Assignment - 2
ISHIKA

Problem - 1

Formula for Continuous 1-D Fourier transform

$$\mathcal{F}[f](u) = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi ux}\, dx$$

a) Linearity

To prove :- $\mathcal{F}[af(x) + bg(x)](u) = a\mathcal{F}[f](u) + b\,\mathcal{F}[g](u)$

Let $h(x) = af(x) + bg(x)$

$$\mathcal{F}[h](u) = \int_{-\infty}^{\infty} (af(x) + bg(x))\, e^{-i2\pi ux}\, dx$$

$$\Rightarrow a\int_{-\infty}^{\infty} f(x)\, e^{-i2\pi ux}\, dx + b\int_{-\infty}^{\infty} g(x)\, e^{-i2\pi ux}\, dx$$

$$\Rightarrow a\,\mathcal{F}[f](u) + b\,\mathcal{F}[g][u]$$ Proved

b) Spatial Shift

To prove :- $\mathcal{F}[f(x-a)](u) = \exp(-i2\pi ua)\cdot\mathcal{F}[f](u)$

$$\mathcal{F}[f(x-a)](u) = \int_{-\infty}^{\infty} f(x-a)\, e^{-i2\pi ux}\, dx$$

Let $x-a = y$, $x = y+a$, $dx = dy$

$$\Rightarrow \int_{-\infty}^{\infty} f(y)\, e^{-2\pi u(y+a)}\, dy$$

$$\Rightarrow \int_{-\infty}^{\infty} f(y) \cdot e^{-2\pi uy - 2\pi ua}\, dy$$

$$\Rightarrow e^{-2\pi ua} \int_{-\infty}^{\infty} f(y)\, e^{-2\pi uy}\, dy$$

$$\Rightarrow e^{-2\pi ua} \cdot \mathcal{F}[f](u)$$ proved

c) Frequency shift

To prove :- $\mathcal{F}[F(x) \cdot \exp(-i2\pi u_0 x)](u) = \mathcal{F}[f](u+u_0)$

$$\mathcal{F}[F](u) = \int_{-\infty}^{\infty} f(x) \cdot e^{-i2\pi u_0 x} \cdot e^{-i2\pi ux}\, dx$$

$$\Rightarrow \int_{-\infty}^{\infty} F(x)\, e^{-i2\pi x[u_0+u]}\, dx$$

$$\Rightarrow \mathcal{F}[F](u+u_0)$$ proved

d) Scaling

To prove:- $\mathcal{F}[f(ax)](u) = \frac{1}{|a|} \cdot \mathcal{F}[f](\frac{u}{a})$

By Formula

$$\mathcal{F}[f](u) = \int_{-\infty}^{\infty} f(ax) \cdot e^{-i2\pi ux} \, dx$$

let $y = ax$ $\Rightarrow x = y/a$ $dx = \frac{1}{a} dy$

$$\Rightarrow \int_{-\infty}^{\infty} f(y) \, e^{-2\pi u(y/a)} \, \frac{1}{a} \cdot dy$$

$$\Rightarrow \frac{1}{|a|} \int_{-\infty}^{\infty} f(y) \, e^{-2\pi u(y/a)} \, dy$$

$$\Rightarrow \frac{1}{|a|} \mathcal{F}[f][u/a]$$

proved

e) Convolution

To prove:- $\mathcal{F}[(f * g)(x)](u) = \mathcal{F}[f](u) + \mathcal{F}[g](u)$

we know convolution formula

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x') \, g(x - x') \, dx'$$

$$\mathcal{F}[(f*g)](u) = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x')\, g(x-x')\, dx'\right) e^{-i2\pi ux}\, dx$$

swap integrals

$$\Rightarrow \int_{-\infty}^{\infty} f(x')\left(\int_{-\infty}^{\infty} g(x-x')\, e^{-i2\pi ux}\, dx\right) dx'$$

let $y = x - x'$ $\quad y + x' = dx$, $dx = dy$

$$\Rightarrow \int_{-\infty}^{\infty} f(x')\left(\int_{-\infty}^{\infty} g(y)\, e^{-i2\pi u(y+x')}\, dy\right) dx'$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x')\left(e^{-i2\pi ux'}\int_{-\infty}^{\infty} g(y)\, e^{-i2\pi uy}\, dy\right) dx'$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x')\cdot e^{-i2\pi ux'}\, dx' \cdot \int_{-\infty}^{\infty} g(y)\, e^{-i2\pi uy}\, dy$$

$$\mathcal{F}[f](u)\cdot \mathcal{F}[g](u)$$

f) Differentiation

To prove:- $\mathcal{F}[f'](u) = i2\pi u\cdot \mathcal{F}[f](u)$

$$\mathcal{F}[f'](u) = \int_{-\infty}^{\infty} f'(x)\, e^{-2\pi u x}\, dx$$

let $u = e^{-i2\pi ux}$ & $dv = f'(x)\,dx$

$du = -i2\pi u . e^{-i2\pi ux}\, dx$ $\quad v = f(x)$

By integration by parts $\left[\int u\,dv = uv - \int v\,du\right.$

$$\Rightarrow \int f'(x)\, e^{-2\pi ux}\, dx = f(x) \cdot e^{-i2\pi ux}\Big|_{-\infty}^{\infty} -$$

$$\int f(x) \cdot (-i\,2\pi u) \cdot e^{-i2\pi ux}\, dx$$

$\Rightarrow$ Now using given limit

$f(x) \cdot e^{-i2\pi ux}$ Vanishes

Remaining Part

$$\Rightarrow -i\,2\pi u \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi ux}\, dx$$

$$\Rightarrow -i\,2\pi u\, F[f](u)$$

Problem 2

given function

$$f(x) = \begin{cases} 0 & (x \le -3) \\ \dfrac{x^2+6x+9}{16} & (-3 < x \le -1) \\ \dfrac{6-2x^2}{16} & (-1 < x \le 1) \\ \dfrac{x^2-6x+9}{16} & (1 < x \le 3) \\ 0 & (x > 3 \end{cases}$$

This is the result what we got when we solved Problem 4 for $h_2(x)$ in HW 1

$$h_2(x) = g * g * g$$

where

$$g(x) = \begin{cases} 1/2 & (-1 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

Fourier transform

$$F[g](u) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i u x}\, dx$$

$$\mathcal{F}[g](u) = \int_{-1}^{1} \frac{1}{2} e^{-2\pi i u x} \, dx$$

$$\Rightarrow \frac{1}{2}\left[\frac{e^{-2\pi i u x}}{-2\pi i u}\right]_{-1}^{1}$$

$$\frac{1}{2}\left[\frac{e^{-2\pi i u} - e^{2\pi i u}}{-2\pi i u}\right]$$

Now we know by Euler formula

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$e^{2\pi i u} - e^{-2\pi i u} = 2i\sin(2\pi u)$$

So

$$\mathcal{F}[g](u) = \frac{1}{2}\left(\frac{-2i\sin(2\pi u)}{-2\pi i u}\right.$$

$$\frac{1}{2}\left(\frac{2\sin(2\pi u)}{2\pi u}\right)$$

$$= \frac{\sin(2\pi u}{2\pi u}$$

normalizing the Function

$$\mathcal{F}[g](u) = \mathrm{sinc}\,(2\pi u)$$

Now $h_g[x] = g * g * g = f(x)$

$$\mathcal{F}[f](u) = \left(\mathrm{sinc}\,(2\pi u)\right)^3$$