J Comb Optim
https://doi.org/10.1007/s10878-017-0235-9

The Wiener index of Sierpiński-like graphs

Chunmei Luo[1] · Liancui Zuo[1] · Philip B. Zhang[1]

Abstract Sierpiński-like graphs constitute an extensively studied family of graphs of fractal nature applicable in topology, mathematics of the Tower of Hanoi, computer science, and elsewhere. In this paper, we focus on the Wiener polarity index, Wiener index and Harary index of Sierpiński-like graphs. By Sierpiński-like graphs' special structure and correlation, their Wiener polarity index and some Sierpiński-like graph's Wiener index and Harary index are obtained.

Keywords Sierpiński-like graphs · Wiener polarity index · Wiener index · Harary index

1 Introduction

In this paper, all graphs considered are simple. Given a graph G, we use $V(G)$, $E(G)$ to denote the vertex set and the edge set of G, respectively. For a real number x, $\lceil x \rceil$ denotes the least integer not less than x. For any integers a and b, we use the symbol $[a, b]$ to denote the set $\{a, a+1, \ldots, b\}$ when $a \leq b$, and $[k]$ to denote $[1, k]$ simply. A complete graph with n vertices is denoted by K_n. For other notations and terminologies which are not mentioned, please see Bollobás (1998)

Graphs of "Sierpiński type" play an important part in many different areas of mathematics as well as in several other scientific fields. The graphs $S(n, 3)$ were generalized

This work is supported by MECF of Tianjin with code 043135202JW1713, NSFC for youth with code 61103073, Tianjin Training Programs of Innovation for Undergraduates with code 201510065052 and Natural Science Foundation of Tianjin with code 16JCYBJC23600.

✉ Liancui Zuo
lczuo@mail.tjnu.edu.cn; lczuo@163.com

1 College of Mathematical Science, Tianjin Normal University, Tianjin 300387, China

Published online: 16 December 2017

to the Sierpiński graphs $S(n, k)$ in Klavžar and Milutinović (1997). The motivation for generalization came from topological studies of the Lipscomb's space (Lipscomb and Perry 1992; Milutinović 1992). In fact, the Sierpiński graphs were also independently studied in Pisanski and Tucker (2001). The graphs $S(n, k)$ have many interesting properties and were studied from different points of view. In Klavžar et al. (2002), unique 1-perfect codes in Sierpiński graphs are studied. Alternative arguments for uniqueness of 1-perfect codes in $S(n, k)$ were presented in Gravier et al. (2005) to determine their optimal $L(2, 1)$-labeling. Recently, covering codes and equitable $L(2, 1)$-labelings of Sierpiński graphs were studied in Beaudou et al. (2010), Fu and Xie (2010), Gravier et al. (2005), Fu and Xie (2010). An appealing relation is that $S(n, 3)$ is isomorphic to the graphs of the Tower of Hanoi puzzle with n disks (Hinz 1992; Klavžar and Milutinović 1997) and had been extensively studied in Hinz et al. (2005), Romik (2006). In Klavžar and Mohar (2005), $S^+(n, k)$ and $S^{++}(n, k)$ are introduced and the crossing numbers of Sierpiński-like graphs are completely determined. In Hinz and Parisse (2012a), Jakavac and Klavžar (2009), vertex, edge and total colorings of Sierpiński-like graphs are studied. Moreover, the hub number of Sierpiński graphs is obtained in Lin et al. (2011). Besides the mentioned properties, several metric properties are also studied in Hinz and Parisse (2012b), Parisse (2009). In Xue et al. (2012, 2015) investigated the hamiltonicity, path t-colorings and the linear colorings of Sierpiński-like graphs. In Luo and Zuo (2017) studied the metric properties of Sierpiński-like graphs.

Definition 1.1 (Klavžar and Milutinović 1997) The Sierpiński graph $S(n, k)$ is defined as follows. For $n \geq 1$ and $k \geq 1$, the vertex set of $S(n, k)$ consists of all n-tuples of integers $1, 2, \ldots, k$, that is, $V(S(n, k)) = [k]^n$. Two different vertices $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ are adjacent if and only if there exists an $h \in [n]$ such that

(a) $u_t = v_t$ for $t \in [h - 1]$;
(b) $u_h \neq v_h$;
(c) $u_t = v_h$ and $v_t = u_h$ for $t \in [h + 1, n]$.

We denote $(u_1, u_2, \ldots, u_n)$ by $< u_1u_2 \cdots u_n >$ or $u_1u_2 \cdots u_n$ briefly. In $u_1u_2 \cdots u_n$, if $u_{i+1} = \cdots = u_j = l$, then it is denoted by $u_1 \cdots u_i l^{j-i} u_{j+1} \cdots u_n$. The vertices i^n, $i \in [k]$, are called the *extreme vertices* of $S(n, k)$. For $i \in [k]$ and $n \geq 2$, let $S_i(n, k)$ denote the subgraph of $S(n, k)$ induced by the vertices of the form $< i \cdots >$, and $S_{ij}(n, k)$ denote the subgraph of $S(n, k)$ induced by the vertices of the form $< ij \cdots >$. Clearly, $S_i(n, k)$ is isomorphic to $S(n - 1, k)$. The edges of $S(n, k)$ that lie in no induced K_k are called *bridge edges*. Note that $S_i(n, k)$ and $S_j(n, k)$, $i \neq j$, are connected by a single bridge edge between vertices ij^{n-1} and ji^{n-1}.

Definition 1.2 (Klavžar and Mohar 2005) The graph $S^+(n, k)$ is defined as follows. For $n \geq 1$ and $k \geq 1$, the graph $S^+(n, k)$ is obtained from Sierpiński graph $S(n, k)$ by adding a new vertex w and edges joining w with k extreme vertices of $S(n, k)$.

Definition 1.3 (Klavžar and Mohar 2005) The graph $S^{++}(n, k)$ is defined as follows. For $n = 1$, let $S^{++}(1, k) = K_{k+1}$. For $n \geq 2$ and $k \geq 1$, the graph $S^{++}(n, k)$ is obtained from the disjoint union of $k + 1$ copies of $S(n - 1, k)$ in which the corresponding extreme vertices in distinct copies of $S(n - 1, k)$ are connected as the complete graph K_{k+1}.

In Jakovac (2014), $S[n, k]$ is introduced, moreover, the hamiltonicity and chromatic number of $S[n, k]$ are studied. Clearly, $S[n, 3]$ is the Sierpiński gasket graph S_n.

$S[n, k]$ is obtained from the Sierpiński graph $S(n, k)$ by contracting all bridge edges. Let $u_1u_2 \cdots u_r jl^{n-(r+1)}$ and $u_1u_2 \cdots u_r lj^{n-(r+1)}$, $1 \leq r \leq n-2$, be two adjacent vertices in $S(n, k)$. In $S[n, k]$, $u_1u_2 \cdots u_r jl^{n-(r+1)}$ and $u_1u_2 \cdots u_r lj^{n-(r+1)}$ are identified in one vertex which is denoted by $u_1u_2 \cdots u_r\{j, l\}$, where $j \neq l$ and $j, l \in [k]$. In particular, vertices lj^{n-1} and jl^{n-1} of $S(n, k)$ are identified in one vertex denoted by $\{l, j\}$ in $S[n, k]$.

Since graph $S(n, k)(n \geq 2)$ is obtained by k copies of $S(n-1, k)$ in which any two different $S(n-1, k)$ are connected by a single bridge edge, the graph $S[n, k]$ is also constructed of k copies of $S[n-1, k]$. Each copy is denoted by $S_i[n, k]$ which is isomorphic to $S[n-1, k]$ for $n \geq 2$, i.e., $S_i[n, k]$ corresponds to $S_i(n, k)$. Clearly, $S_i[n, k]$ and $S_j[n, k]$ share one common vertex $\{i, j\}$ for any $i \neq j$.

Various chemical indexes are caused the attention of graph researchers (Zhou and Trinajstić 2009, 2010; Zuo 2012). In this paper, we mainly study the Wiener polarity index of Sierpiński-like graphs, and the Wiener index and Harary index of Sierpiński graph. In the text, $d(u, v)$ denotes the distance between vertex u and vertex v. In Klavžar and Milutinović (1997), it gives the following result for the distance in $S(n, k)$.

Lemma 1.4 (Klavžar and Milutinović 1997) *For $i, j \in [k], i \neq j, \delta \in [n], \overline{s}, \overline{t} \in [k]^{\delta-1}$ and $\underline{s} \in [k]^{n-\delta}$, let*

$$d_0(\underline{s}i\overline{s}, \underline{s}j\overline{t}) = d(\overline{s}, j^{\delta-1}) + 1 + d(\overline{t}, i^{\delta-1}),$$

and

$$d_l(\underline{s}i\overline{s}, \underline{s}j\overline{t}) = d(\overline{s}, l^{\delta-1}) + 1 + 2^{\delta-1} + d(\overline{t}, l^{\delta-1}),\ for\ all\ l \in [k],$$

then,

$$d(\underline{s}i\overline{s}, \underline{s}j\overline{t}) = \min\left\{d_l(\underline{s}i\overline{s}, \underline{s}j\overline{t}) | l \in [0, k]\right\}.$$

The above minimum can be equivalently written as

$$d(\underline{s}i\overline{s}, \underline{s}j\overline{t}) = \min\left\{d_l(\underline{s}i\overline{s}, \underline{s}j\overline{t}) | l \in [0, k] \setminus \{i, j\}\right\}.$$

2 The Wiener polarity index of Sierpiński-like graphs

In 1947, Wiener polarity index is introduced by Harold Wiener in paper (Trinajstić et al. 2006). The *Wiener polarity index* of a graph G, denoted by $W_p(G)$, is the number of unordered pairs of vertices u, v such that the distance between u and v is 3, i.e.,

$$W_p(G) = |\{\{u, v\} | d(u, v) = 3, u, v \in V(G)\}|.$$

The *eccentricity* $\varepsilon(v)$ of a vertex v in a connected graph G is the maximum graph distance between v and any other vertex u of G. For a disconnected graph, all vertices are defined to have infinite eccentricity. The maximum eccentricity is the graph *diameter*, which is denoted by $d(G)$.

By the definition, if the diameter of graph G is less than 3, that is, there is no vertices in graph G with distance 3, then its Wiener polarity index is equal to 0. In this section, we will discuss the Wiener polarity index of Sierpiński-like graphs.

Lemma 2.1 (Caporossi and Hansen 2000) *For any positive integers $n \geq 1, k \geq 1$, the diameter of graph $S(n, k)$ is equal to $2^n - 1$.*

Lemma 2.2 (Hinz and Parisse 2012b) *For any vertex $u = u_n \ldots u_1$ of graph $S(n, k)$, its eccentricity is*

$$\varepsilon(u) = \max\left\{d\left(u, i^n\right) : i \in [k]\right\},$$

where $d(u, i^n) = \sum_{d=1}^{n} (u_d \neq i) \cdot 2^{d-1}$, and if A is true, then $(A) = 1$, if A is false, then $(A) = 0$. So $d(u, i^n) = 2^n - 1 \Leftrightarrow \forall d \in [n] : u_d \neq i$.

By Lemma 2.2, the diameter of graph $S(n, k)$ is equal to the distance of any two extreme vertices. Bearing in mind that the correlation between graph $S[n, k]$, $S^+(n, k)$ and $S(n, k)$, we can get the diameter of graph $S[n, k]$ and $S^+(n, k)$. After Sierpiński graph $S(n, k)$ contracted all bridge edges, let its vertex u correspond with vertex $[u]$ in graph $S[n, k]$, and $[i^n]$, $i \in [k]$ are called the extreme vertices of $S[n, k]$. $S_i(n, k)$ has the same meaning in $S(n, k)$ and $S^+(n, k)$.

Lemma 2.3 *For any integer $n \geq 1$, the diameter of graph $S[n, k]$ is 0 for $k = 1$ and 2^{n-1} for $k \geq 2$.*

Proof Firstly, $S[n, 1] = K_1$ with diameter 0. For $k \geq 2$, by the definition of graph $S[n, k]$, it is easy to know any two extreme vertices obtain the maximal distance among any two vertices in graph $S[n, k]$. Moreover, the adjacent edges of extreme vertex in the graph $S(n, k)$ are not bridge edges, so the shortest path between two extreme vertices contains bridge edges one less than non bridge edges. Then, the diameter of graph $S[n, k]$ is

$$d\left([i^n], [j^n]\right) = \left\lceil \frac{d(i^n, j^n)}{2} \right\rceil = \left\lceil \frac{2^n - 1}{2} \right\rceil = 2^{n-1},$$

where $i, j \in [k]$, $i \neq j$. The proof is completed. □

Lemma 2.4 *For any positive integers $n \geq 1, k \geq 1$, the diameter of graph $S^+(n, k)$ is*

$$d\left(S^+(n, k)\right) = \begin{cases} 1, & k = 1, \\ 2^{n-1}, & k = 2, \\ 2^{n-1} + \left\lfloor \frac{2^n}{3} \right\rfloor, & k = 3, \\ 2^n - 1, & k \geq 4. \end{cases}$$

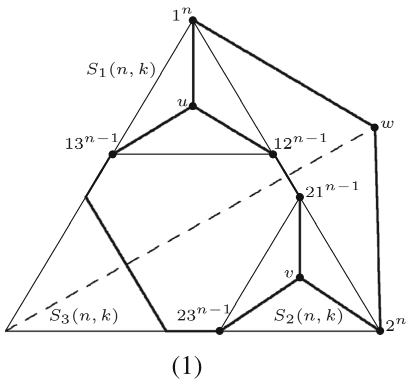

Fig. 1 Graph S(n,3)

Proof We discuss on different cases according to the value of k .

Case 1 $k = 1$. $S^+(n, 1) = K_2$, so its diameter is equal to 1.

Case 2 $k = 2$. $S^+(n, 2) = C_{2^n+1}$, so its diameter is equal to 2^{n-1}.

Case 3 $k \geq 4$. By the definition of graph $S^+(n, k)$ and Lemma 2.2, we can calculate the eccentricity of the new vertex w is 2^{n-1}, so the diameter of $S^+(n, k)$ is no more than $S(n, k)$'s diameter $2^n - 1$. Let $u = 1i^{n-1}$, $v = 2j^{n-1}$, where $i, j \in [3, k]$, $i \neq j$. It is easy to know that the distance of u, v is $2^n - 1$ in graph $S^+(n, k)$, so the diameter of graph $S^+(n, k)$ is $2^n - 1$.

Case 4 $k = 3$.

Obviously, the diameters of $S^+(1, 3) = K_4$ and $S^+(2, 3)$ are 1 and 3, respectively.

If $n \geq 3$, $k = 3$, for every vertex $u \in V\left(S^+(n, k)\right) \setminus \{w\}$, without loss of generality, let $u \in V(S_1(n, k))$. Obviously the vertex v with maximal distance to u satisfies $v \notin V(S_1(n, k))$, let $v \in V(S_2(n, k))$, and d_1, d_2, d_3 denote the shortest distances between u and v that through $e = \left(12^{n-1}, 21^{n-1}\right)$, $S_3(n, k)$ and vertex w, respectively, shown as Fig. 1(1), then

$$d_1 + d_2 + d_3 = \sum_{i=1}^{3} d(u, 1i^{n-1}) + \sum_{i=1}^{3} d(v, 2i^{n-1}) + 2^{n-1} + 4 = 5 \times 2^{n-1}.$$

Note that d_1, d_2, d_3 are integers, so the eccentricity of vertex u in graph $S^+(n, k)$ is

$$\varepsilon(u) = d(u, v) = \min\{d_1, d_2, d_3\} \leq \left\lfloor \frac{5 \times 2^{n-1}}{3} \right\rfloor = 2^{n-1} + \left\lfloor \frac{2^n}{3} \right\rfloor.$$

By the arbitrariness of u and $\varepsilon(w) = 2^{n-1}$, we get

$$d\left(S^+(n, k)\right) = \max_{u \in V\left(S^+(n,k)\right)} \varepsilon(u) \leq 2^{n-1} + \left\lfloor \frac{2^n}{3} \right\rfloor.$$

Let $u_0 \in V(S_1(n,k))$, $v_0 \in V(S_2(n,k))$, shown as Fig. 1(2): if $\lfloor 2^n/3 \rfloor$ is even, then

$$d(u_0, 13^{n-1}) = \frac{1}{2}\left\lfloor \frac{2^n}{3} \right\rfloor, d(v_0, 23^{n-1}) = \frac{1}{2}\left\lfloor \frac{2^n}{3} \right\rfloor - 1;$$

if $\lfloor 2^n/3 \rfloor$ is odd, then

$$d(u_0, 13^{n-1}) = d(v_0, 23^{n-1}) = \frac{1}{2}\left\lfloor \frac{2^n}{3} \right\rfloor - \frac{1}{2}.$$

Thus, the shortest distance between u_0 and v_0 that respectively through the bridge edge $e = \left(12^{n-1}, 21^{n-1}\right)$,$S_3(n,k)$ and vertex w satisfy: if $\lfloor 2^n/3 \rfloor$ is even, then

$$d_{01} = 2^n - \frac{1}{2}\left\lfloor \frac{2^n}{3} \right\rfloor, d_{02} = 2^{n-1} + \left\lfloor \frac{2^n}{3} \right\rfloor, d_{03} = 2^n - \frac{1}{2}\left\lfloor \frac{2^n}{3} \right\rfloor;$$

if $\lfloor 2^n/3 \rfloor$ is odd, then

$$d_{01} = 2^n - \frac{1}{2}\left\lfloor \frac{2^n}{3} \right\rfloor - \frac{1}{2}, d_{02} = 2^{n-1} + \left\lfloor \frac{2^n}{3} \right\rfloor, d_{03} = 2^n + \frac{1}{2} - \frac{1}{2}\left\lfloor \frac{2^n}{3} \right\rfloor,$$

where $d_{02} \le d_{01} \le d_{03}$. Therefore, in graph $S^+(n,k)$,

$$d\left(S^+(n,k)\right) \ge d(u_0, v_0) = \min\{d_{01}, d_{02}, d_{03}\} = d_{02} = 2^{n-1} + \left\lfloor \frac{2^n}{3} \right\rfloor.$$

In summary, the diameter of graph $S^+(n,k)$ is equal to $2^{n-1} + \left\lfloor \frac{2^n}{3} \right\rfloor$. □

Lemma 2.5 *For any positive integer d, it can be only expressed as* $d = 2^{r_1} + 2^{r_2} + \ldots + 2^{r_s}$, *where* $r_1 < r_2 < \cdots < r_s$ *are nonnegative integers. For any integers* $n \ge 1, k \ge 1$, *there are just* $(k-1)^s$ *vertices with distance* $d \in [2^n - 1]$ *to extreme vertex* i^n $(i \in [k])$ *in graph* $S(n,k)$.

Proof Firstly, we show that any positive integer d can be only expressed as $d = 2^{r_1} + 2^{r_2} + \ldots + 2^{r_s}$, where $r_1 < r_2 < \cdots < r_s$ are nonnegative integers. The induction is as follows. Obviously, $d = 1 = 2^0$, assuming that positive integer $d-1$ can only be expressed as $d - 1 = 2^{r_1} + 2^{r_2} + \ldots + 2^{r_s}$, where $r_1 < r_2 < \cdots < r_s$ are nonnegative integers, then integer d can be expressed as

$$d = 2^0 + 2^{r_1} + 2^{r_2} + \ldots + 2^{r_s}.$$

If $r_1 > 0$, then the conclusion is established for d; otherwise, $d = 2^1 + 2^{r_2} + \ldots + 2^{r_s}$, if $r_2 > 1$, then the conclusion is established for d, otherwise, $d = 2^2 + 2^{r_3} + \ldots + 2^{r_s}$. Go on in this way, d can be only expressed as the form described in the lemma. By mathematical induction principle, the conclusion is proved.

For any positive integer $d \in [2^n - 1]$, from the conclusion above and Lemma 2.2, for any $i \in [k]$ and $u = u_n \ldots u_1 \in V(S(n,k))$, if

$$d(u, i^n) = \sum_{p=1}^{n} (u_p \neq i) \cdot 2^{p-1} = d = 2^{r_1} + 2^{r_2} + \ldots + 2^{r_s}$$

where $r_1 < r_2 < \cdots < r_s$ are nonnegative integers, then u_t ($t \in [n]$) are all equal to i, except for $u_{r_j+1} \in [1,k] \setminus \{i\}$, $j \in [s]$. Therefore, vertex u has $(k-1)^s$ possibilities, namely, there are just $(k-1)^s$ vertices with distance $d = 2^{r_1} + 2^{r_2} + \ldots + 2^{r_s}$ to extreme vertex i^n ($i \in [k]$) in graph $S(n,k)$, the lemma is proved. □

By Lemma 2.5, there are respectively $k-1$, $k-1$ and $(k-1)^2$ vertices with distance 1, 2 and 3 to extreme vertex i^n ($i \in [k]$) in graph $S(n,k)$.

Theorem 2.6 *For any positive integer $k \geq 1$, if $n = 1$, then the Wiener polarity index of graph $S(n,k)$ is 0; if $n \geq 2$, then its Wiener polarity index is*

$$W_p(S(n,k)) = \frac{1}{2} k(k-1) \left[k^{n-2}(k-1)^2 + (k+1)\left(k^{n-2} - 1\right)\right].$$

Proof If $n = 1$, graph $S(1,k) = K_k$, its diameter is 0 or 1, so its Wiener polarity index is 0.

If $n = 2$, let $u = u_2 u_1$, $v = v_2 v_1$ denote any two vertices in graph $S(n,k)$. By Lemma 1.4, we know that $d(u,v) = 3$ if and only if $u_2 \neq v_2$, $u_1 \neq v_2$, $v_1 \neq u_2$, so its Wiener polarity index is $W_p(S(2,k)) = C_k^2 \cdot (k-1)^2 = \frac{1}{2} k (k-1)^3$.

If $n \geq 3$, the unordered pairs of vertices u, v whose distance is 3 in graph $S(n,k)$ can be divided into two cases:

(1) $u, v \in V(S_i(n,k))(i \in [k])$, the number of such vertices pairs is $k \cdot W_p(S(n-1,k))$.
(2) $u \in V(S_i(n,k))$, $v \in V(S_j(n,k))(i, j \in [k], i \neq j)$, in this case, $2^{n-1} + 1 \geq 5 > 3$, so the shortest path connects u and v must through the bridge edge $e = \left(ij^{n-1}, ji^{n-1}\right)$. Moreover, u, v satisfy one of the following three cases: $d\left(u, ij^{n-1}\right) = d\left(v, ji^{n-1}\right) = 1$, or $d\left(u, ij^{n-1}\right) = 0$ and $d\left(v, ji^{n-1}\right) = 2$, or $d\left(u, ij^{n-1}\right) = 2$ and $d\left(v, ji^{n-1}\right) = 0$. Considering about all possibilities of i, j, the number of such vertices pairs is

$$C_k^2 \cdot \left[(k-1)^2 + 2(k-1)\right] = C_k^2 \cdot \left(k^2 - 1\right).$$

Thus, for $n \geq 3$, the Wiener polarity index of graph $S(n,k)$ is

$$W_p(S(n,k)) = kW_p(S(n-1,k)) + C_k^2 \cdot \left(k^2 - 1\right).$$

Iterating on the above equation, we get

$$kW_p(S(n-1,k)) = k^2 W_p(S(n-2,k)) + kC_k^2 \cdot \left(k^2 - 1\right),$$

$$\cdots$$

$$k^{n-3}W_p\left(S\left(3,k\right)\right) = k^{n-2}W_p\left(S\left(2,k\right)\right) + k^{n-3}C_k^2 \cdot \left(k^2 - 1\right).$$

Bringing the above $n-2$ equations back one by one, we obtain

$$\begin{aligned} W_p\left(S\left(n,k\right)\right) &= k^{n-2}W_p\left(S\left(2,k\right)\right) + C_k^2 \cdot (k^2-1)\sum_{i=0}^{n-3} k^i \\ &= k^{n-2} \cdot C_k^2 \cdot (k-1)^2 + C_k^2 \cdot (k^2-1) \cdot \frac{1-k^{n-2}}{1-k} \\ &= C_k^2\left[k^{n-2} \cdot (k-1)^2 + (k+1) \cdot (k^{n-2}-1)\right] \\ &= \frac{1}{2}k(k-1)\left[k^{n-2} \cdot (k-1)^2 + (k+1) \cdot (k^{n-2}-1)\right]. \end{aligned}$$

The result is proved. □

Clearly, $S_i\left[n,k\right] \cong S\left[n-1,k\right]$. From the definition of graph $S\left[n,k\right]$ and Lemma 2.5, there are respectively $k-1$ and $(k-1)^2 - C_{k-1}^2$ vertices with distance 1 and 2 to extreme vertex $[i^n]$ $(i \in [k])$ in graph $S\left[n,k\right]$.

Theorem 2.7 *For any integers $n \geq 1, k \geq 1$, the Wiener polarity index of graph $S\left[n,k\right]$ is*

$$W_p\left(S\left[n,k\right]\right) = \begin{cases} 0, if\ n \in [2]\ \ or\ \ k = 1, \\ 2^{n-1} - 2, if\ n \geq 3, k = 2, \\ \frac{1}{2}k\left(k-1\right)\left[k^{n-3}\left(k-2\right)\left(k^2+2k-4\right) + k\left(k-1\right)\left(k^{n-3}-1\right)\right], \\ if\ n \geq 3, k \geq 3. \end{cases}$$

Proof **Case 1** If $n \in [2]$ or $k = 1$, the diameter of graph $S\left[n,k\right]$ is less than 3, so its Wiener polarity index is 0.

Case 2 If $n \geq 3, k = 2$, $S\left[n,k\right] = P_{2^{n-1}+1}$, its wiener polarity index is $2^{n-1}+1-3 = 2^{n-1}-2$.

Case 3 If $n = 3, k \geq 3$, the diameter of $S_i\left[n,k\right]\left(i \in [k]\right)$ is $2^{n-2} = 2 < 3$, any three parts of $S\left[n,k\right]$ are shown as Fig. 2. The unordered pairs of vertices $[u], [v]$ that their distance is 3 in graph $S\left[n,k\right]$ can be divided into two cases:

(1) $[u] = \{i,j\}, [v] \in V(S_p\left[n,k\right])$, where $i,j,p \in [k], i \neq j \neq p$. In this case, $d([v],\{i,p\}) = 1$ or $d\left([v],\{j,p\}\right) = 1$, and there is just one vertex in graph $S_p[n,k]$ that its distances to vertices $\{j,p\}$ and $\{i,p\}$ are both 1, so the number of such vertices pairs is equal to $C_k^2 \cdot C_{k-2}^1 \cdot (2k-3)$.

(2) $[u] \in V(S_i\left[n,k\right]), [v] \in V(S_j\left[n,k\right])$ $(i,j \in [k], i \neq j)$ and $[u],[v]$ are not common vertices of any two parts. Then the shortest path connects $[u]$ and $[v]$ must through the common vertex $\{i,j\}$, and satisfy $d([u],\{i,j\}) = 1$ and $d([v],\{i,j\}) = 2$ or $d([u],\{i,j\}) = 2$ and $d([v],\{i,j\}) = 1$. The number of such vertices pairs is equal to

$$C_k^2 \cdot 2(k-1)\left[(k-1)^2 - C_{k-1}^2 - 1\right] = C_k^2 \cdot (k+1)(k-1)(k-2).$$

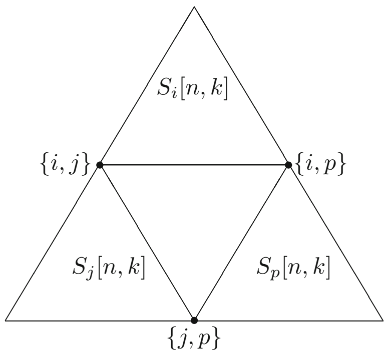

Fig. 2 Three parts' structure in S[n,k]

Adding up the results of the above two cases, we get that for $k \geq 3$, the Wiener polarity index of graph $S[3,k]$ is

$$\begin{aligned} W_p(S[3,k]) &= C_k^2 \cdot (k-2) \cdot (2k-3) + C_k^2 \cdot (k+1)(k-1)(k-2) \\ &= C_k^2 \cdot (k-2)(2k-3+k^2-1) \\ &= C_k^2 \cdot (k-2)(k^2+2k-4) \\ &= \frac{1}{2}k(k-1)(k-2)(k^2+2k-4). \end{aligned}$$

Case 4 If $n \geq 4$ and $k \geq 3$, the diameter of $S_i[n,k]$ $(i \in [1,k])$ is $2^{n-2} \geq 4 > 3$, then the unordered pairs of vertices $[u], [v]$ that their distance is 3 in graph $S[n,k]$ can be divided into two cases:

(1) $[u], [v] \in V(S_i[n,k])$ $(i \in [k])$, because $S_i[n,k] \cong S[n-1,k]$, the number of such vertices pairs is equal to $k \cdot W_P(S[n-1,k])$.
(2) $[u] \in V(S_i[n,k]) \backslash \{i,j\}, [v] \in V(S_j[n,k]) \backslash \{i,j\}$ $(i,j \in [k], i \neq j)$, in this case, the shortest path connects $[u]$ and $[v]$ must through the common vertices $\{i,j\}$. Moreover, $[u], [v]$ can be divided into two cases: or $d([u],\{i,j\}) = 1$ and $d([v],\{i,j\}) = 2$, or $d([u],\{i,j\}) = 2$ and $d([v],\{i,j\}) = 1$. So the number of such vertices pairs is equal to

$$C_k^2 \cdot 2(k-1)\left[(k-1)^2 - C_{k-1}^2\right] = C_k^2 \cdot k(k-1)^2.$$

Adding up the results of the above two cases, we get: for $n \geq 4$, $k \geq 3$, the Wiener polarity index of graph $S[n,k]$ is

$$W_p(S[n,k]) = k \cdot W_p(S[n-1,k]) + C_k^2 \cdot k(k-1)^2.$$

Iterating on the above equation, we get

$$kW_p(S[n-1,k]) = k^2 \cdot W_p(S[n-2,k]) + C_k^2 \cdot k^2(k-1)^2,$$

$$\cdots$$

$$k^{n-4} W_p\left(S[4,k]\right) = k^{n-3} \cdot W_p\left(S[3,k]\right) + C_k^2 \cdot k^{n-3}(k-1)^2.$$

Bringing the above $n-3$ equations back one by one, we obtain

$$\begin{aligned}
W_p\left(S[n,k]\right) &= k^{n-3} W_p(S[3,k]) + C_k^2 \cdot (k-1)^2 \sum_{i=1}^{n-3} k^i \\
&= k^{n-3} \cdot C_k^2 \cdot (k-2)\left(k^2+2k-4\right) + C_k^2 \cdot k(k-1)^2 \cdot \frac{1-k^{n-3}}{1-k} \\
&= C_k^2 \left[k^{n-3} \cdot (k-2)(k^2+2k-4) + k(k-1) \cdot (k^{n-3}-1)\right] \\
&= \frac{1}{2} k(k-1)\left[k^{n-3} \cdot (k-2)(k^2+2k-4) + k(k-1) \cdot (k^{n-3}-1)\right].
\end{aligned}$$

The conclusion is proved. □

Theorem 2.8 *For any positive integer $k \geq 1$, if $n = 1$, the Wiener polarity index of graph $S^+(n,k)$ is 0; if $n = 2$, its Wiener polarity index is $\frac{1}{2}k^2(k-1)(k-2)$; if $n \geq 3$, its Wiener polarity index is*

$$W_p\left(S^+(n,k)\right) = \frac{1}{2} k(k-1)\left[k^{n-2}(k-1)^2 + (k+1)\left(k^{n-2}-1\right) + 2k\right].$$

Proof Since graph $S^+(n,k)$ has a new vertex w with respect to graph $S(n,k)$, any three parts of graph $S^+(n,k)$ have the structure as depicted in Fig. 3. If $n = 1$, graph $S^+(1,k) = K_{k+1}$ with diameter $1 < 3$, then its Wiener polarity index is 0.

For $n = 2$, if the distance of u, v is 3 in graph $S(2,k)$, then their corresponding vertices' distance is still 3 in graph $S^+(2,k)$ except for two extreme vertices, and the eccentricity of vertex w is 2. So the Wiener polarity index of $S^+(2,k)$ is

$$W_p\left(S^+(2,k)\right) = C_k^2 \cdot \left[(k-1)^2 - 1\right] = \frac{1}{2} k^2(k-1)(k-2).$$

When $n \geq 3$, the vertex subset $V\left(S^+(n,k)\right) \setminus \{w\}$ of graph $S^+(n,k)$ can induce a subgraph which is isomorphic to graph $S(n,k)$, and the distance of two extreme vertices in graph $S(n,k)$ is $2^n - 1 \geq 7$, thus, if vertices u, v of graph $S(n,k)$ have distance 3, then their corresponding vertices u', v' in graph $S^+(n,k)$ also have distance 3. Moreover, the increased vertices pairs with distance 3 of $S^+(n,k)$ with respected to graph $S(n,k)$ can be divided into two cases: $u' = w, v' \in V(S_i(n,k))$ $(i \in [k])$, and $d(v', (i^n)') = 2$, there are just $k \cdot (k-1)$ such vertices pairs; $u' = (i^n)', v' \in V(S_j(n,k))$ $(i, j \in [k], i \neq j)$, and $d\left(v', (j^n)'\right) = 1$, there are just $C_k^2 \cdot 2(k-1)$ such vertices pairs. Thus, when $n \geq 3$, the Wiener polarity index of graph $S^+(n,k)$ is

$$W_p\left(S^+(n,k)\right) = W_p\left(S(n,k)\right) + k(k-1) + C_k^2 \cdot 2(k-1).$$

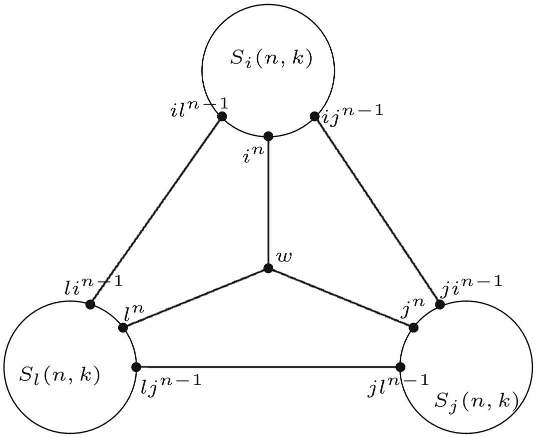

Fig. 3 Three parts of $S^{+}(n,k)$

By Theorem 2.6, we get

$$\begin{aligned}W_p\left(S^{+}(n,k)\right) &= C_k^2\left[k^{n-2}\cdot(k-1)^2+(k+1)\cdot(k^{n-2}-1)\right]\\&\quad +k\,(k-1)+C_k^2\cdot 2(k-1)\\&= C_k^2\left[k^{n-2}\cdot(k-1)^2+(k+1)\cdot(k^{n-2}-1)+2k\right]\\&=\frac{1}{2}k(k-1)\left[k^{n-2}\cdot(k-1)^2+(k+1)\cdot(k^{n-2}-1)+2k\right].\end{aligned}$$

The theorem is proved. □

In the following, the $k+1$ copies $S\,(n-1,k)$ of graph $S^{++}(n,k)$ are respectively denoted by $S^i\,(n-1,k)$ $(i\in[k+1])$, and the vertices of copy $S^i\,(n-1,k)$ are denoted by iu $(u\in[k]^{n-1})$.

Theorem 2.9 *For any positive integer $k\geq 1$, if $n\in[2]$, then the Wiener polarity index of graph $S^{++}(n,k)$ is 0; if $n\geq 3$, then its Wiener polarity index is*

$$W_p\left(S^{++}(n,k)\right)=\frac{1}{2}k\left(k^2-1\right)\left[k^{n-3}(k-1)^2+(k+1)\left(k^{n-3}+k-1\right)\right].$$

Proof For any integer $k\geq 1$, if $n\in[2]$, then graph $S^{++}(n,k)$ with diameter less than 3, so its Wiener polarity index is 0. If $n\geq 3$, then the unordered pairs of vertices u,v that their distance is 3 in graph $S^{++}(n,k)$ can be divided into two cases:

(1) $u,v\in V(S^i(n-1,k))$ $(i\in[k+1])$, there are $(k+1)\cdot W_p\,(S\,(n-1,k))$ such vertices pairs.
(2) $u\in V(S^i(n-1,k)), v\in V(S^j(n-1,k))$ $(i,j\in[k+1], i\neq j)$, any two $S\,(n-1,k)$ copies in graph $S^{++}(n,k)$ have the structure as depicted in Fig. 4. In this case, there must be a $p\in[k]$ such that the shortest path between u

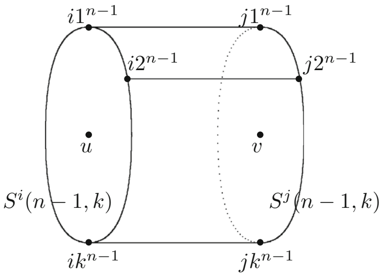

Fig. 4 Two copies of $S(n-1,k)$

and v through the edge $e = \left(ip^{n-1}, jp^{n-1}\right)$, then u, v has three possibilities: $d(u, ip^{n-1}) = 1$ and $d(v, jp^{n-1}) = 1$; or $d(u, ip^{n-1}) = 0$ and $d(v, jp^{n-1}) = 2$; or $d(u, ip^{n-1}) = 2$ and $d(v, jp^{n-1}) = 0$. The number of such vertices pairs is equal to

$$C_{k+1}^2 \cdot k[(k-1)^2 + 2(k-1)] = C_{k+1}^2 \cdot k(k^2-1).$$

Therefore, for $n \geq 3$, the Wiener polarity index of graph $S^{++}(n,k)$ is

$$W_p\left(S^{++}(n,k)\right) = (k+1) \cdot W_p\left(S(n-1,k)\right) + C_{k+1}^2 \cdot k(k^2-1).$$

By Theorem 2.6, we get

$$\begin{aligned}
&W_p\left(S^{++}(n,k)\right) \\
&\quad = (k+1)C_k^2\left[k^{n-3} \cdot (k-1)^2 + (k+1) \cdot (k^{n-3}-1)\right] + C_{k+1}^2 \cdot k\left(k^2-1\right) \\
&\quad = C_{k+1}^2 \cdot (k-1)\left[k^{n-3} \cdot (k-1)^2 + (k+1) \cdot (k^{n-3}-1) + k(k+1)\right] \\
&\quad = \frac{1}{2}k\left(k^2-1\right)\left[k^{n-3} \cdot (k-1)^2 + (k+1) \cdot (k^{n-3}+k-1)\right].
\end{aligned}$$

The theorem is proved. □

3 The Wiener index and Harary index of Sierpiński graph

In this section, we will study another three indexes of Sierpiński graph. Let G be a connected graph, the *Wiener index* of G is defined as the sum of distances between all unordered pairs of vertices in G, that is,

$$W(G) = \sum_{u,v \in V(G)} d(u,v).$$

Harary index, a parallel of Wiener index, is defined as the sum of the inverse of the distances between all unordered pairs of vertices in the graph G, that is,

$$H(G) = \sum_{u,v \in V(G)} \frac{1}{d(u,v)}.$$

While the *hyper-Wiener index* (Klein et al. 1995) is defined as

$$WW(G) = \frac{1}{2}W(G) + \frac{1}{2}\sum_{u,v \in V(G)} d(u,v)^2.$$

In Dobrynin et al. (2001), the Wiener index of trees is obtained, and in Eliasi (2009), the Harary index is studied. In papers Gutman et al. (1998); Gutman (1997), the modification of Wiener index is proposed by Gutman et al., that is,

$$W_\lambda(G) = \sum_{u,v \in V(G)} d(u,v)^\lambda,$$

where $\lambda \neq 0$, which is called λ-*Wiener index*. Then, the hyper-Wiener index of graph G also can be expressed as

$$WW(G) = \frac{1}{2}W(G) + \frac{1}{2}W_2(G).$$

By the definitions above, we obtain that the three indexes of Sierpiński graph are all equal to 0 since it is a isolated vertex for $k = 1$. For $k = 2$, $S(n,k)$ is isomorphic to even path P_{2^n}, its Wiener index is equal to $2^{n-1}\left(2^{2n}-1\right)/3$, its hyper-Wiener index is $\left(2^{n-1}+2^{2n-2}\right)\left(2^{2n}-1\right)/6$, its Harary index is $1 + 2^n\sum_{i=2}^{2^n-1}\frac{1}{i}$. Next, we will discuss the Wiener index, hyper-Wiener index and Harary index of Sierpiński graph for $k \geq 3$.

Lemma 3.1 (Klavžar and Zemljic 2013) *For any positive integers $n \geq 1, k \geq 1$,*

$$\sum_{u \in V(S(n,k))} d\left(u, i^n\right) = k^{n-1}(k-1)\left(2^n-1\right).$$

Lemma 3.2 *For any positive integers $n \geq 1, k \geq 1$,*

$$\sum_{u \in V(S(n,k))} d(u,i^n)^2 = k^{n-2}(k-1)(2^n-1)\left[(k-1)(2^n-1) + \frac{2^n+1}{3}\right].$$

Proof Let $u = u_1u_2\ldots u_n$ be a vertex of graph $S(n,k)$. Since for every $p, s, t \in [n]$ with $s < t$, there are $k^{n-1}(k-1)$ vertices with $u_p \neq i, i \in [k]$ and $k^{n-2}(k-1)^2$ vertices with $u_s \neq i, u_t \neq i$ in graph $S(n,k)$, respectively, it follows by Lemma 2.2,

$$
\begin{aligned}
&\sum_{u\in V(S(n,k))} d(u,i^n)^2 \\
&= \sum_{u\in[k]^n}\left(\sum_{p=1}^{n}(u_p\neq i)\cdot 2^{n-p}\right)^2 \\
&= \sum_{u\in[k]^n}\left[\sum_{p=1}^{n}(u_p\neq i)^2\cdot 2^{2(n-p)}+2\sum_{s<t}(u_s\neq i)(u_t\neq i)\cdot 2^{2n-s-t}\right] \\
&= \sum_{p=1}^{n}\left[\sum_{u\in[k]^n}(u_p\neq i)\cdot 2^{2(n-p)}\right]+2\sum_{s<t}\left[\sum_{u\in[k]^n}(u_s\neq i)(u_t\neq i)\cdot 2^{2n-s-t}\right] \\
&= \sum_{p=1}^{n}k^{n-1}(k-1)\cdot 2^{2(n-p)}+2\sum_{s=1}^{n-1}\sum_{t=s+1}^{n}k^{n-2}(k-1)^2\cdot 2^{2n-s-t} \\
&= k^{n-1}(k-1)\cdot\frac{2^{2n}-1}{3}+2^{2n+1}k^{n-2}(k-1)^2\sum_{s=1}^{n-1}\frac{1}{2^s}\left(\sum_{t=s+1}^{n}\frac{1}{2^t}\right) \\
&= k^{n-1}(k-1)\cdot\frac{2^{2n}-1}{3}+2^{2n+1}k^{n-2}(k-1)^2\left[\frac{1}{3}\left(1-\frac{1}{2^{2n-2}}\right)+\frac{1}{2^{2n-1}}-\frac{1}{2^n}\right] \\
&= k^{n-1}(k-1)\cdot\frac{2^{2n}-1}{3}+2k^{n-2}(k-1)^2\left(\frac{2^{2n}-1}{3}+1-2^n\right) \\
&= k^{n-2}(k-1)(2^n-1)\left[(k-1)(2^n-1)+\frac{2^n+1}{3}\right].
\end{aligned}
$$

The lemma is proved. □

By the definition, the Wiener index and hyper-Wiener index of graph G can be denoted by

$$W(G)=\sum_{t=1}^{d(G)}n_t\cdot t,$$

and

$$WW(G)=\frac{1}{2}\left(\sum_{t=1}^{d(G)}n_t\cdot t+\sum_{t=1}^{d(G)}n_t\cdot t^2\right),$$

and Harary index can be denoted by

$$H(G)=\sum_{t=1}^{d(G)}n_t\cdot\frac{1}{t},$$

where n_t denotes the number of vertices pairs of graph G that their distance is t.

Theorem 3.3 *For any positive integer $k \geq 3$, the Wiener index of graph $S(2,k)$ is*

$$W(S(2,k)) = C_k^2 \cdot \left[3(k-1)^2 + 5(k-1) + 2\right],$$

the hyper-Wiener index is

$$WW(S(2,k)) = C_k^2 \cdot \left[6(k-1)^2 + 7(k-1) + 2\right],$$

and the Harary index is

$$H(S(2,k)) = C_k^2 \cdot \left[\frac{1}{3}(k-1)^2 + 2(k-1) + 2\right].$$

While the Wiener index of graph $S(3,k)$ is

$$W(S(3,k)) = C_k^2 \cdot \left[7(k-1)^4 + 25(k-1)^3 + 30(k-1)^2 + 19(k-1) + 3\right],$$

the hyper-Wiener index is

$$WW(S(3,k)) = 2C_k^2 \cdot \left[14(k-1)^4 + 39(k-1)^3 + 30(k-1)^2 + 21(k-1) + 1\right],$$

and the Harary index is

$$\begin{aligned} &H(S(3,k)) \\ &= C_k^2 \cdot \left[\frac{1}{7}(k-1)^4 + \frac{16}{15}(k-1)^3 + \frac{137}{35}(k-1)^2 + \frac{197}{35}(k-1) + \frac{314}{105}\right]. \end{aligned}$$

Proof Assume that $k \geq 3$.

(1) Obviously, graph $S(2,k)$ has diameter $2^2 - 1 = 3$, and its number of vertices pairs with distance 1 is equal to its edges number, that is, $n_1 = C_k^2 \cdot (k+1)$. Since $S_i(2,k) \cong K_k$ has diameter 1, the vertices u, v of $S(2,k)$ with distance 2 satisfy: $u \in V(S_i(2,k))$, $v \in V(S_j(2,k))$ $(i, j \in [k], i \neq j)$, either $d(u, ij) = 1$ and $d(v, ji) = 0$, or $d(u, ij) = 0$ and $d(v, ji) = 1$, then $n_2 = C_k^2 \cdot 2(k-1)$. By Theorem 2.6, $n_3 = C_k^2 \cdot (k-1)^2$. Thus,

$$W(S(2,k)) = \sum_{t=1}^{3} n_t \cdot t = C_k^2 \cdot \left[3(k-1)^2 + 5(k-1) + 2\right],$$

$$W_2(S(2,k)) = \sum_{t=1}^{3} n_t \cdot t^2 = C_k^2 \cdot \left[9(k-1)^2 + 9(k-1) + 2\right],$$

$$H(S(2,k)) = \sum_{t=1}^{3} n_t \cdot \frac{1}{t} = C_k^2 \cdot \left[\frac{1}{3}(k-1)^2 + 2(k-1) + 2\right],$$

Table 1 The correspondence between $d(u, v)$ and n_d

$d(u, v)$	1	2	3	4	5	6	7
n_d	1	$2(k-1)$	$2(k-1)$ $+(k-1)^2$	$4(k-1)^2$	$2(k-1)^3$ $+(k-1)^2$ $+(k-2)$	$2(k-1)^3$ $+2(k-2)^2$	$(k-1)^4$ $-2(k-2)^2$ $-(k-2)$

and

$$\begin{aligned} WW(S(2,k)) &= \frac{1}{2}W(S(2,k)) + \frac{1}{2}W_2(S(2,k)) \\ &= C_k^2 \cdot \left[6(k-1)^2 + 7(k-1) + 2\right]. \end{aligned}$$

(2) Clearly, graph $S(3,k)$ has diameter 7. For any vertices $u \in V(S_i(3,k))$ and $v \in V(S_j(3,k))$, where $i, j \in [k], i \neq j$, because $2^{3-1} + 1 = 5$, when $d(u,v) \leq 4$, the shortest path between u and v must through the bridge edge $e = \left(ij^2, ji^2\right)$, therefore, $d\left(u, ij^2\right) + d\left(v, ji^2\right) = d(u,v) - 1$; when $5 \leq d(u,v) \leq 7$, the shortest path between u and v must through the edge $e = \left(ij^2, ji^2\right)$ or some $S_p(3,k)$, where $p \in [k]$, $p \neq i \neq j$, thus,

$$d\left(u, ij^2\right) + d\left(v, ji^2\right) = d(u,v) - 1$$

or

$$d\left(u, ip^2\right) + d\left(v, jp^2\right) = d(u,v) - 5.$$

Let there be n_d vertices pairs $u \in V(S_i(3,k))$, $v \in V(S_j(3,k))$ with distance $d(u,v)$, by Lemma 2.5, we get the following Table 1.

From Table 1, we obtain

$$\sum_{\substack{u \in V(S_i(3,k)) \\ v \in V(S_j(3,k))}} d(u,v) = 7(k-1)^4 + 22(k-1)^3 + 22(k-1)^2 + 12(k-1) + 1,$$

$$\sum_{\substack{u \in V(S_i(3,k)) \\ v \in V(S_j(3,k))}} d(u,v)^2 = 49(k-1)^4 + 122(k-1)^3 + 72(k-1)^2 + 54(k-1) - 1,$$

$$\begin{aligned} \sum_{\substack{u \in V(S_i(3,k)) \\ v \in V(S_j(3,k))}} \frac{1}{d(u,v)} &= \frac{1}{7}(k-1)^4 + \frac{11}{15}(k-1)^3 + \frac{166}{105}(k-1)^2 \\ &\quad + \frac{57}{35}(k-1) + \frac{104}{105}. \end{aligned}$$

Then, the Wiener index of graph $S(3,k)$ is

$$\begin{aligned}
W(S(3,k)) &= \sum_{u,v\in V(S(3,k))} d(u,v)\\
&= \sum_{i=1}^{k} \sum_{u,v\in V(S_i(3,k))} d(u,v) + \sum_{i\neq j} \sum_{\substack{u\in V(S_i(3,k))\\ v\in V(S_j(3,k))}} d(u,v)\\
&= kW(S(2,k)) + C_k^2 \cdot \sum_{\substack{u\in V(S_i(3,k))\\ v\in V(S_j(3,k))}} d(u,v)\\
&= C_k^2 \cdot \left[7(k-1)^4 + 25(k-1)^3 + 30(k-1)^2 + 19(k-1) + 3\right],
\end{aligned}$$

the Harary index is

$$\begin{aligned}
H(S(3,k)) &= \sum_{u,v\in V(S(3,k))} \frac{1}{d(u,v)}\\
&= \sum_{i=1}^{k} \sum_{u,v\in V(S_i(3,k))} \frac{1}{d(u,v)} + \sum_{i\neq j} \sum_{\substack{u\in V(S_i(3,k))\\ v\in V(S_j(3,k))}} \frac{1}{d(u,v)}\\
&= kH(S(2,k)) + C_k^2 \cdot \sum_{\substack{u\in V(S_i(3,k))\\ v\in V(S_j(3,k))}} \frac{1}{d(u,v)}\\
&= C_k^2 \cdot \left[\frac{1}{7}(k-1)^4 + \frac{16}{15}(k-1)^3 + \frac{137}{35}(k-1)^2 + \frac{197}{35}(k-1) + \frac{314}{105}\right],
\end{aligned}$$

the 2-Wiener index is

$$\begin{aligned}
W_2(S(3,k)) &= \sum_{u,v\in V(S(3,k))} d(u,v)^2\\
&= \sum_{i=1}^{k} \sum_{u,v\in V(S_i(3,k))} d(u,v)^2 + \sum_{i\neq j} \sum_{\substack{u\in V(S_i(3,k))\\ v\in V(S_j(3,k))}} d(u,v)^2\\
&= kW_2(S(2,k)) + C_k^2 \cdot \sum_{\substack{u\in V(S_i(3,k))\\ v\in V(S_j(3,k))}} d(u,v)^2\\
&= C_k^2 \cdot \left[49(k-1)^4 + 131(k-1)^3 + 90(k-1)^2 + 65(k-1) + 1\right],
\end{aligned}$$

and the hyper-Wiener index is

$$
\begin{aligned}
WW\left(S\left(3,k\right)\right) &= \frac{1}{2}W\left(S\left(3,k\right)\right)+\frac{1}{2}W_2\left(S\left(3,k\right)\right)\\
&= 2C_k^2\cdot\left[14\left(k-1\right)^4+39\left(k-1\right)^3+30\left(k-1\right)^2+21\left(k-1\right)+1\right].
\end{aligned}
$$

The proof is completed. □

In the following, we will consider the Wiener index of graph $S\left(n,3\right)$.

Theorem 3.4 *For any integer $n\geq 0$, the Wiener index of graph $S\left(n+1,3\right)$ is*

$$
\begin{aligned}
W\left(S\left(n+1,3\right)\right) &= 3^{n+1}+\frac{466}{295}\times 3^{n+1}\left(6^n-1\right)-\frac{1}{2}\times 3^{n+1}\left(3^n-1\right)\\
&\quad+\frac{3}{17\times 59\times 2^{n+1}}\cdot\Big\{(221-49\sqrt{17})[(5-\sqrt{17})^n-6^n]\\
&\quad\quad+(221+49\sqrt{17})[(5+\sqrt{17})^n-6^n]\Big\}.
\end{aligned}
$$

Furthermore, a recurrence relation of $W(S(n,3))$ is given as follows,

$$
\begin{aligned}
W(S(n+1,3)) &= 5W(S(n,3))-2W(S(n-1,3))-19\times 3^{2n-3}\\
&\quad+\frac{2}{5}\times 3^n\left(233\times 2^n\times 3^{n-3}+1\right),
\end{aligned}
$$

with the original conditons $W(S(1,3))=3$ and $W(S(2,3))=72$.

Proof We can get the Wiener index of graph $S\left(n+1,3\right)$ from its definition is

$$
\begin{aligned}
W\left(S\left(n+1,3\right)\right) &= \sum_{u,v\in V(S(n+1,3))} d(u,v)\\
&= \sum_{i=1}^{3}\sum_{u,v\in V(S_i(n+1,3))} d(u,v)+\sum_{\substack{i,j\in[3]\\ i\neq j}}\sum_{\substack{u\in V(S_i(n+1,3))\\ v\in V(S_j(n+1,3))}} d(u,v).
\end{aligned}
$$

Because $S_i\left(n+1,3\right), i\in[3]$, are all isomorphic to $S\left(n,3\right)$ and their symmetry in graph $S\left(n+1,3\right)$, the above equation can simplify to

$$
W\left(S\left(n+1,3\right)\right)=3\cdot W\left(S\left(n,3\right)\right)+3\cdot\sum_{\substack{u\in V(S_1(n+1,3))\\ v\in V(S_2(n+1,3))}} d(u,v). \tag{3.1}
$$

From Lemma 1.4, for any two vertices $u\in V(S_1\left(n+1,3\right))$, $v\in V(S_2\left(n+1,3\right))$, their distance is

$$
d(u,v)=\min\{d_0,d_3\}=\frac{d_0+d_3-|d_0-d_3|}{2}. \tag{3.2}
$$

Let $u = 1u_1 \dots u_n = 1u'$, $v = 2v_1 \dots v_n = 2v'$, by Lemma 2.2, we get

$$d_0 = \sum_{p=1}^{n} [(u_p \neq 2) + (v_p \neq 1)] \cdot 2^{n-p} + 1,$$

and

$$d_3 = \sum_{p=1}^{n} [(u_p \neq 3) + (v_p \neq 3)] \cdot 2^{n-p} + 2^n + 1.$$

Bringing them to Eq. (3.2), we get

$$d(u, v) = \frac{1}{2} \left\{ \sum_{p=1}^{n} [(u_p \neq 2) + (v_p \neq 1) + (u_p \neq 3) + (v_p \neq 3)] \cdot 2^{n-p} + 2^n + 2 \right.$$
$$\left. - \left| 2^n + \sum_{p=1}^{n} [(u_p \neq 3) + (v_p \neq 3) - (u_p \neq 2) - (v_p \neq 1)] \cdot 2^{n-p} \right| \right\}.$$

Let

$$b(p) = (u_p \neq 2) + (v_p \neq 1) + (u_p \neq 3) + (v_p \neq 3),$$

and

$$d(p) = (u_p \neq 3) + (v_p \neq 3) - (u_p \neq 2) - (v_p \neq 1).$$

Then

$$\begin{aligned} \sum_{\substack{u \in V(S_1(n+1,3)) \\ v \in V(S_2(n+1,3))}} d(u, v) =& \frac{1}{2} \sum_{u' \in [3]^n} \sum_{v' \in [3]^n} (2^n + 2) + \frac{1}{2} \sum_{u' \in [3]^n} \sum_{v' \in [3]^n} \sum_{p=1}^{n} b(p) \cdot 2^{n-p} \\ & - \frac{1}{2} \sum_{u' \in [3]^n} \sum_{v' \in [3]^n} \left| 2^n + \sum_{p=1}^{n} d(p) \cdot 2^{n-p} \right| \\ =& \frac{1}{2} \cdot 3^{2n} (2^n + 2) + \frac{1}{2} B(n) - \frac{1}{2} D(n). \end{aligned} \tag{3.3}$$

It is not difficult to verify that when u_p, v_p $(p \in [n])$ are determined, the exact values of $b(p), d(p)$ are obtained and shown as Table 2.

From Table 2, for any $p \in [n]$,

$$\sum_{u_p=1}^{3} \sum_{v_p=1}^{3} b(p) = 24, \; \sum_{u_p=1}^{3} \sum_{v_p=1}^{3} d(p) = 0.$$

Table 2 All possible cases of $b(p)$ and $d(p)$

(u_p, v_p)	(1, 1)	(1, 2)	(1, 3)	(2, 1)	(2, 2)	(2, 3)	(3, 1)	(3, 2)	(3, 3)
$b(p)$	3	4	3	2	3	2	2	3	2
$d(p)$	1	0	−1	2	1	0	0	−1	−2

Thus,

$$\begin{aligned}
B(n) &= \sum_{u' \in [3]^n} \sum_{v' \in [3]^n} \sum_{p=1}^{n} b(p) \cdot 2^{n-p} \\
&= \sum_{u_1=1}^{3} \sum_{v_1=1}^{3} \cdots \sum_{u_n=1}^{3} \sum_{v_n=1}^{3} \sum_{p=1}^{n} b(p) \cdot 2^{n-p} \\
&= \sum_{u_2=1}^{3} \sum_{v_2=1}^{3} \cdots \sum_{u_n=1}^{3} \sum_{v_n=1}^{3} \left(\sum_{u_1=1}^{3} \sum_{v_1=1}^{3} b(1) \cdot 2^{n-1} \right) + \ldots \\
&\quad + \sum_{u_1=1}^{3} \sum_{v_1=1}^{3} \cdots \sum_{u_{n-1}=1}^{3} \sum_{v_{n-1}=1}^{3} \left(\sum_{u_n=1}^{3} \sum_{v_n=1}^{3} b(n) \cdot 2^{0} \right) \qquad (3.4) \\
&= 3^{2n-2} \times 24 \left(2^{n-1} + 2^{n-2} + \ldots + 2^{0} \right) \\
&= 24 \times 3^{2n-2} \cdot \left(2^{n} - 1 \right). \\
D(n) &= \sum_{u' \in [3]^n} \sum_{v' \in [3]^n} \left| 2^{n} + \sum_{p=1}^{n} d(p) \cdot 2^{n-p} \right| \\
&= \sum_{u_1=1}^{3} \sum_{v_1=1}^{3} \cdots \sum_{u_n=1}^{3} \sum_{v_n=1}^{3} \left| 2^{n} + \sum_{p=1}^{n} d(p) \cdot 2^{n-p} \right|.
\end{aligned}$$

From Table 2, when (u_1, v_1) is (1, 1) or (2, 2), $d(1) = 1$, the expression inside the absolute value sign of $D(n)$ is greater than 0,

$$\begin{aligned}
Z_1 &= \sum_{u_2=1}^{3} \sum_{v_2=1}^{3} \cdots \sum_{u_n=1}^{3} \sum_{v_n=1}^{3} \left(2^{n} + 2^{n-1} + \sum_{p=2}^{n} d(p) \cdot 2^{n-p} \right) \\
&= \sum_{u_2=1}^{3} \sum_{v_2=1}^{3} \cdots \sum_{u_n=1}^{3} \sum_{v_n=1}^{3} \left(2^{n} + 2^{n-1} \right) + 0 = 2^{n-1} \cdot 3^{2n-1}.
\end{aligned}$$

When (u_1, v_1) is (1, 2) or (2, 3) or (3, 1), $d(1) = 0$, the expression inside the absolute value sign of $D(n)$ is greater than 0,

$$
\begin{aligned}
Z_2 &= \sum_{u_2=1}^{3}\sum_{v_2=1}^{3}\cdots\sum_{u_n=1}^{3}\sum_{v_n=1}^{3}\left(2^n+\sum_{p=2}^{n}d(p)\cdot 2^{n-p}\right)\\
&= \sum_{u_2=1}^{3}\sum_{v_2=1}^{3}\cdots\sum_{u_n=1}^{3}\sum_{v_n=1}^{3}2^n+0=2^n\cdot 3^{2n-2}.
\end{aligned}
$$

When (u_1, v_1) is (2, 1), $d(1) = 2$, the expression inside the absolute value sign of $D(n)$ is greater than 0,

$$
\begin{aligned}
Z_3 &= \sum_{u_2=1}^{3}\sum_{v_2=1}^{3}\cdots\sum_{u_n=1}^{3}\sum_{v_n=1}^{3}\left(2^n+2\cdot 2^{n-1}+\sum_{p=2}^{n}d(p)\cdot 2^{n-p}\right)\\
&= \sum_{u_2=1}^{3}\sum_{v_2=1}^{3}\cdots\sum_{u_n=1}^{3}\sum_{v_n=1}^{3}2^{n+1}+0=2^{n+1}\cdot 3^{2n-2}.
\end{aligned}
$$

When (u_1, v_1) is (1, 3) or (3, 2), $d(1) = -1$,

$$
\begin{aligned}
Z_4 &= \sum_{u_2=1}^{3}\sum_{v_2=1}^{3}\cdots\sum_{u_n=1}^{3}\sum_{v_n=1}^{3}\left|2^n-2^{n-1}+\sum_{p=2}^{n}d(p)\cdot 2^{n-p}\right|\\
&= \sum_{u_2=1}^{3}\sum_{v_2=1}^{3}\cdots\sum_{u_n=1}^{3}\sum_{v_n=1}^{3}\left|2^{n-1}+\sum_{p=2}^{n}d(p)\cdot 2^{n-p}\right|\\
&= \sum_{u_1=1}^{3}\sum_{v_1=1}^{3}\cdots\sum_{u_{n-1}=1}^{3}\sum_{v_{n-1}=1}^{3}\left|2^{n-1}+\sum_{p=1}^{n-1}d(p)\cdot 2^{n-1-p}\right|=D(n-1).
\end{aligned}
$$

When (u_1, v_1) is (3, 3), $d(1) = -2$,

$$
\begin{aligned}
Z_5 &= \sum_{u_2=1}^{3}\sum_{v_2=1}^{3}\cdots\sum_{u_n=1}^{3}\sum_{v_n=1}^{3}\left|2^n-2\cdot 2^{n-1}+\sum_{p=2}^{n}d(p)\cdot 2^{n-p}\right|\\
&= \sum_{u_2=1}^{3}\sum_{v_2=1}^{3}\cdots\sum_{u_n=1}^{3}\sum_{v_n=1}^{3}\left|\sum_{p=2}^{n}d(p)\cdot 2^{n-p}\right|\\
&= \sum_{u_1=1}^{3}\sum_{v_1=1}^{3}\cdots\sum_{u_{n-1}=1}^{3}\sum_{v_{n-1}=1}^{3}\left|\sum_{p=1}^{n-1}d(p)\cdot 2^{n-1-p}\right|.
\end{aligned}
$$

Let

$$F(n)=\sum_{u_1=1}^{3}\sum_{v_1=1}^{3}\cdots\sum_{u_n=1}^{3}\sum_{v_n=1}^{3}\left|\sum_{p=1}^{n}d(p)\cdot 2^{n-p}\right|,$$

then

$$\begin{aligned}D(n)&=2\cdot Z_1+3\cdot Z_2+Z_3+2\cdot Z_4+Z_5\\&=2^n\cdot 3^{2n-1}+2^n\cdot 3^{2n-1}+2^{n+1}\cdot 3^{2n-2}+2D(n-1)+F(n-1)\\&=2^{n+3}\cdot 3^{2n-2}+2D(n-1)+F(n-1).\end{aligned}\tag{3.5}$$

Similar to $D(n)$, we obtain: when $d(1)=2$, the expression inside the absolute value sign of $F(n)$ is greater than 0,

$$\begin{aligned}H_1&=\sum_{u_2=1}^{3}\sum_{v_2=1}^{3}\cdots\sum_{u_n=1}^{3}\sum_{v_n=1}^{3}\left(2\cdot 2^{n-1}+\sum_{p=2}^{n}d(p)\cdot 2^{n-p}\right)\\&=\sum_{u_2=1}^{3}\sum_{v_2=1}^{3}\cdots\sum_{u_n=1}^{3}\sum_{v_n=1}^{3}2^n+0=2^n\cdot 3^{2n-2}.\end{aligned}$$

When $d(1)=1$,

$$\begin{aligned}H_2&=\sum_{u_2=1}^{3}\sum_{v_2=1}^{3}\cdots\sum_{u_n=1}^{3}\sum_{v_n=1}^{3}\left|2^{n-1}+\sum_{p=2}^{n}d(p)\cdot 2^{n-p}\right|\\&=\sum_{u_1=1}^{3}\sum_{v_1=1}^{3}\cdots\sum_{u_{n-1}=1}^{3}\sum_{v_{n-1}=1}^{3}\left|2^{n-1}+\sum_{p=1}^{n-1}d(p)\cdot 2^{n-1-p}\right|=D(n-1).\end{aligned}$$

When $d(1)=0$,

$$\begin{aligned}H_3&=\sum_{u_2=1}^{3}\sum_{v_2=1}^{3}\cdots\sum_{u_n=1}^{3}\sum_{v_n=1}^{3}\left|\sum_{p=2}^{n}d(p)\cdot 2^{n-p}\right|\\&=\sum_{u_1=1}^{3}\sum_{v_1=1}^{3}\cdots\sum_{u_{n-1}=1}^{3}\sum_{v_{n-1}=1}^{3}\left|\sum_{p=1}^{n-1}d(p)\cdot 2^{n-1-p}\right|=F(n-1).\end{aligned}$$

When $d(1) = -1$, due to the symmetry of the value of $d(p)$,

$$\begin{aligned} H_4 &= \sum_{u_2=1}^{3}\sum_{v_2=1}^{3}\cdots\sum_{u_n=1}^{3}\sum_{v_n=1}^{3}\left|-2^{n-1}+\sum_{p=2}^{n} d(p)\cdot 2^{n-p}\right| \\ &= \sum_{u_1=1}^{3}\sum_{v_1=1}^{3}\cdots\sum_{u_{n-1}=1}^{3}\sum_{v_{n-1}=1}^{3}\left|2^{n-1}+\sum_{p=1}^{n-1}[-d(p)]\cdot 2^{n-1-p}\right| \\ &= \sum_{u_1=1}^{3}\sum_{v_1=1}^{3}\cdots\sum_{u_{n-1}=1}^{3}\sum_{v_{n-1}=1}^{3}\left|2^{n-1}+\sum_{p=1}^{n-1} d(p)\cdot 2^{n-1-p}\right| = D(n-1). \end{aligned}$$

When $d(1) = -2$, the expression inside the absolute value sign of $F(n)$ is less than 0,

$$\begin{aligned} H_5 &= \sum_{u_2=1}^{3}\sum_{v_2=1}^{3}\cdots\sum_{u_n=1}^{3}\sum_{v_n=1}^{3}\left|-2\cdot 2^{n-1}+\sum_{p=2}^{n} d(p)\cdot 2^{n-p}\right| \\ &= \sum_{u_2=1}^{3}\sum_{v_2=1}^{3}\cdots\sum_{u_n=1}^{3}\sum_{v_n=1}^{3}\left(2^{n}-\sum_{p=2}^{n} d(p)\cdot 2^{n-p}\right) \\ &= \sum_{u_2=1}^{3}\sum_{v_2=1}^{3}\cdots\sum_{u_n=1}^{3}\sum_{v_n=1}^{3} 2^{n}+0 = 2^{n}\cdot 3^{2n-2}. \end{aligned}$$

Therefore,

$$\begin{aligned} F(n) &= H_1 + 2H_2 + 3H_3 + 2H_4 + H_5 \\ &= 2^n\cdot 3^{2n-2} + 2D(n-1) + 3F(n-1) + 2D(n-1) + 2^n\cdot 3^{2n-2} \\ &= 2^{n+1}\cdot 3^{2n-2} + 4D(n-1) + 3F(n-1). \end{aligned} \tag{3.6}$$

It is not difficult to know that $D(1) = 18$ and $F(1) = 8$. Combining Eqs. (3.5) and (3.6), we shall give explicit expressions for $D(n)$ and $F(n)$.

Let $D(n) = \sum_{n\geq 1} D(n)x^n$ and $F(n) = \sum_{n\geq 1} F(n)x^n$. From the given recurrence relations, it follows that

$$\begin{aligned} D(x) &= 18x + \sum\nolimits_{n\geq 2} 2^{n+3}\cdot 3^{2n-2}x^n + 2xD(x) + xF(x), \\ F(x) &= 8x + \sum\nolimits_{n\geq 2} 2^{n+1}\cdot 3^{2n-2}x^n + 4xD(x) + 3xF(x). \end{aligned}$$

By simplifying these formulas, we obtain a system of linear equations of variables $D(x)$ and $F(x)$,

$$\begin{aligned}(2x-1)D(x)+xF(x)&=-\frac{18x(1-2x)}{1-18x},\\4xD(x)+(3x-1)F(x)&=-\frac{8x(1-9x)}{1-18x}.\end{aligned}$$

By Cramer's rule, we have that

$$\begin{aligned}D(x)&=\frac{2x(18x^2-41x+9)}{(1-18x)(2x^2-5x+1)},\\F(x)&=\frac{8x(1-2x)}{(1-18x)(2x^2-5x+1)}.\end{aligned}$$

Finally, the expressions for $D(n)$ and $F(n)$ are given as follows:

$$\begin{aligned}D(n)&=\frac{1}{17\times 59\times 2^n}\left((31\sqrt{17}-17)(5-\sqrt{17})^n-(31\sqrt{17}+17)(5+\sqrt{17})^n\right)\\&\quad+\frac{61}{59}\times 18^n,\\F(n)&=\frac{1}{59}\times 2^{n+5}\times 9^n-\frac{1}{59\times\sqrt{17}\times 2^{n-3}}\\&\quad\left(\left(2\sqrt{17}-3\right)\left(5-\sqrt{17}\right)^n+\left(3+2\sqrt{17}\right)\left(5+\sqrt{17}\right)^n\right).\end{aligned}$$

Bringing the expressions for $D(n)$ and $B(n)$ to Eq. (3.3), we get

$$\begin{aligned}&\sum_{\substack{u\in V(S_1(n+1,3))\\v\in V(S_2(n+1,3))}}d(u,v)\\&=\frac{1}{2}\cdot 3^{2n}(2^n+2)+\frac{1}{2}B(n)-\frac{1}{2}D(n)\\&=\frac{1}{2}\cdot 3^{2n}(2^n+2)+\frac{1}{2}\times 24\times 3^{2n-2}\cdot\left(2^n-1\right)\\&\quad-\frac{1}{2}\times\frac{1}{17\times 59\times 2^n}\\&\quad\left[(31\sqrt{17}-17)(5-\sqrt{17})^n-(31\sqrt{17}+17)(5+\sqrt{17})^n\right]\\&\quad-\frac{1}{2}\times\frac{61}{59}\times 18^n\\&=3^{2n-1}\left(11\times 2^{n-1}-1\right)-\frac{1}{17\times 59\times 2^{n+1}}\\&\quad\left[(31\sqrt{17}-17)(5-\sqrt{17})^n-(31\sqrt{17}+17)(5+\sqrt{17})^n\right]\\&\quad-\frac{61}{118}\times 18^n.\end{aligned}\tag{3.7}$$

Let

$$M(n) = \sum_{\substack{u \in V(S_1(n+1,3)) \\ v \in V(S_2(n+1,3))}} d(u, v),$$

and bring it to Eq. (3.1), we get

$$\begin{aligned} W(S(n+1,3)) &= 3 \cdot W(S(n,3)) + 3M(n) \\ &= 3^2 \cdot W(S(n-1,3)) + 3M(n) + 3^2 M(n-1) \\ &\cdots \\ &= 3^n \cdot W(S(1,3)) + 3M(n) + 3^2 M(n-1) + \ldots + 3^n M(1) \\ &= 3^{n+1} + \sum_{p=1}^{n} 3^p M(n+1-p). \end{aligned} \tag{3.8}$$

From Eq. (3.7), we obtain

$$\begin{aligned} \sum_{p=1}^{n} 3^p M(n+1-p) &= \sum_{p=1}^{n} 11 \times 3^{2n+1} \times 2^n \times 6^{-p} - \sum_{p=1}^{n} 3^{2n+1} \times 3^{-p} \\ &\quad - \sum_{p=1}^{n} \frac{(31\sqrt{17} - 17)(5 - \sqrt{17})^{n+1}}{17 \times 59 \times 2^{n+2}} \cdot \left(\frac{6}{5 - \sqrt{17}}\right)^p \\ &\quad + \sum_{p=1}^{n} \frac{(31\sqrt{17} + 17)(5 + \sqrt{17})^{n+1}}{17 \times 59 \times 2^{n+2}} \cdot \left(\frac{6}{5 + \sqrt{17}}\right)^p \\ &\quad - \sum_{p=1}^{n} \frac{61}{118} \times 18^{n+1} \times 6^{-p} \\ &= \frac{11}{5} \times 3^{2n+1} \times 2^n \left(1 - \frac{1}{6^n}\right) - 3^{2n+1} \times \frac{1}{2}\left(1 - \frac{1}{3^n}\right) \\ &\quad + \frac{(31\sqrt{17} - 17)(5 - \sqrt{17})^{n+1}}{17 \times 59 \times 2^{n+2}} \cdot \frac{6}{\sqrt{17} + 1} \\ &\quad \cdot \left[1 - \left(\frac{6}{5 - \sqrt{17}}\right)^n\right] \\ &\quad + \frac{(31\sqrt{17} + 17)(5 + \sqrt{17})^{n+1}}{17 \times 59 \times 2^{n+2}} \cdot \frac{6}{\sqrt{17} - 1} \\ &\quad \left[1 - \left(\frac{6}{5 + \sqrt{17}}\right)^n\right] \\ &\quad - \frac{61}{118} \times 18^{n+1} \times \frac{1}{5}\left(1 - \frac{1}{6^n}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{11}{5} \times 3^{n+1}\left(6^n-1\right) - \frac{1}{2} \times 3^{n+1}\left(3^n-1\right) \\
&\quad + \frac{(31\sqrt{17}-17)(5-\sqrt{17})}{17 \times 59 \times 2^{n+2}} \cdot \frac{6}{\sqrt{17}+1}[(5-\sqrt{17})^n - 6^n] \\
&\quad + \frac{(31\sqrt{17}+17)(5+\sqrt{17})}{17 \times 59 \times 2^{n+2}} \cdot \frac{6}{\sqrt{17}-1}[(5+\sqrt{17})^n - 6^n] \\
&\quad - \frac{61 \times 6}{118 \times 5} \times 3^{n+1}\left(6^n-1\right) \\
&= \frac{466}{295} \times 3^{n+1}\left(6^n-1\right) - \frac{1}{2} \times 3^{n+1}\left(3^n-1\right) \\
&\quad + \frac{3}{17 \times 59 \times 2^{n+4}} \cdot \Big\{(31\sqrt{17}-17)(3\sqrt{17}-11) \\
&\quad \cdot [(5-\sqrt{17})^n - 6^n] \\
&\quad + (31\sqrt{17}+17)(3\sqrt{17}+11)[(5+\sqrt{17})^n - 6^n]\Big\} \\
&= \frac{466}{295} \times 3^{n+1}\left(6^n-1\right) - \frac{1}{2} \times 3^{n+1}\left(3^n-1\right) \\
&\quad + \frac{3}{17 \times 59 \times 2^{n+1}} \cdot \Big\{(221-49\sqrt{17})\,[(5-\sqrt{17})^n - 6^n] \\
&\quad + (221+49\sqrt{17})[(5+\sqrt{17})^n - 6^n]\Big\}.
\end{aligned}
$$

Bringing above result to Eq. (3.8), we get

$$
\begin{aligned}
W(S(n+1,3)) =& 3^{n+1} + \frac{466}{295} \times 3^{n+1}\left(6^n-1\right) - \frac{1}{2} \times 3^{n+1}\left(3^n-1\right) \\
&+ \frac{3}{17 \times 59 \times 2^{n+1}} \Big\{(221-49\sqrt{17})[(5-\sqrt{17})^n - 6^n] \\
&+ (221+49\sqrt{17})[(5+\sqrt{17})^n - 6^n]\Big\},
\end{aligned}
$$

which gives an explicit expression of $W(S(n+1,3))$.

Now we shall give a recurrence relation of $W(S(n+1,3))$. Denote

$$W(x) = \sum_{n=1}^{\infty} W(S(n,3))x^n.$$

From Eq. (3.1), we obtain that

$$
\begin{aligned}
\frac{W(x)-3x}{x} =& 3W(x) + 3\Bigg(\sum_{n=1}^{\infty} \frac{1}{2} \times 3^{2n}\left(2^n+2\right)x^n \\
&+ \sum_{n=1}^{\infty} \frac{1}{2} \times 24 \times 3^{2n}\left(2^{2n}+2\right)x^n - \frac{1}{2}D(x)\Bigg)
\end{aligned}
$$

$$= 3W(x) + 3\left(\frac{9x(2-27x)}{(1-9x)(1-18x)} + \frac{12x}{(1-9x)(1-18x)} - \frac{x(18x^2-41x+9)}{(1-18x)(2x^2-5x+1)}\right).$$

A routine computation shows that

$$W(x) = \frac{3\left(24x^3+28x^2-11x+1\right)}{(1-3x)(1-9x)(1-18x)\left(2x^2-5x+1\right)}.$$

Then we have that

$$\begin{aligned}\left(2x^2-5x+1\right)W(x) &= \frac{3\left(24x^3+28x^2-11x+1\right)}{(1-3x)(1-9x)(1-18x)}\\ &= -\frac{4}{27} + \frac{2}{5}\cdot\frac{1}{1-3x} - \frac{19}{27}\cdot\frac{1}{1-9x} + \frac{466}{135}\cdot\frac{1}{1-18x}.\end{aligned}$$

Hence, it follows that

$$\begin{aligned}W(S(n+1,3)) &= 5W(S(n,3)) - 2W(S(n-1,3)) + \frac{2}{5}\times 3^n\\ &\quad - \frac{19}{27}\times 3^{2n} + \frac{466}{135}\times 18^n\\ &= 5W(S(n,3)) - 2W(S(n-1,3)) - 19\times 3^{2n-3}\\ &\quad + \frac{2}{5}\times 3^n\left(233\times 2^n\times 3^{n-3}+1\right).\end{aligned}$$

This completes the proof. □

References

Bollobás B (1998) Modern graph theory. Springer, New York

Beaudou L, Gravier S, Klavžar S, Kovše M, Mollard M (2010) Covering codes in Sierpiński graphs. Discrete Math Theor Comput Sci 12:63–74

Caporossi G, Hansen P (2000) Variable neighborhood search for extremal graphs: 1 The AutoGraphiX system. Discrete Math 212(1–2):29–44

Dobrynin AA, Entringer RC, Gutman I (2001) Wiener index of trees: theory and Applications. Acta Appl Math 6(6):211–249

Eliasi M (2009) Harary index of aigzag polyhex nanotorus. Dig J Nanomater Biostruct 4(4):757–762

Fu HY, Xie DZ (2010) Equitable $L(2,1)$-labelings of Sierpiński graphs. Australas J Comb 46:147–156

Gravier S, Klavžar S, Mollard M (2005) Codes and $L(2,1)$-labelings in Sierpiński-like graphs. Taiwan J Math 9:671–681

Gutman I, Vidović D, Popović L (1998) Graph representation of organic molecules Cayley's plerograms vs. his kenograms. J Chem Soc Faraday Trans 94(7):875-860

Gutman I (1997) A property of the Wiener number and its modifications. Indian J Chem 36A:128–132

Gravier S, Klavžar S, Mollard M (2005) Codes and $L(2,1)$-labelings in Sierpiński-like graphs. Taiwan J Math 9:671–681

Hinz AM (1992) Pascal's triangle and the Tower of Hanoi. Am Math Mon 99:538–544

Hinz AM, Klavžar S, Milutinović U, Parisse D, Petr C (2005) Metric properties of the Tower of Hanoi graphs and Stern's diatomic sequence. Eur J Comb 26:693–708

Hinz AM, Parisse D (2012) Coloring Hanoi and Sierpiński graphs. Discrete Math 312:1521–1535

Hinz AM, Parisse D (2012) The average eccentricity of Sierpiński graphs. Graphs Comb 28(5):1–16

Jakavac M, Klavžar S (2009) Vertex-, edge-, and total-coloring of Sierpiński-like graphs. Discrete Math 309:1548–1556

Jakovac M (2014) A 2-parametric generalization of Sierpiński gasket graphs. Ars Comb 116(3):395–405

Klavžar S, Milutinović U (1997) Graphs $S(n, k)$ and a variant of the Tower of Hanoi problem. J Czechoslov Math 47(122):95–104

Klavžar S, Mohar B (2005) Crossing numbers of Sierpiński-like graphs. J Graph Theory 50:186–198

Klavžar S, Milutinović U, Petr C (2002) 1-Perfect codes in Sierpiński-like graphs. Bull Aust Math Soc 66:369–384

Klavžar S, Zemljic SS (2013) On distances in Sierpinski graphs: almost- extreme vertices and metric dimension. Appl Anal Discrete Math 7(1):72–82

Klein DJ, Lukovits I, Gutman I (1995) On the definition of the hyper-Wiener index for cycle-containing structures. J Chem Inf Comput Sci 35:50–52

Lin CH, Liu JJ, Wang YL, Yen WC (2011) The hub number of Sierpiński-like graphs. Theory Comput Syst 49:588–600

Lipscomb SL, Perry JC (1992) Lipscomb's $L(A)$ space fractalized in Hilbert's $l^2(A)$ space. Proc Am Math Soc 115:1157–1165

Luo C, Zuo L (2017) The metric properties of Sierpiński-like graphs. Appl Math Comput 296:124–136

Milutinović U (1992) Completeness of the Lipscomb space. Glas Ser III 27(47):343–364

Parisse D (2009) On some metric properties of the Sierpiński-like graphs $S(n, k)$. Ars comb 90:145–160

Pisanski T, Tucker TW (2001) Growth in repeated truncations of maps. Atti Sem Mat Fis Univ Modena 49:167–176

Romik D (2006) Shortest paths in the Tower of Hanoi graph and finite automata. SIAM J Discrete Math 20:610–622

Trinajstić N, Li X, Gutman I (2006) Mathematical aspects of Randić-type molecular structure desriptors. Croat Chem Acta 79(3):A31–A32

Xue B, Zuo L, Li GJ (2012) The hamiltonicity and path t-coloring of Sierpiński-like graphs. Discrete Appl Math 160:1822–1836

Xue B, Zuo L, Wang G, Li GJ (2015) The linear t-colorings of Sierpiński-like graphs. Graphs Comb 31:1795–1805

Zhou B, Trinajstić N (2009) On a novel connectivity index. J Math Chem 46(4):1252–1270

Zhou B, Trinajstić N (2010) On general sum-connectivity index. J Math Chem 47(1):210–218

Zuo L (2012) About a conjecture on the Randić Index of graphs. Bull Malays Math Soc 2(2):411–424