The Security of ChaCha20-Poly1305 in the Multi-User Setting

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ABSTRACT

The ChaCha20-Poly1305 AEAD scheme is being increasingly widely deployed in practice. Practitioners need proven security bounds in order to set data limits and rekeying intervals for the scheme. But the formal security analysis of ChaCha20-Poly1305 currently lags behind that of AES-GCM. The only extant analysis (Procter, 2014) contains a flaw and is only for the single-user setting. We rectify this situation. We prove a multi-user security bound on the AEAD security of ChaCha20-Poly1305 and establish the tightness of each term in our bound through matching attacks. We show how our bound differs both qualitatively and quantitatively from the known bounds for AES-GCM, highlighting how subtle design choices lead to distinctive security properties. We translate our bound to the nonce-randomized setting employed in TLS 1.3 and elsewhere, and we additionally improve the corresponding security bounds for GCM. Finally, we provide a simple yet stronger variant of ChaCha20-Poly1305 that addresses the deficiencies highlighted by our analysis.

CCS CONCEPTS

• **Security and privacy → Mathematical foundations of cryptography**; *Block and stream ciphers*; *Hash functions and message authentication codes.*

KEYWORDS

ChaCha20-Poly1305; Multi-User Security; GCM; Nonce Randomization, AEAD; TLS 1.3; Tight Security

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1 INTRODUCTION

ChaCha20-Poly1305 and Galois Counter Mode (GCM) are the two most popular AEAD schemes in use on the Internet today. The TLS 1.3 specification [29] mandates (`MUST`) support for AES128-GCM and strongly recommends (`SHOULD`) support for AES256-GCM and ChaCha20-Poly1305. In addition, ChaCha20-Poly1305 is the default AEAD scheme in OpenSSH, WireGuard, OTRv4, and the Bitcoin Lightning Network. GCM owes much of its popularity to its high parallelizability as well as native support for AES and carry-less multiplication on Intel and AMD CPUs, which allow it to run at record speeds. However, in the absence of such hardware support, ChaCha20-Poly1305 wins the race by a significant margin. Accordingly, ChaCha20-Poly1305 is generally the favored choice on mobile phone hardware and Apple's M1 processor. Moreover, ChaCha20-Poly1305 benefits from Intel's new AVX512 instruction set, which has further narrowed the performance gap between the two schemes on Intel hardware.[1] Its performance benefits derive from the minimalistic ARX design behind the ChaCha20 cipher, the amenability of Poly1305 to fast arithmetic, and the parallelizability in each of these components. Another reason for preferring ChaCha20-Poly1305 on generic hardware platforms is that it is easier to implement in constant time. As a consequence, it is less prone to timing side-channel attacks.

ChaCha20-Poly1305 combines the ChaCha20 stream cipher and the one-time MAC Poly1305 into a nonce-based AEAD scheme. Both were designed independently as separate components by Bernstein [5, 7], and Langley later adapted and combined the two into a nonce-based AEAD scheme and proposed its use in TLS [18]. Despite its popularity, ChaCha20-Poly1305 has received very little formal security analysis. The only extant analysis is in a short, unpublished note by Procter [26]. In contrast, GCM has been the subject of several formal security analyses [4, 15, 16, 19, 21, 24]. In particular, [16] identified a flaw in GCM's original security proof in [21] and provided a corrected proof which was in turn improved upon in [24]. Later works [4, 15, 19] have studied the security of GCM in the *multi-user setting*, where an adversary is given access to an encryption oracle and a decryption oracle for many distinct users and the adversary wins if it can break the security of any single user. Having access to encryptions under distinct keys of the same message and the same nonce can, for instance, facilitate key-recovery attacks on block ciphers [8, 9]. Accordingly, the multi-user setting captures a very practical concern reflecting how a state-actor adversary can benefit from its ability to eavesdrop on TLS connections *en masse.* In fact, TLS 1.3 includes a nonce-randomization

[1]https://bench.cr.yp.to/results-stream.html#amd64-manny1024

mechanism to mitigate against such multi-user attacks, and the rekeying frequencies for GCM in TLS, DTLS, and QUIC were based on its multi-user security bounds [29–31]. In the specific cases of DTLS and QUIC, the multi-user security bounds are also used to determine the maximum number of failed decryptions that can be tolerated before a session is terminated, since decryption failures are not immediately fatal in these protocols. Moreover, in light of the tighter security bounds presented in [15], the QUIC specification [31, Appendix B.1] now allows for larger limits than those prescribed in the TLS 1.3 specification.

Currently, for ChaCha20-Poly1305, the only multi-user security bound available to the IETF is that outlined by Luyks and Paterson in [20]. It leverages the single-user security bound from [26] by applying a standard hybrid reduction technique. The resulting multi-user security bound is essentially the single-user bound, where each term is now multiplied by the number of users. However, this simple approach has three significant shortcomings. Firstly, as described in [19], it generally results in loose bounds, leading to overly conservative security parameter estimates in practical protocols. Secondly, it fails to explicitly quantify the adversary's advantage in terms of its local computational resources. This is because in the standard-model security analysis in [26] this aspect is concealed in a term capturing the PRF security of the ChaCha20 block function, which is not as easy to estimate concretely. Thirdly, the security proof described in Procter's note [26] on which [20] relies is actually incorrect, as we explain in the full version of our paper. As such, the security of an important and increasingly widely deployed AEAD scheme currently rests on shaky foundations.

In this work, we remedy this state of affairs by presenting a new dedicated multi-user security analysis of ChaCha20-Poly1305, on par with that for GCM [15]. A common misconception is that ChaCha20-Poly1305 is structurally equivalent to GCM but instantiated with different primitives. Accordingly, one might be tempted to think that it suffices to simply reuse GCM's security bound with ChaCha20-Poly1305's parameters. While the two constructions share some similarities, they have a number of important differences that preclude such an approach. In addition, their differences are further accentuated in the multi-user security model. For instance, while all multi-user security treatments of GCM are in the ideal-cipher model, ChaCha20-Poly1305 is better analyzed in the (unkeyed) random permutation model. We elaborate on this choice and the differences between GCM and ChaCha20-Poly1305 in Section 3. Below is a summary of our contributions.

1.1 Contributions

Single-User Security. We start off by revisiting Procter's single-user security proof [26]. We point out a flaw in the proof and provide a new proof under the same standard-model assumptions. Our proof retains the same security bound as that originally claimed by Procter. However, fixing the proof is not straightforward and required us to restructure and augment the sequence of games significantly. Due to space constraints, the single-user security analysis of ChaCha20-Poly1305 is located in the full version of our paper.

Multi-User Security. Our main contribution is a tight multi-user security proof for ChaCha20-Poly1305. Through the security bounds we establish, we expose fundamental differences in its security traits compared to GCM [15]. One case in point is that, while the multi-user security of GCM can be improved by rekeying more frequently, i.e., reducing the limit on the maximum amount of data that can be encrypted under a single key, this does not hold in the case of ChaCha20-Poly1305. This is due to the fact that our security bound does not depend on any per-user parameters—such as the number of queries per user or the amount of data per user. On an intuitive level, this distinction stems from the fact that under the hood ChaCha20 is based on a permutation and not a block cipher in counter mode. We provide attacks matching our bounds, showing that our observations are not just an artefact of weak bounds.

Another point of divergence is that for ChaCha20-Poly1305 the dominant term in the security bound depends on the number of verification queries made by the adversary and the tag length, whereas for GCM the dominant term depends on the total number of encrypted data blocks and the block length. Note that the number of verification queries represents the number of valid forgery attempts made by an adversary. In the context of secure communications protocols, this relates to the number of times that an adversary attempts to insert a new ciphertext into the secure channel.[2] Typical protocols running over reliable transport, like TLS [29], shut down upon the first verification error. Protocols running over unreliable transport, like DTLS [30] or QUIC [31], however have to tolerate many invalid ciphertexts due to the network behavior [12], yet their number is still much smaller than the maximum number of messages that can be encrypted. Thus, our analysis shows that these two schemes require protocol designers to tune their security parameters rather differently depending on which scheme they use.

We note that our security proof borrows many ideas from [15], but it also deviates from it substantially and even improves upon it in some respects.

Tightness. We describe attacks to prove the tightness of every term in our security bound. We adapt some of the attacks for GCM and introduce new ones. In particular, adapting the forgery attack on GCM to the case ChaCha20-Poly1305 requires substantially new techniques, due to the different finite field that they employ in their universal hash functions. We describe these attacks in Section 7.1.

Nonce Randomization. Next, we turn our attention to the multi-user security of ChaCha20-Poly1305 when combined with the XN transform for nonce randomization that is currently employed in TLS 1.3. Similarly to [15], we extend our multi-user security bound for plain ChaCha20-Poly1305 to this setting using a balls-into-bins argument. Interestingly, applying the balls-into-bins lemma from [15] to our security bound results in a relatively weak security bound—on the order of 2^{-48}. We present an improved balls-into-bins lemma allowing us to translate our main bound to the XN transformed version of ChaCha20-Poly1305 without any degradation in the security bound. As a noteworthy side-effect, our improved bound for XN also eliminates the corresponding term in the bound of [15] for the nonce-randomized version of GCM, enabling tighter advantage bounds than 2^{-48}. We cover nonce randomization in Section 7.2.

A Stronger Variant. As already observed, the dominant term in the security bound for ChaCha20-Poly1305 relates to the ciphertext

[2]Such a new ciphertext could be obtained by tampering with an existing one, or could result from reordering ciphertexts or creating entirely new ones from scratch.

integrity of the scheme, which in turn depends on the security of the Poly1305 MAC. Indeed, this term dominates all other terms by a substantial margin. Accordingly, our bound shows that the multi-user security of ChaCha20-Poly1305 would be strengthened significantly if Poly1305 were to be replaced with a more secure MAC. In Section 7.4 we explore this possibility by considering a simple variant of ChaCha20-Poly1305 which uses two independently-keyed Poly1305 MAC tags instead of one.

2 PRELIMINARIES

Notation. The empty string is denoted by ε. We denote by $|x|$ the bit length of the bit string x, and by $|x|_n$ its size in n-bit blocks (and the empty string is assigned a block size of 1, i.e., $|x|_n = \max(1, \lceil |x|/n \rceil)$). We call strings whose bit length is a multiple of 8 *byte strings*. For $1 \leq i < j \leq |x|$, let $x[i{:}j]$ denote the substring of x spanning bit i to bit j inclusive, and $\mathrm{trunc}(x, n)$ denote the first n bits of x, i.e. $\mathrm{trunc}(x, n) = x[1{:}n]$. For any two bit strings x and y, their concatenation is denoted by $x \| y$. When $|x| = |y| = c \cdot n$ for some positive integer c, $x \overset{(n)}{+} y$ and $x \overset{(n)}{-} y$ denote the strings that result from individually adding and subtracting their n-bit subwords modulo 2^n when each word is interpreted as an unsigned integer.

If S is a finite set then $|S|$ denotes its cardinality and $y \leftarrow\$ S$ denotes the process of sampling uniformly at random an element from S and assigning it to y. If $\mathcal{A}$ is an algorithm or a procedure, $y \leftarrow \mathcal{A}(x_1, \dots)$ denotes running $\mathcal{A}$ on inputs $x_1, \dots$ and assigning the output to y. By convention, the running time of an adversary is the sum of its running time, the time to answer all of its oracle queries, and the size of its description.

AEAD Syntax. A nonce-based authenticated encryption scheme with associated data $\Pi = (\mathcal{K}, \mathcal{E}, \mathcal{D})$ is a triple of algorithms:

- The key generation algorithm $\mathcal{K}$ takes no input and returns a secret key K. We overload $\mathcal{K}$ to also represent the key space associated to the key generation algorithm.
- The deterministic encryption algorithm $\mathcal{E} : \mathcal{K} \times \mathcal{N} \times \mathcal{AD} \times \mathcal{M} \to \mathcal{C}$ takes as input a secret key $K \in \mathcal{K}$, a nonce $N \in \mathcal{N}$, associated data $AD \in \mathcal{AD}$, and a message $M \in \mathcal{M}$ and returns a ciphertext $C \in \mathcal{C}$. We require $\mathcal{E}$ to have constant expansion, i.e. for any $(K, N, AD, M) \in (\mathcal{K}, \mathcal{N}, \mathcal{AD}, \mathcal{M})$, the expansion $t = |\mathcal{E}(K, N, AD, M)| - |M|$ is constant.
- The deterministic decryption algorithm $\mathcal{D} : \mathcal{K} \times \mathcal{N} \times \mathcal{AD} \times \mathcal{C} \to \mathcal{M} \cup \{\perp\}$ takes as input a secret key $K \in \mathcal{K}$, a nonce $N \in \mathcal{N}$, associated data $AD \in \mathcal{AD}$, and a ciphertext $C \in \mathcal{C}$ and returns either a message $M \in \mathcal{M}$ or the symbol $\perp$ to indicate an invalid ciphertext.

We refer to the associated sets $\mathcal{K}$, $\mathcal{N}$, $\mathcal{AD}$, $\mathcal{M}$, and $\mathcal{C}$ as the key space, the nonce space, the associated-data space, the message or plaintext space and the ciphertext space, respectively. We require every nonce-based AEAD to satisfy correctness, namely for all $(K, N, AD, M) \in (\mathcal{K}, \mathcal{N}, \mathcal{AD}, \mathcal{M})$, it must hold that if $C \leftarrow \mathcal{E}(K, N, AD, M)$ then $M \leftarrow \mathcal{D}(K, N, AD, C)$.

AEAD Multi-User Security. We use the multi-user security definition for authenticated encryption put forward by Bellare and Tackmann [4] adapted to the ideal permutation model. Here, the encryption oracle $\textsc{Enc}(i, N, AD, M)$ and the verification oracle $\textsc{Vf}(i, N, AD, C)$ can be queired for distinct users (identified by the integer i). The adversary $\mathcal{A}$ is also given access to the ideal permutation π through the oracle $\textsc{Prim}(X)$ and its inverse $\textsc{Prim}^{-1}(Y)$.

An adversary is said to be nonce-respecting if it never repeats a pair (i, N) across encryption queries. Throughout, we require that every adversary be nonce-respecting and never asks a verification query $\textsc{Vf}(i, N, AD, C)$ with C being the output of a previous encryption query $\textsc{Enc}(i, N, AD, M)$ (since such queries permit trivial wins). Without loss of generality, we will assume that an adversary does not repeat any queries.

At points in our analysis, we will require that an adversary $\mathcal{A}$ be *d-repeating*, meaning that $\mathcal{A}$ does not repeat the same nonce with more than d users in its encryption queries. $\mathcal{A}$ is *strongly d-repeating* if it is *d-repeating* and additionally does not repeat the same nonce across verification queries for more than d users.

Definition 2.1 (Multi-user AE advantage).
Let $\Pi[\pi] = (\mathcal{K}, \mathcal{E}, \mathcal{D})$ be a nonce-based AEAD scheme constructed from an ideal permutation $\pi : \mathcal{X} \to \mathcal{X}$ with expansion t. We define the multi-user AE advantage of adversary $\mathcal{A}$ against $\Pi[\pi]$ as:

$$\mathsf{Adv}^{\mathrm{muAE}}_{\Pi[\pi]}(\mathcal{A}) = \Pr\left[\mathcal{A}^{G^{\text{Real-muAE}}_{\Pi[\pi]}} \Rightarrow 1\right] - \Pr\left[\mathcal{A}^{G^{\text{Ideal-muAE}}_{\Pi[\pi]}} \Rightarrow 1\right],$$

where games $G^{\text{Real-muAE}}_{\Pi[\pi]}$ and $G^{\text{Ideal-muAE}}_{\Pi[\pi]}$ are defined in Figure 1.

Game $G^{\text{Real-muAE}}_{\Pi[\pi]}$

```
procedure INITIALIZE
1: $K_1, K_2, \cdots \leftarrow\$ \mathcal{K}$

procedure ENC($i, N, AD, M$)
1: $C \leftarrow \mathcal{E}(K_i, N, AD, M)$
2: return $C$

procedure VF($i, N, AD, C$)
1: $M \leftarrow \mathcal{D}(K_i, N, AD, C)$
2: return $(M \neq \perp)$

procedure PRIM($X$)
1: return $\pi(X)$

procedure PRIM$^{-1}$($Y$)
1: return $\pi^{-1}(Y)$
```

Game $G^{\text{Ideal-muAE}}_{\Pi[\pi]}$

```
procedure ENC($i, N, AD, M$)
1: $C \leftarrow\$ \{0,1\}^{|M|+t}$
2: return $C$

procedure VF($i, N, AD, C$)
1: return false

procedure PRIM($X$)
1: return $\pi(X)$

procedure PRIM$^{-1}$($Y$)
1: return $\pi^{-1}(Y)$
```

Figure 1: Games used to define the multi-user AE advantage.

Adversarial Parameters. In the following, for an associated adversary $\mathcal{A}$, we will denote by q_e its maximum number of encryption queries and by σ_e an upper bound on its total number of encrypted blocks. We denote by q_v the maximum number of decryption/verification queries of $\mathcal{A}$ and by σ_v an upper bound on its total number of decrypted/verified blocks. Further, let ℓ_m denote the maximum size in t-bit blocks (including associated data) that $\mathcal{A}$ is allowed to query to its encryption and decryption/verification oracles. Finally, for the multi-user setting, p denotes the maximum number of ideal permutation queries and d denotes the maximum number of users queried with the same nonce during encryption (i.e. for a d-repeating or strongly d-repeating adversary $\mathcal{A}$).

Δ-Universal Hash Functions. Let $\overline{H} : \mathcal{R} \times \mathcal{D} \to \{0,1\}^t$ be a family of keyed hash functions with key space $\mathcal{R}$, domain $\mathcal{D}$ and digest space $\{0,1\}^t$, for some positive integer t. We will consider hash function families over strings and string pairs. When $\mathcal{D} = \{0,1\}^*$, for any positive real number c, we say that $\overline{H}$ is c-almost Δ-universal if for all distinct $M, M' \in \{0,1\}^*$ and all $z \in \{0,1\}^t$, it holds that

$$\Pr_{r \leftarrow\$ \mathcal{R}}\left[\overline{H}_r(M) = \overline{H}_r(M') \overset{(t)}{+} z\right] \le \frac{c \cdot \max(|M|_t, |M'|_t)}{2^t} .$$

Alternatively, when $\mathcal{D} = \{0,1\}^* \times \{0,1\}^*$, for any distinct (AD, C), $(AD', C') \in \{0,1\}^* \times \{0,1\}^*$ and all $z \in \{0,1\}^t$, we require that

$$\Pr_{r \leftarrow\$ \mathcal{R}}\left[H_r(AD, C) = H_r(AD', C') \overset{(t)}{+} z\right] \le \frac{c \cdot \max(|AD|_t + |C|_t, |AD'|_t + |C'|_t)}{2^t} .$$

H-coefficient Technique. The H-coefficient technique [11, 25] is a method for bounding the advantage of a computationally unbounded adversary $\mathcal{A}$, which wlog can be assumed to be deterministic, attempting to distinguish between a real and an ideal game. The technique focusses on the transcripts generated when $\mathcal{A}$ interacts with the oracles in these games, namely, the sequence of input-output pairs $\tau = ((x_1, y_1), (x_2, y_2), \ldots, (x_q, y_q))$. We use $\mathcal{T}_{\text{ideal}}$ to denote the random variable corresponding to the transcript generated by $\mathcal{A}$ in the ideal game. Then, $\mathsf{P}_{\text{ideal}}(\tau)$ and $\mathsf{P}_{\text{real}}(\tau)$ denote the probabilities that a given transcript τ is generated in the corresponding game when interacting with $\mathcal{A}$. A transcript τ is said to be attainable if there exists an adversary such that the probability of generating τ in the ideal game is strictly greater than 0.

The H-coefficient technique relies on identifying a suitable partition of attainable transcripts, applying the following theorem, and then calculating ϵ_1 and ϵ_2 (for a proof, see [11]):

Theorem 2.2 (H-coefficient Technique). *Let $\mathcal{A}$ be a computationally unbounded adversary trying to distinguish between a real game G^{Real} and an ideal game G^{Ideal}. Let $T = T_{\text{good}} \sqcup T_{\text{bad}}$ be a partition of the set of attainable transcripts. If there exist $\epsilon_1, \epsilon_2 \ge 0$ such that $\forall \tau \in T_{\text{good}}$, $\frac{\mathsf{P}_{\text{real}}(\tau)}{\mathsf{P}_{\text{ideal}}(\tau)} \ge 1 - \epsilon_1$ and $\Pr[\mathcal{T}_{\text{ideal}} \in T_{\text{bad}}] \le \epsilon_2$, then*

$$\left|\Pr\left[\mathcal{A}^{G^{\text{Real}}} \Rightarrow 1\right] - \Pr\left[\mathcal{A}^{G^{\text{Ideal}}} \Rightarrow 1\right]\right| \le \epsilon_1 + \epsilon_2 .$$

3 THE CHACHA20-POLY1305 SCHEME

In this section, we provide a brief overview of ChaCha20-Poly1305, as defined in RFC 8439 [23], and lay some of the groundwork for our security analysis.

The AEAD Composition. Pseudocode for ChaCha20-Poly1305 is shown in Figure 3, consisting of the encryption algorithm $\mathcal{E}$, the decryption algorithm $\mathcal{D}$, and their subcomponents: the ChaCha20 stream cipher, the one-time MAC Poly1305_Mac, and the MAC's key-generation algorithm Poly1305_Key_Gen. In turn, these subcomponents are based on the ChaCha20 block function CC_block and the Δ-universal hash function family H over string pairs.

The encryption algorithm $\mathcal{E}$ is represented in Figure 2. It takes as input a 256-bit secret key K, a 96-bit nonce N, a variable-length

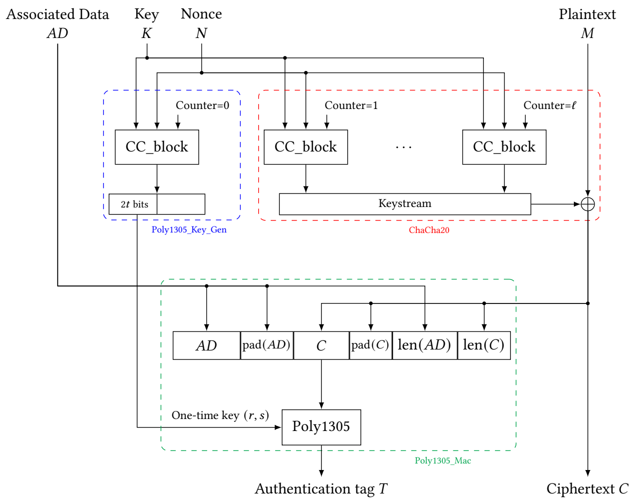

Figure 2: The encryption procedure in ChaCha20-Poly1305**.**

byte string of associated data AD, and a variable-length byte-string message M. It returns a ciphertext C consisting of the ChaCha20 encryption of M, and a 128-bit tag T, computed over AD and C using Poly1305_Mac with a one-time key (r, s) consisting of two 128-bit strings. The one-time key (r, s) is derived anew in each encryption by running the ChaCha20 block function in Poly1305_Key_Gen on K, N, and the counter value zero—which is reserved solely for this purpose. The decryption algorithm $\mathcal{D}$ proceeds analogously: it first derives the one-time key, it recomputes the MAC tag and checks whether it matches that supplied in the ciphertext. If so, it proceeds to decrypt the rest of the ciphertext using ChaCha20 and returns the decrypted message. Otherwise it returns $\bot$, indicating error.

Chacha20. Designed by Bernstein, the ChaCha20 stream cipher [7] is a refinement of the Salsa stream cipher [6]. It uses a 256-bit secret key K and a 96-bit nonce N to encrypt (or decrypt) an arbitrary-length message M (or ciphertext C). As with any stream cipher, it generates a pseudorandom keystream that is XORed to the message. The keystream is generated in blocks of 512 bits through the ChaCha20 block function CC_block. The CC_block function is keyed with K and is evaluated over an input composed of the 96-bit nonce N and a 32-bit block counter i. This way, it is employed as a pseudorandom function, but under the hood it really consists of a 512-bit permutation π in a Davies–Meyer-type configuration. More specifically, the key, counter, and nonce are concatenated and prepended with a constant to form the input to the ChaCha20 permutation and then added once again to the permutation's output using modulo 2^{32} addition on word-by-word basis, i.e., $\text{CC\_block}(K, N, i) = \pi(Z\|K\|i\|N) \overset{(32)}{+} (Z\|K\|i\|N)$, where Z is a 128-bit constant.

Poly1305. The Poly1305 algorithm [5] is a one-time MAC, also designed by Bernstein. It takes a key consisting of two 128-bit strings (r, s); the tag corresponding to a message M is computed as $\overline{H}_r(M) \overset{(t)}{+} s$. Its security relies on the $\bar{c}$-almost Δ-universality of the hash function $\overline{H}$, as shown by Bernstein [5]. The hash function $\overline{H}$

```
procedure $\mathcal{E}(K, N, AD, M)$
1 : $r\|s \leftarrow$ Poly1305_Key_Gen$(K, N)$
2 : $C \leftarrow$ ChaCha20$(K, N, M)$
3 : $T \leftarrow$ Poly1305_Mac$((r, s), AD, C)$
4 : return $C\|T$
procedure $\mathcal{D}(K, N, AD, C\|T)$
1 : $r\|s \leftarrow$ Poly1305_Key_Gen$(K, N)$
2 : $T' \leftarrow$ Poly1305_Mac$((r, s), AD, C)$
3 : if $T \neq T'$ then return $\perp$
4 : return ChaCha20$(K, N, C)$
procedure ChaCha20$(K, N, M)$
1 : $M_1\| \cdots \|M_\ell \leftarrow M$
2 : for $i = 1$ to $\ell - 1$ do
3 :     $Z_i \leftarrow M_i \oplus$ CC_block$(K, N, i)$
4 : $Z_\ell \leftarrow M_\ell \oplus$ trunc(CC_block$(K, N, \ell), |M_\ell|)$
5 : return $Z_1\| \cdots \|Z_\ell$
procedure Poly1305_Key_Gen$(K, N)$
1 : return trunc(CC_block$(K, N, 0), 2t)$
procedure Poly1305_Mac$((r, s), AD, C)$
1 : return $H_r(AD, C) \overset{(t)}{+} s$
```

Figure 3: Pseudocode description of ChaCha20-Poly1305**.**

is outlined in Definition 3.1 and its security is stated in Theorem 3.2, reproduced here from [5] for completeness.

Definition 3.1 (The Hash Function $\overline{H}$ in Poly1305*).* Let t be a positive integer multiple of 8, let $p \geq 2^{t+1}$ be a prime, let r be a t-bit string and let M be any byte string. Parse M as $M = M_1\| \cdots \|M_\ell$ where $|M_i| = t$ for $i < \ell$ and $0 < |M_\ell| \leq t$. Then, $\overline{H}_r(M)$ is given as the t-bit string representation of

$$(c_1 x^\ell + c_2 x^{\ell-1} + \cdots + c_\ell x^1 \mod p) \mod 2^t,$$

where c_i is the integer representation of the $(t+1)$-bit string $M_i\|1$ and x is the integer representation of r after fixing 22 of its bit positions to zero (a process referred to as "clamping").

We refer the reader to [5] for further details on the bit clamping in r and the conversion between integers and strings. Note that for increased generality in our security analysis, we consider a general tag size t, but for ChaCha20-Poly1305 this should be understood to be equal to 128 bits and $p = 2^{130} - 5$. Indeed these values give rise to the constant $\overline{c} = 2^{25}$ in the next theorem.

Theorem 3.2 ($\overline{H}$ is AΔU). *Let $\overline{c} = 2^{25}$, then for any t-bit string s and any pair of distinct byte strings (M, M'), it holds that:*

$$\Pr_{r \leftarrow\$ \{0,1\}^t}\left[\overline{H}_r(M) = \overline{H}_r(M') \overset{(t)}{+} s\right] \leq \frac{\overline{c} \cdot \max(|M|_t, |M'|_t)}{2^t}.$$

The one-time MAC (Poly1305_Mac) that is used in ChaCha20 -Poly1305 (Figure 3) extends the original Poly1305 algorithm to authenticate a pair of strings instead of one. This is accomplished by augmenting the hash function with an appropriate encoding mapping the string pair to a single string. The encoding demarcates the boundary between the two strings. Importantly, this encoding needs to be injective. Definition 3.3 describes how H is constructed from $\overline{H}$. In Theorem 3.4 we show that if $\overline{H}$ is a $\overline{c}$-almost Δ-universal for single strings, then H is also c-almost Δ-universal over string pairs, where $c = 1.5 \cdot \overline{c}$.

Definition 3.3 (The Hash Function H in Poly1305_Mac*).* Let r be a t-bit string and $\overline{H}$ be the hash function used in Poly1305. Then, the hash of any pair of byte strings (AD, C) is given by

$$H_r(AD, C) = \overline{H}_r(AD\|\text{pad}(AD)\|C\|\text{pad}(C)\|\text{len}(AD)\|\text{len}(C)),$$

where pad(X) returns the minimum number of zero bytes such that the bit length of $X\|\text{pad}(X)$ is multiple of t, and len(X) is the $\frac{t}{2}$-bit representation of the byte length of X.

Theorem 3.4 (H is AΔU). *Let $c = 3 \cdot 2^{24}$ and s be a t-bit string, then for any two distinct byte-string pairs (AD, C) and (AD', C') it holds that:*

$$\Pr_{r \leftarrow\$ \{0,1\}^t}\left[H_r(AD, C) = H_r(AD', C') \overset{(t)}{+} s\right] \leq \frac{c \cdot \max(|AD|_t + |C|_t, |AD'|_t + |C'|_t)}{2^t}.$$

Proof. Let $Z = AD\|\text{pad}(AD)\|C\|\text{pad}(C)\|\text{len}(AD)\|\text{len}(C)$ and $Z' = AD'\|\text{pad}(AD')\|C'\|\text{pad}(C')\|\text{len}(AD')\|\text{len}(C')$. If $Z = Z'$, then len(C) = len(C') and len(AD) = len(AD'). Thus $C = C'$ and $AD = AD'$, which is not possible as $(AD, C) \neq (AD', C')$. Therefore $Z \neq Z'$. Using Theorem 3.2,

$$\begin{aligned}
&\Pr_{r \leftarrow\$ \{0,1\}^t}\left[\overline{H}_r(Z) = \overline{H}_r(Z') \overset{(t)}{+} s\right] \\
&\leq \frac{2^{25} \cdot \max(|Z|_t, |Z'|_t)}{2^t} \\
&= \frac{2^{25} \cdot \max(|AD|_t + |C|_t + 1, |AD'|_t + |C'|_t + 1)}{2^t} \\
&\leq \frac{1.5 \cdot 2^{25} \cdot \max(|AD|_t + |C|_t, |AD'|_t + |C'|_t)}{2^t},
\end{aligned}$$

where the last inequality is due to the convention that the block size has minimum value 1. ◻

In the sections that follow, we will consider a generalized form of the ChaCha20-Poly1305 construction parameterized by n, k, t, b, c, and $|Z|$. The first four parameters denote, respectively, the size in bits of the ChaCha20 permutation, the ChaCha20 key K, the Poly1305_Mac tag and hash key r, and the nonce N. The last two parameters denote the security constant of the almost Δ-universal hash function H in Poly1305_Mac and the bit length of the ChaCha20 constant Z. Note from the ChaCha20-Poly1305 construction that $n - (|Z| + k + b)$ is equal to the bit length of the ChaCha20 counter and $n \geq 2t$. The actual values of these parameters, as specified in RFC8439 and then applying Theorem 3.4, are:

$$n = 512, \quad k = 256, \quad t = 128, \quad b = 96, \quad c = 3 \cdot 2^{24}, \quad \text{and} \quad |Z| = 128.$$

4 THE SINGLE-USER SECURITY OF CHACHA20-POLY1305

The single-user security of ChaCha20-Poly1305 can of course be derived from its multi-user security theorem (Theorem 6.1) as a special case. A security proof in the single-user setting was already provided by Procter in [26]. However, Procter's proof is in the standard model and thus relies on a different security assumption on ChaCha20 than our multi-user security proof. Unfortunately his proof is incorrect, but we remedy this in Theorem 4.1 where we recover the same security bound under the same assumptions. Our new security proof together with a note describing the issue in [26] can be found in in the full version of this paper.

Theorem 4.1 (Single-user Security of ChaCha20-Poly1305). *Let n, k, t, c be the parameters of the* ChaCha20-Poly1305 *AEAD scheme. Let $\mathcal{A}$ be a valid nonce-respecting adversary making at most q_v decryption queries, and let ℓ_m be an upper bound on the total size (in t-bit blocks) of every query that it makes. Then there exists a PRF adversary $\mathcal{A}_{prf}$ against the Chacha20 block function* CC_block *such that:*

$$\mathsf{Adv}^{\mathrm{AE}}_{\mathrm{ChaCha20\text{-}Poly1305}}(\mathcal{A}) \leq \mathsf{Adv}^{\mathrm{PRF}}_{\mathrm{CC\_block}}(\mathcal{A}_{prf}) + \frac{q_v \cdot c \cdot \ell_m}{2^t},$$

where $\mathcal{A}_{prf}$ makes the same number of queries as the total number of blocks queried by $\mathcal{A}$.

In the above theorem, the single-user AE advantage and the PRF advantage are defined in the standard way and can be found in the full version of this paper. Note that for small values of ℓ_m this bound is tight, as evidenced by our forgery attack from Proposition 7.1.

5 GENERALIZED BALLS-INTO-BINS

Before delving into the multi-user security of ChaCha20-Poly1305, we present an improved balls-into-bins result. We make extensive use of this result in the proof of our main theorem and in extending the security bound to the nonce-randomization setting. The balls-into-bins problem considers an experiment whereby m balls are thrown into n bins at random. In our case, we will consider a distribution that deviates from the uniform distribution and we will be concerned with bounding the maximum load, i.e., the maximum number of balls landing in any bin. Maximum load results are known in the asymptotic setting [28] and in the concrete setting for uniform and slightly biased distributions [10]. In Theorem 5.1, we give a more general result in the concrete setting that is valid for any biased ball distribution and any number of balls (Case 4). Indeed, through it, we are also able to improve significantly over prior bounds for GCM in the nonce-randomization setting.

Theorem 5.1 (Biased Balls Into Bins). *Consider an experiment where at most Q balls are thrown into a set of bins, where each throw may depend on the outcome of the prior ones. Let $B \in (0, 1]$ be an upper bound on the probability that, when conditioned on prior throws, a ball lands into any bin. Then, for any $m \in \mathbb{N}^*$, $\widetilde{m} > 0$ and $t > 1$ satisfying one of the following conditions:*

(1) $m = \left\lceil \frac{\log_t(B^{-1})+\widetilde{m}}{\log_t((QB)^{-1})} \right\rceil$ *and* $Q \leq \frac{1}{B}$,

(2) $m = \lceil \log_t(B^{-1}) + \widetilde{m} \rceil$ *and* $Q \leq \frac{\log_t(B^{-1})+\widetilde{m}}{Bte}$,

(3) $m = \lceil QBte \rceil$ *and* $Q \geq \frac{\log_t(B^{-1})+\widetilde{m}}{Bte}$,

(4) $m = \left\lceil \frac{\max(QBte, \log_t(B^{-1})+\widetilde{m})}{\max(1, \log_t((QB)^{-1}))} \right\rceil$,

the probability that the heaviest bin contains m balls or more is at most $t^{-\widetilde{m}}$.

The proof can be found in Appendix A. Compared to prior results, the first advantage of this theorem is its generality as it allows more freedom in selecting parameters. Case 4 yields the best bound, giving the smallest maximum load m for a fixed maximum probability bound $t^{-\widetilde{m}}$, and has no restrictions on the maximum number of balls Q. Moreover, Cases 1, 2 and 3 can be derived as subcases of Case 4.

In comparison to the biased balls-into-bins lemmas of Bose, Hoang, and Tessaro, Case 1 improves over [10, Lemma 10] by introducing a trade-off parameter $\widetilde{m}$ between the maximum load and the probability bound instead of a fixed probability bound. This is essentially what allows us to lift the restrictive term of 2^{-48} in the bound for nonce-randomized GCM (cf. Sections 7.2 and 7.3).

In combination, Cases 2 and 3 are roughly equivalent to [10, Lemma 11]. While the latter allows for an unrestricted number of balls, it yields a looser bound than [10, Lemma 10] when the number of balls is small. Thus, when applying the lemmas of [10], one has to choose between a good bound over a restricted range of Q, or a suboptimal bound extending over a larger range of Q. Case 4 improves over both lemmas in [10] by combining their advantages in a single expression while additionally retaining the improvement from Case 1. We employ this improved bound in our multi-user security proof, though Lemma B.1, when bounding the probabilities of bad transcripts in Appendix B.2.

6 THE MULTI-USER SECURITY OF CHACHA20-POLY1305

The following theorem bounds the multi-user security of ChaCha20-Poly1305 in the ideal permutation model, i.e., when the ChaCha20 permutation is assumed to be a random permutation. This allows us to capture the local computation of the adversary expressed as the number of offline queries that it makes to the ChaCha20 permutation.

Theorem 6.1 (Multi-user Security of ChaCha20-Poly1305). *Let* ChaCha20-Poly1305$[\pi]$ *be the AEAD scheme described in Figure 3 having parameters n, k, t, c with its underlying permutation π modelled as a random permutation. Let $\mathcal{A}$ be a d-repeating adversary making at most p ideal permutation queries, q_e encryption queries totaling at most σ_e encrypted blocks, and q_v verification queries. Further, let ℓ_m denote the maximum size in t-bit blocks (including associated data) that it is allowed to query to its encryption and verification oracles. Then:*

$$\mathsf{Adv}^{\mathrm{muAE}}_{\mathrm{ChaCha20\text{-}Poly1305}[\pi]}(\mathcal{A}) \leq \frac{q_v(c\ell_m + 3)}{2^t} + \frac{d(p + q_e)}{2^k} + \frac{2p \cdot (n-k)}{2^k} + \frac{2q_v \cdot (n-k+4t)}{2^k} + \frac{(\sigma_e + q_e)^2}{2^{n+1}} + \frac{1}{2^{2t-2}} + \frac{1}{2^{n-k-2}}.$$

In the above we further require that: $n - k \leq 2^{k-2}$, $\sigma_e \leq \frac{n-k}{6} \cdot 2^{n-k}$, $q_v \leq 2^{n-2}$, $p \leq \min\left(\frac{2t-1}{6} \cdot 2^{2t}, \frac{n-k-1}{6} \cdot 2^{n-k}\right)$, and $d \leq \frac{2t}{3} \cdot 2^{2t}$.

6.1 Proof Overview

The proof of Theorem 6.1 is based on the H-coefficient technique (Theorem 2.2) which we apply to the augmented games described in Figures 4 and 5. These games modify the multi-user AEAD games shown in Figure 1 to give the adversary more information, thereby capturing an even stronger notion of security, but which nevertheless facilitates our proof. Specifically, the adversary is given access to an additional oracle Reveal, that it *must* query exactly once as its last query, right before returning its output—which in turn triggers the Finalize procedure.

In the real world, the Reveal oracle returns the outputs V_j corresponding to all the internal calls made by the encryption oracle to the ChaCha20 block function and all the user keys K_i. In the ideal world, on the other hand, the Reveal oracle returns randomly distributed strings V_j and keys K_i. Note that through $\{V_j\}$ and $\{K_i\}$, the adversary implicitly also learns the direct outputs of the ideal permutation in the real world, as it can easily reconstruct them. Since the augmented games are strictly stronger than the original ones, we can bound any adversary's multi-user advantage by bounding its distinguishing advantage with respect to the augmented games.

In the rest of this section we gradually set up all the components needed to apply the H-coefficient technique. We start in Section 6.2 by specifying the format of transcripts with respect to the augmented games. In Section 6.3 we define six sets of bad transcripts. In Section 6.4 we bound the H-coefficient over good transcripts, and in Section 6.5 we bound the probability of each individual set of bad transcripts. Then plugging the two bounds into Theorem 2.2 yields the desired result.

Finally we note that the bound we obtain here is actually stronger than the simplified version we presented in Theorem 6.1 for better exposition. In particular, the third and fourth terms in the bound in Theorem 6.1 are more accurately given by

$$\frac{2p \cdot \overline{(n-k)}^{\sigma_e}}{2^k} + \frac{2q_v \cdot (\overline{(n-k)}^{p} + \overline{2t}^{p} + \overline{2t}^{d})}{2^k},$$

with the shorthand notation $\overline{i}^{q} = \left\lceil \frac{i}{\max(1, i - \log_2(2q))} \right\rceil$ and $\overline{i}^{0} = 0$, for any $i, q \in \mathbb{N}$. Note that always $\overline{i}^{q} \le i$, yielding the terms in Theorem 6.1.

6.2 Transcripts and Multi-Sets

In the H-coefficient technique, we only need to consider *attainable* transcripts, i.e., ones that have a probability strictly greater than 0 of occuring in the ideal world. Note that here we define transcripts slightly differently, as we include additional information beyond the input-output pairs corresponding to the adversary's queries. We define them this way to facilitate the classification of good and bad transcripts and other aspects in the proof.

Transcripts. A transcript τ of an adversary interacting with an augmented game consists of the following entries:

- **Revealed key entries:** (key, i, K_i).
  Keys are returned as part of the output to the Reveal query. In the real world, these entries correspond to the actual user keys, whereas in the ideal world they correspond to values sampled independently of the rest of the transcript. In the real augmented game these are generated during initialization, whereas in the ideal augmented game they are not sampled until Reveal is queried.
- **Ideal permutation entries:** $(\mathsf{prim}, x, y, +)$ and $(\mathsf{prim}, x, y, -)$.
  An entry $(\mathsf{prim}, x, y, +)$ corresponds to a query $\text{Prim}(x)$ to the ideal permutation oracle with answer y, and an entry $(\mathsf{prim}, x, y, -)$ corresponds to a query $\text{Prim}^{-1}(y)$ to the inverse of the ideal permutation oracle with answer x.
- **Encryption entries:** $(\mathsf{enc}, i, N, AD, M, C\|T, V^{(q)})$.
  These entries contain the values specified in each encryption query $\text{Enc}(i, N, AD, M)$ together with the corresponding response $C\|T$. They additionally include the associated list $V^{(q)}$ of internal ChaCha20 block function calls made by the encryption algorithm in that encryption query. In particular, the $V^{(q)}$ values contain the key material used in Poly1305_Mac. Note, however, that while in the transcript, for convenience, we include $V^{(q)}$ in the encryption entries they are *not* actually returned to the adversary by the encryption oracle. In the augmented games these values are only revealed to the adversary at the end in the Reveal query. In the ideal world, $V^{(q)}$ is instead generated at random, as described in Figure 5, so that all good transcripts (defined below) have a probability strictly greater than 0 of occurring in the real world. This, in turn, ensures that the H-coefficient is not zero. Intuitively, including $V^{(q)}$ in the transcript allows us to better define the set of bad transcripts for our proof and thereby get a better H-coefficient ratio.
- **Verification entries:** $(\mathsf{vf}, i, N, AD, C\|T, \mathsf{false})$.
  Entries of this type correspond to verification queries $\text{Vf}(i, N, AD, C\|T)$ which return false as an answer. In the H-coefficient technique, we only need to be concerned with *attainable* transcripts, and in the ideal world, verification queries always return false as an answer.

Multi-Sets. The H-coefficient technique requires us to bound from below the ratio of the real-world and ideal-world probabilities, for any good transcript, to a value close to one. We accomplish this via a counting argument. In addition to the probability of the user keys being sampled, the probability of a transcript can be reduced to counting the number of distinct ideal permutation calls and random blocks generated (for encryption queries in the ideal world). If their sum is close in either world, we obtain a good H-coefficient ratio that is close to one. To facilitate our counting argument we introduce the following three multi-sets (sets where elements can repeat) and calculate their cardinality:

$$S_1(\tau) = \{(x, y) \mid (\mathsf{prim}, x, y, \cdot) \in \tau\}.$$

This set contains the input-output pairs, associated with the ideal permutation π, called during the ideal permutation queries Prim or Prim^{-1}.

$$\begin{aligned} S_2(\tau) = \{&(Z\|K_i\|0\|N, V_0 \overset{(32)}{-} (Z\|K_i\|0\|N)), \ldots, \\ &(Z\|K_i\|\ell\|N, V_\ell \overset{(32)}{-} (Z\|K_i\|\ell\|N)) \mid \\ &(\mathsf{enc}, i, N, AD, M, C\|T, V_0\| \cdots \|V_\ell) \in \tau\}. \end{aligned}$$

This set contains the input-output pairs, associated to the ideal world with the random blocks generated and in the real world with the ideal permutation π, that was called during the encryption queries Enc.

$$\begin{aligned} S_3(\tau) = \{&(Z\|K_i\|0\|N) \mid ((\mathsf{vf}, i, N, AD, C\|T, \mathsf{false}) \in \tau) \\ &\wedge ((Z\|K_i\|0\|N, \cdot) \notin S_1(\tau) \cup S_2(\tau))\}. \end{aligned}$$

```
Game $G^{\text{augReal-muAE}}_{\Pi[\pi]}$
procedure INITIALIZE
1 :  $K_1, K_2, \cdots \leftarrow\$ \{0,1\}^k, q_e \leftarrow 0$
procedure ENC$(i, N, AD, M)$
1 :  $C \| T \leftarrow \mathcal{E}(K_i, N, AD, M)$
2 :  $q_e \leftarrow q_e + 1$
3 :  $\overline{V}[q_e] \leftarrow (i, N, AD, M, C, T)$
4 :  return $C \| T$
procedure VF$(i, N, AD, C)$
1 :  $M \leftarrow \mathcal{D}(K_i, N, AD, C)$
2 :  return $(M \neq \bot)$
procedure PRIM$(X)$
1 :  return $\pi(X)$
procedure PRIM$^{-1}(Y)$
1 :  return $\pi^{-1}(Y)$
procedure REVEAL    / last query before FINALIZE
1 :  for $q = 1$ to $q_e$ do
2 :      $(i, N, AD, M, C, T) \leftarrow \overline{V}[q]$
3 :      $M_1 \| \cdots \| M_\ell \leftarrow M$
4 :      for $j = 0$ to $\ell$ do
5 :          $V_j \leftarrow \pi(Z \| K_i \| j \| N) \overset{(32)}{+} (Z \| K_i \| j \| N)$
6 :      $V^{(q)} \leftarrow V_0 \| \cdots \| V_\ell$
7 :  return $V^{(1)}, \ldots, V^{(q_e)}, K_1, K_2, \ldots$
procedure FINALIZE$(b)$
1 :  return $b$
```

Figure 4: Real Augmented Game.

```
Game $G^{\text{augIdeal-muAE}}_{\Pi[\pi]}$
procedure INITIALIZE
1 :  $q_e \leftarrow 0$
procedure ENC$(i, N, AD, M)$
1 :  $C \| T \leftarrow\$ \{0,1\}^{|M|+t}$
2 :  $q_e \leftarrow q_e + 1$
3 :  $\overline{V}[q_e] \leftarrow (i, N, AD, M, C, T)$
4 :  return $C \| T$
procedure VF$(i, N, AD, C)$
1 :  return false
procedure PRIM$(X)$
1 :  return $\pi(X)$
procedure PRIM$^{-1}(Y)$
1 :  return $\pi^{-1}(Y)$
procedure REVEAL    / last query before FINALIZE
1 :  for $q = 1$ to $q_e$ do
2 :      $(i, N, AD, M, C, T) \leftarrow \overline{V}[q]$
3 :      $r \leftarrow\$ \{0,1\}^t, W \leftarrow\$ \{0,1\}^{n-2t}$
4 :      $V_0 \leftarrow r \| (T \overset{(t)}{-} H_r(AD, C)) \| W$
5 :      $M_1 \| \cdots \| M_\ell \leftarrow M,\ C_1 \| \cdots \| C_\ell \leftarrow C$
6 :      for $j = 1$ to $\ell - 1$ do
7 :          $V_j \leftarrow M_j \oplus C_j$
8 :      $W' \leftarrow\$ \{0,1\}^{n-|M_\ell|}$
9 :      $V_\ell \leftarrow (M_\ell \oplus C_\ell) \| W'$
10 :     $V^{(q)} \leftarrow V_0 \| \cdots \| V_\ell$
11 :  $K_1, K_2, \cdots \leftarrow\$ \{0,1\}^k$
12 :  return $V^{(1)}, \ldots, V^{(q_e)}, K_1, K_2, \ldots$
procedure FINALIZE$(b)$
1 :  return $b$
```

Figure 5: Ideal Augmented Game.

This set contains the inputs to the ideal permutation π called during verification queries VF in the real world, if they are not also called (or obtained) during a primitive or encryption query.

6.3 Bad Transcripts

Our overarching methodology for defining bad transcripts (i.e., the set $\mathcal{T}_{\text{bad}}$) is to rule out transcripts that: 1) have a different multi-set cardinality in the real world compared to the ideal world or 2) have zero probability of occurring in the real world. This way we ensure that the H-coefficient is close to one. Towards the former, we will ensure that each entry in the first two multi-sets corresponds to a unique and independent call to the ideal permutation π, or a unique and independently generated random block. In the second case, even if the transcripts do not result in repeated multi-set entries they may still be impossible in the real world. Thus a transcript is in $\mathcal{T}_{\text{bad}}$ if it satisfies one of the following:

Case 1 $(x_1, y_1) \in S_1(\tau)$ and $(x_2, y_2) \in S_2(\tau)$ where $x_1 = x_2$. In this case, in the real world, two calls are made to the ideal permutation on the same input, through one ideal permutation query and one encryption query. This case also encompasses the case where $x_1 = x_2$ and $y_1 \neq y_2$, which is impossible in the real world. From this case, we can define the following simplified bad transcript description (which is the only possibility of this case happening):

Bad$_1$**:** There are two entries $(\text{prim}, x, y, \cdot)$ and $(\text{enc}, i, N, AD, M, C \| T, V_0 \| \cdots \| V_\ell)$ such that $x \in \{Z \| K \| 0 \| N, \ldots, Z \| K \| \ell \| N\}$ and $K_i = K$.

Case 2 $(x_1, y_1) \in S_1(\tau)$ and $(x_2, y_2) \in S_2(\tau)$ where $y_1 = y_2$. In this case, in the real world, two calls with the same output are made to the ideal permutation, through one ideal permutation query and one encryption query. This case also encompasses the case where $y_1 = y_2$ and $x_1 \neq x_2$, which is impossible in the real world. From this case, we can define the following simplified bad transcript description:

Bad$_2$**:** There are two entries $(\mathrm{prim}, x, y, \cdot)$ and $(\mathrm{enc}, i, N, AD, M, C\|T, V_0\| \cdots \|V_\ell)$ such that $y \in \{V_0 \overset{(32)}{-} (Z\|K_i\|0\|N), \ldots, V_\ell \overset{(32)}{-} (Z\|K_i\|\ell\|N)\}$.

Case 3 $(x_1, y_1), (x_2, y_2) \in S_2(\tau)$ where $x_1 = x_2$. In this case, in the real world, two calls are made to the ideal permutation on the same input, through one or two encryption queries. This case also encompasses the case where $x_1 = x_2$ and $y_1 \neq y_2$, which is impossible in the real world. From this case, we can define the following simplified bad transcript description:

Bad$_3$**:** There are two entries $(\mathrm{enc}, i, N, AD, M, C\|T, V)$ and $(\mathrm{enc}, j, N', AD', M', C'\|T', V')$ with $N = N'$ (block counter values do not overlap across different nonces), $i \neq j$ (because nonce reuse is not allowed for the same user) and $K_i = K_j$.

Case 4 $(x_1, y_1), (x_2, y_2) \in S_2(\tau)$ where $y_1 = y_2$. In this case, in the real world, two calls with the same output are made to the ideal permutation, through one or two encryption queries. The case where $y_1 = y_2$ and $x_1 = x_2$ being already considered in the previous point, we can restrict this case to $y_1 = y_2$ and $x_1 \neq x_2$, which in fact is impossible in the real world. From this case, we can define the following simplified bad transcript description:

Bad$_4$**:** There are two entries $(\mathrm{enc}, i, N, AD, M, C\|T, V_0\| \cdots \|V_\ell)$ and $(\mathrm{enc}, j, N', AD', M', C'\|T', V'_0\| \cdots \|V'_{\ell'})$ such that $(K_i, s, N) \neq (K_j, t, N')$ and $V_s \overset{(32)}{-} (Z\|K_i\|s\|N) = V'_t \overset{(32)}{-} (Z\|K_j\|t\|N')$ for $0 \leq s \leq \ell$ and $0 \leq t \leq \ell'$.

Case 5 $(x, y) \in S_1(\tau)$ and $(\mathrm{vf}, i, N, AD, C\|T, \mathrm{false}) \in \tau$ where $x = (Z\|K_i\|0\|N)$ and $\exists r \in \{0,1\}^t, W \in \{0,1\}^{n-2t}$ such that $y \overset{(32)}{+} x = (r\|(T \overset{(t)}{-} H_r(AD, C))\|W)$. This case corresponds to an impossible transcript in the real world. As in the real world $\pi(x) = y$ and from this case we obtain $\pi(Z\|K_i\|0\|N) \overset{(32)}{+} (Z\|K_i\|0\|N) = (r\|(T \overset{(t)}{-} H_r(AD, C))\|W)$. Therefore, trunc(CC_block$(K_i, N, 0), 2t) = r\|(T \overset{(t)}{-} H_r(AD, C))$. Thus $(N, AD, C\|T)$ is a valid nonce/AD/ciphertext triplet under key i, and in the real world, the verification query considered would have returned true. From this case, we can define the following simplified bad transcript description:

Bad$_5$**:** There are two entries $(\mathrm{vf}, i, N, AD, C\|T, \mathrm{false})$ and $(\mathrm{prim}, x, y, \cdot)$ such that $x = (Z\|K\|0\|N)$, $K_i = K$ and $\exists r \in \{0,1\}^t, W \in \{0,1\}^{n-2t}$ such that $y \overset{(32)}{+} x = (r\|(T \overset{(t)}{-} H_r(AD, C))\|W)$.

Case 6 $(x, y) \in S_2(\tau)$ and $(\mathrm{vf}, i, N, AD, C\|T, \mathrm{false}) \in \tau$ where $x = (Z\|K_i\|0\|N)$ and $\exists r \in \{0,1\}^t, W \in \{0,1\}^{n-2t}$ such that $y \overset{(32)}{+} x = (r\|(T \overset{(t)}{-} H_r(AD, C))\|W)$. This case corresponds to an impossible transcript in the real world, for a similar reason as the previous case, i.e., in the real world, the verification query considered would have returned true. From this case, we can define the following simplified bad transcript description:

Bad$_6$**:** There are two entries $(\mathrm{vf}, i, N, AD, C\|T, \mathrm{false})$ and $(\mathrm{enc}, j, N, AD', M', C'\|T', V'_0\| \cdots \|V'_u)$ such that $K_j = K_i$ and $\exists r \in \{0,1\}^t, W \in \{0,1\}^{n-2t}$ such that $V'_0 = (r\|(T \overset{(t)}{-} H_r(AD, C))\|W)$.

For $j \in \{1, \ldots, 6\}$, let Bad$_j$ be the set of attainable transcripts satisfying the j-th simplified bad transcript description defined above. Then $\mathcal{T}_{\mathrm{bad}} = \bigcup_{j=1}^{6} \mathrm{Bad}_j$. Note that Bad$_1$, Bad$_2$, and Bad$_3$ contain attainable transcripts calling more than one time the ideal permutation on the same input-output during primitive and encryption queries, and Bad$_4$, Bad$_5$, and Bad$_6$ contain attainable transcripts impossible in the real world.

6.4 Good Transcript Ratio (H-Coefficient)

An attainable transcript that is not in $\mathcal{T}_{\mathrm{bad}}$ is called good, and the set of all good transcripts is denoted by $\mathcal{T}_{\mathrm{good}}$. In the H-coefficient technique, we need to bound the probability ratio of a good transcript being generated in the real world with respect to that of it being generated in the ideal world.

Anomalous transcripts that result in a weak H-coefficient or that make it hard to evaluate have been weeded out as bad transcripts in the previous section. Roughly, these were transcripts that resulted in different multi-set cardinalities across the two worlds and transcripts with a real-world probability of zero. Consequently, we can now easily bound the H-Coefficient (of good transcripts) through a simple counting argument which yields the bound specified in Proposition 6.2. Its proof can be found in Appendix B.1.

Proposition 6.2 (Probability of good transcripts). *For any good transcript $\tau \in T_{\mathrm{good}}$ it holds that:*

$$\frac{\mathsf{P}_{\mathrm{real}}(\tau)}{\mathsf{P}_{\mathrm{ideal}}(\tau)} \geq 1 - \frac{2q_v}{2^t}.$$

6.5 Bad Transcript Probabilities

We now bound the probability that $\mathcal{T}_{\mathrm{ideal}} \in \mathcal{T}_{\mathrm{bad}}$, i.e., the probability that a transcript generated in the ideal world is bad.

The probabilities of events $\mathcal{T}_{\mathrm{ideal}} \in \mathrm{Bad}_j$, for $j \in \{1, \ldots, 6\}$ are bounded in Lemmas B.2–B.7, the proofs of which can be found in Appendix B.2. The corresponding bounds are reproduced below as inequalities (1)–(6). Proposition 6.3 is then obtained by a direct application of the union bound and substituting terms through inequalities (1)–(6). We notably use our improved balls-into-bins theorem to obtain the inequalities (2), (5), and (6).

$$\Pr[\mathcal{T}_{\text{ideal}} \in \text{Bad}_1] \leq \frac{pd}{2^k}. \tag{1}$$

$$\Pr[\mathcal{T}_{\text{ideal}} \in \text{Bad}_2] \leq \frac{p \cdot 2\overline{(n-k)}^{\sigma_e}}{2^k} + \frac{1}{2^{n-k}}. \tag{2}$$

$$\Pr[\mathcal{T}_{\text{ideal}} \in \text{Bad}_3] \leq \frac{q_e(d-1)}{2^k}. \tag{3}$$

$$\Pr[\mathcal{T}_{\text{ideal}} \in \text{Bad}_4] \leq \frac{(\sigma_e + q_e)^2}{2^{n+1}}. \tag{4}$$

$$\Pr[\mathcal{T}_{\text{ideal}} \in \text{Bad}_5] \leq \frac{q_v \cdot 2\left(\overline{(n-k)}^{p} + \overline{2t}^{p}\right)}{2^k} + \frac{1}{2^{n-k-1}} + \frac{1}{2^{2t-1}}. \tag{5}$$

$$\Pr[\mathcal{T}_{\text{ideal}} \in \text{Bad}_6] \leq \frac{q_v}{2^t} + \frac{q_v \cdot c \cdot \ell_m}{2^t} + \frac{q_v \cdot 2 \cdot \overline{2t}^{d}}{2^k} + \frac{1}{2^{2t}}. \tag{6}$$

Proposition 6.3 (Probability of bad transcripts). *Let $\mathcal{T}_{\text{ideal}}$ be the random variable representing the transcript generated by $\mathcal{A}$ in the ideal game. It then holds that:*

$$\Pr[\mathcal{T}_{\text{ideal}} \in \mathcal{T}_{\text{bad}}] \leq \frac{q_v \cdot (c \cdot \ell_m + 1)}{2^t} + \frac{d(p + q_e)}{2^k} + \frac{2 \cdot p\overline{(n-k)}^{\sigma_e}}{2^k}$$
$$+ \frac{q_v \cdot 2\left(\overline{(n-k)}^{p} + \overline{2t}^{p} + \overline{2t}^{d}\right)}{2^k} + \frac{(\sigma_e + q_e)^2}{2^{n+1}} + \frac{1}{2^{2t-2}} + \frac{1}{2^{n-k-2}},$$

again using the shorthand notation $\overline{i}^{q} = \left\lceil \frac{i}{\max(1, i - \log_2(2q))} \right\rceil$ and $\overline{i}^{0} = 0$, for any $i, q \in \mathbb{N}$.

Theorem 6.1 is obtained by combining Propositions 6.2 and 6.3 with Theorem 2.2, where Proposition 6.2 yields ϵ_1 and Proposition 6.3 yields ϵ_2.

7 IMPLICATIONS OF THE MAIN RESULT

In this section, we give some properties and implications of our multi-user security theorem from Section 6. Succinctly, we prove the tightness of the bound and extend the security result to the nonce-randomized setting. Then we explain how our bounds improve over previous results and how they compare and contrast with the security profile of GCM. We further discuss the security limits of ChaCha20-Poly1305 induced by the dominant term in our bound and propose a potential variant scheme that overcomes these limitations.

7.1 Tightness

We establish the tightness of the bound in Theorem 6.1 by providing several attacks matching the theorem bounds via the Propositions 7.1–7.6 below. To the best of our knowledge, the attacks in Propositions 7.1 and 7.6 are new in this context and do not seem to be covered elsewhere in the literature. For space reasons, their details are deferred to Appendix C. Details for the other attacks establishing the stated propositions can be found in the full version of the paper.

Note that all our attacks are valid against the original multi-user game and not only against the augmented version used in the proof. Moreover, for all our attacks except the one in Proposition 7.6, we only need the underlying ChaCha20 permutation π to be a permutation over $\{0,1\}^n$ and not necessarily an ideal permutation.

Hash key recovery / forgery attack. The main idea of the following attack is to retrieve the hash key of Poly1305_Mac and create a forgery, in a similar way as was done for GCM [1, 27]. For this, the attacker tests a set of ℓ_m possible hash keys through each verification query; if the correct hash key is in this set, the forgery attempt will be valid, distinguishing the real from the ideal game.

The attack is described in the multi-user setting but is also valid in the single-user setting as it queries only one user. The main difference with forgery attacks on GCM is that in Poly1305_Mac, 128-bit messages/ciphertexts blocks are encoded (padded with 1) before being transformed into coefficients of a polynomial modulo $2^{130} - 5$, preventing us from accessing the full range of coefficients. We use here the technique from [17] as a workaround, reducing the coefficients of our forgery polynomial by looking for the shortest vector of a carefully constructed lattice.

Proposition 7.1. *Let t be the tag size of* Poly1305_Mac *and H its associated c-almost Δ-universal hash function. Let $\ell_m \geq 5$ be the maximal number of t-bit input blocks in an encryption or verification query to the* ChaCha20-Poly1305 *AEAD scheme. There exists an adversary $\mathcal{A}$ making one encryption query and q_v verification queries such that:*

$$\mathsf{Adv}^{\text{muAE}}_{\text{ChaCha20-Poly1305}[\pi]}(\mathcal{A}) \geq \frac{q_v \cdot c \cdot (\ell_m - 5)}{2^{t+4}}.$$

For simplicity, we queried only one user for a specific encryption query in our attack. The approach can be extended to an attack querying multiple users, which then requires at least one encryption query per user. Moreover, the encryption query we use, in fact, only authenticates the associated data $AD = 0^{t \cdot (\ell_m - 1)}$; the message M queried is empty. The attack should be extendable to arbitrary values AD and M of maximum length ℓ_m by considering a closest vector (CVP) algorithm rather than a shortest vector (SVP) one.

Key recovery attack. The main idea of the two following attacks is to use offline computations (via the permutation oracle in our model) to try to recover one of the user keys. For this, the attacker makes d encryption queries on the same input (N, M, AD) across d different users. It then makes p permutation queries on guessed keys to construct corresponding ciphertext for that input, checking if any ciphertexts match.

Proposition 7.2. *Let k be the key length of the* ChaCha20-Poly1305 *AEAD scheme. There exists a d-repeating adversary $\mathcal{A}$, that makes p permutation queries and $d \leq \frac{2^k}{p}$ encryption queries such that:*

$$\mathsf{Adv}^{\text{muAE}}_{\text{ChaCha20-Poly1305}[\pi]}(\mathcal{A}) \geq \frac{pd}{2^{k+2}}.$$

The previous attack can be adapted to verification queries instead of encryption queries, yielding the following variant.

Proposition 7.3. *Let k be the key length of the* ChaCha20-Poly1305 *AEAD scheme. There exists an adversary $\mathcal{A}$, that makes one permutation query and $q_v \leq 2^k$ verification queries such that:*

$$\mathsf{Adv}^{\text{muAE}}_{\text{ChaCha20-Poly1305}[\pi]}(\mathcal{A}) \geq \frac{q_v}{2^{k+1}}.$$

Key collision attack. The key recovery attacks above can be modified to obtain key collision attacks. Instead of using permutation queries, these rely on encryption queries to detect key collisions between users. While this does not allow key recovery, it suffices to distinguish between the real and ideal games.

This modification applied to the attack from Proposition 7.2 gives us the attack in Proposition 7.4, which can also be seen as an adaptation of the key collision attack of [10] for a d-repeating adversary. We note that, although stated here as an attack against ChaCha20-Poly1305, it extends to a generic attack against any deterministic AE scheme with positive ciphertext expansion in the multi-user setting, similar to the one of [10].

Proposition 7.4. *Let k be the key length of the* ChaCha20-Poly1305 *AEAD scheme. There exists a d-repeating adversary $\mathcal{A}$, that makes $2 \le q_e \le \frac{2^{k+1}}{d-1}$ encryption queries such that:*

$$\mathsf{Adv}^{\mathrm{muAE}}_{\text{ChaCha20-Poly1305}[\pi]}(\mathcal{A}) \ge \frac{q_e(d-1)}{2^{k+3}}.$$

Block collision attack. Finally, the following attacks distinguish the output of the ChaCha20 block function from the output of a random function due to block collisions. By construction, the ChaCha20 block function outputs can have collisions for different inputs, but there cannot be collisions among the outputs of the ChaCha20 permutation. The first attack is a direct application of the idea that, for a single user, we can detect collisions by canceling the feed forward and looking at the difference between two values $C_i \oplus M_i$. In the second attack, we exploit the encryption queries of multiple users, looking only at the non-key parts of $C_i \oplus M_i$. This gives us access to a truncated part of the ChaCha20 permutation and reduces to the problem of distinguishing a truncated permutation from a random function.

Proposition 7.5. *Let n, k be the block and key length of the* ChaCha20-Poly1305 *AEAD scheme. There exists an adversary $\mathcal{A}$ that encrypt at most B blocks per user for a total number of $\sigma_e \le \frac{2^{n+1}}{B-1}$ encrypted blocks across all users such that:*

$$\mathsf{Adv}^{\mathrm{muAE}}_{\text{ChaCha20-Poly1305}[\pi]}(\mathcal{A}) \ge \frac{\sigma_e(B-1)}{2^{n+2}}.$$

Proposition 7.6. *Let n, k be the block and key length of the* ChaCha20-Poly1305$[\pi]$ *AEAD scheme, where the underlying* ChaCha20 *permutation π is modeled as an ideal random permutation. There exists an adversary $\mathcal{A}$ encrypting a total number of $\sigma_e \le \min\left(2^{\frac{n-k}{2}}, 2^{n-k-1}\right)$ blocks such that:*

$$\mathsf{Adv}^{\mathrm{muAE}}_{\text{ChaCha20-Poly1305}[\pi]}(\mathcal{A}) \ge \frac{\sigma_e(\sigma_e-1)}{2^{n+2}}.$$

Matching the bound. The presented attacks closely match the most significant and some further terms in the bound of Theorem 6.1:

Proposition 7.1: first term, $\approx \frac{q_v \cdot c \cdot \ell_m}{2^t}$.
Proposition 7.2: first half of second term, $\approx \frac{pd}{2^k}$,
and third term $\approx \frac{p}{2^k}$, up to a factor of $2(n-k)$.
Proposition 7.4: second half of second term, $\approx \frac{q_e d}{2^k}$.
Proposition 7.3: fourth term, $\approx \frac{q_v}{2^k}$, up to a factor of $2(n-k+4t)$.
Proposition 7.6: fifth term, $\approx \frac{\sigma_e^2}{2^n}$, up to a factor of 4.

7.2 Nonce randomization

The record protocols of TLS 1.3 [29], DTLS 1.3 [30] and QUIC [31] use a nonce-randomization technique to counter large-scale multi-user attacks. Analyzing this technique for the GCM mode, Bellare and Tackmann [4] provided the first multi-user treatment. Hoang, Tessaro, and Thiruvengadam [15] captured the nonce-randomization mechanism as a generic transform XN, which we recall in Figure 6. The XN transform turns an AEAD scheme Π into a nonce-randomized scheme Π^*, and [15] showed that a generic adversary against the multi-user security of Π^* can be reduced to the security of Π against a strongly d-repeating adversary via an adaptive balls-into-bins argument.

We reuse our Lemma A.1 to obtain a generalization of [15, Lemma 4.1 and Theorem 4.2]; their version emerges from our Theorem 7.7, when setting $\delta = 1/2$. The proof of the following theorem can be found in Appendix D.

Theorem 7.7 (Multi-user security of the XN transform). *Let $\Pi[\pi] = (\mathcal{K}, \mathcal{E}, \mathcal{D})$ be a nonce-based AEAD scheme with nonce length b defined on top an ideal permutation π, and let $\Pi^*[\pi] = \mathsf{XN}(\Pi[\pi])$ for the* XN *transform defined in Figure 6. Let $\mathcal{A}$ be a nonce-respecting adversary against Π^* making at most q_e* Enc *queries and q_v* Vf *queries. Then, for some fixed $\delta > 0$, we can construct an adversary $\mathcal{B}$ of the same concrete query complexity as $\mathcal{A}$ which is*

- *d-repeating for $d = \left\lceil \frac{(\delta+1)\cdot b}{\max(1, b-\log_2(q_e))} \right\rceil - 1$, if $q_e \le 2^b \cdot \frac{(\delta+1)\cdot b}{3}$*
- *strongly $\overline{d}$-repeating for $\overline{d} = \left\lceil \frac{(\delta+1)\cdot b}{\max(1, b-\log_2(q_e+q_v))} \right\rceil - 1$, if $q_e + q_v \le 2^b \cdot \frac{(\delta+1)\cdot b}{3}$*

such that

$$\mathsf{Adv}^{\mathrm{muAE}}_{\Pi^*[\pi]}(\mathcal{A}) \le \mathsf{Adv}^{\mathrm{muAE}}_{\Pi[\pi]}(\mathcal{B}) + \frac{1}{2^{\delta b}}.$$

Note that both d and $\overline{d}$ are less or equal to $(\delta+1)\cdot b$, even for a number of queries above 2^b, improving over [15] even when $\delta = 1/2$.

We can combine Theorem 7.7 with our multi-user AE security result for ChaCha20-Poly1305 in Theorem 6.1 to obtain the following bound for the nonce-randomized usage of ChaCha20-Poly1305 (where $n = 512$, $k = 256$, $t = 128$, and $b = 96$), against an adversary that is *not* necessarily d-repeating. We emphasize that to obtain this result, we only need to use Theorem 7.7 with a reduction adversary $\mathcal{B}$ that is d-repeating and not strongly $\overline{d}$-repeating, as this assumption is enough to use Theorem 6.1.

Theorem 7.8 (Multi-user security of nonce-randomized ChaCha20-Poly1305). *Let* ChaCha20-Poly1305$[\pi]$ *be the AEAD scheme described in Figure 3 having parameters n, k, t, c, b and its underlying permutation π modelled as a random permutation. Let $\mathcal{A}$ be an adversary against the multi-user AE security of* XN(ChaCha20-Poly1305$[\pi]$) *making at most p ideal permutation queries, q_e encryption queries totaling at most σ_e encrypted blocks, and q_v verification queries. Further, let ℓ_m denote the maximum size in t-bit blocks (including associated data) that $\mathcal{A}$ is allowed to query to its encryption*

```
procedure $\mathcal{K}^*()$
  1 : $K \leftarrow\$ \mathcal{K}$
  2 : $J \leftarrow\$ \{0,1\}^r$
  3 : return $K \| J$

procedure $\mathcal{E}^*(K\|J, N, AD, M)$
  1 : $N^* \leftarrow N \oplus J$
  2 : $C \leftarrow \mathcal{E}(K, N^*, AD, M)$
  3 : return $C$

procedure $\mathcal{D}^*(K\|J, N, AD, C$
  1 : $N^* \leftarrow N \oplus J$
  2 : $M \leftarrow \mathcal{D}(K, N^*, AD, C)$
  3 : return $M$
```

Figure 6: The XN transform of an AEAD scheme $\Pi = (\mathcal{K}, \mathcal{E}, \mathcal{D})$ into a nonce-randomized AEAD scheme $\Pi^* = (\mathcal{K}^*, \mathcal{E}^*, \mathcal{D}^*)$.

and verification oracles. Then:

$$\mathsf{Adv}^{\mathrm{muAE}}_{\mathsf{XN}(\text{ChaCha20-Poly1305}[\pi])}(\mathcal{A}) \le \frac{q_v(c\ell_m + 3)}{2^t} + \frac{d(p+q_e)}{2^k} + \frac{2p\cdot(n-k)}{2^k} + \frac{2q_v\cdot(n-k+4t)}{2^k} + \frac{(\sigma_e+q_e)^2}{2^{n+1}} + \frac{1}{2^{2t-2}} + \frac{1}{2^{n-k-2}} + \frac{1}{2^{\delta b}},$$

where $d = \left\lceil \frac{(\delta+1)\cdot b}{\max(1, b-\log_2(q_e))} \right\rceil - 1$, for any $0 < \delta \le \frac{t}{b} \cdot 2^{2t-1}$. In the above we further require that: $n-k \le 2^{k-2}$, $\sigma_e \le \frac{n-k}{6} \cdot 2^{n-k}$, $q_e \le 2^b \cdot \frac{(\delta+1)\cdot b}{3}$, $q_v \le 2^{n-2}$, $p \le \min\left(\frac{2t-1}{6} \cdot 2^{2t}, \frac{n-k-1}{6} \cdot 2^{n-k}\right)$.

Note that as long as $\delta \le \frac{t}{b} \cdot 2^{2t-1}$, the restriction on d from Theorem 7.7 is satisfied. We will next discuss and interpret the above bound for nonce-randomized ChaCha20-Poly1305. In particular, we will see how to pick δ such that the term $\frac{1}{2^{\delta b}}$ induced through nonce randomization does not dominate the overall bound.

7.3 Interpreting the Bounds

The dominant term. A closer look at Theorems 6.1 and 7.7 tells us that the most significant term in our bounds for practical scenarios is likely to be $\frac{q_v \cdot c \cdot \ell_m}{2^t}$. This is mainly due to the fact that for current ChaCha20-Poly1305 parameters, the block size is large ($n = 512$) and the key size $k = 256$ of ChaCha20 is twice as big as the tag size $t = 128$ of Poly1305, where the latter is effectively further reduced by the factor $c = 3 \cdot 2^{24}$ due to the clamping of 22 bits in the hash key r (cf. Definition 3.1 and Theorem 3.4). Hence, for the ChaCha20 -Poly1305 parameters, this term in practical scenarios dominates the second-most significant term: $\frac{q_v \cdot c \cdot \ell_m}{2^t} \ge \frac{d(p+q_e)}{2^k}$, as long as the total number of primitive and encryption queries is bounded as $p + q_e \le \frac{\ell_m}{d} \cdot 2^{153}$, even if $q_v = 1$ (when, e.g., applying the bound to a rekeying connection of a reliable-transport Internet security protocol like TLS [29] which terminates upon the first verification error).

Improving the nonce-randomizer bound. When moving from the basic multi-user security result for restricted, d-repeating adversaries to general adversaries, the analysis of the nonce-randomizer transform XN by [15] introduces an additive loss term of $2^{-b/2}$ for nonce length b. Through our improved balls-into-bins lemma, we instead obtain a parameterized term $2^{-\delta b}$, for which $\delta = 1/2$ as in [15] is just one instantiation. Indeed, choosing $\delta = 2$ allows us to ensure that this term is not dominant in the bound of Theorem 7.8 for nonce-randomized ChaCha20-Poly1305 for adversaries making up to even around $q_e \approx 2^b = 2^{96}$ encryption queries.[3]

Notably, our improved result for the XN also readily improves the multi-user security for nonce-randomized GCM [15, Theorem 4.3], allowing improvements to IETF/IRTF draft AEAD limits [14, Section 6.1]: while the advantage bound in [15] cannot become smaller than $2^{-b/2} = 2^{-48}$ (for the GCM nonce length of $b = 96$), our result entirely lifts this restriction, similar to the ChaCha20-Poly1305 case.

Improving over the standard hybrid loss in real-world settings. The only prior multi-user security bound for ChaCha20-Poly1305 is outlined in [20] and is based on a standard hybrid security loss in the number of user u over the single-user bound [26]. This hybrid bound is reflected in IETF/IRTF draft documents on AEAD limits [14, Section 6.2.1] as $\frac{u \cdot q_{v/u} \cdot c \cdot \ell_m}{2^t}$, where $q_{v/u}$ is the maximum failed verification attempts *per user*. In comparison, our bound is more fine-grained, bounding the *total* number q_v of verification attempts across *all* users, where always $q_v \le u \cdot q_{v/u}$; this gap can become relatively large when no tight upper bounds for attempts per user can be derived. Our multi-user bound confirms the approach taken in DTLS 1.3 [30, Section 4.5.3] and QUIC [31, Section 6.6] to derive integrity limits from summing the number of forgery attempts across multiple keys in a connection to counter the security degradation of repeated forgeries over unreliable transports [12].

7.4 Increasing the Hash Size

As discussed in Section 7.3, the dominant term in the multi-user security bounds for ChaCha20-Poly1305 in essentially all practical scenarios is $\frac{q_v \cdot c \cdot \ell_m}{2^t}$, making—relatively speaking—the tag length the scheme's weakest point. The natural question then is whether we can improve this term and obtain a stronger bound by increasing the tag size of ChaCha20-Poly1305. An obvious solution would be to use a wider almost Δ-universal hash function. This appears as an even more appealing solution when considering that, in ChaCha20 -Poly1305, twice as much key material is computed than is used (namely, only half of the first CC_block call output), leaving the other half available as possible extra key material without needing any extra computation. To illustrate one potential approach, we give a construction doubling the tag size and then discuss how this affects the security of ChaCha20-Poly1305.

To double the tag size, we propose as easy solution the following almost Δ-universal hash function that reuses the almost Δ-universal hash function H component of the Poly1305_Mac one-time MAC of ChaCha20-Poly1305. The main idea of our construction $\overline{H}^c$ in

[3]Observe that Theorem 7.8 for $\delta = 2$ allows the adversary to make up to $q_e \le b \cdot 2^b$ encryption queries while ensuring, via Theorem 7.7, that $d \le \lceil b \cdot (\delta + 1) \rceil - 1 = 287 \le 2^{128} = 2^t$ as required for Theorem 6.1.

Definition 7.9 below is to concatenate two instantiations of the hash function H using two independent hash keys. This new hash function $\overline{H}^{c}$, when augmented with a $2t$-bit blinding value, creates a one-time MAC that we call cPoly1305 in reference to it arising from concatenation. This one-time MAC cPoly1305 takes a $2t$-bit hash key and a $2t$-bit blinding value. Note again that due to the unused key material we can obtain the needed key and blinding value for this new almost Δ-universal hash function in the ChaCha20-Poly1305 construction without additional computation and with little modification to the original scheme.

*Definition 7.9 (*cPoly1305 *Hash).* Let r, u be two t-bit strings, H be the c-almost Δ-universal hash function of Poly1305_Mac and (AD, C) be a pair of byte strings. Define the hash function used in cPoly1305 as

$$\overline{H}^{c}_{r\|u}(AD, C) = H_r(AD, C) \| H_u(AD, C).$$

We now give a bound on the almost Δ-universal property of $\overline{H}^{c}$.

Theorem 7.10 (cPoly1305 Hash AΔU). *Let $c = 3 \cdot 2^{24}$ and s be a $2t$-bit string. If (AD, C) and (AD', C') are distinct pairs of byte strings, then:*

$$\Pr_{r\|u \twoheadleftarrow\$ \{0,1\}^{2t}}\left[\overline{H}^{c}_{r\|u}(AD, C) = \overline{H}^{c}_{r\|u}(AD', C') \overset{(t)}{+} s\right] \le \frac{c^2 \cdot \max\left((|AD|_t + |C|_t)^2, (|AD'|_t + |C'|_t)^2\right)}{2^{2t}}.$$

Proof. Let H be the c-almost Δ-universal hash function of Poly1305_Mac. Let (AD, C) and (AD', C') be distinct pairs of byte strings. If $\overline{H}^{c}_{r\|u}(AD, C) = \overline{H}^{c}_{r\|u}(AD', C') \overset{(t)}{+} s$ then

$$H_r(AD, C) = H_r(AD', C') \overset{(t)}{+} s[1{:}t] \qquad \text{and}$$
$$H_u(AD, C) = H_u(AD', C') \overset{(t)}{+} s[t+1{:}2t].$$

Thus if we sample two keys r, u independently and uniformly at random, then

$$\begin{aligned}\Pr_{r\|u \twoheadleftarrow\$ \{0,1\}^{2t}}&\left[\overline{H}^{c}_{r\|u}(AD, C) = \overline{H}^{c}_{r\|u}(AD', C') \overset{(t)}{+} s\right] = \\ &\Pr_{r \twoheadleftarrow\$ \{0,1\}^{t}}\left[H_r(AD, C) = H_r(AD', C') \overset{(t)}{+} s[1{:}t]\right] \\ &\cdot \Pr_{u \twoheadleftarrow\$ \{0,1\}^{t}}\left[H_u(AD, C) = H_u(AD', C') \overset{(t)}{+} s[t+1{:}2t]\right].\end{aligned}$$

The final bound is obtained by applying Theorem 3.4 to each factor of this product. □

We now discuss how using this almost Δ-universal function $\overline{H}^{c}$ in the ChaCha20-Poly1305 construction impacts the security of the scheme (the argument can be generalized to any universal function doubling the tag size). Applying Theorem 6.1 (with a slight modification), we obtain the following upper bound on the multi-user security of ChaCha20-cPoly1305, the AEAD scheme obtained using $\overline{H}^{c}$ in place of H in ChaCha20-Poly1305:

$$\mathsf{Adv}^{\mathrm{muAE}}_{\text{ChaCha20-cPoly1305}[\pi]}(\mathcal{A}) \le \frac{q_v((c\ell_m)^2 + 3)}{2^{2t}} + \frac{d(p + q_e)}{2^k} + \frac{2p \cdot (n-k)}{2^k} + \frac{2q_v \cdot (n - k + 8t)}{2^k} + \frac{(\sigma_e + q_e)^2}{2^{n+1}} + \frac{1}{2^{4t-2}} + \frac{1}{2^{n-k-2}}.$$

The first observation on this changed bound is that by doubling the tag size, we obtain a more uniform bound, with denominators in each term being at least $2^{2t} = 2^k = 2^{256}$, increasing the security by $t - \log_2(c\ell_m)$ bits. The second and perhaps more interesting observation is that the most significant term in the bound would likely become $\frac{d \cdot p}{2^k}$, making offline computation the most probable attack against the scheme. This term corresponds to a key recovery attack (see Proposition 7.2) and is most likely inherent to any nonce based scheme against d-repeating adversaries (see [4] for the equivalent attack in GCM and [8] for block ciphers). Thus choosing a tag size equal to the key length as we have done is probably the best tradeoff in terms of selecting the smallest tag size with the best security.

Our construction ChaCha20-cPoly1305 for doubling the tag size and increasing security allows parallelization and reuse of current implementations of Poly1305. However, it would most likely not have optimal efficiency given the tag size, as it processes each message block twice. A dedicated construction similar in structure to Poly1305 but with a bigger prime number would likely result in a more efficient hash function. A good candidate would be the pseudo-Mersenne prime $p = 2^{255} - 19$. We leave the development of such an alternative construction to future work.

8 CONCLUSIONS

We have given a detailed security analysis of ChaCha20-Poly1305, an increasingly important AEAD scheme. Our analysis is in the multi-user setting and assumes the permutation underlying the scheme is ideal. This enables us to capture offline computation in our model and make a detailed comparison with the corresponding analysis of GCM by Hoang, Tessaro, and Thiruvengadam [15]. Amongst other things, our analysis surfaces that the security limits for ChaCha20-Poly1305 are dominated by the limits of its MAC component. This is in contrast to GCM, where the limiting factor is the AES block size. We have proposed a lightweight way to strengthen ChaCha20-Poly1305 by doubling its MAC tag length. In future work, we plan to investigate alternative MAC constructions and their performance/security characteristics. We will also bring our work to the attention of the TLS and QUIC working groups of the IETF and collaborate with them to establish safe data limits for ChaCha20-Poly1305 in the context of the TLS, DTLS and QUIC protocols.

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A PROOF OF BALLS-INTO-BINS THEOREM

We prove the Balls-Into-Bins theorem in two stages via a preliminary lemma. The improvement over [10] is a direct consequence of this lemma. Lemma A.1 is additionally used to prove Theorem 7.7, resulting in an improved bound for nonce randomization.

A.1 Preliminary Lemma

The proof of Theorem 5.1 and Theorem 7.7 is based on the following lemma.

Lemma A.1. *Let $m, Q \in \mathbb{N}^*$, $B \in (0, 1]$ and $\widetilde{m}, t \in \mathbb{R}_{>0}$, with $t > 1$. For any of the following cases:*

(1) $m = \left\lceil \frac{\log_t(B^{-1})+\widetilde{m}}{\log_t((QB)^{-1})} \right\rceil$ and $Q \le \frac{1}{B}$,

(2) $m = \lceil \log_t(B^{-1}) + \widetilde{m} \rceil$ and $Q \le \frac{\log_t(B^{-1})+\widetilde{m}}{Bte}$,

(3) $m = \lceil QBte \rceil$ and $Q \ge \frac{\log_t(B^{-1})+\widetilde{m}}{Bte}$,

(4) $m = \left\lceil \frac{\max(QBte, \log_t(B^{-1})+\widetilde{m})}{\max(1, \log_t((QB)^{-1}))} \right\rceil$,

it holds that $\binom{Q}{m} \cdot B^{m-1} \le t^{-\widetilde{m}}$.

Note that Cases 2 and 3 can be combined into

$$m = \lceil \max(QBte, \log_t(B^{-1}) + \widetilde{m}) \rceil,$$

and if $Q \le \frac{\log_t(B^{-1})+\widetilde{m}}{Bte}$, Case 4 gives us

$$m = \left\lceil \frac{\log_t(B^{-1}) + \widetilde{m}}{\max(1, \log_t((QB)^{-1}))} \right\rceil \le \lceil \log_t(B^{-1}) + \widetilde{m} \rceil.$$

Proof. We consider the different cases in turn.

Case 1: $m = \left\lceil \frac{\log_t(B^{-1})+\widetilde{m}}{\log_t((QB)^{-1})} \right\rceil$ and $Q \le \frac{1}{B}$.

For this case,

$$\binom{Q}{m} \cdot B^{m-1} \le Q^m \cdot B^{m-1} = B^{-1} \cdot (QB)^m = B^{-1} \cdot t^{-m \cdot \log_t((QB)^{-1})}.$$

As $t > 1$, $QB \le 1$ and $m \ge \frac{\log_t(B^{-1})+\widetilde{m}}{\log_t((QB)^{-1})}$, then

$$t^{-m \cdot log_t((QB)^{-1})} \le t^{-\log_t(B^{-1}) - \widetilde{m}} = B \cdot t^{-\widetilde{m}}.$$

Hence $\binom{Q}{m} \cdot B^{m-1} \le t^{-\widetilde{m}}$.

Cases 2 & 3: $m = \lceil \max(QBte, \log_t(B^{-1}) + \widetilde{m}) \rceil$.
For these cases, as $\binom{Q}{m} < \left(\frac{Q\cdot e}{m}\right)^m$, we have

$$\binom{Q}{m} \cdot B^{m-1} < \left(\frac{Q\cdot e}{m}\right)^m \cdot B^{m-1} = B^{-1} \cdot \left(\frac{QB\cdot e}{m}\right)^m.$$

Moreover, as $m \geq QBte$,

$$B^{-1} \cdot \left(\frac{QB\cdot e}{m}\right)^m \leq B^{-1} \cdot t^{-m}.$$

Finally, as $m \geq \log_t(B^{-1}) + \widetilde{m}$, we obtain

$$B^{-1} \cdot t^{-m} \leq B^{-1} \cdot B \cdot t^{-\widetilde{m}} = t^{-\widetilde{m}}.$$

Hence $\binom{Q}{m} \cdot B^{m-1} \leq t^{-\widetilde{m}}$.

Case 4: $m = \left\lceil \frac{\max(QBte, \log_t(B^{-1})+\widetilde{m})}{\max(1, \log_t((QB)^{-1}))} \right\rceil$.
When $(QB)^{-1} < t$, then $\max(1, \log_t((QB)^{-1})) = 1$. Thus

$$m = \lceil \max(QBte, \log_t(B^{-1}) + \widetilde{m}) \rceil$$

and we can reuse results from Cases 2 and 3.
When $(QB)^{-1} \geq t$, then $\max(1, \log_t((QB)^{-1})) = \log_t((QB)^{-1})$ and $m = \left\lceil \frac{\max(QBte, \log_t(B^{-1})+\widetilde{m})}{\log_t((QB)^{-1})} \right\rceil$.
If we add the assumption that $Q \leq \frac{\log_t(B^{-1})+\widetilde{m}}{Bte}$, then

$$m = \left\lceil \frac{\log_t(B^{-1}) + \widetilde{m}}{\log_t((QB)^{-1})} \right\rceil$$

and we can reuse the result from Case 1.
Finally, if $Q \geq \frac{\log_t(B^{-1})+\widetilde{m}}{Bte}$, when $(QB)^{-1} \geq t$ then

$$m = \left\lceil \frac{QBte}{\log_t((QB)^{-1})} \right\rceil \geq \frac{\log_t(B^{-1}) + \widetilde{m}}{\log_t((QB)^{-1})}.$$

Similarly as in Case 1,

$$\binom{Q}{m} \cdot B^{m-1} \leq B^{-1} \cdot t^{-m\cdot \log_t((QB)^{-1})},$$

$t > 1$, $QB \leq 1$ and $m \geq \frac{\log_t(B^{-1})+\widetilde{m}}{\log_t((QB)^{-1})}$. Hence $\binom{Q}{m} \cdot B^{m-1} \leq t^{-\widetilde{m}}$. □

A.2 Proof of Theorem 5.1

We are now ready to show how Theorem 5.1 is derived from Lemma A.1

Since $m \geq 1$, when $Q = 0$ the probability that the heaviest bin contains m or more balls is clearly 0. Thus, we can focus on the case where $Q > 0$. Let us consider the case where $q \leq Q$ balls are thrown into bins and where m satisfies one of the conditions of the theorem. If $m > q$, the probability that the heaviest bin contains m or more balls is 0. If $m \leq q$ and the heaviest bin contains m or more balls, then at least one of the $\binom{q}{m}$ possible subsets of m balls (from the set of q balls thrown) is in the same bin. Hence, to bound the probability that the heaviest bin contains m or more balls in this case, we only need to bound the probability that at least one of these subsets is in the same bin. For each of these subsets, the probability that all m balls in that subset are thrown into the same bin is at most B^{m-1}. Thus the probability that at least one of the $\binom{q}{m}$ possible subsets of m balls is in the same bin is at most $\binom{q}{m} \cdot B^{m-1} \leq \binom{Q}{m} \cdot B^{m-1}$. Using Lemma A.1, $\binom{Q}{m} \cdot B^{m-1} \leq t^{-\widetilde{m}}$ for the four different cases. Hence the probability that the heaviest bin contains m or more balls is also at most $t^{-\widetilde{m}}$.

B MISSING COMPONENTS IN THE PROOF OF THEOREM 6.1

In this section, we prove the remaining pieces that are necessary to complete the proof of the multi-user security of ChaCha20-Poly1305 outlined in Section 6. In Appendix B.1 we prove Proposition 6.2, which lower bounds the ratio of good transcripts, and in Appendix B.2 we prove Lemmas B.2–B.7, which upper bound the probabilities of bad transcripts in the ideal world.

B.1 Proof of Proposition 6.2 (Good Transcript Ratio)

Let $\tau \in T_{\text{good}}$ be a good transcript and let K_τ, P_τ, E_τ, and V_τ be the sets of revealed-key, ideal-permutation, encryption, and verification entries in τ, respectively. Let $u = |K_\tau|$ be the number of key entries. Let $\mathsf{P}_{\text{ideal}}(\tau)$, resp. $\mathsf{P}_{\text{real}}(\tau)$, be the probability that if we make the queries described in τ (in the same order), we receive in the ideal game, resp. the real game, the corresponding answers recorded in τ.

In the ideal world, the revealed keys and the answers to the ideal permutation, encryption, and verification queries are generated independently. Thus $\mathsf{P}_{\text{ideal}}(\tau) = \mathsf{P}_{\text{ideal}}(K_\tau) \cdot \mathsf{P}_{\text{ideal}}(P_\tau) \cdot \mathsf{P}_{\text{ideal}}(E_\tau) \cdot \mathsf{P}_{\text{ideal}}(V_\tau)$. Recall that for good transcripts, the calls to the ideal permutation π and random blocks induced by ideal permutation and encryption entries are distinct. Therefore, there are exactly $|S_1(\tau)|$ distinct calls to the ideal permutation in P_τ and $|S_2(\tau)|$ independently sampled random blocks generated in E_τ. Hence $\mathsf{P}_{\text{ideal}}(P_\tau) = \prod_{i=0}^{|S_1(\tau)|-1} \frac{1}{2^n - i}$ and $\mathsf{P}_{\text{ideal}}(E_\tau) = \prod_{i=0}^{|S_2(\tau)|-1} \frac{1}{2^n}$. Moreover, as the u-many k-bit user keys are sampled at random and verification queries always return false, we obtain $\mathsf{P}_{\text{ideal}}(K_\tau) = 2^{-ku}$ and $\mathsf{P}_{\text{ideal}}(V_\tau) = 1$. Consequently,

$$\mathsf{P}_{\text{ideal}}(\tau) = 2^{-ku} \cdot \prod_{i=0}^{|S_1(\tau)|-1} \frac{1}{2^n - i} \cdot \prod_{i=0}^{|S_2(\tau)|-1} \frac{1}{2^n}. \tag{7}$$

In the real world, the user keys are also sampled at random and $\mathsf{P}_{\text{real}}(K_\tau) = 2^{-ku}$. Once the user keys have been sampled, the probability of query outputs depends on the number of distinct ideal permutation calls made. For good transcripts in the real world, the calls to the ideal permutation π in ideal permutation and encryption entries are distinct. Therefore, there are exactly $|S_1(\tau)| + |S_2(\tau)|$ distinct calls to the ideal permutation in $P_\tau \cup E_\tau$, and the input-output of these calls are also entirely determined in these sets. Moreover, there are at most $|S_3(\tau)|$ more calls to the ideal permutation done during verification queries and distinct from the ones done during ideal permutation and encryption queries. However, the input-output of these calls is not entirely determined in V_τ, only the inputs are. For the outputs, as τ is a good transcript, they are required to be distinct from the ones in $P_\tau \cup E_\tau$ and not to result in a forgery for queries in V_τ. If q is the exact number of distinct ideal permutation calls in τ, the probability to get these calls is $\prod_{i\in I} \frac{1}{2^n - i} \cdot \prod_{j\in J} \left(1 - \frac{f_j}{2^n - j}\right)$, where I, J is a partition of $\{0, \ldots, q-1\}$ fixed by the order of queries in τ and f_j is the number of values that would result in a forgery while sampling the associated ideal permutation call during verification queries. I contains the index/order in which the associated ideal permutation calls from $P_\tau \cup E_\tau$ are sampled and J contains the remaining ones from V_τ. Note that $|I| = |S_1(\tau)| + |S_2(\tau)|$ and $|J| \leq |S_3(\tau)|$. Thus, $\mathsf{P}_{\text{real}}(\tau) = 2^{-ku} \cdot \prod_{i \in I} \frac{1}{2^n - i} \cdot \prod_{j \in J} \left(1 - \frac{f_j}{2^n - j}\right)$. Furthermore, we can reorder and minimize the products in the following way: $\prod_{i \in I} \frac{1}{2^n - i} \cdot \prod_{j \in J} \left(1 - \frac{f_j}{2^n - j}\right) \geq \prod_{i=0}^{|I|-1} \frac{1}{2^n - i} \cdot \prod_{j=|I|}^{|I|+|S_3(\tau)|-1} \left(1 - \frac{F}{2^n - j}\right)$, where F is a bound on the number of values that would result in a forgery while sampling the associated ideal permutation call of one verification query, i.e. for a verification query $\textsc{Vf}(i, N, AD, C\|T)$, the number of $(r\|s\|W) \in \{0,1\}^n$ such that $H_r(AD, C) \overset{(t)}{+} s = T$. There are 2^{n-2t} possible values for W, and each of the 2^t possible values for r yield exactly one value s such that $H_r(AD, C) \overset{(t)}{+} s = T$. Therefore, we can consider $F = 2^{n-2t} \cdot 2^t = 2^{n-t}$ in the previous inequality. Consequently,

$$\mathsf{P}_{\text{real}}(\tau) \geq 2^{-ku} \cdot \prod_{i=0}^{|S_1(\tau)|+|S_2(\tau)|-1} \frac{1}{2^n - i} \cdot \prod_{j=0}^{|S_3(\tau)|-1} \left(1 - \frac{2^{n-t}}{2^n - |S_1(\tau)| - |S_2(\tau)| - j}\right). \quad (8)$$

We can now calculate the probability ratio:

$$\frac{\mathsf{P}_{\text{real}}(\tau)}{\mathsf{P}_{\text{ideal}}(\tau)} \geq \prod_{j=0}^{|S_3(\tau)|-1} \left(1 - \frac{2^{n-t}}{2^n - |S_1(\tau)| - |S_2(\tau)| - j}\right) \geq \prod_{j=0}^{|S_3(\tau)|-1} \left(1 - \frac{2^{n-t}}{2^n - |S_1(\tau)| - |S_2(\tau)| - |S_3(\tau)|}\right).$$

As $|S_1(\tau)| \leq p \leq \frac{n-k-1}{6} \cdot 2^{n-k}$, $|S_2(\tau)| \leq \sigma_e + q_e \leq \frac{n-k}{3} \cdot 2^{n-k}$ and $|S_3(\tau)| \leq q_v \leq 2^{n-2}$, then

$$|S_1(\tau)| + |S_2(\tau)| + |S_3(\tau)| \leq p + \sigma_e + q_e + q_v \leq (n-k) \cdot 2^{n-k} + 2^{n-2} \leq 2^{n-1}.$$

Hence,

$$\frac{\mathsf{P}_{\text{real}}(\tau)}{\mathsf{P}_{\text{ideal}}(\tau)} \geq \prod_{j=0}^{|S_3(\tau)|-1} \left(1 - \frac{2^{n-t}}{2^{n-1}}\right) = \left(1 - \frac{1}{2^{t-1}}\right)^{|S_3(\tau)|} \geq 1 - \frac{|S_3(\tau)|}{2^{t-1}} \geq 1 - \frac{q_v}{2^{t-1}} = 1 - \frac{2q_v}{2^t}. \quad (9)$$

B.2 Proofs of Bad Transcript Probabilities

In this subsection, we bound the probabilities of the six sets of Bad transcripts in the ideal world through Lemma B.2–B.7. However, we first give a corollary of our balls-into-bins theorem to simplify its application to our lemmas.

Recall that all the transcripts are generated by a valid nonce-respecting adversary $\mathcal{A}$ that is d-repeating. We also recall here that in the ideal world, the keys K_i are uniformly sampled at the end of the execution, during the last oracle query to $\textsc{Reveal}$, and are therefore independent of any other previous queries. Moreover, in the ideal world, all the V_j values are independent and uniformly distributed. As for $V_0 = (r\|(T \overset{(t)}{-} H_r(AD, C))\|W)$, the values r, T and W are uniformly distributed, for $V_j = (M_j \oplus C_j)$, the value C_j is uniformly distributed and for $V_\ell = ((M_\ell \oplus C_\ell)\|W')$, the values C_ℓ and W' are uniformly distributed.

To simplify notations in the following, we denote by $[w]^{K+}$ the key part of a n-bit string w, i.e. $w[|Z|+1{:}|Z|+k]$ and by $[w]^{K-}$ the remaining part of the string, i.e. $w[1{:}|Z|] \,\|\, w[|Z|+k+1{:}n]$. We also denote by $[w]^r$ the part of a n-bit string w that corresponds to the hash key r, i.e. $w[1{:}t]$ and by $[w]^s$ the part that corresponds to the blinding value s of the string, i.e. $w[t+1{:}2t]$.

B.2.1 Balls-Into-Bins Corollary. The following lemma is a direct corollary of our balls-into-bins theorem. It will be used below to bound bad transcript probabilities, specifically in Lemmas B.3, B.6, and B.7. To simplify the computed bounds, we apply our generalized balls-into-bins theorem in the proof of the lemma only for a bounded number of balls. It should be noted that we could lift some of the restrictions on the number of queries in Theorem 6.1 and 7.8 by considering an unrestricted number of balls, however at the expense of a more complicated bound.

Lemma B.1. *Consider the experiment where we throw balls into bins and where each throw may be dependent on the previous ones. Let $Q \in \mathbb{N}$ be the maximum number of balls we throw and $B \in \,]0,1]$ be an upper bound on the probability that, when conditioning on the result of the prior throws, a ball is placed into a specific bin. If $Q \leq B^{-1} \cdot \frac{\log_2(B^{-1})}{3}$, the probability that the heaviest bin contains $m = \left\lceil \frac{2\log_2(B^{-1})}{\max(1, \log_2((QB)^{-1}))} \right\rceil$ or more balls is at most B.*

Proof. We simply apply the Case 4 of Theorem 5.1, with $t = 2$, $\widetilde{m} = \log_t(B^{-1})$ and $Q \leq B^{-1} \cdot \frac{\log_2(B^{-1})}{3}$. □

Note that compared to [10, Lemma 11], our maximum load m can be smaller than $\log_2(B^{-1})$, and compared to [10, Lemma 10], our maximal number of queries can be bigger than B^{-1} (when $\frac{\log_2(B^{-1})}{3} > 1$) and our maximum load m is always smaller than $\lceil 2\log_2(B^{-1}) \rceil$.

B.2.2 Bounding Bad_1 *transcripts probability.*

Lemma B.2 (Bad_1).

$$\Pr[\mathcal{T}_{\text{ideal}} \in \mathsf{Bad}_1] \leq \frac{pd}{2^k}.$$

Proof. Bad_1 is the set of all attainable transcripts τ that contain two entries $(\mathsf{prim}, x, y, \cdot)$ and $(\mathsf{enc}, i, N, AD, M, C\|T, V_0\| \cdots \|V_\ell)$ such that $x \in \{Z\|K\|0\|N, \ldots, Z\|K\|\ell\|N\}$ and $K_i = K$.

If a transcript generated by the adversary $\mathcal{A}$ in the ideal augmented game is in Bad_1, then $\mathcal{A}$ has made an encryption query $\textsc{Enc}(i, N, AD, M)$ and either a query $\textsc{Prim}(Z\|K\| \cdot \|N)$ or a query $\textsc{Prim}^{-1}(y)$ with answer $Z\|K\|\cdot\|N$, and finally a query to $\textsc{Reveal}$ that returned the key K_i to be equal to K. Hence the probability that a transcript generated by $\mathcal{A}$ in the ideal augmented game is in Bad_1 is bounded by the probability that $\mathcal{A}$ makes the previously described queries.

We are going to consider the case where $\mathcal{A}$ is just about to query $\textsc{Reveal}$, but has already made all its other oracle queries. For each of the at most p ideal permutation queries $\textsc{Prim}(Z\|K\| \cdot \|N)$ or $\textsc{Prim}^{-1}(y)$ with answer $Z\|K\| \cdot \|N$ done by $\mathcal{A}$, there are at most d encryption queries $\textsc{Enc}(i, N', AD, M)$ with $N' = N$ done by $\mathcal{A}$. Thus there are at most pd possible pairs of such queries done by $\mathcal{A}$. When querying Reveal, for each of these pairs, the probability that $K_i = K$ is $\frac{1}{2^k}$, as the keys are uniformly sampled, independently from any previous queries. Hence, using a union bound, the probability that for at least one of these pairs $K_i = K$, is at most $\frac{pd}{2^k}$. Thus,

$$\Pr[\mathcal{T}_{\text{ideal}} \in \mathsf{Bad}_1] \leq \frac{pd}{2^k}. \qquad \square$$

B.2.3 Bounding Bad_2 *transcripts probability.*

Lemma B.3 (Bad_2).

$$\Pr[\mathcal{T}_{\text{ideal}} \in \mathsf{Bad}_2] \leq \frac{p \cdot \overline{2(n-k)}^{\sigma_e}}{2^k} + \frac{1}{2^{n-k}}.$$

Proof. Bad_2 is the set of all attainable transcripts τ that contain two entries $(\mathsf{prim}, x, y, \cdot)$ and $(\mathsf{enc}, i, N, AD, M, C\|T, V_0\| \cdots \|V_\ell)$ such that $y \in \left\{V_0 \overset{(32)}{-} (Z\|K_i\|0\|N), \cdots, V_\ell \overset{(32)}{-} (Z\|K_i\|\ell\|N)\right\}$.

Let E_1 be the event that there exist $w \in \{0,1\}^{n-k}$ such that among all encryption queries, there are $m_1 = \left\lceil \frac{2(n-k)}{\max(1, n-k-\log_2(\sigma_e+q_e))} \right\rceil$ or more values of $[V_j]^{K\text{-}} \overset{(32)}{-} (Z\|j\|N)$ that are equal to w, where N is the nonce associated to the encryption query of V_j. Then,

$$\begin{aligned}\Pr[\mathcal{T}_{\text{ideal}} \in \mathsf{Bad}_2] &= \Pr\left[\mathcal{T}_{\text{ideal}} \in \mathsf{Bad}_2 \wedge \overline{E_1}\right] + \Pr[\mathcal{T}_{\text{ideal}} \in \mathsf{Bad}_2 \wedge E_1] \\ &\leq \Pr\left[\mathcal{T}_{\text{ideal}} \in \mathsf{Bad}_2 \;\middle|\; \overline{E_1}\right] + \Pr[E_1].\end{aligned}$$

We will now bound the probability of E_1, so that we only have to consider the event $\mathcal{T}_{\text{ideal}} \in \mathsf{Bad}_2$ conditioned by $\overline{E_1}$ afterward. Recall that in the ideal world, the V_j values are independent and uniformly distributed. We can view each encryption query $\textsc{Enc}(i, N, AD, M_1\| \cdots \|M_\ell)$ with answer $C_1\| \cdots \|C_\ell\|T$ together with the sampling of its associated r, W, W' parameters, as throwing $\ell + 1$ balls $[V_j]^{K\text{-}} \overset{(32)}{-} (Z\|j\|N)$, for $0 \leq j \leq \ell$, uniformly at random into 2^{n-k} bins. Thus if we consider all encryption queries, we throw at most $\sigma_e + q_e$ balls uniformly at random into 2^{n-k} bins. Using Lemma B.1, with $Q = \sigma_e + q_e$ and $B = 2^{-(n-k)}$, the probability that the heaviest bin contains m_1 or more balls is at most $2^{-(n-k)}$. Therefore, the probability of E_1 is bounded by $2^{-(n-k)}$. Hence,

$$\Pr[\mathcal{T}_{\text{ideal}} \in \mathsf{Bad}_2] \leq \Pr\left[\mathcal{T}_{\text{ideal}} \in \mathsf{Bad}_2 \;\middle|\; \overline{E_1}\right] + 2^{-(n-k)}. \qquad (10)$$

Note that $\overline{E_1}$ is the event that for all $w \in \{0,1\}^{n-k}$, among all encryption queries, there are strictly less than m_1 values of $[V_j]^{K\text{-}} \overset{(32)}{-} (Z\|j\|N)$ that are equal to w. To bound the remaining term, we now consider the case where $\mathcal{A}$ is querying the Reveal oracle and all r, W, W' parameters have been sampled, but no user keys K_i have been sampled yet. In the following, we mean by y the value associated to a primitive query and by j the block index associated to an encryption query. The union $\bigcup_{y,j}$ and sum $\sum_y \sum_j$ are over all primitive and encrypted blocks already queried. A transcript is in Bad_2, if there exist a y from a primitive query and a $V_j \overset{(32)}{-} (Z\|K_i\|j\|N)$ from an encryption query that are equal. To bound the probability of this event, we split it into two, one for the event that the key part of $V_j \overset{(32)}{-} (Z\|K_i\|j\|N)$ and y are equal and a second one for the event that the remaining parts are equal. Let $E_2(j, y)$ be the event that $[V_j]^{K+} \overset{(32)}{-} K_i = [y]^{K+}$ and $E_3(j, y)$ be the event that $[V_j]^{K\text{-}} \overset{(32)}{-} (Z\|j\|N) = [y]^{K\text{-}}$. Then, using a union bound,

$$\begin{aligned}\Pr\left[\mathcal{T}_{\text{ideal}} \in \mathsf{Bad}_2 \;\middle|\; \overline{E_1}\right] &\leq \Pr\left[\bigcup_{y,j} E_2(j,y) \wedge E_3(j,y) \;\middle|\; \overline{E_1}\right] \\ &\leq \sum_y \sum_j \Pr\left[E_2(j,y) \;\middle|\; \overline{E_1} \wedge E_3(j,y)\right] \cdot \Pr\left[E_3(j,y) \;\middle|\; \overline{E_1}\right]. \qquad (11)\end{aligned}$$

When querying Reveal, for any pair (j, y), the probability that $[V_j]^{K+} \overset{(32)}{-} K_i = [y]^{K+}$ is the probability that when the key is sampled $K_i = [V_j]^{K+} \overset{(32)}{-} [y]^{K+}$. As the keys are uniformly sampled in Reveal, and independently from any previous queries and parameters, this probability is $\frac{1}{2^k}$. This event is also independent from $\overline{E_1}$ and $E_3(j, y)$. Thus

$$\Pr\left[E_2(j,y) \;\middle|\; \overline{E_1} \wedge E_3(j,y)\right] = \Pr[E_2(j,y)] = \frac{1}{2^k}. \qquad (12)$$

Note that conditioned on event $\overline{E_1}$ there are strictly less than m_1 values $[V_j]^{K\text{-}} \overset{(32)}{-} (Z\|j\|N)$ that are equal to one $[y]^{K\text{-}}$, thus for a fix y, there are strictly less than m_1 block indexes j such that $\Pr\left[E_3(j,y) \;\middle|\; \overline{E_1}\right]$ is not zero. Thus

$$\sum_y \sum_j \Pr\left[E_3(j,y) \;\middle|\; \overline{E_1}\right] < \sum_y m_1 \leq p \cdot m_1. \qquad (13)$$

Moreover, $m_1 = \left\lceil \frac{2(n-k)}{\max(1, n-k-\log_2(\sigma_e+q_e))} \right\rceil \leq 2\left\lceil \frac{(n-k)}{\max(1, n-k-\log_2(2\sigma_e))} \right\rceil = \overline{2(n-k)}^{\sigma_e}$ and in combination with (10), (11), (12), (13), we obtain

$$\Pr[\mathcal{T}_{\text{ideal}} \in \mathsf{Bad}_2] \leq \frac{p \cdot \overline{2(n-k)}^{\sigma_e}}{2^k} + \frac{1}{2^{n-k}}. \qquad \square$$

B.2.4 Bounding Bad_3 *transcripts probability.*

Lemma B.4 (Bad_3).

$$\Pr[\mathcal{T}_{\text{ideal}} \in \mathsf{Bad}_3] \leq \frac{q_e(d-1)}{2^k}.$$

Proof. Bad_3 is the set of all attainable transcripts τ that contain two entries $(\mathsf{enc}, i, N, AD, M, C\|T, V)$ and $(\mathsf{enc}, j, N', AD', M', C'\|T', V')$ with $N = N'$, $i \neq j$ and $K_i = K_j$.

If a transcript generated by the adversary $\mathcal{A}$ in the ideal augmented game is in Bad_3, then this adversary $\mathcal{A}$ has made a pair of encryption queries $\textsc{Enc}(i, N, AD, M)$ and $\textsc{Enc}(j, N', AD', M')$ with $N = N'$ and $i \neq j$, and a query to Reveal that returned the two corresponding K_i and K_j being equal. Hence the probability that a transcript generated by $\mathcal{A}$ in the ideal augmented game is in Bad_3 is bounded by the probability that $\mathcal{A}$ makes the previously described queries.

We are again going to consider the case where $\mathcal{A}$ is just about to query Reveal, but has already made all its other oracle queries. For each of the at most q_e encryption queries $\textsc{Enc}(i, N, AD, M)$ done by $\mathcal{A}$, there are at most $d - 1$ other encryption queries $\textsc{Enc}(j, N', AD', M')$ with $N = N'$ and $i \neq j$ done by $\mathcal{A}$. Thus there are at most $q_e(d-1)$, possible pairs of such encryption queries done by $\mathcal{A}$. When querying Reveal, for each of these pairs, the probability that $K_i = K_j$ is $\frac{1}{2^k}$, as the keys are uniformly sampled, independently from any previous queries. Hence, using a union bound, the probability that for at least one of these pairs $K_i = K_j$, is at most $\frac{q_e(d-1)}{2^k}$. Thus,

$$\Pr[\mathcal{T}_{\text{ideal}} \in \text{Bad}_3] \leq \frac{q_e(d-1)}{2^k}. \qquad \square$$

B.2.5 Bounding Bad$_4$ *transcripts probability.*

Lemma B.5 (Bad$_4$).

$$\Pr[\mathcal{T}_{\text{ideal}} \in \text{Bad}_4] \leq \frac{(\sigma_e + q_e)^2}{2^{n+1}}.$$

Proof. Bad$_4$ is the set of all attainable transcripts τ that contain two entries $(\text{enc}, i, N, AD, M, C\|T, V_0\| \cdots \|V_\ell)$ and $(\text{enc}, j, N', AD', M', C'\|T', V_0'\| \cdots \|V_{\ell'}')$ such that $(K_i, s, N) \neq (K_j, t, N')$ and $V_s \overset{(32)}{-} (Z\|K_i\|s\|N) = V_t' \overset{(32)}{-} (Z\|K_j\|t\|N')$ for $0 \leq s \leq \ell$ and $0 \leq t \leq \ell'$.

If a transcript generated by the adversary $\mathcal{A}$ in the ideal augmented game is in Bad$_4$, then $\mathcal{A}$ has made a Reveal query that returned values verifying $V_s \overset{(32)}{-} (Z\|K_i\|s\|N) = V_t' \overset{(32)}{-} (Z\|K_j\|t\|N')$. As established above, the V_j values are all independent and uniformly distributed. Moreover, the keys K_i are also independent and uniformly distributed. Hence, for each encryption query $\text{Enc}(i, N, AD, M_1\| \cdots \|M_\ell)$, the ℓ values $V_0 \overset{(32)}{-} (Z\|K_i\|0\|N), \cdots, V_\ell \overset{(32)}{-} (Z\|K_i\|\ell\|N)$ are also independent and uniformly distributed. If we consider all encryption queries, we obtain at most $\sigma_e + q_e$ independent and uniformly distributed values, and the probability that at least two of them are equal is at most the birthday bound, i.e., $\frac{(\sigma_e+q_e)^2}{2^{n+1}}$. Thus,

$$\Pr[\mathcal{T}_{\text{ideal}} \in \text{Bad}_4] \leq \frac{(\sigma_e + q_e)^2}{2^{n+1}}. \qquad \square$$

B.2.6 Bounding Bad$_5$ *transcripts probability.*

Lemma B.6 (Bad$_5$).

$$\Pr[\mathcal{T}_{\text{ideal}} \in \text{Bad}_5] \leq \frac{q_v \cdot 2\left(\overline{(n-k)}^{p} + \overline{2t}^{p}\right)}{2^k} + \frac{1}{2^{n-k-1}} + \frac{1}{2^{2t-1}}.$$

Proof. Bad$_5$ is the set of all attainable transcripts τ that contain two entries $(\text{vf}, i, N, AD, C\|T, \text{false})$ and $(\text{prim}, x, y, \cdot)$ such that $x = (Z\|K_i\|0\|N)$ and $\exists r \in \{0,1\}^t, W \in \{0,1\}^{n-2t}$ such that $y \overset{(32)}{+} x = (r\|(T \overset{(t)}{-} H_r(AD, C))\|W)$.

The probability calculation for this case will follow the same outline as for Bad$_2$.
Let E_4 be the event that there exist $w \in \{0,1\}^{n-k}$, such that the number of inverse ideal permutation queries verifying $[x]^{K\text{-}} = w$, is greater or equal to $m_4 = \left\lceil \frac{2(n-k-1)}{\max(1, n-k-1-\log_2(p))} \right\rceil$ and E_5 be the event that there exist $AD^*, C^* \in \{0,1\}^*$ and $T^*, r^* \in \{0,1\}^t$ such that, the number of forward ideal permutation queries verifying $[y \overset{(32)}{+} x]^s \overset{(t)}{+} H_{r^*}(AD^*, C^*) = T^*$ and $r^* = [y \overset{(32)}{+} x]^r$, is greater or equal to $m_5 = \left\lceil \frac{2\cdot(2t-1)}{\max(1, 2t-1-\log_2(p))} \right\rceil$. Then, in a similar way as for Bad$_2$,

$$\begin{aligned}
\Pr[\mathcal{T}_{\text{ideal}} \in \text{Bad}_5] &= \Pr\left[\mathcal{T}_{\text{ideal}} \in \text{Bad}_5 \wedge \overline{E_4} \wedge \overline{E_5}\right] \\
&\quad + \Pr\left[\mathcal{T}_{\text{ideal}} \in \text{Bad}_5 \wedge \overline{\overline{E_4} \wedge \overline{E_5}}\right] \\
&\leq \Pr\left[\mathcal{T}_{\text{ideal}} \in \text{Bad}_5 \;\middle|\; \overline{E_4} \wedge \overline{E_5}\right] + \Pr[E_4 \vee E_5] \\
&\leq \Pr\left[\mathcal{T}_{\text{ideal}} \in \text{Bad}_5 \;\middle|\; \overline{E_4} \wedge \overline{E_5}\right] + \Pr[E_4] + \Pr[E_5].
\end{aligned} \tag{14}$$

We will again bound the probability of E_4 and E_5, leaving us afterward with event $\mathcal{T}_{\text{ideal}} \in \text{Bad}_5$ conditioned by $\overline{E_4} \wedge \overline{E_5}$.

We can view each inverse ideal permutation query $\text{Prim}^{-1}(y)$ with answer x as throwing a ball $[x]^{K\text{-}}$ into one of 2^{n-k} possible bins, with probability at most $\frac{2^k}{2^n - p}$. As $p \leq 2^{n-1}$, then $\frac{2^k}{2^n-p} \leq \frac{2^k}{2^{n-1}} = \frac{1}{2^{n-k-1}}$. Thus if we consider all inverse ideal permutation queries, we throw at most p balls with conditional probability at most $2^{-(n-k-1)}$ into 2^{n-k} bins. Using Lemma B.1, with $Q = p$ and $B = 2^{-(n-k-1)}$, the probability that the heaviest bin contains m_4 or more balls is at most $2^{-(n-k-1)}$. Therefore, the probability of E_4 is bounded by $2^{-(n-k-1)}$:

$$\Pr[E_4] \leq 2^{-(n-k-1)}. \tag{15}$$

For any $AD, C \in \{0,1\}^*$, let Throw(AD, C) be the throwing experiment, where we view each forward ideal permutation $\text{Prim}(x)$ with answer y, as throwing a ball $[y \overset{(32)}{+} x]^r \| ([y \overset{(32)}{+} x]^s \overset{(t)}{+} H_r(AD, C))$, where $r = [y \overset{(32)}{+} x]^r$ into one of 2^{2t} possible bins. For any bin $r\|T \in \{0,1\}^{2t}$, each throw has a conditional probability of at most

$$\begin{aligned}
&\Pr_{y \leftarrow\$ \text{Prim}(x)}\left[\bigcup_{W \in \{0,1\}^{n-2t}} (y \overset{(32)}{+} x) = r\|(T \overset{(t)}{-} H_r(AD, C))\|W\right] \\
&\leq \sum_{W \in \{0,1\}^{n-2t}} \Pr_{y \leftarrow\$ \text{Prim}(x)}\left[(y \overset{(32)}{+} x) = r\|(T \overset{(t)}{-} H_r(AD, C))\|W\right] \\
&\leq \frac{2^{n-2t}}{2^n - p}.
\end{aligned}$$

Therefore, each throw has a conditional probability of at most $\frac{2^{n-2t}}{2^n-p}$. As $p \leq 2^{n-1}$, then $\frac{2^{n-2t}}{2^n-p} \leq \frac{2^{n-2t}}{2^{n-1}} = \frac{1}{2^{2t-1}}$. If we consider all forward ideal permutation queries, we throw at most p balls with conditional probability at most $2^{-(2t-1)}$ into 2^{2t} bins. Using Lemma B.1, with $Q = p$ and $B = 2^{-(2t-1)}$, the probability that the heaviest bin of Throw(AD, C) contains m_5 or more balls is at most $2^{-(2t-1)}$. If E_5 happens, then the bin $r^*\|T^*$ in experiment Throw(AD^*, C^*) contains m_5 or more balls, therefore, the number of balls in the heaviest bin of Throw(AD^*, C^*) contains m_5 or more balls. Thus, the probability of E_5 is also bounded by $2^{-(2t-1)}$:

$$\Pr\left[\overline{E_5}\right] \leq 2^{-(2t-1)}. \tag{16}$$

Combining (14), (15) and (16), we can bound the probability of a transcript being in Bad$_5$:

$$\Pr[\mathcal{T}_{\text{ideal}} \in \text{Bad}_5] \leq \Pr\left[\mathcal{T}_{\text{ideal}} \in \text{Bad}_5 \;\middle|\; \overline{E_4} \wedge \overline{E_5}\right] + \frac{1}{2^{n-k-1}} + \frac{1}{2^{2t-1}}. \tag{17}$$

Note that $\overline{E_4}$ is the event that for all $w \in \{0,1\}^{n-k}$, there are strictly less than m_4 inverse ideal permutation queries such that $[x]^{K\text{-}} = w$ and $\overline{E_5}$ is the event that for any $AD, C \in \{0,1\}^*$ and $T \in \{0,1\}^t$, the number of forward ideal permutation queries verifying $[y \overset{(32)}{+} x]^s \overset{(t)}{+} H_r(AD,C) = T$, where $r = [y \overset{(32)}{+} x]^r$, is strictly less than m_5. To bound the remaining term, we now consider the case where $\mathcal{A}$ is just about to query REVEAL, but has already made all its other oracle queries. In the following, we mean by PRIM and VF, a primitive and verification query already done by $\mathcal{A}$. The union $\bigcup_{\text{VF,PRIM}}$ and sum $\sum_{\text{VF}} \sum_{\text{PRIM}}$ are over all primitive and verification queries already done by $\mathcal{A}$.

A transcript is in Bad_5, if there exists a verification and primitive query such that $x = (Z\|K_i\|0\|N)$ and $[y \overset{(32)}{+} x]^s = T \overset{(t)}{-} H_r(AD,C)$, where $r = [y \overset{(32)}{+} x]^r$. To bound the probability of this event, we split it into two, one for the event that the key part of x is equal to the key of the verification query and a second one for the rest of the event that doesn't depend on the key. Let $E_6(\text{VF}, \text{PRIM})$ be the event that when querying the REVEAL oracle, the key K_i of the verification query VF is equal to the value $[x]^{K+}$ associated to the primitive query PRIM, and $E_7(\text{VF}, \text{PRIM})$ be the event that $[x]^{K\text{-}} = (Z\|0\|N)$ and $[y \overset{(32)}{+} x]^s = T \overset{(t)}{-} H_r(AD,C)$, where $r = [y \overset{(32)}{+} x]^r$, (N, AD, C, T) are the values associated to the verification query VF, and (x,y) are the values associated to the primitive query PRIM. Then, using a union bound,

$$\begin{aligned}
&\Pr\left[\mathcal{T}_{\text{ideal}} \in \text{Bad}_5 \,\middle|\, \overline{E_4} \wedge \overline{E_5}\right] \\
&\le \Pr\left[\bigcup_{\text{VF,PRIM}} E_6(\text{VF}, \text{PRIM}) \wedge E_7(\text{VF}, \text{PRIM}) \,\middle|\, \overline{E_4} \wedge \overline{E_5}\right] \\
&\le \sum_{\text{VF}} \sum_{\text{PRIM}} \Pr\left[E_6(\text{VF}, \text{PRIM}) \,\middle|\, \overline{E_4} \wedge \overline{E_5} \wedge E_7(\text{VF}, \text{PRIM})\right] \\
&\qquad\qquad \cdot \Pr\left[E_7(\text{VF}, \text{PRIM}) \,\middle|\, \overline{E_4} \wedge \overline{E_5}\right]. \qquad (18)
\end{aligned}$$

When querying REVEAL, the users' keys are uniformly sampled, independently from any previous queries and parameters. Thus, for any pair VF, PRIM, the probability that $K_i = [x]^{K+}$ is $\frac{1}{2^k}$. This event is also independent from $\overline{E_4}$, $\overline{E_5}$ and $E_7(\text{VF}, \text{PRIM})$, which both depend only on parameters already fixed before querying REVEAL. Thus

$$\begin{aligned}
\Pr\left[E_6(\text{VF}, \text{PRIM}) \,\middle|\, \overline{E_4} \wedge \overline{E_5} \wedge E_7(\text{VF}, \text{PRIM})\right] &= \Pr[E_6(\text{VF}, \text{PRIM})] \\
&= \frac{1}{2^k}. \qquad (19)
\end{aligned}$$

Note that conditioned on event $\overline{E_4} \wedge \overline{E_5}$, for any verification query VF there are strictly less than $m_4 + m_5$ primitive queries PRIM such that $[x]^{K\text{-}} = (Z\|0\|N)$ and $[y \overset{(32)}{+} x]^s = T \overset{(t)}{-} H_r(AD,C)$, where $r = [y \overset{(32)}{+} x]^r$, (N, AD, C, T) are the values associated to VF, and (x,y) are the values associated to PRIM. Hence, for a fix verification query VF, there are strictly less than $m_4 + m_5$ primitive queries PRIM such that $\Pr\left[E_7(\text{VF}, \text{PRIM}) \,\middle|\, \overline{E_4} \wedge \overline{E_5}\right]$ is not zero. Thus

$$\begin{aligned}
\sum_{\text{VF}} \sum_{\text{PRIM}} \Pr\left[E_7(\text{VF}, \text{PRIM}) \,\middle|\, \overline{E_4} \wedge \overline{E_5}\right] &< \sum_{\text{VF}} (m_4 + m_5) \\
&\le q_v \cdot (m_4 + m_5). \qquad (20)
\end{aligned}$$

Moreover,

$$\begin{aligned}
m_4 = \left\lceil \frac{2(n-k-1)}{\max(1, n-k-1-\log_2(p))} \right\rceil &\le 2 \cdot \left\lceil \frac{(n-k)}{\max(1, n-k-\log_2(2p))} \right\rceil \\
&= 2 \cdot \overline{(n-k)}^{p}
\end{aligned}$$

$$\begin{aligned}
m_5 = \left\lceil \frac{2(2t-1)}{\max(1, 2t-1-\log_2(p))} \right\rceil &\le 2 \cdot \left\lceil \frac{2t}{\max(1, 2t-\log_2(2p))} \right\rceil \\
&= 2 \cdot \overline{2t}^{p}
\end{aligned}$$

and in combination with (17), (18), (19) and (20), we obtain

$$\Pr[\mathcal{T}_{\text{ideal}} \in \text{Bad}_5] \le \frac{q_v \cdot 2\left(\overline{(n-k)}^{p} + \overline{2t}^{p}\right)}{2^k} + \frac{1}{2^{n-k-1}} + \frac{1}{2^{2t-1}}. \quad \square$$

B.2.7 Bounding Bad_6 *transcripts probability.*

LEMMA B.7 (Bad_6).

$$\Pr[\mathcal{T}_{\text{ideal}} \in \text{Bad}_6] \le \frac{q_v}{2^t} + \frac{q_v \cdot c \cdot \ell_m}{2^t} + \frac{q_v \cdot 2 \cdot \overline{2t}^{d}}{2^k} + \frac{1}{2^{2t}}.$$

PROOF. Bad_6 is the set of all transcripts τ that contain two entries $(\text{vf}, i, N, AD, C\|T, \text{false})$ and $(\text{enc}, j, N, AD', M', C'\|T', V_0'\| \cdots \|V_u')$ such that $K_j = K_i$ and $\exists r \in \{0,1\}^t, W \in \{0,1\}^{n-2t}$ such that $V_0' = (r\|(T \overset{(t)}{-} H_r(AD,C))\|W)$.
Recall that for an entry $(\text{enc}, j, N, AD', M', C'\|T', V_0'\| \cdots \|V_u')$, in the ideal world, $V_0' = (r\|(T' \overset{(t)}{-} H_r(AD',C'))\|W)$, with $r \leftarrow\$ \{0,1\}^t$ and $W \leftarrow\$ \{0,1\}^{n-2t}$. Two cases are possible. The first one is when the two queries are made to the same user, i.e., $i = j$, and the second one is when the two queries are made to different users, i.e., $i \neq j$.

$\text{Bad}_{6\text{-}1}$**: case** $i = j$**.**
In the ideal world, Bad_6 for this case could be redefined as the set of all transcripts τ that contain two entries $(\text{vf}, i, N, AD, C\|T, \text{false})$ and $(\text{enc}, i, N, AD', M', C'\|T', V_0'\| \cdots \|V_u')$ such that $T \overset{(t)}{-} H_r(AD,C) = T' \overset{(t)}{-} H_r(AD',C')$, where $r = [V_0']^r$. To bound the probability of this case, we are going to use the c-almost Δ-universal property of the function H. However, this property require for (AD,C) and (AD',C') to be distinct. We are therefore first going to bound the probability of transcripts verifying this case when $(AD,C) = (AD',C')$.

When $(AD,C) = (AD',C')$, if a transcript is in this case then $T \overset{(t)}{-} H_r(AD,C) = T' \overset{(t)}{-} H_r(AD',C')$, i.e. $T = T'$. Depending on the order of the queries, we consider two subcases for when $(AD,C) = (AD',C')$. The first one is for when the verification query is done after the encryption query, and the second one is for the inverse order. For the first subcase, when the verification query is done after the encryption query and $(AD,C) = (AD',C')$, we cannot have that $T = T'$, as it would result in a non valid query. Thus the probability of this subcase is zero. For the second subcase, when the encryption query is done after the verification query and $(AD,C) = (AD',C')$, the probability that $T' = T$ is $\frac{1}{2^t}$, as encryption queries return uniform random strings. For each of the at most q_v verification queries $\textsc{Vf}(i, N, AD, C\|T)$, there can be at most one following encryption query $\textsc{Enc}(i, N, AD', M')$ with answer $C'\|T'$ and the same (i, N), and the probability that $T' = T$ is $\frac{1}{2^t}$. Thus the probability of this subcase is bounded by $\frac{q_v}{2^t}$.

When $(AD, C) \neq (AD', C')$, if a transcript generated by the adversary $\mathcal{A}$ in the ideal augmented game is in this case, then $\mathcal{A}$ has made a verification query $\textsc{Vf}(i, N, AD, C\|T)$, an encryption query $\textsc{Enc}(i, N, AD', M')$ with answer $C'\|T'$, and finally a query to $\textsc{Reveal}$ that returned a V_0' associated to the previous encryption query such that $r = [V_0']^r$ is a uniform random string. Hence the probability that a transcript generated by $\mathcal{A}$ in the ideal augmented game is in this case is bounded by the probability that $\mathcal{A}$ makes the previously described queries. We are going to consider the case where $\mathcal{A}$ is just about to query $\textsc{Reveal}$, but has already made all its other oracle queries.

For each of the at most q_v verification queries $\textsc{Vf}(i, N, AD, C\|T)$ done by $\mathcal{A}$, there are at most one other encryption query $\textsc{Enc}(i, N, AD', M')$ with answer $C'\|T'$ and the same (i, N) done by him. Thus there are at most q_v possible pairs of such queries done by $\mathcal{A}$. When querying $\textsc{Reveal}$, for each of these pairs, the value V_0' associated to the encryption query is computed as $(r\|(T' \overset{(t)}{-} H_r(AD', C'))\|W)$ with r being sampled uniformly at random and independently from any previous queries. Hence, as $(AD, C) \neq (AD', C')$ and H is a c-almost Δ-universal hash function, the probability that $T \overset{(t)}{-} H_r(AD, C) = T' \overset{(t)}{-} H_r(AD', C')$ is bounded by $\frac{c \cdot \max(|AD|_t + |C|_t, |AD'|_t + |C'|_t)}{2^t} \leq \frac{c \cdot \ell_m}{2^t}$, for $r = [V_0']^r$ and V_0' being the associated value to the encryption query. Therefore, using a union bound, the probability that for at least one of these pairs $T \overset{(t)}{-} H_r(AD, C) = T' \overset{(t)}{-} H_r(AD', C')$ for $r = [V_0']^r$, is at most $\frac{q_v \cdot c \cdot \ell_m}{2^t}$.

$$\Pr[\mathcal{T}_{\text{ideal}} \in \mathsf{Bad}_{6\text{-}1}] \leq \frac{q_v}{2^t} + \frac{q_v \cdot c \cdot \ell_m}{2^t}. \tag{21}$$

$\mathsf{Bad}_{6\text{-}2}$**: case** $i \neq j$.
The probability calculation for this case will follow the same outline as for Bad_2.
Let E_8 be the event that there exist $AD^*, C^* \in \{0,1\}^*$ and $(N^*, T^*, r^*) \in \{0,1\}^b \times \{0,1\}^t \times \{0,1\}^t$ such that, the number of encryption queries $\textsc{Enc}(\cdot, N^*, \cdot, \cdot)$ with nonce N^* and associated V_0' verifying $[V_0']^s \overset{(t)}{+} H_{r^*}(AD^*, C^*) = T^*$ and $r^* = [V_0']^r$, is greater or equal to $m_8 = \left\lceil \frac{4t}{\max(1, 2t - \log_2(d))} \right\rceil$. Then,

$$\begin{aligned}
&\Pr[\mathcal{T}_{\text{ideal}} \in \mathsf{Bad}_{6\text{-}2}] \\
&= \Pr\left[\mathcal{T}_{\text{ideal}} \in \mathsf{Bad}_{6\text{-}2} \wedge \overline{E_8}\right] + \Pr[\mathcal{T}_{\text{ideal}} \in \mathsf{Bad}_{6\text{-}2} \wedge E_8] \\
&\leq \Pr\left[\mathcal{T}_{\text{ideal}} \in \mathsf{Bad}_{6\text{-}2} \,\middle|\, \overline{E_8}\right] + \Pr[E_8].
\end{aligned}$$

We will now bound the probability of E_8, so that we only have to consider the event $\mathcal{T}_{\text{ideal}} \in \mathsf{Bad}_{6\text{-}2}$ conditioned by $\overline{E_8}$ afterward. For any $AD, C \in \{0,1\}^*$ and nonce N, let $\mathsf{Throw}(AD, C, N)$ be the throwing experiment, where we view each encryption query $\textsc{Enc}(\cdot, N, \cdot, \cdot)$ with nonce N and associated V_0', as throwing a ball $[V_0']^r\|([V_0']^s \overset{(t)}{+} H_r(AD, C))$ where $r = [V_0']^r$ into 2^{2t} possible bins. For any bin $r\|T \in \{0,1\}^{2t}$, each throw has a conditional probability of at most

$$\begin{aligned}
&\Pr_{V_0' \leftarrow\$ \textsc{Enc}(\cdot, N, \cdot, \cdot)}\left[[V_0']^r \| [V_0']^s = r\|T \overset{(t)}{-} H_r(AD, C)\right] \\
&= \Pr\left[[V_0']^s = T \overset{(t)}{-} H_r(AD, C) \,\middle|\, [V_0']^r = r\right] \cdot \Pr\left[[V_0']^r = r\right] = \frac{1}{2^{2t}}.
\end{aligned}$$

If we consider all encryption queries with the nonce N, we throw at most d balls uniformly at random into 2^{2t} bins. Using Lemma B.1, with $Q = d$ and $B = 2^{-2t}$, the probability that the heaviest bin of $\mathsf{Throw}(AD, C, N)$ contains m_8 or more balls is at most 2^{-2t}. If E_8 happens, then the bin $r^*\|T^*$ in experiment $\mathsf{Throw}(AD^*, C^*, N^*)$ contains m_8 or more balls, therefore, the number of balls in the heaviest bin of $\mathsf{Throw}(AD^*, C^*, N^*)$ contains m_8 or more balls. Thus, the probability of E_8 is also bounded by 2^{-2t}. Hence,

$$\Pr[\mathcal{T}_{\text{ideal}} \in \mathsf{Bad}_{6\text{-}2}] \leq \Pr\left[\mathcal{T}_{\text{ideal}} \in \mathsf{Bad}_{6\text{-}2} \,\middle|\, \overline{E_8}\right] + 2^{-2t}. \tag{22}$$

Note that $\overline{E_8}$ is the event that for any $AD, C \in \{0,1\}^*$ and $(N, T) \in \{0,1\}^b \times \{0,1\}^t$, the number of encryption queries $\textsc{Enc}(\cdot, N, \cdot, \cdot)$ with nonce N and associated V_0' verifying $[V_0']^s \overset{(t)}{+} H_r(AD, C) = T$ where $r = [V_0']^r$, is strictly less than m_8. To bound the remaining term, we now consider the case where $\mathcal{A}$ is querying the $\textsc{Reveal}$ oracle and all parameters r have already been sampled, but no user keys have been sampled yet. In the following, we mean by $\textsc{Enc}$ and $\textsc{Vf}$, an encryption and verification query already done by $\mathcal{A}$. The union $\bigcup_{\textsc{Vf},\textsc{Enc}}$ and sum $\sum_{\textsc{Vf}} \sum_{\textsc{Enc}}$ are over all verification queries already done by $\mathcal{A}$ and all encryption queries with the same nonce as the verification query.

A transcript is in $\mathsf{Bad}_{6\text{-}2}$, if there exists a verification query $\textsc{Vf}(i, N, AD, C\|T)$ and an encryption query $\textsc{Enc}(j, N, AD', M')$ with the same nonce N and an associated V_0' such that $[V_0']^s \overset{(t)}{+} H_r(AD, C) = T$ where $r = [V_0']^r$, and $K_j = K_i$. To bound the probability of this event, we split it into two, one for the event that the keys are equals and a second one for the rest of the event that doesn't depend on the key. Let $E_9(\textsc{Vf}, \textsc{Enc})$ be the event that the key K_i of the verification query $\textsc{Vf}$ is equal to the key K_j of the encryption query $\textsc{Enc}$, and $E_{10}(\textsc{Vf}, \textsc{Enc})$ be the event that $[V_0']^s \overset{(t)}{+} H_r(AD, C) = T$, where $r = [V_0']^r$, (AD, C, T) are the values associated to the verification query $\textsc{Vf}$, and V_0' is the value associated to the encryption query $\textsc{Enc}$. Then, using a union bound,

$$\begin{aligned}
&\Pr\left[\mathcal{T}_{\text{ideal}} \in \mathsf{Bad}_{6\text{-}2} \,\middle|\, \overline{E_8}\right] \\
&\leq \Pr\left[\bigcup_{\textsc{Vf},\textsc{Enc}} E_9(\textsc{Vf}, \textsc{Enc}) \wedge E_{10}(\textsc{Vf}, \textsc{Enc}) \,\middle|\, \overline{E_8}\right] \\
&\leq \sum_{\textsc{Vf}} \sum_{\textsc{Enc}} \Pr\left[E_9(\textsc{Vf}, \textsc{Enc}) \,\middle|\, \overline{E_8} \wedge E_{10}(\textsc{Vf}, \textsc{Enc})\right] \\
&\qquad \cdot \Pr\left[E_{10}(\textsc{Vf}, \textsc{Enc}) \,\middle|\, \overline{E_8}\right].
\end{aligned} \tag{23}$$

When querying $\textsc{Reveal}$, the users' keys are uniformly sampled, independently from any previous queries and parameters. Thus, for any pair $\textsc{Vf}, \textsc{Enc}$, the probability that $K_i = K_j$ is $\frac{1}{2^k}$. This event is also independent from $\overline{E_8}$ and $E_{10}(\textsc{Vf}, \textsc{Enc})$, which both depend only on parameters already fixed before sampling the users' keys.

Thus

$$\Pr\left[E_9(\textsc{Vf}, \textsc{Enc}) \;\middle|\; \overline{E_8} \wedge E_{10}(\textsc{Vf}, \textsc{Enc})\right] = \Pr[E_9(\textsc{Vf}, \textsc{Enc})] = \frac{1}{2^k}. \tag{24}$$

Note that conditioned on event $\overline{E_8}$, for any verification query $\textsc{Vf}$ there are strictly less than m_8 encryption queries $\textsc{Enc}$ with the same nonce N and an associated V_0' such that $[V_0']^s \overset{(t)}{+} H_r(AD, C) = T$, where $r = [V_0']^r$, (AD, C, T) are the values associated to $\textsc{Vf}$, and V_0' is the value associated to $\textsc{Enc}$. Hence, for a fix verification query $\textsc{Vf}$, there are strictly less than m_8 encryption queries $\textsc{Enc}$ such that $\Pr\left[E_{10}(\textsc{Vf}, \textsc{Enc}) \;\middle|\; \overline{E_8}\right]$ is not zero. Thus

$$\sum_{\textsc{Vf}} \sum_{\textsc{Enc}} \Pr\left[E_{10}(\textsc{Vf}, \textsc{Enc}) \;\middle|\; \overline{E_8}\right] < \sum_{\textsc{Vf}} m_8 \le q_v \cdot m_8. \tag{25}$$

Moreover,

$$m_8 = \left\lceil \frac{4t}{\max(1, 2t - \log_2(d))} \right\rceil \le 2 \cdot \left\lceil \frac{2t}{\max(1, 2t - \log_2(2d))} \right\rceil = 2 \cdot \overline{2t}^d$$

and in combination with (22), (23), (24) and (25), we obtain

$$\Pr[\mathcal{T}_{\text{ideal}} \in \text{Bad}_{6\text{-}2}] \le \frac{q_v \cdot 2 \cdot \overline{2t}^d}{2^k} + \frac{1}{2^{2t}}. \tag{26}$$

Combining the two cases (21) and (26) using a union bound, we obtain the following result:

$$\Pr[\mathcal{T}_{\text{ideal}} \in \text{Bad}_6] \le \frac{q_v}{2^t} + \frac{q_v \cdot c \cdot \ell_m}{2^t} + \frac{q_v \cdot 2 \cdot \overline{2t}^d}{2^k} + \frac{1}{2^{2t}}. \qquad \square$$

C ATTACKS AND PROOFS OF LOWER BOUNDS

In this section, we give the details of the two most interesting attacks against ChaCha20-Poly1305 among the ones briefly described in Section 7.1.

C.1 Proof of Proposition 7.1 (Forgery Attack)

We refer to [22] for an introduction to lattices and the terms used in the following proof. We also note that the complexity of this attack may be exponential when using deterministic SVP algorithms (or slightly better when using SVP approximation algorithms), making the attack impractical, especially for large ℓ_m, but still valid in the model we use here, where the adversary is computationally unbounded.

We define an adversary $\mathcal{A}$ that makes one encryption query and q_v verification queries, attempting each time a forgery. If a verification query returns true, then it outputs 1. As a forgery is impossible in the ideal world, then

$$\Pr\left[\mathcal{A}^{G^{\text{Ideal-muAE}}_{\text{ChaCha20-Poly1305}[\pi]}} \Rightarrow 1\right] = 0.$$

Thus

$$\text{Adv}^{\text{muAE}}_{\text{ChaCha20-Poly1305}[\pi]}(\mathcal{A}) = \Pr\left[\mathcal{A}^{G^{\text{Real-muAE}}_{\text{ChaCha20-Poly1305}[\pi]}} \Rightarrow 1\right].$$

Let $\mathcal{A}$ be an adversary against ChaCha20-Poly1305 and $p = 2^{130} - 5$. Note that $p^{\frac{3}{4}} < 2^t < p$. In the following, we will use the integer representation for t-bit strings. The adversary $\mathcal{A}$ makes as its first query an encryption query $\textsc{Enc}(i, N, AD, M)$ with nonce N and inputs of exactly ℓ_m blocks, where $M = \varepsilon$ is an empty string and $AD = AD_1 \| \cdots \| AD_{\ell_m - 1}$, where $|AD_i| = t$ and $AD_i = 2^{t-1}$. It receives a tag T as answer. We denote by r_N and s_N the hash key (after clamping) and the blinding value, associated to the nonce N, i.e., $r_N \| s_N = \text{CC\_block}(K_i, N, 0)[1{:}2t]$. Let $\mathcal{R}$ denote the set of all possible hash keys for Poly1305_Mac (after clamping). Let $\mathcal{R}_1, \ldots, \mathcal{R}_{q_v} \subset \mathcal{R}$ be q_v disjoint sets, each of $3\left\lfloor \frac{\ell_m - 1}{4} \right\rfloor$ hash keys. For each $\mathcal{R}_j$, we will construct a forgery polynomial and the associated pair AD', T, that will be a valid forgery with nonce N, if r_N is in $\mathcal{R}_j$. Let

$$R(x, \mathcal{R}_j) = x^2 \cdot g(x) \cdot \prod_{r \in \mathcal{R}_j} (x - r) \mod p$$

be a polynomial with the set $\mathcal{R}_j$ as roots and where $g(x)$ is a non zero polynomial of degree at most $\left\lfloor \frac{\ell_m - 1}{4} \right\rfloor - 1$ defined such that when we rewrite $R(x, \mathcal{R}_j)$ as $\sum_{i=2}^{\ell_m} a_{\ell_m - i + 1} \cdot x^i \mod p$ (or equivalently as $\sum_{i=2}^{\ell_m} a'_{i-2} \cdot x^i \mod p$ later in the proof), the coefficients a_i satisfies $(2^{t-1} + a_i \mod p) < 2^t$ for all $i < \ell_m$. Note that we consider the coefficients a_i (and a_i') as integers that can take negative values. We will show shortly how to construct such a polynomial $g(x)$ and will assume for now the existence of a $g(x)$ ensuring that $(2^{t-1} + a_i \mod p) < 2^t$ for all $i < \ell_m$. Recall that

$$H_x(AD, \varepsilon) = (c_1 x^{\ell_m} + \cdots + c_{\ell_m - 1} x^2 + c_{\ell_m} x^1 \mod p) \mod 2^t$$

where $c_i = 2^t + AD_i = 2^t + 2^{t-1}$ (i.e., the integer representation of $AD_i \| 1$) for $i < \ell_m$, and c_{ℓ_m} is the integer representation of $\text{len}(AD) \| \text{len}(\varepsilon) \| 1$. Note that $H_{r_N}(AD, \varepsilon) \overset{(t)}{+} s_N = T$ and if r_N is in $\mathcal{R}_j$, then $R(r_N, \mathcal{R}_j) = 0$. We can construct the string AD' and its associated polynomial as $AD' = AD_1' \| \cdots \| AD_{\ell_m - 1}'$, where $|AD_i'| = t$, $AD_i' = (2^{t-1} + a_i \mod p)$, and

$$H_x(AD', \varepsilon) = (c_1' x^{\ell_m} + \cdots + c_{\ell_m - 1}' x^2 + c_{\ell_m}' x^1 \mod p) \mod 2^t,$$

where $c_i' = 2^t + AD_i' = 2^t + (2^{t-1} + a_i \mod p)$ (i.e., the integer representation of $AD_i' \| 1$) for $i < \ell_m$, and c_{ℓ_m}' is the integer representation of $\text{len}(AD') \| \text{len}(\varepsilon) \| 1$. Thanks to our assumption, $(2^{t-1} + a_i \mod p) < 2^t$ and therefore, the string AD' and its associated polynomial is well defined. Moreover, as we constructed our polynomial so that $c_i' = (c_i + a_i \mod p)$ for $i < \ell_m$ and $c_{\ell_m}' = c_{\ell_m}$, then

$$H_x(AD', \varepsilon) = (c_1 x^{\ell_m} + \cdots + c_{\ell_m} x^1 + R(x, \mathcal{R}_j) \mod p) \mod 2^t.$$

Furthermore, since when r_N is in $\mathcal{R}_j$, we obtain $R(r_N, \mathcal{R}_j) = 0$, it follows that, if r_N is in $\mathcal{R}_j$, then:

$$H_{r_N}(AD', \varepsilon) \overset{(t)}{+} s_N = H_{r_N}(AD, \varepsilon) \overset{(t)}{+} s_N = T.$$

Hence, $\textsc{Vf}(i, N, AD', T)$ is a valid forgery query if r_N is in $\mathcal{R}_j$. As r_N is sampled uniformly and independently from the choices of $\mathcal{R}_j$, the probability that $r_N \in \mathcal{R}_j$ is $3\left\lfloor \frac{\ell_m - 1}{4} \right\rfloor \cdot \frac{1}{2^{t-22}} \ge \frac{3(\ell_m - 5)}{2^{t-20}}$. Thus, the probability that $\textsc{Vf}(i, N, AD', T)$ is a valid forgery query is at least $\frac{3(\ell_m - 5)}{2^{t-20}}$. The adversary $\mathcal{A}$ makes q_v such verification queries, one for each $\mathcal{R}_j$. As the $\mathcal{R}_j$ are disjoint, the probability that at least one of these q_v verification queries is a valid forgery is at least $\frac{3q_v(\ell_m-5)}{2^{t-20}}$. Hence the bound in the proposition follows.

We now show how to construct a polynomial $g(x)$ so that the coefficients a'_i of the polynomial $R(x, \mathcal{R}_j) = x^2 \cdot g(x) \cdot \prod_{r\in\mathcal{R}_j}(x-r) \mod p = \sum_{i=2}^{\ell_m} a'_{i-2} \cdot x^i \mod p$ satisfies $(2^{t-1} + a'_i \mod p) < 2^t$ for all $i < \ell_m - 1$. Let $d = \left\lfloor \frac{\ell_m - 1}{4} \right\rfloor$, we can rewrite $\prod_{r\in\mathcal{R}_j}(x-r)$ as $x^{3d} + \sum_{i=0}^{3d-1} b_i \cdot x^i$ and $g(x)$ as $\sum_{j=0}^{d-1} g_j \cdot x^j$. Then

$$g(x) \cdot \prod_{r\in\mathcal{R}_j}(x-r) \mod p = g_0 \cdot \left(x^{3d} + \sum_{i=0}^{3d-1} b_i \cdot x^i\right) + g_1 x^1 \cdot \left(x^{3d} + \sum_{i=0}^{3d-1} b_i \cdot x^i\right) + \cdots + g_{d-1}x^{d-1} \cdot \left(x^{3d} + \sum_{i=0}^{3d-1} b_i \cdot x^i\right) \mod p.$$

We define how to construct the coefficients g_i (and therefore $g(x)$), by using a lattice generated by the d polynomials

$$x^j \cdot \left(x^{3d} + \sum_{i=0}^{3d-1} b_i \cdot x^i\right)$$

where $j < d$. Let $\mathcal{L}$ be the lattice generated by the rows of the following matrix

$$\begin{bmatrix} 1 & b_{3d-1} & b_{3d-2} & \cdots & b_0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & b_{3d-1} & b_{3d-2} & \cdots & b_0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 & b_{3d-1} & b_{3d-2} & \cdots & b_0 \\ & & & & p \cdot I_{4d-1} & & & & \end{bmatrix}$$

where, for $j \le d$, the j-th row represents the polynomial $x^{d-j} \cdot \left(x^{3d} + \sum_{i=0}^{3d-1} b_i \cdot x^i\right)$ (and the columns represents the coefficients of this polynomial) and where the last $4d-1$ rows represents the reductions modulo p of the coefficients, i.e., $p \cdot I_{4d-1}$ where I_{4d-1} is the identity matrix of size $4d-1$. We denote by $(a'_{4d-2}, \cdots, a'_0)$ the shortest vector in this lattice. It arises as a linear combination of the rows of the previous matrix and as such, defines the coefficients g_i (that we don't need to know). More importantly, $R(x, \mathcal{R}_j) = \sum_{i=2}^{\ell_m} a'_{i-2} \cdot x^i \mod p$, where we set $a'_i = 0$ for $i > 4d-2$. We are left to show that $(2^{t-1} + a'_i \mod p) < 2^t$ for all $i \le 4d-2$. We can compute a basis (and determinant) of the lattice $\mathcal{L}$, by looking at the row echelon form of the previous matrix:

$$\begin{bmatrix} I_d & A \\ 0 & p \cdot I_{3d} \end{bmatrix}$$

where A is a $d \times 3d$ matrix. From this matrix, we can observe that the rank of $\mathcal{L}$ is $4d$, and its determinant is p^{3d}. Minkowski's theorem with the infinity norm yields that, for the shortest vector in $\mathcal{L}$, $|a'_i| \le p^{\frac{3d}{4d}}$ for all $i \le 4d-2$. Thus $|a'_i| \le p^{\frac{3}{4}} < 2^{t-1}$ for all $i \le 4d-2$. Hence $-2^{t-1} < a'_i < 2^{t-1}$ and $(2^{t-1} + a'_i \mod p) < 2^t$ for all $i \le 4d-2$.

Therefore, as the adversary $\mathcal{A}$ is unbounded, it can compute the smallest vector of $\mathcal{L}$ and construct $R(x, \mathcal{R}_j) = \sum_{i=2}^{\ell_m} a'_{i-2} \cdot x^i \mod p$ such that $(2^{t-1} + a'_i \mod p) < 2^t$ for all $i < \ell_m - 1$. It then proceeds to the above-described attack. Note that [2] provides an algorithm for solving SVP with the infinity norm used in the proof.

C.2 Proof of Proposition 7.6

The following proof is an adaptation of the lower bound of [13] to a distinguishing attack for ChaCha20-Poly1305. Fix a nonce N. Let $\mathcal{A}$ be an adversary that query the encryption of a total number of σ_e blocks across different users. For each of the σ_e block queried, we denote by W_j the $n-k$ bit string $[C_i \oplus M_i]^{K\text{-}} \overset{(32)}{-} (Z\|i\|N)$, where C_i is the encryption of M_i (the i-th block of a message) with nonce N. The adversary outputs 1, if $W_j \neq W_{j'}$ for all $j \neq j'$.

In the ideal world, the strings W_j are independent and uniformly distributed. Thus, the adversary outputs 1 only if a string doesn't repeat, i.e.,

$$\Pr\left[\mathcal{A}^{G^{\text{Ideal-muAE}}_{\text{ChaCha20-Poly1305}[\pi]}} \Rightarrow 1\right] = \prod_{i=0}^{\sigma_e-1}\left(1 - \frac{i}{2^{n-k}}\right).$$

Using Weierstrass product inequality,

$$\prod_{i=0}^{\sigma_e-1}\left(1 - \frac{i}{2^{n-k}}\right) \ge 1 - \sum_{i=0}^{\sigma_e-1} \frac{i}{2^{n-k}} = 1 - \frac{1}{2} \cdot \frac{\sigma_e(\sigma_e-1)}{2^{n-k}}.$$

As $\sigma_e \le 2^{\frac{n-k}{2}}$, we obtain

$$\Pr\left[\mathcal{A}^{G^{\text{Ideal-muAE}}_{\text{ChaCha20-Poly1305}[\pi]}} \Rightarrow 1\right] \ge \frac{1}{2}. \qquad (27)$$

Recall that in the real world, $W_j = [\pi(Z\|K\|i\|N)]^{K\text{-}}$. Thus, if $W_j \neq W_{j'}$, then the two corresponding calls to the underlying ideal permutation are distinct. As $W_j \neq W_{j'}$ for all $j \neq j'$, the adversary outputs 1 only if it makes σ_e distinct calls to the underlying ideal permutation that doesn't collide on the first $n-k$ bit with the previous queries,

$$\begin{aligned} \Pr\left[\mathcal{A}^{G^{\text{Real-muAE}}_{\text{ChaCha20-Poly1305}[\pi]}} \Rightarrow 1\right] &= \prod_{i=0}^{\sigma_e-1}\left(1 - \frac{i(2^k-1)}{2^n-i}\right) \\ &= \prod_{i=0}^{\sigma_e-1}\left(\frac{2^n - i\cdot 2^k}{2^n-i}\right) = \prod_{i=0}^{\sigma_e-1}\left(1 - \frac{i}{2^{n-k}}\right) \cdot \prod_{i=0}^{\sigma_e-1}\left(\frac{2^n}{2^n-i}\right) \\ &= \Pr\left[\mathcal{A}^{G^{\text{Ideal-muAE}}_{\text{ChaCha20-Poly1305}[\pi]}} \Rightarrow 1\right] \cdot \prod_{i=0}^{\sigma_e-1}\left(\frac{2^n}{2^n-i}\right). \end{aligned}$$

Moreover, as $(2^n - i) \cdot (2^n + i) \le 2^{2n}$, then

$$\prod_{i=0}^{\sigma_e-1}\left(\frac{2^n}{2^n-i}\right) \ge \prod_{i=0}^{\sigma_e-1}\left(\frac{2^n+i}{2^n}\right) = \prod_{i=0}^{\sigma_e-1}\left(1 + \frac{i}{2^n}\right).$$

Using Weierstrass product inequality,

$$\prod_{i=0}^{\sigma_e-1}\left(1 + \frac{i}{2^n}\right) \ge 1 + \sum_{i=0}^{\sigma_e-1} \frac{i}{2^n} = 1 + \frac{\sigma_e(\sigma_e-1)}{2^{n+1}}.$$

Thus, $\Pr\left[\mathcal{A}^{G^{\text{Real-muAE}}_{\text{ChaCha20-Poly1305}[\pi]}} \Rightarrow 1\right]$ is greater or equal to

$$\Pr\left[\mathcal{A}^{G^{\text{Ideal-muAE}}_{\text{ChaCha20-Poly1305}[\pi]}} \Rightarrow 1\right] \cdot \left(1 + \frac{\sigma_e(\sigma_e-1)}{2^{n+1}}\right).$$

Therefore,

$$\mathsf{Adv}^{\text{muAE}}_{\text{ChaCha20-Poly1305}[\pi]}(\mathcal{A}) \geq \frac{\sigma_e(\sigma_e - 1)}{2^{n+1}} \cdot \Pr\left[\mathcal{A}^{G^{\text{Ideal-muAE}}_{\text{ChaCha20-Poly1305}[\pi]}} \Rightarrow 1\right].$$

The final bound is obtained by applying (27) to this inequality.

D PROOF FOR NONCE RANDOMIZATION (XN TRANSFORM)

The proof follows the same strategy as for [15, Theorem 4.2], but then applies our generalized bound from Lemma A.1. The d-repeating reduction $\mathcal{B}$ samples nonce randomizer values J_i for each user i, used to derive the effective nonce $N^* = N \oplus J$ in encryption and verification queries forwarded to its multi-user AE game. It keeps a counter for any used nonce value N^* in encryption queries, counting the number of users for which $\mathcal{B}$ queried this effective nonce to its encryption oracles. When any such counter reaches $d + 1$, $\mathcal{B}$ sets a bad flag bad, stops immediately, and outputs 1 (ensuring that $\mathcal{B}$ is d-repeating[4]); otherwise, $\mathcal{B}$ relays the bit output by $\mathcal{A}$.

In the real world,

$$\Pr\left[\mathcal{B}^{G^{\text{Real-muAE}}_{\Pi[\pi]}} \Rightarrow 1\right] \geq \Pr\left[\mathcal{A}^{G^{\text{Real-muAE}}_{\Pi^*[\pi]}} \Rightarrow 1\right],$$

as $\mathcal{B}$ either outputs 1 or repeats $\mathcal{A}$'s output. Furthermore, $\mathcal{B}$ simulates the ideal world for $\mathcal{A}$ perfectly until the bad flag is set, by the identical-until-bad lemma [3],

$$\Pr\left[\mathcal{B}^{G^{\text{Ideal-muAE}}_{\Pi[\pi]}} \Rightarrow 1\right] \leq \Pr\left[\mathcal{A}^{G^{\text{Ideal-muAE}}_{\Pi^*[\pi]}} \Rightarrow 1\right] + \Pr\left[\mathcal{B} \text{ sets bad in } G^{\text{Ideal-muAE}}_{\Pi[\pi]}\right].$$

Overall, this yields

$$\mathsf{Adv}^{\text{muAE}}_{\Pi[\pi]}(\mathcal{B}) \geq \mathsf{Adv}^{\text{muAE}}_{\Pi^*[\pi]}(\mathcal{A}) - \Pr\left[\mathcal{B} \text{ sets bad in } G^{\text{Ideal-muAE}}_{\Pi[\pi]}\right],$$

so it remains to bound the probability of $\mathcal{B}$ setting bad in the ideal world game.

In $G^{\text{Ideal-muAE}}_{\Pi[\pi]}$, the oracles answer independently of the queried effective nonces and $\mathcal{B}$'s randomizer values J_i. The bad flag is set when across all encryption (and, for the strongly d-repeating case, also verification) queries, an effective nonce derived by $\mathcal{B}$ is used across $d + 1$ users. We can view each effective nonce N^* queried to a user i, as throwing a ball i into one of 2^b bins, where the bin represents the nonce queried. In total, we throw at most q balls, where $q = q_e$ for the d-repeating case and $q = q_e + q_v$ for the strongly d-repeating case. With this perspective, if the flag bad is set, then there exist a set of $d + 1$ balls corresponding to $d + 1$ distinct user in the same bin. Thus we can bound the probability that bad is set by the probability that there exist a set of $d + 1$ balls corresponding to $d + 1$ distinct user in the same bin.

For distinct users, the ball throws are independent and uniformly random distributed through the randomizer values J_i. The probability for any set of $d + 1$ balls of distinct users to hit the same bin is hence 2^{-bd}. Throwing up to q balls means there are at most $\binom{q}{d+1}$ sets of $d + 1$ balls. So, by the union bound,

$$\Pr\left[\mathcal{B} \text{ sets bad in } G^{\text{Ideal-muAE}}_{\Pi[\pi]}\right] \leq \binom{q}{d+1} \cdot 2^{-bd}.$$

We can now apply Case 4 of Lemma A.1 with $m = d+1, Q = q, B = 2^{-b}$, $\widetilde{m} = \delta b$, and $t = 2$, to upper-bound the bad-event probability that any bin contains $m = d + 1$ or more balls at $t^{-\widetilde{m}} = 2^{-\delta b}$ as claimed. For $q \leq 2^b \cdot \frac{(\delta+1)\cdot b}{3}$, we obtain

$$d = m - 1 = \left\lceil \frac{(\delta + 1) \cdot b}{\max(1, b - \log_2(q))} \right\rceil - 1. \quad \square$$

[4]For the *strongly* d-repeating case, $\mathcal{B}$ counts the number of users using a particular effective nonce N^* across both its encryption *and* verification queries. Aborting when a counter reaches $d + 1$ then ensures $\mathcal{B}$ is strongly d-repeating.