Some recent results on enumeration of tableaux and lattice paths

Rosena R. X. Du

East China Normal University

Nankai University, Sep 30, 2020

Outline

1. Definitions and Backgrounds
2. Major and Amajor index for row-increasing tableaux
3. Major index of Schröder n-paths
4. Distribution of descents in row-increasing tableaux
5. Cyclic Sieving

Integer partitions

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partititon of n, i.e.

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k = n,$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$.

The Ferrers diagram of λ is a left-justfied array of cells with λ_i cells in the i-th row, for $1 \leq i \leq k$.

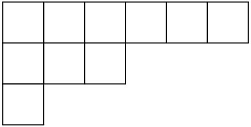


Figure: The Ferrers diagram of a partition $\lambda = (6, 3, 1) \vdash 10$.

Semistandard Young tableau and standard Young tableau

A semistandard Young tableau (SSYT) of shape λ is a filling of the Ferrers diagram of λ with positive integers such that every row is strictly increasing and every column is weakly increasing.

A standard Young tableau (SYT) of shape $\lambda \vdash n$ is a filling of the Ferrers diagram of λ with $\{1, 2, \ldots, n\}$ such that every row and column is strictly increasing.

2	4	6	7	8	9
4	5	6			
8					

1	3	4	5	8	10
2	6	7			
9					

Figure: A semi-standard Young tableau of shape $(6, 3, 1)$ and a standard Young tableau of shape $(6, 3, 1)$.

Major index and amajor index of a tablau

A descent of an SSYT T is any instance of i followed by an $i+1$ in a lower row of T. $D(T)$: the descent set of T. The major index of T is defined by $\mathrm{maj}(T) = \sum_{i \in D(T)} i$. An ascent of T is any instance of i followed by an $i+1$ in a higher row of T than i. $A(T)$: the ascent set of T. The amajor index of T is defined by $\mathrm{amaj}(T) = \sum_{i \in A(T)} i$.

$$\begin{array}{|c|c|c|c|}\hline 1 & \mathbf{2} & \mathbf{5} & 10 \\ \hline 3 & 4 & \mathbf{8} \\ \cline{1-3} \mathbf{6} \\ \cline{1-1} 7 \\ \cline{1-1} 9 \\ \cline{1-1} \end{array} \qquad\qquad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 10 \\ \hline 3 & \mathbf{4} & 8 \\ \cline{1-3} 6 \\ \cline{1-1} \mathbf{7} \\ \cline{1-1} \mathbf{9} \\ \cline{1-1} \end{array}$$

Figure: $T \in \mathrm{SYT}(4,3,1,1,1)$.

$D(T) = \{2,5,6,8\}$, $\mathrm{maj}(T) = 21$. $A(T) = \{4,7,9\}$, $\mathrm{amaj}(T) = 20$.

Major index for standard Young tableaux

Lemma (Stanley's q-hook length formula)
For any partition $\lambda = \sum_i \lambda_i$ of n, we have

$$\sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} = \frac{q^{b(\lambda)}[n]!}{\prod_{u \in \lambda} h(u)}. \tag{1}$$

Here $b(\lambda) = \sum_i (i-1)\lambda_i$.

The famous RSK algorithm is a bijection between permutations of length n and pairs of SYTs of order n of the same shape. Under this bijection, the descent set of a permutation is transferred to the descent set of the corresponding "recording tableau". Therefore many problems involving the statistics descent or major index of pattern-avoiding permutations can be translated to the study of descent or major index of tableaux.

Standard Young tableaux of shape $2 \times n$

For any positive integer n, we have

$$C_q(n) = \sum_{T \in \mathrm{SYT}(2\times n)} q^{\mathrm{maj}(T)} = \frac{q^n}{[n+1]} \begin{bmatrix} 2n \\ n \end{bmatrix}. \qquad (2)$$

Here $[n] = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}$, $[n]! = [n][n-1]\cdots[1]$ and $\begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]!}$.

For example, when $n = 3$, we have

$$C_q(3) = \frac{q^3}{[3+1]} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = q^3 + q^5 + q^6 + q^7 + q^9.$$

And there are five SYT of shape 2×3, with major index 3,6,7,5,9.

1	2	**3**
4	5	6

1	**2**	**4**
3	5	6

1	**2**	**5**
3	4	6

1	3	**4**
2	5	6

1	**3**	**5**
2	4	6

Increasing tableaux

An increasing tableau is an SSYT such that both rows and columns are strictly increasing, and the set of entries is an initial segment of positive integers (if an integer i appears, positive integers less than i all appear).

We denote by $\mathrm{Inc}_k(\lambda)$ the set of increasing tableaux of shape λ with entries are $\{1, 2, \dots, n-k\}$.

1	**2**	3
2	4	5

1	**2**	4
2	3	5

1	2	**3**
3	4	5

1	2	**4**
3	**4**	5

1	3	**4**
2	**4**	5

Figure: There are five increasing tableaux in $\mathrm{Inc}_1(2 \times 3)$.

Increasing tableau is defined by O. Pechenik who studied increasing tableaux in $\mathrm{Inc}_k(2 \times n)$, i.e., increasing tableaux of shape $2 \times n$, with exactly k numbers appeared twice.

O. Pechenik, Cyclic Sieving of Increasing Tableaux and Small Schröder Paths. *J. Combin. Theory Ser. A*, 125: 357–378, 2014.

Major index for Increasing tableau of shape $2 \times n$

Pechinik got the following formula while studying the cyclic sieving of Increasing tableaux.

Theorem (O. Pechenik)
For any positive integer n, and $0 \leq k \leq n$ we have

$$S_q(n,k) = \sum_{T \in \mathrm{Inc}_k(2\times n)} q^{\mathrm{maj}(T)} = \frac{q^{n+k(k+1)/2}}{[n+1]} \begin{bmatrix} n-1 \\ k \end{bmatrix} \begin{bmatrix} 2n-k \\ n \end{bmatrix}. \qquad (3)$$

For example, when $n=3,\ k=1$ we have

1	2	**3**
2	4	5

1	**2**	**4**
2	3	5

1	**2**	**3**
3	4	5

1	**2**	**4**
3	4	5

1	**3**	**4**
2	4	5

$$S_q(3,1) = \sum_{T \in \mathrm{Inc}_1(2\times 3)} q^{\mathrm{maj}(T)} = \frac{q^4}{[3+1]} \begin{bmatrix} 3-1 \\ 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = q^8 + q^7 + q^6 + q^5 + q^4.$$

A refinement of small Schröder number

Setting $q = 1$, we get the cardinality of $\mathrm{Inc}_k(2 \times n)$:

$$s(n,k) = \frac{1}{n+1}\binom{n-1}{k}\binom{2n-k}{n}. \tag{4}$$

$s(n,k)$ is considered as a refinement of the small Schröder number which counts the following sets:

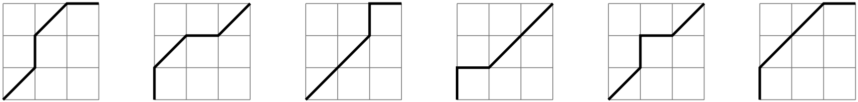
1. Dissections of a convex $(n+2)$-gon into $n-k$ regions;
2. SYTs of shape $(n-k, n-k, 1^k)$;
3. small Schröder n-paths with k flat steps.

In 1996 Stanley gave a bijection between the first two sets.

R. P. Stanley, Polygon dissections and standard Young tableaux. *J. Combin. Theory Ser. A*, 76: 175–177, 1996.

Schröder paths

A Schröder n-path is a lattice path goes from $(0,0)$ to (n,n) with steps $(0,1)$, $(1,0)$ and $(1,1)$ and never goes below the diagonal line $y=x$. If there is no F steps on the diagonal line, it is called a small Schröder path.



There is an obvious bijection between Schröder n-paths and SSYTs of shape $2 \times n$: read the numbers i from 1 to $2n-k$ in increasing order, if i appears only in row 1 (2), it corresponds to a U (D) step, if i appears in both rows, it corresponds to an F step.

1	2	**3**
1	**3**	4

1	**2**	**4**
2	3	**4**

1	**2**	3
1	**2**	4

1	**3**	**4**
2	**3**	**4**

1	2	**4**
1	3	**4**

1	**2**	**3**
2	**3**	4

Motivation: are there any interesting result for these tableaux that correspond to all Schröder n-paths?

Row-increasing tableaux

A row-increasing tableau is an SSYT with strictly increasing rows and weakly increasing columns, and the set of entries is a consecutive segment of positive integers.

We denote by $\mathrm{RInc}_k^m(\lambda)$ the set of row-increasing tableaux of shape λ with set of entries $\{m+1, m+2, \ldots, m+n-k\}$. When $m=0$, we will just denote $\mathrm{RInc}_k^0(\lambda)$ as $\mathrm{RInc}_k(\lambda)$. It is obvious that $\mathrm{Inc}_k(\lambda) \subseteq \mathrm{RInc}_k(\lambda)$.

1	2	**3**
1	**3**	4

1	**2**	**4**
2	3	**4**

1	**2**	3
1	**2**	4

1	**3**	**4**
2	**3**	**4**

1	2	**4**
1	3	**4**

1	**2**	**3**
2	**3**	4

Figure: There are 6 row-increasing tableaux in $\mathrm{RInc}_2(2\times 3)$.

It is not hard to show that $\mathrm{RInc}_k(2\times n)$ is counted by

$$r(n,k) = \frac{1}{n-k+1}\binom{2n-k}{k}\binom{2n-2k}{n-k}. \tag{5}$$

$r(n,k)$ is considered as a refinement of the large Schröder number.

Major index for $\mathrm{Inc}_k(2 \times n)$

Theorem (O. Pechenik)

There exists a bijection γ between $\mathrm{Inc}_k(2 \times n)$ and $\mathrm{SYT}(n-k, n-k, 1^k)$ which preserves the descent set.

Given $T \in \mathrm{Inc}_k(2 \times n)$. Let A be the set of numbers that appear twice. Let B be the set of numbers that appear in the second row immediately right of an element of A. Let $\gamma(T)$ be the tableau of shape $(n-k, n-k)$ formed by deleting all elements of A from the first row of T and all elements of B from the second row of T. It is not hard to prove that γ is a bijection.
E.g., we have $A = \{4, 6, 8\}$ and $B = \{6, 7, 9\}$.

1	2	4	5	6	8
3	4	6	7	8	9

1	2	5
3	4	8
6		
7		
9		

Major index for $\mathrm{RInc}_k(2 \times n)$

There is a bijection $f\colon \mathrm{RInc}_k(2 \times n) \backslash \mathrm{Inc}_k(2 \times n) \mapsto \mathrm{Inc}_{k-1}(2 \times n)$.

Given $T \in \mathrm{RInc}_k(2 \times n) \backslash \mathrm{Inc}_k(2 \times n)$, find the minimal integer j such that $T_{1,j} = T_{2,j}$,. Now we first delete the entry $T_{2,j}$, then move all the entries on the right of $T_{2,j}$ one box to the left and set the last entry as $2n - k + 1$, and define the resulting tableau to be $f(T)$.

$$T : \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 4 & 5 & 6 \\ \hline 2 & 3 & 4 & 6 & 7 \\ \hline \end{array} \quad \mapsto \quad f(T) : \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 4 & 5 & 6 \\ \hline 2 & 4 & 6 & 7 & 8 \\ \hline \end{array}$$

Figure: An example of f with $T \in \mathrm{RInc}_3(2 \times 5) \backslash \mathrm{Inc}_3(2 \times 5)$ and $f(T) \in \mathrm{Inc}_2(2 \times 5)$.

However, f does NOT preserve the major index. In fact we have

Theorem

For any positive integer n, k with $k < n$, we have

$$R_q(n,k) = S_q(n,k) + S_q(n,k-1) + (1 - q^{2n-k})(S_q(n-1,k-1) + S_q(n-1,k-2)).$$

Major index for $\mathrm{RInc}_k(2 \times n)$

Theorem

For any positive integer n, and $0 \leq k \leq n$ we have

$$R_q(n,k) = \sum_{T \in \mathrm{RInc}_k(2\times n)} q^{\mathrm{maj}(T)} = \frac{q^{n+k(k-3)/2}}{[n-k+1]} \begin{bmatrix} 2n-k \\ k \end{bmatrix} \begin{bmatrix} 2n-2k \\ n-k \end{bmatrix}. \quad (6)$$

Using similar method we can also get major index polynomial for row-increasing tableaux of shape $(n-a, a)$:

Theorem

Given positive integers n, a and k with $a \leq \lfloor n/2 \rfloor$, $k \leq a$, we have

$$\sum_{T \in \mathrm{RInc}_k(n-a,a)} q^{\mathrm{maj}(T)} = q^{a+k(k-3)/2} \frac{[n-2a+1]}{[n-a-k+1]} \begin{bmatrix} n-k \\ k \end{bmatrix} \begin{bmatrix} n-2k \\ a-k \end{bmatrix}.$$

Row-increasing tableaux of shape $(n-a, a)$

Summing over a, we get the major index polynomial for all row-increasing tableaux with n cells and at most two rows:

$$\sum_{a=k}^{\lfloor \frac{n}{2} \rfloor} R_{(n-a,a),k}(q) = q^{k(k-1)/2} \begin{bmatrix} n-k \\ k \end{bmatrix} \begin{bmatrix} n-2k \\ \lfloor \frac{n}{2} \rfloor - k \end{bmatrix}.$$

The result for increasing tableaux is more complicated. For example, we have

$$S_{(n-a,a),k} = \frac{2a^2 - 3na - a + n^2 + n - k}{(n-a+1)(n-a)} \binom{n-k}{k} \binom{n-2k}{a-k}.$$

and

$$\sum_{a=k}^{\lfloor \frac{n}{2} \rfloor} S_{(n-a,a),k} = \frac{\lceil \frac{n}{2} \rceil - k}{\lceil \frac{n}{2} \rceil} \binom{n-k}{k} \binom{n-2k}{\lfloor \frac{n}{2} \rfloor - k}.$$

Amajor index for $\mathrm{RInc}_k(2 \times n)$

We also study the amajor index polynomial of SSYTs in $\mathrm{RInc}_k(2 \times n)$.

$$\widetilde{R}_q(n,k) = \sum_{T \in \mathrm{RInc}_k(2\times n)} q^{\mathrm{amaj}(T)} = \frac{q^{k(k-1)/2}}{[n-k+1]} \begin{bmatrix} 2n-k \\ k \end{bmatrix} \begin{bmatrix} 2n-2k \\ n-k \end{bmatrix}.$$

We will prove the above formula by showing that

$$\sum_{T \in \mathrm{RInc}_k(2\times n)} q^{\mathrm{maj}(T)} = q^{n-k} \cdot \sum_{T \in \mathrm{RInc}_k(2\times n)} q^{\mathrm{amaj}(T)}.$$

For example, there are 6 row-increasing tableaux in $\mathrm{RInc}_2(2 \times 3)$, with the $(\mathrm{maj}, \mathrm{amaj})$ pairs $(4,3), (4,5), (2,4), (5,1), (3,3), (6,2)$.

1	2	3
1	2	4

1	3	4
2	3	4

1	2	4
1	3	4

1	2	3
1	3	4

1	2	4
2	3	4

1	2	3
2	3	4

We want to establish a bijection $\Phi : \mathrm{RInc}_k(2 \times n) \mapsto \mathrm{RInc}_k(2 \times n)$ such that

$$\mathrm{maj}(\Phi(T)) = \mathrm{amaj}(T) + n - k.$$

The general result

Theorem

There is a bijection $\Phi : \mathrm{RInc}_k(2 \times n) \mapsto \mathrm{RInc}_k(2 \times n)$ that preserves the second row, and

$$\mathrm{maj}(\Phi(T)) = \mathrm{amaj}(T) + n - k.$$

T :

1	**2**	4	5	**6**	**9**	10	12	**13**	14	**16**	18	**20**
2	3	**6**	7	8	**9**	11	**13**	15	**16**	17	19	**20**

$\Phi(T)$:

1	**3**	4	5	**8**	**9**	10	**11**	12	14	**15**	18	**19**
2	**3**	6	7	**8**	**9**	**11**	13	**15**	16	17	**19**	20

Figure: An example of the map Φ with $n = 13$, $k = 6$, and $l = 3$.

The prime case

A row-increasing tableau T is prime if for each integer j satisfies $T_{1,j+1} = T_{2,j} + 1$, $T_{2,j+1}$ also appears in row 1 in T.

$\mathrm{pRInc}_k^m(\lambda)$: prime row-increasing tableaux of shape λ with set of entries $\{m+1, m+2, \ldots, m+n-k\}$.

For each $T \in \mathrm{pRInc}_k^m(2 \times n)$, let A be the set of numbers that appear twice, and B be the set of numbers that appear in the second row immediately left of an element of A in cyclic order.

Let $g(T)$ be the tableau of shape $2 \times n$ obtained by first deleting all elements in A from the first row and then inserting all elements in B into the first row and list them in increasing order, and keep the entries in row 2 unchanged.

In the following example, we have $A = \{2, 6, 9\}$ and $B = \{3, 8, 9\}$.

T:

1	**2**	4	5	**6**	**9**
2	3	**6**	7	8	**9**

$\xrightarrow{g}$

$g(T)$:

1	**3**	4	5	**8**	**9**
2	**3**	6	7	**8**	**9**

Lemma

The map g is an injection from $\mathrm{pRInc}_k^m(2 \times n)$ to $\mathrm{RInc}_k^m(2 \times n)$ which satisfies the following:

1) If $T_{2,1}$ appears only once in T, then $g(T)_{1,i+1} \leq g(T)_{2,i}$ for each $i, 1 \leq i \leq n-1$;
2) $T_{2,1}$ appears twice in T if and only if $g(T)_{1,n} = g(T)_{2,n}$.

Lemma

For each $T \in \mathrm{pRInc}_k^m(2 \times n)$ we have

$$\mathrm{maj}(g(T)) = \begin{cases} \mathrm{amaj}(T) + n - k, & \text{if } T_{1,1} = T_{2,1}; \\ \mathrm{amaj}(T) + m + n - k, & \text{if } T_{1,1} \neq T_{2,1}. \end{cases} \tag{7}$$

$T:$

5	6	**8**	9	10	**13**
7	**8**	11	12	**13**	14

$\xrightarrow{g}$

$g(T):$

5	6	**7**	9	10	**12**
7	8	11	**12**	13	14

The general case

Given $T \in \mathrm{RInc}_k(2 \times n)$, we can uniquely decompose T into prime row-increasing tableaux $T_1 T_2 \cdots T_l$, and set $\Phi(T) = g(T_1)g(T_2)\cdots g(T_l)$.

An example with $n = 13$, $k = 6$, and $l = 3$. Here $A(T) = \{3, 8, 9, 11, 13, 15, 17, 19\}$, $A(T_1^0) = \{3\}$, $A(T_2^0) = \{11, 13\}$, $A(T_3^0) = \emptyset$, $D(T_1^0) = \{1, 5\}$, $D(T_2^0) = \{10, 12, 14\}$, $D(T_3^0) = \{18\}$. $D(\Phi(T)) = \{1, 5, 8, 10, 12, 14, 15, 18, 19\}$. $\mathrm{amaj}(T) = 95$ and $\mathrm{maj}(\Phi(T)) = 102$.

$T:$

1	**2**	4	5	**6**	**9**	10	12	**13**	14	**16**	18	**20**
2	3	**6**	7	8	**9**	11	**13**	15	**16**	17	19	**20**

			1	4	5			10	12	14		18
2	3	6	7	8	9	11	13	15	16	17	19	20

$\Phi(T):$

1	**3**	4	5	**8**	**9**	10	**11**	12	14	**15**	18	**19**
2	**3**	6	7	**8**	**9**	**11**	13	**15**	16	17	**19**	20

Major index of Schröder n-paths

Let P be a Schröder n-path that goes from the origin $(0,0)$ to (n,n) with k F steps, we can associate with P a word $w = w(P) = w_1 w_2 \cdots w_{2n-k}$ over the alphabet $\{0,1,2\}$ with exactly k 1's.

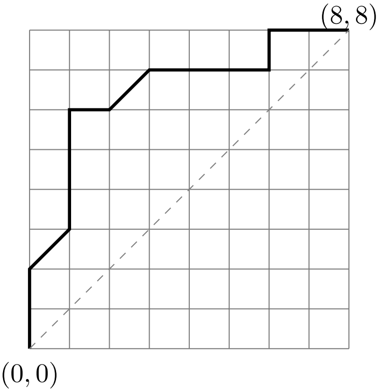


Figure: A Schröder P with $\omega(P) = 00100021222022$.

Major index of Schröder n-paths

The descent set of w is the set of all positions of the descents of w, $D(w) = \{i : 1 \leq i \leq n, w_i > w_{i+1}\}$. The major index of w is defined as $\mathrm{maj}(w) = \sum_{i \in D(w)} i$. And define $\mathrm{maj}(P) = \mathrm{maj}(w(P))$.

In 1993, Bonin, Shapiro and Simion study the major index for Schöder paths and gave the following result:

$$\sum q^{\mathrm{maj}(P)} = \frac{1}{[n-k+1]} \begin{bmatrix} 2n-k \\ k \end{bmatrix} \begin{bmatrix} 2n-2k \\ n-k \end{bmatrix}. \tag{8}$$

Here the sum is over all Schröder n-paths with exactly k F steps.

J. Bonin, L. Shapiro and R. Simion, Some q-analogues of the Schröder numbers arising from combinatorial statistics on lattice paths. *J. Statist. Plann. Inference*, 1993, 35(1): 35–55.

C. Song, The generalized Schröder theory. *Electron. J. Combin.* 12, #53, 2005.

Decents of a Schröder path and ascents of a tableau

1	2	3	4	5	6	8	12
3	7	8	9	10	11	13	14

Figure: The tableau corresponding to P with $\omega(P) = 00100021222022$.

A naive thinking: if the i-th step corresponds to a descent in P, then i is an ascent of T, i.e., $D(P){=}A(T)$ and $\mathrm{maj}(P) = \mathrm{amaj}(T)$. But this is NOT true.

1	2	3	1	3	4	1	2	4	1	2	3	1	2	4	1	2	3
1	2	4	2	3	4	1	3	4	1	3	4	2	3	4	2	3	4

1 1 0 2   0 2 1 1   1 0 2 1   1 0 1 2   0 1 2 1   0 1 1 2

A combinatorial proof is given by Xiaomei Chen, 2019.

Xiaomei Chen, A note on the distribution of major index for Schröder paths, arxiv:1906.09018v1.

Xiaomei Chen, Generalized Schröder paths and Young tableaux with skew shapes, arxiv: 2002.02410.

1. Definitions and Backgrounds
2. Major and Amajor index for row-increasing tableaux
3. Major index of Schröder n-paths
4. Distribution of descents in row-increasing tableaux
5. Cyclic Sieving

Distribution of descents in SYT

Theorem (R. Stanley)
Let $|\lambda/\mu| = n$. For any $1 \leq i \leq n-1$, the number $d_i(\lambda/\mu)$ of SYTs of shape λ/μ for which $i \in D(T)$ is independent of i.

1	2	**3**
4	5	6

1	**2**	**4**
3	5	6

1	**2**	**5**
3	4	6

1	3	**4**
2	5	6

1	**3**	**5**
2	4	6

Similar results hold for increasing tableaux and row-increasing tableaux of shape $2 \times n$.

1	2	***3***
2	4	5

1	***2***	***4***
2	3	5

1	***2***	***3***
3	4	5

1	***2***	***4***
3	4	5

1	***3***	***4***
2	4	5

Distribution of descents in row-increasing tableaux

Theorem

For any positive integers n, a, i and k, with $a \leq \lfloor \frac{n}{2} \rfloor$, $k \leq a$, and $i \leq n-1$, there is a bijection $f : \mathrm{RInc}_k^{(i)}(n-a,a) \mapsto \mathrm{RInc}_k(n-a-1,a-1) \bigcup \mathrm{RInc}_{k-1}(n-a-1,a-1)$.

Corollary

For any positive integers n, a, i and k, with $a \leq \lfloor \frac{n}{2} \rfloor$, $k \leq a$, and $i \leq n-1$, the following numbers are all independent of i.

1. The number of row-increasing tableaux in $\mathrm{RInc}_k^{(i)}(n-a,a)$;
2. The number of increasing tableaux in $\mathrm{Inc}_k^{(i)}(n-a,a)$;
3. The number of SYTs in $SYT^{(i)}(n-a,a)$.

For skew shapes of two rows, the above results are still true.

Counting tableaux with a given integer as a descent

We also gave the exact formula of d_i for incresing tableaux and row-increasing tableaux of shape $2 \times n$.

Theorem

Given positive integers n, a, k and i, with $a \leq \lfloor n/2 \rfloor$, $k \leq a$ and $i \leq n-1$, the number of row-increasing tableaux in $\mathrm{RInc}_k^{(i)}(n-a, a)$ is

$$R^{(i)}_{(n-a,a),k} = \frac{(n-2a+1)(na-a^2+a-2k)}{k(n-k-1)(n-a-k+1)} \binom{n-k-1}{k-1} \binom{n-2k}{a-k}. \tag{9}$$

Corollary

Given positive integers n, i, with $k \leq n$ and $i \leq 2n-1$, the number of row-increasing tableaux in $RInc_k(2 \times n)$ with i as a descent is

$$r(n,k)^{(i)} = \frac{n^2+n-2k}{k(2n-k-1)(n-k+1)} \binom{2n-k-1}{k-1} \binom{2n-2k}{n-k}. \tag{10}$$

Corollary

Given positive integers n, a, k and i, with $a \leq \lfloor n/2 \rfloor$, $k \leq a$ and $i \leq n-1$, the number of increasing tableaux in $\mathrm{Inc}_k^{(i)}(n-a, a)$ is

$$S^{(i)}_{(n-a,a),k} = \frac{n-2a+1}{n-a-k}\binom{n-k-2}{k}\binom{n-2k-2}{a-k-1}. \tag{11}$$

Corollary

Given positive integers n, i, with $k \leq n$ and $i \leq 2n-1$, the number of increasing tableaux in $RInc_k(2 \times n)$ with i as a descent is

$$s(n,k)^{(i)} = \frac{1}{n-k}\binom{2n-k-2}{k}\binom{2n-2k-2}{n-k-1}. \tag{12}$$

Similar results for row-increasing tableaux and increasing tableaux of two-row skew shapes are also obtained.

Cyclic Sieving: definition

- Let X be a set with an action by the cyclic group $\mathcal{C}_n = \langle c \rangle$.
- Let $f \in \mathbb{Z}[q]$ be a polynomial in q.

Definition (Reiner-Stanton-White, 2004)
The triple $(X, \mathcal{C}_n, f)$ has the cyclic sieving phenomenon if for all m, the number of elements of X fixed by c^m is $f(\zeta^m)$, where ζ is any primitive n-th root of unity.

V. Reiner, D. Stanton, and D. White. The cyclic sieving phenomenon. *J. Combin. Theory Ser. A*, 108:17–50, 2004.

V. Reiner, D. Stanton, and D. White. What is … cyclic sieving? *Notices Amer. Math. Soc.* 61 (2014), no. 2, 169–171.

Promotion (Jeu-de-taquin)

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|}\hline & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|}\hline 2 & & 4 \\ \hline 3 & 5 & 6 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|}\hline 2 & 4 & \\ \hline 3 & 5 & 6 \\ \hline \end{array} \mapsto$$

$$\mapsto \begin{array}{|c|c|c|}\hline 2 & 4 & 6 \\ \hline 3 & 5 & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|}\hline 2 & 4 & 6 \\ \hline 3 & 5 & 7 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline \end{array} = \mathcal{P}(T)$$

Theorem (M. Haiman, M.-P. Schutzenberger, . . .)
Promotion induces an action on $\mathrm{SYT}(2 \times n)$ by the cyclic group $\mathcal{C}_{2n}$.

M. Haiman. Dual equivalence with applications, including a conjecture of Proctor. *Discrete Math.*, 99:79–113, 1992.

R. Stanley. Promotion and evacuation. *Electron. J. Combin.*, 16(2):1–24, 2009.

Theorem (D. White, 2007)
The tripe $(\mathrm{SYT}(2\times n), \mathcal{C}_{2n}, \frac{1}{[n+1]}\begin{bmatrix}2n\\n\end{bmatrix})$ has the cyclic sieving phenomenon.

$$\text{Orbit A: } \left\{ \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array} \right\}$$

$$\text{Orbit B: } \left\{ \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array} \right\}$$

$$f(q) = \frac{1}{[3+1]}\begin{bmatrix}6\\3\end{bmatrix} = 1 + q^2 + q^3 + q^4 + q^6.$$

$$f\left(e^{\pi i/3}\right) = 0, \quad f\left((e^{\pi i/3})^2\right) = 2, \quad f\left((e^{\pi i/3})^3\right) = 3,$$
$$f\left((e^{\pi i/3})^4\right) = 2, \quad f\left((e^{\pi i/3})^5\right) = 0, \quad f\left((e^{\pi i/3})^6\right) = 5.$$

Cyclic Sieving of Increasing tableaux

Theorem (C. Pechenik)

For all n and k, there is an action of the cyclic group $\mathcal{C}_{2n-k}$ on $T \in \mathrm{Inc}_k(2 \times n)$, where a generator acts by K-promotion.

Theorem (C. Pechenik)

For all n and k, the triple $(\mathrm{Inc}_k(2 \times n), \mathcal{C}_{2n-k}, f)$ has the cyclic sieving phenomenon, where

$$f(q) = \frac{S_q(n,k)}{q^{n+k(k+1)/2}} = \frac{1}{[n+1]} \begin{bmatrix} n-1 \\ k \end{bmatrix} \begin{bmatrix} 2n-k \\ n \end{bmatrix}.$$

O. Pechenik, Cyclic Sieving of Increasing Tableaux and Small Schröder Paths. *J. Combin. Theory Ser. A*, 125: 357–378, 2014.

Cyclic Sieving of row increasingTableaux

There are similar results for row-increasing tableaux of any rectangular shape.

Theorem

Let $k \geqslant 0$ and let $\lambda \vdash$ n be a rectangular partition. Let $X = CST(\lambda, k)$ and let $C = \mathbb{Z}/k\mathbb{Z}$ act on X via jeu-de-taquin promotion. Then, the triple $(X, C, X(q))$ exhibits the cyclic sieving phenomenon, where $X(q)$ is a q-shift of the principal specialization of the Schur function

$$X(q) := q^{-\kappa(\lambda)} s_\lambda \left(1, q, q^2, \ldots, q^{k-1}\right)$$

B. Rhoades. Cyclic sieving, promotion, and representation theory. *J. Combin. Theory Ser. A*, 117:38–76, 2010.

How about other shapes?

Thank you!