1. $\max \sum_j \alpha_j - k\lambda$.
   $\sum_i \beta_{ij} \leq \lambda$, for all i.
   $\alpha_j \leq \beta_{ij} + c_{ij}$, for all i, j.
   $\alpha_j, \beta_{ij} \geq 0$.

2. $\sum_{cluster c_0} 3\lambda + \sum_{j \in c} c_{ij} \leq 3 \sum_{j \in c} alpha_j$.

3. $3k\lambda + cost \leq 3 \sum_{j \in c} alpha_j$.

4. When $\lambda = 0$, there's no penalty for opening facilities, so you can open all of the facilities, and then assign each client to the closet one.
   When $\lambda = \sum_{j \in c} \sum_{i \in F} c_{ij}$, suppose the optimal solution is S when using exact k facilities, you can drop any one of the k facilities, and the objective function will first decrease by $\sum_{j \in c} \sum_{i \in F} c_{ij}$, and increase by the difference of two assignments, which is smaller than the cost of all edges.

5. $3|S_1|\lambda_1 + cost(S_1) \leq 3 \sum_{j \in c} \alpha_j^1$,
   $3|S_2|\lambda_2 + cost(S_2) \leq 3 \sum_{j \in c} \alpha_j^2$.

6. $$\begin{aligned} cost(S_1) &\leq 3 \sum_{j \in c} \alpha_j^1 - 3|S_1|\lambda_1 \\ &\leq 3 \sum_{j\ inc} \alpha_j^1 - 3|S_1|(\lambda_2 - \frac{\epsilon c_{min}}{3|F|}) \\ &= 3 \sum_{j \in c} \alpha_j^1 - 3\lambda_2|S_1| + \frac{|S_1|}{|F|}\epsilon c_{min} \\ &\leq 3 \sum_{j \in c} \alpha_j^1 - 3\lambda_2|S_1| + \epsilon OPT \end{aligned}$$

7. $$\begin{aligned} \delta_1 cost(S_1) &+ \delta_2 cost(S_2) \\ &\leq \delta_1(3 \sum_{j \in c} \alpha_j^1 - 3\lambda_2|S_1|) + \delta_2(3 \sum_{j \in c} \alpha_j^2 - 3\lambda_2|S_2|) + \delta_1 \epsilon OPT \\ &\leq 3OPT + \delta_1 \epsilon OPT - 3k\lambda_2 \\ &\leq (3 + \delta_1 \epsilon) OPT \end{aligned}$$

8.

$$\begin{aligned}\delta_2 cost(S_2) &\leq (3+\delta_1\epsilon)OPT \\ &\leq (3+\epsilon)OPT \\ &\leq 2(3+\epsilon)OPT\end{aligned}$$

The last inequality holds because $\delta_2 \geq 0.5$.

9. $\frac{k}{|S_1|}$.

10. $c(i, f_2) \leq c(f_1, f_2)$.

11.

$$\begin{aligned}c(i,j) &\leq c(j,f_2)+c(f_2,i) \\ &\leq c(j,f_2)+c(f_1,f_2) \\ &\leq c(j,f_2)+c(f_1,j)+c(j,f_2) \\ &= c(f_1,j)+2c(f_2,j) \qquad (1)\end{aligned}$$

12. $E_j = \frac{k}{|S_1|}c_j^1 + (1-\frac{k}{|S_1|})(c_j^1+2c_j^2)$.
And $\frac{k}{|S_1|} \geq \frac{k-|S_2|}{|S_1|-|S_2|} = \delta_1$.
The sum of two coefficient is 1, but the RHS term has larger coefficient on the large term $c_j^1 + 2c_j^2$.
So, $E \leq \delta_1 c_j^1 + \delta_2(c_j^1 + 2c_j^2)$.

13. We know that $\delta_2 \leq 0.5$, based on Q12, we have

$$\begin{aligned}E_j &\leq \delta_1 c_j^1 + \delta_2(c_j^1+2c_j^2) \\ &\leq 2(\delta_1 c_j^1 + \delta_2 c_j^2)\end{aligned}$$

And the sum of expected cost of each client is the final cost of the assignment, so we have

$$\begin{aligned}E(cost) &\leq 2(\delta_1 cost(S_1) + \delta_2 cost(S_2)) \\ &\leq 2(3+\delta_1\epsilon)OPT \\ &\leq 2(3+\epsilon)OPT\end{aligned}$$

Therefore, we can conclude it's a 6-approximation algorithm, since we can set ϵ to be arbitrarily small.