On the e-positivity of some claw-free graphs

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Symmetric function

- Let $x = (x_1, x_2, \ldots)$ be a set of indeterminates, and let $n \in \mathbb{N}$.
- Let R be a commutative ring with identity.
- A homogeneous symmetric function of degree n over R is a formal power series

$$f(x) = \sum_{\alpha} c_\alpha x^\alpha,$$

where

  - α ranges over all weak compositions $\alpha = (\alpha_1, \alpha_2, \ldots)$ of n of infinite length;
  - $c_\alpha \in R$;
  - x^α stands for the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots$; and
  - $f(x_{w(1)}, x_{w(2)}, \ldots) = f(x_1, x_2, \ldots)$ for every permutation w of the positive integers $\mathbb{P}$.

Monomial symmetric functions

- λ: a partition of a nonnegative integer n

$$(4), \quad (3,1), \quad (2,2), \quad (2,1,1), \quad (1,1,1,1)$$

- Given $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$, define a symmetric function $m_\lambda(x)$ by

$$m_\lambda = \sum_\alpha x^\alpha,$$

  where the sum ranges over all distinct permutations $\alpha = (\alpha_1, \alpha_2, \ldots)$ of the entries of the vector $\lambda = (\lambda_1, \lambda_2, \ldots)$.
- The set $\{m_\lambda : \lambda \vdash n\}$ is a vector space basis for Λ_R^n, the set of all homogeneous symmetric functions of degree n.

Elementary symmetric functions

- If $\lambda = (1^n)$, then

$$e_n = m_{(1^n)} = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}, \quad n \geq 1.$$

- For $\lambda = (\lambda_1, \lambda_2, \ldots)$, let

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots .$$

- The set $\{e_\lambda : \lambda \vdash n\}$ is a vector space basis for $\Lambda_{\mathbb{Q}}^n$.

Chromatic symmetric functions

- Let G be a graph with vertex set $V(G)$ and edge set $E(G)$.
- Let $\mathbb{Z}^+$ be the set of positive integers.
- A *coloring* of $V(G)$ is a function $\kappa : V(g) \mapsto \mathbb{Z}^+$.
- A coloring κ is *proper* if $\kappa(u) \neq \kappa(v)$ whenever $(u, v) \in E(G)$.

Definition (Stanley, 1995)

Given a graph $G = (V, E)$, the chromatic symmetric function is defined by

$$X_G = \sum_{\kappa} \prod_{v \in V} x_{\kappa(v)},$$

where the sum is over all proper colorings of κ of G.

Stanley's isomorphism conjecture for trees

- The chromatic symmetric function X_G generalizes the chromatic polynomial χ_G of G, since $X_G(1^n) = \chi_G(n)$.

Open problem (Stanley, 1995)

If T and T' are nonisomorphic trees, does $X_T \neq X_{T'}$?

- True for trees with at most 29 vertices by Heil and Ji (arXiv:1801.07363v2).
- Caterpillars can be distinguished by X_T due to Loebl and Sereni (arXiv:1405.4132).

Monomial expansion

- If $\lambda = \langle \cdots, 2^{r_2}, 1^{r_1} \rangle$, then let $\tilde{m}_\lambda = r_1! r_2! \cdots m_\lambda$.
- A stable partition (or an independent set) π of G is a set partition of $V(G)$ such that each block of π is totally disconnected.
- The type of π is a partition of $|V(G)|$ whose parts are the sizes of the blocks of π.

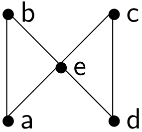
Theorem (Stanley, 1995)

We have

$$X_G = \sum_{\lambda \vdash |V(G)|} a_\lambda \tilde{m}_\lambda,$$

where a_λ is the number of stable partitions of G of type λ.

Monomial expansion

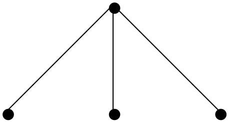


$$X_G = 2\tilde{m}_{(2,2,1)} + 4\tilde{m}_{(2,1,1,1)} + \tilde{m}_{(1,1,1,1,1)}$$

$$\{(a,c),b,d,e\},\quad \{(a,d),b,c,e\},\quad \{(b,d),a,c,e\},\quad \{(b,c),a,d,e\}$$

Stanley's (3+1)-free conjecture

- Let P be a finite poset.
- Let $\mathrm{inc}(P)$ denote the incomparability graph of P.
- Let $\mathbf{3}+\mathbf{1}$ denote the disjoint union of a 3-element chain and 1-element chain. Thus $\mathrm{inc}(\mathbf{3}+\mathbf{1})$ is a claw.



$$X_G = \tilde{m}_{(1,1,1,1)} + 3\tilde{m}_{(2,1,1)} + \tilde{m}_{(3,1)}$$

$$X_G = 4e_{(4)} + 5e_{(3,1)} - 2e_{(2,2)} + e_{(2,1,1)}$$

Stanley's (3+1)-free conjecture

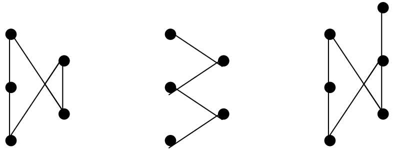


Figure: $(3+1)-free$ $\quad inc(P) \quad$ not $(3+1)-free$

- We say that P is $(\mathbf{3}+\mathbf{1})$-*free* if it contains no induced $\mathbf{3}+\mathbf{1}$.

Stanley's (3+1)-free conjecture

Conjecture (Stanley, 1995)

If P is $(\mathbf{3}+\mathbf{1})$-free, then $X_{\mathrm{inc}(P)}$ is e-positive.

- If P is $\mathbf{3}$-free, then $X_{\mathrm{inc}(P)}$ is e-positive.
- If P is $(\mathbf{3}+\mathbf{1})$-free, then $X_{\mathrm{inc}(P)}$ is claw-free.

Theorem (Guay-Paquet, 2013)

Stanley's (3+1)-free conjecture is true if and only if it is true for $(\mathbf{3}+\mathbf{1})$-free and $(\mathbf{2}+\mathbf{2})$-free posets.

Stanley's (3+1)-free conjecture

Conjecture (Stanley, 1995)

Fix $k \geq 2$. *Let*

$$P_{d,k} = \sum_{i_1,\ldots,i_d} x_{i_1} \cdots x_{i_d},$$

where any k *consecutive terms are distinct. Then* $P_{d,k}$ *is* e*-positive.*

- Stanley (1995) showed that the conjecture is true for $k = 2$.
- Stanley (1995) also showed that cycles are e-positive.
- Dahlberg (2018) showed that $P_{d,3}$ is e-positive.

1. Chromatic symmetric functions

2. Generalized pyramids

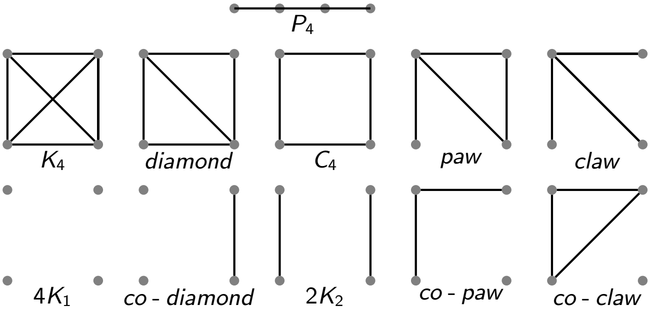
3. $2K_2$-free unit interval graphs

4. Twin vertices

5. Jacobi-Trudi immanants

H-free graphs

- Hamel, Hoang and Tuero (2017) initialed the study of H-free graphs, where $H = \{claw, F\}$ and $H = \{claw, F, co - F\}$, where F is a four-vertex graph.



Some e-positive graphs

Theorem (Tsujie, 2017)

If G is a (claw, P_4)-free graph, then X_G is e-positive.

Theorem (Hamel, Hoang and Tuero, 2017)

If G is (claw, triangle)-free, then each component of G is a path or cycle, and hence X_G is e-positive.

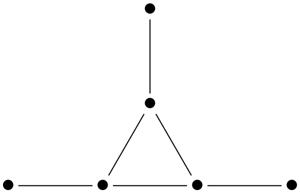
Theorem (Hamel, Hoang and Tuero, 2017)

If G is (claw, paw)-free or (claw, co-paw)-free, then X_G is e-positive.

Not e-positive graphs

Theorem (Hamel, Hoang and Tuero, 2017)

A graph that is H-free for H equal to $\{claw, diamond\}$, $\{claw, K_4\}$, $\{claw, 4K_1\}$, $\{claw, C_4\}$, $\{claw, 2K_2\}$, or $\{claw, co\text{-}claw\}$, is not necessarily e-positive.



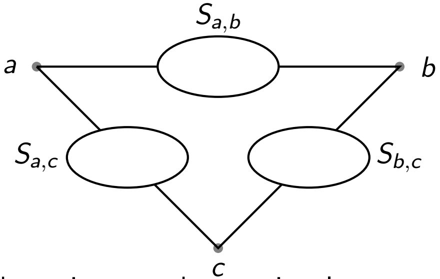
$3 - sun$

$$X_G = 6e_{3,2,1} - 6e_{3,3} + 6e_{4,1,1} + 12e_{4,2} + 18e_{5,1} + 12e_6$$

(claw, co-diamond)-free graphs

Open problem (Hamel, Hoang and Tuero, 2017)

Are (claw, co-diamond)-free graphs e -positive?

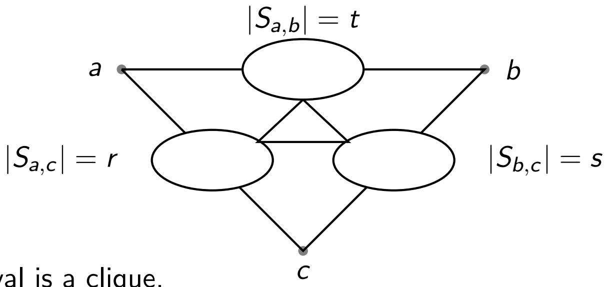


- The three black vertices are the co-triangle.
- Each oval represents a subgraph, with each vertex in subgraph being joined to the two corresponding vertices of the co-triangle.
- At least two ovals are non-empty.

Generalized pyramids

Open problem (Hamel, Hoang and Tuero, 2017)

Are generalized pyramids e-positive?



- Each oval is a clique.
- There are all edges between any two ovals.
- If a graph G is (claw, co-diamond, $2K_2$)-free, then G is the generalized pyramid.

Generalized pyramids

Theorem (Li and Yang, 2019)

Generalized pyramids $GP(r, s, t)$ are e-positive.

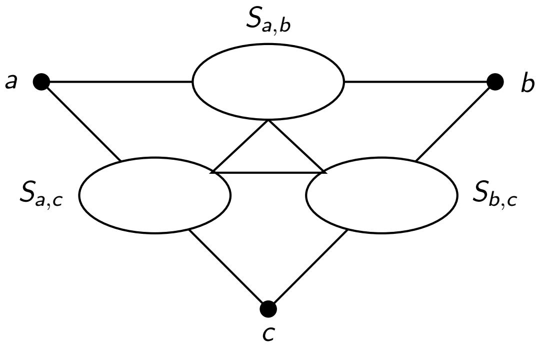


Figure: The generalized pyramid graph $GP(r, s, t)$

Generalized pyramids

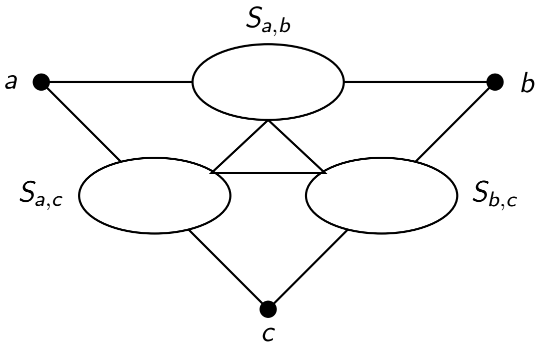
Theorem (Li and Yang, 2019)

For any nonnegative integers r, s, t*, we have*

$$\begin{aligned} X_{GP(r,s,t)} =& \tilde{m}_{(3,1^{r+s+t})} + (rst)\tilde{m}_{(2,2,2,1^{r+s+t-3})} \\ & + (rt + rs + st + r + s + t)\tilde{m}_{(2,2,1^{r+s+t-1})} \\ & + (r + s + t + 3)\tilde{m}_{(2,1^{r+s+t+1})} + \tilde{m}_{(1^{r+s+t+3})}. \end{aligned}$$

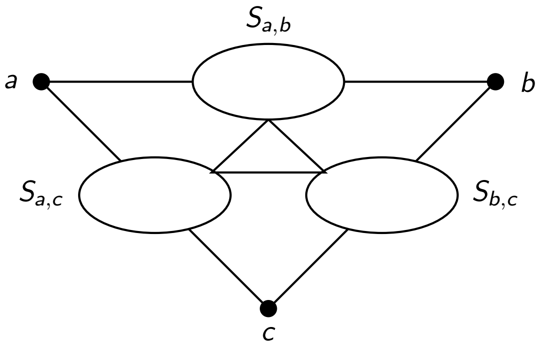
- Any admissible stable partition of $GP(r, s, t)$ is of type $(3, 1^{r+s+t})$, $(2, 1^{r+s+t+1})$, $(2, 2, 1^{r+s+t-1})$, $(2, 2, 2, 1^{r+s+t-3})$ or $(1^{r+s+t+3})$.

Generalized pyramids



- Any admissible stable partition of $GP(r,s,t)$ is of type $(3,1^{r+s+t})$, $(2,1^{r+s+t+1})$, $(2,2,1^{r+s+t-1})$, $(2,2,2,1^{r+s+t-3})$ or $(1^{r+s+t+3})$.

Generalized pyramids



$$[\tilde{m}_{(2,2,1^{r+s+t-1})}]X_{GP(r,s,t)} = (rt + rs + st + r + s + t)$$

Generalized pyramids

Theorem (Li and Yang, 2019)

For any nonnegative integers r, s, t, we have

$$\begin{aligned} X_{GP(r,s,t)} = & \tilde{m}_{(3,1^{r+s+t})} + (rst)\tilde{m}_{(2,2,2,1^{r+s+t-3})} \\ & + (rt + rs + st + r + s + t)\tilde{m}_{(2,2,1^{r+s+t-1})} \\ & + (r + s + t + 3)\tilde{m}_{(2,1^{r+s+t+1})} + \tilde{m}_{(1^{r+s+t+3})}. \end{aligned} \tag{1}$$

Generalized pyramids

Theorem

Let $\lambda \vdash n$. If $e_\lambda = \sum_{\mu \vdash n} M_{\lambda\mu} m_\mu$, then $M_{\lambda\mu}$ is equal to the number of $(0,1)$-matrices $A = (a_{ij})_{i,j \geq 1}$ satisfying $row(A) = \lambda$ and $col(A) = \mu$, where $row(A)$ (resp., $col(A)$) is the vector of row sums (resp., column sums) of A. Moreover, $M_{\lambda\mu} = 0$ unless $\lambda \leq \mu'$, and $M_{\lambda\lambda'} = 1$.

- Given two partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ of $\mathrm{Par}(n)$, we say that $\mu \leq \lambda$ if

$$\mu_1 + \mu_2 + \cdots + \mu_i \leq \lambda_1 + \lambda_2 + \cdots + \lambda_i \quad \text{for all } i \geq 1.$$

- The conjugate of $\lambda = (\lambda_1, \lambda_2, \ldots)$ is defined as the partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ where $\lambda'_i = |\{j : \lambda_j \geq i\}|$.

Generalized pyramids

- Suppose that $r+s+t=i$.
- Let $P=\{(2^3,1^{i-3}),\,(3,1^i),\,(2^2,1^{i-1}),\,(2,1^{i+1}),\,(1^{i+3})\}$.
- In order to give the elementary expansion of $X_{GP(r,s,t)}$, it suffices to consider the monomial expansion of those e_λ's such that $\lambda' \le \mu$ for some $\mu \in P$.
- It is straightforward to verify that the set of such partitions λ is composed of $\{(i,3),\,(i+1,1,1),\,(i+1,2),\,(i+2,1),\,(i+3)\}$.

Generalized pyramids

We have

$$e_{(i,3)} = m_{(2,2,2,1^{i-3})} + (i-1)m_{(2,2,1^{i-1})} + \binom{i+1}{2} m_{(2,1^{i+1})} + \binom{i+3}{3} m_{(1^{i+3})}, \tag{2}$$

$$e_{(i+1,1,1)} = m_{(3,1^{i})} + (2i+3)m_{(2,1^{i+1})} + 2m_{(2,2,1^{i-1})} + 2\binom{i+3}{2} m_{(1^{i+3})}, \tag{3}$$

$$e_{(i+1,2)} = m_{(2,2,1^{i-1})} + (i+1)m_{(2,1^{i+1})} + \binom{i+3}{2} m_{(1^{i+3})}, \tag{4}$$

$$e_{(i+2,1)} = m_{(2,1^{i+1})} + (i+3)m_{(1^{i+3})}, \tag{5}$$

$$e_{i+3} = m_{(1^{i+3})}. \tag{6}$$

Generalized pyramids

Substituting the above m-expansion formulas into (1), we get that

$$X_{GP(r,s,t)} = A\cdot e_{(r+s+t+1,1,1)} + B\cdot e_{(r+s+t,3)} + C\cdot e_{(r+s+t+1,2)} + D\cdot e_{(r+s+t+2,1)} + E\cdot e_{(r+s+t+3)}, \tag{7}$$

where

$$A = (r+s+t)!,$$

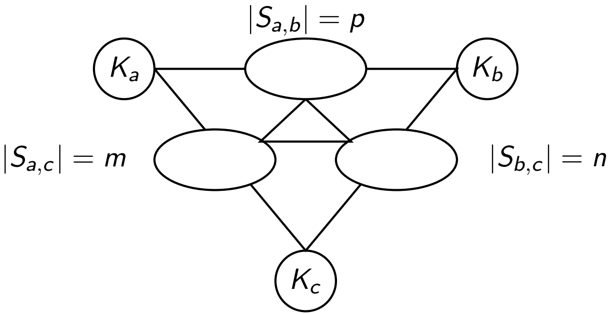
$$B = (r+s+t-3)!\cdot 6rst,$$

$$C = (r+s+t-3)!\cdot 2(r+s+t-1) \cdot [(r^2s+rs^2-2rs)+(rt^2+r^2t-2rt)+(s^2t+st^2-2st)],$$

$$\begin{aligned} D = &(r+s+t-2)!\cdot[(r^4+r^3-2r^2)+(3r^2s-2rs)+(3rs^2-2s^2)\\ &+(3r^2t-2rt)+(9rst-2st)+(3rt^2-2t^2)+3s^2t+5rs^2t\\ &+2s^3t+5r^2st+2r^3t+2r^2t^2+3st^2+5rst^2+2s^2t^2\\ &+t^3+2rt^3+2st^3+t^4+2r^3s+2r^2s^2+s^3+2rs^3+s^4],\end{aligned}$$

Generalization of generalized pyramids

- Replacing the co-triangle by three cliques, e-positivity?



$G(a, b, c; m, n, p)$

Conjecture (Li and Yang, 2019)

The graph $G(a, b, c; m, n, p)$ is e-positive.

Generalized bull graphs

Theorem (Cho and Huh, 2019)

For any positive integers r, s, t, the generalized bull graph $X_{GB(r,s,t)}$ is e-positive.

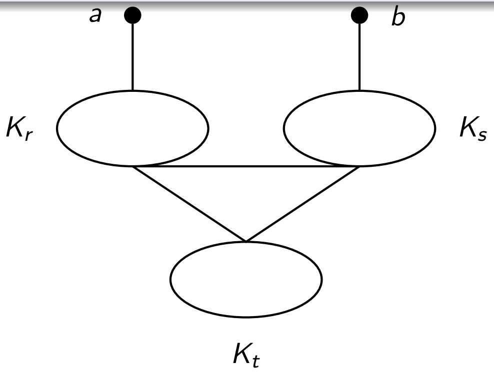


Figure: The generalized bull graph $GB(r, s, t)$

Generalized bull graphs

- Using the same method as before, we get that

$$X_{GB(r,s,t)} = t \cdot \tilde{m}_{(3,1^{r+s+t-1})} + (t(t-1) + tr + sr + st) \cdot \tilde{m}_{(2,2,1^{r+s+t-2})} + (1 + 2t + s + r) \cdot \tilde{m}_{(2,1^{r+s+t})} + \tilde{m}_{(1^{r+s+t+2})}. \tag{8}$$

- Setting $k = r + s + t$ and $i = k - 1$ in (3), (4), (5) and (6), and then substituting these four equations into (8), we obtain

$$X_{GB(r,s,t)} = (r+s+t-2)! \cdot [(r+s+t-1)t \cdot e_{(r+s+t,1,1)} + 2rs \cdot e_{(r+s+t,2)} + (r^3 + r^2 s + rs^2 + s^3 + 2r^2 t + 2rst + 2s^2 t + rt^2 + st^2 - r - s) \cdot e_{(r+s+t+1,1)} + (r+s+t+2)(r+s+t-1)rs \cdot e_{(r+s+t+2)}].$$

$2K_2$-free unit interval graph

- A {claw, $2K_2$}-free graph is not necessarily e-positive.
- The incomparability graph of a $(\mathbf{3}+\mathbf{1})$-free is claw-free.
- The incomparability graph of a $(\mathbf{3}+\mathbf{1})$-free and $(\mathbf{2}+\mathbf{2})$-free poset is called a unit interval graph.

Theorem (Li and Yang, 2019)

If G is a $2K_2$-free unit interval graph, then G is e-positive.

$2K_2$-free unit interval graph

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Theorem (Li and Yang, 2019)

If G is a $2K_2$-free unit interval graph, then G is e-positive.

Theorem (Stanley and Stembridge, 1993)

If G is a co-triangle free graph, then X_G is e-positive.

Theorem (Cho and Huh, 2018)

If G is a generalized bull graph, then X_G is e-positive.

$2K_2$-free unit interval graph

- Given a graph G with vertex set V and edge set E, let $\alpha(G)$ denote the maximum size of stable sets.
- Given a pair of vertices u and v, let $d(u,v)$ denote the number of edges of the shortest path between u and v.
- For any vertex $w \in V$, let $N_i(w) = \{x \in V \mid d(x,w) = i\}$ and $[N_i(w)]$ denote the induced subgraph on $N_i(w)$.
- In particular, $N_1(w)$ is the neighborhood of w, denoted by $N(w)$.

$2K_2$-free unit interval graph

Theorem (Hempel and Kratsch, 2002)

If G is a $2K_2$-free unit interval graph, then there exists a vertex w such that $\alpha([N(w)]) \leq 2$, $N_i(w) = \emptyset$ for $i \geq 4$ and $|N_3(w)| \leq 1$. Moreover, G satisfies one of the following:

(1) $[N(w)]$ *is not connected;*

(2) $[N(w)]$ *is connected and* $|N_3(w)| = 1$;

(3) $[N(w)]$ *is connected,* $|N_3(w)| = 0$ *and* $\alpha([N(w)]) = 1$;

(4) $[N(w)]$ *is connected,* $|N_3(w)| = 0$ *and* $\alpha([N(w)]) = 2$.

- Cases (1), (2) and (3) are solved by Foley, Hoàng and Merkel.
- We prove that for the case of (4), the graph G is a generalized bull or co-triangle free.

$2K_2$-free unit interval graph

- Foley, Hoàng and Merkel pointed out that "The family of $2K_2$-free unit interval graphs that are not known to be e-positive have $[N_1]$ connected, $[N_1]$ contains an induced P_3, $\alpha([N_1]) = 2$, $N_2 \neq \emptyset$, and all $N_i = \emptyset$ for $i \geq 3$."
- Claim 1: $|N_2| \leq 2$.
  (1) Suppose that $|N_2| = s$. Then for any $a \in N_1$, there are at least $s-1$ vertices in N_2 which are adjacent to a. In the contrary case, if there exist $x, y \in N_2$ which are not adjacent to a, then $\{x, y, a, w\}$ induces a $2K_2$ in G since $[N_2]$ is a clique.
  (2) Since $\alpha([N_1]) = 2$. There exist $a, b \in N_1$ such that a and b are not adjacent. Moreover, a, b can not be adjacent to the same vertex x in N_2 (otherwise, $\{x, a, b, w\}$ induce a C_4). (Unit interval graph must be C_4-free due to the poset being $\mathbf{2}+\mathbf{2}$-free.) Hence

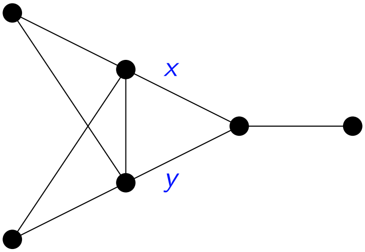
$$s - 1 + s - 1 \leq s,$$

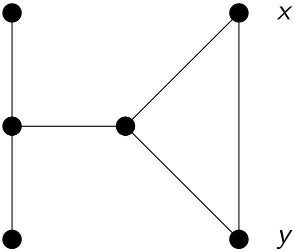
$2K_2$-free unit interval graph

- Without loss of generality, we may assume that G is connected. There are three cases to check:
  (1) $[N(w)]$ is connected, $|N_3(w)| = 0$, $\alpha([N(w)]) = 2$ and $|N_2(w)| = 2$;
  (2) $[N(w)]$ is connected, $|N_3(w)| = 0$, $\alpha([N(w)]) = 2$ and $|N_2(w)| = 1$;
  (3) $[N(w)]$ is connected, $|N_3(w)| = 0$, $\alpha([N(w)]) = 2$ and $|N_2(w)| = 0$;
- For (1) and (3), the graph G is co-triangle free; For (2), G is either a co-triangle free graph or a generalized bull graph.

Twin vertices

- Two vertices x and y are twins if they are adjacent and any vertex z is adjacent to both x and y or non adjacent to both x and y.





Reduction conjecture

- Given a finite simple graph G and a vertex v of G, define G'_v to be the graph obtained from G by replacing v by an edge v_1v_2 and joining the two endpoints v_1, v_2 to all vertices adjacent to v in G.

Conjecture (Foley, Hoang and Merkel, 2019)

If G is e-positive, then G'_v is e-positive.

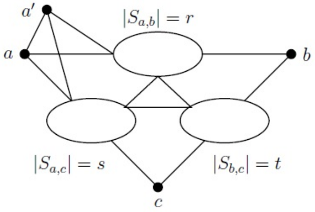
- If the above conjecture is not valid, Ethan Li further considered the following.

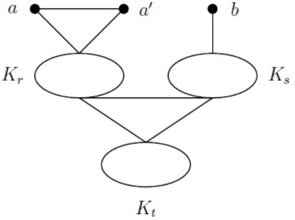
Conjecture

If G is an e-positive unit interval graph, then G'_v is e-positive.

Positive results

- Both generalized pyramids and bull graphs are e-positive.
- Grace Li studied the twining operation on generalized pyramids and bull graphs, and showed that the following graphs are e-positive.





Counterexample

- Ethan Li found a simple counterexample to Foley, Hoang and Merkel's conjecture.
- Define CL_n $(n \geq 2)$ to be the graph obtained from a claw $K_{1,3}$ and a complete graph K_n by identifying one vertex of degree one of the claw and one vertex of the complete graph.

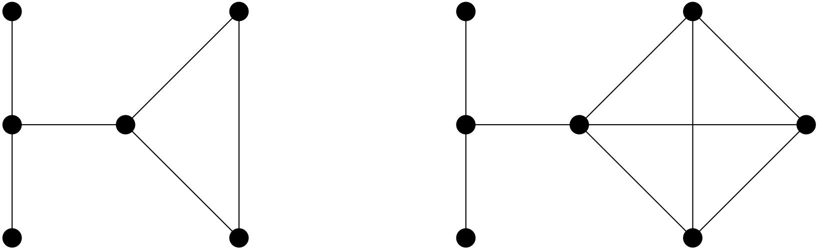


Figure: CL_3 and CL_4

Counterexample

- It is easy to see that CL_2 is e-positive.

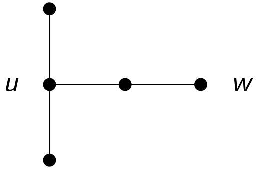
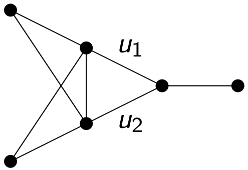
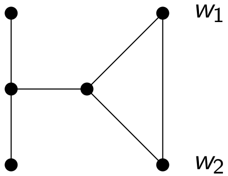


Figure: CL_2

- While neither of the following graphs is e-positive.





Counterexample

Theorem

For any $n \geq 3$, the graph CL_n is not e-positive.

$$
\begin{aligned}
X_{CL_n} =& \tilde{m}_{1^{n+3}} + 3n \cdot \tilde{m}_{2,1^{n+1}} + (n-1)(3n-1)\, \tilde{m}_{2^2,1^{n-1}} \\
& + (n-1)^2(n-2)\, \tilde{m}_{2^3,1^{n-3}} + n \cdot \tilde{m}_{3,1^n} + (n-1)^2\, \tilde{m}_{3,2,1^{n-2}} \\
=& (n+3)!\, m_{1^{n+3}} + 3n(n+1)!\, m_{2,1^{n+1}} \\
& + 2(n-1)(3n-1)(n-1)!\, m_{2^2,1^{n-1}} + 6(n-1)(n-1)!\, m_{2^3,1^{n-3}} \\
& + n \cdot n!\, m_{3,1^n} + (n-1)(n-1)!\, m_{3,2,1^{n-2}}
\end{aligned}
$$

It turns out that

$$[e_{n+1,2}]X_{CL_n} = -2(n-1)! < 0.$$

A conjecture on trees

Conjecture (Dahlberg, She and van Willigenburg)

For every $n \geq 2$ there is a tree T on n vertices, one of which has degree $\lfloor \frac{n}{2} \rfloor$, such that X_T is Schur positive.

- They confirmed the conjecture for $n \leq 19$.
- Rambeloson and Shareshian (arXiv 2006.14415) disproved the conjecture for $n = 20$.
- Can we do more?

- Let $\mu = (\mu_1, \ldots, \mu_n), \nu = (\nu_1, \ldots, \nu_n)$ be two partitions such that $\nu \subseteq \mu$, namely $0 \leq \nu_i \leq \mu_i$ for $1 \leq i \leq n$. Define the *Jacobi-Trudi matrix* as

$$H_{\mu/\nu} = H_{\mu/\nu}(x) = [h_{\mu_i - \nu_j + j - i}]_{1 \leq i,j \leq n},$$

where h_k denotes the kth complete homogeneous symmetric function.

- Moreover, for any partition λ of n, let χ^λ denote the associated irreducible character of the symmetric group S_n. Then the *immanant* of an $n \times n$ matrix $A = [a_{ij}]$ with respect to λ is defined as

$$\mathrm{Imm}_\lambda(A) = \sum_{w \in S_n} \chi^\lambda(w) \prod_{i=1}^{n} a_{i,w(i)}.$$

The Stanley-Stembridge conjecture

- The following expression

$$F_{\mu/\nu}(x,y) = \sum_{\lambda \vdash n} s_\lambda(y) \mathrm{Imm}_\lambda H_{\mu/\nu}(x)$$

can be transformed into

$$F_{\mu/\nu}(x,y) = \sum_{\theta \vdash n} E^{\theta}_{\mu/\nu}(y) s_\theta(x).$$

Conjecture (Stanley and Stembridge, 1993)

$E^{\theta}_{\mu/\nu}$ is a nonnegative linear expansion of h_α's.

- Given $B \subseteq [n]^2$, every placement of n non-attacking rooks on B corresponds to a permutation $w \in S_n$, i.e., $w(i) = j$ is a rook occupies (i, j). We then call w a B-*compatible permutation*.
- Define the *cycle indicator* $Z[B]$ of B to be

$$Z[B] = \sum_{w} p_{\rho(w)} = \sum_{w} p_1^{m_1(\rho(w))} p_2^{m_2(\rho(w))} \cdots,$$

where w ranges over all B-compatible permutations.
- Let S be a placement of (at most n) non-attacking rooks, we can consider it as a directed graph. Then S must be a union of disjoint directed paths and cycles.

Cycle indicator

- Define the *type* of S to be the pair of partitions $(\alpha;\beta)$ such that the parts of α (resp. β) represents the sizes of of the directed paths (resp. cycles). Note that $|\alpha|+|\beta|=n$ and the number of isolated vertices in S is $m_1(\alpha)$.
- Let ω be the involution sending e_λ to h_λ. Then the "forgotten" symmetric functions f_λ are defined by $f_\lambda=\omega(m_\lambda)$.

Theorem (Stanley and Stembridge, 1993)

For any $B\subseteq[n]^2$, we have

$$Z[B]=\sum_{\alpha,\beta}(-1)^{|\beta|}m_1(\alpha)!m_2(\alpha)!\cdots r_{\alpha,\beta}(\bar{B})f_\alpha p_\beta,$$

where $r_{\alpha,\beta}(\bar{B})$ denotes the number of subgraphs of type $(\alpha;\beta)$ in the complement $\bar{B}$.

Combinatorial interpretation

- Note that in a Jacobi-Trudi matrix $H_{\mu/\nu}$, the zero entries form a partition $\sigma \subseteq \delta = (n-1, n-2, \ldots, 1)$ (reading from bottom to top) in the southwest corner. Let B_σ be the complement of the diagram of σ, i.e.,

$$B_\sigma = \{(i,j) \in [n]^2 : j > \sigma_{n-i+1}\}.$$

The diagram of σ is therefore denoted by $\bar{B}_\sigma$.

Theorem (Stanley and Stembridge, 1993)

If $\theta = (N)$ and $\sigma = \sigma(\mu/\nu)$, then $E^\theta_{\mu/\nu} = Z[B_\sigma]$.

Combinatorial interpretation

- Let $\sigma \subseteq \delta$. Then the positions of $[n]^2$ indexed by $\bar{B}_\sigma$ can be used to define a partial order P_σ of $[n]$ in which $i > j$ if and only if $(i,j) \in \bar{B}_\sigma$ ($i \geq j+1$).
- For any partition α of n, define $c_\alpha(P)$ to be the number of ways to partition an n-element poset P into (unordered) chains of cadinality $\alpha_1, \alpha_2, \ldots$, and let $\bar{c}_\alpha(P) := m_1(\alpha)!m_2(\alpha)! \cdots c_\alpha(P)$ denote the number of partitions of P into *ordered* chains of cardinality $\alpha_1, \alpha_2, \ldots$.
- When $B = B_\sigma$, the quantity $m_1(\alpha)!m_2(\alpha)! \cdots r_{\alpha,\varnothing}(\bar{B})$ can be identified as $\bar{c}_\alpha(P_\sigma)$.

Combinatorial interpretation

- Using the two theorems above,

$$E^{(N)}_{\mu/\nu} = Z[B_\sigma] = \sum_{\alpha} m_1(\alpha)! m_2(\alpha)! \cdots r_{\alpha,\varnothing}(\bar{B}_\sigma) f_\alpha = \sum_{\alpha \vdash n} \bar{c}_\alpha(P_\sigma) f_\alpha.$$

Thus, the case $\theta = (N)$ of the initial conjecture is equivalent to the assertion that $\sum_{\alpha \vdash n} \bar{c}_\alpha(P_\sigma) m_\alpha$ is e-positive.

Poset-Chain property

- Naturally, we can generalize the above problem to finding all posets P of $[n]$ such that $\sum_{\alpha\vdash n} \bar{c}_\alpha(P_\sigma) m_\alpha$ is a nonnegative linear combination of the e_λ's. This property is called the *Poset-Chain* property.
- By a theorem of Dean and Keller [DK1968], the posets P_σ are characterized by the fact that they are both (3+1)-free and (2+2)-free (unit interval order $\Leftrightarrow$ Dyck path $\Leftrightarrow$ complement of area). However, there exist some posets satisfying the Poset-Chain property which are not (2+2)-free.

Conjecture (Stanley and Stembridge, 1993)

Any (3+1)-free poset satisfies the Poset-Chain property.

The corresponding conjecture for CSF

- Recall that the *chromatic symmetric function* of a graph G is defined as $X_G = \sum_{\kappa} \prod_{v \in V(G)} x_{\kappa(v)}$, where κ ranges over all proper colorings of G.
- A poset P is said to be $(a+b)$*-free* if it does not contain an induced subposet which is a disjoint union of an a-element chain and a b-element chain. The *incomparability graph* of P, denoted by $\mathrm{inc}(P)$, is defined as follows: the vertices of $\mathrm{inc}(P)$ are elements of P, and two vertices u and v are adjacent if and only if they are incomparable in P.

Conjecture (Stanley, 1995)

If P is (3+1)-free, then $\mathrm{inc}(P)$ is e-positive.

The equivalence of these two conjectures

- Recall that a *stable partition* of a graph G is a set partition of $V(G)$ such that every block is a stable set, i.e., the vertices in the same block are non-adjacent to each other.

Theorem (Stanley, 1995)

Let d_α be the number of stable partitions of G of type α. Then

$$X_G = \sum_{\alpha \vdash n} d_\alpha m_1(\alpha)! m_2(\alpha)! \cdots m_\alpha.$$

- Note that finding partitions of a poset P into (unordered) chains of cardinality $\alpha_1, \alpha_2, \ldots$ is equivalent to finding stable partitions of type α in $\text{inc}(P)$. Hence we have $c_\alpha(P) = d_\alpha$ and $\bar{c}_\alpha(P) = d_\alpha m_1(\alpha)! m_2(\alpha)! \cdots$.

Related references

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Thanks for your attention!