Standard Rothe Tableaux

Neil J.Y. Fan

Department of Mathematics, Sichuan University

对称函数讨论班（南开）

Aug 5, 2020

A Coxeter system is a pair (W, S) consisting of a group W and a set of generators S, subject only to

$$(ss')^{m(s,s')} = 1,$$

where $m(s,s) = 1, m(s,s') = m(s',s) \geq 2$ for $s \neq s'$ in S.

Ex1. The symmetric group S_n, with $S = \{s_1, \ldots, s_{n-1}\}$, where $s_i = (i, i+1)$ and

$$s_i^2 = 1, \text{ for } 1 \leq i \leq n-1;$$
$$(s_i s_j)^2 = 1 \ (or\ s_i s_j = s_j s_i), \text{ for } 1 \leq i, j \leq n-1 \text{ and } |i-j| > 1;$$
$$(s_i s_{i+1})^3 = 1 \ (or\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}), \text{ for } 1 \leq i \leq n-2.$$

An element $w \in W$ can be written as

$$w = s_{i_1} s_{i_2} \cdots s_{i_r},$$

where $s_{i_k} \in S$. If r is minimal, then such an expression is called reduced, and r is called the length of w, denoted $\ell(w)$. The set of reduced expressions of w is denoted by $R(w)$.

Ex2. Let $w = 4213 \in S_4$. Since

$$1234 \xrightarrow{s_3} 1243 \xrightarrow{s_2} 1423 \xrightarrow{s_1} 4123 \xrightarrow{s_2} 4213,$$

we have

$$w = s_3 s_2 s_1 s_2 = s_3 s_1 s_2 s_1 = s_1 s_3 s_2 s_1 \Rightarrow |R(w)| = 3.$$

Question: $\forall w \in S_n,\ |R(w)| = ?$

Theorem (Stanley, Euro. J. Combin., 1984)

Let $w_0 = n \cdots 21$ and $\delta = (n-1, \ldots, 1)$. Then $|R(w_0)| = f^\delta$, where f^δ is the number of SYT of shape δ.

The Stanley symmetric function of a permutation w is defined as

$$F_w = \sum_{a_1 a_2 \cdots a_\ell \in R(w)} \sum_{\substack{1 \le b_1 \le b_2 \le \cdots \le b_\ell \\ a_i < a_{i+1} \Rightarrow b_i < b_{i+1}}} x_{b_1} x_{b_2} \cdots x_{b_\ell}. \tag{1}$$

Since F_w is symmetric,

$$F_w = \sum_\lambda c_\lambda^w s_\lambda \quad \Longrightarrow \quad |R(w)| = \sum_\lambda c_\lambda^w f^\lambda$$

where the coefficients c_λ^w are called the Stanley coefficients or Edelman-Greene coefficients.

Ex3. $F_{214365} = s_{(3)} + 2s_{(2,1)} + s_{(1,1,1)}$ and $F_{4213} = s_{(3,1)}$.

$\mathrm{BYT}(\lambda)$ (resp. $\mathrm{SYT}(\lambda)$): the set of balanced (resp. standard) Young tableaux of shape λ.

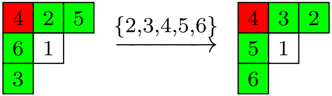
Theorem (Edelman-Greene, Adv. Math. 1987)

Let $\delta = (n-1, \ldots, 1)$ *and* $w_0 = n \cdots 21$. *Then there are bijections*

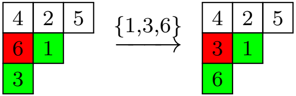
$$\mathrm{BYT}(\delta) \overset{\Phi}{\longleftrightarrow} R(w_0) \overset{\Gamma}{\longleftrightarrow} \mathrm{SYT}(\delta).$$

$$\mathrm{BYT}(\lambda) \overset{\Omega}{\longleftrightarrow} \mathrm{SYT}(\lambda), \quad \forall \lambda \vdash n.$$

balanced hook



not balanced hook



The Rothe diagram $D(w)$ of a permutation w:

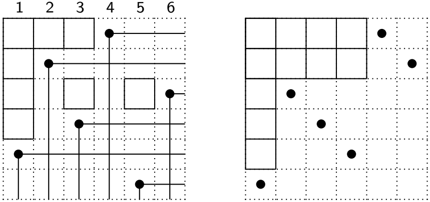


Figure: Rothe diagrams for $w = 426315$ and $w = 562341$

Fact: For a dominant permutation w, i.e., 132-avoiding, the Rothe diagram $\lambda(w)$ of w is a Young diagram. Actually, if w is 132-avoiding, then

$$\mathrm{BYT}(\lambda(w)) \xleftrightarrow{\Omega} \mathrm{SYT}(\lambda(w)) \xleftrightarrow{\Gamma'} R(w).$$

$\mathrm{BRT}(w)$: the set of balanced Rothe tableaux of w.

Theorem (Fomin-Greene-Reiner-Shimozono, Euro.J.C. 1997)

Let $w \in S_n$. Then there is a bijection

$$R(w) \xleftrightarrow{\Phi'} \mathrm{BRT}(w).$$

Let $w = 426315$.

4	8	1	•		
3	•				
6		5		2	•
7		•			
•					
				•	

$T_a \longleftarrow a = s_3 s_5 s_1 s_2 s_4 s_3 s_4 s_1 \in R(w)$

for $i = 1, \ldots, \ell(w)$, let $T_a(p, q) = i$,

where a_i transposes w_p and q $(q < w_p)$.

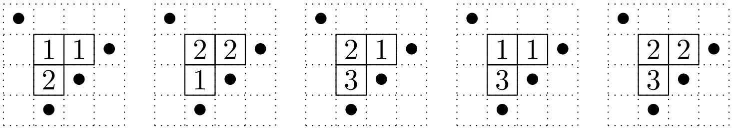
a balanced Rothe tableau

Theorem (Fomin-Greene-Reiner-Shimozono, Euro.J.C. 1997)

Let $w \in S_n$. Then the Schubert polynomial

$$\mathfrak{S}_w(x_1, \ldots, x_n) = \sum_{T \in \mathrm{BFL}(w)} x^T,$$

where $\mathrm{BFL}(w)$ is the set of balanced column-strict labelling of $D(w)$ such that $T(i,j) \leq i$ for $(i,j) \in D(w)$.



Question: set-valued balanced column-strict flag labelling for Grothendieck polynomials?

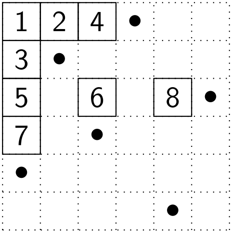
$\mathrm{SRT}(w)$: the set of standard Rothe tableaux of w.

Theorem (F., Discrete Math., 2019)

Let $w \in S_n$. Then

$$|\mathrm{SRT}(w)| \leq |\mathrm{BRT}(w)| = |R(w)|,$$

with equality if and only if w avoids 2413, 2431, 3142 and 4132.



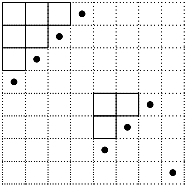
The Case of Equality

The inversion code $c(w) = (c_1, \ldots, c_n)$ of $w = w_1 \cdots w_n$ is defined by where $c_i = |\{j > i : w_j < w_i\}|$.

Given $u = u_1 \cdots u_k \in S_k$ and $v = v_1 \cdots v_{n-k} \in S_{n-k}$, define the direct sum of u and v to be $u \oplus v = u_1 \cdots u_k (v_1 + k) \cdots (v_{n-k} + k)$.

w is called decomposable if it can be expressed as the direct sum of two nontrivial permutations, and indecomposable otherwise.

If w avoids 2413, 2431, 3142 and 4132, then the Rothe diagram of w can be divided into nonintersecting Young diagrams.



Theorem (F., Discrete Math. 2019)

Let $w \in S_n$ be a permutation that avoids the patterns 2413, 2431, 3142 and 4132. Assume that $w = w^1 \oplus \cdots \oplus w^k$, where w^i is indecomposable and $c(w^i) = \lambda^i$ for $1 \le i \le k$. Then each λ^i is a partition and

$$|R(w)| = |\mathrm{BRT}(w)| = |\mathrm{SRT}(w)| = \binom{\ell(w)}{|\lambda^1|, \ldots, |\lambda^k|} \prod_{i=1}^{k} f^{\lambda^i},$$

where f^{λ^i} is the number of standard Young tableaux of shape λ^i.

The General Case

Suppose that w contains one of the patterns 2413,2431,3142,4132. In order to show

$$|\mathrm{SRT}(w)| < |\mathrm{BRT}(w)| = |R(w)|,$$

the key step is to transform w into a 132-avoiding permutation $\widetilde{w}$.

We aim to construct an injection:

$$\Psi : \mathrm{SRT}(w) \xrightarrow{\eta} \mathrm{SRT}(\widetilde{w}) \xleftrightarrow{\Gamma'} R(\widetilde{w}) \longrightarrow R(w).$$

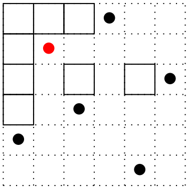
The construction of $\widetilde{w}$:

Keep killing the first ascent of w until we reach a dominant permutation. That is, there exist positive integer sequences $i_1, i_2, \ldots, i_k$, with k minimal, such that i_1 is the first ascent of w, and for $2 \le j \le k$, i_j is the first ascent of $ws_{i_1} \cdots s_{i_{j-1}}$, and

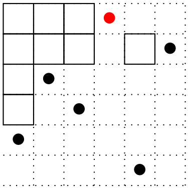
$$\widetilde{w} = ws_{i_1} \cdots s_{i_k}$$

is a dominant permutation.

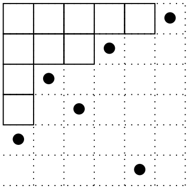
$$w = 426315 \xrightarrow{s_2} 462315 \xrightarrow{s_1} 642315 \Rightarrow \widetilde{w} = ws_2 s_1.$$



$D(w)$



$D(ws_2)$



$D(ws_2 s_1)$

The lifting operation η_i:

Let $w = w_1 \cdots w_n$ be an indecomposable permutation that contains a 132 pattern and i be the first ascent of w. Given $T \in \mathrm{SRT}(w)$, we construct an operation η_i on T, called the lifting operation of T at (i, w_i), as follows.

(1) Apply outward jdt from the empty cell (i, w_i) of T.

(2) Fill $(1, 1)$ with 0, and then add 1 to all the entries of T.

(3) Move the cells of T that are in the $(i+1)$-st row and to the right of the column w_i to the i-th row.

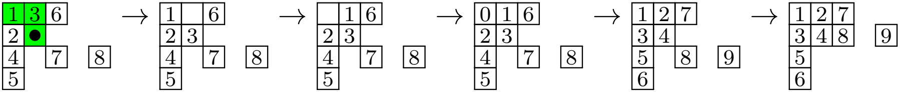
To construct

$$\mathrm{SRT}(w) \xrightarrow{\eta} \mathrm{SRT}(\widetilde{w}),$$

apply a sequence of lifting operations η_i to any $T \in \mathrm{SRT}(w)$, such that $T\eta_{i_1} \cdots \eta_{i_k} \in \mathrm{SRT}(\widetilde{w})$ and $\widetilde{w} = w s_{i_1} \cdots s_{i_k}$ is 132-avoiding.

For example, let $w = 426315$.

The lifting operation η_2 on $T \in \mathrm{SRT}(w)$:



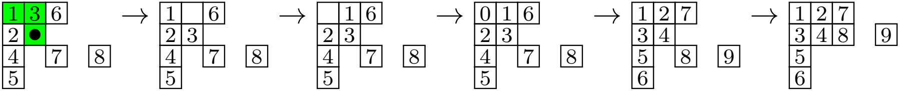
To construct

$$\mathrm{SRT}(w) \xrightarrow{\eta} \mathrm{SRT}(\widetilde{w}),$$

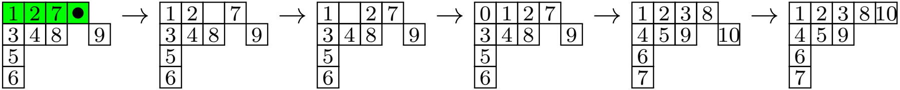
apply a sequence of lifting operations η_i to any $T \in \mathrm{SRT}(w)$, such that $T\eta_{i_1}\cdots\eta_{i_k} \in \mathrm{SRT}(\widetilde{w})$ and $\widetilde{w} = ws_{i_1}\cdots s_{i_k}$ is 132-avoiding.

For example, let $w = 426315$.

The lifting operation η_2 on $T \in \mathrm{SRT}(w)$:



The lifting operation η_1 on $T\eta_2 \in \mathrm{SRT}(ws_2)$:



Given $T \in \mathrm{SYT}(\lambda)$ with $\lambda \vdash n$, the promotion operation on T, denoted by $\partial(T)$, is defined as follows:

$$\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&9\\\hline 6&8&10\\\hline 7\\\cline{1-1}\end{array} \rightarrow \underbrace{\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&9\\\hline 6&8&\\\hline 7\\\cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&\\\hline 6&8&9\\\hline 7\\\cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&&5\\\hline 6&8&9\\\hline 7\\\cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|c|}\hline 1&&4\\\hline 2&3&5\\\hline 6&8&9\\\hline 7\\\cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|c|}\hline &1&4\\\hline 2&3&5\\\hline 6&8&9\\\hline 7\\\cline{1-1}\end{array}}_{\text{jeu de taquin}} \rightarrow \begin{array}{|c|c|c|}\hline 0&1&4\\\hline 2&3&5\\\hline 6&8&9\\\hline 7\\\cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline 7&9&10\\\hline 8\\\cline{1-1}\end{array}$$

Given $T \in \mathrm{SYT}(\lambda)$ with $\lambda \vdash n$, the promotion operation on T, denoted by $\partial(T)$, is defined as follows:

$$\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&9\\\hline 6&8&10\\\hline \multicolumn{1}{|c|}{7}\\\cline{1-1}\end{array} \to \underbrace{\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&9\\\hline 6&8&\\\hline \multicolumn{1}{|c|}{7}\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&\\\hline 6&8&9\\\hline \multicolumn{1}{|c|}{7}\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&&5\\\hline 6&8&9\\\hline \multicolumn{1}{|c|}{7}\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 1&&4\\\hline 2&3&5\\\hline 6&8&9\\\hline \multicolumn{1}{|c|}{7}\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline &1&4\\\hline 2&3&5\\\hline 6&8&9\\\hline \multicolumn{1}{|c|}{7}\\\cline{1-1}\end{array}}_{\text{jeu de taquin}} \to \begin{array}{|c|c|c|}\hline 0&1&4\\\hline 2&3&5\\\hline 6&8&9\\\hline \multicolumn{1}{|c|}{7}\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline 7&9&10\\\hline \multicolumn{1}{|c|}{8}\\\cline{1-1}\end{array}$$

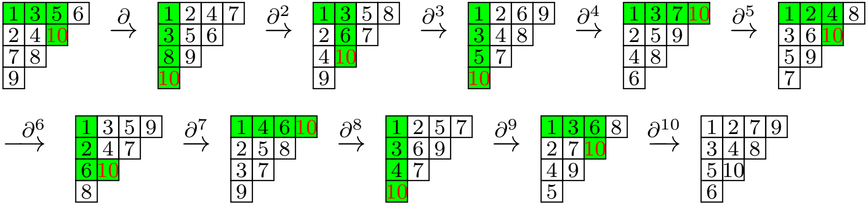
The dual-promotion operation on T, denoted by $\partial^*(T)$, is defined as below:

$$\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline 7&9&10\\\hline \multicolumn{1}{|c|}{8}\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline &2&5\\\hline 3&4&6\\\hline 7&9&10\\\hline \multicolumn{1}{|c|}{8}\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 2&&5\\\hline 3&4&6\\\hline 7&9&10\\\hline \multicolumn{1}{|c|}{8}\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 2&4&5\\\hline 3&&6\\\hline 7&9&10\\\hline \multicolumn{1}{|c|}{8}\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 2&4&5\\\hline 3&6&\\\hline 7&9&10\\\hline \multicolumn{1}{|c|}{8}\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 2&4&5\\\hline 3&6&10\\\hline 7&9&\\\hline \multicolumn{1}{|c|}{8}\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 2&4&5\\\hline 3&6&10\\\hline 7&9&11\\\hline \multicolumn{1}{|c|}{8}\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&9\\\hline 6&8&10\\\hline \multicolumn{1}{|c|}{7}\\\cline{1-1}\end{array}$$

Clearly, $\partial = (\partial^*)^{-1}$.

Let $w_0 = n \cdots 21$ and $\delta = (n-1, \ldots, 1)$. Edelman and Greene constructed the bijection

$$\Gamma : \mathrm{SYT}(\delta) \longrightarrow R(w_0).$$



The coordinates of the first cell in each promotion path are

$$(2,3),(4,1),(3,2),(4,1),(1,4),(2,3),(3,2),(1,4),(4,1),(2,3).$$

Then

$$\Gamma(T) = s_3 s_1 s_2 s_1 s_4 s_3 s_2 s_4 s_1 s_3 \in R(54321).$$

The construction of $\Gamma^*(T)$.

$$\begin{array}{llll}1&3&5&6\\2&4&10\\7&8\\9\end{array}\xrightarrow{\partial^*}\begin{array}{llll}1&2&4&5\\3&7&9\\6&10\\8\end{array}\xrightarrow{(\partial^*)^2}\begin{array}{llll}1&3&4&10\\2&6&8\\5&9\\7\end{array}\xrightarrow{(\partial^*)^3}\begin{array}{llll}1&2&3&9\\4&5&7\\6&8\\10\end{array}\xrightarrow{(\partial^*)^4}\begin{array}{llll}1&2&6&8\\3&4&10\\5&7\\9\end{array}$$

$$\xrightarrow{(\partial^*)^5}\begin{array}{llll}1&3&5&7\\2&6&9\\4&10\\8\end{array}\xrightarrow{(\partial^*)^6}\begin{array}{llll}1&2&4&6\\3&5&8\\7&9\\10\end{array}\xrightarrow{(\partial^*)^7}\begin{array}{llll}1&3&5&10\\2&4&7\\6&8\\9\end{array}\xrightarrow{(\partial^*)^8}\begin{array}{llll}1&2&4&9\\3&6&10\\5&7\\8\end{array}$$

$$\xrightarrow{(\partial^*)^9}\begin{array}{llll}1&3&8&10\\2&5&9\\4&6\\7\end{array}\xrightarrow{(\partial^*)^{10}}\begin{array}{llll}1&2&7&9\\3&4&8\\5&10\\6\end{array}$$

The last cells of the inward jdt paths at each step are

$$(3,2),(1,4),(4,1),(2,3),(3,2),(4,1),(1,4),(2,3),(1,4),(3,2).$$

$$\Gamma^*(T)=s_3s_1s_4s_2s_3s_4s_1s_2s_1s_3.$$

Fact: $\Gamma(T)$ and $\Gamma^*(T)$ are reverses of each other.

To construct the inverse of Γ, Edelman and Greene introduced the Coxeter-Knuth insertion, which is the "right" version of the classical Robinson-Schensted-Knuth correspondence for reduced words.

Insert the sequence 3142341213 by Coxeter-Knuth insertion:

$$\begin{array}{|c|}\hline 3\\\hline\end{array} \xrightarrow{1} \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \xrightarrow{4} \begin{array}{|c|c|}\hline 1&4\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array} \xrightarrow{2} \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array} \xrightarrow{3} \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 3&4&\multicolumn{1}{c}{}\\\cline{1-2}\end{array} \xrightarrow{4} \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 3&4&\multicolumn{1}{c}{}&\multicolumn{1}{c}{}\\\cline{1-2}\end{array} \xrightarrow{1} \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 2&4&\multicolumn{1}{c}{}&\multicolumn{1}{c}{}\\\cline{1-2} 3&\multicolumn{1}{c}{}&\multicolumn{1}{c}{}&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}$$

$$\xrightarrow{2} \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 2&3&\multicolumn{1}{c}{}&\multicolumn{1}{c}{}\\\cline{1-2} 3&4&\multicolumn{1}{c}{}&\multicolumn{1}{c}{}\\\cline{1-2}\end{array} \xrightarrow{1} \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 2&3&\multicolumn{1}{c}{}&\multicolumn{1}{c}{}\\\cline{1-2} 3&4&\multicolumn{1}{c}{}&\multicolumn{1}{c}{}\\\cline{1-2} 4&\multicolumn{1}{c}{}&\multicolumn{1}{c}{}&\multicolumn{1}{c}{}\\\cline{1-1}\end{array} \xrightarrow{3} \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 2&3&4&\multicolumn{1}{c}{}\\\cline{1-3} 3&4&\multicolumn{1}{c}{}&\multicolumn{1}{c}{}\\\cline{1-2} 4&\multicolumn{1}{c}{}&\multicolumn{1}{c}{}&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}$$

Insertion tableau: $\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 2&3&4&\multicolumn{1}{c}{}\\\cline{1-3} 3&4&\multicolumn{1}{c}{}&\multicolumn{1}{c}{}\\\cline{1-2} 4&\multicolumn{1}{c}{}&\multicolumn{1}{c}{}&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}$. Recording tableau: $\begin{array}{|c|c|c|c|}\hline 1&3&5&6\\\hline 2&4&10&\multicolumn{1}{c}{}\\\cline{1-3} 7&8&\multicolumn{1}{c}{}&\multicolumn{1}{c}{}\\\cline{1-2} 9&\multicolumn{1}{c}{}&\multicolumn{1}{c}{}&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}$.

Let w be 132-avoiding, then

$$\Gamma' : \mathrm{SYT}(\lambda(w)) \longleftrightarrow R(w).$$

For example, let $w = 632415$, then $\lambda(w) = (5, 2, 1, 1)$.

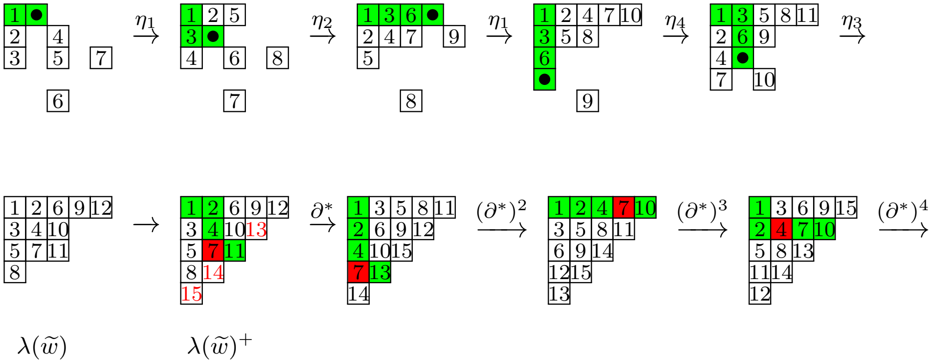
$$\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 5 & 6 & 9 \\ \hline 3 & 4 \\ \cline{1-2} 7 \\ \cline{1-1} 8 \\ \cline{1-1} \end{array} \qquad \longrightarrow \qquad \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 5 & 6 & 9 \\ \hline 3 & 4 & 10 & 11 \\ \cline{1-4} 7 & 12 & 13 \\ \cline{1-3} 8 & 14 \\ \cline{1-2} 15 \\ \cline{1-1} \end{array}$$

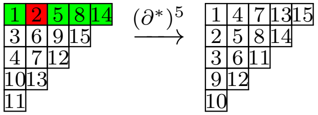
$$\lambda(w) \qquad\qquad\qquad \lambda(w)^+$$

$$\Gamma(\lambda(w)^+) = s_1 s_2 s_3 s_2 s_4 s_3 s_5 s_1 s_2 s_4 s_3 s_2 s_1 s_4 s_2$$

$$\Gamma'(\lambda(w)) = s_5 s_1 s_2 s_4 s_3 s_2 s_1 s_4 s_2.$$

Let $w = 246153$.





$$\widetilde{w} = w{\color{blue}s_1 s_2 s_1 s_4 s_3} = 645213.$$

$$\Gamma(\lambda(\widetilde{w})^+) = {\color{red}s_1 s_2 s_4} s_5 s_3 s_4 s_5 s_1 s_2 s_3 {\color{blue}s_1 s_2 s_1 s_4 s_3}.$$

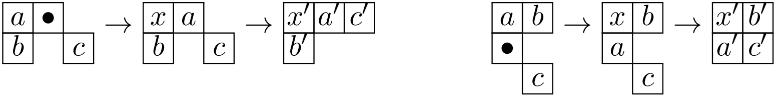
$$\Gamma'(\lambda(\widetilde{w})) = s_5 s_3 s_4 s_5 s_1 s_2 s_3 {\color{blue}s_1 s_2 s_1 s_4 s_3}.$$

$$\Psi(T) = s_5 s_3 s_4 s_5 s_1 s_2 s_3 \in R(w).$$

Theorem (F., Discrete Math. 2019)

Let w be a permutation that contains one of the four patterns 2413, 2431, 3142 and 4132. Then

$$|\mathrm{SRT}(w)| < |R(w)|.$$



(i) (ii)

In either case, $a' \leq c' - 2$. It is easy to construct a tableau $T \in \mathrm{SRT}(\widetilde{w})$ with $a' = c' - 1$.

The Type B Case

Let B_n be the set of signed permutations on $\{1,\ldots,n\}$. (B_n,S) is a Coxeter system, where $S=\{s_0,s_1,\ldots,s_n\}$ and

$$\begin{aligned} s_i^2 &= 1, \text{ for } \ 0\le i\le n;\\ (s_is_j)^2 &= 1 \ (or\ s_is_j=s_js_i), \text{ for } 0\le i,j\le n-1 \text{ and } |i-j|>1;\\ (s_is_{i+1})^3 &= 1 \ (or\ s_is_{i+1}s_i=s_{i+1}s_is_{i+1}), \text{ for } 1\le i\le n-1.\\ (s_0s_1)^4 &= 1 \ (or\ s_0s_1s_0s_1=s_1s_0s_1s_0). \end{aligned}$$

The following theorem was conjectured by Stanley and proved by Haiman.

Theorem (Haiman, Discrete Math., 1989)

The number of reduced expressions of the longest element $w_0^B := [-1,-2,\cdots,-n]$ *in* B_n *is equal to the number SYT of shape* $n\times n$.

Ex5. Let $n=2$. Then $w_0^B=[-1,-2]=s_0s_1s_0s_1=s_1s_0s_1s_0$ and $f^{(2,2)}=2$.

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$$

Ex6. For $n=3$ and $w_0^B=[-1,-2,-3]$. $|R(w_0^B)|=f^{(3,3,3)}=42$.

Question 1: The type B version of standard/balanced Rothe tableaux?

Question 2: $\forall w\in B_n, |R(w)|=?$

Thank You !