JOURNAL OF ALGEBRA **183,** 319–358 (1996)
ARTICLE NO. 0223

The Structure of the Partition Algebras

Paul Martin

Mathematics Department, City University, Northampton Square, London EC1V 0HB, United Kingdom

Communicated by Gordon James

Received February 27, 1995; revised October 16, 1995

1. INTRODUCTION

For Q a complex number and n a natural number the partition algebra $P_n(Q)$ [14, 12] is a generalization both of an algebra $\mathscr{D}_n(Q)$ introduced by Brauer in 1937 [2], and also of the Temperley–Lieb algebra $T_n(Q)$ [19]. It is of current interest because of its role in statistical mechanics [16], specifically as the generalization of the Temperley–Lieb algebra appearing in lattice models (as in Baxter [1]) in high physical dimensions [15].

Recall that the linear basis in the Brauer algebra $\mathscr{D}_n(Q)$ is enumerated by the partitions of the set $\{1, 2, \dots, 2n\}$ into pairs. The linear basis in $P_n(Q)$ may be enumerated by all partitions of the same set. As in the Brauer algebra it is the multiplication in $P_n(Q)$ which depends on the complex parameter Q. It is known [16] that $P_n(Q)$ is semi-simple over the complex field unless Q is an integer $0 \leq Q < 2n - 1$, and this semi-simple generic structure is known [14]. In short, the simple components may be indexed by all partitions λ of the integers 0, $1, \dots, n$. This result, and several aspects of the present work, can be compared instructively with studies of the Brauer algebra by Brown [3] and Hanlon and Wales [9].

This paper is concerned with the structure of $P_n(Q)$ over the complex field for every n and Q. We show that these algebras are quasi-hereditary for $Q \neq 0$, in the sense of Cline *et al*. [4] or Dlab and Ringel [7, Section 4]. For every partition λ we, hence, introduce a Weyl module of $P_n(Q)$ which is irreducible if Q is generic and which has a basis independent of Q. The main result of the paper is a description (Section 2) of the structure of these module for every Q and, hence, of the structure of indecomposable projective modules.

0021-8693/96 $18.00

After a priming operation relying on explicit calculations this is essentially a concrete application of some category theory ideas of Green [8].

1.1. *Basic Definitions*

For M a finite set let $\mathbf{E}_M$ be the set of equivalence relations on M and $\mathbf{S}_M$ the set of partitions of M into disjoint subsets. Recall that $\mathbf{E}_M$ has a natural bijection with $\mathbf{S}_M$. (N.B., we will work interchangeably with these two descriptions without further comment.)

DEFINITION 1. Let ρ be any finite equivalence relation, then $\#(\rho)$ is the number of equivalence classes of ρ.

Let M, N be sets and note that $(\mathbf{E}_M, \subseteq)$ is a lattice. Then

DEFINITION 2.

$$R_N : \mathbf{E}_{M \cup N} \to \mathbf{E}_{M \setminus N} \tag{1}$$

$$R_N : \rho \mapsto \text{largest } \mu \in \mathbf{E}_{M \setminus N} \text{ s.t. } \mu \subseteq \rho \tag{2}$$

(that is, $\mu \subseteq \rho$ as a relation) and

$$C_N : \mathbf{E}_{M \cup N} \to \mathbb{N}_0 \tag{3}$$

$$C_N : \rho \mapsto \#(\rho) - \#(R_N(\rho)). \tag{4}$$

DEFINITION 3. For $\mu \in \mathbf{E}_N, \nu \in \mathbf{E}_N$ define a product $\mu \cdot \nu \in \mathbf{E}_{M \cup N}$ as the smallest $\rho \in \mathbf{E}_{M \cup N}$ s.t. $\rho \supseteq \mu \cup \nu$.

Note that f: $M \to N$ a bijection leads naturally to a bijection $f_{\mathbf{E}}$: $\mathbf{E}_M \to \mathbf{E}_N$.

Let T be a finite set; then define $T_n = T \times \{1, 2, \ldots, n\}$ and if $\alpha \in T$ write α_i for $(\alpha, i) \in T_n$.

If we let $a(\alpha, \beta)$ denote an element of $\mathbf{E}_{\{\alpha, \beta\}_n}$ then by $a(\gamma, \delta) \in \mathbf{E}_{\{\gamma, \delta\}_n}$ we understand the image of $a(\alpha, \beta)$ under the isomorphism $f(\alpha_i) = \gamma_i, f(\beta_i) = \delta_i$ for all i.

Note that $a(\alpha, \gamma) \cdot b(\gamma, \beta) \in \mathbf{E}_{\{\alpha, \gamma, \beta\}_n}$.

DEFINITION 4 (Partition algebra). Let k be a field, $Q \in k$, and $n \in \mathbb{N}$. Then the partition algebra $kP_n(Q)$ is defined as follows. It has basis $\mathbf{E}_{\{\alpha, \beta\}_n}$; i.e., it is k-span$(\mathbf{E}_{\{\alpha, \beta\}_n})$ as a k-space, and multiplication is given by

$$ab = a(\alpha, \beta) * b(\alpha, \beta) = Q^{C_{\{\gamma\}_n}(a(\alpha, \gamma) \cdot b(\gamma, \beta))} R_{\{\gamma\}_n}(a(\alpha, \gamma) \cdot b(\gamma, \beta))$$
$$(a, b \in \mathbf{E}_{\{\alpha, \beta\}_n}). \tag{5}$$

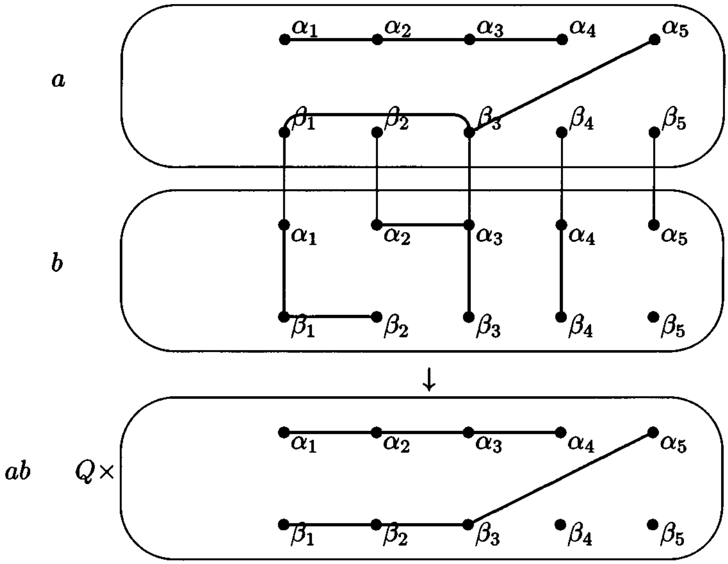


FIG. 1. Pictorial realisations of partitions and their composition.

In short the Partition algebra product is summarized by the example in Fig. 1. This gives a pictorial realization of two partitions of $\{\alpha, \beta\}_5 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}$, namely,

$$a = \{\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, \{\alpha_5, \beta_1, \beta_3\}, \{\beta_2\}, \{\beta_4\}, \{\beta_5\}\}$$

and

$$b = \{\{\alpha_1, \beta_1, \beta_2\}, \{\alpha_2, \alpha_3, \beta_3\}, \{\alpha_4, \beta_4\}, \{\alpha_5\}, \{\beta_5\}\}$$

as clusters, and of composition of partitions ($ab = Q.\{\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, \{\alpha_5, \beta_1, \beta_2, \beta_3\}, \{\beta_4\}, \{\beta_5\}\}$) by an appropriate juxtaposition (cf. p. 868 of [2]).

We define the elements of $\mathbf{S}_{\{\alpha, \beta\}_n}$,

$$\mathbb{1} = \{\{\alpha_1, \beta_1\}, \{\alpha_2, \beta_2\}, \dots \{\alpha_i, \beta_i\}, \dots, \{\alpha_n, \beta_n\}\} \tag{6}$$

$$\mathscr{I}_{ij} = \left\{\{\alpha_1, \beta_1\}, \{\alpha_2, \beta_2\}, \dots \left\{\alpha_i, \beta_j\right\}, \left\{\alpha_j, \beta_i\right\}, \dots, \{\alpha_n, \beta_n\}\right\}, \quad i, j = 1, 2, \dots, n, \tag{7}$$

$$A^i = \{\{\alpha_1, \beta_1\}, \{\alpha_2, \beta_2\}, \dots \{\alpha_i\}, \{\beta_i\}, \dots, \{\alpha_n, \beta_n\}\}, \quad i = 1, 2, \dots, n, \tag{8}$$

$$A^{ij} = \left\{\{\alpha_1, \beta_1\}, \{\alpha_2, \beta_2\}, \dots \left\{\alpha_i, \beta_j, \alpha_j, \beta_i\right\}, \dots, \{\alpha_n, \beta_n\}\right\}, \quad i, j = 1, 2, \dots, n, \tag{9}$$

and we will use the same symbols for their images in $\mathbf{E}_{\{\alpha,\beta\}_n}$, and, hence, in $kP_n(Q)$.

In what follows any proofs which are omitted may be found in [14].

PROPOSITION 1. *The partition algebra is associative and unital, with unit* $\mathbb{1}$, *and the elements in Eq.* (6–9) *generate* $kP_n(Q)$.

Also define elements $F_i^{(n)}$ and idempotents $E_i^{(n)} \in kP_n(Q)$ by $F_n^{(n)} = E_n^{(n)} = \mathbb{1}$ and

$$F_i^{(n)} = A^1 A^2 \cdots A^{n-i}, \qquad E_i^{(n)} = \frac{A^1}{Q}\frac{A^2}{Q} \cdots \frac{A^{n-i}}{Q} \qquad (0 \le i < n; Q \neq 0).$$

DEFINITION 5 (Propagating number).

$$\#^P : \mathbf{E}_{\{\alpha,\beta\}_n} \to \mathbb{N}_0 \tag{10}$$

$$\#^P(\rho) = \#\big(R_{\{\alpha\}_n}(\rho)\big) + \#\big(R_{\{\beta\}_n}(\rho)\big) - \#(\rho). \tag{11}$$

Then $\#^P(F_i^{(n)}) = i$ and extending so that

$$\#^P\left(\sum_{x \in \mathbf{E}_{\{\alpha,\beta\}_n}} c_x x\right) = \max \#^P(x) \text{ s.t. coefficient } c_x \neq 0,$$

we have

PROPOSITION 2.

$$\#^P(ab) \le \min\big(\#^P(a), \#^P(b)\big).$$

1.2. *Basic Properties*

For any n note the algebra injection

$$s : kP_{n-1}(Q) \to kP_n(Q) \tag{12}$$

given in terms of partitions by

$$s : a \mapsto a \cup \big\{\{\alpha_n, \beta_n\}\big\}. \tag{13}$$

Henceforth any reference to *inclusion* of $kP_{n-1}(Q)$ in $kP_n(Q)$ will mean this map. Note the algebra surjection onto the group algebra of the symmetric group

$$t : kP_n(Q) \to k\mathrm{S}_n \tag{14}$$

given by

$$t: A^i, A^{ij} \mapsto 0 \tag{15}$$

$$t: \mathscr{I}_{i\,i+1} \mapsto \sigma_i. \tag{16}$$

The kernel is $kP_n(Q)\, A^n\, kP_n(Q)$ (consider Proposition 2) and there is a corresponding injection

$$\psi: k\mathrm{S}_n \to kP_n(Q) \tag{17}$$

such that $t\psi$ is the identity map on $k\mathrm{S}_n$. Here any reference to *inclusion* of $k\mathrm{S}_n \subset kP_n(Q)$ will mean the injection ψ.

Let $\mathscr{A}$ and $\mathscr{B}$ be algebras, and let $\theta: \mathscr{A} \to \mathscr{B}$ be any algebra homomorphism. Then any left $\mathscr{B}$ module ${}_{\mathscr{B}}M$ is also a left $\mathscr{A}$ module, with action given by $am = \theta(a)m$. In this paper we call this the *restriction* via θ (or, where no ambiguity arises, just the restriction) of ${}_{\mathscr{B}}M$ to $\mathscr{A}$, denoted ${}_{\mathscr{B}}M \downarrow_{\mathscr{A}}$.

Recall from Cline *et al*. [4] (or more directly from Dlab and Ringel [7, Section 4]) that for an algebra $\mathscr{A}$ an idempotent $e \in \mathscr{A}$ is a heredity idempotent if $e\mathscr{A}e$ is semi-simple and the map $\mathscr{A}e \otimes_{e\mathscr{A}e} e\mathscr{A} \to \mathscr{A}e\mathscr{A}$ is bijective; and that a list of idempotents $(e_1, e_2, \dots, e_l)$ in $\mathscr{A}$ is a heredity chain if $\mathscr{A}e_1\mathscr{A} = \mathscr{A}$,

$$\mathscr{A}e_i\mathscr{A} \subset \mathscr{A}e_{e-1}\mathscr{A} \tag{18}$$

for all i, and e_i is a heredity idempotent mod $\mathscr{A}e_{i+1}\mathscr{A}$ for all i (with $e_{l+1} = 0$). Now put $\mathscr{A}_i = \mathscr{A}/\mathscr{A}e_{i+1}\mathscr{A}$ and recall also that a heredity chain is maximal if $e_i\mathscr{A}_ie_i$ is a division ring for each i. If $(e_1, e_2, \dots, e_l)$ is a maximal heredity chain for $\mathscr{A}$ then in this context the left $\mathscr{A}$ module obtained by restriction from the left $\mathscr{A}_i$ module $\mathscr{A}_ie_i$ will be called a Weyl module Δ_i. A Weyl module is either simple or has a unique proper maximal submodule. Inequivalent Weyl modules have inequivalent simple tops, and all simple $\mathscr{A}$ modules arise in this way.

Recall now from Green [8, Section 6.2] that for $\mathscr{A}$ an algebra and idempotent $e \in \mathscr{A}$ there are functors

$$(e\mathscr{A}e - \mathrm{mod}) \xrightarrow{g} (\mathscr{A} - \mathrm{mod}) \xrightarrow{f} (e\mathscr{A}e - \mathrm{mod}) \tag{19}$$

$$g(N) = \mathscr{A}e \otimes_{e\mathscr{A}e} N, \qquad f(M) = eM.$$

Note that fg is the isomorphism functor on $e\mathscr{A}e$, f is exact, and g takes projectives to projectives.

Let us combine these ideas and note that if $\mathscr{A}$ has heredity chain $(e_1, e_2, \ldots, e_l)$ then $e_i \mathscr{A} e_i$ has heredity chain $(e_i, e_{i+1}, \ldots, e_l)$ and Weyl modules

$$\left\{\Delta_j^{e_i} = e_i \Delta_j \mid \Delta_j \text{ a Weyl module of } \mathscr{A};\ e_i \Delta_j \neq 0\right\}$$

and

$$f(\Delta_j) = \begin{cases} \Delta_j^{e_i} & \text{if it exists,} \\ 0 & \text{otherwise;} \end{cases} \qquad g\left(\Delta_j^{e_i}\right) = \Delta_j,$$

PROPOSITION 3. *For $k = \mathbb{C}$ and $Q \neq 0$ the list $(E_n^{(n)}, E_{n-1}^{(n)}, \ldots, E_0^{(n)})$ is a heredity chain for $kP_n(Q)$.*

Proof. Equation (18) follows in our case from Proposition 2 on noting that $kP_n(Q)\, E_i^{(n)}\, kP_n(Q)$ has a basis

$$\left\{x \in \mathbf{E}_{\{\alpha,\, \beta\}_n} \mid \#^P(x) \leq i\right\}. \tag{20}$$

The semi-simplicity property follows on noting that the quotient algebra in our case is isomorphic to the group algebra over k of some permutation group on i objects (in fact it is the symmetric group S_i). The bijective property is proved similarly. ∎

Indeed for G any simple unoriented graph with vertices $1, 2, \ldots, n$, and $kD_G(Q)$ the subalgebra of $kP_n(Q)$ generated by $\{\mathbb{1}, A^1, A^2, \ldots, A^n\} \cup \{A^{ij}: (i, j) \text{ and edge of } G\}$ then we have

PROPOSITION 4. *For $k = \mathbb{C}, kD_G(Q)$ is also quasi-hereditary with the same heredity chain.*

The proof is virtually identical on replacing $P_n(Q)$ with $D_G(Q)$, except that the permutation group on i objects is not always the whole symmetric group!

1.3. *Properties of $P_n(Q)$* − mod

The structure of $P_n(Q) = \mathbb{C}P_n(Q)$ is all but determined by the following readily proven results (see [14] for details).

PROPOSITION 5. *For $Q \neq 0$ there are isomorphisms of algebras*

$$\kappa_n: A^n\, kP_n(Q)\, A^n \to kP_{n-1}(Q) \tag{21}$$

given in terms of partitions by

$$\kappa_n: a \cup \{\{\alpha_n\}, \{\beta_n\}\} \mapsto Q\, a \tag{22}$$

and

$$kP_n(Q)/kP_n(Q)\ A^n\ kP_n(Q) \cong kS_n. \tag{23}$$

Then (cf. Green [8, Section 6.2]) there are functors between categories

$$(P_{n-1}(Q) - \text{mod}) \xrightarrow{\mathscr{G}} (P_n(Q) - \text{mod}) \xrightarrow{\mathscr{F}} (P_{n-1}(Q) - \text{mod}). \tag{24}$$

(N.B., we will use the same symbols $\mathscr{F}$, $\mathscr{G}$ for any n) such that $\mathscr{F}\mathscr{G}$ is an isomorphism on $(P_{n-1}(Q) - \text{mod})$ and $\mathscr{G}\mathscr{F}(P_n(Q)) = P_n(Q)\ A^n\ P_n(Q)$. For example, for M an ideal of $P_{n-1}(Q)$

$$\begin{aligned}\mathscr{G}(M) &= P_n(Q)\ A^n \otimes_{A^n\ P_n(Q)\ A^n} \kappa_n^{-1}(M) = P_n(Q)\,A^n\kappa_n^{-1}(M)\\ &= P_n(Q)\ A^n s(M).\end{aligned} \tag{25}$$

Note that $A^nP_n(Q)A^n = QA^ns(P_{n-1}(Q))$ is an intermediate *identity* in Eq. (21). Note also that both $P_n(Q)$ and $P_n(Q)A^n$ can be viewed as right $P_{n-1}(Q)$ modules by virtue of the algebra injection s. The former is a restriction, and in the sense of the letter we may write

$$\mathscr{G}(M) = P_n(Q)\,A^n \otimes_{P_{n-1}(Q)} M \tag{26}$$

for any left $P_{n-1}(Q)$ module M.

Thus, in particular, to classify simple/Weyl modules of $P_n(Q)$ given those of $P_{n-1}(Q)$ we note that they are in correspondence with those of $P_{n-1}(Q)$, except that modules of $P_n(Q)/P_n(Q)A^nP_n(Q)$ have no image in $P_{n-1}(Q) -$ mod and must be added "by hand" using Eq. (23). Consequently by induction on n with $P_0 \cong k$ as base—and hereafter we will be concerned exclusively with $k = \mathbb{C}$—we have the following.

COROLLARY 5.1. *For $k = \mathbb{C}$ and $Q \neq 0$, irreducible representations of $P_n(Q)$ are indexed by the set of partitions (of integers, i.e., in the sense of Macdonald* [11]) $\mathscr{L}_n = \{\lambda \vdash i : i = 0, 1, 2, \dots, n\}$.

For example, the left module $P_n(Q)F_0^{(n)}$ is a Weyl module, and it and the corresponding simple molule will be indexed by the single "empty partition" $\lambda \vdash 0$. More generally, we use

DEFINITION 6. Let S(λ) be the simple module of $\mathbb{C}S_n$ usually associated to the partition λ, and let $J_\lambda \in \mathbb{C}S_n$ be any primitive idempotent such that $\mathbb{C}S_nJ_\lambda \cong S(\lambda)$ [10].

Then for $n > 0$, $\psi(\mathbb{C}S_nJ_\lambda)$ is the simple $P_n(Q)/P_n(Q)\ A^n\ P_n(Q)$ module associated to partition $\lambda \vdash n$ by the isomorphism with $\mathbb{C}S_n$ (and therefore having *no* explicity Q dependence).

DEFINITION 7. Then $\mathcal{S}_\lambda = \mathcal{S}_\lambda(n)$ is the left $P_n(Q)$ module obtained by restricting $\psi(\mathbb{C}S_n J_\lambda)$. The $P_{n+1}(Q)$ module obtained by applying a functor of type $\mathcal{G}$ to $\mathcal{S}_\lambda(n)$ is $\mathcal{S}_\lambda(n+1)$ (also written $\mathcal{S}_\lambda$ where no ambiguity arises). Corresponding $P_{n+m}(Q)$ modules are defined by iterating the above process.

We will examine these modules in detail in Section 3, where we will see the following.

PROPOSITION 6. *The left modules $\mathcal{S}_\lambda(n)$ are the Weyl modules of $P_n(Q)$.*

For M a left $P_n(Q)$ module let $M\downarrow_{P_{n-1}}$ represent the restriction to $P_{n-1}(Q) \subset P_n(Q)$ (i.e., via $s(P_{n-1}(Q))$), and $M\uparrow_{P_{n+1}}$ similarly represent induction to $P_{n+1}(Q)$.

PROPOSITION 7. *Let M be a left $P_n(Q)$ module. Then induction and restriction are related by the isomorphism of left $P_{n+1}(Q)$ modules $(\mathcal{G}\mathcal{G}(M))\downarrow_{P_{n+1}} \cong M\uparrow_{P_{n+1}}$, that is,*

$$(\mathcal{G}(\mathcal{G}(M)))\downarrow_{P_{n+1}} \cong P_{n+1}(Q) \otimes_{P_n(Q)} M. \tag{27}$$

Proof. From above (see also [14])

$$\mathcal{G}(M) = P_{n+1}(Q)\, A^{n+1} \otimes_{P_n(Q)} M \tag{28}$$

and the isomorphism of left $P_{n+1}(Q)$ modules is

$$\left(P_{n+2}(Q)\, A^{n+2} A^{n+1} \otimes_{P_n(Q)} M\right)\downarrow_{P_{n+1}} \cong P_{n+1}(Q) \otimes_{P_n(Q)} M \tag{29}$$

given, with $m \in M$, by

$$A^{n+1\,n+2} A^{n+1} A^{n+2} m \mapsto 1m \tag{30}$$

since

$$A^{n+1\,n+2} A^{n+1} A^{n+2} = \{\{\alpha_1, \beta_1\}\{\alpha_2, \beta_2\} \ldots \{\alpha_n, \beta_n\}\{\alpha_{n+1}, \alpha_{n+2}\}\{\beta_{n+1}\}\{\beta_{n+2}\}\}, \tag{31}$$

whereupon β_{n+1}, β_{n+2} may be ignored and α_{n+2} plays the role of β_{n+1}. ∎

1.4. *Overview of This Paper*

Note that if, for some given n, a left module $\mathcal{S}_\lambda(n)$ has a simple proper submodule in some specialization of Q, then the corresponding module $\mathcal{S}_\lambda(m)$ at any $m > n$ will also have a proper submodule by Eq. (24).

Crucially, at the *lowest* value of n for which $\mathscr{S}_\lambda(n)$ has a submodule M this submodule will be isomorphic to some left $P_n(Q)/P_n(Q)A^nP_n(Q)$ module. This is because in any other case the morphism $M \hookrightarrow \mathscr{S}_\lambda(n)$ would have an image in level $P_{n-1}(Q)$, giving a contradiction. Thus all occurrences of submodules can be located by locating those isomorphic to left $P_n(Q)/P_n(Q)A^nP_n(Q)$ modules and then applying the functor $\mathscr{G}$. This is the main computational device of the present paper. That is to say, we search for invariant subspaces isomorphic to these simple modules.

We will call the one dimensional $P_n(Q)$ module $\mathscr{S}_{(n)}$ the trivial module. For example, for $m \in \mathscr{S}_{(n)}$

$$A^i m = A^{ij} m = 0 \tag{32}$$

and

$$\mathscr{I}_{ij} m = \mathbb{1} m = m. \tag{33}$$

The main result of this paper, on the structures of the exceptional cases, is given in Section 2. After a brief review (cf. [14]) of the explicit construction of bases for Weyl modules in Section 3 we proceed to a proof by looking in all Weyl modules $\mathscr{S}_\lambda(n)$ (with $\lambda \vdash l < n$) for invariant subspaces in which Eq. (32) is obeyed at some Q. We do this first by explicit calculation (Section 4) and then using Frobenius reciprocity (Section 5). Eventually (in Section 5.1) we are able to deduce invariant factors for the Gram matrices of Weyl modules over the ring $Z[Q]$ and, hence, to confirm the structure of every submodule.

2. THE MAIN RESULT

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be an integer partition represented as a Young diagram, and let λ_{ij} be the j^{th} box in the i^{th} row of the diagram. For L an integer the L-diagram of λ is obtained from the Young diagram by placing a number in each box such that the content of box λ_{ij} is $L - i + j$. Note that the content of the first box in the first row is L.

For example,

$$\begin{array}{|c|c|} \hline 3 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

is the 3-diagram of (2, 1).

Let λ, μ be partitions such that $\lambda \supset \mu$. Then, for Q an integer, (μ, λ) is a Q-pair of partitions if the $|\mu|$-diagram of λ differs from the $|\mu|$-diagram of μ by a horizontal row of boxes of which the last (rightmost) contains Q.

The following table allows the determination of all Q-pairs (μ, λ) for $\mu \in \mathscr{L}_3$:

$$\begin{array}{c} 0 \\ \hline \end{array}\quad 1\quad 2\quad 3\quad 4\quad 5\ \cdots$$

$$\begin{array}{|c|l} \cline{1-1} 1 & 2\ \ 3\ \ 4\ \ 5\ \cdots \\ \cline{1-1} \multicolumn{1}{c}{0} & \end{array}$$

$$\begin{array}{|c|l} \cline{1-1} 2 & 3\ \ 4\ \cdots \\ \cline{1-1} 1 & \\ \cline{1-1} \multicolumn{1}{c}{0} & \end{array} \qquad \begin{array}{|c|c|l} \cline{1-2} 2 & 3 & 4\ \ 5\ \ 6\ \cdots \\ \cline{1-2} \end{array}$$

$$\begin{array}{|c|l} \cline{1-1} 3 & 4\ \ 5\ \cdots \\ \cline{1-1} 2 & \\ \cline{1-1} 1 & \\ \cline{1-1} \multicolumn{1}{c}{0} & \end{array} \qquad \begin{array}{|c|c|l} \cline{1-2} 3 & 4 & 5\ \ 6\ \cdots \\ \cline{1-2} 2 & \multicolumn{1}{|c}{3} & \\ \cline{1-1} \multicolumn{1}{c}{1} & & \end{array} \qquad \begin{array}{|c|c|c|l} \hline 3 & 4 & 5 & 6\ \ 7\ \ 8\ \cdots \\ \hline \multicolumn{1}{c}{2} & \multicolumn{1}{c}{3} & \multicolumn{1}{c}{4} & \end{array}$$

For example, the middle diagram on the bottom row may be used to see that $(\mu = (2,1),\ \lambda = (4,1))$ is a 6-pair.

Recall that a *tableau* is a sequence of partitions such that each skew diagram of adjacent partitions is a horizontal strip. A Q-tableau is a tableau such that each adjacent pair of partitions is also a Q-pair.

Remark 1. Note that a horizontal strip is not necessarily a horizontal row (see MacDonald [11]), but in Q-tableau the skew diagrams are horizontal rows and, for instance, in the graphical representation of a Q-tableau as a tableau (with the boxes of $\lambda^{(0)}$ left blank), each number appears in one row only. Suppose a number k has leftmost appearance in box λ_{ij}, then the number of (consecutive) appearances, r_k say, in that row is $Q + i - j - z + 1$, where $z = |\lambda^{(0)}| + \sum_{l=1}^{k-1} r_l$.

A maximal Q-tableau, for given n, is one which cannot be extended by the addition of any $\lambda \in \mathscr{L}_n$. It is straightforward that each partition in $\mathscr{L}_n$ occurs at most once each on the left and right of Q-pairs, thus, we have

PROPOSITION 8. *Maximal Q-tableaux* (*including those of length* 1) *partition* $\mathscr{L}_n$.

DEFINITION 8. For given Q and n let I be the maximal proper submodule of $\mathscr{S}_\lambda$ (I unique by the quasi-heredity property). Then define

$\mathscr{S}_\lambda^Q$ as the simple top of $\mathscr{S}_\lambda$, i.e., such that

$$0 \to I \to S_\lambda \to S_\lambda^Q \to 0 \tag{34}$$

is a short exact sequence.

In particular for Q generic (not an integer, say) then $\mathscr{S}_\lambda^Q = \mathscr{S}_\lambda$.

PROPOSITION 9. *For given Q and n let*

$$\lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \lambda^{(3)} \subset \cdots \subset \lambda^{(r)}$$

be a maximal Q-tableau with $|\lambda^{(r)}| \leq n$. Then there is an exact sequence of left $\mathbb{C}P_n(Q)$ modules

$$0 \to \mathscr{S}_{\lambda^{(r)}} \to \mathscr{S}_{\lambda^{(r-1)}} \to \mathscr{S}_{\lambda^{(r-2)}} \to \ \ldots \to \mathscr{S}_{\lambda^{(0)}} \tag{35}$$

and, conversely, every sequence of (non-zero) morphisms of distinct left modules of this type corresponds to a Q-tableau. The images of the morphisms are the associated simple modules $\mathscr{S}_\lambda^Q$, so that these $\mathscr{S}_\lambda$ modules have Loewy structure

$$\mathscr{S}_{\lambda^{(i)}} = \begin{array}{c} \mathscr{S}^Q_{\lambda^{(i)}} \\ \mathscr{S}^Q_{\lambda^{(i+1)}} \end{array} \tag{36}$$

and the associated indecomposable projectives have structure

$$P_{\lambda^{(i)}} = \begin{array}{ccc} & \mathscr{S}^Q_{\lambda^{(i)}} & \\ \mathscr{S}^Q_{\lambda^{(i+1)}} & & \mathscr{S}^Q_{\lambda^{(i-1)}} \\ & \mathscr{S}^Q_{\lambda^{(i)}} & \end{array}. \tag{37}$$

The dimensions of all simple and indecomposable projective modules then follow imediately from the generic dimensions, which we review in the next section. For example, from the table above for $n = 5$

$$\dim\left(\mathscr{S}^2_{(0)}\right) = \dim\left(\mathscr{S}_{(0)}\right) - \left(\dim\left(\mathscr{S}_{(3)}\right) - \left(\dim\left(\mathscr{S}_{(3,1)}\right) - \dim\left(\mathscr{S}_{(3,1^2)}\right)\right)\right). \tag{38}$$

Blocks of the type described above are not new (see [13 pp. 169–174]).

In order to prove Proposition 9 we will establish the morphisms in Eq. (35) by categorical techniques (Section 4) and the structure by examining the Gram matrices of generic simple modules. We give explicit bases in Section 3 and compute the determinants of the Gram matrices in Section 5.1.

3. LEFT IDEALS AND BASES

In what follows it will be convenient to use a streamlined notation for partitions $a \in \mathbf{S}_M$. Where unambiguous, we will strip off all commas and replace curly brackets with round, thus

$$\{\{\alpha, \beta, \gamma\}, \{\delta\}, \{\epsilon\}\} = ((\alpha\beta\gamma)(\delta)(\epsilon)).$$

Each $a_i \in a \in \mathbf{S}_M$ is a subset of M and is called a "part" of a. Each part of degree 1 is a "singleton," and the partition with all parts singletons corresponds to the "empty" equivalence relation in $\mathbf{E}_M$.

Bases of the indecomposable (and, for $Q \neq 0$, projective) left module $\mathscr{S}_{(0)} = P_n(Q)F_0^{(n)}$ have an obvious isomorphism with $\mathbf{S}_{\{\alpha\}_n}$. In particular, let $b \in \mathbf{S}_{\{\beta\}_n}$ be the partition corresponding to the empty equivalence relation, i.e., $b = ((\beta_1)(\beta_2)\cdots(\beta_n))$. Then a basis of $\mathscr{S}_{(0)}$ is given by

$$B_p = \left\{a \cdot b : a \in \mathbf{S}_{\{\alpha\}_n}, b = ((\beta_1)(\beta_2)\ldots(\beta_n))\right\}.$$

Note that here $R_{\{\beta\}_n}(a \cdot b) = a$. This B_p is the unique basis of $\mathscr{S}_{(0)}$ which is a subset of $\mathbf{S}_{\{\alpha, \beta\}_n}$.

Define a map from partitions of a set M to partitions of the integer $|M|$

$$\zeta : \mathbf{S}_M \to \{\lambda : \lambda \vdash |M|\} \tag{39}$$

by in each partition ordering the subsets M_i of M such that $|M_{i+1}| \leq |M_i|$ and then

$$\zeta : \{M_1, M_2, \ldots\} \mapsto (|M_1|, |M_2|, \ldots). \tag{40}$$

It will be convenient to write λ^a for $\zeta(a)$, and call it the partition "shape" of a.

For $W \in B_p$ then $\lambda^{R_{\{\beta\}_n}(W)} \vdash n$ describes the partition shape of the image of W in $\mathbf{S}_{\{\alpha\}_n}$. For example, if $R_{\{\beta\}_n}(W) = a = ((\alpha_1\alpha_2)(\alpha_3))$ the partition a has shape $\lambda^a = (2, 1)$.

It is useful to sort basis elements $(a \cdot ((\beta_1)(\beta_2)\cdots(\beta_n))) \in B_p$ first by the number of parts in a

$$|a| \stackrel{\text{def}}{=} (\lambda^a)'_1$$

(i.e., the first component of the conjugate partition) and then by shape λ^a. The set of shapes with a given number of parts is easily compiled, and the number of basis elements with each shape is a simple combinatorial calculation. The first few bases may be obtained as follows (we use tableau form for the partitions so that the number of parts may be read at a glance, i.e., as the number of rows):

$$
\begin{array}{c}
\begin{array}{|c|}\hline 1 \\ \hline\end{array} \\[1em]
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \\[1em]
\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array} \\[1em]
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \\[1em]
\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}
\end{array}
\tag{41}
$$

Note from the definition of multiplication in Eq. (5), or from Proposition 2, that propagating number $\#^P$ is non-increasing in any multiplication. There is thus a filtration of invariant suspaces of $P_n(Q)$:

$$P_n(Q)F_0^{(n)}P_n(Q) \subset P_n(Q)F_1^{(n)}P_n(Q) \subset P_n(Q)F_2^{(n)}P_n(Q) \subset \cdots \subset P_n(Q).$$

For each $a \in \mathbf{S}_{\{\alpha\}_n}$ with $|a| \geq i$ parts there are $|a|!/(|a|-i)!$ ways of generating elements of $\mathbf{S}_{\{\alpha\}_n \cup \{\beta\}_i}$ by *inserting* the i elements of $\{\beta\}_i$ into distinct parts of a (i.e., so that no two β_j's appear in the same part, but each β_j is in a part with at least one α_k). Then we have

DEFINITION 9. For M a set of degree i let $\mathscr{W}_M \subset \mathbf{S}_{\{\alpha\}_n \cup M}$ be the set of all possible ways of inserting the i elements of M into distinct parts of all objects $a \in \mathbf{S}_{\{\alpha\}_n}$, then define

$$B_p[i] = \left\{ w \cdot ((\beta_1)(\beta_2)\dots(\beta_{n-i})) : w \in \mathscr{W}_{\{\beta\}_n - \{\beta\}_{n-i}} \right\}.$$

For example, with $i = 2$ there are no contributions to the set $B_p[2]$ coming from $a = ((\alpha_1\alpha_2 \cdots \alpha_n))$, since this a does not have enough distinct parts. More explicitly, with $n = 3$,

$$
\begin{aligned}
B_p[2] = \{ & ((\alpha_1\beta_2)(\alpha_2\beta_3)(\alpha_3)(\beta_1)), ((\alpha_1\beta_3)(\alpha_2\beta_2)(\alpha_3)(\beta_1)), \\
& ((\alpha_1\beta_2)(\alpha_2)(\alpha_3\beta_3)(\beta_1)), ((\alpha_1)(\alpha_2\beta_3)(\alpha_3\beta_2)(\beta_1)), \\
& ((\alpha_1)(\alpha_2\beta_2)(\alpha_3\beta_3)(\beta_1)), ((\alpha_1\beta_3)(\alpha_2)(\alpha_3\beta_2)(\beta_1)), \\
& ((\alpha_1\alpha_2\beta_2)(\alpha_3\beta_3)(\beta_1)), ((\alpha_1\alpha_2\beta_3)(\alpha_3\beta_2)(\beta_1)), \\
& ((\alpha_1\alpha_3\beta_2)(\alpha_2\beta_3)(\beta_1)), ((\alpha_1\alpha_3\beta_3)(\alpha_2\beta_2)(\beta_1)), \\
& ((\alpha_3\alpha_2\beta_2)(\alpha_1\beta_3)(\beta_1)), ((\alpha_3\alpha_2\beta_3)(\alpha_1\beta_2)(\beta_1))\}.
\end{aligned}
$$

DEFINITION 10. Let $B_p^2[i] \subset \mathbf{S}_{\{\alpha,\beta\}_n}$ be the set of partitions with propagating number i.

DEFINITION 11. Define $B'_p[i]$ as the subset of $B_p[i]$ such that $a \in B'_p[i]$ implies $|R_{\{\beta\}_n}(a)| = n$ (i.e., no two α_i's in the same part).

Note that $B_p[i] \subset B_p^2[i]$. When writing elements of $B_p[i]$ we may unambiguously abbreviate by writing $(i)^j$ for $(\alpha_i\,\beta_j)$ and omit (β_k). Thus $((1)(2)^3(3)^2) = ((\alpha_1)(\alpha_2\,\beta_3)(\alpha_3\,\beta_2)(\beta_1))$.

The subquotient

$$P_n[i] = P_n(Q)F_i^{(n)}P_n(Q)/P_n(Q)F_{i-1}^{(n)}P_n(Q)$$

(with $F_{-1}^{(n)} = 0$ so that $P_n[0] = P_n(Q)F_0^{(n)}P_n(Q)$ and $F_n^{(n)} = 1$ so $P_n[n] \cong \mathbb{C}S(n)$ (Eq. (23))) has basis $B_p^2[i]$. Each element $a(\alpha,\beta) \in B_p^2[i]$ can be described in three stages:

(i) the partition of $\{\alpha\}_n$ it effects, i.e. $R_{\{\beta\}_n}(a(\alpha,\beta)) \in \mathbf{S}_{\{\alpha\}_n}$;

(ii) the details of the i distinct connections between $\{\alpha\}_n$ and $\{\beta\}_n$;

(iii) the partition of $\{\beta\}_n$ it effects, i.e., $R_{\{\alpha\}_n}(a(\alpha,\beta))$.

For example

$$((\alpha_1\alpha_2\,\beta_3)(\alpha_3\,\beta_2)(\beta_1)) = \begin{array}{|c|c|c|}\hline \alpha_1 & \alpha_2 & \beta_3 \\ \hline \alpha_3 & \beta_2 & \multicolumn{1}{c}{} \\ \cline{1-2} \beta_1 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \to \begin{array}{|c|c|l}\hline \alpha_1 & \alpha_2 & \cdots{}_2 \\ \hline \alpha_3 & \multicolumn{2}{l}{\cdots{}_1} \\ \cline{1-1} \end{array} \qquad \begin{array}{r|c|}\cline{2-2} & \beta_1 \\ \cline{2-2} {}_1\cdots & \beta_2 \\ \cline{2-2} {}_2\cdots & \beta_3 \\ \cline{2-2} \end{array},$$

where the dotted lines indicate that these parts are connected as numbered.

Because of the quotient by $P_n(Q)F_{i-1}^{(n)}P_n(Q)$, action of $P_n(Q)$ on the left of such a basis element cannot change (iii). As a left $P_n(Q)$ module $P_n[i]$ is thus a direct sum of submodules each with basis characterized by fixed $R_{\{\alpha\}_n}(x)$ and fixed $K \subset \{\beta\}_n$ (such that $\beta_j \in K$ if and only if there exists α_k such that $\beta_j \sim^x \alpha_k$) for every x in the basis. Each such submodule is isomorphic to the module $P_n[i]' = P_n(Q)F_i^{(n)}$ (modulo $P_n(Q)F_{i-1}^{(n)}P_n(Q)$), with basis $B_p[i]$.

There is a "right" action of the symmetric group S_i on $P_n[i]'$, realized by permuting the indices on the set $\{\beta_{n-i+1},\ldots,\beta_n\}$ as they appear in basis elements. With respect to this action $P_n[i]'$ is a projective S_i module. Thus $P_n[i]'$ may be decomposed as a direct sum of modules corresponding to irreducible representations of S_i. The action of S_i commutes with the action of $P_n(Q)$ (by associativity of $P_n(Q)$), and from the construction of

$P_n[i]'$ we have $\mathscr{G}(P_{n-1}[i]') \cong P_n[i]'$; thus finally [14]

$$P_n[i]' \cong \bigoplus_{\lambda \vdash i} (\mathrm{S}(\lambda) \otimes \mathscr{S}_\lambda), \tag{42}$$

where $\mathrm{S}(\lambda)$ is the λ simple right module of S_i.

For example, a basis $B_p[1]$ for the generically simple module $\mathscr{S}_{(1)}$ can be constructed by writing down all the ways of including β_n into a part from the pure $\{\alpha\}_n$ parts of $B_p = B_p[0]$, which are illustrated in our table (Eq. (41)). The number of ways for each element from B_p here is just the number of parts—so the dimensions of the irreducible representations are the row sums in

	No. Parts	$= 1$	2	3	4	...
$n = 1$		1×1				
2		1×1	1×2			
3		1×1	3×2	1×3		
4		1×1	7×2	6×3	1×4	
5		1×1	15×2	25×3	10×4	1×5

(43)

(and so on), where the first number in each product is the total number of partitions with that many parts, and the second is the number of parts.

Let $\binom{n}{i} = n!/i!(n-i)!$ for $n \geq i$ and $\binom{n}{i} = 0$ otherwise. The dimensions of irreducible representations corresponding to one-dimensional irreducible representations of S_i at each level for higher i are then given by the row sums in:

$$\begin{array}{llll}
1 \times 0 & & & \\
1 \times 0 & 1 \times \binom{2}{i} & & \\
1 \times 0 & 3 \times \binom{2}{i} & 1 \times \binom{3}{i} & \\
1 \times 0 & 7 \times \binom{2}{i} & 6 \times \binom{3}{i} & 1 \times \binom{4}{i} \\
\ldots & & &
\end{array} \tag{44}$$

$(i \geq 2)$ and so on. For the purposes of computing dimensions we can think of $\mathscr{S}_\lambda$ for $\lambda \vdash i$ as a product $\mathscr{S}_{(i)} \otimes \mathrm{S}(\lambda)$.

3.1. *Inner Product on* $\mathscr{S}_\lambda$

Recall that for $\lambda \vdash i$, J_λ is a primitive idempotent of $\mathbb{C}\mathrm{S}_i$, so $\psi(J_\lambda)$ is a primitive idempotent of $P_i[i]$. For $n > i$ let Γ: $\{\alpha\}_i \to \{\alpha\}_n$ be inclusion, then there is a corresponding algebra injection

$$\mathscr{A}_\Gamma\colon P_i(Q) \to P_n(Q) \tag{45}$$

given by

$$\mathscr{A}_\Gamma\colon a \mapsto a \cup \{\{\alpha_{i+1}, \beta_{i+1}\}, \{\alpha_{i+2}, \beta_{i+2}\}, \dots, \{\alpha_n, \beta_n\}\} \tag{46}$$

which, generalizing our earlier convention, we call inclusion. For every other injection $\tau\colon \{\alpha\}_i \to \{\alpha\}_n$ there is an algebra injection $\mathscr{A}_\tau\colon P_i(Q) \to P_n(Q)$ by the obvious generalization of $\mathscr{A}_\Gamma$. Now let $I_\lambda \in P_n(Q)$ be the image of $\psi(J_\lambda) \in P_i(Q)$ under the injection $\mathscr{A}_\chi$, where

$$\chi(\alpha_j) = \alpha_{n-i+j}$$

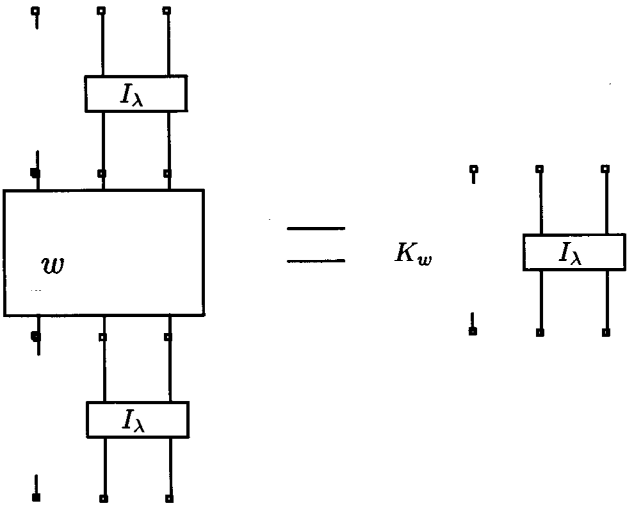
(i.e., "acting" on the last i objects rather than the first i). Then we have

Proposition 10. *For $Q \neq 0$, $(1/Q^{n-i})F_i^{(n)}I_\lambda$ is a primitive idempotent of $P_n[i]$ and $\mathscr{S}_\lambda = P_n[i]'I_\lambda$.*

This a consequence of Proposition 5. It may also be seen as follows. Schematically, for $w \in P_n(Q)$ we can represent

$$F_i^{(n)}I_\lambda w F_i^{(n)}I_\lambda = K_w F_i^{(n)}I_\lambda \qquad \left(\operatorname{mod} P_n(Q)F_{i-1}^{(n)}P_n(Q)\right)$$

as



(explicitly, this is the $n = 3$, $i = 2$ case). Here if w connects α_2, α_3 to β_2, β_3 as anything other than *scalar* $\times$ *permutation* then $k_w = 0$ by the $P_n[i]$ quotient. If it is a permutation then

$$I_\lambda \,(\text{permutation})\; I_\lambda = (\text{scalar})\; I_\lambda$$

by definition of I_λ.

Now let $w_\lambda = \{\epsilon_1 J_\lambda, \epsilon_2 J_\lambda, \ldots, \epsilon_{\dim(S(\lambda))} J_\lambda\}$ be a basis of $S(\lambda)$ (as a simple left module) orthogonal with respect to an inner product (i.e., $J_\lambda \epsilon_i^T \epsilon_j J_\lambda = \delta_{ij} J_\lambda$). Then, for elements $W_j \in P_n$ such that

$$\left\{ W_j F_i^{(n)} w : j = 1, 2, \ldots, \dim\left(\mathscr{S}_{(n)}\right); w \in \mathrm{S}_i \right\}$$

is a basis for $P_n[i]'$, we have that

$$B_\lambda = \left\{ W_j F_i^{(n)} w : j = 1, 2, \ldots, \dim\left(\mathscr{S}_{(n)}\right); w \in \mathscr{A}_\chi\left(\psi(w_\lambda)\right) \right\}$$

is a basis of the Weyl module $\mathscr{S}_\lambda$, and we have the inner product on $\mathscr{S}_\lambda$ defined by

$$\langle jk \mid lm \rangle\, F_i^{(n)} I_\lambda = I_\lambda \epsilon_j^T F_i^{(n)} W_k^T W_m F_i^{(n)} \epsilon_l I_\lambda,$$

where $a(\alpha, \beta)^T = a(\beta, \alpha)$. For example, using the picture in our schematic above, the Gram matrices for $\lambda = (2)$ and $\lambda = (1^2)$ may be extracted from a simple 6×6 table of diagram combinations, giving

$$\begin{vmatrix} Q & 0 & 0 & 1 & 1 & 0 \\ 0 & Q & 0 & 1 & 0 & 1 \\ 0 & 0 & Q & 0 & \pm 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & \pm 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{vmatrix} = \begin{cases} (Q-1)^2 (Q-4), & \lambda = (2), \\ Q(Q-3)^2, & \lambda = (1^2). \end{cases}$$

Note that if $R_{\{\beta\}_n}(W_m F_i^{(n)})$ has q parts and $\lambda \vdash p\,(q \geq p)$ then the diagonal entries are

$$\langle im \mid im \rangle = Q^{q-p}. \tag{47}$$

Note that these determinants are consistent with Proposition 9. The Smith normal form of these matrices is diagonal $(1, 1, 1, 1, (Q-1), (Q-1) \cdot (Q-4))$ and diagonal$(1, 1, 1, 1, (Q-3), (Q)(Q-3))$, respectively, revealing, for example, that for $\lambda = (2)$ at $Q = 1$ the module aquires a two-dimensional invariant subspace, as claimed in the proposition. The normalization is such that this kind of analysis works even for $Q = 0$.

3.2. *Induction and Restriction of* $\mathscr{S}_\lambda$ *for* $P_n(Q) \supset P_{n-1}(Q)$

The form of the inner product above shows that $\mathbb{C}P_n(Q)$ is generically (with respect to Q) semisimple and $\mathscr{S}_\lambda$ simple. The generic restriction rule for simple modules here is given by

PROPOSITION 11 (see [14]). *For* $\triangleright$ *meaning "one box added to"* ([6]) *the generic restriction from* $P_n(Q)$ *to* $P_{n-1}(Q)$ *is*

$$\mathscr{S}_\lambda(n)\downarrow_{P_{n-1}} = \left(\bigoplus_{\lambda' \triangleright \lambda} \mathscr{S}_{\lambda'}(n-1)\right) \oplus \left(\bigoplus_{\lambda' \triangleleft \triangleright \lambda} \mathscr{S}_{\lambda'}(n-1)\right) \oplus \bigoplus_{\lambda' \triangleleft \lambda} \mathscr{S}_{\lambda'}(n-1). \tag{48}$$

and the induction rule follows from this by Frobenius reciprocity.

For example, denoting multiplicitity 3 by $3 \cdot ()$,

$$\mathscr{S}_{(2,1)}\downarrow \cong \mathscr{S}_{(1^2)} \oplus \mathscr{S}_{(2)} \bigoplus \mathscr{S}_{(1^3)} \oplus 3.\mathscr{S}_{(2,1)} \oplus \mathscr{S}_{(3)} \bigoplus \mathscr{S}_{(2,1^2)} \oplus \mathscr{S}_{(2^2)} \oplus \mathscr{S}_{(3,1)}. \tag{49}$$

4. SUBMODULES AT EXCEPTIONAL Q VALUES

4.1. *Submodules of the Left Module* $\mathscr{S}_0 = P_n(Q)\, F_0^{(n)}$

It is known that the generically simple modules $P_n(Q)\; F_0^{(n)}$ develop invariant subspaces at some integer Q values [16]. To find these explicitly note

DEFINITION 12. Let

$$k_W = \prod_{i=1}^{(\lambda^W)'_1} \left(-((\lambda^W)_i - 1)!\right). \tag{50}$$

Then $\chi_n \in P_n(Q)\; F_0^{(n)}$ is

$$\chi_n = \sum_{W \in \mathbf{S}_{\{\alpha\}_n}} k_W\left(W \cdot ((\beta_1)(\beta_2)\ldots(\beta_n))\right). \tag{51}$$

PROPOSITION 12. *There is a unique subspace V_n of $P_n(Q)\ F_0^{(n)}$ such that $A^{jk}V_n = 0$ for all j, k, and V_n is one dimensional and spanned by χ_n.*

Proof. Consider the coefficient of some word W in $A^{jk}\chi_n$. This will be zero unless $a_j \sim^W \alpha_k$. If $\alpha_j \sim^W \alpha_k$ then the coefficient is

$$k_W + \sum_{W' <_{jk} W} k_{W'}, \tag{52}$$

where the sum is over $W' \neq W$ such that $A^{jk}W' = W$.

There is only one W whose coefficient is automatically zero for all j, k (that is, $((\alpha_1)(\alpha_2)\ldots(\alpha_n)))$, so (noting the form of Eq. (52)) we have at least $|B_p[0]| - 1$ independent constraints here. Thus the solution, if it exists, is unique up to an overall scalar factor.

Now

$$(\lambda^{W'})'_1 = (\lambda^W)'_1 + 1$$

so if the length of the part of W containing α_j, α_k is m then there are $m-1$ possible lengths l for the part of W' containing α_j (that is, the part containing α_j may have length $l = 1, 2, \ldots, m-1$, with the part containing α_k then having length $m - l$). Furthermore, for given l there are

$$\frac{(m-2)!}{(m-2-l+1)!(l-1)!}$$

possible ways of filling in the other components of this part. In such a case $k_{W'}/k_W$ is

$$\frac{(l-1)!(m-l-1)!}{(m-1)!}$$

(the contributions of unchanged parts divide out) so overall the coefficient vanishes if

$$(m-1)! = \sum_{l=1}^{m-1} (m-2)!.$$

This is true, so

$$k_W = -\sum_{W' <_{jk} W} k_{W'} \tag{53}$$

for all W. ∎

For example, in $P_2(Q)$ note that

$$((\alpha_1\alpha_2\,\beta_1\,\beta_2))\,\big(((\alpha_1)(\alpha_2)(\beta_1)(\beta_2)) - ((\alpha_1\alpha_2)(\beta_1)(\beta_2))\big)$$
$$= ((\alpha_1\alpha_2)(\beta_1)(\beta_2)) - ((\alpha_1\alpha_2)(\beta_1)(\beta_2)) = 0.$$

PROPOSITION 13. *For* $\lambda \vdash n$

$$0 \to \mathscr{S}_\lambda \to \mathscr{S}_{(0)}$$

is an exact sequence of left $P_n(Q)$ *modules iff* $Q = n-1$ *and* $\lambda = (n)$. *The submodule is spanned by* $\chi_n \mid_{Q=n-1}$.

Proof. First note that

$$\mathscr{I}_{ij}\,\chi_n = \chi_n.$$

Now consider the coefficient of W in $A^l\chi_n$. This will be zero unless α_l is in a part on its own. If so then the coefficient is

$$Qk_W + \sum_{W' >_l W} k_{W'},$$

where the sum is over $W' \neq W$ such that $A^l W' = W$. There are $(\lambda^W)'_1 - 1$ possibilities for W', corresponding to the $(\lambda^W)'_1 - 1$ different parts of W to which the singleton α_l may be added, each to make a different W'. Obviously

$$(\lambda^{W'})'_1 = (\lambda^W)'_1 - 1$$

so the coefficient vanishes when

$$Q = \frac{\sum_{W' >_l W} \prod_{i=1}^{(\lambda^{W'})'_1} \big(((\lambda^{W'})_i - 1)!\big)}{\prod_{i=1}^{(\lambda^W)'_1} \big(((\lambda^W)_i - 1)!\big)} = \sum_{i=1}^{(\lambda^W)'_1 - 1} \frac{((\lambda^{W'})_i - 1)!}{((\lambda^W)_i - 1)!(1-1)!}$$

(where the sum is over the possible parts added to in making W' out of W, i.e., the parts different in W' and, hence, with k_W factors not cancelling)

$$Q = \sum_{i=1}^{(\lambda^W)'_1 - 1} \big(((\lambda^W)_i + 1) - 1\big) = n - 1 \tag{54}$$

(independent of W). ∎

It follows from a computation of the determinant of the Gram matrix for this module [16], or from our category theory arguments (Proposition 5), that this is the *only* submodule of $\mathscr{S}_{(0)}$ at $Q = n-1$.

4.2. *Submodules of* $P_n[1]'$

DEFINITION 13. *For* $W \in B_p[1]$ *let* L_W *be the length of the part of* W *containing* β_n (*not counting* β_n *itself*) *and*

$$k_W[1] = L_W\, k_W$$

then

$$\chi_n[1] = \sum_{(W\ldots)\in B_p[1]} k_W[1]\big(W \cdot ((\beta_1)(\beta_2)\ldots(\beta_{n-1}))\big).$$

PROPOSITION 14.

(1) $A^{jk}\chi_n[1] = 0$ *for all* j, k;

(2) *More generally, let* v *be an* n*-tuple of scalars and*

$$k_W(v) = L_W(v) k_W,$$

where

$$L_W(v) = \sum_{i\ s.t.\ \alpha_i \sim^W \beta_n} v_i.$$

Then all solutions $X \in P_n[1]$ *to*

$$A_{jk} X = 0 \qquad \textit{for all } j, k \tag{55}$$

are of the form

$$\chi_n(v) = \sum_{(W\ldots)\in B_p[1]} k_W(v)\big(W \cdot ((\beta_1)(\beta_2)\ldots(\beta_{n-1}))\big). \tag{56}$$

Proof. The proof is similar to the $P_n(Q)F_0^{(n)}$ case.

The only β_i which plays any role here is β_n, so it will be sufficient in notation for partitions to ignore them all, except to record which part β_n is in by ‘‘marking’’ that part.

Consider the coefficient of some word W in $A_{jk}\,\chi_n(v)$. This will be zero unless $\alpha_j \sim^W \alpha_k$.

There are thus (at most) n words for which there is no contraint, so if the proposed set of solutions (from Eq. (56)) is a subspace of the space of all solutions then it is the whole space.

If $\alpha_j \sim^W \alpha_k$ then the coefficient of W is

$$k_W(v) + \sum_{W' <_{jk} W} k_{W'}(v),$$

where the sum is over $W' \neq W$ such that $A^{jk}W' = W$. Now

$$(\lambda^{W'})'_1 = (\lambda^W)_1{}' + 1$$

so if the length of the part of W containing α_j, α_k is m then there are $m-1$ possible lengths l for the part of W' containing α_j (that is, the part containing α_j may have length $l = 1, 2, \ldots, m-1$, with the part containing α_k then having length $m - l$).

There are now two cases to consider. Either the part of W containing α_j, α_k is the marked one (i.e., also containing β_n), or not.

In the former case then for each l there are two possibilities for W', i.e., with the part containing α_j marked, or the part containing α_k marked. If $L_{W'}$ in the first case is f_l (say) then in the second case it must be $L_W(v) - f_l$, thus altogether (using Eq. (50))

$$k_W(v) = -\sum_{l=1}^{m-1} \big(f_l + (L_W(v) - f_l)\big)k_{W'} = L_W(v)k_W \tag{57}$$

as required.

In the latter case $L_W(v) = L_{W'}(v)$. Let us take $L_W(v) = p$, then

$$k_W[1] = pk_W = -\sum_{l=1}^{m-1} pk_{W'}$$

again as required. ∎

For example, in $P_2(Q)$ note that

$$\begin{aligned} A^{12}\big(((\alpha_1)(\alpha_2\,\beta_2)(\beta_1)) + ((\alpha_1\,\beta_2)(\alpha_2)(\beta_1)) - 2.((\alpha_1\alpha_2\,\beta_2)(\beta_1))\big) \\ = ((\alpha_1\alpha_2\,\beta_2)(\beta_1)) + ((\alpha_1\alpha_2\,\beta_2)(\beta_1)) - 2.((\alpha_1\alpha_2\,\beta_2)(\beta_1)) = 0. \end{aligned}$$

PROPOSITION 15. *For* $\lambda \vdash n$

$$0 \to \mathscr{S}_\lambda \to \mathscr{S}_{(1)}$$

is an exact sequence of left $P_n(Q)$ *modules iff* $Q = n$ *and* $\lambda = (n)$. *The submodule is spanned by* $\chi_n[1]\,|_{Q=n}$.

Proof. Unless $Q = 0$ and $n = 2$ there is no solution to

$$A^l \chi_n(v) = 0 \qquad \text{for all } l \tag{58}$$

unless v is a scalar multiple of $(1, 1, 1, \dots, 1)$. We can see this as follows:

For $n = 1$ $P_n[1]$ *is* the trivial module. For $n = 2$ Eq. (58) gives the system

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = Q \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

and so for $Q = 0$ we define a new simple module from $\mathscr{S}_{(1)}$ by the short exacts sequence

$$0 \to \mathscr{S}_{(1^2)} \to \mathscr{S}_{(1)} \to \mathscr{S}^0_{(1)} \to 0$$

while for $Q = 2$ the short exact sequence is

$$0 \to \mathscr{S}_{(2)} \to \mathscr{S}_{(1)} \to \mathscr{S}^2_{(1)} \to 0.$$

For $n \geq 3$ the coefficients of n part partitions in Eq. (58) give, for example, in the case $((1)^3(2)(3))$,

$$(l = 2) \qquad Qv_1 - (v_1 + v_2) = 0$$

and

$$(l = 3) \qquad Qv_1 - (v_1 + v_3) = 0$$

(i.e., $v_2 = v_3$)—similarly $v_1 = v_2$, and so on.

Thus we may restrict attention to $\chi_n[1]$, and the solution will be unique if it exists, and at $Q = n$ it will give a short exact sequence

$$0 \to \mathscr{S}_{(n)} \to \mathscr{S}_{(1)} \to \mathscr{S}_{(1)} / \big(P_n(Q)\, \chi_n[1] \,|_{Q=n} \big) \to 0,$$

generalizing the $Q = 2$ but not the $Q = 0$ case at $n = 2$. It remains to prove existence.

Consider the coefficient of W in $A^l \chi_n[1]$. This will be zero (modulo $P_n(Q) F_0^{(n)}$) unless α_l is in an (unsuperscripted) part on its own. If so then the coefficient is

$$Qk_W + \sum_{W' >_i W} k_{W'},$$

where the sum is over $W' \neq W$ such that $A^i W' = W$. Their are $(\lambda^W)_1' - 1$ possibilities for W', corresponding to the $(\lambda^W)_1' - 1$ different parts of W to which the singleton α_i may be added, each to make a different W'.

Obviously,

$$(\lambda^{W'})'_1 = (\lambda^{W})'_1 - 1$$

so the coefficient vanishes when

$$Q = \frac{\sum_{W' >_i W}\left(L_{W'}\prod_{i=1}^{(\lambda^{W'})_1'}\left(((\lambda^{W'})_i - 1)!\right)\right)}{L_W\prod_{i=1}^{(\lambda^W)'_1}\left(((\lambda^W)_i - 1)!\right)}$$

$$= \sum_{i=1}^{(\lambda^W)'_1 - 1}\left(\frac{L_{W'}((\lambda^{W'})_i - 1)!}{L_W((\lambda^W)_i - 1)!(1-1)!}\right),$$

where the sum is over the possible parts added to, in making W' out of W, i.e., the parts different in W' and, hence, with respective $k_W[1]$ factors not cancelling. In one of these summands l is in the superscripted part, so that $L_{W'} = L_W + 1$; otherwise $L_{W'} = L_W$. Thus we have

$$Q = \left(\sum_{i=1}^{(\lambda^W)'_1 - 1}\left(((\lambda^W)_i + 1) - 1\right)\right) + \frac{+1.((\lambda^{W'})_i - 1)!}{L_W((\lambda^W)_i - 1)!},$$

but for the latter summand, $(\lambda^{W'})_i = (\lambda^W)_i + 1 = L_W + 1$. So altogether we have

$$Q = (n-1) + 1 = n \tag{59}$$

(independent of W). ∎

4.3. *A Remark on Frobenius Reciprocity*

Recall that for an algebra $\mathscr{A}$ then $\mathrm{Hom}_{\mathscr{A}}({}_{\mathscr{A}}V, {}_{\mathscr{A}}W)$ is the vector space of $\mathscr{A}$ homomorphisms from ${}_{\mathscr{A}}V \to {}_{\mathscr{A}}W$. If $\mathscr{B}$ is a subalgebra of $\mathscr{A}$, and ${}_{\mathscr{B}}N$ and ${}_{\mathscr{A}}M$ are left modules as indicated, and $({}_{\mathscr{B}}N)\uparrow_{\mathscr{A}} = \mathscr{A} \otimes_{\mathscr{B}\mathscr{B}} N$, then Frobenius reciprocity [5] is the isomorphism

$$\mathrm{Hom}_{\mathscr{A}}\left(({}_{\mathscr{B}}N)\uparrow_{\mathscr{A}}, {}_{\mathscr{A}}M\right) \cong \mathrm{Hom}_{\mathscr{B}}\left({}_{\mathscr{B}}N, ({}_{\mathscr{A}}M)\downarrow_{\mathscr{B}}\right). \tag{60}$$

In our case let $\mathscr{B} = P_{m-1}(Q)$ and $\mathscr{A} = P_m(Q)$. The restriction of ${}_{\mathscr{A}}M = \mathscr{S}_0$ is $\mathscr{S}_0 \oplus \mathscr{S}_{(1)}$, and the generic induction of ${}_{\mathscr{B}}N = \mathscr{S}_{(m-1)}$ is [14]

$$\left({}_{\mathscr{B}}\mathscr{S}_{(m-1)}\right)\uparrow_{\mathscr{A}} \cong \left(\mathscr{S}_{(m)} \oplus \mathscr{S}_{(m-1,1)}\right) \oplus 2\cdot\mathscr{S}_{(m-1)} \oplus \mathscr{S}_{(m-2,1)} \oplus \mathscr{S}_{(m-2)}, \tag{61}$$

where $m\cdot$ denotes m copies. Generically the isomorphism in Eq. (60) is satisfied with both spaces empty (for sufficiently large m), but when

$m = Q + 1$ we have a homomorphism in $\mathrm{Hom}_{\mathscr{A}}({}_{\mathscr{A}}\uparrow{}_{\mathscr{B}}N, {}_{\mathscr{A}}M)$ by Eq. (54). This implies a $\mathscr{B}$ module morphism from $\mathscr{S}_{(m-1)}$ either to $\mathscr{S}_{(1)}$ or to $\mathscr{S}_0$. The latter is excluded by Proposition 5 (via our argument that $m = Q + 1$ is the *first* place such a morphism occurs), and we recover Eq. (59), i.e., a morphism to $\mathscr{S}_{(1)}$ at $n = m - 1 = Q$.

4.4. *Submodules of $P_n[i]'$ for $i \geq 2$*

The new feature at $i = 2$ is that S_2 itself has a structure, so $P_n[2]'$ is a direct sum of two Weyl modules, $\mathscr{S}_{(1^2)}$ and $\mathscr{S}_{(2)}$. The submodules of each must be found. It is also possible that the inclusion will be the image of an S_n module other than the trivial module.

DEFINITION 14. Let

$$\Psi_n[i] = \left\{x \in P_n[i]' \mid A^{jk}x = 0 \text{ for all } j, k\right\}.$$

PROPOSITION 16. *The space $\Psi_n[i]$ is spanned by elements of the form*

$$\chi_n(v^1, v^2, \dots, v^i) = \sum_{W \in B_p[i]} \left(\prod_{j=1}^{i} L_W^j(v^j) \right) k_W W, \tag{62}$$

where $v^1, v^2, \dots, v^i$ are n-tuples and

$$L_W^m(v) = \sum_{k \text{ such that } \alpha_k \sim^W \beta_{n-i+m}} v_k.$$

Proof. First, there are at most $n!/(n-i)!$ words (i.e., elements of $B_p[i]$) for which there is no independent constraint on coefficient given by the vanishing of $A^{jk}X$, namely those in $B_p'[i]$. For these words the product of L's in Eq. (62) is just a product of the form $v_{k_1}^1 v_{k_2}^2 \cdots v_{k_i}^i$, where the k_j's are distinct. We can construct a standard ordered basis for the $n!/(n-i)!$-dimensional space of unconstrained coefficients of these words using appropriate v^j's. Let ϵ^k be a vector such that $(\epsilon^k)_j = \delta_{kj}$. Then a suitable choice is

$$\epsilon_n[i] = \left\{(v^1, v^2, \dots, v^i) = (\epsilon^{j_1}, \epsilon^{j_2}, \dots, \epsilon^{j_i}) \mid \{j_1, j_2, \dots, j_i\} \subset \{1, 2, \dots, n\}\right\}.$$

Thus these elements $\chi_n(v^1, v^2, \dots, v^i)$ span if they are in the set.

On the other hand, several components contribute, in general, to the coefficient of a word in $A^{jk}\chi_n(v^1, v^2, \dots, v^i)$. As usual there is the word itself for which $\alpha_j \sim^W \alpha_k$, together with words of one more part. There are two cases to consider:

If the part containing j, k is not primed then it is not primed in any word in the sum, and we are reduced to within an overall factor of equation (53). If it is primed then one of the L factors changes, and we are reduced, up to an overall factor, to the calculation in equation (57). ∎

To summarize, an arbitrary element of $\Psi_n[i]$ may be formed by choosing arbitrary coefficients of $B'_p[i]$ elements (i.e. those of the form $((1)(2)\ldots(j_1)^{k_1}\ldots(j_2)^{k_i}\ldots(j_i)^{k_i}\ldots(n)))$, with the coefficients of all other words then being completely determined via Eq. (62).

It will be convenient to represent the basis for $\Psi_n[i]$ given by the v^j's in $\epsilon_n[i]$ as

$$X_n[i] = \left\{ \chi_n(\epsilon^{l_1}, \ldots, \epsilon^{li}) \mid \{l_1, l_2, \ldots, l_i\} \subset \{1, 2, \ldots, n\} \right\}$$

and introduce shorthand

$$\underline{l_1 l_2 \ \cdots \ l_i} \stackrel{\text{def}}{=} \chi_n(\epsilon^{l_1}, \ldots, \epsilon^{l_i}),$$

that is,

$$\underline{l_1 l_2 \ \cdots \ l_i} = \big((\alpha_1)(\alpha_2) \cdots (\alpha_{l_1} \beta_1)(\alpha_{l_2} \beta_2) \cdots (\alpha_{l_i} \beta_i) \cdots (\alpha_n) \big) + \cdots$$

(we have omitted singleton β_j's) and $+\cdots$ signifies elements of $B_p[i]$ with fewer parts, whose coefficients are determined as above. For example, $\underline{251} \in X_5[3]$ contains

$$\begin{aligned}
&((\alpha_1 \beta_3)(\alpha_2 \beta_1)(\alpha_3)(\alpha_4)(\alpha_5 \beta_2)) - ((\alpha_1 \alpha_3 \beta_3)(\alpha_2 \beta_1)(\alpha_4)(\alpha_5 \beta_2)) \\
&\quad - ((\alpha_1 \beta_3)(\alpha_2 \alpha_3 \beta_1)(\alpha_4)(\alpha_5 \beta_2)) - ((\alpha_1 \beta_3)(\alpha_2 \beta_1)(\alpha_4)(\alpha_5 \alpha_3 \beta_2)) \\
&\quad - ((\alpha_1 \alpha_4 \beta_3)(\alpha_2 \beta_1)(\alpha_3)(\alpha_5 \beta_2)) - ((\alpha_1 \beta_3)(\alpha_2 \alpha_4 \beta_1)(\alpha_3)(\alpha_5 \beta_2)) \\
&\quad - ((\alpha_1 \beta_3)(\alpha_2 \beta_1)(\alpha_3)(\alpha_4 \alpha_5 \beta_2)) \\
&\quad - ((\alpha_1 \beta_3)(\alpha_2 \beta_1)(\alpha_3 \alpha_4)(\alpha_5 \beta_2)) + \cdots .
\end{aligned}$$

Note that

$$A^j \, \underline{l_1 l_2 \ \cdots \ l_i} = 0$$

if $j \in \{l_1, l_2, \ldots, l_i\}$.

PROPOSITION 17. *The space $\Psi_n[i]$ is*

(i) *a left module of the subalgebra $kS_n \subset P_n(Q)$;*

(ii) *a submodule of $P_n[i]'$ with respect to the right S_i action.*

Proof.

(i) Note that $A^{jk}x = 0$ for all j, k implies

$$A^{jk}\mathscr{I}_{lm}x = 0 \qquad \forall j, k, l, m.$$

(ii) By associativity. ∎

Recalling that $B'_p[i]$ indexes a basis of $\Psi_n[i]$, it follows that the actions of S_n and S_i on the basis states are permutations among them given by the corresponding permutations among the elements of $B'_p[i]$.

PROPOSITION 18. *For $\lambda \vdash n$ and $\mu \vdash i$ the existence of a nontrivial morphism (of Weyl modules) $\mathscr{S}_\lambda \to \mathscr{S}_\mu$ implies*

(i) $\mu \subset \lambda$

(ii) *that the skew diagram $\lambda - \mu$ contains no (1^2) subdiagram.*

Proof.

(i) Suppose $\mu \not\subset \lambda$, then

$$I_\lambda\big((1)(2)\dots(n-i)(n-i+1)^{k_1}(n-i+2)^{k_2}\cdots(n)^{k_n}\big)I_\mu$$
$$= I_\lambda I_\mu\big((1)(2)\cdots(n)^{k_n}\big) = 0$$

(the first expression to be understood in the sense of Proposition 17) so

$$I_\lambda\Psi_n[i]I_\mu = 0.$$

But in fact

$$\Psi_n[i]I_\mu \subset P_n[i]'I_\mu = \mathscr{S}_\lambda(n)$$

and $I_\lambda\mathscr{S}_\lambda(n) \neq 0$, contradicting the supposition.

(ii) Note that as a left S_n, right S_i module

$$\Psi_n[i] \cong \bigoplus_{\mu \vdash i}\big((\mathrm{S}(\mu) \otimes S((n-i)))\uparrow_{\mathrm{S}_n}\big)\mathrm{S}(\mu). \quad ∎$$

Finally we look for constraints on Q such that there is a space $\Phi_n^Q[i] = \{x \in \Psi_n[i] \mid A^j x = 0\ \forall j\}$. The constraints coming from coefficients of words of the form $((1)(2)\cdots(j_1)^{k_1}\cdots(j_2)^{k_2}\cdots(j_i)^k(n))$ in

$$A^h\chi_n(v^1, v^2, \dots, v^i) = 0 \tag{63}$$

($h \neq j_l$ for any l) now take the form

$$Q\left(\prod_{l=1}^{i} v_{j_l}^{k_l}\right) - \left(\sum_{m=1}^{i}\left(\prod_{l\neq m} v_{j_l}^{k_l}\right)\left(v_{j_m}^{k_m} + v_h^{k_m}\right)\right)$$
$$-(n-(i+1))\left(\prod_{l=1}^{i} v_{j_l}^{k_l}\right) = 0. \tag{64}$$

Although not every element of $\Psi_n[i]$ can be written in the form $\chi_n(v^1, v^2, \ldots, v^i)$ (so we are imposing possibly overstrict conditions here), this is nonetheless an illuminating case to start with, as we will see, and it has the enormous merit that in this case the system of $n!/(n-i-1)!$ equations represented by Eq. (64) (that is, $n!/(n-i)!$ choices for $j_1, \ldots, j_i$ in order, $(n-i)$ choices for h) is sufficient for Eq. (63) to hold.

For example, the coefficient of $((1)(2)\cdots(j_1 g)^{k_1}\cdots(j_2)^{k_2}\cdots(n))$ in $A^h\chi_n(v^1, v^2, \ldots, v^i)$ is

$$Q\left(\left(v_{j_1}^{k_1} + v_g^{k_1}\right)\left(\prod_{l=2}^{i} v_{j_l}^{k_l}\right)\right) - 2\left(\prod_{l=2}^{i} v_{j_l}^{k_l}\right)\left(v_{j_1}^{k_1} + v_g^{k_1} + v_h^{k_1}\right)$$
$$-\left(\left(v_{j_1}^{k_1} + v_g^{k_1}\right)\sum_{m=2}^{i}\left(\prod_{\substack{l=2\\(l\neq m)}}^{i} v_{j_l}^{k_l}\right)\left(v_{j_m}^{k_m} + v_h^{k_m}\right)\right)$$
$$-(n-i-2)\left(v_{j_1}^{k_1} + v_g^{k_1}\right)\left(\prod_{l=2}^{i} v_{j_l}^{k_l}\right) \tag{65}$$

which is linearly dependent on the set of equations at Eq. (64) (specifically it is Eq. (64) as shown plus Eq. (64) with j_1 and g interchanged).

For $n \gg i$ we can save ourselves the trouble of solving this system of multilinear equations by using Frobenius reciprocity again (see Section 5). First note that there is clearly a solution of the form

$$v_j^k = v_l^k \qquad \text{for all } j, k, l$$

for

$$Q = n + i - 1.$$

This corresponds to the short exact sequence

$$0 \to \mathscr{S}_{(n)} \to \mathscr{S}_{(i)} \to \mathscr{S}_{(i)}^{Q} \to 0 \tag{66}$$

at that Q value.

The Frobenius reciprocity arguments we will use to conclude this analysis in Section 5 lose efficacy for $n - i \leq 2$, since then the Hom spaces in Eq. (60) are not generically empty and the identification of morphisms becomes more complicated. There is not quite enough information to "bootstrap" the answer, and we must make explicit the calculation. We will get this out of the way now.

4.5. *Submodules of* $P_n[n-1]'$

The task of finding submodules is in general slightly complicated by the non-linear (multi-linear) realization of the solution to the constraints above. A relatively straightforward and illuminating case, however, is $i = n-1$. In fact this and $i = n-2$ are the two most useful cases since, as we have said, the others can be deduced from them using Frobenius reciprocity.

Let us write the words with n parts (as before, not counting the singleton β_n) in the basis for $P_n[n-1]'$ as follows:

First, for $i > j$ define $\Gamma(i,j)$ as the $(i-j+1)$-tuple of cyclic elements of $P_n(Q)$,

$$\left(\mathbb{1}, \mathscr{I}_{i-1\,i}, \mathscr{I}_{i-2\,i-1}\mathscr{I}_{i-1\,i}, \ldots, I_{j\,j+1}\ldots\mathscr{I}_{i-2\,i-1}\mathscr{I}_{i-1\,i}\right);$$

then the $n!$-tuple

$$\bigoplus_{k=0}^{n-2} \Gamma(n-k,1)$$

gives an ordered basis for the $\mathbb{C}S_n$ subalgebra of $P_n(Q)$, and we may operate with this on the n part $P_n[n-1]'$ basis element

$$\left((1)^1(2)^2 \cdots (n-1)^{n-1}(n)\right)$$

to generate all the words with n parts in a specific order. Note that this may be written

$$\left(\Gamma(n,1) \otimes \underline{\left(\bigotimes_{k=1}^{n-2} \Gamma(n-k,1)\right)}\right)\left((1)^1(2)^2\ldots(n-1)^{n-1}(n)\right). \quad (67)$$

Then the underlined part permutes the superscripted elements only, and so it could be thought of as acting on the $n-1$ connections to the β_i's (i.e., rearranging the α indices in the pairs $(i)^i = (\alpha_i, \beta_i)$).

In this form the case $n = 3$, for example, may be written in S_n cycle notation as

$$\begin{aligned}((1,(23),(123)) \otimes (1,(12)))\left((1)^1(2)^2(3)\right)\\ = ((1,(12),(23),(23)(12),(123),(123)(12))\left((1)^2(2)^2(3)\right)\end{aligned}$$

and so comes out in the order

$$\Big\{\big((1)^1(2)^2(3)\big),\big((1)^2(2)^1(3)\big),\big((1)^1(2)(3)^2\big),\big((1)^2(2)(3)^1,\\ \big((1)(2)^1(3)^2\big),\big((1)(2)^2(3)^1\big)\Big\}. \quad (68)$$

Naturally associated to this ordered set is the ordered set U_X of $n!$ elements of $X_n[n-1]$, and the requirement $A^l X = 0$ for all l may be written as a matrix equation

$$\mathscr{R}^n_{[n-1]}\left(\sum_{i=1}^{n} A^i\right)V = 0, \quad (69)$$

where V is the vector of coefficients (i.e., $U_X.V = X$). This is sufficient because the action of each A^l contributes non-trivially to $(n-1)!$ rows, with no overlap between distinct l's.

Definition 15. Let $\mathscr{M}_n$ be an $n \times n$ matrix with entries in $\mathrm{S}_{n-1} \cup \{0\}$ as follows:

$$\mathscr{M}_n = \begin{pmatrix} 0 & 1 & \sigma_1 & \sigma_2\sigma_1 & \sigma_3\sigma_2\sigma_1 & \dots & \sigma_{n-2}\sigma_{n-3}\dots\sigma_1 \\ 1 & 0 & 1 & \sigma_2 & \sigma_3\sigma_2 & \dots & \sigma_{n-2}\dots\sigma_3\sigma_2 \\ \sigma_1 & 1 & 0 & 1 & \sigma_3 & \dots & \sigma_{n-2}\dots\sigma_3 \\ \sigma_1\sigma_2 & \sigma_2 & 1 & 0 & 1 & \dots & \sigma_{n-2}\dots\sigma_4 \\ \vdots & & & & & & \\ \sigma_1\sigma_2\dots\sigma_{n-2} & \dots & & & & & 0 \end{pmatrix} \quad (70)$$

and for ν a representation of $\mathbb{C}\mathrm{S}_n$ let $R_\nu(\mathscr{M}_n)$ be the matrix obtained from $\mathscr{M}_n$ by using the representation ν for the matrix elements. In particular, we will put $\nu = S_{n-1}$ to denote the use of the regular representation, and $\nu \vdash n-1$ to denote the use of the usual irreducible representation $S(\nu)$ (the choice of basis in this case will not be important here).

Writing $\dim(\nu)$ for $\dim(\mathrm{S}(\nu))$, note that $\dim(R_\nu(\mathscr{M}_n)) = n\dim(\nu)$.

The matrix $\mathscr{R}^n_{[n-1]}(\sum_{i=1}^n A^i)$ is readily determined using Eq. (67) to be given by

$$\mathscr{R}^n_{[n-1]}\left((Q-(n-1))\,\mathbb{1} - \sum_{i=1}^{n} A^i\right) = R_{\mathrm{S}_{n-1}}(\mathscr{M}_n). \quad (71)$$

Up to similarity transformations we can then write

$$R_{\mathrm{S}_{n-1}}(\mathscr{M}_n) \cong \bigoplus_{\nu \vdash n-1} 1_{\dim(\nu)} \otimes R_\nu(\mathscr{M}_n). \quad (72)$$

Equation (72) applied to Eq. (71) corresponds to the decomposition of $P_n[n-1]'$ into Weyl modules, so from now on we can look for submodules not just of $P_n[n-1]'$, but of its $\mathscr{S}_\lambda(n)$ (with $\lambda \vdash n-1$) components individually, as we require. On the other hand, $R_{S_{n-1}}(\mathscr{M}_n)$ commutes with the regular representation of S_n for which the $n!$ n-part words are a basis (with the action of permuting the α_is). Thus, the eigenvalue multiplicities for our eigenvalue equation (from Eq. (71))

$$R_{S_{n-1}}(\mathscr{M}_n)V = (Q-(n-1))V \tag{73}$$

are given by the dimensions of simple S_n modules, and these $\lambda \vdash n$ modules will identify the $\lambda \vdash n$ Weyl modules which *are* the submodules we seek. In particular there is a largest eigenvalue $n-1$, giving $Q = 2n - 2$, with eigenvector given by $V_i = 1$ for all i.

As noted above, each block in Eq. (72) is n times the size of an irreducible representation of S_{n-1}. This is the dimension of the representation of S_n induced from that irreducible representation, just as the regular representation at $n-1$ induces to the regular representation at n. Indeed the first factor in the formal "factorization" of the basis in Eq. (67) corresponds precisely to inducing from S_{n-1} to S_n. It then follows from our arrangement of Eq. (67) that the S_n content of these blocks, in the sense of eigenvalue multiplicities, is determined by the usual induction rule (each S_n representation corresponding to a single yet to be determined Q value, through the eigenvalue in Eq. (73)).

We now move to the main result of this section.

PROPOSITION 19. *For given $Q, \lambda \vdash n, \nu \vdash n-1$, there is an exact sequence of left $P_n(Q)$ modules*

$$0 \to \mathscr{S}_\lambda \to \mathscr{S}_\nu \tag{74}$$

iff (ν, λ) is a Q-pair

Proof. We prove, first, for ν restricted to rectangular partitions ρ (say) and then for all $\mu \subset \rho$.

Note that $\mathscr{M}_n$ from Eq. (70) obeys

$$\operatorname{Tr}(\mathscr{M}_n) = 0 \tag{75}$$

$$\operatorname{Tr}(\mathscr{M}_n^2) = (n-1)\dim(\mathscr{M}_n) \tag{76}$$

and that, as well as $\mathscr{M}_n$ itself, each $R_\rho(\mathscr{M}_n)$ will obey a form of these equations separately. Thus if $\rho'_1 = i$ (that is, i is the depth of the rectangle), so that the S_n induced module decomposes as

$$\rho\uparrow = \chi^{(1)} \oplus \chi^{(2)} = (\rho + e_1) \oplus (\rho + e_{i+1}). \tag{77}$$

Then the two eigenvalues k_1, k_2 obey

$$\dim(\rho + e_1)(k_1) + \dim(\rho + e_{i+1})(k_2) = 0 \tag{78}$$

and

$$\dim(\rho + e_1)(k_1)^2 + \dim(\rho + e_{i+1})(k_2)^2 = (n-1)n\dim(\rho) \tag{79}$$

so k_1, k_2 are determined up to signs (it is not obvious that either of the signs gives the claimed result!).

Now consider $R_\lambda(\mathscr{M}_n)$ in this case and note that $R_{\lambda\downarrow}(\mathscr{M}_{n-1})$ (i.e., part of the level $P_{n-1}(Q)$ eigenvalue problem) is embedded in it in the top left-hand corner. Hence we could look for eigenvectors of the form

$$V = \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} \tag{80}$$

such that, with simple content of the restriction $\lambda\downarrow = \bigoplus_i \mu^{(i)}$,

$$R_\lambda(\mathscr{M}_n)V = \left(\begin{array}{c|c} \oplus_i R_{\mu^{(i)}}(\mathscr{M}_{n-1}) & \vdots \\ \hline R_\lambda(\sigma_1\sigma_2..\sigma_{n-2})\ R_\lambda(\sigma_2..\sigma_{n-2}) \ldots R_\lambda(\sigma_{n-2})\ R_\lambda(1) & 0 \end{array} \right) \begin{pmatrix} v \\ 0 \\ \hline 0 \end{pmatrix}, \tag{81}$$

where v is any eigenvector associated to $\mu^{(1)}$ a summand of the restriction of λ, i.e., such that with $(R_\lambda(\sigma_i\sigma_{i+1} \ldots \sigma_{n-2}))_{jk}$ the jk^{th} matrix element,

$$\sum_{i=1}^{n-2} v_i(\sigma_i\sigma_{i+1} \ldots \sigma_{n-2})_{j1} + v_{n-1}1 = 0 \tag{82}$$

for all $j = 1, 2, \ldots, \dim(\lambda)$. Here are $\dim(\lambda)$ constraints on $\dim(\uparrow\mu) = n\dim(\mu)$ unknowns. Thus there are at least $\dim(\uparrow\mu) - \dim(\lambda)$ eigenvalues in common between $R_\lambda(\mathscr{M}_n)$ and $R_{\mu^{(1)}}(\mathscr{M}_{n-1})$ (note Eq. (73) implies that the Q values obtained from these eigenvalues will be shifted by $+1$).

Now stripping off boxes from the diagram in rows we eventually obtain the desired result, since the other sign gives a contradiction with the result of a straightforward explicit calculation at $\lambda = (n)$ (cf. Eq. (66)).

The full proposition now follows by the same means, stripping off boxes from the now known rectangular cases. ∎

4.6. *The Case $P_n[n-2]'$*

PROPOSITION 20. *For given $Q, \lambda \vdash n, \nu \vdash n-2$, there is an exact sequence of left $P_n(Q)$ module morphisms given by*

$$0 \to \mathscr{S}_\lambda \to \mathscr{S}_\nu \tag{83}$$

iff (ν, λ) is a Q-pair.

Proof. We return to Eq. (63) and replace it with the more general requirement $A^h x = 0\,(x \in \Psi_n[i])$, which can be written as

$$A^h \sum_{n!/(n-i)!\text{elements}} \alpha(v^1, v^2, \dots, v^i)\,\chi_n(v^1, v^2, \dots, v^i) = 0 \tag{84}$$

(where the sum is over an appropriate basis of $\chi_n(v^1, v^2, \dots, v^i)$'s—see the proof of Proposition 16). Here

$$w_{j_1 j_2 \dots j_i} = \sum_{\text{elements}} \alpha(v^1, v^2, \dots, v^i)\, v^1_{j_1} v^2_{j_2} \dots v^i_{j_i}$$

is the coefficient of $((j_1)^1 (j_2)^2 \cdots (j_i)^i (j_{i+1}) \dots (j_n))$ in x (with $(j_1 j_2 \cdots j_n)$ a permutation of $(12 \cdots n)$). Thus, the constraints generalizing Eq. (64) are

$$\begin{aligned}
&\sum_{\text{elements}} \alpha(v^1, v^2, \dots, v^i)\left(Q\left(\prod_{l=1}^{i} v^{k_l}_{j_l} \right) - \left(\sum_{m=1}^{i} \left(\prod_{l \neq m} v^{k_l}_{j_l} \right)\left(v^{k_m}_{j_m} + v^{k_m}_{h} \right) \right)\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. -(n-(i+1))\left(\prod_{l=1}^{i} v^{k_l}_{j_l} \right) \right) \\
&\quad = Q w_{j_1 j_2 \dots j_i} - \left((i) w_{j_1 j_2 \dots j_i} + w_{h j_2 \dots j_2} + w_{j_1 h \dots j_i} + \cdots + w_{j_1 j_2 \dots h} \right) \\
&\qquad -(n-(i+1)) w_{j_1 j_2 \dots j_i} = 0.
\end{aligned} \tag{85}$$

We must solve this system in case $i = n - 2$. First let us check that there are no more constraints. Formally, there are further constraints associated with coefficients of words with fewer parts, such as $((j_1 j_3 j_4)^1 (j_2)^2)$ in the case $n = 4$. But as in Eq. (65) the coefficient of $(\cdots (j_1 g)^1 \cdots (j_2)^2 \cdots)$, say, is a linear combination of an equation the form of Eq. (85), together with a similar equation with the roles of j_1 and g interchanged, which vanishes if every equation of the form of Eq. (85) does. All other constraints are linearly dependent on the Eq. (85) in the same way, so we may restrict attention to those. This means that the $w_{j_1 j_2 \dots j_{n-2}}$'s should be treated as independent unknowns, and the system is linear.

Altogether Eq. (85) represents $n!/(n-i-1)!$ equations in $n!/(n-i)!$ unknowns. With $i = n - 2$ these may be arranged into n interlocking groups of $(n-1)!$ equations in $(n-1)!$ unknowns, with each group equivalent to a system from Section 4.5, except for replacing $Q \to Q + 1$. For example, consider the coefficients of the six basis elements in $B'_4[2]$ in which a_4 is isolated. Among these six, the action of each A^i $(i = 1, 2, 3)$ gives a non-trivial constraint with just two elements (for example, A^1 gives constraints on $((1)(2)^1(3)^2(4))$ and $((1)(2)^2(3)^1(4)))$. The six constraints generated in this way give Eq. (73) (with $n = 4$), once a_4 is discarded (and, similarly, cycling the roles $4 \to 3 \to 2 \to 1$ to give the other three groups). These systems are solved uniquely in Section 4.5 as separate systems, but we require consistent solutions (here there are unknowns in common).

Now suppose $w^1 \in \Psi_n[i]$ is such that $\mathrm{S}_n w^1 \cong \mathrm{S}(\lambda)$ as a left S_n module, and $w^1 \mathrm{S}_i \cong \mathrm{S}(\mu)$ as a right S_i module, and let $\{w^1, w^2, \dots, w^l\} \subset \Psi_n[i]$ be a basis for $\mathrm{S}_n w^1$. Then the restriction to S_{n-1} is

$$\left(\mathrm{S}_n w^1\right)\downarrow_{\mathrm{S}_{n-1}} \cong \bigoplus_{\rho = \lambda - \square} \mathrm{S}(\rho)$$

and projecting onto the space spanned by elements $\underline{j_1 \dots j_{n-2}}$ with no $j_i = n$ (N.B. This space is fixed by S_{n-1}) we have

$$\mathrm{Proj}\left(\left(\mathrm{S}_n w^1\right)\downarrow_{S_{n-1}}\right) \cong \bigoplus_{\{\rho | \rho = \lambda - \square\,;\, \rho \supset \mu\}} S(\rho).$$

To see this note that $\mathrm{Proj}(\{w^1, \dots, w^l\})$ still spans an S_{n-1} module, but the projected vectors are not necessarily linearly independent.

In case $i = n - 2$ we can locate a solution x to

$$A^i \mathrm{Proj}(x) = 0 \qquad \text{all } i \neq n$$

for each pair (ρ, μ) (each case fixes some $Q = Q_{\rho\mu}$) from an effectively identical level $n-1$ result (Section 4.5). In fact for $i, n \notin j_1, \dots, j_{n-2}$ there is no mixing between the projection and the complement, so the $\underline{j_1 \dots j_{n-2}}$ component of $A^i x$ vanishes at $Q = Q_{\rho\mu}$. By restricting to different subgroup of S_n isomorphic to S_{n-1} then $A^i x = 0$ for all i if and only if each $Q_{\rho\mu}$ is the same. From the definition of L-diagrams this is only the case if there is exactly *one* pair (ρ, μ), and, taking into account proposition 18(ii), this implies that $\lambda - \mu = (2)$. ∎

5. PROOF OF THE MAIN RESULT

In what follows $\lambda + e_i = (\lambda_1, \lambda_2, \dots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, \dots)$ (so $\lambda + e_1$ exists iff $\lambda_{i-1} > \lambda_i$ or $i = 1$).

Proposition 9 may now be proved by induction on n for each sequence. More precisely, we will show first that the Weyl module morphisms are as implied by this proposition. Essentially we have to show that a pair of modules may be entered in the sequence in Eq. (35) iff it is indexed by a Q-pair. Using Proposition 1 (and functors $\mathscr{G}$ and $\mathscr{F}$) we need only check the cases involving $|\lambda^{(r)}| = n$.

We have seen in Proposition 19 that the "if" part is true by explicit computation in the cases for which $|\lambda^{(r)}| = n$ and $|\lambda^{(r-1)}| = n - 1$. The only if part is also verified *within this sector* (i.e., there are no other morphisms such that $|\lambda^{(r-1)}| = n - 1$).

Now suppose the proposition is true at level $P_n(Q)$. Let $\psi \vdash n$ and $\mu \vdash n - L$ for $L > 1$. For $L > 1$ there are no occurrences of the left $P_{n+1}(Q)$ module $S_\mu(n+1)$ as a composition factor of the induction of left $P_n(Q)$ module $\mathscr{S}_\psi(n)$ to $P_{n+1}(Q)$; and no occurrences of $\mathscr{S}_\psi(n)$ as a composition factor of the restriction of $\mathscr{S}_\mu(n+1)$. Frobenius reciprocity tells us that

$$\mathrm{Hom}_{P_{n+1}}\big(\mathscr{S}_\psi(n)\uparrow_{P_{n+1}}, \mathscr{S}_\mu(n+1)\big) \cong \mathrm{Hom}_{P_n}\big(\mathscr{S}_\psi(n), \mathscr{S}_\mu(n+1)\downarrow_{P_n}\big). \tag{86}$$

We can consider the circumstances under which there is a morphism on the left, by examining those on the right. There are various possibilties to deal with.

PROPOSITION 21. *There is a map $\mathscr{S}_{\psi+e_j} \to \mathscr{S}_\mu$ for some j if $(\mu, \psi + e_i)$ is a $Q =$ pair.*

Proof. There necessarily exist some L and x such that

$$\mu = \psi + e_i - Le_x. \tag{87}$$

We can consider $L > 2$, since $L = 1, 2$ are dealt with explicitly in Propositions 19 and 20. But then either $(\mu - e_i + e_x, \psi)$ or $(\mu + e_x, \psi)$ (if $i = x$) is a Q-pair. Neither $\mu - e_i + e_x$ in the first case nor $\mu + e_x$ in the second can form a pair with any other component of $\mathscr{S}_\mu(n+1)\downarrow$ here (from the definition of an L-diagram), so they are direct summands. Hence, there is a morphism on the right of Eq. (86). The corresponding morphism on the left cannot come from any component of $\mathscr{S}_\psi(n)\uparrow$, except of the claimed form, by the inductive assumption. ∎

PROPOSITION 22. *If $(\mu, \psi + e_i)$ is a Q-pair and $\phi \neq \psi + e_i$ then there is no non-trivial map $\mathscr{S}_\phi \to \mathscr{S}_\mu$.*

Proof. Consider $\chi \vdash n$ such that $\lambda = \psi + e_i$ is not a component of $\mathscr{S}_\chi(n)\uparrow$. Then $\chi \not\subset \lambda$ and there exists j such that $\chi_j > \lambda_j$ (and elsewhere at least two boxes are missing in χ, cf. λ). We can write

$$\chi = \lambda + e_j - e_k - e_l + \cdots,$$

where k, l are distinct from j and $+\ldots$ similarly gives no cancelation. For (μ, λ) a Q-pair and $\mu = \lambda - Le_x$ we have

$$Q = |\mu| - x + \mu_x + L. \tag{88}$$

We will check that there is no component $\mathscr{S}_\rho(n)$ of $\mathscr{S}_\mu(n+1)\downarrow$ such that (ρ, χ) is a Q-pair. There are three generic possibilities for ρ:

1. For $(\lambda - Le_x + e_z, \chi)$ a Q'-pair for some Q' we require $k, l = x$ (else $\rho \not\subset \chi$) and, hence, $z = j$. Then

$$Q' = (|\mu| + 1) - x + \mu_x + (L-2)$$

but then $Q' \neq Q$ so (ρ, χ) is not a Q-pair.

2. For $(\lambda - Le_x + e_z - e_t, \chi)$ a Q'-pair for some Q' we require $k = x, l = t$ (else $\rho \not\subset \chi$) and, hence, $z = j$. Then

$$Q' = |\mu| - x + \mu_x + (L-1)$$

$(t \neq x)$

$$Q' = |\mu| - x + (\mu_x - 1) + (L-1)$$

$(t = x)$ but either way $Q' \neq Q$.

3. For $(\lambda - Le_x - e_t, \chi)$ there is no way to form a pair, since $j = x$ is forced and then $\rho \not\subset \chi$.

This tells us that we can eliminate all candidates for the morphism in proposition 21 except $\psi + e_i$ itself and (possibly) a rectangular tableau (if one is in the list). This is because by choosing χ to contain $\psi + e_j$ (say), but not $\psi + e_i$ we get no map. Since $\psi + e_j$ cannot be covered in $\mathscr{S}_\psi(n)\uparrow$ (meaning it cannot have another module glued above it, because it is a quotient) we deduce that it has no map into μ. For $n > 3$ we can always choose ψ to have no rectangular $\psi + e_j$. Cases up to $n = 3$ have already been dealt with explicitly in examples [17]. ∎

Proposition 23. *For given μ,*

(i) *if there is no pair (μ, λ) then $\mathscr{S}_\mu$ is simple,*

(ii) *if there is no pair (ϕ, μ) and $|\mu| \leq n$ then $\mathscr{S}_\mu$ is projective.*

Proof. If there is no pair (μ, λ) then there is no map into $\mathscr{S}_\mu$ except possibly from some $\lambda \vdash n+1$ by inductive assumption, but then if there is such a map it appears in $\mathrm{Hom}_{P_{n+1}}(\mathscr{S}_\psi(n)\uparrow, \mathscr{S}_\mu(n+1))$ for some $\psi \vdash n$, since $\mathscr{S}_\lambda(n+1)$ will be a quotient of any $\mathscr{S}_\psi(n)\uparrow$ for which it is a composition factor. But in fact there is no map in $\mathrm{Hom}_{P_{n+1}}(\mathscr{S}_\psi(n)\uparrow, \mathscr{S}_\mu(n+1))$, since there is no (uncovered) pair in $\mathrm{Hom}_{P_n}(\mathscr{S}_\psi(n), \mathscr{S}_\mu(n+1)\downarrow)$. To see this note first that cases with $L < 3$ are already dealt with explicitly, so we can restrict attention to $|\psi| - |\mu| > 1$. Then there are three cases to consider (corresponding to the three types of component of $\mathscr{S}_\mu(n+1)\downarrow$). For $\mu - e_t$ there is no pair since either $\mu - e_t$ is not the left-hand side of *any* pair or it is in a pair with $L < 3$ (and hence not with

ψ). For $\mu - e_t + e_j$ there is no pair similarly. For $\mu + e_j$ there is no pair or there *is* a pair $(\mu + e_j, \psi)$ (say), but $\mu + e_j$ is on the *right*-hand side of another pair, with the left-hand side (ν say) another component of $\mathcal{S}_\mu(n+1)\downarrow$. In this case μ is not a right-hand side and $\mathcal{S}_{\mu+e_j}(n)$ is not a submodule of $\mathcal{S}_\mu(n+1)\downarrow$ (since by assumption the $P_n(Q)$ module $\mathcal{S}_\mu(n)$ is projective and, hence, by Proposition 7, the restriction $\mathcal{S}_\mu(n+1)\downarrow$ is a quotient of a projective module by some irrelevant modules—thus $\mathcal{S}_\nu(n)$ is glued under $\mathcal{S}_{\mu+e_j}(n)$) and there is no morphism in $\mathrm{Hom}_{P_n}(\mathcal{S}_\psi(n), \mathcal{S}_\mu(n+1)\downarrow)$.

If no pair (ϕ, μ) and $|\mu| \leq n$ then $\mathcal{S}_\mu(n)$ is projective by assumption, but $\mathcal{G}$ takes projectives to projectives. ∎

N.B. In fact if $\mu \vdash n+1$ then (ii) still holds, as we will prove in the next section.

From Propositions 21, 22, and 23 we have that $\mathcal{S}_\lambda \to \mathcal{S}_\mu$ iff (μ, λ) is a Q-pair. Thus, further, the image of $\mathcal{S}_\lambda$ here is simple (i.e., it is $\mathcal{S}_\lambda^Q$ from Eq. (36)) since otherwise there is some ρ such that $\mathcal{S}_\rho \to \mathcal{S}_\lambda$ and then $\mathcal{S}_\rho \to \mathcal{S}_\mu$, a contradiction. If $\mu \subset \lambda \subset \rho$ gives a Q-tableau then

$$\mathcal{S}_\rho \xrightarrow{f} \mathcal{S}_\lambda \xrightarrow{g} \mathcal{S}_\mu$$

and the composite gf has image zero similarly (thus the kernel of g is $\mathcal{S}_\rho^Q$).

It remains to check that $\mathcal{S}_\mu / \mathcal{S}_\lambda^Q$ is simple (if so then Eq. (36) follows using Serre's cde triangle [18]). We do this in the next section.

5.1. *The Determinant of the Gram Matrix for* $\mathcal{S}_\lambda$

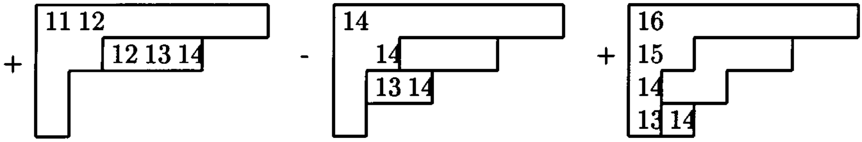
We have a complete set of generic simple modules $\mathcal{S}_\lambda$, each with inner product. Let us work with the partition basis B_λ and write $M_n(\lambda)$ for the Gram matrix. We can determine

$$\det(M_n(\lambda)) = \prod_{R=0}^{\infty} (Q - R)^{L_n^\lambda(R)}, \tag{89}$$

where $L_n^\lambda(R)$ is an alternating sum of dimensions of Weyl modules with index partition $\lambda(d)$ $(d = 1, 2, \ldots, d_{\max})$, where $\lambda(d) \supset \lambda$ is such that $\lambda(d) - \lambda$ is the rim of $\lambda(d)$ of depth d (and the rightmost entry of the row $\lambda(d) - \lambda(d-1)$ in the $|\lambda(d-1)|$-diagram of $\lambda(d)$ is R—it is sufficient that $\lambda(d_{\max})$ has this property). Note that a $\lambda(1)$ may not exist for given R, λ (then $\mathcal{S}_\lambda$ is simple at $Q = R$) and $d_{\max}$ is the maximum depth such that $|\lambda(d)| \leq n$:

$$L_n^\lambda(R) = \sum_{d=1}^{d_{\max}} (-1)^{d-1} |\mathcal{S}_{\lambda(d)}|. \tag{90}$$

EXAMPLE. For $n = 18$ the $\lambda(d)$s appearing in the alternating sum for $\lambda = (7,2,1,1)$ and $R = 14$ are illustrated with their appropriate signs in the following picture:



Note that the result in Eq. (89) completes the proof of Proposition 9, since $\det(M_n(\lambda))$ is a product of invariant factors, so $L_n^\lambda(R)$ is an upper bound on the dimension of the maximal proper submodule of $\mathscr{S}_\lambda$ at $Q = R$ (as if it were, at least one factor of $1/(Q-R)$ for each generic primitive idempotent which cannot survive the specialization $Q \to R$).

Proof. First we show that the degree of $\det(M_n(\lambda))$ is as in Eq. (89):

Each entry in M is an algebraic number times a power of Q. The highest power of Q in each row occurs on the diagonal (Eq. (47)). The highest power may also occur elsewhere in the row in some, but not all, rows (easy proof) so the degree is given by the product of diagonal elements. For given n let N_q be the number of partitions of n into exactly q parts. Then

$$\text{degree}\big(M_n((0))\big) = \sum_{q=1}^{n} N_q . q = |\mathscr{S}_{(1)}|. \tag{91}$$

The first identity follows since the diagonal element of the Gram matrix for a partition with q parts is Q^q, and the second identity is from Eq. (43). Similarly,

$$\text{degree}\big(M_n((1))\big) = \sum_{q=2}^{n} \underbrace{N_q \cdot (q)}_{Eq.\,(43)} \quad \underbrace{(q-1)}_{\text{contrib. to degree}} = |\mathscr{S}_{(2)}| + |\mathscr{S}_{(1^2)}| \tag{92}$$

and if $\lambda \vdash p$

$$\begin{aligned}
\text{degree}(M_n(\lambda)) &= \dim(S(\lambda)) \sum_{q=p+1}^{n} N_q \cdot \frac{q!}{(q-p)!p!} \cdot (q-p) \\
&= (p+1) \dim(S(\lambda)) \sum_q N_q \cdot \frac{q!}{(q-(p+1))!(p+1)!} \\
&= (p+1) \dim(S(\lambda)) |\mathscr{S}_{(p+1)}| = \sum_{\mu \rhd \lambda} |\mathscr{S}_\mu|. \qquad (93)
\end{aligned}$$

This is consistent with Eq. (89) since it is identical to the sum over rims of length 1 in $\sum_{R=0}^{\infty} L_n^\lambda(R)$, and all other sums over rims of fixed length cancel (readily, but not trivially, proven by reference to Littlewood−Richardson rules).

The remaining details of the determinant are fixed as follows:

If the submodules we already found are maximal then the submodule for each $Q = R$ (if any) contributes at least a factor $(Q - R)^{L_n^\lambda(R)}$ to the determinant (in a basis of unnormalized primitive idempotents—such as Smith normal form—exactly d of the invariant factors must vanish at $Q = R$ for there to be a d-dimensional submodule, and each of these must vanish like $(Q - R)$ or a higher power). Suppose n, λ are the lowest n and "rightmost" (i.e., largest or joint largest degree) λ for which some Qs develop a submodule other than as given in Proposition 9. The only mechanism possible for this which is consistent with the results of the previous section is for $\mathscr{S}_\lambda$ to develop a Loewy structure

$$\begin{matrix} \mathscr{S}_\lambda^Q \\ x \\ \mathscr{S}_\nu^Q \end{matrix}, \qquad (94)$$

where $\mathscr{S}_\nu^Q$ is the submodule identified in the proposition and $\nu \vdash m < n$ and x contains components with $\mu \vdash n$. This structure is forced because the x part never shows up in any Hom (by the analysis in the previous section). By this construction then for all other Qs and for x and ν the dimensions are standard (i.e., as implied by Proposition 9) and, hence, the non-standard submodule(s) are *larger* than standard (being the standard part *plus* something else). The overall degree of $\det(M_n(\lambda))$ would be strictly larger than the actual value. Thus we have a contradiction, so finally there can be no such n, λ and Proposition 9 is proven.

ACKNOWLEDGMENTS

I thank Dave Woodcock for giving me his useful perspective on quasi-heredity. I thank EPSRC (Grants GRJ25758 and GRJ29923) and the Nuffield Foundation for financial support of parts of this project. I also thank the referee for some helpful suggestions.

REFERENCES

1. R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, New York, 1982).
2. R. Brauer, *Ann. Math.* **38**, 857 (1937).
3. W. P. Brown, *Michigan Math. J.* **3**, 1 (1955−56).
4. E. Cline, B. Parshall, and L. Scott, *J. Reine Angew Math.* **391**, 85 (1988).
5. P. M. Cohn, *Algebra*, Vol. 2, (Wiley, New York, 1982), p. 255.
6. G. deB. Robinson, *Representation Theory of the Symmetric Group* (Univ. of Toronto Press, Toronto, 1961).
7. V. Dlab and C. M. Ringel, *Compositio Math.* **70**, 155 (1989).
8. J. A. Green, *Polynomial Representations of* GL_n (Springer-Verlag, Berlin, 1980), Section 6.2.
9. P. Hanlon and D. Wales, *J. Algebra* **164**, 773 (1994).
10. G. D. James and A. Kerber, *The Representation Theory of the Symmetric Group* (Addison−Wesley, London, 1981).
11. I. Macdonald, *Symmetric Functions and Hall Polynomials* (Oxford, Univ. Press, Oxford, 1979).
12. P. P. Martin, *Publ. Res. Inst. Math. Sci.* **26**, 485 (1990).
13. P. P. Martin, *Potts Models and Related Problems in Statistical Mechanics* (World Scientific, Singapore, 1991).
14. P. P. Martin, *J. Knot Theory Ramifications* **3**, 51 (1994).
15. P. P. Martin and H. Saleur, *Commun. Math. Phys.* **158**, 155, (1993).
16. P. P. Martin and H. Saleur, *Lett. Math. Phys.* **30**, 179 (1994).
17. N.B. The reader may care to consider the ‘‘worked example’’ filled version of this paper at http://web.city.ac.uk/∼ sg387/hotmet.two.
18. J.-P. Serre, *Linear Representations of Finite Groups* (Springer-Verlag, New York, 1977), p. 124.
19. H. N. V. Temperley and E. H. Lieb, *Proc. R. Soc. A, 1971*, p. 251.