

# 行列式的计算方法

1. 利用定义计算.

$$\begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 2 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & n-1 & \cdots & 0 & 0 \\ n & 0 & \cdots & 0 & 0 \end{vmatrix} = (-1)^{\tau(n \cdots 321)} n! = (-1)^{\frac{n(n-1)}{2}} n! \quad \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & n-1 \\ n & 0 & 0 & \cdots & 0 \end{vmatrix} = (-1)^{\tau(23 \cdots n1)} n! = (-1)^{n-1} n!$$

2. 利用行列式性质把行列式化为上、下三角形行列式.

$$\begin{vmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{vmatrix} = 6 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{vmatrix} = 6 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix} = 48.$$

$$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix} = \begin{vmatrix} 1+x & 1 & 1 & 1 \\ -x & -x & 0 & 0 \\ 1 & 1 & 1+y & 1 \\ 0 & 0 & -y & -y \end{vmatrix} = \begin{vmatrix} x & 1 & 1 & 1 \\ 0 & -x & 0 & 0 \\ 0 & 1 & y & 1 \\ 0 & 0 & 0 & -y \end{vmatrix} = x^2 y^2$$

2. 行列式按一行(一列)展开,或按多行(多列)展开(Laplace 定理).

公式:  $\begin{vmatrix} A & 0 \\ C_1 & B \end{vmatrix} = \begin{vmatrix} A & C_2 \\ 0 & B \end{vmatrix} = |A||B|, \begin{vmatrix} 0 & A \\ B & D_1 \end{vmatrix} = \begin{vmatrix} D_2 & A \\ B & 0 \end{vmatrix} = (-1)^{mm} |A||B|$ , 其中  $A, B$  分别是  $m, n$  阶的方阵.

例子:  $D_{2n} = \begin{vmatrix} a & & & b \\ & \ddots & & \ddots \\ & & a & b \\ & & b & a \\ & & & \ddots \\ b & & & & a_{2n} \end{vmatrix}.$

利用 Laplace 定理,按第  $n, n+1$  行展开,除 2 级子式  $\begin{vmatrix} a & b \\ b & a \end{vmatrix}$  外其余由第  $n, n+1$  行所得的 2 级子式均为零.

故  $D_{2n} = \begin{vmatrix} a & b \\ b & a \end{vmatrix} (-1)^{n+n+1+n+n+1} D_{2n-2} = (a^2 - b^2) D_{2n-2}$ , 此为递推公式,应用可得  $D_{2n} = (a^2 - b^2)^n$ .

$$\begin{aligned} D_{2n} &= \begin{vmatrix} a & & & b \\ & \ddots & & \ddots \\ & & a & b \\ & & b & a \\ & & & \ddots \\ b & & & & a \end{vmatrix} = \begin{vmatrix} a & & & b \\ & \ddots & & \ddots \\ & & a & b \\ & & a+b & a+b \\ & & & \ddots \\ a+b & & & & a+b \end{vmatrix} = \begin{vmatrix} a-b & & & b \\ & \ddots & & \ddots \\ & & a-b & b \\ & & 0 & a+b \\ & & & \ddots \\ 0 & & & & a+b \end{vmatrix} \\ &= (a-b)^n (a+b)^n = (a^2 - b^2)^n. \end{aligned}$$

3. 箭头形行列式或者可以化为箭头形的行列式.

$$\text{例: } \begin{vmatrix} a_0 & 1 & 1 & \cdots & 1 \\ 1 & a_1 & 0 & \cdots & 0 \\ 1 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & a_n \end{vmatrix} = \begin{vmatrix} a_0 - \sum_{i=1}^n \frac{1}{a_i} & 1 & 1 & \cdots & 1 \\ 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{vmatrix} = (a_0 - \sum_{i=1}^n \frac{1}{a_i}) a_1 a_2 \cdots a_n.$$

$$\begin{aligned} \text{例: } & \begin{vmatrix} x_1 - m & x_2 & x_3 & \cdots & x_n \\ x_1 & x_2 - m & x_3 & \cdots & x_n \\ x_1 & x_2 & x_3 - m & \cdots & x_n \\ \vdots & \vdots & \vdots & & \vdots \\ x_1 & x_2 & x_3 & \cdots & x_n - m \end{vmatrix} = \begin{vmatrix} x_1 - m & x_2 & x_3 & \cdots & x_n \\ m & -m & 0 & \cdots & 0 \\ m & 0 & -m & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ m & 0 & 0 & \cdots & 0 - m \end{vmatrix} \\ & = \begin{vmatrix} \sum_{i=1}^n x_i - m & x_2 & x_3 & \cdots & x_n \\ 0 & -m & 0 & \cdots & 0 \\ 0 & 0 & -m & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 - m \end{vmatrix} = (\sum_{i=1}^n x_i - m)(-m)^{n-1}. \end{aligned}$$

$$\text{例: } \begin{vmatrix} x_1 & a_2 & a_3 & \cdots & a_n \\ a_1 & x_2 & a_3 & \cdots & a_n \\ a_1 & a_2 & x_3 & \cdots & a_n \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & a_3 & \cdots & x_n \end{vmatrix} = \begin{vmatrix} x_1 & a_2 & a_3 & \cdots & a_n \\ a_1 - x_1 & x_2 - a_2 & 0 & \cdots & 0 \\ a_1 - x_1 & 0 & x_3 - a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 - x_1 & 0 & 0 & \cdots & x_n - a_n \end{vmatrix} \quad (\text{第一行的倍加到其余各行})$$

$$\begin{aligned} & = \prod_{i=1}^n (x_i - a_i) \cdot \begin{vmatrix} \frac{x_1}{x_1 - a_1} & \frac{a_2}{x_2 - a_2} & \frac{a_3}{x_3 - a_3} & \cdots & \frac{a_n}{x_n - a_n} \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{vmatrix} \quad (\text{每一列提出相应的公因子 } x_i - a_i) \\ & = \prod_{i=1}^n (x_i - a_i) \cdot \begin{vmatrix} \frac{x_1}{x_1 - a_1} + \sum_{i=2}^n \frac{a_i}{x_i - a_i} & \frac{a_2}{x_2 - a_2} & \frac{a_3}{x_3 - a_3} & \cdots & \frac{a_n}{x_n - a_n} \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix} = (1 + \sum_{i=1}^n \frac{a_i}{x_i - a_i}) \prod_{i=1}^n (x_i - a_i). \end{aligned}$$

$$\text{例: } \begin{vmatrix} 1 & 1 & \cdots & 1 & a_0 \\ 0 & 0 & \cdots & a_1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a_{n-1} & \cdots & 0 & 1 \\ a_n & 0 & \cdots & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & \cdots & 1 & a_0 - \sum_{i=1}^n \frac{1}{a_i} \\ 0 & 0 & \cdots & a_1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a_{n-1} & \cdots & 0 & 0 \\ a_n & 0 & \cdots & 0 & 0 \end{vmatrix} = (-1)^{\frac{n(n+1)}{2}} a_1 a_2 \cdots a_n (a_0 - \sum_{i=1}^n \frac{1}{a_i}).$$

其它例子：特点是除了主对角线,其余位置上的元素各行或各列都相同.

$$\begin{vmatrix} a+x_1 & a & a & \cdots & a \\ a & a+x_2 & a & \cdots & a \\ a & a & a+x_3 & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ a & a & a & \cdots & a+x_n \end{vmatrix}, \begin{vmatrix} x+a_1 & a_2 & a_3 & \cdots & a_n \\ a_1 & x+a_2 & a_3 & \cdots & a_n \\ a_1 & a_2 & x+a_3 & \cdots & a_n \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & a_3 & \cdots & x+a_n \end{vmatrix}.$$

4. 逐行逐列相加减.行列式特点是每相邻两行(列)之间有许多元素相同.用逐行(列)相减可以化出零.

$$\text{例: } \begin{vmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 2 & 2 & \cdots & 2 \\ 2 & 2 & 3 & \cdots & 2 \\ \vdots & \vdots & \vdots & & \vdots \\ 2 & 2 & 2 & \cdots & n \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 \\ 2 & 2 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & n-2 \end{vmatrix} = (-2)(n-2)!.$$

$$\text{例: } \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 2 & 3 & \cdots & n-1 & n \\ 3 & 3 & 3 & \cdots & n-1 & n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-1 & n-1 & n-1 & \cdots & n-1 & n \\ n & n & n & \cdots & n & n \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & 0 \\ n & n & n & \cdots & n & n \end{vmatrix} = (-1)^{n-1} n.$$

$$\text{例: } \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-1 & n & 1 & \cdots & n-3 & n-2 \\ n & 1 & 2 & \cdots & n-2 & n-1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 1 & 1 & \cdots & 1 & 1-n \\ 1 & 1 & 1 & \cdots & 1-n & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1-n & \cdots & 1 & 1 \\ 1 & 1-n & 1 & \cdots & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{n(n+1)}{2} & 2 & 3 & \cdots & n-1 & n \\ 0 & 1 & 1 & \cdots & 1 & 1-n \\ 0 & 1 & 1 & \cdots & 1-n & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & 1-n & \cdots & 1 & 1 \\ 0 & 1-n & 1 & \cdots & 1 & 1 \end{vmatrix} = \frac{n(n+1)}{2} \begin{vmatrix} 1 & 1 & \cdots & 1 & 1-n \\ 1 & 1 & \cdots & 1-n & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1-n & \cdots & 1 & 1 \\ 1-n & 1 & \cdots & 1 & 1 \end{vmatrix}$$

$$\begin{aligned}
&= \frac{n(n+1)}{2} \begin{vmatrix} 1 & 1 & \cdots & 1 & 1-n \\ 0 & 0 & \cdots & -n & n \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & -n & \cdots & 0 & n \\ -n & 0 & \cdots & 0 & n \end{vmatrix} = \frac{n(n+1)}{2} \begin{vmatrix} 1 & 1 & \cdots & 1 & 1-n \\ 0 & 0 & \cdots & -n & n \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & -n & \cdots & 0 & n \\ -n & 0 & \cdots & 0 & n \end{vmatrix} = \frac{n(n+1)}{2} \begin{vmatrix} 1 & 1 & \cdots & 1 & -1 \\ 0 & 0 & \cdots & -n & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & -n & \cdots & 0 & 0 \\ -n & 0 & \cdots & 0 & 0 \end{vmatrix} \\
&= \frac{n(n+1)}{2} (-1)^{\frac{(n-1)(n-2)}{2}} (-1)^{n-1} n^{n-2} = (-1)^{\frac{n(n-1)}{2}} \frac{n+1}{2} n^{n-1}.
\end{aligned}$$

$$\begin{aligned}
\text{例: } & \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ x & 1 & 2 & \cdots & n-2 & n-1 \\ x & x & 1 & \cdots & n-3 & n-2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x & x & x & \cdots & 1 & 2 \\ x & x & x & \cdots & x & 1 \end{vmatrix} = \begin{vmatrix} 1-x & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1-x & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1-x & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1-x & 1 \\ x & x & x & \cdots & x & 1 \end{vmatrix} \\
&= \begin{vmatrix} 1-x & x & 0 & \cdots & 0 & 0 \\ 0 & 1-x & x & \cdots & 0 & 0 \\ 0 & 0 & 1-x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1-x & x \\ x & 0 & 0 & \cdots & 0 & 1-x \end{vmatrix} = (1-x)^n + (-1)^{n+1} x^n.
\end{aligned}$$

5. 升阶法(或加边法, 添加一行一列, 利于计算, 但同时保持行列式不变).

$$\text{例: } \begin{vmatrix} a+x_1 & a & a & \cdots & a \\ a & a+x_2 & a & \cdots & a \\ a & a & a+x_3 & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ a & a & a & \cdots & a+x_n \end{vmatrix} = \begin{vmatrix} 1 & a & a & a & \cdots & a \\ 0 & a+x_1 & a & a & \cdots & a \\ 0 & a & a+x_2 & a & \cdots & a \\ 0 & a & a & a+x_3 & \cdots & a \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & a & a & a & \cdots & a+x_n \end{vmatrix}$$

$$\begin{aligned}
&= \begin{vmatrix} 1 & a & a & a & \cdots & a \\ -1 & x_1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & x_2 & 0 & \cdots & 0 \\ -1 & 0 & 0 & x_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & 0 & \cdots & x_n \end{vmatrix} = \begin{vmatrix} 1 + \sum_{i=1}^n \frac{a}{x_i} & a & a & a & \cdots & a \\ 0 & x_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & x_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & x_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x_n \end{vmatrix} = x_1 x_2 \cdots x_n \left( 1 + \sum_{i=1}^n \frac{a}{x_i} \right).
\end{aligned}$$

$$\text{例: } \begin{vmatrix} 1+a_1+b_1 & a_1+b_2 & \cdots & a_1+b_n \\ a_2+b_1 & 1+a_2+b_2 & \cdots & a_2+b_n \\ \vdots & \vdots & & \vdots \\ a_n+b_1 & a_n+b_2 & \cdots & 1+a_n+b_n \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -b_1 & -b_2 & \cdots & -b_n \\ -a_1 & 0 & 1+a_1+b_1 & a_1+b_2 & \cdots & a_1+b_n \\ -a_2 & 0 & a_2+b_1 & 1+a_2+b_2 & \cdots & a_2+b_n \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ -a_n & 0 & a_n+b_1 & a_n+b_2 & \cdots & 1+a_n+b_n \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & -b_1 & -b_2 & \cdots & -b_n \\ -a_1 & 1 & 1 & 0 & \cdots & 0 \\ -a_2 & 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ -a_n & 1 & 0 & 0 & \cdots & 1 \end{vmatrix} = \begin{vmatrix} 1+\sum_{i=1}^n a_i & -n & 1 & 1 & \cdots & 1 \\ -\sum_{i=1}^n a_i b_i & 1+\sum_{i=1}^n b_i & -b_1 & -b_2 & \cdots & -b_n \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix} = \begin{vmatrix} 1+\sum_{i=1}^n a_i & -n \\ -\sum_{i=1}^n a_i b_i & 1+\sum_{i=1}^n b_i \end{vmatrix}$$

$$= (1+\sum_{i=1}^n a_i)(1+\sum_{i=1}^n b_i) - n \sum_{i=1}^n a_i b_i = 1 + \sum_{i=1}^n a_i + \sum_{i=1}^n b_i + \sum_{j=1}^n \sum_{\substack{i=1 \\ i \neq j}}^n b_j (a_i - a_j).$$

6. 利用范德蒙德行列式.

$$\text{计算行列式: } D = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ x_1^{n-2} & x_2^{n-2} & x_3^{n-2} & \cdots & x_n^{n-2} \\ x_1^n & x_2^n & x_3^n & \cdots & x_n^n \end{vmatrix}$$

$$\text{解: 令: } D_1 = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ x_1 & x_2 & \cdots & x_n & y \\ x_1^2 & x_2^2 & \cdots & x_n^2 & y^2 \\ \vdots & \vdots & & \vdots & \vdots \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} & y^{n-2} \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} & y^{n-1} \\ x_1^n & x_2^n & \cdots & x_n^n & y^n \end{vmatrix}, \text{这是一个 } n+1 \text{ 级范德蒙德行列式.}$$

一方面,由范德蒙德行列式得  $D_1 = \prod_{1 \leq j < i \leq n} (x_i - x_j) \cdot (y - x_1)(y - x_2) \cdots (y - x_n)$ . 可看做是关于  $y$  的一个  $n$  次多项式.

另一方面,将  $D_1$  按最后一列展开,可得一个关于  $y$  的多项式  $D_1 = p_n y^n + p_{n-1} y^{n-1} + \cdots + p_1 y + p_0$ , 其中

$y^{n-1}$  的系数  $p_{n-1}$  与所求行列式  $D$  的关系为  $D = -p_{n-1}$ .

由  $D_1 = \prod_{1 \leq j < i \leq n} (x_i - x_j) \cdot (y - x_1)(y - x_2) \cdots (y - x_n)$  来计算  $y^{n-1}$  的系数  $p_{n-1} = - \prod_{1 \leq j < i \leq n} (x_i - x_j) \cdot \sum_{i=1}^n x_i$ ,

故有  $D = -p_{n-1} = \prod_{1 \leq j < i \leq n} (x_i - x_j) \cdot \sum_{i=1}^n x_i$

其它的例子:

$$\begin{vmatrix} a_1^n & a_1^{n-1}b_1 & a_1^{n-2}b_1^2 & \cdots & a_1b_1^{n-1} & b_1^n \\ a_2^n & a_2^{n-1}b_2 & a_2^{n-2}b_2^2 & \cdots & a_2b_2^{n-1} & b_2^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n+1}^n & a_{n+1}^{n-1}b_{n+1} & a_{n+1}^{n-2}b_{n+1}^2 & \cdots & a_{n+1}b_{n+1}^{n-1} & b_{n+1}^n \end{vmatrix} \quad (\text{每一行提公因子 } a_i^n)$$

$$= a_1^n a_2^n \cdots a_{n+1}^n \begin{vmatrix} 1 & \frac{b_1}{a_1} & (\frac{b_1}{a_1})^2 & \cdots & (\frac{b_1}{a_1})^{n-1} & (\frac{b_1}{a_1})^n \\ 1 & \frac{b_2}{a_2} & (\frac{b_2}{a_2})^2 & \cdots & (\frac{b_2}{a_2})^{n-1} & (\frac{b_2}{a_2})^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & \frac{b_{n+1}}{a_{n+1}} & (\frac{b_{n+1}}{a_{n+1}})^2 & \cdots & (\frac{b_{n+1}}{a_{n+1}})^{n-1} & (\frac{b_{n+1}}{a_{n+1}})^n \end{vmatrix} = a_1^n a_2^n \cdots a_{n+1}^n \prod_{1 \leq j < i \leq n} (\frac{b_i}{a_i} - \frac{b_j}{a_j}).$$

7. 利用数学归纳法证明行列式.(对行列式的级数归纳)

例: 证明当  $\alpha \neq \beta$  时,  $D_n = \begin{vmatrix} \alpha + \beta & \alpha\beta & 0 & \cdots & 0 & 0 \\ 1 & \alpha + \beta & \alpha\beta & \cdots & 0 & 0 \\ 0 & 1 & \alpha + \beta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha + \beta & \alpha\beta \\ 0 & 0 & 0 & \cdots & 1 & \alpha + \beta \end{vmatrix} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$

证明: 利用第二数学归纳法: 若  $n=1$ , 则  $D_1 = \alpha + \beta = \frac{\alpha^2 - \beta^2}{\alpha - \beta}$ , 成立, 假设结论对所有小于  $n$  的都成立, 则将

$D_n$  按第一行(或第一列)展开得  $D_n = (\alpha + \beta)D_{n-1} - \alpha\beta D_{n-2}$ , 利用归纳假设可得

$$D_n = (\alpha + \beta)D_{n-1} - \alpha\beta D_{n-2} = (\alpha + \beta) \frac{\alpha^n - \beta^n}{\alpha - \beta} - \alpha\beta \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}. \text{成立.}$$

8. 利用递推公式.

例: 计算行列式  $D_n = \begin{vmatrix} \alpha + \beta & \alpha\beta & 0 & \cdots & 0 & 0 \\ 1 & \alpha + \beta & \alpha\beta & \cdots & 0 & 0 \\ 0 & 1 & \alpha + \beta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha + \beta & \alpha\beta \\ 0 & 0 & 0 & \cdots & 1 & \alpha + \beta \end{vmatrix}$

解: 按第一行展开得  $D_n = (\alpha + \beta)D_{n-1} - \alpha\beta D_{n-2}$ , 将此式化为:

$$(1) D_n - \alpha D_{n-1} = \beta(D_{n-1} - \alpha D_{n-2}) \text{ 或 } (2) D_n - \beta D_{n-1} = \alpha(D_{n-1} - \beta D_{n-2})$$

利用公式(1)得  $D_n - \alpha D_{n-1} = \beta(D_{n-1} - \alpha D_{n-2}) = \beta^2(D_{n-2} - \alpha D_{n-3}) = \cdots = \beta^{n-2}(D_2 - \alpha D_1) = \beta^n$ , 即

$$D_n = \alpha D_{n-1} + \beta^n. (3)$$

利用公式(2)得  $D_n - \beta D_{n-1} = \alpha(D_{n-1} - \beta D_{n-2}) = \alpha^2(D_{n-2} - \beta D_{n-3}) = \cdots = \alpha^{n-2}(D_2 - \beta D_1) = \alpha^n$ , 即

$$D_n = \beta D_{n-1} + \alpha^n. (4)$$

$$\text{由(3)(4) 解得: } D_n = \begin{cases} \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}, & \alpha \neq \beta \\ (n+1)\alpha^n, & \alpha = \beta \end{cases}$$

$$\text{例: 计算行列式 } D_n = \begin{vmatrix} a & b & \cdots & 0 & 0 \\ c & a & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & a & b \\ 0 & 0 & \cdots & c & a \end{vmatrix}_n$$

解: 按第一行展开可得  $D_n = aD_{n-1} - bcD_{n-2}$ , 此时令  $\alpha + \beta = a, \alpha\beta = bc$ , 则  $D_n = (\alpha + \beta)D_{n-1} - \alpha\beta D_{n-2}$ ,

变形为  $D_n - \alpha D_{n-1} = \beta(D_{n-1} - \alpha D_{n-2})$ , 此为递推公式. 利用刚才的例子可求得结果.

这里  $\alpha + \beta = a, \alpha\beta = bc$ , 即  $\alpha, \beta$  是方程  $x^2 - ax + bc = 0$  的两个根.

$$\text{例: } D_n = \begin{vmatrix} 2 & 1 & & & \\ 1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & 1 \\ & & & 1 & 2 \end{vmatrix}_n = 2D_{n-1} - D_{n-2}, \text{ 故 } D_n - D_{n-1} = D_{n-1} - D_{n-2}, \text{ 递推公式.}$$

$$\text{例: } D_n = \begin{vmatrix} 7 & 2 & & & \\ 5 & 7 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 7 & 2 \\ & & & 5 & 7 \end{vmatrix}_n = 7D_{n-1} - 10D_{n-2}.$$

9. 分拆法. 将行列式的其中一行或者一列拆成两个数的和, 将行列式分解成两个容易求的行列式的和.

$$\text{例: } D = \begin{vmatrix} 1+a_1+b_1 & a_1+b_2 & \cdots & a_1+b_n \\ a_2+b_1 & 1+a_2+b_2 & \cdots & a_2+b_n \\ \vdots & \vdots & & \vdots \\ a_n+b_1 & a_n+b_2 & \cdots & 1+a_n+b_n \end{vmatrix}.$$

解: 取  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ , 记  $\alpha = (a_1, a_2, \dots, a_n)^T, \beta_i = (b_i, b_i, \dots, b_i)^T$ , 则

$$\begin{aligned}
 D &= |\varepsilon_1 + \alpha + \beta_1, \varepsilon_2 + \alpha + \beta_2, \dots, \varepsilon_n + \alpha + \beta_n| \\
 &= |\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n| + \sum_{i=1}^n |\varepsilon_1, \dots, \varepsilon_{i-1}, \alpha, \varepsilon_{i+1}, \dots, \varepsilon_n| + \sum_{i=1}^n |\varepsilon_1, \dots, \varepsilon_{i-1}, \beta_i, \varepsilon_{i+1}, \dots, \varepsilon_n| \\
 &\quad + \sum_{1 \leq i < j \leq n} |\varepsilon_1, \dots, \varepsilon_{i-1}, \alpha, \varepsilon_{i+1}, \dots, \varepsilon_{j-1}, \beta_j, \varepsilon_{j+1}, \dots, \varepsilon_n| + \sum_{1 \leq i < j \leq n} |\varepsilon_1, \dots, \varepsilon_{i-1}, \beta_i, \varepsilon_{i+1}, \dots, \varepsilon_{j-1}, \alpha, \varepsilon_{j+1}, \dots, \varepsilon_n| \\
 &= 1 + \sum_{i=1}^n a_i + \sum_{i=1}^n b_i + \sum_{1 \leq i < j \leq n} (a_i - a_j) b_j + \sum_{1 \leq i < j \leq n} (a_i - a_j) b_i = \dots
 \end{aligned}$$

$$\text{例: } D_n = \begin{vmatrix} a & b & b & \cdots & b & b \\ c & a & b & \cdots & b & b \\ c & c & a & \cdots & b & b \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c & c & c & \cdots & a & b \\ c & c & c & \cdots & c & a \end{vmatrix} = \begin{vmatrix} c+a-c & b & b & \cdots & b & b \\ c & a & b & \cdots & b & b \\ c & c & a & \cdots & b & b \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c & c & c & \cdots & a & b \\ c & c & c & \cdots & c & a \end{vmatrix}$$

$$= \begin{vmatrix} c & b & b & \cdots & b & b \\ c & a & b & \cdots & b & b \\ c & c & a & \cdots & b & b \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c & c & c & \cdots & a & b \\ c & c & c & \cdots & c & a \end{vmatrix} + \begin{vmatrix} a-c & b & b & \cdots & b & b \\ 0 & a & b & \cdots & b & b \\ 0 & c & a & \cdots & b & b \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & c & c & \cdots & a & b \\ 0 & c & c & \cdots & c & a \end{vmatrix} = V_1 + V_2$$

$V_1$ : 第一列提出公因子  $c$  后, 第一列的  $-b$  倍加到其余各列上,  $V_2$  按第一列展开.

$$V_1 = c \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & a-b & 0 & \cdots & 0 & 0 \\ 1 & c-b & a-b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & c-b & c-b & \cdots & a-b & 0 \\ 1 & c-b & c-b & \cdots & c-b & a-b \end{vmatrix} = c(a-b)^{n-1}$$

$$V_2 = (a-c)D_{n-1}, \text{ 故 } D_n = c(a-b)^{n-1} + (a-c)D_{n-1},$$

由  $b, c$  的对称性质, 亦可得  $D_n = b(a-c)^{n-1} + (a-b)D_{n-1}$ , 这两个式子中消去  $D_{n-1}$ , 可得结论,

$$D_n = \frac{c(a-b)^n - b(a-c)^n}{c-b}.$$

注: (1) 同一个行列式, 可有多种计算方法. 要利用行列式自身元素的特点, 选择合适的计算方法.

(2) 以上的各种方法并不是互相独立的, 计算一个行列式时, 有时需要综合运用以上方法,