The g-indexes of standard Young tableaux

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- Derivatives
- Eulerian polynomials; des, exc permutation statistics
- Stirling permutation; k-Stirling permutation
- Inversion sequences
- Second-order Eulerian polynomials
- André polynomials, γ-coefficients
- Number of involutions
- Context-free grammar

$\Longrightarrow$

Standard Young tableaux

Eulerian polynomials

Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$. A *descent* of π is an index $i \in [n]$ such that $\pi(i) > \pi(i+1)$ or $i = n$. Let $\mathrm{des}\,(\pi)$ be the number of descents of π. The number $\left\langle {n \atop i} \right\rangle = \{\pi \in \mathfrak{S}_n : \mathrm{des}\,(\pi) = i\}$ is called the Eulerian number, and the polynomial

$$A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{des}\,(\pi)}$$

is called the Eulerian polynomial. The historical origin of Eulerian polynomial is the following summation formula :

$$\left(x\frac{d}{dx}\right)^n \frac{1}{1-x} = \sum_{k=0}^{\infty} k^n x^k = \frac{A_n(x)}{(1-x)^{n+1}}. \tag{1}$$

Generalizations.

A k-Stirling permutation of order n is a permutation of the multiset $\{1^k, 2^k, \ldots, n^k\}$ such that for each i, $1 \leq i \leq n$, all entries between any two occurrences of i are at least i.

When $k = 2$, the *k-Stirling permutation* is reduced to the ordinary Stirling permutation (Gessel-Stanley, 1978).

Let $\mathcal{Q}_n(k)$ be the set of k-Stirling permutations of order n. The k-order Eulerian polynomials are defined by

$$C_n(x;k) = \sum_{\sigma \in \mathcal{Q}_n(k)} x^{\operatorname{des}(\pi)}, \;\; C_0(x;k) = 1.$$

Dzhumadil'daev-Yeliussizov(2014):

The polynomials $C_n(x;k)$ satisfy the recurrence relation

$$C_{n+1}(x;k) = (kn+1)x C_n(x;k) + x(1-x)C_n'(x;k), \qquad (2)$$

- $k=1$: $C_n(x;1) = A_n(x)$.

- $k=2$, the polynomial $C_n(x;k)$ is reduced to the *second-order Eulerian polynomial* $C_n(x)$, i.e., $C_n(x;2) = C_n(x)$.

$[1,2,3]$	x^1
$[1,3,2]$	x^2
$[2,1,3]$	x^2
$[2,3,1]$	x^2
$[3,1,2]$	x^2
$[3,2,1]$	x^3

$$A_3(x) = x^3 + 4x^2 + x$$

$$A_3(1) = 6 = 3!$$

$[1,1,2,2,3,3]$	x^1
$[1,1,2,3,3,2]$	x^2
$[1,1,3,3,2,2]$	x^2
$[1,2,2,1,3,3]$	x^2
$[1,2,2,3,3,1]$	x^2
$[1,2,3,3,2,1]$	x^3
$[1,3,3,1,2,2]$	x^2
$[1,3,3,2,2,1]$	x^3
$[2,2,1,1,3,3]$	x^2
$[2,2,1,3,3,1]$	x^3
$[2,2,3,3,1,1]$	x^2
$[2,3,3,2,1,1]$	x^3
$[3,3,1,1,2,2]$	x^2
$[3,3,1,2,2,1]$	x^3
$[3,3,2,2,1,1]$	x^3

$$C_3(x) = 6x^3 + 8x^2 + x$$

$$C_3(1) = 15 = 5!!$$

Theorem

Let k be a positive integer. For $n \geq 1$, we have

$$\left(\frac{x}{(1-x)^k}\frac{d}{dx}\right)^n \frac{1}{1-x} = \frac{C_n(x; k+1)}{(1-x)^{n+kn+1}}.$$

In particular, we have

$$\left(x\frac{d}{dx}\right)^n \frac{1}{1-x} = \frac{A_n(x)}{(1-x)^{n+1}}, \tag{3}$$

$$\left(\frac{x}{1-x}\frac{d}{dx}\right)^n \frac{1}{1-x} = \frac{C_n(x)}{(1-x)^{2n+1}}. \tag{4}$$

Next ?

5 steps!

(Step 1) study a much more general problem

(Step 2) inversion sequences

(Step 3) integer partitions

(Step 4) k-Young tableaux

(Step 5) Standard Young tableaux

(Step 1) study a much more general problem

$$\left(c(x)\frac{d}{dx}\right)^n f(x) = ?$$

Notations:

$$c := c(x)$$

$$f := f(x)$$

$$D = \frac{d}{dx}$$

$$\left(c(x)\frac{d}{dx}\right)^n f(x) = ?$$

$$\begin{aligned}
(cD)f &= cDf \\
(cD)^2 f &= cD(cDf) = c(Dc)(Df) + c^2(D^2 f) \\
(cD)^3 f &= cD\big(c(Dc)(Df) + c^2(D^2 f)\big) \\
&= c\big(D(c(Dc))(Df) + c(Dc)D^2 f + D(c^2)(D^2 f) + c^2 D^3 f\big) \\
&= c\big(((Dc)(Dc) + cD^2 c)(Df) + c(Dc)D^2 f + D(c^2)(D^2 f) + c^2 D^3 f\big) \\
&= c((Dc)(Dc) + cD^2 c)(Df) + c^2(Dc)D^2 f + cD(c^2)(D^2 f) + c^3 D^3 f \\
&= [c(Dc)^2 + c^2 D^2 c](Df) + [c^2(Dc) + 2c^2 Dc](D^2 f) + [c^3]D^3 f
\end{aligned}$$

Notations:

$$\mathsf{f}_k = D^k f$$

$$c_k = D^k c$$

In particular, $\mathsf{f}_0 = f$ and $c_0 = c$.

$$
\begin{aligned}
(cD)f &= (c)\mathrm{f}_1,\\
(cD)^2 f &= (c c_1)\mathrm{f}_1 + (c^2)\mathrm{f}_2,\\
(cD)^3 f &= (c {c_1}^2 + c^2 c_2)\mathrm{f}_1 + (3c^2 c_1)\mathrm{f}_2 + (c^3)\mathrm{f}_3,\\
(cD)^4 f &= (c {c_1}^3 + 4c^2 c_1 c_2 + c^3 c_3)\mathrm{f}_1 + (7c^2 {c_1}^2 + 4c^3 c_2)\mathrm{f}_2 + (6c^3 c_1)\mathrm{f}_3 + (c^4)\mathrm{f}_4,\\
(cD)^5 f &= (c {c_1}^4 + 11c^2 {c_1}^2 c_2 + 4c^3 {c_2}^2 + 7c^3 c_1 c_3 + c^4 c_4)\mathrm{f}_1 + (15c^2 {c_1}^3\\
&\qquad + 30c^3 c_1 c_2 + 5c^4 c_3)\mathrm{f}_2 + (25c^3 {c_1}^2 + 10c^4 c_2)\mathrm{f}_3 + (10c^4 c_1)\mathrm{f}_4 + (c^5)\mathrm{f}_5.
\end{aligned}
$$

So many things we can do:

- special coefficents, sum of coefficients, OEIS
- special $c(x)$,
- special $f(x)$,
- special relation between $c(x)$ and $f(x)$, like $c = f$ or $c = f^2$.
- formal derivative, context-free grammar
- general formula for all coefficients (!)

For $n \geq 1$, we define

$$(cD)^n f = \sum_{k=1}^{n} A_{n,k} \mathrm{f}_k. \tag{5}$$

$A_{n,k}$ is a function of $c, c_1, \ldots, c_{n-k}$:

$$A_{n,k} := A_{n,k}(c, c_1, c_2, \ldots, c_{n-k}).$$

By induction, $A_{n+1,1} = cDA_{n,1}$, $A_{n,n} = c^n$ and for $2 \leq k \leq n$,

$$A_{n+1,k} = cA_{n,k-1} + cDA_{n,k}. \tag{6}$$

Proposition (Comtet, 1973)

For $1 \leq k \leq n$, we have

$$A_{n,k} = \frac{c}{k!} \sum (2-k_1)(3-k_1-k_2)\cdots(n-k_1-k_2-\cdots-k_{n-1}) \frac{c_{k_1}}{k_1!} \cdots \frac{c_{k_{n-1}}}{k_{n-1}!},$$

where the summation is over all sequences $(k_1, k_2, \ldots, k_{n-1})$ of nonnegative integers such that $k_1 + k_2 + \cdots + k_{n-1} = n - k$ and $k_1 + \cdots + k_j \leq j$ for any $1 \leq j \leq n-1$.

(Step 2) inversion sequences

An integer sequence $\mathrm{e} = (e_1, e_2, \ldots, e_n)$ is an *inversion sequence* of length n if $0 \leq e_i < i$ for all $1 \leq i \leq n$.

Let $\mathcal{I}_n$ be the set of inversion sequences of length n.

There is a natural bijection ψ between $\mathcal{I}_n$ and $\mathfrak{S}_n$ defined by $\psi(\pi) = \mathrm{e}$, where $e_i = \#\{j \mid 1 \leq j < i \text{ and } \pi(j) > \pi(i)\}$.

Definition

For $\mathrm{e} \in \mathcal{I}_n$, let

$$|\mathrm{e}|_j = \#\{i \mid e_i = j,\ 1 \le i \le n\}.$$

Then we define

$$\phi(\mathrm{e}) = c \cdot c_{|\mathrm{e}|_1} c_{|\mathrm{e}|_2} \cdots c_{|\mathrm{e}|_{n-1}} \cdot \mathrm{f}_{|\mathrm{e}|_0}. \tag{7}$$

Example. Take $n = 9$ and $\mathrm{e} = (0, 0, 1, 0, 4, 2, 4, 0, 1)$, then $|\mathrm{e}|_0 = 4, |\mathrm{e}|_1 = 2, |\mathrm{e}|_2 = 1, |\mathrm{e}|_3 = 0, |\mathrm{e}|_4 = 2$ and $|\mathrm{e}|_j = 0$ for $5 \le j \le 8$.

So that $\phi(\mathrm{e}) = c \cdot c_2 c_1 c c_2 cccc \cdot f_4 = c^6 c_1 {c_2}^2 \cdot \mathrm{f}_4$.

Theorem

For $n \geq 1$, we have

$$(cD)^n f = \sum_{\mathrm{e} \in \mathcal{I}_n} \phi(\mathrm{e}). \tag{8}$$

Example. When $n = 3$, the correspondence between $\mathrm{e} \in \mathcal{I}_3$ and $\phi(\mathrm{e})$ is illustrated as follows:

e	000	001	002	010	011	012
$\lvert\mathrm{e}\rvert_0\ \lvert\mathrm{e}\rvert_1\ \lvert\mathrm{e}\rvert_2$	300	210	201	210	120	111
$\phi(\mathrm{e})$	$ccc\mathrm{f}_3$	$cc_1c\mathrm{f}_2$	$ccc_1\mathrm{f}_2$	$cc_1c\mathrm{f}_2$	$cc_2c\mathrm{f}_1$	$cc_1c_1\mathrm{f}_1$

So that

$$\sum_{\mathrm{e} \in \mathcal{I}_3} \phi(\mathrm{e}) = (c{c_1}^2 + c^2 c_2)\mathrm{f}_1 + (3c^2 c_1)\mathrm{f}_2 + c^3 \mathrm{f}_3.$$

Proof. Assume that (8) holds for n. Let $\mathcal{I}_{n,k} = \{\mathrm{e} \in \mathcal{I}_n : |\mathrm{e}|_0 = k\}$. Then for any $\mathrm{e} \in \mathcal{I}_{n,k}$, we have

$$\phi(\mathrm{e}) = c \cdot c_{|\mathrm{e}|_1} \cdot c_{|\mathrm{e}|_2} \cdots c_{|\mathrm{e}|_{n-1}} \cdot \mathrm{f}_k.$$

Let e' be obtained from $\mathrm{e} = (e_1, e_2, \ldots, e_n)$ by appending e_{n+1}.

We distinguish three cases:

(*i*) If $e_{n+1} = 0$, then $\phi(\mathrm{e}') = c \cdot c_{|\mathrm{e}|_1} \cdot c_{|\mathrm{e}|_2} \cdots c_{|\mathrm{e}|_{n-1}} \cdot c \cdot \mathrm{f}_{k+1}$;

(*ii*) If $e_{n+1} = i$ and $1 \le i \le n-1$, then

$$\phi(\mathrm{e}') = c \cdot c_{|\mathrm{e}|_1} \cdot c_{|\mathrm{e}|_2} \cdots c_{|\mathrm{e}|_i+1} \cdots c_{|\mathrm{e}|_{n-1}} \cdot c \cdot \mathrm{f}_k;$$

(*iii*) If $e_{n+1} = n$, then $\phi(\mathrm{e}') = c \cdot c_{|\mathrm{e}|_1} \cdot c_{|\mathrm{e}|_2} \cdots c_{|\mathrm{e}|_{n-1}} \cdot c_1 \cdot \mathrm{f}_k$.

The first case accounts for the term $cA_{n,k-1}$ and the last two cases account for the term $cDA_{n,k}$. Then $\sum_{\mathrm{e} \in I_{n+1,k}} \phi(\mathrm{e}) = (cA_{n,k-1} + cDA_{n,k})\mathrm{f}_k = A_{n+1,k}\mathrm{f}_k$, which follows from (6).

We can derive Comtet's formula by using Theorem 3. For $\mathrm{e} \in \mathcal{I}_n$, let $k = |\mathrm{e}|_0$ and $k_i = |\mathrm{e}|_{n-i}$ for $1 \leq i \leq n-1$. Note that

$$k_1 + k_2 + \cdots + k_{n-1} = n - k$$

and $k_1 + \cdots + k_j \leq j$ for each j. Therefore, the number of such e is equal to

$$\binom{1}{k_1}\binom{2-k_1}{k_2}\binom{3-k_1-k_2}{k_3}\cdots\binom{n-k_1-k_2-\cdots-k_{n-1}}{k}$$
$$= \frac{(2-k_1)(3-k_1-k_2)\cdots(n-k_1-k_2-\cdots-k_{n-1})}{k!k_1!k_2!\cdots k_{n-1}!}.$$

(Step 3) integer partitions

An integer partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a weakly decreasing sequence of nonnegative integers.

size: $|\lambda| = \sum_j \lambda_j$. If $|\lambda| = n$, we write $\lambda \vdash n$.

We denote by m_i the number of parts equal i. By using the multiplicities, we also denote λ by $(1^{m_1} 2^{m_2} \cdots n^{m_n})$.

The length of λ, denoted $\ell(\lambda)$, is the maximum subscript j such that $\lambda_j > 0$.

The Ferrers diagram of λ is graphical representation of λ with λ_i boxes in its ith row and the boxes are left-justified.

Since the $c_{k_1}, c_{k_2}, \ldots, c_{k_{n-1}}$ are commutative, we have to group the terms in (7) and (8) which produce the same product $c_{k_1} c_{k_2} \cdots c_{k_{n-1}}$.

The type of n is a pair (k, μ), denoted by $(k, \mu) \vdash n$, where $k \in [n]$ and $\mu = (\mu_1, \ldots, \mu_{n-1})$ is a partition of $n-k$, i.e., μ is written up to $n-1$ terms by appending 0's at the end.

$$\mathrm{Set}\,(\mu) = \{\mu_j \mid 1 \leq j \leq n-1\},$$

$$|\mu|_j = \#\{i \mid \mu_i = j, 1 \leq i \leq n-1\}.$$

Let $(|\mathsf{e}|_0, \mu(\mathsf{e}))$ be the type of $\mathsf{e} \in \mathcal{I}_n$, where $\mu(\mathsf{e})$ is the decreasing order of $|\mathsf{e}|_1, \ldots, |\mathsf{e}|_{n-1}$.

For each type (k, μ) of n, let $p_{k,\mu}$ be the number of inversion sequences of type (k, μ). It follows from Theorem 3 that

$$(cD)^n f = \sum_{(k,\mu) \vdash n} p_{k,\mu} c c_{\mu_1} c_{\mu_2} \cdots c_{\mu_{n-1}} \mathsf{f}_k, \tag{9}$$

where the summation is taken over all types (k, μ) of n.

$$
\begin{aligned}
(cD)f &= (c)\mathrm{f}_1,\\
(cD)^2 f &= (c{c_1})\mathrm{f}_1 + (c^2)\mathrm{f}_2,\\
(cD)^3 f &= (c{c_1}^2 + c^2 c_2)\mathrm{f}_1 + (3c^2 c_1)\mathrm{f}_2 + (c^3)\mathrm{f}_3,\\
(cD)^4 f &= (c{c_1}^3 + 4c^2 c_1 c_2 + c^3 c_3)\mathrm{f}_1 + (7c^2 {c_1}^2 + 4c^3 c_2)\mathrm{f}_2 + (6c^3 c_1)\mathrm{f}_3 + (c^4)\mathrm{f}_4
\end{aligned}
$$

Example

$p_{1,(0)} = 1$, $p_{2,(0)} = 1$, $p_{1,(1)} = 1$,

$p_{3,(0,0)} = 1$, $p_{2,(1,0)} = 3$, $p_{1,(2,0)} = 1$ and $p_{1,(1,1)} = 1$.

Lemma

By convention, set $p_{0,\mu} = 0$.

If $(k, \mu) = (1, (1, 1, \ldots, 1))$, *then let* $p_{k,\mu} = 1$.

For other type (k, μ) *of* n, *we have*

$$p_{k,\mu} = \sum_{j \in \mathrm{Set}\,(\mu) \setminus \{0\}} (|\mu|_{j-1} + 1) p_{k,\mu^{(j)}} + p_{k-1,\mu^{(0)}}, \qquad (10)$$

where $\mu^{(j)}$ *is obtained from* μ *by replacing the last occurrences of the part* j *by* $j-1$ *and by deleting the last* 0

and $\mu^{(0)}$ *is obtained from* μ *by deleting the last* 0. *Thus* $(k, \mu^{(j)}) \vdash (n-1)$ *and* $(k-1, \mu^{(0)}) \vdash (n-1)$.

Proof.

Take an inversion sequence $\mathrm{e} \in \mathcal{I}_n$ of type (k, μ). Let $\mathrm{e}' = (e_1, e_2, \ldots, e_{n-1}) \in \mathcal{I}_{n-1}$ be obtained from e by deleting the last e_n. If $e_n = 0$, then, the type of e' is $(k-1, \mu^{(0)})$. This operation is reversible. If $e_n = i$ $(1 \leq i \leq n-1)$ and $|\mathrm{e}|_i = j \in \mathrm{Set}\,(\mu) \setminus \{0\}$, then the type of e' is $(k, \mu^{(j)})$. In this case, the operation is not reversible. We have exactly $(|\mu|_{j-1} + 1)$ ways to do the inverses. In fact we can append $e_n = i' \neq i$ at the end of e' with the condition of $|\mathrm{e}|_i - 1 = j - 1 = |\mathrm{e}|_{i'}$ to obtain an inversion sequence in $\mathcal{I}_n$ of type (k, μ). $\square$

As an illustration of (10), in order to get inversion sequences of type $(k, \mu) = (3, (2, 1, 1, 0, 0, 0))$, we distinguish three cases:

(i) For each $\mathrm{e} \in \mathcal{I}_6$ that counted by $p_{2,(2,1,1,0,0)}$, we can get exactly one inversion sequence of type (k, μ) by appending $e_7 = 0$ at the end of e;

(ii) Let $\mathrm{e} \in \mathcal{I}_6$ be an inversion sequence counted by $p_{3,(1,1,1,0,0)}$. If $|\mathrm{e}|_i = 1$ then we can append $e_7 = i$ at the end of e. As we have three choices for i, we get the term $3p_{3,(1,1,1,0,0)}$;

(iii) Let $\mathrm{e} \in \mathcal{I}_6$ be an inversion sequence counted by $p_{3,(2,1,0,0,0)}$. If $|\mathrm{e}|_i = 0$ or $i = 6$ then we can append $e_7 = i$ at the end of e. As we have four choices for i, we get the term $4p_{3,(2,1,0,0,0)}$.

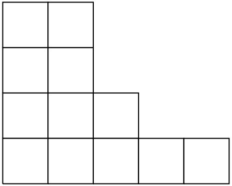
$$p_{3,(2,1,1,0,0,0)} = 4p_{3,(2,1,0,0,0)} + 3p_{3,(1,1,1,0,0)} + p_{2,(2,1,1,0,0)}$$

(Step 4) k-Young tableaux

- (traditional) partitions and Standard Young tableaux

For a Ferrers diagram $\lambda \vdash n$ (we will often identify a partition with its Ferrers diagram), a *(standard) Young tableau* (SYT, for short) of shape λ is a filling of the n boxes of λ with the integers $1, 2, \ldots, n$ such that each number is used, and all rows and columns are increasing (from left to right, and from bottom to top, respectively). Given a Young tableau, we number its rows starting from the bottom and going above. Let $\mathrm{SYT}(n)$ be the set of standard Young tableaux of size n.

Example. partition $\lambda = (5, 3, 2, 2)$ and a STY of shape λ

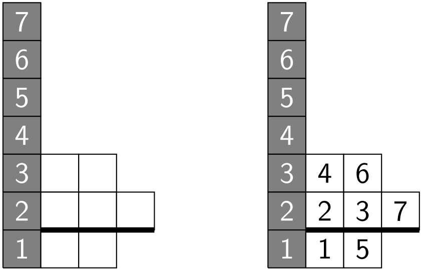


10	12			
4	8			
3	6	7		
1	2	5	9	11

- *k*-Standard Young tableaux

Each type (k, μ) of n can be represented by a picture which contains k boxes in the bottom row, and the Young diagram of the partition μ in the top. Such picture is called a (k, μ)-diagram.

Example. type=(2,(3,2,0,0,0,0))



Definition

Let (k, μ) be a type of n. A k-Young tableau Z of shape (k, μ) is a filling of the n boxes of the (k, μ)-diagram by the integers $1, 2, \ldots, n$ such that

(i) each number is used,

(ii) all rows and columns in the top Young diagram are increasing (from left to right, and from bottom to top, respectively),

(iii) the bottom row becomes an increasing sequence of lenght k, starting with 1.

The filling of the top Young diagram of the partition μ is called the top Young tableau of the k-Young tableau. Unlike the ordinary Young tableau, there is no condition between the bottom row and the top Young tableau.

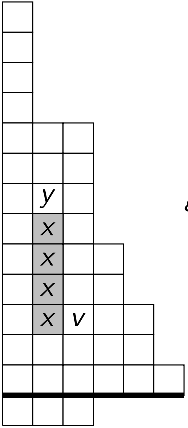
We always put a special column of n boxes at the left of k-Young tableaux, and labelled by the integers $1, 2, \ldots, n$ from bottom to top.

Definition

Let Z be a k-Young tableau of shape (k, μ), where $k + |\mu| = n$. For each $v \in [n]$, suppose that v is in the box (i, j) of the top Young diagram, we define the g-index of v, denoted by $g_Z(v)$, to be the number of boxes $(i-1, j')$ such that $j' \geq j$ and the letter in this box is less than or equal to v.

If v is in the bottom row, then we define $g_Z(v) = 1$.

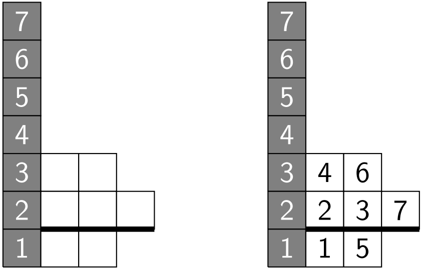
The g-index of Z is given by $G_Z = g_Z(1)g_Z(2)\cdots g_Z(n)$.



$$x \leq v < y$$

$$g_Z(v) = \#x$$

Example.



$$\color{red}{g_Z}(1) = 1,\ \color{red}{g_Z}(2) = 1,\ \color{red}{g_Z}(3) = 1,\ \color{red}{g_Z}(4) = 2,$$

$$\color{red}{g_Z}(5) = 1,\ \color{red}{g_Z}(6) = 1,\ \color{red}{g_Z}(7) = 2.$$

Theorem

If $(k, \mu) \vdash n$, then we have

$$p_{k,\mu} = \sum_{Z} G_Z \tag{11}$$

where the summation is taken over all k-Young tableaux of shape (k, μ).

Proof.

Identity (11) is obtained from Lemma 5 by induction on n. The maximum letter n in the k-Young tableaux Z can be at the end of the bottom row, or a corner in the top Young tableau of Z. In the first case, $g_Z(n) = 1$, and removing the letter n yields a $(k-1)$-Young tableau of shape $(k-1, \mu)$. In the second case, $g_Z(n) = |\mu|_{j-1} + 1$, and removing the letter n yields a k-Young tableaux of shape $(k, \mu^{(j)})$, where j is the length of the row contained n. We recover all terms in (10). $\square$

A complete example: Stirling numbers

The Stirling numbers of the first kind $\left[{n \atop k}\right]$ can be defined as follows:

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k = x(x+1)\cdots(x+n-1).$$

we have (Blasiak, Flajolet, 2010)

$$(e^x D)^n f = e^{nx} \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \mathsf{f}_k.$$

We replace $c = e^x$ and $c_j = e^x$ in (9). By Theorem 8, we obtain

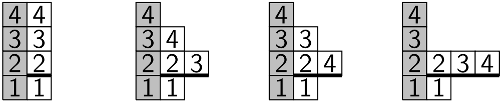
$$(e^x D)^n f = e^{nx} \sum_{(k,\mu)\vdash n} \sum_{Z} G_Z \mathsf{f}_k,$$

Hence

$$\sum_{(k,\mu)\vdash n} \sum_{Z} G_Z x^k = x(x+1)(x+2)\cdots(x+n-1), \tag{12}$$

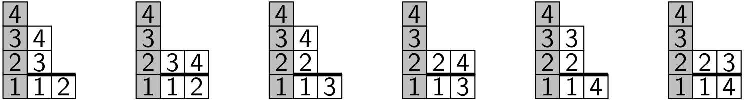
where Z is taken over all k-Young tableaux of shape (k, μ).

$k=1$



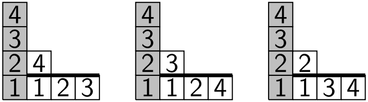
$\sum G = 6$ $\quad G=1\cdot1\cdot1\cdot1 \quad G=1\cdot1\cdot1\cdot2 \quad G=1\cdot1\cdot1\cdot2 \quad G=1\cdot1\cdot1\cdot1$

$k=2$



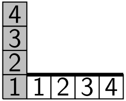
$\sum G = 11$ $\quad G=1\cdot1\cdot2\cdot2 \quad G=1\cdot1\cdot2\cdot1 \quad G=1\cdot1\cdot1\cdot2 \quad G=1\cdot1\cdot1\cdot1 \quad G=1\cdot1\cdot1\cdot1 \quad G=1\cdot1\cdot1\cdot1$

$k=3$



$\sum G = 6$ $\quad G=1\cdot1\cdot1\cdot3 \quad G=1\cdot1\cdot2\cdot1 \quad G=1\cdot1\cdot1\cdot1$

$k=4$



$\sum G = 1$ $\quad G=1\cdot1\cdot1\cdot1$

Another example.

Proposition.

Let $\left\{ {n \atop k} \right\}$ be the Stirling numbers of the second kind. Then we have

$$\left\{ {n \atop k} \right\} = \sum_{Z} G_Z,$$

where the summation is taken over all k-Young tableaux of shape $(k, (1^{n-k}0^{k-1}))$.

Proof.

Let $c = x$ and $f = 1/(1-x)$.

Then $c_1 = 1$ and $c_j = 0$ for $j \geq 2$, and $\mathsf{f}_k = k!/(1-x)^{k+1}$.

It follows from (9) that

$$\begin{aligned}(xD)^n \frac{1}{1-x} &= \sum_{(k,\mu)\vdash n} p_{k,\mu} c c_{\mu_1} c_{\mu_2} \cdots c_{\mu_{n-1}} \mathsf{f}_k \\ &= \sum_{(k,\mu=(1^{n-k}0^{k-1}))\vdash n} p_{k,\mu} \cdot \frac{k! x^k}{(1-x)^{k+1}} \\ &= \frac{1}{(1-x)^{n+1}} \sum_{(k,\mu=(1^{n-k}0^{k-1}))\vdash n} p_{k,\mu} \cdot k! x^k (1-x)^{n-k}.\end{aligned}$$

By Theorem 8, we have

$$
\begin{aligned}
A_n(x) &= \sum_{k=0}^{n} p_{k,(1^{n-k}0^{k-1})} \cdot k!x^k(1-x)^{n-k} \\
&= \sum_{k=0}^{n} \sum_{Z} G_Z \cdot k!x^k(1-x)^{n-k}, \qquad (13)
\end{aligned}
$$

where the second summation is taken over all k-Young tableaux of shape $(k,(1^{n-k}0^{k-1}))$. Recall that the Frobenius formula for Eulerian polynomials is given as follows (Chow, 2008):

$$
A_n(x) = \sum_{k=0}^{n} k! \left\{ {n \atop k} \right\} x^k(1-x)^{n-k}.
$$

By comparing with (24), we get the desired result.

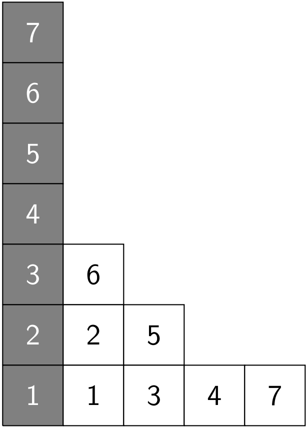
(Step 5) Standard Young tableaux

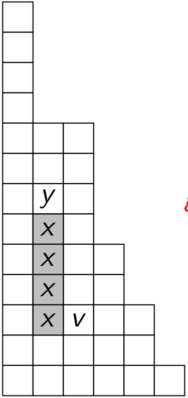
Let T be a standard Young tableau of shape λ. We always put a special column of n boxes at the left of T, and labelled by $1, 2, 3, \ldots, n$ from bottom to top. For each $v \in [n]$, suppose that v is in the box (i, j), we define the g-index of v, denoted by $g_T(v)$, to be the number of boxes $(i-1, j')$ such that $j' \geq j$ and the letter in this box is less than or equal to v.

The g-index of T is defined by

$$G_T = g_T(1)g_T(2)\cdots g_T(n).$$

$$g_T(1) = 1,\ g_T(2) = 1,\ g_T(3) = 2,\ g_T(4) = 1,$$
$$g_T(5) = 1,\ g_T(6) = 4,\ g_T(7) = 1.$$





$$x \leq v < y$$

$$g_T(v) = \#x$$

Let $\lambda(T)$ be the corresponding partition of the Young tableau T.
If $\lambda(T) = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, then let $\lambda(T)! = \lambda_1!\lambda_2! \cdots \lambda_\ell!$.

$$\left(x\frac{d}{dx}\right)^n \frac{1}{1-x} = \frac{A_n(x)}{(1-x)^{n+1}},$$

$$\left(\frac{x}{1-x}\frac{d}{dx}\right)^n \frac{1}{1-x} = \frac{C_n(x)}{(1-x)^{2n+1}}.$$

Main Theorems.

$$(2n-1)!! = \sum_{T \in \mathrm{SYT}(n)} G_T \ \lambda!; \tag{14}$$

$$C_n(x) = \sum_{T \in \mathrm{SYT}(n)} G_T \ \lambda! \ x^{n+1-\ell(\lambda)}; \tag{15}$$

$$n! = \sum_{T \in \mathrm{SYT}(n)} G_T; \tag{16}$$

$$A_n(x) = \sum_{T \in \mathrm{SYT}(n)} G_T \ x^{n+1-\ell(\lambda)}; \tag{17}$$

$$\# involutions(n) = \sum_{T \in \mathrm{SYT}(n)} 1; \tag{18}$$

$$V_n(x) = \sum_{T \in \mathrm{SYT}(n)} x^{n+1-\ell(\lambda)}, \tag{19}$$

where $A_n(x)$ is the Eulerian polynomial, and $C_n(x)$ is the Eulerian polynomial of second kind. $V_n(x)$ is the generating function of involutions for the length of its longest increasing subsequence.

$$x \qquad \begin{array}{cc} 4 & 4 \\ 3 & 3 \\ 2 & 2 \\ 1 & 1 \end{array} \qquad \begin{aligned} g &= 1,1,1,1 \\ {\color{red}G_T} &= 1 \\ \lambda(T)! &= 1 \end{aligned}$$

$$x^2 \qquad \begin{array}{ccc} 4 & & \\ 3 & 3 & \\ 2 & 2 & \\ 1 & 1 & 4 \end{array} \qquad \begin{aligned} g &= 1,1,1,3 \\ {\color{red}G_T} &= 3 \\ \lambda(T)! &= 2 \end{aligned} \qquad \begin{array}{ccc} 4 & & \\ 3 & 4 & \\ 2 & 2 & \\ 1 & 1 & 3 \end{array} \qquad \begin{aligned} g &= 1,1,2,2 \\ {\color{red}G_T} &= 4 \\ \lambda(T)! &= 2 \end{aligned} \qquad \begin{array}{ccc} 4 & & \\ 3 & 4 & \\ 2 & 3 & \\ 1 & 1 & 2 \end{array} \qquad \begin{aligned} g &= 1,1,2,2 \\ {\color{red}G_T} &= 4 \\ \lambda(T)! &= 2 \end{aligned}$$

$$x^3 \qquad \begin{array}{ccc} 4 & & \\ 3 & & \\ 2 & 3 & 4 \\ 1 & 1 & 2 \end{array} \qquad \begin{aligned} g &= 1,1,2,1 \\ {\color{red}G_T} &= 2 \\ \lambda(T)! &= 4 \end{aligned} \qquad \begin{array}{ccc} 4 & & \\ 3 & & \\ 2 & 2 & 4 \\ 1 & 1 & 3 \end{array} \qquad \begin{aligned} g &= 1,1,2,1 \\ {\color{red}G_T} &= 2 \\ \lambda(T)! &= 4 \end{aligned}$$

$$x^3 \qquad \begin{array}{cccc} 4 & & & \\ 3 & & & \\ 2 & 4 & & \\ 1 & 1 & 2 & 3 \end{array} \qquad \begin{aligned} g &= 1,1,1,3 \\ {\color{red}G_T} &= 3 \\ \lambda(T)! &= 6 \end{aligned} \qquad \begin{array}{cccc} 4 & & & \\ 3 & & & \\ 2 & 3 & & \\ 1 & 1 & 2 & 4 \end{array} \qquad \begin{aligned} g &= 1,1,2,1 \\ {\color{red}G_T} &= 2 \\ \lambda(T)! &= 6 \end{aligned} \qquad \begin{array}{cccc} 4 & & & \\ 3 & & & \\ 2 & 2 & & \\ 1 & 1 & 3 & 4 \end{array} \qquad \begin{aligned} g &= 1,1,2,1 \\ {\color{red}G_T} &= 2 \\ \lambda(T)! &= 6 \end{aligned}$$

$$x^4 \qquad \begin{array}{ccccc} 4 & & & & \\ 3 & & & & \\ 2 & & & & \\ 1 & 1 & 2 & 3 & 4 \end{array} \qquad \begin{aligned} g &= 1,1,1,1 \\ {\color{red}G_T} &= 1 \\ \lambda(T)! &= 24 \end{aligned}$$

$$A_4(x) = x^4 + 11x^3 + 11x^2 + x.$$

$$C_4(x) = 24x^4 + 58x^3 + 22x^2 + x.$$

Proofs

First, we prove

$$C_n(x) = \sum_{T \in \mathrm{SYT}\,(n)} G_T \ \lambda! \ x^{n+1-\ell(\lambda)}$$

Then, we prove

$$A_n(x) = \sum_{T \in \mathrm{SYT}\,(n)} G_T \ x^{n+1-\ell(\lambda)}$$

Proof for $C_n(x)$

Setting $c = x/(1-x)$ and $f = 1/(1-x)$, then we have

$$c_j = \frac{j!}{(1-x)^{j+1}} \quad (j \geq 1); \qquad \mathrm{f}_k = \frac{k!}{(1-x)^{k+1}} \quad (k \geq 0).$$

By using (9), we obtain

$$\begin{aligned}
&\left(\frac{x}{1-x}D\right)^n \frac{1}{1-x} \\
&= \sum_{(k,\mu)\vdash n} p_{k,\mu} \cdot c c_{\mu_1} c_{\mu_2} \cdots c_{\mu_{n-1}} \mathrm{f}_k \\
&= \sum_{(k,\mu)\vdash n} p_{k,\mu} \cdot \frac{x^{|\mu|_0+1}}{1-x} \frac{\mu_1!}{(1-x)^{\mu_1+1}} \cdots \frac{\mu_{n-1}!}{(1-x)^{\mu_{n-1}+1}} \frac{k!}{(1-x)^{k+1}} \\
&= \frac{1}{(1-x)^{2n+1}} \sum_{(k,\mu)\vdash n} p_{k,\mu} \cdot k!\mu_1! \cdots \mu_{n-1}! x^{|\mu|_0+1},
\end{aligned}$$

where the summation is taken over all types (k, μ) of n.

Combining (4) and Theorem 8, we have

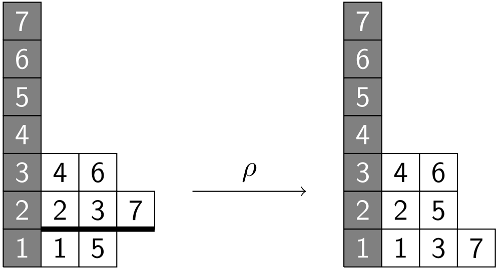
$$
\begin{aligned}
C_n(x) &= \sum_{(k,\mu)\vdash n} p_{k,\mu} \cdot k!\mu_1! \cdots \mu_{n-1}! x^{|\mu|_0+1} \\
&= \sum_{(k,\mu)\vdash n} \sum_{Z} G_Z \cdot k!\mu_1! \cdots \mu_{n-1}! x^{|\mu|_0+1}. \qquad (20)
\end{aligned}
$$

In view of (15) and (20), we need to establish some relations between k-Young tableaux and standard Young tableaux.

From k-Young tableaux to standard Young tableaux

Let Z be a k-Young tableau of shape (k, μ). We define $T = \rho(Z)$ to be the unique standard Young tableau such that the sets of the letters in the j-th column in Z and T are the same for all j. Let us list some basic facts of this map $Z \mapsto T = \rho(Z)$:

- (i) We can obtain T from Z by ordering the letters in each column in increasing order. One can check that if T is obtained in this way, then T is a standard Young tableau;
- (ii) The partition $\lambda(T)$ is the decreasing ordering of the sequence $(k, \mu_1, \ldots, \mu_{n-1})$, removing the 0's at the end. Hence, $\lambda(T)! = k!\mu_1!\mu_2! \cdots \mu_{n-1}!$;
- (iii) We have $n - \ell(\lambda(T)) = |\mu|_0$;
- (iv) In general $G_Z \neq G_T$.


ρ

However the map ρ is not bijective. Let

$$\rho^{-1}(T) = \{(k, \mu, Z) \mid \rho(Z) = T\}.$$

By the above properties of ρ and (20), we have

$$\begin{aligned} C_n(x) &= \sum_{T \in \mathrm{SYT}(n)} \sum_{(k,\mu,Z) \in \rho^{-1}(T)} G_Z \cdot k! \mu_1! \cdots \mu_{n-1}! x^{|\mu|_0 + 1} \\ &= \sum_{T \in \mathrm{SYT}(n)} \lambda(T)! x^{n+1-\ell(\lambda(T))} \sum_{(k,\mu,Z) \in \rho^{-1}(T)} G_Z. \end{aligned} \tag{21}$$

It suffices to prove the following lemma.

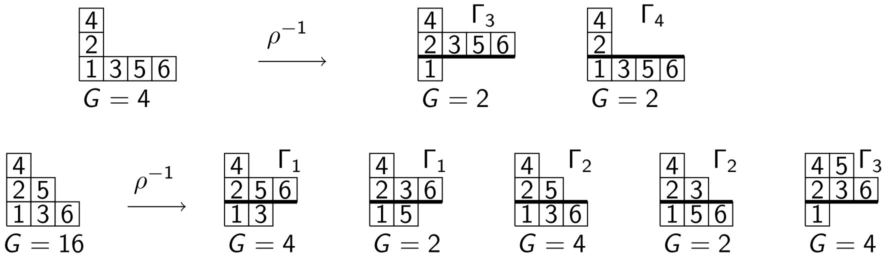
Lemma

For each standard Young tableau T, we have

$$\sum_{Z \in \rho^{-1}(T)} G_Z = G_T, \tag{22}$$

where we write $Z \in \rho^{-1}(T)$ instead of $(k, \mu, Z) \in \rho^{-1}(T)$ since we can recover (k, μ) from Z.

Example.



ρ^{-1}
Γ_3
Γ_4
$G = 4$
$G = 2$
$G = 2$
ρ^{-1}
Γ_1
Γ_1
Γ_2
Γ_2
Γ_3
$G = 16$
$G = 4$
$G = 2$
$G = 4$
$G = 2$
$G = 4$

Proof. We will proof (22) by induction on the size of T. Suppose that (22) is true for all standard Young tableau T of size $n-1$. Given a $T \in \mathrm{SYT}\,(n)$. Let T' is a standard Young tableau of size $n-1$ obtained from T by removing the letter n. This operation is reversible if $\lambda(T)$ is known. The hypothesis of induction:

$$\sum_{Z' \in \rho^{-1}(T')} G_{Z'} = G_{T'}, \tag{23}$$

It should be noted that

$$G_T = G_{T'} \times g_T(n).$$

On the other hand, for a k-Young tableau $Z \in \rho^{-1}(T)$ of size n, if we remove the letter n, we obtain a k'-Young tableau $Z' \in \rho^{-1}(T')$ of sie $n-1$. However, unlike Young tableau, this operation is not always reversible.

Let β be the length of the row containing the letter n in k-Young tableau $Z \in \rho^{-1}(T)$ with shape (k, μ) if n is in the top Young tableau of Z. The set $\rho^{-1}(T)$ can be divided into four subsets:

$$\rho^{-1}(T) = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4,$$

$\Gamma_1 = \{Z \in \rho^{-1}(T) : n \text{ is in the top Young tableau and } k = \beta - 1\}$,

$\Gamma_2 = \{Z \in \rho^{-1}(T) : n \text{ is in the bottom row and } k-1 \in \mu\}$,

$\Gamma_3 = \{Z \in \rho^{-1}(T) : n \text{ is in the top Young tableau and } k \neq \beta - 1\}$,

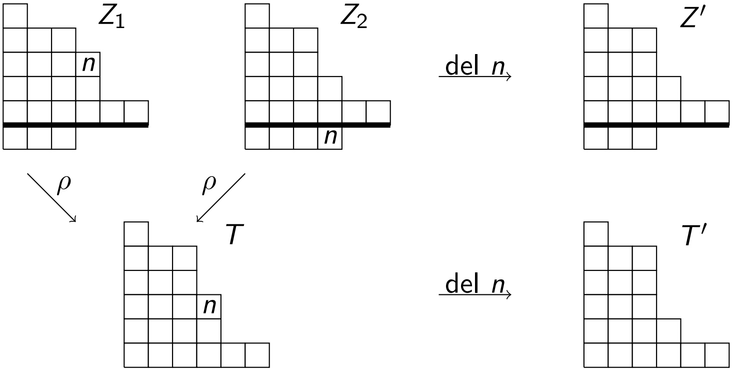
$\Gamma_4 = \{Z \in \rho^{-1}(T) : n \text{ is in the bottom row and } k-1 \notin \mu\}$.

It should be noted that some of the Γ_i may be empty according to T.

$$\begin{array}{c} \begin{array}{|c|c|c|c|}\hline 4 \\ \cline{1-1} 2 \\ \hline 1 & 3 & 5 & 6 \\ \hline\end{array} \\ G=4 \end{array} \xrightarrow{\rho^{-1}} \begin{array}{c} \Gamma_3 \\ \begin{array}{|c|c|c|c|}\hline 4 \\ \hline 2 & 3 & 5 & 6 \\ \hline 1 \\ \cline{1-1}\end{array} \\ G=2 \end{array} \quad \begin{array}{c} \Gamma_4 \\ \begin{array}{|c|c|c|c|}\hline 4 \\ \cline{1-1} 2 \\ \hline 1 & 3 & 5 & 6 \\ \hline\end{array} \\ G=2 \end{array}$$

$$\begin{array}{c} \begin{array}{|c|c|c|}\hline 4 \\ \cline{1-2} 2 & 5 \\ \hline 1 & 3 & 6 \\ \hline\end{array} \\ G=16 \end{array} \xrightarrow{\rho^{-1}} \begin{array}{c} \Gamma_1 \\ \begin{array}{|c|c|c|}\hline 4 \\ \hline 2 & 5 & 6 \\ \hline 1 & 3 \\ \cline{1-2}\end{array} \\ G=4 \end{array} \quad \begin{array}{c} \Gamma_1 \\ \begin{array}{|c|c|c|}\hline 4 \\ \hline 2 & 3 & 6 \\ \hline 1 & 5 \\ \cline{1-2}\end{array} \\ G=2 \end{array} \quad \begin{array}{c} \Gamma_2 \\ \begin{array}{|c|c|c|}\hline 4 \\ \cline{1-2} 2 & 5 \\ \hline 1 & 3 & 6 \\ \hline\end{array} \\ G=4 \end{array} \quad \begin{array}{c} \Gamma_2 \\ \begin{array}{|c|c|c|}\hline 4 \\ \cline{1-2} 2 & 3 \\ \hline 1 & 5 & 6 \\ \hline\end{array} \\ G=2 \end{array} \quad \begin{array}{c} \Gamma_3 \\ \begin{array}{|c|c|c|}\hline 4 & 5 \\ \hline 2 & 3 & 6 \\ \hline 1 \\ \cline{1-1}\end{array} \\ G=4 \end{array}$$

We claim that the set Γ_1 and Γ_2 have the same carnality.
Moreover, for each $Z_1 \in \Gamma_1$, there exists $Z_2 \in \Gamma_2$ in a unique manner, such that $Z_1' = Z_2' \in \rho^{-1}(T')$,



Moreover, we have the relations for the g-indexes :

$$g_{Z_1}(n) = g_T(n) - 1$$

$$g_{Z_2}(n) = 1$$

For $Z_3 \in \Gamma_3$ and $Z_4 \in \Gamma_4$ we have

$$g_{Z_3}(n) = g_T(n)$$

$$g_{Z_4}(n) = g_T(n)$$

By all these observations, we have

$$\begin{aligned}
\sum_{Z\in\rho^{-1}(T)} G_Z &= \sum_{Z_1\in\Gamma_1, Z_2\in\Gamma_2} (G_{Z_1}+G_{Z_2}) + \sum_{Z_3\in\Gamma_3} G_{Z_3} + \sum_{Z_4\in\Gamma_4} G_{Z_4} \\
&= \sum_{Z_1\in\Gamma_1, Z_2\in\Gamma_2} (g_{Z_1}(n)G_{Z'} + g_{Z_2}(n)G_{Z'}) \\
&\quad + \sum_{Z_3\in\Gamma_3} g_T(n)G_{Z_3'} + \sum_{Z_4\in\Gamma_4} g_T(n)G_{Z_4'} \\
&= g_T(n) \sum_{Z'\in\rho^{-1}(T')} G_{Z'} \\
&= g_T(n)G_{T'} \\
&= G_T.
\end{aligned}$$

Proof for $A_n(x)$

Recall:

$$\left(x\frac{d}{dx}\right)^n \frac{1}{1-x} = \frac{A_n(x)}{(1-x)^{n+1}},$$

$$\left(\frac{x}{1-x}\frac{d}{dx}\right)^n \frac{1}{1-x} = \frac{C_n(x)}{(1-x)^{2n+1}}.$$

Theorems:

$$C_n(x) = \sum_{T\in\mathrm{SYT}(n)} G_T\ \lambda(T)!\ x^{n+1-\ell(\lambda)}$$

$$A_n(x) = \sum_{T\in\mathrm{SYT}(n)} G_T\ x^{n+1-\ell(\lambda)}$$

• try same proof as $C_n(x)$, fail.

Recall the proof for $C_n(x)$.

Setting $c = x/(1-x)$ and $f = 1/(1-x)$, then we have

$$c_j = \frac{j!}{(1-x)^{j+1}} \quad (j \geq 1); \qquad \mathrm{f}_k = \frac{k!}{(1-x)^{k+1}} \quad (k \geq 0).$$

By using (9), we obtain

$$\left(\frac{x}{1-x}D\right)^n \frac{1}{1-x}$$

$$= \sum_{(k,\mu)\vdash n} p_{k,\mu} \cdot c c_{\mu_1} c_{\mu_2} \cdots c_{\mu_{n-1}} \mathrm{f}_k$$

$$= \sum_{(k,\mu)\vdash n} p_{k,\mu} \cdot \frac{x^{|\mu|_0+1}}{1-x} \frac{\mu_1!}{(1-x)^{\mu_1+1}} \cdots \frac{\mu_{n-1}!}{(1-x)^{\mu_{n-1}+1}} \frac{k!}{(1-x)^{k+1}}$$

$$= \cdots$$

For $A_n(x)$ by the same proof as for $C_n(x)$:

Let $c = x$ and $f = 1/(1-x)$.

Then $c_1 = 1$ and $c_j = 0$ for $j \geq 2$, and $\mathrm{f}_k = k!/(1-x)^{k+1}$.

It follows from (9) that

$$\begin{aligned}(xD)^n \frac{1}{1-x} &= \sum_{(k,\mu)\vdash n} p_{k,\mu} c c_{\mu_1} c_{\mu_2} \cdots c_{\mu_{n-1}} \mathrm{f}_k \\ &= \sum_{(k,\mu=(1^{n-k}0^{k-1}))\vdash n} p_{k,\mu} \cdot \frac{k! x^k}{(1-x)^{k+1}} \\ &= \cdots\end{aligned}$$

We have

$$A_n(x) = \sum_{k=0}^{n} \sum_{Z} G_Z \cdot k! x^k (1-x)^{n-k}, \tag{24}$$

where the second summation is taken over all k-Young tableaux of shape $(k, (1^{n-k}0^{k-1}))$.

But we want to proof:

$$A_n(x) = \sum_{T \in \mathrm{SYT}\,(n)} G_T \; x^{n+1-\ell(\lambda)}$$

Fail!

$$c_k = D^k c, \quad \mathsf{f}_k = D^k f$$

- Proof for $C_n(x)$.

$$C_n(x) = \sum_{T \in \mathrm{SYT}\,(n)} G_T \ \lambda(T)! \ x^{n+1-\ell(\lambda)}$$

Setting $c = x/(1-x)$ and $f = 1/(1-x)$, then we have

$$c_j = \frac{j!}{(1-x)^{j+1}} \quad (j \geq 1); \qquad \mathsf{f}_k = \frac{k!}{(1-x)^{k+1}} \quad (k \geq 0).$$

- Proof for $A_n(x)$. We want something like:

$$A_n(x) = \sum_{T \in \mathrm{SYT}\,(n)} G_T \ x^{n+1-\ell(\lambda)}$$

$$c_j = \frac{1}{(1-x)^{j+1}} \quad (j \geq 1); \qquad \mathsf{f}_k = \frac{1}{(1-x)^{k+1}} \quad (k \geq 0).$$

Impossible !

context-free grammars and formal derivative

W.Y.C. Chen (1993)

For an alphabet V, let $\mathbb{Q}[[V]]$ be the ring of the rational commutative ring of formal power series in monomials formed from letters in V.

A context-free grammar over V is a function $G : V \to \mathbb{Q}[[V]]$ that replaces each letter in V with an element of $\mathbb{Q}[[V]]$. The formal derivative D_G is a linear operator defined with respect to the grammar G.

For example, if $V = \{x, y\}$ and $G = \{x \to xy, y \to y\}$, then

$$D_G(x) = xy$$
$$D_G^2(x) = D_G(xy) = xy^2 + xy.$$

For two formal functions u and v, we have

$$D_G(u + v) = D_G(u) + D_G(v)$$
$$D_G(uv) = D_G(u)v + uD_G(v).$$

It follows from *Leibniz's rule* that

$$D_G^n(uv) = \sum_{k=0}^{n} \binom{n}{k} D_G^k(u) D_G^{n-k}(v).$$

Setting $u_i = D_G^i(u)$, it follows from (9) and (11) that

$$(uD_G)^n = \sum_{(k,\mu)\vdash n} \sum_{Z} G_Z u u_{\mu_1} u_{\mu_2} \cdots u_{\mu_{n-1}} D_G^k, \tag{25}$$

where the first summation is taken over all types (k, μ) of n and the second summation is taken over all k-Young tableaux of shape (k, μ).

- RHS: OK
- LHS: Bad

It is well-known that Eulerian polynomials are symmetric, i.e., $A_0(x) = 1$ and

$$A_n(x) = \sum_{i=1}^{n} \left\langle {n \atop i} \right\rangle x^i = \sum_{i=1}^{n} \left\langle {n \atop i} \right\rangle x^{n+1-i} \text{ for } n \geq 1.$$

There is a grammatical interpretation of Eulerian numbers due to Dumont (1996), which can be restated as follows.

Proposition.

If $G = \{x \to y, y \to y\}$, then we have

$$(xD_G)^n(y) = \sum_{i=1}^{n} \left\langle {n \atop i} \right\rangle x^{n+1-i} y^i \text{ for } n \geq 1.$$

Proof for $A_n(x)$.

Let $G = \{x \to y, y \to y\}$. From (25), we have

$$(xD_G)^n(y) = \sum_{(k,\mu)\vdash n} \sum_{Z} G_Z x x_{\mu_1} x_{\mu_2} \cdots x_{\mu_{n-1}} D_G^k(y),$$

where $x_0 = x$ and $x_i = D_G^i(x) = y$ for $i \geq 1$ and $D_G^k(y) = y$ for $k \geq 0$. Hence

$$(xD_G)^n(y) = \sum_{(k,\mu)\vdash n} \sum_{Z} G_Z y^{n-|\mu|_0} x^{|\mu|_0+1}.$$

Comparing this with Dumont's result, we get

$$A_n(x) = \sum_{i=1}^{n} \left\langle {n \atop i} \right\rangle x^{n+1-i} = (xD_G)^n(y)\big|_{y=1} = \sum_{(k,\mu)\vdash n} \sum_{Z} G_Z x^{|\mu|_0+1},$$

where the first summation is taken over all types (k,μ) of n and the second summation is taken over all k-Young tableaux of shape (k,μ).

We have finally

$$A_n(x) = \sum_{T \in \mathrm{SYT}\,(n)} G_T \; x^{n+1-\ell(\lambda)},$$

by the Lemma from the proof for $C_n(x)$:

$$\sum_{Z \in \rho^{-1}(T)} G_Z = G_T.$$

Thank you for your attention!