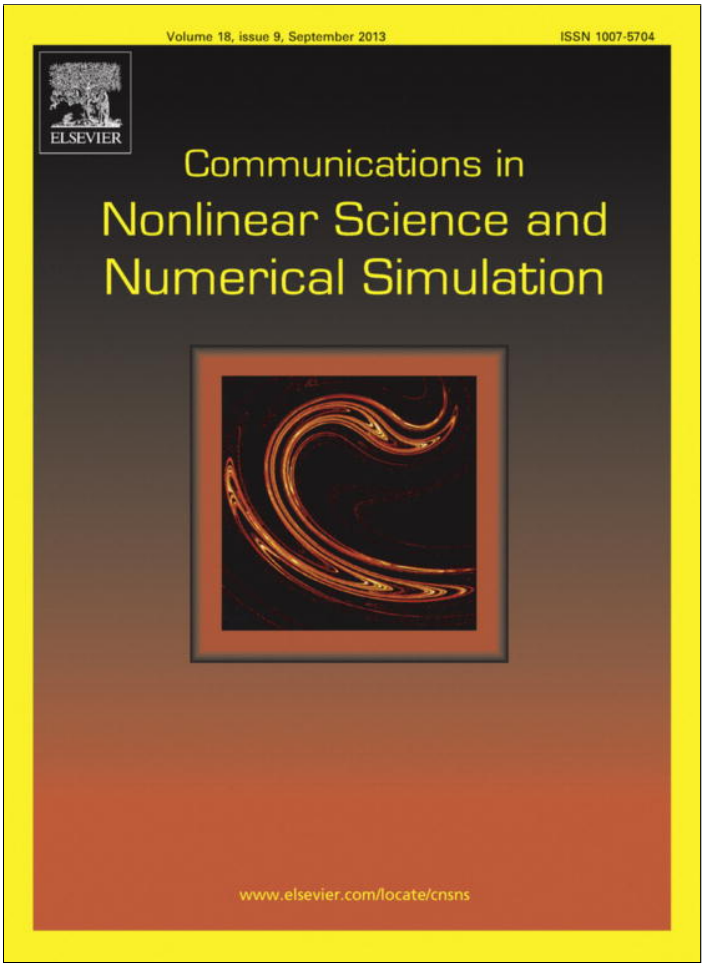
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Lie symmetry analysis to the time fractional generalized fifth-order KdV equation

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ABSTRACT

In this paper, using the Lie group analysis method, we study the invariance properties of the time fractional generalized fifth-order KdV equation. It shows that this equation can be reduced to an equation which is related to the Erdélyi–Kober fractional derivative. Of course, this method can also be applied to other nonlinear fractional partial differential equations.

1. Introduction

Symmetry plays a very important role in various fields of nature. As is known to all, Lie method is an effective method and a large number of equations [1–8] are solved with the aid of this method. There are still many authors using this method to find the exact solutions [9–15] of nonlinear partial differential equation. Most recently, fractional differential equations (FDEs) have been increasingly used in mathematical modeling of physical to biology, chemistry, mechanics processes, etc. In the past, there are a huge number of papers and a lot of excellent books (see, e.g., [16–39] and papers cited therein) devoted to such applications. It is necessary to point out that some methods are used to construct numerical [17,18], exact and explicit solutions of nonlinear FDEs, such as Adomian decomposition method [19–21], transform method [22,23], homotopy perturbation method [24], variational iteration method [25], sub-equation method [26–28], and so on. However, the authors know only some papers [29–33] in which group analysis has been applied for the investigation of FDEs. The primary objective of this article is to investigate the Lie symmetry analysis to the time fractional fifth-order KdV equation.

In this paper, by means of the Lie symmetry group method, we will consider the following time fractional fifth-order KdV equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = u_{xxxxx} + Mu^p u_x, \tag{1.1}$$

where $0 < \alpha \leqslant 1, p > 0$ and M is a constant. When $\alpha = 1, p = 1$, this equation can be reduced to the general simplified Kawahara equation. When $\alpha = 1, p = 2$, it can be reduced to the general simplified modified Kawahara equation. These fifth-order KdV types of equations have been derived to model many physical phenomena [15]. Now, we briefly review the main

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definitions and properties from the recent fractional calculus proposed by Jumarie [34,35] which will be used in the following sections. The modified Riemann-Liouville derivative defined by Jumarie [34]

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, & 0<\alpha<1, \\ \left[f^{(n)}(t)\right]^{(\alpha-n)}, & n\leqslant\alpha<n+1,\ n\geqslant 1. \end{cases} \tag{1.2}$$

Some useful formulas and properties of Jumarie's modified Riemann–Liouville derivative were summarized in [34], three useful formulas of them are

$$D_t^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}t^{\gamma-\alpha}, \quad \gamma>0, \tag{1.3}$$

$$D_t^\alpha[u(t)v(t)] = u(t)D_t^\alpha v(t) + v(t)D_t^\alpha u(t), \tag{1.4}$$

$$D_t^\alpha[f(u(t))] = f'_u[u(t)]D_t^\alpha u(t) = D_u^\alpha f[u(t)](u'_t)^\alpha, \tag{1.5}$$

which will be used in the following sections.

Our aim in the present work is to discuss the time fractional fifth-order KdV equations with the help of Lie's symmetry group method. We get the corresponding infinitesimals, Lie algebra, and show that the time fractional generalized fifth-order KdV equation can be transformed into a nonlinear ODE of fractional order. The plan of the paper is as follows. Section 2 discusses the Lie symmetry analysis of the fractional partial differential equation (FPDE). Then in Section 3, the similarity method is applied to reduce the time fractional generalized fifth-order KdV equation into an ordinary differential equation. Finally, we present conclusions in the last section.

2. Lie symmetry analysis of fractional partial differential equations

In this section, according to the Lie theory we will study the time fractional fifth-order KdV equation. Next, we present below brief details of the Lie symmetry analysis to FPDE about two independent variables. Consider a scalar time FPDE having the following form

$$\frac{\partial^\alpha u}{\partial t^\alpha} = F(x,t,u,u_x,u_{xx},\ldots). \tag{2.1}$$

If (2.1) is invariant under a one parameter Lie group of point transformations

$$\begin{aligned} t^* &= t + \epsilon\tau(x,t,u) + O(\epsilon^2), \\ x^* &= x + \epsilon\xi(x,t,u) + O(\epsilon^2), \\ u^* &= u + \epsilon\eta(x,t,u) + O(\epsilon^2), \\ \frac{\partial^\alpha u^*}{\partial t^*} &= \frac{\partial^\alpha u}{\partial t^\alpha} + \epsilon\eta_\alpha^0(x,t,u) + O(\epsilon^2), \\ \frac{\partial u^*}{\partial x^*} &= \frac{\partial u}{\partial x} + \epsilon\eta^x(x,t,u) + O(\epsilon^2), \\ \frac{\partial^2 u^*}{\partial x^{*2}} &= \frac{\partial^2 u}{\partial x^2} + \epsilon\eta^{xx}(x,t,u) + O(\epsilon^2), \\ &\vdots \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \eta^x &= D_x(\eta) - u_xD_x(\xi) - u_tD_x(\tau), \\ \eta^{xx} &= D_x(\eta^x) - u_{xt}D_x(\tau) - u_{xx}D_x(\xi), \\ \eta^{xxx} &= D_x(\eta^{xx}) - u_{xxt}D_x(\tau) - u_{xxx}D_x(\xi), \\ \eta^{xxxx} &= D_x(\eta^{xxx}) - u_{xxxt}D_x(\tau) - u_{xxxx}D_x(\xi), \\ &\vdots \end{aligned} \tag{2.3}$$

Here, D_x denotes the total derivative operator and is defined by

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + \cdots$$

with infinitesimal generator

$$V = \tau(x,t,u)\frac{\partial}{\partial t} + \xi(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{2.4}$$

If the vector field (2.4) generates a symmetry of (1.1), then V must satisfy Lie's symmetry condition

$$pr^{(n)}V(\Delta_1)|_{\Delta_1=0} = 0,$$

where $\Delta_1 = \frac{\partial^\alpha u}{\partial t^\alpha} - F(x,t,u,u_x,u_{xx},\ldots)$.

To obtain the group transformation which is generated by the infinitesimal generators V, we need to solve the following initial problems (Lie equations)

$$\begin{aligned} &\frac{d(\bar{x}(\varepsilon))}{d\varepsilon} = \xi(\bar{x}(\varepsilon),\bar{t}(\varepsilon),\bar{u}(\varepsilon)), \quad \bar{x}(0) = x,\\ &\frac{d(\bar{u}(\varepsilon))}{d\varepsilon} = \eta(\bar{x}(\varepsilon),\bar{t}(\varepsilon),\bar{u}(\varepsilon)), \quad \bar{u}(0) = u. \end{aligned} \tag{2.5}$$

Now, we consider transformations of the form (2.2) which conserve the structure of fractional derivative operator (1.2). In (1.2) the lower limit of the integral is fixed and, therefore, it should be invariant with respect to such transformations (2.2). The invariance condition gets to

$$\tau(x,t,u)|_{t=0} = 0. \tag{2.6}$$

The αth extended infinitesimal have to do with Riemann–Liouville fractional time derivative with (2.6) reads (see [29–33])

$$\eta_\alpha^0 = D_t^\alpha(\eta) + \xi D_t^\alpha(u_x) - D_t^\alpha(\xi u_x) + D_t^\alpha(D_t(\tau)u) - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u), \tag{2.7}$$

where the operator D_t^α express the total fractional derivative operator.

In order to discuss (2.7), it is necessary for us to recall the generalized Leibnitz rule [36,37] given by

$$D_t^\alpha[u(t)v(t)] = \sum_{n=0}^{\infty}\binom{\alpha}{n}D_t^{\alpha-n}u(t)D_t^n v(t), \quad \alpha > 0, \tag{2.8}$$

where

$$\binom{\alpha}{n} = \frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}. \tag{2.9}$$

Making use of the Leibnitz rule (2.8), (2.7) becomes

$$\eta_\alpha^0 = D_t^\alpha(\eta) - \alpha D_t(\tau)\frac{\partial^\alpha u}{\partial t^\alpha} - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)D_t^{\alpha-n}(u_x) - \sum_{n=1}^{\infty}\binom{\alpha}{n+1}D_t^{n+1}(\tau)D_t^{\alpha-n}(u). \tag{2.10}$$

According to the compound function of the chain rule [33,38], one can get

$$\frac{d^m f(g(t))}{dt^m} = \sum_{k=0}^{m}\sum_{r=0}^{k}\binom{k}{r}\frac{1}{k!}[-g(t)]^r\frac{d^m}{dt^m}[g(t)^{k-r}]\frac{d^k f(g)}{dg^k}. \tag{2.11}$$

Now, using the chain rule (2.11) and the generalized Leibnitz rule (2.8) with $f(t) = 1$, one can get

$$D_t^\alpha(\eta) = \frac{\partial^\alpha \eta}{\partial t^\alpha} + \eta_u\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \sum_{n=1}^{\infty}\binom{a}{n}\frac{\partial^n \eta_u}{\partial t^n}D_t^{\alpha-n}(u) + \mu, \tag{2.12}$$

where

$$\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{a}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}[-u]^r\frac{\partial^m}{\partial t^m}[u^{k-r}]\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}. \tag{2.13}$$

Thus, (2.10) can become [30]

$$\eta_\alpha^0 = \frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta_u - \alpha D_t(\tau))\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \mu + \sum_{n=1}^{\infty}\left[\binom{a}{n}\frac{\partial^\alpha \eta_u}{\partial t^\alpha} - \binom{a}{n+1}D_t^{n+1}(\tau)\right]D_t^{\alpha-n}(u) - \sum_{n=1}^{\infty}\binom{a}{n}D_t^n(\xi)D_t^{\alpha-n}(u_x). \tag{2.14}$$

According to the Lie theory, we obtain

Theorem 1. *A solution $u = \theta(x,t)$ is an invariant solution of (2.1) if and only if*

(i) *$u = \theta(x,t)$ is an invariant surface, in other words,*
$V\theta = 0 \Longleftrightarrow (\tau(x,t,u)\frac{\partial}{\partial t} + \xi(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial u})\theta = 0,$

(ii) *$u = \theta(x,t)$ is the solution of FPDE (2.1).*

3. Time fractional generalized fifth-order KdV equation

In this section, we will investigate the invariance properties of the time fractional generalized fifth-order KdV equation.

According to the Lie theory, applying the fifth prolongation $pr^{(5)}V$ to the Eq. (2.1), one can find the following system of symmetry equation reads as

$$\eta_\alpha^0 - \eta^{xxxxx} - Mu^p\eta^x - Mp\eta u^{p-1}u_x = 0, \tag{3.1}$$

Solving (3.1) along with (2.3), one can get

$$\begin{aligned}
&\xi_u = \tau_u = \xi_t = \xi_{xxxxx} = \tau_x = \eta_{uu} = 0,\\
&\frac{\partial^\alpha \eta}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} - \eta_{xxxxx} - Mu^p\eta_x = 0,\\
&Mu^p\xi_x - Mp\eta u^{p-1} - 5\eta_{xxxxu} - \alpha\tau_t Mu^p = 0,\\
&5\xi_x - \alpha\tau_t = 0, \quad \eta_{xu} - 2\xi_{xx} = 0, \quad 2\eta_{xxxu} - \xi_{xxxx} = 0, \quad \eta_{xxu} - \xi_{xxx} = 0,\\
&\binom{a}{n}\partial_t^n(\eta_u) - \binom{a}{n+1}D_t^{n+1}(\tau) = 0, \quad \text{for } n = 1, 2, \ldots
\end{aligned} \tag{3.2}$$

Solution of this system gives

$$\xi = c_1x + c_2, \quad \tau = \frac{5c_1t}{\alpha}, \quad \eta = \frac{-4c_1u}{p}, \tag{3.3}$$

where c_1 and c_2 are arbitrary constants. Thus, we can get the corresponding infinitesimal operator

$$V = \frac{5c_1t}{\alpha}\frac{\partial}{\partial t} + (c_1x + c_2)\frac{\partial}{\partial x} - \frac{4c_1u}{p}\frac{\partial}{\partial u}. \tag{3.4}$$

One can obtain the corresponding two-dimensional Lie algebra

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{5t}{\alpha}\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - \frac{4u}{p}\frac{\partial}{\partial u}. \tag{3.5}$$

It is easy to check that the symmetry generators found in (3.5) form a closed Lie algebra

$$[V_1, V_2] = V_1, \quad [V_2, V_1] = -V_1.$$

The similarity variables for the infinitesimal generator V_2 can be found by solving the corresponding characteristic equations

$$\frac{dx}{x} = \frac{\alpha dt}{5t} = \frac{-pdu}{4u}. \tag{3.6}$$

Integration of (3.6) provides the following similarity variable and function

$$\xi = xt^{\frac{-\alpha}{5}}, \quad u = t^{\frac{-4\alpha}{5p}}g(\xi). \tag{3.7}$$

From the above process, one can see that (1.1) can be reduced into a nonlinear ODE of fractional order. Consequently, we have the following theorem.

Theorem 2. *The transformation (3.7) reduces (1.1) to the following nonlinear ordinary differential equation of fractional order*

$$\left(P_{\frac{\alpha}{5}}^{1-\frac{4\alpha}{5p}-\alpha,\alpha}g\right)(\xi) = g_{\xi\xi\xi\xi\xi} + Mg^pg_\xi, \tag{3.8}$$

with the Erdelyi–Kober fractional differential operator $P_\beta^{\tau,\alpha}$ of order [39]

$$(P_\beta^{\tau,\alpha}g) := \prod_{j=0}^{n-1}\left(\tau + j - \frac{1}{\beta}\xi\frac{d}{d\xi}\right)(K_\beta^{\tau+\alpha,n-\alpha}g)(\xi), \tag{3.9}$$

$$n = \begin{cases} [\alpha] + 1, & \alpha \notin \mathrm{N}, \\ \alpha, & \alpha \in \mathrm{N}, \end{cases} \tag{3.10}$$

where

$$(K_\beta^{\tau,\alpha}g)(\xi) := \begin{cases} \frac{1}{\Gamma(\alpha)}\int_1^\infty (u-1)^{\alpha-1}u^{-(\tau+\alpha)}g(\xi u^{\frac{1}{\beta}})du, & \alpha > 0, \\ g(\xi), & \alpha = 0 \end{cases} \tag{3.11}$$

is the Erdélyi–Kober fractional integral operator.

Proof. Let $n-1<\alpha<n, n=1,2,3,\ldots$ According to the Riemann–Liouville fractional derivative, one can get

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}s^{\frac{-4\alpha}{5p}}g(xs^{\frac{-\alpha}{5}})ds\right]. \tag{3.12}$$

Let $\nu=\frac{t}{s}$, one can have $ds=-\frac{t}{\nu^2}d\nu$, so (3.12) can be expressed as

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^n}{\partial t^n}\left[t^{n-\alpha-\frac{-4\alpha}{5p}}\frac{1}{\Gamma(n-\alpha)}\int_1^\infty (\nu-1)^{n-\alpha-1}\nu^{-(n-\alpha+1-\frac{4\alpha}{5p})}g(\xi\nu^{\frac{\alpha}{5}})d\nu\right]. \tag{3.13}$$

On the basis of the Erdélyi–Kober fractional integral operator (3.11). One can get

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^n}{\partial t^n}\left[t^{n-\alpha-\frac{-4\alpha}{5p}}\left(K_{\frac{\alpha}{5}}^{1-\frac{4\alpha}{5p},n-\alpha}g\right)(\xi)\right]. \tag{3.14}$$

In view of the relation ($\xi=xt^{\frac{-\alpha}{5}}$), we can get

$$t\frac{\partial}{\partial t}\phi(\xi)=tx\left(-\frac{\alpha}{5}\right)t^{-\frac{\alpha}{5}-1}\phi'(\xi)=-\frac{\alpha}{5}\xi\frac{\partial}{\partial\xi}\phi(\xi). \tag{3.15}$$

One can arrive at

$$\begin{aligned}\frac{\partial^n}{\partial t^n}\left[t^{n-\alpha-\frac{-4\alpha}{5p}}\left(K_{\frac{\alpha}{5}}^{1-\frac{4\alpha}{5p},n-\alpha}g\right)(\xi)\right]&=\frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\alpha-\frac{-4\alpha}{5p}}\left(K_{\frac{\alpha}{5}}^{1-\frac{4\alpha}{5p},n-\alpha}g\right)(\xi)\right)\right]\\&=\frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\alpha-\frac{4\alpha}{5p}}\left(n-\alpha-\frac{4\alpha}{5p}-\frac{\alpha}{5}\xi\frac{\partial}{\partial\xi}\left(K_{\frac{\alpha}{5}}^{1-\frac{4\alpha}{5p},n-\alpha}g\right)(\xi)\right)\right].\end{aligned} \tag{3.16}$$

Repeating the same way for $n-1$ times, one can get

$$\begin{aligned}\frac{\partial^n}{\partial t^n}\left[t^{n-\alpha-\frac{-4\alpha}{5p}}\left(K_{\frac{\alpha}{5}}^{1-\frac{4\alpha}{5p},n-\alpha}g\right)(\xi)\right]&=\frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\alpha-\frac{-4\alpha}{5p}}\left(K_{\frac{\alpha}{5}}^{1-\frac{4\alpha}{5p},n-\alpha}g\right)(\xi)\right)\right]\\&=\frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\alpha-\frac{4\alpha}{5p}}\left(n-\alpha-\frac{4\alpha}{5p}-\frac{\alpha}{5}\xi\frac{\partial}{\partial\xi}\left(K_{\frac{\alpha}{5}}^{1-\frac{4\alpha}{5p},n-\alpha}g\right)(\xi)\right)\right]\\&=\cdots=t^{-\alpha-\frac{4\alpha}{5p}}\prod_{j=0}^{n-1}\left(1-\frac{4\alpha}{5p}-\alpha+j-\frac{\alpha}{5}\xi\frac{d}{d\xi}\right)\left(K_{\frac{\alpha}{5}}^{1-\frac{4\alpha}{5p},n-\alpha}g\right)(\xi).\end{aligned} \tag{3.17}$$

Now we make use of (3.9), and find

$$\frac{\partial^n}{\partial t^n}\left[t^{n-\alpha-\frac{-4\alpha}{5p}}\left(K_{\frac{\alpha}{5}}^{1-\frac{4\alpha}{5p},n-\alpha}g\right)(\xi)\right]=t^{-\alpha-\frac{4\alpha}{5p}}\left(P_{\frac{\alpha}{5}}^{1-\frac{4\alpha}{5p},n-\alpha}g\right)(\xi), \tag{3.18}$$

Substituting the expression (3.18) into (3.14), one can get

$$\frac{\partial^\alpha u}{\partial t^\alpha}=t^{-\alpha-\frac{4\alpha}{5p}}\left(P_{\frac{\alpha}{5}}^{1-\frac{4\alpha}{5p},n-\alpha}g\right)(\xi). \tag{3.19}$$

Thus, the time fractional generalized fifth-order KdV equation (1.1) can be reduced into an ordinary differential equation of fractional order

$$\left(P_{\frac{\alpha}{5}}^{1-\frac{4\alpha}{5p}-\alpha,\alpha}\quad g\right)(\xi)=g_{\xi\xi\xi\xi\xi}+Mg^pg_\xi. \qquad \square \tag{3.20}$$

4. Summary and discussion

Lie group analysis method is successfully to investigate the symmetry properties of time fractional generalized fifth-order KdV equation. At the same time, the Lie algebra and similarity reduction are obtained. However, the obtained point transformation groups for time fractional fifth-order KdV equation are narrower than this for generalized fifth-order KdV equation. Using the Lie point symmetries, we have shown that this equation can be transformed into a nonlinear ODE of fractional order. In a word, the symmetry analysis based on the Lie group method is a very powerful method and is worthy of studying further.

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