二、计算 n 级行列式 $D_n = \begin{vmatrix} 1 & 1 & \cdots & 1 & 2-n \\ 1 & 1 & \cdots & 2-n & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 2-n & 1 & \cdots & 1 & 1 \end{vmatrix}$.

法一

解

$$D_n = \begin{vmatrix} 1 & 1 & \cdots & 1 & 2-n \\ 0 & 0 & \cdots & 1-n & n-1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1-n & \cdots & 0 & 0 \\ 1-n & n-1 & \cdots & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & \cdots & n-1 & 1 \\ 0 & 0 & \cdots & 1-n & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1-n & \cdots & 0 & 0 \\ 1-n & 0 & \cdots & 0 & 0 \end{vmatrix}$$

$$= (-1)^{inv(n\cdots 21)}(1-n)^{n-1}$$

$$= (-1)^{\frac{n(n-1)}{2}}(1-n)^{n-1}$$

法二

解

$$D_n = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & \cdots & 1 & 2-n & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 2-n & 1 & \cdots & 1 & 1 & 0 \end{vmatrix}_{n+1}$$

$$= \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1-n & -1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 1-n & 0 & \cdots & 0 & 0 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & 1+\frac{n}{1-n} \\ 0 & 0 & \cdots & 0 & 1-n & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 1-n & 0 & \cdots & 0 & 0 & 0 \end{vmatrix} = \frac{1}{1-n}$$

$$= (-1)^{\frac{n(n-1)}{2}}(1-n)^{n-1}.$$

三、计算行列式 $D = \begin{vmatrix} a+x_1 & a & a & \cdots & a \\ a & a+x_2 & a & \cdots & a \\ a & a & a+x_3 & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ a & a & a & \cdots & a+x_n \end{vmatrix}$.

解

$$D = \begin{vmatrix} 1 & a & a & \cdots & a \\ 0 & a+x_1 & a & \cdots & a \\ 0 & a & a+x_2 & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & a & a & \cdots & a+x_n \end{vmatrix} = \begin{vmatrix} 1 & a & a & \cdots & a \\ -1 & x_1 & 0 & \cdots & 0 \\ -1 & 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \cdots & x_n \end{vmatrix} = \begin{vmatrix} 1+a\sum_{i=1}^{n}\frac{1}{x_i} & a & a & \cdots & a \\ 0 & x_1 & 0 & \cdots & 0 \\ 0 & 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & x_n \end{vmatrix} = \left(1+a\sum_{i=1}^{n}\frac{1}{x_i}\right)\left(\prod_{i=1}^{n}x_i\right)$$

四、问 λ,μ 取何值时？齐次线性方程组 $\begin{cases} \lambda x_1 + x_2 + x_3 = 0, \\ x_1 + \mu x_2 + x_3 = 0, \\ x_1 + 2\mu x_2 + x_3 = 0 \end{cases}$ 有非零解.

解

$$D = \begin{vmatrix} \lambda & 1 & 1 \\ 1 & \mu & 1 \\ 1 & 2\mu & 1 \end{vmatrix} = \begin{vmatrix} \lambda-1 & 0 & 1 \\ 0 & \mu-1 & 1 \\ 0 & 2\mu-1 & 1 \end{vmatrix} = (\lambda-1)\begin{vmatrix} \mu-1 & 1 \\ 2\mu-1 & 1 \end{vmatrix} = (\lambda-1)(-\mu) = 0$$

当 $\lambda=1$ 或 $\mu=0$ 时，这个方程组有非零解.