# Continuum Modelling of Freeway Traffic Flows in the Era of Connected and Automated Vehicles: A Critical Perspective and Research Needs

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## Abstract

This paper provides a comprehensive review of continuum traffic flow models. A comprehensive review of models for conventional traffic is presented that classifies the models into various families regarding their derivation bases. Previous discussions and debates over the performance of models developed for conventional traffic are covered in detail, and wherever applicable, new insights are provided on the properties and interpretations of the existing models. A review of the recent attempts at incorporating connected and automated vehicles (CAVs) traffic flow into the continuum framework is also conducted. The paper also analyses the strengths, limitations, and properties of the existing model families for CAV traffic flow. Research gaps and the issues inherent to CAVs are highlighted, and future directions in the era of connected and automated vehicles are discussed.

**Keywords:** continuum model; macroscopic model; traffic flow; fundamental diagram; connected and automated vehicles; driver behaviour

## 1. Introduction

Continuum models of traffic flow are hyperbolic systems and predict the evolution of traffic states based only on the initial and boundary conditions (Daganzo, 1995b). These models are also known as macroscopic models since they can study traffic flow's collective behaviour at an aggregated level using fluid-like state variables such as density and flow. Continuum models require less calibration effort and are computationally faster than microscopic models, and moreover, their input data are already consistent with the aggregated data from the loop detectors (Kotsialos et al., 2002; Papageorgiou, 1998).

Since the seminal contributions by Lighthill and Whitham (1955) and Richards (1956), numerous developments have been made on the subject with respect to the theoretical considerations and empirical observations of real-world traffic. To be considered as desirable, models must be theoretically consistent, behaviourally sound, and capable of explaining/predicting a wide range of empirical observations, while having minimal practical and implementation issues (Helbing, 2001). However, debates and discussions are still ongoing over existing models' performances and properties, and to the best of the authors' knowledge, no existing model has yet built a strong consensus amongst researchers regarding all aspects discussed above.

With respect to theoretical consistency, in some cases, such as the issues around faster-than-traffic waves (Daganzo, 1995b), researchers' viewpoints are diverging into separate schools of thought (Helbing, 2009; Helbing and Johansson, 2009; Zhang, 2009). On the other hand, much less has been understood about the usefulness of the proposed models against real-world traffic, since compared to the extent of literature, the empirical studies of continuum models are extremely limited. The outcomes of recent real-world evaluations of the models (Kontorinaki et al., 2017; Mohammadian et al., 2021c; Spiliopoulou et al., 2014) are conflicting regarding some aspects, such as capacity drop. Furthermore, it has been shown that some models with supposed theoretical consistency (e.g., regarding anisotropy property (Daganzo, 1995b)) can exhibit significant behavioural issues of the same kind when implemented on real-world traffic (Mohammadian et al., 2021c).

Issues applicable to the existing models should be revisited in the context of connected and automated vehicles (CAVs), which will share the road with conventional vehicles in the upcoming decades. Properties of such mixed traffic conditions will be much more complex because of the more pronounced role of human factors and the specific aspects of CAVs, such as information/connectivity, automation, lane choice. Therefore, the underlying assumptions of the existing models for conventional traffic may no longer be valid to describe the properties of the mixed traffic condition regarding driver behaviour and physics of traffic. There is a strong need to develop new continuum models as we transition into the era of CAVs.

A comprehensive review of continuum models is constructive in identifying the potentials and limitations of existing models as well as revealing the research needs for the development of new models. While there are already several instructive reviews on the subject (Bellomo and Dogbe, 2011; Ferrara et al., 2018 chapters 3 & 4; Helbing, 2001; Hoogendoorn and Bovy, 2001; Piccoli and Tosin, 2009; Seo et al., 2017; Treiber and Kesting, 2013 chapters 6 to 9; van Wageningen-Kessels et al., 2015; Zhang et al., 2001), each one has its own specific limitations. Some notable review efforts date back to twenty years ago, and there has been substantial developments in the subject in the last two decades. Table 1 summarizes the strengths and limitations of more recent reviews.

Table 1- Summary of strengths and limitations of existing reviews on continuum traffic flow models

| Review | Strengths | Limitations |
|---|---|---|
| **Piccoli & Tosin** (2009) | • Contains adequate coverage of non-equilibrium model families<br>• Contains in-depth discussion of the mathematical properties of models | • Limited coverage and discussion on equilibrium models<br>• Limited discussions on real-world applications<br>• Limited discussions about driving behaviours and human factors |
| **Bellomo & Dogbe** (2011) | • Provides a robust mathematical discussion of non-equilibrium models<br>• Discusses extended continuum models for pedestrian traffic | • Despite covering the empirical observations of traffic flow, contains limited discussions about practical implications of models |
| **Treiber & Kesting** (2013) Chapters 6 to 9 | • Provides strong physical discussions about derivation bases of models<br>• Comprehensively discusses the applications of models on real-world traffic scenarios<br>• Discusses the physical properties of models regarding how they treat driving behaviors<br>• Provide some discussions on the Lagrangian description of the continuum framework<br>• Contains detailed discussion on numerical solutions of models | • Focuses only on a few continuum models<br>• Ignores the class of non-equilibrium models devoid of faster-than traffic waves<br>• Does not discuss previous critiques and debates over models in detail |
| **van Wageningen-Kessels et al.** (2015) | • Contains comprehensive coverage of multi-class equilibrium models<br>• Provides a historical description of the relationships between different model families<br>• Discusses Lagrangian variations of models | • Discusses non-equilibrium models briefly<br>• Issues with modelling frameworks are not investigated and covered in detail |
| **Seo** et al. (2017) | • Discusses the real-world application of models and their connection to data-driven approaches for traffic prediction<br>• Discusses the relationship between the nature of traffic data and traffic models | • Limited coverage of major continuum model families<br>• Does not discuss the issues with the models in detail |
| **Ferrara** et al. Chapters 3 & 4 | • Provides a wide coverage of continuum models<br>• Provides valuable insight into the real-world application of models for control purposes<br>• Contains rich discussion on optimization and calibration models | • Does not discuss the theoretical and implementation issues with the models in detail |

Besides the points listed in Table 1, none of the existing efforts have provided a comprehensive review of the recent developments in modelling CAV traffic. As such, the critical issues inherent to modelling mixed CAV traffic flows have not been explored in the context of continuum modelling.

This paper attempts to bridge these gaps by putting CAVs as the main ingredient of future directions. Table 2 presents the structure of this review paper and elaborates the primary focus of each section and their linkages to CAVs. To this end, first, a comprehensive review of different modelling families for conventional traffic is presented in an effort to address the shortcomings of previous review papers as identified in Table 1. In particular, the previous discussions and debates about these models are covered in detail. Wherever applicable, we provide new insights on the properties and interpretations of the existing continuum models for traditional traffic while also discussing the implications for the CAV traffic. Next, the properties of traffic flow in a connected and automated environment are discussed, and the paper reviews some existing efforts for continuum modelling of CAV traffic flow. Then, we revisit the different modelling frameworks and their inherent issues both for the conventional and CAV traffic. In doing so, this paper also attempts to clarify some confusions and controversies in the literature regarding several empirical and

theoretical aspects of traffic flow, e.g., the effectiveness and consistency of several model equilibrium families for complex traffic phenomena (e.g., capacity drop, hysteresis, etc), the controversies around faster-than-traffic waves (Helbing, 2009; Helbing and Johansson, 2009; Zhang, 2009), and the issues that arises both in the absence and presence of the relaxation terms in the non-equilibrium models. Finally, we review the existing models of CAV traffic flow, discuss the issues arising for modelling mixed CAVs, and the research needs in the era of connected and automated vehicles.

Table 2- Structure of this review paper

| Section | Primary focus and specific linkages to the CAV traffic flows |
|---|---|
| Section 2 | • Review of major existing model families for conventional traffic flows within the equilibrium and non-equilibrium frameworks<br>• Covering theoretical and empirical considerations underlying each model family<br>• Providing new interpretations and perspective about existing models wherever applicable<br>• Revisiting specific models' potentials and limitations for CAV traffic flows |
| Section 3 | • Review of major continuum models developed for CAV traffic flows |
| Section 4 | • Revisiting specific challenges of the equilibrium and non-equilibrium frameworks and their applicability for modelling traffic flow of CAVs<br>• Critical perspectives on the existing models developed for CAVs with respect to empirical and theoretical considerations as well as human factors<br>• Elaborating on the research gaps and drawing future directions of continuum modelling in the era of CAVs |

The paper is organized as follows. Section 2 comprehensively reviews the existing continuum models for conventional traffic and revisits the potentials of existing models for CAV traffic flows wherever applicable. Section 3 discusses the properties of CAVs and reviews existing models developed for CAV traffic flow. Section 4 presents critical perspectives on the issues applicable to the existing continuum models while highlighting the research gaps and discussing research needs in the era of CAVs. Section 5 presents some concluding remarks.

## 2. Review of continuum models for conventional traffic

This section provides a comprehensive review of continuum models developed for traffic flows of conventional vehicles. Our primary focus is on the vehicular interactions at the freeway level, and an in-depth discussion of network traffic flow theory and node models are beyond the scope of this paper (refer to Garavello and Piccoli, 2006b; Jin, 2021 for a review of concepts, definitions, and modeling frameworks).

Models are reviewed and classified into distinct families based on their physical properties and the underlying premises. Wherever applicable, this section provides new insights into the interpretations of the existing models and sheds more light on the potential issues and challenges applicable to the existing models. As well, this section occasionally revisits the potentials and challenges of some specific models or model families for modelling traffic flow of CAVs.

All continuum models can be categorized into two broad families of equilibrium and non-equilibrium models. In the equilibrium models, traffic flow dynamics are primarily reliant on the law of conservation of vehicles in a network, and the so-called fundamental diagram (FD) plays an integral part. Equilibrium models, however, differ in how they define and approach the concept of the FD. A secondary classification system is therefore utilized within this group to compare models based on their rationale towards the FD.

The second framework includes non-equilibrium (NE) models. All models in this group consider additional dynamic mechanisms for speed adaptation in conjunction with the law of conservation of vehicles. Some of these models are phenomenological, meaning that their governing equations have been proposed mainly based on the similarities between fluid flows and traffic flows. Others are derived from either the car-following (CF) relationships or the gas-kinetic theory (GKT). This classification is also adopted in this paper, and models within each family are discussed separately. Some NE models can also be categorized based on secondary criteria (e.g., number of vehicle classes). Further classification is, however, avoided to prevent overlap between the three groups of models.

The models are discussed based on their underlying motivations and their connection with other models in their families. Therefore, the review does not necessarily follow a chronological order. To save space, mathematical formulations of the models are presented only for some of the models.

### 2.1. Equilibrium Models for freeways

This section reviews the major equilibrium models of freeway traffic flows. We start with the kinematic wave theory and elaborate, in some detail, on its mathematical properties, underlying physical considerations, and practical shortcomings concerning empirical observations. Such elaborations are then revisited when reviewing various equilibrium models, and later on, non-equilibrium models, wherever applicable.

#### 2.1.1. The kinematic wavey theory and the cell transmission model (CTM)

Lighthill and Whitham (1955) and Richards (1956) independently proposed the first continuum model (known as the LWR model for short) based on the concept of mass conservation from fluid dynamics. For a homogeneous section of a road, the differential form of the conservation equation is:

$$\partial\rho(x,t)/\partial t + \partial Q(x,t)/\partial x = 0 \tag{2.1}$$

where $\rho(x,t)$ is density ($veh/km$) and $Q(x,t)$ is flow ($veh/hr$) at position $x$ and time $t$. These two parameters are linked to traffic speed ($V, km/hr$) through the hydrodynamic relationship ($Q = \rho V$). Eq. 2.1 is a non-linear hyperbolic partial differential (PDE) equation that treats density and flow as kinematic waves. The model is, therefore, sometimes called the kinematic wave ($KW$) or hydrodynamic model. The LWR model assumes that speed is an explicit function of density ($V = V_e(\rho)$), and several functions have been proposed for the equilibrium speed $V_e(\rho)$ (e.g., del Castillo, 2012; Del Castillo and Benítez, 1995; Drake et al., 1967; Greenberg, 1959). The essential physical properties of the LWR model can be analytically studied by substituting $V_e(\rho)$ in Eq 2.1, and rewriting it in quasi-linear form as:

$$\partial\rho/\partial t + Q_e'(\rho)\, \partial\rho/\partial x = 0 \tag{2.2}$$

where $Q_e'(\rho) = V_e(\rho) + \rho V_e'(\rho)$ is the speed of the information propagation (also called the propagation velocity or characteristic speed[1]) and represents the propagation velocity of smooth variations in the density profile. Since $V_e'(\rho) \leq 0$, the propagation velocity is always less than the traffic speed, and all waves propagate backward from the driver's point of view (Treiber and Kesting, 2013).

As the LWR model is a hyperbolic wave equation, its solution can be studied by analyzing the Riemann problems (Lebacque, 1996), i.e., problems involving a jump in otherwise continuous initial conditions (LeVeque, 2002a). In this case, all possible Riemann problems can be categorized into finite groups such that for each group, the solution structure remains similar regarding the type of waves arising at the initial discontinuity (e.g., shockwave or rarefaction wave) and their direction of propagation. The functional form of the fundamental diagram significantly affects the structure of the solutions arising for the Riemann problems.

If the fundamental diagram is strictly concave, the LWR model produces shockwaves in the deceleration situations and rarefaction waves in the acceleration situation. From a behavioural perspective, a deceleration shockwave dictates that vehicles going through the discontinuity adapt their speed to the downstream traffic condition instantaneously, i.e., the deceleration rate is unbounded. On the other hand, a rarefaction acceleration means that vehicles perform the speed adaptation gradually over a certain range, and thus, the initial discontinuity gets dispersed over time while expanding in space.

A strictly concave $FD$ can result in several unrealistic performances of the LWR model in terms of driving behaviours. First, a strictly concave FD requires that vehicles always adapt their speed to the small changes in density due to the genuine nonlinearity of the characteristic speed (Zhang, 2001), and this behaviour is unrealistic for small density. For example, empirical observations suggest that in free-flow traffic, density variations do not affect drivers' speed (see e.g., Treiber and Kesting, 2013 chapter 4). Second, a strictly concave $FD$ results in a characteristic speed strictly decreasing with respect to density, whereas several empirical observations have found that beyond certain congested states, congestion propagation speed is often relatively insensitive to traffic density (e.g., Edie and Baverez, 1967; Treiber and Kesting, 2013 chapter 8). Furthermore, the rarefaction waves generated by a strictly concave $FD$ can cover a large space, inidicating that vehicles adapt their speed in an unrealistically long transition zone, which is unrealitic in some circumstances. For example, vehicles leaving a highly congested zone to enter an empty road often accelerate in a transition zone with a length of a few hunderd meters (Treiber and Kesting, 2013 chapter 8).

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[1] In the rest of the paper, these terms may be used interchangeably.

There are two alternatives to a strictly concave FD, i.e., concave-convex FDs (e.g., the FDs used by Kühne, 1984; Messmer and Papageorgiou, 1990), where at some point in the congestion region, an inflection point appears, and afterward, the magnitude of the characteristic speed gradually decreases to that of the highly congested traffic. While such concave-convex FDs may sometimes even provide a better fit against empirical observations of flow-density plots (Li and Zhang, 2011), they result in unrealistic and 'non-physical' driving behaviours from the LWR model (Ansorge, 1990). For example, for transitions from free flow to congested regimes, nonclassical shockwaves can appear that consist of a shockwave followed by a rarefaction wave (see LeVeque, 2002a chapter 16; Mohammadian, 2021), meaning that drivers first sharply decelerate to reach an intermediate density, and then gradually decelerate over a large space to adapt to the congested region ahead, which is unrealistic. Therefore, the concavity of the fundamental diagram is necessary for the LWR model to ensure consistency. Ansorge (1990) proved that the concavity condition of the FD automatically guarantees the solution uniqueness. Another alternative is that FD remains strictly concave in certain intermediate density ranges but non-strictly concave elsewhere (e.g., in very light traffic and beyond certain congested states) (e.g., the FDs proposed by del Castillo, 2012; Del Castillo and Benítez, 1995). These types of FDs result in more realistic behaviour and additional solution structures for the Riemann problems (e.g., accelerations waves can be shockwaves (see Zhang, 2001).

With a well-defined FD, the LWR model has several promising properties from an analytical and practical perspective including, being parsimonious as the only model parameters are those given by the FD, d) Usefulness against the basic empirical observations: the LWR model can capture aspects such as queue formation and dissolutions, propagation of shockwaves, and convection of stop-and-go waves provided they are imposed at the boundaries (Treiber and Kesting, 2013 chapter 8).

Besides, Newell (1998) showed that the LWR model could be extended to capture the moving bottleneck phenomenon, first recognized by Gazis and Herman (1992), where a single slow-moving vehicle can cause queue formation on traffic behind. The theory proposed by Newell (1998) is grounded on the assumption that from the perspective of a moving observer whose speed differ from traffic speed, the perceived relative flow can be given by an explicit function of the local traffic density. This assumption allows to model slow-moving vehicles through a fundamental diagram with smaller capacity, which makes it possible to reproduce the moving bottleneck internally using the KW theory. The same logic can be adopted to reproduce congestion formation and wave propagation for lane-drop or accident-induced bottlenecks (see e.g., Treiber and Kesting, 2013 chapter 8).

The standard LWR model describes the evolution of traffic states from a fixed-point observer standpoint (Eulerian time-space domain). It has been shown that the main concept of the LWR model, i.e., the law of conservation of vehicles, can be described through other parameters and different coordinates (Laval and Leclercq, 2013; Leclercq, 2007). Newell (1993) proposed a simplified version of the LWR model for straightforward engineering applications, where the primary state variable is cumulative flow. Daganzo (2005a) extended this concept and proposed a Lagrangian (also known as variational) formulation of the model, where the solutions are the trajectories of moving observers with the average traffic speed (Daganzo, 2005b).

Lebacque (1996) showed that the LWR model could be solved numerically with the Godunov scheme, where the computational procedure can be summarized into steps: a) identifying and categorizing all the possible Riemann problems and their corresponding solution structure b) solving the Riemann problem for each category analytically as an iterative process. The computational procedure can, therefore, depend on the functional form of the FD.

Daganzo (1994) proposed the cell transmission model (CTM), which is a simpler representation of the LWR model and involves less computational effort. CTM adopts a piecewise linear FD (aka triangular), which simplifies the nonlinear PDE of the model to a piecewise linear PDE with a singularity at the interface of the free-flow and congestion regimes. CTM is commonly presented in discretised form as:

$$\rho_k(t + \Delta t) = \rho_k(t) + \Delta t / \Delta x \, (Q_k^{up} - Q_k^{down})$$
$$Q_k(t + \Delta t) = Q_e\big(\rho_k(t + \Delta t)\big)$$

(2.3)

where $\Delta t$ is the simulation time step, and $Q_k^{up}$ and $Q_k^{down}$ are respectively the inflow and outflow from cell $k$ and are determined based on the minimum supply and demand method (Daganzo, 1994), which is a special case of the Godunov scheme (see Lebacque, 1996).

The triangular FD introduces distinct properties to the CTM model. Daganzo (2006) showed that CTM is equivalent to some elementary cellular automata models and also to the simplified car-following model by Newell (2002) in which vehicles trajectories in car-following scenarios are shifted in time and space. This is because, in a triangular FD, characteristic speeds are linearly degenerate (LD) in both free-flow and congested regimes, meaning that they are constant and insensitive to density variations ($\partial \lambda / \partial \rho = 0$). The Triangular FD resolves several behavioural issues associated with a strictly concave FD. First, congestion propagation speed becomes constant for all congest states, which is more realistic compared to a strictly concave FD. Second, in a free-flow state, all vehicles travel at the same speed and regardless of density. Consequently, initial discontinuities in the density are preserved as both the upstream and downstream states travel at the same speed, and therefore, no flow passes through the discontinuity (LeVeque, 2002b). However, with a triangular FD, both acceleration and deceleration manoeuvres are completed through shockwaves and through an unbounded rate (Zhang, 2001).

Due to its parsimony and mathematical tractability, CTM is inarguably the most widely-used model for traffic networks with several junctions. This is because due to the linear nature of CTM, the entering and exiting flow at road cells could easily be adapted concerning in- and out- flows at the network elements (e.g., merge). Daganzo (1995a) extended the model to account for various network elements such as nodes, merges, and diverges. When applying CTM to networks with multiple junctions and links, the key question is to identify how entering flows to a node are distributed among the exiting links. Jin and Zhang (2003) proposed a parsimonious model for assigning the merging flows based on the proportion of demand in the upstream links. Such a distribution scheme, together with the first-in-first-out rule (see e.g., Papageorgiou, 1990) has been utilized to develop the multi-commodity variation of CTM on network traffic, where vehicles have predefined route choices (Jin, 2012; Jin and Zhang, 2004). Although the first-in-first-out property is not guaranteed in real-world traffic due to over-taking manoeuvres (see e.g., Jin and Li, 2007 ), it has turned out that ensuring the first-in-first-out property is more realistic, and in this regard, Carey et al. (2014) proposed an analytical framework to ensure this property is always guaranteed when applying CTM on the network level.

Both CTM and the generic LWR model are based on a set of simplified assumptions about real-world traffic and suffer from a number of limitations with regard to the empirical observations, such as:

1- Transitions between any two traffic states occur on the same phase trajectory in the density-flow plane as that of the FD. Consequently, the model cannot capture the hysteresis effect, which is observable in acceleration and deceleration cycles (Treiterer and Myers, 1974).
2- Shockwaves are modelled as discontinuous transitions, which is unrealistic. In real-world traffic, vehicles decelerate in a finite spatial range with a finite deceleration rate, and thus, realistic shockwaves must have a finite range.
3- The wide scattering in the flow-density diagram cannot be captured since for a given density, only a unique flow is possible.
4- Due to the equilibrium traffic condition assumption, the model cannot replicate the emergence of stop-and-go waves in the presence of a perturbation in initially steady-state traffic.
5- The capacity-drop (CD) phenomenon cannot be captured because transitions from congestion to free-flow traffic occur at the maximum density.

Earlier attempts to improve the original LWR model focused on smoothening the shockwaves introduced a diffusion term to the right-hand side Eq. 2.1 (Nelson and Sopasakis, 1999; Whitham, 1974), and it has been argued that such treatments lead to a more realistic behaviour of LWR-type models (e.g., shockwaves are smoothed) (Treiber and Kesting, 2013 chapter 8). Nelson (2000) argued that such an extended LWR model can describe the scattering in the flow-density diagram. From a physical point of view, however, the scattering derived from such an approach cannot be interpreted meaningfully and seems to result from artifacts of systematic diffusion of traffic states. For instance, the range of scattering obtained in Nelson (2000) is implausibly large, where for some cases congested traffic exhibits a higher flow than free traffic. The latter could be a consequence of the parabolic nature of the diffusion term, which introduces "gas-like" behaviour (Daganzo, 1995b). These issues could make questionable any conclusion drawn from these scatterings regarding complex phenomena such as the capacity drop (CD) and hysteresis.

Numerous approaches have also been proposed to incorporate the CD into the CTM. (Srivastava et al., 2015) introduced a modified CTM in which demand function linearly decreased in congested traffic states

and argued that this approach could replicate CD. Shirke et al. (2019) extended CTM for arterial traffic and incorporated the loss of flow occurring due to drivers' reaction time in the initial period after a traffic light turns green. Some approaches introduced phenomenological mechanisms to modify the inflow and outflow of the cells near bottleneck areas (e.g., Ferrara et al., 2015; Muralidharan and Horowitz, 2015; Torné et al., 2014); however, a recent study by Kontorinaki et al. (2017) has argued that such approaches, although applying flow drop downstream of the bottleneck, fail to replicate congestion as the flow drop occurs in the free-flow branch of the FD.

Some researchers have adopted discontinuous FDs (Koshi, 1981) in which the free-flow and congestion regimes have distinct capacities (e.g., Lu et al., 2009; Srivastava and Geroliminis, 2013). This approach, however, results in an infinite characteristic speed at the interface of the free-flow and congestion regimes (Li and Zhang, 2013). Evidence supporting the existence of continuous equilibrium states (Cassidy, 1998) also calls the nature of a discontinuous FD into question. To remedy this issue, Jin et al. (2015) derived a model from the KW theory with a continuous FD, where the CD is considered a calibration parameter. The authors argued that the model can replicate the CD phenomenon downstream of a lane-drop bottleneck. Another analytical approach for capturing CD in merge bottlenecks is presented by Leclercq et al. (2011), where merging vehicles act as a moving bottleneck, and the merge flow directly affects the magnitude of the CD.

So far, we reviewed the kinematic wave theory and CTM models while discussing their analytical properties as well as their inadequacies against the complex empirical features of traffic (e.g., scattering, capacity drop, hysteresis, etc.). This section also reviewed several existing efforts to incorporate the capacity drop phenomenon into the KW theory and CTM. The following subsections discuss the extended equilibrium models, where the proposed models extend the KW theory to incorporate several behavioural and empirical aspects of real-world traffic.

### 2.1.2. Extended equilibrium models

This section reviews the existing attempts to improve the behaviour of the original LWR model with respect to the behavioural, empirical, physical aspects of real-world traffic. The following subsections provide an organized review of these models, classifying them into distinct groups according to their development's underlying rationales. Section 2.1.3 revisits these different model families with a critical perspective.

(1) The LWR model with bounded acceleration

One of the main drawbacks of the original LWR model is its unrealistic speed adaptation mechanism. To elaborate, the macroscopic speed adaption of the vehicles is considered, which is:

$$dV_e(t)/dt = -\rho(V_e'(\rho))^2 \, \partial\rho/\partial x \qquad (2.4)$$

Eq 2.4 imposes an unbounded speed adaptation rate near discontinuities, both for deceleration and acceleration manoeuvres. However, most studies have focused on the latter issue and proposed models to incorporate boundedness of the acceleration rate into the continuum models (e.g., Jin and Laval, 2018; Laurent-Brouty et al., 2021; Lebacque, 2002; Lebacque, 1997; Leclercq, 2007). Two primary reasons can explain why bounded acceleration has attracted more attention than bounded deceleration. First, many complex traffic phenomena such as capacity drop and hysteresis have been frequently observed downstream bottlenecks where traffic flow is in acceleration. Empirical studies have often associated such phenomena with the fact that vehicles leaving bottlenecks often have a smaller speed adaptation rate than those entering congestion zone from behind. However, perhaps the more important reason is the inherent difficulties that incorporating bounded deceleration might bring to the mathematical tractability of the continuum models (Giorgi et al., 2002). Several consistency conditions for which hyperbolic systems, such as the lax inequalities and the Rankine-Hugoniot jump condition (LeVeque, 2002a), cannot be satisfied, if the deceleration rate near discontinuities is bounded. Therefore, we argue that any continuum model that leads to the formation of shockwaves cannot endogenously capture the bounded deceleration property. As we shall see in the next sections, this issue also applies to the majority of non-equilibrium models

Perhaps the first effort to incorporate bounded acceleration into the LWR model was made by Lebacque (1997). The author proposed some adjustments for the numerical solution of the LWR model, where a systematic bound is applied for the cells experiencing unbounded acceleration. In Lebacque (2002), a two-phase extension of the LWR model with bounded acceleration (BA-LWR for short) is

developed, which directly incorporates boundedness of the acceleration into the physics of the LWR model. The model defines a maximum value as an upper bound for the acceleration. If the acceleration predicted by the LWR model is beyond the maximum, a dynamic equation for speed ($dV/dt = A_{max}$) is coupled with the LWR model, where $A_{max}$ is the bounded acceleration rate. The BA-LWR model can be presented as:

$$
\begin{cases}
\partial\rho/\partial t + \partial\rho V/\partial x = 0 \quad , \ V = V_e(\rho) & if \ \dfrac{dV_e(\rho)}{dt} \leq A_{max} \qquad LWR\ phase \\[2mm]
\begin{cases} \partial\rho/\partial t + \partial\rho V/\partial x = 0 \\ \partial V/\partial t + V\,\partial V/\partial x = A_{max} \end{cases} & otherwiswe \qquad\qquad BA\ phase
\end{cases}
\tag{2.5}
$$

For the bounded acceleration phase, two PDEs are used simultaneously, but the system has only one characteristic speed ($\lambda = V$), suggesting that traffic density propagates within the driving directions. Mathematically, this means that the system is not strictly hyperbolic at the BA phase since, for strict hyperbolicity, distinct characteristics must be obtained for each PDE (LeVeque, 2002a). For a detailed discussion on the analytical solutions and physical properties of the model, the reader is referred to Lebacque (2003). It has been argued that the BA-LWR model can capture CD and hysteresis (Khoshyaran and Lebacque, 2015). Recently, it has been shown that with a triangular FD, BA-LWR is equivalent to an extension of the car-following model by Newell (2002) for the bounded acceleration (Jin and Laval, 2018). Furthermore, Jin and Laval (2018) the issue of the unbounded acceleration can also apply to a range of non-equilibrium models.

Leclercq (2007) utilized the BA-LWR framework for studying moving bottlenecks and, argued that the effects of slow-moving vehicles were underestimated in previous studies of moving bottlenecks within the LWR model (e.g., Newell, 1998) due to the unbounded acceleration. It has been argued that the BA-LWR model by Leclercq (2007) can also be applied for noise and emission modelling of moving bottlenecks (Can et al., 2010). However, the approach proposed by Leclercq (2007) applies the bounded acceleration to almost all vehicles in a single-class traffic flow, whereas moving bottlenecks are often caused by one or group of slow-moving vehicles. As well slow moving and heavy vehicles are the major causes of aspects such as noise and emission. Therefore, we argue that bounded acceleration framework applied to single-class equilibrium models may not have significant implications for aspects such as moving bottlenecks and noise and emission modelling.

while the BA-LWR framework parsimoniously improves the acceleration behaviour of the LWR model, several shortcomings remain, which affects its real-world performance. Our recent benchmarking study of continuum model against real-world traffic (Mohammadian et al., 2021c) shows that the BA-LWR model may have the limited capability of capturing capacity drop and hysteresis due to several reasons.

First, this approach may not comprehensively capture CD since it still dismisses other potential contributing factors such as the LC manoeuvres in the merge zones or the proportion flow of merging/diverging vehicles. Therefore, since in BA-LWR the magnitude of CD is a property of the upper acceleration bound, realistic values for the acceleration bound may cause unrealistic value for the capacity drop and vice versa. In this regard, it should also be noted that the average acceleration rate of vehicles in the macroscopic setting could be different than that of the individual vehicles. Likewise, BA-LWR may not be capable of comprehensively describing hysteresis loops since its predicted hysteresis loops are always of the same clockwise pattern; however, recent studies have identified various structures for hysteresis loops, including a counter-clockwise pattern (Chen et al., 2014; Laval, 2011; Saifuzzaman et al., 2017). These findings have been explained by aggressive/timid driving (Laval, 2011) or the difficulty of driving task in the framework of Task–Capability Interface model (Saifuzzaman et al., 2017).

The shortcomings of BA-LWR could be more significant for CAVs traffic flows. Several empirical studies have found that CAVs can decelerate much more smoothly than conventional vehicles (Kontar et al., 2021) due to proactive response to traffic conditions regarding ahead, considering safety and communication (Milanés and Shladover, 2014; Shladover, 2018). In other words, the unbounded deceleration and shockwave issues with continuum models can be more pronounced for CAVs, but the BA-LWR framework behaves just the same as the LWR model and produces shockwaves.

(2) The LWR-type models incorporating lane-changing manoeuvres.

Another research direction has focused on multi-lane traffic flows where lane-changing (LC) manoeuvres or their effects are considered. Many of these models commonly consider a separate LWR model for each lane, and the equations are related through an interchange term accounting for the LC

manoeuvres. The underlying logic behind the LC manoeuvres, however, differs in these models. Throughout this paper, these models are referred to as multi-lane LWRs (ML-LWRs). The models in this category assume that traffic consists of a single-vehicle class.

Munjal and Pahl (1969) proposed the first ML-LWR model in which LC manoeuvres are defined as a function of the density difference between adjacent lanes, assuming that characteristic speeds are the same for all lanes. A two-dimensional extension of the LWR model is developed in another study (Michalopoulos et al., 1984), where lateral coordination is introduced to consider LC manoeuvres. This approach also calls for the development of a two-dimensional FD, which makes the model intricate. Holland and Woods (1997) proposed another ML-LWR model similar to that of Munjal and Pipes (1971), where each lane can have a distinct characteristic speed. It has been argued, however, that the above models cannot address the adverse effects of LC manoeuvres on following vehicles because they assume that LC vehicles perform instantaneous manoeuvres (Laval and Daganzo, 2006).

To capture the adverse effects of LC vehicles, Laval and Daganzo (2006) proposed another ML-LWR model where LC vehicles are defined as moving bottlenecks subject to bounded acceleration. The authors argued that their ML-LWR model could capture the CD phenomenon observed downstream of lane-drop bottlenecks and be useful for studying the capacity flow of moving bottlenecks. Jin (2010) argues that LC vehicles hinder traffic flow not only in the destination lane but also in their current lane. To account for such effects, Jin (2010) proposed another LWR-type model in which the effect of LC vehicles in all lanes is reflected as a capacity drop in the FD. The model is validated in Gan and Jin (2013). Jin (2017b) developed a model for lane-drop bottleneck and analytically investigated the properties of kinematic waves in lane-drop scenarios. The model was further developed in Jin (2017a) to account for the capacity drop, where LC vehicles inside lane-drop zones are assumed to adopt a bounded acceleration rate.

Overall, despite their theoretical potentials, existing LWR-type models accounting for lane-changing maneuvers dismiss major factors involved in the lane-changing process. Many existing models have primarily focused on mechanisms to capture the after-effects of lane-changing maneuvers on traffic phenomena, while the mechanisms for lane-changing occurrences are often oversimplified. For instance, lane-changing occurrence is often modelled as a deterministic function of relative density difference between the adjacent lane (Laval and Daganzo, 2006), or the fundamental diagram and traffic compositions (Jin, 2010, 2013). While such mechanisms may be effective near merge areas or lane-drop bottlenecks, they are likely to have little implications for discretionary lane-changing maneuvers elsewhere. The stochasticity and the human factors involved in the lane-changing maneuvers have not been incorporated rigorously in the existing models, whereas recent driving simulator studies have found out that such factors could be more pronounced in connected environments (Ali et al., 2018; Ali et al., 2019).

(3) The LWR-type models involving multiple user/vehicle classes.

The original LWR model treats traffic flow as if it consists of only a single user class, and therefore, heterogeneities between vehicles and users and their consequential effects on traffic flow are neglected. Several efforts have been made over time to incorporate such heterogeneities into the LWR framework, and these models are referred to as multi-class LWRs (MC-LWRs) in the rest of this paper. As stated by Logghe and Immers (2003), a general formulation can describe MC-LWR models as:

$$\begin{cases} \partial \rho_u / \partial t + \partial (\rho_u V_u) / \partial x = 0 \\ \quad V_u = V(\rho_{1,\dots}\rho_{nu}) \end{cases} \qquad (2.6)$$

where the subscript $u$ identifies the class-specific state variables for each user/vehicle class, in Multi-class models, each class vehicle class has its own characteristic speed. As expressed in Eq. 2.6, MC-LWR models define the LWR model for each user class separately, where different classes are related to one another through the FD. These models vary primarily in the ways in which they define such relationships. In this section, a review of the MC-LWR models is presented.

The logic behind some MC-LWR models is to consider separate route-choice behaviours for various classes. Such models can also be classified as multi-commodity models (e.g., Jin, 2012, 2013), which are most suitable for modelling traffic networks. Daganzo (1997) presented the first MC-LWR model consisting of two user classes, in which one class is subject to a specific lane. The model can, therefore, replicate the route-choice behaviour of diverging vehicles near off-ramps and their adverse effects on traffic flow. Daganzo (1997) argued that the model can capture the queuing effects of diverging traffic in such areas. Another MC-LWR model was presented by Daganzo (2002) in which two groups of user classes, i.e.,

"rabbits" and "slugs" with "aggressive and "timid" driving behaviours, respectively, are considered on a two-lane highway. "Rabbits" are considered as aggressive drivers who always occupy high-speed lanes and overtake traffic whenever possible, whereas "slugs" display "timid" driving behaviour and never engage in passing. The main premise is that the traffic condition does not change the personalities of drivers. The model considers two traffic regimes, defined as "1-pipe" and "2-pipe". In the "1-pipe" regime, a multi-lane highway behaves as if it were a single-lane highway, and is dominant when the traffic condition in both lanes is the same so that all vehicles drive at the same speed. In the 2-pipe traffic state, the traffic condition in different lanes varies, and it allows one class to experience the free-flow condition, while the other class experiences congestion. Daganzo (2002) argued that the model can replicate the reversed lambda-pattern observed in empirical flow-density scattered data.

Wong and Wong (2002) proposed another MC-LWR model in which different user classes (of the same vehicle length) obeyed the same form of FD, but varied in terms of their desired speed. Consequently, various user classes can adopt different speeds for a given spacing in this model. The authors argued that the model could replicate capacity-drop, hysteresis, and platoon dispersion. Zhang and Jin (2002) proposed an MC-LWR model in which different classes vary both in vehicles' lengths and desired speeds. The key premise is that only in the free-flow condition can various classes adopt different speeds. In other words, all vehicles drive at the same speed in the congestion regime. The same logic is adopted in the model by Chanut and Buisson (2003), where user classes also vary in terms of vehicle length and desired speed and where maximum density and critical density, at which traffic breakdown occurs, are dynamic and depend on the traffic composition.

Some MC-LWR models describe the dynamics of various classes based on passenger cars. Logghe and Immers (2003) presented the first MC-LWR model in this category, where the effects of heavy vehicles on traffic flow are scaled in terms of fixed passenger-car equivalents (PCEs). As a result, vehicle classes can adopt various spacing at the same speed, where PCEs are not sensitive to the traffic regime. The concept of PCE is also utilized in another MC-LWR model developed by Ngoduy and Liu (2007), in which various classes adopt different speeds in the free-flow condition but drive at the same speed after the onset of congestion. The authors argued that the model could replicate the CD phenomenon, hysteresis, and platoon dispersion. In Ngoduy (2010), the model is extended to network traffic, and in-homogeneities such as lane drops are included. Another MC-LWR model developed by van Lint et al. (2008), known as FASTLANE, is also based on the PCE concept, where PCEs are <u>determined</u> as a function of speed, spacing, and temporal headway of various classes are used to determine the effective density. The speed of each class is then determined based on the effective density. This means that both the PCEs and FD in FASTLANE are implicit, as they depend on one another. A generic variation of FASTLANE is proposed by van Wageningen-Kessels et al. (2014) which can be simplified to some of the previous MC-LWR models as special cases depending on the choice of parameters.

Logghe and Immers (2003) adopted a distinct approach and derived their MC-LWR model from the user-equilibrium theory (Wardrop, 1952). Their model is derived from two major premises, i.e., that vehicles do not occupy more than the necessary space and that faster vehicles do not affect the slower ones. Because of the latter, the semi-congested regime appears, in which slower vehicles can still drive at their free-flow speed, whereas faster vehicles are already in the congestion state. This model is recently further developed and generalized by Qian et al. (2017) for more than two vehicle classes, and the lateral interactions of vehicles are incorporated as the concept of perceived equivalent density. The model uses lateral interactions to determine how each class perceives the effects of other classes for a given traffic condition.

(4) The LWR-type models incorporating stochasticity
In recent years, there have been empirical and theoretical studies that endorse the stochastic behaviour of traffic flow. For instance, Kim and Zhang (2008) presented a stochastic theory of kinematic waves, focusing on the heterogeneities in time gaps between different drivers, and suggested that the differences between heterogeneous time gaps and reaction times would result in stochastic wave speeds. The theory was also applied to real-world vehicle trajectories, and the authors found that the theory can explain the scattering phenomenon in the flow-density diagrams. Other examples include the studies on modelling the flow-density diagrams through stochastic processes (e.g., Jabari et al., 2014; Ni et al., 2018; Qu et al., 2017; Siqueira et al., 2016) that have explained the scattering phenomenon through probabilistic models of the fundamental diagram.

In parallel, another research direction has focused on incorporating stochasticity into the continuum framework. The models of these types can be classified as stochastic equilibrium models or equilibrium models with stochastic fundamental diagrams. The stochastic models are motivated by improving the operational performances of the original LWR model as well as capturing some observed phenomena, e.g., scattering. The underlying mechanism for stochasticity is, however, different between the models.

Boel and Mihaylova (2006) modified the supply and demand flows of the cell transmission model by specifying variations in traffic dynamics within each cell. They considered extremely congested and light traffic and assumed that, for these cases, cell interface flows follow Gaussian and binomial distributions, respectively. Later, Sumalee et al. (2011) proposed the stochastic cell transmission model (SCTM), where the stochasticity in supply and demand flows are characterized by defining noise parameters for the parameters of the triangular FD (i.e., free-flow speed, maximum density, and wavespeed). Five operational modes are defined based on the possible regimes for any set of upstream and downstream cells, and their corresponding emergence probabilities are postulated. The model then uses a finite mixture of these probabilistic modes to calculate the stochastic flow, and thereby, the new state for density. The logic in SCTM has been explained as considering the variance in demand and supply flows in order to calculate the variance in traffic density (Zhong et al., 2014). The authors show the performance of the proposed model against real-world traffic by calibrating the model. However, a question remains about the robustness of the stochastic models in terms of validation. Our recent benchmarking study showed that models that explicitly do not consider the congestion wave speed as a parameter perform poorly with slight changes in the boundary condition. Another issue is that many of the existing model performance assessments of the stochastic models is based on cross-sectional data. It is not clear how stochastic models would perform regarding spatiotemporal propagation of the congestion regions. In Zhong et al. (2013), SCTM is extended for freeways for network elements such as junctions, merges and diverges.

Ngoduy (2011) presented another stochastic model based on the MC-LWR model by Ngoduy and Liu (2007), where the capacity of each class is considered as a stochastic variable, subject to Weibull distributions. Ngoduy (2011) implemented the model on real-world traffic and suggested that the stochastic model captures a wider range of scattering in the flow-density diagram compared to the original MC-LWR model. Another stochastic model is presented by Li et al. (2012), where a random free-flow speed is defined, which leads to a generalization of the FD in the density-flow plane, and thereby, potentials for replicating the scattering phenomenon.

A challenge of the stochastic models is their analytical properties within the hyperbolic framework regarding the structure of the solutions to the Riemann problems. For instance, while the original LWR framework cannot accept a Riemann problem as $\rho_{UP} > \rho_{DW}$ and $V_{UP} > V_{DW}$ (because traffic speed is a unique and decreasing function of density), such a Riemann problem can be possible in the stochastic models, where in general, the structure of the solutions may not necessarily be in line with the behavioural properties. As will be explained in more detail in later sections, such a Riemann problem can result in a shockwave travelling downstream and forcing vehicles to speed up in some non-equilibrium models (Zhang, 2000a; Zhang, 2009), and such behaviour is not realistic because drivers primarily pay attention to the traffic condition ahead. The analytical properties of the stochastic framework have not been investigated adequately in the stochastic models discussed above.

Laval and Chilukuri (2014) analytically investigated a generic class of stochastic equilibrium models with triangular fundamental diagrams with a focus on the solutions to the Riemann problems. The results suggested that the structure of the solutions for the stochastic case is consistent with the deterministic models and that for a given location over time, the solutions to the stochastic model converged to the deterministic solutions.

Jabari and Liu (2012) argued that defining stochasticity by means of noise terms might only be suitable at the road boundaries (e.g., inflows/outflows at merge or diverge) but not for the homogeneous sections, where the hyperbolic conservation laws define dynamics of traffic state. Furthermore, they argued that noise terms in the previous models did not adequately reflect the intrinsic mechanisms for the stochasticity in traffic, that is, the heterogeneity between driving behaviours. To address these points, Jabari and Liu (2012) derived a generic variation of CTM with stochastic processes, where the source of stochasticity is the randomness in the time gap, which can follow various distribution forms. They investigated the analytical properties of the model and suggested that for the hydrodynamic limit of stochastic processes, the solutions to the stochastic model would converge to the solutions to deterministic CTM. In Jabari and Liu (2013), a Gaussian approximation of the model by Jabari and Liu (2012) is proposed, where the primary focus is on the real-world application of the model for queue estimation of congested traffic.

## 2.2. Non-equilibrium (NE) models for freeways
### 2.2.1. Definitions and concepts

Non-equilibrium[2] (NE) models drop the assumption of equilibrium in general and apply it only the steady-state traffic. These models define a separate dynamic equation for acceleration and therefore consist of more than one (often two) hyperbolic PDEs. The FD in these models applies only to the equilibrium conditions in which the state variables are stationary. The general form for many[3] of the NE models for a section of homogeneous road without on/off ramps is:

$$\begin{cases} \partial \rho / \partial t + \partial (\rho V)/\partial x = 0 \\ \partial V / \partial t + V\, \partial V/\partial x = g(\rho, V, V_e(\rho), \dots) \end{cases} \qquad (2.7)$$

where the second equation accounts for the acceleration mechanism, which is derived from distinct premises in each NE model. To obtain the characteristic speeds, these models must be written in a quasi-linear form as:

$$\partial \vec{U}/\partial t + A(\vec{U})\partial \vec{U}/\partial x = S(\vec{U}) \qquad (2.8)$$

where $\vec{U}$ is the vector of the conserved variables (commonly density ($\rho$) and flow-type variables), $A(\vec{U})$ the Jacobian matrix with and $S(\vec{U})$ being the source term. The characteristics of NE models can then be obtained by solving for $\det[A(\vec{U}) - \vec{\lambda}\vec{I}] = 0$. The same as with the equilibrium models, the GNL characteristics fit into the condition[4] ($\partial \lambda / \partial \rho \neq 0$), and produces shock and rarefaction waves, whereas the LD characteristics satisfy the condition ($\partial \lambda / \partial \rho = 0$), and produces contact waves.

For all NE continuum models, the Jacobian matrix always exist uniquely and can be obtained as $A(\vec{U}) = \partial F(\vec{U})/\partial \vec{U}$ assuming that $F(\vec{U})$ is the global flux term. If such a global is obtained as an explicit analytic function, the NE's governing equations can then be formulated in the conservative form as follows:

$$\partial \vec{U}/\partial t + \partial F(\vec{U})/\partial x = S(\vec{U}) \qquad (2.9)$$

Conventionally, the conservative form in Eq. 2.8 has been central to studying the Riemann problems and approximating the numerical solutions of NE models (Mammar et al., 2009b; Mohammadian et al., 2021b; Zhang, 2000b). Solutions to Riemann problems often involves the emergence of intermediate states between the upstream and downstream states. However, it is not always easy to formulate the governing equations of NE models in terms of explicit state variables with corresponding global fluxes, and in such cases, the models are non-conservative and can only be written in the quasi-linear form as in Eq. 2.8. Recent developments in hyperbolic systems and numerical methods have shown that both Riemann problems and numerical solutions can still be studied and approximated based on the quasi-linear form in Eq. 2.8 (see e.g., Delis et al., 2014; Diaz et al., 2019; Serezhkin and Menshov, 2020).

Heidemann (1999) questioned the theoretical consistency of NE models suggesting that these models cannot preserve the first equation of Eq. 2.7 for ($Q = Q_e(\rho)$). Heidemann argued that simultaneous consideration of this condition and Eq. 2.y introduces extra terms that violate the law of conservation of vehicles. However, Zhang (2003c) posited that phase transitions between two traffic states are hysteretic and accompanied by a deviation from the equilibrium state, and thus, the first equation of Eq. 2.7 should not necessarily be conserved for the condition ($Q = Q_e(\rho)$). Non-equilibrium models must, however, be consistent with the LWR model regarding the steady-state condition and regarding the zero-relaxation time limit in the case of models with a relaxation term.

There are four conventional approaches to develop non-equilibrium models:
1- The phenomenological approach in which hyperbolic conservation laws are directly borrowed from fluid dynamic concepts and implemented on traffic flow. In this case, the speed adaptation

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[2] Because non-equilibrium models commonly have two dynamic equations, they are typically classified as "second-order" models in the literature. In this paper, however, such a classification is not adopted in order to avoid confusion with the mathematical definition of "$n^{th} - order$" PDE, i.e., the PDE which has at least one "$n^{th} - order$" derivative term.
[3] A few NE models include more than two equations and cannot be explained by this general form.
[4] More generally, a GNL characteristic must satisfy $\nabla \lambda(U) r(U) \neq 0$, where $r$ is the right eigenvector corresponding to the characteristics (Leveque, 2002b)

mechanisms may not necessarily be meaningful with respect to driving behaviours (Treiber and Kesting, 2013).

2- Derivation from car-following relations using the certain approximation of car-following variables in the Eulerian coordinates (Helbing and Johansson, 2009). In this case, it is often difficult to establish a direct relationship between the non-equilibrium and the underlying car-following models (Papageorgiou, 1998).

3- Derivation from the gas-kinetic approach, where vehicular interactions for a finite number of vehicles at the mesoscopic scale are transformed to the macroscopic scale (Helbing, 1996b). In this case, the speed adaptation mechanism does not capture the dynamics of any individual vehicle, but it represents the average vehicular interactions within a moving window travelling with the average traffic speed (Treiber and Kesting, 2013). As will be discussed in Section 2.2.3, this approach often requires some variance between the individual vehicles within the moving window, even in the steady-state traffic.

4- Derivation from the equivalence theory (Jin, 2016), where continuum models are derived based on a direct transformation of car-following variables in the Lagrangian co-ordinates. In this case, continuum models describe the traffic flow dynamics assuming all vehicles have the same car-following behaviours, even in the non-steady traffic condition.

While the primary motivation behind NE models is to explain non-steady traffic conditions and the associated phenomena, desirable NE models must also capture steady-state traffic conditions consistently. For instance, considering the linkage between the fundamental diagram and steady-state traffic (Cassidy, 1998), it is expected that the steady-state condition of NE models encodes the concept of the fundamental diagram explicitly. This aspect will be much more critical in the case of CAV traffic flows, where the steady-state traffic and string stability is explicitly incorporated in the modeling and control (Yao et al., 2019; Zhou and Zhu, 2020). As will be discussed in the following, many models also incorporate relaxation as a mechanism to approach steady-state equilibrium traffic, and in such cases, desirable NE models must be consistent with the LWR model at the zero-relaxation time limit.

### 2.2.2. phenomenological and elementary car-following (CF)-based NE models

The first NE continuum model was independently proposed by Payne (1971), and later on by Whitham (1974) (known as the PW model for short). The PW model defines acceleration as a function of both deviations from the equilibrium state and changes in the traffic condition ahead. With regard to Eq. 2.8, the acceleration mechanism in the PW model is:

$$\partial V / \partial t + V \partial V / \partial x = (V_e(\rho) - V)/\tau - C^2(\rho)/\rho \, (\partial \rho / \partial x) \qquad (2.10)$$

with the characteristic speeds $\lambda_{1,2} = V \pm C(\rho)$, where $C(\rho) = \sqrt{-V_e'(\rho)/2\tau}$ is the so-called sonic velocity, and $\tau$ is the relaxation time, during which the equilibrium state is recovered. The first term on the right side of Eq. 2.10 is called the relaxation term and accounts for the parts of speed adaptation that occur because of deviation from the equilibrium condition. The second term is the anticipation term and represents the speed adaptations occurring due to changes in the traffic condition ahead, which is defined as a density gradient. In other words, even if vehicles are already driving at the equilibrium speed, they will accelerate or decelerate if the traffic condition ahead changes.

The PW model allows for the occurrence of a wide range of traffic flows at a given density and vice versa. As a result, the scattering in the flow-density diagram can be simulated (Payne, 1979). It has been argued that the model can also replicate the capacity-drop phenomenon since the assumption of equilibrium traffic is relaxed (Papageorgiou, 1998).

Nevertheless, the model has been criticized for its drawbacks. Depending on the choice of relaxation time, solutions of the model for initially homogeneous traffic subject to small perturbations can be unconditionally stable, which means that the model cannot adequately replicate traffic instabilities and stop-and-go waves (Helbing, 1996a). To capture traffic instabilities, Kühne (1984) and Kerner and Konhäuser (1993) modified the PW model and assumed that the sonic velocity in the anticipation term is constant. They also introduced a concave-convex FD to replicate stop-and-go waves in a specific density range of congested traffic. Consequently, the stability of solutions is guaranteed not only for light traffic but also for highly congested traffic above a specific density value. The acceleration mechanism in this model with respect to Eq. 2.6 is:

$$\partial V / \partial t + V \partial V / \partial x = (V_e(\rho) - V)/\tau - C_0^2/\rho \, (\partial \rho / \partial x) + \eta \, (\partial^2 V / \partial x^2) \qquad (2.11)$$

where $C_0$ is constant and $\eta$ is the viscosity coefficient. The third term on the right-hand side of Eq. 2.11 is called the viscosity term and is derived from a translation of Navier-Stokes' fluid dynamics equations (see Helbing, 1996b for more discussion on Navier-Stokes' equations ). It has been argued that the viscosity term is not physically meaningful for traffic flow but that it can improve the numerical properties of the model (Treiber and Kesting, 2013 chapter 9). It is worth noting that today, the numerical solution of continuum models is less of a challenge as improved numerical schemes have been presented (Delis et al., 2014; Helbing and Treiber, 1999; Jin and Zhang, 2001; Mammar et al., 2009b; Mohammadian and van Wageningen-Kessels, 2018).

It has been argued that the PW model is numerically challenging to simulate (Treiber and Kesting, 2013 chapter 9). Inconsistent accounts of the performance of the model for similar scenarios (e.g., Cremer and May, 1987; Leo and Pretty, 1992; Papageorgiou, 1983) have also been linked with the choice of numerical schemes (Zhang, 1998). To overcome these issues, Messmer and Papageorgiou (1990) proposed METANET, which is based on a simplified discretised form of the PW model. METANET has been widely used for control purposes because of its simplicity (see, e.g., Ferrara et al., 2018 chapter 8, and references therein). It is worth noting that although METANET is commonly considered as a discretised version of the PW model, it is a phenomenological model, and cannot be drived from car-following relations.

Some other phenomenological models cannot, on the other hand, be explained by driving behaviours; e.g., the model proposed by Ross (1988) does not include an anticipation term and ignores the FD. Therefore, the acceleration equation is independent of traffic density, and all vehicles not driving at their desired speed can accelerate to obtain it regardless of the traffic condition ahead (Newell, 1989). It has been argued that without the anticipation term, any NE model is unconditionally unstable (Treiber and Kesting, 2013 chapter 15). Another phenomenological NE model was presented by Michalopoulos et al. (1993). The desired speed in this model is a function of the geometrical changes in the road, and the relaxation term is active only in the presence of such changes.

### 2.2.3. Improved car-following-based and phenomenological NE models

In a famous paper, Daganzo (1995b) sharply criticized the NE models at the time for being inconsistent with driving rules. Daganzo (1995b) argued that some of the previous models (e.g., Michalopoulos et al., 1993) assumed that the personalities of drivers could be affected by traffic conditions. The second argument by Daganzo (1995b) targeted the viscosity terms (as in the model by Kerner and Konhäuser, 1993) for increasing the width of traffic shockwave, which is commonly of an order of only a few vehicles, to unrealistically large values. Daganzo (1995b) showed that these terms lead to the "wrong-way-travel" effect, meaning that vehicles in congestion zones can reverse if the free-flow condition appears upstream.

Furthermore, Daganzo (1995b) argued that the existing NE models did not preserve the anisotropic property of traffic flow, i.e., that vehicles must only react to "frontal stimuli" and not be affected by the traffic condition behind. Daganzo (1995b) argued that since the second characteristic speed in these models were faster than traffic speed, they could inform downstream vehicles about the upstream traffic condition, forcing them to accelerate or decelerate accordingly.

Several efforts have been made to address these criticisms by Daganzo (1995b). Zhang (1998) showed that the anticipation mechanism of the PW model can itself produce the "wrong-way travel" effect, regardless of the presence of the viscosity term. The observation by Zhang (1998) is consistent with the zero-relaxation time limit of the PW, which results in the LWR model with a diffusion term (Treiber and Kesting, 2013). To resolve the issue, Zhang (1998) derived the PW model from an improved CF relationship in which drivers respond to the frontal stimuli with a delay. The acceleration mechanism in the resulting NE model is:

$$\partial V/\partial t + V\partial V/\partial x = (V_e(\rho) - V)/\tau - C(\rho)^2/\rho \, (\partial\rho/\partial x) \qquad (2.12)$$

where $C(\rho) = \alpha\rho V_e'(\rho)$ being the generalized form of the sonic velocity, with $\alpha$ as a constant, is the main difference between this model and the standard PW model. Zhang (1998) showed that such a modification prevents the "wrong-way-travel" effect. The model was then extended to a generalized one in Zhang (2003b), where the main focus was to derive the viscosity effect from the CF relationships. Viscosity in this model is linked to drivers' memories and is derived from the concept of drivers' resistance to adapt their speed instantaneously in the CF scenarios. The acceleration mechanism in the resulting model is then:

$$\partial V/\partial t + V\partial V/\partial x = (V_e(\rho) - V)/\tau - C(\rho)^2/\rho \, (\partial\rho/\partial x) - 2\beta C(\rho) \, (\partial V/\partial x) + \eta(\rho) \, (\partial^2 V/\partial x^2) \qquad (2.13)$$

in which, $\beta$ is a positive constant and $\eta(\rho) = 2\beta\tau C(\rho)^2$ is the viscosity coefficient. The model's characteristic speeds are obtained as $\lambda_{1,2} = V + \left(\beta \pm \sqrt{\beta^2 + 1}\right)C(\rho)$. Zhang (2003b) argued that with $\beta \gg 1$, the faster characteristic in this model converges to the traffic speed, and that the model could be classified as anisotropic. However, for such a condition, the first characteristic speed becomes infeasibly large, which could be a more serious inconsistency. Furthermore, recent numerical experiments have found that, for a given Riemann problem, slight changes to the choice of the parameter $\beta$ can alter the structure of solutions significantly (Chu et al., 2022)

Another NE model is derived from the continuum approximation to the OVM model by Berg et al. (2000). The authors derived the density from a high-order Taylor expansion of the definition in integral form in the microscopic setting instead of using the standard definition of traffic density as the inverse of the spatial headway. The authors argued that such a rationale resulted in the same stability criterion for the continuum model as that of the OVM. This model, however, includes several high-order derivative terms for traffic density. Gupta and Katiyar (2006a) argued that this model could produce the wrong-way-travel effect and adapted the model to eliminate this effect. The resulting model is nevertheless quite similar to the model by Zhang (2003b). In Gupta and Katiyar (2007), a multi-class extension of their model (Gupta and Katiyar, 2006b) is presented where similar to the model by Wong and Wong (2002), it is assumed that various user-classes follow the same functional form of the same speed-density relationship but with different desired speeds.

Except for some simplified fundamental diagram, the models above (Berg et al., 2000; Gupta and Katiyar, 2006b, 2007; Zhang, 1998, 2003b) cannot be written in the conservative form (see, Eq. 2.8) with an explicit flux term $F(\vec{U})$. This is because in all these models, the square of $V_e'(\rho)$ is present in the gradient terms which complicates defining the models' equations in vector forms through global state variables ($\vec{U}$) with global flux ($F(\vec{U})$).

While addressing several the issues raised by Daganzo (1995b), the models discussed above still have a characteristic speed faster than traffic speed under specific conditions.

Meanwhile, alternative interpretations were proposed for the waves travelling faster or slower than traffic. The quantity $C^2$ has been interpreted as speed variance between vehicles (see, e.g., Hoogendoorn, 1999), and consequently, the characteristics $\lambda = V \pm C$ have been linked with the speed range of vehicles travelling slower or faster than the average speed. This interpretation is in line with the underlying premises of the models derived from the gas-kinetic theory, which will be discussed in more detail in Section 2.2.3. However, such an interpretation is not conclusive because first, in many NE models, the first characteristic ($\lambda_1 = V - C$) becomes negative in the congestion region, and therefore, cannot be interpreted as a driving speed[5]. Besides, in many NE models with the characteristic speeds $\lambda = V \pm C$, the second characteristic speed $\lambda_2 = V + C$ is often genuinely nonlinear (GNL). It has been shown that for specific Riemann problems, such GNL faster-than-traffic characteristic speeds can generate shock and rarefaction waves travelling downstream, and thereby reaching vehicles from behind and affecting their dynamics (Zhang, 2000b, 2009).

Aw and Rascle (2000) argued that faster-than-traffic waves appeared in previous models because the anticipation term was derived from the standpoint of a fixed-point observer and did not take the convection of vehicles into account. They suggested that the anticipation term must be computed from the standpoint of the moving observer travelling with the average traffic speed. Such a rationale results in the following acceleration mechanism with respect to Eq. 2. 7:

$$\partial V/\partial t + V\partial V/\partial x = \rho P'(\rho)\,\partial V/\partial x \qquad (2.14)$$

where $P(\rho)$ is a smooth increasing function, known as the so-called pressure variable. The characteristics of this model are $\lambda_1 = V - \rho P'(\rho)$ and $\lambda_2 = V$, where the second characteristic always travels at the speed of the traffic. Therefore, the model is anisotropic in Daganzo's (1995b) sense. The original Aw-Rascle model is phenomenological and is not derived from CF relationships. However, Aw et al. (2002) showed that the Aw-Rascle model can be viewed as the hydrodynamic limit of "follow-the-leader'' (also known as stimulus-response) car-following models (see e.g., Gazis et al., 1961).

---

[5] As will be discussed in more detail in Section 2.2.3., this issue is not applicable to the non-local models because in such model the speed $\lambda_1 = V - C$ is always positive

Zhang (2002) independently derived a similar continuum model from a car-following relationship, assuming that drivers' response time is an explicit function of the inter-vehicular spacing. A direct consequence of this assumption is that the relaxation term does not appear in the resulting continuum model, which is in line with the model by Aw and Rascle (2000). It can be shown that Zhang's model is a special case of Aw and Rascle's (2000) model with the condition $P(\rho) = -V_e(\rho)$. These models are collectively referred to as ARZ in the rest of this paper. As the relaxation term is absent in the ARZ model, the FD cannot be uniquely derived since according to Eq. 2.14, any homogeneous state satisfying $\partial V/\partial x = 0$ could be viewed as equilibrium condition. Lebacque et al. (2007b) argued that the ARZ model can be viewed as the LWR model with "variable fundamental diagram".

Although the model by Zhang (2002) incorporates the concept of the FD, the ARZ model does not explicitly consider a maximum density, and the model can simulate an initial condition where vehicles are bumper-to-bumper but can travel at any speed. Greenberg (2002) coupled the ARZ model with a relaxation term to resolve this issue so that the non-physical solutions can approach the FD. However, this approach addresses the issue only partially and requires a small relaxation time to be effective. Colombo (2002) argued that the ARZ model does not include an explicit maximum density, and proposed another NE model to address this issue. Colombo's (2002) model is known as the phase-transition model since it considers distinct hyperbolic equations for the free flow and congestion phases, respectively. For the free flow phase, traffic is in equilibrium, and the LWR model is selected, whereas for the congestion phase, a phenomenological non-equilibrium model is proposed. The mathematical formulation of the phase transition model can be expressed as:

$$
\begin{cases}
\quad \partial\rho/\partial t + \partial(\rho V_f(\rho))/\partial x = 0 & in\ free\ flow\ (\Omega_f) \\
\begin{cases} \partial\rho/\partial t + \partial(\rho V_c(\rho,q))/\partial x = 0 \\ \partial q/\partial t + \partial((q-q_*)V_c(\rho,q))/\partial x = 0 \end{cases} & in\ congestion\ (\Omega_c)
\end{cases}
\tag{2.15}
$$

where $q$ is a flow-type variable, which is clarified in more detail below, $q_*$ is a constant parameter, and $\Omega_f$ and $\Omega_c$ are the solution domains in the free flow and congestion regions, respectively, given as:

$$
\begin{cases}
\Omega_f = \{(\rho,q) \in [0,\rho_{max}] \times [0,+\infty]\ ,\quad V_f(\rho) \geq V_{f-}\ ,\quad q = \rho V\} \\
\Omega_c = \left\{ \begin{array}{c} (\rho,q) \in [0,\rho_{max}] \times [0,+\infty]\ ,\quad V_c(\rho,q) \leq V_{c+}\ , \\ \dfrac{(Q^- - q^*)}{\rho_{max}} < \dfrac{(q-q^*)}{\rho_{max}} < \dfrac{(Q^+ - q^*)}{\rho_{max}} \end{array} \right\}
\end{cases}
\tag{2.16}
$$

where $V_{f-} = V_f(\rho_{cr})$ is the minimum speed in free flow with $\rho_{cr}$ being the density corresponding to the interface between the free flow and congestion regions, and $V_{c+}$ is the maximum speed in congestion, and $Q^-$ and $Q^+$ are the minimum and the maximum values for $q$. Traffic speed in the free flow and congestion regions is given as:

$$
V_f(\rho) = \left(1 - \frac{\rho}{\rho_{jam}}\right) V_{max} \quad and \quad V_c(\rho,q) = q\left(\frac{1}{\rho} - \frac{1}{\rho_{jam}}\right)
\tag{2.17}
$$

where $V_{max}$ is the maximum speed, $\rho_{jam}$ is the maximum density under the jam condition. Characteristic speeds of the model in the congestion region are obtained as:

$$
\lambda_1 = (2/\rho_* - 1/\rho)(q_* - q) - q_*/\rho_* \ and\ \lambda_2 = V > \lambda_1
\tag{2.18}
$$

where $\rho_*$ is another calibration parameter subject to the condition $(q/q_* + \rho/\rho_* > 1)$. The model by Colombo (2002), has some promising potentials with respect to the empirical observations as well as the analytical properties of traffic flow. To elaborate, we shall interpret the quantity $q$ from a physical perspective. From micro-to-macro transforms, the average spatial gap (inter-vehicular spacing) between vehicles can be approximated as:

$$
S = \frac{1}{\rho} - \frac{1}{\rho_{jam}}
\tag{2.19}
$$

Meanwhile, by comparing Eq. 2.19 and traffic speed in the congestion phase of the phase-transition model (see Eq. 2.17), one can obtain the average spatial gap as $S = V_c/q$. This leads to obtaining the average time

gap $T_g$ as $T_g = S/V_C = 1/q$, and consequently, the flow-type quantity $q$ in the phase-transition model can be interpreted physically as traffic flow corresponding to the time gap. Considering this physical interpretation, we posit that the quantity $q_*$ in Eq. 2.15 must be interpreted as the inverse of the equilibrium time gap, rather than as 'characteristic of the road under consideration', the interpretation given originally by Colombo (2002). Our proposed interpretation is consistent since with $q = q_*$, the second dynamic equation in the congestion phase is eliminated, and one obtains the steady-state condition of the model in the congestion region. Under the steady-state condition, traffic flow is obtained as $Q = q_*(1 - \rho/\rho_{jam})$, which is equivalent to triangular fundamental diagram in the congestion region. Therefore, it could be argued that under the steady-state condition, the phase-transition model by Colombo (2002) reduces to the cell-transmission model.

Blandin et al. (2011) proposed a general phase-transition model in which the triangular fundamental diagram is adopted for the free-flow phase, and for the congestion phase, the solution domain is defined as an area surrounding the triangular fundamental diagram as well. Blandin et al. (2013) investigate the performance of the phase transition model against the NGSIM data and suggest that the model can explain many traffic phenomena such as hysteresis and traffic instabilities. However, since the spatiotemporal range of the NGSIM data is quite limited and dominated by the congestion phase, the finding in Blandin et al. (2013) mainly demonstrates the model's performance regarding convective propagation of congested states. The question remains about the performance of the phase-transiton models regarding aspects such as congestion emergence, queue formation, and dissolution of jam fronts.

The model by Colombo (2002) is perhaps the first non-equilibrium model that explicitly restricts the solution domain to specific ranges ($\Omega_f$ and $\Omega_c$) in the flow-density diagram. This property makes the model potentially more realistic, but there are two potential issues with the way the solution domains are introduced. First, they are discontinuous at the interface between the free flow and congestion, meaning that, the model systematically dismisses the possibility of continuous stead-state traffic from the free flow to congestion, observed by Cassidy (1998). Second, the admissible range is imposed externally and highly depends on the choice of the parameters $Q^-$ and $Q^+$. As a result, the intermediate states arising in the Riemann problems may not necessarily fall within the solution's admissible range, and imposing restrictions in the numerical solutions might be necessary, which can, in turn, result in unrealistic behavioural implications. For instance, numerical examples presented in Chalons and Goatin (2008) shows that multiple intermediate states and jumps arise for a typical deceleration corresponding to a transition from free flow to congestion."

Goatin (2006) used the phase-transition concept to improve the ARZ model. Goatin (2006) argued that the ARZ model is not well-posed near the zero density due to loss of strict hyperbolicity. As a result, the solutions no longer depend on the initial conditions which cause mathematical instabilities. To remedy this issue, Goatin (2006) proposed another phase transition model in which the LWR model determines the free-flow traffic, and the ARZ model determines the congested condition. (Lebacque et al., 2007a) argued that the ARZ model could return non-physical solutions for both very small and near jam densities (e.g., densities about physically maximum) and proposed some treatments for the function $P(\rho)$ in these density ranges to resolve the issues. Based on these treatments, Mammar et al. (2009a) derived the analytical solutions of the ARZ model for all possible phase transitions and proposed a Godunov-type scheme for the numerical solution of the model.

Lebacque et al. (2007b) proposed the generic "second-order" model (GSOM), which includes both the ARZ and Colombo's (2002) model as special cases. GSOM is also a phenomenological model which can include an arbitrary source term as a special case. The second dynamic equation in GSOM in non-standard form as:

$$\partial(\rho\mathcal{L})/\partial t + \partial(\rho\mathcal{L}V)/\partial x = \varphi f(\mathcal{L}) \tag{2.20}$$

where $\mathcal{L} = \mathcal{L}(\rho, V)$ is an arbitrary conserved variable satisfying the condition $d\mathcal{L}/dt = 0$, meaning it is conserved along its trajectory. The parameter $\varphi \in \{0,1\}$ allows one to use the model with or without the source term. It should be mentioned that the source term in this model does not, in general, equal the relaxation term as in previous PW-type models. This is because the generic quantity $f(\mathcal{L})$ does not necessarily represent the relaxation acceleration. However, with $\mathcal{L} = (V - V_e(\rho))$, and $f(\mathcal{L}) = -\rho\mathcal{L}/\tau$, and $\varphi = 1$, GSOM is identical to the model by Greenberg (2002) and the model by Zhang (2002) with a relaxation term. Likewise, the model is the same as Colombo's (2002) model with $\mathcal{L} = (q - q_*)/\rho$ and $\varphi = 0$. Lebacque and Khoshyaran (2013) presented the Lagrangian formulation of GSOM from the perspective

of a moving observer. Zhang et al. (2009) proposed another GSOM-like model in which the second equation is based on an artificial variable, called "pseudo density", which is transformed from speed.

The ARZ model has been interpreted as the LWR model with a "variable fundamental diagram", and several studies have suggested that such a GSOM model can theoretically explain the wide scattering of the traffic states by placing them through multiple flow-density curves (Fan and Seibold, 2013; Lebacque et al., 2007b; Würth et al., 2021). However, this theoretical property may not be achieved when the model parameters are estimated from minimizing the error between simulation and observation. Our recent benchmarking studies of continuum models against real-world traffic show that GSOM models can have limited performance in replicating the observed scattering (Mohammadian et al., 2021c). This observation could in part, be related to the model's stability behaviour. A simple linear stability analysis following the procedure in Helbing and Johansson (2009) shows that the most popular variant of GSOM models, i.e., the model by Zhang (2002) with a relaxation term, is unconditionally stable and cannot replicate the emergence of stop-and-go waves from small disturbances in otherwise steady-state equilibrium traffic. This property is interesting because it contradicts the very motivation underlying the use of non-equilibrium models. Therefore, we argue that the present version of the GSOM-like models could have little implications for CAV traffic flows as investigating the instability of CAV traffic flow is a major aspect (Ciuffo et al., 2021; Gunter et al., 2020; Sun et al., 2020).

Meanwhile, a distinct approach was adopted by Jiang et al. (2002) for eliminating "faster-than-traffic" waves, where another NE model was developed from the continuum approximation of the full velocity difference CF model (Jiang et al., 2001). The acceleration mechanism in this model is:

$$\partial V/\partial t + V \partial V/\partial x = (V_e(\rho) - V)/\tau + C_0\, \partial V/\partial x \qquad (2.21)$$

where $C_0$ is a constant and $\lambda_1 = V - C_0$ and $\lambda_2 = V$ are the characteristic speeds. This model is proposed as the continuum equivalent of the FVDM car-following model, and in recent years, several empirical and theoretical studies have shown the capability of the FVDM for modelling AVs (Gunter et al., 2020). Therefore, at first sight, the model in Eq. 2.21 could be a good candidate for continuum modelling of the automated vehicles' traffic flow. However, this model is derived based on oversimplifying derivation processes and the constant propagation speed $C_0$ results in several theoretical and behavioural issues. For instance, for sufficiently large relaxation times, the acceleration equation gets decoupled from the continuity equation, and when applied to real-world traffic, the model can violate the anisotropy property near the ramp areas (Mohammadian et al., 2021c). Such issues have significant implications for the CAV traffic flow. From the perspective of this model, vehicles always react to the speed differences ahead regardless of the available spacing. This aspect is in conflict with how CAVs respond based on collision avoidance systems and time to collision, taking into account both speed and spacing (Van Arem et al., 2006; Xiao et al., 2017). Therefore, we argue that the present version of the model above has little implication for CAVs.

### 2.2.4. The gas kinetic theory and relevant continuum models

(1) The gas-kinetic theory

Since the seminal contribution by Prigogine (1961), many continuum models have been derived from the gas-kinetic theory (GKT) for modelling traffic flow. The GKT framework is an intermediate approach between the microscopic and macroscopic approaches. The main question in this class of models is to identify the existence probability of vehicles sharing the same physical properties (e.g., speed) at a specific time and location. To this end, the standard spatiotemporal coordinate $(X, t)$ is replaced with a phase-space co-ordinate $\vec{F}(x(t), V(t), ..., t)$ in which all state variables are parameterized with respect to time. The general formulation of GKT-based models is the Boltzmann equation:

$$\partial \tilde{\rho}(\vec{F})/\partial t + \nabla_{\vec{F}}\big(\tilde{\rho}(\vec{F})\, d\vec{F}/dt\big) = \big(\partial \tilde{\rho}(\vec{F})/\partial t\big)_{interation} \qquad (2.22)$$

where $\tilde{\rho}(\vec{F})$ is the phase-space density, which represents the probability distribution of vehicles in the spatio-temporal region $[x - dx/2, x + dx/2] \times [t - dt/2, t + dt/2]$, whose speed are in the range $[V - dV/2, V + dV/2]$. The left-hand-side of Eq. 2.22 is like the acceleration equation in continuum models. The physical interpretation of Eq. 2.22 is that the total change in the density of vehicles sharing the same physical properties is dependent on how they interact with one another. The right-hand-side of Eq. 2.20, the interaction term, is postulated differently in all gas-kinetic-based models.

Prigogine and Andrews (1960) defines phase-space density as $\tilde{\rho}(x, V, t)$, where they assumed that there exists a distribution for the desired speed with respect to time and space. (Munjal and Pahl, 1969) defined the desired speed as a function of traffic density. Paveri-Fontana (1975) assumed that there exists an individual characteristic desired speed, which is considered time-independent during a journey. The phase-space density in this model is defined as $\tilde{\rho}(x, V, V_d, t)$ in which $V_d$ represents the desired speed of each driver. Nelson (1995) improved the underlying assumption of this model by introducing a correlation between various vehicle classes. It has been argued that Prigogine's (1961) model can predict scattering in the flow-density diagram for congested traffic (Nelson and Sopasakis, 1998).

Continuum models can be derived from gas-kinetic models by the method of moments, as in probability theory (Hoogendoorn, 1999). As will be discussed in Section 3, many of the existing models for CAV traffic flow are GKT-based. The following section reviews the GKT-based NE models along with their underlying assumptions. To avoid overlapping, mathematical formulations of some of these models are excluded and discussed when only reviewing models developed for CAV traffic flow.

(2) NE models based on the gas kinetic theory

Phillips (1979) derived the first GKT-based NE model from a modified version of the model by Prigogine and Herman (1971), where the length of vehicles is also considered and the relaxation time is considered a function of density. The model by Phillips (1979) included a third dynamic PDE for the conservation of velocity variance. Helbing (1996b) showed that a hierarchy of continuum models including the LWR and existing PW-like models at the time could be consistently derived from the GKT model by Paveri-Fontana (1975). Helbing (1996b) showed that the basic LWR model could also be derived from the gas-kinetic theory, where in this case, the functional form of the FD is no longer predefined. Instead, the FD is derived from the definition of steady-state traffic, where the resulting functional form is based on some parameters, such as the probability of LC manoeuvres, which cannot be easily postulated in real-world traffic. Helbing (1996b) also proposed a generalized form of NE models with a third dynamic equation for the velocity variance. It is worth noting that the third equation can allow for new families of traffic waves when both speed and density are homogeneous ($\partial \rho / \partial x = \partial V / \partial x = 0$), which may not be practical in real-world traffic[6].

Treiber et al. (1999) proposed another GKT-based NE model, where a CF relationship also determines driver behaviour at the microscopic level. Central to the underlying CF relationship is that the traffic condition ahead ("interaction zone") also affects the steady-state speed which is derived implicitly. The acceleration equation in this model is defined as:

$$\partial V / \partial t + V \partial V / \partial x = (V_e^*(\rho, V, \rho_a, V_a) - V)/\tau - 1/\rho \, \partial(\sigma_V^2(\rho, V))/\partial x \qquad (2.23)$$

where $\sigma^2(\rho, V) = A(\rho)V^2$ with $A(\rho) = A_0 + \Delta A[1 + \tanh((\rho - \rho_{cr})/\Delta \rho)]$ being the variance perfector in which $A_0$ and $\Delta A$ are constants, $\rho_{cr}$ is the critical density, and $\Delta \rho$ is the width of transition from free flow to congestion. $V_e^*$ is the "target speed" which is found by:

$$V_e^*(\rho, V, \rho_a, V_a) = V_{max}(1 - \frac{A(\rho)}{A(\rho_{jam})}(\frac{\rho_a T}{1 - \rho_a/\rho_{jam}})^2 B(\delta V)) \qquad (2.24)$$

where $\rho_a$ and $V_a$ are the non-local state variables at the interaction zone ($X_a = X + \gamma(1/\rho_{jam} + VT)$) in which $\gamma$ is the anticipation factor, and $B(\delta_V) = 2[1/\sqrt{2\pi} \exp(-\delta_V{}^2/2) + (1 + \delta_V{}^2) \int_{-\infty}^{\delta_V} 1/\sqrt{2\pi} \exp(-\delta_V{}^2/2)]$ is the Boltzmann factor with $\delta_V = V - V_a \, /\sigma_V$.

Like diffusion terms, the non-local interaction in this model introduces smoothing effects on sharp shockwaves and subject them to a finite physical width. However, the effects of non-localities are propagated only forward which prevents the model from exhibiting the wrong-way-travel effect (Helbing and Johansson, 2009). The steady-state equilibrium condition in this model is implicitly given based on the definition $\partial \rho / \partial x = \partial V / \partial x = 0$ which is obtained with $\rho = \rho_a$ and $V = V_a$. While the implicit steady-state condition in this model is consistent with the underlying behavioural premises, the non-local GKT model suffers from two behavioural issues.

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[6] To elaborate, it should be mentioned that the term speed variance in GKT-based models is comparable to temperature for other physical system e.g., gas flows. For such systems, kinematic waves can still occur if density and speed are the same for a given set of the upstream and downstream states, but the gas temperature is not the same. A similar case in traffic flow could be a situation where speed and density are the same at the upstream and downstream, but speed variance is different. However, such a condition may not be widely observable and therefore, its prediction by the GKT-based models could be of little practical relevance.

First, the model is inconsistent with the LWR model at the zero-relaxation time limit. To elaborate, we multiply the acceleration equation given in Eq. 2.23 by relaxation time ($\tau$) and take the limit $\lim \tau \to 0$, which results in $V = V_e^*(\rho, V, \rho_a, V_a)$, which does not necessarily satisfy the steady-state equilibrium condition unless. Furthermore, the function $V_e^*$ can take negative values because some parts of the non-local speed adaptation are placed in this term. The non-local GKT model can, therefore, yield non-physical results (i.e., $V = V_e^*(\rho, V, \rho_a, V_a) < 0$) at the zero-relaxation time limit.

Meanwhile, GKT-based models with two dynamic equations are prone to a conceptual inconsistency with the stead-state traffic. To elaborate let $\rho_{st}$ and $V_{st}$ be the density and speed in the steady-state traffic, which, by definition, is any traffic condition subject to $\partial \rho_{st}/\partial t = \partial \rho_{st}/\partial x = V_{st}/\partial t = \partial V_{st}/\partial x = 0$ for a sizeable window in time-space diagram (Zhang (1999-hysteresis)). From the perspective of the GKT model, the steady-state traffic condition is restricted to those traffic conditions, where there exists speed variance among the underlying vehicles given as $\sigma_{V_{st}}^2 = A(\rho_{st})V_{st}^2$ subject to $\partial \sigma_{V_{st}}^2/\partial t = \partial \sigma_{V_{st}}^2/\partial x = 0$. This restriction brings two downsides. First, such a steady-state traffic condition may not frequently be observable in real-world traffic because in practice, speed variance leads to the spatial dispersion of vehicles over time violating both $\partial V_{st}/\partial t = 0$ and $\partial V_{st}/\partial x = 0$ for a moving window travelling with the average speed. Second, the steady-state condition of the GKT model is inconsistent with the car-following stead-state traffic where vehicle's speed should be invariant across different vehicles. In Section 4.3., we discuss how this issue is especially critical for modelling CAV traffic flows.

Nevertheless, it has been shown that the non-local GKT model can replicate many empirical observations of real-world traffic, such as the widely scattered traffic states and traffic instabilities in the flow-density diagram and scattering (Helbing, 2001; Mohammadian et al., 2021c; Treiber and Helbing, 1999). Ngoduy (2012b) extended the model by Treiber et al. (1999) to incorporate "aggressive" and "timid" driving behaviours, and the results suggested that traffic is more stable with aggressive driving, whereas the width and amplitude of traffic instabilities increase by timid driving. Hoogendoorn and Bovy (2000) derived another GKT-based model for multi-class traffic, where the underlying gas-kinetic model (i.e., the model by Hoogendoorn and Bovy, 1998) is a multi-class extension of the model by Paveri-Fontana (1975). Hoogendoorn and Bovy (2000) showed that the model is generic in the sense that with a proper choice of parameters, the model can be identical to some previous models (e.g., Helbing, 1996b; Payne, 1971; Phillips, 1979).

Meanwhile, Tampère et al. (2003) followed a distinct approach, focusing on incorporating human factors in the continuum framework. They derived a GKT-based model, known as the "human-kinetic" model, in which driver alertness has been introduced as an additional parameter. Driver alertness in this model explicitly affects the car-following behaviour of vehicles, i.e., that vehicles tend to maintain the desired spacing and to synchronize their speed with that of their leading vehicles.

## 3. Review of continuum models incorporating connected and automated vehicles

CAVs are expected to reshape the properties of the conventional traffic flow because of the use of communication and automated-driving technologies.

Automated vehicles can perform a wide variety of driving tasks at five different automation levels (SAE, 2014). At lower levels, the driving tasks performed by CAVs primarily include acceleration/deceleration as well as adaptive cruise control (ACC), where a vehicle can regularly adjust its speed concerning the available spacing to its immediately preceding vehicle. At the higher levels of automation, more complex driving tasks are performed by CAVs, such as steering and lane-changing, which requires the communication of the vehicle and the surrounding environment. The highest automation level is a situation where the automated vehicles can substitute human drivers in all the complex aspects of driving (e.g., real-time decision making, route choice, etc).

Connected vehicles communicate information about traffic conditions in a broader range than observable by human drivers, and the information can be used by either human drivers or the automated driving system. If the information is used by human drivers, then the influence of the connected environment on traffic flow depends on how drivers cooperate and respond to the information, meaning human factors will play a major role in the impact of the connected vehicles on traffic flow. Such a complex situation will be discussed more critically in Sections 4.3.

On the other hand, as long as the information is utilized and responded to by the automated driving system, then the role of human drivers will be controlled or eliminated. For instance, cooperative ACC driving (CACC) is a special case of CAV traffic at lower levels of automation, where a vehicle can communicate information with more than one preceding vehicle. ACC and CACC driving will probably be

the most prevalent level of CAVs at the early stages of introducing the CAVs into the traffic stream. Another aspect of CAVs is platoon-based driving in which automated vehicles can platoon and adopt a steady speed and constant inter-vehicular spacing (Sharma and Zheng, 2021).

In the following, we present a review of the existing continuum models for CAV traffic flow, where most of these models focus on the ACC or CACC aspect of CAVs.

### 3.1. Continuum models developed for traffic flow of CAVs

Studies on the continuum modelling of CAVs date back to the work of Darbha and Rajagopal (1999), where the PW model (Payne, 1971; Whitham, 1974) was modified to investigate the effects of vehicles equipped with adaptive cruise control (ACC) on traffic flow. Through stability analysis and numerical experiments, the authors suggested that with a constant-headway policy, ACC vehicles could aggravate traffic instabilities. Later on, Li and Shrivastava (2002) argued that the findings in Darbha and Rajagopal (1999) could be biased in that the external factors in the numerical experiments, such as flow exchanges at the boundaries, were not controlled. To eliminate this effect, they investigated traffic flow behaviour on a homogeneous circular road, and the numerical experiments and stability analysis suggested improved stability in the traffic flow. Another study by Yi and Horowitz (2006) derived more comprehensive stability criteria for traffic flow in the presence of ACC-equipped vehicles through performing a non-linear stability analysis on the extended version of the PW model. These studies assumed that the traffic flow is fully populated with ACC-equipped vehicles.

Tampère et al. (2009) proposed a multi-class variation of the human-kinetic model by Tampère (2004) for mixed traffic flows involving conventional and connected vehicles. The model assumes that conventional and connected vehicles have the same car-following properties, but the level of driver alertness is higher for connected vehicles, due to receiving information about the traffic ahead in advance. The authors suggested that traffic flow stability increased with an increase in the penetration rate of connected vehicles.

Ngoduy et al. (2009) presented another continuum model for mixed traffic flows involving conventional and connected vehicles, where the information can travel between the connected vehicles. The model assumes that when receiving the information, connected vehicles can start decelerating smoothly before perceiving the congestion in their visible range. The propagation of information for vehicles follows the form:

$$\partial \phi_u / \partial t + \xi \, \partial \phi_u / \partial x = -\gamma \phi_u \qquad (3.1)$$

where $\phi_u$ is a smooth normalized function in the range [0 1] and defines the probability of vehicles upstream receiving information ($\phi_u = 0$ for a conventional vehicle), and $\xi$ is the propagation velocity of the message, which reflects the transmission range. The term $\gamma \phi_u$ is introduced to nullify the effects of information beyond a certain spatial range, with $\gamma$ acting as the damping rate. The function is then incorporated into the acceleration equation at the CF level. Ngoduy et al. (2009) derived a multi-class NE model from the gas-kinetic model by Hoogendoorn and Bovy (2000). With the same continuity equation as in Eq. 2.8, the acceleration equation is:

$$\partial V_u / \partial t + V_u \partial V_u / \partial x = \left(V_{e,u} - V_u\right)/\tau_u - 1/\rho_u \, \partial P_u / \partial x \qquad (3.2)$$

where $P_u = \rho_u \Theta_u$ is the so-called pressure function defined based on the velocity variance and $V_{e,u}$ is the equilibrium speed of class $u$, defined as:

$$V_{e,u} = V_{max,u} - (1 - \eta_u) \, \tau_u \sum_1^U \Pi_{us} \qquad (3.3)$$

with $\eta_u$ being the probability of LC manoeuvres for class $u$ and $\Pi_{us}$ is the braking rate for class $u$ and any vehicle class ($s$). Ngoduy et al. (2009) used the model to study the mixed traffic flow for different traffic compositions, and the results suggest that both the capacity and stability of traffic flow improve as the penetration rate of connected vehicles increases. Ngoduy (2012a) presented another continuum model for mixed-traffic flows of conventional and automated vehicles from the gas-kinetic model by Hoogendoorn and Bovy (2000), in which the acceleration mechanisms of ACC-equipped vehicles obey the CF relationship provided by Davis (2004). The acceleration equation for this model is:

$$\partial V_u / \partial t + V_u \partial V_u / \partial x = \left(V_{e,u} - V_u\right)/\tau_u - 1/\rho_u \, \partial P_u / \partial x + (\gamma_u / \tau^*) \, \partial V / \partial x \qquad (3.4)$$

where $\gamma_u = 1$ for ACC vehicles, and $\gamma_u = 0$ for conventional vehicles. Similar to previous works, the numerical tests for this model also suggest that the stability and the capacity of traffic flow improve with increases in the penetration rate of ACC-equipped vehicles.

Ngoduy (2013a) extended the model by Treiber et al. (1999) for studying traffic flows of CACC-equipped vehicles, where the cooperative driving element is derived from a multi-anticipative CF relationship. The acceleration equation in this model is presented based on traffic flow as:

$$\partial V/\partial t + V\partial V/\partial x = (V_e^*(\rho, V, \rho_a, V_a) - V)/\tau - 1/\rho\, \partial(AV^2)/\partial x + v_{Acc} \tag{3.5}$$

where $v_{Acc} = 1/\rho\partial V/\partial x \sum_{n=1}^{N} n/\tau_n^*$ incorporates the acceleration of CACC vehicles, with $N$ being the number of preceding vehicles that a follower can interact with, and $N = 1$ accounting for ACC traffic flow without cooperation. $\tau_n^*$ is the state-dependent relaxation time of vehicles to adapt their speed to their $n^{th}$ leader and is defined as:

$$1/\tau_n^* = 1/(2\tau_m^0)[1 + \tanh((\rho - \rho_{cr})/\Delta\rho)] \tag{3.6}$$

where $\tau_m^0$ is the maximum relaxation time of vehicles under the jam condition. It is assumed that the condition $\tau_1^* < \tau_2^* < \cdots < \tau_N^*$ is satisfied within a platoon. Numerical simulations and stability analysis suggested that both CACC and ACC traffic have greater flow capacity and stability range than conventional traffic, with CACC traffic performing the best. Another continuum model for automated vehicle platoon-based driving was presented by Ngoduy (2013b). The model is based on the GKT-based model by Hoogendoorn and Bovy (2000), where the CF model by Helbing and Tilch (1998) was extended for a platoon of vehicles. It was established in the aforementioned study that platoon-based driving significantly increases the stability of traffic.

Delis et al. (2015) proposed a generic variation of the GKT-based model for comparing the traffic flow of ACC and CACC vehicles with that of conventional traffic. The acceleration equation in this model is:

$$\partial V/\partial t + V\partial V/\partial x = (V_e^*(\rho, V, \rho_a, V_a) - V)/\tau\,[1 - \beta F(\rho)] + (\alpha v_{Acc} - 1/\rho\, \partial(AV^2)/\partial x) \tag{3.7}$$

where $\alpha = \beta = 1$ defines CACC driving, $\alpha = 1$ and $\beta = 0$ define ACC driving, and $\alpha = \beta = 0$ describes conventional traffic as in Treiber et al. (1999), where $F(\rho) = [1 + \tanh((\rho - \rho_{cr})/\Delta\rho)]$ and the term $v_{Acc}$ is derived by:

$$v_{Acc} = \frac{1}{\rho}0.5\sum_{n=1}^{N}\frac{\rho^* V_n^* - \rho V}{\tau_n^*}F(\rho) \tag{3.8}$$

where $V_n^*$ refers to the non-local speed for vehicle $n$ computed at $X_n^* = X + n.*\gamma^*(1/\rho_{jam} + T^*V)$, and $\rho^* = 1/(1/\rho_{jam} + T^*V_n^*)$ and other parameters are previously defined. While the model by Ngoduy (2013a) defines the acceleration of ACC in terms of gradients, Delis et al. (2015) adopted a non-local approach and incorporated the acceleration of ACC/CACCs into the relaxation term. Similar to the work of Ngoduy (2013a), the study concluded that both ACC and CACC vehicles improve the stability of traffic flow. Compared to the model by Ngoduy (2013a), the stabilizing effects of ACC/CACC vehicles are more significant in the model by Delis et al. (2015). The instability range of the model is analytically discussed in more detail in Porfyri et al. (2015).

Delis et al. (2018) proposed a multi-lane variation of the model, where each lane has its own dynamic equation, allowing vehicles to change lane depending on the relative traffic condition between the lanes. Such an approach can be suitable for CAV traffic flow as it can address some of the issues of continuum modelling associated with the single-pipe treatment of traffic flow. However, the model by Delis et al. (2018) can become intricate for freeways with many lanes. Furthermore, the mechanism for LC manoeuvres in this model is phenomenological and based entirely on the relative traffic condition between the adjacent lanes, which may not adequately reflect the human factors of LC manoeuvers and their effects on traffic flow properties (Zheng, 2014).

Zheng et al. (2015) derived a NE model accounting for a situation in which drivers receive information about traffic conditions behind with potential applicability to the connected environment. The resulting acceleration equation in this model is derived from the CF found by Helly (1959):

$$\partial V/\partial t + (V - C_0)\partial V/\partial x = r_f \alpha_f(1/\rho - 1/\rho_e^f(V)) - r_b \alpha_b(1/\rho - 1/\rho_e^b(V)) \tag{3.9}$$

where $C_0$ is a constant sonic velocity, indices $f$ and $b$ refer to traffic condition ahead and behind respectively, $\rho_e(V)$ defines the equilibrium density, r is a weight factor defined for balancing the attention to traffic condition ahead and behind, and $\alpha(1/s^2)$ is the sensitivity coefficient which is comparable to relaxation time. In this model, the effects of the available information on the acceleration are placed in source terms based on the local traffic condition. However, the available information on both sides is expected to affect the anticipatory acceleration in real-world traffic. Ngoduy and Jia (2017) incorporated the multi-anticipative driving in the CF model by Helly (1959) and derived a generic variation of the model by Zheng et al. (2015), where the effects of multi-anticipative driving are placed into a gradient term. The authors suggested that multi-anticipative driving improved the operational properties and the stability of traffic flow.

It has been argued that the models above (i.e., the models by Ngoduy and Jia, 2017; Zheng et al., 2015) are "anisotropic" in that, under the condition $C_0 > 0$, no wave can travel faster than traffic speed. However, depending on the choice of parameters, the traffic condition behind can affect the local acceleration in both the models. This suggests that the definition of the anisotropy by Daganzo (1995b) i.e., individual vehicles are not in general affected by traffic conditions behind, is no longer valid for these models. Section 4.2.2 comprehensively investigates the issues around faster-than-traffic waves and the anisotropic property with a focus on their applicability and significance for CAV traffic flow.

Meanwhile, Jin and Yang (2013) adopted a distinct, parsimonious approach to incorporate platoon-based driving into the cell transmission model. Their model tracks the position of a platoon in the time-space domain and reconstructs road cells based on the space occupied by the platoon. This feature allows the model to adapt the supply and demand function of road cells based on the platoon-based driving.

## 4. Discussion: existing issues, research gaps, and future direction

This section provides a critical perspective on various continuum modelling frameworks and concludes by highlighting research needs and future directions in the era of CAVs.

To this end, first, the essential properties and challenges of continuum models are revisited for modelling the conventional traffic, and then, their applicability for CAV traffic flow are discussed. Wherever applicable, empirical evidence or numerical tests are utilized to back up the discussions. The numerical method used is based on a class of approximate Riemann solvers (see Mohammadian et al., 2021b for performance analyses for continuum traffic flow models), but the details of the method are not presented in order to save space and to keep the discussion focused. The discussions are mainly aimed to highlight the problems and point out research needs, but the paper also discusses the potential solutions wherever applicable, which we hope can inspire other researchers to adopt and test them.

Next, we discuss the limitations of the existing continuum models of CAV traffic and highlight the issues inherent to CAV traffic flow, considering recent empirical findings on CAVs and driver behaviour. The discussions are concluded by highlighting important research needs and drawing some future directions.

### 4.1. Revisiting extended equilibrium models against complex traffic phenomena and implications for CAVs

Multi-class LWR (MC-LWR) models are another family of extended equilibrium models that are motivated primarily by the heterogeneity of diver-vehicle units. It has been argued that overall, these models have more predictive accuracy compared to the standard LWR model (Qian et al., 2017). Many of the MC-LWR models can account for truck and passenger-car flow separately and thereby consider the critical density as a dynamic property of the traffic composition. Some of these models define three-state traffic in which it is possible for faster classes to experience congestion while heavy vehicles still experience a free-flow regime (Logghe and Immers, 2008; Qian et al., 2017). It has been argued that this approach can also be useful for modelling moving bottlenecks (Logghe and Immers, 2008). However, the existence of class-specific FDs might be open to debate, as a distinct observation of steady-state traffic for each class may not be easily obtained from detector data. Even if complete vehicle trajectories are available, the percentage of heavy vehicles experiencing steady traffic states, for a wide range of traffic densities, could be too limited to postulate a valid fundamental diagram.

Accounting for traffic composition also enables the MC models to exhibit some levels of scattering in the resulting flow-density diagram (e.g., Ngoduy and Liu, 2007). Therefore, several researchers have argued that these models can reproduce the associated complex phenomena with scattering such as capacity drop and hysteresis (Ngoduy and Liu, 2007; Wong and Wong, 2002). However, the scattering in the resulting flow-density diagrams cannot be ascribed to non-equilibrium traffic conditions. This is

because all traffic states on the flow-density diagram could be placed on a set of equilibrium curves depending on the traffic composition.

Another important question is whether the discontinuities observed in the resulting flow-density diagrams of an MC-LWR model could be interpreted as a capacity drop. Several researchers have described the discontinuities in the flow-density diagrams of the MC-LWR model as consequences of the interaction between vehicle classes (e.g., Ngoduy and Liu, 2007; Wong and Wong, 2002). Although there is no consensus about the factors affecting capacity drop, the reason why the LWR model cannot describe it is an established fact due to the equilibirium traffic and symmetric phase trajectories over the FD. In other words, traffic flow for a given transition from congestion to a free-flow zone is at a maximum, which is the same as the transition from a free-flow to a congestion zone. MC-LWR models also apply the entropy condition for vehicles within each class. However, in a majority of these models, the speed of vehicle classes is different only in free-flow traffic. Therefore, when vehicle classes are considered altogether, the average flow upstream of a congestion zone could be slightly higher compared to downstream because faster vehicles reach congestion upstream sooner than slower ones. Such a flow drop could stem from averaging, which is distinctively different from the capacity drop frequently reported in the literature, which can happen in traffic flow with a single-vehicle class because of the difference between acceleration and deceleration rates.

Similarly, MC-LWR models may not be suitable for studying the hysteresis phenomenon in traffic flows. It has been argued that the MC-LWR models show some closed loops in the phase diagram of the upstream of the bottleneck (van Wageningen-Kessels et al., 2015). However, the loops produced by the MC-LWR models are not necessarily the genuine hysteresis loops because they cannot be explained by the difference between acceleration/deceleration rates. This is because MC-LWR models all belong to the equilibrium framework and their speed adaption mechanism depends primarily on the continuity equation. Furthermore, the phase trajectories predicted by MC-LWR models are collected from fixed-point locations, which means variable groups of vehicles with different compositions of vehicle classes are compared with one another. It is worth noting that even if such loops are observable on the trajectory of moving observers, their patterns may not necessarily be in line with observed in real-world data due to the role of single-pipe treatment in changing the structure of lane-based hysteresis loops.

Finally, it has been widely acknowledged that these nonlinear phenomena are consequences of traffic instabilities in non-equilibrium regimes (Newell, 1962). Recent studies have suggested that the role of human factors (e.g., aggressive/timid driving and task difficulty) in the emergence of these phenomena is significant (Laval, 2011; Laval and Leclercq, 2010; Saifuzzaman et al., 2017; Tampere et al., 2005). MC-LWR models cannot account for either non-equilibrium traffic conditions or the impact of human factors, as the heterogeneity in these models is mainly defined as the operational properties of vehicle classes such as the desired speed and the gap acceptance behaviour, and all classes have the same acceleration behaviour as the standard LWR model.

The issues discussed here could be more pronounced for the case of CAV traffic flow. On the surface, the MC-LWR models may be the most suitable framework for mixed traffic with CAVs because a mixed CAV traffic flow is indeed multi-class. However, such a traffic condition has unique properties, which may not be easily accommodated in MC-LWR models. Unlike conventional multi-class traffic, where the main difference between vehicle classes is their desired speed and vehicle lengths, CAVs and conventional passenger cars are the same in this regard. Instead, some other properties of CAVs (e.g., response time, acceleration and deceleration behaviour, etc.) distinguish them from conventional passenger cars (Sun et al., 2018, 2020). Such differences and their behavioural implications may not be easily accounted for in MC-LWR models. Furthermore, one of the key questions regarding mixed CAV traffic flow is their stability, which is clearly beyond the capability of the MC-LWR framework.

### 4.2.    Revisiting the challenging aspects of non-equilibrium models

Section 3 discussed that many existing continuum models for CAV traffic flows are non-equilibrium models based on the gas-kinetic theory (e.g., Delis et al., 2015; Delis et al., 2018; Ngoduy, 2013a, b; Ngoduy et al., 2009; Ngoduy and Jia, 2017; Ngoduy and Wilson, 2014). In this section, we examine two critical issues applicable to such non-equilibrium modelling frameworks, namely, the issue of faster-than-traffic waves and the relaxation terms, while revisiting their significance for CAV traffic.

### 4.2.1. Revisiting the confusions and controversies around faster-than-traffic waves

The concept of the anisotropy in traffic flow and whether the earlier NE models can preserve it has been a matter of controversy ever since the famous paper by Daganzo (1995b). The anisotropic property suggests that drivers are mainly affected by the traffic condition ahead and should generally not be influenced by what happens behind them. Daganzo (1995b) argued that the NE models existing at the time violated the anisotropic property through two separate mechanisms, one of which being the emergence of faster-than-traffic waves[7] (Daganzo, 1995b; Del Castillo et al., 1994). It has been demonstrated that these waves can send information about the upstream traffic condition to downstream vehicles and force them to accelerate or decelerate (Daganzo, 1995b; Zhang, 2000a). These findings encouraged researchers to propose models devoid of faster-than-traffic waves (e.g., Aw and Rascle, 2000; Colombo, 2002; Lebacque et al., 2007b).

While there is now a consensus in the literature that in the absence of faster-than-traffic waves, the information from the characteristic speeds cannot reach vehicles from behind and affect their dynamics, there have been other sorts of confusion and controversies around the linkage between such waves and the anisotropy property.

First, in the literature, the absence of faster-than traffic waves has been confused with a sufficient condition for guaranteeing the anisotropy property. For instance, in the model by Zheng et al. (2015), drivers can adapt their speed with respect to bi-directional traffic information, including traffic condition behind, but since such adaptations are placed in the source terms, and the model is devoid of a faster-than-traffic characteristic speed, it has been considered as an 'anisotropic' continuum model. However, the definition of the anisotropic property by Daganzo (1995b) requires the models' car-following behaviour in traffic flow not to be affected by traffic conditions behind in any form. Another example is the model by Jiang et al. (2002), which is devoid of faster-than-traffic waves, but can exhibit non-anisotropic behaviours (e.g., dispersing congestion regions downstream of the ramp areas) when implemented on real-world traffic (Mohammadian et al., 2021c) because its distinct mathematical structure requires the use of central numerical scheme (see Mohammadian et al., 2021c for a detailed discussion). These examples show that the absence of faster-than-traffic waves should not be considered sufficient for preserving the anisotropy property.

Second, there has been a controversy around whether the presence of faster-than-traffic waves is a sufficient condition for violating the anisotropy property. Helbing and Johansson (2009) suggested that faster-than-traffic waves did not necessarily violate the anisotropic property (Helbing, 2009; Helbing and Johansson, 2009; Zhang, 2009). The authors associated these waves with the group velocity, i.e., the velocity at which the overall shape of traffic waves travel. Through theoretical analyses and numerical tests, they showed that these waves could not be artifacts of the derivation of continuum models from microscopic relationships because they are also observable in the underlying CF relationships. Zhang (2009) countered that the numerical example given by Helbing and Johansson (2009) could be unrealistic in real-world traffic because it suggests that a line of vehicles can be triggered to accelerate beginning from back to front. Furthermore, Zhang (2009) showed that faster-than-traffic waves could result in convexities in the flow-density diagram, meaning information carried by faster-than-traffic characteristics could reach vehicles from behind and affect their dynamics. In the following, we re-examine the effects of faster-than-traffic waves on both NE and MC-LWR models.

A simple numerical test can verify whether a faster-than-traffic characteristic in the NE models can reach vehicles from behind and affect their dynamics. As per the anisotropic property, vehicles in ongoing traffic (Figure 1-a) should not react to the state of traffic light upstream. In other words, if a traffic light upstream turns red, the speed of vehicles downstream should not be affected. Figure 1-b illustrates this situation through numerical simulation of the ARZ model (Aw et al., 2002; Zhang, 2002). When a shockwave emerges upstream of a traffic light, a contact discontinuity separates the empty road and the downstream vehicles. Traffic speed and density are not affected downstream when a traffic light turns red upstream.

To illustrate the effects of the faster characteristic, the same test case is applied to the improved PW-like model by Zhang (1998) for the case of sufficiently large relaxation time, to control the effects of relaxation term on eliminating faster-than-traffic waves (see, e.g., Zhang, 2009). As discussed, the anticipation term in the model by Zhang (1998) does not produce the wrong-way travel effect, and the only

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[7] The other mechanism is the "wrong-way travel" effect, which is not discussed in detail in this section because it can stem from other factors such as the structure of the anticipation and diffusion terms (see, e.g., Zhang 1998)

concern in the sense of Daganzo (1995b) is the existence of a faster-than-traffic characteristic. Figure 1-c illustrates the road density profile one minute after the traffic light turns red at position $x = 4$km, and it is shown that both density and speed downstream are affected by a deceleration fan due to the obstruction of traffic flow upstream. These results are not in line with the physics of traffic flows, as they suggest that upstream vehicles can determine the movement of traffic downstream, similar to water flow.

To elaborate on this point, a comparable numerical test is applied, in the current paper, to water flow in an open channel, where a water gate closes at a specific time (Figure 1-c, d). It is clear from Figure 1-d that the closure of the water gate results in the emergence of a deceleration fan downstream of the gate, which affects the depth and speed of the water. This is because the movement of water is affected by the upstream momentum, and the obstruction of flow upstream can block this momentum. These results can help illustrate how the faster-than-traffic characteristic in the PW-like models violates the anisotropic property. However, does this necessarily mean that any sort of faster-than-traffic waves can impose undesirable effects on any model?

To investigate this question, we refer once again to the definition of the anisotropy in the sense of Daganzo (1995b). This definition suggests that the anisotropic property is violated if the following vehicles can affect their leaders in any form. One possible situation for this case is the overtaking manoeuvres, where vehicles performing LC manoeuvres may temporarily consider traffic conditions behind in their decision-making process (Zhang, 2003a). Another potential context is multi-class traffic, where some classes are faster than other classes. It has been argued that early MC-LWR models may not preserve the anisotropy because faster vehicles can systematically affect the speed of slower classes (Logghe and Immers, 2003). On the other hand, it has been argued that recent MC-LWR models can preserve the anisotropic property since slower classes are not affected by faster vehicles (Logghe and Immers, 2003; Qian et al., 2017).
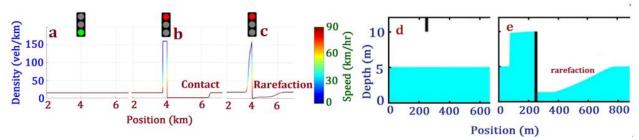


Figure 1- Effects of faster-than-traffic waves in traffic flow. a) Initial condition; i.e., ongoing steady-state traffic. The density profile at $t = 100s$ after the traffic light turns red is given by: b) the ARZ model; c) Zhang's (1998) model with increased relaxation time. For both models, the FD by Del Castillo and Benítez (1995) is used with the parameters: $V_{max} = 90km/hr$, $\rho_{max} = 160$, and $c_{jam} = 15km/hr$; d) Steady-state super-critical water flow in an open channel; this flow regime is comparable with free-flow traffic; e) Depth profile of water flow at t = 100s after the gate closure. To keep the discussion focused on traffic flow, the mathematical details of shallow waters are not presented. See Toro (1998) for more discussion.

Nevertheless, in MC-LWR models, faster vehicles can sometimes have a greater characteristic speed than the average traffic speed. It has been argued that this could be a violation of the anisotropic property for traffic flow as a whole (van Wageningen-Kessels et al., 2013). This is because, in the literature, the existence of any faster-than-traffic wave has been linked with the unconditional violation of the anisotropic property. For instance, we refer to the argument by Ngoduy and Liu (2007), that: "The anisotropic property is satisfied within a vehicle class. However, due to overtaking, a certain characteristic (of faster vehicles) can travel faster than the speed of slower vehicles, which explains the violation of anisotropic property between vehicle classes in the context of multilane traffic flow."

However, although overtaking vehicles in a multi-lane traffic pay attention to traffic condition behind and occasionally drop the anisotropic property (Zhang, 2003a), their traveling faster than the average speed does not necessarily violate the anisotropic property for the downstream vehicles, as long as the slower vehicles downstream are not affected. Mathematically, this condition can be guaranteed if the class-specific characteristic speeds for faster classes are linearly degenerate with respect to the density of slower classes, as in the recent three-state MC-LWR models (Logghe and Immers, 2008; Qian et al., 2017).

Likewise, faster-than-traffic characteristics in MC-LWR models do not necessarily violate the anisotropic property for traffic flow as a whole. This is because the curvature of the resulting FD for all vehicles is the determining factor in checking whether the anisotropic property is preserved (see Zhang, 2009 for some illustrative examples). Suppose the resulting FD is strictly concave in free flows. In that case, no information carried by the overtaking vehicles can affect downstream vehicles, and the anisotropic

property can be preserved for the traffic flow as a whole as well. This means that the existence of faster-than-traffic waves, although necessary, is not a sufficient condition for the violation of the anisotropic property in the framework of MC-LWR models. On the other hand, since MC-LWR models are commonly single-pipe, faster-than-traffic waves in these models can introduce another issue that is the violation of the first-in-first-out property. In other words, MC-LWR models allow faster vehicles to overtake slower ones even in a single-lane highway, which can potentially affect the reliability of the class-specific travel times estimated by these models.

Meanwhile, the adverse effects of faster-than-traffic waves on the performance of continuum models may not be, overall, significant, both from a theoretical and practical perspective.

From a theoretical perspective, faster-than-traffic waves generally do not last long due to the relaxation term (Zhang, 2009), which is present in many models with a characteristic speed faster than traffic. Furthermore, in many cases, faster-than-traffic waves are of the form $\lambda_2 = V + \alpha|C(\rho)|$, where $C(\rho)$ is a property of the fundamental diagram given as $C(\rho) = \rho V_e'(\rho)$ in CF-based models (e.g., Payne, 1971; Whitham, 1974; Zhang, 1998, 2003b) and as $C(\rho) = \sqrt{\partial\Theta/\partial\rho}$ in the GKT-based models (e.g., Ngoduy, 2013a; Tampère et al., 2009; Treiber et al., 1999). Therefore, in both cases, faster-than-traffic waves can be eliminated in certain density ranges, where the condition $C(\rho) = 0$ or $C(\rho) \approx 0$ applies. For instance in the case of CF-based models, if the triangular FD or the FD by del Castillo (2012) are used, the faster-than-traffic speeds are eliminated in the entire free-flow regime. Likewise, in the non-local GKT-based models, the condition $C(\rho) = 0$ is guaranteed in certain density range where speed variance is insensitive to density ($\partial\Theta/\partial\rho \approx 0$), namely, the density ranges below certain small thresholds in the free-flow states and beyond certain thresholds in congested states. Such a density range is closely related to the density rage in which the non-local GKT model is stable (see e.g., Treiber et al., 2010).

From a practical perspective, faster-than-traffic may not affect the models' performances for real-world traffic significantly. For instance, the outcomes of our recent benchmarking study of continuum models against real-world (Mohammadian et al., 2021c) suggest that some models with faster-than-traffic waves (e.g., METANET (Papageorgiou et al., 1990) and the non-local GKT (Treiber et al., 1999)) outperformed many 'anisotropic' models regarding many empirical aspects of real-world traffic (e.g., replicating the scattering phenomenon and travel time errors).

Additional factors may come into play for CAV traffic flow, which could also potentially alleviate the adverse effects of faster-than-traffic waves. For instance, if CAVs significantly stabilize traffic flow, then the instability range in the flow-density plane would shrink. Therefore, the active range of faster-than-traffic waves for the non-local models would also shrink since in these models, faster-than-traffic waves appear primarily in the instability range of the models.

Furthermore, the anisotropic property of traffic flow should be revisited for CAV traffic flow since, in a connected environment, drivers can receive information about traffic conditions more frequently than in conventional traffic, and such information may affect their driving behaviours. For instance, recent driving-simulator studies have found that when preparing for LC manoeuvres in a connected environment, drivers tend to increase their spacing and time gap more significantly than in the conventional environment (Ali et al., 2018; Ali et al., 2019; Ali et al., 2020). If such findings are widely confirmed when CAVs are in, then it could be argued that the issue of faster-than-traffic will be less significant. Therefore, the models involving such waves should not be disqualified, especially if they can overall provide more insight into the behavioural and empirical properties of CAV traffic flow.

### 4.2.2. The importance of the relaxation term and the issues with relaxation time

From Section 2.2, we recall that many CF-based and GKT-based non-equilibrium continuum models employ a relaxation term in the acceleration equation to capture the speed adaptation caused by the tendency of traffic flow to approach the steady-state equilibrium condition. In contrast, ever since the models by Aw and Rascle (2000) and Zhang (2002), there has been a distinct class of continuum models (those without faster-than-traffic characteristic speeds), where the relaxation term is not present in the original form of the models (e.g., Blandin et al., 2011; Garavello and Goatin, 2011; Goatin, 2006). In this section, we discuss the dilemma of the relaxation term in non-equilibrium models and elaborate on the advantages and the issues that arises in the presence and the absence of the relaxation term.

The absence of the relaxation term brings about several benefits from the mathematical and computational perspective. For instance, all possible Riemann problems that can arise at the discontinuities can be studied analytically, and Godunov-type schemes can be developed for the numerical solution of the models (see Chalons and Goatin, 2008; Mammar et al., 2009a). This property can be

significantly advantageous for modelling traffic flow on networks because the Godunov scheme can be viewed as supply-demand analysis (Lebacque et al., 2005), and therefore, be coupled with node models to determine the exact distribution of traffic flow at junctions (see e.g., Garavello and Piccoli, 2006a; Garavello and Piccoli, 2006b).

However, several conceptual problems arise in the continuum models without a relaxation term, which can ultimately cast doubt on whether such models should be considered non-equilibrium. First, any initial condition satisfying $\partial V / \partial x = 0$ is a steady-state condition, suggesting that there is no unique fundamental diagram in the models devoid of a relaxation term. To resolve this issue, Lebacque et al. (2007b) argued that the models by Aw and Rascle (2000) and Zhang (2002) can be interpreted as an extension of the LWR model with 'variable fundamental diagram'. However, such an interpretation does not completely reconcile the concept of the fundamental diagram with such models. This is because another consequence of the absence of relaxation term is that any initial condition satisfying $\partial V / \partial x \approx 0$ results in a long-lasting travelling wave. As a result, deviations from any initially steady-state condition can grow in amplitude and span in the time-space diagram, which is unrealistic with respect to traffic instabilities. Empirical observations and analytical studies have posited that traffic instabilities can only occur in certain intermediately congested states (Treiber and Kesting, 2011; Zheng et al., 2011a; Zheng et al., 2011b), and that not all deviations from steady-state traffic can grow in magnitude and span in the time-space diagram (Helbing et al., 1999; Helbing et al., 2009). Therefore, it could be argued that models devoid of a relaxation term cannot consistently replicate the non-equilibrium traffic and the associated phenomena.

On the other hand, in the presence of the relaxation term, one obtains a unique fundamental diagram, and the conceptual problems discussed do not apply. However, the choice of the relaxation time in such models can impose other issues about replicating traffic instabilities. In many continuum models with a relaxation term, relaxation time is treated as a constant parameter, meaning that relaxation acceleration is only dependent on the difference from the equilibrium condition and is insensitive to the traffic regime. To some extent, constant relaxation time has some benefits for the mathematical tractability of the models. For instance, state-dependent relaxation time can complicate the derivation of the instability range in NE models through nonlinear stability analysis (Yi et al., 2003). Meanwhile, constant relaxation time can introduce other issues.

The shorter the relaxation time, the higher is the relaxation term, and therefore, equilibrium states are recovered sooner. For zero relaxation time limit, consistent NE models should reduce to the LWR model or its variations (see Treiber and Kesting, 2013 chapter 9), which is incapable of capturing traffic instabilities. Similarly, a given NE model may perform poorly in reproducing traffic instabilities depending on the magnitude of the relaxation time. Both the range and amplitude of traffic instabilities predicted by NE models can be affected by the magnitude of the relaxation time.

To elaborate on these points, the nonlocal GKT model by Treiber et al. (1999) is considered. We simulate the emergence of traffic instabilities on a ring road, where the initial traffic density is determined by the following perturbation function.

$$\rho(x, 0) = \overline{\rho} + \Delta\rho \left[ cosh^{-2} \left( \frac{x - x_0}{w^+} \right) - \frac{w^+}{w^-} cosh^{-2} \left( \frac{x - x_0 - \Delta x_0}{w^-} \right) \right]$$

where $\overline{\rho}$ is the initial steady-state density, $\Delta\rho$ is the initial amplitude of perturbation, and all other parameters are shape factors. Two simulation scenarios, $\overline{\rho}_1 = 40 \, veh/km$ and $\overline{\rho}_2 = 32 \, veh/km$, are considered, where $\Delta\rho = 10 \, veh/km$ is applied to both (Figure 2-a, b). Other parameters for the perturbation function and the model are the same as in the original study (Treiber et al., 1999). These scenarios are simulated for three levels of relaxation time: $\tau = 30s$, $\tau = 20s$, and $\tau = 10s$. Figure 2 (c-e) presents the results of the simulation for the first scenario, $\overline{\rho}_1 = 40 \, veh/km$, where the density is relatively high. It is obvious that the structures of the traffic instabilities for the three levels of relaxation time are significantly different. Both the amplitude and the width of traffic instabilities significantly decrease with decreases in the relaxation time, where some levels of traffic instabilities can still be observed for $\tau = 10s$.

The second scenario, $\overline{\rho}_2 = 32 \, veh/km$, clearly highlights the impact of relaxation time on smoothing traffic instabilities (Figure 2 (f-h)). The results show that for $\tau = 30s$, the initial disturbance turns into a local cluster traveling upstream at roughly around $t = 5 \, min$ (Figure 2(f)). However, with $\tau = 20s$, the localized disturbance grows less significantly in amplitude. Although the initial disturbance decreases its speed over time, it does not travel backward until nearly the end of the simulation period. Interestingly, when $\tau = 10s$ is applied, the initial disturbance in the density profile is dampened quickly and traffic becomes stable. The significant differences in the second scenario can be explained by the initial

homogeneous density, $\overline{\rho}_2 = 32\ veh/km$, which is close to the verge of instability for the selected parameter set. Unlike the first one, this scenario is not so congested that some traffic instability can emerge for any value of relaxation time. Thus, the relaxation term in this scenario becomes a determining factor in whether a given non-equilibrium traffic condition can be effectively simulated. These effects of the relaxation time can potentially affect the performance of continuum models in reproducing stop-and-go waves.

When it comes to real-world traffic, the relaxation time can be considered a physically meaningful and observable parameter (Laval and Leclercq, 2008). However, relaxation time is commonly treated as a calibration parameter in many continuum-modelling applications (e.g., Cremer and Papageorgiou, 1981; Papageorgiou et al., 1983; Porfyri et al., 2016; Spiliopoulou et al., 2014). In this regard, among the possibilities, the optimal relaxation time value is selected such that the overall predictive accuracy of the model is maximized. This approach can, however, affect the capability of non-equilibrium models to reproduce traffic instabilities. First, the wavelength and amplitude of stop-and-go waves can significantly differ from the observations, and furthermore, it is not guaranteed if the calibrated relaxation time can even preserve the range of traffic instabilities within the observations.

Relaxation time can also affect the speed of congestion propagation, which is an observable operational parameter. To elaborate on this point, the first scenario (Figure 2 (c-e)) is considered again. It is clear that for different values of the relaxation time, the resulting clusters from the initial disturbance adopt different speeds when propagating backward. With the decrease in relaxation time, the negative slope of the congestion zone in the $x - t$ plane becomes steeper, which means the resulting congestions propagate backward faster. The mechanism behind this is intertwined and twofold. The first one is that different relaxation times can result in different amplitudes of traffic instabilities, each of which is associated with different propagation speeds in the flow-density diagram. The second one is related to the mathematical effects of source terms, which are commonly excluded from the formulations of continuum models when deriving the characteristic speeds. In other words, source terms act as external forces and can affect the propagation speeds, and their effects can become significant for shorter relaxation times. Similar effects of source terms on other physical systems have been widely discussed (see LeVeque, 2002a chapter 19). However, the physical interpretation of these effects for traffic flow has not been discussed in detail.

It is worth noting that the speed of the congestion propagation is an observable parameter of traffic flow, and is of a universal order of $14 - 20$ km/hr (Zheng et al., 2011a; Zielke et al., 2008). This means that from the applicational point of view, the choice of relaxation time must be such that the speed of the congestion propagation is as close as possible to that of the observations. However, this may not necessarily be guaranteed by calibrating the relaxation time, as commonly done in the literature.
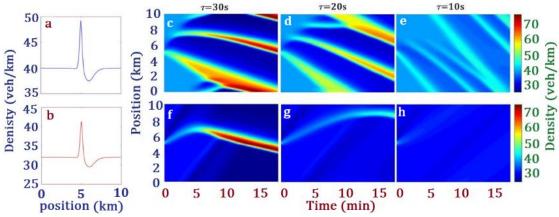


Figure 2- simulation of stop-and-go traffic with the non-local GKT model by Treiber et al. (1999), where the simulation parameters are $V_{max} = 90km/hr$, $\rho_{jam} = 150\ veh/km$, and $T = 1.2s$, and all other parameters are the same as in the original paper. a,b) The initial condition for scenarios one and two respectively; c-e) Scenario one: $\overline{\rho}_1 = 40 veh/km$ f-h) Scenario two: $\overline{\rho}_2 = 32 veh/km$

The issues around the relaxation term and relaxation time could be more significant for the case of mixed CAV traffic flow. On the one hand, automated vehicles adopt a headway control policy. Recent field studies on the automated vehicles have found that constant time-headway policy results in a unique fundamental diagram in the steady-state traffic condition (Shi and Li, 2021). Besides, headway control aspect of adaptive cruise control is based on a desired headway to be set by drivers, meaning that automated vehicles adjust their headway even when there is no speed difference between the AVs and their

leader. These findings further endorse the idea that a relaxation term is needed in non-equilibrium models for modelling automated vehicles.

On the other hand, due to headway control, the relaxation time for CAVs could differ from that for conventional vehicles. Furthermore, for mixed CAV flows, response time could differ between conventional vehicles operated by human drivers, CAVs operated by the automated system, and CAVs operated by human drivers (Sharma et al., 2019b). Such differences are likely to be more pronounced across different traffic condition, and consequently, constant relaxation time for all classes across all traffic condition could be more questionable.

Furthermore, even if distinct relaxation time is defined for different classes, the issue of relaxation time could still affect other essential aspects of modelling mixed CAV flows. For instance, a key motivation behind continuum modelling of mixed CAV flows, both in the existing literature and future directions, is to understand how CAVs would affect the stability of traffic flow (Sun et al., 2018). In the existing models for CAVs (e.g., Delis et al., 2015; Ngoduy et al., 2009), the relaxation time is often selected arbitrarily from a typical range, and stability of the mixed traffic is studied under various scenarios. As discussed, however, even in the typical range, different values of relaxation time can have different determining effects on whether traffic instabilities would emerge or dissipate. As a result, the models may not be able to adequately capture traffic instabilities in mixed traffic. This is especially the case when it comes to implementing models to real-world scenarios, where the relaxation time is commonly treated as a calibration parameter and optimized with respect to calibration considerations rather than stability criteria.

### 4.3. Revisiting the continuum models incorporating CAVs: research gaps and future directions

The primary motivation behind having CAVs in traffic stream is to improve the efficiency and safety of traffic flow which, if achieved, can bring other side benefits such as eco-friendly driving and economic profits (Guanetti et al., 2018; Lu et al., 2014). The positive impacts of CAVs on traffic flow are hypothesized primarily based on the assumption of keeping the role of human drivers to a minimum and full cooperation of drivers with CAVs (Kim, 2015). For instance, by taking control of driving tasks, automated vehicles are expected to maintain smaller time gaps and spacings (Jia et al., 2014), which will directly influence traffic flow capacity. This would also enable CAVs to perform platoon-based driving, which is expected to prevent or alleviate the stop-and-go waves by reducing the time lag, and therefore, stabilizing traffic flow (Sun et al., 2020). However, since CAVs are not yet widely introduced in traffic, their expected potential benefits would be subject to uncertainty. Therefore, it is essential to consider comprehensive scenarios covering different stages and operation levels of CAVs to investigate whether the existing continuum models can adequately capture the potential effects of CAVs on traffic flow.

### 4.3.1 Traffic flow of CAVs

We first consider a scenario where CAVs fully populate traffic flows and operate at the highest automation level effectively. In this case, many basic assumptions underlying the existing models are likely to be overall more plausible than they are for current traffic conditions where the human drivers are in charge, and heterogeneities are neglected. After all, many of the existing continuum models already treat drivers as programmed units whose driving behaviours are determined based entirely on physical laws. For such fully automated traffic, the main question will be whether the existing continuum models adequately reflect the longitudinal and lateral interactions of CAVs.

To this end, we shall first revisit the longitudinal interactions in the existing continuum models for CAVs. It seems that several aspects of AVs such as ACC- and CACC- driving have been already somehow embedded in many existing non-equilibrium models, even in those primarily developed for conventional traffic. Any non-equilibrium model with a relaxation term accounts for speed regulation with respect to the fundamental diagram and speed-spacing relations, and therefore, somehow considers the ACC component of AVs. Meanwhile, the anticipation component captures the speed adaptations with respect to changes in traffic conditions ahead. Many of the recent developments in continuum modelling of CAVs are, as such, based on further refinements of the relaxation and anticipation terms in the GKT-based models (see Section 3). However, in the existing models (including those developed for CAVs), the relaxation and anticipation mechanisms are not consistent with the automated driving components of the commercially available ACC-equipped vehicles.

One of the essential criteria for a desirable continuum model of AVs is the capability to describe the steady-state traffic in a theoretically consistent and valid way against empirical observations. However,

many of the existing models of CAVs are GKT-based models, and their corresponding steady-state conditions are not consistent with those of CAVs. In Section 2.2.3 we showed that the steady-state condition for many GKT-based models results in non-zero and significant velocity variance between the individual vehicles. However, from a microscopic perspective, the steady-state condition requires that microscopic quantities such as speed to be invariant between all the vehicles.

From an empirical perspective, there also exist a discrepancy between the steady-state condition in AVs and the fundamental diagram in many non-equilibrium models. Many existing AVs adopt a constant time headway policy (CTH) in the steady-state traffic, which is equivalent to the triangular fundamental diagram (Shi and Li, 2021). However, the fundamental diagram in many existing non-equilibrium models of CAVs are concave-convex non-linear functions and do not reflect the CTH policy.

Second, AVs are equipped with sensors to measure both distance and speed difference to their leaders, and collision avoidance strategies partially determine the cooperative component of their acceleration behaviours. However, in many existing GKT-based models of CAVs, the anticipation term does not explicitly account for the speed difference, as opposed to the GSOM and phase-transition models. Furthermore, safety considerations are largely dismissed in the existing continuum models.

A potential solution to resolve these issues is to utilize phase-transition models (Blandin et al., 2011; Colombo, 2002) as a base to incorporate automated vehicles. While the present versions of these models lack a relaxation term and dismiss safety considerations, they still explicitly account for speed difference and constant time gap in their governing equation. Continuum approximation of safety surrogate measures can be incorporated in the continuum models to adapt driving strategies and acceleration mechanisms with respect to safety considerations (Mohammadian et al., 2021a). The phase transition models are especially promising to incorporate such aspects because they explicitly account for the time gap as an explicit state variable (see Eq. 2.2519 and the relevant thereafter).

Finally, a more serious discrepancy is the continuous nature of speed adaption in the existing continuum models as opposed to the hierarchical activation levels in the longitudinal motion of AVs. For instance, at the car-following level, different components of automated driving (e.g., headway control, cruise control, and collision avoidance, etc.) are activated at different conditions (e.g., acceleration, steady-state, deceleration, critical time-to-collision, etc., (Milanés and Shladover, 2014; Xiao et al., 2017 for a detailed discussion). However, in many existing continuum models, speed adaptation components are systematic in the sense that both relaxation and anticipation contribute to the speed adaption, regardless of the traffic condition. Such speed adaptation mechanisms are inconsistent with the hierarchical driving strategies utilized in automated driving.

At first sight, the lack of continuum models based on complex driving strategies may be explained by the fact that continuum models are supposed to describe traffic flow at an aggregate level and not at the level of nuances in car-following behaviours. However, the equivalence between car-following and continuum models (see Jin, 2016 for a detailed discussion) suggests that for every continuum model, there should exist an equivalent car-following model that describes collective interactions of identical vehicles as the corresponding continuum model does. Thus, it is reasonable to expect that for a desirable continuum model of CAVs, its equivalent car-following model should incorporate the aspects mentioned above. Therefore, further development of continuum models with respect to incorporating the hierarchical aspects of automated driving remains a significant research need.

However, a serious question remains about whether it is feasible to incorporate complex aspects of automated diving and yet derive a continuum model that is mathematically tractable and practically relevant within the constraints of hyperbolic systems. For example, by incorporating hierarchical driving strategies in the speed adaptation equation, it will be difficult to present the resulting continuum models in the conservative form as in Eq. 8. This is because the acceleration equation will not be based on a conservation principle (Zhang, 2000a), and in the literature, several behavioural continuum models also lack the general conservative property (e.g., Zhang, 1998; Zhang, 2003b). While lack of conservative property used to pose important challenges to studying Riemann problems and approximating numerical solutions (Abgrall and Karni, 2010; Zhang, 2000a), in recent years, there has been substantial development in studying Riemann problems and numerical approximation of non-conservative hyperbolic systems (Diaz et al., 2019; Serezhkin and Menshov, 2020). Therefore, preserving the conservative property should not be considered as an insurmountable constraint, and non-conservative continuum models of CAVs can be used to incorporate complex automated driving strategies in a hierarchical manner.

### 4.3.2 Mixed traffic flow of CAVs and conventional vehicles

So far, we have discussed a scenario where CAVs fully populate freeways, and full automation controls their longitudinal interactions. In the upcoming decades, however, traffic flow will be more likely to be in a mixed condition, where CAVs share the road infrastructure with conventional vehicles. Many existing multi-class models developed for conventional and CAV traffic may not be suitable for accommodating the properties of such a mixed CAV traffic for several main reasons:

(1) They are single-pipe and treat the multi-lane traffic as if it were single-lane. Consequently, in all single-pipe multi-class models, faster vehicles either can overtake the slower ones or queue behind the slowest class. In the former case, the violation of the first-in-first-out property occurs, and the impact of the lane-changing manoeuvres is not adequately modelled. In the latter case, the advantage is to allow for modelling moving bottleneck. However, this property will be inadequate for modelling freeways with lane restrictions where moving bottlenecks caused by slow vehicles are restricted to specific lanes, whereas single-pipe treatment causes all the lanes to experience the moving bottleneck.

Besides these issues, additional properties of mixed CAV traffic can further call into question the use of single-pipe multi-class models. For example, it is likely that at early stages, road authorities will allow the drivers of CAVs to enable automated driving mode only on specifically allocated lanes (Mahmassani, 2016), such that in case of driver intervention for manual driving, exiting such allocated lanes will be mandatory. Under such scenarios, the assumption of single-pipe traffic will no longer be valid due to varying traffic conditions and different underlying premises across the lanes. As such, the benefits of the single-pipe models (e.g., parsimony and simplicity) may no longer outweigh the complexities of the multi-lane framework, such as designing the lane-changing mechanisms.

Therefore, it is expected that multi-lane models, instead of single-pipe models, are likely to attract more research attention for modelling the mixed traffic flow of CAVs, especially given that some of the complexities associated with the multi-lane framework might be less challenging than before. For instance, one of the main challenges used to be a lack of reliable data and adequate knowledge about the mechanisms behind discretionary lane-changing manoeuvres. In recent years, on the other hand, there has been significant development in understanding lane-changing behaviours (e.g., Ali et al., 2019; Ali et al., 2020; Zheng, 2014; Zheng et al., 2013) with a focus on human factors. These understandings could be utilized for designing more reliable lane-changing mechanisms in the continuum framework.

(2) Existing models reviewed in Section 3 are not flexible enough to accommodate a wide range of potential impacts of different CAV penetration rates on traffic flow dynamics. Almost all the existing studies on continuum modelling of AVs suggest that with an increase in their penetration rate, AVs can systematically bring positive outcomes (e.g., increased capacity) to traffic flow, and their stabilizing effect can emerge at small penetration rates (less than 30%) (e.g., Ngoduy et al., 2009). However, some recent microscopic simulations have suggested that there could be a significantly large threshold (e.g., 70%) for the penetration rates of AVs, below which automated driving may not bring significant improvements to traffic flow (Calvert et al., 2017). Therefore, it is essential to ensure the underlying premises of the continuum models regarding the penetration rate of AVs are flexible and adaptable to the new findings.

(3) Existing models dismiss the possibility of negative effects of AVs on traffic flow properties, which have been observed in some field experiments. Several recent field experiments with ACC-equipped vehicles have suggested that such vehicles are string unstable (Ciuffo et al., 2021; Gunter et al., 2020). However, in many existing continuum models of AVs (Delis et al., 2015; Delis et al., 2018; Ngoduy, 2013a, b; Ngoduy et al., 2009; Ngoduy and Jia, 2017; Ngoduy and Wilson, 2014), AVs systematically stabilize traffic flow, which could be caused by the discrepancy between systematic speed adaptation in continuum models and hierarchical control at the car-following level, discussed earlier in this section, or by the issues with the choice of the relaxation time and source terms, discussed in Section 4.2.2.

(4) Finally, existing models are based on the notion that drivers fully cooperate with AVs, which contradicts with the empirical findings that drivers' cooperation with automated driving depends on traffic conditions and human factors. For instance, through field experiments with AVs, Viti et al. (2008) found that drivers tend to deactivate ACC driving in congested traffic conditions. Varotto et al. (2015a) conducted a driving-simulator study and found that such interferences with AVs negatively affected traffic flow properties (e.g., increased time gap), and thereby could undermine the expected positive effects of CAVs. Recent field experiments with AVs and driver behaviour modelling have suggested that drivers' perceived risk and task difficulty level are major determining factors to driver's intervention with AVs and transitions from manual to automated driving (Varotto et al., 2017; Varotto et al., 2018).

Therefore, to address these issues above, there is a significant research need to develop hybrid continuum models that incorporate both automated and manual driving for the class of automated vehicles.

Such hybrid models must distinctively incorporate robust and realistic speed adaptation mechanisms for both automated and manual driving phases. The most critical component of such a hybrid modelling approach would be designing transition mechanisms from automated to manual driving and vice versa. For example, robust mechanisms must consider major human psychological factors involved in such transitions (e.g., drivers' boredom, risk perception, etc) as well as their impacts on driver behaviour (e.g., take-over time).

Driver behaviour and human factors will also be pronounced in connected environments when automated driving is not active. In such a condition, the potential impacts of the connectivity on traffic flow depend primarily on how drivers process and react to the information. For instance, recent driving-simulator studies on driving in connected environments have found that drivers' response time may be increased in certain car-following situations (Sharma et al., 2019b), and that such behavioural adaptations can be explained with theories of human factors (Sharma et al., 2019a; Sharma et al., 2020). These findings demonstrate that to understand the collective properties of mixed traffic flow, it is necessary to adequately incorporate major human factors of the connected environment into the continuum framework. Therefore, the need to develop continuum models based on well-established human factors theories will be more pronounced for the mixed CAV traffic.

## 5. Conclusion

This paper comprehensively reviewed different families of continuum models for conventional traffic as well as the recent attempts for incorporating CAVs into the continuum framework. The strengths and limitations of each family in describing the observations from real-world traffic were discussed in detail, and a summary is provided in Table 2 and Table 3.

The paper also revisited some essential properties and challenges of different model families for conventional traffic and investigated their implications for CAVs traffic. For example, the paper discussed that multi-class equilibrium models might not be suitable for accommodating the complexities of mixed traffic flows of CAVs and conventional vehicles even though such a traffic condition is multi-class. For mixed CAV traffic, different vehicle classes are of the same length, and their space occupancy is the same, but the different vehicle classes differ mainly regarding factors such as speed adaptation, response time, etc. Besides, one of the key questions regarding mixed CAV traffic flow is their stability, which is clearly beyond the capability of the MC-LWR framework. In contrast, non-equilibrium models are more suitable for modelling mixed CAVs, and many existing models proposed for CAV traffic flows are non-equilibrium models based on the gas-kinetic theory (e.g., Delis et al., 2015; Delis et al., 2018; Ngoduy, 2013a, b; Ngoduy et al., 2009; Ngoduy and Jia, 2017; Ngoduy and Wilson, 2014).

Next, this paper revisited two critical issues applicable to many non-equilibrium models, including the recent models developed for CAVs, namely, the issue of faster-than-traffic waves and the relaxation terms. Regarding the former, the paper provided a comprehensive discussion and attempted to clarify some confusions and controversies around faster-than-traffic waves, while revisiting their significance for mixed CAV traffic flows. By implementing a simple numerical experiment on the model by Zhang (1998), which is devoid of all the issues discussed by Daganzo (1995b) except faster-than-traffic waves, we demonstrated the violation of the anisotropy and settled the controversy (Helbing, 2009; Helbing and Johansson, 2009; Zhang, 2009). Meanwhile, the paper discussed that this issue is, overall, likely to be less significant for CAV traffic flows due to several reasons, including bidirectional communication. The paper also highlighted the need for relaxation terms in non-equilibrium models for ensuring consistency with the steady-state and obtaining plausible instability behaviours. However, the choice of the relaxation time can potentially counteract the capability of non-equilibrium models for replicating traffic instabilities, and this issue could be more pronounced for mixed CAV traffic flows since relaxation time will be closely related to and affected by headway control aspects of AVs.

Finally, this paper discussed several limitations of existing continuum models for modelling pure CAV flows and mixed CAV traffic flows, highlighting the need to develop new models. Many limitations of existing models are linked with oversimplified assumptions about driver behaviour and neglecting complex human psychological factors. This huge gap could be explained by the mathematical considerations within the hyperbolic systems, such as the conservative form, which has been conventionally treated as a priority. However, in recent years, there has been substantial development in studying Riemann problems and numerical approximation of non-conservative hyperbolic systems (Diaz et al., 2019; Serezhkin and Menshov, 2020).

Therefore, preserving the conservative property should no longer be considered a constraint, and the priorities in developing new continuum models of CAV traffic must shift towards incorporating the complex human factors and driving strategies. For instance, numerous recent field experiments with AVs have identified that drivers frequently intervene with the system and transition between ACC and manual driving (Varotto et al., 2015b), and that such interventions are strongly linked with psychological factors such as risk perception (Varotto et al., 2018; Zhang et al., 2019). Therefore, a significant research need is developing hybrid continuum models that incorporate both automated and manual driving. Such hybrid models must distinctively incorporate robust and realistic mechanisms for both automated and manual driving phases and the transition from automated to manual driving and vice versa to study the collective consequences of driver's intervention with CAVs on traffic flow dynamics.

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