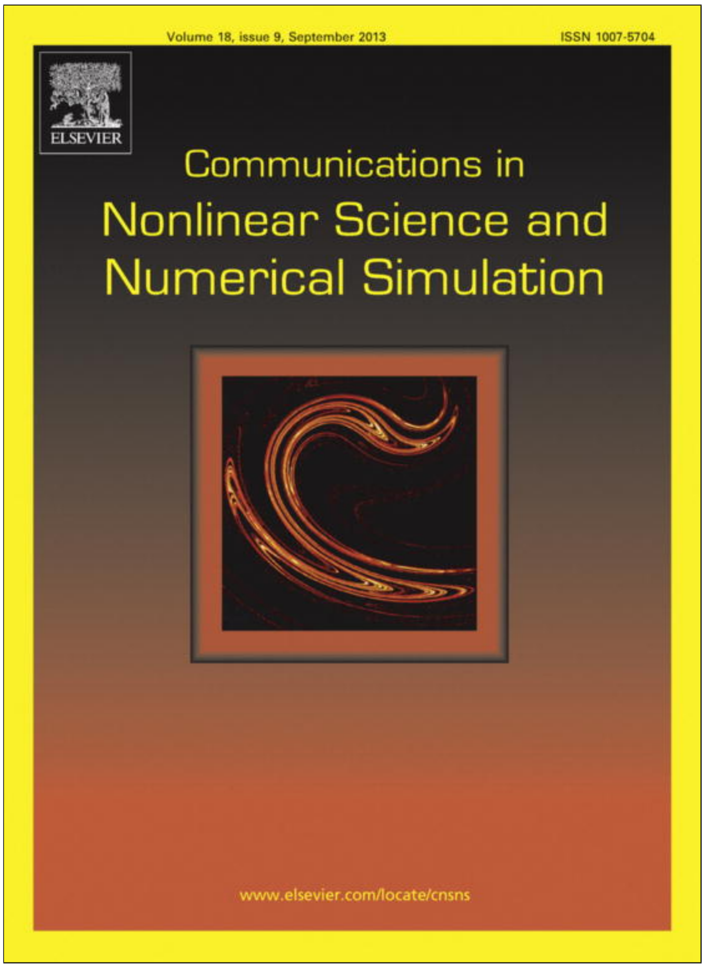
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Symmetry reduction, exact solutions and conservation laws of a new fifth-order nonlinear integrable equation

Gang-wei Wang, Xi-qiang Liu *, Ying-yuan Zhang

School of Mathematics Sciences, Liaocheng University, Liaocheng, Shandong, PR China

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ABSTRACT

In this paper, by applying Lie symmetry method, we get the corresponding Lie algebra and similarity reductions of a new fifth-order nonlinear integrable equation. At the same time, the explicit and exact analytic solutions are obtained by means of the power series method. At last, we also give the conservation laws.

1. Introduction

Symmetry plays a very important role in various fields of nature, especially in integrable systems for the existence of infinitely many symmetries. The task of finding an increasing number of solutions of systems of PDEs is related to the group properties of these differential equations. In order to find the Lie point symmetry of a nonlinear equation, various effective methods have been proposed, such as the classical Lie group method, the nonclassical Lie group method [1–5], and so on. Lie's method is an effective and the simplest method among group theoretic techniques and a large number of equations [6–18] are solved with the aid of this method.

In this paper, by means of the Lie symmetry group method, we will consider the following new fifth-order equation [19]

$$u_{ttt} - u_{txxxx} - 4(u_x u_t)_{xx} - 4(u_x u_{xt})_x = 0, \tag{1.1}$$

where $u = u(x,t)$. In [19], the author examined a new fifth-order nonlinear evolution of the integrability and obtained some multiple soliton solutions by using Hereman–Nuseri method.

Our aim in the present work is to perform the new fifth-order equation with the help of Lie's method. Then we get the corresponding Lie algebra, similarity reductions, conservation laws and new exact solutions.

2. Lie symmetry analysis of (1.1)

In this section, we will perform Lie group method for Eq. (1.1).

If (1.1) is invariant under a one parameter Lie group of point transformations

$$t^* = t + \epsilon\tau(x,t,u) + O(\epsilon^2), \quad x^* = x + \epsilon\xi(x,t,u) + O(\epsilon^2), \quad u^* = u + \epsilon\eta(x,t,u) + O(\epsilon^2), \tag{2.1}$$

* Corresponding author. Tel.: +86 15965256591.
E-mail address: liuxiq@sina.com (X.-Q. Liu).

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with infinitesimal generator

$$V = \tau(x,t,u)\frac{\partial}{\partial t} + \xi(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{2.2}$$

If the vector field (2.2) generates a symmetry of (1.1), then V must satisfy Lie's symmetry condition

$$pr^{(5)}V(\Delta_1)|_{\Delta_1=0} = 0,$$

where $\Delta_1 = u_{ttt} - u_{txxxx} - 4(u_x u_t)_{xx} - 4(u_x u_{xt})_x$. Applying the fifth prolongation $pr^{(5)}V$ to Eq. (1.1), we find the following system of symmetry equations then the invariant condition reads as

$$\eta^{ttt} - \eta^{txxxx} - 4\eta^t u_{xxx} - 4\eta^{xxx}u_t - 12\eta^{tx}u_{xx} - 12\eta^{xx}u_{tx} - 8\eta^x u_{txx} - 8\eta^{txx}u_x = 0, \tag{2.3}$$

where

$$\begin{aligned}
&\eta^t = D_t(\eta) - u_x D_t(\xi) - u_t D_t(\tau),\\
&\eta^x = D_x(\eta) - u_x D_x(\xi) - u_t D_x(\tau),\\
&\eta^{xx} = D_x(\eta^x) - u_{xt}D_x(\tau) - u_{xx}D_x(\xi),\\
&\eta^{xt} = D_t(\eta^x) - u_{xt}D_t(\tau) - u_{xx}D_t(\xi),\\
&\eta^{xxx} = D_x(\eta^{xx}) - u_{xxt}D_x(\tau) - u_{xxx}D_x(\xi),\\
&\eta^{xxt} = D_t(\eta^{xx}) - u_{xxt}D_t(\tau) - u_{xxx}D_t(\xi),\\
&\eta^{xxxx} = D_x(\eta^{xxx}) - u_{xxxt}D_x(\tau) - u_{xxxx}D_x(\xi),\\
&\eta^{txxxx} = D_t(\eta^{xxxx}) - u_{xxxxt}D_t(\tau) - u_{xxxxx}D_t(\xi).
\end{aligned} \tag{2.4}$$

Here, D_i denotes the total derivative operator and is defined by

$$D_i = \frac{\partial}{\partial x^i} + u_i\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + \cdots, \quad i = 1,2$$

and $(x^1, x^2) = (t, x)$.

Solving (2.3) along with (2.4), one can get

$$\tau = 2c_1 t + c_2, \quad \xi = c_1 x + c_3, \quad \eta = -(c_1 u + c_4), \tag{2.5}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants.

We obtain the corresponding four-dimensional Lie algebra

$$V_1 = \frac{\partial}{\partial t}, \quad V_2 = \frac{\partial}{\partial x}, \quad V_3 = \frac{\partial}{\partial u}, \quad V_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}.$$

Their commutator table is given Table 1.

To obtain the group transformation which is generated by the infinitesimal generators V_i for $i = 1,2,3,4$, we need to solve the following initial problems

$$\begin{aligned}
&\frac{d(\bar{x}(\varepsilon))}{d\varepsilon} = \xi(\bar{x}(\varepsilon), \bar{t}(\varepsilon), \bar{u}(\varepsilon)), \quad \bar{x}(0) = x,\\
&\frac{d(\bar{t}(\varepsilon))}{d\varepsilon} = \tau(\bar{x}(\varepsilon), \bar{t}(\varepsilon), \bar{u}(\varepsilon)), \quad \bar{t}(0) = t,\\
&\frac{d(\bar{u}(\varepsilon))}{d\varepsilon} = \eta(\bar{x}(\varepsilon), \bar{t}(\varepsilon), \bar{u}(\varepsilon)), \quad \bar{u}(0) = u,
\end{aligned} \tag{2.6}$$

where ε is a parameter. So we can obtain the Lie symmetry group

$$g : (x,t,u) \to (\bar{x}, \bar{t}, \bar{u}). \tag{2.7}$$

Exponentiating the infinitesimal symmetries of Eq. (2.1), we get the one-parameter groups $g_i(\varepsilon)$ generated by V_i for $i = 1,2,3,4$

$$\begin{aligned}
&g_1 : (x,t,u) \mapsto (x, t+\varepsilon, u),\\
&g_2 : (x,t,u) \mapsto (x+\varepsilon, t, u),\\
&g_3 : (x,t,u) \mapsto (x, t, u+\varepsilon),\\
&g_4 : (x,t,u) \mapsto (e^{-\varepsilon}x, e^{-2\varepsilon}t, e^{\varepsilon}u).
\end{aligned} \tag{2.8}$$

The symmetry groups g_1 and g_2 demonstrate the time- and space-invariance of the equation. The well-known scaling symmetry turns up in g_4. Consequently, we can obtain the corresponding Theorem:

Table 1
Commutator table of the Lie algebra of Eq. (1.1).

$[V_i, V_j]$	V_1	V_2	V_3	V_4
V_1	0	0	0	$2V_1$
V_2	0	0	0	V_2
V_3	0	0	0	$-V_3$
V_4	$-2V_1$	$-V_2$	V_3	0

Theorem 1. *If u = f (x,t) is a solution of fifth-order Eq. (1.1), so are the functions*

$$\begin{aligned} &g_1(\varepsilon)\cdot f(x,t) = f(x-\varepsilon,t),\\ &g_2(\varepsilon)\cdot f(x,t) = f(x,t-\varepsilon),\\ &g_3(\varepsilon)\cdot f(x,t) = f(x,t)-\varepsilon,\\ &g_4(\varepsilon)\cdot f(x,t) = \mathrm{e}^{-\varepsilon} f(\mathrm{e}^{-\varepsilon}x, \mathrm{e}^{-2\varepsilon}t). \end{aligned} \tag{2.9}$$

If taking the following single-soliton solution (see Fig. 1) [17] of Eq. (1.1)

$$u = \frac{k_1 \mathrm{e}^{k_1 x \pm k_1^2 t}}{1 + k_1 \mathrm{e}^{k_1 x \pm k_1^2 t}}. \tag{2.10}$$

One can obtain new exact solution of Eq. (1.1) by applying g_1 as follows

$$u = \frac{k_1 \mathrm{e}^{k_1 (x-\varepsilon) \pm k_1^2 t}}{1 + k_1 \mathrm{e}^{k_1 (x-\varepsilon) \pm k_1^2 t}}. \tag{2.11}$$

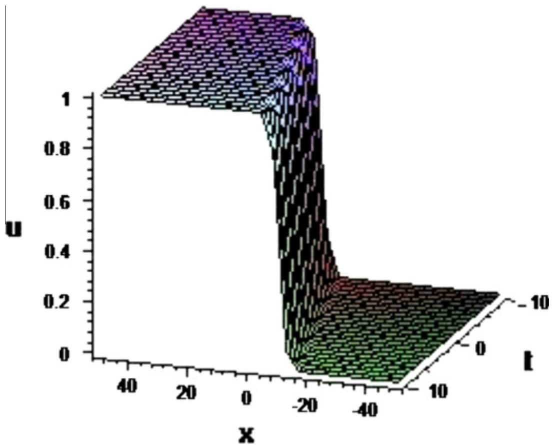

Fig. 1. The special curve solution of (2.10) for $k = 1,\ u = \frac{e^{x+2t}}{1+e^{x+2t}}$.

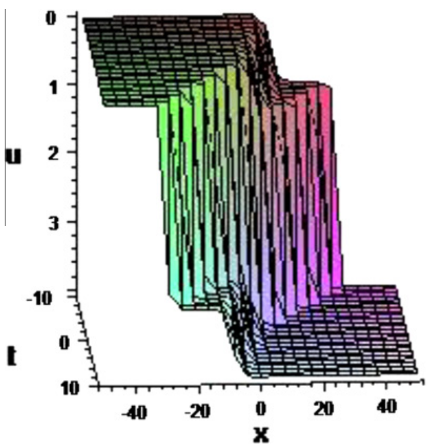

Fig. 2. The special curve solution of (2.12).

If taking the following two-soliton solution (see Fig. 2)

$$u = \frac{5e^{x+t} + 15e^{3x+9t} + 8e^{4x+10t}}{5 + 5e^{x+t} + 5e^{3x+9t} + 2e^{4x+10t}}. \tag{2.12}$$

One can get new exact solution of Eq. (1.1) by applying g_2 as follows

$$u = \frac{5e^{x+t-\varepsilon} + 15e^{3x+9(t-\varepsilon)} + 8e^{4x+10(t-\varepsilon)}}{5 + 5e^{x+t-\varepsilon} + 5e^{3x+9(t-\varepsilon)} + 2e^{4x+10(t-\varepsilon)}}, \tag{2.13}$$

where k_1 and ε are arbitrary constants.

By selecting the arbitrary constants, one can obtain many solutions.

Remark 1. A lot of new invariant solutions can be found through given solutions [19] for (1.1). Thus we generalize the corresponding results in [19].

3. Generalized symmetries for (1.1)

In Section 2, we have got the symmetry group of Eq. (1.1). Now we consider the symmetries of Eq. (1.1) by using the generalized symmetry method. This method is also called the method of undetermined coefficient [20].

For a nonlinear evolution equation

$$F(t, x, y, u, u_t, u_x, u_y, \ldots) = 0, \tag{3.1}$$

namely a function σ is a symmetry of Eq. (3.1), if it satisfies

$$F'(u)\sigma = 0, \tag{3.2}$$

for all solutions u, where

$$F'(u)\sigma = \frac{\partial F}{\partial u}\sigma + \frac{\partial F}{\partial u_t}\sigma_t + \frac{\partial F}{\partial u_x}\sigma_x + \frac{\partial F}{\partial u_y}\sigma_y + \frac{\partial F}{\partial u_{tx}}\sigma_{tx} + \frac{\partial F}{\partial u_{xx}}\sigma_{xx} + \cdots$$

From (3.2), the symmetry equation of Eq. (1.1) satisfies

$$\sigma_{ttt} - \sigma_{txxxx} - 4\sigma_t u_{xxx} - 4\sigma_{xxx}u_t - 12\sigma_{tx}u_{xx} - 12\sigma_{xx}u_{tx} - 8\sigma_x u_{txx} - 8\sigma_{txx}u_x = 0. \tag{3.3}$$

To solve Eq. (3.3), let

$$\sigma = \alpha u_t + \beta u_x + eu + \gamma, \tag{3.4}$$

where α, β, e, and γ are functions of $t,\ x$ to be determined.

Substituting (3.4) into (3.3) with the help of (1.1), one can get the associated determining equations and solve them, we obtain

$$\alpha = 2c_1t + c_2, \quad \beta = c_1x + c_3, \quad e = c_1, \quad \gamma = c_4, \tag{3.5}$$

where $c_i\ (i = 1, 2, \ldots, 4)$ are arbitrary constants. Substituting (3.5) into (3.4), we get

$$\sigma = (c_1x + c_3)u_x + (2c_1t + c_2)u_t + c_1u + c_4. \tag{3.6}$$

So we can get

$$\sigma = c_1(u_x + 2tu_t + u) + c_2u_t + c_3u_x + c_4. \tag{3.7}$$

The equivalent vector expression of the above symmetry is

$$V = (c_1x + c_3)\frac{\partial}{\partial x} + (2c_1t + c_2)\frac{\partial}{\partial t} - (c_1u + c_4)\frac{\partial}{\partial u},$$

which coincide precisely with the vector field $V_i\ (i = 1, 2, \ldots, 4)$ are obtained in Section 2.

4. Symmetry reductions and exact group-invariant solutions

In order to obtain similarity reductions and solutions of Eq. (1.1), one first solves the equation $\sigma = 0$, to obtain invariant transformations and then substitutes these results into Eq. (1.1) to determine the corresponding reduced equations. Finally, similarity solutions can be obtained. We write the characteristic equation in the form

$$\frac{dx}{c_1x + c_3} = \frac{dt}{2c_1t + c_2} = \frac{du}{-c_1u - c_4}.$$

Here we discuss the following cases:

4.1. V_1

The group-invariant solution corresponding to V_1 is $u = f(\xi)$, where $\xi = x$ is the group-invariant, the substitution of this solution into Eq. (1.1) gives the trivial solution $u(x,t) = C, C$ is a constant.

4.2. V_2

The group-invariant solution corresponding to V_2 is $u = f(\xi)$, where $\xi = t$ is the group-invariant, the substitution of this solution into Eq. (1.1) gives the trivial solution $u(x,t) = C_1t^2 + 2C_2t + C_3, C_1, C_2, C_3$ are constants.

4.3. $V_1 + \lambda V_2$

For the linear combination $V_1 + \lambda V_2$, we have

$$u = f(\xi), \tag{4.1}$$

where $\xi = x - \lambda t$ is the group-invariant. Substitution of (4.1) into the (1.1), we reduce it to the following ODE

$$f^{(5)} + 12f'f''' + 12(f'')^2 - \lambda^2 f''' = 0, \tag{4.2}$$

4.4. V_4

For the generator V_4, we obtain

$$u = t^{-\frac{1}{2}}f(\xi), \tag{4.3}$$

where $\xi = xt^{-\frac{1}{2}}$ is the group-invariant. Substituting (4.3) into (1.1), we reduce it to the following ODE

$$\frac{1}{2}\xi f^{(5)} + \frac{15}{8}f - \frac{33}{8}\xi f - \frac{3}{2}\xi^2 f'' - \frac{1}{8}\xi^3 f''' + \frac{5}{2}\xi f'''' + 2f'''f + 6\xi f'''f' + 24f''f' + 6\xi(f'')^2 = 0, \tag{4.4}$$

where $f' = \frac{df}{d\xi}$.

5. The exact power series solutions

In this section, we will give the exact analytic solutions to the reduced Eq. (4.2) by using the power series method. Now, we seek a solution of Eq. (4.2) in a power series of the followings form

$$f(\xi) = \sum_{n=0}^{\infty} c_n \xi^n. \tag{5.1}$$

Substituting (5.1) into (4.2), we get

$$\begin{aligned}
&120c_5 + \sum_{n=1}^{\infty}(n+1)(n+2)(n+3)(n+4)(n+5)c_{n+5}\xi^n + 72c_1c_3 \\
&\quad + 12\sum_{n=1}^{\infty}\sum_{k=0}^{n}(k+1)(n+1-k)(n+2-k)(n+3-k)c_{k+1}c_{n+3-k}\xi^n + 48c_2^2 \\
&\quad + 12\sum_{n=1}^{\infty}\sum_{k=1}^{n}(k+1)(n+2-k)(n+3-k)c_{k+1}c_{n+3-k}\xi^n - 6\lambda^2 c_3 - \lambda^2\sum_{n=1}^{\infty}(n+3)(n+2)(n+1)\xi^n = 0.
\end{aligned} \tag{5.2}$$

Now from (5.2), comparing coefficients, for $n = 0$, one can get

$$c_5 = \frac{\lambda^2 c_3 - 12c_1c_3 - 8c_2^2}{20}. \tag{5.3}$$

Generally, for $n \geqslant 1$, we obtain

$$\begin{aligned}
c_{n+5} = \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)}\Bigg(&\lambda^2(n+3)(n+2)(n+1) \\
&-12\sum_{k=0}^{n}(k+1)(n+1-k)(n+2-k)(n+3-k)c_{k+1}c_{n+3-k} - \sum_{k=1}^{n}(k+1)(n+2-k)(n+3-k)c_{k+1}c_{n+3-k}\Bigg).
\end{aligned} \tag{5.4}$$

From (5.3) and (5.4), one can get all the coefficients $c_n (n \geqslant 5)$ of the power series (5.1). For arbitrary chosen constant numbers c_0, c_1, c_2, c_3 and c_4, the other terms also can be determined successively from (5.3) and (5.4) in a unique way. In addition, it is easy to prove that the convergence of the power series (5.1) with the coefficients give by (5.3) and (5.4) [17,18]. we omit it here. Thus this power series solution is an exact analytic solution.

Thus, the power series solution of Eq. (4.2) can be written as following

$$\begin{aligned} f(\xi) &= c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + c_4\xi^4 + c_5\xi^5 + \sum_{n=1}^{\infty} c_{n+5}\xi^{n+5} \\ &= c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + c_4\xi^4 + \frac{\lambda^2 c_3 - 12c_1c_3 - 8c_2^2}{20}\xi^5 \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)} \Big(\lambda^2(n+3)(n+2)(n+1) \\ &\quad - 12\sum_{k=0}^{n}(k+1)(n+1-k)(n+2-k)(n+3-k)c_{k+1}c_{n+3-k} - \sum_{k=1}^{n}(k+1)(n+2-k)(n+3-k)c_{k+1}c_{n+3-k}\Big)\xi^{n+5}. \end{aligned}$$

Thus, the exact power series solution of Eq. (1.1) is

$$\begin{aligned} u(x,t) &= \left[c_0 + c_1(x-\lambda t) + c_2(x-\lambda t)^2 + c_3(x-\lambda t)^3 + c_4(x-\lambda t)^4 + \frac{\lambda^2 c_3 - 12c_1c_3 - 8c_2^2}{20}(x-\lambda t)^5 + \sum_{n=1}^{\infty} c_{n+5}(x-\lambda t)^{n+5}\right] \\ &= \Bigg[c_0 + c_1(x-\lambda t) + c_2(x-\lambda t)^2 + c_3(x-\lambda t)^3 + c_4(x-\lambda t)^4 + \frac{\lambda^2 c_3 - 12c_1c_3 - 8c_2^2}{20}(x-\lambda t)^5 \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)} \Bigg(\lambda^2(n+3)(n+2)(n+1) \\ &\quad - 12\sum_{k=0}^{n}(k+1)(n+1-k)(n+2-k)(n+3-k)c_{k+1}c_{n+3-k} - \sum_{k=1}^{n}(k+1)(n+2-k)(n+3-k)c_{k+1}c_{n+3-k}\Bigg)(x-\lambda t)^{n+5}\Bigg], \end{aligned}$$

where c_i $(i = 0, 1, 2, 3, 4)$ are arbitrary constants, the other coefficients $c_n(n \geqslant 5)$ can be determined successively from (5.3) and (5.4).

Of course, in physical applications, it will be write the approximate form

$$u(x,t) = c_0 + c_1(x-\lambda t) + c_2(x-\lambda t)^2 + c_3(x-\lambda t)^3 + c_4(x-\lambda t)^4 + \frac{\lambda^2 c_3 - 12c_1c_3 - 8c_2^2}{20}(x-\lambda t)^5 + \cdots.$$

Remark 2. The exact solution of (4.4) can be derived in the same way. Here we do not list them for simplicity.

Remark 3. From the above procedure, it is easy to see that the power series method [16–18] is very powerful for higher-order nonlinear or nonautonomous DEs. To the best of our knowledge, the solutions obtained in this paper have not been reported in previous literature. Thus, these solutions are new.

6. Conservation laws of (1.1)

In this section, in order to get conservation laws of (1.1), we briefly present the main notations and theorems used in this paper.

6.1. Preliminaries

Consider a sth-order nonlinear evolution equation

$$F(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(s)}) = 0, \tag{6.1}$$

with n independent variables $x = (x_1, x_2, \ldots, x_n)$ and a dependent variable u, where $u_{(1)}, u_{(2)}, \ldots, u_{(s)}$ denote the collection of all first, second, $\ldots$, sth-order partial derivatives. $u_i = D_i(u)$, $u_{ij} = D_jD_i(u), \ldots$ Here

$$D_i = \frac{\partial}{\partial x_i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, 2, \ldots, n$$

is the total differential operator with respect to x_i.

Definition 1 (*See Ref. [21]*). The adjoint equation of Eq. (6.1) is defined by

$$F^*(x, u, v, u_{(1)}, v_{(1)}, u_{(2)}, v_{(2)}, \ldots, u_{(s)}, v_{(s)}) = 0, \tag{6.2}$$

with

$$F^*(x,u,v,u_{(1)},v_{(1)},u_{(2)},v_{(2)},\dots,u_{(s)},v_{(s)}) = \frac{\delta(vF)}{\delta u},$$

where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{m=1}^{\infty}(-1)^m D_{i_1}\cdots D_{i_m}\frac{\partial}{\partial u_{i_1 i_2\cdots i_m}},$$

denotes the Euler–Lagrange operator, v is a new dependent variable, $v = v(x)$.

Theorem 2 (*See Ref. [21]*). *The system consisting of Eq. (6.1) and its adjoint Eq. (6.2)*

$$\begin{cases} F(x,u,u_{(1)},u_{(2)},\dots,u_{(s)}) = 0, \\ F^*(x,u,v,u_{(1)},v_{(1)},u_{(2)},v_{(2)},\dots,u_{(s)},v_{(s)}) = 0, \end{cases} \tag{6.3}$$

has a formal Lagrangian, namely

$$L = vF(x,u,u_{(1)},u_{(2)},\dots,u_{(s)}). \tag{6.4}$$

In the following we recall the "new conservation theorem" given by Ibragimov in Ref. [22].

Theorem 3. *Every Lie point, Lie–Bäcklund and non-local symmetry of Eq. (1.1) provides a conservation law for Eq. (1.1) and the adjoint equation. Then the elements of conservation vector (C^1, C^2) are defined by the following expression*

$$C^i = \xi^i L + W^\alpha\left[\frac{\partial L}{\partial u_i^\alpha} - D_j\left(\frac{\partial L}{\partial u_{ij}^\alpha}\right) + D_jD_k\left(\frac{\partial L}{\partial u_{ijk}^\alpha}\right)\right] + D_j(W^\alpha)\left[\left(\frac{\partial L}{\partial u_{ij}^\alpha}\right) - D_k\left(\frac{\partial L}{\partial u_{ijk}^\alpha}\right) + \cdots\right], \tag{6.5}$$

where $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$.

6.2. Conservation laws of (1.1)

Now, we will study the conservation laws by using the adjoint equation and symmetries of (1.1). For (1.1), the adjoint equation has the form

$$F = -v_{ttt} + v_{txxxx} + 4u_{xxx}v_t + 4u_t v_{xxx} - 12u_{xx}v_{xt} - 12u_{tx}v_{xx} + 8u_{txx}v_x + 8u_x v_{txx} \tag{6.6}$$

and the Lagrangian in the symmetrized form

$$L = v(u_{ttt} - u_{txxxx} - 4u_{xxx}u_t - 12u_{xx}u_{xt} - 8u_x u_{txx}). \tag{6.7}$$

On the basis of Theorem 2, the conserved vector corresponding to an operator

$$V = \xi^1(x,t,u)\frac{\partial}{\partial t} + \xi^2(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial u}.$$

The operator V yields the conservation law $D_t(C^1) + D_x(C^2) = 0$, where the conserved vector $C = (C^1, C^2)$ is given by (6.5) and has the components

$$\begin{aligned} C_1 = {} & \xi^1 L + W[-4vu_{xxx} + D_x(12vu_{xx}) + D_{xx}(-8vu_x) + D_tD_t(v) + D_xD_xD_xD_x(-v)] \\ & + D_x(W)[-12u_{xx}v - D_x(-8u_x v) + D_xD_xD_x(-v)] + D_t(W)[-D_t v] + D_xD_x(W)[-8u_x v + D_xD_x(-v)] \\ & + D_tD_t(W)v + D_xD_xD_x(W)[-D_x(-v)] + D_xD_xD_xD_x(W)(-v), \end{aligned} \tag{6.8}$$

$$\begin{aligned} C_2 = {} & \xi^2 L + W[-8vu_{txx} - D_x(-12vu_{xt}) - D_t(-12vu_{xx}) + D_xD_x(-4vu_t) + D_xD_t(-8vu_x) - D_tD_xD_xD_x(-v)] \\ & + D_t(W)[-12vu_{xx} - D_x(-8u_x v) - D_xD_xD_x(-v)] + D_x(W)[-12u_{xt}v - D_x(-4u_t v) - D_t(-8vu_x)] \\ & + D_x(W)D_t(W)[-8u_x v + D_xD_x(-v)] + D_xD_x(W)(-4u_t v) + D_tD_xD_x(W)[-D_x(-v)] + D_tD_xD_xD_x(W)(-v). \end{aligned} \tag{6.9}$$

So, (6.8) and (6.9) define the corresponding components of a non-local conservation law for the system of (1.1) and (6.6) corresponding to any operator v admitted by (1.1).

Now, let us make calculations for the operator $v = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}$ in detail. For this operator, one can get $W = -u - (2tu_t + xu_x)$, we can get the conservation vector of Eq. (1.1)

$$\begin{aligned} C_1 = {} & 2tv(u_{ttt} - u_{txxxx} - 4u_{xxx}u_t - 12u_{xx}u_{xt} - 8u_x u_{txx}) + (-u - (2tu_t + xu_x))[-4vu_{xxx} + D_x(12vu_{xx}) \\ & + D_{xx}(-8vu_x) + D_tD_t(v) + D_xD_xD_xD_x(-v)] + D_x(-u - (2tu_t + xu_x))[-12u_{xx}v - D_x(-8u_x v) \\ & + D_xD_xD_x(-v)] + D_t(-u - (2tu_t + xu_x))[-D_t v] + D_xD_x(-u - (2tu_t + xu_x))[-8u_x v + D_xD_x(-v)] \\ & + D_tD_t(-u - (2tu_t + xu_x))v + D_xD_xD_x(-u - (2tu_t + xu_x))[-D_x(-v)] + D_xD_xD_xD_x(-u - (2tu_t + xu_x))(-v), \end{aligned} \tag{6.10}$$

$$\begin{aligned}C_2 = {} & xv(u_{ttt} - u_{txxxx} - 4u_{xxx}u_t - 12u_{xx}u_{xt} - 8u_x u_{txx}) + (-u - (2tu_t + xu_x))[-8vu_{txx} - D_x(-12vu_{xt}) \\ & -D_t(-12vu_{xx}) + D_xD_x(-4vu_t) + D_xD_t(-8vu_x) - D_tD_xD_xD_x(-v)] + D_t(-u - (2tu_t + xu_x))[-12vu_{xx} \\ & -D_x(-8u_xv) - D_xD_xD_x(-v)] + D_x(-u - (2tu_t + xu_x))[-12u_{xt}v - D_x(-4u_tv) - D_t(-8vu_x)] \\ & + D_x(-u - (2tu_t + xu_x))D_t(-u - (2tu_t + xu_x))[-8u_xv + D_xD_x(-v)] + D_xD_x(-u - (2tu_t + xu_x))(-4u_tv) \\ & + D_tD_xD_x(-u - (2tu_t + xu_x))[-D_x(-v)] + D_tD_xD_xD_x(-u - (2tu_t + xu_x))(-v).\end{aligned} \tag{6.11}$$

This vector involves an arbitrary solution v of the adjoint Eq. (6.6) and provides an infinite number of the conservation laws.

Remark 4. With the aid of Maple15, we have checked that the above vector (C^1, C^2) is the conservation vector of Eq. (1.1).

7. Conclusions

We have performed Lie symmetry analysis for a new fifth-order nonlinear integrable equation. Then the Lie algebra, similarity reductions and exact solutions are obtained. Moreover, the power series solution of the reduced equation are given simultaneously. These are new solutions for the new fifth-order nonlinear integrable equation. At last, we also give the conservation laws of the new fifth-order nonlinear integrable equation which have not been found in the existent papers until now. The symmetry analysis based on the Lie group method is a very powerful method and is worthy of studying further.

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