第六章练习题答案

一. 填空题

1. $\dfrac{n(n+1)}{2}$;　　2. n;　　3. $V_1\cap V_2=\{0\}$;　　4. $(0,0,\cdots,0),-X$;　　5. $A^{-1}X$;

6. $n-r$;

7. $(x+1,x-1,x^2+1,x^2-1)=(x^3,x^2,x,1)\underbrace{\begin{pmatrix}0&0&0&0\\0&0&1&1\\1&1&0&0\\1&-1&1&-1\end{pmatrix}}_{=A}$,矩阵$A$的秩就是向量组的秩.

8. 同时包含V_1,V_2的V的最小子空间是V_1+V_2,同时包含于V_1,V_2的V的最大子空间是$V_1\cap V_2$.

9. $\alpha_3=(0,0,1,0),\alpha_4=(0,0,0,1)$,添加后的向量组$\alpha_1,\alpha_2,\alpha_3,\alpha_4$线性无关即可.

10. $\dim V=\dim W$;　　11. $n+1$;

13. $d_1+d_2+d_3-d$.

$$\begin{aligned}\dim(V_1+V_2+V_3)&=\dim(V_1+V_2)+\dim V_3-\dim((V_1+V_2)\cap V_3)\\&=\dim V_1+\dim V_2-\dim((V_1\cap V_2)+\dim V_3-\dim((V_1+V_2)\cap V_3)\\&=d_1+d_2+d_3-\dim((V_1\cap V_2)-\dim((V_1+V_2)\cap V_3).\end{aligned}$$

20. $x^2+2x+3=ax^3+b(x^3+x)+c(x^2+1)+d(x+1)=(a+b)x^3+cx^2+(b+d)x+c+d$,

则$a+b=0,c=1,b+d=2,c+d=3$,故$a=-b=0,c=1,d=2$,得坐标$(0,0,1,2)$.

二. 计算题

1. 解　$A=E+\begin{pmatrix}0&0&1\\0&0&1\\0&2&1\end{pmatrix}=E+T$,设$B=\begin{pmatrix}a&b&c\\d&e&f\\x&y&z\end{pmatrix}$满足$AB=BA$,而$AB=BA$等价于$TB=BT$.

即$\begin{pmatrix}0&0&1\\0&0&1\\0&2&1\end{pmatrix}\begin{pmatrix}a&b&c\\d&e&f\\x&y&z\end{pmatrix}=\begin{pmatrix}a&b&c\\d&e&f\\x&y&z\end{pmatrix}\begin{pmatrix}0&0&1\\0&0&1\\0&2&1\end{pmatrix}$,从而$\begin{pmatrix}x&y&z\\x&y&z\\2d+x&2e+y&2f+z\end{pmatrix}=\begin{pmatrix}0&2c&a+b+c\\0&2f&d+e+f\\0&2z&x+y+z\end{pmatrix}$,

得$\begin{cases}d=x=0\\y=2c=2f\\a+b+c=z\\e+f=z\end{cases}$,取$a,b,c$自由,则$\begin{cases}d=x=0\\y=2f=2c\\z=a+b+c\\e=a+b\end{cases}$,故$B=\begin{pmatrix}a&b&c\\0&a+b&c\\0&2c&a+b+c\end{pmatrix}$,

$W=\left\{\begin{pmatrix}a&b&c\\0&a+b&c\\0&2c&a+b+c\end{pmatrix}\middle|\,a,b,c\in P\right\}$,$\dim W=3$,基:$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}0&1&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}0&0&1\\0&0&1\\0&2&1\end{pmatrix}$.

2. 解 $(f_1, f_2, f_3)=(1,x,x^2)\underbrace{\begin{pmatrix}1&1&0\\-1&0&1\\0&1&2\end{pmatrix}}_{=A}$,$(g_1,g_2,g_3)=(1,x,x^2)\underbrace{\begin{pmatrix}0&1&1\\1&0&-1\\0&-1&1\end{pmatrix}}_{=B}$.

且 $A^{-1}=\begin{pmatrix}-1&-2&1\\2&2&-1\\-1&-1&1\end{pmatrix}, B^{-1}=\begin{pmatrix}\frac{1}{2}&1&\frac{1}{2}\\\frac{1}{2}&0&-\frac{1}{2}\\\frac{1}{2}&0&\frac{1}{2}\end{pmatrix}$.

(1) $|A|=1$, $|B|=-2$,都可逆,满秩,故 $r(f_1, f_2, f_3)=r(g_1,g_2,g_3)=3$,而线性空间的维数为 3,故 f_1, f_2, f_3 和 g_1,g_2,g_3 都是 $P[x]_3$ 的基.

(2) $(g_1,g_2,g_3)=(1,x,x^2)B=(f_1, f_2, f_3)A^{-1}B$,计算可得 $A^{-1}=\begin{pmatrix}-1&-2&1\\2&2&-1\\-1&-1&1\end{pmatrix}$,则 $A^{-1}B=\begin{pmatrix}-2&-2&2\\2&3&-1\\-1&-2&1\end{pmatrix}$.

(3) $f=1+2x+3x^2=(1,x,x^2)\begin{pmatrix}1\\2\\3\end{pmatrix}=(f_1, f_2, f_3)A^{-1}\begin{pmatrix}1\\2\\3\end{pmatrix}=(f_1, f_2, f_3)\begin{pmatrix}-2\\3\\0\end{pmatrix}$.

$f=1+2x+3x^2=(1,x,x^2)\begin{pmatrix}1\\2\\3\end{pmatrix}=(g_1,g_2,g_3)B^{-1}\begin{pmatrix}1\\2\\3\end{pmatrix}=(g_1,g_2,g_3)\begin{pmatrix}4\\-1\\2\end{pmatrix}$.

3 解 设 $\varepsilon_1=(1,0,0),\varepsilon_2=(0,1,0),\varepsilon_3=(0,0,1)$,则

$(\alpha_1,\alpha_2,\alpha_3)=(\varepsilon_1,\varepsilon_2,\varepsilon_3)\underbrace{\begin{pmatrix}1&0&3\\2&1&0\\1&-1&2\end{pmatrix}}_{=A}$,$(\beta_1,\beta_2,\beta_3)=(\varepsilon_1,\varepsilon_2,\varepsilon_3)\underbrace{\begin{pmatrix}6&-1&3\\6&-1&1\\5&-2&1\end{pmatrix}}_{=B}$.

$A^{-1}=\begin{pmatrix}-\frac{2}{7}&\frac{3}{7}&\frac{3}{7}\\\frac{4}{7}&\frac{1}{7}&-\frac{6}{7}\\\frac{3}{7}&-\frac{1}{7}&-\frac{1}{7}\end{pmatrix}, B^{-1}=\begin{pmatrix}-\frac{1}{14}&\frac{5}{14}&-\frac{1}{7}\\\frac{1}{14}&\frac{9}{14}&-\frac{6}{7}\\\frac{1}{2}&-\frac{1}{2}&0\end{pmatrix}$.

(1) $(\beta_1,\beta_2,\beta_3)=(\alpha_1,\alpha_2,\alpha_3)A^{-1}B=(\alpha_1,\alpha_2,\alpha_3)\begin{pmatrix}3&-1&0\\0&1&1\\1&0&1\end{pmatrix}$.

(2) $\xi=(\alpha_1,\alpha_2,\alpha_3)Y=(\beta_1,\beta_2,\beta_3)\frac{1}{2}Y$.则 $[(\alpha_1,\alpha_2,\alpha_3)-\frac{1}{2}(\beta_1,\beta_2,\beta_3)]Y=0$,

$2(\alpha_1,\alpha_2,\alpha_3)-(\beta_1,\beta_2,\beta_3)=\begin{pmatrix}-4&1&3\\-2&3&-1\\-3&0&3\end{pmatrix}\to\begin{pmatrix}1&0&-1\\0&1&-1\\0&0&0\end{pmatrix}$,解得 $X=\begin{pmatrix}1\\1\\1\end{pmatrix}$,即 $\xi=(\alpha_1,\alpha_2,\alpha_3)\begin{pmatrix}1\\1\\1\end{pmatrix}$ 满足条件.

4 解

(1) 由 $aE_1+bE_2+cE_3+dE_4=\begin{pmatrix} a & b \\ c & d \end{pmatrix}=0$,得 $a=b=c=d=0$,从而 E_1,E_2,E_3,E_4 线性无关,是 $P^{2\times 2}$ 的一组基.

而 $(F_1,F_2,F_3,F_4)=(E_1,E_2,E_3,E_4)\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}}_{=A}$,可得 F_1,F_2,F_3,F_4 线性无关,也是 $P^{2\times 2}$ 的一组基.

(3) $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}=aE_1+bE_2+cE_3+dE_4$.

$$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}=(E_1,E_2,E_3,E_4)\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}=(F_1,F_2,F_3,F_4)A^{-1}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}=(F_1,F_2,F_3,F_4)\begin{pmatrix} a \\ b-a \\ c-b \\ d-c \end{pmatrix}.$$

5 解

方法一. $(\alpha_1,\alpha_2,\alpha_3)=\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}\to\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,则 $\dim V_1=2$,取 α_1,α_2 为基.

$(\beta_1,\beta_2,\beta_3)=\begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -3 & -1 \\ 1 & -1 & 1 \end{pmatrix}\to\begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,则 $\dim V_2=2$,取 β_1,β_2 为基.

而 $V_1+V_2=L(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)$,则

$$(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)=\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 1 & -1 \\ 0 & 1 & 1 & 1 & -1 & 1 \\ -1 & 2 & 0 & 1 & -3 & -1 \\ 0 & 1 & 1 & 1 & -1 & 1 \end{array}\right)\to\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 1 & -1 \\ 0 & 1 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

则 $\dim(V_1+V_2)=3$,$\alpha_1,\alpha_2,\beta_1$ 是一组基.且 $\dim(V_1\cap V_2)=2+2-3=1$.

任取 $\xi\in V_1\cap V_2$,则 $\xi=x_1\alpha_1+x_2\alpha_2=y_1\beta_1+y_2\beta_2$,即 $x_1\alpha_1+x_2\alpha_2-y_1\beta_1-y_2\beta_2=0$.系数矩阵

$$(\alpha_1,\alpha_2,-\beta_1,-\beta_2)=\left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ -1 & 2 & -1 & 3 \\ 0 & 1 & -1 & 1 \end{array}\right)\to\left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)\to\left(\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

求解基础解系得 $\eta=(1,-1,0,1)$,则 $\xi=\alpha_1-\alpha_2=\beta_2$ 就是 $V_1\cap V_2$ 的基.

6. 解 (1) 求(I)和(II)的解空间V_1和V_2的维数和一组基,即求(I)和(II)的基础解系.

(I) $\begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{pmatrix}$,得基础解系为$\eta_1 = (1,-1,1,0)^T, \eta_2 = (2,-1,0,-2)^T$.

(II) $\begin{pmatrix} -2 & 1 & 6 & -1 \\ -1 & 2 & 5 & -1 \end{pmatrix} \to \begin{pmatrix} -1 & 2 & 5 & -1 \\ 0 & -3 & -4 & 1 \end{pmatrix}$,得基础解系为$\eta_3 = (7,-4,3,0)^T, \eta_4 = (-1,1,0,3)^T$.

(2) $V_1 + V_2 = L(\eta_1, \eta_2, \eta_3, \eta_4)$,而

$$(\eta_1, \eta_2, \eta_3, \eta_4) = \begin{pmatrix} 1 & 2 & 7 & -1 \\ -1 & -1 & -4 & 1 \\ 1 & 0 & 3 & 0 \\ 0 & -2 & 0 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 7 & -1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

故$\dim(V_1 + V_2) = 3$,基η_1, η_2, η_3.

$V_1 \cap V_2$的一组基就是方程组$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ 2x_1 + 2x_2 + x_4 = 0 \\ -2x_1 + x_2 + 6x_3 - x_4 = 0 \\ -x_1 + 2x_2 + 5x_3 - x_4 = 0 \end{cases}$的一个基础解系.求得一组基础解系为

$\eta_0 = (3,-2,1,-2)^T$.

7 解$(\eta_1, \eta_2, \eta_3) = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$,

$$\alpha = \varepsilon_1 + 2\varepsilon_2 + 3\varepsilon_3 = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = (\eta_1, \eta_2, \eta_3) \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = (\eta_1, \eta_2, \eta_3) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

三 证明题

6. 证明: 任给$\alpha_1 \in W_1, \alpha_2 \in W_2$,则$\alpha_1 + \alpha_2 \in W_1 + W_2 = W_1 \cup W_2$,即$\alpha_1 + \alpha_2 \in W_1$或者$\alpha_1 + \alpha_2 \in W_2$,

故由$\alpha_1 + \alpha_2 \in W_1$可知$\alpha_2 \in W_1$;由$\alpha_1 + \alpha_2 \in W_2$可知$\alpha_1 \in W_2$,即得$W_1 \subseteq W_2$或$W_2 \subseteq W_1$.

7. 证明: 假若不成立,则$\dim W \geq 2$,即W中至少存在两个向量$\alpha = (a_1, a_2, \cdots, a_n), \beta = (b_1, b_2, \cdots, b_n)$线性无关,

其中a_i, b_i全非零,α, β线性无关,不妨设$\dfrac{a_1}{b_1} \neq \dfrac{a_2}{b_2}$,则$b_2\alpha - a_2\beta = (b_2a_1 - a_2b_1, b_2a_2 - a_2b_2, \cdots, b_2a_n - a_2b_n) \in W$,

但是$b_2a_1 - a_2b_1 \neq 0, b_2a_2 - a_2b_2 = 0$,矛盾.

9. 证明: 半正定,存在非退化线性替换$X = CY$,使得$f(X)$化为规范形$g(Y) = Y^T C^T ACY = y_1^2 + y_2^2 + \cdots + y_r^2$,

故$f(X) = 0 \Leftrightarrow g(Y) = 0 \Leftrightarrow y_1^2 + y_2^2 + \cdots + y_r^2 = 0 \Leftrightarrow y_1 = y_2 = \cdots = y_r = 0$,对$X = CY$,设$C^{-1} = (d_{ij})_n$,则

(*) $\begin{cases} y_1 = d_{11}x_1 + d_{12}x_2 + +d_{1n}x_n = 0 \\ y_2 = d_{21}x_1 + d_{22}x_2 + +d_{2n}x_n = 0 \\ \cdots \\ y_r = d_{r1}x_1 + d_{r2}x_2 + +d_{rn}x_n = 0 \end{cases}$,系数矩阵的秩为 r ,基础解系含有向量的个数为 $n-r$,(*)的解空间的维数为 $n-r$, $X'AX = 0$ 的全部解构成实数域上的线性空间就是(*)的解空间.

10. 证明: 取 W 的一组基 $\alpha_1, \alpha_2, \cdots, \alpha_m$,扩充成 V 的一组基 $\alpha_1, \alpha_2, \cdots, \alpha_m, \alpha_{m+1}, \cdots, \alpha_n$,则 $U = L(\alpha_{m+1}, \cdots, \alpha_n)$ 就是 W 在 V 中的补子空间.对如上的基 $\alpha_1, \alpha_2, \cdots, \alpha_m, \alpha_{m+1}, \cdots, \alpha_n$,向量组 $\alpha_1, \alpha_2, \cdots, \alpha_m, \alpha_{m+1} + \alpha_1, \alpha_{m+2}, \cdots, \alpha_n$ 仍然是一组基, $U_1 = L(\alpha_{m+1} + \alpha_1, \alpha_{m+2}, \cdots, \alpha_n)$ 也是 W 在 V 中的一个补子空间,但是 $U \neq U_1$.

11 解: (1) 任取 $f(x), g(x) \in W$,首先 $f(1) = 0, g(1) = 0$,则

对 $f(x) + g(x)$, $f(1) + g(1) = 0$,故 $f(x) + g(x) \in W$,对加法封闭.

对 $kf(x)$,有 $kf(1) = 0$,即 $kf(x) \in W$,对数乘封闭,从而 W 是 $\mathbf{R}[x]_n$ 的子空间.

(2) $x-1, (x-1)^2, \cdots, (x-1)^{n-1}$ 线性无关,且若 $f(x)$ 是一个 $n-1$ 次多项式满足 $f(1) = 0$.则 $f(x)$ 在1点泰勒展开有 $f(x) = f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \cdots + \frac{f^{(n-1)}(1)}{(n-1)!}(x-1)^{n-1}$.故 W 是 $n-1$ 维子空间,

$x-1, (x-1)^2, \cdots, (x-1)^{n-1}$ 是一组基.