Symmetric functions and log-concavity

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Outline

1. The equivariant characteristic polynomial
2. The longest increasing subsequence
3. The Kazhdan-Lusztig polynomial of a matroid
4. The equivariant Kazhdan-Lusztig polynomial of a matroid

Outline

Matroid

Let E be a finite set and $\mathcal{I} \subset 2^E$. A matroid M is an ordered pair $(E, \mathcal{I})$ satisfying

(1) $\emptyset \in \mathcal{I}$;

(2) If $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$; (hereditary property)

(3) If $A, B \in \mathcal{I}$ and $|B| > |A|$, then $\exists\, x \in B$ such that $A \cup x \in \mathcal{I}$. (exchange property)

The set E is said to be the ground set. A set $A \in \mathcal{I}$ is called an independent set.

Example	Ground set	Independent set
Graphic matroid	edge set of a graph	forest (no cycle)
Uniform matroid $U_{m,d}$	a finite set $[m+d]$	subset of cardinality $\leq d$
Representable matroid	a set of vectors over a field	linearly independent vectors

In particular, the unifrom matroid $U_{0,d}$ is the Boolean matorid.

Lattice of flats

The rank of $A \subset E$, denoted $r(A)$, is defined as the cardinality of its maximal independent set.

A subset $A \subset E$ is a flat if $\mathrm{cl}(A) = A$, where $\mathrm{cl}(A) := \{x : r(A \cup x) = r(A)\}$.

The flats of a matroid M form a geometric lattice under inclusion and we denote it by $L(M)$.

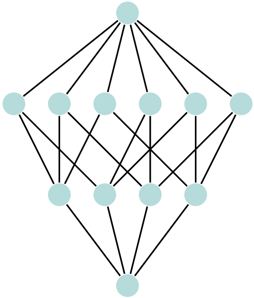

Figure: $L(U_{1,3})$

Localization and restriction

Let F be a flat of a matroid M.

The **restriction** M_F, is the matroid on the ground set F consisting of subsets of F which are independent in M.

The **contraction** M/F is the matroid on the ground set $E \setminus F$ consisting of subsets of $E \setminus F$ whose union with a basis for F are independent in M.

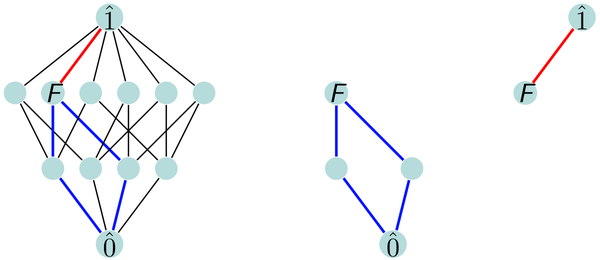

Figure: $L(M), L(M_F)$, and $L(M/F)$

The characteristic polynomial of a matroid

The characteristic polynomial of a matroid M is defined to be

$$\chi_M(t) = \sum_{F \in L(M)} \mu(\emptyset, F) t^{r(M)-r(F)}$$

or

$$\chi_M(t) = \sum_{S \subset E} (-1)^{|S|} t^{r(M)-r(S)}.$$

Example

For a graph G, $\chi_{M(G)}(t) = t^{-c}\chi_G(t)$, where $\chi_G(t)$ is the chromatic polynomial of G and c is the number of connected components of G.

The log-concavity

A polynomial $f(t) = a_0 + a_1 t + \cdots + a_n t^n$ with real coefficients is said to be log-concave if $a_i^2 \geq a_{i-1}a_{i+1}$ for any $0 < i < n$, and it is said to have no internal zeros if there are not three indices $0 \leq i < j < k \leq n$ such that $a_i, a_k \neq 0$ and $a_j = 0$.

Conjecture (Heron (1972), Rota (1971), Welsh (1976))

For any matroid M, the characteristic polynomial $\chi_M(t)$ is a log-concave polynomial with no internal zeros.

K. Adiprasito, J. Huh, and E. Katz. Hodge theory for combinatorial geometries, Ann. of Math. (2) 188 (2018),381–452.

The Orlik-Solomon algebra of a matroid

Let M be a matroid with the ground set E.

Let $\Lambda = \Lambda(E)$ be the free exterior algebra generated by $\mathbf{e}_i$ for all $i \in E$.

Define $\partial : \Lambda(E) \to \Lambda(E)$ by

$$\partial(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_k) = \sum_{i=1}^{k} (-1)^{i-1} \mathbf{e}_1 \wedge \cdots \wedge \hat{\mathbf{e}}_i \wedge \cdots \mathbf{e}_k,$$

where $\hat{}$ indicates an omitted factor, and extending to a linear map.

Let $J = J(M)$ be the ideal of $\Lambda(E)$ generated by $\partial(\mathbf{e}_S)$ for all dependent sets S.

The Orlik-Solomon algebra of M is the quotient $\Lambda(E)/J(M)$.

P. Orlik and L. Solomon. Topology and combinatorics of complements of hyperplanes. Inventiones mathematicae, 56(1980) 167–189.

The equivariant characteristic polynomial

Let M be a matroid on the ground set E, and let W be a finite group acting on E and preserving M. We will refer to this collection of data as an equivariant matroid $W \curvearrowright M$. Let $OS^W_{M,i} \in Rep(W)$ be the degree i part of the Orlik-Solomon algebra of M.

The equivariant characteristic polynomial is defined by

$$H^W_M(t) = \sum_{i=0}^{r(M)} (-1)^i t^{r(M)-i} OS^W_{M,i}.$$

It is known that $\dim H^W_M(t)$ is the usual characteristic polynomial of the matroid M.

K. Gedeon, N. Proudfoot, and B. Young. The equivariant Kazhdan-Lusztig polynomial of a matroid. *J. Combin. Theory Ser. A*, 150(2017) 267–294, .

dimension, but a formula for the class in $\mathrm{Rep}(\mathfrak{S}_l \times \mathfrak{S}_t)$, we need an equivariant version of Möbius inversion. We find it clearest to explain this result in the context of a general poset with a group action. Because we want to reserve μ for partitions, we will denote the Möbius function of a poset by $\mathfrak{m}$.

Let P be a poset with unique minimal element $\hat{0}$, and let G be a group acting on P. Let V be a G-representation with a direct sum decomposition $V = \bigoplus_{p \in P} U_p$ such that $g(U_p) = U_{gp}$ for each $g \in G$ and $p \in P$. For $q \in P$, put $V_q := \bigoplus_{r \geq q} U_r$.

For $p \in P$, let $(\hat{0}, p) = \{q \in P : \hat{0} < q < p\}$. Let $\Delta(\hat{0}, p)$ be the order complex of $(\hat{0}, p)$ – the simplicial complex on the ground set $(\hat{0}, p)$ whose faces are the totally ordered subsets of $(\hat{0}, p)$. A classical result of P. Hall states [4] that the Möbius function $\mathfrak{m}(p)$ is the reduced Euler characteristic $\tilde{\chi}(\Delta(\hat{0}, p))$.

We will define an equivariant version of $\mathfrak{m}$. Namely, let G_p be the stabilizer of p and let $\mathrm{Rep}(G_p)$ be its representation ring. We define

$$\mathfrak{m}_{\mathrm{eq}}(p) = \sum_j (-1)^{j+1} \left[\widetilde{H}_j(\Delta(\hat{0}, p)) \right]$$

where $\widetilde{H}_j$ is the reduced homology group. So, under the map $\mathrm{Rep}(G_p) \to \mathbb{Z}$ sending a representation to its dimension, $\mathfrak{m}_{\mathrm{eq}}(p)$ is sent to the Möbius function $\mathfrak{m}(p)$. Among group theorists, $\mathfrak{m}_{\mathrm{eq}}(p)$ is called the "Lefschetz element".

Let $G \backslash P$ be a set of orbit representatives for the action of G on P. Our equivariant Möbius inversion formula is the following.

Theorem 13. *With the above definitions, we have the equality*

$$[U_{\hat{0}}] = \sum_{p \in G \backslash P} \left[\mathrm{Ind}_{G_p}^{G} \left(\mathfrak{m}_{\mathrm{eq}}(p) \otimes V_p \right) \right] \tag{30}$$

in the representation ring $\mathrm{Rep}(G)$.

Equivariant log-concavity

Denote $VRep(W)$ by the representation ring((or Green ring) of W, which is the ring formed from all the (isomorphism classes of the) finite-dimensional linear representations of the group W. Fix a finite group W. We define a sequence $(C_0, C_1, C_2, \ldots)$ in $VRep(W)$ to be log-concave if, for all $i > 0$, $C_i^{\otimes 2} - C_{i-1} \otimes C_{i+1} \in Rep(W)$, where $Rep(W)$ is the representations of W.

Conjecture (Gedeon, Proudfoot, and Young)

Let $W \curvearrowright M$ be an equivariant matroid. Then the equivariant characteristic polynomial $H_M^W(t)$ is log-concave.

K. Gedeon, N. Proudfoot, and B. Young. The equivariant Kazhdan-Lusztig polynomial of a matroid. *J. Combin. Theory Ser. A*, 150(2017) 267–294.

The Frobenius characteristic map

Consider the Frobenius characteristic map

$$\mathrm{ch} : \mathrm{VRep}(S_n) \longrightarrow \Lambda^n,$$

where Λ^n is the $\mathbb{Z}$-module of symmetric functions of degree n in the variables $\mathbf{x} = (x_1, x_2, \ldots)$.

Given two virtual representations $V_1 \in VRep(S_{n_1})$ and $V_2 \in VRep(S_{n_2})$, we have

$$\mathrm{ch}\left(\mathrm{Ind}_{S_{n_1} \times S_{n_2}}^{S_{n_1+n_2}} V_1 \otimes V_2\right) = \mathrm{ch}(V_1)\,\mathrm{ch}(V_2) \in \Lambda^{n_1+n_2}.$$

Given two virtual representations $V_1, V_2 \in VRep(S_n)$, we have

$$\mathrm{ch}\,(V_1 \otimes V_2) = \mathrm{ch}(V_1) * \mathrm{ch}(V_2) \in \Lambda^n.$$

The image of the irreducible representation V_λ under ch is the Schur function s_λ and, in particular, the image of the trivial representation $V_{(n)}$ is the complete symmetric function h_n.

I. G. Macdonal. Symmetric functions and Hall polynomials. Oxford University Press.

Kronecker coefficients

If $V_\mu \otimes V_\nu = \bigoplus_\lambda g^\lambda_{\mu\nu} V_\lambda$ then $s_\mu * s_\nu = \sum_\lambda g^\lambda_{\mu\nu} s_\lambda$, where $g^\lambda_{\mu\nu}$ is the Kronecker coefficient.

If $\mathrm{Ind}^{S_{|\lambda|}}_{S_{|\mu|}\times S_{|\nu|}} (V_\mu \otimes V_\nu) = \bigoplus_\lambda c^\lambda_{\mu\nu} V_\lambda$, then $s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$, where $c^\lambda_{\mu\nu}$ is the Littlewood–Richardson coefficient.

Problem

If $C_i \in Rep(S_n)$, then what is the relation between $C_i^{\otimes 2} - C_{i-1} \otimes C_{i+1} \in Rep(S_n)$ and $\mathrm{Ind}^{S_{2n}}_{S_n\times S_n}(C_i \otimes C_i - C_{i-1} \otimes C_{i+1}) \in Rep(S_{2n})$?

$$\dim(C_i^{\otimes 2} - C_{i-1} \otimes C_{i+1}) = (\dim C_i)^2 - \dim C_{i+1} \dim C_{i-1}.$$

$$\dim(\mathrm{Ind}^{S_{2n}}_{S_n\times S_n}(C_i \otimes C_i - C_{i-1} \otimes C_{i+1})) = \frac{|S_{2n}|}{|S_n|^2}((\dim C_i)^2 - \dim C_{i+1} \dim C_{i-1}).$$

Example (Gedeon, Proudfoot, and Young)

Let S_n be the symmetric group of order n. Let B_n be the Boolean matroid on a ground set of n elements. Then

$$OS^{S_n}_{B_n,i} = \bigwedge^i \mathbb{C}^n \quad (\textit{exterior power})$$

and thus

$$chOS^{S_n}_{B_n,i} = s_{n-i,1^i} + s_{n-i+1,1^{i-1}} = e_i h_{n-i} \ (EC2\ Ex.7.72).$$

Therefore,

$$chH^{S_n}_{B_n}(t) = \sum_{i=0}^{n} (-1)^i t^{n-i} e_i h_{n-i} = h_n[(t-1)X].$$

The square bracket denotes the plethystic substitution and it is a convention that $X = x_1 + x_2 + \cdots$.

$H_{B_n}^{S_n}(t)$ is log-concave

Theorem

The equivariant characteristic polynomial $H_{B_n}^{S_n}(t)$ is log-concave.

Let $a_i = s_{n-i,1^i}$. We need to show that

$$(a_i + a_{i-1}) * (a_i + a_{i-1}) \geq_s (a_{i+1} + a_i) * (a_{i-1} + a_{i-2}).$$

Expand both sides, we have

$$a_i * a_i + a_{i-1} * a_{i-1} + 2a_i * a_{i-1} \geq_s a_{i+1} * a_{i-1} + a_{i+1} * a_{i-2} + a_i * a_{i-1} + a_i * a_{i-2}$$

It suffices to show that

$$a_i * a_i \geq_s a_{i+1} * a_{i-1},$$

$$a_{i-1} * a_{i-1} \geq_s a_i * a_{i-2},$$

$$a_i * a_{i-1} \geq_s a_{i+1} * a_{i-2}.$$

THEOREM 2.1. *Let* $c_\lambda = \langle S_{(1^{n-s},s)} \odot S_{(1^{n-t},t)}, S_\lambda \rangle$ *where* $s \leqslant t$, $s+t \geqslant n+1$, *and* λ *is a straight shape. Then*

(a) $c_\lambda = 0$ *if* λ *is not a hook or double hook shape,*

(b) $c_\lambda = \chi(s+t-n-1 \leqslant r \leqslant s+n-t)$ *if* λ *is hook shape* $(1^{n-r}, r)$,

(c) *finally if* λ *is a double hook shape* $(1^l, 2^k, p, q)$

where $k, l \geqslant 0$, *and* $2 \leqslant p \leqslant q$, *we let* $u = \max(p, s+t-n)$, $v_0 = \min(q, s-k-1)$, $v_1 = \min(q, s-k)$, $\omega = s+t-l-2k$, *and* $x = \lfloor \omega/2 \rfloor$. *Then*

$$c_\lambda = \begin{cases} 0 & \text{if } p+k > s \\ \chi(u \leqslant x-1 \leqslant v_0) + \chi(u \leqslant x \leqslant v_1) & \text{if } p+k \leqslant s \text{ and } \omega \text{ even} \\ \chi(u \leqslant x \leqslant v_0) + \chi(u \leqslant x \leqslant v_1) & \text{if } p+k \leqslant s \text{ and } \omega \text{ odd.} \end{cases}$$

Figure: Remmel,Theorem 2.1

J. B. Remmel, A formula for the Kronecker products of Schur functions of hook shapes, *J. Algebra* 120 (1989), no. 1, 100–18.

Induced equivariant log-concavity?

Since

$$e_i^2 - e_{i-1}e_{i+1} = s_{2^i}$$

and

$$h_{n-i}^2 - h_{n-i-1}h_{n-i+1} = s_{n-i,n-i},$$

we have

$$(e_i h_{n-i})^2 >_s e_{i+1}h_{n-i-1} \cdot e_{i-1}h_{n-i+1}.$$

Example (Gedeon, Proudfoot, and Young)

For Uniform matroid $U_{m,d}$*, let* $n = m + d$*. If* $j < d$ *Then*

$$OS^{S_n}_{U_{m,d},j} = \bigwedge^j \mathbb{C}^n \quad (\textit{exterior power})$$

and $OS^{S_n}_{U_{m,d},d} = V_{m+1,1^{d-1}}$.

Outline

For a given permutation $\pi \in \mathfrak{S}_n$, let $\ell(\pi)$ denote the length of a longest increasing subsequence of π. Define $L_{n,k}$ to be the set of permutations $\pi \in \mathfrak{S}_n$ with $\ell(\pi) = k$ for $1 \leq k \leq n$. Let $\ell_{n,k} = |L_{n,k}|$. Chen proposed the following conjecture.

Conjecture (Chen)

For any fixed n, the sequence $\{\ell_{n,k}\}_{k=1}^{n}$ is log-concave, namely, $\ell_{n,k}^2 \geq \ell_{n,k+1}\ell_{n,k-1}$ for $1 < k < n$.

Bóna, Lackner and Sagan further made a companion conjecture for involutions. Define $I_{n,k}$ to be the set of involutions $\pi \in \mathfrak{S}_n$ with $\ell(\pi) = k$ for $1 \leq k \leq n$. Let $i_{n,k} = |I_{n,k}|$. They proposed the following conjecture.

Conjecture (Bóna, Lackner and Sagan)

For any fixed n, the sequence $\{i_{n,k}\}_{k=1}^{n}$ is log-concave.

- W.Y.C. Chen, Log-concavity and q-log-convexity conjectures on the longest increasing subsequences of permutations, arXiv:0806.3392.
- M. Bóna, M.-L. Lackner, and B.E. Sagan, Longest increasing subsequences and log concavity, Annals of Combinatorics, 2017,21(4), 535-549.

It is well known that each permutation $\pi \in \mathfrak{S}_n$, under the Robinson-Schensted correspondence, is mapped to a pair of standard Young tableaux of the same partition shape, say $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$. Moreover, there holds $\ell(\pi) = \lambda_1$. In that case, we also say that π is of shape λ, denoted $\mathrm{sh}\,\pi = \lambda$.

Bóna, Lackner and Sagan proposed a new way to look at these problems. Given a set Λ of partitions of n, for $1 \leq k \leq n$ let

$$L_{n,k}^{\Lambda} = \{\pi \in L_{n,k} \mid \mathrm{sh}\,\pi \in \Lambda\}, \quad \ell_{n,k}^{\Lambda} = |L_{n,k}^{\Lambda}|;$$
$$I_{n,k}^{\Lambda} = \{\pi \in I_{n,k} \mid \mathrm{sh}\,\pi \in \Lambda\}, \quad i_{n,k}^{\Lambda} = |I_{n,k}^{\Lambda}|.$$

Thus, the sequence $\{\ell_{n,k}\}_{k=1}^{n}$ (resp. $\{i_{n,k}\}_{k=1}^{n}$) is just $\{\ell_{n,k}^{\Lambda}\}_{k=1}^{n}$ (resp. $\{i_{n,k}^{\Lambda}\}_{k=1}^{n}$) when taking Λ to be the set of all partitions of n.

Theorem (Gao,Xie,Yang)

Suppose that m, n are two positive integers.

(1) For $\Lambda = \{(j^m, (n-j)^m) \mid \lceil \frac{n}{2} \rceil \leq j \leq n\}$, both $\{\ell^{\Lambda}_{mn,k}\}_{k=1}^{mn}$ and $\{i^{\Lambda}_{mn,k}\}_{k=1}^{mn}$ are log-concave.

(2) For $\Lambda = \{(j^m, 1^{m(n-j)}) \mid 1 \leq j \leq n\}$, both $\{\ell^{\Lambda}_{mn,k}\}_{k=1}^{mn}$ and $\{i^{\Lambda}_{mn,k}\}_{k=1}^{mn}$ are log-concave.

when $m = 1, 2$, this result was first obtained by Bóna, Lackner, and Sagan.

A. L. L. Gao, M. H. Y. Xie, A. L. B. Yang. Schur positivity and log-concavity related to longest increasing subsequences. Discrete Mathematics, 2019, 342(9): 2570-2578.

Given a partition λ, let f^λ denote the number of standard Young tableaux of shape λ.

Theorem

Suppose that m, n are two positive integers.

(1) For $\lceil \frac{n}{2} \rceil < k < n$ we have

$$(f^{(k^m,(n-k)^m)})^2 \geq f^{((k+1)^m,(n-k-1)^m)} f^{((k-1)^m,(n-k+1)^m)}.$$

(2) For $1 < k < n$ we have

$$(f^{(k^m,1^{m(n-k)})})^2 \geq f^{((k+1)^m,1^{m(n-k-1)})} f^{((k-1)^m,1^{m(n-k+1)})}.$$

Theorem

Suppose that m and n are two positive integers.

(1) For $\lceil \frac{n}{2} \rceil < k < n$*, the difference*

$$s^2_{(k^m,(n-k)^m)} - s_{((k+1)^m,(n-k-1)^m)} s_{((k-1)^m,(n-k+1)^m)}$$

is Schur positive.

(2) For $1 < k < n$*, the difference*

$$s^2_{(k^m,1^{m(n-k)})} - s_{((k+1)^m,1^{m(n-k-1)})} s_{((k-1)^m,1^{m(n-k+1)})}$$

is Schur positive.

Theorem (Lam, Postnikov and Pylyavaskyy, Conjectured by Okounkov)

Given any two skew partitions λ/μ and ν/ρ, the difference

$$s_{\lfloor\frac{\lambda+\nu}{2}\rfloor/\lfloor\frac{\mu+\rho}{2}\rfloor}s_{\lceil\frac{\lambda+\nu}{2}\rceil/\lceil\frac{\mu+\rho}{2}\rceil} - s_{\lambda/\mu}s_{\nu/\rho}$$

is Schur positive.

Given two partitions λ and μ, let $\lambda \cup \mu = (\nu_1, \nu_2, \nu_3, \ldots)$ be the partition obtained by rearranging all parts of λ and μ in the weakly decreasing order. Let $\mathrm{sort}_1(\lambda,\mu) := (\nu_1, \nu_3, \nu_5, \ldots)$ and $\mathrm{sort}_2(\lambda,\mu) := (\nu_2, \nu_4, \nu_6, \ldots)$.

Theorem (Lam, Postnikov and Pylyavaskyy, conjectured by Fomin, Fulton, Li and Poon)

For any two partitions λ and μ, the difference

$$s_{\mathrm{sort}_1(\lambda,\mu)}s_{\mathrm{sort}_2(\lambda,\mu)} - s_\lambda s_\mu$$

is Schur positive.

T. Lam , A. Postnikov A, P. Pylyavskyy . Schur positivity and Schur log-concavity. American journal of mathematics, 2007, 129(6): 1611-1622.

Let $P_{w,v}(q)$ denote the *Kazhdan-Lusztig polynomials* and $Q_{v,w}(q) = P_{w_\circ w, w_\circ v}(q)$, for the longest permutation $w_\circ \in S_n$. For $w \in S_n$ and a $n \times n$ matrix $X = (x_{ij})$, the *Kazhdan-Lusztig immanant* was defined as

$$Imm_w(X) := \sum_{v \in S_n} (-1)^{l(vw)} Q_{w,v}(1)\, x_{1,v(1)} \cdots x_{n,v(n)}.$$

Rhoades and Skandera defined the *Temperley-Lieb immanant* $Imm_w^{\mathrm{TL}}(x)$ by

$$Imm_w^{\mathrm{TL}}(X) := \sum_{v \in S_n} f_w(v)\, x_{1,v(1)} \cdots x_{n,v(n)}.$$

Theorem (Rhoades-Skandera)

For a 321-avoiding permutation $w \in S_n$, we have $Imm_w^{\mathrm{TL}}(X) = Imm_w(X)$.

- Andrei Okounkov. Why would multiplicities be log-concave? In: The orbit method in geometry and physics. Birkhäuser Boston, Boston, MA, 2003, 329–347.
- S. Fomin, W. Fulton, C.-K. Li and Y.-T. Poon, Eigenvalues, singular values, and Littlewood-Richardson coefficients, Amer. J. Math. 127 (2005), 101–127.
- B. Rhoades, M. Skandera. Temperley-lieb immanants. Annals of Combinatorics, 2005, 9(4): 451-494.

For $\lceil \frac{n}{2} \rceil < k < n$, taking $\lambda = ((k+1)^m, (n-k-1)^m)$, $\nu = ((k-1)^m, (n-k+1)^m)$, and $\mu = \rho = \emptyset$ in the first theorem, we obtain the Schur positivity of

$$s^2_{(k^m,(n-j)^m)} - s_{((k+1)^m,(n-k-1)^m)} s_{((k-1)^m,(n-k+1)^m)}.$$

For $1 < k < n$, taking $\lambda = ((k+1)^m, 1^{m(n-k-1)})$, $\nu = ((k-1)^m, 1^{m(n-k+1)})$ and $\mu = \rho = \emptyset$ in the first theorem, we obtain the Schur positivity of

$$s_{(k^m,1^{m(n-k+1)})} s_{(k^m,1^{m(n-k-1)})} - s_{((k+1)^m,1^{m(n-k-1)}} s_{((k-1)^m,1^{m(n-k+1)}}. \tag{1}$$

Taking $\lambda = (k^m, 1^{m(n-k+1)})$ and $\mu = (k^m, 1^{m(n-k-1)})$ in the second theorem, we obtain the Schur positivity of

$$s^2_{(k^m,1^{m(n-k)})} - s_{(k^m,1^{m(n-k+1)})} s_{(k^m,1^{m(n-k-1)})}. \tag{2}$$

Combining (1) and (2) we obtain the Schur positivity of

$$s^2_{(k^m,1^{m(n-k)})} - s_{((k+1)^m,1^{m(n-k-1)})} s_{((k-1)^m,1^{m(n-k+1)})}.$$

Conjecture

For $1 \leq k \leq n$, *let*

$$f_{n,k} = \sum_{\lambda \vdash n, \lambda_1 = k} s_\lambda^2,$$

then $f_{n,k}^2 - f_{n,k+1} f_{n,k-1}$ *is Schur positive with the convention that* $f_{n,0} = f_{n,n+1} = 0$.

Conjecture

For $1 \leq k \leq n$, *let*

$$g_{n,k} = \sum_{\lambda \vdash n, \lambda_1 = k} s_\lambda,$$

then $g_{n,k}^2 - g_{n,k+1} g_{n,k-1}$ *is Schur positive with the convention that* $g_{n,0} = g_{n,n+1} = 0$.

Chen also put forward some log-concavity conjecture about perfect matchings. For any fixed n, let Θ be the set of partitions of n all of whose column lengths are even. Chen's conjecture can be restated as follows.

Conjecture

For any fixed n, the sequence $\{i_{n,k}^{\Theta}\}_{k=1}^{n}$ *is log-concave.*

Conjecture

For $1 \leq k \leq n$*, let*

$$g_{n,k}^{\Theta} = \sum_{\lambda \in \Theta, \lambda_1 = k} s_\lambda,$$

then $(g_{n,k}^{\Theta})^2 - g_{n,k+1}^{\Theta} g_{n,k-1}^{\Theta}$ *is Schur positive with the convention that* $g_{n,0}^{\Theta} = g_{n,n+1}^{\Theta} = 0$.

Conjecture (Bóna, Lackner and Sagan)

For any fixed n, the sequence $\{\ell_{n,k}^{\Theta}\}_{k=1}^{n}$ *is log-concave.*

In general, the difference $(f_{n,k}^{\Theta})^2 - f_{n,k+1}^{\Theta} f_{n,k-1}^{\Theta}$ is not Schur positive.

Let

$$h_{n,k} = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=k}} f^\lambda s_\lambda.$$

Conjecture (Jonathan Novak, Brendon Rhoades)

For any $n \geq 3$, we have

$$h_{n,k-1} * h_{n,k+1} \leq h_{n,k} * h_{n,k}$$

for all $2 \leq k \leq n-1$.

Jonathan Novak, Brendon Rhoades. Increasing Subsequences and Kronecker Coefficients. arXiv:2006.13146.

Open Problems in Algebraic Combinatorics, May 17-21, 2021, University of Minnesota.

Outline

The Kazhdan-Lusztig polynomial of a matroid

Elias, Proudfoot, and Wakefield proved that there is a unique way to associate to each loopless matroid M a polynomial $P_M(t) \in \mathbb{Z}[t]$ satisfying the following properties:

1. If $r(M) = 0$, then $P_M(t) = 1$.
2. If $r(M) > 0$, then $\deg P_M(t) < \frac{1}{2} r(M)$.
3. For every M, $t^{r(M)} P_M(t^{-1}) = \sum_{F \in L(M)} \chi_{M_F}(t) P_{M/F}(t)$.

B. Elias, N. Proudfoot, and M. Wakefield. The Kazhdan-Lusztig polynomial of a matroid. *Adv. Math.*, 299:36–70, 2016.

Conjectures on the KL polynomials of matroids

Conjecture (Elias, Proudfoot, and Wakefield)

For any matroid M, the coefficients of the Kazhdan–Lusztig polynomial $P_M(t)$ are non-negative.

Theorem (Elias, Proudfoot, and Wakefield)

If a matroid M is representable, then $P_M(t)$ has non-negative coefficients.

Remark

This conjecture may be confirmed by Tom Braden, June Huh, Jacob P. Matherne, Nicholas Proudfoot, and Botong Wang(王博潼).

B. Elias, N. Proudfoot, and M. Wakefield. The Kazhdan-Lusztig polynomial of a matroid. *Adv. Math.*, 299:36–70, 2016.

Tom Braden, June Huh, Jacob P. Matherne, Nicholas Proudfoot, and Botong Wang, Singular Hodge theory for combinatorial geometries, in preparation.

The log-concavity

Conjecture (Elias, Proudfoot, Wakefield)

For any matroid M the Kazhdan-Lusztig polynomial $P_M(t)$ is a log-concave polynomial with no internal zeros.

B. Elias, N. Proudfoot, and M. Wakefield. The Kazhdan-Lusztig polynomial of a matroid. *Adv. Math.* 299 (2016), 36–70.

Conjecture (Gedeon, Proudfoot and Young)

The polynomial $P_M(t)$ has only negative zeros for any matroid M.

By the well known Newton inequality, if $f(t)$ has only negative zeros, then it must be a log-concave polynomial without internal zeros.

K. Gedeon, N. Proudfoot, and B. Young. Kazhdan-Lusztig polynomials of matroids: a survey of results and conjectures. *Sém. Lothar. Combin.*, 78B:Article 80, 2017.

Outline

The equivariant Kazhdan-Lusztig polynomial of a matroid

Given an equivariant matroid $W \curvearrowright M$, there is a unique way to define a polynomial $P_M^W(t)$ such that the following conditions are satisfied:

1. If $r(M) = 0$, then $P_M^W(t)$ is equal to the trivial representation in degree 0.
2. If $r(M) > 0$, then $\deg P_M^W(t) < \frac{1}{2} r(M)$.
3. For every M, $t^{r(M)} P_M^W(t^{-1}) = \sum_{[F] \in L/W} \mathrm{Ind}_{W_F}^W \left(H_{M_F}^{W_F}(t) \otimes P_{M/F}^{W_F}(t) \right)$.

The polynomial $P_M^W(t)$ is called the equivariant Kazhdan-Lusztig polynomial, whose coefficients are virtual representations of W.

By sending every virtual representation to its dimension, we can obtain the non-equivariant Kazhdan-Lusztig polynomial.

B. Elias, N. Proudfoot, and M. Wakefield. The Kazhdan-Lusztig polynomial of a matroid. *Adv. Math.*, 299(2016) 36–70.

K. Gedeon, N. Proudfoot, and B. Young. The equivariant Kazhdan-Lusztig polynomial of a matroid. *J. Combin. Theory Ser. A*, 150(2017) 267–294.

Conjecture (Gedeon, Proudfoot, and Young)

Let $W \curvearrowright M$ be an equivariant matroid. Then the equivariant Kazhdan-Lusztig polynomial $P_M^W(t)$ is log-concave.

The equivariant analogue of being real rooted is the statement that the minors of the Toeplitz matrix are honest (rather thanvirtual) representations. The equivariant analogue of (strictly) interlacing is that the principal minors of the Bézoutian are honest representations.

Conjecture (Proudfoot)

The polynomial $P_M^W(t)$ is equivariantly real rooted.

Theorem (Gedeon, Proudfoot, and Young)

For any $n \geq 2$, we have

$$\mathcal{P}_{U_{1,n-1}}(t) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor - 1} s_{(n-2i,2^i)} t^i.$$

Thanks for your attention!