A COUNTEREXAMPLE TO A CONJECTURE ON SCHUR POSITIVITY OF CHROMATIC SYMMETRIC FUNCTIONS OF TREES

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ABSTRACT. We show that no tree on twenty vertices with maximum degree ten has Schur positive chromatic symmetric function, thereby providing a counterexample to a conjecture from [1].

Among the many nice results on chromatic symmetric functions in the paper [1] of Dahlberg, She and van Willigenburg is Theorem 39 therein, which says that no bipartite graph on n vertices with a vertex of degree more than $\lceil \frac{n}{2} \rceil$ has Schur positive chromatic symmetric function. In particular, Theorem 39 applies to trees. A near-converse to Theorem 39 for trees is posed in [1, Conjecture 42], which says that for every $n \geq 2$, there is a tree T on n vertices, one of which has degree $\lfloor \frac{n}{2} \rfloor$, such that the chromatic symmetric function of T is Schur positive. The authors of [1] confirmed this conjecture for $n \leq 19$, using computer calculations. Sadly, the conjecture is false for $n = 20$, as we show here. We use SageMath [2] calculations after a preparatory proposition that reduces the number of trees that we must examine.

We give the requisite definitions and reiterate more formally. Given a (finite, loopless, simple) graph $G = (V, E)$, a **proper coloring** of G is a function κ from V to the set $\mathbb{P}$ of positive integers such that $\kappa(v) \neq \kappa(w)$ whenever $\{v, w\} \in E$. We fix an infinite set $\mathbf{x} := \{x_i : i \in \mathbb{P}\}$ of pairwise commuting variables, and write $\mathbf{K}(G)$ for the set of all proper colorings of G. To each proper coloring κ one associates a monomial

$$\mathbf{x}^\kappa := \prod_{v \in V} x_{\kappa(v)}.$$

The **chromatic symmetric function** X_G of G is the sum of all such monomials,

$$X_G(\mathbf{x}) := \sum_{\kappa \in \mathbf{K}(G)} \mathbf{x}^\kappa.$$

Chromatic symmetric functions were introduced by Stanley in [5] and have drawn considerable attention. Various results and conjectures, including the above-mentioned theorem and conjecture from [1], relate the structure of G to the expansion of X_G in terms of one or more familiar bases for the algebra Λ of symmetric functions. Recall that if B is a basis for Λ and $f \in \Lambda$, we call f B-positive if, when we expand $f = \sum_{b \in B} \alpha_b b$, each α_b is non-negative. The Schur basis for Λ is a fundamental object in symmetric function theory. See for example [3, Chapter 7] for basic properties of Schur functions and other rudimentary facts about symmetric functions that will be used herein without reference.

We prove the following result, thereby disproving Conjecture 42 of [1].

Theorem 1. *If T is a tree on twenty vertices, one of which has degree ten, then $X_T(\mathbf{x})$ is not Schur positive.*

A stable partition of G is a set partition $\pi : V = \bigcup_{j=1}^{k} \pi_j$ with each π_j an independent set in G. We assume without loss of generality that $|\pi_j| \geq |\pi_{j+1}|$ for each $j \in [n-1]$. Setting $\lambda_j = |\pi_j|$ for each j, we get that $\lambda := (\lambda_1, \ldots, \lambda_k)$ is a partition of the integer $|V|$. We call λ the type of π. Given another partition $\mu = (\mu_1, \ldots, \mu_\ell)$ of $|V|$, we write $\mu \preceq \lambda$ if λ dominates μ, that is, if $\sum_{j=1}^{m} \mu_j \leq \sum_{j=1}^{m} \lambda_j$ for all $m \in [k]$. Our proof of Theorem 1 rests on the following basic result, due to Stanley. This result follows quickly from the fact that if $\mu \preceq \lambda$, then when the Schur function s_λ is expanded in the monomial basis, the coefficient of m_μ is positive.

Lemma 2 (Proposition 1.5 of [4])**.** *If $X_G(\mathbf{x})$ is Schur positive and G admits a stable partition of type λ, then G admits a stable partition of type μ whenever $\mu \preceq \lambda$.*

Corollary 3. *Assume that $T = (V, E)$ is a tree on $2n$ vertices and $v \in V$ has degree n in T. If $X_T(\mathbf{x})$ is Schur positive, then every $x \in V$ that is neither v nor a neighbor of v is a leaf in T.*

Proof. As T is connected and bipartite, T has a unique bipartition $\pi : V = \pi_1 \cup \pi_2$. If $X_T(\mathbf{x})$ is Schur positive, then π has type (n, n) by Lemma 2. We assume without loss of generality that $v \in \pi_1$. Then the neighborhood $N_T(v)$ is contained in π_2 and so $\pi_2 = N_T(v)$. Were the claim of the corollary false, some $z \in V$ would be at distance three from v in T and therefore lie in π_2, which is impossible. $\square$

For each partition $\nu = (\nu_1, \ldots, \nu_t)$ of $n-1$, let $T(\nu)$ be a tree on $2n$ vertices in which one vertex v has exactly n neighbors $v_1, \ldots, v_n$, and for $1 \leq i \leq t$, v_i has exactly ν_i neighbors other than v (each of which is necessarily a leaf). The next result follows immediately from Corollary 3.

Corollary 4. *If T is a tree on $2n$ vertices, one of which has degree n, and $X_T(\mathbf{x})$ is Schur positive, then there is some partition ν of $n-1$ such that T is isomorphic with $T(\nu)$.*

Theorem 1 follows from the next result, which we prove by inspection using SageMath calculations.

Proposition 5. *If ν is a partition of the integer nine, then $X_{T(\nu)}$ is not Schur positive.*

Our computations reveal in particular that if $n = 10$ and $\nu_1 \geq 6$, then the coefficient of $s_{(9,9,2)}$ in the Schur expansion of $X_{T(\nu)}(\mathbf{x})$ is negative; and if $n = 10$ and $\nu_1 \leq 5$, then the coefficient of $s_{(3,3,2,2,2,2,2,2,2)}$ in the Schur expansion of $X_{T(\nu)}(\mathbf{x})$ is negative. This Schur expansion has can have as few as four negative coefficients (when ν is one of $(6,2,1)$, $(6,1,1,1)$ or $(5,4)$) and as many as thirty (when ν is one of $(2,2,2,2,1)$, $(2,2,2,1,1,1)$ or $(1,1,1,1,1,1,1,1,1)$). Our programs, along with the complete Schur expansion of $X_{T(\nu)}(\mathbf{x})$ for each partition ν of nine, can be found at `https://github.com/emmanuellasa/Schur_Decomposition_20`.

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