Integral Bases for P-Recursive Sequences

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July 24, 2020, Tianjin, China

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joint work with Lixin Du, Manuel Kauers, and Thibaut Verron

Integral bases: algebraic case

Notation.

- A: an integral domain (e.g. $\mathbb{Z}$, $\mathbb{C}[x]$);
- K: the quotient field of A (e.g. $\mathbb{Q}$, $\mathbb{C}(x)$);
- L: a separable extension of K with $[L:K] = r \leq +\infty$;
- B: a ring extension of A with $B \subseteq L$.

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Definition. An element $\beta \in B$ is integral over A if

$$\beta^n + a_{n-1}\beta^{n-1} + \cdots + a_0 = 0 \quad \text{with } a_i \in A.$$

Theorem. The set

$$\mathcal{O}_{B/A} := \{\beta \in B \mid \beta \text{ is integral over } A\}$$

forms an A-module which is called the integral closure of A in B.

Problem. When is the A-module $\mathcal{O}_{B/A}$ free?

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Definition. Let $\{\beta_1,\ldots,\beta_r\}$ be a basis of L over K. The trace map $\mathrm{Tr}_{L/K}:L\to K$ is defined by for any $\alpha\in L$,

$$\mathrm{Tr}_{L/K}(\alpha)\triangleq\mathrm{Tr}(M_\alpha)=a_{1,1}+a_{2,2}+\ldots+a_{r,r},$$

where $M_\alpha=(a_{i,j})\in K^{r\times r}$ with

$$\alpha\cdot\beta_i=\sum_{j=1}^{r}a_{i,j}\beta_j\quad\text{for } i=1,2,\ldots,r.$$

Remark. The $\mathrm{Tr}_{L/K}$ is independent of the choice of bases.

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- B: a ring extension of A with $B \subseteq L$.

Definition. The discriminant of $\alpha_1, \ldots, \alpha_r \in L$ is

$$\mathrm{Disc}_{L/K}(\alpha_1, \ldots, \alpha_r) \triangleq \det((\mathrm{Tr}_{L/K}(\alpha_i \alpha_j))) \in K.$$

Proposition.

1. $\{\alpha_1, \ldots, \alpha_r\}$ is a basis of L over K $\Leftrightarrow$ $\mathrm{Disc}_{L/K}(\alpha_1, \ldots, \alpha_r) \neq 0$;
2. If $\alpha_1, \ldots, \alpha_r \in \mathcal{O}_{L/A}$, then $\mathrm{Disc}_{L/K}(\alpha_1, \ldots, \alpha_r) \in \mathcal{O}_{K/A}$;
3. If $\beta_i = \sum_{j=1}^{r} b_{i,j} \alpha_j$ for $i = 1, \ldots, r$, then

$$\mathrm{Disc}_{L/K}(\beta_1, \ldots, \beta_r) = \mathrm{Disc}_{L/K}(\alpha_1, \ldots, \alpha_r) \cdot \det((b_{i,j}))^2.$$

Existence of integral bases: algebraic case

Definition. If $\mathcal{O}_{L/A}$ is free, any basis of $\mathcal{O}_{L/A}$ is called an integral basis of L over K.

Theorem. If A is PID, then $\mathcal{O}_{L/A}$ is a free A-module of rank $[L:K]$.

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Proof.

1. $(A, \preceq)$ is a partially ordered set, where $a \preceq b$ if $a \mid b$ for $a, b \in A$. Since A is PID, any nonempty set of A has a minimum.
2. Since L is separable, $L = K(\theta)$ for some $\theta \in \mathcal{O}_{L/A}$. Then $\{1, \theta, \ldots, \theta^{r-1}\} \subseteq \mathcal{O}_{L/A}$ is a basis of L over K and so

$$\Lambda := \{\mathrm{Disc}_{L/K}(\alpha_1, \ldots, \alpha_r) \mid \alpha_1, \ldots, \alpha_r \in \mathcal{O}_{L/A}\} \subseteq A$$

is nonempty. Let $\beta_1, \ldots, \beta_r \in \mathcal{O}_{L/A}$ be such that their discriminant is minimal in Λ.

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Proof.

3. Claim. $\{\beta_1,\ldots,\beta_r\}$ is an integral basis. If not, then $\exists\,\beta \in \mathcal{O}_{L/A}\setminus\{0\}$ such that

$$\beta = \frac{b_1\beta_1 + \cdots b_r\beta_r}{d},$$

where $b_1,\ldots,b_r,d \in A$ and d is not a unit in A. W.L.O.G, we may assume that $b_1 = 1$. Then

$$\mathrm{Disc}_{L/K}(\beta,\beta_2\ldots,\beta_r) = \frac{1}{d^2}\cdot \mathrm{Disc}_{L/K}(\beta_1,\ldots,\beta_r) \in A,$$

which contradicts with the minimality of $\mathrm{Disc}_{L/K}(\beta_1,\ldots,\beta_r)$.

Existence of integral bases: algebraic case

Definition. If $\mathcal{O}_{L/A}$ is free, any basis of $\mathcal{O}_{L/A}$ is called an integral basis of L over K.

Theorem. If A is PID, then $\mathcal{O}_{L/A}$ is a free A-module of rank $[L:K]$.

Examples.

1 Let $A=\mathbb{Z}$ and $K=\mathbb{Q}$. Then any algebraic number field has an integral basis. Let $L=\mathbb{Q}(\sqrt{m})$ with $m\in\mathbb{Z}$. Then an integral basis of L over $\mathbb{Q}$ is $\{1,\sqrt{m}\}$ if $m\equiv 2$ or $3 \mod 4$ and $\{1,(1+\sqrt{m})/2\}$ if $m\equiv 1 \mod 4$.

2 Let $A=\mathbb{C}[x]$ and $K=\mathbb{C}(x)$. Then any finite algebraic extension of K has an integral basis. Let $L=K(\beta)$ with β being a root of $P=(\frac{25}{16}x^3+2x^4)-x^3y-(2x+1)y^2+y^3$. Then an integral basis of L over $\mathbb{C}(x)$ is $\{1,\beta,(-\beta+\beta^2)/x\}$.

Problem. How to construct integral bases?

Computation of integral bases: van Hoeij's algorithm

Input. $M \in C[x,y]$ monic irreducible over $C(x)$ with $r = \deg_y(M)$;
output. an integral basis $\{B_0, \cdots, B_{r-1}\}$.

1. Start with $(B_0, \ldots, B_{r-1}) := (1, \beta, \ldots, \beta^{r-1})$.
2. For $d \in \{0, 1, \ldots, r-1\}$
3. While there exist $a_0, \ldots, a_{d-1} \in C[x]$ such that

$$A = \frac{a_0 B_0 + \cdots + a_{d-1} B_{d-1} + B_d}{p(x)}$$

is integral and $p(x) \in C[x] \setminus C$; replace B_d by A.

4. Return $B_0, \ldots, B_{r-1}$.

Computation of integral bases: van Hoeij's algorithm

Example. $M = (\frac{25}{16}x^3 + 2x^4) - x^3y - (2x+1)y^2 + y^3$ and $M(\beta) = 0$.

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integral closure $\mathcal{O}_{C(x)(\beta)/C[x]}$

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$\mathcal{O}_0$

$C[x] + C[x]\beta + C[x]\beta^2$

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$\alpha = 0, \quad B_2 := \dfrac{-\beta+\beta^2}{x}$ $\mathcal{O}_1$

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$\alpha \in C, \quad B_d := \frac{a_0 B_0 + \cdots a_{d-1} B_{d-1} + B_d}{x - \alpha}$ $\qquad \mathcal{O}_n$

$\cdots$

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$C[x] + C[x]\beta + C[x]\beta^2$

$\{1, \beta, \frac{1}{x}(-\beta + \beta^2)\}$ is an integral basis of $C(x)(\beta)$.

Integral bases: general framework

Definition. Let k be a field. The map $\nu : k \to \mathbb{Z} \cup \{\infty\}$ is called a valuation if for all $a, b \in k$

- $\nu(a) = \infty$ iff $a = 0$;
- $\nu(ab) = \nu(a) + \nu(b)$;
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Example. For a nonzero $f \in C(x)$, define $\nu_z(f) = m$ if

$$f = (x-z)^m \frac{a}{b} \quad \text{where } (x-z) \nmid a, b.$$

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The corresponding valuation ring is

$$C[x]_{x-z} = \left\{ \frac{a}{b} \in C(x) \,\middle|\, (x-z) \nmid b \right\}.$$

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Example. $(C(x), \nu_z)$ is a valued field.

Fact. The valuation ring is integrally closed.

Integral Bases: general framework

Definition. Let V be a vector space over (k, ν). The map $\mathrm{val} : V \to \mathbb{Z} \cup \{\infty\}$ is called a value function if for all $B, B_1, B_2 \in V$ and $u \in k$

- $\mathrm{val}(B) = \infty$ iff $B = 0$;
- $\mathrm{val}(u \cdot B) = \nu(u) + \mathrm{val}(B)$;
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An element B of V is called integral if $\mathrm{val}(B) \geq 0$.

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Fact. The integral elements of V form a $\mathcal{O}_{(k,\nu)}$-module.

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An element B of V is called integral if $\mathrm{val}(B) \geq 0$.

Fact. The integral elements of V form a $\mathcal{O}_{(k,\nu)}$-module.

Problem 1. When is this module free, i.e., when does there exist an integral basis?

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An element B of V is called integral if $\mathrm{val}(B) \geq 0$.

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Problem 1. When is this module free, i.e., when does there exist an integral basis?

Problem 2. How to compute a basis if it exists?

Discriminant functions

Notation.

- (k, v): a valued field with value group $\mathbb{Z}$;
- (V, val): a valued vector space over (k, v) of dimension r.

Definition. Let $x \in k$ with $v(x) = 1$ and $\mathbb{B}_V$ denote the set of all bases of V. A map $\mathrm{Disc} : \mathbb{B}_V \to \mathbb{Z}$ is a discriminant function on V if for a basis $B_1, \ldots, B_r$ of V,

(i) $\gamma := \mathrm{Disc}(\{B_1, \ldots, B_r\}) \geq 0$ if all B_i are integral;

(ii) for all $\alpha_1, \ldots, \alpha_{d-1} \in k$ with $d \leq r$,

$$\mathrm{Disc}(B_1, \ldots, B_{d-1}, \alpha_1 B_1 + \cdots + \alpha_{d-1} B_{d-1} + B_d, B_{d+1}, \ldots, B_r) = \gamma$$

(iii) $\mathrm{Disc}(B_1, \ldots, B_{d-1}, x^{-1} B_d, B_{d+1}, \ldots, B_r) < \gamma$.

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(iii) $\mathrm{Disc}(B_1, \ldots, B_{d-1}, x^{-1} B_d, B_{d+1}, \ldots, B_r) < \gamma$.

Theorem.

$$(V, \mathrm{val}) \text{ has a discriminant function}$$

$$\Updownarrow$$

$$(V, \mathrm{val}) \text{ has an integral basis}$$

Computation of integral bases: general case

Input. a vector space basis $\{B_1, \ldots, B_r\}$ of (V, val) over (k, ν).
output. an integral basis.

1. For $d \in \{1, \ldots, r\}$ do:
2. Replace B_d by $(x)^{-\mathrm{val}(B_d)} B_d$, where $\nu(x) = 1$.
3. While there exist $a_1, \ldots, a_{d-1} \in \mathcal{O}_k$ such that

$$A = \frac{a_1 B_1 + \cdots + a_{d-1} B_{d-1} + B_d}{x}$$

is integral; replace B_d by A.

4. Return $B_1, \ldots, B_r$.

Existence of integral bases: general case

Theorem. Let (V, val) be a valued vector space over (k, ν).
TFAE.

- (a) There is an integral basis of (V, val).
- (b) There is a discriminant function $\mathrm{Disc}: \mathbb{B}_V \to \mathbb{Z}$, where $\mathbb{B}_V$ is the set of all bases of V.
- (c) The algorithm terminates.
- (d) The completion of V w.r.t ν is of dimension $r = \dim_k(V)$.

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integral bases

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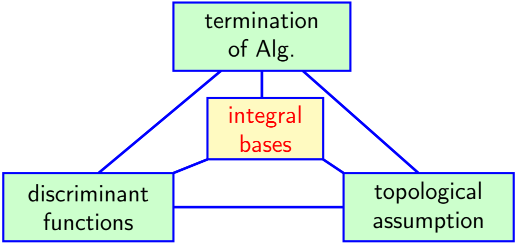

Integral Bases: three cases

Algebraic case

- $K = C(x)[y]/\langle M\rangle$, where $M \in C(x)[y]$ irreducible
- The integral elements of K form a free $C[x]$-module.
- Computation: van Hoeij's algorithm 1994, etc.

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D-finite case

- $V = C(x)[D]/\langle L\rangle$, $Dx = xD + 1$, where $L \in C(x)[D]$ admits a fundamental system of solutions in $\bar{C}[[[x-\alpha]]]]$.
- The integral elements of V form a free $C[x]$-left module.
- Computation: Kauers-Koutschan's algorithm 2015.

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P-recursive case

- Question. What are integral elements?

D-finite functions

Definition. A function $f(x)$ is called D-finite over C if

$$p_0(x)\,f(x)+p_1(x)f'(x)+\cdots+p_r(x)f^{(r)}(x)=0 \quad \text{for } p_i\in C[x].$$

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Examples.

$$\frac{1}{x^2+2x}, \quad \frac{1}{\sqrt{x+1}}, \quad \exp(x), \quad \log(x), \quad J_\alpha(x), \quad {}_2F_1(a,b;c;z), \quad \cdots$$

D-finite functions: solution space

Setting.

- $L = p_0(x) + p_1(x)D + \cdots + p_r(x)D^r \in C[x][D]$ with $p_r \neq 0$.
- $V = C(x)[D]/\langle L \rangle$, $D \cdot x = x \cdot D + 1$.

D-finite functions: solution space

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- $L = p_0(x) + p_1(x)D + \cdots + p_r(x)D^r \in C[x][D]$ with $p_r \neq 0$.
- $V = C(x)[D]/\langle L \rangle$, $D \cdot x = x \cdot D + 1$.

Let $\alpha \in \bar{C}$. For any function $f(x)$

- Operator action:

$$L \cdot f = p_0(x)f(x) + p_1(x)f'(x) + \cdots + p_r(x)f^{(r)}(x).$$

D-finite functions: solution space

Setting.

- $L = p_0(x) + p_1(x)D + \cdots + p_r(x)D^r \in C[x][D]$ with $p_r \neq 0$.
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Fact. If $x - \alpha \nmid p_r(x)$, then L admits r linearly independent solution in

$$C[[x-\alpha]].$$

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Question. Does there always exist r linearly independent solutions?

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Theorem. Let $L \in C(x)[D]$. Then L admits a fundamental system of generalized series solutions of the form

$$\exp(P((x-\alpha)^{\frac{1}{s}}))(x-\alpha)^{\nu}Q((x-\alpha)^{\frac{1}{s}},\log(x-\alpha))$$

for some $s \in \mathbb{N}$, $P \in \bar{C}[x]$, $\nu \in \bar{C}$ and $Q \in \bar{C}[[x]][y]$.

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Notation.

$$\bar{C}[[[x-\alpha]]] := \bigcup_{\nu \in C} (x-\alpha)^{\nu}\bar{C}[[x-\alpha]][\log(x-\alpha)]$$

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$$\bar{C}[[[x-\alpha]]] := \bigcup_{\nu \in C} (x-\alpha)^{\nu}\bar{C}[[x-\alpha]][\log(x-\alpha)]$$

Definition. Such a series is called integral at α if $\nu \geq 0$.

Integral bases: D-finite case

Definition. An operator $B \in V = C(x)[D]/\langle L \rangle$ is called integral if

$$B \cdot f \quad \text{is integral}$$

for every series solution f of L at any $\alpha \in \bar{C}$.

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Remark. Let $\{b_1, \cdots .b_r\}$ be a fundamental system of L at α. Then

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Theorem. The integral elements of V form a free $C[x]$-module.

$$\mathrm{wr}_{L,\alpha}(B) := \det(((B_i \cdot b_j))_{i,j=1}^{r}) \in \bar{C}[[[x-\alpha]]].$$

Integral bases: D-finite case

Example. $L = (2x+1) - (4x^2)D + 2x(2x-1)D^2$

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$x=0$	1		
1st sol	$1 + x + \frac{1}{2}x^2 + \cdots$		
2nd sol	$x^{1/2} + \cdots$		

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1st sol	$1 + x + \frac{1}{2}x^2 + \cdots$	$1 + x + \cdots$	
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1 and xD are integral elements of $C(x)[D]/\langle L\rangle$, but D is not.

Integral bases: D-finite case

Example. $L = (2x+1) - (4x^2)D + 2x(2x-1)D^2$

$x = 1/2$	1	xD	
1st sol	$\frac{1}{2} + (x - \frac{1}{2}) + \cdots$	$\frac{1}{4} + \frac{3}{4}(x - \frac{1}{2}) + \cdots$	
2nd sol	$1 + (x - \frac{1}{2}) + \cdots$	$\frac{1}{2} + \frac{3}{2}(x - \frac{1}{2}) + \cdots$	

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Example. $L = (2x+1) - (4x^2)D + 2x(2x-1)D^2$

$x = 1/2$	1	xD	$2xD - 1$
1st sol	$\frac{1}{2} + (x - \frac{1}{2}) + \cdots$	$\frac{1}{4} + \frac{3}{4}(x - \frac{1}{2}) + \cdots$	$\frac{1}{2}(x - \frac{1}{2}) + \cdots$
2nd sol	$1 + (x - \frac{1}{2}) + \cdots$	$\frac{1}{2} + \frac{3}{2}(x - \frac{1}{2}) + \cdots$	$2(x - \frac{1}{2}) + \cdots$

Integral bases: D-finite case

Example. $L = (2x+1) - (4x^2)D + 2x(2x-1)D^2$

$x = 1/2$	1	xD	$\frac{1}{2x-1}(2xD-1)$
1st sol	$\frac{1}{2} + (x - \frac{1}{2}) + \cdots$	$\frac{1}{4} + \frac{3}{4}(x - \frac{1}{2}) + \cdots$	$\frac{1}{4} + \cdots$
2nd sol	$1 + (x - \frac{1}{2}) + \cdots$	$\frac{1}{2} + \frac{3}{2}(x - \frac{1}{2}) + \cdots$	$1 + \cdots$

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$\frac{1}{2x-1}(2xD-1)$ is an integral element of $C(x)[D]/\langle L\rangle$, but D does not belong to $C[x] + C[x]xD$.

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2nd sol	$1 + (x - \frac{1}{2}) + \cdots$	$\frac{1}{2} + \frac{3}{2}(x - \frac{1}{2}) + \cdots$	$1 + \cdots$

$\frac{1}{2x-1}(2xD-1)$ is an integral element of $C(x)[D]/\langle L\rangle$, but D does not belong to $C[x] + C[x]xD$.

In fact, $\left\{1, \frac{1}{2x-1}(2xD-1)\right\}$ is an integral basis.

Applications of integral bases: D-finite case

Example. $L = (2x+1) - (4x^2)D + 2x(2x-1)D^2$.

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Let y be a solution of L and set

$$\omega_0 := y, \quad \omega_1 := \frac{1}{2x-1}(2xD-1)\cdot y.$$

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Consider

$$f = \frac{a_0\omega_0 + a_1\omega_1}{uv^m}$$

where

$$a_0 = 4x^2 + 37x - 11, \quad a_1 = -28x^3 + 40x^2 - x - 1,$$
$$u = 4, \qquad v = (x-1)x, \qquad m = 2.$$

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Question. How to calculate $\int f dx$?

Applications of integral bases: D-finite case

Example. $L = (2x+1) - (4x^2)D + 2x(2x-1)D^2$.

Hermite reduction. Find $b_0, b_1, c_0, c_1 \in \mathbb{C}[x]$ such that

$$\frac{a_0\omega_0 + a_1\omega_1}{uv^m} = \left(\frac{b_0\omega_0 + b_1\omega_1}{v^{m-1}}\right)' + \frac{c_0\omega_0 + c_1\omega_1}{uv^{m-1}}.$$

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It follows that

$$a_0\omega_0 + a_1\omega_1 \equiv b_0 uv^m \left(\frac{\omega_0}{v^{m-1}}\right)' + b_1 uv^m \left(\frac{\omega_1}{v^{m-1}}\right)' \bmod v$$

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where

$$\omega_0' = \frac{1}{2x}\omega_0 - \frac{1-2x}{2x}\omega_1, \quad \omega_1' = \omega_1.$$

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Since $\{\omega_0, \omega_1\}$ is an integral basis of $C(x)[D]/\langle L\rangle$, we have

$$\begin{pmatrix} 41x - 11 \\ 11x - 1 \end{pmatrix} = \begin{pmatrix} 2-6x & 2-2x \\ 0 & 4-8x \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} \bmod v$$

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So $b_0 = \frac{1}{2}(4x+11)$, $b_1 = \frac{5}{2}(2x-1)$.

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Therefore

$$\int f dx = \frac{(11+4x)\omega_0 + 5(2x-1)\omega_1}{8(1-x)^2x^2} = \frac{5}{x-1}y' - \frac{2x+3}{(x-1)x}y.$$

P-recursive sequences

Definition. A sequence $f : \mathbb{N} \to C$ is called P-recursive if

$$p_0(n)f(n) + p_1(n)f(n+1) + \cdots + p_r(n)f(n+r) = 0 \quad \text{for } p_i \in C[x].$$

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Example. The Harmonic sequence $f(n) = \sum_{k=1}^{n} \frac{1}{k}$ satisfies

$$(n+1)f(n) - (2n+3)f(n+1) + (n+2)f(n+2) = 0.$$

P-recursive sequences: solution space

Setting.

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Let $\alpha \in C$. For a sequence $f \in C^{\alpha+\mathbb{Z}} := \{ u : \alpha + \mathbb{Z} \to C \}$,

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Question. How to decide the solution space $\mathrm{Sol}(L)$?

P-recursive sequences: solution space

Example 1. $f(n+1)+f(n)-f(n+2)=0$

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$\alpha=0$	$\cdots$	-2	-1					
1st sol	$\cdots$	1	0					
2nd sol	$\cdots$	0	1					

P-recursive sequences: solution space

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$\alpha=0$	$\cdots$	-2	-1	0	1	2	3	$\cdots$
1st sol	$\cdots$	1	0	1	1	2	3	$\cdots$
2nd sol	$\cdots$	0	1	1	2	3	5	$\cdots$

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sol	$\cdots$	0	1	0	0	?	$\cdots$

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$\alpha=0$	$\cdots$	-2	-1	0	1	2	$\cdots$
sol	$\cdots$	0	1	0	0	?	$\cdots$

$$f(-1)+2\cdot f(0)+0\cdot f(2)=0$$

P-recursive sequences: solution space

Example 1. $f(n+1)+f(n)-f(n+2)=0$

$\alpha=0$	$\cdots$	-2	-1	0	1	2	3	$\cdots$
1st sol	$\cdots$	1	0	1	1	2	3	$\cdots$
2nd sol	$\cdots$	0	1	1	2	3	5	$\cdots$

Example 2. $nf(n)+2n^2f(n+1)+(n+1)^2f(n+3)=0$

$\alpha=0$	$\cdots$	-2	-1	0	1	2	$\cdots$
sol	$\cdots$	0	1	0	0	?	$\cdots$

$$f(-1)+2\cdot f(0)+0\cdot f(2)=0$$

Contradiction!

Deformed P-recursive sequences

Setting [van Hoeij1999].

- $L = p_0(n) + p_1(n)S + \cdots + p_r(n)S^r \in C[n][S]$ with $p_0, p_r \neq 0$.

Let q be a new parameter. For a sequence

$$f \in C^{\alpha+\mathbb{Z}} := \{u : \alpha + \mathbb{Z} \to C((q))\},$$

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- Operator action:

$$L \cdot f = p_0(n+q)f(n) + p_1(n+q)f(n+1) + \cdots + p_r(n+q)f(n+r).$$

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- Solution space:

$$\mathrm{Sol}(L) := \{ f : \alpha + \mathbb{Z} \to C((q)) \mid L \cdot f = 0\}.$$

Deformed P-recursive sequences: solution space

Example. $L = n + 2n^2 S + (n+1)^2 S^3$.

Deformed P-recursive sequences: solution space

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$\alpha = 0$	$\cdots$	-2	-1	0				
1st sol	$\cdots$	1	0	0				
2nd sol	$\cdots$	0	1	0				
3rd sol	$\cdots$	0	0	1				

Deformed P-recursive sequences: solution space

Example. $L = n + 2n^2 S + (n+1)^2 S^3$.

$\alpha = 0$	$\cdots$	-2	-1	0	1	2	3	$\cdots$
1st sol	$\cdots$	1	0	0	$\frac{-q+2}{(q-1)^2}$	$\frac{2q-4}{q^2}$	$\frac{-4q+8}{(q+1)^2}$	$\cdots$
2nd sol	$\cdots$	0	1	0	0	$\frac{-q+1}{q^2}$	$\frac{2q-2}{(q+1)^2}$	$\cdots$
3rd sol	$\cdots$	0	0	1	$\frac{-2q^2+8q-8}{(q-1)^2}$	$\frac{4q^2-16q+16}{q^2}$	$\frac{-8q^2+31q-32}{(q+1)^2}$	$\cdots$

Fact. $\mathrm{Sol}(L)$ is a $C((q))$-vector space of dimension $r = \mathrm{ord}(L)$.

Deformed P-recursive sequences: solution space

Example. $L = n + 2n^2 S + (n+1)^2 S^3$.

$\alpha = 0$	$\cdots$	-2	-1	0	1	2	3	$\cdots$
1st sol	$\cdots$	1	0	0	$2+\cdots$	$-4q^{-2}+2q^{-1}$	$8+20q+\cdots$	$\cdots$
2nd sol	$\cdots$	0	1	0	0	$q^{-2}-q^{-1}$	$-1+6q+\cdots$	$\cdots$
3rd sol	$\cdots$	0	0	1	$-8+\cdots$	$16q^{-2}-16q^{-1}+\cdots$	$-32+95q+\cdots$	$\cdots$

Question. How to define a value function on $V = C(x)[S]/\langle L\rangle$?

Value functions: P-recursive case

Definition. For an operator $B \in V = C(n)[S]/\langle L \rangle$, we define $\mathrm{val}_z \colon V \to \mathbb{Z} \cup \{\infty\}$ by

$$\mathrm{val}_z(B) := \min_{b \in \mathrm{Sol}(L)} \Big(\nu_q((B \cdot b)(z)) - \liminf_{n \to \infty} \nu_q(b(z-n)) \Big)$$

for any $z \in \alpha + \mathbb{Z}$.

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Remark. For a normalized basis $\{b_1, \cdots, b_r\}$ of $\mathrm{Sol}(L)$, we have

$$\mathrm{val}_z(B) = \min_{j=1}^{r} \Big(\nu_q\big((B \cdot b_j)(z)\big) \Big).$$

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Theorem. val_z is indeed a value function on V.

Definition. An operator B of V is called integral at z if $\mathrm{val}_z(B) \geq 0$.

Theorem. The integral elements of V form a free $C[n]_{n-z}$-module.

$$\mathrm{Disc}_z(B_1, \cdots, B_r) := \nu_q\big(\det(((B_i \cdot b_j)(z))_{i,j=1}^r)\big)$$

Integral bases: P-recursive case

Example. $L = n + 2n^2 S + (n+1)^2 S^3$

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1st sol	$\cdots$	1	0	0	$2+\cdots$	$-4q^{-2}+2q^{-1}$	$8+20q+\cdots$	$\cdots$
2nd sol	$\cdots$	0	1	0	0	$q^{-2}-q^{-1}$	$-1+6q+\cdots$	$\cdots$
3rd sol	$\cdots$	0	0	1	$-8+\cdots$	$16q^{-2}-16q^{-1}+\cdots$	$-32+95q+\cdots$	$\cdots$

Integral bases: P-recursive case

Example. $L = n + 2n^2 S + (n+1)^2 S^3$

$n=1$	1			
1st sol	$2+\cdots$			
2nd sol	0			
3rd sol	$-8+\cdots$			

Integral bases: P-recursive case

Example. $L = n + 2n^2 S + (n+1)^2 S^3$

$n=1$	1	S		
1st sol	$2+\cdots$	$-4q^{-2}+2q^{-1}$		
2nd sol	0	$q^{-2}-q^{-1}$		
3rd sol	$-8+\cdots$	$16q^{-2}-16q^{-1}+\cdots$		

1 is an integral element of $C(n)[S]/\langle L\rangle$, but S not.

Integral bases: P-recursive case

Example. $L = n + 2n^2 S + (n+1)^2 S^3$

$n=1$	1	$(n-1)^2 S$		
1st sol	$2+\cdots$	$-4+2q$		
2nd sol	0	$1-q$		
3rd sol	$-8+\cdots$	$16-16q+\cdots$		

Integral bases: P-recursive case

Example. $L = n + 2n^2S + (n+1)^2S^3$

$n=1$	1	$(n-1)^2S$	S^2	
1st sol	$2+\cdots$	$-4+2q$	$8+20q+\cdots$	
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Integral bases: P-recursive case

Example. $L = n + 2n^2 S + (n+1)^2 S^3$

$n=1$	1	$(n-1)^2 S$	S^2	$S^2 - 2(n-1)^2 S$
1st sol	$2+\cdots$	$-4+2q$	$8+20q+\cdots$	$24q+\cdots$
2nd sol	0	$1-q$	$-2+6q+\cdots$	$4q+\cdots$
3rd sol	$-8+\cdots$	$16-16q+\cdots$	$-32+95q+\cdots$	$63q+\cdots$

Integral bases: P-recursive case

Example. $L = n + 2n^2 S + (n+1)^2 S^3$

$n=1$	1	$(n-1)^2 S$	S^2	$-2(n-1)S + \frac{1}{n-1}S^2$
1st sol	$2+\cdots$	$-4+2q$	$8+20q+\cdots$	$24+\cdots$
2nd sol	0	$1-q$	$-2+6q+\cdots$	$4+\cdots$
3rd sol	$-8+\cdots$	$16-16q+\cdots$	$-32+95q+\cdots$	$63+\cdots$

integral closure

$$B_2 := \frac{1}{n-1}((n-1)^2 S + S^2)$$

$$\boxed{C[n]_{n-1} + C[n]_{n-1}(n-1)^2 S + C[n]_{n-1} S^2}$$

Integral bases: P-recursive case

Example. $L = n + 2n^2 S + (n+1)^2 S^3$

$n=1$	1	$(n-1)^2 S$	S^2	$-2(n-1)S + \frac{1}{n-1} S^2$
1st sol	$2 + \cdots$	$-4 + 2q$	$8 + 20q + \cdots$	$24 + \cdots$
2nd sol	0	$1 - q$	$-2 + 6q + \cdots$	$4 + \cdots$
3rd sol	$-8 + \cdots$	$16 - 16q + \cdots$	$-32 + 95q + \cdots$	$63 + \cdots$

$\left\{\ 1,\ (n-1)^2 S,\ -2(n-1)S + \frac{1}{n-1} S^2\ \right\}$ is an integral basis of $C(n)[S]/\langle L \rangle$ at $z = 1$.

Summary

Main results.

- Extend van Hoeij's algorithm to valued vector spaces.
- Construct integral bases for P-recursive sequences.

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Thank you!