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A (2+1)-dimensional sine-Gordon and sinh-Gordon equations with symmetries and kink wave solutions

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Abstract

In this paper, a (2+1)-dimensional sine-Gordon equation and a sinh-Gordon equation are derived from the well-known AKNS system. Based on the Hirota bilinear method and Lie symmetry analysis, kink wave solutions and traveling wave solutions of the (2+1)-dimensional sine-Gordon equation are constructed. The traveling wave solutions of the (2+1)-dimensional sinh-Gordon equation can also be provided in a similar manner. Meanwhile, conservation laws are derived.

1. Introduction

It is well-known that the classical $(1+1)$-dimensional sine-Gordon (sG) equation

$$u_{tt} = u_{xx} + \sin u \tag{1}$$

or equivalent form

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$$u_{xt} = \sin u \tag{2}$$

appears in many scientific fields [1,2,12,5–7,34], such as quantum-field and differential geometry theory [1,2,12,10,24,5–7]. Many mathematicians and physicists studied this well-known equation from different aspects. The authors in [2] discussed the sG equation through using the inverse scattering method. Leibbrandt [12] studied solutions of the sine-Gordon equation in higher dimensions. Klein [10] considered geometric interpretation as surfaces of constant negative curvature. Rubinstein [24] presented a model of field theory and studied in detail. Gu and Hu [5] provided explicit solutions to the intrinsic generalization for the wave and sine-Gordon equations, and Hu [7] investigated the relationship between soliton and differential Geometry through the sG equation. In [25], the authors studied symmetry groups of the intrinsic generalized wave and sine-Gordon equations. A quantum-mechanical system is constructed over a Fock space of particles in [31] based on N-soliton solutions.

In paper [35], one of the authors derived a (2+1)-dimensional KdV and mKdV equation from positive case. This paper is the continuation of that one. In this paper, from the extend AKNS system, we derive a (2+1)-dimensional sine-Gordon equation as well as a (2+1)-dimensional sinh-Gordon equation. Kink wave solutions and their interactional wave propagation are constructed for the (2+1)-dimensional sine-Gordon equation. Furthermore, Lie symmetries approach is employed to reduce the (2+1)-dimensional sine-Gordon and sinh-Gordon equations so that their traveling wave solutions are obtained.

2. Derivation of (2+1)-dimensional sine-Gordon and sinh-Gordon equations

It is well-known that the AKNS system [1] is one of the classical well-known integrable systems from which a great many of nonlinear evolution equations can be derived, such as the famous KdV equation, the MKdV equation, the nonlinear Schrödinger equation (NLS), the Burgers equation, the (1+1)-dimensional sine-Gordon equation, etc. Based on the AKNS system, let us consider the following (2+1)-dimensional zero curvature equation [1,33,11,34,35],

$$X_t - X_x + T_x - T_y + [X, T] = 0, \tag{3}$$

where $[X, T] = XT - TX$,

$$X = \begin{pmatrix} -i\zeta & q \\ r & i\zeta \end{pmatrix}, \quad T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{4}$$

Here in Eq. (4), $i^2 = -1$, ζ is an eigenparameter independent of time t (i.e. $\zeta_t = 0$), q, r are two potential functions of x, t, and A, B, C, D are the functions to be determined. Substituting Eq. (4) into Eq. (3) leads to the following equations

$$\begin{cases} A_x - A_y = rB - qC, \\ B_x - B_y = q_x - q_t + 2iB\zeta + Aq - qD, \\ C_x - C_y = r_x - r_t - 2iC\zeta + rD - rA, \\ D_x - D_y = qC - rB. \end{cases} \tag{5}$$

From the first and the last equations, we can choose $D = -A$. Hence, Eq. (5) becomes

$$\begin{cases} A_x - A_y = rB - qC, \\ B_x - B_y = q_x - q_t + 2iB\zeta + 2Aq, \\ C_x - C_y = r_x - r_t - 2iC\zeta - 2rA. \end{cases} \tag{6}$$

In order to solve for A, B, C, let us target at expanding A, B, C in the form of truncated power series with regard to the eigenvalue ζ. Since the positive cases have already been studied in the literature, here in our paper we do the negative case. The negative order of integrable equations originated from the work in [15,16], and thereafter some interesting integrable equations with the properties of generalized Lax representations and algebraic structure were developed from the negative hierarchy [32,17,19,21,22]. The negative case may generate some new equations which have different physical meanings [18,20,23].

Therefore let us try employing the following expansions

$$A=\sum a_n(x,y,t)\zeta^{-n},\ B=\sum b_n(x,y,t)\zeta^{-n},\ C=\sum c_n(x,y,t)\zeta^{-n}. \tag{7}$$

Substituting Eq. (7) into Eq. (6), it immediately generates

$$\begin{cases} a_{nx}-a_{ny}=rb_n-qc_n, \\ b_{n-1,x}-b_{n-1,y}=2ib_n+2qa_{n-1}, \\ c_{n-1,x}-c_{n-1,y}=-2ic_n-2ra_{n-1}, \\ q_x-q_t=b_{nx}-b_{ny}-2a_nq, \\ r_x-r_t=c_{nx}-c_{ny}+2a_nr. \end{cases} \tag{8}$$

In the special case $n=1$, we have

$$A=a_1(x,y,t)\zeta^{-1},\ B=b_1(x,y,t)\zeta^{-1},\ C=c_1(x,y,t)\zeta^{-1}, \tag{9}$$

and

$$\begin{cases} a_{1x}-a_{1y}=rb_1-qc_1, \\ b_{1x}-b_{1y}=2a_1q, \\ q_x-q_t=-2ib_1, \\ c_{1x}-c_{1y}=-2a_1r, \\ r_x-r_t-2ic_1=0. \end{cases} \tag{10}$$

Eq. (10) admits the following special solutions:

$$\begin{cases} a_1=\frac{i}{4}\cos u, \\ b_1=c_1=\frac{i}{4}\sin u, \\ q=-r=\frac{u_x-u_y}{2}, \end{cases} \tag{11}$$

and subsequently yields the following (2+1)-dimensional sine-Gordon equation

$$u_{xx}-u_{xy}-u_{xt}+u_{yt}=\sin u. \tag{12}$$

In a similar way, we may select the following special solutions of Eq. (10)

$$\begin{cases} a_1=\frac{i}{4}\cosh u, \\ b_1=-c_1=\frac{i}{4}\sinh u, \\ q=r=\frac{u_x-u_y}{2}, \end{cases} \tag{13}$$

to get the (2+1)-dimensional sinh-Gordon equation below

$$u_{xx}-u_{xy}-u_{xt}+u_{yt}=\sinh u. \tag{14}$$

Remark.

1. One may obtain other integrable (2+1)-dimensional equations through choosing different solutions of Eq. (10). Here in our paper, we just focus on those which have physical applications.
2. Under the transformation $\xi = x + y + t, \eta = y, \tau = t$, Eq. (12) and Eq. (14) equivalent to Eqs. $U_{\eta\tau} = \sin U$ and $U_{\eta\tau} = \sinh U$, this is just the case of (1+1)-dimensional.

3. Multi-kink wave solutions

Let us consider the following transformation [4]

$$u(x, y, t) = 2i \ln \frac{f^*}{f}, \tag{15}$$

where f^* is the complex conjugate of function f. Since $\sin u = \frac{e^{iu} - e^{-iu}}{2i}$, substituting Eq. (15) into Eq. (12) yields

$$\begin{aligned}
&2\left(\frac{f^*_{xx} f^* - f^*_x f^*_x}{(f^*)^2} - \frac{f_{xx} f - f_x f_x}{f^2}\right) - 2\left(\frac{f^*_{xy} f^* - f^*_y f^*_x}{(f^*)^2} - \frac{f_{yx} f - f_y f_x}{f^2}\right) \\
&-2\left(\frac{f^*_{tx} f^* - f^*_t f^*_x}{(f^*)^2} - \frac{f_{tx} f - f_t f_x}{f^2}\right) + 2\left(\frac{f^*_{ty} f^* - f^*_t f^*_y}{(f^*)^2} - \frac{f_{ty} f - f_t f_y}{f^2}\right) \\
&\qquad = \frac{1}{2}\left(\frac{(f^*)^2 - f^2}{(f^*)^2} - \frac{f^2 - (f^*)^2}{f^2}\right),
\end{aligned} \tag{16}$$

which implies the following bilinear forms [8]

$$D_x D_x f \cdot f - D_y D_x f \cdot f - D_t D_x f \cdot f + D_t D_y f \cdot f = \frac{f^2 - (f^*)^2}{2}, \tag{17}$$

where the operator D is defined by

$$D^{n_1}_{x_1} \cdots D^{n_l}_{x_l} f \cdot g = \left(\partial_{x_1} - \partial_{x'_1}\right)^{n_1} \cdots \left(\partial_{x_l} - \partial_{x'_l}\right)^{n_l} f(x_1, \cdots x_l)\, g\left(x'_1, \cdots x'_l\right)|_{x'_l = x_1, \cdots x'_l = x_l}. \tag{18}$$

Let us assume that f can be expanded in the power of ε as follows,

$$f = 1 + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \cdots. \tag{19}$$

Substituting Eq. (19) into Eq. (17), we have

$$\begin{aligned}
&\text{coefficients} \quad (\varepsilon): \\
&f^{(1)}_{xx} - f^{(1)}_{yx} - f^{(1)}_{tx} + f^{(1)}_{ty} = \frac{1}{2}\left(f^{(1)} - (f^{(1)})^*\right),
\end{aligned} \tag{20}$$

$$\begin{aligned}
&\text{coefficients} \quad (\varepsilon^2): \\
&2\left(f^{(2)}_{xx} - f^{(2)}_{yx} - f^{(2)}_{tx} + f^{(2)}_{ty}\right) \\
&= -\left(D_x D_x f^{(1)} \cdot f^{(1)} - D_y D_x f^{(1)} \cdot f^{(1)} - D_t D_x f^{(1)} \cdot f^{(1)} + D_t D_y f^{(1)} \cdot f^{(1)}\right) \\
&\quad + f^{(2)} - f^{(2*)} + \frac{1}{2}\left((f^{(1)})^2 - \left((f^{(1)})^{(*)}\right)^2\right),
\end{aligned} \tag{21}$$

$$
\begin{aligned}
&\text{coefficients} \quad (\varepsilon^3): \\
&2\left(f^{(3)}_{xx} - f^{(3)}_{yx} - f^{(3)}_{tx} + f^{(3)}_{ty}\right) \\
=&-2\left(D_x D_x f^{(1)} \cdot f^{(2)} - D_y D_x f^{(1)} \cdot f^{(2)} - D_t D_x f^{(1)} \cdot f^{(2)} + D_t D_y f^{(1)} \cdot f^{(2)}\right) \\
&+f^{(3)} - f^{(3*)} + (f^{(1)})(f^{(2)}) - (f^{(1)})^*(f^{(2)})^*, \qquad (22) \\
&\cdots .
\end{aligned}
$$

In order to get some exact solutions, let us set up

$$f^{(1)} = ie^{\xi_1}, \ \xi_1 = c_1 t + k_1 x + l_1 y, \tag{23}$$

with the following constraint condition

$$k_1^2 - k_1 c_1 - k_1 l_1 + l_1 c_1 = 1. \tag{24}$$

Then, we have

$$f_1(x, y, t) = 1 + ie^{\xi_1}, \tag{25}$$

which gives us the following single kink wave solution

$$u = 2i \ln \frac{1 + ie^{\xi_1}}{1 - ie^{\xi_1}} = 4 \arctan e^{\xi_1}. \tag{26}$$

Because of its linearity, Eq. (20) admits the following solutions

$$f^{(1)} = ie^{\xi_1} + ie^{\xi_2}, \ \xi_1 = c_1 t + k_1 x + l_1 y, \ \xi_2 = c_2 t + k_2 x + l_2 y, \tag{27}$$

where $c_j, k_j, l_j (j = 1, 2)$ are constants. Apparently, substituting Eq. (27) into Eq. (21) produces

$$f^{(2)} = -e^{\xi_1 + \xi_2 + A_{12}}, \tag{28}$$

where

$$e^{A_{12}} = \frac{(k_1 - k_2)(l_1 - l_2) + (c_1 - c_2)(k_1 - k_2) - (k_1 - k_2)^2 - (c_1 - c_2)(l_1 - l_2)}{(k_1 + k_2)^2 - (k_1 + k_2)(l_1 + l_2) - (c_1 + c_2)(k_1 + k_2) + (c_1 + c_2)(l_1 + l_2)}. \tag{29}$$

That is to say,

$$f_2(x, y, t) = 1 - e^{\xi_1 + \xi_2 + A_{12}} + ie^{\xi_1} + ie^{\xi_2}. \tag{30}$$

Therefore, two kink wave solution is given by

$$u = 4 \arctan \frac{e^{\xi_1} + e^{\xi_2}}{1 - e^{\xi_1 + \xi_2 + A_{12}}}. \tag{31}$$

Adopting the same procedure shown above, we could obtain a 3-kink wave solution

$$u = 2i \arctan \frac{e^{\xi_1} + e^{\xi_2} + e^{\xi_3} - e^{\xi_1 + \xi_2 + \xi_3 + A_{12} + A_{13} + A_{23}}}{1 - (e^{\xi_1 + \xi_2 + A_{12}} + e^{\xi_1 + \xi_3 + A_{13}} + e^{\xi_2 + \xi_3 + A_{23}})}, \tag{32}$$

where

$$e^{A_{ij}} = \frac{(k_i - k_j)(l_i - l_j) + (c_i - c_j)(k_i - k_j) - (k_i - k_j)^2 - (c_i - c_j)(l_i - l_j)}{(k_i + k_j)^2 - (k_i + k_j)(l_i + l_j) - (c_i + c_j)(k_i + k_j) + (c_i + c_j)(l_i + l_j)} \quad (i < j, \quad i, j = 1, 2, 3). \tag{33}$$

Repeated the similar procedure N times, we can construct N-kink wave solution:

$$u = 2i \ln \frac{\sum\limits_{\mu=0,1} \exp\Big(\sum\limits_{i=1}^{N} \mu_i(\xi_i - i\frac{\pi}{2}) + \sum\limits_{1\le i<j}^{N} \mu_i \mu_j A_{ij}\Big)}{\sum\limits_{\mu=0,1} \exp\Big(\sum\limits_{i=1}^{N} \mu_i(\xi_i + i\frac{\pi}{2}) + \sum\limits_{1\le i<j}^{N} \mu_i \mu_j A_{ij}\Big)}. \tag{34}$$

4. Determinant representation of the N-kink wave solution

The N-kink wave solution could be represented in the terms of determinants. Let us consider the following determinant associated with a parameter λ

$$p(\lambda) = \det\left(\lambda\delta_{ij} + \frac{4A_iB_i}{(A_i + A_j)(B_i + B_j)} e^{\frac{\xi_i+\xi_j-i\pi}{2}}\right), \tag{35}$$

where δ_{ij} is a characteristic function

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}; \tag{36}$$

and $A_i = c_i - k_i$, $A_j = c_j - k_j$, $B_i = k_i - l_i$, $B_j = k_j - l_j$. Apparently, $p(\lambda)$ is a N-th degree polynomial. So, let

$$p(\lambda) = \lambda^N + a_1\lambda^{N-1} + a_2\lambda^{N-2} + \ldots + a_N, \tag{37}$$

where $a_1, a_2, ..., a_N$ are N coefficients. Obviously,

$$a_N = p(0) = \begin{vmatrix} e^{\xi_1 - \frac{i\pi}{2}} & \frac{4A_1B_1}{(A_1+A_2)(B_1+B_2)} e^{\frac{\xi_1+\xi_2-i\pi}{2}} & \cdots & \frac{4A_1B_1}{(A_1+A_N)(B_1+B_N)} e^{\frac{\xi_1+\xi_N-i\pi}{2}} \\ \frac{4A_2B_2}{(A_2+A_1)(B_2+B_1)} e^{\frac{\xi_2+\xi_1-i\pi}{2}} & e^{\xi_2 - \frac{i\pi}{2}} & \cdots & \frac{4A_2B_2}{(A_2+A_N)(B_2+B_N)} e^{\frac{\xi_2+\xi_N-i\pi}{2}} \\ \vdots & \vdots & & \vdots \\ \frac{4A_NB_N}{(A_N+A_1)(B_N+B_1)} e^{\frac{\xi_N+\xi_1-i\pi}{2}} & \frac{4A_NB_N}{(A_N+A_2)(B_N+B_2)} e^{\frac{\xi_N+\xi_2-i\pi}{2}} & \cdots & e^{\xi_N - \frac{i\pi}{2}} \end{vmatrix} \tag{38}$$

$$= (-4i)^N A_1B_1A_2B_2 \cdots A_NB_N e^{\xi_1+\xi_2+\cdots\xi_N} \tag{39}$$

$$\times \begin{vmatrix} \frac{1}{4A_1B_1} & \frac{1}{(A_1+A_2)(B_1+B_2)} & \cdots & \frac{1}{(A_1+A_N)(B_1+B_N)} \\ \frac{1}{(A_2+A_1)(B_2+B_1)} & \frac{1}{4A_2B_2} & \cdots & \frac{1}{(A_2+A_N)(B_2+B_N)} \\ \vdots & \vdots & & \vdots \\ \frac{1}{(A_N+A_1)(B_N+B_1)} & \frac{1}{(A_N+A_2)(B_N+B_2)} & \cdots & \frac{1}{4A_NB_N} \end{vmatrix} \tag{40}$$

$$= (-4i)^N A_1B_1A_2B_2 \cdots A_NB_N e^{\xi_1+\xi_2+\cdots\xi_N} \left(\frac{1}{4A_1B_1} \prod_{i=2}^{N} \frac{(A_i - A_j)(B_i - B_j)}{(A_i + A_j)(B_i + B_j)}\right) \tag{41}$$

$$\times \begin{vmatrix} 1 & \frac{1}{(A_1+A_2)(B_1+B_2)} & \cdots & \frac{1}{(A_1+A_N)(B_1+B_N)} \\ 0 & \frac{1}{4A_2B_2} & \cdots & \frac{1}{(A_2+A_N)(B_2+B_N)} \\ \vdots & \vdots & & \vdots \\ 0 & \frac{1}{(A_N+A_2)(B_N+B_2)} & \cdots & \frac{1}{4A_NB_N} \end{vmatrix} \tag{42}$$

$$= \cdots \quad \cdots \tag{43}$$

$$= (-4i)^N A_1B_1A_2B_2\cdots A_NB_N e^{\xi_1+\xi_2+\cdots\xi_N} \times \left(\frac{1}{4^N A_1B_1A_2B_2\cdots A_NB_N} \prod_{1\le i<j}^{N} \frac{(A_i-A_j)(B_i-B_j)}{(A_i+A_j)(B_i+B_j)} \right) \tag{44}$$

$$= (-i)^N e^{\xi_1+\xi_2+\cdots\xi_N} \prod_{1\le i<j}^{N} \frac{(A_i-A_j)(B_i-B_j)}{(A_i+A_j)(B_i+B_j)}. \tag{45}$$

Repeating the above procedure, we shall obtain

$$\begin{aligned} &a_N = p(0), \\ &a_{N-1} = p'(0), \\ &a_{N-2} = \tfrac{1}{2}p''(0), \\ &\cdots, \end{aligned} \tag{46}$$

where $'$ stands for $\frac{dp(\lambda)}{d\lambda}$.

Comparing all the coefficients (46) with the numerator of (34), we can readily find that

$$\sum_{\mu=0,1} \exp\left(\sum_{i=1}^{N} \mu_i(\xi_i - i\frac{\pi}{2}) + \sum_{1\le i<j}^{N} \mu_i\mu_j A_{ij} \right) = a_N + a_{N-1} + a_{N-2} + \cdots + a_1 = p(1), \tag{47}$$

and

$$u = 2i \ln \frac{\det\left(\lambda\delta_{ij} + \frac{4A_iB_i}{(A_i+A_j)(B_i+B_j)} e^{\frac{\xi_i+\xi_j-i\pi}{2}} \right)}{\det\left(\lambda\delta_{ij} + \frac{4A_iB_i}{(A_i+A_j)(B_i+B_j)} e^{\frac{\xi_i+\xi_j+i\pi}{2}} \right)}, \tag{48}$$

where δ_{ij} is defined in (36).

5. Interaction of kink waves

Now, let us consider the two-kink wave solution

$$u = 2i \ln \frac{1 - e^{\xi_1+\xi_2+A_{12}} - i(e^{\xi_1} + e^{\xi_2})}{1 - e^{\xi_1+\xi_2+A_{12}} - i(e^{\xi_1} + e^{\xi_2})}$$

where

$$\xi_j = c_jt + k_jx + l_jy, \quad k_j^2 - k_jc_j - k_jl_j + l_jc_j = 1, \quad j = 1, 2,$$

and investigate the interaction of two-kink waves solutions. Without loss of a generality, let us assume that $k_1 = mk_2$, $l_1 = ml_2$, where m is a non-zero real number, then $c_1 = \frac{k_1^2 - k_1 l_1 - 1}{k_1 - l_1}$ and $c_2 = \frac{m^2 k_1^2 - m^2 k_1 l_1 - 1}{m(k_1 - l_1)}$. Obviously, ξ_2 can be rewritten in terms of ξ_1, that is,

$$\xi_2 = m\xi_1 - \frac{m-1}{m(k_1 - l_1)} t. \tag{49}$$

Therefore, in the obit of constant ξ_1, when $t \to -\infty$ both $e^{\xi_1 + \xi_2 + A_{12}}$ and e^{ξ_1} are approaching 0 and

$$u \sim 2i \ln \frac{1 - ie^{\xi_1}}{1 + ie^{\xi_1}} = 4 \arctan e^{\xi_1}. \tag{50}$$

When $t \to \infty$, apparently ie^{ξ_1}, $e^{\xi_1 + \xi_2 + A_{12}}$ and e^{ξ_2} are dominant terms. Hence, we have

$$u \sim 2i \ln \frac{e^{\xi_1 + \xi_2 + A_{12}} - ie^{\xi_1}}{e^{\xi_1 + \xi_2 + A_{12}} + ie^{\xi_1}} = 4 \arctan e^{-(\xi_1 + A_{12})}. \tag{51}$$

Adopting the similar procedure as above and playing the same scenario in the obit of constant ξ_2, we then know that ξ_1 can be rewritten by ξ_2 as

$$\xi_1 = \frac{\xi_2}{m} + \frac{1 - m^2}{m^2(k_1 - l_1)} t. \tag{52}$$

Therefore, in the obit of constant ξ_2, when $t \to -\infty$ and $t \to \infty$, we obtain the following two asymptotic formulations

$$u \sim 2i \ln \frac{e^{\xi_1 + \xi_2 + A_{12}} - ie^{\xi_2}}{e^{\xi_1 + \xi_2 + A_{12}} + ie^{\xi_2}} = 4 \arctan e^{-(\xi_2 + A_{12})} \tag{53}$$

and

$$u \sim 2i \ln \frac{1 - ie^{\xi_2}}{1 + ie^{\xi_2}} = 4 \arctan e^{\xi_2}, \tag{54}$$

respectively.

In light of the above asymptotic analysis, we can conclude that when the kink waves travel alone with the x-axis, the one on the left travels faster and interacts with the one on the right. After their interactions, two kink waves interchange their position.

6. Lie symmetries analysis and traveling wave solutions of the (2+1)-dimensional sine-Gordon equation (12)

Obviously, the following simple transformation

$$v = e^{iu}, \tag{55}$$

sends

$$\sin u = \frac{v - v^{-1}}{2i}, \cos u = \frac{v + v^{-1}}{2}, \tag{56}$$

and

$$u = \arccos \frac{v + v^{-1}}{2}. \tag{57}$$

Substituting Eqs. (55) and (56) into Eq. (12) leads Eq. (12) to the following equation

$$-i\frac{v_{xx}v - v_x^2}{v^2} + i\frac{v_{xt}v - v_x v_t}{v^2} + i\frac{v_{xy}v - v_x v_y}{v^2} - i\frac{v_{yt}v - v_y v_t}{v^2} = \frac{v - v^{-1}}{2i}, \tag{58}$$

which can be reduced to

$$2\left(v_{xx}v - v_x^2 - v_{xt}v + v_x v_t - v_{xy}v + v_x v_y + v_{yt}v - v_y v_t\right) - v^3 + v = 0. \tag{59}$$

As per the Lie group method shown in [13,14,3,9,26,29,30,27,28], Eq. (59) has the following vector fields,

$$V = \xi_t(x,y,t,u)\frac{\partial}{\partial t} + \xi_x(x,y,t,u)\frac{\partial}{\partial x} + \xi_y(x,y,t,u)\frac{\partial}{\partial y} + \xi_u(x,y,t,u)\frac{\partial}{\partial u}. \tag{60}$$

Then a direct but lengthy computation generates the following results

$$\xi_t = (x+y)F_2 + F_3,\ \xi_x = (-x-2y)F_2 + F_4,\ \xi_y = yF_2 + F_1,\ \eta_v = 0, \tag{61}$$

where F_1, F_2, F_3, F_4 are arbitrary functions of x, y, t. Thus, Eq. (59) has the following symmetries

$$V_1 = \frac{\partial}{\partial t}, V_2 = \frac{\partial}{\partial x}, V_3 = \frac{\partial}{\partial y}, V_4 = (x+y)F_2\frac{\partial}{\partial t} + (-x-2y)F_2\frac{\partial}{\partial x} + yF_2\frac{\partial}{\partial y}, \tag{62}$$

the first three of which are apparently the basic geometry symmetries. It is clear that Eq. (59) definitely has traveling wave solutions. Substituting the traveling wave setting $v(\xi) = f(k_1x + k_2y - k_3t)$ into Eq. (12) yields the following ordinary differential equation

$$Af''f - Af'^2 - f^3 + f = 0, \tag{63}$$

where $A = 2k_1^2 + 2k_1k_3 - 2k_1k_2 - 2k_2k_3$. Let us assume that Eq. (63) admits special solutions in the form of

$$f = a_0 + a_1\phi + a_2\phi^2, \tag{64}$$

where a_0, a_1, a_2 are constants and ϕ satisfies the following well-known Riccati equation

$$\phi' = R + \phi^2, \tag{65}$$

with the following solutions:

$$\phi = -\sqrt{-R}\tanh\sqrt{-R}\xi,\ \phi = -\sqrt{-R}\coth\sqrt{-R}\xi,\ R < 0 \tag{66}$$

and

$$\phi = \sqrt{R}\tan\sqrt{R}\xi,\ \phi = -\sqrt{R}\cot\sqrt{R}\xi,\ R > 0. \tag{67}$$

Substituting Eqs. (64) and (65) into Eq. (63), we have

$$a_2 = \pm\frac{1}{R},\ A = \pm\frac{1}{2R},\ a_0 = a_1 = 0. \tag{68}$$

So, we obtain the traveling wave solutions of the (2+1)-dimensional sine-Gordon equation (12):

$$u = \arccos\frac{v + v^{-1}}{2}. \tag{69}$$

Case i) when $R < 0$,

$$v = \mp \tanh^2 \sqrt{-R}\xi, \quad v = \mp \coth^2 \sqrt{-R}\xi, \tag{70}$$

and

$$u = \arccos\left[\frac{\mp(\cosh(4\sqrt{-R}\xi) - 3)}{2\sinh^2(2\sqrt{-R}\xi)}\right] \tag{71}$$

where $\xi = k_1 x + k_2 y - k_3 t$.

Case ii) when $R > 0$,

$$v = \pm \tan^2 \sqrt{R}\xi, \; v = \pm \cot^2 \sqrt{R}\xi, \tag{72}$$

and

$$u = \arccos\left[\frac{\pm(\cos(4\sqrt{-R}\xi) - 3)}{2\sin^2(2\sqrt{-R}\xi)}\right] \tag{73}$$

where $\xi = k_1 x + k_2 y - k_3 t$.

Remark. Let

$$v = e^u, \tag{74}$$

then

$$\sinh u = \frac{v - v^{-1}}{2}, \; \cosh u = \frac{v + v^{-1}}{2}, \tag{75}$$

and

$$u = \operatorname{arccosh}\frac{v + v^{-1}}{2}. \tag{76}$$

Substituting Eqs. (74) and (76) into the (2+1)-dimensional sinh-Gordon equation (12) yields the same equation as Eq. (59)

$$2\left(v_{xx}v - v_x^2 - v_{xt}v + v_x v_t - v_{xy}v + v_x v_y + v_{yt}v - v_y v_t\right) - v^3 + v = 0. \tag{77}$$

So, just substituting (62) and (64) into

$$u = \operatorname{arccosh}\frac{v + v^{-1}}{2} = \begin{cases} \operatorname{arccosh}\left[\frac{\mp(\cosh(4\sqrt{-R}\xi)-3)}{2\sinh^2(2\sqrt{-R}\xi)}\right], & R < 0 \\ \operatorname{arccosh}\left[\frac{\pm(\cos(4\sqrt{-R}\xi)-3)}{2\sin^2(2\sqrt{-R}\xi)}\right], & R > 0 \end{cases}, \tag{78}$$

which is the exact traveling wave solutions of the (2+1)-dimensional sinh-Gordon equation (14), with $\xi = k_1 x + k_2 y - k_3 t$.

7. Conservation laws

Below, we present the multipliers Q with the corresponding conserved forms $T^t dxdy + T^y dxdt + T^x dydt$ (where $(T^t, T^x, -T^y)$ is the conserved vector). The computed multipliers are up to first order in derivatives of u. Given here, there are infinitely many as all functions $f_i(x + y + t)$, $i = 1, 2, 3$, of Eq. (12) viz.,

$$Q = [(x+t)u_y - (2t+x)u_x + tu_t)]f_1(x+y+t)$$
$$+ (u_t - u_y)f_2(x+y+t) + (u_x - u_y)f_3(x+y+t) + ku_y,$$

where k is a constant.

We construct the conserved forms for some special cases in which the term $T^t dxdy$ leads to the 'conserved density'.

1. $Q_1 = u_x$:

$$[\tfrac{1}{4}u_x u_y - \tfrac{1}{4}u_x{}^2 - \tfrac{1}{4}uu_{xy} + \tfrac{1}{4}uu_{xx}]dxdy$$
$$+[-\tfrac{1}{4}u_t u_x + \tfrac{1}{4}u_x{}^2 + \tfrac{1}{4}uu_{xt} - \tfrac{1}{4}uu_{xx}]dxdt$$
$$+[-1 - \tfrac{1}{4}u_t u_x - \tfrac{1}{4}u_x u_y + \tfrac{1}{2}u_x{}^2 - \tfrac{1}{4}uu_{xt} - \tfrac{1}{4}uu_{xy} + \tfrac{1}{2}uu_{yt} + \cos u]dydt.$$

2. $Q_2 = u_y$:

$$[\tfrac{1}{4}u_y{}^2 - \tfrac{1}{4}u_x u_y - \tfrac{1}{4}uu_{yy} + \tfrac{1}{4}uu_{xy}]dxdy$$
$$+[1 - \tfrac{1}{4}u_t u_y + \tfrac{1}{4}u_x u_y - \tfrac{1}{4}uu_{yt} + \tfrac{1}{4}uu_{xy} - \tfrac{1}{2}uu_{xx} + \tfrac{1}{2}uu_{xt} - \cos u]dxdt$$
$$+[-\tfrac{1}{4}u_t u_y - \tfrac{1}{4}u_y{}^2 + \tfrac{1}{2}u_x u_y + \tfrac{1}{4}uu_{yt} + \tfrac{1}{4}uu_{yy} - \tfrac{1}{2}uu_{xy}]dydt.$$

3. $Q_3 = u_t$:

$$[-1 + \tfrac{1}{4}u_t u_y - \tfrac{1}{4}u_t u_x + \tfrac{1}{4}uu_{yt} - \tfrac{1}{4}uu_{xt} + \tfrac{1}{2}uu_{xx} - \tfrac{1}{2}uu_{xy} + \cos u]dxdy$$
$$+[-\tfrac{1}{4}u_t{}^2 + \tfrac{1}{4}u_t u_x + \tfrac{1}{4}uu_{tt} - \tfrac{1}{4}uu_{xt}]dxdt$$
$$+[-\tfrac{1}{2}u_t{}^2 + \tfrac{3}{4}u_t u_x + \tfrac{1}{2}uu_{tt} - \tfrac{3}{4}uu_{xt} - \tfrac{1}{4}u_t u_y + \tfrac{1}{4}uu_{yt}]dydt.$$

For the following special cases, we only present the conserved density component of the conserved form, T^t due to the cumbersome nature of it.

4. $Q_4 = (x+y+t)(u_x - u_y)$:

$$\tfrac{1}{2}u_x u_y x - \tfrac{1}{4}u_y{}^2 x + \tfrac{1}{2}u_y y u_x - \tfrac{1}{4}y u_y{}^2$$
$$+\tfrac{1}{2}u_x u_y t - \tfrac{1}{4}u_y{}^2 t - \tfrac{1}{4}u_x{}^2 x - \tfrac{1}{4}y u_x{}^2 - \tfrac{1}{4}u_x{}^2 t$$
$$-\tfrac{1}{2}uxu_{xy} - \tfrac{1}{2}uyu_{xy} - \tfrac{1}{2}utu_{xy} + \tfrac{1}{4}uu_{yy}x$$
$$+\tfrac{1}{4}uu_{yy}y + \tfrac{1}{4}uu_{yy}t + \tfrac{1}{4}uxu_{xx} + \tfrac{1}{4}uyu_{xx} + \tfrac{1}{4}utu_{xx}.$$

5. $Q_5 = (x+y+t)(u_t - u_y)$:

$$
\begin{aligned}
&-x-t-y+\tfrac{1}{4}u_xu_yx+\tfrac{1}{4}u_yyu_x+\tfrac{1}{4}u_xu_yt-\tfrac{3}{4}uxu_{xy}\\
&-\tfrac{3}{4}uyu_{xy}-\tfrac{3}{4}utu_{xy}+\tfrac{1}{4}uu_{yy}x+\tfrac{1}{4}uu_{yy}y\\
&+\tfrac{1}{4}uu_{yy}t+\tfrac{1}{2}uxu_{xx}+\tfrac{1}{2}uyu_{xx}+\tfrac{1}{2}utu_{xx}\\
&-\tfrac{1}{4}{u_y}^2x-\tfrac{1}{4}y{u_y}^2-\tfrac{1}{4}{u_y}^2t+y\cos u+x\cos u+t\cos u-\tfrac{1}{4}utu_{xt}\\
&+\tfrac{1}{4}u_tu_yt-\tfrac{1}{4}uyu_{xt}-\tfrac{1}{4}uxu_{xt}-\tfrac{1}{4}u_tu_xt\\
&+\tfrac{1}{4}u_tu_yx-\tfrac{1}{4}u_tu_xx-\tfrac{1}{4}u_tyu_x+\tfrac{1}{4}u_tyu_y\\
&+\tfrac{1}{4}uxu_{yt}+\tfrac{1}{4}uyu_{yt}+\tfrac{1}{4}utu_{yt}.
\end{aligned}
$$

6. $Q_6 = -u_yx - u_yt + 2u_xt + u_xx - tu_t$:

$$
\begin{aligned}
&-t+\tfrac{1}{4}{u_y}^2x+\tfrac{1}{4}{u_y}^2t+\tfrac{1}{4}{u_x}^2x+\tfrac{1}{2}{u_x}^2t\\
&+\tfrac{1}{4}uu_y-\tfrac{1}{4}uu_x-\tfrac{1}{2}u_xu_yx+t\cos u\\
&-\tfrac{3}{4}u_xu_yt-\tfrac{1}{4}uu_{yy}x-\tfrac{1}{4}uu_{yy}t-\tfrac{1}{4}uxu_{xx}\\
&+\tfrac{1}{4}u_tu_yt-\tfrac{1}{4}u_tu_xt-\tfrac{1}{4}utu_{xt}+\tfrac{1}{4}utu_{yt}\\
&+\tfrac{1}{2}uxu_{xy}+\tfrac{1}{4}utu_{xy}.
\end{aligned}
$$

8. Conclusions

In this paper, under the extended Lax pair (3) for the (2+1)-dimensions, we derived a (2+1)-dimensional sine-Gordon and a (2+1)-dimensional sinh-Gordon equation. Kink wave solutions, multi-kink wave interactions, and traveling wave solutions are derived. At last, some conservation laws are presented for some special cases. In this paper, we got the (2+1)-dimensional sin-Gordon equation and sinh-Gordon equation. Conservation laws and kink wave solutions are derived, these results also provides a good basis for the effectiveness of some numerical methods, such as invariant discretization schemes [36,37], structure-preserving method [38–45] and so on. It is worth mention that, at this point, there are some issues need to be studied further, such as nonlocal symmetry, symmetry reductions, more exact solutions as well as their versions with variable coefficients. They will be reported in future works.

Declaration of competing interest

The authors declare that they have no conflict of interest.

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