第二章 矩阵的分解

2.5 谱分解

1. 正规矩阵的谱分解

设A是正规矩阵, 则$\exists U \in U^{n\times n}$, 满足: $A = U\mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}U^H$, 若令$U = (\alpha_1, \ldots, \alpha_n)$, 则

$$A = (\alpha_1, \ldots, \alpha_n)\mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}\begin{pmatrix} \alpha_1^H \\ \alpha_2^H \\ \vdots \\ \alpha_n^H \end{pmatrix}$$

$$= \lambda_1\alpha_1\alpha_1^H + \ldots + \lambda_n\alpha_n\alpha_n^H \qquad (1)$$

其中α_i是矩阵A的特征值λ_i所对应的单位特征向量，$\alpha_i\alpha_i^H$是n阶矩阵，(1)式称为A的谱分解. 由于λ_i可能是重根，所以上式可以化简.

设正规矩阵A有r个互异特征值$\lambda_1, \ldots, \lambda_r$，特征值$\lambda_i$对应的代数重数是$n_i$，则$\lambda_i$所对应的$n_i$个单位特征向量记为$\alpha_i^1, \ldots, \alpha_i^{n_i}$，则$A$的谱分解可写成

$$A = \sum_{j=1}^{r} \lambda_j \sum_{i=1}^{n_j} \alpha_j^i (\alpha_j^i)^H = \sum_{j=1}^{r} \lambda_j E_j,$$

其中$E_j=\sum_{i=1}^{n_j}\alpha_j^i(\alpha_j^i)^H$, 显然, $E_j^H=E_j=(E_j)^2, E_jE_i=0(i\neq j)$

$E_1=\alpha_1\alpha_1^H+\alpha_2\alpha_2^H,$

$E_1{}^2=(\alpha_1\alpha_1^H+\alpha_2\alpha_2^H)(\alpha_1\alpha_1^H+\alpha_2\alpha_2^H)$

$=\alpha_1\alpha_1^H\alpha_1\alpha_1^H+\alpha_1\alpha_1^H\alpha_2\alpha_2^H+\alpha_2\alpha_2^H\alpha_1\alpha_1^H+\alpha_2\alpha_2^H\alpha_2\alpha_2^H=\alpha_1\alpha_1^H+\alpha_2\alpha_2^H=E_1,$

$E_2=\alpha_3\alpha_3^H+\alpha_4\alpha_4^H+\alpha_5\alpha_5^H,$

$E_1E_2=(\alpha_1\alpha_1^H+\alpha_2\alpha_2^H)(\alpha_3\alpha_3^H+\alpha_4\alpha_4^H+\alpha_5\alpha_5^H)=0.$

定理1: 设n阶矩阵A有r个互异特征值$\lambda_1, \dots, \lambda_r$, 特征值$\lambda_i$的代数重数是$n_i$, 则$A$为正规矩阵$\Leftrightarrow$存在$r$个$n$阶矩阵$E_1, \dots, E_r$, 满足:

1)$A = \sum_{j=1}^{r} \lambda_j E_j$; 2)${E_j}^H = E_j = \left(E_j\right)^2$; 3) $E_j E_i = 0 (i \neq j)$; 4) $\sum_{j=1}^{r} E_j = I$; 5)满足上述性质的E_j唯一; 6) $r\left(E_j\right) = n_j$.

证明: $\Rightarrow$:1),2),3)已证.

4)令$U_j = \left(\alpha_j^1, \dots, \alpha_j^{n_j}\right)$, 则$E_j = U_j {U_j}^H$.

于是$E_1+E_2+\ldots+E_r=U_1{U_1}^H+U_2{U_2}^H+\ldots+$

$$U_r{U_r}^H=(U_1,U_2,\ldots,U_r)\begin{pmatrix}{U_1}^H\\{U_2}^H\\\vdots\\{U_r}^H\end{pmatrix}=UU^H=I.$$

5) 现证E_j是唯一的. 不难证明$E_jA=\lambda_jE_j=AE_j$, 又若G_j满足1)-4), 故G_j满足$G_jA=\lambda_jG_j=AG_j$, 因此

$$(\lambda_i-\lambda_j)E_jG_i=\lambda_iE_jG_i-\lambda_jE_jG_i$$

$= E_j(\lambda_i G_i) - (\lambda_j E_j)G_i = E_j A G_i - E_j A G_i = 0$, 因为 $\lambda_i - \lambda_j \neq 0$, 所以$E_j G_i = 0$, 于是:

$E_j = E_j I = E_j \sum_{i=1}^{r} G_i = E_j G_j = \left(\sum_{i=1}^{r} E_i\right) G_j = G_j$.

6) 因为$E_j = U_j {U_j}^H$, 由上节引理知$r(E_j) = r(U_j) = n_j$.

$\Leftarrow$:$AA^H = \left(\sum_{j=1}^{r} \lambda_j E_j\right)\left(\sum_{j=1}^{r} \overline{\lambda_j} {E_j}^H\right) = \sum_{i=1}^{r} \lambda_i \overline{\lambda_i} E_i {E_i}^H = \sum_{i=1}^{r} \lambda_i \overline{\lambda_i} E_i$, 且$A^H A = \sum_{i=1}^{r} \lambda_i \overline{\lambda_i} E_i$, 所以$AA^H = A^H A$.

正规矩阵谱分解步骤:

1. 求A的r个互异特征值$\lambda_1, \ldots, \lambda_r$;
2. 对每个λ_i, 求特征子空间$E(\lambda_i)$的一组标准正交基 $\alpha_i^1, \ldots, \alpha_i^{n_i}$,
3. 令$E_i = \sum_{j=1}^{n_j} \alpha_i^j \left(\alpha_i^j\right)^H$, 则$A = \sum_{i=1}^{r} \lambda_i E_i$.

例: 设$A=\begin{pmatrix}1&0&-2\\0&0&0\\-2&0&4\end{pmatrix}$, 求$A$的谱分解，并求$\sum_{i=1}^{100}A^i$.(课本例题)

解：因A为对称阵，故正规，由$|\lambda I-A|=\lambda^2(\lambda-5)=0$得$\lambda_1=0$, (二重) $\lambda_2=5$.

取$E(0)$的标基为$x_1=\left(\frac{2}{\sqrt{5}},0,\frac{1}{\sqrt{5}}\right)^T$, $x_2=(0,1,0)^T$,

取$E(5)$的标基为$x_3=\left(\frac{1}{\sqrt{5}},0,\frac{-2}{\sqrt{5}}\right)^T$,

则$E_1 = x_1{x_1}^T + x_2{x_2}^T = \begin{pmatrix} \frac{4}{5} & 0 & \frac{2}{5} \\ 0 & 1 & 0 \\ \frac{2}{5} & 0 & \frac{1}{5} \end{pmatrix}, E_2 = x_3{x_3}^T =$

$\begin{pmatrix} \frac{1}{5} & 0 & \frac{-2}{5} \\ 0 & 0 & 0 \\ \frac{-2}{5} & 0 & \frac{4}{5} \end{pmatrix}$, 所以$A = \lambda_1 E_1 + \lambda_2 E_2$, 从而

$$\sum_{i=1}^{100} A^i = \left(\sum_{i=1}^{100} {\lambda_1}^i\right) E_1 + \left(\sum_{i=1}^{100} {\lambda_2}^i\right) E_2 = \left(\sum_{i=1}^{100} 5^i\right) E_2$$

$$= \frac{5^{101} - 5}{4} \begin{pmatrix} \frac{1}{5} & 0 & \frac{-2}{5} \\ 0 & 0 & 0 \\ \frac{-2}{5} & 0 & \frac{4}{5} \end{pmatrix}$$

$$= \frac{5^{100}-1}{4}\begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}.$$

例: $A=\begin{pmatrix}-2i & 4 & -2\\ -4 & -2i & -2i\\ 2 & -2i & -5i\end{pmatrix}$, 验证$A$是正规矩阵, 写出$A$的谱分解.

解: 由于$A^H=\begin{pmatrix}2i & -4 & 2\\ 4 & 2i & 2i\\ -2 & 2i & 5i\end{pmatrix}=-A$, 所以$A$为反Hermite阵, A正规.

$|\lambda I - A| = \begin{vmatrix} \lambda+2i & -4 & 2 \\ 4 & \lambda+2i & 2i \\ -2 & -2i & -5i \end{vmatrix} = (\lambda+6i)^2(\lambda - 3i)$, 所以$\lambda_1 =- 6i$, (二重) $\lambda_2 = 3i$, 取$E(-6i)$的标基为$x_1 = \left(0,\ \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)^T$, $x_2 = \left(\frac{5i}{3\sqrt{5}},\ \frac{4}{3\sqrt{5}}, \frac{-2}{3\sqrt{5}}\right)^T$; 取$E(3i)$的标基为$x_3 = \left(\frac{2}{3}i,\ -\frac{2}{3}, \frac{1}{3}\right)^T$.

则$E_1 = x_1 {x_1}^H + x_2 {x_2}^H = \begin{pmatrix} \frac{5}{9} & \frac{4i}{9} & \frac{-2i}{9} \\ \frac{-4i}{9} & \frac{5}{9} & \frac{2}{9} \\ \frac{2i}{9} & \frac{2}{9} & \frac{8}{9} \end{pmatrix}$, $E_2 =$

${x_3}{x_3}^H = \begin{pmatrix} \frac{4}{9} & \frac{-4i}{9} & \frac{2i}{9} \\ \frac{4i}{9} & \frac{4}{9} & \frac{-2}{9} \\ \frac{-2i}{9} & \frac{-2}{9} & \frac{1}{9} \end{pmatrix}$, 所以$A =- 6iE_1 + 3iE_2$.

2. 单纯矩阵的谱分解

设A是n阶单纯矩阵(可对角化), 特征值为$\lambda_1, \dots, \lambda_n$, 对应特征向量为$\alpha_1, \dots, \alpha_n$, 若令$P=(\alpha_1, \dots, \alpha_n)$, 则

$$A = P\mathrm{diag}\{\lambda_1, \dots, \lambda_n\}P^{-1}$$

$$= (\alpha_1, \dots, \alpha_n)\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$= \lambda_1\alpha_1\beta_1 + \dots + \lambda_n\alpha_n\beta_n \qquad (2)$$

其中β_i是P^{-1}的第i个行向量, $\alpha_i\beta_i$是n阶矩阵.

由于$PP^{-1}=I$, 故$\alpha_1\beta_1+\ldots+\alpha_n\beta_n=I$, 由于$PP^{-1}=I$, 故$\beta_i\alpha_j=\delta_{ij}$. 由于$\lambda_i$可能是重根，所以上式可以化简.

$$
\begin{aligned}
&A\\
&=\lambda_1\left(\alpha_{11}\beta_{11}+\ldots+\alpha_{1n_1}\beta_{1n_1}\right)\\
&+\lambda_2\left(\alpha_{21}\beta_{21}+\ldots+\alpha_{2n_2}\beta_{2n_2}\right)+\ldots\\
&+\lambda_r\left(\alpha_{r1}\beta_{r1}+\ldots+\alpha_{rn_r}\beta_{rn_r}\right)
\end{aligned}
$$

$$= \lambda_1 E_1 + \lambda_2 E_2 + \ldots + \lambda_r E_r.$$

我们有以下定理:

定理2: 设n阶单纯矩阵A有r个互异特征值$\lambda_1, \ldots, \lambda_r$, 特征值$\lambda_i$的代数重数是$n_i$, 则存在$r$个$n$阶矩阵$E_1, \ldots, E_r$, 满足:

1) $A = \sum_{j=1}^{r} \lambda_j E_j$; 2) $\left(E_j\right)^2 = E_j$; 3) $E_j E_i = 0 (i \neq j)$

4) $\sum_{j=1}^{r} E_j = I$; 5) 满足上述性质的E_j唯一;

6) $r\left(E_j\right) = n_j$.

证明: 略.

单纯矩阵谱分解步骤:

1. 求A的r个互异特征值$\lambda_1, \ldots, \lambda_r$;
2. 对每个λ_i, 求特征子空间$E(\lambda_i)$的一组基$\alpha_i{}^1, \ldots, \alpha_i{}^{n_i}$;
3. 将所有基组成矩阵P, 求P^{-1}. 令P^{-1}的行向量为$\beta_1, \ldots, \beta_n$;
4. 按照λ_i的重数计算$E_i = \sum_{j=1}^{n_i} \alpha_i{}^j \beta_i{}^j$, 则$A = \sum_{i=1}^{r} \lambda_i E_i$.

推论: 在定理2的条件下，设A的谱阵为E_i, $1 \le i \le r$, 则

1) $E_i = \frac{1}{\varphi_i(\lambda_i)} \varphi_i(A)$, $i = 1, \ldots, r$. 其中

$$\varphi_i(\lambda) = \prod_{l=1,\, l \neq i}^{n} (\lambda - \lambda_l).$$

2) 若$f(\lambda)$为任一多项式, 则

$$f(A) = f(\lambda_1)E_1 + f(\lambda_2)E_2 + \ldots + f(\lambda_r)E_r.$$

特别地有$A = {\lambda_1}^m E_1 + {\lambda_2}^m E_2 + \ldots + {\lambda_r}^m E_r.$

例: 求$A=\begin{pmatrix}1 & 4 & 2\\ 0 & -3 & 4\\ 0 & 4 & 3\end{pmatrix}$的谱分解并计算$A^{100}$.

解: $f_A(\lambda) = |\lambda I - A| = (\lambda - 1)(\lambda - 5)(\lambda + 5)$, 所以 $\lambda_1 = 1$, $\lambda_2 = 5$, $\lambda_3 =- 5$, 特征值互异, 故A单纯. 取 $\lambda_1 = 1$的一个特征向量$a_1 = (1,0,0)^T$, $\lambda_2 = 5$的一个特征向量$a_2 = (2,1,2)^T$,$\lambda_3 =- 5$的一个特征向量 $a_3 = (1, -2,1)^T$.

令$P=(a_1,a_2,a_3)=\begin{pmatrix}1&2&1\\0&1&-2\\0&2&1\end{pmatrix}$, 求得$P^{-1}=$

$$\begin{pmatrix}1&0&-1\\0&\frac{1}{5}&\frac{2}{5}\\0&\frac{-2}{5}&\frac{1}{5}\end{pmatrix},$$

令$\beta_1=(1,0,-1),\beta_2=\left(0,\frac{1}{5},\frac{2}{5}\right),\beta_3=\left(0,\frac{-2}{5},\frac{1}{5}\right)$.

则$E_1=\alpha_1\beta_1=\begin{pmatrix}1\\0\\0\end{pmatrix}(1,0,-1)=\begin{pmatrix}1&0&-1\\0&0&0\\0&0&0\end{pmatrix}$,

$E_2=\alpha_2\beta_2=\begin{pmatrix}0&\frac{2}{5}&\frac{4}{5}\\0&\frac{1}{5}&\frac{2}{5}\\0&\frac{2}{5}&\frac{4}{5}\end{pmatrix}$, $E_3=\alpha_3\beta_3=\begin{pmatrix}0&\frac{-2}{5}&\frac{1}{5}\\0&\frac{4}{5}&\frac{-2}{5}\\0&\frac{-2}{5}&\frac{1}{5}\end{pmatrix}$,

所以A的谱分解为$A=\lambda_1E_1+\lambda_2E_2+\lambda_3E_3=E_1+5E_2-5E_3$.

因此$A^{100}=E_1+5^{100}E_2+(-5)^{100}E_3=E_1+5^{100}(E_2+E_3)$.

命题: $E \in C^{n\times n}$为幂等阵, 则

1) E为单纯矩阵且其Jordan标准型为$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$;

2) E的特征值只能是1或者0;

3) $r(E) = tr(E)$;

4) $Ex = x \Leftrightarrow x \in R(E)$.

证明: 由$E^2 = E$, 知$\varphi(\lambda) = \lambda^2 - \lambda = \lambda(\lambda - 1)$将$A$零化, 故$m_E(\lambda)|\lambda(\lambda - 1)$, 所以$m_E(\lambda)$无重根, 且其根只能是1或者0, 故存在可逆阵$P$, 使$P^{-1}EP = \mathrm{diag}\{1, \dots, 1, 0, \dots, 0\}$. 1)2)3)得证.

对4) $x = Ex$, 则$x \in R(E)$. 反之, $x \in R(E)$, 则有$x = Ey$, 故$Ex = E(Ey) = E^2y = Ey = x$.

定义: 设$C^n = V_1 \oplus V_2$, $\forall x \in C^n$可唯一的分解成$x = y + z$, 其中$y \in V_1, z \in V_2$, 此时称y为x在V_1上的投影. 令$E \in L(C^n, C^n)$, $Ex = y$, 称E为投影变换.

命题: E是投影变换$\Leftrightarrow$ E是幂等阵.

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证明: $\Rightarrow$: 若E是投影变换, 则$\forall x \in C^n$, 由$x = y + z$, 其中$y \in V_1, z \in V_2, Ex = y$. 所以$E^2x = E(Ex) = Ey = y = Ex$, 即$E^2 = E$.

$\Leftarrow$: 若$E^2 = E$, 令$V_1 = \{y|y = Ex, x \in C^n\} = R(E)$, $V_2 = \{x|Ex = 0\} = N(E)$, 则只需要证明$C^n = V_1 \oplus V_2$, 就可以得$Ex$是$x$在$R(E)$上的投影变换. 下证$C^n = V_1 \oplus V_2$.

任取$z \in V_1 \cap V_2$, 有$z = Ex, Ez = \theta$, 这里θ为原点对应的向量. 则$\theta = Ez = E^2x = Ex = z$, 所以$V_1 \cap V_2 = \{\theta\}$,

$\forall x \in C^n$, 有$x = Ex + (I - E)x$, 其中$Ex \in V_1, (I - E)x \in V_2$, 所以$C^n = V_1 \oplus V_2$.