On the unimodality of over-(q, t)-binomial coefficients

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Communication with F. Brenti

Talk at Nankai University

October 9, 2020

Outline

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Unimodality and Log-concavity

Let $\{a_i\}_{i \geqslant 0}$ be a sequence of positive real numbers.

- It is unimodal if $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_m \geqslant a_{m+1} \geqslant \cdots$.
- It is log-concave if $a_i^2 \geqslant a_{i-1}a_{i+1}$ for $i \geqslant 1$.

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Log-concavity $\Longrightarrow$ Unimodality.

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Example

Each row in Pascal's triangle $\{\binom{n}{k}\}_{k=0}^{n}$ is log-concave.

Unimodality problems in combinatorics are surveyed by Stanley (1989), Brenti (1994) and Brändén (2014).

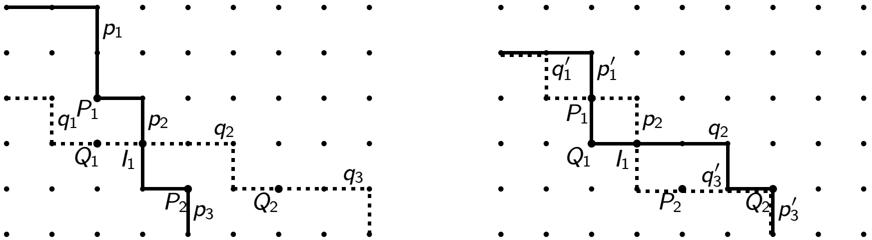
Unimodality in Pascal's triangle

Theorem (Su-Wang, Elec JC 2008)

The sequence $\{\binom{n_0-ia}{k_0+ib}\}_{i\geqslant 0}$ *is log-concave, where* $n_0 \geqslant k_0 \geqslant 0$, $a, b \geqslant 0$.

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

Combinatorial proof of the log-concavity



TP and PF

- An infinite matrix M is totally positive (TP) if every minor is nonnegative.
- A nonnegative sequence $\{a_i\}_{i \geqslant 0}$ is a Pólya frequency (PF) sequence if the matrix $(a_{j-i})_{i,j}$ is TP, where $a_i \equiv 0$ for $i < 0$.
- A finite sequence $a_0, a_1, \ldots, a_n$ is PF if the infinite sequence $a_0, a_1, \ldots, a_n, 0, 0, \ldots$ is PF.

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Example

Each row in Pascal's triangle $\{\binom{n}{k}\}_{k=0}^{n}$ is PF.

Theorem (Yu, Adv.Appl. Math. 2009, conjectured by Su-Wang)

The finite sequence $\{\binom{n_0-ia}{k_0+ib}\}_{i\geqslant 0}$ is PF, where $n_0 \geqslant k_0 \geqslant 0$, $a, b \geqslant 0$.

Gaussian polynomials (q-binomial coefficients)

The q-binomial coefficient is defined to be

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1)(q^{n-1} - 1)\cdots(q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1)\cdots(q - 1)}.$$

Recurrence:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q .$$

It is the generating function for the partitions fitting inside an $(n-k)\times k$ rectangle.

Unimodality of q-binomial coefficients

Theorem

Every q-binomial coefficient is a unimodal polynomial in q.

- This was conjectured by Cayley (1856) and proved first by Sylvester (1878) using invariant theory.
- Stanley (1980) proved it via the hard Lefschetz theorem.
- Macdonald (1980) generalize it to Schur function.
- O'Hara (1990) gave a combinatorial proof. Her chain construction was modeled by Zeilberger (1989).
- Pak and Panova (2013) proved the strict unimodality of q-multinomial coefficients.
- Andrews (1998) showed the unimodality of q-multinomial coefficients.

Unimodality of the difference of q-binomial coefficients

Theorem

If n is odd and $2k \leqslant n+1$, then $\begin{bmatrix} n \\ k \end{bmatrix}_q - \begin{bmatrix} n \\ k-1 \end{bmatrix}_q$ is a unimodal polynomial in q with non-negative coefficients.

- The above polynomial is called Kostka polynomial.
- Andrews (1993) proved the non-negativity.
- Reiner and Stanton (1998) showed the unimodality.

An application to q-ballot numbers

Allen gave the following q-analogue of ballot numbers:

$$B_q(n,r) = \frac{[2n-1]!_q[2r]_q}{[n+r]_q![n-r]_q!} = \frac{1}{q^{n-r}}\left(\begin{bmatrix} 2n-1 \\ n+r-1 \end{bmatrix}_q - \begin{bmatrix} 2n-1 \\ n+r \end{bmatrix}_q \right).$$

Corollary (conjectured by Allen, thesis 2014)

The polynomial $B_q(n,r)$ is unimodal.

The q-analogue of unimodality and log-concavity

Write $f(q) \geqslant_q g(q)$ if $f(q) - g(q)$ is a polynomial in q with nonnegative coefficients. Let $\{a_i(q)\} \geqslant 0$ be a sequence of polynomials with nonnegative coefficients.

- It is q-unimodal if

$$a_0(q) \leqslant_q a_1(q) \leqslant_q \cdots \leqslant_q a_m(q) \geqslant_q a_{m+1}(q) \geqslant_q \cdots .$$

- It is q-log-concave (introduced by Butler) if for $i \geqslant 1$,

$$a_i^2(q) \geqslant_q a_{i-1}(q)a_{i+1}(q).$$

- It is strongly q-log-concave (introduced by Stanley) if for $j \geqslant i \geqslant 1$,

$$a_i(q)a_j(q) \geqslant_q a_{i-1}(q)a_{j+1}(q).$$

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Example

For fixed n, the sequence $\left\{ \begin{bmatrix} n \\ k \end{bmatrix}_q \right\}_k$ is q-unimodal.

Strong q-log-concavity of q-binomial coefficients

Theorem (Butler, JCTA 1990)

For fixed n, the sequence $\left\{ \begin{bmatrix} n \\ k \end{bmatrix}_q \right\}_k$ is strongly q-log-concave.

Theorem (Sagan, Trans. AMS 1992)

For fixed k, the sequence $\left\{ \begin{bmatrix} n \\ k \end{bmatrix}_q \right\}_n$ is strongly q-log-concave.

Let $\mathbb{X} = \{x_1, x_2, \ldots\}$ be a countably infinite set of variables. The elementary and complete homogeneous symmetric functions of degree k in $x_1, x_2, \ldots, x_n$ are defined by

$$\begin{aligned} e_k(n) := e_k(x_1, x_2, \ldots, x_n) &= \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}, \\ h_k(n) := h_k(x_1, x_2, \ldots, x_n) &= \sum_{1 \le i_1 \le i_2 \le \ldots \le i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}, \end{aligned}$$

where $e_0(n) = h_0(n) = 1$ and $e_k(n) = 0$ for $k > n$.
Set $e_k(n) = h_k(n) = 0$ unless $k, n \ge 0$, and $e_k(0) = h_k(0) = \delta_{0,k}$, where $\delta_{0,k}$ is the Kronecker delta.

Theorem (Su-Wang-Yeh, EJC 2011)

Let $\{x_i\}_{i\geq 1}$ be a sequence of polynomials in q with nonnegative coefficients. If the sequence $\{x_i\}_{i\geq 1}$ is strongly q-log-concave, then for the fixed integers a, b, n_0 and k_0 satisfying $ab \geq 0$, $n_0 \geq k_0$, the sequences

$$\{e_{k_0-ib}(n_0+ia)\}_{i\in\mathbb{Z}}, \qquad \{h_{k_0-ib}(n_0+ia)\}_{i\in\mathbb{Z}}$$

are strongly q-log-concave respectively.

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Remark

- This answers a question proposed by Sagan (Trans. AMS 1992).
- Some special cases follow from Jacobi-Trudi identity.

Strong q-log-concavity of q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix} = q^{-\binom{k}{2}} e_k(1, q, \ldots, q^{n-1}) = h_k(1, q, \ldots, q^{n-k}).$$

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Theorem (Su-Wang-Yeh, EJC 2011)

For $n_0 \geqslant k_0 \geqslant 0$, $a, b \geqslant 0$, the sequence

$$\left\{ \begin{bmatrix} n_0 - ia \\ k_0 + ib \end{bmatrix}_q \right\}_{i \geqslant 0} \quad \text{and} \quad \left\{ q^{\binom{k_0+ib}{2}} \begin{bmatrix} n_0 - ia \\ k_0 + ib \end{bmatrix} \right\}_{i \geqslant 0}$$

is strongly q-log-concave.

the q-Stirling numbers of two kinds

The q-Stirling numbers of two kinds in terms of symmetric function:

$$\begin{aligned} c[n,k] &= e_{n-k}([1],[2],\ldots,[n-1]), \\ S[n,k] &= h_{n-k}([1],[2],\ldots,[k]), \end{aligned}$$

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Note that the sequence$\{[i]\}_{i\geq 1}$ are strongly q-log-concave.

Corollary (Su-Wang-Yeh, EJC 2011)

Let n_0, k_0, a, b be four nonnegative integers, where $n_0 \geq k_0$. The following sequences are all strongly q-log-concave:

(i) $\{c[n_0+ia, k_0+ib]\}_{i\geq 0}$, *with* $b \geq a \geq 0$;

(ii) $\{S[n_0-ia, k_0+ib]\}_{i\geq 0}$, *with* $a, b \geq 0$;

(iii) $\{S[n_0+ia, k_0+ib]\}_{i\geq 0}$, *with* $b \geq a \geq 0$.

Strong q-log-concavity of q-multinomial coefficients

$$\begin{bmatrix} m_1 + m_2 + \cdots + m_n \\ m_1, m_2, \ldots, m_n \end{bmatrix}_q = \begin{cases} \frac{[m_1+m_2+\cdots+m_n]!}{[m_1]![m_2]!\cdots[m_n]!}, & \text{if } m_k \in \mathbb{N} \text{ for all } k; \\ 0, & \text{otherwise.} \end{cases}$$

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Theorem (q-analogue of a result by Su-Wang-Yeh, 2011)

For $d_1 \geqslant \sum_{k=1}^n d_k \geqslant 0$, the sequence

$$\left\{ \begin{bmatrix} \sum_{k=1}^n m_k + i\sum_{k=1}^n d_k \\ m_1+id_1, m_2+id_2, \ldots, m_n+id_n \end{bmatrix}_q \right\}_{i\geqslant 0}$$

is strongly q-log-concave.

Outline

An overpartition analogue of the q-binomial coefficients

An overpartition is a partition in which the last occurrence of each distinct number may be overlined. The eight overpartitions of 3 are

$$(3), (\overline{3}), (2,1), (\overline{2},1), (2,\overline{1}), (\overline{2},\overline{1}), (1,1,1), (1,1,\overline{1}).$$

Dousse and Kim (JCTA, 2018) defined over-(q,t)-binomial coefficients by

$$\overline{\begin{bmatrix} m+n \\ n \end{bmatrix}}_{q,t} := \sum_{k,N\geqslant 0} \bar{p}(m,n,k,N) t^k q^N,$$

where $\bar{p}(m,n,k,N)$ counts the number of overpartitions of N, with k overlined parts, fitting inside an $m \times n$ rectangle.

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The over-(q, t)-binomial coefficients reduce to

- q-binomial coefficients when $t = 0$.
- over-q-binomial coefficients (Dousse and Kim, 2017) when $t = 1$.
- Delannoy numbers when $q = t = 1$.

Around over-(q, t)-binomial coefficients

$$\overline{\begin{bmatrix} m+n \\ n \end{bmatrix}}_{q,t} = \overline{\begin{bmatrix} n+m \\ m \end{bmatrix}}_{q,t},$$

$$\overline{\begin{bmatrix} m+n \\ n \end{bmatrix}}_{q,t} = \sum_{k=0}^{\min\{m,n\}} t^k q^{\frac{k(k+1)}{2}} \frac{(q)_{m+n-k}}{(q)_k (q)_{m-k} (q)_{n-k}},$$

$$\overline{\begin{bmatrix} m+n \\ n \end{bmatrix}}_{q,t} = \overline{\begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}}_{q,t} + q^n \overline{\begin{bmatrix} m+n-1 \\ n \end{bmatrix}}_{q,t} + tq^n \overline{\begin{bmatrix} m+n-2 \\ n-1 \end{bmatrix}}_{q,}$$

Strong (q, t)-log-concavity of over-(q, t)-binomial coefficients

Theorem (Dousse-Kim, JCTA 2018)

For fixed n, $\left\{ \overline{\begin{bmatrix} n \\ k \end{bmatrix}}_{q,t} \right\}_{k\geqslant 0}$ *is strongly (q, t)-log-concave. That is,*

$$\overline{\begin{bmatrix} n \\ k \end{bmatrix}}_{q,t} \overline{\begin{bmatrix} n \\ \ell \end{bmatrix}}_{q,t} - \overline{\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}}_{q,t} \overline{\begin{bmatrix} n+1 \\ \ell+1 \end{bmatrix}}_{q,t}$$

is a bivariate polynomial in q and t with nonnegative coefficients.

A unified result

Theorem (Su, 2020)

For $n_0 \geqslant k_0 \geqslant 0$, $a, b \geqslant 0$, the sequence $\left\{ \overline{\begin{bmatrix} n_0 - ia \\ k_0 + ib \end{bmatrix}}_{q,t} \right\}_{i \geqslant 0}$ is strongly $(q^u, tq^v, 1)$-log-concave. That is, for $j \geqslant i \geqslant 1$,

$$\overline{\begin{bmatrix} n_0 - ia \\ k_0 + ib \end{bmatrix}}_{q,t} \overline{\begin{bmatrix} n_0 - ja \\ k_0 + jb \end{bmatrix}}_{q,t} - \overline{\begin{bmatrix} n_0 - ia + a \\ k_0 + ib - b \end{bmatrix}}_{q,t} \overline{\begin{bmatrix} n_0 - ja - a \\ k_0 + jb + b \end{bmatrix}}_{q,t}$$

is a polynomial with nonnegative coefficients, where each monomial is the product of q^u's and tq^v's.

Connection to Delannoy numbers

Note that

$$\overline{\begin{bmatrix} m+n \\ n \end{bmatrix}}_{q,t} = \sum_{k=0}^{\min\{m,n\}} t^k q^{\frac{k(k+1)}{2}} \begin{bmatrix} m+n-k \\ k, m-k, n-k \end{bmatrix}_q .$$

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The over-(q, t)-binomial coefficients reduce to

- Delannoy numbers when $t = q = 1$.
- q-Delannoy numbers (defined by Sagan) when $q = 1$ after exchanging q and t.

Main corollaries

Corollary (Dousse-Kim, JCTA 2018)

For $1 \leqslant k \leqslant \ell \leqslant n-1$,

$$\overline{\begin{bmatrix} n \\ k \end{bmatrix}}_{q,t} \overline{\begin{bmatrix} n \\ \ell \end{bmatrix}}_{q,t} - \overline{\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}}_{q,t} \overline{\begin{bmatrix} n+1 \\ \ell+1 \end{bmatrix}}_{q,t}$$

has non-negative coefficients as a polynomial in q and t.

Corollary (Dousse-Kim, JCTA 2018)

For $1 \leqslant k \leqslant \ell \leqslant n-1$, *The q-Delannoy numbers satisfy*

$$\begin{aligned} D_q(n-k,k)D_q(n-\ell,\ell) &\geqslant_q D_q(n-k+1,k-1)D_q(n-\ell-1,\ell+1), \\ D_q(n-k,k)D_q(n-\ell,\ell) &\geqslant_q D_q(n-k,k-1)D_q(n-\ell,\ell+1). \end{aligned}$$

A unimodality problem of over-(q, t)-binomial coefficients

Recall

$$\overline{\begin{bmatrix} m+n \\ n \end{bmatrix}}_{q,t} = \sum_{k=0}^{\min\{m,n\}} t^k q^{\frac{k(k+1)}{2}} \begin{bmatrix} m+n-k \\ k, m-k, n-k \end{bmatrix}_q .$$

Conjecture (Dousse-Kim, JCTA 2018)

For every positive m and n, $\overline{\begin{bmatrix} m+n \\ n \end{bmatrix}}_{q,1}$ *is unimodal in q.*

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Proposition (Brenti-Sentinelli, arXiv:2008.02383, 2020)

The polynomial $\sum_{\{\lambda \in \overline{\mathcal{P}} : \lambda_1 \le n, \ell(\lambda) = m\}} q^{|\lambda|}$ *is symmetric and unimodal.*

Outline

Q-binomial coefficients and Schur functions

Let λ be a partition. Stanley (1989) proved that the principal specialization of Schur function $s_\lambda(1, q, \ldots, q^{n-1})$ is a symmetric and unimodal polynomial in q.
It follows that

$$s_k(1, q, \ldots, q^{n-1}) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}$$

is symmetric and unimodal.

Super-Schur functions

Let $\mathbf{x} = \{x_1, x_2, \dots\}$ and $\mathbf{y} = \{y_1, y_2, \dots\}$ be two sequences of independent variables and λ be a partition. The super-Schur function (or hook Schur function) corresponding to λ is defined to be

$$s_\lambda(x_1, x_2, \dots / y_1, y_2, \dots).$$

It arises in Lie superalgebras (Kač, Adv. Math. 1977, Rota et al, a new class of symmetric functions, 1982). It is known that

$$s_k(x_1, \dots x_m / y_1, \dots, y_n) = \sum_{j=0}^{k} h_j(x_1, \dots x_m) e_{k-j}(y_1, \dots, y_n).$$

Note that $e_k(y_1, \dots, y_n) = s_k(\mathbf{0}/y_1, \dots, y_n)$ and $h_k(x_1, \dots, x_n) = s_k(x_1, \dots, x_n/\mathbf{0})$.

Y-invariant diagraphs

Let D be a locally finite, weighted digraph. For a path $\pi = u_0 u_1 \cdots u_k$ in D, $w(\pi) := \prod_{i=1}^{k} w(u_{i-1}, u_i)$. For a pair of vertices u and v, $P_D(u, v) := \sum_\pi w(\pi)$, where the sum is over all paths π in D from u to v.

y-invariant diagraph (Brenti, Adv. Math.1993):
For any two pairs u,v and u',v',

$$P_D(u, v) = P_D(u', v'),$$

where u (resp. v) and u' (resp. v') have the same x-coordinate.

Y-invariant diagraphs and super-Schur functions

Theorem (Brenti, Adv. Math.1993)

There exists a y-invariant, locally finite, weighted digraph D such that

$$P_D((0,0),(n,k)) = s_k(x_1,\ldots,x_n/y_1,\ldots,y_n)$$

for all $(n,k) \in \mathbb{N} \times \mathbb{N}$.

$$\begin{aligned} P_D((0,0),(n,k)) &= x_n P_D((0,0),(n,k-1)) + y_n P_D((0,0),(n-1,k-1)) \\ &\quad + P_D((0,0),(n-1,k)). \end{aligned}$$

Brenti's framework

Theorem (Brenti, JCTA 1995)

For $(n, k) \in \mathbb{N} \times \mathbb{N}$, *define an infinite matrix* $M = (M_{n,k})_{n,k \in \mathbb{N}}$ *by*

$$M_{n,k} := P_D((0,0),(n,k)).$$

Then

(i) *M is TP.*

(ii) *every row of M is a PF sequence.*

Application to over-(q,t)-binomial coefficients

Recall

$$\overline{\begin{bmatrix} m+n \\ n \end{bmatrix}}_{q,t} = q^n \overline{\begin{bmatrix} m+n-1 \\ n \end{bmatrix}}_{q,t} + tq^n \overline{\begin{bmatrix} m+n-2 \\ n-1 \end{bmatrix}}_{q,t} + \overline{\begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}}_{q,}$$

Then

$$\begin{aligned} \overline{\begin{bmatrix} m+n \\ n \end{bmatrix}}_{q,t} &= P((0,0),(n,m)) \\ &= q^n P((0,0),(n,m-1)) + tq^n P((0,0),(n-1,m-1)) \\ &\quad + P((0,0),(n-1,m)) \\ &= s_k(1,q,\ldots,q^{n-1}/tq,tq^2,\ldots,tq^n). \end{aligned}$$

Brenti's framework

Lemma (Brenti, JCTA 1995)

Let t *be a nonnegative integer, and* $\mathbf{x} = \{x_n\}, \mathbf{y} = \{y_n\}, \mathbf{z} = \{z_n\}$ *be three sequences. Define a matrix* $M = (M_{n,k})_{n,k\in\mathbb{N}}$ *by*

$$M_{n,k} := z_n M_{n-t,k-1} + y_n M_{n-1-t,k-1} + x_n M_{n-1,k}$$

if $n+k$ *is positive (where* $M_{n,k}=0$ *if* $n<0$ *or* $k<0$*), and* $M_{0,0}= 1$*. Then*

(i) M *is* $(\mathbf{x}, \mathbf{y}, \mathbf{z})$*-TP.*

(ii) *every row of* M *is an* $(\mathbf{x}, \mathbf{y}, \mathbf{z})$*-PF sequence.*

About strong q-log-concavity

Theorem (Su, 2020)

Let t be a nonnegative integer, and $\mathbf{x} = \{x_n\}, \mathbf{y} = \{y_n\}, \mathbf{z} = \{z_n\}$ be three sequences. Define a matrix $M = (M_{n,k})_{n,k\in\mathrm{N}}$ by

$$M_{n,k} = z_n M_{n-t,k-1} + y_n M_{n-1-t,k-1} + x_n M_{n-1,k}$$

if $n+k$ is positive (where $M_{n,k}$=0 if $n<0$ or $k<0$), and $M_{0,0}$= 1. If M is symmetric, i.e., $M_{n,k} = M_{k,n}$, then $\{M_{n_0-ia,k_0+ib}\}_{i\geqslant 0}$ is strongly $(\mathbf{x},\mathbf{y},\mathbf{z})$-log-concave for fixed $n_0 \geqslant k_0 \geqslant 0$ and $a, b \geqslant 0$.

About q-PF

Theorem (Su, 2020)

Let $\{x_i\}_{i\geq 1}$ be a sequence of polynomials in q with nonnegative coefficients. If $x_{k-1}x_{\ell+1} = x_k x_\ell$ for $\ell \geqslant k$, then for the fixed integers a, b, n_0 and k_0 satisfying $ab \geqslant 0$, $n_0 \geqslant k_0$, the sequences $\{e_{k_0-ib}(n_0+ia)\}_{i\in\mathbb{Z}}$ are q-PF.

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Corollary (Su, 2020)

For $n_0 \geqslant k_0 \geqslant 0$, $a, b \geqslant 0$, the sequence

$$\left\{ q^{\binom{k_0+ib}{2}} \begin{bmatrix} n_0 - ia \\ k_0 + ib \end{bmatrix} \right\}_{i\geqslant 0}$$

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- It doesn't hold for $\left\{ \begin{bmatrix} n_0 - ia \\ k_0 + ib \end{bmatrix}_q \right\}_{i\geqslant 0}$

A generalization of symmetric functions

Brenti (JCTA 1995) defined

$$e_k^{(t)}(n) := e_k^{(t)}(x_1, x_2, \ldots, x_n) = \sum_{i_1, i_2, \ldots, i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$$

where the sum runs over all $i_1, i_2, \ldots, i_k$ such that $i_{j+1} - i_j \geq t$ for all $0 \leq j \leq k-1$.

Note that $e_k^{(0)}(n) = h_k(n)$ and $e_k^{(1)}(n) = e_k(n)$.

Is there any application of this generalization?

Thank you!