第一章　多项式

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$$
\begin{array}{r|cccc|l}
x^2 - 3x + 1 & 3x^3 & +4x^2 & -5x & +6 & 3x + 13 \\
 & 3x^3 & -9x^2 & +3x & & \\
\hline
 & & 13x^2 & -8x & +6 & \\
 & & 13x^2 & -39x & +13 & \\
\hline
 & & & 31x & -7 &
\end{array}
$$

$$f(x) = q(x)g(x) + r(x)$$

例 1

$$f(x) = x^4 + 3x^3 - x^2 - 4x - 3$$
$$g(x) = 3x^3 + 10x^2 + 2x - 3$$

求 $(f(x), g(x))$, 并求 $u(x), v(x)$ 使

$$(f(x), g(x)) = u(x)f(x) + v(x)g(x)$$

辗转相除法可按下面的格式来作:

$$
\begin{array}{r|r|l}
3x^3+10x^2+2x-3 & x^4+3x^3-x^2-4x-3 & \frac{1}{3}x-\frac{1}{9} \\
 & x^4+\frac{10}{3}x^3+\frac{2}{3}x^2-x & =q_1(x) \\
\cline{2-2}
 & -\frac{1}{3}x^3-\frac{5}{3}x^2-3x-3 & \\
 & -\frac{1}{3}x^3-\frac{10}{9}x^2-\frac{2}{9}x+\frac{1}{3} & \\
\cline{2-2}
 & r_1(x)=-\frac{5}{9}x^2-\frac{25}{9}x-\frac{10}{3} &
\end{array}
$$

$$f(x)=q_1(x)g(x)+r_1(x)$$

$-\frac{27}{5}x+9$	$3x^3+10x^2+2x-3$	$x^4+3x^3-x^2-4x-3$	$\frac{1}{3}x-\frac{1}{9}$
$=q_2(x)$	$3x^3+15x^2+18x$	$x^4+\frac{10}{3}x^3+\frac{2}{3}x^2-x$	$=q_1(x)$
	$-5x^2-16x-3$	$-\frac{1}{3}x^3-\frac{5}{3}x^2-3x-3$	
	$-5x^2-25x-30$	$-\frac{1}{3}x^3-\frac{10}{9}x^2-\frac{2}{9}x+\frac{1}{3}$	
	$r_2(x)=9x+27$	$r_1(x)=-\frac{5}{9}x^2-\frac{25}{9}x-\frac{10}{3}$	

$-\frac{27}{5}x+9$	$3x^3+10x^2+2x-3$	$x^4+3x^3-x^2-4x-3$	$\frac{1}{3}x-\frac{1}{9}$
$=q_2(x)$	$3x^3+15x^2+18x$	$x^4+\frac{10}{3}x^3+\frac{2}{3}x^2-x$	$=q_1(x)$
	$-5x^2-16x-3$	$-\frac{1}{3}x^3-\frac{5}{3}x^2-3x-3$	
	$-5x^2-25x-30$	$-\frac{1}{3}x^3-\frac{10}{9}x^2-\frac{2}{9}x+\frac{1}{3}$	
	$r_2(x)=9x+27$	$r_1(x)=-\frac{5}{9}x^2-\frac{25}{9}x-\frac{10}{3}$	

$$f(x)=q_1(x)g(x)+r_1(x)$$

$$g(x)=q_2(x)r_1(x)+r_2(x)$$

$-\frac{27}{5}x+9$	$3x^3+10x^2+2x-3$	$x^4+3x^3-x^2-4x-3$	$\frac{1}{3}x-\frac{1}{9}$
$=q_2(x)$	$3x^3+15x^2+18x$	$x^4+\frac{10}{3}x^3+\frac{2}{3}x^2-x$	$=q_1(x)$
	$-5x^2-16x-3$	$-\frac{1}{3}x^3-\frac{5}{3}x^2-3x-3$	
	$-5x^2-25x-30$	$-\frac{1}{3}x^3-\frac{10}{9}x^2-\frac{2}{9}x+\frac{1}{3}$	
	$r_2(x)=9x+27$	$r_1(x)=-\frac{5}{9}x^2-\frac{25}{9}x-\frac{10}{3}$	$-\frac{5}{81}x-\frac{10}{81}$
		$-\frac{5}{9}x^2-\frac{5}{3}x$	$=q_3(x)$
		$-\frac{10}{9}x-\frac{10}{3}$	
		$-\frac{10}{9}x-\frac{10}{3}$	
		0	

$$f(x)=q_1(x)g(x)+r_1(x)$$

$$g(x)=q_2(x)r_1(x)+r_2(x)$$

$$r_1(x)=q_3(x)r_2(x)$$

因此

$$(f(x), g(x)) = \frac{1}{9} r_2(x) = x + 3$$

由

$$\begin{aligned} f(x) &= q_1(x) g(x) + r_1(x) \\ g(x) &= q_2(x) r_1(x) + r_2(x) \\ r_1(x) &= q_3(x) r_2(x) \end{aligned}$$

可知

$$\begin{aligned} r_2(x) &= g(x) - q_2(x) r_1(x) \\ &= g(x) - q_2(x) \left(f(x) - q_1(x) g(x) \right) \\ &= -q_2(x) f(x) + \left(1 + q_1(x) q_2(x) \right) g(x) \end{aligned}$$

于是，令

$$\begin{aligned} u(x) &= -\frac{1}{9} q_2(x) = \frac{3}{5} x - 1, \\ v(x) &= \frac{1}{9} \left(1 + q_1(x) q_2(x) \right) = -\frac{1}{5} x^2 + \frac{2}{5} x, \end{aligned}$$

就有

$$(f(x), g(x)) = u(x) f(x) + v(x) g(x).$$