# Analytic aspects of Delannoy numbers

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(Joint work with X. Chen and S.-N. Zheng)

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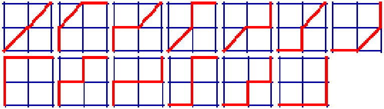
# Contents

# Outline

# Delannoy numbers

The Delannoy numbers $D(n,k)$ count the number of lattice paths from $(0,0)$ to $(n,k)$ using steps $(1,0),(0,1)$ and $(1,1)$.

- $D(2,2)=13$:



- $D(n,k)=D(n-1,k)+D(n-1,k-1)+D(n,k-1)$.

| $n\setminus k$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 3 | 5 | 7 | 9 |
| 2 | 1 | 5 | 13 | 25 | 41 |
| 3 | 1 | 7 | 25 | 63 | 129 |
| 4 | 1 | 9 | 41 | 129 | 321 |

- The central Delannoy numbers $D(n,n)$.

# Why Delannoy numbers?

Henri-Auguste Delannoy (28 September 1833–5 February 1915) was a French army officer and amateur mathematician.


- Banderier and Schwer, Why Delannoy numbers? J. Statist. Plann. Inference, 2005.
- Comtet, Advanced Combinatorics, 1974.

# Formulas

1. $D(n,k) = D(n-1,k) + D(n-1,k-1) + D(n,k-1)$.
2. $\sum_{n,k\geq 0} D(n,k)x^n y^k = (1-x-y-xy)^{-1}$.
3. $D(n,k) = \sum_i \binom{n+k-i}{i,n-i,k-i} = \sum_i \binom{n+k-i}{k}\binom{k}{i}$.
4. $D(n,k) = \sum_j \binom{n}{j}\binom{k}{j}2^j$.
5. The matrix has the decomposition

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 1 & 5 & 13 & 25 \\ 1 & 7 & 25 & 63 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 4 & \\ & & & 8 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ & 1 & 2 & 3 \\ & & 1 & 3 \\ & & & 1 \end{pmatrix}.$$

# The central Delannoy numbers

- Let $D_n = D(n,n)$ denote the central Delannoy numbers. Then
  1. $D_n = \sum_i \binom{n}{i}\binom{n+i}{i} = \sum_i \binom{n+i}{n-i}\binom{2i}{i} = \sum_i \binom{n}{i}^2 2^i$.
  2. $D_n = P_n(3)$, where $P_n(x) = \sum_i \binom{n}{i}\binom{n+i}{i}\left(\frac{x-1}{2}\right)^i$. (Legendre polynomials).
  3. $nD_n = 3(2n-1)D_{n-1} - (n-1)D_{n-2}, \quad D_0 = 1,\ D_1 = 3.$
  4. $\sum_{n\geq 0} D_n x^n = \frac{1}{\sqrt{1-6x+x^2}}$.
  5. $$\sum_{n\geq 0} D_n x^n = \cfrac{1}{1-3x-\cfrac{4x^2}{1-3x-\cfrac{2x^2}{1-3x-\cfrac{2x^2}{1-3x-\cdots}}}}.$$
  6. The central Delannoy numbers bear the same relation to the Schröder numbers as the central binomial coefficients do to the Catalan numbers.
- Sulanke listed 29 objects counted by the central Delannoy numbers.

Sulanke, Objects counted by the central Delannoy numbers, 2003.

# Log-behavior of Delannoy numbers

1
1 1
1 3 1
1 5 5 1
1 7 13 7 1
1 9 25 25 9 1
1 11 41 63 41 11 1
1 13 61 129 129 61 13 1
1 15 85 231 321 231 85 15 1

Kiselman (2012) proposed the following conjectures:

(1) $D(n,n)$ is log-convex in $n$.

(2) $D(n,k)$ is log-concave in $k$ for fixed $n$.

(3) $D(n-k,k)$ is log-concave in $k$ for fixed $n$.

# Outline

# Delannoy square (A008288)

Arrange Delannoy numbers in a square array

$$D^{\ulcorner} = [D(n,k)]_{n,k\geq 0} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 3 & 5 & 7 & \cdots \\ 1 & 5 & 13 & 25 & \cdots \\ 1 & 7 & 25 & 63 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Recall that $\sum_{n,k\geq 0} D(n,k)x^n y^k = \frac{1}{1-x-y-xy}$.
Hence the generating function of the $k$th column of $D^{\ulcorner}$ is

$$\sum_{n\geq 0} D(n,k)x^n = \frac{(1+x)^k}{(1-x)^{k+1}}.$$

# Delannoy triangle (tribonacci triangle)

Let $d(n,k) = D(n-k,k)$ and define the Delannoy triangle

$$D = [d(n,k)] = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 1 & & & \\ 1 & 5 & 5 & 1 & & \\ 1 & 7 & 13 & 7 & 1 & \\ \vdots & & & & & \ddots \end{bmatrix}.$$

Then the generating function of the $k$th column of $D$ is

$$\sum_{n\geq k} d(n,k)x^n = \frac{x^k(1+x)^k}{(1-x)^{k+1}}.$$

The Delannoy polynomial $d_n(x) = \sum_{k=0}^{n} d(n,k)x^k$ is defined as the row-generating function of $D$.

# Half of Delannoy triangle (A113139)

Define the half of Delannoy triangle

$$D^{\mathrm{h}} = [D^{\mathrm{h}}(n,k)] = \begin{bmatrix} 1 & & & & \\ 3 & 1 & & & \\ 13 & 5 & 1 & & \\ 63 & 25 & 7 & 1 & \\ \vdots & & & & \ddots \end{bmatrix}, \quad D^{\mathrm{h}}(n,k) = D(n, n-k).$$

The generating function of the $k$th column of $D^{\mathrm{h}}$ is $D(x)x^k S^k(x)$, where

$$D(x) = \frac{1}{\sqrt{1-6x+x^2}} = 1 + 3x + 13x^2 + 63x^3 + \cdots$$

and

$$S(x) = \frac{1 - x - \sqrt{1-6x+x^2}}{2x} = 1 + 2x + 6x^2 + 22x^3 + \cdots$$

are the generating functions of the central Delannoy numbers $D(n,n)$ and the large Schröder numbers $S_n$ respectively.

# Delannoy recursive matrix (A118384)

The Delannoy recursive matrix $D^r = [r_{n,k}]$ is defined by

$$\begin{cases} r_{0,0} = 1, \quad r_{n+1,0} = 3r_{n,0} + 4r_{n,1}, & \text{for } n \geq 0; \\ r_{n+1,k} = r_{n,k-1} + 3r_{n,k} + 2r_{n,k+1}, & \text{for } k \geq 1. \end{cases}$$

Then

$$D^r = \begin{pmatrix} 1 & & & & \\ 3 & 1 & & & \\ 13 & 6 & 1 & & \\ 63 & 33 & 9 & 1 & \\ \vdots & & & & \ddots \end{pmatrix}.$$

The generating function of the $k$th column of $D^{\mathrm{h}}$ is $D(x)x^k\bar{s}^k(x)$, where

$$\bar{s}(x) = \frac{1-3x-\sqrt{1-6x+x^2}}{4x^2} = 1 + 3x + 11x^2 + 45x^3 + \cdots$$

is the generating functions of the shifted little Schröder numbers $s_{n+1}$.

# Riordan arrays

**Definition**

A (proper) Riordan array, denoted by $(d(x), h(x))$, is an infinite lower triangular matrix whose generating function of the $k$th column is

$$d(x)h^k(x)$$

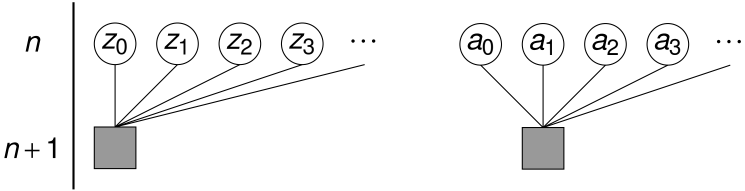
for $k = 0, 1, 2, \ldots$, where $d(0) = 1, h(0) = 0$ and $h'(0) \neq 0$.
It is called an improper Riordan array if $d(0) = 1$ and $h(0) \neq 0$.

- The Delannoy square $D^{\ulcorner} = \left(\frac{1}{1-x}, \frac{1+x}{1-x}\right)$ is improper.
- The Delannoy triangle $D = \left(\frac{1}{1-x}, \frac{x(1+x)}{1-x}\right)$.
- The half of Delannoy triangle $D^{\mathrm{h}}(n,k) = D(n, n-k) = (D(x), xS(x))$.
- The Delannoy recursive matrix $D^r = (D(x), x\bar{s}(x))$.

# $A$- and $Z$-sequences of a Riordan array

- A Riordan array $R=(d(x),h(x))=[r_{n,k}]_{n,k\geq 0}$ can be characterized by two sequences $A=(a_n)_{n\geq 0}$ and $Z=(z_n)_{n\geq 0}$ such that

$$r_{0,0}=1,\quad r_{n+1,0}=\sum_{j\geq 0} z_j r_{n,j},\quad r_{n+1,k+1}=\sum_{j\geq 0} a_j r_{n,k+j}.$$




- $Z=(3,4,0,\ldots)$ and $A=(1,3,2,0,\ldots)$ for the recursive matrix $D^r$.
- Denote $Z(x)=\sum_{n\geq 0} z_n x^n$ and $A(x)=\sum_{n\geq 0} a_n x^n$. Then

$$d(x)=\frac{1}{1-xZ(h(x))},\qquad h(x)=xA(h(x)).$$

# Product matrix

- Let $R$ be a Riordan array with $A = (a_n)$ and $Z = (z_n)$. Then $\overline{R} = RJ$, where $\overline{R}$ is obtained from $R$ by deleting the 0th row and

$$J(R) = \begin{bmatrix} z_0 & a_0 & & & & \\ z_1 & a_1 & a_0 & & & \\ z_2 & a_2 & a_1 & a_0 & & \\ z_3 & a_3 & a_2 & a_1 & a_0 & \\ \vdots & \vdots & & & & \ddots \end{bmatrix}.$$

is the product matrix of $R$.

- For the half of Delannoy triangle $D^h = (D(x), xS(x))$, we have $Z(x) = \frac{3+x}{1-x} = 3 + 4x + 4x^2 + \cdots$, $\quad A(x) = \frac{1+x}{1-x} = 1 + 2x + 2x^2 + \cdots$, and

$$J(D^h) = \begin{bmatrix} 3 & 1 & & & & \\ 4 & 2 & 1 & & & \\ 4 & 2 & 2 & 1 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

# Outline

# Totally positive matrices (TP)

- A matrix is TP ($\mathrm{TP}_r$) if its minors of all orders ($\leq r$) are nonnegative.
- $V(x_1, x_2, \ldots) = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots \\ 1 & x_2 & x_2^2 & \\ 1 & x_3 & x_3^2 & \\ \vdots & & & \ddots \end{bmatrix}$ is TP if $0 \leq x_1 \leq x_2 \leq \cdots$.
- Hilbert matrix $H = \left[\frac{1}{i+j-1}\right]_{i,j\geq 1} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \\ \vdots & & & \ddots \end{bmatrix}$ is TP.
  - Hilbert (1894) evaluated $\det H_n = \dfrac{c_n^4}{c_{2n}}$, where $c_n = \prod_{i=1}^{n-1} i!$.
  - Cauchy determinant $\det\left[\frac{1}{x_i+y_j}\right]_{1\leq i,j\leq n} = \dfrac{\prod_{i=2}^{n}\prod_{j=1}^{i-1}(x_i-x_j)(y_i-y_j)}{\prod_{i=1}^{n}\prod_{j=1}^{n}(x_i+y_j)}$.

# Toeplitz matrix and Hankel matrix of a sequence

Let $\alpha = (a_n)_{n\geq 0}$ be a finite or infinite sequence of nonnegative numbers. Define its Toeplitz matrix $T(\alpha)$ and Hankel matrix $H(\alpha)$

$$T(\alpha) = \begin{bmatrix} a_0 & & & & \\ a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ a_3 & a_2 & a_1 & a_0 & \\ \vdots & & & & \ddots \end{bmatrix}, \quad H(\alpha) = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \\ a_2 & a_3 & a_4 & a_5 & \\ a_3 & a_4 & a_5 & a_6 & \\ \vdots & & & & \ddots \end{bmatrix}.$$

We say that $\alpha = (a_n)_{n\geq 0}$ is

(i) a Pólya frequency sequence (**PF**) if $T(\alpha)$ is TP.

(ii) a log-concave sequence (**LC**) if $T(\alpha)$ is $\mathrm{TP}_2$: $\quad a_{n-1}a_{n+1} \leq a_n^2$.

(iii) a Stieltjes moment sequence (**SM**) if $H(\alpha)$ is TP.

(iv) a log-convex sequence (**LCX**) if $H(\alpha)$ is $\mathrm{TP}_2$: $\quad a_{n-1}a_{n+1} \geq a_n^2$.

# Characterizations of PF sequences

## Theorem (Schoenberg and Edrei, 1952)

*Let $a_i > 0$. Then*

$$a_0 = 1, a_1, a_2, \ldots \text{ is PF} \iff \sum_{i\geq 0} a_i x^i = \frac{\prod_{j\geq 1}(1+\alpha_j x)}{\prod_{j\geq 1}(1-\beta_j x)}\, e^{\gamma x},$$

*where $\gamma, \alpha_j, \beta_j \geq 0$ and $\sum(\alpha_j + \beta_j) < \infty$.*

- $1 + \alpha x$;
- $\frac{1}{1-\beta x} = 1 + \beta x + \beta^2 x^2 + \cdots + \beta^n x^n + \cdots$;
- $e^{\gamma x} = 1 + \gamma x + \frac{\gamma^2}{2!} x^2 + \cdots + \frac{\gamma^n}{n!} x^n + \cdots$.

## Theorem (Aissen, Schoenberg and Whitney, 1952)

*Let $a_i \geq 0$. Then $a_0, \ldots, a_n$ is PF $\iff \sum_{i=0}^n a_i x^i$ has only real zeros.*


Karlin, Total Positivity, Vol. I, 1968.

# Characterizations of SM sequences

**Theorem (folklore)**

*The following conditions are equivalent.*

1. $\alpha = (a_n)_{n\geq 0}$ *is SM.*
2. $a_n = \int_0^{+\infty} x^n d\mu(x)$, *where* $\mu$ *is a non-negative measure on* $[0, +\infty)$.
3. $H(\alpha) = [a_{i+j}]_{i,j\geq 0}$ *is TP.*
4. $\det[a_{i+j}]_{0\leq i,j\leq n} \geq 0$ *and* $\det[a_{i+j+1}]_{0\leq i,j\leq n} \geq 0$ *for all* $n \geq 0$.
5. $\sum_{n=0}^{N} c_n a_n \geq 0$ *for each polynomial* $\sum_{n=0}^{N} c_n x^n \geq 0$ *on* $[0, +\infty)$.


- Pólya and Szegö, Problems and Theorems in Analysis II, 1976.
- Shohat and Tamarkin, The Problem of Moments, 1943.
- Widder, The Laplace Transform, 1946.

# Total positivity of Delannoy triangle and square

**Theorem (Chen and W., 2019, LAA)**

*Let $R = (d(x), h(x))$ be a proper or improper Riordan array. If both $d(x)$ and $h(x)$ are PF, then $R = (d(x), h(x))$ is TP.*

**Corollary (W., Zheng and Chen, 2019, DM)**

*Both the Delannoy triangle and the Delannoy square are TP.*

- Delannoy triangle $D = [d(n,k)] = \left(\frac{1}{1-x}, \frac{x(1+x)}{1-x}\right)$.
- Delannoy square $D^{\ulcorner} = [D(n,k)] = \left(\frac{1}{1-x}, \frac{1+x}{1-x}\right)$.
- $\sum_{n\geq 0} D(n,k)x^n = \frac{(1+x)^k}{(1-x)^{k+1}} \Longrightarrow D(0,k), D(1,k), D(2,k), \ldots$ is PF for each $k$
  $\Longrightarrow D(0,k), D(1,k), D(2,k), \ldots$ is log-concave for each $k$.

# Total positivity of the half of Delannoy triangle

**Theorem (Chen, Liang and W., EuJC, 2015)**

*If the product matrix $J(R)$ of a Riordan array $R$ is TP, then so is $R$.*

**Theorem (W., Zheng and Chen, 2019, DM)**

*The half Delannoy triangle $D^h$ and Delannoy recursive matrix $D^r$ are TP.*

$$J(D^h) = \begin{bmatrix} 3 & 1 & & & & \\ 4 & 2 & 1 & & & \\ 4 & 2 & 2 & 1 & & \\ 4 & 2 & 2 & 2 & 1 & \\ \vdots & \vdots & & & & \ddots \end{bmatrix}, \quad J(D^r) = \begin{bmatrix} 3 & 1 & & & & \\ 4 & 3 & 1 & & & \\ & 2 & 3 & 1 & & \\ & & 2 & 3 & 1 & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

# Total positivity of the Delannoy recursive matrix

Let $R = [r_{n,k}]$ be an infinite lower triangular matrix defined by

$$\begin{cases} r_{0,0} = 1, \quad r_{n+1,0} = pr_{n,0} + qr_{n,1}, & \text{for } n \geq 0; \\ r_{n+1,k} = r_{n,k-1} + sr_{n,k} + tr_{n,k+1}, & \text{for } k \geq 1. \end{cases}$$

**Theorem (Liang, Mu and W., DM., 2016)**

*If $s^2 \geq 4t$ and $p(s + \sqrt{s^2 - 4t}) \geq 2q$, then $R$ is TP and $(r_{n,0})_{n\geq 0}$ is SM.*

- $p = s = 3, q = 4, t = 2$ for the Delannoy recursive matrix $D^r$.
- $D^r$ is a totally positive matrix.
- $(D(n,n))_{n\geq 0}$ is a Stieltjes moment sequence.

  $\Longrightarrow$ $(D(n,n))_{n\geq 0}$ is log-convex in $n$.

# Delannoy polynomials

The Delannoy polynomial $d_n(x) = \sum_{k=0}^{n} d(n,k)x^k$ is the row-generating function of the Delannoy triangle

$$D = [d(n,k)] = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 1 & & & \\ 1 & 5 & 5 & 1 & & \\ 1 & 7 & 13 & 7 & 1 & \\ \vdots & & & & & \ddots \end{bmatrix}.$$

1. $d_n(x) = (x+1)d_{n-1}(x) + xd_{n-2}(x), \qquad d_0(x) = 1, \quad d_1(x) = x+1.$
2. $\sum_{n\geq 0} d_n(x)y^n = \frac{1}{1-(x+1)y-xy^2}.$
3. $d_n(x) = \frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1-\lambda_2}$, where $\lambda_1$ and $\lambda_2$ are roots of $\lambda^2-(x+1)\lambda-x=0$.
4. $d_n(x) = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n-k}{k} x^k (x+1)^{n-2k}. \qquad$ (the $\gamma$-positivity)

# Zeros of Delannoy polynomials

Now $d_n(x) = (x+1)d_{n-1}(x) + xd_{n-2}(x)$ and $d_n(x) = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2}$.

## Lemma

1. $d_n(x) = \prod_{k=1}^{n}(x + r_{n,k})$, *where* $r_{n,k} = -\left(\sqrt{1+\cos^2\frac{k\pi}{n+1}} + \cos\frac{k\pi}{n+1}\right)^2$.
2. $r_{n+1,1} < r_{n,1} < r_{n+1,2} < \cdots < r_{n+1,n} < r_{n,n} < r_{n+1,n+1}$.
3. $(r_{n,1})_{n\geq 1}$ *are strictly decreasing and* $\lim_{n\to+\infty} r_{n,1} = -(\sqrt{2}+1)^2$.
4. $(r_{n,n})_{n\geq 1}$ *are strictly increasing and* $\lim_{n\to+\infty} r_{n,n} = -(\sqrt{2}-1)^2$.

## Theorem (W., Zheng and Chen, 2019, DM)

*Zeros of* $d_n(x)$ *are real, distinct and in the interval* $(-3-2\sqrt{2}, -3+2\sqrt{2})$. *Furthermore, zeros of all* $d_n(x)$ *are dense in* $[-3-2\sqrt{2}, -3+2\sqrt{2}]$.

- $d(n,0), d(n,1), \ldots, d(n,n)$ is a unimodal and log-concave sequence.

# Asymptotic normality

Let $a(n,k)$ be a double-indexed sequence of nonnegative numbers and

$$p(n,k) = \frac{a(n,k)}{\sum_{j=0}^{n} a(n,j)}.$$

We say that $a(n,k)$ is asymptotically normal by a central limit theorem if

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \sum_{k\le \mu_n + x\sigma_n} p(n,k) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt \right| = 0,$$

where $\mu_n$ and $\sigma_n^2$ are the mean and variance of $a(n,k)$, respectively.
We say that $a(n,k)$ is asymptotically normal by a local limit theorem if

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \sigma_n p(n, \lfloor \mu_n + x\sigma_n \rfloor) - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right| = 0.$$

In this case,

$$a(n,k) \sim \frac{e^{-x^2/2} \sum_{j=0}^{n} a(n,j)}{\sigma_n \sqrt{2\pi}} \text{ as } n\to\infty,$$

where $k = \mu_n + x\sigma_n$ and $x = O(1)$.

# A criterion for asymptotic normality

**Lemma (Bender, JCTA, 1973)**

*Suppose that $A_n(x)=\sum_{k=0}^{n} a(n,k)x^k$ have only real zeros. Let $A_n(x)=\prod_{i=1}^{n}(x+r_{n,i})$. Denote $\mu_n=\sum_{i=1}^{n}\frac{1}{1+r_{n,i}}$ and $\sigma_n^2=\sum_{i=1}^{n}\frac{r_{n,i}}{(1+r_{n,i})^2}$. If $\sigma_n^2\to+\infty$, then the numbers $a(n,k)$ are asymptotically normal (by central and local limit theorems) with the mean $\mu_n$ and variance $\sigma_n^2$.*

- the binomial coefficients $\binom{n}{k}$; (De Moivre-Laplace theorem)
- the (signless) Stirling numbers of the first kind $c(n,k)$;
- the Stirling numbers of the second kind $S(n,k)$;
- the Eulerian numbers $A(n,k)$;
- the Narayana numbers $N(n,k)$. (Chen, Mao and W., 2020)


Harper, Stirling behavior is asymptotically normal, Ann. Math. Statist., 1967.

Bender, Central and local limit theorems applied to asymptotic enumeration, J. Combin. Theory Ser. A, 1973.

# Asymptotic normality of Delannoy numbers

**Theorem (W., Zheng and Chen, 2019, DM)**

*The Delannoy numbers $d(n,k)$ are asymptotically normal with the mean $\mu_n = n/2$ and variance $\sigma_n^2 \sim \sqrt{2}n/8$.*

**Proof.** Clearly, $\mu_n = n/2$ by the symmetry $d(n,k) = d(n, n-k)$.

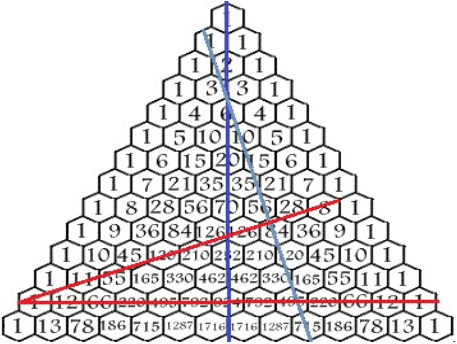
$$\begin{aligned}
\sigma_n^2 &= \sum_{k=1}^{n} \frac{\left(\cos\frac{k\pi}{n+1} + \sqrt{1+\cos^2\frac{k\pi}{n+1}}\right)^2}{\left[1+\left(\cos\frac{k\pi}{n+1} + \sqrt{1+\cos^2\frac{k\pi}{n+1}}\right)^2\right]^2} = \sum_{k=1}^{n} \frac{1}{4\left(1+\cos^2\frac{k\pi}{n+1}\right)} \\
&\to \frac{n}{4\pi}\int_0^{\pi} \frac{1}{1+\cos^2\theta}\, d\theta = \frac{\sqrt{2}}{8}n.
\end{aligned}$$

**Corollary**

$D(n,n) = d(2n,n) \sim (\sqrt{2}+1)^{2n+1}/2\sqrt{n\pi\sqrt{2}}.$

# Outline

# Log-behavior in the Pascal triangle



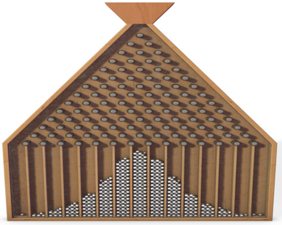

- log-concave by Su and W. and PF by Yu.
- log-convex by Su and W. Actually, SM!
- first log-concave and then log-convex, by Su and W. and by Yu.

Su and W., On unimodality problems in Pascal's triangle, EJC, 2008.

Yaming Yu, Confirming two conjectures of Su and Wang on binomial coefficients, Adv. in Appl. Math., 2009.

# Asymptotic normality in the Pascal triangle

**De Moivre-Laplace:** $\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}$ is asymptotically normal.



1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

**Godsil:** $\binom{n}{0}, \binom{n-1}{1}, \binom{n-2}{2}, \dots$ is asymptotically normal.

## Conjecture (Shapiro, Adv. in Appl. Math., 2001)

$\binom{n}{0}, \binom{n-a}{b}, \binom{n-2a}{2b}, \dots$ *is asymptotically normal if* $0 \le a \le b$.

- Yu (2009) showed the polynomial $\sum_k \binom{n-ak}{bk} x^k$ has only real zeros.
- Hou et al. recently settled the conjecture.

# Log-behavior in the Delannoy triangle

1
1 1
1 3 1
1 5 5 1
1 7 13 7 1
1 9 25 25 9 1
1 11 41 63 41 11 1
1 13 61 129 129 61 13 1
1 15 85 231 321 231 85 15 1

(1) $d(2n,n) = D(n,n)$ is log-convex in $n$. Actually, SM!

(2) $d(n+k,k) = D(n,k)$ is log-concave in $k$ for fixed $n$. Actually, PF!

(3) $d(n,k) = D(n-k,k)$ is log-concave in $k$ for fixed $n$. Actually, PF!

# Several problems in the Delannoy triangle

1
1 1
1 3 1
1 5 5 1
1 7 13 7 1
1 9 25 25 9 1
1 11 41 63 41 11 1
1 13 61 129 129 61 13 1
1 15 85 231 321 231 85 15 1
1 17 113 377 681 681 377 113 17 1

Let $a(n,k) := d(n+ak, m+bk)$. We suggest two problems as follows.

1. If $0 \le a < b$ and $0 \le m < b$, then $a(n,k)$ is asymptotically log-convex in $k$.
2. If $n \ge m \ge 0, b > a > 0$ and $m < b$, then $a(n,k)$ is asymptotically normal.
   Yu (2009) showed the polynomial $\sum_k a(n,k)x^k$ has only real zeros.

Thank you for your attention!