Combinatorics and Topology of Posets

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Matroid

- Let E be a finite set and $\mathcal{I} \subset 2^E$. A matroid M is an ordered pair $(E, \mathcal{I})$ satisfying
  (1) $\emptyset \in \mathcal{I}$;
  (2) If $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$; (hereditary property)
  (3) If $A, B \in \mathcal{I}$ and $|B| > |A|$, then $\exists\, x \in B$ such that $A \cup x \in \mathcal{I}$. (exchange property)

  The set E is said to be the ground set. A set $A \in \mathcal{I}$ is called an independent set. ($\Rightarrow$ dependent set)
- For any set $A \subset E$, define the rank of A as the cardinality of its maximal independent set, denoted by $r(A)$.

  - A set $A \subset E$ is a flat if $\mathrm{cl}(A) = A$, where $\mathrm{cl}(A) := \{x : r(A \cup x) = r(A)\}$.
  - The flats of a matroid M form a geometric lattice under inclusion and we denote it by $L(M)$.

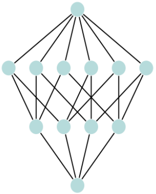


Figure: $L(U_{1,3})$

Orlik-Solomon algebra of a matroid

- Let M be a simple matroid with ground set $\mathcal{I} = \{1, \ldots, n\}$.
- Let $\mathcal{E} = \wedge(e_1 \ldots, e_n)$ be the graded exterior algebra on elements e_i of degree one corresponding to the points of M. $(\Rightarrow \mathcal{E} = \bigoplus_{p=0}^{n} \mathcal{E}^p)$
- Define the linear mapping $\partial : \mathcal{E}^P \to \mathcal{E}^{P-1}$ by

$$\partial(e_{i_1} \wedge \ldots \wedge e_{i_p}) := \sum_{k=1}^{p} (-1)^{k-1} e_{i_1} \wedge \ldots \wedge \widehat{e_{i_k}} \wedge \ldots \wedge e_{i_p}.$$

- If $S = (i_1, \ldots, i_p)$ is an ordered p-tuple, we denote the product by e_S.
- Let $\mathcal{J}$ denote the ideal of $\mathcal{E}$ generated by $\{\partial e_S | S \text{ is depedent}\}$.
- The Orlik-Solomon algebra of M is the quotient $\mathcal{E}/\mathcal{J}$. $(\Rightarrow$ graded)

M. Falk. Combinatorial and Algebraic Structure in Orlik–Solomon Algebras. *Europ. J. Combinatorics.* 2001, **22**: 687-698.

Equivariant characteristic polynomial

- Let W be a finite group acting on $\mathcal{I}$ and preserving M. We will refer to this collection of data as an equivariant matroid $M \curvearrowright W$.
- Given an equivalent matroid $M \curvearrowright W$, Gedeon, Proudfoot and Young first defined the equivariant characteristic polynomial

$$H_M^W(t) := \sum_{i=0}^{\operatorname{rk} M} (-1)^i t^{\operatorname{rk} M - i} OS_{M,i}^W \in \operatorname{grVRep}(W).$$

where $OS_{M,i}^W \in \operatorname{Rep}(W)$ is the degree i part of the Orlik-Solomon algebra of M.

K. Gedeon, N. Proudfoot, and B. Young. The equivariant Kazhdan-Lusztig polynomial of a matroid. *J. Combin. Theory Ser. A.* 2017, **150**: 267-294.

Equivariant Kazhdan-Lusztig polynomial

Theorem

There is a unique way to assign to each equivariant matroid $M \curvearrowright W$ an element $P_M^W(t) \in \mathrm{grVRep(W)}$, called the equivariant Kazhdan-Lusztig polynomial, such that the following conditions are satisfied:

(1). If $\mathrm{rk}\ M = 0$, then $P_M^W(t) = \rho_\emptyset$, where $\rho_\emptyset$ is the trivial representation in degree 0.

(2). If $\mathrm{rk}\ M > 0$, then $\deg\ P_M^W(t) < \frac{1}{2}\ \mathrm{rk}\ M$.

(3). For every M,

$$t^{\mathrm{rk}\ M} P_M^W(t^{-1}) = \sum_{[F] \in L(M)/W} \mathrm{Ind}_{W_F}^W \left(H_{M_F}^{W_F}(t) \otimes P_{M^F}^{W_F}(t) \right), \tag{1}$$

where $W_F \subset W$ is the stabilizer of F.

(4). Given a homomorphism $\varphi : W' \to W$, $P_M^{W'}(t) = \varphi^* P_M^W(t)$.

Abstract simplicial complex

- An abstract simplicial complex Δ on finite vertex set V is a nonempty collection of subsets of V such that
  (1) $\{v\} \in \Delta$ for all $v \in V$;
  (2) If $G \in \Delta$ and $F \subset G$, then $F \in \Delta$.
- The elements of Δ are called faces (or simplices) of Δ and the maximal faces are called facets.
- We say that a face F has dimension d and write $\dim F = d$ if $d = |F| - 1$. Faces of dimension d are referred to as d-faces.
- We refer to 0-dimensional faces as vertices and to 1-dimensional faces as edges.
- Empty simplicial complex: the (-1)-dimensional complex $\{\emptyset\}$. Degenerate empty complex: the empty set $\emptyset$. We say that it has dimension -2.

M. Wachs. Poset topology: tools and applications. arxiv.org/pdf/math/0602226, 2006.

J. Jonsson. Introduction to simplicial homology. https://people.kth.se/ jakobj/doc/homology/homology.pdf

Geometric realization: rough procedure

- One may realize a simplicial complex as a geometric object in $\mathbb{R}^n$, and the procedure is roughly the following.
  (1) Identify each vertex with a point.
  (2) For each edge ab, draw a line segment between the points realizing the vertices a and b.
  (3) Next, for each 2-dimensional face abc, fill the triangle with sides given by the line segments realizing ab, ac, and bc.
  (4) Continue in this manner in higher dimensions....

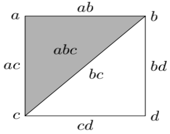


Figure: Geometric realization of E_1, where E_1....

- Note that the full realization of an abstract simplicial complex is determined by how we realize the vertices of the complex.

Geometric realization: formal definition

- Let Δ be an abstract simplicial complex with vertex set V and let $f : V \to \mathbb{R}^n$ be any map.
- For any $d \geq 0$, define the standard d-simplex to be the set

$$X_d := \{(\lambda_0, \ldots, \lambda_d) : \lambda_i \geq 0 \text{ for } 0 \leq i \leq d, \lambda_0 + \cdots + \lambda_d = 1\} \subset \mathbb{R}^{d+1}.$$

- For any nonempty d-face $\sigma = \{a_0, \ldots, a_d\}$ of Δ, we have that f induces a map $f_\sigma : X_d \to \mathbb{R}^n$ by

$$f_\sigma(\lambda_0, \ldots, \lambda_d) := \lambda_0 f(a_0) + \cdots + \lambda_d f(a_d).$$

- We say that f induces a geometric realization of Δ if the following hold:
  (1) The map f_σ is injective for each $\sigma \in \Delta \setminus \{\emptyset\}$;
  (2) For any nonempty $\sigma, \tau \in \Delta$, we have that

$$\mathrm{im}\, f_\sigma \cap \mathrm{im}\, f_\tau = \mathrm{im}\, f_{\sigma\cap\tau}.$$

- The actual geometric realization is the union

$$\|\Delta\| = \bigcup_{\sigma \in \Delta\setminus\{\emptyset\}} \mathrm{im}\, f_\sigma.$$

Chain group $\tilde{C}_n(\Delta, \mathbb{F})$

Let $\mathbb{F}$ be a commutative ring and Δ be a simplicial complex.

- For each $n \geq -1$, we form a free $\mathbb{F}$-module $\tilde{C}_n(\Delta; \mathbb{F})$ with a basis indexed by the n-dimensional faces of Δ.
- Specifically, for each face $a_0 a_1 \cdots a_n$, we have a basis element $e_{a_0 a_1 \cdots a_n}$. We refer to a basis element as an oriented simplex.
- We refer to $\tilde{C}_n(\Delta; \mathbb{F})$ is the chain group of degree n.

Running examples. For $E_1 = \{\emptyset, a, b, c, d, ab, ac, bc, bd, cd, abc\}$, we get that

$$
\begin{aligned}
\tilde{C}_{-1}(E_1) &= \{\lambda \mathbf{e}_\emptyset : \lambda \in \mathbb{F}\} \cong \mathbb{F}, \\
\tilde{C}_0(E_1) &= \{\lambda_a \mathbf{e}_a + \lambda_b \mathbf{e}_b + \lambda_c \mathbf{e}_c + \lambda_d \mathbf{e}_d : \lambda_a, \lambda_b, \lambda_c, \lambda_d \in \mathbb{F}\} \cong \mathbb{F}^4, \\
\tilde{C}_1(E_1) &= \{\lambda_{ab} \mathbf{e}_{a,b} + \cdots + \lambda_{cd} \mathbf{e}_{c,d} : \lambda_{ab}, \ldots, \lambda_{cd} \in \mathbb{F}\} \cong \mathbb{F}^5, \\
\tilde{C}_2(E_1) &= \{\lambda \mathbf{e}_{a,b,c} : \lambda \in \mathbb{F}\} \cong \mathbb{F}.
\end{aligned}
$$

- For $n \geq 0$, we write $a_0 \wedge a_1 \ \cdots \wedge a_n = e_{a_0 a_1 \cdots a_n}$ to denote oriented simplices. The symbol $\wedge$ denotes exterior product,

$$b \wedge a = -a \wedge b;$$

$$a \wedge a = 0$$

- In degree -1, we stick to the notation $e_\emptyset$.

Simplicial chain complex

- We define boundary map ∂_n on a given oriented simplex $a_0 \wedge a_1 \ \cdots \wedge a_n$ by

$$\partial_n(a_0 \wedge a_1 \ \cdots \wedge a_n) := \sum_{r=0}^{n} (-1)^r a_0 \wedge \cdots \wedge a_{r-1} \wedge \hat{a}_r \wedge a_{r+1} \wedge \cdots \wedge a_n$$

for each n, where $\hat{a}_r$ denotes removal of the element a_r.
- In the special case $n = 0$, we let $\partial_0(a) = e_\emptyset$ for each vertex a.
- To obtain a homomorphism, we extend ∂_n linearly to the whole of $\tilde{C}_n(\Delta)$.
- We have that $\partial_n \partial_{n+1} = 0$ for every n.
- The sequence

$$\mathsf{C}(\Delta) : \cdots \xrightarrow{\partial_{n+2}} \tilde{C}_{n+1}(\Delta) \xrightarrow{\partial_{n+1}} \tilde{C}_n(\Delta) \xrightarrow{\partial_n} \tilde{C}_{n-1}(\Delta) \xrightarrow{\partial_{n-1}} \cdots$$

defines a chain complex. We refer to $C(\Delta)$ as a simplicial chain complex.

Running example 1. The complex E_1 has dimension 2, which means that we get the following simplicial chain complex:

$$\mathsf{C}(E_1) : 0 \longrightarrow \tilde{C}_2(E_1) \xrightarrow{\partial_2} \tilde{C}_1(E_1) \xrightarrow{\partial_1} \tilde{C}_0(E_1) \xrightarrow{\partial_0} \tilde{C}_{-1}(E_1) \longrightarrow 0.$$

Simplicial homology

- We define the $\mathbb{F}$-module $Z_n(\Delta; \mathbb{F})$ of cycles and the $\mathbb{F}$-module $B_n(\Delta; \mathbb{F})$ of boundaries by the following formulas:

$$\begin{aligned} Z_n(\Delta; \mathbb{F}) &= \ker \partial_n \\ &= \{z \in \tilde{C}_n(\Delta, \mathbb{F}) : \partial_n(z) = 0\}, \\ B_n(\Delta; \mathbb{F}) &= \operatorname{im} \partial_{n+1} \\ &= \{z \in \tilde{C}_n(\Delta, \mathbb{F}) : z = \partial_{n+1}(x) \text{ for some } x \in \tilde{C}_{n+1}(\Delta, \mathbb{F})\}. \end{aligned}$$

- We define the simplicial homology in degree n of Δ to be the quotient

$$\tilde{H}_n(\Delta, \mathbb{F}) := Z_n(\Delta; \mathbb{F}) / B_n(\Delta; \mathbb{F}).$$

Running example 1. The complex E_1 has dimension 2, which means that we get the following simplicial chain complex

$$\mathsf{C}(E_1) : 0 \longrightarrow \tilde{C}_2(E_1) \xrightarrow{\partial_2} \tilde{C}_1(E_1) \xrightarrow{\partial_1} \tilde{C}_0(E_1) \xrightarrow{\partial_0} \tilde{C}_{-1}(E_1) \longrightarrow 0.$$

Special cases

- The empty simplicial complex $\{\emptyset\}$. We only have

$$\tilde{C}_{-1}(\Delta;\mathbb{C}) = \mathbb{C}\cdot e_\emptyset.$$

Then $Z_{-1}(\Delta;\mathbb{C}) = \ker\partial_{-1} = \mathbb{C}\cdot e_\emptyset$ and $B_{-1}(\Delta;\mathbb{C}) = \mathrm{Im}\,\partial_0 = 0$. Hence

$$\tilde{H}_{-1}(\Delta;\mathbb{C}) \cong \mathbb{C}.$$

- The degenerate empty complex $\emptyset$.
- The reduced homology and unreduced homology.
  (1) Reduced homology $\tilde{H}_n(\Delta,\mathbb{F})$: $\tilde{C}_{-1}(\Delta,\mathbb{F}) = \mathbb{F}\cdot e_\emptyset$ and ∂_0 is defined by $\partial_0(a) = e_\emptyset$ for each vertex a.
  (2) Unreduced homology $H_n(\Delta,\mathbb{F})$: $C_{-1}(\Delta,\mathbb{F}) = 0$ and ∂_0 is defined to be the zero map.
  (3) In all degrees $n \geq 1$, we always have that $H_n(\Delta,\mathbb{F}) = \tilde{H}_n(\Delta,\mathbb{F})$.

Order complex of a poset

- To every poset P, one can associate an abstract simplicial complex $\Delta(P)$ called the order complex of P.
- The vertices of $\Delta(P)$ are the elements of P and the faces of $\Delta(P)$ are the chains (i.e., totally ordered subsets) of P.

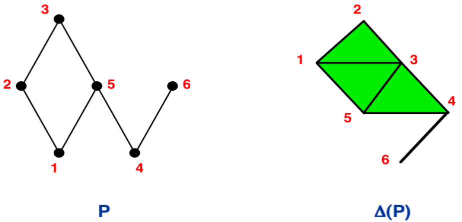


Figure 1.1.1. Order complex of a poset

- The order complex of the empty poset is the empty simplicial complex $\{\emptyset\}$.

Face lattice of a simplicial complex

- To every simplicial complex Δ one can associate a poset $P(\Delta)$ called the face poset of Δ, which is defined to be the poset of nonempty faces ordered by inclusion.
- The face lattice $L(\Delta)$ is $P(\Delta)$ with a smallest element $\hat{0}$ and a largest element $\hat{1}$ attached.

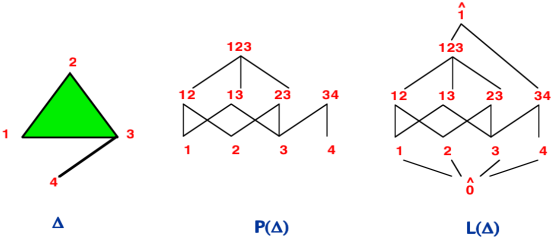


Figure 1.1.2. Face poset and face lattice of a simplicial complex

Poset homology

By (co)homology of a poset, we usually mean the reduced simplicial (co)homology of its order complex.

- For each poset P and integer j, where $j \geq -1$, define the chain space
  $$C_j(P;\mathbf{k}) := \mathbf{k}\text{-module freely generated by } j\text{-chains of } P,$$
  where $\mathbf{k}$ is a field or the ring of integers.
- The boundary map $\partial_j := C_j(P;\mathbf{k}) \to C_{j-1}(P;\mathbf{k})$ is defined by
  $$\partial_j(x_1 < \ldots < x_{j+1}) = \sum_{i=1}^{j+1}(-1)^i(x_1 < \ldots < \hat{x}_i < \ldots < x_{j+1}),$$
  where the $\hat{}$ denotes deletion.
- We have that $\partial_{j-1}\partial_j = 0$, which makes $(C_j(P;\mathbf{k}), \partial_j)$ an algebraic complex.
- Define the cycle space $Z_j(P;\mathbf{k}) := \ker \partial_j$ and the boundary space $B_j(P;\mathbf{k}) := \mathrm{im}\ \partial_{j+1}$.
- Homology of the poset P in dimension j is defined by
  $$\widetilde{H}_j(P;\mathbf{k}) := Z_j(P;\mathbf{k})/B_j(P;\mathbf{k}).$$

Poset cohomology

- The coboundary map $\delta_j := C_j(P;\mathbf{k}) \to C_{j+1}(P;\mathbf{k})$ is defined by

$$\delta_j(x_1 < \ldots < x_j) = \sum_{i=1}^{j+1}(-1)^i \sum_{x\in(x_{i-1},x_i)} (x_1 < \ldots < \hat{x}_i < \ldots < x_{j+1})$$

- Define the cocycle space to be $Z^j(P;\mathbf{k}) := \ker \partial_j$ and the coboundary space to be $B^j(P;\mathbf{k}) := \operatorname{im} \partial_{j-1}$.
- Cohomology of the poset P in dimension j is defined by

$$\widetilde{H}^j(P;\mathbf{k}) := Z^j(P;\mathbf{k})/B^j(P;\mathbf{k}).$$

- When $\mathbf{k}$ is a field, $\widetilde{H}^j(P;\mathbf{k})$ and $\widetilde{H}_j(P;\mathbf{k})$ are isomorphic vector spaces.

Reduced Euler characteristic

- The reduced Euler characteristic $\tilde{\chi}(\Delta)$ of a simplicial complex Δ is defined to be

$$\tilde{\chi}(\Delta) := \sum_{i=-1}^{\dim \Delta} (-1)^i f_i(\Delta),$$

where $f_i(\Delta)$ is the number of i-faces of Δ.

Theorem (Euler-Poincaré formula.)

For any simplicial complex Δ,

$$\tilde{\chi}(\Delta) = \sum_{i=-1}^{\dim \Delta} (-1)^i \tilde{\beta}_i(\Delta),$$

where $\tilde{\beta}_i(\Delta)$ is the i-th reduced Betti number of Δ, i.e., the rank, as an abelian group, of the i-th reduced homology of Δ over $\mathbb{Z}$.

Reduced Euler characteristic of $\Delta(P)$

- The j-th (reduced) Betti number of P is given by

$$\tilde{\beta}_j(P) = \dim \tilde{H}_j(P; \mathbb{C}),$$

which is the same as the rank of the free part of $\tilde{H}_j(P; \mathbb{Z})$.

- For $|P| \geq 0$, we have

$$\tilde{\chi}(\Delta(P)) = \sum_{i=-1}^{\dim \Delta(P)} (-1)^i \tilde{\beta}_i\Big(\Delta(P)\Big) = \sum_{i=-1}^{\dim \Delta(P)} (-1)^i \dim \tilde{H}_i(\Delta(P); \mathbb{C}).$$

- It follows from the Euler-Poincaré formula that the Euler characteristic is a topological invariant.

- Need to consider: What's the reduced Euler characteristic of the degenerate empty complex $\emptyset$? It should be 1.

Bounded poset

- A poset P is said to be bounded if it has a top element $\hat{1}$ and a bottom element $\hat{0}$.
- Define the proper part of P, for which $|P| \geq 2$, to be

$$\bar{P} := P - \{\hat{0}, \hat{1}\}.$$

  If $|P| = 1$, define $\Delta(\bar{P})$ to be the degenerate empty complex $\emptyset$. We also say $\Delta((x, y)) = \emptyset$ and $\ell(x, y) = -2$ if $x = y$.
- Given a poset P, we define the bounded extension

$$\hat{P} := P \cup \{\hat{0}, \hat{1}\},$$

  where new elements $\hat{0}$ and $\hat{1}$, are adjoined (even if P already has a bottom or top element).

Möbius function

- The Möbius function $\mu_P (= \mu)$ of a poset P defined recursively on closed intervals of P as follows:

$$\mu(x,x) = 1, \qquad \text{for all } x \in P$$

$$\mu(x,y) = -\sum_{x \leq z < y} \mu(x,z), \quad \text{for all } x < y \in P$$

For a bounded poset P, define the Möbius invariant

$$\mu(P) := \mu_P(\hat{0}, \hat{1}).$$

Philip Hall Theorem

Theorem (Philip Hall Theorem)

For any poset P,

$$\mu(\hat{P}) = \tilde{\chi}(\Delta(P)).$$

- Given a bounded poset P with $|P| \geq 2$, we have

$$\mu(P) = \tilde{\chi}(\Delta(\bar{P})).$$

Thus, from $\tilde{\chi}(\Delta(P)) = \sum_{i=-1}^{\dim \Delta(P)} (-1)^i \dim \tilde{H}_i(\Delta(P); \mathbb{C})$, we have

$$\mu(P) = \sum_{i=-1}^{\dim \Delta(\bar{P})} (-1)^i \dim \tilde{H}_i(\Delta(\bar{P}); \mathbb{C}).$$

- $\mu_P(x, y)$ depends only on the topology of the open interval (x, y) of P.

Characteristic function

- A weak rank function r on $\mathrm{Int}(P)$, is a function satisfying the following conditions:
  (1) $r_{xy} \in \mathbb{Z}$ for all $x \leq y \in P$,
  (2) if $x < y$, then $r_{xy} > 0$,
  (3) if $x \leq y \leq z$, then $r_{xy} + r_{yz} = r_{xz}$.
- The characteristic function $\chi_P\ (= \chi)$ of a poset P defined recursively on closed intervals of P as follows:

$$\chi(x,z) = \sum_{x \leq y \leq z} \mu(x,y) t^{r_{yz}}, \quad \text{for all } x \leq z \in P$$

  where r is a weak rank function.
- For $x < y$ in P, we write $\tilde{H}_j(x,y)$ for the complex homology of the open interval (x,y) of P;
- When $x = y$, define $\tilde{H}_j(x,y)$ to be $\mathbb{C}$ and $\dim \tilde{H}_j(x,y)$ to be 1 if $j = -2$, and to be 0 for all other j.

Characteristic function

For a bounded poset P with $|P| \geq 2$, since

$$\mu(P) = \sum_{i=-1}^{\dim \Delta(\bar{P})} (-1)^i \dim \tilde{H}_i(\bar{P}; \mathbb{C})$$

$$\Rightarrow \; \mu(\hat{0}, x) = \sum_{j=-1}^{\dim \Delta(\overline{P_{\hat{0},x}})} (-1)^j \dim \tilde{H}_j\big(\hat{0}, x\big) \text{ for } x \neq \hat{0};$$

$$\chi(P) = \sum_{\hat{0} \leq x \leq \hat{1}} \mu(\hat{0}, x) t^{r_{x,\hat{1}}} = \sum_{i=0}^{r_{\hat{0}\hat{1}}} \Big(\sum_{r_{\hat{0},x}=i} \mu(\hat{0}, x) \Big) t^{r_{\hat{0}\hat{1}}-i}$$

$$= \sum_{i=1}^{r_{\hat{0}\hat{1}}} \Big(\sum_{r_{\hat{0},x}=i} \sum_{j=-1}^{\dim \Delta(\overline{P_{\hat{0},x}})} (-1)^j \dim \tilde{H}_j\big(\hat{0}, x\big) \Big) t^{r_{\hat{0}\hat{1}}-i} + \mu(\hat{0}, \hat{0}) t^{r_{\hat{0}\hat{1}}}.$$

Group representation

We restrict our discussion to finite groups G and finite dimensional vector spaces over the field $\mathbb{C}$.

- A finite dimensional vector space V over $\mathbb{C}$ is said to be a representation of G if there is a group homomorphism

$$\phi : G \to GL(V).$$

- For $g \in G$ and $v \in V$, we write gv instead of $\phi(g)(v)$ and view V as a module over the ring $\mathbb{C}G$ (G-module for short).
- The dimension of the representation V is defined to be the dimension of V as a vector space.
- We say that V_1 and V_2 are isomorphic representations of G and write $V_1 \cong_G V_2$ if there is a vector space isomorphism $\psi : V_1 \to V_2$ such that

$$\psi(gv) = g\psi(v).$$

Hopf trace formula

- A *G-poset* is a poset together with a G-action on its elements that preserves the partial order; i.e., $x < y \Rightarrow gx < gy$.
- A *G-simplicial complex* is a simplicial complex together with an action of G on its vertices that takes faces to faces.
- If P is a G-poset then its order complex $\Delta(P)$ is a G-simplicial complex, and if Δ is a G-simplicial complex then its face poset $P(\Delta)$ is a G-poset.

Theorem (Hopf trace formula)

For any G-simplicial complex Δ,

$$\bigoplus_{i=-1}^{\dim \Delta} (-1)^i C_i(\Delta; \mathbb{C}) \cong_G \bigoplus_{i=-1}^{\dim \Delta} (-1)^i \tilde{H}_i(\Delta; \mathbb{C}).$$

Question 1

For a bounded poset P, recall that

$$\mu(P) = \sum_{i=-1}^{\dim \Delta(\bar{P})} (-1)^i \dim \tilde{H}_i(\bar{P}; \mathbb{C}) \quad \text{for} \quad |\mathrm{P}| \geq 2;$$

$$\mu(P) = 1 \quad \text{for} \quad |\mathrm{P}| = 1.$$

Question: Can we use

$$\bigoplus_{i=-1}^{\dim \Delta(\bar{P})} (-1)^i \tilde{H}_i(\Delta(\bar{P}); \mathbb{C}) \quad \text{for} \quad |\mathrm{P}| \geq 2;$$

$$\mathbb{C} \quad \text{for} \quad |\mathrm{P}| = 1.$$

to denote the equivariant version of the Möbius function μ_P?

S. Assaf and S. David. Specht modules decompose as alternating sums of restrictions of Schur modules. *Proc. Amer. Math. Soc.* 2020. **148**(3): 1015-1029.

Question 2

For a bounded poset P, recall that

$$\chi(P) = \sum_{i=1}^{r_{\hat{0}\hat{1}}} \Big(\sum_{r_{\hat{0},x}=i} \sum_{j=-1}^{\dim \Delta(\overline{P_{\hat{0},x}})} (-1)^j \dim \tilde{H}_j(\hat{0}, x) \Big) t^{r_{\hat{0}\hat{1}}-i} + t^{r_{\hat{0}\hat{1}}} \quad \text{for} \quad |P| \geq 2;$$

$$\Big(\text{ or } \sum_{i=1}^{r_{\hat{0}\hat{1}}} \Big(\sum_{j=-1}^{\dim \Delta(\overline{P_{\hat{0},\hat{1}}})} (-1)^j \sum_{r_{\hat{0},x}=i} \dim \tilde{H}_j(\hat{0}, x) \Big) t^{r_{\hat{0}\hat{1}}-i} + t^{r_{\hat{0}\hat{1}}} \Big)$$

$$1 \qquad \text{for} \quad |P| = 1.$$

Question: Can we use

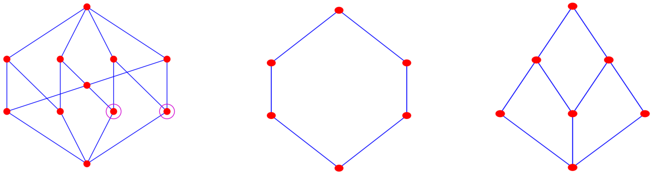
$$\sum_{i=1}^{r_{\hat{0}\hat{1}}} \Big(\bigoplus_{r_{\hat{0},x}=i} \bigoplus_{j=-1}^{\dim \Delta(\overline{P_{\hat{0},x}})} (-1)^j \tilde{H}_j(\hat{0}, x) \Big) t^{r_{\hat{0}\hat{1}}-i} + \mathbb{C}\, t^{r_{\hat{0}\hat{1}}} \quad \text{for} \quad |P| \geq 2;$$

$$\mathbb{C} \qquad \text{for} \quad |P| = 1.$$

to denote the equivariant version of the characteristic function χ_P?

Geometric lattice

- A lattice is a poset L for which any two elements have a meet and join, where a join of x and y, denoted $x \vee y$, is an upper bound z such that $z \leq z'$ for all upper bounds z'. (meet, $x \wedge y$); ($\Rightarrow \hat{0}$ and $\hat{1}$ exist)



- Given a poset P, we say that P is graded of rank n if every maximal chain of P has length n. ($\Rightarrow$ rank function)
- A finite graded lattice L satisfying condition (1) or (2) above is called semimodular.
  (1) For all $x, y \in L$, we have $\mathrm{rk}(x) + \mathrm{rk}(y) \geq \mathrm{rk}(x \vee y) + \mathrm{rk}(x \wedge y)$.
  (2) If x and y both cover $x \wedge y$, then $x \vee y$ covers both x and y.
- A finite lattice is geometric if it is both semimodular and atomic.

Basic definition

- Given a geometric lattice L, let $\mathbf{A}$ be the set of atoms of L.
- Let $\mathbf{S}_p$ be the set of all p-tuples $S = (a_1, \ldots, a_p)$ where $a_i \in \mathbf{A}$. For $p = 0$, we agree that $\mathbf{S}_0$ consists of the empty tuple $(\,)$.
- Let
$$\mathbf{S} = \bigcup_{p \geq 0} \mathbf{S}_p.$$
- If $S = (a_1, \ldots, a_p)$ write $\vee S = a_1 \vee \ldots \vee a_p$ and for $p = 0$ write $\vee(\,) = \hat{0}$.
- If $x \in L$, let $\mathbf{S}_x$ consists of all $S \in \mathbf{S}$ with $\vee S = x$.
- We say that $S \in \mathbf{S}_p$ is independent if $r(\vee S) = p$, and dependent if $r(\vee S) < p$.

P. Orlik and L. Solomon. Combinatories and Topology of Complements of Hyperplanes. Inventiones math. 56, 167-189 (1980).

Orlik-Solomon algebra of a geometric lattice

- Let $\mathscr{E} = \bigoplus_{p\geq 0} \mathscr{E}_p$ be the exterior algebra of the vector space which has a basis consisting of elements e_a in one to one correspondence with the elements $a \in \mathbf{A}$.
- If $S = (a_1, \ldots, a_p)$, let $e_S = e_{a_1} \ldots e_{a_p}$. If $S = (\,)$ write $e_S = 1$. Thus $\mathscr{E}$ has a basis consisting of all e_S with S standard.
- Define a $\mathbb{C}$-linear map $\partial : \mathscr{E} \to \mathscr{E}$ by $\partial\, 1 = 0$, $\partial\, e_a = 1$ and for $S = (a_1, \ldots, a_p)$,

$$\partial\, e_S = \sum_{k=1}^{p} (-1)^{k-1} e_{a_1} \ldots \widehat{e_{a_k}} \ldots e_{a_p}.$$

- Let $\mathscr{I}$ be the ideal of $\mathscr{E}$ generated by all elements ∂e_S where S is depedent.
- Let

$$\mathscr{A} := \mathscr{E} / \mathscr{I}.$$

- Since $\mathscr{I}$ is generated by homogeneous elements, $\mathscr{A} = \bigoplus_{p\geq 0} \mathscr{A}_p$ is a graded anticommutative $\mathbb{C}$-algebra.

Hermitian inner product

- Define a Hermitian inner product $\langle \, , \, \rangle$ on $\mathscr{E}$ by requiring that the standard basis elements e_S form an orthonormal basis.
- If $u \in \mathscr{E}$, we write u uniquely in the form $u = \sum_{c_S e_S}$ where $c_S \in \mathbb{C}$ and the S is standard. Then the support of u, denoted by $\mathrm{supp}(u)$, is defined to be the set of S with $c_S \neq 0$.
- The support, $\mathrm{supp}(\mathscr{M})$ of a subspace $\mathscr{M}$ of $\mathscr{E}$ is the union of the supports of its elements. $\Rightarrow$ Two subspaces with disjoint supports are orthogonal.
- If $x \in L$, let $\mathscr{E}_x := \sum_{S \in S_x} \mathbb{C} e_S$. Thus $\mathscr{E} = \oplus_{x \in L} \mathscr{E}_x$.
- Let $\alpha_S = \varphi\, e_S$ and let $\mathscr{A}_x = \varphi\, \mathscr{E}_x$, where $\varphi : \mathscr{E} \to \mathscr{A}$ is the natural homomorphism.

Theorem

$$\mathscr{A} = \bigoplus_{x \in L} \mathscr{A}_x \quad \text{and} \quad \mathscr{A}_p = \bigoplus_{r(x)=p} \mathscr{A}_x.$$

$$\mathscr{A}_x(L_x) \simeq \mathscr{A}_x(L) \;\Rightarrow\; \mathscr{A}_p \simeq \bigoplus_{r(x)=p} \mathscr{A}_x(L_x).$$

An Algebra Defined by Shuffles

- If $S = (\,)$ let $\beta_S = 1$ and for $S = (a_1, \ldots, a_p)$ define β_S by

$$\beta_S := \sum_{\pi} \operatorname{sgn} \pi (a_{\pi_1}, a_{\pi_1} \vee a_{\pi_2}, \ldots, a_{\pi_1} \vee a_{\pi_2} \vee \ldots \vee a_{\pi_p}),$$

  where $\pi = \pi_1 \ldots \pi_p$ sums over all permutations of length p.
- Let

$$\mathscr{B} := \sum_{S \in \mathbf{S}} \mathbb{C}\, \beta_S.$$

- We can show that $\mathscr{B} = \bigoplus_{p \geq 0} \mathscr{B}_p$ is also a graded $\mathbb{C}$-algebra.
- If $x \in L$, let $\mathscr{B}_x := \sum_{S \in S_x} \mathbb{C}\beta_S$.
- Then

$$\mathscr{B} = \oplus_{x \in L} \mathscr{B}_x.$$

Isomorphism $\theta : \mathscr{A} \to \mathscr{B}$

Theorem

Let L be a finite geometric lattice. There exists an isomorphism $\theta : \mathscr{A} \to \mathscr{B}$ of algebras such that $\theta\, \alpha_S = \beta_S$. The map $\theta : \mathscr{A} \to \mathscr{B}$ defines a natural transformation of functors.

The map $\theta : \mathscr{A}_x \to \mathscr{B}_x$ is an isomorphism.

Since $\mathscr{B}_p = \bigoplus_{r(x)=p} \mathscr{B}_x$ we have a commutative diagram of exact sequences

$$
\begin{array}{ccccccccccccc}
0 & \to & \mathscr{A}_1 & \to & \bigoplus_{r(x)=\ell-1} \mathscr{A}_x & \to & \bigoplus_{r(x)=\ell-2} \mathscr{A}_x & \to & \cdots & \to & \mathscr{A}_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \to & \mathscr{B}_1 & \to & \bigoplus_{r(x)=\ell-1} \mathscr{B}_x & \to & \bigoplus_{r(x)=\ell-2} \mathscr{B}_x & \to & \cdots & \to & \mathscr{B}_0 & \to & 0
\end{array}
\tag{3.12}
$$

where the vertical maps are isomorphisms. The Poincaré polynomial of $\mathscr{B}$ is

Homology Groups of a geometric lattice

Let L be a geometric lattice with rank $\ell \geq 2$,

- Define a simplicial complex K to be $K = \Delta(\bar{L})$.
- Let $\tilde{K}$ be the augmented complex obtained from K by adjoining a simplex of dimension -1 on which G acts trivially.
- Then the reduced homology $\tilde{H}(K;\mathbb{C})$ of K is the homology $H(\tilde{K};\mathbb{C})$ of $\tilde{K}$.
- According to Folkman and Rota, the homology of $\tilde{K}$ is given by

$$\dim H_i(\tilde{K};\mathbb{C}) = 0 \quad \text{if} \quad i \neq \ell - 2$$

$$\dim H_{\ell-2}(\tilde{K};\mathbb{C}) = (-1)^{\ell}\mu(L).$$

J. Folkman. The homology groups of a lattice. *J. Math. and Mech.* 15, 631-636 (1966).

G.-C. Rota. On the Foundations of Combinational Theory I. Theory of Möbius Functions. Z. Wahrscheinlichkeitsrechnung 2, 340-368 (1964).

Verification: non-equivariant version

Given a geometric lattice L with rank $\ell \geq 2$,

- From $\mu(P) = \sum_{i=-1}^{\dim \Delta(\bar{P})} (-1)^i \dim \tilde{H}_i(\Delta(\bar{P}); \mathbb{C})$, we obtain

$$\mu(L) = \sum_{j=-1}^{\ell-2} (-1)^j \dim \tilde{H}_j\big(\Delta(\bar{L}); \mathbb{C}\big) = \sum_{j=-1}^{\ell-2} (-1)^j \dim H_j\big(\tilde{K}; \mathbb{C}\big)$$

$$= (-1)^{\ell-2} \dim H_{\ell-2}\big(\tilde{K}; \mathbb{C}\big) = \mu(L);$$

- From

$$\chi(P) = \sum_{i=1}^{r_{\hat{0}\hat{1}}} \Bigg(\sum_{r_{\hat{0},x}=i} \sum_{j=-1}^{\dim \Delta(\overline{P_{\hat{0},x}})} (-1)^j \dim \tilde{H}_j\big(\hat{0}, x\big) \Bigg) t^{r_{\hat{0}\hat{1}}-i} + t^{r_{\hat{0}\hat{1}}},$$

we get

$$\chi(L) = \sum_{i=1}^{\ell} \sum_{\mathrm{rank}(x)=i} \mu(\hat{0}, x) t^{\ell-i} + t^{\ell} = \sum_{i=0}^{\ell} \sum_{\mathrm{rank}(x)=i} \mu(\hat{0}, x) t^{\ell-i}.$$

Let G be a group which acts as a group of automorphisms of L (rank $\ell \geq 2$), then

$$\bigoplus_{i=-1}^{\dim \Delta(\bar{L})} (-1)^i \tilde{H}_i(\Delta(\bar{L});\mathbb{C}) \quad \Rightarrow \quad (-1)^{\ell-2} H_{\ell-2}\big(\tilde{K};\mathbb{C}\big);$$

and

$$\sum_{i=1}^{r_{\hat{0}\hat{1}}} \Bigg(\bigoplus_{r_{\hat{0},x}=i} \bigoplus_{j=-1}^{\dim \Delta(\overline{P_{\hat{0},x}})} (-1)^j \tilde{H}_j\big(\hat{0},x\big) \Bigg) t^{r_{\hat{0}\hat{1}}-i} + \mathbb{C}\, t^{r_{\hat{0}\hat{1}}}$$

$$\Rightarrow \quad \sum_{i=2}^{\ell} \Bigg(\bigoplus_{r_{\hat{0},x}=i} (-1)^{i-2} H_{i-2}\big(\tilde{K}(\hat{0},x);\mathbb{C}\big) \Bigg) t^{\ell-i}$$

$$+ \Bigg(\bigoplus_{r_{\hat{0},x}=1} (-1)^{-1}\mathbb{C} \Bigg) t^{\ell-1} + \mathbb{C}\, t^{\ell}.$$

Why identifiable?

The group G is represented by linear transformations of the graded vector spaces $\mathscr{A}$ and $\mathscr{B}$.

Theorem

Let L be a finite geometric lattice of rank $\ell \geq 2$. Then $\mathscr{B}_{\hat{1}}$ and $H_{\ell-2}(\tilde{K})$ are isomorphic G-modules.

$$\sum_{i=2}^{\ell}\Big(\bigoplus_{r_{\hat{0},x}=i}(-1)^{i-2}H_{i-2}\big(\tilde{K}(\hat{0},x);\mathbb{C}\big)\Big)t^{\ell-i}+\Big(\bigoplus_{r_{\hat{0},x}=1}(-1)^{-1}\mathbb{C}\Big)t^{\ell-1}+\mathbb{C}\,t^{\ell}$$

$$\Rightarrow \sum_{i=2}^{\ell}(-1)^{i}\mathscr{A}_i t^{\ell-i}+(-1)\mathscr{A}_1 t^{\ell-1}+\mathscr{A}_0 t^{\ell}=H_M^G(t),$$

where the second last equation is from $\mathscr{A}_1 \cong \mathbb{C}^{|\mathbf{A}|}$ and $\mathscr{A}_0 \cong \mathbb{C}$.

Thanks for your attention!