NATIONAL UNIVERSITY OF SINGAPORE

MA3209 - Mathematical Analysis III

(Semester 1 : AY2014/2015)

Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. Please write your matriculation/student number only. Do not write your name.

2. This examination paper contains **6** questions and comprises **5** printed pages.

3. Answer **ALL** questions.

4. Please start each question on a new page.

5. This is a CLOSED BOOK (with helpsheet) examination.

6. Students are allowed to use two handwritten, A4 size, double-sided helpsheets.

7. Calculators are not necessary and not allowed.

1. (a) (10 marks) Show that the equation

$$f(x) = x, \quad \text{where} \quad f(x) = \log(3 + \sin x)$$

has a unique real solution.

Solution We need only to show that $f : \mathbb{R} \to \mathbb{R}$ is a contraction with respect to the usual metric. To check it, we have

$$|f'(x)| = \left| \frac{\cos x}{3 + \sin x} \right| \le \frac{1}{3 - 1} = \frac{1}{2}.$$

So for all $x_1, x_2 \in \mathbb{R}$, by the mean value theorem,

$$|f(x_1) - f(x_2)| \le \frac{1}{2}|x_1 - x_2|.$$

(b) (10 marks) Let $\{g_n(x)\}_{n=1}^{\infty}$ be a sequence of continuous functions on $[0, 16]$. Show that the sequence of functions $\{h_n(x)\}_{n=1}^{\infty}$, which are continuous on $[0, 4]$ with

$$h_n(x) = \int_0^{x^2} \log(3 + \sin(g_n(t)))dt,$$

has a uniformly convergent subsequence.

Solution We need only to show that $\{h_n\}_{n=1}^{\infty}$ is both uniformly bounded and equicontinuous. To show the uniform boundedness, we check that for any $x \in [0, 4]$,

$$\begin{aligned} |h_n(x)| &= \left| \int_0^{x^2} \log(3 + \sin(g_n(t)))dt \right| \\ &\le \left| \int_0^{x^2} \log(3 + 1)dt \right| \\ &\le 16 \log 4. \end{aligned}$$

To show the equicontinuity, we check that for any $x \in (0, 4)$,

$$\begin{aligned} |h_n'(x)| &= \left| \frac{d}{dx} \int_0^{x^2} \log(3 + \sin(g_n(t)))dt \right| \\ &= 2x \left|\log(3 + \sin(g_n(t)))dt\right| \\ &\le 2x \log 4 \le 8 \log 4. \end{aligned}$$

Thus for all $\epsilon > 0$, we take $\delta = \epsilon/(8\log 4)$ and by the mean value theorem have that $|h_n(x_1) - h_n(x_2)| < \epsilon$ for all $x_1, x_2 \in [0,4]$ and $|x_1 - x_2| < \delta$.

2. (15 marks) Let M be an arbitrary set in a metric space $R = (X, \rho)$. Show that the set of all limit points of M (called the derived set of M and is denoted by M') is a closed set.

   Solution To show that M' is a closed set, we take any convergent sequence $\{x_n\} \subseteq M'$ whose limit is denoted as x_0 and show that $x_0 \in M'$. Since $x_n \in M'$, there are infinitely many points in M such that they are in the $1/n$ neighbourhood centered at x_n. Thus inductively, we can take $y_n \in M$ such that $\rho(x_n, y_n) < 1/n$ and let y_n be distinct to $y_1, y_2, \ldots, y_{n-1}$. Hence $\{y_n\}$ is a sequence in M with distinct elements. We have that $\{y_n\}$ has the same limit x_0 as $\{x_n\}$. Therefore x_0 is a limit point of M, or equivalently, $x_0 \in M'$.

3. (15 marks) Let the metric space $R = (X, \rho)$ be a subspace of $C[0,1]$, with the point set

$$X = \{x(t) : x(t) \text{ is continuous on } [0,1] \text{ and } \int_0^1 x(t)dt = 0\}$$

   and the distance

$$\rho(x(t), y(t)) = \max_{t\in[0,1]} |x(t) - y(t)|.$$

   Determine if R is complete. Justify your answer.

   Solution A result learnt in class is that if X is a closed set in the complete metric space (Y, ρ), then the subspace (X, ρ) is complete. Thus we only need to show that the set X is closed in the metric space $C[0,1]$. Suppose $\{x_n\} \subseteq X$, and converges to $x_0(t) \in C[0,1]$. Then we have that $x_n(t)$ converges uniformly on the interval $[0,1]$. Thus

$$\int_0^1 x_0(t)dt = \int_0^1 \lim_{n\to\infty} x_n(t)dt = \lim_{n\to\infty} \int_0^1 x_n(t)dt = \lim_{n\to\infty} 0 = 0.$$

   We conclude that $x_0(t) \in X$.

4. (15 marks) In a metric space (X, ρ), we define the diameter of a set M by

$$\operatorname{diam}(M) = \sup_{x,y\in M} \rho(x,y).$$

   Show that if M is a compact set, then there are $x_0, y_0 \in M$ such that $\operatorname{diam}(M) = \rho(x_0, y_0)$.

Solution Suppose $\{x_n\}, \{y_n\} \subseteq M$ are sequences such that $\lim_{n\to\infty} \rho(x_n, y_n) = \text{diam}(M)$. Then there is a subsequence $\{x_{n_k}\}$ that converges to $x_0 \in M$, and a subsequence of $\{y_{n_k}\}$, which we denoted by $\{y_{n_{k(l)}}\}$, that converges to $y_0 \in M$. Then $\rho(x_0, y_0) = \lim_{l\to\infty} \rho(x_{n_{k(l)}}, y_{n_{k(l)}}) = \text{diam}(M)$.

5. (15 marks) Let M be a compact set in a metric space R and $\{G_\alpha\}$ be an open covering of M. Show that there exists $\delta > 0$ such that for any $x \in M$, the open ball $B(x, \delta)$ is covered by a G_α in the family of the open covering.

   Solution We prove by contradiction. Suppose for any n, there is a point x_n such that $N_{1/n}(x_n)$ is not covered by any G_α. Then the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ that converges to $x \in M$. We have that x is covered by a certain $G_{\alpha'}$, and then there is $\epsilon > 0$ such that $N_\epsilon(x) \subseteq G_{\alpha'}$. Since $x_{n_k} \to x$, there is N_ϵ such that for all $k > N_\epsilon$, we have $x_{n_k} \in N_{\epsilon/2}(x)$, and thus $N_{\epsilon/2}(x_{n_k}) \subseteq N_\epsilon(x) \subseteq G_{\alpha'}$ for all $k > N_\epsilon$. This is contradictory to our assumption since if k is large enough, $\epsilon/2 > 1/n_k$.

6. We assume that A and B are nonempty sets in the Euclidean n-space. Define

$$A + B = \{x \in \mathbb{R}^n \mid x = a + b, \text{ where } a \in A \text{ and } b \in B\}.$$

   (a) (10 marks) Show that if A and B are compact sets, then $A + B$ is a compact set.

   Solution To show that $A + B$ is compact, we check that $A + B$ is sequentially compact, that is, any sequence in $A+B$ has a subsequence that converges to a point in $A+B$. Let $\{x_n\} = \{a_n + b_n\}$ be a sequence in $A + B$, where $a_n \in A$ and $b_n \in B$. By the compactness of A we have that a subsequence of $\{a_n\}$, say $\{a_{n_k}\}$, converges to $a \in A$. Similarly by the compactness of B, a subsequence of $\{b_{n_k}\}$, say $\{b_{n_{k(l)}}\}$, converges to a point $b \in B$. Then the subsequence $\{x_{n_{k(l)}}\} = \{a_{n_{k(l)}} + b_{n_{k(l)}}\}$ of $\{x_n\}$ converges to $a + b \in A + B$.

   (b) (10 marks) Show that if A and B are connected sets, then $A + B$ is a connected set.

   Solution To show that $A + B$ is connected, we note that for any $a \in A$, the set

$$a + B = \{a + b \mid b \in B\}$$

   is connected. Suppose to the contrary, $a + B$ is not connected, then there exists open sets U and V in the Euclidean n-space such that

$(a+B)\cap U$ and $(a+B)\cap V$ are nonempty disjoint sets and $((a+B)\cap U)\cup((a+B)\cap V)) = a+B$. Then we have that $-a+U$ and $-a+V$ are opensets in the Euclidean n-space and $B\cap(-a+U)$ and $B\cap(-a+V)$ are nonempty disjoint sets and $(B\cap(-a+U))\cup(B\cap(-a+V)) = B$. It is contradictory to the connectedness of B. Similarly, for any $b\in B$, the set

$$b + A = \{a + b \mid a \in A\}$$

is connected.

We fix $a_0 \in A$, and have that for any $b \in B$, the connected sets $a_0 + B$ and $b + A$ has nonempty intersection including $a_0 + b$. Thus the set

$$C(b) = (a_0 \in A) \cup (b \in B)$$

is connected. Then note that for any $b_1, b_2 \in B$, the sets $C(b_1)$ and $C(b_2)$ has nonempty intersection containing $a_0 + A$. Thus the union of all the $C(b)$

$$\bigcup_{b\in B} C(b) = A + B$$

is connected.