三、计算 n 阶行列式 $D_n = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{vmatrix}$.

解

法一 $D_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & 1 & \cdots & 0 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & & 1 & 1 \\ -1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & -1 & 0 \\ -1 & 0 & 0 & \cdots & 0 & -1 \end{vmatrix} = (1-n)(-1)^n = (-1)^n(1-n)$

法二 $D_n = \begin{vmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{vmatrix} = \begin{vmatrix} -1 & 0 & & & & \\ & -1 & 0 & & & \\ & & -1 & \ddots & & \\ & & & \ddots & -1 & 0 \\ 1 & 2 & 3 & \cdots & n-1 & n-1 \end{vmatrix} = (-1)^{n-1}(n-1)$

四、利用克拉默法则求解线性方程组 $\begin{cases} x_1 + 2x_2 - x_3 = 2, \\ x_1 - 2x_2 + 2x_3 = 3, \\ 2x_1 - x_2 + x_3 = 3. \end{cases}$

解 $D = \begin{vmatrix} 1 & 2 & -1 \\ 1 & -2 & 2 \\ 2 & -1 & 1 \end{vmatrix} = 3$

$D_1 = \begin{vmatrix} 2 & 2 & -1 \\ 3 & -2 & 2 \\ 3 & -1 & 1 \end{vmatrix} = 3, \quad D_2 = \begin{vmatrix} 1 & 2 & -1 \\ 1 & 3 & 2 \\ 2 & 3 & 1 \end{vmatrix} = 6, \quad D_3 = \begin{vmatrix} 1 & 2 & 2 \\ 1 & -2 & 3 \\ 2 & -1 & 3 \end{vmatrix} = 9$

$x_1 = \frac{D_1}{D} = 1, \quad x_2 = \frac{D_2}{D} = 2, \quad x_3 = \frac{D_3}{D} = 3.$

五、n 阶行列式 D 中每个数 a_{ij} 分别用 2^{i-j} 乘所得的行列式记为 D_1，求行列式 D_1 的值.

解 $D_1 = \begin{vmatrix} a_{11} & 2^{-1}a_{12} & 2^{-2}a_{13} & \cdots & 2^{1-n}a_{1n} \\ 2a_{21} & a_{22} & 2^{-1}a_{23} & \cdots & 2^{2-n}a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ 2^{n-1}a_{n1} & 2^{n-2}a_{n2} & 2^{n-3}a_{n3} & \cdots & a_{nn} \end{vmatrix}$

$= 2^{1+2+\cdots+n} \begin{vmatrix} a_{11} & 2^{-1}a_{12} & 2^{-2}a_{13} & \cdots & 2^{1-n}a_{1n} \\ a_{21} & 2^{-1}a_{22} & 2^{-2}a_{23} & \cdots & 2^{1-n}a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & 2^{-1}a_{n2} & 2^{-2}a_{n3} & \cdots & 2^{1-n}a_{nn} \end{vmatrix}$

$= 2^{1+2+\cdots+n}\, 2^{-1-2\cdots-n}\, D$

$= D$

$$A = \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & & \vdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix}$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-n+1 & 1 & \cdots & 1 \\ 1 & \lambda-n+1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & \lambda-n+1 \end{vmatrix}_n$$

加边法

$$= \begin{vmatrix} 1 & 0 & 0 & & 0 \\ 1 & \lambda-n+1 & 1 & \cdots & 1 \\ 1 & 1 & \lambda-n+1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & \lambda-n+1 \end{vmatrix}_{n+1}$$

$$= \begin{vmatrix} 1 & -1 & -1 & \cdots & -1 \\ 1 & \lambda-n & 0 & \cdots & 0 \\ 1 & 0 & \lambda-n & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & \lambda-n \end{vmatrix}_{n+1}$$

$$\frac{\lambda}{\lambda-n} \longrightarrow$$

$$= \begin{vmatrix} 1+\frac{n}{\lambda-n} & 0 & 0 & \cdots & 0 \\ 1 & \lambda-n & 0 & \cdots & 0 \\ 1 & 0 & \lambda-n & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & \lambda-n \end{vmatrix}_{n+1}$$

$$= \lambda(\lambda-n)^{n-1}$$