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Lie symmetry analysis, nonlinear self-adjointness and conservation laws to an extended (2+1)-dimensional Zakharov–Kuznetsov–Burgers equation ☆

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ABSTRACT

This paper addresses an extended (2+1)-dimensional Zakharov–Kuznetsov–Burgers (ZKB) equation. The Lie symmetry analysis leads to many plethora of solutions to the equation. The nonlinear self-adjointness condition for the ZKB equation is established and subsequently used to construct simplified but infinitely many nontrivial and independent conserved vectors. In particular, we also get conservation laws of the equation with the corresponding Lie symmetry.

1. Introduction

The nonlinear evolution equations (NLEEs) are encountered in a variety of scientific fields, such as physics, chemistry, engineering and others. A vast amount of research work has been investigated in the study of exact solutions of the NLEEs, in particular, the solitary wave solutions largely due to their frequent occurrence in nature. Besides their physical relevances, they serve as a bench mark for accuracy of numerical schemes as well as prove to be quiet handy in testing of computer algorithms. Due to the increased interest in the NLEEs, a broad range of analytical and numerical methods have been developed to construct exact solutions to NLEEs. Some of these efficient methods are the Lie symmetry method [1–3], Darboux transformation method [4], Jacobi elliptic method [5], Painleve analysis [6], the inverse scattering method [7], the Baklund transformation method [8], the conservation law method [9], the Hirota bilinear method [10], the ansatz method [11] and many other methods.

Zakharov and Kuznetsov [12] established an equation which is related to nonlinear ion-acoustic waves (IAWs) in magnetized plasma including cold ions and hot isothermal electrons. Few related studies concerning Zakharov and Kuznetsov and its generalized form are discussed in [13–19]. The quantum hydrodynamic (QHD) model is a generalization of the classical fluid model of plasmas where QHD transport equations are expressed with reference to conservation laws for particle momentum and energy. Several authors for instance, Stenflo et al. [20], Khan et al. [21] etc., have used QHD model to study the linear and nonlinear waves in quantum plasma. Further, the existence of a small number of ions along with the electrons and positron in many astrophysical environments, attracted much attention and consequently lot of work available in the literature (e.g.,[22,23]). It is well know that the small amplitude wave propagation in a medium which posses both the characteristics dispersive and dissipative terms can be best modeled by Korteweg–deVries–Burgers (KdVB) equation. Mamun and Shukla [24] and Shukla and Mamun [25] in their study observed that the dissipative Burger term in KdVB was due to the presence of kinematic viscosity in the plasma. Also, they were able to generate dispersive shock wave in the plasma. El-Bedwehy and Moslem derived the Zakharov Kuznetsov Burgers (ZKB) equation [26] in an electron–positron–ion (e–p–i) plasma and applied their numerical results to the electrostatic fluctuations in the interstellar medium. Masood et al. [27] studies the obliquely propagating nonlinear quantum ion acoustic shock wave in a viscous quantum

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(e–p–i) magnetoplasma. They have used QHD model with the small amplitude expansion method to independently derive the ZKB equation. They have used tanh method to obtain the results.

In this paper, we consider the (2+1)-dimensional ZKB equation and investigate it from the point view of Lie symmetries and conservation laws. By omitting the details of derivation, we directly write ZKB equation in the form

$$u_t + auu_x + bu_{xxx} + cu_{xyy} - du_{xx} - eu_{yy} = 0, \tag{1}$$

where a, b, c, d and e, with d and e being positive, are constant quantities which involves the physical quantities like: mass; density; magnetic field; kinematic viscosity; plasma frequency; superthermality; ion gyrofrequency, etc., while x, y and t are the independent variables that represent the spatial and temporal variables respectively, where as $u(x,y,t)$ is the dependent variable that represents the wave profile. For detail discussions reader is refer to [26,27].

The paper is organized in the following manner. In Section 2, similarity reductions and explicit solutions are derived. In Section 3, we will show that this equation is nonlinearly self-adjoint. On the basis of the point symmetries, conservation laws are constructed. Finally, the main findings of the paper are recapitulated in Section 4.

2. Similarity reductions and exact solutions

The corresponding vector fields of (1) are

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial y}, \quad V_3 = \frac{\partial}{\partial t}, \quad V_4 = t\frac{\partial}{\partial x} + \frac{1}{a}\frac{\partial}{\partial u}. \tag{2}$$

We make some discussions on the ZKB equation based on the vector fields.

(I) V_1

For the generator V_1, we have

$$u = f(g,h), \tag{3}$$

where $g = y, h = t$ are the group-invariants. Substituting (3) into (1), one can get

$$f_h - ef_{gg} = 0. \tag{4}$$

It is important to note that (4) is the celebrated Heat equation.

(II) V_2

For the generator V_2, we get

$$u = f(r,h), \tag{5}$$

where $r = x, h = t$ are the group-invariants. Substituting (5) into (1), we reduce it to the following PDE

$$f_h + aff_r + bf_{rrr} - df_{rr} = 0. \tag{6}$$

(III) V_3

For the generator V_3, we get

$$u = f(r,g), \tag{7}$$

where $r = x, g = y$ are the group-invariants. Putting (7) into (1), one can obtain

$$aff_r + bf_{rrr} + cf_{rgg} - df_{rr} - ef_{gg} = 0. \tag{8}$$

(IV) V_4

In this case, one can obtain

$$u = f(h,g) + \frac{x}{at}, \tag{9}$$

where $h = t, g = y$ are the group-invariants. Substituting (9) into (1), one can obtain

$$hf_h + f - ehf_{gg} = 0. \tag{10}$$

(V) $V_1 + V_2$

The symmetry $V_1 + V_2$ yields the following invariants

$$h = t, \quad s = x - y, \quad u = f(h,s). \tag{11}$$

Treating f as the new dependent variable and h and s as new independent variables, the ZKB Eq. (1) transforms to

$$f_h + aff_s + (b+c)f_{sss} - (d+e)f_{ss} = 0. \tag{12}$$

It is should been noted that all above reduced equation are (1+1)-dimensional PDEs. It is also difficult to get solutions of them. In order to get their solutions, once again, by using Lie symmetry method. For the sake of simplicity, in what follows, we only consider (12) in details.

The symmetry algebra of (12) is generated by the vector fields

$$\Upsilon_1 = \frac{\partial}{\partial h}, \quad \Upsilon_2 = \frac{\partial}{\partial s}, \quad \Upsilon_3 = h\frac{\partial}{\partial s} + \frac{1}{a}\frac{\partial}{\partial f}. \tag{13}$$

The combination $\Upsilon_1 + \lambda\Upsilon_2$ of the two symmetries Υ_1 and Υ_2 yields the following invariants

$$f = \Psi(\theta), \quad \theta = s - \lambda h, \tag{14}$$

and using (14) and (12) is transformed to the nonlinear ODE

$$-\lambda\Psi' + a\Psi\Psi' + (b+c)\Psi''' - (d+e)\Psi'' = 0. \tag{15}$$

Now, we search a solution of (15) in a power series of the form [28,29]

$$\Psi = \sum_{n=0}^{\infty} c_n\theta^n. \tag{16}$$

Substituting (16) into (15), one can get

$$\begin{aligned}
&-\lambda c_1 - \lambda\sum_{n=1}^{\infty}(n+1)c_{n+1}\theta^n + ac_0c_1 + a\sum_{n=1}^{\infty}\sum_{j=0}^{n}(n+1-j)c_jc_{n+1-j}\theta^n \\
&\quad - 2(d+e)c_2 - (d+e)\sum_{n=1}^{\infty}(n+1)(n+2)c_{n+2}\theta^n + 6(b+c)c_3 \\
&\quad + (b+c)\sum_{n=1}^{\infty}(n+3)(n+2)(n+1)c_{n+3}\theta^n = 0.
\end{aligned} \tag{17}$$

Next, from (17), for the case of $n = 0$, one gets

$$c_3 = \frac{\lambda c_1 - ac_0c_1 + 2(d+e)c_2}{6(b+c)}. \tag{18}$$

Generally, for $n \geqslant 1$, one obtains

$$\begin{aligned}
c_{n+3} = &\frac{1}{(b+c)(n+1)(n+2)(n+3)}\Bigg(\lambda(n+1)c_{n+1} + (d+e)(n+1)(n+2) \\
&c_{n+2} - a\sum_{j=0}^{n}(n+1-j)c_jc_{n+1-j}\Bigg).
\end{aligned} \tag{19}$$

Thus, the power series solution of (16) is as follows

$$\begin{aligned}
\Psi(\theta) &= c_0 + c_1\theta + c_2\theta^2 + c_3\theta^3 + \sum_{n=1}^{\infty} c_{n+3}\theta^{n+3} \\
&= c_0 + c_1\theta + c_2\theta^2 + \frac{\lambda c_1 - ac_0c_1 + 2(d+e)c_2}{6(b+c)}\theta^3 \\
&\quad + \sum_{n=1}^{\infty}\frac{1}{(b+c)(n+1)(n+2)(n+3)}\Bigg(\lambda(n+1)c_{n+1} \\
&\quad + (d+e)(n+1)(n+2)c_{n+2} - a\sum_{j=0}^{n}(n+1-j)c_jc_{n+1-j}\Bigg)\theta^{n+3}.
\end{aligned} \tag{20}$$

Consequently, the exact power series solution of (1) can be written as follows

$$
\begin{aligned}
u(x,y,t) &= c_0 + c_1(x-y-\lambda t) + c_2(x-y-\lambda t)^2 + c_3(x-y-\lambda t)^3 \\
&+ \sum_{n=1}^{\infty} c_{n+3}(x-y-\lambda t)^{n+3} = c_0 + c_1(x-y-\lambda t) + c_2(x-y-\lambda t)^2 \\
&+ \frac{\lambda c_1 - ac_0c_1 + 2(d+e)c_2}{6(b+c)}(x-y-\lambda t)^3 \\
&+ \sum_{n=1}^{\infty} \frac{1}{(b+c)(n+1)(n+2)(n+3)} \Bigg(\lambda(n+1)c_{n+1} \\
&+ (d+e)(n+1)(n+2)c_{n+2} - a\sum_{j=0}^{n}(n+1-j)c_jc_{n+1-j} \Bigg)(x-y-\lambda t)^{n+3},
\end{aligned} \tag{21}
$$

where $c_i(i = 0, 1, 2, 3)$ are arbitrary constants, the other coefficients $c_n(n \geqslant 3)$ also can derived.

Remark 1. The explicit solutions of other equations can also be derived in the same way. Here we do not list them for simplicity.

In order to search for others explicit solutions of Eq. (1), we make use of the auxiliary equation, i.e., the Riccati equation of the following "general form"

$$
\varphi' = r + p\varphi + q\varphi^2, \tag{22}
$$

and use its solution to construct the solutions for ZKB equation. Here, p, q, r are real constant. By omitting the details, we directly write the general solution of (22) as

$$
\varphi = \frac{\sqrt{4rq - p^2}}{2q} \frac{C_1 e^{\frac{\theta}{2}\sqrt{4rq-p^2}} - C_2 e^{-\frac{\theta}{2}\sqrt{4rq-p^2}}}{C_1 e^{\frac{\theta}{2}\sqrt{4rq-p^2}} + C_2 e^{-\frac{\theta}{2}\sqrt{4rq-p^2}}} - \frac{p}{2q}, \tag{23}
$$

where C_1, C_2 are arbitrary constants. By balancing the highest derivative and nonlinear terms in (15), we assume the solution of (1) of the form

$$
f(\theta) = a_0 + a_1\varphi + a_2\varphi^2, \tag{24}
$$

where a_0, a_1, a_2 are constants to be determined.

Substituting the ansatz (24) along with (22) into (15), collecting coefficients of monomials of φ with the aid of Maple, and then setting each coefficients equal to zero, one gets

$$
\begin{aligned}
&a_2 = -12\,\frac{q^2(b+c)}{a}, \quad a_1 = -\frac{12}{5}\frac{(5bp + 5cp - d - e)q}{a}, \\
&\lambda = \frac{5aa_0 - 6p(d+e) + 60qr(b+c)}{5},
\end{aligned} \tag{25}
$$

where a_0 is a arbitrary constant. Also,

$$
(d+e)^2 = 25p^2(b+c)^2 + 100qr(b+c)^2. \tag{26}
$$

From the asatz (24) and making use of Eqs. (25) and (23), one can get the explicit solution of (1)

$$
\begin{aligned}
u(x,y,t) = &-12\frac{q^2(b+c)}{a}\left(\frac{\sqrt{4rq-p^2}}{2q}\frac{C_1 e^{\frac{\theta}{2}\sqrt{4rq-p^2}} - C_2 e^{-\frac{\theta}{2}\sqrt{4rq-p^2}}}{C_1 e^{\frac{\theta}{2}\sqrt{4rq-p^2}} + C_2 e^{-\frac{\theta}{2}\sqrt{4rq-p^2}}} - \frac{p}{2q}\right)^2 \\
&-\frac{12}{5}\frac{(5bp+5cp-d-e)q}{a} \\
&\times\left(\frac{\sqrt{4rq-p^2}}{2q}\frac{C_1 e^{\frac{\theta}{2}\sqrt{4rq-p^2}} - C_2 e^{-\frac{\theta}{2}\sqrt{4rq-p^2}}}{C_1 e^{\frac{\theta}{2}\sqrt{4rq-p^2}} + C_2 e^{-\frac{\theta}{2}\sqrt{4rq-p^2}}} - \frac{p}{2q}\right) + a_0,
\end{aligned} \tag{27}
$$

here $\theta = x - y - \lambda t$, and λ is given by (25).

The solution (27) is the general solution of ZKB equation and therefore several independently real solutions can be obtained. For instance, the exact solution (21) which was obtained by "tanh method" in Ref. [27], can be retrieved from (27) by choosing $C_1 = C_2 = 1, p = 0, a_0 = \frac{9(d+e)^2}{25a(b+c)}$, and $qr = \frac{(d+e)^2}{100(b+c)^2}$. Similarly, one can compare the accuracy of the numerical results obtained in [26] via (27).

In particular, if we let $C_1 = 1, C_2 = -1, p = 0$, and $4rq - p^2 > 0$, one can get

$$
\begin{aligned}
u(x,y,t) = &-12\frac{q^2(b+c)}{a}\left(\frac{\sqrt{4rq-p^2}}{2q}\cot h\left(\frac{\theta}{2}\sqrt{4rq-p^2}\right) - \frac{p}{2q}\right)^2 \\
&-\frac{12}{5}\frac{(5bp+5cp-d-e)q}{a}\left(\cot h\left(\frac{\theta}{2}\sqrt{4rq-p^2}\right) - \frac{p}{2q}\right) + a_0.
\end{aligned} \tag{28}
$$

In particular, if we set $C_1 = 1, C_2 = 1$, and $4rq - p^2 > 0$, one can derive

$$
\begin{aligned}
u(x,y,t) = &-12\frac{q^2(b+c)}{a}\left(\frac{\sqrt{4rq-p^2}}{2q}\tan h\left(\frac{\theta}{2}\sqrt{4rq-p^2}\right) - \frac{p}{2q}\right)^2 \\
&-\frac{12}{5}\frac{(5bp+5cp-d-e)q}{a}\left(\tan h\left(\frac{\theta}{2}\sqrt{4rq-p^2}\right) - \frac{p}{2q}\right) + a_0.
\end{aligned}
$$

3. Conservation laws of (1)

In this section, we obtain conservation laws for the ZKB equation.

Theorem 1 ([30,31]). *The system and its adjoint equation*

$$
\begin{aligned}
&F(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(s)}) = 0, \\
&F^*(x, u, v, u_{(1)}, v_{(1)}, u_{(2)}, v_{(2)}, \ldots, u_{(s)}, v_{(s)}) = 0,
\end{aligned} \tag{29}
$$

has a formal Lagrangian, namely

$$
L = vF(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(s)}). \tag{30}
$$

In the following we recall the "new conservation theorem" given by Ibragimov in [32].

Definition 1 ([30,31]). The first equation of (29) is said to be nonlinearly self-adjoint if for some arbitrary function $\phi(x, u) \neq 0$, we have

$$
F^*|_{v=\phi} = \lambda(x, u, u_{(1)}, \ldots)F, \tag{31}
$$

where λ is an indeterminate variable coefficient.

Theorem 2 [32]. *Every Lie point, Lie-Bäcklund and non-local symmetry of Eq. (1) provides a conservation law for Eq. (1) and the adjoint equation. Then the elements of conservation vector (C^1, C^2, C^3) are given by*

$$
\begin{aligned}
C^i = \xi^i L + W^\alpha &\left[\frac{\partial L}{\partial u_i^\alpha} - D_j\left(\frac{\partial L}{\partial u_{ij}^\alpha}\right) + D_jD_k\left(\frac{\partial L}{\partial u_{ijk}^\alpha}\right)\right] \\
&+ D_j(W^\alpha)\left[\left(\frac{\partial L}{\partial u_{ij}^\alpha}\right) - D_k\left(\left(\frac{\partial L}{\partial u_{ijk}^\alpha}\right)\right) + \ldots\right],
\end{aligned} \tag{32}
$$

where $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$.

3.1. Nonlinear self-adjointness

For (1), the adjoint equation has the form

$$
F^* = -(v_t + auv_x + bv_{xxx} + cv_{xyy} + dv_{xx} + ev_{yy}) = 0, \tag{33}
$$

and the Lagrangian in the symmetrized form

$$
L = v(u_t + auu_x + bu_{xxx} + cu_{xyy} - du_{xx} - eu_{yy}). \tag{34}
$$

If we substitute u instead of v in Eq. (33), one can find that Eq. (1) is not recovered. Therefore, we can say Eq. (1) is not self adjoint. Next, we look for an explicit form of $\phi(x,y,t,u) \neq 0$ for Eq. (1) that holds Eq. (33).

$$v_t + auv_x + bv_{xxx} + cv_{xyy} + dv_{xx} + ev_{yy}|_{v=\phi(x,u)} = \lambda(u_t + auu_x + bu_{xxx} + cu_{xyy} - du_{xx} - eu_{yy}). \tag{35}$$

If we set $v = \phi(t,x,y,u)$, one can get

$$\begin{aligned}
v_t &= \phi_t + \phi_u u_t,\\
v_x &= \phi_x + \phi_u u_x,\\
v_{xx} &= \phi_{xx} + 2\phi_{xu}u_x + \phi_{uu}u_x^2 + \phi_u u_{xx},\\
v_{yy} &= \phi_{yy} + 2\phi_{yu}u_y + \phi_{uu}u_y^2 + \phi_u u_{yy},\\
v_{xxx} &= \phi_{xxx} + 3\phi_{xxu}u_x + 3\phi_{xuu}u_x^2 + 3\phi_{xu}u_{xx} + \phi_{uuu}u_x^3\\
&\quad + 3\phi_{uu}u_x u_{xx} + \phi_u u_{xxx},\\
v_{xyy} &= \phi_{xyy} + \phi_{yyu}u_x + 2\phi_{xyu}u_y + 2\phi_{yuu}u_x u_y + 2\phi_{yu}u_{xy}\\
&\quad + \phi_{uux}u_y^2 + \phi_{uuu}u_x u_y^2 + 2\phi_{uu}u_y u_{yx} + \phi_{ux}u_{yy}\\
&\quad + \phi_{uu}u_x u_{yy} + \phi_u u_{xyy}.
\end{aligned} \tag{36}$$

Plugging them into (35), one can arrive at the following self-adjointness condition

$$\begin{aligned}
&\phi_t + \phi_u u_t + a u(\phi_x + \phi_u u_x) + d(\phi_{xx} + 2\phi_{xu}u_x + \phi_{uu}u_x^2 + \phi_u u_{xx})\\
&+ e\left(\phi_{yy} + 2\phi_{yu}u_y + \phi_{uu}u_y^2 + \phi_u u_{yy}\right) + b(\phi_{xxx} + 3\phi_{xxu}u_x + 3\phi_{xuu}u_x^2\\
&+ 3\phi_{xu}u_{xx} + \phi_{uuu}u_x^3 + 3\phi_{uu}u_x u_{xx} + \phi_u u_{xxx}) + c(\phi_{xyy} + \phi_{yyu}u_x + 2\phi_{xyu}u_y\\
&+ 2\phi_{yuu}u_x u_y + 2\phi_{yu}u_{xy} + \phi_{uux}u_y^2 + \phi_{uuu}u_x u_y^2 + 2\phi_{uu}u_y u_{yx}\\
&+ \phi_{ux}u_{yy} + \phi_{uu}u_x u_{yy} + \phi_u u_{xyy}) = -\lambda(u_t + auu_x + bu_{xxx} + cu_{xyy} - du_{xx} - eu_{yy}).
\end{aligned} \tag{37}$$

From the coefficient of the term u_t, one can get

$$\lambda = -\phi_u. \tag{38}$$

Note that the other coefficients of u and all of the derivative yields

$$\begin{aligned}
&b\phi_{uuu} = 0,\ c\phi_{uuu} = 0,\ 3b\phi_{uu} = 0,\ 2c\phi_{yuu} = 0,\ \phi_{yu} = 0,\\
&3b\phi_{xxu} + 2d\phi_{xu} + c\phi_{yyu} = 0,\ 2ec\phi_{yu} + 2c\phi_{xyu} = 0,\\
&3b\phi_{xu} + 2d\phi_u = 0,\ c\phi_{xu} + 2e\phi_u = 0,\\
&3b\phi_{xuu} + d\phi_{uu} = 0,\ c\phi_{xuu} + e\phi_{uu} = 0,\\
&\phi_t + a u\phi_x + b\phi_{xxx} + d\phi_{xx} + e\phi_{yy} + c\phi_{xyy} = 0.
\end{aligned} \tag{39}$$

Solve them, one can get the solution

$$\phi = \frac{c_3\left(\left(e^{\sqrt{k_1}y}\right)^2 c_1 + c_2\right)}{e^{\sqrt{k_1}y}e^{k_1 et}}. \tag{40}$$

In particular, (39) has a special solution

$$\phi = c_1 y + c_2, \tag{41}$$

where c_1, c_2 and c_3 are constants. We, thus, have the following statement.

Theorem 3. The ZKB Eq. (1) is nonlinearly self-adjoint with the substitution $v = \phi$ and ϕ given by (40) or (41).

3.2. Conservation laws of (1)

We now construct the conservation laws by using the adjoint equation and symmetries of (1). For (1), the adjoint equation is given by

$$F = v_t + auv_x + bv_{xxx} + cv_{xyy} + dv_{xx} + ev_{yy}, \tag{42}$$

and the Lagrangian in the symmetrized form

$$L = v\left(u_t + auu_x + bu_{xxx} + \frac{1}{3}c(u_{xyy} + u_{yxy} + u_{yyx}) - du_{xx} - eu_{yy}\right). \tag{43}$$

Consider Theorem 2, the corresponding vector fields is given by

$$V = \xi^1(x,y,t,u)\frac{\partial}{\partial t} + \xi^2(x,y,t,u)\frac{\partial}{\partial x} + \xi^3(x,y,t,u)\frac{\partial}{\partial y} + \phi(x,y,t,u)\frac{\partial}{\partial u}. \tag{44}$$

The conservation law is decided by

$$D_t(C^1) + D_x(C^2) + D_y(C^3) = 0. \tag{45}$$

Here the conserved vector $C = (C^1, C^2, C^3)$ are given by (32) and the components given by

$$C^1 = \xi^1 L + W\frac{\partial L}{\partial u_t}, \tag{46}$$

$$\begin{aligned}
C^2 &= \xi^2 L + W\left[\frac{\partial L}{\partial u_x} - D_x\frac{\partial L}{\partial u_{xx}} + D_{xx}\frac{\partial L}{\partial u_{xxx}} + D_{yy}\frac{\partial L}{\partial u_{xyy}}\right]\\
&\quad + W_x\left[\frac{\partial L}{\partial u_{xx}} - D_x\frac{\partial L}{\partial u_{xxx}}\right] + W_y\left[-D_y\frac{\partial L}{\partial u_{xyy}}\right] + W_{xx}\left[\frac{\partial L}{\partial u_{xxx}}\right]\\
&\quad + W_{yy}\left[\frac{\partial L}{\partial u_{xyy}}\right],
\end{aligned} \tag{47}$$

$$\begin{aligned}
C^3 &= \xi^3 L + W\left[-D_y\frac{\partial L}{\partial u_{yy}} + D_{xy}\frac{\partial L}{\partial u_{yxy}} + D_{yx}\frac{\partial L}{\partial u_{yyx}}\right]\\
&\quad + W_x\left[-D_y\frac{\partial L}{\partial u_{yxy}}\right] + W_y\left[\frac{\partial L}{\partial u_{yy}} - D_x\frac{\partial L}{\partial u_{yyx}}\right] + W_{xy}\left[\frac{\partial L}{\partial u_{yxy}}\right]\\
&\quad + W_{yx}\left[\frac{\partial L}{\partial u_{yyx}}\right],
\end{aligned} \tag{48}$$

that is

$$C^1 = \xi^1 L + Wv, \tag{49}$$

$$\begin{aligned}
C^2 &= \xi^2 L + W\left[auv - D_x(-dv) + D_{xx}(bv) + D_{yy}\left(\frac{1}{3}cv\right)\right]\\
&\quad + W_x(-dv - D_x(bv)) + W_y\left(-D_y\left(\frac{1}{3}cv\right)\right) + W_{xx}bv\\
&\quad + W_{yy}\frac{1}{3}cv\\
&= \xi^2 L + W\left(auv + dv_x + bv_{xx} + \frac{1}{3}cv_{yy}\right) - W_x(dv + bv_x)\\
&\quad - W_y\left(\frac{1}{3}cv_y\right) + bvW_{xx} + \frac{1}{3}cvW_{yy},
\end{aligned} \tag{50}$$

$$\begin{aligned}
C^3 &= \xi^3 L + W\left[-D_y(-ev) + D_{xy}\left(\frac{2}{3}cv\right)\right] + W_x\left[-D_y\left(\frac{1}{3}cv\right)\right]\\
&\quad + W_y\left[-ev - D_x\left(\frac{1}{3}cv\right)\right] + W_{xy}\left[\frac{2}{3}cv\right]\\
&= \xi^3 L + W\left(ev_y + \frac{2}{3}cv_{xy}\right) - \frac{1}{3}cv_y W_x - W_y\left(ev + \frac{1}{3}cv_x\right)\\
&\quad + \frac{2}{3}cvW_{xy},
\end{aligned} \tag{51}$$

with

$$W = \phi - \xi^1 u_t - \xi^2 u_x - \xi^3 u_y. \tag{52}$$

Next, we consider following cases.
Case 1.
For the operator $V = \frac{\partial}{\partial t}$, we have

$$W = -u_t, \tag{53}$$

we can get the conservation vector of (1)

$$C^1 = -u_t v, \tag{54}$$

$$C^2 = -u_t\left(auv + dv_x + bv_{xx} + \frac{1}{3}cv_{yy}\right) + u_{tx}(dv + bv_x) + \frac{1}{3}cu_{ty}v_y - bvu_{txx} - \frac{1}{3}cvu_{tyy}, \tag{55}$$

$$C^3 = -u_t\left(ev_y + \frac{2}{3}cv_{xy}\right) + \frac{1}{3}cv_y u_{tx} + u_{ty}\left(ev + \frac{1}{3}cv_x\right) - \frac{2}{3}cvu_{txy}. \tag{56}$$

Case 2.

For the generator $V = \frac{\partial}{\partial x}$, we get

$$W = -u_x, \tag{57}$$

one can obtain the conservation vector of (1)

$$C^1 = -u_x v, \tag{58}$$

$$C^2 = -u_x\left(auv + dv_x + bv_{xx} + \frac{1}{3}cv_{yy}\right) + u_{xx}(dv + bv_x) + \frac{1}{3}cu_{xy}v_y - bvu_{xxx} - \frac{1}{3}cvu_{xyy}, \tag{59}$$

$$C^3 = -u_x\left(ev_y + \frac{2}{3}cv_{xy}\right) + \frac{1}{3}cv_y u_{xx} + u_{xy}\left(ev + \frac{1}{3}cv_x\right) - \frac{2}{3}cvu_{xxy}. \tag{60}$$

Case 3.

For the Lie algebra $V = \frac{\partial}{\partial y}$, one can arrive at

$$W = -u_y, \tag{61}$$

we can reach the conservation vector of (1)

$$C^1 = -u_y v, \tag{62}$$

$$C^2 = -u_y\left(auv + dv_x + bv_{xx} + \frac{1}{3}cv_{yy}\right) + u_{yx}(dv + bv_x) + \frac{1}{3}cu_{yy}v_y - bvu_{yxx} - \frac{1}{3}cvu_{yyy}, \tag{63}$$

$$C^3 = -u_y\left(ev_y + \frac{2}{3}cv_{xy}\right) + \frac{1}{3}cv_y u_{yx} + u_{yy}\left(ev + \frac{1}{3}cv_x\right) - \frac{2}{3}cvu_{yxy}. \tag{64}$$

Case 4.

For the operator $V = t\frac{\partial}{\partial x} + \frac{1}{a}\frac{\partial}{\partial u}$, we have

$$W = \frac{1}{a} - tu_x, \tag{65}$$

we derive the conservation vector of (1)

$$C^1 = \frac{1}{a}v - tu_x v, \tag{66}$$

$$C^2 = \left(\frac{1}{a} - tu_x\right)\left(auv + dv_x + bv_{xx} + \frac{1}{3}cv_{yy}\right) + tu_{xx}(dv + bv_x) + \frac{1}{3}ctu_{xy}v_y - btvu_{xxx} - \frac{1}{3}ctvu_{xyy}, \tag{67}$$

$$C^3 = \left(\frac{1}{a} - tu_x\right)\left(ev_y + \frac{2}{3}cv_{xy}\right) + \frac{1}{3}ctv_y u_{xx} + tu_{xy}\left(ev + \frac{1}{3}cv_x\right) - \frac{2}{3}ctvu_{xyx}. \tag{68}$$

It is clear that they are involves an arbitrary solution v of the adjoint Eq. (33), and they presents an infinite number of the conservation laws.

4. Conclusions

In the present paper, by using the Lie symmetry groups, we studied the symmetry properties, similarity reduction forms and explicit solutions of the (2+1)-dimensional ZKB equation. Moreover, we also constructed the power series solution of the equation. At last, we derived the nonlinear self-adjointness of Eq. (1), by virtue of this fact, infinitely many conservation laws of the equation are exhibited. Furthermore, the results obtained here can be useful in enhancing the understanding of nonlinear propagation of small amplitude electrostatic structures in dense, dissipative (e–p–i) magnetoplasmas.

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