The Sperner Property and Representation of $\mathfrak{sl}(2, \mathbb{C})$

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The Sperner property

In 1927, Emanuel Sperner proved that if $S_1, \ldots, S_m$ are distinct subsets of an n-element set such that we never have $S_i \subset S_j$, then $m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Moreover, the equality is achieved by taking all subsets of S with $\lfloor \frac{n}{2} \rfloor$ elements. This result spawned a host of generalizations, most conveniently stated in the language of partially ordered poset.

A finite poset P is ranked (graded) if for every $x \in P$ every maximal chain with x as top element has the same length. We say that P is graded of rank n if every maximal chain of P has length n. Thus P has a unique rank function $\rho : P \to \{0, 1, 2, \ldots, n\}$ such that

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- $\rho(x) = 0$ if x is a minimal element of P;
- $\rho(y) = \rho(x) + 1$ if y covers x in P, denoted by $x < y$.

Define $P_i := \{x \in P : \rho(x) = i\}$ and set $p_i = p_i(P) = Card\, P_i$. The rank-generating function of P:

$$F(P, q) = p_0 + p_1 q + p_2 q^2 + \cdots + p_n q^n.$$

The sequence $\{p_0, p_1, p_2, \ldots, p_n\}$ is called the sequence of Whitney numbers of P. Let M_i denote the $(i+1)$st largest Whitney number in P. So the sequence $\{M_0, M_1, \ldots, M_n\}$ is the sequence of Whitney numbers arranged in nonincreasing order.

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We say that P is rank-symmetric if $p_i = p_{n-i}$ for all i. P is rank-unimodal if

$$p_0 \leq p_1 \leq \cdots \leq p_i \geq p_{i+1} \geq \cdots \geq p_n$$

for some i.

An antichain (Sperner family) is a subset A of P such that no two distinct elements of A are comparable. The poset P is said to have the Sperner property (property S_1) if the largest size of an antichain is equal to $max\{p_i : 0 \leq i \leq n\}$. More generally, if k is a positive integer then P is said to have the k-Sperner property (property S_k) if the largest subset of P containing no $(k+1)$-element chain has cardinality $max\{p_{i_1} + \cdots + p_{i_k} : 0 \leq i_1 < \cdots < i_k \leq n\}$. If P has property S_k for all $k \leq n$, we say that P has property S (strongly Sperner).

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Suppose that P is graded of rank n and is rank-symmetric. We say that P has property T if for all $0 \leq i \leq [\frac{n}{2}]$ there exists p_i pairwise disjoint saturated chains $x_i < x_{i+1} < \cdots < x_{n-i}$ where $x_j \in P_j$.

A ranked poset P satisfies condition T'_k, $1 \leq k \leq n$, if there exists M_k disjoint chains in P which each intersect each of the $k+1$ largest ranks. P satisfies condition T' if it satisfies condition T'_k for all k. Thus the condition T' means that for all k there are disjoint chains which cover the $(k+1)$st largest rank and intersect every larger rank.

In fact, property T is a special case of condition T'.

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Theorem

Let P be a finite graded rank-symmetric poset of rank n. The following three conditions are equivalent:

- *P is rank-unimodal and has property S;*
- *P has property T;*
- *Let V_i be the complex vector space with basis P_i. Then for $0 \leq i < n$, there exists linear transformations $\varphi_i : V_i \to V_{i+1}$ satisfying the following two properties:*
  1. *If $0 \leq i \leq [\frac{n}{2}]$, then the composite transformation $\varphi_{n-i-1}\varphi_{n-i-2}\cdots\varphi_i : V_i \to V_{n-i}$ is invertible.*
  2. *Let $x \in P_i$ and $\varphi_i(x) = \sum_{y \in P_{i+1}} c_y y$. Then $c_y = 0$ unless $x < y$.*

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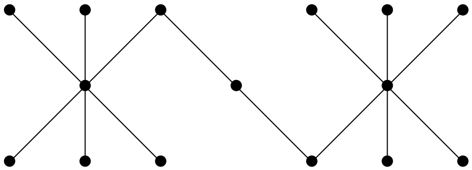
R. P. Stanley, Weyl groups, the hard Lefschetz theorem, and the Sperner property, SIAM J. Algebraic Discrete Methods, 1.2 (1980), pp. 168-184.

J. R. Griggs, On chains and Sperner *k*-families in ranked posets, J. Combinatorial Theory.



The Sperner property

Let X be a complex projective variety of complex dimension n. Suppose that there are finitely many pairwise disjoint subsets C_i of X, each isomorphic as an algebraic variety to complex affine space of some dimension n_i such that

1. the union of C_i's is X,
2. $\overline{C}_i - C_i$ is a union of some of the C_j's ($\overline{C}_i$ denotes the closure of C_i either in Hausdorff or Zariski topology).

We then say C_i's form a cellular decomposition of X.

Given a cellular decomposition $\{C_i\}$ of X, define a partial ordering $Q^X = Q^X(C_1, C_2 \ldots)$ on the C_i's by setting $C_i \geq C_j$ in Q^X if $C_i \subset \overline{C}_j$. If X is irreducible of dimension n, it can be shown that Q^X is graded of rank n, with the rank function given by $\rho(C) = n - dim C$. If X is nonsingular, then *Poincaré* duality implies that Q^X is rank-symmetric.

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Theorem

Let X be a nonsingular irreducible complex projective variety of complex dimension n with a cellular decomposition $\{C_i\}$. Then Q^X is graded of rank n, rank symmetric, rank unimodal and has property S.

A ranked poset is Peck if it is rank symmetric, rank unimodal and strongly Sperner.

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Representation of $\mathfrak{sl}(2, \mathbb{C})$

Recall that the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is

$$\mathfrak{sl}(2, \mathbb{C}) = \{A \in Mat_2(\mathbb{C}) : tr(A) = 0\}.$$

It has basis $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and we have $[e, f] = h$, $[h, e] = 2e$ and $[h, f] = -2f$.

By a representation V of a Lie algebra $\mathfrak{g}$ we mean a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$, the Lie algebra of all linear transformations of some complex vector space V. A subspace of V that is stable under the action of $\mathfrak{g}$ is called a subrepresentation of V. A representation V is irreducible if its only subrepresentations are either 0 or V and is called completely irreducible if it is the direct sum of irreducible representations.

Representation of $\mathfrak{sl}(2, \mathbb{C})$

Definition

Let V be a representation of $\mathfrak{sl}(2, \mathbb{C})$. A vector $v \in V$ is called vector of weight λ, $\lambda \in \mathbb{C}$, if it is an eigenvector for h with eigenvalue λ:

$$hv = \lambda v.$$

We denote by $V[\lambda] \subset V$ the subspace of vectors of weight λ.

Let λ be a weight of V(i.e. $V[\lambda] \neq 0$) which is maximal in the following sense:

$$Re\, \lambda \geq Re\, \lambda' \quad \text{for every weight } \lambda' \text{ of } V.$$

Such a weight will be called a "highest weight of V", and vectors $v \in V[\lambda]$ will be called highest weight vectors.

Representation of $\mathfrak{sl}(2, \mathbb{C})$

Lemma

The actions of e and f on $V[\lambda]$ is given by

$$eV[\lambda] \subset V[\lambda + 2]$$
$$fV[\lambda] \subset V[\lambda - 2].$$

Proof. Let $v \in V[\lambda]$. Then

$$hev = [h, e]v + ehv = 2ev + e \cdot \lambda v = (\lambda + 2)ev,$$

so $ev \in V[\lambda + 2]$. The proof of f is similar. $\square$

Representation of $\mathfrak{sl}(2, \mathbb{C})$

Lemma

Let V be a representation of $\mathfrak{sl}(2, \mathbb{C})$ with highest weight λ and $v_0 \in V[\lambda]$ a highest weight vector. Define

$$v_k = f^k v_0, \; k \geq 0.$$

Then

1. $ev_k = k(\lambda - k + 1)v_{k-1}$ *for* $k > 0$ *and* $ev_0 = 0$;
2. $hv_k = (\lambda - 2k)v_k$.

Representation of $\mathfrak{sl}(2,\mathbb{C})$

Theorem

1. *For any $n \geq 0$, let V_n be the finite-dimensional vector space with basis $v_0, v_1, \ldots, v_n$. Define the action of $\mathfrak{sl}(2,\mathbb{C})$ by*

$$hv_k = (n-2k)v_k, \quad 0 \leq k \leq n;$$
$$fv_k = v_{k+1}, \quad 0 \leq k \leq n,\ fv_n = 0;$$
$$ev_k = k(n+1-k)v_{k-1}, \quad 0 \leq k \leq n,\ ev_0 = 0.$$

   Then V_n is an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$; we will call the irreducible representation with highest weight n.

2. *For $n \neq m$, representation V_n, V_m are non-isomorphic.*
3. *Every finite-dimensional irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ is isomorphic to one of representations V_n.*

Representation of $\mathfrak{sl}(2,\mathbb{C})$

Theorem

Any finite dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$ is completely reducible.

Theorem

Every finite-dimensional representation V of $\mathfrak{sl}(2,\mathbb{C})$ can be written in the form

$$V = \bigoplus_{n\in\mathbb{Z}} V[n],$$

This decomposition is called the weight decomposition of V.

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An action of $\mathfrak{sl}(2, \mathbb{C})$ on posets

Associate to any ranked poset

$$P = \bigcup_{i=0}^{n} P_i$$

a graded complex vector space

$$\tilde{P} = \bigoplus_{i=0}^{n} \tilde{P}_i,$$

where $\tilde{P}_i$ is the complex vector space freely generated by vectors $\tilde{a}$ corresponding to elements of P_i.

An action of $\mathfrak{sl}(2,\mathbb{C})$ on posets

A linear operator X on $\tilde{P}_i$ is a lowering operator if $X\tilde{P}_i \subset \tilde{P}_{i-1}$. It is a raising operator if $X\tilde{P}_i \subset \tilde{P}_{i+1}$. A raising operator defined by

$$X\tilde{a} = \sum \theta(a,b)\tilde{b}$$

is an order raising operator if $\theta(a,b) \neq 0$ implies b covers a. For any poset P of length n, define a linear operator H on $\tilde{P}$ by

$$H\tilde{a} = (2i - n)\tilde{a}$$

when $a \in P_i$.

A representation of $\mathfrak{sl}(2,\mathbb{C})$ on a complex vector space V can be thought of as a choice of three linear operators X, Y and H such that $XY - YX = H$, $HX - XH = 2X$ and $HY - YH = -2Y$.

An action of $\mathfrak{sl}(2,\mathbb{C})$ on posets

An eigenvector for H with eigenvalue λ is referred to as a "weight vector" of the representation of "weight" λ.
Any $(d+1)$-dimensional irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ has as a basis a "string" of vectors $v_0, v_1, \ldots, v_d$ with

$$\begin{aligned} Hv_j &= (2j-d)v_j; \\ Xv_j &= v_{j+1}; \\ Yv_j &= j(d-j+1)v_{j-1}. \end{aligned}$$

Definition

Let P be a ranked poset of length n. The poset P carries a representation of $\mathfrak{sl}(2,\mathbb{C})$ if there exists a lowering operator Y and an order raising operator X on $\tilde{\mathrm{P}}$ such that $XY - YX = H$.

An action of $\mathfrak{sl}(2, \mathbb{C})$ on posets

If P carries a representation of $\mathfrak{sl}(2, \mathbb{C})$, then the rank subspace $\tilde{P}_i$ is the weight space of weight $(2i - n)$ for the representation.

Lemma

A ranked poset P of length n is Peck if and only if there exists an order raising operator X on $\tilde{P}$ such that

$$X^{n-2i}|_{\tilde{P}_i} : \tilde{P}_i \to \tilde{P}_{n-i}$$

is an isomorphism for every $0 \leq i < \frac{n}{2}$.

An action of $\mathfrak{sl}(2, \mathbb{C})$ on posets

Theorem

A ranked poset is Peck if and only if it carries a representation of $\mathfrak{sl}(2, \mathbb{C})$.

R. A. Proctor, Representations of $\mathfrak{sl}(2, \mathbb{C})$ on posets and the Sperner property, SIAM J. Algebraic Discrete Methods, 3.2 (1980), pp. 275-280.

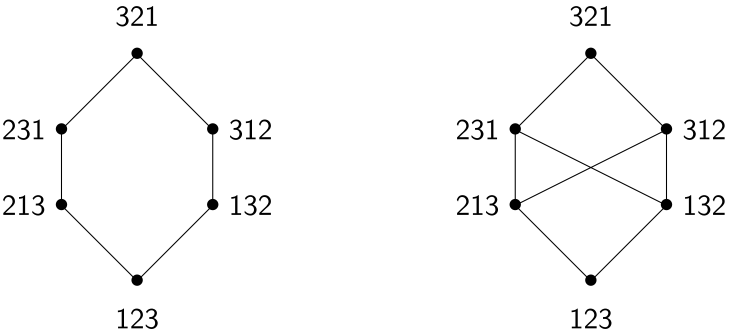
Proof. ($\Leftarrow$) Let P be a ranked poset carrying a representation of $\mathfrak{sl}(2, \mathbb{C})$ with order raising operator X. Since the completely reducibility, the representation can be expressed as a direct sum of irreducible representations, $V \cong \bigoplus_i V_i$.

If one of the irreducible representations has dimension $d+1$, then exactly one of its $d+1$ basis vectors falls in each of the middle $d+1$ consecutive rank subspaces $\tilde{P}_{\frac{n-d}{2}}, \tilde{P}_{\frac{n-d}{2}+1}, \ldots, \tilde{P}_{\frac{n+d}{2}}$.

Then the given string of the set has a member falling in $\tilde{P}_{n-i}$ if and only if it also has a member falling in $\tilde{P}_i$. Since X^{d-2j} is an isomorphism from jth to $(d-j)$th weight space in any irreducible $(d+1)$ dimensional representation where $0 \leq j < \frac{d}{2}$. By the lemma above, P is Peck. $\square$

Let S_n denote the symmetric group of permutations of n elements, viewed as a Coxeter group with respect to the simple transpositions $s_i = (i\; i+1)$ for $i = 1, \ldots, n-1$. The weak order $W_n = (S_n, \leq)$ is the poset structure on S_n whose cover relations are defined as follows: $u \lessdot w$ if and only if $w = us_i$ for some i and $\ell(w) = \ell(u) + 1$, where ℓ denotes Coxeter length.

This definition is in contrast to the strong order on S_n which has cover relations corresponding to the right multiplication by any $t_{ij} = (i\, j)$, rather than just the simple transpositions s_i.



Theorem

For all $n \geq 1$ the weak order W_n is strongly Sperner, and therefore Peck.

Christian Gaetz and Yibo Gao, A combinatorial $\mathfrak{sl}(2,\mathbb{C})$-action and the Sperner property for the weak order. 2018. arXiv: 1811.05501 [math.CO].

Proof. Define the operators $U, D, H : CW_n \to CW_n$,

$$U \cdot w = \sum_{i:\ell(ws_i)=\ell(w+1)} i \cdot ws_i,$$

$$D \cdot w = \sum_{\substack{1\leq i<j\leq n \\ \ell(wt_{ij})=\ell(w)-1}} (2(w_i - w_j - a(w, wt_{ij})) - 1) \cdot wt_{ij},$$

An action of $\mathfrak{sl}(2, \mathbb{C})$ on posets

$$H(w) = \left(2\ell - \binom{n}{2}\right) \cdot w.$$

where $a(w, wt_{ij}) := \#\{k < i : w_j < w_k < w_i\}$.

$$UD - DU = H.$$

$\square$

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Further directions

The weak and strong Bruhat orders generalize naturally to any finite Coxeter group C with the role of the simple transpositions $(i\, i+1)$ replaced by any choice of simple reflections, and the set of all transpositions $(i\, j)$ replaced by the set of all reflections in C.
Problem 1. Is the weak order on any finite Coxeter group strongly Sperner?

A ranked poset P has a symmetric chain decomposition if P can be decomposed into a disjoint union of saturated chains, each of which occupies a set of ranks which is symmetric about the middle rank of P. For example, a symmetric chain decomposition of the posets W_3 and S_3 is given by $\{123, 213, 231, 321\}$ and $\{132, 312\}$.
Problem 2. Which Coxeter group weak orders admit a symmetric chain decomposition? Do all Coxeter group strong orders admit one?