An introduction to symmetric functions in non-commuting variables

Yinhao Li

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Introduction

Let $\mathbb{Q}[[x_1, x_2, \ldots,]] = \mathbb{Q}[[\mathbf{x}]]$ be the algebra of formal power series over $\mathbb{Q}$ in a countably infinite set of commuting variables x_i. We say that an element f in $\mathbb{Q}[[\mathbf{x}]]$ is symmetric if f is and

$$f(x_1, x_2, \ldots) = f(x_{\omega(1)}, x_{\omega(2)}, \ldots) \tag{1}$$

holds for any permutation w of the positive integers $\mathbb{P}$. Define the *algebra of symmetric functions in commuting variables*, $\Lambda = \Lambda(x)$, to be the subalgebra consisting of all elements which are symmetric and of bounded degree in $\mathbb{Q}[[\mathbf{x}]]$.

M. Rosas, B. Sagan, Symmetric functions in noncommuting variables, Trans. Amer. Math. Soc. 358 (1) (2006) 215–232.

Introduction

Let $\mathbb{Q}\langle\langle x_1, x_2, \ldots, \rangle\rangle = \mathbb{Q}\langle\langle \mathbf{x} \rangle\rangle$ be the algebra of formal power series over $\mathbb{Q}$ in a countably infinite set of non-commuting variables x_i. We say that an element f in $\mathbb{Q}\langle\langle \mathbf{x} \rangle\rangle$ is symmetric if f is invariant under (1). Define the *algebra of symmetric functions in non-commuting variables*, $\Pi = \Pi(x)$, to be the subalgebra consisting of all elements which are symmetric and of bounded degree in $\mathbb{Q}\langle\langle \mathbf{x} \rangle\rangle$.

This algebra was first studied by M.C. Wolf in 1936, and is different from the algebra of noncommutative symmetric functions of Gelfand et al. and the partially commutative symmetric functions studied by Lascoux, Schtzenberger, Fomin and Greene.

Basic definitions

Let n be a positive integer. A *partition* $\lambda = (\lambda_1, \ldots, \lambda_l)$ is a sequence of non-increasing positive integers whose sum is n. Here $l = l(\lambda)$ denotes the length of λ. We also use the notation $\lambda = \langle 1^{m_1}, \ldots, n^{m_n} \rangle$.

Define $[n] = \{1, \ldots, n\}$. A *set partition* is a collection of disjoint set (called *blocks*) $B_1, \ldots, B_l$ such that the union of them is $[n]$. We write $\pi = B_1/B_2/\cdots/B_l$, and here $l = l(\pi)$ is said to be the length of π. The *type* $\lambda(\pi)$ of π is defined to be the partition of n whose parts are the sizes of the blocks of π.

The set partitions of $[n]$ forms the *partition lattice* π_n. In this poset, $\pi \leq \sigma$ iff each block of π is contained in some block of σ. The meet and join operations are denoted by $\wedge$ and $\vee$, respectively.

Basic definitions

Now given $\pi \vdash n$, we present the analogous bases for $\Pi(\mathbf{x})$.

The *monomial symmetric function in non-commuting variables* m_π is defined by

$$m_\pi = \sum_{(i_1, i_2, \ldots, i_n)} x_{i_1} x_{i_2} \cdots x_{i_n},$$

where the sum is over all n-tuples $(i_1, i_2, \ldots, i_n)$ with $i_j = i_k$ if and only if j, k are in the same block in π. For example,

$$m_{13/24} = x_1x_2x_1x_2 + x_2x_1x_2x_1 + x_1x_3x_1x_3 + x_3x_1x_3x_1 + x_2x_3x_2x_3 + x_3x_2x_3x_2 + \cdots.$$

Basic definitions

The *power sum function in non-commuting variables* p_π is defined by

$$p_\pi = \sum_{(i_1, i_2, \ldots, i_n)} x_{i_1} x_{i_2} \cdots x_{i_n},$$

where $i_j = i_k$ if j, k are in the same block in π. For instance,

$$p_{13/24} = x_1 x_2 x_1 x_2 + x_2 x_1 x_2 x_1 + x_1^4 + x_2^4 + \cdots = m_{13/24} + m_{1234}.$$

The *elementary symmetric function in non-commuting variables* e_π is defined by

$$e_\pi = \sum_{(i_1, i_2, \ldots, i_n)} x_{i_1} x_{i_2} \cdots x_{i_n},$$

where $i_j \neq i_k$ if j, k are in the same block in π. For example,

$$\begin{aligned} e_{13/24} =& x_1x_1x_2x_2 + x_2x_2x_1x_1 + x_1x_2x_2x_1 + x_2x_1x_2x_1 + \cdots \\ =& m_{12/34} + m_{14/23} + m_{12/3/4} + m_{14/2/3} + m_{1/23/4} \\ &+ m_{1/2/34} + m_{1/2/3/4}. \end{aligned}$$

Basic definitions

It is a little complicated to define the complete homogeneous symmetric functions, and hence we first introduce another way of looking at the previous definitions.

Any two sets D, R and a function $f : D \to R$ determine a *kernel* set partition, $\ker f \vdash D$, whose blocks are the nonempty preimages $f^{-1}(r)$ for $r \in R$. For $f : [n] \to \mathbf{x}$ we denote by M_f the corresponding monomial

$$M_f = f(1)f(2)\cdots f(n).$$

Then it follows that

$$m_\pi = \sum_{\ker f = \pi} M_f.$$

For example, if $\pi = 13/24$, then the functions with $\ker f = \pi$ are exactly those of the form $f(1) = f(3) = x_{i_0}$ and $f(2) = f(4) = x_{j_0}$, where $i_0 \neq j_0$. This f gives rise to the monomial $M_f = x_{i_0}x_{j_0}x_{i_0}x_{j_0}$ in the sum for $m_{13/24}$.

Basic definitions

Define

$$h_\pi = \sum_{(f,L)} M_f,$$

where $f : [n] \to \mathbf{x}$ and L is a linear ordering of the elements of each block of $(\ker f) \wedge \pi$. For example,

$$\begin{aligned} h_{13/24} =& m_{1/2/3/4} + m_{12/3/4} + 2m_{13/2/4} + m_{14/2/3} + m_{1/23/4} + 2m_{1/24/3} \\ &+ m_{1/2/34} + m_{12/34} + 4m_{13/24} + m_{14/23} + 2m_{123/4} + 2m_{124/3} \\ &+ 2m_{134/2} + 2m_{1/234} + 4m_{1234}. \end{aligned}$$

Basic definitions

Define the *projection* map $\rho : \mathbb{Q}\langle\langle x \rangle\rangle \to \mathbb{Q}[[x]]$ which merely lets the variables commute. For $\lambda = \langle 1^{m_1} 2^{m_2} \cdots n^{m_n} \rangle$, define

$$\lambda! = \lambda_1! \cdots \lambda_l!, \lambda^! = m_1! \cdots m_n!.$$

Moreover, for any set partition π, we let $\pi! = \lambda(\pi)!$ and $\pi^! = \lambda(\pi)^!$.

Theorem

We have

$$\rho(m_\pi) = \pi^! m_{\lambda(\pi)},$$
$$\rho(p_\pi) = p_{\lambda(\pi)},$$
$$\rho(e_\pi) = \pi! e_{\lambda(\pi)},$$
$$\rho(h_\pi) = \pi! h_{\lambda(\pi)}.$$

Basic definitions

Note that the number of set partitions of type λ, denoted by $\binom{n}{\lambda}$, may be expressed with the above notations:

$$\binom{n}{\lambda} = \frac{n!}{\lambda! \lambda^!}.$$

Besides, we give an action on set partitions (or places) rather than variables. We define

$$g \circ m_\pi = m_{g\pi},$$

or equivalently,

$$g \circ (x_{i_1} \cdots x_{i_n}) = x_{i_{g^{-1}(1)}} \cdots x_{i_{g^{-1}(n)}}.$$

From this definition, it can be verified the formula $g \circ b_\pi = b_{g\pi}$ is also valid for $b_\pi = p_\pi, e_\pi$ or h_π.

Change of basis

Recall that the meet (greatest lower bound) of two set partitions σ and π are denoted by $\sigma \wedge \pi$, and we use $\hat{0}$ and $\hat{1}$ to represent the unique minimal element $1/2/\cdots/n$ and maximal element $12\cdots n$, respectively.

Theorem

We have the following change-of-basis formulae.

$$p_\pi = \sum_{\sigma \geq \pi} m_\sigma,$$

$$e_\pi = \sum_{\sigma \wedge \pi = \hat{0}} m_\sigma,$$

$$h_\pi = \sum_{\sigma} (\sigma \wedge \pi)! m_\sigma.$$

Change of basis

We use the following simple form to define the Möbius function.

$$\sum_{a \leq c \leq b} \mu(a, c) = \delta_{a,b}.$$

There is a nice formula for the Möbius functions of Π_n. It is easy to see that if $\sigma \leq \pi$, then $[\sigma, \pi] = \prod_i \Pi_{c_i}$. Then the Möbius functions are given by

$$\mu(\sigma, \pi) = \prod_i (-1)^{c_i - 1}(c_i - 1)!.$$

We shall use $\lambda(\sigma, \pi)$ to denote the integer partition whose parts are these c_i's.

Change of basis

Theorem (Möbius inversion formula)

Let P be a poset such that every principal order ideal is finite. Let $f, g : P \to K$, where K is a field. Then

$$g(t) = \sum_{s \leq t} f(s), \quad \forall t \in P$$

iff

$$f(t) = \sum_{s \leq t} \mu(s,t) g(s) \quad \forall t \in P.$$

The dual form of this formula changes $s \leq t$ to $s \geq t$ and $\mu(s,t)$ to $\mu(t,s)$.

Change of basis

Corollary

Let P be a poset and $f, g, h : P \to K$, where K is a field and $g(a) \neq 0$ for all $a \in P$. Then

$$f(a) = \sum_{b \leq a} g(b) \sum_{c \geq b} h(c), \quad \forall a \in P$$

iff

$$h(a) = \sum_{c \leq a} \frac{\mu(a, c)}{g(c)} \sum_{b \leq c} \mu(b, c) f(b), \quad \forall a \in P.$$

Proof. First, use the Möbius inversion formula to obtain

$$g(a) \sum_{c \geq a} h(c) = \sum_{b \leq a} f(b).$$

Then divide by $g(a)$ and apply the dual Möbius inversion formula.

Change of basis

Theorem

We have

$$m_\pi = \sum_{\sigma \geq \pi} \mu(\pi, \sigma) p_\sigma,$$

$$m_\pi = \sum_{\sigma \geq \pi} \frac{\mu(\pi, \sigma)}{\mu(\hat{0}, \sigma)} \sum_{\tau \leq \sigma} \mu(\tau, \sigma) e_\tau,$$

$$m_\pi = \sum_{\sigma \geq \pi} \frac{\mu(\pi, \sigma)}{|\mu(\hat{0}, \sigma)|} \sum_{\tau \leq \sigma} \mu(\tau, \sigma) h_\tau.$$

Proof of the second formula. We have

$$e_\pi = \sum_{\sigma} \Big(\sum_{\tau \leq \sigma \wedge \pi} \mu(\hat{0}, \tau) \Big) m_\sigma = \sum_{\tau \leq \pi} \mu(\hat{0}, \tau) \sum_{\sigma \geq \tau} m_\sigma.$$

Then use the above Corollary.

Change of basis

Theorem

We have the following formulae:

$$e_\pi = \sum_{\sigma \le \pi} \mu(\hat{0}, \sigma) p_\sigma, \qquad p_\pi = \frac{1}{\mu(\hat{0}, \pi)} \sum_{\sigma \le \pi} \mu(\sigma, \pi) e_\sigma,$$

$$h_\pi = \sum_{\sigma \le \pi} |\mu(\hat{0}, \sigma)| p_\sigma, \qquad p_\pi = \frac{1}{|\mu(\hat{0}, \pi)|} \sum_{\sigma \le \pi} \mu(\sigma, \pi) h_\sigma,$$

$$e_\pi = \sum_{\sigma \le \pi} (-1)^\sigma \lambda(\sigma, \pi)! h_\sigma, \qquad h_\pi = \sum_{\sigma \le \pi} (-1)^\sigma \lambda(\sigma, \pi)! e_\sigma,$$

where $(-1)^\sigma = (-1)^{n - l(\lambda(\sigma))}$*, or equivalently,* $\mu(\hat{0}, \sigma) = (-1)^\sigma |\mu(\hat{0}, \sigma)|$.

Change of basis

To give an analogue of the involution ω on $\Lambda(\mathbf{x})$ which sends e_λ to h_λ, by abuse of notation, we define $\omega : \Pi(\mathbf{x}) \to \Pi(\mathbf{x})$ by $\omega(e_\pi) = h(\pi)$ extending it linearly.

Theorem

The map ω has the following properties.

(i) *It is an involution.*

(ii) $\omega(p_\pi) = (-1)^\pi p_\pi$ *for all* π.

(iii) *We have* $\omega\rho = \rho\omega$.

Proof. These can be easily verified by using the change-of-basis formulae among p_π, e_π and h_π.

The lifting map and inner products

We will now introduce a right inverse $\tilde{\rho}$ for the projection map ρ and an inner product for which $\tilde{\rho}$ is an isometry. Define the *lifting* map $\tilde{\rho} : \Lambda(\mathbf{x}) \to \Pi(\mathbf{x})$ by linearly extending

$$\tilde{\rho}(m_\lambda) = \frac{\lambda!}{n!} \sum_{\lambda(\pi)=\lambda} m_\pi .$$

Proposition

The map $\rho\tilde{\rho}$ is the identity map on $\Lambda(\mathbf{x})$.

The standard inner products on $\Lambda(\mathbf{x})$ are defined by $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda,\mu}$. Now we define the analogue on $\Pi(\mathbf{x})$ by

$$\langle m_\pi, h_\sigma \rangle = n! \delta_{\pi,\sigma},$$

where $\pi \vdash [n]$.

The lifting map and inner products

Theorem

The bilinear form $\langle \cdot, \cdot \rangle$ satisfies:

(i) $\langle f, g \rangle = \langle g, f \rangle$.

(ii) $\langle f, f \rangle \geq 0$ *and the equality holds only if* $f = 0$.

(iii) $\langle h \circ f, h \circ g \rangle = \langle f, g \rangle$ *for any permutation* h.

Theorem

The map $\tilde{\rho}$ *is an isometry, i.e.,* $\langle f, g \rangle = \langle \tilde{\rho}(f), \tilde{\rho}(g) \rangle$ *for* $f, g \in \Lambda(\mathbf{x})$.

The lifting map and inner products

Theorem

The following formulae define equivalent bilinear forms:

$$\langle e_\pi, e_\sigma\rangle = n!(\pi\wedge\sigma)!, \qquad \langle e_\pi, h_\sigma\rangle = n!\delta_{\pi\wedge\sigma,\hat{0}},$$

$$\langle e_\pi, p_\sigma\rangle = (-1)^\sigma n!\zeta(\sigma,\pi), \qquad \langle e_\pi, m_\sigma\rangle = (-1)^\sigma n!\lambda(\sigma,\pi)!\zeta(\sigma,\pi),$$

$$\langle h_\pi, h_\sigma\rangle = n!(\pi\wedge\sigma)!, \qquad \langle h_\pi, p_\sigma\rangle = n!\zeta(\sigma,\pi),$$

$$\langle h_\pi, m_\sigma\rangle = n!\delta_{\pi,\sigma}, \qquad \langle p_\pi, p_\sigma\rangle = n!\frac{\delta_{\pi,\sigma}}{|\mu(\hat{0},\pi)|},$$

$$\langle p_\pi, m_\sigma\rangle = n!\frac{\mu(\sigma,\pi)\zeta(\sigma,\pi)}{|\mu(\hat{0},\pi)|}, \qquad \langle m_\pi, m_\sigma\rangle = n!\sum_{\tau\geq\pi\vee\sigma}\frac{\mu(\pi,\tau)\mu(\sigma,\tau)}{|\mu(\hat{0},tau)|}.$$

Here $\zeta(\sigma,\pi)$ equals 1 if $\sigma\leq\pi$, and equals 0 otherwise.

MacMahon symmetric functions

In order to give the analogue of Schur functions, we need to introduce the MacMahon symmetric functions. Consider the following commuting variables

$$\dot{\mathbf{x}} = \{\dot{x}_1, \dot{x}_2, \ldots\}, \ddot{\mathbf{x}} = \{\ddot{x}_1, \ddot{x}_2, \ldots\}, \ldots, \mathbf{x}^{(n)} = \{x_1^{(n)}, x_2^{(n)}, \ldots\}.$$

For each positive integer m, the symmetric group $\mathfrak{S}_m$ acts on $\mathbb{Q}[[\dot{\mathbf{x}}, \ddot{\mathbf{x}}, \ldots, \mathbf{x}^{(n)}]]$ in the following way:

$$gf(\dot{x}_1, \ddot{x}_1, \ldots, \dot{x}_2, \ddot{x}_2, \ldots) = f(\dot{x}_{g(1)}, \ddot{x}_{g(1)}, \ldots, \dot{x}_{g(2)}, \ddot{x}_{g(2)}, \ldots),$$

where $g(i) = i$ for $i > m$. We say that f is symmetric if it is invariant under the actions of $\mathfrak{S}_m$ for all $m \geq 1$.

MacMahon symmetric functions

Consider a monomial

$$M = \dot{x}_1^{a_1} \ddot{x}_1^{b_1} \cdots (x_1^{(n)})^{c_1} \dot{x}_2^{a_2} \ddot{x}_2^{b_2} \cdots (x_2^{(n)})^{c_2} \cdots .$$

We define the *multiexponent* of M to be the vector partition

$$\vec{\lambda} = \{\lambda^1, \lambda^2, \ldots\} = \{[a_1, b_1, \ldots, c_1], [a_2, b_2, \ldots, c_2], \ldots\},$$

the *multidegree* of M to be the vector

$$\vec{m} = [m_1, m_2, \ldots, m_n] = [a_1, b_1, \ldots, c_1] + [a_2, b_2, \ldots, c_2] + \cdots ,$$

and the *degree* of M to be $m = \sum_i m_i$. Here we write $\vec{\lambda} \vdash \vec{m}, \vec{m} \vdash m$. Then we define the *algebra of MacMahon symmetric functions*, $\mathcal{M} = \mathcal{M}(\dot{\mathbf{x}}, \ddot{\mathbf{x}}, \ldots, \mathbf{x}^{(n)})$, to be the subalgebra of $\mathbb{Q}[[\dot{\mathbf{x}}, \ddot{\mathbf{x}}, \ldots, \mathbf{x}^{(n)}]]$ consisting of all f which are symmetric and of bounded degree.

MacMahon symmetric functions

The *monomial MacMahon symmetric functions* are defined by

$$m_{\vec{\lambda}} = \text{sum of all the monomials with multiexponent } \vec{\lambda}.$$

We define the *power sum*, *elementary*, and *complete homogeneous MacMahon symmetric functions* to be multiplicative with

$$\begin{aligned} p_{[a,b,\dots,c]} &= m_{[a,b,\dots,c]}, \\ \sum_{a,b,\dots,c} e_{[a,b,\dots,c]} q^a r^b \cdots s^c &= \prod_{i \geq 1} (1 + \dot{x}_i q + \ddot{x}_i r + \cdots + x_i^{(n)} s), \\ \sum_{a,b,\dots,c} h_{[a,b,\dots,c]} q^a r^b \cdots s^c &= \prod_{i \geq 1} \frac{1}{(1 - \dot{x}_i q - \ddot{x}_i r - \cdots - x_i^{(n)} s)}, \end{aligned}$$

where a basis $b_{\vec{\lambda}}$ is said to be multiplicative if $b_{\vec{\lambda}} = b_{\lambda^1} \cdots b_{\lambda^l}$.

MacMahon symmetric functions

The connection between the MacMahon symmetric functions and the symmetric functions in non-commuting variables are described as follows. Let $[1^n]$ be the vector of n ones, and $\mathcal{M}_{[1^n]}$ be the subspace spanned by all the $m_{\vec{\lambda}}$, where $\vec{\lambda} \vdash [1^n]$. Define a linear map

$$\Phi : \bigoplus_{n \geq 0} \mathcal{M}_{[1^n]} \to \Pi, \quad \dot{x}_i \ddot{x}_j \cdots x_k^{(n)} \mapsto x_i x_j \cdots x_k.$$

Theorem

The map Φ is an isomorphism of vector spaces. Furthermore, for the known four basis of Π,

$$\Phi(b_{\lambda^1, \ldots, \lambda^l}) = b_{B_1/\ldots/B_n},$$

where $j \in B_i$ if and only if the j-th component of λ^i is nonzero.

Schur functions

We now give a combinatorial definition of an analogue of a Schur function. Consider the alphbet

$$A = \{\dot{1}, \dot{2}, \ldots\} \uplus \{\ddot{1}, \ddot{2}, \ldots\} \uplus \cdots \uplus \{1^{(n)}, 2^{(n)}, \ldots\}.$$

Partially order A by $i^{(k)} < j^{(l)}$ iff $i < j$. Given $\lambda \vdash m$, $\vec{m} = [m_1, \ldots, m_n] \vdash m$, then we define a *dotted Young tableaux $\dot{T}$ of shape λ and multidegree $\vec{m}$* to be a "semistandard" Young tableaux of shape λ on the alphabet A and there are m_k entries with k dots.

Now we define the *MacMahon Schur function* to be

$$S_\lambda^{\vec{m}} = \sum_{\lambda(\dot{T})=\lambda} M_{\dot{T}} \quad \text{where} \quad M_{\dot{T}} = \prod_{i^{(j)} \in \dot{T}} x_i^{(j)}.$$

For example, if $\lambda = (3,1)$, $\vec{m} = [2,2]$, then $[\dot{x}_1^2 \ddot{x}_1 \ddot{x}_2] S_\lambda^{\vec{m}} = 3$, the corresponding tableaux are given as follows.

$$T_1 = \begin{array}{|c|c|c|}\hline \dot{1} & \dot{1} & \ddot{1} \\ \hline \ddot{2} & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}, \quad T_2 = \begin{array}{|c|c|c|}\hline \dot{1} & \ddot{1} & \dot{1} \\ \hline \ddot{2} & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}, \quad T_3 = \begin{array}{|c|c|c|}\hline \ddot{1} & \dot{1} & \dot{1} \\ \hline \ddot{2} & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}.$$

Theorem

The function $S_\lambda^{\vec{m}}$ is a MacMahon symmetric function.

Proof. We only need to verify that $S_\lambda^{\vec{m}}$ is invariant under $(i, i+1)$. For the columns which contain a pair $i^{(k)}, (i+1)^{(l)}$, replace the pair by $i^{(l)}, (i+1)^{(k)}$. For the columns which contain exactly one $i^{(k)}$ or $(i+1)^{(l)}$, replace them by $(i+1)^{(k)}$ or $(i+1)^{(l)}$, respectively. Then we get an involution exchanging the number of $i^{(k)}$ and $(i+1)^{(k)}$.

Schur functions

If $\vec{m} = [1^n]$, then we will write S_λ for $S_\lambda^{\vec{m}}$ and make no distinction between S_λ and $\Phi(S_\lambda)$. In particular, if $\vec{m} = [m]$, then $S_\lambda^{\vec{m}} = s_\lambda$.

Note that S_λ do not form a basis for $\Pi(\mathbf{x})$ since the number of such functions is $p(m)$, the number of integer partitions of m. However, it has several properties similar to the ordinary Schur functions.

Theorem

The functions S_λ has the following properties.

(i) $S_\lambda = \sum_{\mu \trianglelefteq \lambda} \mu! K_{\lambda,\mu} \sum_{\lambda(\sigma)=\mu} m_\sigma$.

(ii) *The S_λ are linearly independent.*

(iii) $\rho(S_\lambda) = n! s_\lambda$, $\tilde{\rho}(n! s_\lambda) = S_\lambda$.

(iv) $\langle S_\lambda, S_\mu \rangle = n!^2 \delta_{\lambda,\mu}$.

Here $\trianglelefteq$ denotes the dominance order, $K_{\lambda,\mu}$ denotes the Kostka number.

Jacobi-Trudi determinants

If $f \in \mathbb{Q}[[\dot{\mathbf{x}}, \ddot{\mathbf{x}}, \dots, \mathbf{x}^{(n)}]]$, then let $\langle \vec{m} \rangle f$ denote the sum of all monomials M of multidegree $\vec{m}$ in f.

Theorem

Given a partition λ and vector $\vec{m}$ with $\lambda, \vec{m} \vdash m$, we have

$$S_{\lambda}^{\vec{m}} = \langle \vec{m} \rangle \det \left(\sum_{\vec{t} \vdash \lambda_i - i + j} h_{\vec{t}} \right) \quad \textit{and} \quad S_{\lambda'}^{\vec{m}} = \langle \vec{m} \rangle \det \left(\sum_{\vec{t} \vdash \lambda_i - i + j} e_{\vec{t}} \right),$$

where λ' is the conjugate partition of λ.

Proof. Analogues to the proof of the ordinary Jacobi-Trudi identity by lattice paths, see EC2.

Corollary

We have $\omega(S_{\lambda}) = S_{\lambda'}$.

The RSK map

A *biword of length n over A* is a $2 \times n$ array β of elements of A such that if the dots are removed, then it becomes a generalized permutation as defined in EC2. For example,

$$\beta = \begin{matrix} \dot{1} & \dot{2} & \ddot{2} & \dot{2} & \ddot{3} & \ddot{4} \\ \dot{2} & \ddot{1} & \ddot{3} & \dot{3} & \ddot{2} & \dot{1} \end{matrix}.$$

The the map $\beta \overset{RSK}{\mapsto} (\dot{T}, \dot{U})$ is defined by just ignoring the dots and performing the ordinary RSK algorithm.

The RSK map

Theorem

The map $\beta \overset{RSK}{\mapsto} (\dot{T}, \dot{U})$ is a bijection between biwords and pairs of dotted Young tableaux of the same shape such that $\vec{m}(\beta) = (\vec{m}(\dot{T}), \vec{m}(\dot{U}))$, where the operator $\vec{m}$ takes the multidegree of β, $\dot{T}$ and $\dot{U}$.

Similarly, we are able to give an analogue of the Cauchy identity.

Theorem

We have

$$\sum_{m \geq 0} \sum_{\lambda, \vec{m}, \vec{p} \vdash m} S_{\lambda}^{\vec{m}}(\dot{\mathbf{x}}, \ddot{\mathbf{x}}, \dots, \mathbf{x}^{(n)}) S_{\lambda}^{\vec{p}}(\dot{\mathbf{y}}, \ddot{\mathbf{y}}, \dots, \mathbf{y}^{(n)}) = \prod_{i,j \geq 1} \frac{1}{1 - \sum_{k,l=1}^{n} x_i^{(k)} y_j^{(l)}}.$$

Comments and questions

(I) Rosas computed the specializations of symmetric functions in non-commuting variables and MacMahon symmetric functions.

(II) Is there an expression for $S_{\lambda}^{\vec{m}}$ analogous to Jacobi's bialternant formula $(s_{\lambda}(x_1, \ldots, x_n) = a_{\lambda+\delta}/a_{\delta})$?

(III) Is there a connection between $\Pi(\mathbf{x})$ and the partition algebra $P_n(x)$?

(IV) Is there a way to define S_{π} for $\pi \vdash [n]$?

(V) Use symmetric functions in non-commuting variables to deal with the positivity problems in ordinary symmetric functions, like the usage of chromatic symmetric function in non-commuting variables.

P. Martin, The structure of partition algebras, J. Algebra 183 (1996), 319C358.

M.H. Rosas, Specializations of MacMahon symmetric functions and the polynomial algebra, Discrete Math. 246 (2002), 285C293.

Thank you for your attention!