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Singular solitons, shock waves, and other solutions to potential KdV equation

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Abstract This paper addresses the potential Korteweg–de Vries equation. The singular 1-soliton solution is obtained by the aid of ansatz method. Subsequently, the G'/G-expansion method and the exp-function approach also gives a few more interesting solutions. Finally, the Lie symmetry analysis leads to another plethora of solution to the equation. These results are going to be extremely useful and applicable in applied mathematics and theoretical physics.

Keywords Solitons · Lie symmetry · Integrability

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1 Introduction

The mathematical models of shallow water waves are a growing area of research in mechanical engineering. There have been several models that are proposed to study this dynamics. A few of these models are the well-known Korteweg–de Vries equation (KdV), Boussinesq equation, Benjamin–Bona–Mahoney equation that is also known as the Peregrine equation or regularized long wave (RLW) equation, Kawahara equation, and many others. All these models govern the dynamics of water waves along ocean shores, canals, and other low-lying areas of water. These models are all well studied and very well understood. However, in the recent past there are several other models that are being reported. A few of such models are the Rosenau–Kawahara equation, Rosenau–RLW equation and these models dispersive water waves along ocean shores. Another model that is believed to replicate tsunami waves is the potential KdV (pKdV) equation. This paper will study this equation in order to extract solitary wave solutions by the application of several integration tools. These models fall in the category of nonlinear evolution equations (NLEEs) that arise in mathematical physics.

The mathematical analysis of NLEEs is growing at an alarming rate. There are several aspects of NLEEs which are being addressed by several scientists across the globe [1–25]. One of the primary focus is the integrability issue. The integration of these several NLEEs leads to a plethora of solutions. These solutions are extremely important in various areas of applied mathematics and theoretical physics especially where these equations appear. It is due to this pressing reason, there are different forms of integration architecture, which have been developed in the last couple of decades.

A few of these integration tools are variational iteration method, Adomian decomposition method, first integral method, differential transform method, Lie symmetry analysis, mapping method, and several others. However, it is imperative to exercise caution while implementing these techniques of integration. While these methods reveal results and several forms of solutions to these equations, there are a few hindrances that one experiences. For example, none of these integration tools lead to the computation of soliton radiation or phonons which is an important component of soliton solutions. In addition, most of these techniques do not lead to multiple soliton solutions that is also important in various areas of engineering sciences where the study soliton dynamics is extremely important, such as optical fiber technology.

This paper addresses one such NLEE that is known as the pKdV equation. This equation is studied in applied mathematics including the dynamics of shallow water waves. The integrability aspect of this equation will be the primary focus of this paper. The ansatz method will be first employed to extract the singular 1-soliton solution to this equation. Subsequently, three integration tools will be applied to integrate the pKdV equation that will display a range of solutions. The three additional methods of integration that will be applied are the G'/G-expansion method, exp-function approach, and the Lie symmetry analysis. These results will be discussed in the following sections.

2 Singular soliton solution

The dimensionless structure of the pKdV equation that is going to be studied in this paper is given by [3]

$$q_t + a\,(q_x)^2 + bq_{xxx} = 0, \tag{1}$$

where a and b are constant quantities, while x and t are the independent variables that represent the spatial and temporal variables, respectively, while $q(x,t)$ is the dependent variable that represents the wave profile. This section will focus on the singular 1-soliton solution to this equation that will be obtained by the ansatz method. The starting ansatz therefore is given by

$$q(x,t) = A\coth^p[B(x-vt)], \tag{2}$$

where A and B are free parameters while v is the velocity of the soliton. The unknown exponent is p, where $p > 0$, whose value will fall out during the course of derivation of the soliton solution. Substituting (2) into (1) and simplifying, we get

$$\begin{aligned}
&v\left(\coth^{p+2}\tau - \coth^p\tau\right)\\
&+apAB\left(\coth^{2p+4}\tau - 2\coth^{2p+2}\tau + \coth^{2p}\tau\right)\\
&-bB^2\Big[(p+2)(p+4)\coth^{p+6}\tau\\
&-(p+2)\{2(p+1)+(p+4)\}\coth^{p+4}\tau\\
&+\Big\{p^2+2(p+1)(p+2)\Big\}\coth^{p+2}\tau\\
&-p^2\coth^p\tau\Big] = 0,
\end{aligned} \tag{3}$$

where the notation

$$\tau = B(x-vt) \tag{4}$$

was introduced.

By the balancing principle, equating the exponents $2p$ and $p+2$ leads to

$$2p = p+2, \tag{5}$$

so that

$$p = 2. \tag{6}$$

It needs to be noted here that the same value of p is recovered when the exponent pairs $2p+2$ and $p+4$ or $2p+4$ and $p+6$ are equated. Now from (3), setting the coefficients of the linearly independent functions $\coth^{p+j}\tau$, for $j = 0, 2, 4$ and 6, to zero, we get

$$v = 4bB^2, \tag{7}$$

$$v = 28bB^2 - 2aAB, \tag{8}$$

$$A = \frac{12b}{a}B. \tag{9}$$

Now, equating the two values of the soliton velocity from (7) and (9) also gives the relation between the soliton free parameters as seen in (9). The only constraint condition that is evident from (9) is given by

$$a \neq 0. \tag{10}$$

Hence, finally, the singular 1-soliton solution to (1) is given by

$$q(x,t) = A\coth^2[B(x-vt)], \tag{11}$$

where the relation between the free parameters of the soliton is given by (9) while the velocity of the soliton is given in (7) or (9). The following two sections display other solutions that are obtained by two other integration techniques.

3 G'/G-expansion method

In this section, we will demonstrate the G'/G-expansion method on pKdV equation. The pKdV equation that will be studied is given by [3]

$$q_t + a\left(q_x\right)^2 + bq_{xxx} = 0, \tag{12}$$

which by the wave transformation

$$q(x,t) = q(\tau), \tau = B(x - vt), \tag{13}$$

is converted to

$$-vq'(\tau) + aB\left(q'\right)^2(\tau) + bB^2 q'''(\tau) = 0, \tag{14}$$

where a and b are arbitrary constants. By balancing princing principle, we get $2(m+1) = m+3$, hence $m = 1$. We then suppose that Eq. (14) has the following formal solutions:

$$q = A_1\left(\frac{G'}{G}\right) + A_0, \tag{15}$$

where A_0 and A_1 are unknown constants to be determined by solving a set of algebraic equation by Maple and this leads to [4]

$$A_1 = \frac{6bB}{a}, \quad v = bB^2(\lambda^2 - 4\mu), \tag{16}$$

where μ and A_0 are arbitrary constants.

When $\Delta = \lambda^2 - 4\mu > 0$, we obtain hyperbolic function solutions:

$$\begin{aligned} q(x,t) &= q(\tau) \\ &= A_0 - \frac{6b\lambda B}{2a} \\ &\quad + \frac{6b\, B\varepsilon}{a}\left(\frac{c_1\sinh(\varepsilon\tau) + c_2\cosh(\varepsilon\tau)}{c_1\cosh(\varepsilon\tau) + c_2\sinh(\varepsilon\tau)}\right) \end{aligned} \tag{17}$$

and $\varepsilon = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}$.

When $\Delta = \lambda^2 - 4\mu < 0$, we obtain trigonometric function solutions:

$$\begin{aligned} q(x,t) &= q(\tau) \\ &= A_0 - \frac{6b\lambda B}{2a} \\ &\quad + \frac{6b\, B\varepsilon}{a}\left(\frac{-c_1\sin(\varepsilon\tau) + c_2\cos(\varepsilon\tau)}{c_1\cos(\varepsilon\tau) + c_2\sin(\varepsilon\tau)}\right), \end{aligned} \tag{18}$$

where $\varepsilon = \frac{1}{2}\sqrt{4\mu - \lambda^2}$.

When $\Delta = \lambda^2 - 4\mu = 0$, we obtain trigonometric function solutions:

$$\begin{aligned} q(x,t) &= q(\tau) \\ &= A_0 - \frac{6b\lambda B}{2a} + \frac{6b\, B}{a}\left(\frac{c_2}{c_1 + c_2 Bx}\right). \end{aligned} \tag{19}$$

4 Exp-function method

In this approach, it is assumed that the governing equation can be expressed in the following form [5]:

$$q(\tau) = \frac{\sum_{n=-c}^{d} a_n e^{n\tau}}{\sum_{m=-p}^{s} b_m e^{m\tau}}, \tag{20}$$

where $c, d, p,$ and s are positive integers which are unknown to be further determined; a_n and b_m are unknown constants. In this case, the pKdV equation that will be studied is given by [3]

$$q_t + a\left(q_x\right)^2 + bq_{xxx} = 0, \tag{21}$$

which by the wave transformation is converted to

$$-vq'(\tau) + aB\left(q'\right)^2(\tau) + bB^2 q'''(\tau) = 0, \tag{22}$$

where a and b are arbitrary constants. Using the ansatz (20), for the linear term of highest order q'^2 and q''' by simple calculation, we have

$$\left(q'\right)^2 = \frac{c_1 e^{2(d+s)\tau} + \cdots}{c_2 e^{4s\tau} \cdots} \tag{23}$$

and

$$q''' = \frac{c_3 e^{(d+7s)\tau} + \cdots}{c_4 e^{8s\tau} + \cdots}, \tag{24}$$

where c_i are determined coefficients only for simplicity. Balancing highest order of exp-function in q'^2 and q''', we have

$$2d + 2s - 4s = d + 7s - 8s, \tag{25}$$

which leads to the result

$$d = s. \tag{26}$$

Similarly, to determine the values of c and p, we balance the linear term of lowest order in Eq. (22)

$$\left(q'\right)^2 = \frac{d_1 e^{-2(c+p)\tau} + \cdots}{d_2 e^{-4p\tau} + \cdots} \tag{27}$$

and

$$q''' = \frac{\cdots + d_3 e^{-(c+7p)\tau}}{\cdots + d_4 e^{-8p\tau}}, \tag{28}$$

where d_i are determined coefficients only for simplicity. Balancing lowest order of exp-function in Eqs. (27) and (28), we have

$$-(2c+2p)+4p=-(c+7p)+8p, \tag{29}$$

which leads to the result

$$c=p. \tag{30}$$

Since the final solution does not strongly depend upon the choice of values of c and d, so for simplicity, we set $p=c=1$, $d=s=1$, the function Eq. (20) becomes

$$q(\tau)=\frac{a_{-1}e^{-\tau}+a_0+a_1e^{\tau}}{b_{-1}e^{-\tau}+b_0+b_1e^{\tau}}. \tag{31}$$

Substituting Eq. (31) into Eq. (22), and equating to zero, the coefficients of all powers of $e^{n\tau}$ yield a set of algebraic equations for $a_1, a_0, a_{-1}, b_1, b_0, v, B$, and b_{-1}. Solving the system of algebraic equations with the aid of Maple, we obtain the following several set of solutions.

First set:

$$a_1=b_1=0,\ v=bB^2,\ a_{-1}=-\frac{b_{-1}(-aa_0+6bBb_0)}{ab_0}, \tag{32}$$

where $a, b_0\neq 0$, $\tau=B(x-bB^2t)$. Here, $a, b, a_0, a_{-1}, b_0, b_{-1}$, and B are free parameters.

Second set:

$$a_{-1}=\frac{-6(6a_1Bbb_{-1}+aa_0^2)bBb_{-1}}{a^2a_0^2}, \tag{33}$$

$$v=bB^2,\ b_1=\frac{-a^2a_0^2}{36B^2b^2b_{-1}},\ b_0=0,$$

where $a, a_0, b, b_{-1}, B\neq 0$, $\tau=B(x-bB^2t)$ and a, b, a_0, b_{-1}, a_1 are free parameters.

Third set:

$$b_{-1}=-az,\ v=bB^2,\ a_{-1}=\frac{(6bBb_1-aa_1)z}{b_1}, \tag{34}$$

where

$$z=\frac{(a_0b_1-a_1b_0)(6b_0b_1bB+a_0ab_1-aa_1b_0)}{36(B^2b^2b_1^3)} \tag{35}$$

and a, b, a_0, b_0, a_1, b_1, B are free parameters with the restriction $bBb_1\neq 0$.

Fourth set:

$$a_0=0,\ b_0=0,\ v=4bB^2,\ a_{-1}=\frac{-b_{-1}(12bBb_1-aa_1)}{ab_1}, \tag{36}$$

where a, b, B, a_1, b_1, b_{-1} are free parameters with the condition $ab_1\neq 0$.

5 Lie symmetry analysis

This section will focus on the Lie symmetry analysis to the pKdV equation. In the first subsection, the basic mathematical phenomena of Lie symmetry analysis will be described. Next, this analysis will be applied to the pKdV equation and several solutions will be retrieved.

5.1 Lie symmetry analysis to (1)

In this section, we will perform Lie group method for Eq. (1). If (1) is invariant under a one-parameter Lie group of point transformations

$$\begin{aligned} t^* &= t+\epsilon\tau(x,t,q)+O(\epsilon^2),\\ x^* &= x+\epsilon\xi(x,t,q)+O(\epsilon^2),\\ q^* &= q+\epsilon\eta(x,t,q)+O(\epsilon^2), \end{aligned} \tag{37}$$

with infinitesimal generator

$$V=\tau(x,t,q)\frac{\partial}{\partial t}+\xi(x,t,q)\frac{\partial}{\partial x}+\eta(x,t,q)\frac{\partial}{\partial q}. \tag{38}$$

If the vector field (38) generates a symmetry of (1), then V must satisfy Lie's symmetry condition:

$$pr^{(3)}V(\Delta_1)|_{\Delta_1=0}=0, \tag{39}$$

where $\Delta_1=q_t+a\,(q_x)^2+bq_{xxx}$. Applying the third prolongation $pr^{(3)}V$ to Eq. (1), we find the following system of symmetry equations then the invariant condition reads as

$$\eta^t+2a\eta^xq_x+\eta^{xxx}=0, \tag{40}$$

where

$$\begin{aligned} \eta^t &= D_x(\eta)-q_xD_x(\xi)-q_tD_x(\tau)\\ &= D_x(\eta-\xi q_x-\tau q_t)+\xi q_{xx}+\tau q_{xt}\\ &= \eta_x+(\eta_q-\xi_x)q_x-\tau_xq_t-\xi_qq_x^2-\tau_qq_xq_t, \end{aligned} \tag{41}$$

$$\begin{aligned} \eta^t &= D_t(\eta)-q_xD_t(\xi)-q_tD_t(\tau)\\ &= D_t(\eta-\xi q_x-\tau q_t)+\xi q_{xt}+\tau q_{tt}\\ &= \eta_t-\xi_tq_x+(\eta_q-\tau_t)q_x-\tau_tq_t-\xi_qq_xq_t-\tau_qq_t^2, \end{aligned} \tag{42}$$

$$\eta^{xx}=D_x(\eta^x)-q_{xt}D_x(\tau)-q_{xx}D_x(\xi)\ldots, \tag{43}$$

$$\eta^{xxx}=D_x(\eta^{xx})-q_{xxt}D_x(\tau)-q_{xxx}D_x(\xi)\ldots. \tag{44}$$

Here, D_i denotes the total derivative operator and is defined by

$$D_i = \frac{\partial}{\partial x^i} + q_i \frac{\partial}{\partial q} + q_{ij} \frac{\partial}{\partial q_j} + \cdots, \qquad i = 1, 2, \tag{45}$$

and $(x^1, x^2) = (t, x)$.

Then, in terms of the Lie symmetry analysis method, one can obtain

$$\tau = 3c_1 t + c_2, \;\; \xi = c_1 x + c_3, \;\; \eta = -c_1 q + c_4, \tag{46}$$

where c_1, c_2, c_3, and c_4 are arbitrary constants.

We obtain the corresponding (four-dimensional Lie algebra) geometric vector fields of (1) is as follows:

$$\begin{aligned} &V_1 = \frac{\partial}{\partial t}, V_2 = \frac{\partial}{\partial x}, \\ &V_3 = \frac{\partial}{\partial q}, V_4 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - q\frac{\partial}{\partial q}. \end{aligned} \tag{47}$$

It is easy to check that the symmetry generators found in (47) form a closed Lie algebra

$$\begin{aligned} &[V_1, V_2] = [V_2, V_1] = [V_1, V_3] \\ &\quad = [V_3, V_1] = [V_3, V_2] = [V_2, V_3] = 0. \\ &[V_1, V_4] = -[V_4, V_1] = 3V_1, [V_2, V_4] \\ &\quad = -[V_4, V_2] = V_2, \\ &[V_3, V_4] = -[V_4, V_3] = -V_3. \end{aligned} \tag{48}$$

To obtain the group transformation which is generated by the infinitesimal generators V_i for $i = 1, 2, 3, 4$, we need to solve the following initial problems

$$\begin{aligned} &\frac{\mathrm{d}(\bar{x}(\varepsilon))}{\mathrm{d}\varepsilon} = \xi\Big(\bar{x}(\varepsilon), \bar{t}(\varepsilon), \bar{q}(\varepsilon)\Big), \bar{x}(0) = x, \\ &\frac{\mathrm{d}(\bar{t}(\varepsilon))}{\mathrm{d}\varepsilon} = \tau\Big(\bar{x}(\varepsilon), \bar{t}(\varepsilon), \bar{q}(\varepsilon)\Big), \bar{t}(0) = t, \\ &\frac{\mathrm{d}(\bar{u}(\varepsilon))}{\mathrm{d}\varepsilon} = \eta\Big(\bar{x}(\varepsilon), \bar{t}(\varepsilon), \bar{q}(\varepsilon)\Big), \bar{q}(0) = q, \end{aligned} \tag{49}$$

where ε is a parameter. Thus, one can obtain the Lie symmetry group

$$g : (x, t, q) \to (\bar{x}, \bar{t}, \bar{q}). \tag{50}$$

Exponentiating the infinitesimal symmetries of Eq. (1), we get the one-parameter groups $g_i(\varepsilon)$ generated by V_i for $i = 1, 2, 3, 4$

$$\begin{aligned} &g_1 : (x, t, q) \mapsto (x, t + \varepsilon, q), \\ &g_2 : (x, t, q) \mapsto (x + \varepsilon, t, q), \\ &g_3 : (x, t, q) \mapsto (x, t, q + \varepsilon), \\ &g_4 : (x, t, q) \mapsto (\mathrm{e}^{-\varepsilon} x, \mathrm{e}^{-3\varepsilon} t, \mathrm{e}^{\varepsilon} q). \end{aligned} \tag{51}$$

The symmetry groups g_1 and g_2 demonstrate the time- and space-invariance of the equation. The well-known scaling symmetry turns up in g_4. Consequently, we can obtain the corresponding theorem:

Theorem 1 *If* $q = f(x,t)$ is a solution of (1), so are the functions

$$\begin{aligned} &g_1(\varepsilon) \cdot f(x, t) = f(x - \varepsilon, t), \\ &g_2(\varepsilon) \cdot f(x, t) = f(x, t - \varepsilon), \\ &g_3(\varepsilon) \cdot f(x, t) = f(x, t) - \varepsilon, \\ &g_4(\varepsilon) \cdot f(x, t) = \mathrm{e}^{-\varepsilon} f(\mathrm{e}^{-\varepsilon} x, \mathrm{e}^{-3\varepsilon} t). \end{aligned} \tag{52}$$

5.2 Symmetry reductions and exact group-invariant solutions

5.2.1 V_1

The group-invariant solution corresponding to V_1 is $q = f(\xi)$, where $\xi = t$ is the group-invariant, the substitution of this solution into Eq. (1) gives the trivial solution $q(x, t) = C$, C is a constant.

5.2.2 V_2 (Stationary solution)

For the generator V_2, we have

$$q = f(\xi), \tag{53}$$

where $\xi = x$ is the group-invariant. Substituting Eq. (53) into Eq. (1), we reduce it to the following ODE

$$a(f')^2 + bf''' = 0, \tag{54}$$

where $f' = \frac{\mathrm{d}f}{\mathrm{d}\xi}$.

5.2.3 $V_2 + \lambda V_1$ (Traveling wave solutions)

For the linear combination $V_2 + \lambda V_1$, we have

$$q = f(\xi), \tag{55}$$

where $\xi = x - \lambda t$ is the group-invariant. Substitution of (55) into the (1), we reduce it to the following ODE

$$\lambda f' + a(f')^2 + bf''' = 0, \tag{56}$$

where $f' = \frac{\mathrm{d}f}{\mathrm{d}\xi}$.

It should be noted that if $\lambda \neq 0$, then (56) is the traveling wave solution to the equation, where $\lambda \neq 0$ denotes the wave speed. In particular, if $\lambda = 0$, then we get (53). Now, we consider the traveling wave solutions of the (56).

By balancing the highest order partial derivative term and the nonlinear term in (56), we get the value of m,

$$(u(\xi))^{2m+2} = (u(\xi))^{m+3}, \tag{57}$$

so one can get $m = 1$. In this case, we get

$$f(\xi) = a_0 + a_1\varphi, \tag{58}$$

where a_0, a_1 are constants to be determined, and $\varphi(\xi)$ satisfies

$$\varphi' = A + B\varphi + C\varphi^2. \tag{59}$$

Substituting the ansatz (58) along with Eq. (59) into Eq. (55), collecting coefficients of monomials of φ with the aid of Maple, then setting each coefficients to zero, we can deduce

$$A = A,\ B = B,\ C = C,\ a = a,\ b = \frac{\lambda}{4\,AC - B^2},$$
$$\lambda = \lambda,\ a_1 = -\frac{6C\lambda}{\left(4\,AC - B^2\right)a},\ a_0 = a_0. \tag{60}$$

In view of Eq. (59) which has a lot of fundamental solutions (27 solutions) [16] , one can find a number of exact traveling wave solutions for Eq. (1), some of which are listed as follows:

Family 1 : When $B^2 - 4AC > 0$ and $BC \neq 0$ (or $AC \neq 0$),

$$q(x,t) = a_0 - a_1\frac{1}{2C} \times \left[B + \sqrt{B^2 - 4AC}\tanh\left(\frac{\sqrt{B^2 - 4AC}}{2}\xi\right)\right]. \tag{61}$$

$$q(x,t) = a_0 - a_1\frac{1}{2C} \times \left[B + \sqrt{B^2 - 4AC}\coth\left(\frac{\sqrt{B^2 - 4AC}}{2}\xi\right)\right]. \tag{62}$$

$$q(x,t) = a_0 - a_1\frac{1}{2C}\left[B + \sqrt{B^2 - 4AC} \times\left(\tanh\left(\sqrt{B^2-4AC}\xi\right) \pm i\,\mathrm{sech}\left(\sqrt{B^2-4AC}\xi\right)\right)\right]. \tag{63}$$

$$q(x,t) = a_0 - a_1\frac{1}{2C}\left[B + \sqrt{B^2 - 4AC} \times\left(\coth\left(\sqrt{B^2-4AC}\xi\right) \pm i\,\mathrm{csch}\left(\sqrt{B^2-4AC}\xi\right)\right)\right]. \tag{64}$$

$$q(x,t) = a_0 - a_1\frac{1}{4C}\left[2B + \sqrt{B^2 - 4AC} \times\left(\tanh\left(\frac{\sqrt{B^2-4AC}}{4}\xi\right) + \coth\left(\frac{\sqrt{B^2-4AC}}{4}\xi\right)\right)\right]. \tag{65}$$

$$q(x,t) = a_0 + a_1\frac{1}{2C}\left[-B + \frac{\sqrt{(E^2+F^2)(B^2-4AC)} - E\sqrt{B^2-4AC}\cosh(\sqrt{B^2-4AC}\xi)}{E\sinh(\sqrt{B^2-4AC}\xi)+F}\right]. \tag{66}$$

$$q(x,t) = a_0 + a_1\frac{1}{2C}\left[-B - \frac{\sqrt{(F^2-E^2)(B^2-4AC)} + E\sqrt{B^2-4AC}\sinh(\sqrt{B^2-4AC}\xi)}{E\cosh(\sqrt{B^2-4AC}\xi)+F}\right], \tag{67}$$

where E and F are two non-zero real constants and satisfies $F^2 - E^2 > 0$.

$$q(x,t) = a_0 + a_1\left[\frac{2A\cosh(\frac{\sqrt{B^2-4AC}}{2}\xi)}{\sqrt{B^2-4AC}\sinh(\frac{\sqrt{B^2-4AC}}{2}\xi) - B\cosh(\frac{\sqrt{B^2-4AC}}{2}\xi)}\right]. \tag{68}$$

$$q(x,t) = a_0 + a_1\left[\frac{-2A\sinh(\frac{\sqrt{B^2-4AC}}{2}\xi)}{-\sqrt{B^2-4AC}\cosh(\frac{\sqrt{B^2-4AC}}{2}\xi) + B\sinh(\frac{\sqrt{B^2-4AC}}{2}\xi)}\right]. \tag{69}$$

$$q(x,t) = a_0 + a_1\left[\frac{2A\cosh(\sqrt{B^2 - 4AC}\xi)}{\sqrt{B^2 - 4AC}\sinh(\sqrt{B^2 - 4AC}\xi) - B\cosh(\sqrt{B^2 - 4AC}\xi) \pm i\sqrt{B^2 - 4AC}\xi}\right]. \tag{70}$$

$$q(x,t)=a_0+a_1\left[\frac{2A\sinh(\sqrt{B^2-4AC}\xi)}{\sqrt{B^2-4AC}\cosh(\sqrt{B^2-4AC}\xi)-B\sinh(\sqrt{B^2-4AC}\xi)\pm\sqrt{B^2-4AC}\xi}\right]. \tag{71}$$

$$q(x,t)=a_0+a_1\left[\frac{4A\sinh(\frac{\sqrt{B^2-4AC}}{4}\xi)\cosh(\frac{\sqrt{B^2-4AC}}{4}\xi)}{-2B\sinh(\frac{\sqrt{B^2-4AC}}{4}\xi)\cosh(\frac{\sqrt{B^2-4AC}}{4}\xi)+2\sqrt{B^2-4AC}\cosh^2(\frac{\sqrt{B^2-4AC}}{4}\xi)-\sqrt{B^2-4AC}}\right]. \tag{72}$$

Family 2 : When $B^2-4AC<0$ and $BC\neq 0$ (or $AC\neq 0$)

$$q(x,t)=a_0+a_1\frac{1}{2C}\times\left[-B+\sqrt{4AC-B^2}\tan\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right)\right]. \tag{73}$$

$$q(x,t)=a_0-a_1\frac{1}{2C}\times\left[B+\sqrt{4AC-B^2}\cot\left(\frac{\sqrt{4AC-B^2}}{2}\xi\right)\right]. \tag{74}$$

$$q(x,t)=a_0+a_1\frac{1}{2C}\left[-B+\sqrt{4AC-B^2}\left(\tan\left(\sqrt{4AC-B^2}\xi\right)\pm\sec\left(\sqrt{4\mathrm{AC}-\mathrm{B}^2}\xi\right)\right)\right]. \tag{75}$$

$$q(x,t)=a_0-a_1\frac{1}{2C}\left[B+\sqrt{4AC-B^2}\left(\cot\left(\sqrt{4AC-B^2}\xi\right)\pm\csc\left(\sqrt{4AC-B^2}\xi\right)\right)\right]. \tag{76}$$

$$q(x,t)=a_0-a_1\times\frac{1}{4C}\left[-2B+\sqrt{4AC-B^2}\left(\tan\left(\frac{\sqrt{4AC-B^2}}{4}\xi\right)-\cot\left(\frac{\sqrt{4AC-B^2}}{4}\xi\right)\right)\right]. \tag{77}$$

$$q(x,t)=a_0+a_1\frac{1}{2C}\left[-B+\frac{\pm\sqrt{(F^2-E^2)(4AC-B^2)}-E\sqrt{4AC-B^2}\cos(\sqrt{4AC-B^2}\xi)}{E\sin(\sqrt{4\mathrm{AC}-\mathrm{B}^2}\xi)+\mathrm{F}}\right]. \tag{78}$$

$$q(x,t)=a_0+a_1\frac{1}{2C}\left[-B+\frac{\pm\sqrt{(F^2-E^2)(4AC-B^2)}+E\sqrt{4AC-B^2}\sinh(\sqrt{4AC-B^2}\xi)}{E\cos(\sqrt{4AC-B^2}\xi)+F}\right], \tag{79}$$

where E and F are two non-zero real constants and satisfies $F^2-E^2>0$.

$$q(x,t)=a_0+a_1\left[\frac{-2A\cos(\frac{\sqrt{4AC-B^2}}{2}\xi)}{\sqrt{4AC-B^2}\sin(\frac{\sqrt{4AC-B^2}}{2}\xi)+B\cos(\frac{\sqrt{4AC-B^2}}{2}\xi)}\right]. \tag{80}$$

$$q(x,t)=a_0+a_1\times\left[\frac{2A\sin(\frac{\sqrt{4AC-B^2}}{2}\xi)}{\sqrt{4AC-B^2}\cos(\frac{\sqrt{4AC-B^2}}{2}\xi)-B\sin(\frac{\sqrt{4AC-B^2}}{2}\xi)}\right]. \tag{81}$$

$$q(x,t)=a_0+a_1\left[\frac{-2A\cos(\sqrt{4AC-B^2}\xi)}{\sqrt{4AC-B^2}\sin(\sqrt{4AC-B^2}\xi)+B\cos(\sqrt{4AC-B^2}\xi)\pm i\sqrt{4AC-B^2}\xi}\right]. \tag{82}$$

$$q(x,t) = a_0 + a_1\left[\frac{2A\sin(\sqrt{4AC-B^2}\xi)}{\sqrt{4AC-B^2}\cos(\sqrt{4AC-B^2}\xi) - B\sin(\sqrt{4AC-B^2}\xi) \pm \sqrt{4AC-B^2}\xi}\right]. \tag{83}$$

$$q(x,t) = a_0 + a_1\left[\frac{4A\sin(\frac{\sqrt{4AC-B^2}}{4}\xi)\cos(\frac{\sqrt{4AC-B^2}}{4}\xi)}{-2B\sin(\frac{\sqrt{4AC-B^2}}{4}\xi)\cos(\frac{\sqrt{4AC-B^2}}{4}\xi) + 2\sqrt{4AC-B^2}\cos^2(\frac{\sqrt{4AC-B^2}}{4}\xi) - \sqrt{4AC-B^2}}\right]. \tag{84}$$

Family 3 : When $A = 0$ and $BC \neq 0$

$$q(x,t) = a_0 + a_1\left(\frac{-Bd}{B(d+\cosh(B\xi) - \sinh(B\xi))}\right), \tag{85}$$

where d is an arbitrary constant.

Family 4 : When $A = B = 0$ and $C \neq 0$

$$q(x,t) = a_0 + a_1\left(\frac{-1}{B(\xi)+k}\right), \tag{86}$$

where k is an arbitrary constant.

5.2. 3. V_4 (Scalar group-invariant solutions)

For the generator V_3, we have

$$q = t^{-\frac{1}{3}} f(\xi), \tag{87}$$

where $\xi = xt^{-\frac{1}{3}}$ is the group-invariant. Substitution of (87) into the (1), we get

$$-\frac{1}{3}\xi f' - \frac{1}{3}f' + a(f')^2 + bf''' = 0, \tag{88}$$

where $f' = \frac{\mathrm{d}f}{\mathrm{d}\xi}$.

We note that the reduced equations (88) are nonautonomous ODE. In general, it is difficult to get the exact solutions. However, the power series method [17,18] is an effective tool for dealing with such nonautonomous ODE.

Now, we seek a solution of Eq. (88) in a power series of the following form:

$$f(\xi) = \sum_{n=0}^{\infty} c_n \xi^n. \tag{89}$$

Substituting (89) into (88), we get

$$\begin{aligned} &6bc_3 + b\sum_{n=1}^{\infty}(n+1)(n+2)(n+3)c_{n+3}\xi^n \\ &-\frac{1}{3}c_1 - \frac{1}{3}\sum_{n=1}^{\infty}(n+1)c_{n+1}\xi^n \\ &-\frac{1}{3}\sum_{n=1}^{\infty} nc_n\xi^n + ac_1c_1 \\ &+a\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(n-k+2)c_kc_{n-k+2}\right)\xi^n = 0. \end{aligned} \tag{90}$$

Now from (94), comparing coefficients, for $n = 0$, one can get

$$c_3 = \frac{\frac{1}{3}c_1 - ac_1^2}{6b}. \tag{91}$$

Generally, for $n \geq 1$, we obtain

$$\begin{aligned} c_{n+3} = \frac{1}{b(n+1)(n+2)(n+3)}\Bigg(&\frac{1}{3}(n+1)c_{n+1} \\ &-\frac{1}{3}nc_n - a\sum_{k=1}^{n}(n-k+2)c_kc_{n-k+2}\Bigg). \end{aligned} \tag{92}$$

From (91) and (92), one can get all the coefficients $c_n (n \geq 5)$ of the power series (89). For arbitrary chosen constant numbers c_0, c_1, and c_2, the other terms also can be determined successively from (91) and (92) in a unique way. In addition, it is easy to prove that the convergence of the power series (89) with the coefficients given by (91) and (92) [17,18]. we omit it here. Thus this power series solution is an exact analytic solution.

Therefore, the power series solution of Eq. (87) can be written as follows:

$$
\begin{aligned}
f(\xi) &= c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + \sum_{n=1}^{\infty} c_{n+3}\xi^{n+3} \\
&= c_0 + c_1\xi + c_2\xi^2 + \frac{\frac{1}{3}c_1 - ac_1^2}{6b}\xi^3 \\
&\quad + \frac{1}{b(n+1)(n+2)(n+3)}\left(\frac{1}{3}(n+1)c_{n+1}\right. \\
&\quad \left. -\frac{1}{3}nc_n - a\sum_{k=1}^{n}(n-k+2)c_k c_{n-k+2}\right)\xi^{n+3}.
\end{aligned} \tag{93}
$$

Therefore, we get the solution of Eq. (1)

$$
\begin{aligned}
&q(x,t) \\
&= \left[c_0 + c_1 xt^{-\frac{1}{3}} + c_2(xt^{-\frac{1}{3}})^2 + c_3(xt^{-\frac{1}{3}})^3 + \sum_{n=1}^{\infty} c_{n+3}\xi^{n+3}\right. \\
&= c_0 + c_1 xt^{-\frac{1}{3}} + c_2(xt^{-\frac{1}{3}})^2 + \frac{\frac{1}{3}c_1 - ac_1^2}{6b}(xt^{-\frac{1}{3}})^3 \\
&\quad + \frac{1}{b(n+1)(n+2)(n+3)}\left(\frac{1}{3}(n+1)c_{n+1}\right. \\
&\quad \left.\left. -\frac{1}{3}nc_n - a\sum_{k=1}^{n}(n-k+2)c_k c_{n-k+2}\right)(xt^{-\frac{1}{3}})^{n+3}\right]t^{-\frac{1}{3}}.
\end{aligned} \tag{94}
$$

Of course, in physical applications, it will be written in the approximate form

$$
q(x,t) = \left(c_0 + c_1 xt^{-\frac{1}{3}} + c_2(xt^{-\frac{1}{3}})^2 + c_3(xt^{-\frac{1}{3}})^3 + \cdots\right)t^{-\frac{1}{3}}. \tag{95}
$$

Remark 1 The exact solution of the (58) and (60) can be derived in the same way. Here, we do not list them for simplicity.

Remark 2 Through the above discussion, it is easily seen that the power series method [17,18] is very effective for higher-order nonlinear or nonautonomous DEs. To the best of our knowledge, the solutions obtained in this paper have not been reported in previous literature. Thus, these solutions are new solutions of (1). And above all, these power series play an important role in the investigation of physical phenomena and other natural phenomenon.

6 Conclusions

This paper is essentially a continuation of an earlier published paper on pKdV equation where the mapping method and Lie symmetry analysis were predominantly the integration architectures [3]. This paper used a set of different integration tools, namely the G'/G-expansion method, the exp-function method, and Lie symmetry analysis in order to retrieve additional set of solutions to the pKdV equation. The singular soliton solution was also obtained by the ansatz method. All these results are very concrete and consequently stand on a strong footing to launch further and deeper investigation into this equation, possibly in presence of perturbation terms.

The soliton perturbation theory will be developed for this equation, in future, and consequently the adiabatic parameter dynamics will be displayed. In addition, in future, the stochastic perturbation term will be included and thus mean free velocity of the soliton will be determined by formulating the corresponding Langevin equation. In addition, the perturbed pKdV equation will be integrated using the same tools or perhaps different tools in order to extract further information. This is just a foot in the door.

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