行列式的计算方法

1.利用定义计算.

$$\begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 2 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & n-1 & \cdots & 0 & 0 \\ n & 0 & \cdots & 0 & 0 \end{vmatrix} = (-1)^{\tau(n\cdots 321)} n! = (-1)^{\frac{n(n-1)}{2}} n!.\ \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & n-1 \\ n & 0 & 0 & \cdots & 0 \end{vmatrix} = (-1)^{\tau(23\cdots n1)} n! = (-1)^{n-1} n!$$

2. 利用行列式性质把行列式化为上、下三角形行列式.

$$\begin{vmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{vmatrix} = 6 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{vmatrix} = 6 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix} = 48.$$

$$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix} = \begin{vmatrix} 1+x & 1 & 1 & 1 \\ -x & -x & 0 & 0 \\ 1 & 1 & 1+y & 1 \\ 0 & 0 & -y & -y \end{vmatrix} = \begin{vmatrix} x & 1 & 1 & 1 \\ 0 & -x & 0 & 0 \\ 0 & 1 & y & 1 \\ 0 & 0 & 0 & -y \end{vmatrix} = x^2 y^2$$

3. 行列式按一行(一列)展开,或按多行(多列)展开(Laplace 定理).

公式: $\begin{vmatrix} A & 0 \\ C_1 & B \end{vmatrix} = \begin{vmatrix} A & C_2 \\ 0 & B \end{vmatrix} = |A||B|$, $\begin{vmatrix} 0 & A \\ B & D_1 \end{vmatrix} = \begin{vmatrix} D_2 & A \\ B & 0 \end{vmatrix} = (-1)^{mn}|A||B|$,其中 A, B 分别是 m, n 阶的方阵.

例子: $D_{2n} = \begin{vmatrix} a & & & & & b \\ & \ddots & & & \iddots & \\ & & a & b & & \\ & & b & a & & \\ & \iddots & & & \ddots & \\ b & & & & & a \end{vmatrix}_{2n}$.

利用 Laplace 定理,按第 $n, n+1$ 行展开,除 2 级子式 $\begin{vmatrix} a & b \\ b & a \end{vmatrix}$ 外其余由第 $n, n+1$ 行所得的 2 级子式均为零.

故 $D_{2n} = \begin{vmatrix} a & b \\ b & a \end{vmatrix} (-1)^{n+n+1+n+n+1} D_{2n-2} = (a^2-b^2) D_{2n-2}$,此为递推公式,应用可得 $D_{2n} = (a^2-b^2)^n$.

$$D_{2n} = \begin{vmatrix} a & & & & & b \\ & \ddots & & & \iddots & \\ & & a & b & & \\ & & b & a & & \\ & \iddots & & & \ddots & \\ b & & & & & a \end{vmatrix} = \begin{vmatrix} a & & & & & b \\ & \ddots & & & \iddots & \\ & & a & b & & \\ & & a+b & a+b & & \\ & \iddots & & & \ddots & \\ a+b & & & & & a+b \end{vmatrix} = \begin{vmatrix} a-b & & & & & b \\ & \ddots & & & \iddots & \\ & & a-b & b & & \\ & & 0 & a+b & & \\ & \iddots & & & \ddots & \\ 0 & & & & & a+b \end{vmatrix}$$

$= (a-b)^n (a+b)^n = (a^2-b^2)^n$.

4. 箭头形行列式或者可以化为箭头形的行列式.

例: $$\begin{vmatrix} a_0 & 1 & 1 & \cdots & 1 \\ 1 & a_1 & 0 & \cdots & 0 \\ 1 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & a_n \end{vmatrix} = \begin{vmatrix} a_0-\sum_{i=1}^{n}\frac{1}{a_i} & 1 & 1 & \cdots & 1 \\ 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{vmatrix} = (a_0-\sum_{i=1}^{n}\frac{1}{a_i})a_1a_2\cdots a_n .$$

例: $$\begin{vmatrix} x_1-m & x_2 & x_3 & \cdots & x_n \\ x_1 & x_2-m & x_3 & \cdots & x_n \\ x_1 & x_2 & x_3-m & \cdots & x_n \\ \vdots & \vdots & \vdots & & \vdots \\ x_1 & x_2 & x_3 & \cdots & x_n-m \end{vmatrix} = \begin{vmatrix} x_1-m & x_2 & x_3 & \cdots & x_n \\ m & -m & 0 & \cdots & 0 \\ m & 0 & -m & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ m & 0 & 0 & \cdots & 0-m \end{vmatrix}$$

$$= \begin{vmatrix} \sum_{i=1}^{n}x_i-m & x_2 & x_3 & \cdots & x_n \\ 0 & -m & 0 & \cdots & 0 \\ 0 & 0 & -m & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0-m \end{vmatrix} = (\sum_{i=1}^{n}x_i-m)(-m)^{n-1}.$$

例: $$\begin{vmatrix} x_1 & a_2 & a_3 & \cdots & a_n \\ a_1 & x_2 & a_3 & \cdots & a_n \\ a_1 & a_2 & x_3 & \cdots & a_n \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & a_3 & \cdots & x_n \end{vmatrix} = \begin{vmatrix} x_1 & a_2 & a_3 & \cdots & a_n \\ a_1-x_1 & x_2-a_2 & 0 & \cdots & 0 \\ a_1-x_1 & 0 & x_3-a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_1-x_1 & 0 & 0 & \cdots & x_n-a_n \end{vmatrix}$$ (第一行的倍加到其余各行)

$$= \prod_{i=1}^{n}(x_i-a_i)\cdot \begin{vmatrix} \frac{x_1}{x_1-a_1} & \frac{a_2}{x_2-a_2} & \frac{a_3}{x_3-a_3} & \cdots & \frac{a_n}{x_n-a_n} \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{vmatrix}$$ (每一列提出相应的公因子 x_i-a_i)

$$= \prod_{i=1}^{n}(x_i-a_i)\cdot \begin{vmatrix} \frac{x_1}{x_1-a_1}+\sum_{i=2}^{n}\frac{a_i}{x_i-a_i} & \frac{a_2}{x_2-a_2} & \frac{a_3}{x_3-a_3} & \cdots & \frac{a_n}{x_n-a_n} \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix} = (1+\sum_{i=1}^{n}\frac{a_i}{x_i-a_i})\prod_{i=1}^{n}(x_i-a_i).$$

例: $\begin{vmatrix} 1 & 1 & \cdots & 1 & a_0 \\ 0 & 0 & \cdots & a_1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a_{n-1} & \cdots & 0 & 1 \\ a_n & 0 & \cdots & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & \cdots & 1 & a_0-\sum_{i=1}^{n}\dfrac{1}{a_i} \\ 0 & 0 & \cdots & a_1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a_{n-1} & \cdots & 0 & 0 \\ a_n & 0 & \cdots & 0 & 0 \end{vmatrix} = (-1)^{\frac{n(n+1)}{2}} a_1a_2\cdots a_n(a_0-\sum_{i=1}^{n}\dfrac{1}{a_i})$.

其它的例子：特点是除了主对角线,其余位置上的元素各行或各列都相同.

$$\begin{vmatrix} a+x_1 & a & a & \cdots & a \\ a & a+x_2 & a & \cdots & a \\ a & a & a+x_3 & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ a & a & a & \cdots & a+x_n \end{vmatrix}, \begin{vmatrix} x+a_1 & a_2 & a_3 & \cdots & a_n \\ a_1 & x+a_2 & a_3 & \cdots & a_n \\ a_1 & a_2 & x+a_3 & \cdots & a_n \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & a_3 & \cdots & x+a_n \end{vmatrix}.$$

5. 逐行逐列相加减.行列式特点是每相邻两行(列)之间有许多元素相同.用逐行(列)相减可以化出零.

例: $\begin{vmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 2 & 2 & \cdots & 2 \\ 2 & 2 & 3 & \cdots & 2 \\ \vdots & \vdots & \vdots & & \vdots \\ 2 & 2 & 2 & \cdots & n \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 \\ 2 & 2 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & n-2 \end{vmatrix} = (-2)(n-2)!$.

例: $\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 2 & 3 & \cdots & n-1 & n \\ 3 & 3 & 3 & \cdots & n-1 & n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-1 & n-1 & n-1 & \cdots & n-1 & n \\ n & n & n & \cdots & n & n \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & 0 \\ n & n & n & \cdots & n & n \end{vmatrix} = (-1)^{n-1}n$.

例: $\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-1 & n & 1 & \cdots & n-3 & n-2 \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 1 & 1 & \cdots & 1 & 1-n \\ 1 & 1 & 1 & \cdots & 1-n & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1-n & \cdots & 1 & 1 \\ 1 & 1-n & 1 & \cdots & 1 & 1 \end{vmatrix}$

$$= \begin{vmatrix} \dfrac{n(n+1)}{2} & 2 & 3 & \cdots & n-1 & n \\ 0 & 1 & 1 & \cdots & 1 & 1-n \\ 0 & 1 & 1 & \cdots & 1-n & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & 1-n & \cdots & 1 & 1 \\ 0 & 1-n & 1 & \cdots & 1 & 1 \end{vmatrix} = \frac{n(n+1)}{2} \begin{vmatrix} 1 & 1 & \cdots & 1 & 1-n \\ 1 & 1 & \cdots & 1-n & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1-n & \cdots & 1 & 1 \\ 1-n & 1 & \cdots & 1 & 1 \end{vmatrix}$$

$$=\frac{n(n+1)}{2}\begin{vmatrix}1&1&\cdots&1&1-n\\0&0&\cdots&-n&n\\\vdots&\vdots&&\vdots&\vdots\\0&-n&\cdots&0&n\\-n&0&\cdots&0&n\end{vmatrix}=\frac{n(n+1)}{2}\begin{vmatrix}1&1&\cdots&1&-1\\0&0&\cdots&-n&0\\\vdots&\vdots&&\vdots&\vdots\\0&-n&\cdots&0&0\\-n&0&\cdots&0&0\end{vmatrix}$$

$$=\frac{n(n+1)}{2}(-1)^{\frac{(n-1)(n-2)}{2}}(-1)^{n-1}n^{n-2}=(-1)^{\frac{n(n-1)}{2}}\frac{n+1}{2}n^{n-1}.$$

例: $$\begin{vmatrix}1&2&3&\cdots&n-1&n\\x&1&2&\cdots&n-2&n-1\\x&x&1&\cdots&n-3&n-2\\\vdots&\vdots&\vdots&&\vdots&\vdots\\x&x&x&\cdots&1&2\\x&x&x&\cdots&x&1\end{vmatrix}=\begin{vmatrix}1-x&1&1&\cdots&1&1\\0&1-x&1&\cdots&1&1\\0&0&1-x&\cdots&1&1\\\vdots&\vdots&\vdots&&\vdots&\vdots\\0&0&0&\cdots&1-x&1\\x&x&x&\cdots&x&1\end{vmatrix}$$

$$=\begin{vmatrix}1-x&x&0&\cdots&0&0\\0&1-x&x&\cdots&0&0\\0&0&1-x&\cdots&0&0\\\vdots&\vdots&\vdots&&\vdots&\vdots\\0&0&0&\cdots&1-x&x\\x&0&0&\cdots&0&1-x\end{vmatrix}=(1-x)^n+(-1)^{n+1}x^n.$$

6. 升阶法(或加边法, 添加一行一列,利于计算,但同时保持行列式不变).

例: $$\begin{vmatrix}a+x_1&a&a&\cdots&a\\a&a+x_2&a&\cdots&a\\a&a&a+x_3&\cdots&a\\\vdots&\vdots&\vdots&&\vdots\\a&a&a&\cdots&a+x_n\end{vmatrix}=\begin{vmatrix}1&a&a&a&\cdots&a\\0&a+x_1&a&a&\cdots&a\\0&a&a+x_2&a&\cdots&a\\0&a&a&a+x_3&\cdots&a\\\vdots&\vdots&\vdots&\vdots&&\vdots\\0&a&a&a&\cdots&a+x_n\end{vmatrix}$$

$$=\begin{vmatrix}1&a&a&a&\cdots&a\\-1&x_1&0&0&\cdots&0\\-1&0&x_2&0&\cdots&0\\-1&0&0&x_3&\cdots&0\\\vdots&\vdots&\vdots&\vdots&&\vdots\\-1&0&0&0&\cdots&x_n\end{vmatrix}=\begin{vmatrix}1+\sum_{i=1}^{n}\frac{a}{x_i}&a&a&a&\cdots&a\\0&x_1&0&0&\cdots&0\\0&0&x_2&0&\cdots&0\\0&0&0&x_3&\cdots&0\\\vdots&\vdots&\vdots&\vdots&&\vdots\\0&0&0&0&\cdots&x_n\end{vmatrix}=x_1x_2\cdots x_n(1+\sum_{i=1}^{n}\frac{a}{x_i}).$$

例: $\begin{vmatrix} 1+a_1+b_1 & a_1+b_2 & \cdots & a_1+b_n \\ a_2+b_1 & 1+a_2+b_2 & \cdots & a_2+b_n \\ \vdots & \vdots & & \vdots \\ a_n+b_1 & a_n+b_2 & \cdots & 1+a_n+b_n \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -b_1 & -b_2 & \cdots & -b_n \\ -a_1 & 0 & 1+a_1+b_1 & a_1+b_2 & \cdots & a_1+b_n \\ -a_2 & 0 & a_2+b_1 & 1+a_2+b_2 & \cdots & a_2+b_n \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ -a_n & 0 & a_n+b_1 & a_n+b_2 & \cdots & 1+a_n+b_n \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & -b_1 & -b_2 & \cdots & -b_n \\ -a_1 & 1 & 1 & 0 & \cdots & 0 \\ -a_2 & 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ -a_n & 1 & 0 & 0 & \cdots & 1 \end{vmatrix} = \begin{vmatrix} 1+\sum_{i=1}^n a_i & -n & 1 & 1 & \cdots & 1 \\ -\sum_{i=1}^n a_i b_i & 1+\sum_{i=1}^n b_i & -b_1 & -b_2 & \cdots & -b_n \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix} = \begin{vmatrix} 1+\sum_{i=1}^n a_i & -n \\ -\sum_{i=1}^n a_i b_i & 1+\sum_{i=1}^n b_i \end{vmatrix}$$

$$= (1+\sum_{i=1}^n a_i)(1+\sum_{i=1}^n b_i) - n\sum_{i=1}^n a_i b_i = 1+\sum_{i=1}^n a_i + \sum_{i=1}^n b_i + \sum_{j=1}^n \sum_{\substack{i=1 \\ i\neq j}}^n b_j (a_i - a_j)\,.$$

7. 利用范德蒙德行列式.

计算行列式: $D = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ x_1^{n-2} & x_2^{n-2} & x_3^{n-2} & \cdots & x_n^{n-2} \\ x_1^n & x_2^n & x_3^n & \cdots & x_n^n \end{vmatrix}$

解: 令: $D_1 = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ x_1 & x_2 & \cdots & x_n & y \\ x_1^2 & x_2^2 & \cdots & x_n^2 & y^2 \\ \vdots & \vdots & & \vdots & \vdots \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} & y^{n-2} \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} & y^{n-1} \\ x_1^n & x_2^n & \cdots & x_n^n & y^n \end{vmatrix}$,这是一个 $n+1$ 级范德蒙德行列式.

一方面,由范德蒙德行列式得 $D_1 = \prod_{1\le j<i\le n}(x_i - x_j)\cdot(y-x_1)(y-x_2)\cdots(y-x_n)$.可看做是关于 y 的一个 n 次多项式.

另一方面,将 D_1 按最后一列展开,可得一个关于 y 的多项式 $D_1 = p_n y^n + p_{n-1}y^{n-1} + \cdots + p_1 y + p_0$,其中

y^{n-1}的系数p_{n-1}与所求行列式D的关系为$D=-p_{n-1}$.

由$D_1=\prod_{1\le j<i\le n}(x_i-x_j)\cdot(y-x_1)(y-x_2)\cdots(y-x_n)$来计算$y^{n-1}$的系数$p_{n-1}=-\prod_{1\le j<i\le n}(x_i-x_j)\cdot\sum_{i=1}^{n}x_i$,

故有$D=-p_{n-1}=\prod_{1\le j<i\le n}(x_i-x_j)\cdot\sum_{i=1}^{n}x_i$

其它的例子:

$$\begin{vmatrix} a_1^n & a_1^{n-1}b_1 & a_1^{n-2}b_1^2 & \cdots & a_1b_1^{n-1} & b_1^n \\ a_2^n & a_2^{n-1}b_2 & a_2^{n-2}b_2^2 & \cdots & a_2b_2^{n-1} & b_2^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n+1}^n & a_{n+1}^{n-1}b_{n+1} & a_{n+1}^{n-2}b_{n+1}^2 & \cdots & a_{n+1}b_{n+1}^{n-1} & b_{n+1}^n \end{vmatrix} \text{(每一行提公因子} a_i^n\text{)}$$

$$=a_1^na_2^n\cdots a_{n+1}^n\begin{vmatrix} 1 & (\frac{b_1}{a_1}) & (\frac{b_1}{a_1})^2 & \cdots & (\frac{b_1}{a_1})^{n-1} & (\frac{b_1}{a_1})^n \\ 1 & (\frac{b_2}{a_2}) & (\frac{b_2}{a_2})^2 & \cdots & (\frac{b_2}{a_2})^{n-1} & (\frac{b_2}{a_2})^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & (\frac{b_{n+1}}{a_{n+1}}) & (\frac{b_{n+1}}{a_{n+1}})^2 & \cdots & (\frac{b_{n+1}}{a_{n+1}})^{n-1} & (\frac{b_{n+1}}{a_{n+1}})^n \end{vmatrix}=a_1^na_2^n\cdots a_{n+1}^n\prod_{1\le j<i\le n}(\frac{b_i}{a_i}-\frac{b_j}{a_j}).$$

8.利用数学归纳法证明行列式.(对行列式的级数归纳)

例：证明当$\alpha\ne\beta$时，$D_n=\begin{vmatrix} \alpha+\beta & \alpha\beta & 0 & \cdots & 0 & 0 \\ 1 & \alpha+\beta & \alpha\beta & \cdots & 0 & 0 \\ 0 & 1 & \alpha+\beta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha+\beta & \alpha\beta \\ 0 & 0 & 0 & \cdots & 1 & \alpha+\beta \end{vmatrix}=\dfrac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}$

证明:利用第二数学归纳法:若$n=1$,则$D_1=\alpha+\beta=\dfrac{\alpha^2-\beta^2}{\alpha-\beta}$,成立,假设结论对所有小于$n$的都成立,则将

D_n按第一行(或第一列)展开得$D_n=(\alpha+\beta)D_{n-1}-\alpha\beta D_{n-2}$,利用归纳假设可得

$D_n=(\alpha+\beta)D_{n-1}-\alpha\beta D_{n-2}=(\alpha+\beta)\dfrac{\alpha^n-\beta^n}{\alpha-\beta}-\alpha\beta\dfrac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}=\dfrac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}$.成立.

9. 利用递推公式.

例：计算行列式$D_n=\begin{vmatrix} \alpha+\beta & \alpha\beta & 0 & \cdots & 0 & 0 \\ 1 & \alpha+\beta & \alpha\beta & \cdots & 0 & 0 \\ 0 & 1 & \alpha+\beta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha+\beta & \alpha\beta \\ 0 & 0 & 0 & \cdots & 1 & \alpha+\beta \end{vmatrix}$

解: 按第一行展开得 $D_n=(\alpha+\beta)D_{n-1}-\alpha\beta D_{n-2}$,将此式化为:

(1) $D_n-\alpha D_{n-1}=\beta(D_{n-1}-\alpha D_{n-2})$ 或 (2) $D_n-\beta D_{n-1}=\alpha(D_{n-1}-\beta D_{n-2})$

利用公式(1)得 $D_n-\alpha D_{n-1}=\beta(D_{n-1}-\alpha D_{n-2})=\beta^2(D_{n-2}-\alpha D_{n-3})=\cdots=\beta^{n-2}(D_2-\alpha D_1)=\beta^n$,即

$D_n=\alpha D_{n-1}+\beta^n$. (3)

利用公式(2)得 $D_n-\beta D_{n-1}=\alpha(D_{n-1}-\beta D_{n-2})=\alpha^2(D_{n-2}-\beta D_{n-3})=\cdots=\alpha^{n-2}(D_2-\beta D_1)=\alpha^n$,即

$D_n=\beta D_{n-1}+\alpha^n$. (4)

由(3)(4) 解得: $D_n=\begin{cases}\dfrac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}, & \alpha\neq\beta\\ (n+1)\alpha^n, & \alpha=\beta\end{cases}$,

例: 计算行列式 $D_n=\begin{vmatrix}a & b & \cdots & 0 & 0\\ c & a & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & a & b\\ 0 & 0 & \cdots & c & a\end{vmatrix}_n$

解: 按第一行展开可得 $D_n=aD_{n-1}-bcD_{n-2}$,此时令 $\alpha+\beta=a,\alpha\beta=bc$, 则 $D_n=(\alpha+\beta)D_{n-1}-\alpha\beta D_{n-2}$,

变形为 $D_n-\alpha D_{n-1}=\beta(D_{n-1}-\alpha)D_{n-2}$,此为递推公式.利用刚才的例子可求得结果.

这里 $\alpha+\beta=a,\alpha\beta=bc$, 即 α,β 是方程 $x^2-ax+bc=0$ 的两个根.

例: $D_n=\begin{vmatrix}2 & 1 & & & \\ 1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & 1\\ & & & 1 & 2\end{vmatrix}_n=2D_{n-1}-D_{n-2}$,故 $D_n-D_{n-1}=D_{n-1}-D_{n-2}$,递推公式.

例: $D_n=\begin{vmatrix}7 & 2 & & & \\ 5 & 7 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 7 & 2\\ & & & 5 & 7\end{vmatrix}_n=7D_{n-1}-10D_{n-2}$.

10. 分拆法.将行列式的其中一行或者一列拆成两个数的和,将行列式分解成两个容易求的行列式的和.

例: $D=\begin{vmatrix}1+a_1+b_1 & a_1+b_2 & \cdots & a_1+b_n\\ a_2+b_1 & 1+a_2+b_2 & \cdots & a_2+b_n\\ \vdots & \vdots & & \vdots\\ a_n+b_1 & a_n+b_2 & \cdots & 1+a_n+b_n\end{vmatrix}$.

解: 取 $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$,记 $\alpha=(a_1,a_2,\cdots,a_n)^T$, $\beta_i=(b_i,b_i,\cdots,b_i)^T$,则

$$D=\left|\varepsilon_1+\alpha+\beta_1,\varepsilon_2+\alpha+\beta_2,\cdots,\varepsilon_n+\alpha+\beta_n\right|$$

$$=\left|\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n\right|+\sum_{i=1}^{n}\left|\varepsilon_1,\cdots,\varepsilon_{i-1},\alpha,\varepsilon_{i+1},\cdots,\varepsilon_n\right|+\sum_{i=1}^{n}\left|\varepsilon_1,\cdots,\varepsilon_{i-1},\beta_i,\varepsilon_{i+1},\cdots,\varepsilon_n\right|$$

$$+\sum_{1\le i<j\le n}\left|\varepsilon_1,\cdots,\varepsilon_{i-1},\alpha,\varepsilon_{i+1},\cdots,\varepsilon_{j-1},\beta_j,\varepsilon_{j+1},\cdots,\varepsilon_n\right|+\sum_{1\le i<j\le n}\left|\varepsilon_1,\cdots,\varepsilon_{i-1},\beta_i,\varepsilon_{i+1},\cdots,\varepsilon_{j-1},\alpha,\varepsilon_{j+1},\cdots,\varepsilon_n\right|$$

$$=1+\sum_{i=1}^{n}a_i+\sum_{i=1}^{n}b_i+\sum_{1\le i<j\le n}(a_i-a_j)b_j+\sum_{1\le i<j\le n}(a_i-a_j)b_i=\cdots$$

例: $D_n=\begin{vmatrix} a & b & b & \cdots & b & b \\ c & a & b & \cdots & b & b \\ c & c & a & \cdots & b & b \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c & c & c & \cdots & a & b \\ c & c & c & \cdots & c & a \end{vmatrix}=\begin{vmatrix} c+a-c & b & b & \cdots & b & b \\ c & a & b & \cdots & b & b \\ c & c & a & \cdots & b & b \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c & c & c & \cdots & a & b \\ c & c & c & \cdots & c & a \end{vmatrix}$

$$=\begin{vmatrix} c & b & b & \cdots & b & b \\ c & a & b & \cdots & b & b \\ c & c & a & \cdots & b & b \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c & c & c & \cdots & a & b \\ c & c & c & \cdots & c & a \end{vmatrix}+\begin{vmatrix} a-c & b & b & \cdots & b & b \\ 0 & a & b & \cdots & b & b \\ 0 & c & a & \cdots & b & b \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & c & c & \cdots & a & b \\ 0 & c & c & \cdots & c & a \end{vmatrix}=V_1+V_2$$

V_1: 第一列提出公因子 c 后,第一列的 $-b$ 倍加到其余各列上,V_2 按第一列展开.

$$V_1=c\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & a-b & 0 & \cdots & 0 & 0 \\ 1 & c-b & a-b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & c-b & c-b & \cdots & a-b & 0 \\ 1 & c-b & c-b & \cdots & c-b & a-b \end{vmatrix}=c(a-b)^{n-1}$$

$V_2=(a-c)D_{n-1}$, 故 $D_n=c(a-b)^{n-1}+(a-c)D_{n-1}$,

由 b,c 的对称性质,亦可得 $D_n=b(a-c)^{n-1}+(a-b)D_{n-1}$,这两个式子中消去 D_{n-1},可得结论,

$$D_n=\frac{c(a-b)^n-b(a-c)^n}{c-b}.$$

注: (1) 同一个行列式,可有多种计算方法.要利用行列式自身元素的特点,选择合适的计算方法.

(2) 以上的各种方法并不是互相独立的,计算一个行列式时,有时需要综合运用以上方法,