CSE 202: Algorithm Design and Analysis
Homework Assignment 1

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Problem 1

1. True. Proof by example.

   First we say the n `union` operations are formatted in the following way:

   at first each variable is in its separate set and after that every `union` is to union the roots of two subsets of variables next to each other. This means that There will be only one last set of variables after operations, and it will form a well-balanced binary tree. Then we conduct the `find` operations.

   Suppose all m external `find` operations are called to find the root of the leaves of this well-balanced binary tree, then each `find` operation takes $\Omega(\texttt{log}n)$ and in total the such a sequence takes $\Omega(m\texttt{log}n)$.

2. Proof by contradiction.

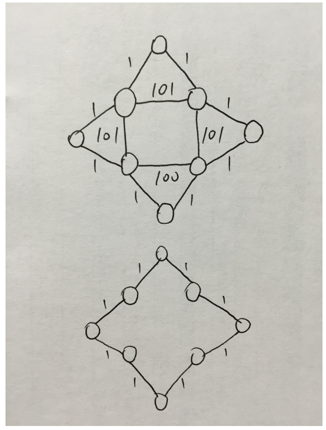
   Assume there is a MST T and e is not in it. Then we can add e into T. This will lead to a cycle C which contains e.

   We could always remove any other edge from cycle C since every other edge in c has a bigger weight than e. After that, we will still have a tree but this tree will have a smaller weight compared with T. Therefore T is not a MST and we have found a contradiction.

3. False.

   We could have this cycle C with 4 nodes. Each edge has weight bigger than 100. (Let's assume that the smallest weight edge in C has a weight of 100). And between each 2 neighbor nodes in this cycle C, there is another path which contains one more node and forms a triangle cycle with these 2 nodes. Let's further assume that each edge in this triangle has a weight of 1. In this situation, the MST will have no edge from C.

   Below is the picture illustration, the center cycle is C:



4. Same as problem 3.

   False.

   We could have this cycle C with 4 nodes. Each edge has weight bigger than 100. And there is another path including one more node between each 2 nodes which only cause 2 more weight. In this situation, the MST will have no edge from C. Details can be seen in the above picture. The center cycle will never have any edge in any MST, thus false.

5. Proof by contradiction.

   Assume T_1 and T_2 are two different MSTs. Lets say e_1 is the edge with smallest weight that is in T_1 but not in T_2.

   Then we add e_1 into T_2 and we get a cycle. Therefore one edge in the cycle we call it e_2 is not in T_1.

   We know that the weight of e_2 is bigger than the weight of e_1, and therefore we know T_2 = $T_1 \cup \{e_2\} \setminus \{e_1\}$ has a total weight bigger than T_1. Therefore it is not a valid MST. Contradiction found.

Problem 2

1. 1, 3, 3, 1
2. (a) Assume that $i \leq d_1 + 1, j > d_1 + 1$ and there is a link between v_1 and v_j. Then we can have $(v_1, v_i) \notin E$ and $(v_1, v_j) \in E$.
   Another thing is that we know that the degree of v_i should be bigger than or equal to v_j, and since we already have $(v_1, v_i) \notin E$ and $(v_1, v_j) \in E$. We know that there must be another node u that $(u, v_i) \in E$ and $(u, v_j) \notin E$.

   (b) The changes need to be made could be from part(a). Namely we could first remove the edge (v_1, v_j), then we pick any v_i such that $2 \leq i \leq d_1 + 1$, and we add the edge (v_1, v_i). Then we find a node u of neighbor v_i which is not a neighbor of v_j, and we remove the edge (u, v_i) and replace with another edge (u, v_j).
   By doing this, we could have each node degree unchanged but generated the required edge (v_1, v_i).

   (c) By combining prat(a) and part(b). We could know that for each neighbor of v_1 that is not belongs to v_2 to v_{d_1+1}, we would have another $u \in V$ that $(u, v_i) \in E$ and $(u, v_j) \notin E$, $(v_1, v_i) \notin E$ and $(v_1, v_j) \in E$. Therefore according to part(b), we could always transform the above pair and get rid of edge (u, v_j) and get a new edge (u, v_i).
3. Algorithm

Algorithm 1 Check Graph Existence

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function GRAPH–EXIST(D)
    $n \leftarrow \texttt{size}(D)$
    for d in D **do**
        if $d > n - 1$ **then return** false
    if $\texttt{sum}(D)$ is not even **then return** false
    Sort D in descending order and record the mapping
    for i in $(1, \ldots, n-1)$ **do**
        if $d_i < 0$ **then return** false
        for j in $(1, \ldots, d_i)$ **do**
            Create edge $(i, i + j)$
            $d_{i+j} \leftarrow d_{i+j} - 1$
    if $d_n \neq 0$ **then return** false
    return true

- Proof of correctness: If there exists an degree that is larger than the limit $(n - 1)$ or the total degree is odd, such a graph must not exist. Then, based on the results in part 2, we conclude that the original problem is equivalent as finding such a degree sequence with descending order in which for any node, all its connected nodes are its close neighbors. Hence, if we loop over all nodes, create corresponding edges between neighbors, and find out that some nodes' degrees are negative or the last node still has free degree at the end, such a graph must not exist as well.
- Time Complexity: Based on our algorithm, the running time is $O(2n + n\texttt{log}n + 2m) = O(m + n\texttt{log}n)$.

Problem 3

1. We can treat each intersection as a node v in a undirected connected graph $G(V, E)$. And each road as a edge e between node in G. We can have a initialized set of all the edges R. Every time we pick up a node v from the graph G, we can remove all the edges linked v from R.

   - Inputs: An undirected, connected graph $G(V, E)$ and the corresponding edge set R. This graph G should not contain self-loops (edges with both endpoints equal to the same node) or multiple edges between the same pair of nodes.
   - Constraints: Pick up as few as possible nodes to make R finally empty. Every time we pick up a node v from the graph G, we can remove all the edges linked v from R.
   - Outputs: A set S of nodes we picked up as well as the size of the S.

2. False.

   The example is attached below:

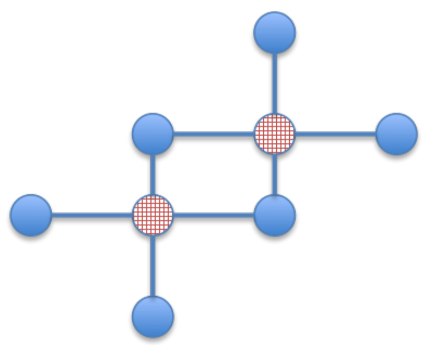

Figure 1: optimal choices of intersections

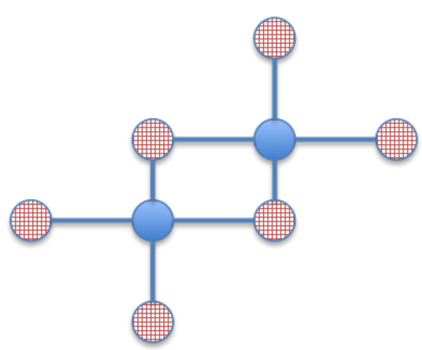

Figure 2: choices of intersections of this algorithm

3. This algorithm does not works correctly either.

   The example is attached below:

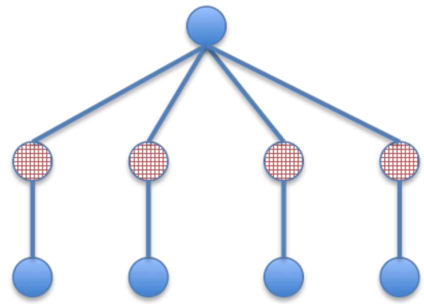

Figure 3: optimal choices of intersections

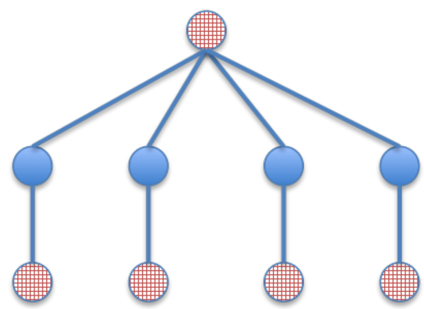

Figure 4: choices of intersections of this algorithm