$$\frac{1}{16}\left[n^2 + 4 + 4n + 2n + 5\right] = \frac{1}{16}\left[n^2 + 6n + 9\right]$$

$$\frac{1}{16}(n+3)^2$$

Case 2: $-1 \le x \le 1$

here

for $t \in [x-0, x+2]$, $h_1(x-t) = \frac{1}{4}(2-(x-t))$

for $t \in [x-2, x,]$ $h_1(x-t) = \frac{1}{4}((x-t+2))$

$$h_2(x) = \frac{1}{2}\left( \int_{-1}^{x} \frac{1}{4}(x-t+2)\,dt + \int_{x}^{1} \frac{1}{4}(2-x+t)\,dt \right)$$

$$\frac{1}{2}\times\frac{1}{4}\left( \left[ xt+2t-\frac{t^2}{2} \right]_{-1}^{x} + \left[ 2t-xt+\frac{t^2}{2} \right]_{x}^{1} \right)$$

$$\frac{1}{8}\left[ (x+2)t-\frac{t^2}{2} \right]_{-1}^{x} + \left[ (2-x)t+\frac{t^2}{2} \right]_{x}^{1}$$

$$\frac{1}{8}\left[ \left((x+2)x - \frac{x^2}{2}\right) - \left((x+2)(-1) - \frac{1}{2}\right) \right] +$$

$$\left[ (2-x)(1) + \frac{1}{2} - \left((2-x)(x) - \frac{x^2}{2}\right) \right]$$

$$\Rightarrow \frac{1}{8}\left[ \left(x^2+2x-\frac{x^2}{2}\right) + \left(x+2+\frac{1}{2}\right) + \left(2-x+\frac{1}{2}\right) - (2x - x^2 + \frac{x^2}{2}) \right]$$

$$\frac{1}{8}\left[ \frac{x^2}{2} + 2x + x + 2 + \frac{1}{2} + 2 - x + \frac{1}{2} - 2x + \frac{x^2}{2} \right)$$

$$= \frac{1}{8}(3-x^2)$$

Case 3:-

$1 \le x \le 3$

$$h_2(n) = \frac{1}{2}\int_{n-2}^{1} \frac{1}{4}(2-(n-t))\,dt$$

$$\frac{1}{2}\times\frac{1}{4}\left[2t - nt + \frac{t^2}{2}\right]_{n-2}^{1}$$

$$\frac{1}{8}\left[(2-n) + \frac{t^2}{2}\right]_{n-2}^{1}$$

$$\Rightarrow \frac{1}{8}\left[(2-n)\cdot 1 + \frac{1}{2} - \left[(2-n)(n-2) + \frac{(n-2)^2}{2}\right]\right.$$

$$\frac{1}{8}\left[2-n + \frac{1}{2} - \left[-(n-2)^2 + \frac{(n-2)^2}{2}\right]\right)$$

$$\frac{1}{8}\left[2-n + \frac{1}{2} + (n-2)^2 - \frac{(n-2)^2}{2}\right)$$

$$\frac{1}{16}\left[4 - 2n + 1 + 2(n-2)^2 - (n-2)^2\right]$$

$$\frac{1}{16}\left[5 - 2n + n^2 + 4 - 4n\right]$$

$$\frac{1}{16}\left[3-n\right]^2$$

Result

$$h_2(n) = \begin{cases} \frac{1}{16}(n+3)^2 & -3 \le n \le -1 \\ \frac{1}{8}(3-n^2) & -1 \le n \le 1 \\ \frac{1}{16}(3-n)^2 & 1 \le n \le 3 \\ 0 & \text{else} \end{cases}$$

(b)

for $h_1(n)$ at points $-2, -1, 0, 1, 2$

$h_1(-2) = 0$

$h_1(-1) = 1/4$

$h_1(0) = 1/2$

$h_1(1) = 1/4$

$h_1(2) = 0$

for $h_2(n)$ at points $-3, -2, -1, 0, 1, 2, 3$

$h_2(-3) = 0$

$h_2(-2) = 1/16$

$h_2(-1) = 1/4$

$h_2(0) = 3/8$

$h_2(1) = 1/4$

$h_2(2) = 1/16$

$h_2(3) = 0$

c)

Both sequence of convolution smooths the signal but in continuous convolution smooth curves are due to integration and for discrete convolution there are stepwise curves due to summation