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§1. DEFINITION AND EXAMPLES OF METRIC SPACES

DEF A *metric space* $R = (X, \rho)$ is the pair of two things: a set X, whose elements are called points, and a distance, i.e., a single-valued, nonnegative real function $\rho(x, y)$, defined for arbitrary x and y in X, such that

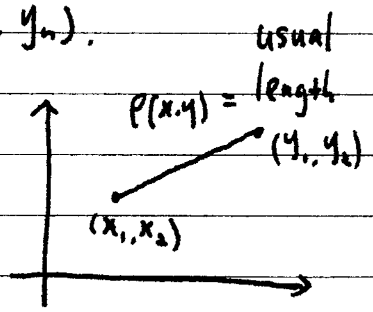
1. $\rho(x, y) = 0$ if and only if $x = y$
2. $\rho(x, y) = \rho(y, x)$ (axiom of symmetry)
3. $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$ (triangle axiom).

EXP $X = \mathbb{R}$, $\rho(x, y) = |x - y|$.

EXP $X = \mathbb{R}^n$, for $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n)$,

$$\rho(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

(Euclidean n-space)



$n = 2$ case.

Conditions ① and ② are obvious. To validate condition ③, Let $y - x = a = (a_1, a_2, \cdots, a_n)$, $z - y = b = (b_1, b_2, \cdots, b_n)$.

Then $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$

$$\Leftrightarrow \sqrt{(a_1 + b_1)^2 + \cdots + (a_n + b_n)^2} \leq \sqrt{a_1^2 + \cdots + a_n^2} + \sqrt{b_1^2 + \cdots + b_n^2}$$

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$$\iff (a_1+b_1)^2+\cdots+(a_n+b_n)^2 \le (a_1^2+\cdots+a_n^2)+(b_1^2+\cdots+b_n^2) + 2\sqrt{a_1^2+\cdots+a_n^2}\,\sqrt{b_1^2+\cdots+b_n^2}$$

$$\iff 2a_1b_1+2a_2b_2+\cdots+2a_nb_n \le 2\sqrt{a_1^2+\cdots+a_n^2}\,\sqrt{b_1^2+\cdots+b_n^2}$$

which is a consequence of the Schwarz inequality:

$$\boxed{\begin{array}{l}(a_1b_1+a_2b_2+\cdots+a_nb_n)^2 \le (a_1^2+\cdots+a_n^2)(b_1^2+\cdots+b_n^2) \\ \text{for all real numbers } a_1,\dots,a_n,b_1,\dots,b_n\end{array}}$$

PRF of the Schwarz inequality:

$$(a_1^2+a_2^2+\cdots+a_n^2)(b_1^2+b_2^2+\cdots+b_n^2)-(a_1b_1+a_2b_2+\cdots+a_nb_n)^2$$

$$=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_ib_j-a_jb_i)^2 \ge 0. \qquad \square.$$

EXP $X=\mathbb{R}^n$. $\rho_\infty(x,y) = \max\{|y_k-x_k| : k=1,\dots,n\}$

Validity of the three conditions: exercise.

EXP $X=\mathbb{R}^n$, $\rho_p(x,y) = \left(\sum_{k=1}^{n}|y_k-x_k|^p\right)^{1/p}$ for $p\ge 1$.

Conditions ① and ②: obvious. Condition ③: we validate for $p>1$. Need to show for any $\vec{a}$, $\vec{b}$, the Minkowski inequality

$$\boxed{(|a_1+b_1|^p+\cdots+|a_n+b_n|^p)^{1/p} \le (|a_1|^p+\cdots+|a_n|^p)^{1/p}+(|b_1|^p+\cdots+|b_n|^p)^{1/p}}$$

holds. The Minkowski inequality follows from Hölder's

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inequality

$$\boxed{\sum_{k=1}^{n}|x_k y_k| \le \left(\sum_{k=1}^{n}|x_k^p|\right)^{1/p}\left(\sum_{k=1}^{n}|y_k|^q\right)^{1/q}, \text{ where } \frac{1}{p}+\frac{1}{q}=1,\ p,q>0, \text{ for all real } x_1, \ldots x_n, y_1, \ldots y_n}$$

PRF of that the Minkowski inequality follows from Hölder's inequality:

Write $(|a|+|b|)^p = (|a|+|b|)^{p-1}|a| + (|a|+|b|)^{p-1}|b|$. Then

$$\sum_{k=1}^{n}(|a_k|+|b_k|)^p = \sum_{k=1}^{n}(|a_k|+|b_k|)^{p-1}|a_k| + \sum_{k=1}^{n}(|a_k|+|b_k|)^{p-1}|b_k|.$$

By Hölder's inequality.

$$\sum_{k=1}^{n}(|a_k|+|b_k|)^{p-1}|a_k| \le \left(\sum_{k=1}^{n}\left[(|a_k|+|b_k|)^{p-1}\right]^q\right)^{1/q}\left(\sum_{k=1}^{n}|a_k|^p\right)^{1/p}$$

(Since $q=\frac{p}{p-1}$) $$= \left(\sum_{k=1}^{n}(|a_k|+|b_k|)^p\right)^{1-\frac{1}{p}}\left(\sum_{k=1}^{n}|a_k|^p\right)^{1/p}$$

Similarly.

$$\sum_{k=1}^{n}(|a_k|+|b_k|)^{p-1}|b_k| \le \left(\sum_{k=1}^{n}(|a_k|+|b_k|)^p\right)^{1-\frac{1}{p}}\left(\sum_{k=1}^{n}|b_k|^p\right)^{1/p}.$$

Sum up:

$$\sum_{k=1}^{n}(|a_k|+|b_k|)^p \le \left(\sum_{k=1}^{n}(|a_k|+|b_k|)^p\right)^{1-\frac{1}{p}}\left[\left(\sum_{k=1}^{n}|a_k|^p\right)^{1/p}+\left(\sum_{k=1}^{n}|b_k|^p\right)^{1/p}\right]$$

$$\Longleftrightarrow \left(\sum_{k=1}^{n}(|a_k|+|b_k|)^p\right)^{1/p} \le \left(\sum_{k=1}^{n}|a_k|^p\right)^{1/p}+\left(\sum_{k=1}^{n}|b_k|^p\right)^{1/p} \qquad \square$$

PRF of Hölder's inequality:

First we show a short inequality

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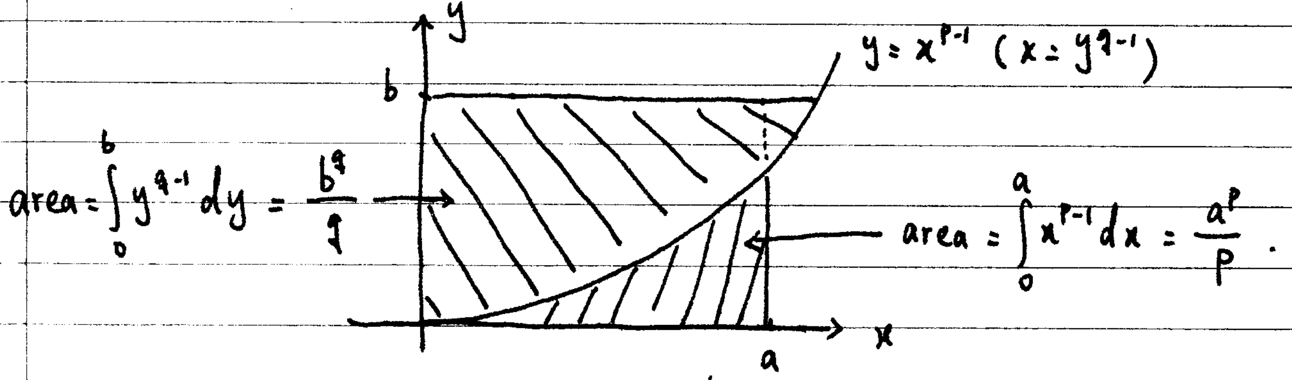
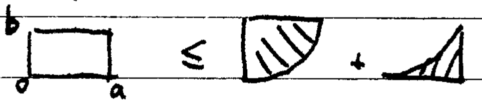
$ab \le a^p/p + b^q/q$ for $a, b > 0$, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

The proof of the inequality is explained by the graph:



and the geometric fact that .

Note that Hölder's inequality is <u>homogeneous</u> in the sense that if it holds for $\mathbf{x}$ and $\mathbf{y}$, then also for $\lambda\mathbf{x} = (\lambda\mathbf{x}_1, \ldots, \lambda\mathbf{x}_n)$ and $\mu\mathbf{y} = (\mu\mathbf{y}_1, \ldots, \mu\mathbf{y}_n)$ for any $\lambda, \mu \neq 0$. Thus we need only to prove it under the condition $\sum_{k=1}^n |\mathbf{x}_k|^p = \sum_{k=1}^n |\mathbf{y}_k|^q = 1$. In this case it follows from

$$\sum_{k=1}^n |x_k y_k| \le \sum_{k=1}^n \frac{|x_k|^p}{p} + \frac{|y_k|^q}{q} = \frac{1}{p}\sum_{k=1}^n |x_k|^p + \frac{1}{q}\sum_{k=1}^n |y_k|^q = \frac{1}{p} + \frac{1}{q} = 1. \quad \square$$

RMK The distance in Euclidean n-space is ρ_2 and the distance in last example is, in some sense, the limit of ρ_p as $p \to \infty$:

$$\rho_\infty(x, y) = \lim_{p\to\infty} \rho_p(x, y) \quad \text{for all } x, y \text{ in } \mathbb{R}^n \qquad \text{(Exercise)}.$$

EXP X: all possible sequences $x = (x_1, x_2, \cdots, x_n, \cdots)$ of real numbers, Subject :

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which satisfy the condition $\sum_{k=1}^{\infty} |x_k|^p < \infty$, for $p \geq 1$.

$\rho_p(x, y) = \left(\sum_{k=1}^{\infty} |y_k - x_k|^p\right)^{1/p}$. (ℓ^p space).

First we show that $\rho_p(x, y)$ is well defined. For any n,

$$\left(\sum_{k=1}^{n} |y_k - x_k|^p\right)^{1/p} \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{1/p} \quad \text{(Minkowski)}$$

$$\leq \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{\infty} |y_k|^p\right)^{1/p} = \text{const} < \infty \quad \text{(assumption)}$$

Since $\left(\sum_{k=1}^{n} |y_k - x_k|^p\right)^{1/p}$ increases monotonically as $n \to \infty$. $\rho_p(x, y)$, as the limit of the sequence, is well defined by the <u>monotone convergence theoreom</u>.

Then we show condition ③ holds, that is,

$$\left(\sum_{k=1}^{\infty} |z_k - x_k|^p\right)^{1/p} - \left[\left(\sum_{k=1}^{\infty} |y_k - x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{\infty} |z_k - y_k|^p\right)^{1/p}\right] \leq 0.$$

It is the $n \to \infty$ limit of the inequality

$$\left(\sum_{k=1}^{n} |z_k - x_k|^p\right)^{1/p} - \left[\left(\sum_{k=1}^{n} |y_k - x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |z_k - y_k|^p\right)^{1/p}\right] \leq 0.$$

EXP X: all possible sequences $x = (x_1, x_2, \cdots, x_n, \cdots)$ of real numbers, which satisfy $\sup_{k=1}^{\infty}(|x_k|) < \infty$. $\rho_\infty(x, y) = \sup_{k=1}^{\infty} |x_k - y_k|$.

(ℓ^∞ space).

EXP X: all continuous function on interval $[a, b]$.

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$\rho_p(x, y) = \left(\int_a^b |x(t) - y(t)|^p \, dt\right)^{1/p}$ for $p \geq 1$. $(C^p([a,b]))$

ρ_p is well defined since $x(t)$ and $y(t)$ are bounded (why?).

To show condition ③ is satisfied, we need the integral form of the Minkowski inequality. The proof is a homework problem.

EXP X: The same as in $C^p([a,b])$. $\rho_\infty(x, y) = \max_{t \in [a,b]} (|x(t) - y(t)|)$

$(C^\infty([a,b]))$.

Show condition ③ is satisfied : exercise.