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二、计算 n 级行列式 $D_n = \begin{vmatrix} 1 & 1 & \cdots & 1 & 2-n \\ 1 & 1 & \cdots & 2-n & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 2-n & 1 & \cdots & 1 & 1 \end{vmatrix}$.

法一

解 $D_n = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 2-n & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 2-n & 1 & \cdots & 1 & 1 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 1-h & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 1-h & 0 & \cdots & 0 & 0 \end{vmatrix}$$

$$= (-1)^{inv(n \cdots 21)} (1-h)^{n-1}$$

$$= (-1)^{\frac{n(n-1)}{2}} (1-h)^{n-1}$$

法二

$$D_n = \begin{vmatrix} 1 & 1 & \cdots & 1 & 2-n \\ 0 & 0 & \cdots & 1-n & n-1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1-h & \cdots & 0 & 0 \\ 1-h & n-1 & \cdots & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & \cdots & n-1 & 1 \\ 0 & 0 & \cdots & 1-n & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1-h & \cdots & 0 & 0 \\ 1-h & 0 & \cdots & 0 & 0 \end{vmatrix}$$

$$= (-1)^{inv(n \cdots 21)} (1-h)^{n-1}$$

$$= (-1)^{\frac{n(n-1)}{2}} (1-h)^{n-1}$$

法三

$$D_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 2-n \\ 0 & 1 & 1 & \cdots & 2-n & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 2-n & 1 & \cdots & 1 & 1 \end{vmatrix}_{n+1}$$

$$= \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ -1 & 0 & \cdots & 0 & 1-h \\ \vdots & \vdots & & \vdots & \vdots \\ -1 & 1-h & \cdots & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{1-h} & 1 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1-h \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1-h & \cdots & 0 & 0 \end{vmatrix}$$

$$= (-1)^{\frac{n(n-1)}{2}} (1-h)^{n-1}$$

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三、计算行列式 $D = \begin{vmatrix} a+x_1 & a & a & \cdots & a \\ a & a+x_2 & a & \cdots & a \\ a & a & a+x_3 & \cdots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a & a & a & \cdots & a+x_n \end{vmatrix}.$

解 $D = \begin{vmatrix} 1 & a & a & \cdots & a \\ 0 & a+x_1 & a & \cdots & a \\ 0 & a & a+x_2 & \cdots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a & a & \cdots & a+x_n \end{vmatrix} = \begin{vmatrix} 1 & a & a & \cdots & a \\ -1 & x_1 & 0 & \cdots & 0 \\ -1 & 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & x_n \end{vmatrix} = \begin{vmatrix} 1+a\sum_{i=1}^n \frac{1}{x_i} & a & a & \cdots & a \\ 0 & x_1 & 0 & \cdots & 0 \\ 0 & 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_n \end{vmatrix} = \left(1+a\sum_{i=1}^n \frac{1}{x_i}\right) \left(\prod_{i=1}^n x_i\right)$

四、问 λ, μ 取何值时？齐次线性方程组 $\begin{cases} \lambda x_1 + x_2 + x_3 = 0, \\ x_1 + \mu x_2 + x_3 = 0, \\ x_1 + 2\mu x_2 + x_3 = 0 \end{cases}$ 有非零解。

解 $D = \begin{vmatrix} \lambda & 1 & 1 \\ 1 & \mu & 1 \\ 1 & 2\mu & 1 \end{vmatrix} = \begin{vmatrix} \lambda-1 & 0 & 1 \\ 0 & \mu-1 & 1 \\ 0 & 2\mu-1 & 1 \end{vmatrix} = (\lambda-1) \begin{vmatrix} \mu-1 & 1 \\ 2\mu-1 & 1 \end{vmatrix} = (\lambda-1)(-\mu) = 0$

当 $\lambda=1$ 或 $\mu=0$ 时，这个方程组有非零解。

五、证明 $D_{2n} = \begin{vmatrix} a_n & & & & b_n \\ & \ddots & & & \vdots \\ & & a_1 & b_1 & \\ & & c_1 & d_1 & \\ & \ddots & & & \ddots \\ c_n & & & & & d_n \end{vmatrix} = \prod_{i=1}^n (a_i d_i - b_i c_i).$

证 对 n 用数学归纳法。

$n=1$ 时， $D_2 = \begin{vmatrix} a_1 & b_1 \\ c_1 & d_1 \end{vmatrix}.$

假设等式对 $n-1$ 成立。

$$\begin{aligned} D_{2n} &= \begin{vmatrix} a_n & b_n \\ c_n & d_n \end{vmatrix} D_{2n-2} \\ &= (a_n d_n - b_n c_n) \prod_{i=1}^{n-1} (a_i d_i - b_i c_i) \\ &= \prod_{i=1}^n (a_i d_i - b_i c_i). \end{aligned}$$