# Distribution properties of some combinatorial polynomials

Lily Li Liu

School of Mathematical Sciences
Qufu Normal University

liulily@qfnu.edu.cn

9 Oct. 2020

# Contents

# Outline

# Unimodality, symmetry and log-concavity

## Definition

Let $(a_0, a_1, \ldots, a_n)$ be a sequence of positive numbers.

- It is unimodal (UM) if $a_0 \leq \cdots \leq a_{m-1} \leq a_m \geq \cdots \geq a_n$. ($m$ is called a mode of the sequence.)
- It is symmetric (SYM) if $a_k = a_{n-k}$ for all $0 \leq k \leq n$.
- It is log-concave (LC) if $a_{k-1}a_{k+1} \leq a_k^2$ for all $1 < k < n$.
- It is spiral (SP) if $a_n \leq a_1 \leq a_{n-1} \leq a_2 \leq a_{n-2} \leq \cdots \leq a_{\lfloor \frac{n-3}{2} \rfloor} \leq a_{\lceil \frac{n+3}{2} \rceil} \leq a_{\lfloor \frac{n-1}{2} \rfloor} \leq a_{\lceil \frac{n+1}{2} \rceil}$, and $a_{\frac{n}{2}+1} \leq a_{\frac{n}{2}}$ if $n$ is even.

- Clearly, LC $\iff \frac{a_{k+1}}{a_k} \leq \frac{a_k}{a_{k-1}} \implies$ UM, and SP $\implies$ UM.

Combinatorialists love to prove counting sequences are unimodal.
— Zeilberger

# Pascal triangle

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 1 | | | | | | |
| | | | | | 1 | | 1 | | | | | |
| | | | | 1 | | 2 | | 1 | | | | |
| | | | 1 | | 3 | | 3 | | 1 | | | |
| | | 1 | | 4 | | 6 | | 4 | | 1 | | |
| | 1 | | 5 | | 10 | | 10 | | 5 | | 1 | |
| 1 | | 6 | | 15 | | 20 | | 15 | | 6 | | 1 |

- Each row in the Pascal triangle is UM, SYM and LC.

# Generating functions

## Definition

We say that a polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$ is UM (SYM, LC, SP) if the coefficients sequence $a_0, a_1, \ldots, a_n$ has such a property.

## Newton inequality

If $f(t) = \sum_{i=0}^{n} a_i t^i$ is real-rooted, then

$$a_i^2 \geq a_{i-1} a_{i+1} \frac{(i+1)(n-i+1)}{i(n-i)}.$$

$\Longrightarrow (a_0, a_1, \ldots, a_n)$ is LC and UM with at two modes.

Generating functions are a bridge between discrete mathematics and continuous analysis. — Wilf

# $\gamma$-positivity

## Definition

A symmetric polynomial $f(t) = \sum_{i=0}^{n} a_i t^i$ is called $\gamma$-positivity if

$$f(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_k t^k (1+t)^{n-2k},$$

and $\gamma_k \geq 0$ for each $k$.

- Clearly, $\gamma$-positivity $\Longrightarrow$ UM and SYM.

D. Foata, M.-P. Schützenberger, Théorie géométrique des polynômes eulériens, Lecture Notes in Math., vol. 138, Springer-Verlag, Berlin, 1970.

# Eulerian polynomials $A_n(t)$

$n = 5:$

$$
\begin{array}{r|ccccc}
 & 1 & 26 & 66 & 26 & 1 \\
\hline
{\color{blue}1}\times & (1 & 4 & 6 & 4 & 1) \\
{\color{blue}22}\times & & (1 & 2 & 1) & \\
{\color{blue}16}\times & & & (1) & &
\end{array}
$$

$$
\begin{aligned}
& A_5(t) \\
= \; & t^4 + 26t^3 + 66t^2 + 26t + 1 \\
= \; & {\color{blue}1}(1+t)^4 + {\color{blue}22}t(1+t)^2 + {\color{blue}16}t^2
\end{aligned}
$$

# Eulerian polynomials $A_n(t)$

$n = 5$ :

$$
\begin{array}{r|ccccc}
 & 1 & 26 & 66 & 26 & 1 \\
\hline
1\times & (1 & 4 & 6 & 4 & 1) \\
22\times & & (1 & 2 & 1) & \\
16\times & & & (1) & &
\end{array}
$$

$$
\begin{aligned}
& A_5(t) \\
= {} & t^4 + 26t^3 + 66t^2 + 26t + 1 \\
= {} & 1(1+t)^4 + 22t(1+t)^2 + 16t^2
\end{aligned}
$$

$n = 6$ :

$$
\begin{array}{r|cccccc}
 & 1 & 57 & 302 & 302 & 57 & 1 \\
\hline
1\times & (1 & 5 & 10 & 10 & 5 & 1) \\
52\times & & (1 & 3 & 3 & 1) & \\
136\times & & & (1 & 1) & &
\end{array}
$$

$$
\begin{aligned}
& A_6(t) \\
= {} & t^5 + 57t^4 + 302t^3 + 302t^2 + 57t + 1 \\
= {} & 1(1+t)^5 + 52t(1+t)^3 + 136t^2(1+t)
\end{aligned}
$$

# Relations

**Proposition**

*If both $f(t)$ and $g(t)$ are $\gamma$-positive, then so is $f(t)g(t)$.*

- RZ and SYM $\Longrightarrow \gamma$-positivity.

In fact, suppose that $f(t)$ has nonnegative and symmetric coefficients, and all its zeros are real. Then its zeros apart 0 and $-1$ come in reciprocal pairs $a, 1/a$, i.e.

$$\begin{aligned} f(t) &= t^p(1+t)^q \prod_{i\geq 1}(t-a_i)(t-1/a_i) \\ &= t^p(1+t)^q \prod_{i\geq 1}((1+t)^2-(2+a_i+1/a_i)) \end{aligned}$$

Note that, since $a_i < 0$,

$$2+a_i+1/a_i = \frac{(a_i+1)^2}{a_i} < 0.$$

# References


- Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, Jinan, 1986.
- Brenti, Unimodal, log-concave, and Pólya frequency sequences in combinatorics, Mem. Amer. Math. Soc., 1989.
- Brenti, Log-concave and unimodal sequences in algebra, combinatorics, and geometry: an update, 1994.
- Brändén, Unimodality, log-concavity, real-rootedness and beyond, 2015. (♦ Ann. of Math.; ♦ Invent. Math.; ♦ J. Amer. Math. Soc., 2009)

# Background

Recently, $\gamma$-positivity attracted attention after the work of

- Bränden (EJC, JAC, 2004) on $P$-Eulerian polynomials;
- Gal (DCG, 2005) on flag triangulations of spheres.

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- Bränden (EJC, JAC, 2004) on $P$-Eulerian polynomials;
- Gal (DCG, 2005) on flag triangulations of spheres.

$\gamma$-positivity polynomials arise often in enumerative, algebraic and geometric contexts. Refer to

- Athanasiadis, Gamma-positivity in combinatorics and geometry, Sem. Lothar. Combin. 77 (2018) 64pp.
- Petersen, Eulerian Numbers, Birkhaüser, 2015.

# Outline

# Eulerian polynomials

Let $S_n$ be the group of permutations of $[n] = \{1, 2, \ldots, n\}$.

For $\pi = \pi_1\pi_2 \ldots \pi_n \in S_n$, let

- $\mathrm{des}(\pi) := |\{i \in [n-1] : \pi_i > \pi_{i+1}\}|$
- $\mathrm{exc}(\pi) := |\{i \in [n-1] : \pi_i > i\}|$

be the number of descents and excedances of $\pi$, respectively.

## Eulerian polynomial

$$A_n(t) = \sum_{\pi \in S_n} t^{\mathrm{des}(\pi)} = \sum_{\pi \in S_n} t^{\mathrm{exc}(\pi)}.$$

# Properties of Eulerian polynomials

**Theorem**

*Eulerian polynomial $A_n(t)$*

- (Kurtz, JCTA 1972; Gasharov, JCTA 1998) *is symmetric and unimodal,*
- (Comtet, 1974; L.-Wang, AAM 2007) *has only real zeros.*

L., Wang, A unified approach to polynomial sequences with only real zeros, Adv. in Appl. Math. 39 (2007) 542–560.

# Properties of Eulerian polynomials

## Theorem (Foata-Strehl, 1976)

*Eulerian polynomial $A_n(t)$ can be written as*

$$A_n(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_{n,k} t^k (1+t)^{n-1-2k},$$

*where $\gamma_{n,k}$ is the number of $\pi \in S_n$ with $\mathrm{des}(\pi) = k$ for which*

- *there is no $i \in \{2, \ldots, n-1\}$ such that $\pi(i-1) > \pi(i) > \pi(i+1)$, $\pi(n-1) < \pi(n)$.*

Foata, Strehl, Euler numbers and variations of permutations. Colloquio Internazionale sulle Teorie Combinatorie (Roma, 1973), Tomo I, 119–131. Atti dei Convegni Lincei, No. 17, Accad. Naz. Lincei, Rome, 1976.

# Eulerian polynomials of type $B$

- Denote $B_n$ by the hyperoctahedral group on $[n]$.
- An element $\pi \in B_n$ is $\pi_1\pi_2\ldots\pi_n$, with $|\pi| = |\pi_1||\pi_2|\ldots|\pi_n| \in S_n$ and $\pi(-i) = -\pi_i$.
- The total order
$$-1 <_r -2 <_r -3 <_r \cdots <_r 0 <_r 1 <_r 2 <_r 3 <_r \cdots .$$
- $\mathrm{des}_B(\pi)$: # of $i \in \{0, 1, \ldots, n-1\}$ s.t. $\pi_i >_r \pi_{i+1}$, with $\pi_0 = 0$.

## Eulerian polynomial of type $B$

$$B_n(t) = \sum_{\pi \in B_n} t^{\mathrm{des}_B(\pi)}.$$

# Properties of Eulerian polynomials

**Theorem**

*Eulerian polynomial $B_n(t)$*

- (Brenti, EurJC 1994) *is symmetric and unimodal,*
- (Brenti, EurJC 1994; L.-Wang, AAM 2007) *has only real zeros.*
- (Peterson, AM 2007) *Eulerian polynomia of type B $B_n(t)$ can be written as*

$$B_n(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_{n,k} t^k (1+t)^{n-1-2k},$$

*where $\gamma_{n,k}$ is the number of $\pi \in B_n$ with $\mathrm{des}_B(\pi) = k$ such that*

- $|\pi| \in S_n$ *has $k$ descenting runs of size at least two.*

# Outline

# Involution

An involution is a permutation $\pi \in S_n$ such that $\pi^{-1} = \pi$.

Let $I_n$ be the set of all involutions in $S_n$ and let

$$I_n(t) = \sum_{\pi \in I_n} t^{\mathrm{des}(\pi)}.$$

**Theorem**

*The polynomial $I_n(t)$ is*

- *symmetric* (conjectured Dumout; proved Strehl, SLC 1980)
- *unimodal* (partially Dukes, 2007; fully Guo-Zeng, JCTA 2006)
- $\gamma$*-positive* (Wang, JCTA 2019)

- $I_n(t)$ is not LC (Marilena-Flavio-Matteo, EurJC 2009).

# Involution of hyperoctahedral group

- Denote $B_n$ by the hyperoctahedral group on $[n]$.
- Let $I_n^B$ be the set of all involutions in $B_n$ .
- $B_n$-analogue of $I_n(t)$ is

$$I_n^B(t) = \sum_{\pi \in I_n^B} t^{\mathrm{des}_B(\pi)}.$$

# Initial values

$$I_n^B(t) = \begin{cases} 1+t, & \text{if } n=1, \\ 1+4t+t^2, & \text{if } n=2, \\ 1+9t+9t^2+t^3, & \text{if } n=3, \\ 1+17t+40t^2+17t^3+t^4, & \text{if } n=4, \\ 1+28t+127t^2+127t^3+28t^4+t^5, & \text{if } n=5, \\ 1+43t+331t^2+634t^3+331t^4+43t^5+t^6, & \text{if } n=6. \end{cases}$$

**Theorem (Moustakas, GC 2019)**

*$I_n^B(t)$ is symmetric and unimodal for $n \geq 1$.*

- $I_n^B(t)$ is not LC (Moustakas, GC 2019).

# Initial values

$$I_n^B(t) = \begin{cases} 1+t, & \text{if } n=1, \\ (1+t)^2+2t, & \text{if } n=2, \\ (1+t)^3+6t(1+t), & \text{if } n=3, \\ (1+t)^4+13t(1+t)^2+8t^2, & \text{if } n=4, \\ (1+t)^5+23t(1+t)^3+48t^2(1+t), & \text{if } n=5, \\ (1+t)^6+37t(1+t)^4+168t^2(1+t)^2+56t^3, & \text{if } n=6. \end{cases}$$

**Theorem (Conjectured by Moustakas, GC 2019; Cao-L., $2020^+$)**

*The polynomial $I_n^B(t)$ is $\gamma$-positive for $n \geq 1$.*

# Proof

Let the expansion of $I_n^B(t)$ be

$$I_n^B(t) = \sum_{i=0}^{n} I_{n,i}^B t^i, \tag{1}$$

where $I_{n,i}^B$ is the number of involutions having $i$ descents of $B_n$.
Since the symmetry

$$I_{n,i}^B = I_{n,n-i}^B,$$

we rewrite $I_n^B(t)$ as follows

$$I_n^B(t) = \begin{cases} \sum_{i=0}^{(n-1)/2} I_{n,i}^B t^i (1 + t^{n-2i}), & \text{if } n \text{ is odd,} \\ I_{n,n/2}^B t^{n/2} + \sum_{i=0}^{n/2-1} I_{n,i}^B t^i (1 + t^{n-2i}), & \text{if } n \text{ is even.} \end{cases}$$

# Proof

Applying the well-known formula

$$x^n + y^n = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} (xy)^j (x+y)^{n-2j},$$

the polynomial $I_n^B(x)$ has the $\gamma$-expansion

$$I_n^B(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} C_{n,k} t^k (1+t)^{n-2k},$$

where

$$C_{n,k} = \begin{cases} \sum_{i=0}^{k} (-1)^{k-i} \frac{n-2i}{n-k-i} \binom{n-k-i}{k-i} I_{n,i}^B, & \text{if } n > 2k, \\ I_{n,k}^B + \sum_{i=0}^{k-1} (-1)^{k-i} \frac{n-2i}{n-k-i} \binom{n-k-i}{k-i} I_{n,i}^B, & \text{if } n = 2k. \end{cases}$$

# Proof

Set $x = x(t) = t/(1+t)^2$ and

$$I_n^B(t) = (1+t)^n P_n(x).$$

Then

$$P_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} C_{n,k} x^k.$$

## Theorem

*For $n \geq 3$ and $k \geq 0$, the numbers $C_{n,k}$ satisfy*

$$\begin{aligned} nC_{n,k} &= (2k+1)C_{n-1,k} + 4(n-2k+1)C_{n-1,k-1} \\ &+(2k^2+2k+n-1)C_{n-2,k} \\ &+4\big(2k(n-2k)+k-n+1\big)C_{n-2,k-1} \\ &+8(n-2k+1)(n-2k+2)C_{n-2,k-2}, \end{aligned}$$

*where $C_{n,k} = 0$ if $k < 0$ or $k > n/2$.*

# Problems

**Problem (Moustakas, GC 2019)**

*Find the explicit combinatorial interpretations of the $\gamma$-coefficients of $I_n(t)$ and $I_n^B(t)$.*

# Fixed-point free involution

Let $J_n$ be the set of all fixed-point free involutions in $S_n$. Define

$$J_n(t) = \sum_{\pi \in J_n} t^{\mathrm{des}(\pi)}.$$

**Theorem (Conjectured by Guo-Zeng, JCTA 2006; Proved by Wang, JCTA 2019)**

*The polynomial $J_{2n}(t)$ is $\gamma$-positive.*

# Fixed-point free involution in $B_n$

Let $J_n^B$ be the set of all fixed-point free involutions in $B_n$. Define

$$J_n^B(t) = \sum_{\pi \in J_n^B} t^{\mathrm{des}_B(\pi)}.$$

$$J_n^B(t) = \begin{cases} t(1+t), & \text{if } n = 2, \\ t(1+t)^3 + 2t^2(1+t), & \text{if } n = 4, \\ t(1+t)^5 + 8t^2(1+t)^3 + 12t^3(1+t), & \text{if } n = 6, \\ t(1+t)^7 + 19t^2(1+t)^5 + 100t^3(1+t)^3 & \\ \qquad +72t^4(1+t), & \text{if } n = 8. \end{cases}$$

# Problems

**Problem (Cao-L., $2020^+$)**

*If*

$$J^B_{2n}(t) = \sum D^B_{2n,k} t^k (1+t)^{2n+1-2k},$$

*then* $D^B_{2n,k} \geq 0$ *for* $n \geq 1$.

- Main problem:

$$\sum_{n\geq 0} J^B_n(t) \frac{x^n}{(1-t)^{n+1}} = \sum_{m\geq 0} \frac{t^m}{(1-x^2)^{m^2}}.$$

# Outline

# Derangement polynomials

- A permutation $\pi \in S_n$ is called a derangement if $\pi_i \neq i$ for all $i \in [n]$.
- Let $D_n$ denote the set of all derangements of $S_n$.
- Derangement polynomials is

$$d_n(t) = \sum_{\pi \in D_n} t^{\mathrm{exc}(\pi)} = \sum_{k=0}^{n} d_{n,k} t^k,$$

for $n \geqslant 1$ and $d_0(t) = 1$.

**Theorem (Brenti, PAMS 1990)**

*The derangement polynomial $d_n(t)$ is symmetric and unimodal.*

# $\gamma$-positivity of $d_n(t)$

**Theorem (Athanasiadis-Savvidou, SLC 2012)**

*The derangement polynomial $d_n(t)$ can be written as*

$$d_n(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} \xi_{n,k} t^k (1+t)^{n-2k},$$

*where $\xi_{n,k}$ is the number of $\pi \in S_n$ with $\mathrm{des}(\pi) = k-1$ for which*

- *there is no $i \in \{2, \ldots, n-1\}$ s.t. $\pi_{i-1} > \pi_i > \pi_{i+1}$,*
- *$\pi_1 < \pi_2$ and $\pi_{n-1} < \pi_n$.*

# Derangement polynomials of type $B$

Let $D_n^B$ denote the set of all derangements of $B_n$.

**Definition (Brenti, EurJC 1994)**

We say that $i \in [n]$ is a type $B$ excedance of $\pi \in B_n$ if $\pi_i = -i$ or $\pi_{|\pi_i|} > \pi_i$.

We denote $\mathrm{exc}_B(\pi)$ by the number of type $B$ excedances of $\pi$.

**Definition (Chen-Tang-Zhao, EJC 2009)**

$B$ analogue of the derangement polynomial is defined by

$$d_n^B(t) = \sum_{\pi \in D_n^B} t^{\mathrm{exc}_B(\pi)} = \sum_{k=0}^{n} d_{n,k}^B t^k$$

for $n \geqslant 1$, where $d_{n,k}$ is the number of signed derangements in $D_n^B$ with exactly $k$ excedances of type $B$. Set $d_0^B(x) = 1$.

# Properties of $d_n^B(t)$

**Theorem (Chen-Tang-Zhao, EJC 2009)**

*The derangement polynomial $d_n^B(t)$ has*

- *the reality of zeros;*
- *the unimodality;*
- *the asymptotic normality;*
- *the spiral property.*

# Wreath product

- The wreath product $\mathcal{C}_r \wr S_n$ is the usual action of a cyclic group $\mathcal{C}_r = \{0, 1, \ldots, r-1\}$ by a permutation of $S_n$.
- Elements in $\mathcal{C}_r \wr S_n$ are called $r$-colored permutations, and represented by
$$\pi^\xi = ({\pi_1}^{\xi_1}, {\pi_2}^{\xi_2}, \ldots, {\pi_n}^{\xi_n}),$$
where $\pi = \pi_1\pi_2\ldots\pi_n \in S_n$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathcal{C}_r^n$.
- The descent set and the excedance set of $\pi^\xi$ are defined by
$$\begin{aligned}\mathrm{Des}(\pi^\xi) &= \{i \in [n] \mid \xi_i > \xi_{i+1} \text{ or } \xi_i = \xi_{i+1} \text{ and } \pi_i > \pi_{i+1}\}\\ \mathrm{Exc}(\pi^\xi) &= \{i \in [n] \mid \pi_i > i \text{ or } \pi_i = i \text{ and } \xi_i > 0\},\end{aligned}$$
with $\pi_{n+1} = n+1$ and $\xi_{n+1} = 0$.
- Let $\mathrm{des}(\pi^\xi) = |\mathrm{Des}(\pi^\xi)|$ and $\mathrm{exc}(\pi^\xi) = |\mathrm{Exc}(\pi^\xi)|$.

**Example**

If $\pi^\xi = (4^3, 2^0, 6^3, 1^3, 5^1, 3^2) \in \mathcal{C}_4 \wr S_6$, then $\pi^\xi$ has descents at $1, 3, 4, 6$ and excedences at $1, 3, 5$. So $\mathrm{des}(\pi^\xi) = 4$, $\mathrm{exc}(\pi^\xi) = 3$.

# Cyclic derangement polynomial

- A fixed point of $\pi^\xi \in \mathcal{C}_r \wr S_n$ is an integer $i \in [n]$ such that $\pi_i = i$ and $\xi_i = 0$.
- A permutation $\pi^\xi \in \mathcal{C}_r \wr S_n$ is called a cyclic derangement if it has no fixed points.
- Let $D_n^{(r)}$ denote the set of all cyclic derangements of $\mathcal{C}_r \wr S_n$ and denote $d_n^{(r)} = |D_n^{(r)}|$.

## Definition (Chow, JCTA 2009)

The cyclic derangement polynomial is defined by

$$d_n^{(r)}(x) = \sum_{\pi^\xi \in D_n^{(r)}} x^{\mathrm{exc}(\pi^\xi)} = \sum_{k=0}^{n} d_{n,k}^{(r)} x^k$$

for $n \geq 1$, where $d_{n,k}^{(r)}$ is the number of cyclic derangements in $\mathcal{C}_r \wr S_n$ with exactly $k$ excedances.

# Initial values

$$d_n^{(r)}(t) = \begin{cases} (r-1)t, & \text{if } n=1, \\ r^2 t + (r-1)^2 t^2, & \text{if } n=2, \\ r^3 t + r^2(4r-3)t^2 + (r-1)^3 t^3, & \text{if } n=3, \\ r^4 t + r^3(11r-4)t^2 + r^2(11r^2-16r+6)t^3 & \\ +(r-1)^4 t^4, & \text{if } n=4, \\ r^5 t + r^4(26r-9t^2 + r^3(66r^2-55r+10)t^3 & \\ +r^2(26r^3-55r^2+40r+10)t^4 + (r-1)^5 t^5, & \text{if } n=5. \end{cases}$$

In particular,

- $d_n^{(1)}(x) = d_n(x)$ the derangement polynomials.
- $d_n^{(2)}(x) = d_n^B(x)$ the derangement polynomials of type $B$.

# Unimodality

**Theorem (Steingrímsson, MIT 1992; Chow-Mansour, IJM 2010)**

*For $n \geq 2$, the cyclic derangement polynomial $d_n^{(r)}(x)$ has only real zeros and zeros of $d_n^{(r)}(x)$ are separated by zeros of $d_{n-1}^{(r)}(x)$.*

**Corollary**

*The cyclic derangement polynomials $d_n^{(r)}(x)$ is unimodal for $r \geq 1$.*

# Spiral property

**Theorem (L.-Dong, DM 2020)**

*The cyclic derangement polynomials $d_n^{(r)}(x)$ possess the spiral property for $r \geq 2$. Precisely speaking, for $n, r \geq 2$, coefficients of $d_n^{(r)}(x)$ satisfy*

$$d_{n,n}^{(r)} < d_{n,1}^{(r)} < d_{n,n-1}^{(r)} < \cdots < d_{n,\lceil \frac{n+3}{2} \rceil}^{(r)} < d_{n,\lfloor \frac{n-1}{2} \rfloor}^{(r)} < d_{n,\lceil \frac{n+1}{2} \rceil}^{(r)},$$

*and*

$$d_{n,\frac{n}{2}+1}^{(r)} < d_{n,\frac{n}{2}}^{(r)}$$

*if $n$ is even.*

L., Dong, Cyclic derangement polynomials of the wreath product $\mathcal{C}_r \wr S_n$, Discrete Math. 343 (2020) 112109.

Thank you for your attention!