Combinatorial models for Schubert Polynomials

Sophie C.C. Sun

Department of Mathematics
Tianjin University of Finance and Economics

September 1, 2020

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Background

- In 1879, Hermann Schubert: "How many lines in space meet four given lines".
- In 1902, David Hilbert's 15th Problem: Give rigorous foundation of Schubert's Enumerative Calculus. (Intersection Theory)

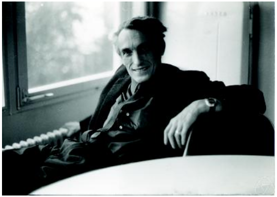
Hermann Schubert (1848–1911)

David Hilbert (1862–1943)

Background

In 1982, Alain Lascoux and M.-P. Schützenberger introduced Schubert polynomials as polynomial representatives of the Schubert classes in the cohomology ring of the flag manifold.

Alain Lascoux (1944–2013)
ICM invited speaker, 1998



M.-P. Schützenberger (1920–1996)
法国科学院院士

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Permutation

Definition

- A permutation w is a bijection from $[n]$ to $[n]$, where $[n] := \{1, 2, \ldots, n\}$.
- Let S_n denote the symmetric group of permutations of $[n]$. For $1 \le i \le n-1$, let $s_i = (i, i+1)$ denote an adjacent transposition. Then $\{s_1, s_2, \ldots, s_{n-1}\}$ is a set of generators satisfying the following relations:

$$\begin{cases} s_i^2 = 1; & \\ s_i s_j = s_j s_i, & \text{if } |i-j| > 1; \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}. & \end{cases}$$

Reduced word and Length

Definition

A permutation $w \in S_n$ can be written as a product $w = s_{i_1} s_{i_2} \cdots s_{i_r}$. If r is minimal, then $\ell(w) = r$ is called the *length* of w, and the sequence $a = (i_1, i_2, \ldots, i_r)$ is called *reduced word* of w. The set of reduced words of w is denoted by $R(w)$.

Example

For $w = 4213 \in S_4$, we have $w = s_3 s_2 s_1 s_2 = s_3 s_1 s_2 s_1 = s_1 s_3 s_2 s_1$, and so

$$R(w) = \{(3, 2, 1, 2), (3, 1, 2, 1), (1, 3, 2, 1)\}.$$

The definition of Schubert Polynomials

For $f \in \mathbb{Z}[x]$, define a *divided difference operator* ∂_i by

$$\partial_i f = \frac{f - s_i f}{x_i - x_{i+1}}.$$

The *Schubert polynomials* $\mathfrak{S}_w(x)$ can be defined as follows:

(1) For the longest permutation $w_0 = n \cdots 21$, set

$$\mathfrak{S}_{w_0}(x) = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}.$$

(2) If $w \neq w_0$, then set

$$\mathfrak{S}_w(x) = \partial_i \, \mathfrak{S}_{ws_i}(x),$$

where s_i is an adjacent transposition such that $\ell(ws_i) > \ell(w)$.

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Billey-Jockusch-Stanley's Model

Given a reduced word $\mathbf{a} = (a_1, a_2, \ldots, a_p)$ for w, we say a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)$ is **a**-compatible if

$$\begin{aligned} &\alpha_1 \le \alpha_2 \le \cdots \le \alpha_p, \\ &\alpha_i \le a_i \qquad \text{for } 1 \le i \le p, \\ &\alpha_i < \alpha_{i+1} \quad \text{if } a_i < a_{i+1}. \end{aligned}$$

Theorem (Billey-Jockusch-Stanley, J. Algebraic Combin., 1993)

For a permutation $w \in S_n$, the Schubert polynomials of w can be written as

$$\mathfrak{S}_w(x) = \sum_{(a_1,a_2,\ldots,a_\ell)\in R(w)} \sum_{\substack{1\le\alpha_1\le\alpha_2\le\cdots\le\alpha_\ell \\ \alpha_i\le a_i \\ a_i<a_{i+1}\Rightarrow\alpha_i<\alpha_{i+1}}} x_{\alpha_1}x_{\alpha_2}\cdots x_{\alpha_\ell}.$$

S. Billey, W. Jockusch and R.P. Stanley, Some combinatorial properties of Schubert polynomials, J. Algebraic Combin. 2 (1993), 345–374.

An example

Example

For $w = 1432 \in S_4$, we have

$$w = s_3 s_2 s_3 = s_2 s_3 s_2.$$

$$R(w) = \{(3,2,3), (2,3,2)\}$$

a	(3,2,3)	(2,3,2)
α	(1,1,2) (1,1,3) (1,2,3) (2,2,3)	(1,2,2)

$$\mathfrak{S}_{1432}(x) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3 + x_1 x_2^2.$$

The Stanley Symmetric Function

$$\mathfrak{S}_w(x) = \sum_{(a_1,a_2,\ldots,a_\ell)\in R(w)} \sum_{\substack{1\le\alpha_1\le\alpha_2\le\cdots\le\alpha_\ell \\ \alpha_i\le a_i \\ a_i<a_{i+1}\Rightarrow\alpha_i<\alpha_{i+1}}} x_{\alpha_1}x_{\alpha_2}\cdots x_{\alpha_\ell}.$$

Definition (Stanley, European J. Combin., 1984)

For a permutation w, the Stanley symmetric function *$F_w(x)$ is defined as*

$$F_w(x) = \sum_{(a_1,a_2,\ldots,a_\ell)\in R(w)} \sum_{\substack{1\le\alpha_1\le\alpha_2\le\cdots\le\alpha_\ell \\ a_i<a_{i+1}\Rightarrow\alpha_i<\alpha_{i+1}}} x_{\alpha_1}x_{\alpha_2}\cdots x_{\alpha_\ell}.$$

R.P. Stanley, On the number of reduced decompositions of elements of Coxeter groups, European J. Combin. 5 (1984), 359–372.

Stanley Symmetric Function

Since F_w is symmetric, one may write

$$F_w = \sum_{\lambda} c_{\lambda}^{w} s_{\lambda},$$

where the coefficients c_{λ}^{w} are called the Stanley coefficients or Edelman-Greene coefficients.

Example

$$F_{214365} = s_{(3)} + 2s_{(2,1)} + s_{(1,1,1)},$$

$$F_{4213} = s_{(3,1)}.$$

Reduced Word Tableaux

- The *column reading word* of T, denoted $\mathrm{column}(T)$, is obtained by reading the entries of T along the columns from top to bottom, right to left.
- An increasing tableau T is called a *reduced word tableau* for w if $\mathrm{column}(T) \in R(w)$.

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 2 & 4 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$$

Figure 3.1: Reduced word tableaux of $w = 321654$

Edelman-Greene coefficent

Theorem (Edelman-Greene, Adv. Math., 1987)

Assume that $F_\lambda = \sum_\lambda c_\lambda^w s_\lambda$. *Then*

$$c_\lambda^w = \#\{\textit{reduced word tableaux of } w \textit{ with shape } \lambda\}.$$

Theorem (Lam-Lee-Shimozono, 2018)

$$c_\lambda^w = \#\{\textit{EG-pipedreams of } w \textit{ with shape } \lambda\}.$$

P. Edelman and C. Greene, Balanced tableaux, Adv. Math. 63 (1987), 42–99.

T. Lam, S. Lee and M. Shimozono, Back stable Schubert calculus, arXiv:1806.11233v1.

Problem (Lam-Lee-Shimozono, 2018)

Find a shape-preserving bijection between:

$$\{\textit{Reduced word tableaux of } w\} \longleftrightarrow \{\textit{EG-pipedreams of } w\}.$$

1	2	4	5
2	5		

1	2	4
2	5	
4		

1	2	5
2	4	
5		

Figure 3.2: Reduced word tableaux of $w = 321654$

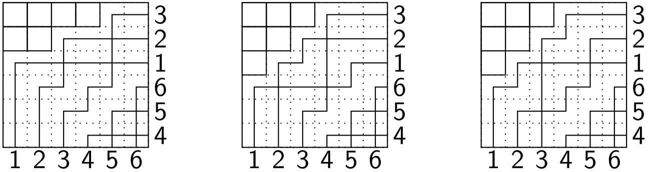

Figure 3.3: The EG-pipedreams of $w = 321654$.

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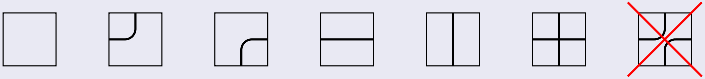
Bumpless Pipedream

Definition (Lam-Lee-Shimozono, 2018)

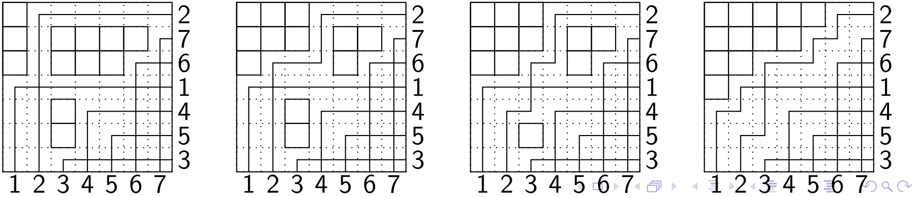
A bumpless pipedream *of w is a tiling of the $n \times n$ square with*



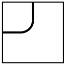
such that

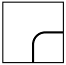
(1) *the pipe labeled i enters from the south boundary in column i and exits from the east boundary in row $w^{-1}(i)$;*

(2) *no two pipes overlap any step or cross more than once.*



Def.

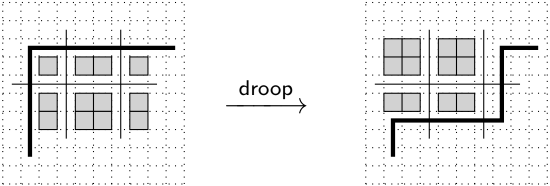
 NW-elbow

 SE-elbow

A **droop** is a local move that swaps a SE elbow e with an empty box t, when the SE-elbow lies strictly to the northwest of the empty box. Let R be the rectangle with northwest corner e and southeast corner t and let p be the pipe passing through e.

A droop is allowed only if

(1) the westmost column and northmost row of R contains p;
(2) the rectangle R contains only one elbow: the SE elbow which is at e;
(3) after the droop we obtain another bumpless pipedream.



Proposition (Lam-Lee-Shimozono, 2018)

For a permutation w, every bumpless pipedream of w can be obtained from the Rothe pipedream $D(w)$ by a sequence of droops.

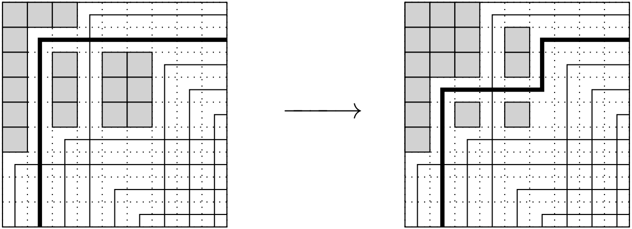


Figure 3.4: A droop operation on $D(w)$.

Bumpless pipedream

For a bumpless pipedream P, define the weight $\mathrm{wt}(P)$ of P to be the product of $x_i - y_j$ over all empty boxes of P in row i and column j.

Theorem (Lam-Lee-Shimozono, 2018)

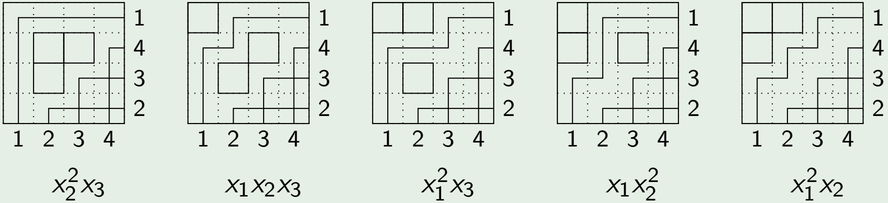
For any permutation w,

$$\mathfrak{S}_w(x; y) = \sum_P \mathrm{wt}(P), \tag{3.1}$$

where the sum is over the bumpless pipedreams of w.

T. Lam, S. Lee and M. Shimozono, Back stable Schubert calculus, arXiv:1806.11233v1.

Example



$$\mathfrak{S}_{1432}(x_1, x_2, x_3) = x_2^2x_3 + x_1x_2x_3 + x_1^2x_3 + x_1^2x_2 + x_1x_2^2$$

EG-pipedream

An *EG-pipedream* of w is a pipedream P such that all the boxes of P are at the northwest corner and form a Young diagram.

Example

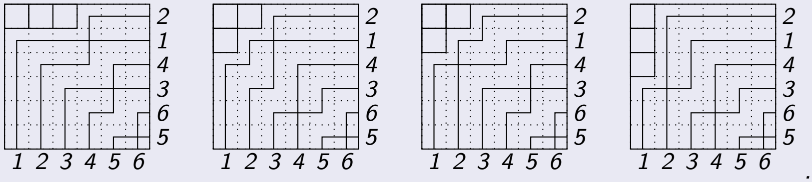


Figure 3.5: Four EG-pipedreams of $w = 214365$.

Reduced Word Tableaux and EG-pipedreams

Theorem (Fan-Guo-Sun, arXiv:1810.11916.)

There is a shape preserving one-to-one correspondence between reduced word tableaux and EG-pipedreams of w.

$$\begin{array}{ccc} & & \text{Lascoux-Schützenberger tree} \\ & & \Downarrow \\ \{\text{reduced word tableaux}\} & \xleftrightarrow[(D.Little, Adv.Math., 2003)]{\text{Little map}} & \{\text{leaves of Modified LS-tree}\} \\ & & \updownarrow \\ \{\text{EG-pipedreams}\} & \longleftrightarrow & \{\text{leaves of Edelman-Greene tree}\} \end{array}$$

Neil J.Y. Fan, Peter L. Guo, Sophie C.C. Sun, Bumpless pipedreams, reduced word tableaux and Stanley Symmetric Functions, arXiv:1810.11916.

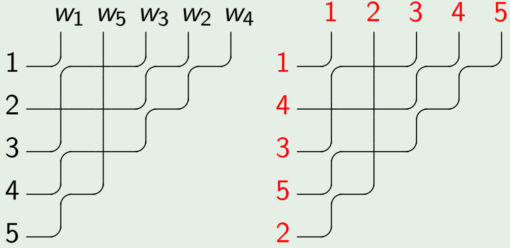
Contents

Given a reduced word $\mathbf{a} = (a_1, a_2, \ldots, a_p)$ and an $\mathbf{a}$-compatible sequence $\alpha = \alpha_1\alpha_2\cdots\alpha_p$, an rc-graph is defined by placing crossings at all the positions $(\alpha_k, a_k - \alpha_k + 1)$, which form a subsect of $\{1, 2, \ldots\} \times \{1, 2, \ldots\}$.

Example

The rc-graph for $w = 14352$ with reduced word $(2, 3, 2, 4)$ and compatible sequence $(1, 2, 2, 3)$.

$$\begin{array}{ccccc} \bullet & + & \bullet & \bullet & \bullet \\ + & + & \bullet & \bullet & \\ \bullet & + & \bullet & & \\ \bullet & \bullet & & & \\ \bullet & & & & \end{array}$$



For $w \in S_\infty$ and $D \in \mathcal{RC}(w)$, a **ladder move** L_{ij} is a change of the following

$$
\begin{array}{ccc}
 & j & j{+}1 \\
i{-}m & \cdot & \cdot \\
 & + & + \\
 & + & + \\
 & + & + \\
i & + & \cdot
\end{array}
\quad \longmapsto \quad
\begin{array}{ccc}
 & j & j{+}1 \\
i{-}m & \cdot & + \\
 & + & + \\
 & + & + \\
 & + & + \\
i & \cdot & \cdot
\end{array}
$$

A **chute move** C_{ij} is a change of the following type

$$
\begin{array}{cccccc}
 & j{-}m & & & & j \\
i & \cdot & + & + & + & + \\
i{+}1 & \cdot & + & + & + & \cdot
\end{array}
$$

$$\downarrow$$

$$
\begin{array}{cccccc}
 & j{-}m & & & & j \\
i & \cdot & + & + & + & \cdot \\
i{+}1 & + & + & + & + & \cdot
\end{array}
$$

Lemma (Bergeron and Billey)

Ladder and chute moves preserve the permutation associated with an rc-graph.

N. Bergeron and S. Billey, RC-graphs and Schubert polynomials, Experiment. Math. 2 (1993), 257–269.

Bottom rc-graph and top rc-graph

$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 \\ 1 & + & + & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & \cdot & \cdot & \\ 3 & + & \cdot & \cdot & & \\ 4 & + & + & & & \\ 5 & + & & & & \end{array}$$

bottom rc-graph $D_{bot}(w)$

$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 \\ 1 & + & + & \cdot & \cdot & + \\ 2 & \cdot & + & \cdot & \cdot & \\ 3 & \cdot & + & \cdot & & \\ 4 & \cdot & + & & & \\ 5 & \cdot & & & & \end{array}$$

top rc-graph $D_{top}(w)$

Let $\mathcal{L}(D)$ (respectively, $\mathcal{C}(D)$) be the set of rc-graphs that can be derived from D by some sequence of ladder moves (respectively, chute moves).

Theorem (Bergeron and Billey)

$$\mathfrak{S}_w = \sum_{D \in \mathcal{L}(D_{bot}(w))} x_D = \sum_{D \in \mathcal{C}(D_{top}(w))} x_D.$$

RC-graph and Schubert polynomial

Theorem (Bergeron and Billey, Fomin and Kirillov)

Let $w \in S_n$ be a permutation, then

$$\mathfrak{S}_w(x) = \sum_{D \in \mathcal{RC}(w)} x^D,$$

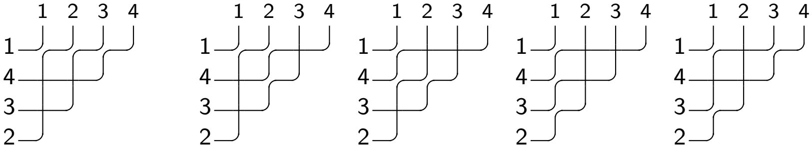
where $\mathcal{RC}(w)$ be the set of all rc-graphs corresponding to w.

N. Bergeron and S. Billey, RC-graphs and Schubert polynomials, Experiment. Math. 2 (1993), 257–269.

S. Fomin and A. N. Kirillov, The Yang-Baxter equation, symmetric functions, and Schubert polynomials, Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics (Florence 1993), Discrete Math. 153 (1996), 123–143.

An example

For $w = 1432$, the corresponding rc-graphs are



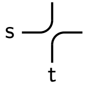
$\mathbf{a} = 323$	$\mathbf{a} = 323$	$\mathbf{a} = 323$	$\mathbf{a} = 323$	$\mathbf{a} = 323$
$\alpha = (2,2,3)$	$\alpha = (1,2,3)$	$\alpha = (1,1,3)$	$\alpha = (1,1,2)$	$\alpha = (1,2,2)$

$$\mathfrak{S}_{1432}(x) = x_1^2x_2 + x_1^2x_3 + x_1x_2x_3 + x_2^2x_3 + x_1x_2^2.$$

Algorithm on rc-graph

Algorithm on rc-graph: Label each strand by the row where it starts from.

(1) Set $i_0 = i$. Starting with row i_0, find the rightmost position (i_0, j_0) where the configuration is shown as



where $s \leq r < t$. $(\star)$

(2) For $D \in \mathcal{RC}(w)$, add a crossing in the position (i_0, j_0), and let $s_0 = s$ and $t_0 = t$ be the strands that cross there.

(3) (a) If the resulting graph is a legal rc-graph, stop the algorithm.
  (b) If s_0 and t_0 also cross at (i_1, j_1'), then delete this second crossing from D. (Since $s_0 < t_0$, we must have $i_1 < i_0$.)

(4) Find $j_1 < j_1'$ maximal such that the configuration at (i_1, j_1) is as in $(\star)$. (Such a j_1 must exist since $i_1 < i_0 \leq r$.)

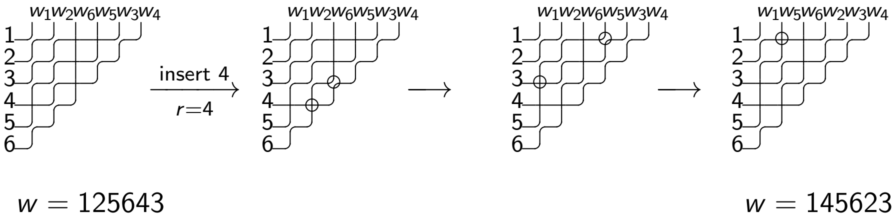
(5) Add the crossing (i_1, j_1) to D, and let s_1 and t_1 be the strands that cross there.

(6) (c) If the result is an rc-graph, stop.
  (d) Otherwise continue deleting and inserting crossings in the manner just explained.

An example

If p is the last step of the process, set $k = s_p$ and $l = t_p$, and let D' be the resulting graph.

Example Given an rc-graph $D \in \mathcal{RC}(w)$, where $w = 125643$, $r = 4$. Insert $i = 4$ into D we have



Then we have

$$(D, 4) \xrightarrow{\text{insertion algorithm}} (D', 2, 5).$$

RC-graph and Monk's Rule

- $D \in \mathcal{RC}(w)$, $w \in S_\infty$,
- r and i are integers with $0 < i \leq r$,
- k and l are positive integers,
- D' is an rc-graph for a permutation of length $\ell(w) + 1$.

$$(D, i) \xrightarrow{\text{Insertion algorithm}} (D', k, l).$$

The algorithm results in

$$x_{D'} = x_D x_i.$$

Theorem (Monk's Rule)

Given $w \in S_\infty$ and a simple transposition s_r, we have

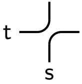
$$\mathfrak{S}_{s_r}\mathfrak{S}_w = \sum_{\substack{k \leq r < l \\ \ell(wt_{kl}) = \ell(w)+1}} \mathfrak{S}_{wt_{kl}}.$$

Reverse algorithm on rc-graph

Reverse algorithm on rc-graph: Given a permutation w', integers $0 < k \leq r < l$ and rc-graph $D' \in \mathcal{RC}(w')$, there is a unique position $(i_0, j_0) \in D'$ such that the two strands k and l cross.

(1) Delete that crossing from D'.

(2) (a) If there is no position to the right of (i_0, j_0) where the configuration is shown in Figure $(\star\star)$, then stop the process.



where $s \leq r < t$ $\qquad (\star\star)$

(b) Otherwise, let (i_0, j_0') be the position minimizing $j_0' > j_0$ and having the configuration in Figure $(\star\star)$.

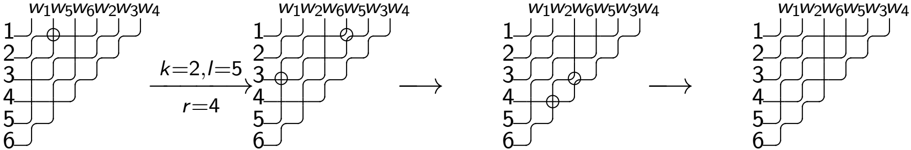
(3) Let $s_1 = s$ and $t_1 = t$ be the strands there. It is easy to see that s_1 and t_1 must cross at some position (i_1, j_1) where $i_1 > i_0$ and $j_1 < j_0'$.

(4) Add a crossing to the position (i_0, j_0'), and delete (i_1, j_1).

(5) If possible, find the minimal position (i_1, j_1') satisfying the configuration in Figure $(\star\star)$ and such that $j_1' > j_1$.

(6) Continue this way until there is no such position.

Ex. Given an rc-graph $D' \in \mathcal{RC}(w')$, where $w' = 145623$. Let $k = 2$ and $l = 5$. By the reverse insertion algorithm, we obtain the following rc-graph D and record the integer $i = 4$.



Then we have

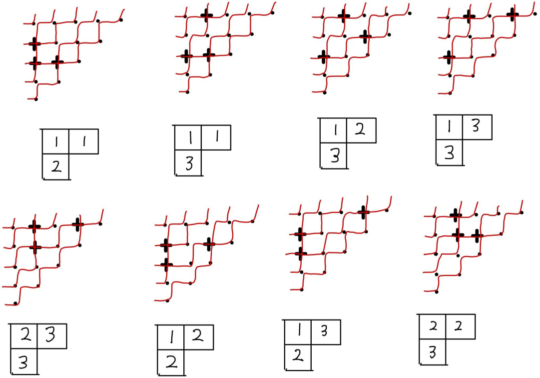
$$(D', 2, 5) \xrightarrow{\text{reverse insertion algorithm}} (D, 4).$$

Grassmannian perrmutation

Def. A permutation w is Grassmannian if it has only one descent.

Remark The rc-graphs of a grassmannian permutation can correspond to some semistandard young tableaux.

Ex. For $w = 13524$, the rc-graphs are shown as follows.



Remark

When w is Grassmannian, the insertion algorithm corresponds to the usual RSK-algorithm on semistandard Young tableaux.

RC-graph and Littlewood-Richardson rule

Theorem (Kogan, 2001)

For a partition λ, let $w(\lambda, d)$ denote the Grassmannian permutation for λ with descent at d. If the permutation v has no descents after position d, the coefficient $c^u_{v,w(\lambda,d)}$ in

$$\mathfrak{S}_v \mathfrak{S}_{w(\lambda,d)} = \sum_u c^u_{v,w(\lambda,d)} \mathfrak{S}_u,$$

is equal to the number of pairs (R, Y) such that $R \leftarrow Y = U$, where R is an rc-graph of permutation v, Y is a semistandard Young tableau of shape λ, and U is any rc-graph of permutation u.

M. Kogan, RC-graphs and a generalized Littlewood-Richardson rule, Internat. Math. Res. Notices 2001, no. 15, 765–782.

The word of Young tableaux

For each Young tableau Y, associate a row reading word $\mathbf{w}(Y)$, which is given by reading the entries of the tableau from left to right in each row, starting from the bottom row and going to the top one.

Example

The words of the following Young diagrams are $\mathbf{w}(Y_1) = 112$, $\mathbf{w}(Y_2) = 313$ and $\mathbf{w}(Y_3) = 42122$, respectively.

$$Y_1 = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline \end{array} \qquad Y_2 = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 3 \\ \cline{1-1} \end{array} \qquad Y_3 = \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}$$

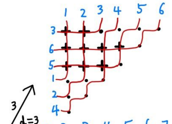
Ex. $\mathfrak{G}_{25314} \times \mathfrak{G}_{23514} = \mathfrak{G}_{365124} + \mathfrak{G}_{3741256}$

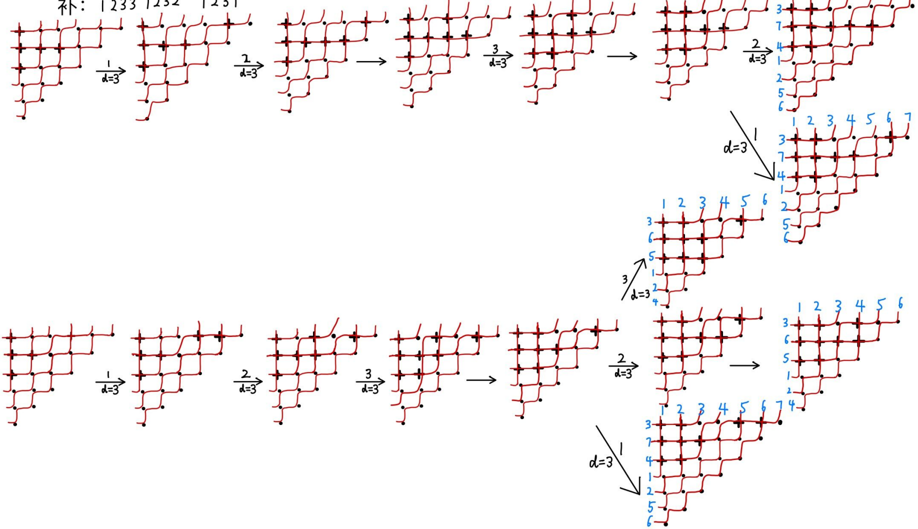
$w(\lambda, d) = 32415$. $\lambda = 211$, $d = 3$. $v = 25314$

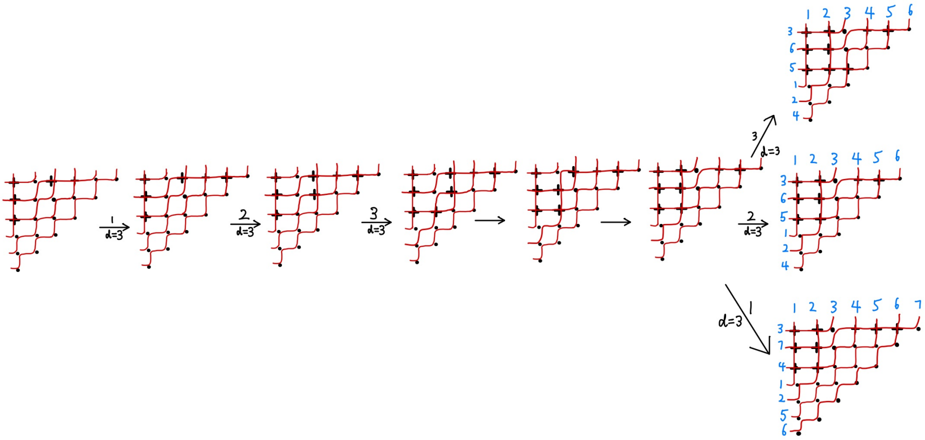
Y: $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$

W(Y): 3211 3212 3213

补: 1233 1232 1231







1
d=3
2
d=3
3
d=3
3
d=3
2
d=3
d=3
1
1 2 3 4 5 6
3
6
5
1
2
4
1 2 3 4 5 6
3
6
5
1
2
4
1 2 3 4 5 6 7
3
7
4
1
2
5
6

Contents

Kohnert move

A Kohnert move on a diagram selects the rightmost cell of a given row and moves the cell to the first available position above, jumping over other cells in its way as needed.

Conjecture (Kohnert, 1991)

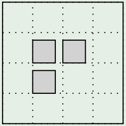
Let w be a permutation. Then

$$\mathfrak{S}_w(x_1, x_2, \ldots) = \sum_{D \in \mathcal{KM}(w)} x^D,$$

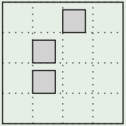
where $\mathcal{KM}(w)$ denotes the set of all diagrams that can be obtained by applying a series of Kohnert moves to the Rothe diagram $D(w)$.

A. Kohnert. Weintrauben, Polynome, Tableaux. Bayreuther Mathematische Schriften, 38:1–97, 1991.

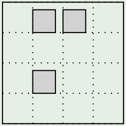
Example



$x_2^2 x_3$



$x_1 x_2 x_3$



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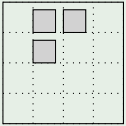
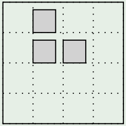


$x_1^2 x_2$



$x_1 x_2^2$

$$\mathfrak{S}_{1432}(x_1, x_2, x_3) = x_2^2 x_3 + x_1 x_2 x_3 + x_1^2 x_3 + x_1^2 x_2 + x_1 x_2^2$$

Contents

More combinatorial models

Balanced tableaux

Prism tableaux

Flagged Weyl module

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Thank you for your attention!