16-662: Robot Autonomy

Lecture 2: Modeling and Control

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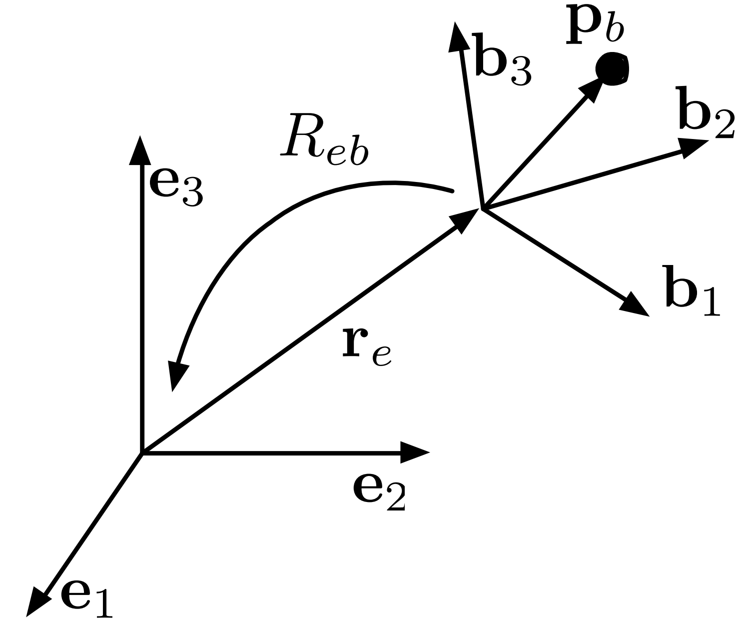
Lecture Outline

- Model
  - Propeller model
  - Force and moments generation
  - Equations of motion
  - Simplifying assumptions
- Control
  - Concept review
  - Attitude control (inner loop)
  - Position control (outer loop)
  - Gain selection
- Project 1A discussion and suggestions

Rigid-Body Transform

$$\mathbf{p}_e = R_{eb}\mathbf{p}_b + \mathbf{r}_e$$

rotation translation



Rotation (special orthogonal group):

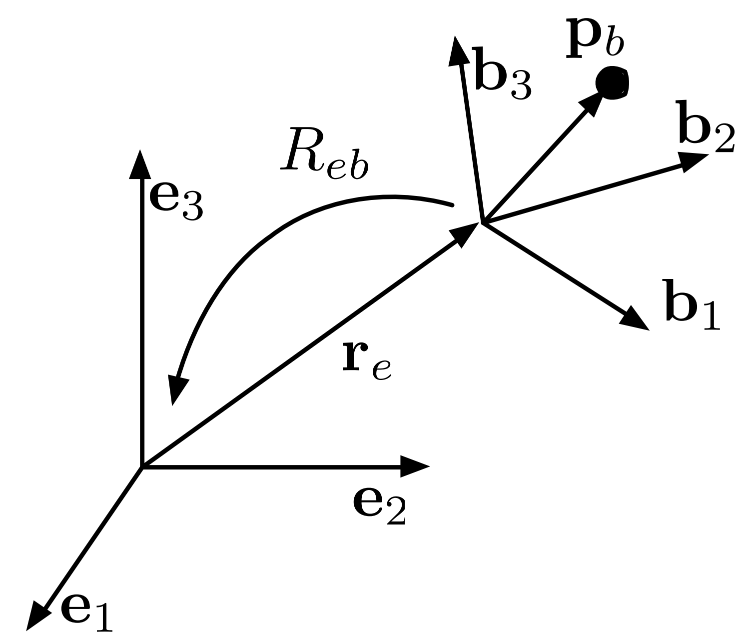
$$SO(3) = \{A \in R^3 | ATA = I, \det(A) = 1\}$$

Parameterizations:

- Euler angles
- Quaternions

Rigid-Body Transform

Euler angle parameterization of rotation:



$$R_{eb} = R_z(\psi) R_y(\theta) R_x(\phi)$$

yaw pitch roll

ZYX (321) form

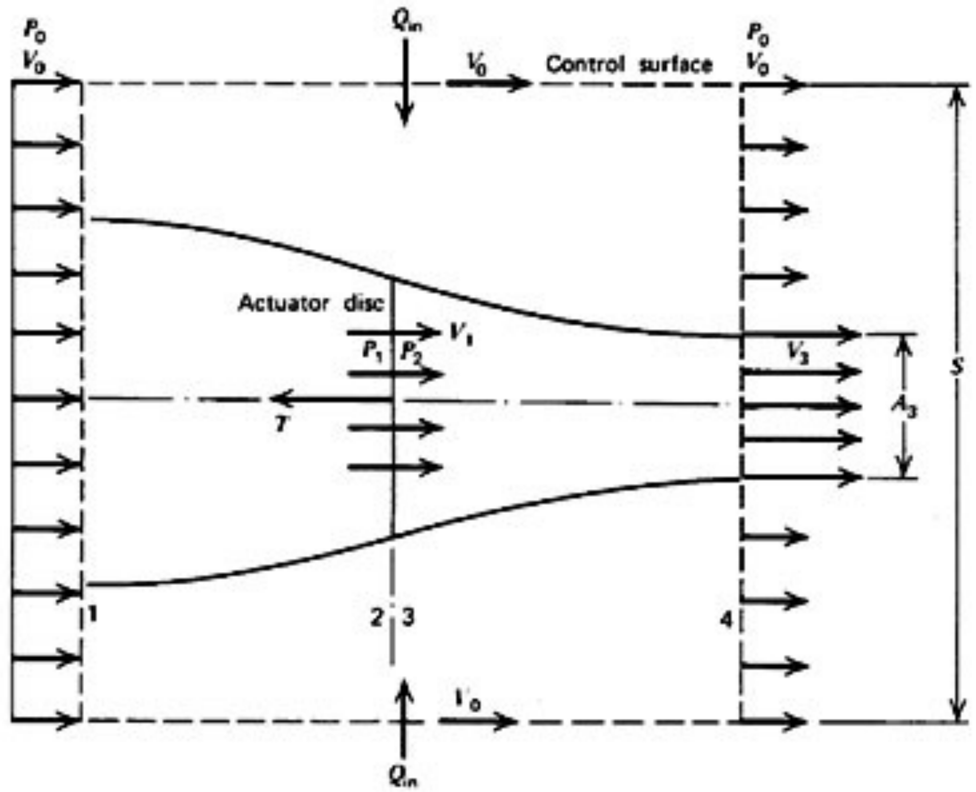
$$R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix} \quad R_y(\theta) = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \quad R_z(\psi) = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Coordinates (6 DoF):

$$\mathbf{q} = [x,\ y,\ z,\ \phi,\ \theta,\ \psi]^{\mathrm{T}}$$

Propeller and Motor Model

From dimensional analysis:



(McCormick, 1979)

air density (kg/m^3)

diameter (m)

Thrust $$T = k_T \rho n^2 D^4$$

speed (rev/s)

Moment $$Q = k_Q \rho n^2 D^5$$

Motor is a controlled sub-system (model as first-order):

$$\dot{w}_i = k_m \left(w_i^d - w_i \right)$$

Overloaded notation: $\{\omega_x,\, \omega_y,\, \omega_z\}$

Note labels vs enumeration (x vs i)

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Newton-Euler Equations

total force   mass   linear acceleration

linear velocity

$$\begin{bmatrix} \mathbf{F} \\ \tau \end{bmatrix} = \begin{bmatrix} m\mathbb{1}_3 & \mathbb{0}_3 \\ \mathbb{0}_3 & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \alpha \end{bmatrix} + \begin{bmatrix} \omega \times m\mathbf{v} \\ \omega \times \mathbf{I}_3\omega \end{bmatrix}$$

total torque

inertia tensor

angular velocity

angular acceleration

We could apply the *Euler-Lagrange* equations to the *Lagrangian* for the system but go directly to this form to save time.

Newton-Euler Equations

$$\begin{bmatrix} \mathbf{F} \\ \tau \end{bmatrix} = \begin{bmatrix} m\mathbb{1}_3 & \mathbb{0}_3 \\ \mathbb{0}_3 & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \alpha \end{bmatrix} + \begin{bmatrix} \omega \times m\mathbf{v} \\ \omega \times \mathbf{I}_3\omega \end{bmatrix}$$

Recall matrix form (from 16-642):

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau$$

forces/torques

inertia (mass)

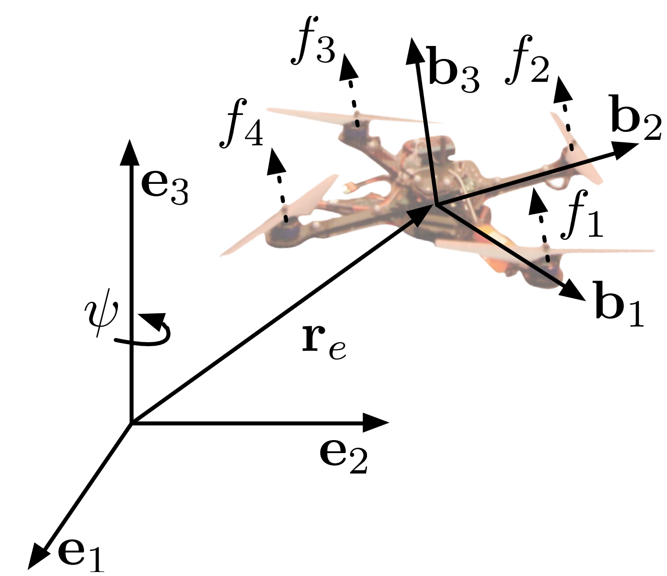
gravity

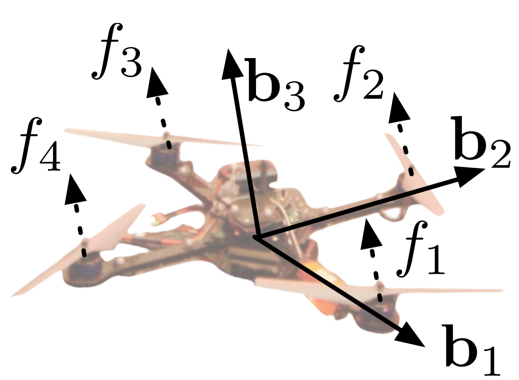
Coriolis/centrifugal

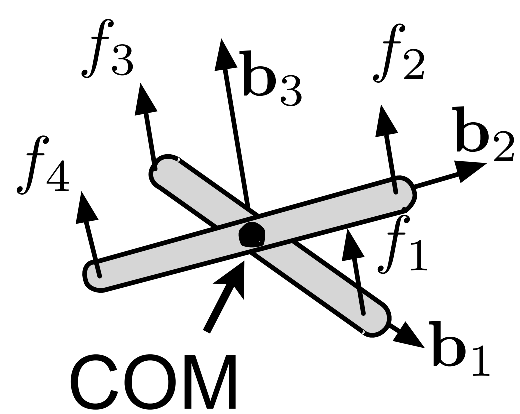
Force-Moment Generation

Newton-Euler equations:

$$\begin{bmatrix} \mathbf{F} \\ \tau \end{bmatrix} = \begin{bmatrix} m\mathbb{1}_3 & \mathbb{0}_3 \\ \mathbb{0}_3 & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \alpha \end{bmatrix} + \begin{bmatrix} \omega \times m\mathbf{v} \\ \omega \times \mathbf{I}_3\omega \end{bmatrix}$$







Total force:

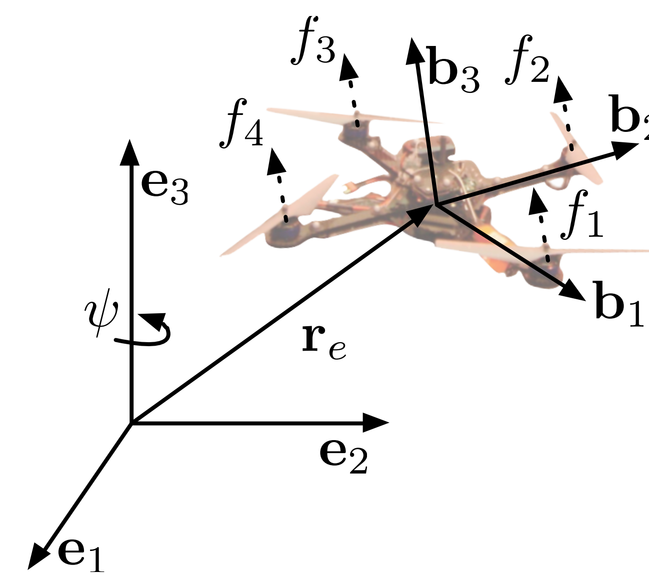
Body: $f = \sum_{i=1}^{4} f_i \xrightarrow[\text{along } \mathbf{b}_3]{} \mathbf{F}_b = \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix}$

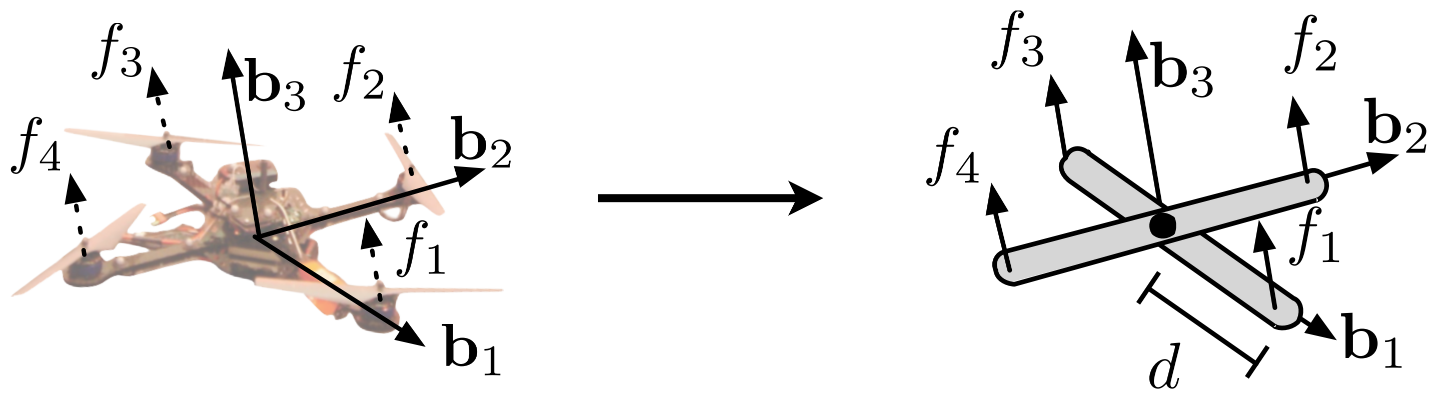
Inertial: $\mathbf{F}_e = R_{eb}\mathbf{F}_b - m\mathbf{g} \longleftarrow$ gravity

Force-Moment Generation

Newton-Euler equations:

$$\begin{bmatrix} \mathbf{F} \\ \tau \end{bmatrix} = \begin{bmatrix} m\mathbb{1}_3 & \mathbb{0}_3 \\ \mathbb{0}_3 & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \alpha \end{bmatrix} + \begin{bmatrix} \omega \times m\mathbf{v} \\ \omega \times \mathbf{I}_3\omega \end{bmatrix}$$



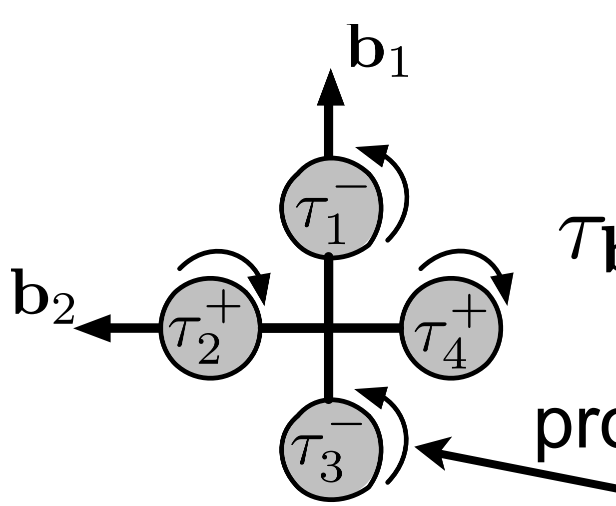


Total torque:

Recall: $\tau = \mathbf{r} \times \mathbf{F}$

$$\tau_{\mathbf{b}_1} = d\,(f_2 - f_4)$$

$$\tau_{\mathbf{b}_2} = d\,(f_3 - f_1)$$



propeller direction of rotation

induced moments

$$\tau_{\mathbf{b}_3} = -\tau_1 + \tau_2 - \tau_3 + \tau_4$$

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Equations of Motion (Summary)

Simplifying assumption: aligned center of mass and body frame

$$\begin{bmatrix} m\mathbb{1}_3 & \mathbb{0}_3 \\ \mathbb{0}_3 & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \alpha \end{bmatrix} + \begin{bmatrix} \cancel{\omega \times m\mathbf{v}} \\ \omega \times \mathbf{I}_3\omega \end{bmatrix} = \begin{bmatrix} \mathbf{F}_e \\ \tau \end{bmatrix} = \begin{bmatrix} R_{eb}\mathbf{F}_b - m\mathbf{g} \\ \left[\tau_{\mathbf{b}_1}, \tau_{\mathbf{b}_2}, \tau_{\mathbf{b}_3}\right]^{\mathrm{T}} \end{bmatrix}$$

$$\mathbf{F}_e = R_{eb}\mathbf{F}_b - m\mathbf{g}$$

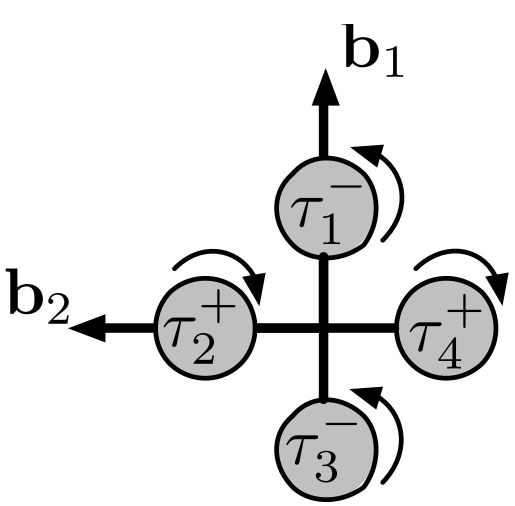
$$\mathbf{F}_b = \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix}$$

$$\begin{aligned} \tau_{\mathbf{b}_1} &= d\left(f_2 - f_4\right) \\ \tau_{\mathbf{b}_2} &= d\left(f_3 - f_1\right) \\ \tau_{\mathbf{b}_3} &= -\tau_1 + \tau_2 - \tau_3 + \tau_4 \end{aligned}$$

Prop model:

$$f_i = c_{\mathrm{T}}\bar{\omega}_i^2$$

$$\tau_i = \pm c_{\mathrm{Q}}\bar{\omega}_i^2$$



$$\begin{bmatrix} f \\ \tau_{\mathbf{b}_1} \\ \tau_{\mathbf{b}_2} \\ \tau_{\mathbf{b}_3} \end{bmatrix} = \begin{bmatrix} c_{\mathrm{T}} & c_{\mathrm{T}} & c_{\mathrm{T}} & c_{\mathrm{T}} \\ 0 & dc_{\mathrm{T}} & 0 & -dc_{\mathrm{T}} \\ -dc_{\mathrm{T}} & 0 & dc_{\mathrm{T}} & 0 \\ -c_{\mathrm{Q}} & c_{\mathrm{Q}} & -c_{\mathrm{Q}} & c_{\mathrm{Q}} \end{bmatrix} \begin{bmatrix} \bar{w}_1^2 \\ \bar{w}_2^2 \\ \bar{w}_3^2 \\ \bar{w}_4^2 \end{bmatrix}$$

Simplifying Model Assumptions

- Aligned center of mass and body frame
- Motor follows first-order model
- Ignore:
  - Power model (battery properties)
  - Rotor/blade flapping
  - Drag:
    - Induced: advancing blade generates more lift than retreating blade
    - Momentum: induced velocity of airflow through rotor
    - Parasitic: non-lifting (e.g., body) surfaces
    - Profile: transverse velocity of the rotor blade moving through air

For a more complete description of power and drag effects:

M. Bangura and R. Mahony, “Nonlinear Dynamic Modeling for High Performance Control of a Quadrotor,” presented at the Proc. of Australasian Conf. on Robot. and Autom., Wellington, New Zealand, 2012.

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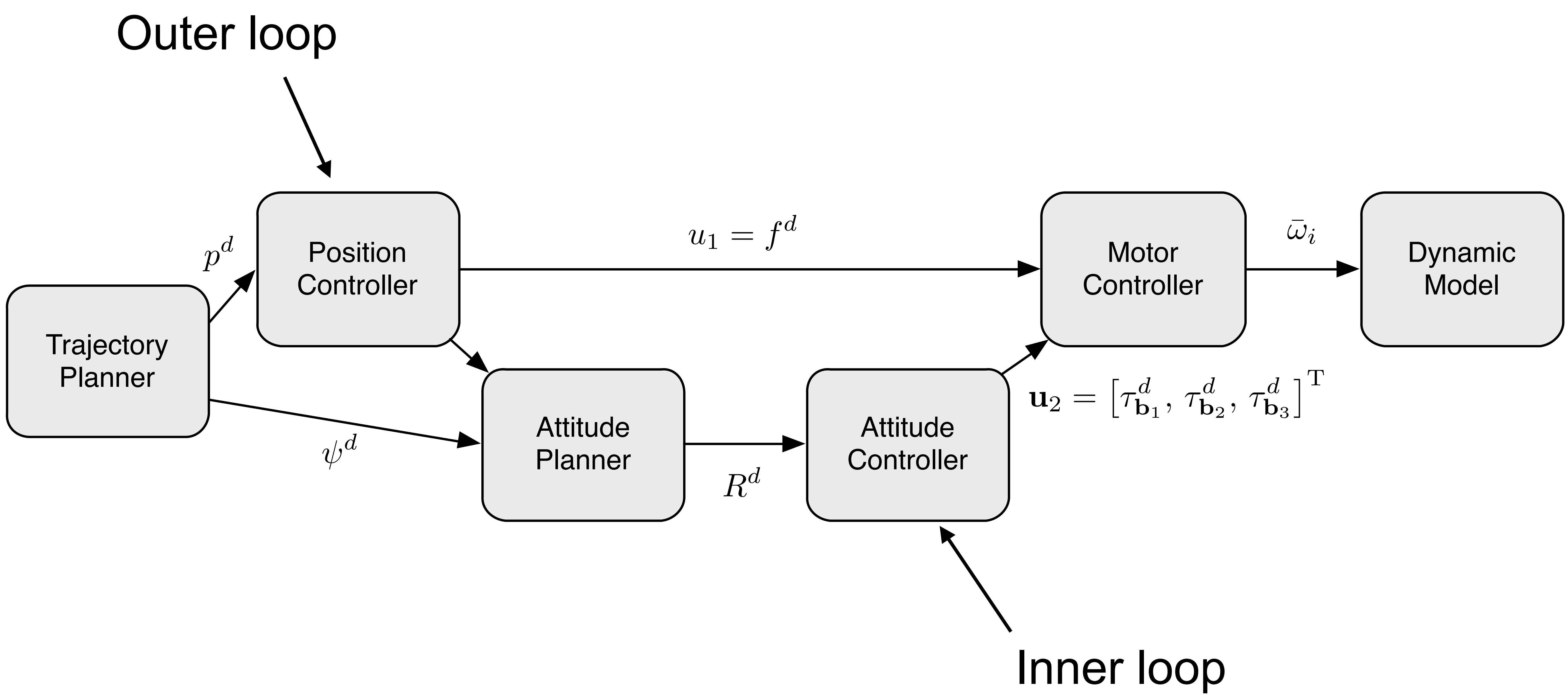
Control System Diagram



R. Mahony, V. Kumar, and P. Corke. Multirotor aerial vehicles: Modeling, estimation, and control of quadrotor. IEEE Robot. Autom. Mag., 19(3):20–32, Sept. 2012.

Proportional-Derivative Feedback Control

From 16-642 (or undergrad), define the linear time invariant (LTI) system:

$$\dot{x} = Ax + Bu$$

with the control law (regulator):

$$u = -Kx$$

The closed loop system is:

$$\dot{x} = (A - BK)x$$

Or with respect to a constant (or slowly varying) reference:

$$u = -Kx \longrightarrow u = -K(x - x^d) = -Ke_x$$

For a second order system:

$$u = -K_p e_x - K_d \dot{e}_x$$

Linearization of Nonlinear Model

Given a nonlinear system:

$$\dot{x} = f(x, u)$$
$$y = h(x)$$

We can approximate the system as locally linear (about an equilibrium point)

$$z = x - x_e$$

The system becomes:

$$\dot{z} = \dot{x} = f(z + z_e) = f_z(z)$$
$$y = h(z + x_e) = h_z(z)$$

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Linearization of Nonlinear Model

The linear approximation is:

$$\dot{z} = Az + Bu$$

$$A = \left.\frac{\partial f_z}{\partial z}\right|_{\substack{z=x_e\\u=u_e}} \qquad B = \left.\frac{\partial f_z}{\partial u}\right|_{\substack{z=x_e\\u=u_e}}$$

Assume identity feedback for now (so *y=h(x)* is already linear).

Linearize about hover:

$$R_0 = R\left(\phi_0 = 0,\ \theta_0 = 0,\ \psi_0\right)$$

$$R^d = R_z\left(\psi_0 + \Delta\psi\right) R_{yx}\left(\Delta\phi, \Delta\theta\right)$$

$$\mathbf{F}_{b,0} = \left[0,\ 0,\ mg\right]^{\mathrm{T}}$$

Attitude Control

PD control law:

$$\mathbf{u}_2 = -k_R \mathbf{e}_R - k_\omega \mathbf{e}_\omega$$

$$\mathbf{e}_\omega = \omega - \omega^d$$

Rotation error metric:

nonlinear

$$\mathbf{e}_R = \frac{1}{2}\left(\left(R^d\right)^{\mathrm{T}} R - R^{\mathrm{T}} R^d\right)^{\vee}$$

hat operator:

$$\hat{\mathbf{a}} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

vee operator:

$$(\hat{\mathbf{a}})^{\vee} = \mathbf{a}$$

Attitude Control

Rotation error metric:

$$\mathbf{e}_R = \frac{1}{2}\left(\left(R^d\right)^{\mathrm{T}} R_0 - R_0^{\mathrm{T}} R^d\right)^{\vee}$$

after linearization $\longrightarrow$

$$\approxeq \begin{bmatrix} 0 & \Delta\psi & -\Delta\theta \\ -\Delta\psi & 0 & -\Delta\phi \\ \Delta\theta & -\Delta\phi & 0 \end{bmatrix}^{\vee}$$

$$= [\Delta\phi,\ \Delta\theta,\ \Delta\psi]^{\mathrm{T}}$$

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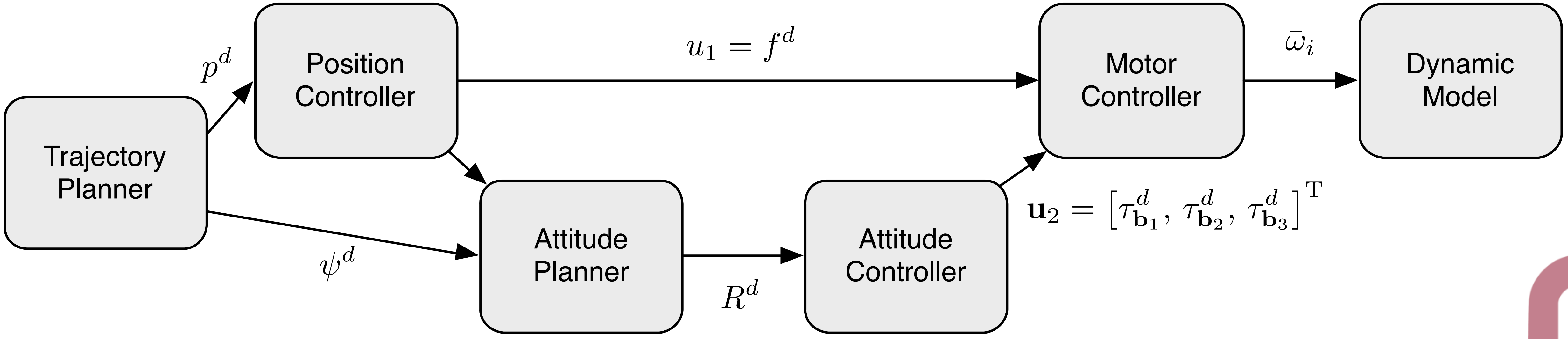
Attitude Control

PD control law:

$$\mathbf{u}_2 = -k_R \mathbf{e}_R - k_\omega \mathbf{e}_\omega$$

$$\mathbf{e}_R = [\Delta\phi,\ \Delta\theta,\ \Delta\psi]^{\mathrm{T}}$$

$$\mathbf{e}_\omega = \omega - \omega^d$$



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Position Control

Linearized model:

$$\ddot{x} = g\left(\Delta\theta\cos\psi_0 + \Delta\phi\sin\psi_0\right)$$

$$\ddot{y} = g\left(\Delta\theta\sin\psi_0 - \Delta\phi\cos\psi_0\right)$$

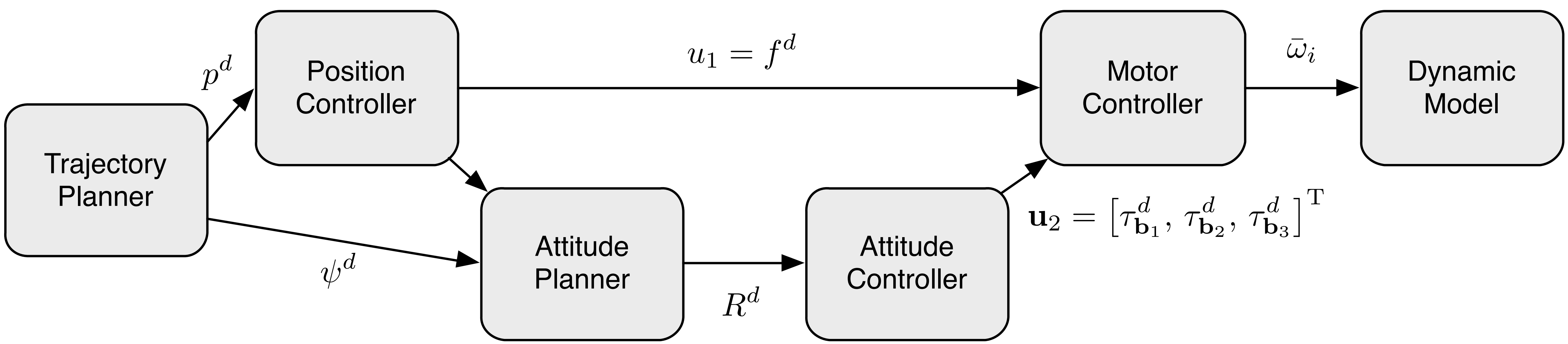
$$\ddot{z} = \frac{1}{m}u_1 - g$$

Mathematica derivation

Position Control

PD control law:

$$u_1 = m\mathbf{b}_3^{\mathrm{T}} \left(\mathbf{g} + \mathbf{a}^d + K_d\mathbf{e}_v + K_p\mathbf{e}_p\right)$$



How do we pick the gains?

Proportional-Derivative Feedback Control

Recall for a second order system:

$$\ddot{x} + 2\gamma\omega_n\dot{x} + \omega_n^2 x = 0$$

Three response cases:

Under-damped:

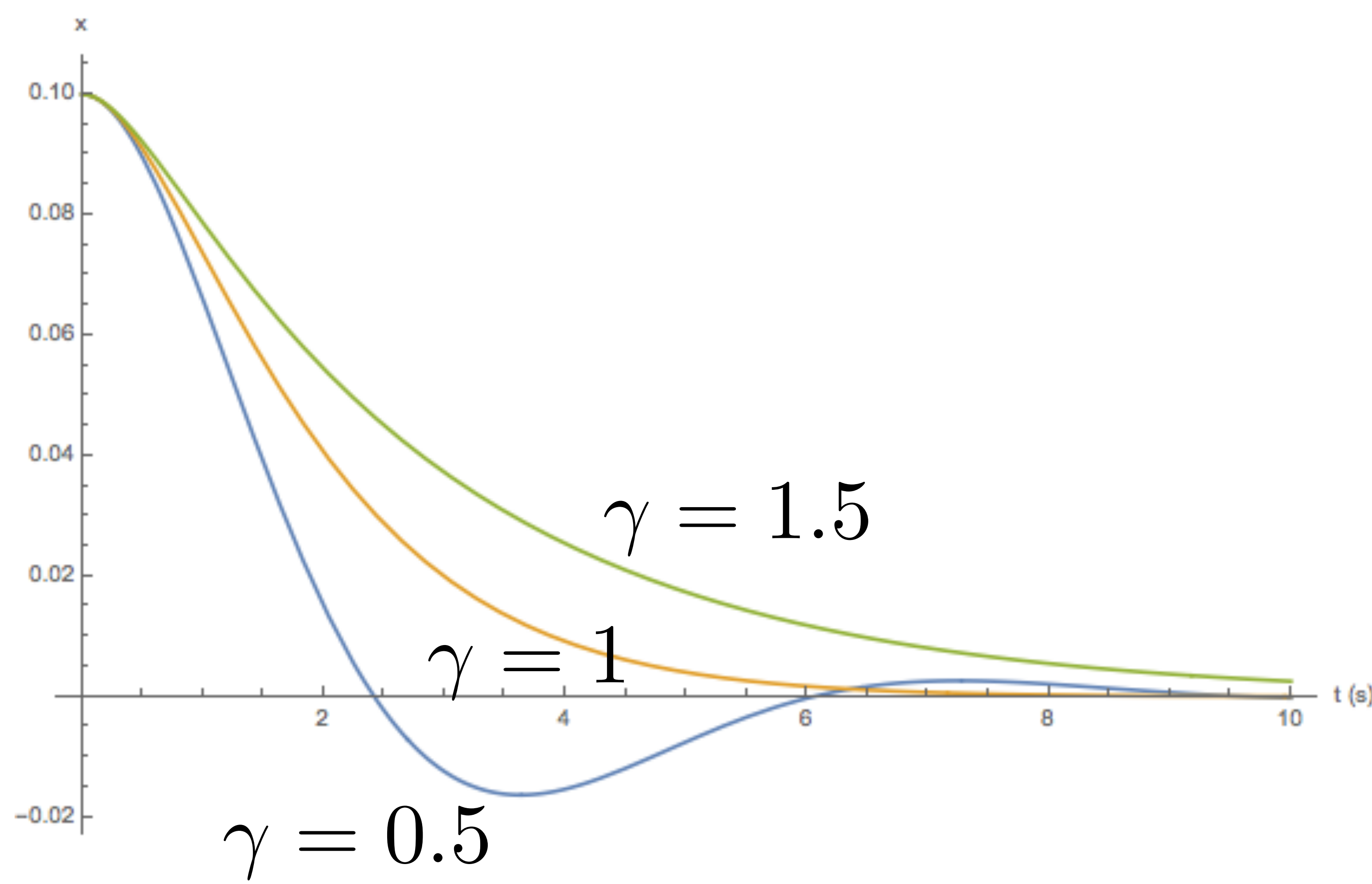
$$0 \leq \gamma < 1$$

Critically damped:

$$\gamma = 1$$

Over-damped:

$$\gamma > 1$$



Model error response as second order and design accordingly

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LQR (alternative gain selection approach)

Penalty for not getting to goal

$$J(\mathbf{x}, \mathbf{u}) = \int_0^\infty \left(\mathbf{x}^\mathrm{T} Q \mathbf{x} + \mathbf{u}^\mathrm{T} R \mathbf{u}\right) dt$$

Penalty for expending energy