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Determinants and characteristic polynomials of Lie algebras

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For an s-tuple $\mathbb{A} = (A_1, \ldots, A_s)$ of square matrices of the same size, the (joint) determinant of $\mathbb{A}$ and the characteristic polynomial of $\mathbb{A}$ are defined by

$$\det(\mathbb{A})(z) = \det(z_1 A_1 + z_2 A_2 + \cdots + z_s A_s)$$

and

$$p_{\mathbb{A}}(z) = \det(z_0 I + z_1 A_1 + z_2 A_2 + \cdots + z_s A_s),$$

respectively. This paper calculates determinant of the finite dimensional irreducible representations of $\mathfrak{sl}(2, \mathbb{F})$, which is either zero or a product of some irreducible quadratic polynomials. Moreover, it shows that a finite dimensional Lie algebra is solvable if and only if the characteristic polynomial is completely reducible.

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1. Introduction

In this paper $\mathbb{F}$ denotes an algebraically closed field of characteristic 0 and all the questions are discussed over $\mathbb{F}$. Let $A_1, \dots, A_s$ be $n \times n$ matrices and I be the identity matrix of the same size. Their joint determinant $\det(z_1 A_1 + z_2 A_2 + \cdots + z_s A_s)$ and characteristic polynomial $\det(z_0 I + z_1 A_1 + z_2 A_2 + \cdots + z_s A_s)$ are natural generalizations of the classical characteristic polynomial $\det(zI - A)$ for a single square matrix A. It turns out that they are interesting and important mathematical quantities that can be used in many branches of mathematics, such as matrix theory, operator theory, group theory and Lie algebra theory. To the authors' knowledge, after Dedekind and Frobenius' work on finite group determinant (see [9,6,7,3,4,8]), little attention is paid to these generalizations. Some related studies can be seen in [1,11]. In 2009, the notion of projective spectrum was defined by R. Yang in [18] through the multiparameter pencil $z_1 A_1 + z_2 A_2 + \cdots + z_s A_s$, and many results have been obtained since then (cf. [2,10,12,17]). In particular, in [13] the first author and R. Yang investigated the characteristic polynomial associated with finite dimensional representations of finitely generated group G. It was shown that a unitary representation of G contains a one-dimensional representation if and only if the associated characteristic polynomial of its generators contains a linear factor.

This paper continues with the study with a focus on finite dimensional representation of finite Lie algebras. The main results are as follows (cf. Theorem 4.1).

Main Theorem 1. *The joint determinant of a finite dimensional irreducible representation of $\mathfrak{sl}(2, \mathbb{F})$ is either zero or a product of some irreducible quadratic polynomials.*

This result is very different from Dedekind and Frobenius' Theorem, which says that the determinant of a finite dimensional representation of a finite group is irreducible if and only if the representation is irreducible. A Lie algebra L is said to be solvable if it has a finite derivative sequence

$$L = L^{(0)} \supset L^{(1)} \supset \cdots \supset L^{(k)} \supset L^{(k+1)} = \{0\},$$

where $L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$ for $i = 1, \dots, k+1$. Using the characteristic polynomial of representations of a Lie algebra, we give the following characterization of a solvable Lie algebra.

Main Theorem 2. *Let L be a finite dimensional Lie algebra over $\mathbb{F}$. Then L is solvable if and only if the characteristic polynomial of L is a product of linear polynomials.*

This paper is organized as follows. In Section 2 we show some basic facts about the determinant and characteristic polynomial of representations of a Lie algebra. In Section 3, we calculate the determinant associated with some classical tridiagonal matrices. Section 4 and Section 5 are devoted to the proof of the two main theorems.

2. The characteristic polynomial of Lie algebras

In this section, we define the characteristic polynomials of representations of a Lie algebra and give some elementary properties. We first recall the notation of minor of a matrix. Let $A = (a_{ij})$ be an $n \times n$ matrix. If the rows and columns chosen are given by subscripts

$$1 \le i_1 < \cdots < i_p \le n, \quad 1 \le j_1 < \cdots < j_p \le n,$$

respectively, then the corresponding $p \times p$ minor of A is denoted by

$$A\begin{pmatrix} I' \\ J' \end{pmatrix} = A\begin{pmatrix} i_1 \cdots i_p \\ j_1 \cdots j_p \end{pmatrix} := \det(a_{i_k j_l})_{k,l=1}^{p},$$

where multi-ordered-index notation $I' = (i_1 \cdots i_p)$ and $J' = (j_1 \cdots j_p)$. And $|I'| := p$. In the case of two matrices A_1 and A_2, the coefficients in $\det(z_1A_1 + z_2A_2)$ are computed by a formula in [13]. Similar formula holds for several matrices.

Proposition 2.1. *Let $\mathbb{A} = (A_1, \ldots, A_s)$ be an s-tuple of $n \times n$ matrices. Then*

$$\det(\mathbb{A})(z) = \sum_{n_1+\cdots+n_s=n} a_{n_1,\ldots,n_s} z_1^{n_1} \cdots z_s^{n_s},$$

where

$$a_{n_1,\ldots,n_s} = \sum_{I,J\in S_n} (-1)^{\tau(I)+\tau(J)} A_1\begin{pmatrix} I_1 \\ J_1 \end{pmatrix} \cdots A_s\begin{pmatrix} I_s \\ J_s \end{pmatrix}.$$

Here $|I_i| = |J_i| = n_i$, $I = (I_1, \ldots, I_s), J = (J_1, \ldots, J_s)$, and $\tau(I)$ is the number of inversions in the permutation I.

Proof. The power $z_1^{n_1} \cdots z_s^{n_s}$ occurs only in the product of some special $n_i \times n_i$ minor of z_iA_i, say,

$$A_1\begin{pmatrix} I_1 \\ J_1 \end{pmatrix} \cdots A_s\begin{pmatrix} I_s \\ J_s \end{pmatrix} z_1^{n_1} \cdots z_s^{n_s}, \tag{2.1}$$

where $|I_i| = |J_i| = n_i$ and $I = (I_1, \ldots, I_s), J = (J_1, \ldots, J_s) \in S_n$. On the other hand, any term of the product which occurs in the expanding of the determinant $\det(\mathbb{A})(z)$ must be multiplied by $(-1)^{\tau(I)+\tau(J)}$. In fact, if $I = J = (1, 2, \ldots, n)$, the result is obvious. For the general case I and J, we can switch rows in the original matrix $\mathbb{A}(z) := z_1A_1 + z_2A_2 + \cdots + z_sA_s$ such that the rows of the new matrix $\mathbb{B}(z)$ are the rows I of the former matrix $\mathbb{A}(z)$, then switch columns in the matrix $\mathbb{B}(z)$ such that the columns of the new matrix

$\mathbb{C}(z)$ are the columns J of $\mathbb{B}(z)$. Therefore $\det(\mathbb{C})(z) = (-1)^{\tau(I)+\tau(J)} \det(\mathbb{A})(z)$. The fact already proved shows that the product (2.1) occurs in the determinant $\det(\mathbb{C})(z)$. The result also follows. $\square$

Let L be a finite dimensional Lie algebra with basis $e_1, \ldots, e_s$ and $\phi : L \to \mathfrak{gl}(n, \mathbb{F})$ be a finite dimensional representation. We shall call

$$\det(L_\phi(z)) := \det(z_1\phi(e_1) + \cdots + z_s\phi(e_s))$$

the determinant of L with respect to ϕ and the basis $e_1, \ldots, e_s$, while

$$p_{L_\phi}(z) := \det(z_0 I + z_1\phi(e_1) + \cdots + z_s\phi(e_s))$$

is called the characteristic polynomial of L with respect to ϕ and the basis $e_1, \ldots, e_s$. It is clear that if $z_0 = 0$, then $p_{L_\phi}(z) = \det(L_\phi(z))$, and that the characteristic polynomial $p_{L_\phi}(z)$ is a homogeneous polynomial of degree n.

Example 2.2. Let $L = \mathfrak{gl}(2, \mathbb{F})$ with the standard basis consisting of the matrices e_{ij} which has 1 in the (i, j) position and 0 elsewhere. Considering the natural representation of $\mathfrak{gl}(2, \mathbb{F})$, we have

$$\begin{aligned} p_L(z) &= \det(z_0 I + z_1 e_{11} + z_2 e_{12} + z_3 e_{21} + z_4 e_{22}) \\ &= z_0^2 + z_0 z_1 + z_0 z_4 + z_1 z_4 - z_2 z_3. \end{aligned}$$

We say that the characteristic polynomial $p_L(z)$ of L is completely reducible if $p_L(z)$ can be factored into a product of linear polynomials.

Proposition 2.3. *Let L be a Lie algebra, and let $\phi : L \to \mathfrak{gl}(n, \mathbb{F})$ be a finite dimensional representation. If $e_1, \ldots, e_s$ and $e'_1, \ldots, e'_s$ are two bases of L, then the characteristic polynomials*

$$p_\phi(z) := \det(z_0 I + z_1\phi(e_1) + \cdots + z_s\phi(e_s))$$

and

$$\widetilde{p}_\phi(\widetilde{z}) := \det(\widetilde{z}_0 I + \widetilde{z}_1\phi(e'_1) + \cdots + \widetilde{z}_s\phi(e'_s))$$

have the same reducibility. In particular, if $p_\phi(z)$ is completely reducible, so is $\widetilde{p}_\phi(\widetilde{z})$.

Proof. Write $e'_i = \sum_j a_{ji} e_j$, then the matrix (a_{ij}) is invertible. We have

$$\widetilde{p}_\phi(\widetilde{z}) = \det(\widetilde{z}_0 I + \sum_i \widetilde{z}_i \phi(\sum_j a_{ji} e_j))$$

$$
\begin{aligned}
&= \det(\widetilde{z_0} I + \sum_i \sum_j a_{ji} \widetilde{z_i} \phi(e_j)) \\
&= \det(\widetilde{z_0} I + \sum_j (\sum_i a_{ji} \widetilde{z_i}) \phi(e_j)) \\
&= p_\phi(\widetilde{z_0}, \sum_i a_{1i} \widetilde{z_i}, \dots, \sum_i a_{si} \widetilde{z_i}),
\end{aligned}
$$

which proves the proposition. □

3. A class of tridiagonal matrices

A matrix $(a_{ij}) \in \mathbb{F}^{n\times n}$ is called tridiagonal if $a_{ij} = 0$ for $|i-j| \geq 2$. This type of matrices appear frequently in many areas of science and engineering. Determinant, inversion and eigenvalue problem about tridiagonal matrices have been investigated by many authors, for example in [15,5,16]. This section aims to compute the determinant for some classical tridiagonal matrices.

Let

$$
J_{n+1} = \begin{pmatrix} a_1 & b_1 & & & \\ c_1 & a_2 & b_2 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_n \\ & & & c_n & a_{n+1} \end{pmatrix}
$$

be a tridiagonal matrix. The following basic facts are well-known.

Proposition 3.1. *Let J_{n+1} be defined as above. Then*

(1) J_{n+1} satisfies the recursion

$$
\det(J_{n+1}) = a_{n+1} \det(J_n) - b_n c_n \det(J_{n-1}), \quad (n \geq 2),
$$

which shows that for any i the determinant $\det(J_{n+1})$ is only related to the product $b_i c_i$ and independent of a single b_i or c_i.

(2) If $a_1 = \cdots = a_{n+1} = 0$, then the possible nonzero principal minors of J_{n+1} are those whose rows are given by subscripts $i_1 < i_1+1 < i_2 < i_2+1 < \cdots < i_k < i_k+1$, and

$$
J_{n+1} \begin{pmatrix} i_1(i_1+1)\cdots i_k(i_k+1) \\ i_1(i_1+1)\cdots i_k(i_k+1) \end{pmatrix} = (-1)^k \prod_{j=1}^{k} (b_{i_j} c_{i_j}).
$$

Proof. It is easy to get (1). For simplicity, write $J := J_{n+1}$. To show (2), we declare that if the principal minor $J\begin{pmatrix} i_1 \cdots i_s \\ i_1 \cdots i_s \end{pmatrix} \neq 0$, then $s = 2k, i_{2j} = i_{2j-1}+1, j = 1, \dots, k$. In fact, the elements of the first row in the minor are all zero if $i_2 > i_1 + 1$. So $i_2 = i_1 + 1$. By Laplace's Theorem, we have

$$J\begin{pmatrix} i_1 \cdots i_s \\ i_1 \cdots i_s \end{pmatrix} = J\begin{pmatrix} i_1 i_{1+1} \\ i_1 i_{1+1} \end{pmatrix} J\begin{pmatrix} i_3 \cdots i_s \\ i_3 \cdots i_s \end{pmatrix} = -b_{i_1} c_{i_1} J\begin{pmatrix} i_3 \cdots i_s \\ i_3 \cdots i_s \end{pmatrix}.$$

By similar argument, we confirm the declaration and get (2) easily. $\square$

Next let m be a fixed natural number, and suppose z_0 and z_1 are indeterminates. Let

$$A = \begin{pmatrix} 0 & m & & & \\ & 0 & (m-1) & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 2 & \ddots & & \\ & & \ddots & \ddots & \\ & & & m & 0 \end{pmatrix},$$

$$C := A + B,$$

and

$$H = \operatorname{diag}(m, m-2, m-4, \ldots, -m+2, -m).$$

The matrices A, B and H are related to representations of $\mathfrak{sl}(2, \mathbb{F})$ which we will study in the next section. Here we will consider the determinant of the tridiagonal matrices

$$J_m := z_0 I + z_1 H + C.$$

It is obvious that $f_m(z_0, z_1) := \det(J_m)$ is a polynomial of degree $m+1$ in z_0 and z_1.

Lemma 3.2. *The following hold.*

(1) $f_m(z_0, -z_1) = f_m(z_0, z_1)$.

(2) $f_m(-z_0, z_1) = (-1)^{m+1} f_m(z_0, z_1)$.

Proof. Let

$$S = e_{1,m+1} + e_{2,m} + \cdots + e_{m+1,1}.$$

We have $S^2 = I_{m+1}$ and $Se_{ij}S = e_{m+2-i,m+2-j}$. Due to

$$SCS = C, \quad SHS = -H,$$

we get

$$f_m(z_0, z_1) = \det(SJ_mS) = f_m(z_0, -z_1).$$

So

$$
\begin{aligned}
& f_m(-z_0, z_1) \\
& \quad = f_m(-z_0, -z_1) \\
& \quad = (-1)^{m+1} \det \begin{pmatrix} z_0 + mz_1 & -m & & & \\ -1 & z_0 + (m-2)z_1 & -(m-1) & & \\ & -2 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -m & z_0 - mz_1 \end{pmatrix} \\
& \quad = (-1)^{m+1} f_m(z_0, z_1).
\end{aligned}
$$

Here we have used Proposition 3.1 (1) in the last equation. □

We conjecture that the following factorization holds:

$$
f_m(z_0, z_1) = \prod_{i=0}^{m} \left(z_0 - (m-2i)\sqrt{z_1^2 + 1} \right).
$$

Observe that when m is odd, f_m is a product of $(m+1)/2$ irreducible quadratic polynomials, and when m is even it is a product of z_0 with $m/2$ irreducible quadratic polynomials. However, we can only prove in the two special cases $z_0 = 0$ and $z_1 = 0$.

Proposition 3.3.

$$
f_m(z_0, 0) = \begin{cases} \prod_{i=0}^{\frac{m-1}{2}} (z_0^2 - (m-2i)^2), & m \textit{ is odd}, \\ z_0 \prod_{i=0}^{\frac{m}{2}-1} (z_0^2 - (m-2i)^2), & m \textit{ is even.} \end{cases}
$$

Proof. For simplicity, we write $f_m(z_0, 0)$ by $f_m(z_0)$. Adding the $(i+1)$-th,..., $(m+1)$-th column to the column $i(i = 1, \ldots, m)$, subtracting the $(i-1)$-th row from row $i(i = m+1, \ldots, 2)$, and expanding from the first column, we obtain

$$
f_m(z_0) = (z_0 + m) f_{m-1}(z_0 - 1).
$$

By induction, we get

$$
f_m(z_0) = \prod_{i=0}^{m} (z_0 - (m-2i)),
$$

and the result follows. □

Corollary 3.4. *For any two natural numbers k and m with $k \leq [\frac{m+1}{2}]$ we have that*

(1) if m is odd, then

$$\sum_{1 \leq i_1 < \cdots < i_k \leq \frac{m+1}{2}} (2i_1 - 1)^2 \cdots (2i_k - 1)^2 = \sum_{\substack{1 \leq j_1 < \cdots < j_k \leq m \\ j_{l+1} - j_l \geq 2}} \prod_{s=1}^{k} j_s(m + 1 - j_s);$$

(2) if m is even, then

$$\sum_{1 \leq i_1 < \cdots < i_k \leq \frac{m}{2}} 2^{2k} i_1^2 \cdots i_k^2 = \sum_{\substack{1 \leq j_1 < \cdots < j_k \leq m \\ j_{l+1} - j_l \geq 2}} \prod_{s=1}^{k} j_s(m + 1 - j_s).$$

Proof. We prove the case m is even only; the case m is odd is similar. In this case, we have

$$\det(z_0 I + C) = z_0 \prod_{i=0}^{\frac{m-2}{2}} (z_0^2 - (m - 2i)^2).$$

Comparing the coefficients of z_0^{m+1-2k}, we obtain

$$\sum_{i_1 < \cdots < i_{2k}} C \begin{pmatrix} i_1 \cdots i_{2k} \\ i_1 \cdots i_{2k} \end{pmatrix} = (-1)^k \sum_{1 \leq i_1 < \cdots < i_k \leq \frac{m}{2}} 2^{2k} i_1^2 \cdots i_k^2.$$

By Proposition 3.1 (2), we get

$$\sum_{i_1 < \cdots < i_{2k}} C \begin{pmatrix} i_1 \cdots i_{2k} \\ i_1 \cdots i_{2k} \end{pmatrix} = (-1)^k \sum_{\substack{1 \leq j_1 < \cdots < j_k \leq m \\ i_{l+1} - i_l \geq 2}} \prod_{s=1}^{k} j_s(m + 1 - j_s),$$

and this completes the proof. □

Proposition 3.5.

$$f_m(0, z_1) = \begin{cases} (-1)^{\frac{m+1}{2}} (m!!)^2 (z_1^2 + 1)^{\frac{m+1}{2}}, & m \text{ is odd}, \\ 0, & m \text{ is even}. \end{cases}$$

Proof. Let t_k be the leading principal minor of order k of $z_1 H + C$. We now verify by induction that

$$t_{k+1} = \sum_{i=0}^{[\frac{k+1}{2}]} (-1)^i (2i - 1)!! C_{k+1}^{2i} (\prod_{j=0}^{k-i} (m - 2j)) z_1^{k+1-2i}, \quad k = 0, 1, \ldots, m.$$

Here we set $(-1)!! = 1$ and $(-3)!! = 0$. It is easy to check that the formula holds when $k = 0, 1$. Suppose it is holds for nonnegative integers up to k. Set $C_i^j = 0$ if $j > i$ or $j < 0$. Then

$$\begin{aligned}t_{k-1} &= \sum_{i=0}^{[\frac{k-1}{2}]} (-1)^i (2i-1)!! C_{k-1}^{2i} (\prod_{j=0}^{k-2-i} (m-2j)) z_1^{k-1-2i} \\ &= \sum_{i=1}^{[\frac{k+1}{2}]} (-1)^{i-1} (2i-3)!! C_{k-1}^{2i-2} (\prod_{j=0}^{k-1-i} (m-2j)) z_1^{k+1-2i} \\ &= \sum_{i=0}^{[\frac{k+1}{2}]} (-1)^{i-1} (2i-3)!! C_{k-1}^{2i-2} (\prod_{j=0}^{k-1-i} (m-2j)) z_1^{k+1-2i},\end{aligned}$$

and

$$\begin{aligned}t_k &= \sum_{i=0}^{[\frac{k}{2}]} (-1)^i (2i-1)!! C_k^{2i} (\prod_{j=0}^{k-1-i} (m-2j)) z_1^{k-2i} \\ &= \sum_{i=0}^{[\frac{k+1}{2}]} (-1)^i (2i-1)!! C_k^{2i} (\prod_{j=0}^{k-1-i} (m-2j)) z_1^{k-2i}.\end{aligned}$$

Using the recursion $t_{k+1} = (m-2k) z_1 t_k - k(m+1-k) t_{k-1}$, we obtain

$$\begin{aligned}t_{k+1} = \sum_{i=0}^{[\frac{k+1}{2}]} (-1)^i (\prod_{j=0}^{k-1-i} (m-2j)) z_1^{k+1-2i} \\ ((m-2k)(2i-1)!! C_k^{2i} + k(m+1-k)(2i-3)!! C_{k-1}^{2i-2}).\end{aligned}$$

Direct computation shows

$$\begin{aligned}(m-2k)(2i-1)!! C_k^{2i} + k(m+1-k)(2i-3)!! C_{k-1}^{2i-2} \\ = (2i-1)!! C_{k+1}^{2i} (m-2k+2i).\end{aligned}$$

Thus

$$t_{k+1} = \sum_{i=0}^{[\frac{k+1}{2}]} (-1)^i (2i-1)!! C_{k+1}^{2i} (\prod_{j=0}^{k-i} (m-2j)) z_1^{k+1-2i}.$$

In particular, if $k = m$, we have

$$t_{m+1} = \sum_{i=0}^{[\frac{m+1}{2}]} (-1)^i (2i-1)!!C_{m+1}^{2i} (\prod_{j=0}^{m-i} (m-2j)) z_1^{m+1-2i}.$$

Notice that for any $0 \le i \le [\frac{m+1}{2}]$,

$$\prod_{j=0}^{m-i} (m-2j) = \begin{cases} (-1)^{\frac{m-2i+1}{2}} m!!(m-2i)!!, & m \text{ is odd}, \\ 0, & m \text{ is even}. \end{cases}$$

Next we consider the case m is odd only. By the fact $(2j)! = 2^j j!(2j-1)!!$, we can get

$$(2i-1)!!C_{m+1}^{2i}(m-2i)!! = m!!C_{\frac{m+1}{2}}^i.$$

Therefore

$$\begin{aligned} t_{m+1} &= \sum_{i=0}^{[\frac{m+1}{2}]} (-1)^{\frac{m+1}{2}} (m!!)^2 C_{\frac{m+1}{2}}^i z_1^{m+1-2i} \\ &= (-1)^{\frac{m+1}{2}} (m!!)^2 (z_1^2+1)^{\frac{m+1}{2}}, \end{aligned}$$

which is what we want. □

4. The characteristic polynomial of $\mathfrak{sl}(2, \mathbb{F})$

Let L denote the Lie algebra $\mathfrak{sl}(2, \mathbb{F})$ with standard basis consisting of

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then $[x, y] = h, [h, x] = 2x, [h, y] = -2y$. For any natural number m, we let $\phi : L \to \mathfrak{gl}(V)$ be an irreducible representation of dimension $m+1$. It is well-known that up to representation equivalence there exists only one such representation. We can take a basis $v_0, v_1, \dots, v_m$ of V such that (see [14])

$$\phi(h)v_i = (m-2i)v_i, \quad \phi(y)v_i = (i+1)v_{i+1}, \quad \phi(x)v_i = (m+1-i)v_{i-1},\ i \ge 0.$$

Here we assume $v_{-1} = 0$. Relative to this basis, we have

$$\phi(h) = H, \quad \phi(x) = A, \quad \phi(y) = B,$$

and the characteristic polynomial of L is

$$p_{L_\phi}(z) = \det(z_0 I + z_1\phi(h) + z_2\phi(x) + z_3\phi(y)).$$

Notice that $z_0 I + z_1\phi(h) + z_2\phi(x) + z_3\phi(y)$ is a tridiagonal matrix. By Proposition 3.1 (1), z_2 and z_3 must appear as the product z_2z_3 in $p_{L_\phi}(z)$. So

$$p_{L_\phi}(z_0, z_1, z_2, z_3) = p_{L_\phi}(z_0, z_1, \sqrt{z_2z_3}, \sqrt{z_2z_3}).$$

It is obvious that $p_{L_\phi}(z)$ is a homogeneous polynomial of degree $m+1$. Thus

$$p_{L_\phi}(z_0, z_1, \sqrt{z_2z_3}, \sqrt{z_2z_3}) = (\sqrt{z_2z_3})^{m+1} p_{L_\phi}(\frac{z_0}{\sqrt{z_2z_3}}, \frac{z_1}{\sqrt{z_2z_3}}, 1, 1).$$

Observe that when $z_2 = z_3 = 1$ we have $p_{L_\phi}(z_0, z_1, 1, 1) = f_m(z_0, z_1)$. Therefore

$$p_{L_\phi}(z_0, z_1, z_2, z_3) = (\sqrt{z_2z_3})^{m+1} f_m(\frac{z_0}{\sqrt{z_2z_3}}, \frac{z_1}{\sqrt{z_2z_3}}).$$

By Proposition 3.3 and Proposition 3.5, we get the following theorem.

Theorem 4.1. *Let $\phi : \mathfrak{sl}(2, \mathbb{F}) \to \mathfrak{gl}(V)$ be an irreducible representation of dimension $m + 1$. Then*

$$\det(z_1\phi(h) + z_2\phi(x) + z_3\phi(y)) = \begin{cases} (-1)^{\frac{m+1}{2}} (m!!)^2 (z_1^2 + z_2z_3)^{\frac{m+1}{2}}, & m \text{ is odd}, \\ 0, & m \text{ is even}, \end{cases} \tag{4.2}$$

and

$$\det(z_0 I + z_2\phi(x) + z_3\phi(y)) = \begin{cases} \prod_{i=0}^{\frac{m-1}{2}} (z_0^2 - (m-2i)^2 z_2z_3), & m \text{ is odd}, \\ z_0 \prod_{i=0}^{\frac{m}{2}-1} (z_0^2 - (m-2i)^2 z_2z_3), & m \text{ is even}. \end{cases} \tag{4.3}$$

Proof. The results follow from

$$\det(z_1\phi(h) + z_2\phi(x) + z_3\phi(y)) = (\sqrt{z_2z_3})^{m+1} f_m(0, \frac{z_1}{\sqrt{z_2z_3}})$$

and

$$\det(z_0 I + z_2\phi(x) + z_3\phi(y)) = (\sqrt{z_2z_3})^{m+1} f_m(\frac{z_0}{\sqrt{z_2z_3}}, 0),$$

respectively. □

Since $\mathfrak{sl}(2, \mathbb{F})$ is semisimple, by Weyl's Theorem any representation of $\mathfrak{sl}(2, \mathbb{F})$ is completely reducible. Recall that a representation V is completely reducible if V is a direct sum of irreducible subrepresentations. One can deduce the following corollaries easily.

Corollary 4.2. *Let $\phi : \mathfrak{sl}(2, \mathbb{F}) \to \mathfrak{gl}(V)$ be a finite dimensional representation. Then the determinant* $\det(z_1\phi(h)+z_2\phi(x)+z_3\phi(y))$ *is zero if and only if V contains an irreducible odd-dimensional subrepresentation.*

Corollary 4.3. *Let $\phi : \mathfrak{sl}(2, \mathbb{F}) \to \mathfrak{gl}(V)$ be a finite dimensional representation. Then* $\det(z_0 I + z_2\phi(x) + z_3\phi(y))$ *is completely reducible if and only if ϕ is a trivial representation.*

5. Solvable Lie algebra

In this section, we look into the characteristic polynomials of solvable Lie algebras. First we consider linear solvable Lie algebras.

Theorem 5.1. *Let L be a subalgebra of $\mathfrak{gl}(V)$ with a basis $e_1, \dots, e_s$. Then L is solvable if and only if the characteristic polynomial* $\det(z_0 I + z_1 e_1 + \cdots + z_s e_s)$ *is completely reducible.*

Proof. By Lie's Theorem there exists a basis of V such that the matrices of L are upper triangular relative to the basis. Thus the necessity is obvious.

Now we prove the sufficiency part. Levi decomposition asserts that L is a semidirect product of a solvable ideal and a semisimple subalgebra, i.e.,

$$L = S + R,$$

where S is a semisimple subalgebra of L and R is the radical-the maximal solvable ideal. We only need to prove $S = 0$. Suppose otherwise, then there exists a 3-dimensional simple subalgebra S_1 of S that is isomorphic to $\mathfrak{sl}(2, \mathbb{F})$ with standard basis h, x, y (see Humphreys [14], Page 37). In other words, the isomorphism $\phi : \mathfrak{sl}(2, \mathbb{F}) \to S_1$ is a faithful representation. By Corollary 4.3, $\det(z_0 I+z_1\phi(x)+z_2\phi(y))$ can not be completely reducible. On the other hand, in view of Proposition 2.3, setting $e_1 = \phi(x), e_2 = \phi(y)$ and taking $z_3 = \cdots = z_s = 0$, we see that the condition of this theorem implies that $\det(z_0 I + z_1\phi(x) + z_2\phi(y))$ is completely reducible, which is a contradiction. □

Corollary 5.2. *If $A, B \in \mathfrak{gl}(V)$ are such that* $\det(z_0 I + z_1 A + z_2 B)$ *is not completely reducible, then A and B cannot be simultaneously contained in any solvable subalgebra of $\mathfrak{gl}(V)$.*

Example 5.3. Let

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\det(z_0I + z_1A_1 + z_2A_2) = z_0(z_0^2 + z_0z_2 - z_1z_2).$$

So A_1 and A_2 can not include in any solvable subalgebra of $\mathfrak{gl}(3, \mathbb{F})$. Notice that the converse of the corollary is not correct. For example, let

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\det(z_0I + z_1A_1 + z_2A_2) = z_0^3.$$

But A_1 and A_2 cannot be simultaneously contained in any solvable subalgebra of $\mathfrak{gl}(3, \mathbb{F})$.

Theorem 5.4. *Let L be a Lie algebra with finite dimension over $\mathbb{F}$. Then L is solvable if and only if the characteristic polynomial of L is completely reducible with respect to any finite dimensional representation and any basis.*

Proof. The sufficiency comes from the above theorem and Ado Theorem which says that every Lie algebra is isomorphic to some linear Lie algebra. The necessity follows from Theorem 5.1 above and the fact that homomorphic image of a solvable Lie algebra is also solvable. □

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References

[1] A. Amitsur, On the characteristic polynomial of a sum of matrices, Linear Multilinear Algebra 8 (3) (1980) 177–182.
[2] I. Chagouel, M. Stessin, K. Zhu, Geometric spectral theory for compact operators, Trans. Amer. Math. Soc. 368 (3) (2016) 1559–1582.
[3] K. Conrad, The origin of representation theory, Enseign. Math. 44 (3–4) (1998) 361–392.
[4] C. Curtis, Representation theory of finite groups: from Frobenius to Brauer, Math. Intelligencer 14 (1992) 48–57.
[5] C.M. Da Fonseca, On the eigenvalues of some tridiagonal matrices, J. Comput. Appl. Math. 200 (2007) 283–286.
[6] R. Dedekind, Gesammelte Mathematische Werke, vol. II, Chelsea, New York, 1969.
[7] L.E. Dickson, An elementary exposition of Frobenius theory of group characters and group-determinants, Ann. Math. 4 (1902) 25–49;
L.E. Dickson, Mathematical Papers, vol. II, Chelsea, New York, 1975, pp. 737–761.

[8] E. Formanek, D. Sibley, The group determinant determines the group, Proc. Amer. Math. Soc. 112 (3) (1991) 649–656.
[9] F.G. Frobenius, Über vertauschbare Matrizen, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin, 1896, pp. 601–614;
F.G. Frobenius, Gesammelte Abhandlungen, Band II, Springer-Verlag, New York, 1968, pp. 705–718.
[10] R. Grigorchuk, R. Yang, Joint spectrum and the infinite dihedral group, Proc. Steklov Inst. Math. 297 (1) (2017) 145–178.
[11] C. Hanselka, Characteristic polynomials of symmetric matrices over the univariate polynomial ring, J. Algebra 487 (2017) 340–356.
[12] W. He, R. Yang, Projective spectrum and kernel bundle, Sci. China Math. 57 (2014) 1–10.
[13] Z. Hu, R. Yang, On the characteristic polynomials of multiparameter pencils, Linear Algebra Appl. 558 (2018) 250–263.
[14] J.E. Humphreys, Introduction to Lie Algebras and Representation Theory, Springer Verlay, New York, 1972.
[15] R.K. Mallik, The inverse of a tridiagonal matrix, Linear Algebra Appl. 325 (2001) 109–139.
[16] S. Noschese, L. Pasquini, L. Reichel, Tridiagonal Toeplitz matrices: properties and novel applications, Numer. Linear Algebra Appl. 20 (2) (2007) 302–326.
[17] M. Stessin, R. Yang, K. Zhu, Analyticity of a joint spectrum and a multivariable analytic Fredholm theorem, New York J. Math. 17A (2011) 39–44.
[18] R. Yang, Projective spectrum in Banach algebras, J. Topol. Anal. 1 (3) (2009) 289–306.