Counterexamples of the twins conjecture on chromatic symmetric functions

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Outline

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Chromatic symmetric function

Given a finite simple graph $G = (V, E)$, a proper coloring of G is a function κ from V to $\mathbb{P} = \{1, 2, \ldots\}$ such that $\kappa(u) \neq \kappa(v)$ whenever $uv \in E$.

Definition (Stanley, 1995)

Let $V = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. The chromatic symmetric function is defined by

$$X_G = X_G(x_1, x_2, \ldots) = \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)},$$

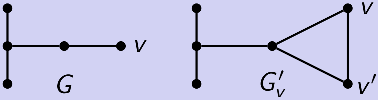
where the sum ranges over all proper colorings $\kappa : V \to \mathbb{P}$.

R.P. Stanley, A symmetric function generalization of the chromatic polynomial of a graph, Adv. Math., 111 (1995), pp. 166–194.

Twins conjecture

- Two vertices x and y are twins if they are adjacent and any vertex z is either adjacent to both x and y or non adjacent to both x and y.
- Given a graph G and a vertex v of G, define G'_v to be the graph obtained from G by adding a vertex v' that is a twin of v.

Example



Conjecture (Foley, Hoàng and Merkel, 2018)

If G is e-positive, then G'_v is e-positive for any vertex $v \in V$.

A.M. Foley, C.T. Hoàng and O.D. Merkel, Classes of graphs with *e*-positive chromatic symmetric function, Electron. J. Combin., 26 (2019), no. 3, Paper 3.51, 19 pp.

Introduction

Theorem (Hermosillo de la Maza, Jing and Masjoody, 2018)

A connected graph G is (claw, bull)-free graph if and only if it belongs to one of the following (disjoint) classes of graphs:

- *the class of graphs which are expansions of paths of length at least* 4,
- *the class of graphs which are expansions of cycles of length at least* 6,
- *the class of connected graphs which are complements of triangle-free graphs.*

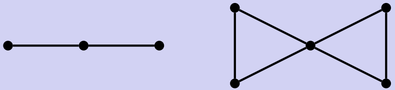
S.G. Hermosillo de la Maza, Y. Jing, M. Masjoody, On the structure of (claw, bull)-free graphs, arXiv:1901.00043.

Introduction

Definition

An expansion of a graph $G = (V, E)$ *with vertex set* $V = \{v_1, v_2, \ldots, v_n\}$ *is any graph* H *obtained from* G *by substituting its vertices with disjoint cliques* $K^{[i]}$, $i = 1, \ldots, n$, *and adding the edges of the complete bipartite graphs with the partite sets* $V(K^{[i]})$ *and* $V(K^{[j]})$ *for each* $v_i v_j \in E$.

Example



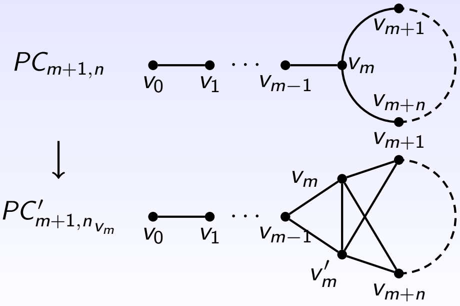
P_3 $\quad$ K_2 $\quad$ K_1 $\quad$ K_2

Counterexample

Define $PC_{m+1,n}$ to be the graph obtained from a path P_{m+1} and a cycle C_{n+1} by identifying one vertex of degree one of the path and one vertex of the cycle.

Theorem

For any $m \geq 0, n \geq 1$, we have $PC_{m+1,n}$ is e-positive. Moreover, for $n = 3$ and $m \geq 1$, we have $PC'_{m+1,3_{v_m}}$ is not e-positive.



Outline

Symmetric functions in noncommuting variables

- Let Π_d denote the lattice of set partitions π of $[d]$, ordered by refinement.
- Let $\{x_1, x_2, x_3, \dots\}$ be a set of noncommuting variables.

Definition

Let $\pi \in \Pi_d$. Define the monomial symmetric functions m_π by

$$m_\pi = \sum_{i_1, i_2, \dots, i_d} x_{i_1} x_{i_2} \dots x_{i_d},$$

where the sum is over all sequences $i_1, i_2, \dots, i_d$ of positive integers such that $i_j = i_k$ if and only if j and k are in the same block of π.

- The monomial symmetric functions, $\{m_\pi : \pi \in \Pi_d, d \in \mathbb{N}\}$, are linearly independent over $\mathbb{C}$, and we call their span the algebra of symmetric functions in noncommuting variables.

Symmetric functions in noncommuting variables

Definition

The elementary symmetric functions e_π is defined by

$$e_\pi = \sum_{\sigma:\sigma\wedge\pi=\hat{0}} m_\sigma = \sum_{i_1,i_2,\ldots,i_d} x_{i_1}x_{i_2}\ldots x_{i_d},$$

where the second sum is over all sequences $i_1, i_2, \ldots, i_d$ of positive integers such that $i_j \neq i_k$ if j and k are both in the same block of π.

- The set $\{e_\pi : \pi \in \Pi_d, d \in \mathbb{N}\}$ is a basis of the algebra of symmetric functions in noncommuting variables.
- For $\pi \in \Pi_d$ we define $\lambda(\pi) = (1^{r_1}2^{r_2}\ldots d^{r_d})$ to be the integer partition of d whose parts are the block sizes of π.
- Allowing the variables to commute transforms e_π into $1!^{r_1}2!^{r_2}\ldots d!^{r_d}e_{\lambda(\pi)}$.

Chromatic symmetric functions in noncommuting variables

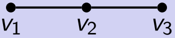
Definition (Gebhard and Sagan, 2001)

For any graph G with vertices labeled $v_1, v_2, \ldots, v_d$ in a fixed order, define

$$Y_G = \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_d)} = \sum_{\kappa} x_{\kappa},$$

where again the sum is over all proper colorings κ of G, but the x_i are now noncommuting variables.

Example



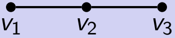
$$Y_{P_3} = m_{13/2} + m_{1/2/3}$$

- Y_G depends not only on G, but also on the labeling of its vertices.

D. Gebhard, B. Sagan, A chromatic symmetric function in noncommuting variables, J. Algebraic Combin. 2 (2001) 227 – 255.

Some e-positivity results

Example



$$Y_{P_3} = \frac{1}{2}e_{12/3} - \frac{1}{2}e_{13/2} + \frac{1}{2}e_{1/23} + \frac{1}{2}e_{123}$$

- Let $B_{\sigma,i}$ denote the block of σ containing i and let $B_{\tau,i}$ denote the block of τ containing i, we define

$$\sigma \equiv_i \tau \text{ iff } \lambda(\sigma) = \lambda(\tau) \text{ and } |B_{\sigma,i}| = |B_{\tau,i}|$$

and extend this definition so that

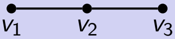
$$e_\sigma \equiv_i e_\tau \;\text{ iff } \sigma \equiv_i \tau.$$

- Let (τ) and $e_{(\tau)}$ denote the equivalence classes of τ and e_τ
- $\sum_{\sigma\in\Pi_d} c_\sigma e_\sigma \equiv_i \sum_{(\tau)\subseteq\Pi_d} c_{(\tau)} e_{(\tau)}$ where $c_{(\tau)} = \sum_{\sigma\in(\tau)} c_\sigma$.

Some e-positivity results

- We say that a labeled graph G (and similarly Y_G) is (e)-positive if all the $c_{(\tau)}$ are non-negative for some labeling of G and suitably chosen congruence.
- Notice that the expansion of Y_G for a labeled graph may have all non-negative amalgamated coefficients for congruence modulo i , but not for congruence modulo j.
- Clearly (e)-positivity results for Y_G specialize to e-positivity results for X_G.

Example



$$Y_{P_3} = \frac{1}{2}e_{12/3} - \frac{1}{2}e_{13/2} + \frac{1}{2}e_{1/23} + \frac{1}{2}e_{123}$$

$$Y_{P_3} \equiv_3 \frac{1}{2}e_{(12/3)} + \frac{1}{2}e_{(123)}$$

$$Y_{P_3} \equiv_2 e_{(12/3)} - \frac{1}{2}e_{(13/2)} + \frac{1}{2}e_{(123)}$$

The e-positivity of $PC_{m+1,n}$

Theorem (Gebhard and Sagan, 2001)

If Y_G is (e)-positive, then Y_{G+K_m} is also (e)-positive.

- Given any graph G with vertices $\{v_1, v_2, \ldots, v_d\}$, define $G + K_m$ to be the graph with

$$V(G+K_m) = V(G) \cup \{v_{d+1}, \ldots, v_{d+m-1}\}$$

and

$$E(G+K_m) = E(G) \cup \{e = v_i v_j : i, j \in [d, d+m-1]\}.$$

- For $\pi \in \Pi_d$, we let $\pi + i$ denote the partition given by π with the additional i elements $d+1, d+2, \ldots, d+i$ added to B_π.
- $\langle m \rangle_i \overset{def}{=} m(m-1)\ldots(m-i+1)$.
- $(m)_i \overset{def}{=} m(m+1)\ldots(m+i-1)$.

The e-positivity of $PC_{m+1,n}$

Lemma (Gebhard and Sagan, 2001)

If $m \geq 1$, *and*

$$Y_G \equiv_d \sum_{(\pi)\subseteq \Pi_d} c_{(\pi)} e_{(\pi)},$$

then

$$Y_{G+K_{m+1}} \equiv_{d+m} \sum_{(\pi)\subseteq \Pi_d} \sum_{i=0}^{m-1} \frac{c_{(\pi)} \langle m-1 \rangle_i}{(b)_{i+1}} [(b-m+i) e_{(\hat{\pi})} + (i+1) e_{(\bar{\pi})}]$$

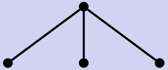
where $b = |B_\pi|$ *and*

$$\hat{\pi} = \pi + i/d + i + 1, \ldots, d+m,$$

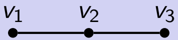
$$\bar{\pi} = \pi + i + (d+m)/d + i + 1, \ldots, d+m-1.$$

The e-positivity of $PC_{m+1,n}$

Example



$$X_{K_{1,3}} = 4e_{(4)} + 5e_{(3,1)} - 2e_{(2,2)} + e_{(2,1,1)}$$



$$Y_{P_3} = \frac{1}{2}e_{12/3} - \frac{1}{2}e_{13/2} + \frac{1}{2}e_{1/23} + \frac{1}{2}e_{123}$$

$$Y_{P_3} \equiv_3 \frac{1}{2}e_{(12/3)} + \frac{1}{2}e_{(123)}$$

$$Y_{P_3} \equiv_2 e_{(12/3)} - \frac{1}{2}e_{(13/2)} + \frac{1}{2}e_{(123)}$$

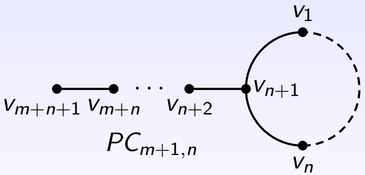
The e-positivity of $PC_{m+1,n}$

Proposition (Gebhard and Sagan, 2001)

For all $d \geq 2$, Y_{C_d} is (e)-positive.

Corollary

For any $m \geq 0, n \geq 1$, we have $PC_{m+1,n}$ is e-positive.



Outline

Decomposition techniques for graphs

Theorem (Orellana and Scott, 2014)

Let G be a graph where $e_1, e_2, e_3 \in E(G)$ form a triangle. Furthermore, define

- $G_{2,3} = (V(G), E(G) - \{e_1\})$
- $G_{1,3} = (V(G), E(G) - \{e_2\})$
- $G_3 = (V(G), E(G) - \{e_1, e_2\})$.

Then

$$X_G = X_{G_{2,3}} + X_{G_{1,3}} - X_{G_3}.$$

- M. Guay-Paquet (2013) has proved the same modular relation for the special case when G is an incomparability graph of $(3+1)$-free posets.

R. Orellana and G. Scott, Graphs with equal chromatic symmetric function, Discrete Math., 320 (2014), pp. 1 – 14.

M. Guay-Paquet, A modular relation for the chromatic symmetric function of (3+1)-free posets. preprint, Arxiv: 1306.2400.

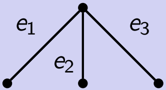
Sketch of the proof

Theorem (Stanley, 1995)

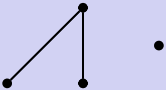
$$X_G = \sum_{S \subseteq E} (-1)^{\#S} p_{\lambda(S)}.$$

where $\lambda(s)$ is the partition whose parts are the orders of the connected components of the spanning subgraphs of G induced by S.

Example



claw



$S = \{e_1, e_2\}$
$\lambda(S) = (3, 1)$

Sketch of the proof

Consider the following partition of the set of spanning subgraphs of G:

- $G^1 = \{S \subseteq E(G) : e_1, e_2, e_3 \in S\}$
- $G^2 = \{S \subseteq E(G) : e_1, e_2 \in S, e_3 \notin S\}$
- $G^3 = \{S \subseteq E(G) : e_1, e_3 \in S, e_2 \notin S\}$
- $G^4 = \{S \subseteq E(G) : e_2, e_3 \in S, e_1 \notin S\}$
- $G^5 = \{S \subseteq E(G) : e_1 \in S, e_2, e_3 \notin S\}$
- $G^6 = \{S \subseteq E(G) : e_2 \in S, e_1, e_3 \notin S\}$
- $G^7 = \{S \subseteq E(G) : e_3 \in S, e_1, e_2 \notin S\}$
- $G^8 = \{S \subseteq E(G) : e_1, e_2, e_3 \notin S\}$

Sketch of the proof

Then

$$
\begin{aligned}
X_G &= \sum_{S\subseteq E} (-1)^{\#S} p_{\lambda(S)} \\
&= \sum_{i=1}^{8} \sum_{S\subseteq G^i} (-1)^{\#S} p_{\lambda(S)} \\
&= \sum_{i\in 4,6,7,8} \sum_{S\subseteq G^i} (-1)^{\#S} p_{\lambda(S)} + \sum_{i\in 3,5,7,8} \sum_{S\subseteq G^i} (-1)^{\#S} p_{\lambda(S)} \\
&- \sum_{i\in 7,8} \sum_{S\subseteq G^i} (-1)^{\#S} p_{\lambda(S)} + \sum_{i\in 1,2} \sum_{S\subseteq G^i} (-1)^{\#S} p_{\lambda(S)} \\
&= X_{G_{2,3}} + X_{G_{1,3}} - X_{G_3} + \sum_{i\in 1,2} \sum_{S\subseteq G^i} (-1)^{\#S} p_{\lambda(S)}.
\end{aligned}
$$

It is easy to see that $\sum_{i\in 1,2} \sum_{S\subseteq G^i} (-1)^{\#S} p_{\lambda(S)} = 0$, and the proof follows.

Decomposition techniques for graphs

Corollary (Orellana and Scott, 2014)

Let $G_{1,2}$ *be a graph with the adjacent edges* $e_1 = vv_1$, $e_2 = vv_2$ *and* $e_3 = v_1v_2 \notin E(G_{1,2})$. *Define*

- $G_{1,3} = (V(G_{1,2}), (E(G_{1,2}) - \{e_2\}) \cup \{e_3\})$
- $G_{2,3} = (V(G_{1,2}), (E(G_{1,2}) - \{e_1\}) \cup \{e_3\})$
- $G_1 = (V(G_{1,2}), E(G_{1,2}) - \{e_2\})$
- $G_3 = (V(G_{1,2}), (E(G_{1,2}) - \{e_1, e_2\}) \cup \{e_3\})$

Then

$$X_{G_{1,2}} = X_{G_{2,3}} + X_{G_1} - X_{G_3}.$$

Proof. Let $G_{1,2,3} = (V(G_{1,2}), E(G_{1,2}) \cup \{e_3\})$. Then we have

$$X_{G_{1,2,3}} = X_{G_{2,3}} + X_{G_{1,3}} - X_{G_3}$$

and

$$X_{G_{1,2,3}} = X_{G_{1,3}} + X_{G_{1,2}} - X_{G_1}.$$

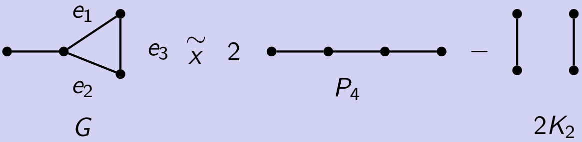
An equivalence relation

Definition

Let $\{G_i\}_{i\leq p}$ and $\{H_i\}_{i\leq k}$ be sets of graphs, and let $\{c_i\}_{i\leq p}$ and $\{d_i\}_{i\leq k}$ be real numbers. Define an equivalence relation *$\underset{X}{\sim}$ on linear combinations of graphs to be*

$$\sum_{i\leq p} c_i G_i \underset{X}{\sim} \sum_{i\leq k} d_i H_i, \quad \textit{if} \quad \sum_{i\leq p} c_i X_{G_i} = \sum_{i\leq k} d_i X_{H_i}.$$

Example



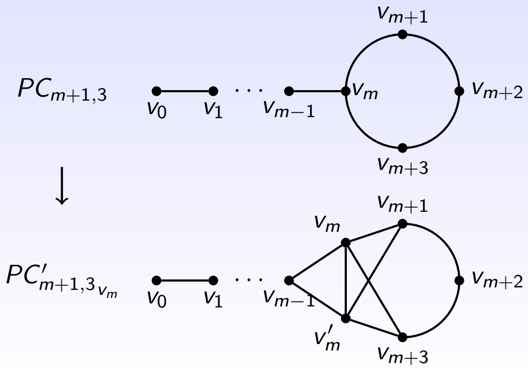
$$X_G = 2X_{P_4} - X_{2K_2}$$

The non-e-positivity of $PC'_{{m+1,3}_{v_m}}$

Theorem

For any $m \geq 1$, $PC'_{{m+1,3}_{v_m}}$ is not e-positive.

Proof. The chromatic symmetric function of $PC'_{{m+1,3}_{v_m}}$ can be acquired by repeated application of the decomposition techniques.



Sketch of the proof

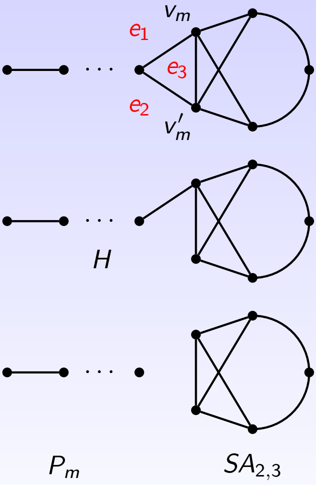
$$PC'_{m+1,3_{v_m}}$$

$$\Big\downarrow \tilde{X}$$

$$2PC'_{m+1,3_{v_m 1,3}}$$

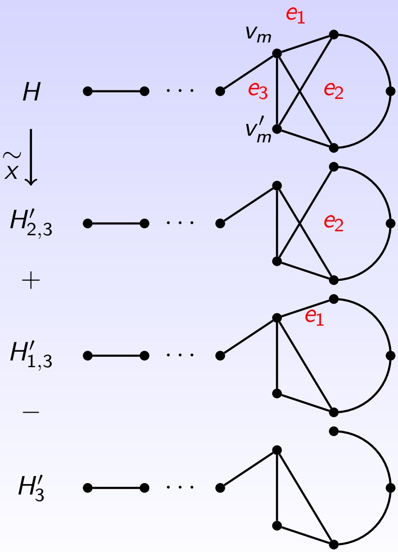
$$-$$

$$PC'_{m+1,3_{v_m 3}}$$



$$X_{PC'_{m+1,3_{v_m}}} = 2X_H - X_{P_m} X_{SA_{2,3}}$$

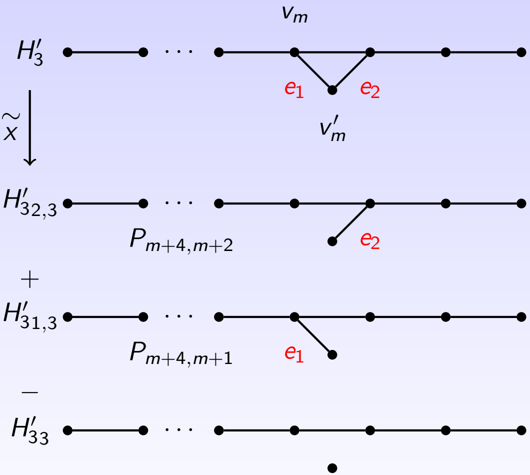
Sketch of the proof



$$X_{H'_{2,3}} = 2X_{PC_{m+2,3}} - X_{P_{m+1}}X_{C_4}$$

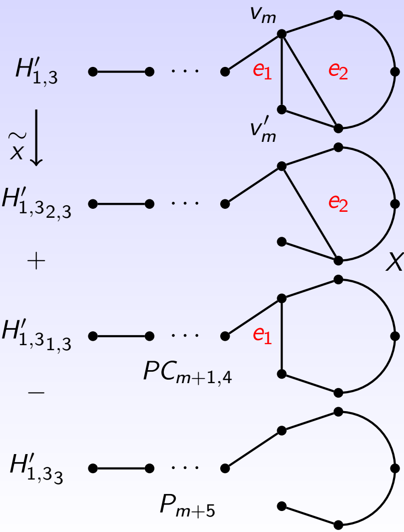
$$X_H = X_{H'_{2,3}} + X_{H'_{1,3}} - X_{H'_3}$$

Sketch of the proof



$$X_{H'_3} = X_{P_{m+4,m+2}} + X_{P_{m+4,m+1}} - e_1 X_{P_{m+4}}$$

Sketch of the proof



$$X_{H'_{1,3_{2,3}}} = X_{PC_{m+1,4}} + e_1 X_{PC_{m+1,3}} - X_{P_{m+4,m+1}}$$

$$X_{H'_{1,3}} = X_{H'_{1,3_{2,3}}} + X_{PC_{m+1,4}} - X_{P_{m+5}}$$

Sketch of the proof

Hence we conclude:

$$\begin{aligned}X_{PC'_{m+1,3_{v_m}}} =& 4X_{PC_{m+2,3}} + 4X_{PC_{m+1,4}} + 2e_1X_{PC_{m+1,3}} + 2e_1X_{P_{m+4}} \\ & - 2X_{P_{m+1}}X_{C_4} - 4X_{P_{m+4,m+1}} - 2X_{P_{m+4,m+2}} - X_{P_m}X_{SA_{2,3}} - 2X_{P_{m+5}}\end{aligned}$$

Using the same method the following results can be proved.

- $X_{PC_{m,n}} = nX_{P_{m+n}} - \sum_{i=2}^{n} X_{P_{m-2+i}}X_{C_{n+2-i}}$.
- $X_{P_{m,n}} = X_{P_{m+1}} + e_1X_{P_m} - X_{P_n}X_{P_{m-n+1}}$, where $n \leq m$.
- $X_{SA_{2,n}} = 4X_{C_{n+2}} + 2e_1X_{C_{n+1}} + 2e_2X_{P_n} - 6X_{P_{n+2}}$.

Then, we have

$$\begin{aligned}X_{PC'_{m+1,3_{v_m}}} =& 20X_{P_{m+5}} + 2e_1X_{P_{m+4}} - 8X_{C_2}X_{P_{m+3}} \\ & + (2X_{P_3} - 8X_{C_3} - 2e_1X_{C_2})X_{P_{m+2}} + (4X_{P_4} - 6X_{C_4} - 2e_1X_{C_3})X_{P_{m+1}} \\ & + (6X_{P_5} - 4X_{C_5} - 2e_1X_{C_4} - 2e_2X_{P_3})X_{P_m} \\ =& 20X_{P_{m+5}} + 2e_1X_{P_{m+4}} - 16e_2X_{P_{m+3}} - (2e_{(2,1)} + 42e_3)X_{P_{m+2}} \\ & - (56e_4 + 4e_{(2,2)} + 4e_{(3,1)})X_{P_{m+1}} - (6e_{(4,1)} + 4e_{(3,2)} + 50e_5)X_{P_m}.\end{aligned}$$

Sketch of the proof

From

$$
\begin{aligned}
X_{PC'_{m+1,3_{v_m}}} =& 20X_{P_{m+5}} + 2e_1X_{P_{m+4}} - 16e_2X_{P_{m+3}} - (2e_{(2,1)} + 42e_3)X_{P_{m+2}} \\
& - (56e_4 + 4e_{(2,2)} + 4e_{(3,1)})X_{P_{m+1}} - (6e_{(4,1)} + 4e_{(3,2)} + 50e_5)X_{P_m},
\end{aligned}
$$

we derive that

$$
\begin{aligned}
\sum_{m=0}^{\infty} X_{PC'_{m+1,3_{v_m}}} t^{m+5} =& 20\sum_{m=0}^{\infty} X_{P_{m+5}} t^{m+5} + 2e_1\sum_{m=0}^{\infty} X_{P_{m+4}} t^{m+5} \\
& - 16e_2\sum_{m=0}^{\infty} X_{P_{m+3}} t^{m+5} - (2e_{(2,1)} + 42e_3)\sum_{m=0}^{\infty} X_{P_{m+2}} t^{m+5} \\
& - (56e_4 + 4e_{(2,2)} + 4e_{(3,1)})\sum_{m=0}^{\infty} X_{P_{m+1}} t^{m+5} \\
& - (6e_{(4,1)} + 4e_{(3,2)} + 50e_5)\sum_{m=0}^{\infty} X_{P_m} t^{m+5}.
\end{aligned}
$$

Sketch of the proof

Proposition (Stanley, 1995)

$$\sum_{d\geq 0} X_{P_d} t^d = \frac{\sum_{i\geq 0} e_i t^i}{1-\sum_{i\geq 1}(i-1)e_i t^i} = \frac{E(t)}{E(t)-tE'(t)},$$

where $E(t)=\sum_{i\geq 0} e_i t^i$.

By direct calculation, we derive that

$$\sum_{m=0}^{\infty} X_{PC'_{m+1,3_{v_m}}} t^{m+5} = \frac{F(t)}{E(t)-tE'(t)},$$

where

$$\begin{aligned} F(t) = & 4e_{(2,2)}t^5E'(t) + 24e_4t^5E'(t) + 6e_{(2,1)}t^4E'(t) + 18e_3t^4E'(t) \\ & + 2e_{(1,1)}t^3E'(t) + 24e_2t^3E'(t) + 22e_1t^2E'(t) - 4e_{(3,2)}t^5E(t) \\ & - 6e_{(4,1)}t^5E(t) - 50e_5t^5E(t) - 8e_{(2,2)}t^4E(t) - 4e_{(3,1)}t^4E(t) \\ & - 80e_4t^4E(t) - 8e_{(2,1)}t^3E(t) - 60e_3t^3E(t) - 2e_{(1,1)}t^2E(t) \\ & - 40e_2t^2E(t) - 20e_1tE(t) + 20tE'(t). \end{aligned}$$

Sketch of the proof

Let $\lambda = (3^k)$, where $k \geq 2$. We proceed to consider the coefficient of $e_\lambda t^{|\lambda|}$. Observe that $e_\lambda t^{|\lambda|}$ can only appear in the expansion of the following terms:

$$\frac{18e_3 t^4 E'(t) + 20tE'(t) - 60e_3 t^3 E(t)}{E(t) - tE'(t)}.$$

Recall that

$$E(t) = \sum_{n=0}^{\infty} e_n t^n,$$

we have the numerator equals

$$\sum_{n=0}^{\infty} (20n + 18ne_3 t^3 - 60e_3 t^3) e_n t^n.$$

Sketch of the proof

Notice that $\lambda = (3^k)$, so we need only to consider the terms $n = 0$ and $n = 3$ in

$$\sum_{n=0}^{\infty} (20n + 18ne_3 t^3 - 60e_3 t^3) e_n t^n.$$

It is trivial to check that

$$-60e_3 t^3 + (60 + 54e_3 t^3 - 60e_3 t^3) e_3 t^3 = -6e_{(3,3)} t^6.$$

Thus we proceed to consider the formula

$$\frac{-6e_{(3,3)} t^6}{1 - \sum_{n=1}^{\infty} (n-1) e_n t^n} = -6e_{(3,3)} t^6 \sum_{i=0}^{\infty} \left(\sum_{n=1}^{\infty} (n-1) e_n t^n \right)^i .$$

Since λ has k parts, we see that $i = k - 2$, and that $n = 3$ in each term of

$$\sum_{n=1}^{\infty} (n-1) e_n t^n.$$

Hence the coefficient of $e_\lambda t^{|\lambda|}$ equals $-6 \cdot 2^{k-2}$.

Sketch of the proof

Next we consider the coefficient of $e_\mu t^{|\mu|}$, where $\mu = (3^k, 1)$ and $k \geq 2$. Note that $e_\mu t^{|\mu|}$ can only appear in the expansion of the following terms:

$$\frac{18e_3 t^4 E'(t) + 20tE'(t) - 60e_3 t^3 E(t) + 22e_1 t^2 E'(t) - 4e_{(3,1)} t^4 E(t) - 20e_1 tE(t)}{E(t) - tE'(t)}.$$

The numerator equals

$$\sum_{n=0}^{\infty}(18ne_3 t^3 + 20n - 60e_3 t^3)e_n t^n + \sum_{n=0}^{\infty}(22ne_1 t - 4e_{(3,1)} t^4 - 20e_1 t)e_n t^n.$$

In order to get $e_\mu t^{|\mu|}$, it suffices to consider the terms $n = 0, 1, 3$ in $\sum_{n=0}^{\infty}(18ne_3 t^3 + 20n - 60e_3 t^3)e_n t^n$ and $n = 0, 3$ in $\sum_{n=0}^{\infty}(22ne_1 t - 4e_{(3,1)} t^4 - 20e_1 t)e_n t^n$.

Sketch of the proof

Namely,

$$
\begin{aligned}
&-60e_3t^3+(18e_3t^3+20-60e_3t^3)e_1t+(60+54e_3t^3-60e_3t^3)e_3t^3\\
&-4e_{(3,1)}t^4-20e_1t+(66e_1t-4e_{(3,1)}t^4-20e_1t)e_3t^3\\
=&\,20e_1t-42e_{(3,1)}t^4-6e_{(3,3)}t^6-20e_1t+42e_{(3,1)}t^4-4e_{(3,3,1)}t^7\\
=&\,-6e_{(3,3)}t^6-4e_{(3,3,1)}t^7.
\end{aligned}
$$

Hence $e_\mu t^{|\mu|}$ lies in

$$\frac{-6e_{(3,3)}t^6-4e_{(3,3,1)}t^7}{1-\sum_{n=1}^{\infty}(n-1)e_nt^n}.$$

Sketch of the proof

Notice that

$$1-\sum_{n=1}^{\infty}(n-1)e_nt^n = 1-\sum_{n=2}^{\infty}(n-1)e_nt^n,$$

we see that the denominator does not contain e_1. Thus in order to get $e_\mu t^{|\mu|}$, we need only to check

$$\frac{-4e_{(3,3,1)}t^7}{1-\sum_{n=1}^{\infty}(n-1)e_nt^n} = -4e_{(3,3,1)}t^7\sum_{i=0}^{\infty}\left(\sum_{n=1}^{\infty}(n-1)e_nt^n\right)^i.$$

Since $\mu=(3^k,1)$, we see that $i=k-2$ and $n=3$ for each term

$$\sum_{n=1}^{\infty}(n-1)e_nt^n.$$

Therefore the coefficient of $e_\mu t^{|\mu|}$ equals -2^k.

Sketch of the proof

Finally, we consider the coefficient of $e_\theta t^{|\theta|}$, where $\theta = (3^k, 2)$ and $k \geq 2$. Note that $e_\theta t^{|\theta|}$ can only appear in the expansion of the following terms:

$$\frac{18e_3t^4E'(t) + 20tE'(t) - 60e_3t^3E(t) + 24e_2t^3E'(t) - 4e_{(3,2)}t^5E(t) - 40e_2t^2E(t)}{E(t) - tE'(t)}.$$

The numerator equals

$$\sum_{n=0}^{\infty}(18ne_3t^3 + 20n - 60e_3t^3)e_nt^n + \sum_{n=0}^{\infty}(24ne_2t^2 - 4e_{(3,2)}t^5 - 40e_2t^2)e_nt^n.$$

It suffices to consider the terms $n = 0, 2, 3$ in $\sum_{n=0}^{\infty}(18ne_3t^3 + 20n - 60e_3t^3)e_nt^n$ and $n = 0, 3$ in $\sum_{n=0}^{\infty}(24ne_2t^2 - 4e_{(3,2)}t^5 - 40e_2t^2)e_nt^n$ in order to get $e_\theta t^{|\theta|}$.

Sketch of the proof

Namely,

$$\begin{aligned}
&-60e_3t^3+(36e_3t^3+40-60e_3t^3)e_2t^2+(60+54e_3t^3-60e_3t^3)e_3t^3\\
&-4e_{(3,2)}t^5-40e_2t^2+(72e_2t^2-4e_{(3,2)}t^5-40e_2t^2)e_3t^3\\
=&\,40e_2t^2-24e_{(3,2)}t^5-6e_{(3,3)}t^6-40e_2t^2+28e_{(3,2)}t^5-4e_{(3,3,2)}t^8\\
=&\,4e_{(3,2)}t^5-6e_{(3,3)}t^6-4e_{(3,3,2)}t^8.
\end{aligned}$$

Hence $e_\theta t^{|\theta|}$ lies in

$$\frac{4e_{(3,2)}t^5-6e_{(3,3)}t^6-4e_{(3,3,2)}t^8}{1-\sum_{n=1}^{\infty}(n-1)e_nt^n}.$$

Sketch of the proof

It is clear that

$$\frac{4e_{(3,2)}t^5 - 6e_{(3,3)}t^6 - 4e_{(3,3,2)}t^8}{1 - \sum_{n=1}^{\infty}(n-1)e_n t^n} = 4e_{(3,2)}t^5 \sum_{i=0}^{\infty}\left(\sum_{n=1}^{\infty}(n-1)e_n t^n\right)^i$$

$$- 6e_{(3,3)}t^6 \sum_{i=0}^{\infty}\left(\sum_{n=1}^{\infty}(n-1)e_n t^n\right)^i - 4e_{(3,3,2)}t^8 \sum_{i=0}^{\infty}\left(\sum_{n=1}^{\infty}(n-1)e_n t^n\right)^i.$$

Since $\theta = (3^k, 2)$, in the first summation we take $i = k-1$ and $n = 3$ in each term of

$$\sum_{n=1}^{\infty}(n-1)e_n t^n.$$

Hence the coefficient of $e_\theta t^{|\theta|}$ in the first summation equals 2^{k+1}.

Sketch of the proof

In the second summation, we take $i = k - 1$. Moreover, we take $n = 3$ in $k - 2$'s

$$\sum_{n=1}^{\infty}(n-1)e_n t^n,$$

and $n = 2$ in one

$$\sum_{n=1}^{\infty}(n-1)e_n t^n.$$

There are $k - 1$ choices. Thus the coefficient of $e_\theta t^{|\theta|}$ in the second summation equals:

$$-6 \cdot (k-1) \cdot 2^{k-2} = -3(k-1)2^{k-1}.$$

Sketch of the proof

Finally, in the third summation, we take $i = k - 2$ and $n = 3$ in each term of

$$\sum_{n=1}^{\infty} (n-1) e_n t^n.$$

Hence the coefficient of $e_\theta t^{|\theta|}$ in the third summation equals -2^k. Thus the coefficient of $e_\theta t^{|\theta|}$ in $\sum_{m=0}^{\infty} X_{PC'_{m+1,3_{v_m}}} t^{m+5}$ equals

$$2^{k+1} - 3(k-1)2^{k-1} - 2^k = 2^{k-1}(5 - 3k).$$

Thus for $k \geq 2$, the coefficient of $e_\theta t^{|\theta|}$ is negative.

THANK YOU!