# A Toeplitz property of ballot permutations and odd order permutations

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这是与张珈瑞合作的工作

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# Outline

1. Introduction
2. A short proof for $|\mathscr{B}_n| = |\mathscr{P}_n|$
3. Spiro's Conjecture
4. The Toeplitz property
5. Summary

# André permutations

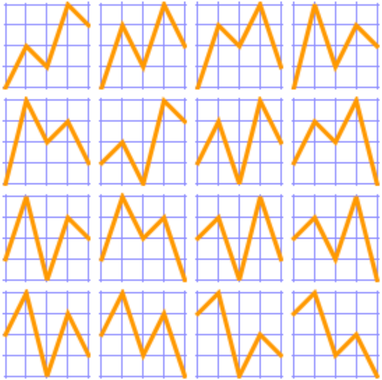
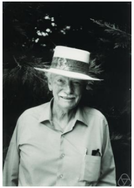


Figure: All 16 alternating permutations of [5].

# André Permutations

- 1881 André: enum. alternating perms.
- 1968 Niven: enum. perms. w. a given signature.

**Definition**

The signature of $\pi$ is the seq. $(q_1, q_2, \ldots, q_{n-1})$ where

$$q_i = \begin{cases} -1, & \text{if } \pi_i > \pi_{i+1}; \\ 1, & \text{if } \pi_i < \pi_{i+1}. \end{cases}$$

# A determinantal formula

For any $\{-1,1\}$-seq. $Q=(q_1,\,\ldots,\,q_{n-1})$, denote by

$$[Q]=[n;\ k_1,k_2,\ldots,k_r]$$

the $\#$ of $\pi\in\mathfrak{S}_n$ having the sig. $Q$, where $k_1<\cdots<k_r$ are the subscripts of those $q$'s having value $-1$. Niven showed

$$[Q]=\det\left(\binom{k_i}{k_{j-1}}\right)_{i,j=1}^{r+1},$$

where $k_0=0$ and $k_{r+1}=n$, and showed that

$$[Q] \text{ attains its max.} \iff Q \text{ is the sig. of an André perm.}$$

André, Sur les permutations alternées, J. de Math. 1881.

Niven, A combinatorial problem of finite sequences, Nieuw Arch. Wiskd. 1968.

# Ballot Permutation

## Definition

Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$. The height of $\pi$:

$$h(\pi) = \mathrm{asc}(\pi_1\pi_2\cdots\pi_n) - \mathrm{des}(\pi_1\pi_2\cdots\pi_n)$$

$\pi$ is a ballot permutation if $h(\pi_1\pi_2\cdots\pi_i) \geq 0\ \forall\ i \in [n]$.

# Example of Ballot Permutation

## Example

A ballot perm. of final height 0 is a Dyck permutation. The # of Dyck perms. in $\mathfrak{S}_{2n+1}$ is the Eulerian-Catalan #

$$2\sum_{j=0}^{n}(-1)^j\binom{2n+1}{j}(n+1-j)^{2n}.$$

Bidkhori and Sullivant, Eulerian-Catalan numbers, Electron. J. Combin., 2011.

# The # of ballot permutations



**Theorem (Bernardi, Duplantier and Nadeau)**

*The # of ballot perms. of length n is* $|\mathscr{B}_n| = \begin{cases} (n-1)!!^2, & \text{if } n \text{ is even;} \\ n!! \cdot (n-2)!!, & \text{if } n \text{ is odd.} \end{cases}$

Bernardi, Duplantier, and Nadeau, A bijection between well-labelled positive paths and matchings, SLC, 2010.

# Odd Order Permutation

$\mathscr{P}_n$: the set of odd order (every cycle is of odd length) perms. of $[n]$. Considering the neighbours of the letter $n$, we see that

$$|\mathscr{P}_n| = |\mathscr{P}_{n-1}| + (n-1)(n-2)|\mathscr{P}_{n-2}|.$$

So

$$|\mathscr{P}_n| = \begin{cases} (n-1)!!^2, & \text{if } n \text{ is even;} \\ n!! \cdot (n-2)!!, & \text{if } n \text{ is odd.} \end{cases}$$

**Question**

*Combinatorial proof for* $|\mathscr{B}_n| = |\mathscr{P}_n|$?

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# $\omega$-decomposable

## Definition

Let $\omega \in \mathscr{B} \setminus \{\epsilon\}$. A ballot perm. $\pi \in \mathscr{B}_n$ is $\omega$-decomposable if $\pi = \alpha\omega\gamma\delta$ s.t.

$$h(\alpha\omega\gamma) = h(\omega),$$

where $\alpha$, $\gamma$ and $\delta$ are allowed to be empty. Denote by $X_n(\omega)$ the set of $\omega$-decomposable ballot perms. Define the $\omega$-decomposition of $\pi \in X_n(\omega)$ to be $(\alpha, \omega, \gamma, \delta)$ s.t. $\pi = \alpha\omega\gamma\delta$ and $\gamma$ is the longest word s.t. $\gamma'\omega_{-1}$ is ballot, where $\gamma'$ is the reversal of $\gamma$.

# A bijection between $\omega$-decomposable

**Lemma**

*Let* $\lambda = i\,n\,(j-1)\,j$ *and* $\mu = (j-1)\,j\,n\,i$*, where* $i+2 \le j \le n-1$*. Then*

$$(\alpha,\,\lambda,\,\gamma,\,\delta) \mapsto (\gamma',\,\mu,\,\alpha',\,\delta)$$

*is a bijection between the sets* $X_n(\lambda)$ *and* $X_n(\mu)$*, with the inverse*

$$(\alpha,\,\mu,\,\gamma,\,\delta) \mapsto (\gamma',\,\lambda,\,\alpha',\,\delta).$$

*Then* $|X_n(\lambda)| = |X_n(\mu)|$.

# $|\mathscr{B}_n|$ and $|\mathscr{P}_n|$ satisfy the same recurrence

Let $n \geq 4$, $i \geq 1$, and $i + 2 \leq j \leq n - 1$.

$\mathscr{B}_n(i, j)$: the set of perms. in $\mathscr{B}_n$ having the factor $inj$.

Consider the involution $j - 1 \leftrightarrow j$ on $\mathscr{B}_n(i, j-1) \setminus X_n(\lambda)$. One may obtain

$$b_n(i, j-1) - b_n(i, j) = |X_n(\lambda)| = |X_n(\mu)| = b_n(j, i) - b_n(j-1, i).$$

Then

$b_n(i, j) + b_n(j, i) = b_n(i, j-1) + b_n(j-1, i) = \cdots = b_n(i, i+1) + b_n(i+1, i) = 2b_{n-2}$.

Since the $\#$ of ballot perms. in $\mathscr{B}_n$ ending with $n$ is $b_{n-1}$, we obtain

$$b_n = b_{n-1} + \sum_{i,j} b_n(i, j) = b_{n-1} + (n-1)(n-2)b_{n-2}.$$

# $|\mathscr{B}_n|$ and $|\mathscr{P}_n|$ satisfy the same initial values

$b_1 = b_2 = p_1 = p_2 = 1.$

Hence, $b_n = p_n$.

# Outline

# Spiro's Conjecture

Spiro: what statistics in $\mathscr{P}_n$ corresponds to the descents in $\mathscr{B}_n$?

- $\mathscr{B}_{n,d}$: the set of perms. in $\mathscr{B}_n$ with $d$ descents.
- $\mathscr{P}_{n,d}$: the set of perms. in $\mathscr{P}_n$ s.t. $d = \sum_{\text{cycles } c \text{ of } \pi} \min\big(\mathrm{cdes}(c), \mathrm{casc}(c)\big)$, where

$$\mathrm{cdes}(c) = \Big|\{i \in [k] \colon c_i > c_{i+1} \text{ with } c_{k+1} = c_1\}\Big|,$$
$$\mathrm{casc}(c) = \Big|\{i \in [k] \colon c_i < c_{i+1} \text{ with } c_{k+1} = c_1\}\Big|.$$

**Example**

Let $\pi = (21893)(457)(6)$. Then $\sum = 2 + 1 = 3$.

S. Spiro, Ballot permutations and odd order permutations, Discrete Math. 343(6) (2020), 111869.

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Let $\pi = (21893)(457)(6)$. Then $\sum = 2 + 1 = 3$.

**Conjecture (Spiro, 2020)**

$b_{n,d} = p_{n,d}$.

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# Example for Spiro's Conjecture

## Conjecture (Spiro, 2020)

$b_{n,d} = p_{n,d}$.

## Example

$b_{4,1} = p_{4,1} = 8$. In fact,

$$\begin{aligned}\mathscr{B}_{4,1} &= \{1243, 1324, 1342, 1423, 2314, 2341, 2413, 3412\} \quad \text{and} \\ \mathscr{P}_{4,1} &= \{(234), (243), (123), (124), (132), (134), (142), (143)\} \\ &= \{1342, 1423, 2314, 2431, 3124, 3241, 4132, 4213\}.\end{aligned}$$

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- On Jul. 21st, Zhao claimed a generating function proof for $b_{n,d} = p_{n,d}$.
- We are still looking for a combin. proof.

# Outline

# Toeplitz matrix

## Definition

A square matrix $(a_{ij})$ is said to be Toeplitz if

$$a_{i+1,j+1} = a_{i,j}$$

for all well defined entries.

$$\begin{bmatrix} a & b & c & d & e \\ f & a & b & c & d \\ g & f & a & b & c \\ h & g & f & a & b \\ i & h & g & f & a \end{bmatrix}$$

# The Toeplitz matrices for $\mathscr{B}_n$ for $3 \le n \le 8$

## Example

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 3 & 2 & 1 \\ 3 & 0 & 3 & 2 \\ 4 & 3 & 0 & 3 \\ 5 & 4 & 3 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 9 & 6 & 3 & 1 \\ 9 & 0 & 9 & 6 & 3 \\ 12 & 9 & 0 & 9 & 6 \\ 15 & 12 & 9 & 0 & 9 \\ 17 & 15 & 12 & 9 & 0 \end{bmatrix}.$$

# The Toeplitz matrices for $\mathscr{B}_n$ for $3 \leq n \leq 8$

## Example

$$\left|\mathscr{B}_7(i,j)\right|_{i,j} = \begin{bmatrix} 0 & 45 & 36 & 27 & 19 & 13 \\ 45 & 0 & 45 & 36 & 27 & 19 \\ 54 & 45 & 0 & 45 & 36 & 27 \\ 63 & 54 & 45 & 0 & 45 & 36 \\ 71 & 63 & 54 & 45 & 0 & 45 \\ 77 & 71 & 63 & 54 & 45 & 0 \end{bmatrix}.$$

# The Toeplitz matrices for $\mathscr{B}_n$ for $3 \leq n \leq 8$

## Example

$$\left|\mathscr{B}_8(i,j)\right|_{i,j} = \begin{bmatrix} 0 & 225 & 182 & 139 & 99 & 65 & 38 \\ 225 & 0 & 225 & 182 & 139 & 99 & 65 \\ 268 & 225 & 0 & 225 & 182 & 139 & 99 \\ 311 & 268 & 225 & 0 & 225 & 182 & 139 \\ 351 & 311 & 268 & 225 & 0 & 225 & 182 \\ 385 & 351 & 311 & 268 & 225 & 0 & 225 \\ 412 & 385 & 351 & 311 & 268 & 225 & 0 \end{bmatrix}.$$

# A further refined Toeplitz property for ballot permutations

- $\mathscr{B}_{n,d}(i,j)$: the set of perms. in $\mathscr{B}_{n,d}$ having the factor $inj$.
- $\mathscr{P}_{n,d}(i,j)$: the set of perms. in $\mathscr{P}_{n,d}$ having the factor $inj$ in some cycle.
- Write $|\mathscr{B}_{n,d}(i,j)| = b_{n,d}(i,j)$ and $|\mathscr{P}_{n,d}(i,j)| = p_{n,d}(i,j)$.

**Theorem (W. & Zhang)**

*The matrix $\left(b_{n,d}(i,j)\right)_{i,j=1}^{n-1}$ is Toeplitz for all n and d.*

We will establish a bijection

$$T\colon \mathscr{B}_{n,d}(i,j) \to \mathscr{B}_{n,d}(i+1,\, j+1) \qquad \text{for } \max(i,j) \le n-2.$$

# Notation

Let $m = \min(i,j)$ and $M = \max(i,j) \le n-2$. For $x \in [n]$, define

$$\overline{x} = \begin{cases} x+1, & \text{if } x = M, \\ x, & \text{else,} \end{cases} \quad \text{and} \quad \underline{x} = \begin{cases} x-1, & \text{if } x = m+1, \\ x, & \text{else,} \end{cases}$$

and define the words

$$\overline{(k;\ell)} = \overline{k(k+1)\cdots(k+\ell-1)} \quad \text{and} \quad \underline{(k;\ell)} = \underline{k(k-1)\cdots(k-\ell+1)}.$$

## Example

If $(m,M) = (4,8)$, then $\overline{(6;3)} = \overline{678} = 679$ and $\underline{(6;2)} = \underline{65} = 64$.

# Notion: the core

For $\pi \in \mathscr{B}_{n,d}(i,j)$, define the lower width

$$\ell = \begin{cases} 0, & \text{if } M \sim M+1 \text{ (adjacent)} \\ \text{length of the longest factor of the form } \overline{(m;\ell)} \text{ or } \overline{(m;\ell)}', \text{else,} \end{cases}$$

where $\sigma'$ is the reversal of $\sigma$. Define the core of $\pi$ to be

$$\kappa = \begin{cases} \overline{(i;\ \ell)}'\, n\, j, & \text{if } i < j, \\ i\, n\, \overline{(j;\ \ell)}, & \text{if } i > j. \end{cases}$$

**Example**

The cores are red: 162549738, 326549781.

# The bijection $T$: $\mathscr{B}_{n,d}(i, j) \to \mathscr{B}_{n,d}(i+1, j+1)$

1. Core replacement (to produce the factor $(i+1)\, n\, (j+1)$):

$$\kappa \mapsto \begin{cases} (i+1)\, n\, \underline{(j+1;\ \ell)}, & \text{if } i < j \\ \underline{(i+1;\ \ell)'}\, n\, (j+1), & \text{if } i > j. \end{cases}$$

**Example**

If $\kappa = 5497$, then $\kappa \mapsto 5987$. If $m = 3$ and $\kappa = 97$, then $\kappa \mapsto 49$.

2. Straightening s.t. the letters in $[m, M+1] \backslash \mathcal{A}(\kappa)$ remian in order, where $\mathcal{A}(\kappa)$ is the alphabet of $\kappa$.

# Example 1 for the bijection $T$: $\mathscr{B}_n(i,j) \to \mathscr{B}_n(i+1,j+1)$

Since $M \leq n-2$, we find $n \geq 4$. When $m = 1$ and $M = 2$,

**Example**

$3142 \mapsto 2431$. In fact,

1. $M = 2 \not\sim M+1$.
2. $\ell = 2$.
3. $\kappa = \pi \mapsto 2431$.

**Example**

$1423 \mapsto 1243$. In fact,

1. $M = 2 \sim M+1$.
2. $\ell = 0$.
3. $1423 \mapsto 1243$.

# Exmaple 2 for the bijection $T$: $\mathscr{B}_n(i,j) \to \mathscr{B}_n(i+1,j+1)$

## Example

$1\,6\,2\,5\,4\,9\,7\,3\,8 \mapsto 1\,4\,2\,5\,9\,8\,7\,3\,6$. In fact,

1. $n = 9$ and $m = 4$.
2. $M = 7 \not\sim M+1$.
3. $\ell = 2$.
4. $\kappa = 5\,4\,9\,7 \mapsto 5\,9\,8\,7$.
5. Since $[m,\ M+1] = [4,8]$ and $\mathcal{A}(\kappa) = \{5,4,9,7\}$, we map $[4,8]\backslash\mathcal{A}(\kappa) = \{6,8\}$ to $[4,8]\backslash\{5,9,8,7\} = \{4,6\}$.
6. $162549738 \mapsto 142598736$.

# Exmaple 3 for the bijection $T$: $\mathscr{B}_n(i,j) \to \mathscr{B}_n(i+1,j+1)$

## Example

$3\,2\,6\,5\,4\,9\,7\,8\,1 \mapsto 3\,2\,7\,6\,4\,5\,9\,8\,1$. In fact,

1. $n = 9$ and $m = 4$.
2. $M = 7 \sim M + 1$.
3. $\ell = 0$.
4. $\kappa = 9\,7 \mapsto 5\,9$.
5. Since $[m,\ M+1] = [4,8]$ and $\mathcal{A}(\kappa) = \{9,7\}$, we map $[4,8]\backslash\mathcal{A}(\kappa) = \{4,5,6,8\}$ to $[4,8]\backslash\{5,9\} = \{4,6,7,8\}$, that is, $(5,6) \mapsto (6,7)$.
6. $3\,2\,6\,5\,4\,9\,7\,8\,1 \mapsto 3\,2\,7\,6\,4\,5\,9\,8\,1$

# Proof outline for that $T$ is a bijection

1. Show that $T(\pi) \in \mathscr{B}_{n,d}(i+1, j+1)$. More precisely,
    1. $T(\pi)$ contains the factor $(i+1)n(j+1)$;
    2. $\mathrm{des}(T(\pi)) = \mathrm{des}(\pi)$; and
    3. $T(\pi)$ is ballot.
2. Define $T' : \mathscr{B}_{n,d}(i+1, j+1) \to B_{n,d}(i,j)$ in a similar way.
   One may show that both compositions $TT'$ and $T'T$ are identities.

# A Toeplitz property of odd order permutations

**Theorem**

*The matrix $P(n, d)$ is Toeplitz for all $n$ and $d$.*

# Exmaple for the bijection $\mathscr{P}_n(i,j) \to \mathscr{P}_n(i+1,j+1)$

## Example

$(1\,6\,8\,2\,A)(3\,C\,9\,B\,7\,5\,4) \mapsto (1\,3\,6\,2\,7)(4\,C\,A\,9\,8\,B\,5)$. In fact,

1. $n = C$. We use the notation $(A, B, C) = (10, 11, 12)$.
2. $m = 3$
3. $M = 9 \not\sim M+1$
4. $\ell = 3$
5. $\kappa = 5\,4\,3\,C\,9 \mapsto 4\,C\,A\,9\,8$.
6. We map $[m,\ M+1] \backslash \mathcal{A}(\kappa) = [3, A] \backslash \mathcal{A}(\kappa) = \{6, 7, 8, A\}$ to $[3, A] \backslash \{4\,C\,A\,9\,8\} = \{3, 5, 6, 7\}$.
7. $(1\,6\,8\,2\,A)(3\,C\,9\,B\,7\,5\,4) \mapsto (1\,3\,6\,2\,7)(4\,C\,A\,9\,8\,B\,5)$.

# A refinement of Spiro's conjecture

**Conjecture**

$b_{n,d}(1,j) + b_{n,d}(j,1) = 2p_{n,d}(1,j)$, *for all* $n$, $d$, *and* $2 \le j \le n-1$.

**Theorem**

*The above conjecture refines Spiro's conjecture.*

**Proof.**

- $p_{n,d}(i,j) = p_{n,d}(j,i)$.
- If $|i-j| = 1$, then $b_{n,d}(i,j) = b_{n-2,\,d-1}$ and $p_{n,d}(i,j) = p_{n-2,\,d-1}$.
- If the conjecture is true, then $b_{n,d}(i,j) + b_{n,d}(j,i) = 2p_{n,d}(i,j)$.
- $b_{n,d} = b_{n-1,\,d} + \sum_{i\neq j} b_{n,d}(i,j)$ and $p_{n,d} = p_{n-1,\,d} + \sum_{i\neq j} p_{n,d}(i,j)$.

$\square$

# Outline

# Summary

1. A new semi-combinatorial proof for $|\mathscr{B}_n| = |\mathscr{P}_n|$.
2. A bijection to establish the Toeplitz property $b_{n,d}(i,j) = b_{n,d}(i+1, j+1)$.
3. In the same fashion, $p_{n,d}(i,j) = p_{n,d}(i+1, j+1)$.
4. We refined Spiro's conjecture by tracking the neighbors of $n$.

# Further study

## Question

*Find a generating function for $b_{n,d}(i,j)$, eg.*

$$\sum_{n,d,i,j} b_{n,d}(i,j)\frac{x^i}{i!}\frac{y^j}{j!}z^n t^d.$$

# Further observations

**Conjecture**

*Suppose that n is odd. Then* $b_{n,(n-1)/2}(i,j) = p_{n,(n-1)/2}(i,j)$ *for all* $i, j$.

**Conjecture**

*The sequence* $\{b_{n,d}(1,j)\}_{j=2}^{n-1}$ *is decreasing.*

**Conjecture**

*The sequence* $\{b_{n,d}(j,1)\}_{j=2}^{n-1}$ *is unimodal (log-concave).*

# *Thanks!*