MA3209, Homework 1

Due at the beginning of your tutorial session, Monday 27 August or Wednesday 29 August.

1. Consider the metric spaces $(\mathbb{R}^n, \rho_p(x,y) = (\sum_{k=1}^n |x_k - y_k|^p)^{1/p})$ and $(\mathbb{R}^n, \rho_\infty(x,y) = \max_{k=1}^n |x_k - y_k|)$.

   (a) Show that the function $\rho_\infty(x,y)$ satisfies the triangle axiom.

   (b) Show that for any $p \geq 1$ and x, y in $\mathbb{R}^n$
   $$\rho_\infty(x,y) \leq \rho_p(x,y) \leq n\rho_\infty(x,y).$$

   (c) Show that for any $p > 1$ and x, y in $\mathbb{R}^n$
   $$\rho_p(x,y) \leq \rho_1(x,y) \leq n^{1-\frac{1}{p}}\rho_p(x,y).$$
   (Hint: For the left inequality, show $a^p + b^p \leq (a+b)^p$ for all $a, b > 0$ first. For the right inequality, use Hölder's inequality.)

2. Show that the distance function in metric space $C^p([c_1, c_2])$
   $$\rho_p(x(t), y(t)) = \left(\int_{c_1}^{c_2} |x(t) - y(t)|^p\right)^{\frac{1}{p}}$$
   satisfies the triangle inequality when $p > 1$, in the following steps.

   (a) Show that the triangle inequality follows from the integral form of the Minkowski inequality for any continuous functions $a(t), b(t)$ on $[c_1, c_2]$
   $$\left(\int_{c_1}^{c_2} |a(t) + b(t)|^p dt\right)^{\frac{1}{p}} \leq \left(\int_{c_1}^{c_2} |a(t)|^p dt\right)^{\frac{1}{p}} + \left(\int_{c_1}^{c_2} |b(t)|^p dt\right)^{\frac{1}{p}}. \quad (1)$$

   (b) Show that the integral form of the Minkowski inequality (1) follows from the integral form of Hölder's inequality for any continuous functions $x(t), y(t)$ on $[c_1, c_2]$, and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$
   $$\int_{c_1}^{c_2} |x(t)y(t)| dt \leq \left(\int_{c_1}^{c_2} |x(t)|^p dt\right)^{\frac{1}{p}} \left(\int_{c_1}^{c_2} |y(t)|^q dt\right)^{\frac{1}{q}}. \quad (2)$$
   (Hint: Use the trick
   $$\int_{c_1}^{c_2} |a(t) + b(t)|^p dt = \int_{c_1}^{c_2} |a(t) + b(t)|^{p-1}|a(t)| dt + \int_{c_1}^{c_2} |a(t) + b(t)|^{p-1}|b(t)| dt.$$
)

(c) Prove the integral form of Hölder's inequality (2).
(Hint: Reduce it to the special case that $\int_{c_1}^{c_2} |x(t)|^p dt = 1$, $\int_{c_1}^{c_2} |y(t)|^q dt = 1$.)

3. Let (X, ρ) be a metric space. Define the function $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)}.$$

Show that (X, d) is also a metric space.

4. Let V be a vector space, and let $\|\cdot\| : V \to \mathbb{R}$ be a function (called a *norm* on V) satisfying the following conditoins:

(i) $\|x\| \geq 0$ for all x in V.

(ii) $\|x\| = 0$ if and only if $x = 0$.

(iii) $\alpha\|x\| = |\alpha|\|x\|$ for all x in V and all scalar α.

(iv) $\|x + y\| \leq \|x\| + \|y\|$ for all x, y in V.

Define $\rho : V \times V \to \mathbb{R}$ by $\rho(x, y) = \|x - y\|$ for x, y in V. Show that (V, ρ) is a metric space.

5. (Extra credit) Let p be a prime number. For any integer $n > 0$ we define $v_p(n)$ as the exponent of p in the decomposition of n into prime numbers. Let $x = \pm\frac{r}{s}$ be any rational number $\neq 0$, with r and s integers > 0. Define

$$v_p(x) = v_p(r) - v_p(s). \tag{3}$$

Then define the function $\rho : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ by ($\mathbb{Q}$ stands for the set of rational numbers)

$$\rho(x, y) = \begin{cases} p^{-v_p(x-y)} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

(a) Show that $v_p(x)$ is well defined, that is, $v_p(x)$ in (3) does not depend on the particular expression of x as a fraction.

(b) Show that $(\mathbb{Q}, \rho)$ is a metric space. (ρ is the p-adic distance useful in number theory.)