An introduction to Stanley symmetric functions

郭龙

Center for Combinatorics, Nankai University

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Motivation of Stanley: Count the number of reduced decompositions of a permutation.

- For $1 \leq i \leq n-1$, let $s_i = (i, i+1)$ denote the **adjacent (or, simple) transposition**. They satisfy the braid relations:

$$s_i s_j = s_j s_i, \qquad \text{for } |i-j| > 1;$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}.$$

- Each permutation w of $\{1, 2, \ldots, n\}$ can be written as a product of these s_i.

Example: $321 = s_1 s_2 s_1 = s_2 s_1 s_2.$

- A product of minimum length is called a **reduced expression**.

  Fact 1: Two given reduced repressions of w can be transformed to each other by applying braid relations.

  Fact 2: A given reduced repression of w can be transformed to a given reduced expression by applying braid relations & $s_i^2 = 1$.

- Let $R(w)$ denote the set of reduced words of w, namely,

  $$R(w) = \{(a_1, \ldots, a_p) \colon s_{a_1} \cdots s_{a_p} \text{ is a reduced expression of } w\}.$$

  For example, $R(321) = \{(1,2,1),(2,1,2)\}$.

How to count $\#R(w)$?

It was observed for some small values of n that

$$\#R(w_0 = n \cdots 21) = f^{(n-1,\ldots,2,1)}.$$

Stanley defined a class of functions in terms of Quasisymmtric Functions:

$$F_w(x) = \sum_{a \in R(w)} \sum_{(a,i)} x_{i_1} \cdots x_{i_p},$$

where, for $a = (a_1, \dots, a_p)$ and $i = (i_1, \dots, i_p)$,

$$a_k < a_{k+1} \Rightarrow i_k < i_{k+1}.$$

Notice that

$$\#R(w) = [x_1 \cdots x_p] F_w(x).$$

$F_w(x)$ is now known as the Stanley symmetric function.

R. Stanley, On the number of reduced decompositions of elements of Coxeter groups, European J. Combin. 5 (1984), 359–372.

Why is $F_w(x)$ symmetric?

One should check that $F_w(x) = s_i \cdot F_w(x)$. Find a bijection $(a, i) \mapsto (a', i')$ such that

$$\cdots x_i^s \, x_{i+1}^t \cdots \quad \mapsto \quad \cdots x_i^t \, x_{i+1}^s \cdots$$

(For Schur functions: **The Bender-Knuth involution**)

Stanley found such a bijection. Very technical!

Wishful Thinking as a Proof Technique

Use **Schubert polynomials** or **nil-Coxeter algebras** to investigate the symmetry property of F_w.

Schubert polynomials $\{\mathfrak{S}_w(x) \colon w \in S_n\}$ can be defined inductively in terms of "divided difference operator".

(1) For $w_0 = n \cdots 21$, $\mathfrak{S}_{w_0}(x) = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$;

(2) If $w \neq w_0$, there exists a position k such that $w_k < w_{k+1}$. Set

$$\mathfrak{S}_w(x) = \partial_k \mathfrak{S}_{ws_k}(x) = \frac{\mathfrak{S}_{ws_k}(x) - \mathfrak{S}_{ws_k}(x_1, \ldots, x_{k+1}, x_k, \ldots, x_{n-1})}{x_k - x_{k+1}}.$$

(independent of the choice of k since $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$, and $\partial_i \partial_j = \partial_j \partial_i$ for $|i - j| > 1$)

Billey-Jockusch-Stanley formula:

$$\mathfrak{S}_w(x) = \sum_{a \in R(w)} \sum_{(a,i)\ \&\ i_k \le a_k} x_{i_1} \cdots x_{i_p}.$$

对比:

$$F_w(x) = \sum_{a \in R(w)} \sum_{(a,i)} x_{i_1} \cdots x_{i_p}.$$

S. Billey, W. Jockusch and R. Stanley, Some combinatorial properties of Schubert polynomials, J. Algebraic Combin. 2 (1993), 345–374.

Stanley symmetric functions are stable Schubert polynomials.

Let

$$1^m \otimes w = 1 \cdots m\,(w_1 + m) \cdots (w_n + m).$$

An obvious bijection between $R(w)$ and $R(1^m \otimes w)$:

$$(a_1, \ldots, a_p) \quad \mapsto \quad (a_1 + m, \ldots, a_p + m).$$

Therefore,

$$F_w(x) = \lim_{m \to \infty} \mathfrak{S}_{1^m \otimes w}(x).$$

Fact. If $w_i < w_{i+1}$, then $\mathfrak{S}_w(x)$ is symmetric in x_i and x_{i+1}.

Check

$$\partial_i \mathfrak{S}_w = \frac{\mathfrak{S}_w - s_i \cdot \mathfrak{S}_w}{x_i - x_{i+1}} = 0.$$

By definition,

$$\partial_i \mathfrak{S}_w = \partial_i(\partial_i \mathfrak{S}_{ws_i}),$$

which vanishes using the fact that $\partial_i^2(f) = 0$ for any polynomial f.

Recalling

$$F_w(x) = \lim_{m \to \infty} \mathfrak{S}_{1^m \otimes w}(x),$$

we see that $F_w(x) = s_i \cdot F_w(x)$ for any i.

Hecke algebra: a vector space V + multiplication $\times$

$\mathcal{H}(S_n)$: $V = \mathrm{Span}_{\mathbb{C}}\{T_w \colon w \in S_n\}$. The multiplication is determined by

$$T_w T_{s_i} = T_{ws_i}, \qquad \text{if } w_i < w_{i+1};$$

$$T_w T_{s_i} = a_i T_w + b_i T_{ws_i}, \qquad \text{if } w_i > w_{i+1};$$

When $a_i = b_i = 0$ for any i, $\mathcal{H}(S_n)$ is called a nil-Coxeter algebra, denoted $\mathcal{N}_n$.

S. Fomin and R.P. Stanley, Schubert polynomials and the nilcoxeter algebra, Adv. Math. 103 (1994), 196–207.

Equivalently, $\mathcal{N}_n$ is an algebra determined by generators $T_1, T_2, \ldots, T_{n-1}$ and relations

$$T_i^2 = 0;$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1};$$

$$T_i T_j = T_j T_i, \qquad \text{for } |i-j| > 1.$$

Viewing T_i as T_{s_i}, the set $\{T_w : w \in S_n\}$ is a basis.

Idea of Fomin and Stanley:

Let

$$A(x) = (I + xT_{n-1})(I + xT_{n-2}) \cdots (I + xT_1),$$

where I is the identity element. Consider

$$\begin{aligned} F(x_\infty) =& A(x_1)\, A(x_2)\, A(x_3) \cdots \\ =& (I + x_1 T_{n-1})(I + x_1 T_{n-2}) \cdots (I + x_1 T_1) \\ & (I + x_2 T_{n-1})(I + x_2 T_{n-2}) \cdots (I + x_2 T_1) \\ & \cdots \end{aligned}$$

Expand $F(x_\infty)$:

$$F(x_\infty) = \sum_{w \in S_n} \overline{F}_w(x)\, T_w.$$

It can be easily checked that

$$\overline{F}_w(x) = \sum_{a \in R(w)} \sum_{(a,i)} x_{i_1} \cdots x_{i_p},$$

where, for $a = (a_1, \ldots, a_p)$ and $i = (i_1, \ldots, i_p)$,

$$a_k < a_{k+1} \Rightarrow i_k < i_{k+1}.$$

Theorem (Fomin-Stanley, 1994)

For $w \in S_n$, $F_w(x) = \overline{F}_w(x)$

To show $F_w(x) = s_i \cdot F_w(x)$, it is equivalent to proving

$$A(x_i)A(x_{i+1}) = A(x_{i+1})A(x_i).$$

In fact, it can be readily verified from the definition that

$$A(x)A(y) = A(y)A(x).$$

For example, when $n = 2$,

$$A(x)A(y) = (I + xT_1)(1 + yT_1) = I + xT_1 + yT_1 = A(y)A(x).$$

Now we have seen that $F_w(x)$ is indeed a symmetric function. Hence $F_w(x)$ can be expanded in the basis of Schur functions:

$$F_w(x) = \sum_\lambda c_{w,\lambda} s_\lambda(x).$$

Stanley conjectured that $c_{w,\lambda}$ are nonnegative (in other words, $F_w(x)$ is **Schur-positive**). This conjecture was confirmed by Edelman and Greene by developing the **Coxeter-Knuth insertion algorithm**.

P. Edelman and C. Greene, Balanced tableaux, Adv. Math. 63 (1987), 42–99.

RSK vs Hecke

Both RSK and Hecke can be applied to a word $a = (a_1, a_2, \ldots, a_n)$ to generate an insertion tableau.

The Hecke algorithm was introduced in order to expand a stable Grothendieck polynomial in the basis of stable Grothendieck polynomials indexed by Grassmannian permutations.

When a is a **reduced word**, the Hecke algorithm specializes to the Coxeter-Knuth algorithm.

A.S. Buch, A. Kresch, M. Shimozono, H. Tamvakis and A. Yong, Stable Grothendieck polynomials and K-theoretic factor sequences, Math. Ann. 340 (2008), 359–382.

RSK

Let $a = (a_1, a_2, \ldots, a_n)$ be a word of positive integers. We first use RSK to generate an insertion tableau $P_{\mathrm{RSK}}(a)$. Construct a sequence

$$(P_0 = \emptyset, P_1, P_2, \ldots, P_n = P_{\mathrm{RSK}}(a)),$$

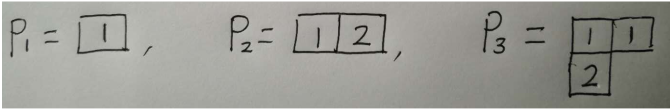
where P_i is the tableau by inserting a_i into P_{i-1}.

Suppose that P_{i-1} has been constructed. Let us generate the first row of P_i as follows.

Case 1. a_i is $\geq$ each entry in the first row. In this case, put a_i at the end of the first row. The insertion process terminates.

Case 2. a_i is $<$ some entry in the first row. In this case, locate the leftmost one $> a_i$, say x. Replace x by a_i, and then insert x into the second row using the same procedure.

Example. For $a = (1, 2, 1)$, the tableaux P_i are illustrated as follows.



Hecke

Let $(a_1, a_2, \ldots, a_n)$ be a word of positive integers. We next use the Hecke algoritm to generate an insertion tableau $P_{\mathrm{Hecke}}(a)$. Construct a sequence

$$(P'_0 = \emptyset, P'_1, P'_2, \ldots, P'_n = P_{\mathrm{Hecke}}(a)),$$

where P_i is the tableau by inserting a_i into P_{i-1}.

Suppose that P'_{i-1} has been constructed. Let us generate the first row of P'_i as follows.

Case 1. a_i is $>$ each entry in the first row. In this case, put a_i at the end of the first row. The insertion process terminates.

Case 2. $a_i =$ the largest entry in the first row. The insertion process terminates.

Case 3. a_i is $<$ some entry in the first row. In this case, locate the leftmost one $> a_i$, say x. If the first row is strictly increasing by replacing x by a_i, then replace x by a_i. Otherwise, keep the first row unchanged. Insert x into the second row by the same procedure.

Example. For $a = (1, 2, 1)$,

$$P_1' = \begin{array}{|c|}\hline 1 \\ \hline\end{array}, \quad P_2' = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}, \quad P_3' = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$$

对比：

$P_{\mathrm{RSK}}(a)$ is a **semistandard tableau**. That is, each row $P_{\mathrm{RSK}}(a)$ is weakly increasing, and each column of $P_{\mathrm{RSK}}(a)$ is strictly increasing.

$P_{\mathrm{Hecke}}(a)$ is an **increasing tableau**. That is, both rows and columns of $P_{\mathrm{Hecke}}(a)$ are strictly increasing.

Theorem (Edelman-Greene, 1987)

The coefficient $c_{w,\lambda}$ equals the number of increasing tableaux of shape λ whose column reading words are reduced words of w.

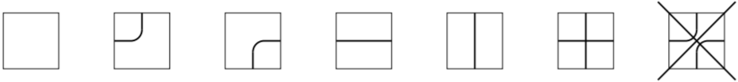
Recently, Thomas Lam, Seung Jin Lee and Mark Shimozono found a new combinatorial explanation for $c_{w,\lambda}$ in terms of **bumpless pipedreams**.

T. Lam, S. Lee and M. Shimozono, Back stable Schubert calculus, arXiv:1806.11233v1, 63 pages.

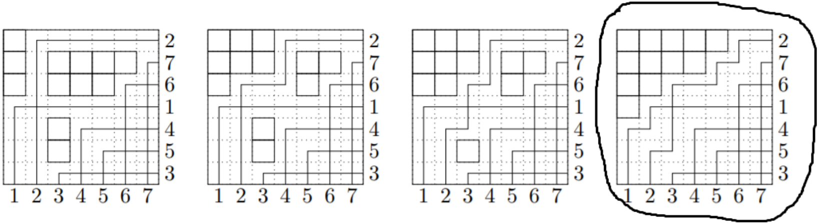
(1) pipes go north or east;

(2) no two pipes overlap or cross more than once.

(3) each box looks like one of the following six tiles:



Example. Some bumpless pipedreams for $w = 2761453$.



Theorem (Lam-Lee-Shimozono, 2018)

The coefficient $c_{w,\lambda}$ equals the number of **Edelman-Greene bumpless pipedreams** *for w of shape λ (that is, bumpless pipedreams for w such that all the empty squares lie at the northwest corner, where they form a Young diagram λ).*

Problem (Lam-Lee-Shimozono, 2018)

Find a shape-preserving bijection between reduced words tableaux and Edelman-Greene bumpless pipedreams.

Neil Fan, Peter Guo and Sophie Sun, Bumpless pipedreams, reduced word tableaux and Stanley symmetric functions, arXiv:1810.11916.

Set

$$u(n) := \max_{w \in S_n} \mathfrak{S}_w(1, 1, \ldots, 1).$$

Stanley conjectured that the limit

$$\lim_{n \to \infty} \frac{1}{n^2} \log_2 u(n)$$

exists.

Richard P. Stanley, Some Schubert shenanigans, arXiv:1704.00851.

Macdonald's identity:

$$\mathfrak{S}_w(1,\ldots,1) = \frac{1}{\ell!} \sum_{(a_1,\ldots,a_\ell)\in R(w)} a_1 \cdots a_\ell,$$

where $\ell = \ell(w)$ is length of w:

$$\ell(w) = \#\{(i,j)\colon 1 \le i < j \le n, \quad w_i > w_j\}.$$

Merzon and Smirnov conjectured that if $\mathfrak{S}_w(1, 1, \ldots, 1)$ attains the maximum value, then w is a **layered permutation**.

A layered permutation is of the following form:

$$(b_1, \ldots, 1, b_2, \ldots, b_1 + 1, \ldots, b_k, \ldots, b_{k-1} + 1),$$

where

$$1 \leq b_1 < b_2 < \cdots < b_k = n$$

G. Merzon and E. Smirnov, Determinantal identities for flagged Schur and Schubert polynomials, European J. Math. 2 (2016), 227-245.

Morales, Pak and Panova proved that if the Merzon-Smirnov conjecture is true, then

$$\lim_{n\to\infty} \frac{1}{n^2} \log_2 u(n) = \frac{\gamma}{\log 2} \approx 0.2932362762,$$

where $\gamma \approx 0.2032558981$.

A. Morales, I. Pak and G. Panova, Asymptotics of principal evaluations of Schubert polynomials for layered permutations, Proc. Amer. Math. Soc. 147 (2019), 1377–1389.

A most recent formula for $\mathfrak{S}_w(1,\ldots,1)$:

Theorem 1.5. *Let $\sigma \in S_n$, and choose any 132-avoiding permutation π which is greater than σ in the right weak order (such a π always exists, since w_0 is 132-avoiding). Then*

$$\mathfrak{S}_\sigma(1,\ldots,1) = \frac{1}{(\ell(\pi)-\ell(\sigma))!} \sum_{C:\sigma\to\pi} \mathrm{wt}^\pi(C),$$

where the sum is over all saturated chains from σ to π in the strong Bruhat order on $[e,\pi]_R$, and where wt^π is defined by (1).

C. Gaetz and K. Tung, The Sperner property for 132-avoiding intervals in the weak order, arXiv:2006.16359v1.

Thank you!