Lecture notes of MA3209: Metric space

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New topics will be added to the notes after each class, and typos will be fixed anytime they are spotted.

1 Definition and examples of metric spaces

Definition 1.1. A *metric space* $R = (X, \rho)$ is the pair of two things: a set X, whose elements are called points, and a distance, i.e., a single-valued, nonnegative real function $\rho(x, y)$, defined for arbitrary x and y in X, such that

1. $\rho(x, y) = 0$ if and only if $x = y$,
2. $\rho(x, y) = \rho(y, x)$ (axiom of symmetry),
3. $\rho(x, y) + \rho(y, z) \geq (x, z)$ (triangle axiom).

Example 1.2. $X = \mathbb{R}$ and $\rho(x, y) = |x - y|$.

Example 1.3. (Euclidean n-space) $X = \mathbb{R}^n$ and for $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$,

$$\rho(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}.$$

In that space, conditions 1 and 2 are obviously satisfied. To validate condition 3, we denote

$$\begin{aligned} a &= (a_1, a_2, \ldots, a_n) = (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n), \\ b &= (b_1, b_2, \ldots, b_n) = (y_1 - z_1, y_2 - z_2, \ldots, y_n - z_n). \end{aligned}$$

then

$$\begin{aligned} & \rho(x, z) \leq \rho(x, y) + \rho(y, z) \\ \iff & \sqrt{(a_1 + b_1)^2 + \ldots + (a_n + b_n)^2} \leq \sqrt{a_1^2 + \ldots + a_n^2} + \sqrt{b_1^2 + \ldots + b_n^2} \\ \iff & (a_1 + b_1)^2 + \ldots + (a_n + b_n)^2 \\ & \leq (a_1^2 + \ldots + a_n^2) + (b_1^2 + \ldots + b_n^2) + 2\sqrt{a_1^2 + \ldots + a_n^2}\sqrt{b_1^2 + \ldots + b_n^2} \\ \iff & 2(a_1 b_1 + a_2 b_2 + \ldots + a_n b_n) \leq 2\sqrt{a_1^2 + \ldots + a_n^2}\sqrt{b_1^2 + \ldots + b_n^2}. \end{aligned}$$

Then it is a consequence of the following theorem.

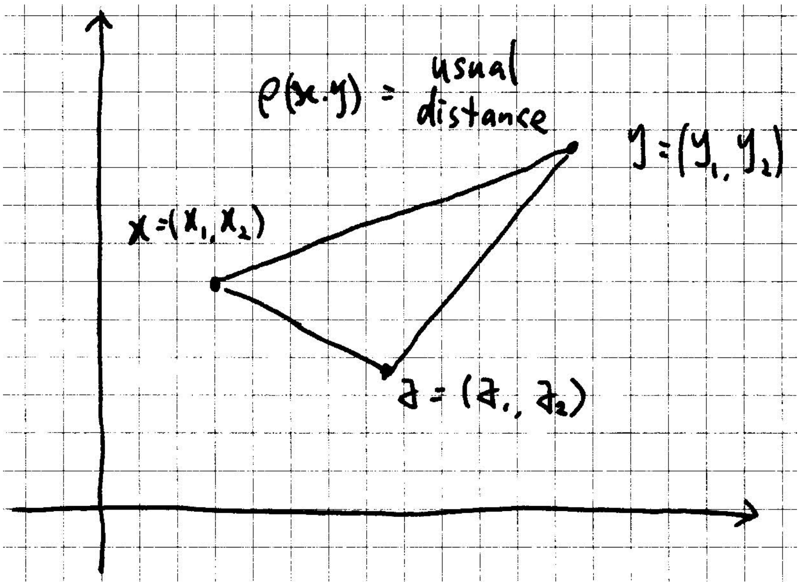

Figure 1: When $n = 2$, we see from the figure that $\rho(x, y)$ is the usual distance and the triangle axiom has a simple geometric meaning.

Theorem 1.4 (Schwarz inequality)**.** *For all real numbers* $a_1, a_2, \dots, a_n$ *and* $b_1, b_2, \dots, b_n$*,*

$$(a_1b_1 + a_2b_2 + \dots + a_nb_n)^2 \le (a_1^2 + a_2^2 + \dots + a_n^2)(b_1^2 + b_2^2 + \dots + b_n^2). \tag{1}$$

Proof. The right-hand side of (1) can be expanded as

$$\begin{aligned}(a_1^2 + a_2^2 + \dots + a_n^2)(b_1^2 + b_2^2 + \dots + b_n^2) &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i^2 b_j^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} a_j^2 b_i^2 \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_i^2 b_j^2 + a_j^2 b_i^2.\end{aligned}$$

On the other hand, the left-hand side of (1) can be expanded as

$$(a_1b_1 + a_2b_2 + \dots + a_nb_n)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}(a_ib_i)(a_jb_j) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} 2(a_ib_j)(a_jb_i).$$

Thus we have

$$(a_1^2+a_2^2+\dots+a_n^2)(b_1^2+b_2^2+\dots+b_n^2)-(a_1b_1+a_2b_2+\dots+a_nb_n)^2 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_ib_j-a_jb_i)^2 \ge 0,$$

and finish the proof. □

Example 1.5. $X = \mathbb{R}^n$ and

$$\rho_p(x, y) = \left(|x_1 - y_1|^p + |x_2 - y_2|^p + \dots + |x_n - y_n|^p\right)^{\frac{1}{p}}.$$

In this example, conditions 1 and 2 are obviously satisfied, and condition 3, based on arguments similar to those in the for Example 1.3, is equivalent to the following theorem.

Theorem 1.6 (Minkowski inequality). *For all real numbers $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$, and $p \geq 1$,*

$$(|a_1 + b_1|^p + \ldots + |a_n + b_n|^p)^{\frac{1}{p}} \leq (|a_1|^p + \ldots + |a_1|^p)^{\frac{1}{p}} + (|b_1|^p + \ldots + |b_1|^p)^{\frac{1}{p}}.$$

The proof of the Minkowski inequality when $p > 1$ depends on Hölder's inequality, while the $p = 1$ case turns out to be much easier.

Theorem 1.7 (Hölder's inequality). *For all real numbers $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n$, and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$,*

$$\sum_{k=1}^{n} |x_k y_k| \leq \left(\sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |y_k|^q \right)^{\frac{1}{q}}.$$

Proof of the Minkowski inequality ($p > 1$) from Hölder's inequality. We starts from a simple identity

$$(|a| + |b|)^p = |a|(|a| + |b|)^{p-1} + |b|(|a| + |b|)^{p-1}. \tag{2}$$

By Hölder's inequality,

$$\begin{aligned}
\sum_{k=1}^{n} |a_k|(|a_k| + |b_k|)^{p-1} &\leq \left(\sum_{k=1}^{n} |a_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} \left[(|a_k| + |b_k|)^{p-1} \right]^q \right)^{\frac{1}{q}} \\
&= \left(\sum_{k=1}^{n} |a_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} (|a_k| + |b_k|)^p \right)^{1-\frac{1}{p}},
\end{aligned}$$

where we use the relation $q = \frac{p}{p-1}$ in the last step. Similarly,

$$\begin{aligned}
\sum_{k=1}^{n} |b_k|(|a_k| + |b_k|)^{p-1} &\leq \left(\sum_{k=1}^{n} |b_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} \left[(|a_k| + |b_k|)^{p-1} \right]^q \right)^{\frac{1}{q}} \\
&= \left(\sum_{k=1}^{n} |b_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} (|a_k| + |b_k|)^p \right)^{1-\frac{1}{p}}.
\end{aligned}$$

Sum up the two identities above, we obtain with the help of the simple identity (2)

$$\sum_{k=1}^{n} (|a_k| + |b_k|)^p \leq \left[\left(\sum_{k=1}^{n} |a_k|^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |b_k|^p \right)^{\frac{1}{p}} \right] \left(\sum_{k=1}^{n} (|a_k| + |b_k|)^p \right)^{1-\frac{1}{p}}.$$

Dividing $\left(\sum_{k=1}^{n} (|a_k| + |b_k|)^p\right)^{1-\frac{1}{p}}$ on both sides, we prove the Minkowski inequality. □

Proof of Hölder's inequality. First we show a short inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \tag{3}$$

for $a, b > 0$, $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. The proof of the inequality relies on the integral formula

$$\begin{aligned} \frac{a^p}{p} &= \int_0^a x^{p-1} dx, \\ \frac{b^q}{q} &= \int_0^b y^{q-1} dy = \int_0^b y^{\frac{1}{p-1}} dy, \end{aligned}$$

where in the second formula we used the relation $q-1 = \frac{1}{p-1}$ as a consequence of $\frac{1}{p}+\frac{1}{q} = 1$. Hence $\frac{a^p}{p}$ and $\frac{b^q}{q}$ are expressed geometrically as the following areas of regions.

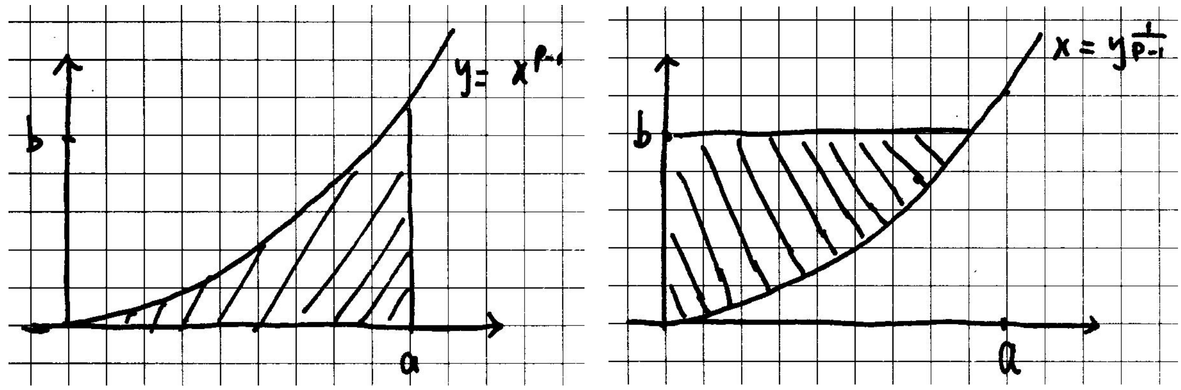

Figure 2: The areas of the two shaded regions represent $\frac{a^p}{p}$ and $\frac{b^q}{q}$ respectively.

Noting that the graph of $x = y^{\frac{1}{p-1}}$ coincides with that of $y = x^{p-1}$, we put the two graphs in one figure:

Then it is clear that ab, geometrically represented by the area of the rectangle, is smaller than the sum of $\frac{a^p}{p}$ and $\frac{b^q}{q}$ by a "corner", see Figure 1. If the value of b is larger,

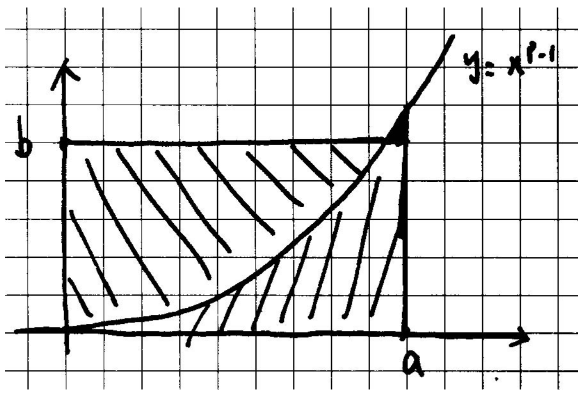

Figure 3: $\frac{a^p}{+}\frac{b^q}{q}$ is larger than ab by the area of the area of the solid coner.

the "corner" occurs in a different place, see Figure 1.

Note that Hölder's inequality is *homogeneous*, in the sense that if it holds for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, then it also holds for $\lambda x = (\lambda x_1, \ldots, \lambda x_n)$ and $\mu y =$

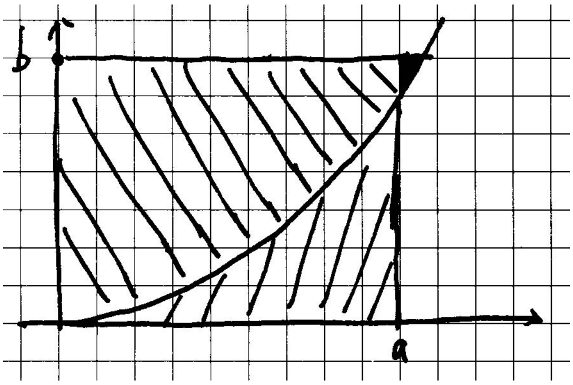

Figure 4: $\frac{a^p}{p} + \frac{b^q}{q}$ is larger than ab by the area of the area of the solid coner.

$(\mu y_1, \dots, \mu y_n)$:

$$\sum_{k=1}^{n} |x_k y_k| \le \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |y_k|^q\right)^{\frac{1}{q}}$$
$$\iff \sum_{k=1}^{n} |\lambda x_k \mu y_k| \le \left(\sum_{k=1}^{n} |\lambda x_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |\mu y_k|^q\right)^{\frac{1}{q}}.$$

Thus we need only to prove it under the condition $\sum_{k=1}^{n} |x_k|^p = 1$ and $\sum_{k=1}^{n} |y_k|^q = 1$. In that case, the short inequality (3) implies that for each k, $|x_k y_k| \le \frac{|x_k|^p}{p} + \frac{|y_k|^q}{q}$. Sum up for k from 1 to n, we have

$$\sum_{k=1}^{n} |x_k y_k| \le \sum_{k=1}^{n} \frac{|x_k|^p}{p} + \frac{|y_k|^q}{q} = \frac{1}{p} \sum_{k=1}^{n} |x_k|^p + \frac{1}{q} \sum_{k=1}^{n} |y_k|^q = \frac{1}{p} + \frac{1}{q}$$
$$= 1 = \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |y_k|^q\right)^{\frac{1}{q}},$$

and prove the special case. The theorem in general setting is then proved by the homogeneous property. □

Example 1.8. $X = \mathbb{R}^n$ and $\rho(x, y) = \max_{k=1}^{n}(|x_k - y_k|)$.

The verification that the space is a metric space is left as an exercise.

Remark 1.9. The distance in Euclidean n-space is ρ_2, and the distance in Example 1.8 is, in some sense, the limit of ρ_p as $p \to \infty$:

$$\rho_\infty(x, y) = \lim_{p \to \infty} \rho_p(x, y), \quad \text{for all } x, y \in \mathbb{R}^n.$$

The proof of the above limit identity is left for exercise.

Example 1.10 (ℓ^p space)**.** X is the set of all possible sequences $x = (x_1, x_2, \dots, x_n, \dots)$ of real numbers, which satisfies the condition $\sum_{k=1}^{\infty} |x_k|^p < \infty$. The distance function is $(p \ge 1)$

$$\rho_p(x, y) = \left(\sum_{k=1}^{\infty} |x_k - y_k|^p\right)^{\frac{1}{p}}.$$

First we show that $\rho_p(x,y)$ is well defined. For any n,

$$\left(\sum_{k=1}^{n}|x_k-y_k|^p\right)^{\frac{1}{p}} \le \left(\sum_{k=1}^{n}|x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n}|y_k|^p\right)^{\frac{1}{p}}$$
$$\le \left(\sum_{k=1}^{\infty}|x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty}|y_k|^p\right)^{\frac{1}{p}} = \text{const} < \infty,$$

where in the first step we use the Minkowski inequality. Since $(\sum_{k=1}^{n}|x_k-y_k|^p)^{\frac{1}{p}}$ increases monotonically as $n\to\infty$, the limit exists by the *monotone convergence theorem*, and it is $\rho_p(x,y)$.

Then we show the condition 3 holds, that is,

$$\left(\sum_{k=1}^{\infty}|x_k-z_k|^p\right)^{\frac{1}{p}} - \left[\left(\sum_{k=1}^{\infty}|x_k-y_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty}|y_k-z_k|^p\right)^{\frac{1}{p}}\right] \le 0.$$

This is the $n\to\infty$ limit of the inequality

$$\left(\sum_{k=1}^{n}|x_k-z_k|^p\right)^{\frac{1}{p}} - \left[\left(\sum_{k=1}^{n}|x_k-y_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n}|y_k-z_k|^p\right)^{\frac{1}{p}}\right] \le 0.$$

Example 1.11 (ℓ^∞ space)**.** X is the set of all possible sequences $x=(x_1,x_2,\dots,x_n,\dots)$ of real numbers, which satisfies the condition $\sup_{k=1}^{\infty}|x_k-y_k|$. The distance function is

$$\rho_p(x) = \sup_{k=1}^{\infty}|x_k-y_k|.$$

Example 1.12 ($C^p([a,b])$ space)**.** X is the set of all continuous functions on interval $[a,b]$. The distance is $(p\ge 1)$

$$\rho_p(x,y) = \left(\int_a^b |x(t)-y(t)|^p\right)^{\frac{1}{p}}.$$

The distance function $\rho_p(x,y)$ is well defined since $x(t)$ and $y(t)$ are bounded, which is a consequence of the extreme value theorem. To show condition 3 holds, we need the integral form of the Minkowski inequality. The proof is a homework problem.

Example 1.13 ($C^\infty([a,b])$ space)**.** X is the set of all continuous functions on interval $[a,b]$. The distance is

$$\rho_\infty(x,y) = \max_{t\in[a,b]}(|x(t)-y(t)|).$$

It is an exercise to show that the distance function $\rho_\infty(x,y)$ satisfies the triangle axiom (condition 3).

2 Convergence of sequences. Limit points

Definition 2.1. An *open ball* $B(x_0, r)$ in the metric space R is the set of all points x in R which satisfy the condition $\rho(x, x_0) < r$. x_0 is called the *centre* and r the *radius.* The open ball is also called a *neighbourhood* of x_0, denoted as $N_r(x_0)$.

Example 2.2. The open balls in $\mathbb{R}^2$ with origin as their center and radius 1, in metric spaces with distance function ρ_1, ρ_2 (Euclidean 2-space), ρ_4 and ρ_∞ are shown below.

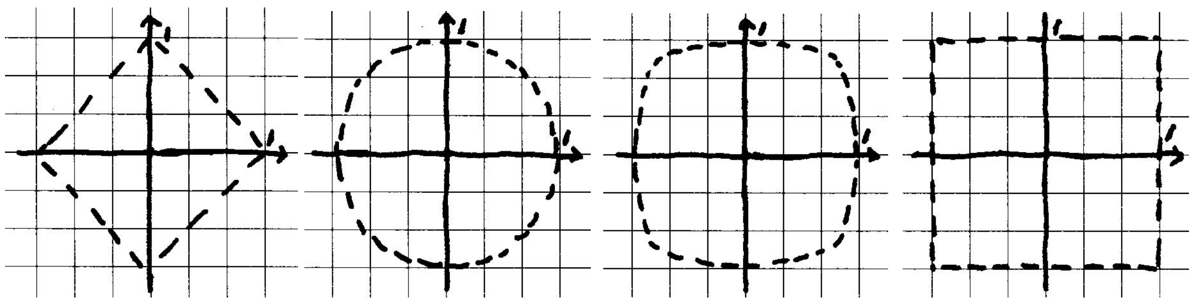

Figure 5: The open balls with center $(0, 0)$ and radius 1 in $(\mathbb{R}^2, \rho_1)$, $(\mathbb{R}^2, \rho_2)$, $(\mathbb{R}^2, \rho_4)$ and $(\mathbb{R}^2, \rho_\infty)$ respectively.

Definition 2.3. A *closed ball* $B[x_0, r]$ is the set of all points x in R which satisfy the condition $\rho(x, x_0) \leq r$.

Example 2.4. The closed balls in $\mathbb{R}^2$ with origin as their center and radius 1, in metric spaces with distance function ρ_1, ρ_2 (Euclidean 2-space), ρ_4 and ρ_∞ are shown below.

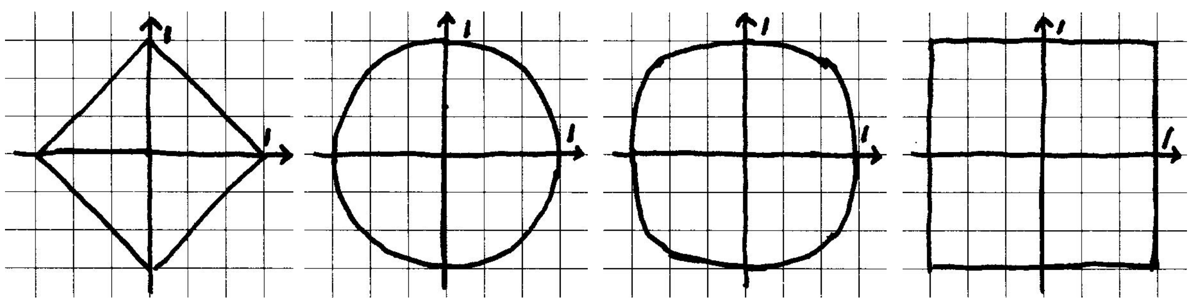

Figure 6: The closed balls with center $(0, 0)$ and radius 1 in $(\mathbb{R}^2, \rho_1)$, $(\mathbb{R}^2, \rho_2)$, $(\mathbb{R}^2, \rho_4)$ and $(\mathbb{R}^2, \rho_\infty)$ respectively.

Definition 2.5. A point x is called a *contact point* of the set M if every neighbourhood of x contains at least one point of M. The set of all contact points of the set M is the *closure* of M, denoted by $\overline{M}$.

In last example, the closure of $B((0, 0), 1)$, the open ball, is the closed ball $B[(0, 0), 1]$. See the figure below for an explanation in the Euclidean 2-space, and other cases are left for exercise.

However, is the closure of the open ball $B(x_0, r)$ the closed ball $B[x_0, r]$ in *all* matric spaces? The answer is NO, and we have a counterexample as follows.

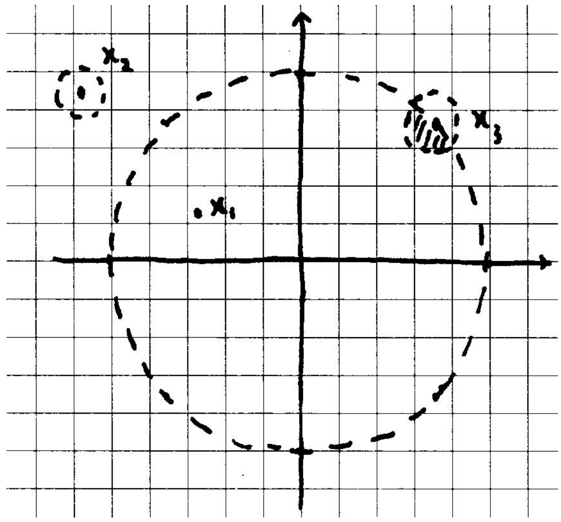

Figure 7: The point x_1, in the open ball, is in the closure of the open ball since all its neighbourhoods at least contains itself; the point x_2, outside of the closed ball, is not in the closure of the open ball since it has a small neighbourhood disjoint to the open ball; the point x_3, lying on the boundary of the ball, is in the closure, since all its neibourhoods, however small, intersect with the open ball.

Example 2.6 (discrete metric space)**.** Let X be any set (of more than one element), and

$$\rho(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Then for any x in X, $B(x, 1) = \{x\}$, and $B[x, 1] = X$. Since for any $y \neq x$, $x \notin N_{\frac{1}{2}}(y)$, we have $\overline{B(x, 1)} = \{x\}$, and then $\overline{B(x, 1)} \neq B[x, 1]$.

Theorem 2.7. *The closure of the closure of M is equal to the closure of M: $\overline{\overline{M}} = \overline{M}$.*

Proof. Every point of M is a contact point of M, so $\overline{M} \supseteq M$, and also $\overline{\overline{M}} \supseteq \overline{M}$. Then we need to show that if $x \in \overline{\overline{M}}$, it is also in $\overline{M}$. If x is in $\overline{\overline{M}}$ but not in $\overline{M}$, there exists a neighbourhood $N_\epsilon(x)$ of x such that it does not intersect with M. Since $x \in \overline{\overline{M}}$, $N_{\frac{\epsilon}{3}}(x)$ contains a point x_1 in $\overline{M}$. Then the neighbourhood $N_{\frac{\epsilon}{3}}(x_1)$ of x_1 contains a point x_2 in M since $x_1 \in \overline{M}$. By the triangle inequality,

$$\rho(x, x_2) \leq \rho(x, x_1) + \rho(x_1, x_2) < \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2}{3}\epsilon < \epsilon,$$

which is contradictory to the assumption that $N_\epsilon(x)$ does not intersect with M. □

Theorem 2.8. *If $M_1 \subseteq M_2$, then $\overline{M_1} \subseteq \overline{M_2}$.*

Proof. Exercise. □

Theorem 2.9. *The closure of a sum is equal to the sum of clusures: $\overline{M_1 \cup M_2} = \overline{M_1} \cup \overline{M_2}$.*

Proof. Be Theorem 2.9, $\overline{M_1 \cup M_2} \supseteq \overline{M_1}$ since $M_1 \cup M_2 \supseteq M_1$, and similarly $\overline{M_1 \cup M_2} \supseteq \overline{M_2}$. Thus $\overline{M_1 \cup M_2} \supseteq \overline{M_1} \cup \overline{M_2}$.

Now we show that $\overline{M_1 \cup M_2} \subseteq \overline{M_1} \cup \overline{M_2}$. If there exists x in $\overline{M_1 \cup M_2}$ but not in $\overline{M_1} \cup \overline{M_2}$, then there exists $N_{\epsilon_1}(x)$ that is disjoint to M_1, and $N_{\epsilon_2}(x)$ that is disjoint to M_2. We have $N_{\min(\epsilon_1,\epsilon_2)}(x)$, the intersection between the two neighbourhood, is disjoint to both M_1 and M_2, which means $N_{\min(\epsilon_1,\epsilon_2)}(x) \cap (M_1 \cup M_2) = \emptyset$. This is contradictory to the assumption that $x \in \overline{M_1 \cup M_2}$. □

Definition 2.10. The point x is called a *limit point* of the set M if an arbitrary neighbourhood of x contains an infinite number of points of M, and is called an *isolated point* of M if x is in M and has a neighbourhood $N_\delta(x)$ that does not contain any point of M different from x.

Theorem 2.11. *Every contact point of the set M is either a limit point or an isolated point of M.*

Proof. Let x be a contact point of M, but not a limit point. Then there is a neighbourhood $N_\epsilon(x)$ such that it contains only finitely many points $x_1, x_2, \dots, x_k$ in M. Let $\rho(x, x_i) = \epsilon_i$ for $i = 1, 2, \dots, k$, and take $\bar{\epsilon}$ be the smallest one among nonzero values of ϵ_i. (One of ϵ_i may be 0, which means that x itself is among the k points.) Then the neighbourhood $N_{\bar{\epsilon}}(x)$ contains no points among $x_1, \dots, x_k$ that are different from x, in other words, it contains no points in M that are different from x. We conclude that x is an isolated point of M. □

Remark 2.12. As a conclusion of the proof, we see that the point x is in M. But in the intermediate steps, this fact is not clear at all, and we cannot make use of it.

Thus the closure of a set can be divided into

$$\overline{M} = \{\text{isolated points}\} \cup \{\text{limit points that belong to } M\} \\ \cup \{\text{limit points that does not belong to } M\}.$$

Definition 2.13. Let $x_1, x_2, \dots$ be a sequence of points in the metric space R, we say this sequence *converges to* the point x if every neighbourhood $N_\epsilon(x)$ contains all points x_n starting with some one of them. The point x is called the *limit* of the sequence $\{x_n\}$.

Two simple properties:

- No sequence can have two distinct limits.
- If $\{x_n\}$ converges to x, so does any of its subsequence.

Theorem 2.14.
- *The point x is a contact point of the set M if and only if there exists a sequence $\{x_n\}$ of points of the set M which converges to x.*
- *The point x is a limit point of M if and only if there exists a sequence $\{x_n\}$ of* distinct *points of the set M which converges to x.*

Proof. If $\{x_n\} \subset M$ converges to x, then in any neighbourhood $N_\epsilon(x)$ there are points $\{x_n, x_{n+1}, x_{n+2}, \dots\} \subset M$ lying in $N_\epsilon(x)$. (Here n depends on ϵ). So x is a contact point. Furthermore, if $\{x_n, x_{n+1}, \dots\}$ are distinct points, then $N_\epsilon(x)$ contains infinitely many points of M, and it is a limit point.

Conversely, if x is a contact point of M, then we construct $x_1, x_2, \dots$ as follows:

Step 1: Choose x_1 as a point in M that lies in $N_{\frac{1}{1}}(x)$.

Step 2: Choose x_2 as a point in M that lies in $N_{\frac{1}{2}}(x)$.

$\dots\dots$

Step n: Choose x_n as a point in M that lies in $N_{\frac{1}{n}}(x)$.

$\dots\dots$

and we construct the desired sequence in M that converges to x. If x is further a limit point, in Step n ($n > 1$) we "choose x_n as a point in M that lies in $N_{\frac{1}{n}}(x)$ that is *distinct* from $x_1, x_2, \dots, x_{n-1}$. □

Definition 2.15. Let A and B be two sets in the metric cpace R. The set A is *dense* in B if $\overline{A} \supseteq B$, In particular, A is *everywhere dense* if $\overline{A} = R$.

Definition 2.16. The metric space R is *separable* if it has an everywhere dense subset that is countable.

Example 2.17. The metric space $R = (\mathbb{R}, \rho(x, y) = |x - y|)$ is separable, and the countable set of rational numbers is everywhere dense.

Proof. For any real number x, let x_n be the largest rational number in the form $\frac{m}{10^n}$ (m is an integer) that is less than or equal to x. Then $\{x_1, x_2, \dots\}$ converges to x. By Theorem 2.14, x is a contact point of $\mathbb{Q} = \{\text{rational numbers}\}$, and $\mathbb{Q}$ is everywhere dense.

One example of the construction: if $x = \pi$, then $x_1 = \frac{31}{10^1} = 3.1$, $x_2 = \frac{314}{10^2} = 3.14$, $x_3 = \frac{3141}{10^3} = 3.141, \dots$. □

Example 2.18. The metric space $R = (\mathbb{R}^m, \rho_\infty(x, y) = \max_{k=1}^n(|x_k - y_k|))$ is separable, and the countable set $A = \{x = (x_1, x_2, \dots, x_m) : x_k \in \mathbb{Q}\}$ is everywhere dense.

Proof. For any point $\bar{x} = (\bar{x}_1, \dots, \bar{x}_m)$, let the sequence $\{x_k^{(n)}\}$ be the sequence constructed in Example 2.17. Then as $n \to \infty$, $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \dots, x_m^{(n)})$ converges to $\bar{x}$. □

Example 2.19. ℓ^∞ space is not separable.

To show that all everywhere dense subset of ℓ^∞ space are uncountable, we need the fact that

Proposition 2.20. *The subsets of the set of all positive integers ($\mathbb{Z}_+ = \{1, 2, 3, \dots\}$) is an uncountable set.*

The proof of this proposition will be given in Appendix A.

Proof of that ℓ^∞ is not separable. Let S be any subset of $\mathbb{Z}_+$. Define $x^{(S)} \in \ell^\infty$ as

$$x^{(S)} = (x_1^{(S)}, \dots, x_m^{(S)}), \quad \text{where} \quad x_k^{(S)} = \begin{cases} 1 & \text{if } k \in S, \\ 0 & \text{if } k \notin S. \end{cases}$$

Then we see that

$$\rho_\infty(x^{(S)}, x^{(S')}) = \begin{cases} 1 & \text{if } S \neq S', \\ 0 & \text{if } S = S'. \end{cases} \tag{4}$$

Suppose A is an everywhere dense subset of R. For each $x^{(S)}$, there is an $a^{(S)} \in A$ lying in $N_{\frac{1}{3}}(x^{(S)})$. The points $a^{(S)}$ are distinct. Otherwise if $a^{(S)} = a^{(S')}$ with $S \neq S'$, then

$$\rho_\infty(x^{(S)}, x^{(S')}) \leq \rho_\infty(x^{(S)}, a^{(S)}) + \rho_\infty(a^{(S)}, x^{(S')}) \leq \frac{1}{3} + \frac{1}{3} = \frac{2}{3},$$

which is contradictory to (4).

This the set A has no less elements than the subsets of $\mathbb{Z}_+$, and it is an uncountable set. □

Definition 2.21. Let x be a point in the metric space R and A be a subset of R, then the *distance* from x to A is defined as

$$\rho(A, x) = \inf\{\rho(a, x) : a \in A\}.$$

If A and B are subsets of R, then the *distance* between A and B is defined as

$$\rho(A, B) = \inf(\rho(a, b) : a \in A,\ b \in B\}.$$

We can show that $\rho(A, x) = 0$ if and only if x is a contact point of A.

Definition 2.22. If A is a set in the metric space R, then all limit points of A constitute the *derived set* of A, denoted by A'.

Unlike the closure of a set, the derived set of a derived set is not the derived set, that is, $(M')' \neq M'$ generally.

Example 2.23. In the metric space $R = (\mathbb{R}, \rho(x, y) = |x - y|)$, $M' = \{0\}$, and $(M')' = \emptyset$.

3 Open and closed sets

Definition 3.1. A set M in a metric space R is said to be *closed* if it coincides with its closure: $\overline{M} = M$.

Example 3.2. The closure of any set M is closed, since $\overline{\overline{M}} = \overline{M}$.

Example 3.3. The closed ball $B[x_0, r]$ is a closed set.

Proof. It is clear that $B[x_0, r] \subseteq \overline{B[x_0, r]}$. We need to show $B[x_0, r] \supseteq \overline{B[x_0, r]}$. Suppose x is a point not in $B[x_0, r]$, then $\rho(x, x_0) > r$, say, $\rho(x, x_0) = r + \epsilon$ with $\epsilon > 0$. Then the neighbourhood $N_{\frac{\epsilon}{2}}(x)$ contains no points in $B[x_0, r]$. This can be shown by the triangle inequality: if x' is a point in $N_{\frac{\epsilon}{2}}(x)$, then $\rho(x', x_0) \leq r$, and $\rho(x, x_0) \leq \rho(x, x') + \rho(x', x_0) \leq r + \frac{\epsilon}{2}$, which is contradictory to the assumption that $\rho(x, x_0) = r + \epsilon$.

Thus any point x outside of $B[x_0, r]$ has a neighbourhood that is disjoint to $B[x_0, r]$ and is not a contact point. We prove that $B[x_0, r] \supseteq \overline{B[x_0, r]}$. □

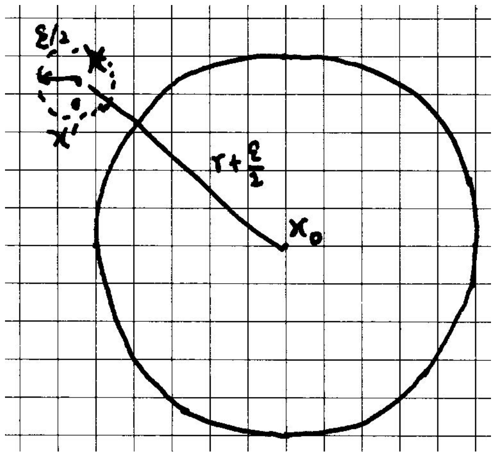

Figure 8: The closed ball is closed.

Theorem 3.4. *The intersection of an arbitrary number of closed sets is a closed set; the sum of a finite number of closed sets is a closed set.*

Proof. First, let $\{M_\alpha\}$ be a family of arbitrary number of closed sets. Need to show that if x is not in the intersection $\bigcap_\alpha M_\alpha$, then x is not a contact point of $\bigcap_\alpha M_\alpha$, that is, x has a neighbourhood disjoint to it.

x is not in $\bigcap_\alpha M_\alpha$ means that x is not in at least one closed set M_α. Then there exists a neighbourhood $N_\epsilon(x)$ that is disjoint to that M_α. This implies that $N_\epsilon(x)$ is disjoint to the intersection $\bigcap_\alpha M_\alpha$.

Next we assume $M_1, M_2, \ldots, M_n$ are closed sets, and need to show that if x is not in the sum of these sets, it has a neighbourhood that is disjoint to the sum. For any $i = 1, 2, \ldots, n$, since x is outside of the closed set M_i, there is a neighbourhood $N_{\epsilon_i}(x)$ that is disjoint to M_i. Taking the intersection of these neighbourhoods, which is $N_\epsilon(x)$ with $\epsilon = \min(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$, then we can verify that $N_\epsilon(x)$ is disjoint to the sum $\bigcup_{i=1}^n M_i$. $\square$

You can think why the second part of the proof does not work for infinite sum of closed sets.

Example 3.5. The single point set $\{x_i\}$ is closed (think is as $B[x_i, 0]$). Therefore the finite set $\{x_1, x_2, \ldots, x_n\}$ is closed, by Theorem 3.4. But an arbitrary infinite set is not nesessarily closed. (A good example of a nonclosed countable set is $\mathbb{Q} \subset \mathbb{R}$, with the usual distance function.)

Definition 3.6. The point x is said to be an *interior point* of the set M if there exists a neighbourhood $N_\epsilon(x)$ of x that is contained in M. The set M is called an *open set* if all of its points are interior points.

Example 3.7. Any point in an open ball $B(x_0, r)$ is an interior point of $B(x_0, r)$, so the open ball $B(x_0, r)$ is an open set.

Proof. Suppose x is a point in $B(x_0, r)$. Then $\rho(x, x_0) < r$, say, $\rho(x, x_0) = r - \epsilon$ with $\epsilon > 0$. Then the neighbourhood $N_{\frac{\epsilon}{2}}(x)$ is contained in $B(x_0, r)$. This can be shown by the triangle inequality: Any point x' in the neighbourhood $N_{\frac{\epsilon}{2}}(x)$ satisfies $\rho(x', x_0) \leq$

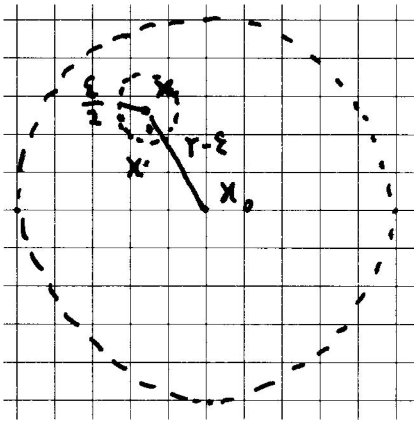

Figure 9: The open ball is open.

$\rho(x', x) + \rho(x, x_0) \leq (r - \epsilon) + \frac{\epsilon}{2} = r - \frac{\epsilon}{2} < r$, so x' is in the open ball. Thus we have shown that any point x in the open ball is an interior point. □

Theorem 3.8. *The set M in metric space R is an open set if and only if its complement $R \setminus M$ is a closed set.*

Proof. The property that M is open is equivalent to that any point x in M has a neibourhood $N_\epsilon(x)$ contained in M. A simple rephrase of this statement is that any point x out of $R \setminus M$ has a neighbourhood $N_\epsilon(x)$ disjoint to $R \setminus M$. The rephrased statement is equivalent to say $R \setminus M$ is a closed set.

Since all steps of the argument are equivalency, the argument above proves both the "if" and "only if" parts. □

Example 3.9. The whole space R is both closed and open (think it informally as $B[x_0, \infty]$ and $B(x_0, \infty)$). Thus its complement, the empty set $\emptyset$, is both open and closed.

Theorem 3.10. *The sum of an arbitrary number of open sets is an open set; the intersection of a finite number of open sets are open sets.*

Proof. A direct proof like that of Theorem 3.4 is not very hard and is left as an exercise. Below we show a proof that relies on the result of Theorem 3.4.

We use the identities

$$R \setminus \left(\bigcup_\alpha M_\alpha \right) = \bigcap_\alpha (R \setminus M_\alpha), \quad R \setminus \left(\bigcap_{i=1}^n M_i \right) = \bigcap_{i=1}^n (R \setminus M_i).$$

The statement that $\{M_\alpha\}$ are open sets is equivalent to that $\{R \setminus M_\alpha\}$ are closed sets (Theorem 3.10). Then by Theorem 3.4, the intersection $\bigcap_\alpha (R \setminus M_\alpha) = R \setminus (\bigcup_\alpha M_\alpha)$ is closed, which implies by Theorem 3.10 again that $\bigcup_\alpha M_\alpha$ is open.

The proof of the latter part of the theorem is similar. The statement that $\{M_i\}$ are open sets is equivalent to that $\{R \setminus M_i\}$ are closed sets. Then by Theorem 3.4, the sum $\bigcup_{i=1}^n (R \setminus M_i) = R \setminus (\bigcap_{i=1}^n M_i)$ is closed, which implies that $\bigcap_{i=1}^n M_i$ is open. □

Definition 3.11. A family of open sets $\{G_\alpha\}$ is called a *basis* if every open set in R can be represented as the sum of (possibly infinitely many) sets belonging to this family.

Theorem 3.12. *A family of open sets $\{G_\alpha\}$ is a basis of the metric space R is for any open set G and every point x in G, a set G_α can be found in this family such that $x \in G_\alpha \subset G$.*

Proof. If $\{G_\alpha\}$ is a basis and $\bigcup_{\alpha \in A} G_\alpha = G$, an open set, then for any $x \in G$, it is contained by at least one G_α. On the other hand, if for any $x \in G$, there is a G_x in the family $\{G_\alpha\}$ such that $G_x \subseteq G$ and $x \in G_x$, then the sum $\bigcup_{x \in G} G_x = G$. □

Example 3.13. The family of all open balls form a basis, since any x in the open set G is an interior point, so there is a neighbourhood of x such that $x \in N_\epsilon(x) \subseteq G$.

Example 3.14. The family of all open balls with rational radii form a basis. To see this, note that in the above mentioned setting, we can take $x \in N_{\epsilon'}(x) \subseteq G$ where ϵ' is a positive rational number less than ϵ.

Theorem 3.15. *The metric space R has a countable basis if and only if it is separable (that is, R has a countable everywhere dense subset).*

In the proof we apply a result as follows:

Proposition 3.16. *The union of countably many countable sets is a countable set.*

The proof of the proposition is given in Appendix B

Proof. If $\{x_1, x_2, \dots\}$ is a countable everywhere dense set, then we show that the countable family $A = \{N_{\frac{1}{n}}(x_m) : m, n = 1, 2, 3, \dots\}$ is a basis. To show that A is countable, we use Proposition 3.16 and that A is the union of the countably many countable sets $A_n = \{N_{\frac{1}{n}}(x_m) : m = 1, 2, 3, \dots\}$.

For any open set G and $x \in G$, there exists $N_\epsilon(x) \subseteq G$. Since $\{x_i\}$ is everywhere dense, there is an x_n in $N_{\frac{\epsilon}{3}}(x)$. Then take an integer k such that $\frac{\epsilon}{3} < \frac{1}{k} < \frac{2\epsilon}{3}$. We have that $\rho(x, x_n) < \frac{\epsilon}{3} < \frac{1}{k}$, so $x \in N_{\frac{1}{k}}(x_n)$. On the other hand, any point x' in $N_{\frac{1}{k}}(x_n)$ satisfies $\rho(x', x) \leq \rho(x', x_n) + \rho(x_n, x) < \frac{1}{k} + \frac{\epsilon}{3} < \epsilon$. It implies that $N_{\frac{1}{k}}(x_n) \subseteq N_\epsilon(x) \subseteq G$. There fore $x \in N_{\frac{1}{k}}(x_n) \subseteq G$, and by Theorem 3.12 A is a basis.

Conversely, if $\{G_1, G_2, \dots\}$ is a countable basis, we take x_i as an arbitrary point in G_i, and show that $\{x_i\}$ is everywhere dense. Suppose x is a point in G. For any positive integer k, $N_{\frac{1}{k}}(x)$ is an open set, and then it contains one open set in the basis $\{G_i\}$, say G_{n_k}, that contains x. Then the point x_{n_k} in G_k satisfies $\rho(x_{n_k}, x) < \frac{1}{k}$. Therefore the sequence $\{x_{n_1}, x_{n_2}, \dots\}$ converges to x, and then $\{x_n\}$ is an everywhere dense subset since any $x \in R$ is its contact point. □

Definition 3.17. A family of sets $\{M_\alpha\}$ is a *covering* of the metric space R if $\bigcup_\alpha M_\alpha = R$. If all M_α are open (closed), the covering is called an *open covering* (*closed covering*).

Theorem 3.18. *If R is a separable metric space, and $\{M_\alpha\}$ is an open covering, then we can select a countable subcovering.*

Proof. Let $\{G_\alpha\}$ be a countable basis. For any $k = 1, 2, 3, \dots$, if G_k is contained by some M_α, then we pick up one that covers G_k and name it M_k; otherwise we leave M_k undefined. Then we have a countable family of open sets, like $\{M_1, M_3, M_4, M_6, \dots\}$

(assuming that $G_2, G_5, \dots$ are not contained by any M_α). We claim that this family of open sets cover R.

For any $x \in R$, there is at least one M_α that contains x. Since M_α is open, there is a G_n such that $x \in G_n \subseteq M_\alpha$, by Theorem 3.12. Then M_n, which is one M_α that contains G_n, is well defined, and it contains x. (But M_n may not be the M_α we considered above.) $\square$

4 Application of the principle of contraction mappings. Picard's theorem

One important application of the principle of contraction mappings is the proof of the following theorem of differential equations.

Theorem 4.1 (Picard's). *Let $\frac{dy}{dx} = f(x, y)$ be a given differential equation with the initial condition $y(x_0) = y_0$, where $f(x, y)$ is defined and continuous in some plane region G which contains the point (x_0, y_0), and satisfies the Lipschitz condition with respect to y*

$$|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2|, \quad \text{for } (x, y_1), (x, y_2) \in G,$$

then on some closed interval $I = [x_0 - d, x_0 + d]$ there exists a unique solution $y = \psi(x)$ of the differential equation with the initial condition.

5 Compact sets in metric spaces

Definition 5.1. A set M in the metric space R is *relatively compact* if every sequence of elements in M contains a subsequence that converges to some point $x \in R$. M is *compact* if it is relatively compact and closed, or equivalently, if every sequence of elements in M contains a subsequence that converges to some point $x \in M$.

Example 5.2. A bounded set in Euclidean n-space is relatively compact, and a bounded closed set in Euclidean n-space is compact. This statement is a consequence of the Bolzano-Weierstrass theorem.

Definition 5.3. The set M in a metric space R is *bounded* if and only if its *diameter* $\inf_{a,b \in M} \rho(a, b) < \infty$.

Example 5.4. The closed ball $B[(0, 0, \dots), 1]$ in ℓ^2 space is bounded, but it is not (relatively) compact. The sequence $\{x^{(n)}\}$ where $x^{(n)} = (\underbrace{0, 0, \dots, 0}_{n-1}, 1, 0, 0, \dots)$ does not have any convergent subsequence. The reason is that $\rho_2(x^{(m)}, x^{(n)}) = \sqrt{2}$ for all $m \neq n$, so no subsequence is a fundamental sequence, and then no subsequence converges.

Definition 5.5. For any set M in the metric space R, the set A is an *ϵ-net* with respect to M if for any $x \in M$, there exists an $a \in A$ such that $\rho(a, x) < \epsilon$. The set M is *totally bounded* if for any $\epsilon > 0$, there exists a finite ϵ-net with respect to M.

A totally bounded set is bounded (exercise). But a bounded set is not necessarily totally bounded. In Example 5.4, the bounded closed ball $B[(0,0,\dots),1]$ is not totally bounded. If A is an $\frac{1}{2}$-net of the closed ball, then for each $x^{(n)}$, there is a point $a_n \in A$ such that $a_i \in B[x^{(n)}, \frac{1}{2}]$. Since the closed balls $B[x^{(n)}, \frac{1}{2}]$ are disjoint, a_n have to be distinct, and then A has at least contably infinite number of points.

Example 5.6. A bounded set in Euclidean n-space is totally bounded.

Proof. We show that for each $\epsilon > 0$, we can construct a finite ϵ-net. First we cover M by a large cube. Then divide the large cube in to small enough cubicles, say, all sides of the cubicles are less than $\epsilon/\sqrt{n}$. Then each point in M is contained in a cubicle and the vertices of the cubicle has distance with this point less than ϵ. The set of vertices of the cubicles constitute a finite ϵ-net. □

Theorem 5.7. *A set M in a complete metric space R is relatively compact if and only if M is totally bounded.*

Proof. First we orive the "only if" part by contradiction. Suppose M is not totally bounded, then there exsits an $\epsilon > 0$ such that no finite set of R is an ϵ-net for M. We construct a sequence $\{x_n\} \subseteq M$ inductively as follows.

1. Select an arbitrary point $x_1 \in M$.

2. After having $x_1, \dots, x_n$, we select $x_{n+1} \in M$ such that $\rho(x_{n+1}, x_i) \geq \epsilon$ for all $i = 1, 2, \dots, n$. This selection is possible since $x_1, \dots, x_n$ do not form an ϵ-net of M.

Then $\rho(x_m, x_n) \geq \epsilon$ for all $m \neq n$, and no subsequence of $\{x_n\}$ is a fundamental sequence, let alone a convergent one. Thus M is not relatively compact.

Then suppose M is totally bounded, and we prove the "if" part. Let $X = \{x_n\} \subseteq M$ be a sequence, and $A_k = \{a_1^{(k)}, \dots, a_{n_k}^{(k)}\}$ be a finite $\frac{1}{2^k}$ net for M. We select a subsequence $\{x'_n\}$ of X inductively as follows.

1. The closed balls $B[a_i^{(1)}, \frac{1}{2}]$ $(i = 1, \dots, n_1)$ cover M, so at least one such closed ball contains infinitely many terms of X. Denote the subsequence consisting of these terms by X_1, and select x'_1 as the first term in X_1.

2. After the construction of subsequence X_k of X, we note that the closed balls $B[a_i^{(k+1)}, \frac{1}{2}]$ $(i = 1, \dots, n_{k+1})$ cover M, and also the sequence X_k. Then at least one such closed bal contains infinitely many terms of X_k. Denote the subsequence of X_k consisting of these terms by X_{k+1}, and select x'_{k+1} as the first term in X_{k+1} that is not already selected as $x'_1, \dots, x'_k$.

Then we have a subsequence $\{x'_n\}$ of X. Note that for any k, the terms $x'_k, x'_{k+1}, \dots$ are all in X_k. Since the distance between any pair of terms of X_k is $\leq 2 \cdot \frac{1}{2^k}$, we have that $\rho(x'_m, x'_n) \leq frac12^k$ for all $m, n > k$. Thus $\{x'_n\}$ is a fundamental sequence and it converges, since the metric space is complete. □

Note that the "only if" part of the proof does not depend on the completeness of the space, and then a relatively compact set is totally bounded is true in any metric space.

Corollary 5.8. *A set M in a complete metric space is relatively compact if and only if for any $\epsilon > 0$ there exists a compact ϵ-net fpr M.*

Proof. If M is relatively compact, then it has a finite ϵ-net for any $\epsilon > 0$. Since any finite set in a metric space is compact (exercise), M has compact an ϵ-net for any $\epsilon > 0$. On the other hand, suppose for any $\epsilon > 0$, M has a compact $\frac{\epsilon}{2}$-net, denoted as A. Then A has a finite $\frac{\epsilon}{2}$-net, denoted as B, since A is compact. Then we find that B is an ϵ-net for M, and then we conclude that M is totally bounded. □

In Corollary 5.8, the condition of compact ϵ-net can be replaced by relatively compact ϵ-net.

A Proof of Proposition 2.20

Here we give the proof of Proposition 2.20 that the set of subsets of positive integers is uncountable. This is a special form of the so called Cantor's theorem.

Suppose $S = \{\text{all subsets of } \mathbb{Z}_+ = \{1, 2, 3, \dots\}\}$ is countable. Then there exists an 1-1 correspondence between S and $\mathbb{Z}_+$. (A possible form of the correspondence is as follows:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \dots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \dots \\ \{1,2\} & \{1,2,4\} & \text{all even numbers} = \{2,4,6,\dots\} & \text{all prime numbers} & \dots \end{array} \tag{5}$$

)

We consider the subset s of $\mathbb{Z}_+$, which is an element of S, defined as

$$s = \{\text{all positive integers that are not in the corresponding subsets of } \mathbb{Z}_+\}.$$

(For example, if the correspondence is constructed as in (5), then $1 \notin s$, $2 \notin s$, $3 \in s$, $4 \in s$, $\dots$.)

Then s, as an element of S, has a corresponding integer in $\mathbb{Z}_+$, denoted as n_s. Below we discuss if n_s belongs to s.

If $n_s \in s$, then by the definition of s, n_s is not in s, whici is a contradiction; if $n_s \notin s$, then by the definition of s, $n_s \in s$, which is also a contradiction.

The two contradictions imply that the initial assumption that S is countable is false.

B Proof of Proposition 3.16

Here we give the proof to Proposition 3.16 that the countable union of sets $A_1, A_2, A_3, \dots$ which are countable sets individually is also a countable sets.

Without loss of generality, we assume that there are no redundant elements in A_n, $n = 1, 2, 3, \dots$, (otherwise the totall number of elements in the union can only be fewer). Then label elements in A_n as

$$A_n = \{a_{n,1}, a_{n,2}, a_{n,3}, \dots\}, \tag{6}$$

and arrange the elements of the union $\bigcup_{n=1}^{\infty} A_n$ as an infinite array

$$\begin{array}{ccccc}
a_{11} & a_{12} & a_{13} & a_{14} & \ldots \\
a_{21} & a_{22} & a_{23} & a_{24} & \ldots \\
a_{31} & a_{32} & a_{33} & a_{34} & \ldots \\
a_{41} & a_{42} & a_{43} & a_{44} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array} \tag{7}$$

Then we “count” the entries in the two dimension array in the “snake” way:

$$\begin{array}{ccccccccc}
a_{11} & \rightarrow & a_{12} & & a_{13} & \rightarrow & a_{14} & & \\
& \swarrow & & \nearrow & & & \swarrow & & \nearrow \\
a_{21} & & a_{22} & & a_{23} & & & a_{24} & \\
\downarrow & \nearrow & & \swarrow & & & \nearrow & & \swarrow \\
a_{31} & & a_{32} & & a_{33} & & & a_{34} & \\
& \swarrow & & \nearrow & & & \swarrow & & \nearrow \\
a_{41} & & a_{42} & & a_{43} & & & a_{44} & \\
\downarrow & \nearrow & & \swarrow & & & \nearrow & & \swarrow
\end{array} \tag{8}$$

and find that these elements has an 1-1 mapping with positive integers.