Bijective proofs of proper coloring theorems

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October 25, 2020

Outline

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Chromatic polynomial

Let $G = (V, E)$ be a graph with vertex set V and edge set E. A $\mathbb{P}$-coloring of G is a function $\kappa : V \to \mathbb{P}$, and a $[t]$-coloring is a function $\kappa : V \to [t]$, where $[t] = \{1, 2, \cdots, t\}$.

The edge $e = uv \in E$ is monochromatic in the coloring κ if $\kappa(u) = \kappa(v)$, and a coloring is proper if none of its edges are monochromatic.

Definition

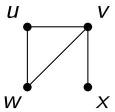
The chromatic polynomial of the graph G is defined as

$$\chi_G(t) = \textit{the number of proper } [t]\textit{-colorings of } G,$$

for all positive integers t.

- The chromatic polynomial of the graph beyond is

$$\chi_G(t) = t(t-1)^2(t-2).$$



Background

A spanning subgraph of a graph $G = (V, E)$ is a graph on the same set of vertices, but with a subset of the edges. We associate each subset $S \subset E$ of edges with the spanning subgraph of G with edge set S.

A connected component, or simply component, K of a spanning subgraph S is a maximal subset of vertices such that one can reach any vertex in K from any other by traveling vertex-to-vertex via the edges of S.

Let $c(S) =$ the number of components of S.

The components of any graph (or spanning subgraph) form a partition of its vertices.

- Example:

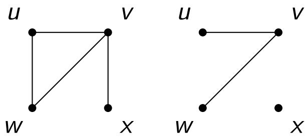


Figure: A graph and a spanning subgraph $S = \{uv, vw\}$, $c(S) = 2$, and the induced partition $\lambda(S) = (3, 1)$

Background

Given a spanning subgraph $S \in E$ and a coloring κ, we say that κ is monochromatic on the components of S if $\kappa(u) = \kappa(v)$ for all vertices u and v in the same component of S.

If κ is a proper coloring, it follows that κ can only be monochromatic on the components of S if $S = \emptyset$.

Lemma

For every graph $G = (V, E)$ and spanning subgraph $S \in E$, the number of $[t]$-colorings of G that are monochromatic on the components of S is $t^{c(S)}$.

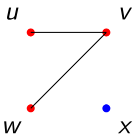


Figure: A monochromatic coloring of $S = \{uv, vw\}$

Chromatic symmetric function

Let $\mathbf{x} = \{x_1, x_2, \dots\}$ be an infinite set of commuting variables indexed by the positive integers. To every $\mathbb{P}$-coloring of the graph $G = (V, E)$, we associate the monomial or weight

$$x^\kappa = \prod_{v \in V} x_{\kappa(v)}.$$

Definition (Stanley, 1995)

$$X(G; \mathbf{x}) = \sum_\kappa x^\kappa,$$

where the sum is over all proper $\mathbb{P}$-colorings of G.

$$x_i = \begin{cases} 1 & \text{if } i \in [n], \\ 0 & \text{if } i \notin [n]. \end{cases}$$

By making this substitution, $X(G; x)$ reduces to $\chi(G; t)$, so the chromatic symmetric function is indeed a generalization of the chromatic polynomial.

Power sum symmetric function

Definition

The Power sum symmetric function is defined by

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_\ell},$$

where

$$p_n = m_n = \sum_{j=1}^{\infty} x_j^n.$$

Example

Let $\lambda = (2,1)$*, then*

$$p_{(2,1)} = p_2 p_1 = \sum_{i=1}^{\infty} x_i^2 \sum_{j=1}^{\infty} x_j.$$

Sign-reversing involution

Let S be a finite set with an associated sign function

$$\mathrm{sgn} : S \to \{+1, -1\}.$$

A sign-reversing involution on S is an involution $\iota : S \to S$ that satisfies the following two conditions

- If $\iota(s) = s$, then $\mathrm{sgn}(s) = +1$.
- If $\iota(s) = t$ and $s \neq t$, then $\mathrm{sgn}(s) = -\mathrm{sgn}(t)$.

Immediately from the definition we see that

$$\sum_{s \in \mathcal{S}} \mathrm{sgn}(s) = |\mathrm{Fix}\, \iota|,$$

where $\mathrm{Fix}\, \iota = \{s \in S | \iota(s) = s\}$ is the set of fixed points of ι, and the vertical bars denote cardinality.

Preliminaries

Lemma

For every graph $G = (V, E)$ and spanning subgraph $S \in E$, we have

$$\sum_{\kappa} x^{\kappa} = p_{\lambda(S)},$$

where the sum is over all $\mathbb{P}$-colorings of G that are monochromatic on the components of S.

The cancellation we desire, which results in equality

$$\sum_{s \in \mathcal{S}} \operatorname{sgn}(s)\operatorname{wt}(s) = \sum_{s \in \mathsf{Fix}\ \iota} \operatorname{wt}(s).$$

It holds since the number of elements in $\mathcal{S}$ of a given weight is finite and ι is weight-preserving, meaning that $\operatorname{wt}(s) = \operatorname{wt}(\iota(s))$ for all $s \in \mathcal{S}$.

Outline

Main results

Both $\chi(G;t)$ and $X(G;\mathbf{x})$ can be expressed as sums over the spanning subgraphs of G.

Theorem (Birkhoff,1912)

For every graph $G=(V,E)$ and every positive integer t, we have

$$\chi(G,t)=\sum_{S\subseteq E}(-1)^{|S|}t^{c(S)}. \tag{1}$$

Theorem (Stanley,1995)

For every graph $G=(V,E)$, we have

$$X(G,\mathbf{x})=\sum_{S\subseteq E}(-1)^{|S|}p_{\lambda(S)}. \tag{2}$$

Sketch of the proof (1)

We start by giving combinatorial interpretation to the right-hand side.

Since the number of $[t]$-colorings of V that are monochromatic on the components of S is $t^{c(S)}$, we have

$$\sum_{S\subseteq E}(-1)^{|S|}t^{c(S)} = \sum_{(S,\kappa)\in\mathcal{S}}(-1)^{|S|},$$

where

$\mathcal{S} = \{(S,\kappa) : S \subseteq E \text{ and } \kappa : V \to [t] \text{ is monochromatic on the components of } S\}.$

Define the ordering of edges of G as being first, last, earlier, or later, referring to their positions in this order.

We also define a sign function on the pairs in $\mathcal{S}$ by

$$\mathrm{sgn}(S,\kappa) = (-1)^{|S|},$$

thus we have

$$\sum_{S\subseteq E}(-1)^{|S|}t^{c(S)} = \sum_{(S,\kappa)\in\mathcal{S}}(-1)^{|S|} = \sum_{(S,\kappa)\in\mathcal{S}}\mathrm{sgn}(S,\kappa).$$

Sketch of the proof (1)

Define the sign-reversing involution ι by

- $\iota(S, \kappa) = (S, \kappa)$, if κ is a proper coloring;
- $\iota(S, \kappa) = (S \triangle e, \kappa)$, if κ has at least one monochromatic edge, where e is the last such edge, and $\triangle$ is the symmetric difference operator, removing e from S if it is present and adding it to S otherwise.

Lemma

The mapping ι is an sign-reversing involution on $\mathcal{S}$ with

$$\mathrm{Fix}\,\iota = \{(\emptyset, \kappa) : \kappa \textit{ is a proper coloring of } G\}.$$

By applying the above lemma, we can deduce that

$$\sum_{S \subseteq E} (-1)^{|S|} t^{c(S)} = \sum_{(S,\kappa) \in \mathcal{S}} \mathrm{sgn}(S, \kappa) = |\mathrm{Fix}\,\iota| = \chi(G; t).$$

Sketch of the proof (2)

Lemma

For every graph $G = (V, E)$ and spanning subgraph $S \in E$, we have

$$\sum_{\kappa} x^{\kappa} = p_{\lambda(S)},$$

where the sum is over all $\mathbb{P}$-colorings of G that are monochromatic on the components of S.

Defining the weight of a pair $(S, \kappa) \in \mathcal{S}$ as

$$\mathrm{wt}(S, \kappa) = x^{\kappa},$$

and appealing to the above Lemma, we see that

$$\sum_{S \subseteq E} (-1)^{|S|} p_{\lambda(S)} = \sum_{(S,\kappa) \in \mathcal{S}} \mathrm{sgn}(S, \kappa) \mathrm{wt}(S, \kappa).$$

Thus

$$\sum_{S \subseteq E} (-1)^{|S|} p_{\lambda(S)} = \sum_{(S,\kappa) \in \mathcal{S}} \mathrm{sgn}(S, \kappa) \mathrm{wt}(S, \kappa) = \sum_{(S,\kappa) \in \mathrm{Fix}\, \iota} \mathrm{wt}(S, \kappa) = X(G; \mathbf{x}).$$

Outline

Background

- A walk from the vertex v_0 to the vertex v_k in the graph $G = (V, E)$ is an alternating sequence $v_0, e_1, v_1, e_2, v_2, \cdots, e_k, v_k$ of vertices and edges such that for all $i \in [k]$ we have $e_i = v_{i-1}v_i \in E$.
- A path is a walk satisfies that it does not repeat any vertices or edges, except possibly the first and the last vertices. Note that we allow bothwalks and paths to be edgeless.
- A walk or a path is closed if $v_0 = v_k$, that is, if its first and last vertices are the same.
- A cycle, also known as a circuit, is a closed path with more than one vertex.
- Graphs not containing cycles can be referred to as acyclic, but are more commonly called forests.

Background

- Given a fixed total ordering on the edges E of G, a broken circuit is a subset $B \subseteq E$ of the form
$$B = C - \max C,$$
where C is a cycle and $\max C$ is the last edge of C.
- A spanning subgraph $S \subseteq E$ is NBC (short for no broken circuits) if it does not contain any broken circuits.

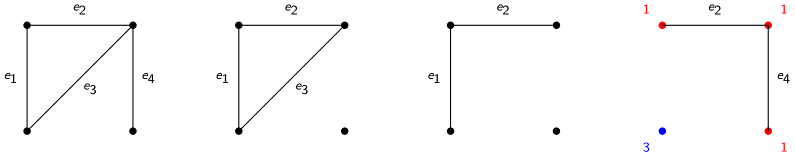


Figure: A graph with ordered edges $e_1 < e_2 < e_3 < e_4$; a cycle in this graph; the corresponding broken circuit; an NBC spanning subgraph with color

Sign-reversing involution

Consider a particular pair (S, κ) from $\mathcal{S}$, i.e. S is an arbitrary spanning subgraph of G and κ is monochromatic on the components of S.

Define the sign-reversing involution ι by

- $\iota(S, \kappa) = (S, \kappa)$ if $S = \emptyset$;
- $\iota(S, \kappa) = (S \triangle e, \kappa)$ (where $e = \max C$) if S contains a broken circuit;
- $\iota(S, \kappa) = (S \triangle e, \kappa)$ (where $e =$ the last edge) if S not contains a broken circuit.

It follows that if the broken circuit B is contained in S, then it is also contained in $S \triangle e$, and the involution ι will not create a broken circuit if S not contains a broken circuit at first.

We also define the sign function on the pairs in $\mathcal{S}$ by

$$\mathrm{sgn}(S, \kappa) = (-1)^{|S|},$$

Then the summation in the previous theorem leaves only the NBC spanning subgraphs.

Main results

Theorem (Whitney, 1932)

For every graph $G = (V, E)$, every total ordering of its edges, and every positive integer t, we have

$$\chi(G;t) = \sum_{\substack{S \subseteq E \\ S \text{ is } \overline{\mathrm{NBC}}}} (-1)^{|S|} t^{c(S)}.$$

Theorem (Stanley, 1995)

For every graph $G = (V, E)$, every total ordering of its edges, and every positive integer t, we have

$$X(G;\mathbf{x}) = \sum_{\substack{S \subseteq E \\ S \text{ is } \overline{\mathrm{NBC}}}} (-1)^{|S|} p_{\lambda(S)}.$$

Outline

Background

- An oriented (or directed) edge is an arc which connects two vertices from u to v, denoted by $\overrightarrow{uv}$.
- An orientation of the (undirected) graph $G = (V, E)$ is obtained by replacing each edge $uv \in E$ by one of the arcs $\overrightarrow{uv}$ or $\overrightarrow{vu}$.
- An oriented graph is an undirected graph with an orientation, which we denote by $O = (V, A)$, where A is the set of arcs.
- An acyclic orientation of the (undirected) graph $G = (V, E)$ is an orientation which do not create a directed cycle.

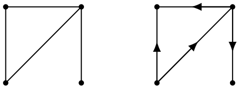


Figure: A graph, and an acyclic orientation

Mixed graph

- A mixed graph is a triple $M = (V, E, A)$ where V is a set of vertices, E is a set of edges, and A is a set of arcs.
- A walk from v_0 to v_k in the mixed graph $M = (V, E, A)$ is an alternating sequence $v_0, c_1, v_1, c_2, v_2, \cdots, c_k, v_k$ of vertices and edges/arcs such that for all $i \in [k]$, either $c_i = v_{i-1}v_i \in E$ or $c_i = \overrightarrow{v_{i-1}v_i} \in A$.
- We further call this walk a path if it does not repeat any vertices, edges, or arcs, except possibly the first and last vertices.
- A walk is closed if its first and last vertices are the same.
- A closed path is a cycle if it has more than one vertex.
- A mixed graph is acyclic if it does not contain a cycle.

Example

The sequence u, uv, v, vu, u is not considered to be a path because it repeats the edge uv, while the sequence $u, \overrightarrow{uv}, v, vu, u$ is considered to be a path, and furthernore, a cycle.

Main results

Lemma

For every graph $G = (V, E)$, every total ordering of its edges, and every positive integer t, we have

$$\chi(G; -t) = (-1)^{|V|} \sum_{\substack{S \subseteq E \\ S \text{ is } \overline{\mathrm{NBC}}}} t^{c(S)}.$$

Proposition

If a mixed graph contains a closed walk that traverses at least one arc, then it also contains a cycle.

Theorem (Stanley, 1973)

For every graph $G = (V, E)$,

$$(-1)^{|V|}\chi(G; -1) = \text{ the number of acyclic orientations of } G.$$

Sketch of the proof

Since we have

$$\chi(G;-1) = (-1)^{|V|} \sum_{\substack{S \subseteq E \\ S \text{ is } \overline{\mathrm{NBC}}}} 1.$$

It suffices to exhibit a bijection between the NBC spanning subgraphs of G and its acyclic orientations.

Define $\mathcal{M}_i$ to be the set of acyclic mixed graphs comprised of

- an NBC subset of the edges $\{e_1, \cdots, e_i\}$ and
- an acyclic orientation of the edges $\{e_{i+1}, \cdots, e_m\}$.

We simply fix one orientation of each edge e_i as normal, denoted by $\overrightarrow{e_i}$, and call the opposite orientation $\overleftarrow{e_i}$ abnormal.

We define $\phi_i : \mathcal{M}_{i-1} \to \mathcal{M}_i$ for all indices $1 \leq i \leq m$.

Take $M \in \mathcal{M}_{i-1}$ and suppose that the edge e_i of G appears as the arc a_i in M. The mixed graph $\phi_i(M)$ is obtained by either unorienting a_i (replacing the arc a_i with the edge e_i) or removing it (deleting the arc a_i). We unorient a_i if both

- (A) a_i is the normal orientation of e_i and
- (B) unorienting a_i does not create a cycle.

Otherwise, we remove a_i.

We define the inverse of ϕ_i, which denoted by $\psi_i : \mathcal{M}_i \to \mathcal{M}_{i-1}$.

Given a mixed graph $M' \in M_i$, the mixed graph $M = \psi_i(M')$ is obtained by adding one of the orientations of edge e_i to M' (and removing e_i if it is present in M'). We give e_i the abnormal orientation if both

- (A') e_i is not an edge of M' and
- (B') adding $\overleftarrow{e_i}$ to M' does not create a cycle.

Otherwise, we give e_i the normal orientation.

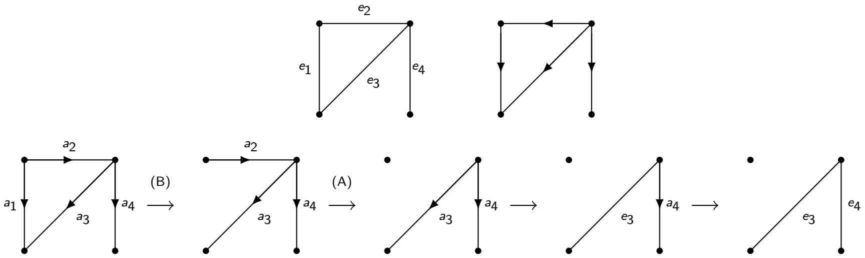


Figure: Applying the mappings ϕ_1, ϕ_2, ϕ_3, and ϕ_4 to an acyclic orientation of the graph shown in the top left, where the normal orientations of the edges are as shown in the top right.

Outline

Multi-colored acyclic orientations

Suppose that $O = (V, A)$ is an orientation of the graph $G = (V, E)$ and that κ is a coloring of G (by $[t]$ or by $\mathbb{P}$). We say that O and κ are compatible if $\overrightarrow{uv} \in A$ implies $\kappa(u) \leq \kappa(v)$, that is, if all the arcs of O point in the direction of weakly increasing values of κ.

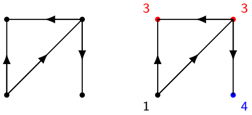


Figure: An oriented graph, and a compatible coloring

Theorem (Stanley, 1973)

For every graph $G = (V, E)$ and every positive integer t, we have

$$(-1)^{|V|}\chi(G; t) = \\ |\{(O, \kappa) : O \text{ is an acyclic orientation of } G \text{ and } \kappa \text{ a compatible } [t]\text{-coloring}\}|.$$

Sketch of the proof

We would like to construct a bijection between (S, κ) and (O, κ), where O is an acyclic orientation of G and κ is a compatible $[t]$-coloring.

We first construct, for every $[t]$-coloring κ of G, a bijection Φ between the sets

- $\mathcal{O}_\kappa = \{O : O \text{ is an acyclic orientation of } G \text{ compatible with } \kappa\}$ and
- $\mathcal{S}_\kappa = \{S : S \subseteq E \text{ is NBC and } \kappa \text{ is monochromatic on the components of } S\}$

For each $i \in [t]$, define

$$V_i = \{v \in V : \kappa(v) = i\},$$

and let E_i denote the set of (monochromatic) edges of G between vertices in V_i.

Take some orientation $O \in \mathcal{O}_\kappa$, and for each $i \in [t]$, let O_i denote the sub-orientation consisting of all arcs of O between vertices of color i, so O_i is an orientation of the spanning subgraph E_i.

Let $\phi^{(i)}$ denote the bijection from acyclic orientations of E_i to its NBC spanning subgraphs.

Define $\Phi(O)$by

$$\Phi(O) = \phi^{(1)}(O_1) \cup \phi^{(2)}(O_2) \cup \cdots \cup \phi^{(t)}(O_t).$$

For each $i \in [t]$, let $\psi^{(i)}$ denote the inverse of the bijection $\psi^{(i)}$.

We then define $\Psi(S)$ by

$$\Psi(S) = C \cup \psi^{(1)}(S_1) \cup \psi^{(2)}(S_2) \cup \cdots \cup \psi^{(t)}(S_t)$$

Chromatic symmetric function

When $\lambda = (\lambda_1, \cdots, \lambda_m)$ is a partition of n into m parts, we define

$$\omega(p_\lambda) = (-1)^{n-m} p_\lambda.$$

Theorem (Stanley, 1995)

For every graph G, we have

$$\omega(X(G; \mathbf{x})) = \sum_{(O,\kappa)} x^\kappa,$$

where the sum is over all pairs (O, κ) where O is an acyclic orientation of G that is compatible with the $\mathbb{P}$-coloring κ.

Strong chromatic polynomial

Definition

The strong chromatic polynomial *of the mixed graph* $G = (V, E, A)$ *to be the number of* $[t]$*-colorings* κ *satisfying*

- *(a)* $\kappa(u) \neq \kappa(v)$ *if* $uv \in E$ *and*
- *(b)* $\kappa(u) < \kappa(v)$ *if* $\overrightarrow{uv} \in A$.

Beck, Blado, Crawford, Jean-Louis, and Young introduced the weak chromatic polynomial of a mixed graph, which counts colorings satisfying the weak version of the inequality in (b).

M. Beck, T. Bogart, and T. Pham. Enumeration of Golomb rulers and acyclic orientations of mixed graphs. Electron. J. Combin., 19(3): Paper 42, 13 pp., 2012.

M. Beck, D. Blado, J. Crawford, T. Jean-Louis, and M. Young. On weak chromatic polynomials of mixed graphs. Graphs Combin., 31(1):91–98, 2015.

THANK YOU!