第四章 部分习题

2. $\begin{pmatrix}1&1\\0&1\end{pmatrix}^n=\begin{pmatrix}1&n\\0&1\end{pmatrix}$, $\begin{pmatrix}1&-1&-1&-1\\-1&1&-1&-1\\-1&-1&1&-1\\-1&-1&-1&1\end{pmatrix}^2=\begin{pmatrix}4&0&0&0\\0&4&0&0\\0&0&4&0\\0&0&0&4\end{pmatrix}$.

$$\begin{pmatrix}\lambda&1&0\\0&\lambda&1\\0&0&\lambda\end{pmatrix}^2=\begin{pmatrix}\lambda^2&2\lambda&1\\0&\lambda^2&2\lambda\\0&0&\lambda^2\end{pmatrix},\begin{pmatrix}\lambda&1&0\\0&\lambda&1\\0&0&\lambda\end{pmatrix}^3=\begin{pmatrix}\lambda^3&3\lambda^2&3\lambda\\0&\lambda^3&3\lambda^2\\0&0&\lambda^3\end{pmatrix},$$

$$\begin{pmatrix}\lambda&1&0\\0&\lambda&1\\0&0&\lambda\end{pmatrix}^4=\begin{pmatrix}\lambda^4&4\lambda^3&6\lambda^2\\0&\lambda^4&4\lambda^3\\0&0&\lambda^4\end{pmatrix},\begin{pmatrix}\lambda&1&0\\0&\lambda&1\\0&0&\lambda\end{pmatrix}^5=\begin{pmatrix}\lambda^5&5\lambda^4&10\lambda^3\\0&\lambda^5&5\lambda^4\\0&0&\lambda^5\end{pmatrix}.$$

猜想$\begin{pmatrix}\lambda&1&0\\0&\lambda&1\\0&0&\lambda\end{pmatrix}^n=\begin{pmatrix}\lambda^n&n\lambda^{n-1}&\frac{n(n-1)}{2}\lambda^{n-2}\\0&\lambda^n&n\lambda^{n-1}\\0&0&\lambda^n\end{pmatrix}$.再证明.只要证 $A^{n+1}=AA^n$ 即可.

$$\begin{pmatrix}\lambda&1&0\\0&\lambda&1\\0&0&\lambda\end{pmatrix}\begin{pmatrix}\lambda^n&n\lambda^{n-1}&\frac{n(n-1)}{2}\lambda^{n-2}\\0&\lambda^n&n\lambda^{n-1}\\0&0&\lambda^n\end{pmatrix}=\begin{pmatrix}\lambda^{n+1}&(n+1)\lambda^n&(\frac{n(n-1)}{2}+n)\lambda^{n-1}\\0&\lambda^{n+1}&(n+1)\lambda^n\\0&0&\lambda^{n+1}\end{pmatrix}$$

$$=\begin{pmatrix}\lambda^{n+1}&(n+1)\lambda^n&\frac{n(n+1)}{2}\lambda^{n-1}\\0&\lambda^{n+1}&(n+1)\lambda^n\\0&0&\lambda^{n+1}\end{pmatrix}=A^{n+1}.$$

4.设 $A=\begin{pmatrix}1&0&0\\0&1&2\\3&1&2\end{pmatrix}$,设 $B=\begin{pmatrix}a&b&c\\d&e&f\\x&y&z\end{pmatrix}$,并设 $AB=BA$.则

$$\begin{pmatrix}a&b&c\\d+2x&e+2y&f+2z\\3a+d+2x&3b+e+2y&3c+f+2z\end{pmatrix}=\begin{pmatrix}a+3c&b+c&2b+2c\\d+3f&e+f&2e+2f\\x+3z&y+z&2y+2z\end{pmatrix}.$$

或者 $A=E+\begin{pmatrix}0&0&0\\0&0&2\\3&1&1\end{pmatrix}=E+T$,则 $AB=BA$,即为 $TB=BT$.

从而$TB=\begin{pmatrix}0&0&0\\2x&2y&2z\\3a+d+x&3b+e+y&3c+f+z\end{pmatrix}=BT=\begin{pmatrix}3c&c&2b+c\\3f&f&2e+f\\3z&z&2y+z\end{pmatrix}$.则

$\begin{cases}3a+d+x-3z=0\\e+y-z=0\\2x=3f=6y\\b=c-0\end{cases}$ 即系数矩阵$\begin{pmatrix}3&1&0&0&1&0&-3\\0&0&1&0&0&1&-1\\0&0&0&3&-2&0&0\\0&0&0&0&1&-3&0\end{pmatrix}$,解得

有 3 个自由的.设为d,e,f,则得$\eta_1=(-\frac{1}{3},1,0,0,0,0,0)$, $\eta_2=(1,0,1,0,0,0,1)$, $\eta_2=(0,0,0,1,\frac{3}{2},\frac{1}{2},\frac{1}{2})$, .

则$B=\begin{pmatrix}-\frac{1}{3}d+e&0&0\\d&e&f\\\frac{3}{2}f&\frac{1}{2}f&e+\frac{1}{2}f\end{pmatrix}=d\begin{pmatrix}-\frac{1}{3}&0&0\\1&0&0\\0&0&0\end{pmatrix}+e\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+f\begin{pmatrix}0&0&0\\0&0&1\\\frac{3}{2}&\frac{1}{2}&\frac{1}{2}\end{pmatrix}$

可再设自由的为d,y,z,则得$\eta_1=(-\frac{1}{3},1,0,0,0,0,0)$, $\eta_2=(1,0,-1,2,3,1,0)$, $\eta_2=(1,0,1,0,0,0,1)$.

则$B=\begin{pmatrix}-\frac{1}{3}d+y+z&0&0\\d&z-y&2y\\3y&y&z\end{pmatrix}=d\begin{pmatrix}-\frac{1}{3}&0&0\\1&0&0\\0&0&0\end{pmatrix}+z\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+y\begin{pmatrix}1&0&0\\0&-1&2\\3&1&0\end{pmatrix}$,

例 设$A=\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}$,设$B=\begin{pmatrix}a&b&c\\d&e&f\\x&y&z\end{pmatrix}$,并设$AB=BA$.则

$\begin{pmatrix}d&e&f\\x&y&z\\0&0&0\end{pmatrix}=\begin{pmatrix}0&a&b\\0&d&e\\0&x&y\end{pmatrix}$. $d=x=y=0$, $a=e=z$, $b=f$,则$B=\begin{pmatrix}a&b&c\\0&a&b\\0&0&a\end{pmatrix}$.

设$A=\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}$,设$B=\begin{pmatrix}a&b&c\\d&e&f\\x&y&z\end{pmatrix}$,并设$AB=BA$.则

$\begin{pmatrix}0&0&0\\a&b&c\\d&e&f\end{pmatrix}=\begin{pmatrix}b&c&0\\e&f&0\\y&z&0\end{pmatrix}$. $b=c=f=0$, $a=e=z$, $d=y$,则$B=\begin{pmatrix}a&0&0\\d&a&0\\x&d&a\end{pmatrix}$.

5. 假设矩阵$B=\begin{pmatrix}b_{11}&b_{12}&\cdots&b_{1n}\\b_{21}&b_{22}&\cdots&b_{2n}\\\vdots&\vdots&&\vdots\\b_{n1}&b_{n2}&\cdots&b_{nn}\end{pmatrix}$满足$AB=BA$.则

$$AB=\begin{pmatrix} a_1b_{11} & a_1b_{12} & \cdots & a_1b_{1n} \\ a_2b_{21} & a_2b_{22} & \cdots & a_2b_{2n} \\ \vdots & \vdots & & \vdots \\ a_nb_{n1} & a_nb_{n2} & \cdots & a_nb_{nn} \end{pmatrix}=\begin{pmatrix} a_1b_{11} & a_2b_{12} & \cdots & a_nb_{1n} \\ a_1b_{21} & a_2b_{22} & \cdots & a_nb_{2n} \\ \vdots & \vdots & & \vdots \\ a_1b_{n1} & a_2b_{n2} & \cdots & a_nb_{nn} \end{pmatrix}=BA.$$

则有 $a_ib_{ij}=a_jb_{ij}$,对任意 i,j,从而若 $i\neq j$,则 $b_{ij}=0$,故矩阵 $B=\begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{pmatrix}$.

6. 解: 假设 $B=\begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1r} \\ B_{21} & B_{22} & \cdots & B_{2r} \\ \vdots & \vdots & & \vdots \\ B_{r1} & B_{r2} & \cdots & B_{rr} \end{pmatrix}$满足 $AB=BA$.

则 $AB=\begin{pmatrix} a_1B_{11} & a_1B_{12} & \cdots & a_1B_{1r} \\ a_2B_{21} & a_2B_{22} & \cdots & a_2B_{2r} \\ \vdots & \vdots & & \vdots \\ a_rB_{r1} & a_rB_{r2} & \cdots & a_rB_{rr} \end{pmatrix}=\begin{pmatrix} a_1B_{11} & a_2B_{12} & \cdots & a_rB_{1r} \\ a_1B_{21} & a_2B_{22} & \cdots & a_rB_{2r} \\ \vdots & \vdots & & \vdots \\ a_1B_{r1} & a_2B_{r2} & \cdots & a_rB_{rr} \end{pmatrix}=BA.$

则有 $a_iB_{ij}=a_jB_{ij}$,对任意 i,j,从而若 $i\neq j$,则 $B_{ij}=0$,故矩阵 $B=\begin{pmatrix} B_{11} & 0 & \cdots & 0 \\ 0 & B_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & B_{nn} \end{pmatrix}$.

7. 8. 10 解: $A^T=A$,且 $A^2=0$,则设 $A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$,则

$$A^2=A^TA=\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}=\begin{pmatrix} \sum_{i-1}^{n}a_{i1}^2 & * & \cdots & * \\ * & \sum_{i-1}^{n}a_{i2}^2 & \cdots & * \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & \sum_{i-1}^{n}a_{in}^2 \end{pmatrix}=0.$$

从而 $\sum_{i-1}^{n}a_{ij}^2=0$,从而 $a_{ij}=0$.任意 i,j,则 $A=0$.

11. 若 $(AB)^T=AB$,则 $B^TA^T=AB$,即 $BA=AB$.反之亦成立. 12. $A=\dfrac{A+A^T}{2}+\dfrac{A-A^T}{2}$

13. 解:$|A|=\begin{vmatrix} n & \sum_{i=1}^{n}x_i & \cdots & \sum_{i=1}^{n}x_i^{n-1} \\ \sum_{i=1}^{n}x_i & \sum_{i=1}^{n}x_i^2 & \cdots & \sum_{i=1}^{n}x_i^{n} \\ \vdots & \vdots & & \vdots \\ \sum_{i=1}^{n}x_i^{n-1} & \sum_{i=1}^{n}x_i^{n} & \cdots & \sum_{i=1}^{n}x_i^{2n-2} \end{vmatrix}=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}\begin{vmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{vmatrix}$

14. 证明: $\Rightarrow$. 假设存在非零矩阵 B ,满足 $AB=0$,则矩阵 B 的列向量组可以看成是线性方程组 $Ax=0$ 的解,而矩阵 B 非零,从而 $Ax=0$ 有非零解,则系数矩阵的行列式为零.

$\Leftarrow$. 反之,若 $|A|=0$,则 $Ax=0$ 有非零解,取基础解系 $\eta_1,\eta_2,\cdots,\eta_{n-r}$,则令 $B=(\eta_1,\eta_2,\cdots,\eta_{n-r},0,\cdots,0)$. 则有 $AB=0$.

15. 证明: 由题意,对任一 n 维列向量 X ,都有 $AX=0$,则取特殊的一组 $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$,则有 $A\varepsilon_i=0$,

从而 $A(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)=AE=A=0$.

或者 $AX=0$ 的基础解系含有 n 个向量,从而系数矩阵的秩 0 ,即 $A=0$.

16. 证明:1) $BC=0$,从而有 $C^TB^T=0$,则 B^T 的列向量组,即 B 的行向量组是方程组 $C^Tx=0$ 的解.而

$r(C)=r$,行满秩, C^T 列满秩,从而 $C^Tx=0$ 只有零解,而 B^T 的列向量组就是 $C^Tx=0$ 的解,从而 $B=0$.

2) $BC=C$,从而 $(B-E)C=0$,由1) 得到 $B=E$.

17. 19. 由 $A^k=0$,则 $A^k-E=-E$,即 $(A-E)(A^{k-1}+A^{k-2}+\cdots+A+E)=-E$.

21. 对 $X=\begin{pmatrix} 0 & A \\ C & 0 \end{pmatrix}$,设 $X^{-1}=\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$,则 $\begin{pmatrix} 0 & A \\ C & 0 \end{pmatrix}\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}=\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}$,即

$\begin{pmatrix} AX_3 & AX_4 \\ CX_1 & CX_2 \end{pmatrix}=\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}$,则 $X_3=A^{-1},X_1=X_4=0,X_2=C^{-1}$,则 $X^{-1}=\begin{pmatrix} 0 & C^{-1} \\ A^{-1} & 0 \end{pmatrix}$.

23. 1) A 对称,且可逆,则 $(A^{-1})^T=(A^T)^{-1}=A^{-1}$,从而逆矩阵也对称,

若 A 反对称,且可逆,则 $(A^{-1})^T=(A^T)^{-1}=(-A)^{-1}=-A^{-1}$,从而逆矩阵也反对称,

2) 假设矩阵 A 是一个奇数阶的可逆反对称矩阵,则首先 $A^T=-A$,从而行列式为 $|A^T|=|-A|$,即 $|A|=-|A|$,从而 $|A|=0$,与 A 可逆矛盾.

24. 29 $\begin{vmatrix} E_m & B \\ A & E_n \end{vmatrix}=\begin{vmatrix} E_m-BA & 0 \\ A & E_n \end{vmatrix}=|E_m-BA||E_n|=|E_m-BA|$.同时

$$\begin{vmatrix} E_m & B \\ A & E_n \end{vmatrix} = \begin{vmatrix} E_m & B \\ 0 & E_n - AB \end{vmatrix} = |E_n - AB||E_m| = |E_n - AB|.$$

30. 首先,$\begin{vmatrix} E_m & B \\ A & \lambda E_n \end{vmatrix} = \begin{vmatrix} E_m & B \\ 0 & \lambda E_n - AB \end{vmatrix} = |\lambda E_n - AB||E_m| = |\lambda E_n - AB|$.

$$\begin{vmatrix} E_m & B \\ A & \lambda E_n \end{vmatrix} = \begin{vmatrix} E_m - \frac{1}{\lambda}BA & 0 \\ A & \lambda E_n \end{vmatrix} = \left|E_m - \frac{1}{\lambda}BA\right||\lambda E_m| = \left|\frac{1}{\lambda}(\lambda E_m - BA)\right||\lambda E_m| = \lambda^{n-m}|\lambda E_m - BA|.$$

则$|\lambda E_n - AB| = \lambda^{n-m}|\lambda E_m - BA|$.

补充题.

1. $r(A)=1$,从而 $A\neq 0$,则存在元素 $a_{ij}\neq 0$,设矩阵 A 的列向量组为 $\alpha_1,\alpha_2,\cdots,\alpha_n$,则 $\alpha_j\neq 0$,由于 $r(A)=1$,则列向量组 $\alpha_1,\alpha_2,\cdots,\alpha_n$ 的秩为1,从而各列向量都是 $\alpha_j=\begin{pmatrix} b_1 \\ b_i \\ \vdots \\ b_n \end{pmatrix}$ 的数量乘积.设 $\alpha_i = a_i\alpha_j$,

则 $A=(a_1\alpha_j, a_2\alpha_j,\cdots,a_n\alpha_j)=(a_1,a_2,\cdots,a_n)\begin{pmatrix} b_1 \\ b_i \\ \vdots \\ b_n \end{pmatrix}$.

2. 由于 $A^l=0$,则 $0\le r(A)\le 1$,若 $r(A)=0$,则 $A=0$,从而 $A^2=0$,若 $r(A)=1$,则由 1 题可得 $A=\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}(b_1,b_2)$,则 $A^l=((b_1,b_2)\begin{pmatrix} a_1 \\ a_2 \end{pmatrix})^{l-1}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}(b_1,b_2)=(a_1b_1+a_2b_2)^{l-1}A=0$,而 $A\neq 0$,从而 $a_1b_1+a_2b_2=0$.则 $A^2=(a_1b_1+a_2b_2)A=0$

6. $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \to \begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix} \to \begin{pmatrix} A & 0 \\ 0 & D-CA^{-1}B \end{pmatrix}$,

即$\begin{pmatrix} E & 0 \\ -CA^{-1} & E \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} E & -B \\ 0 & E \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D-CA^{-1}B \end{pmatrix}$,两边取行列式为

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} A & 0 \\ 0 & D-CA^{-1}B \end{vmatrix} = |A||D-CA^{-1}B| = |A||D-A^{-1}CB| = |AD-CB|$$

7. 由 $r(A)=r$ 可知,存在可逆矩阵 P,Q,使得 $PAQ=\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}$,则 $PAP^{-1}=\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}Q^{-1}P^{-1}$.

设 $Q^{-1}P^{-1}=\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$,则 $PAP^{-1}=\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}=\begin{pmatrix} X_1 & X_2 \\ 0 & 0 \end{pmatrix}$,后 $n-r$ 行为零.

8. $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\to\begin{pmatrix} a & 0 \\ 1 & a^{-1} \end{pmatrix}\to\begin{pmatrix} 1 & -1+a^{-1} \\ 1 & a^{-1} \end{pmatrix}\to\begin{pmatrix} 1 & -1+a^{-1} \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & -1+a^{-1} \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1-a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a^{-1} & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$,则

$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ -a^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & a-1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1+a^{-1} \\ 0 & 1 \end{pmatrix}$

2) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\to\begin{pmatrix} a & b \\ 0 & d-\frac{c}{a}b \end{pmatrix}=\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\to\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$,则

$\begin{pmatrix} 1 & 0 \\ -\frac{c}{a} & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -\frac{b}{a} \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ -a^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & a-1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1+a^{-1} \\ 0 & 1 \end{pmatrix}$,从而

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -a^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & a-1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1+a^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}$

11. 设 A 列满秩,则存在可逆矩阵 P,Q ,使得 $A=P\begin{pmatrix} E_n \\ 0 \end{pmatrix}Q$,

$A=P\begin{pmatrix} E_nQ \\ 0 \end{pmatrix}=P\begin{pmatrix} Q \\ 0 \end{pmatrix}=P\begin{pmatrix} Q & 0 \\ 0 & E_{s-n} \end{pmatrix}\begin{pmatrix} E_n \\ 0 \end{pmatrix}=P_1\begin{pmatrix} E_n \\ 0 \end{pmatrix}$.

$A=P\begin{pmatrix} E_s & 0 \end{pmatrix}Q=\begin{pmatrix} P & 0 \end{pmatrix}Q=\begin{pmatrix} E_s & 0 \end{pmatrix}\begin{pmatrix} P & 0 \\ 0 & E_{n-s} \end{pmatrix}Q=\begin{pmatrix} E_s & 0 \end{pmatrix}Q_1$.

12. 对 A ,存在可逆矩阵 P_1,Q_1 ,使得 $A=P_1\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}Q_1=P_1\begin{pmatrix} E_r \\ 0 \end{pmatrix}\begin{pmatrix} E_r & 0 \end{pmatrix}Q_1=PQ$.