Introduction to Lorentzian polynomials

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Outline

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Matroid

Let E be a finite set and $\mathcal{I} \subset 2^E$. A matroid M is an ordered pair $(E, \mathcal{I})$ satisfying

(1) $\emptyset \in \mathcal{I}$;

(2) If $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$; (hereditary property)

(3) If $A, B \in \mathcal{I}$ and $|B| > |A|$, then $\exists x \in B$ such that $A \cup x \in \mathcal{I}$. (exchange property)

The set E is said to be the ground set. A set $A \in \mathcal{I}$ is called an independent set.

For example, if E is a finite set of vectors in some vector space, then the collection of linearly independent vectors from E form the independent sets of a matroid.

Example	Ground set	Independent set
Graphic matroid	edge set of a graph	forest (no cycle)
Uniform matroid $U_{m,d}$	a finite set $[m+d]$	subset of cardinality $\leq d$
Representable matroid	a set of vectors over a field	linearly independent vectors

In particular, the unifrom matroid $U_{0,d}$ is the Boolean matorid.

Rank and submodularity

Let M be a matroid on E with independent sets $\mathcal{I}$, and $X, Y \subseteq E$.

Definition

The rank of X is the maximum size of an independent set in X.

We denote rank by $r(X)$.

The rank function is monotonic:

$$X \subseteq Y \Rightarrow r(X) \leq r(Y)$$

and submodular:

$$r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$$

Flats of a matroid

Definition

Let M be a matroid on ground set E, and $F \subseteq E$, then F is a flat if for every $x \in E \backslash F$

$$r(F) < r(F \cup x)$$

For a vector configuration, the flats correspond to the spans of subsets of vectors.

The set of flats of M forms a lattice, which we denote by $\mathcal{L}(M)$.

Characteristic polynomial of a matroid

The characteristic polynomial of a matroid M is defined to be

$$\chi_M(t) = \sum_{F \in L(M)} \mu(\emptyset, F) t^{r(M)-r(F)}$$

or

$$\chi_M(t) = \sum_{S \subset E} (-1)^{|S|} t^{r(M)-r(S)}.$$

Example

For a graph G, $\chi_{M(G)}(t) = t^{-c}\chi_G(t)$, where $\chi_G(t)$ is the chromatic polynomial of G and c is the number of connected components of G.

Log-concavity

A polynomial

$$f(t) = a_0 + a_1 t + \cdots + a_n t^n$$

with real coefficients is said to be log-concave if

$$a_i^2 \geq a_{i-1} a_{i+1}$$

for any $0 < i < n$, and it is said to have no internal zeros if there are not three indices $0 \leq i < j < k \leq n$ such that $a_i, a_k \neq 0$ and $a_j = 0$.

- Richard P. Stanley. Log-concave and unimodal sequences in algebra, combinatorics, and geometry. In: Graph theory and its applications: East and West (Jinan, 1986). Vol. 576. Ann. New York Acad. Sci. New York Acad. Sci., New York, 1989, 500–535.
- Francesco Brenti, Log-concave and Unimodal sequences in Algebra, Combinatorics, and Geometry: an update, Contemporary Math., 178 (1994), 71-89.
- Andrei Okounkov. Why would multiplicities be log-concave? In: The orbit method in geometry and physics (Marseille, 2000). Vol. 213. Progr. Math. Birkhäuser Boston, Boston, MA, 2003, 329–347

Conjecture (Heron (1972), Rota (1971), Welsh (1976))

For any matroid M, the characteristic polynomial $\chi_M(t)$ is a log-concave polynomial with no internal zeros.

Solved.

The proof of log-concavity follows from an application of the Hodge-Riemann relations in degree one (one positive eigenvalue condition).

- June Huh(许埈珥), Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs, Journal of the American Mathematical Society 25 (2012), 907–927.
- Karim Adiprasito, June Huh, and Eric Katz. Hodge theory for combinatorial geometries, Ann. of Math. (2) 188 (2018),381–452.
- Tom Braden, June Huh, Jacob P. Matherne, Nicholas Proudfoot, Botong Wang(王博潼), A semi-small decomposition of the Chow ring of a matroid, arXiv:2002.03341.

Conjecture (Mason(1972))

For any matroid M on $[n]$ and any positive integer k

(i) $I_k(M)^2 \geqslant I_{k-1}(M)I_{k+1}(M)$,

(ii) $I_k(M)^2 \geqslant \frac{k+1}{k} I_{k-1}(M)I_{k+1}(M)$,

(iii) $I_k(M)^2 \geqslant \frac{k+1}{k}\frac{n-k+1}{n-k} I_{k-1}(M)I_{k+1}(M)$, *i.e.* $\frac{I_k(M)^2}{\binom{n}{k}^2} \geqslant \frac{I_{k+1}(M)}{\binom{n}{k+1}}\frac{I_{k-1}(M)}{\binom{n}{k-1}}$

where $I_k(M)$ is the number of k-element independent sets of M.

(i) was proved in

- Karim Adiprasito, June Huh, and Eric Katz, Hodge theory for combinatorial geometries. Ann. of Math. (2) 188 (2018), no. 2, 381–452.

(ii) was prove in

- June Huh, Benjamin Schroter and Botong Wang, Correlation bounds for fields and matroids. arXiv:1806.02675.

(iii), called ultra log-concavity,

$$\frac{I_k(M)^2}{\binom{n}{k}^2} \geqslant \frac{I_{k+1}(M)}{\binom{n}{k+1}} \frac{I_{k-1}(M)}{\binom{n}{k-1}}$$

was proved by

- Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant, Log-Concave Polynomials III: Mason's Ultra-Log-Concavity Conjecture for Independent Sets of Matroids, arXiv:1811.01600.
- Petter Brändén, June Huh, Hodge-Riemann relations for Potts model partition functions, arXiv:1811.01696
- Petter Brändén, June Huh, Lorentzian polynomials, Annals of Mathematics 192 (2020), to appear. arXiv:1902.03719.

The equality holds for the Boolean matroid, which is the uniform matroid of rank n on an n element ground set.

A generalization was in

- Christopher Eur, June Huh, Logarithmic concavity for morphisms of matroids, Advances in Mathematics 367 (2020), 107094.

Outline

Newton's inequalities

Newton's inequalities on the coefficients of a polynomial look very similar to Mason's inequalities.

To state these inequalities, let $P(x, y) = \prod_{i=1}^{n} (x + \alpha_i y)$ be a bivariate homogeneous polynomial with all $\alpha_i \in \mathbb{R}$.

We can write the coefficients in the expansion of $P(x, y)$ using the elementary symmetric functions as

$$P(x, y) = \prod_{i=1}^{n} (x + \alpha_i y) = \sum_{k=0}^{n} e_k (\alpha_1, \ldots, \alpha_n) x^{n-k} y^k$$

Briefly, let e_k denote $e_k(\alpha_1, \ldots, \alpha_n)$. Then, Newton's Inequalities say that for all $0 < k < n$, we have

$$\frac{e_k^2}{\binom{n}{k}^2} \geq \frac{e_{k-1}}{\binom{n}{k-1}} \cdot \frac{e_{k+1}}{\binom{n}{k+1}} \tag{1}$$

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Furthermore, the partial derivatives $\partial_x P(x,y)$ and $\partial_y P(x,y)$ both factor into a product of linear factors with real coefficients of the same form, so we can continue to apply partial derivatives until we get a quadratic polynomial.

It is easy to show

$$\partial_y^{k-1}\partial_x^{n-k-1}P(x,y) = n!\left(\frac{e_{k-1}x^2}{\left(\binom{n}{k}\right)} + \frac{2e_k xy}{\binom{n}{k}} + \frac{e_{k+1}y^2}{\binom{n}{k}}\right)$$

since $\partial_y^{k-1}\partial_x^{n-k-1}P(x,y)$ at $y=1$ has only real roots, the discriminant of this quadratic is nonnegative which implies (1).

The Hessian of a function $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ is $H_f = (\partial_i \partial_j f)_{i,j=1}^n$.

If we let $\tilde{e}_k = e_k/\binom{n}{k}$, then the Hessian of

$$Q =: \partial_y^{k-1}\partial_x^{n-k-1}P(x,y) = n!\left(\tilde{e}_{k-1}x^2 + 2\tilde{e}_{k-1}xy + \tilde{e}_{k+1}y^2\right)$$

is

$$H_Q = 2n!\begin{pmatrix} \tilde{e}_{k-1} & \tilde{e}_k \\ \tilde{e}_k & \tilde{e}_{k+1} \end{pmatrix}.$$

Observation: If $e_i \geq 0$ for all i, then H_Q has signature $(+,-),(+,0)$, or $(0,0)$.

Proof. H_Q is a symmetric matrix with nonnegative real entries, so it has two real eigenvalues.

If H_Q is not identically zero, then it has at least one positive eigenvalue since $H_Q\begin{pmatrix} 1 \\ 1 \end{pmatrix} > 0$. (The trace of a matrix is the sum of its eigenvalues)

The eigenvalues of such a 2×2 matrix can only come in three types $(+,+),(+,0)$, or $(+,-)$. If $\det H_Q \leq 0$, then H_Q has at most one positive eigenvalue. (The determinant of a matrix is the product of its eigenvalues)

More generally, for any real symmetric matrix A with nonzero eigenvalues, we say A has Lorentz signature $(+,-,-,\dots,-)$ if it has one positive eigenvalue and the rest are all negative.

Equivalently, if and only if the quadratic form $q = x^T Ax$ may be written as

$$q = \ell_1^2 - \ell_2^2 - \ell_3^2 - \cdots - \ell_n^2$$

where $\ell_i = x^T v_i$ for each $i, \{v_1, \dots, v_n\}$ is a basis for $\mathbb{R}^n$, and $x = (x_1, \dots, x_n)^T$ is the column vector of coordinates on $\mathbb{R}^n$

Outline

Let n and d be nonnegative integers, and set $[n] = \{1, \ldots, n\}$.

Let H_n^d be the set of degree d homogeneous polynomials in $\mathbb{R}[w_1, \ldots, w_n]$.

We define a topology on H_n^d using the Euclidean norm for the coefficients.

Let $\mathrm{P}_n^d \subseteq \mathrm{H}_n^d$ be the open subset of polynomials all of whose coefficients are positive.

Definition (Lorentzian polynomials)

We set $\mathring{\mathrm{L}}_n^0 = \mathrm{P}_n^0$, $\mathring{\mathrm{L}}_n^1 = \mathrm{P}_n^1$, *and*

$$\mathring{\mathrm{L}}_n^2 = \Big\{ f \in \mathrm{P}_n^2 \mid H_f \textit{ is nonsingular and has exactly one positive eigenvalue} \Big\}.$$

For d *larger than* 2, *we define* $\mathring{\mathrm{L}}_n^d$ *recursively by setting*

$$\mathring{\mathrm{L}}_n^d = \Big\{ f \in \mathrm{P}_n^d \mid \partial_i f \in \mathring{\mathrm{L}}_n^{d-1} \textit{ for all } i \in [n] \Big\}.$$

The polynomials in $\mathring{\mathrm{L}}_n^d$ *are called strictly Lorentzian, and the limits of strictly Lorentzian polynomials are called Lorentzian.*

Example

Let $f = \sum_{k=0}^{d} a_k x^k y^{d-k}$ be a homogeneous polynomial of degree $d \geq 2$, with all $a_k > 0$. *Under what conditions is $f \in \mathring{L}_n^2$?*

By definition, $f \in L_n^d$ if and only if every possible way to successively differentiate f down to a quadratic $Q = \partial_{i_1} \partial_{i_2} \cdots \partial_{i_{d-2}} f \in \mathring{L}_n^2$.

This is equivalent to requiring det $H_Q < 0$ since we have assumed $a_k > 0$ for all k.

As we saw, the condition $\det H_Q < 0$ is equivalent to saying the coefficients $a_1, \ldots, a_n$ is ultra log-concave.

Note that if f is a Lorentzian polynomial, then f has no internal zeros. (later)

Example

Consider the cubic form

$$f = 2w_1^3 + 12w_1^2 w_2 + 18w_1 w_2^2 + \theta w_2^3,$$

where θ is a real parameter.

A straightforward computation shows that

$$f \text{ is Lorentzian if and only if } 0 \leq \theta \leq 9$$

and

$$f \text{ is stable if and only if } 0 \leq \theta \leq 8.$$

Clearly, if f is in the closure of $\mathring{\mathrm{L}}_n^d$ in H_n^d, then f has nonnegative coefficients and

$$\partial^\alpha f \text{ has at most one positive eigenvalue for every } \alpha \in \Delta_n^{d-2}.$$

Example

The bivariate cubic

$$f = w_1^3 + w_2^3$$

shows that the converse fails.
In this case, $\partial_1 f$ and $\partial_2 f$ are Lorentzian, but f is not Lorentzian.

Outline

M-convex set

Denote by e_i the i-th standard basis vector of $\mathbb{N}^n$.

Definition

A collection $J \subset \mathbb{N}^n$ is M-convex or matroid-convex if it satisfies any one of the following equivalent conditions

- *For any $\alpha, \beta \in \mathrm{J}$ and any index i satisfying $\alpha_i > \beta_i$, there is an index j satisfying*

$$\alpha_j < \beta_j \;\; \textit{and} \;\; \alpha - e_i + e_j \in J.$$

- *For any $\alpha, \beta \in J$ and any index i satisfying $\alpha_i > \beta_i$, there is an index j satisfying*

$$\alpha_j < \beta_j \;\; \textit{and} \;\; \alpha - e_i + e_j \in J \;\; \textit{and} \;\; \beta - e_j + e_i \in J.$$

The first condition is called the exchange property for M-convex sets, and the second condition is called the symmetric exchange property for M-convex sets.

For example, $\{$(0,0),(-1,1),(-2,2)$\}$ is *M*-convex, but $\{$(0,0),(-2,2)$\}$ is not.

If $J \subset \{0,1\}^n$, then J is *M*-convex if and only if J is the set of bases of a matroid.

The convex hull of an M-convex set is a polytope also called a generalized permutahedron.

Characterization

The support of a multivariate polynomial $f = \sum a_\alpha x^\alpha$ where $x^\alpha = \prod x_i^{\alpha_i}$, is

$$\mathrm{supp}(f) = \{\alpha \in \mathbb{N}^n : a_\alpha \neq 0\}$$

Theorem

Let $f \in H_n^d$ be a homogeneous polynomial with nonnegative coefficients. Then f is Lorentzian if and only if

1. *The support of f is M-convex.*
2. *The Hessian of $\partial_{i_1}\partial_{i_2}\cdots\partial_{i_{d-2}} f$ has at most one positive eigenvalue for all $1 \leq i_1, i_2, \ldots, i_{d-2} \leq n$.*

Recall that a bivariate homogeneous polynomial $\sum_{k=0}^{d} a_k w_1^k w_2^{d-k}$ is strictly Lorentzian if and only if the sequence a_k is positive and strictly ultra log-concave.

Example

The above theorem says that, in this case, the polynomial $\sum_{k=0}^{d} a_k w_1^k w_2^{d-k}$ is Lorentzian if and only if the sequence a_k is nonnegative, ultra log-concave, and has no internal zeros.

Example

By the above theorem, it is straightforward to check that elementary symmetric polynomials are Lorentzian (stable indeed).

Define a generating polynomial for any finite subset $J \subset \mathbb{N}^n$ by

$$f_J := \sum_{\alpha \in J} \frac{x^\alpha}{\alpha!} := \sum_{\alpha \in J} \frac{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}{\alpha_1! \ldots \alpha_n!}$$

The property of a subset being *M*-convex is completely characterized by the Lorentzian property.

Theorem

If $J \subset \mathbb{N}^n$ is finite, then f_J is Lorentzian if and only if f J is M-convex.

Hodge–Riemann relation for Lorentzian polynomials

Theorem

Let f be a nonzero homogeneous polynomial in $\mathbb{R}[w_1, \ldots, w_n]$ of degree $d \geq 2$.

- *If f is in $\mathring{\mathrm{L}}^d_n$, then $H_f(w)$ is nonsingular for all $w \in \mathbb{R}^n_{>0}$.*
- *If f is in L^d_n, then $H_f(w)$ has exactly one positive eigenvalue for all $w \in \mathbb{R}^n_{>0}$.*

Note that for any nonzero degree $d \geq 2$ homogeneous polynomial f with nonnegative coefficients, the following conditions are equivalent:

- The function $f^{1/d}$ is concave on $\mathbb{R}^n_{>0}$.
- The function $\log f$ is concave on $\mathbb{R}^n_{>0}$.
- The Hessian of f has exactly one positive eigenvalue on $\mathbb{R}^n_{>0}$.

Let f be a polynomial in n variables with nonnegative coefficients.

Gurvits defines f to be strongly log-concave if, for all $\alpha \in \mathbb{N}^n$,

$$\partial^\alpha f \text{ is identically zero or } \log(\partial^\alpha f) \text{ is concave on } \mathbb{R}^n_{>0}.$$

Anari *et al.* define f to be completely log-concave if, for all $m \in \mathbb{N}$ and any $m \times n$ matrix (a_{ij}) with nonnegative entries,

$$\left(\prod_{i=1}^m D_i\right) f \text{ is identically zero or } \log\left(\left(\prod_{i=1}^m D_i\right) f\right) \text{ is concave on } \mathbb{R}^n_{>0},$$

where D_i is the differential operator $\sum_{j=1}^n a_{ij}\partial_j$.

Theorem

The following conditions are equivalent for any homogeneous polynomial f.

- *f is completely log-concave.*
- *f is strongly log-concave.*
- *f is Lorentzian.*

Theorem

If $f = \sum_{\alpha \in \Delta_n^d} \frac{c_\alpha}{\alpha!} w^\alpha$ *is a Lorentzian polynomial, then*

$$c_\alpha^2 \geq c_{\alpha+e_i-e_j} c_{\alpha-e_i+e_j} \quad \text{for any } i,j \in [n] \text{ and any } \alpha \in \Delta_n^d.$$

Proof Consider the Lorentzian polynomial $\partial^{\alpha-e_i-e_j} f$. Substituting w_k by zero for all k other than i and j, we get the bivariate quadratic polynomial

$$\frac{1}{2} c_{\alpha+e_i-e_j} w_i^2 + c_\alpha w_i w_j + \frac{1}{2} c_{\alpha-e_i+e_j} w_j^2.$$

The displayed polynomial is Lorentzian and hence $c_\alpha^2 \geq c_{\alpha+e_i-e_j} c_{\alpha-e_i+e_j}$. ∎

Linear operators preserving Lorentzian polynomials

Let κ be an element of $\mathbb{N}^n$, let γ be an element of $\mathbb{N}^m$, and set $k = |\kappa|_1$. Fix a linear operator

$$T : \mathbb{R}_\kappa[w_i] \to \mathbb{R}_\gamma[w_i],$$

and suppose that the linear operator T is *homogeneous of degree* ℓ for some $\ell \in \mathbb{Z}$:

$$\Big(0 \le \alpha \le \kappa \text{ and } T(w^\alpha) \neq 0\Big) \Longrightarrow \deg T(w^\alpha) = \deg w^\alpha + \ell.$$

The symbol of T is a homogeneous polynomial of degree $k + \ell$ in $m + n$ variables defined by

$$\mathrm{sym}_T(w, u) = \sum_{0 \le \alpha \le \kappa} \binom{\kappa}{\alpha} T(w^\alpha) u^{\kappa - \alpha}.$$

We show that the homogeneous operator T preserves the Lorentzian property if its symbol sym_T is Lorentzian.

Theorem

If $\mathrm{sym}_T \in \mathrm{L}^{k+\ell}_{m+n}$ *and* $f \in \mathrm{L}^d_n \cap \mathbb{R}_\kappa[w_i]$, *then* $T(f) \in \mathrm{L}^{d+\ell}_m$.

When $n = 2$, it provides a large class of linear operators that preserve the ultra log-concavity of sequences of nonnegative numbers with no internal zeros.

Examples

Consider the linear operator T which makes a nonnegative change of variables encoded by an $n \times n$ matrix $A = (a_{i,j})$ with nonnegative entries. By the usual matrix action on polynomials,

$$T(f) = f(Ax) = f\left(\sum_j a_{1,j}x_j, \sum_j a_{2,j}x_j, \dots, \sum_j a_{n,j}x_j\right)$$

In this case, if $T : P_\kappa \longrightarrow \mathbb{R}[x_1, \dots, x_n]$, then we claim T preserves the Lorentzian property. Observe,

$$G_T = T\left[(x_1 + y_1)^{\kappa_1}(x_2 + y_2)^{\kappa_2} \dots (x_n + y_n)^{\kappa_n}\right] = \prod_{i=1}^{n}\left(y_i + \sum_j a_{i,j}x_j\right)^{\kappa_i}$$

is homogeneous and stable since it does not vanish on the intersection of the positive imaginary halfplanes in $\mathbb{C}^n$. Hence G_T is Lorentzian. So, the claim holds.

We record some useful operators that preserves the Lorentzian property. The *multi-affine part* of a polynomial $\sum_{\alpha \in \mathbb{N}^n} c_\alpha w^\alpha$ is the polynomial $\sum_{\alpha \in \{0,1\}^n} c_\alpha w^\alpha$.

Corollary

The multi-affine part of any Lorentzian polynomial is a Lorentzian polynomial.

Let N be the linear operator defined by the condition $N(w^\alpha) = \frac{w^\alpha}{\alpha!}$. The normalization operator N turns generating functions into exponential generating functions.

Corollary

If f is a Lorentzian polynomial, then $N(f)$ is a Lorentzian polynomial.

Corollary below extends the classical fact that the convolution product of two log-concave sequences with no internal zeros is a log-concave sequence with no internal zeros.

Corollary

If $N(f)$ and $N(g)$ are Lorentzian polynomials, then $N(fg)$ is a Lorentzian polynomial.

If $f = f(w_1, w_2, \ldots, w_n)$ is a stable multi-affine polynomial with nonnegative coefficients, then the multi-affine polynomial $\Phi_\theta^{1,2}(f)$ defined by

$$\Phi_\theta^{1,2}(f) = (1-\theta) f(w_1, w_2, w_3, \ldots, w_n) + \theta f(w_2, w_1, w_3, \ldots, w_n)$$

is stable for all $0 \leq \theta \leq 1$.

An analog for Lorentzian polynomials is stated as follows.

Corollary

Let $f = f(w_1, w_2, \ldots, w_n)$ be a multi-affine polynomial with nonnegative coefficients. If the homogenization of f is a Lorentzian polynomial, then the homogenization of $\Phi_\theta^{1,2}(f)$ is a Lorentzian polynomial for all $0 \leq \theta \leq 1$.

Outline

Examples of Lorentzian polynomials

- homogeneous stable polynomials
- volume polynomials of convex bodies
- volume polynomials of projective varieties
- homogeneous multivariate Tutte polynomials of matroids (Mason's conjecture)
- multivariate characteristic polynomials of M-matrixs
- normalized Schur polynomials

Outline

Whitney numbers

One really challenging problem is a conjecture of Rota and Welsh.

Let W_k be the number of flats of rank k in a matroid M on a ground set of size n.

The numbers W_k are called Whitney numbers.

Conjecture (Rota(1971), Welsh(1976))

For any matroid M on $[n]$ and any positive integer $1 \leq k \leq n-1$

$$\frac{W_k^2}{\binom{n}{k}^2} \geq \frac{W_{k-1}}{\binom{n}{k-1}} \cdot \frac{W_{k+1}}{\binom{n}{k+1}}$$

Lots of conjectured unimodal or log concave families of polynomials are yet to be "Lorentzianized".

Let P be a finite poset and $e_k(P)$ be the number of order preserving surjections $\sigma : P \to \{1, 2, \ldots, k\}$.

Is the sequence $(e_k(P) : k \geq 1)$ always log-concave? Note, this polynomial is not necessarily real-rooted.

This sequence is related to the Neggers-Stanley conjecture which Branden and Stembridge found a counter example.

This conjecture asserted that the univariate polynomial counting the linear extensions of a partially ordered set by their number of descents has real zeros.

THANK YOU!