Background, conjectures and questions on Stanley's chromatic symmetric functions

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- She ... come to my office and describe it. At the end, She would turn to me and say “Is it right?” I was very flattered that she thought I would know. — McMullen (Mirzakhani 的博导, Fields medal winner in 1998, when he became a full professor) at Harvard.
- She is very optimistic, and that’s infectious. When you work with her, you feel you have a much better chance of solving problems that at first seem hopeless. — Eskin (Mirzakhani 的合作者).

Mirzakhani, A simple proof of a theorem of Schur, Amer. Math. Monthly, 1998.

Mathematicians Of The Day, `https://mathshistory.st-andrews.ac.uk/Biographies/Mirzakhani/`.

Symmetric functions

- m_λ is the sum of the monomials whose exponents can be rearranged to λ.
- $e_\lambda = e_{\lambda_1} e_{\lambda_2} \ldots e_{\lambda_r}$, where $e_n = m_{1^n}$.
- $p_\lambda = p_{\lambda_1} p_{\lambda_2} \ldots p_{\lambda_r}$, where $p_n = m_n$.
- $h_\lambda = h_{\lambda_1} h_{\lambda_2} \ldots h_{\lambda_r}$, where $h_n = \sum_{\lambda \vdash n} m_\lambda$.
- $s_\lambda = \sum_{T \in CS_\lambda} w(T)$.

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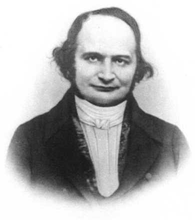
$$s_{21}(x_1, x_2, x_3) = 2x_1x_2x_3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2.$$

Origin for Schur symmetric functions

1. 1815 Cauchy.
2. 1841 Jacobi: the Jacobi-Trudi identity.

$$s_\lambda = \det(h_{\lambda_i+j-i})_{i,j=1}^{\ell(\lambda)} = \det(e_{\lambda'_i+j-i})_{i,j=1}^{\ell(\lambda')}.$$

3. 1901 Schur: applications of Schur polynomials in the representation theory of the symmetric group.

Schur symmetric functions in combinatorics

1. Kostka number $K_{\lambda,\mu} = [m_\mu]s_\lambda$: the number of semi-standard Young tableaux of shape λ and type μ.
2. Semi-standard Young tableaux occurs in lattice paths, trees, partitions, ...

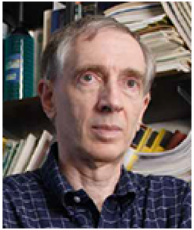
Macdonald, Symmetric Functions and Hall Polynomials, 2nd ed., 1995.

Stanley, Book review for Macdonalds's Symmetric functions and Hall polynomials, 1981.

Schur symmetric functions in representation theory

1. One cares class functions f: $S_n \to \mathbb{Q}$ and their characteristics $\mathrm{ch}(f)$.
2. "The reason for dealing with $\mathrm{ch}(f)$ rather than directly with f is that important class functions have simple characteristics."
3. $\mathrm{ch}(f)$ are symmetric functions.
4. There is a natural indexing of the irreducible characters χ^λ of S_n by partitions $\lambda \vdash n$ such that $\mathrm{ch}(\chi^\lambda) = s_\lambda$.
5. The character table of S_n is the transition matrix between $\{p_\lambda\}$ and $\{s_\lambda\}$:

What is a chromatic symmetric function?



- 1912 Birkhoff: chromatic symmetric polynomial.
- 1995 Stanley: chromatic symmetric function.

Birkhoff, A determinant formula for the number of ways of coloring a map, Ann. 1912.

Stanley, A symmetric function generalization of the chromatic polynomial of a graph, Adv. 1995.

Definition 1.1 (Stanley, 1995, Adv. Math.)

The chromatic symmetric function of a graph G:

$$X_G(x_1, x_2, \ldots) = \sum_{\kappa} \prod_{v} x_{\kappa(v)}$$

where the sum is over all proper colorings κ.

93 arXiv manuscripts on CSF

Cornell University

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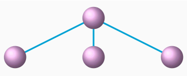
1. arXiv:2006.08418 [pdf, ps, other] math.CO
   A symmetric function of increasing forests
   Authors: Alex Abreu, Antonio Nigro
   Submitted 15 June, 2020; **originally announced** June 2020.
   Comments: 18 pages
   MSC Class: 05A05; 05E05
2. arXiv:2006.00657 [pdf, ps, other] math.CO
   Chromatic symmetric functions from the modular law
   Authors: Alex Abreu, Antonio Nigro
   Submitted 31 May, 2020; **originally announced** June 2020.
   Comments: 19 pages
   MSC Class: 05A05; 05E05
3. arXiv:2004.09198 [pdf, other] math.CO math.RT
   A combinatorial expansion of vertical-strip LLT polynomials in the basis of elementary symmetric functions
   Authors: Per Alexandersson, Robin Sulzgruber
   Submitted 20 April, 2020; **originally announced** April 2020.
   Comments: 47 pages
   MSC Class: 05E05; 06A07; 05A05
4. arXiv:2004.04599 [pdf, ps, other] math.RT math.CO
   Combinatorial Hopf algebras from representations of families of wreath products
   Authors: Tyrone Crisp, Caleb Kennedy Hill
   Submitted 9 April, 2020; **originally announced** April 2020.
   Comments: 26 pages
5. arXiv:2001.00181 [pdf, ps, other] math.CO
   Non-Schur-positivity of chromatic symmetric functions
   Authors: David G. L. Wang, Monica M. Y. Wang

The (3+1)-conjecture

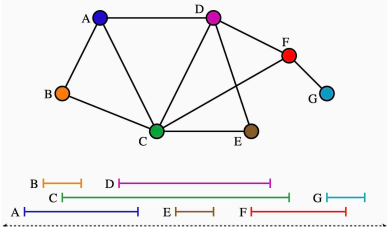
Conjecture 1.2 (Stanley and Stembridge, 1993)

The CSF of every claw-free interval graph is e-positive.

Stanley and Stembridge, On immanants of Jacobi-Trudi matrices and permutations with restricted position, JCTA, 1993.



The claw.



An interval graph (right, stolen from wiki).

The Schur positivity conjecture

Conjecture 1.3 (Stanley, 1998)

The CSF of every claw-free graph is Schur positive.

Stanley, Graph colorings and related symmetric functions: ideas and applications: A description of results, interesting applications, & notable open problems, DM, 1998.

Gasharov, On Stanley's chromatic symmetric function and clawfree graphs, DM, 1999.

- 1998 Stanley: ... was first raised by Gasharov (unpublished).
- 1999 Gasharov: Stanley conjectured ...

Notion summary

- The CSF
- Claw-free graphs
- Interval graphs

Graph invariants determined by its CSF

The csf X_G determines:

- # of vertices
- # of edges
- # of triangles
- $\sum \deg^2 v$, over vertices v
- girth: length of a shortest cycle
- $\sum x^{|A|}$, over stable sets A

The csf X_T of a tree determines:

- # of paths of i edges
- # of vertices of degree d
- # of subtrees of j vertices
- # of leaves

Martin, Morin, and Wagner, On distinguishing trees by their chromatic symmetric functions, JCTA, 2008.

Orellana and Scott, Graphs with equal chromatic symmetric functions, DM, 2014.

More connections of the CSF in combinatorics

- stable partitions of G
- acyclic orientations of G
- the lattice of contractions (connected partitions ordered by refinement) of G
- Kazhdan-Lusztig theory [2, Page 187]: One may deduce easily from a result of Haiman [1] that indifference graphs are Schur positive. Haiman's proof used KL conjectures (proved later).

Haiman, Hecke algebra characters and immanant conjectures, JAMS, 1993.

Stanley, A symmetric function generalization of the chromatic polynomial of a graph, Adv., 1995.

Claw-free graphs

- Why do we study claw-free graphs [1]:
  - Every even-order claw-free connected graph has a perfect matching.
  - An algorithm for finding a max. stable set in a claw-free graph is discoverd.
  - Claw-free perfect graphs are characterized.
- A characterization of claw-free graphs [2].
- A Fulkerson prize [3]: solving a conjecture using the characterzation.

Faudree, Flandrin and Ryjáček, Claw-free graphs—A survey, DM, 1997.

Chudnovsky and Seymour, The structure of claw-free graphs, In: Surveys in Combinatorics 2005.

Chudnovsky, Robertson, Seymour, and Thomas, The strong perfect graph theorem, Ann., 2006.

Chudnovsky, Robertson, Seymour, and Thomas




Interval graphs


- Introduced by Hajós (full member of MTA) in 1957.
- Benzer: used interval graphs to model genetic structure.
- Applications: archaeology, biology, psychology, sociology, management, genetics, engineering, scheduling, transportation and others; see Corneil et al. [3].
- 闫桂英老师的报告

Hajós, Über eine Art von Graphen, Internat. Math. Nachr., 1957.

Benzer, On the topology of the genetic fine structure, PNAS, 1959.

Corneil, Olariu, and Stewart, The LBFS structure and recognition of interval graphs, SIDMA, 2010.

Some progress on the (3+1) conjecture

1. 1996 Gasharov proved the claw-free interval graphs are Schur positive.
2. 2001 Gebhard and Sagan introduced CSF in noncommutative variables and proved the e-positivity of K-chain graphs.
3. 2004 Guay-Paquet, Morales and Rowland counted the claw-free interval graphs approximately and proved the (3+1) conjecture holds for almost all claw-free interval graphs.
4. 2012 Shareshian and Wachs introduced chromatic quasisymmetric functions, refined Gasharov's results, and established connections between the conjecture and representation theory.
5. ...

Some progress on the Schur positivity conjecture

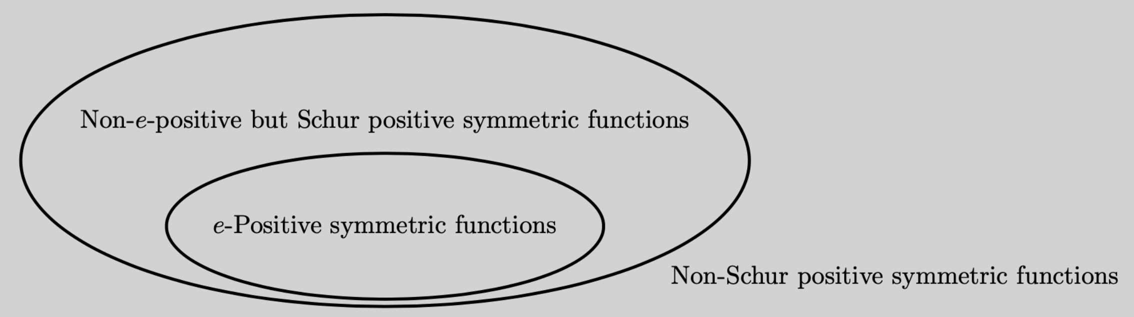
Almost all progress are for special graph classes.

Else?

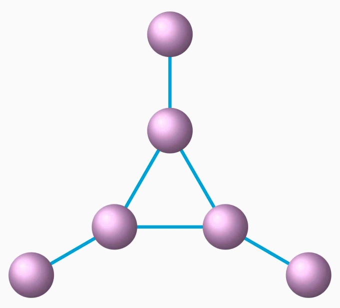
Methods of proving the Schur positivity

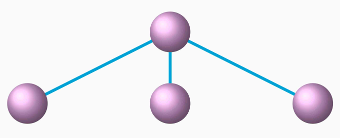
1. Express the target as the sum of Schur positive terms; basis transformation.
2. RSK: in the language of semi-standard Young tableaux; combin. interpretation for the Schur coeffs. using bijections.
3. The crystal method: consider another graph.
4. The dual equivalent graph method: Assaf, Trans. AMS 370, 8777, 2018. ??
5. Representation method: Construct a graded S_n-module and show that the Frobenius map of the irreducible decomposition of the module is the target symmetric function. Then the Schur coeff. is the # of occurrences of irreducible characteristics under certain action, which is certainly nonnegative.
6. Geometry method: Construct a variant which deforms to the union of Grassmanians. Then a Schur coeff. is the dimension of some subspace, which is certainly nonnegative.

e-positivity VS Schur positivity



Eg. for graph positivities





$$X_{\text{net}} = 6s_{321} + 12s_{31^3} + 24s_{2^21^2} + 48s_{21^4} + 48s_{1^6}$$
$$= 12e_6 + 18e_{51} + 12e_{42} + 6e_{41^2} - 6e_{3^2} + 6e_{321},$$
$$X_{\text{claw}} = s_{31} - s_{2^2} + 5s_{21^2} + 8s_{1^4}.$$

How to compute the Schur expansion of the CSF?

1. Definition + basis transformation.
2. Stanley's p_λ-method + basis transformation.
3. The GF method using recurrences and the structure of graph classes.
4. Gessel's fundamental quasisymmetric functions + Gessel-to-Schur lifting + the slinky rule.
5. What else?

The tree classification problem

Question 1.4 (The tree classification problem)

Can we give a criterion to determine whether a tree is e-positive, and whether a tree is Schur positive?

We know only a little results, eg., the Dynkin graph D_n and E_n are not e-positive. I don't think doing more is hard. More data would be helpful.

Line graphs are e-positive or not

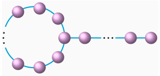
Question 1.5

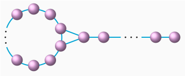
Is a line graph always e-positive?

Line graphs are claw-free. I think solving it out must be exciting.

The tadpoles and their line graphs

Tadpoles (蝌蚪) constitute a special subclass of squids (鱿鱼) defined by Martin et al.





Pawlowski introduced pointed chromatic symmetric function $X_{G,v}$ and showed that the wedge sum $G \vee H$ is Schur positive if $X_{G,v}$ and $X_{H,v}$ are pointed Schur positive in the sense of pointed Schur funtions which is due to Strahov. As a consequence, the tadpoles are Schur positive.

Martin, Morin, and Wagner, On distinguishing trees by their chromatic symmetric functions, JCTA, 2008.

Pawlowski, Chromatic symmetric functions via the group algebra of s_n, arXiv, 2018.

Strahov, Generalized characters of the symmetric group, Adv. Math., 2007.

The GF of paths and cycles

Proposition 1.6 (Stanley, 1995)

$$\sum_{n\ge 0} X_{P_n} z^n = \frac{E(z)}{F(z)} = 1 + e_1 z + 2e_2 z^2 + (3e_3 + e_{21}) z^3 + \cdots, \quad \textit{and}$$

$$\sum_{n\ge 2} X_{C_n} z^n = \frac{z^2 E''(z)}{F(z)} = 2e_2 z^2 + 6e_3 z^3 + (12e_4 + 2e_{2^2}) z^4 + \cdots,$$

where

$$E(z) = \sum_{n\ge 0} e_n z^n \quad \textit{and} \quad F(z) = E(z) - zE'(z).$$

As a consequence, paths and cycles are e-positive.

Stanley's p_λ-method to compute X_G

We write $\kappa! = \prod_{i\ge 1} \kappa_i!$ and $\kappa^! = \prod_{i\ge 1} k_i!$, if $\kappa = (1^{k_1}2^{k_2}\ldots)$.

Proposition 1.7 (Stanley, 1995)

$$X_G = \sum_{S\subseteq E} (-1)^{|S|} p_{\lambda(S)}$$

where ρ runs over stable partitions of G, and $\lambda(S)$ is the integer partition consisting of the component orders of the graph (V, S).

Proof for the GF of tadpoles

Let $v_0, v_l, \ldots, v_2, v_1$ be the vertices on the path of a tadpole, where v_1 is the leaf and v_0 is of degree 3. Let $S_k = \{v_1 v_2,\ v_2 v_3,\ \ldots,\ v_{k-1} v_k\}$. Consider when $S_k \subseteq S$ and $S_{k+1} \not\subseteq S$. Then

$$\sum_{\substack{S_k \subseteq S \subseteq E \\ v_k v_{k+1} \notin S}} (-1)^{|S|} p_{\lambda(S)} = (-1)^{k-1} p_k X_{G_{l-k}} \qquad \text{for } 1 \le k \le l.$$

When $S_{l+1} \subseteq S$, considering the number c of edges on the cycle which are contained in the component of (V, S) that contains S_{l+1}, we find

$$\sum_{S_{l+1} \subseteq S \subseteq E} (-1)^{|S|} p_{\lambda(S)} = (-1)^{l+m} p_{l+m} + \sum_{c=0}^{m-1} (c+1)(-1)^{l+c} p_{l+c+1} X_{P_{m-c-1}}.$$

Continuing the proof.

By Stanley's p_λ-method, we obtain

$$X_{G_l} - \sum_{k=1}^{l} (-1)^{k-1} p_k X_{G_{l-k}} = (-1)^{l+m} p_{l+m} + \sum_{c=1}^{m} c(-1)^{l+c-1} p_{l+c} X_{P_{m-c}}. \tag{1}$$

Recast it in terms of GF, using the fact

$$\sum_{j\geq 0} p_j (-z)^j = \frac{F(z)}{E(z)},$$

and by a one-page routine calculation, we obtain the GF for the CSFs of tadpoles.

The GF for the CSF of the line graph of tadpoles

Proposition 1.8 (W. and Wang, 2020+)

$$\sum_{m\ge 2}\sum_{l\ge 0} X_{\mathrm{Tp}_{m,l}} x^m y^l = \frac{x^2}{(x-y)^2}\left[\frac{x(x-y)E''(x)E(y)}{F(x)F(y)} - y\left(\frac{E'(x)}{F(x)} - \frac{E'(y)}{F(y)}\right)\right].$$

$$\sum_{m\ge 2}\sum_{l\ge 0} X_{L(\mathrm{Tp}_{m,l})} x^m y^l = \frac{x^2}{(x-y)^2}\left[\frac{(x^2-y^2)E''(x)E(y)}{F(x)F(y)} - 2y\left(\frac{E'(x)}{F(x)} - \frac{E'(y)}{F(y)}\right)\right].$$

We obtained the positivity for $l \le 6$, but have't succeeded in general yet.

Question 1.9

Can we read the e-positivity of the tadpoles and their line graphs from the above GFs?

A way to extracting the e-coeff.

Proposition 1.10

Let $\{G_n\}_{n\ge0}$ be a sequence of graphs. Suppose that

$$\sum_{n\ge0} X_{G_n} z^{|G_n|} = A_z\left(\sum_{\mu\in\Lambda} \frac{a_\mu e_\mu z^{|\mu|}}{F(z)}\right),$$

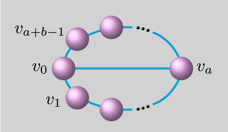
where Λ is the set of partitions of nonnegative integers and $a_\mu \in \mathbb{R}$. Then for any partition $\lambda = 2^{l_2}3^{l_3}\cdots \vdash |G_n|$,

$$[e_\lambda]X_{G_n} = M_\lambda \sum_{\mu=2^{u_2}3^{u_3}\cdots\subseteq\lambda} \frac{a_\mu}{\ell_\lambda(\ell_\lambda-1)\cdots(\ell_\lambda-\ell_\mu+1)} \prod_{u_i\ge1} \frac{l_i(l_i-1)\cdots(l_i-u_i+1)}{(i-1)^{u_i}},$$

where $M_\lambda = \binom{\ell_\lambda}{l_2,\, l_3,\, \ldots} \prod_{l_i\ge1}(i-1)^{l_i}$.

The cycle-chord graph

A special kind of theta graphs.

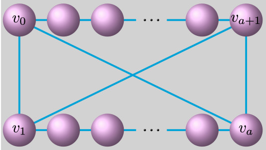


Proposition 1.11 (W. and Wang, 2020+)

$$\sum_{a,b\ge 1} X_{CC_{a,b}} x^a y^b = \frac{xy}{(x-y)^2}\left[\frac{x^2E''(x)E(y)+y^2E''(y)E(x)}{F(x)F(y)} - \frac{2xy}{x-y}\left(\frac{E'(x)}{F(x)} - \frac{E'(y)}{F(y)}\right)\right].$$

Bondy, The 'graph theory' of the Greek alphabet, Graph theory and applications (Proc. Conf., Western Michigan Univ., Kalamazoo, Mich., 1972; dedicated to the memory of J.W.T. Youngs), LNM 303, Springer, pp. 43–54, 1972.

The saltire graph



Proposition 1.12 (W. and Wang, 2020+)

$$\sum_{a,b\geq 1} X_{\mathrm{SA}_{a,b}} x^a y^b = \frac{xy\big[x(2x+y)E''(x)E(y)+y(2y+x)E(x)E''(y)\big]}{(x-y)^2F(x)F(y)} - \frac{6x^2y^2}{(x-y)^3}\left[\frac{E'(x)}{F(x)}-\frac{E'(y)}{F(y)}\right] - \frac{xy}{x-y}\left[\frac{xE''(x)}{F(x)}-\frac{yE''(y)}{F(y)}\right].$$

Proof: $\mathsf{SA}_{2,n}$ is e-positive

$$\sum_{n\geq 0} X_{\mathsf{SA}_{2,\,n+2}} z^n = A_z\left(\frac{(4z^2+2e_1z^3)E''(z)+(2e_2z^2-6)E(z)}{z^4F(z)}\right)$$

$$= A_z\left(\frac{(4z^2+2e_1z^3)2\sum_{n\geq 0}\binom{n+2}{2}e_{n+2}z^n+(2e_2z^2-6)\sum_{n\geq 0}e_nz^n}{z^4F(z)}\right).$$

Proof continued

Let $\lambda = 2^{l_2} 3^{l_3} \cdots$. By the extraction proposition, we find

$$\frac{[e_{\lambda 1}]X_G}{M_\lambda} = \frac{2l_2}{l} + \sum_{i\ge 0} 4\binom{i+2}{2}\frac{l_{i+2}}{(i+1)l} - 6 = \frac{2}{l}\sum_{i\ge 4} l_i(i-3) \ge 0, \quad \text{and}$$

$$\begin{aligned}\frac{[e_\lambda]X_G}{M_\lambda} =& \sum_{i\ge 0} 8\binom{i+2}{2}\frac{l_{i+2}}{l(i+1)} + 2\left(\frac{l_2}{l} + \frac{l_2(l_2-1)}{l(l-1)} + \sum_{i\ge 3}\frac{l_i l_2}{l(l-1)(i-1)}\right)\\ &-6-6\sum_{i\ge 2}\frac{l_i}{l(i-1)}\\ =& \frac{2}{l(l-1)}\sum_{i\ge 3}\frac{l_i}{i-1}\left[i(2i-5)(l-1-l_2)+2l_2(i^2-3i+1)\right] \ge 0.\end{aligned}$$

This completes the proof.

Conjectures

Conjecture 1.13

The saltire graphs $\mathsf{SA}(n,\ n+1)$ *and* $\mathsf{SA}(n,\ n+2)$ *are e-positive.*

Checked for $n \le 8$.

Conjecture 1.14

The generalized Petersen graph $P(2n, 1)$ *is not Schur positive.*

One may ask the positivity classification question for any graph. Would one really do it? Or, which ones deserve a concentration?

Thanks!