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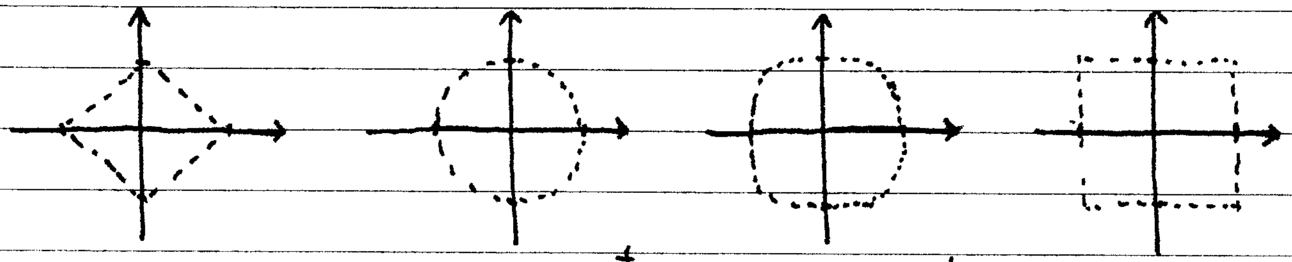
§2. CONVERGENCE OF SEQUENCES. LIMIT POINTS.

DEF An <u>open ball</u> $B(x_0, r)$ in the metric space R is the set of all points x in R which satisfy the condition $\rho(x, x_0) < r$. x_0 is called the <u>centre</u> and r the <u>radius</u>. Sometimes the open ball is called a <u>neighbourhood</u> of x_0, denoted as $N_r(x_0)$.

EXP The open balls with $x_0 =$ origin, $r=1$ in metric spaces $(\mathbb{R}^2, \rho_1)$, $(\mathbb{R}^2, \rho_2)$ = Euclidean 2-space, $(\mathbb{R}^2, \rho_4)$, and $(\mathbb{R}^2, \rho_\infty)$.



$(\mathbb{R}^2, \rho_1)$: $|x|+|y|<1$　$(\mathbb{R}^2, \rho_2)$: $(x^2+y^2)^{\frac{1}{2}}<1$　$(\mathbb{R}^2, \rho_4)$: $(x^4+y^4)^{\frac{1}{4}}<1$　$(\mathbb{R}^2, \rho_\infty)$: $\max(|x|,|y|)<1$

DEF A closed ball $B[x_0, r]$ is the set of all points x in R which satisfy the condition $\rho(x, x_0) \leq r$.

DEF A point x is called a <u>contact point</u> of the set M if every neighbourhood of x contains at least one point of M. The set of all contact points of the set M is the

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<u>closure</u> of M, denoted by $\bar{M}$.

EXP In last example, the closure of $B((0,0),1)$, the open ball, is $B[(0,0),1]$, the closed ball, in each of the four metric spaces.

QUE Is $\overline{B(x_0,r)} = B[x_0,r]$ true in all metric spaces?

ANS No. Counterexample: the discrete metric space. Let X be any set of more than one element and $\rho(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$.

THM The closure of the closure of M is equal to the closure of M: $\bar{\bar{M}} = \bar{M}$.

PRF Every point of M is a contact point of M, so $\bar{M} \supseteq M$. $\bar{\bar{M}} \supseteq \bar{M}$. Need to show that if $x \in \bar{\bar{M}}$, then $x \in \bar{M}$. If not, suppose the neighbourhood $N_\varepsilon(x)$ does not intersect M. But since $x \in \bar{\bar{M}}$, $N_{\varepsilon/2}(x)$ contains a point x_1 in $\bar{M}$, and then $N_{\varepsilon/2}(x_1)$ contains a point x_2 in M. Hence

$$\begin{aligned} \rho(x,x_2) &\leq \rho(x,x_1) + \rho(x_1,x_2) \qquad \text{(triangle axiom)} \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

and $x_2 \in N_\varepsilon(x) \cap M$. Contradiction $\square$.

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THM If $M_1 \subseteq M_2$, then $\bar{M}_1 \subseteq \bar{M}_2$.

PRF Exercise.

THM The closure of a sum is equal to the sum of closures:

$\overline{M_1 \cup M_2} = \bar{M}_1 \cup \bar{M}_2$.

PRF By the theorem above, $\overline{M_1 \cup M_2} \supseteq \bar{M}_1$ and $\overline{M_1 \cup M_2} \supseteq \bar{M}_2$.

Hence $\overline{M_1 \cup M_2} \supseteq \bar{M}_1 \cup \bar{M}_2$. if $x \notin \bar{M}_1 \cup \bar{M}_2$. $\Leftrightarrow x \notin \bar{M}_1$

and $x \notin \bar{M}_2$, then there exist $N_{\varepsilon_1}(x)$ that does not intersect M_1.

and $N_{\varepsilon_2}(x)$ that does not intersect M_2. Hence $N_{\min(\varepsilon_1, \varepsilon_2)}(x)$

does not intersect $M_1 \cup M_2$, which means $x \notin \overline{M_1 \cup M_2}$. Thus

we also obtain $\overline{M_1 \cup M_2} \subseteq \bar{M}_1 \cup \bar{M}_2$. $\square$.

DEF The point x is called a <u>limit point</u> of the set M if an arbitrary neighbourhood of x contains an infinite number of points of M, and is called an <u>isolated point</u> of M if x is in M and has a neighbourhood $N_\varepsilon(x)$ that does not contain any point of M different from x.

THM Every contact point of the set M is either a limit point or an

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isolated point of M.

PRF Let x be a contact point of M but not a limit point. That is, there exists a neighbourhood $N_\varepsilon(x)$ that contains only finitely many points $x_1, x_2, \cdots, x_n$ in M that are distinct from x. Let $\rho(x, x_i) = \varepsilon_i$ and take $\varepsilon_0 > 0$ to be less than any ε_i. Then $N_{\varepsilon_0}(x)$ contains no points in M other than x itself, and it is an isolated point of M. □.

Thus $\bar{M} = \{\text{isolated points}\} \cup \{\text{limit points that belong to } M\}$
$\cup \{\text{limit points that does not belong to } M\}$.

DEF Let $x_1, x_2, \cdots$ be a sequence of points in the metric space R. We say this sequence <u>converges to the point</u> x if every neighbourhood $N_\varepsilon(x)$ contains all points x_n starting with some one of them. The point x is called the <u>limit</u> of the sequence $\{x_n\}$.

Two simple properties : ① No sequence can have two distinct limits
② If $\{x_n\}$ converges to x, so does any of its subsequence.

THM ① The point x is a contact point of the set M if any only if there exists a sequence $\{x_n\}$ of points of the set M which

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converges to x.

② The point x is a limit point of M if any only if there exists a sequence $\{x_n\}$ of <u>distinct</u> points of the set M which converges to x.

PRF If $\{x_n\} \subset M$ converges to x, then in any neighbourhood $N_\varepsilon(x)$ there are points $\{x_n, x_{n+1}, x_{n+2}, \cdots\}$ which are in M lying in $N_\varepsilon(x)$. (Here n depends on ε). So x is a contact point. Further more, if $\{x_n, x_{n+1}, \cdots\}$ are distinct points, then $N_\varepsilon(x)$ contains infinitely many points of M, and it is a limit point.

Conversely, if x is a contact point of M, then for $n=1,2,3,\cdots$ each $N_{1/n}(x)$ contains a point, say x_n, in M. Then $\{x_1, x_2, \cdots x_n, \cdots\}$ is a sequence converging to x. Furthermore, if x is a limit point, we can choose $x_1, x_2, \cdots$ consecutively such that x_n is distinct from $x_1, \cdots, x_{n-1}$ (since $N_{1/n}(x)$ contains infinitely many points in M), and make the points in the sequence $\{x_n\}$ distinct.

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DEF Let A and B be two sets in the metric space R. The set A is *dense* in B if $\bar{A} \supseteq B$. In particular, A is *everywhere dense* if $\bar{A} = R$.

DEF The metric space R is *separable* if it has an everywhere dense subset that is countable.

EXP The metric space $R = (\mathbb{R}, \rho(x,y) = |x-y|)$ is separable and the countable set of rational numbers is everywhere dense.

PRF For any real number x, let x_n be the largest rational number in the form of $\frac{m}{10^n}$ (m is an integer) that is $\le x$. Then $\{x_1, x_2, \cdots\}$ converges to x. By last theorem, $x \in \overline{\{\text{rational number}\}}$. $\square$

EXP The metric space $R = (\mathbb{R}^m, \rho_\infty(x,y) = \max\limits_{k=1}^{m}(|x_k - y_k|))$ is separable and the countable set $A = \{x = (x_1, \cdots, x_m) : x_k \text{ is rational for } k = 1, 2, \cdots, m\}$ is everywhere dense

PRF For any point $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_m)$, let the sequence $\{x_k^{(n)}\}$ be that constructed for $\bar{x}_k$ in last example. Then as $n \to \infty$, $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \cdots, x_m^{(n)}) \in A$ convergen to $\bar{x}$. $\square$

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EXP ℓ^∞ space is NOT seperable.

To show that we need the fact

The subsets of the set of all positive integers constitute an uncountable set.

PRF of the example : Let S be any subset of $\{1, 2, 3, \cdots\}$. Define $x^{(S)} = (x_1^{(S)}, x_2^{(S)}, \cdots)$ as $x_k^{(S)} = \begin{cases} 1 & \text{if } k \in S \\ 0 & \text{if } k \notin S \end{cases}$. Then it is not hard to see that $\rho_\infty(x^{(S)}, x^{(S')}) = \begin{cases} 1 & \text{if } S \neq S' \\ 0 & \text{if } S = S'. \end{cases}$

Suppose A is an everywhere dense subset of R. For each $x^{(S)}$, there is an $a^{(S)} \in A$ such that $\rho_\infty(a^{(S)}, x^{(S)}) < \frac{1}{3}$. If $S \neq S'$, then $a^{(S)} \neq a^{(S')}$, otherwise $\rho(x^{(S)}, x^{(S')}) \leq \rho(x^{(S)}, a^{(S)}) + \rho(a^{(S)}, x^{(S')}) < \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$. Thus A contains uncountably many elements $a^{(S)}$. $\square$

DEF Let A and B be arbitrary sets in the metric space R, and x be a point in R. The <u>distance</u> from x to A is defined as

$$\rho(A, x) = \inf\{\rho(a, x) : a \in A\}$$

and the <u>distance</u> between A and B is defined as

$$\rho(A, B) = \inf\{\rho(a, b) : a \in A, b \in B\}.$$

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We can show that $\rho(A,x)=0$ if and only if x is a contact point of A.

DEF If A is a set in the metric space R, then all limit points of A constitute the <u>derived</u> <u>set</u> of A, denoted by A'.

Note: $(M')' = M'$ is NOT true. For example, $R = (\mathbb{R}, \rho(x,y)=|x-y|)$, $M = \{ \frac{1}{n} : n=1,2,\cdots \}$.