Lattice graphs and Schubert polynomials

Hui Lin[1] and Arthur L. B. Yang[2]
Center for Combinatorics, LPMC
Nankai University, Tianjin 300071, P. R. China
Email: [1]lin_linhui@eyou.com, [2]arthurlbyang@eyou.com

October 8, 2003

Abstract. We give a combinatorial interpretation for the Schubert polynomials in two sets of commutative variables based on lattice graphs (defined in Section 2), and then directly get some new properties of these polynomials. RC-graphs are easily reformulated from the lattice graphs. In the particular cases where the Schubert polynomials are of skew type, we recover the lattice path interpretation. Moreover, it becomes more transparent to explain the factorization properties of the NilCauchy Kernel. With a special class of polynomials indexed by reduced column strict tableaux, we generalize the expansion of Schubert polynomials in terms of key polynomials.

Keywords: Schubert polynomial, lattice graph, key polynomial, NilCauchy kernel.

Running title: Lattice graphs and Schubert polynomials

AMS Classification: 05E05, 05A15.

Corresponding Author: Arthur L. B. Yang, ArthurLbYang@eyou.com

1. Introduction

Motivated by the work of Bernstein-Gelfand-Gelfand [3] and Demazure [7], Lascoux and Schützenberger defined Schubert polynomials [17, 18, 19], which was extensively studied by many authors. We first review here the basic facts of the theory needed for our investigation, and more details can be found in [24].

Let $w = (w_1, w_2, \ldots, w_n)$ be a permutation in the symmetric group S_n on n elements, let s_i denote the transposition that interchanges the i-th and $(i+1)$-th entries and fixes all other elements. It is well known that $s_1, s_2, \ldots, s_{n-1}$ generate S_n, with the following relations

$$\begin{aligned} s_i^2 &= 1, \\ s_i s_j &= s_j s_i \quad for\ |i-j| > 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}. \end{aligned} \tag{1.1}$$

For $w \in S_n$, let $c_i(w) = \#\{j : i < j \text{ and } w(i) > w(j)\}$, then $c(w) = (c_1(w), c_2(w), \ldots, c_n(w))$ is called the code of w, and $\ell(w) = \sum_{i=1}^{n} c_i(w)$ is the length of w, i.e., the number of inversions of w. We say that a decomposition $w = s_{r_1} s_{r_2} \ldots s_{r_p}$ is reduced if $p = \ell(w)$ and call $r_1 r_2 \ldots r_p$ a reduced word for w. Let $R(w)$ denote the set of all reduced words for a permutation w. $w_0 = [n, n-1, \ldots, 1]$ is the longest element of S_n, we call it *maximal permutation*, and it plays a very important role in the definition of Schubert polynomials. One may check that

$$(s_{n-1} \ldots s_1)(s_{n-1} \ldots s_2) \ldots (s_{n-1} s_{n-2})(s_{n-1})$$

is a reduced decomposition of w_0 (we add parentheses in order to distinguish some factors), and the following lemma is trivial:

Lemma 1.1 *Given a permutation $w \in S_n$, let $c = [c_1, c_2, \ldots, c_n]$ be its code, then the concatenation of right factors of $(s_{n-1} \ldots s_1)(s_{n-1} \ldots s_2) \ldots (s_{n-1})()$ of respective lengths $c_1, c_2, \ldots, c_n$ is a reduced decomposition of w.*

For example, $w = [3, 1, 6, 2, 5, 4]$, its code $c(w) = [2, 0, 3, 0, 1, 0]$, then $w = (s_2 s_1)(s_5 s_4 s_3)(s_5)$.

Let $\mathbb{Z}[a_1, a_2, \ldots, a_n]$ denote the ring of polynomials in n commutative variables with coefficients in $\mathbb{Z}$. For any $1 \leqslant i \leqslant n-1$, we define the *divided difference operators*

$$\partial_i f(a_1, a_2, \ldots, a_n) = \frac{f(\ldots, a_i, a_{i+1}, \ldots) - f(\ldots, a_{i+1}, a_i, \ldots)}{a_i - a_{i+1}} = \frac{f - f^{s_i}}{a_i - a_{i+1}}.$$

It's easy to check that $\partial_i^2 = 0$, $\partial_i \partial_j = \partial_j \partial_i$ if $|i - j| > 1$ and $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$, from this one may deduce that $\partial_w = \partial_{r_1} \partial_{r_2} \ldots \partial_{r_p}$ is well defined for any reduced word $r_1 r_2 \ldots r_p \in R(w)$. Moreover, $\partial_{r_1} \partial_{r_2} \ldots \partial_{r_p} = 0$ if $r_1 r_2 \ldots r_p$ is not reduced. Notice that the product of the divided difference operators or the simple transpositions act on the polynomial according to the order from right to left.

Divided differences satisfy Leibnitz formulas, as easily seen from the definition:

$$\partial_i(fg) = (\partial_i f)g + f^{s_i}(\partial_i g) = (\partial_i g)f + g^{s_i}(\partial_i f), \tag{1.2}$$

where f and g are polynomials. In particular, symmetric functions in x_i, x_{i+1} are scalars for ∂_i, i.e.

$$g = g^{s_i} \Longrightarrow \partial_i(fg) = g(\partial_i f).$$

Schubert polynomials in two sets of commutative variables were defined by Lascoux [18], and they are written in terms of divided difference operators only acting on the first alphabet $\mathbb{A}$ as follows, starting from a top element $\mathbb{X}_{w_0}(\mathbb{A},\mathbb{B})$:

Definition 1.2

$$\begin{cases} \mathbb{X}_{w_0}(\mathbb{A},\mathbb{B}) & := & \prod_{1\leqslant i,j\leqslant n,\ i+j\leqslant n}(a_i-b_j) \\ \mathbb{X}_{w}(\mathbb{A},\mathbb{B}) & = & \partial_{w^{-1}w_0}\mathbb{X}_{w_0}(\mathbb{A},\mathbb{B}) \end{cases}$$

The preceding definition implies the recursions

$$\partial_i\mathbb{X}_w(\mathbb{A},\mathbb{B}) \;=\; \begin{cases} \mathbb{X}_{ws_i}(\mathbb{A},\mathbb{B}) & if\ w_i > w_{i+1}, \\ 0 & if\ w_i < w_{i+1}. \end{cases} \tag{1.3}$$

We can also write Schubert polynomials indexed by code of a permutation, denoted by $\mathbb{Y}_{c(w)}$, i.e., $\mathbb{Y}_{c(w)} = \mathbb{X}_w$. For $m > n$, we can associate with a permutation $w = [w_1, w_2, \ldots, w_n] \in S_n$ the permutation $v = [w_1, \ldots, w_n, n+1, \ldots, m] \in S_m$. Now v and w have the same length and the same set of reduced words. The stability of Schubert polynomials shows that $\mathbb{X}_w$ is well-defined for each permutations $w \in S_\infty = \bigcup_n S_n$. Specializing the second set of variables into $\mathbf{0}$, we get Schubert polynomial in one set of commutative variables, and denote it by $X_w(\mathbb{A}) = \mathbb{X}_w(\mathbb{A}, \mathbf{0})$.

Key polynomials are very important to study Schubert polynomials, which were introduced by Demazure [8] for Weyl groups and investigated combinatorially by Lascoux and Schützenberger (who called them *standard bases*) [20, 21] in the case of symmetric group. They give several combinatorial descriptions of the key polynomials as well as an explicit formula for the expansion of Schubert polynomials $\mathbb{X}_w$ as a positive sum of key polynomials. In order to be distinguished from the general case, they also define free key polynomials when the variables are noncommutative. Now we first recall some related definitions before we give combinatorial descriptions of the key polynomials.

The *Knuth* or *plactic* equivalence $\sim$ is defined on the set of all words by the transitive closure of the relations

$$\begin{aligned} xzy &\sim zxy \quad for \quad x \leqslant y < z \\ yxz &\sim yzx \quad for \quad x < y \leqslant z, \end{aligned}$$

while the *Coxeter-Knuth* or *Nilplactic* equivalence $\equiv$ is defined on reduced words [9, 20] by the symmetric transitive closure of the relations

$$\begin{aligned} i(i+1)i &\equiv (i+1)i(i+1) \\ xzy &\equiv zxy \quad for \quad x < y < z \\ yxz &\equiv yzx \quad for \quad x < y < z, \end{aligned}$$

where $x < y < z$ are letters. Respectively we call the $\sim$ equivalence classes plactic classes and the $\equiv$ equivalence classes Nilplactic classes.

A *column strict tableau* is a filling of a Ferrers diagram with positive integers which is weakly increasing in each row and strictly increasing down each column. The column reading word of the column strict tableau T is $col(T) = v^{(1)}v^{(2)}\cdots$, where $v^{(j)}$ is the strictly decreasing column word comprising the j-th column of T. The content of T is the composition whose i-th part is the number of occurrences of the letter i in T. Now we project the word i into s_i. If the column reading word of T can be look on as a reduced decomposition for a permutation, we call tableau T as *reduced column strict tableau*. Similarly, we can define the *row reading word* of T is $row(T) = \cdots u^{(2)}u^{(1)}$, where $u^{(i)}$ is the weakly increasing row word comprising the i-th row. Of course, the row reading word of *reduced column strict tableau* is also a reduced word of the same permutation. The *content* of T is the composition $(\gamma_1, \gamma_2, \ldots)$, where γ_i is the number of occurrences of the letter i in T.

We recall that a *key* is a column strict tableau such that its columns are pairwise comparable for the inclusion order. Now we can associate a composition to a key tableau. In fact, there is an obvious bijection between compositions and keys given by $\alpha \rightarrowtail key(\alpha)$, where $key(\alpha)$ is the column strict tableau of shape $\lambda(\alpha)$ (decreasing rearrangement of α) whose first α_j columns contains the letter j for all j. The inverse map is given by $T \rightarrowtail content(T)$.

Following [21, 25, 23], we now define the left and right keys of a tableau. The left key $K^-(T)$ is the tableau of the same shape as T whose j-th column is the first column of any skew tableau in the plactic class of T with the following properties:

- its sequence of column lengths is a permutation of the corresponding sequence for T;
- its first column has the same length as the j-th column of T.

Now if we replace the first column by the last one, we get the right key $K^+(T)$. Further, if we replace the plactic relations with the nilplactic ones, we similarly the define the left nil key $K^\circ_-(T')$ and the right nil key $K^\circ_+(T')$ for reduced column strict tableau T'. It's easy to verify that $K^-(T)$, $K^+(T)$, $K^\circ_-(T')$ and $K^\circ_+(T')$ are indeed key tableaux.

Definition 1.3 *For any $\alpha \in \mathbb{N}^n$, the free key polynomial $\mathbf{K}_\alpha(\mathbb{A})$ is the sum in free algebra of all the words, which are column words of all tableaux having right key less or equal to $key(\alpha)$.*

Fix an integer i, and consider the row words only containing the two letters i and $i+1$. Then take any factor $(i+1, i)$, ignore it, and iterate. At last, we get an increasing word. For this word, there are two natural neighbors. One is obtained by changing the rightmost unpaired i into $i+1$, and the other is obtained by changing the leftmost unpaired $i+1$ into i. Iterating, one gets the *i-string* containing the word. For example, given a word 1121221, the 1-string is

$$11\mathbf{21}1\mathbf{21} \to 11\mathbf{21}2\mathbf{21} \to 12\mathbf{21}2\mathbf{21} \to 22\mathbf{21}2\mathbf{21}.$$

It is worth mentioning that the pairing is unique, so the string is well defined. The transposition s_i act on the string by exchanging the number of occurrences of unpaired i and $i+1$ but keeping the increasing order. Given a word w in the string, denote w^{s_i} as the symmetrical part of w. Now we can define the action on the element w of the string of the operator π_i:

• If w is in the middle of the string, then $\pi_i w = w$;

• If w is in the left part of the string, then $\pi_i w$ is the sum of all the words between w and w^{s_i};

• If w is in the right part of the string, then $\pi_i w$ is equal to minus the sum of all words contained in the symmetrical part of the string finishing just before w.

For the general word, we ignore the letters $\neq i, \neq i+1$ and define the operators s_i and π_i in the same meaning. Although the operators π do not satisfy the braid relations, but we have the following theorem due to Lascoux and Schützenberger :

Theorem 1.4 *[21] Let λ be a partition, and σ be a permutation such that $\alpha = \lambda^\sigma$. Let $s_i s_j \cdots s_k$ be a decomposition of σ, then*

$$\mathbf{K}_\lambda(\mathbb{A}) = \cdots 3^{\lambda_3} 2^{\lambda_2} 1^{\lambda_1} \quad \& \quad \mathbf{K}_\alpha(\mathbb{A}) = \pi_k \cdots \pi_j \pi_i \mathbf{K}_\lambda. \tag{1.4}$$

In [22], Lascoux also define the double key polynomials $\mathbf{DK}_\alpha$ in two sets of commutative variables. The definition is similar to the free key polynomials. Given a partition λ, we associate $\mathbf{DK}_\lambda$ to a biword $w^\lambda = \cdots \binom{1}{2}\binom{2}{2} \cdots \binom{\lambda_2}{2}\binom{1}{1}\binom{2}{1} \cdots \binom{\lambda_1}{1}$ instead of a word. When i is changed into $i+1$, we change $\binom{j}{i}$ into $\binom{j+1}{i+1}$. Suppose $\alpha = \lambda^\sigma$, We have

Definition 1.5

$$\mathbf{DK}_\alpha(\mathbb{A}, \mathbb{B}) = \theta(\pi_{\sigma^{-1}}(w^\lambda)) \quad \& \quad \theta\left(\binom{j}{i}\right) = a_i - b_j. \tag{1.5}$$

Denote π_i^a as the *isobaric operators* on polynomials (in commutative variables a_i) induced by π_i:

$$\begin{aligned}\pi_i^a f(a_1, a_2, \ldots, a_n) &= \frac{a_i f(\ldots, a_i, a_{i+1}, \ldots) - a_{i+1} f(\ldots, a_{i+1}, z_i, \ldots)}{a_i - a_{i+1}} \\ &= \frac{a_i f - a_{i+1} f^{s_i}}{a_i - a_{i+1}}.\end{aligned}$$

It's easy to check that $\pi_i^a = \partial_i a_i$ and π_i^a satisfies the braid relation (1.1), which makes it accessible to give a recursive definition of key polynomials $\kappa_\alpha(\mathbb{A})$ as follows:

Definition 1.6

$$\kappa_\lambda(\mathbb{A}) = a_1^{\lambda_1} a_2^{\lambda_2} \ldots \quad \& \quad \kappa_{\alpha s_i}(\mathbb{A}) = \pi_i^a \kappa_\alpha \textit{ if } \alpha_i > \alpha_{i+1}, \tag{1.6}$$

where λ is a decreasing reordering of α.

In fact, if we look on every letter i in free algebra as the corresponding commutative variable a_i, key polynomial κ_α is just the commutative image of free key polynomial $\mathbf{K}_\alpha$.

Now we can restate a theorem which gives a new combinatorial description of key polynomials by V. Reiner and M. Shimozono [25] as follows :

Theorem 1.7 *[25]*

$$\kappa_\alpha(\mathbb{A}) = \sum_{\mathbf{rev(r)} \equiv T,\ \mathbf{cp}\, is\, \mathbf{rev(r)}\text{-}compatible} a_{cp_1} a_{cp_2} \cdots a_{cp_\ell}, \tag{1.7}$$

where T is a reduced column strict tableau and content$(K_-^\circ(T)) = \alpha$.

For the compatible word, we have:

Definition 1.8 *For any sequence $\mathbf{r} = (r_1 \ldots r_p)$, we say that a p-tuple $\mathbf{cp} = (cp_1, \ldots, cp_p)$ of (strictly) positive integers is $\mathbf{r}$-compatible if*

$$\begin{gathered} cp_1 \geq cp_2 \geq \cdots \geq cp_p, \\ cp_j \leq r_j \ ,\ for\ 1 \leq j \leq p, \\ cp_j > cp_{j+1} \ ,\ if\ r_j > r_{j+1}. \end{gathered}$$

There are also several other interesting combinatorial descriptions of key polynomials. Kohnert [15] gave a combinatorial interpretation of κ_α in terms of key diagram $D(\alpha)$, which is an array of squares having α_i squares left-justified in row i. Based on the work of Lascoux [22], C. Learnt [23] investigate the key polynomials through the crystal structure on RC-graphs.

This paper is organized as follows. In Section 2., we will describe the lattice configuration interpretation to Schubert polynomials and give an explicit proof. In Section 3., we will stress the link between lattice graphs and nonintersecting paths for the skew Schubert polynomials. The connections among NilCauchy kernel, NilPlactic Kernel, key polynomials and lattice graphs are stated in Section 4.. At last, we will give a program to compute the lattice configurations in Section 5..

2. Lattice Configuration

About the combinatorial construction of Schubert polynomial, Fomin and Kirillov [10] introduced RC-graph (for reduced word compatible sequence graph). There is also another complicated construction given by Bergeron based on the Rothe diagram in [2]. Probably the simplest way to generate Schubert polynomial was given first by Kohnert [14, 15], and he proved the truth for the vexillary permutation. The attempt to prove the general case has been given by Learnt [23] and Winkel [27, 28]. Bergeron and Billey gave an algorithm for generating the simple polynomials in [1], they also introduced the double RC-graph for the case in two sets of variables, but we think it's a little complicated. Kohnert gave another description for the Schubert polynomials based on the above work in [16]. In this section we define a new combinatorial object, we call it lattice graph, and introduce an algorithm for computing Schubert polynomials by "L-move", it's based on the reduced words and the braid relations (1.1).

First, we give the definition of lattice graph as follows:

Definition 2.1 *In south-east coordinate, a graph, which is composed of some vertical steps from point (i, j) to point $(i, j+1)$ which are denoted by element $[i, j]$, is called a lattice graph of permutation w if*
• *the sequence $\mathbf{r}$ obtained by reading Y-coordinate of each element from bottom to top and left to right is a reduced word of w.*
• *the sequence $\mathbf{cp}$ obtained by reading X-coordinate of each element from bottom to top and left to right satisfies that $rev(\mathbf{cp})$ is $rev(\mathbf{r})$-compatible.*

All lattice graphs of a permutation w is called the lattice configuration of w, and we denote the set by $L(w)$. Let $C(\mathbf{r})$ denote the set of all $\mathbf{r}$-compatible sequences. From Definition 2.1, $L(w)$ is in one-to-one correspondence with the set $\{(\mathbf{r}, \mathbf{cp}) : rev(\mathbf{r}) \in R(w)\ , \mathbf{cp} \in C(\mathbf{r})\}$, $L(w)$ is the graphical expression of the latter one. Since $cp_j \leq r_j$, all elements are in the triangle area which is determined by x=1 and y=x, we may only draw this area. Notice that if $C(\mathbf{r}) = \emptyset$ for reduced word $rev(\mathbf{r})$, then we can't find a lattice graph from which we can read $rev(\mathbf{r})$. For example, let $w = [3, 1, 6, 2, 5, 4]$, then $\mathbf{r} = [2, 1, 5, 4, 3, 5]$ is a reduced word of w, $\mathbf{cp} = [4, 3, 2, 2, 1, 1]$ is $rev(\mathbf{r})$-compatible, the corresponding lattice graph is

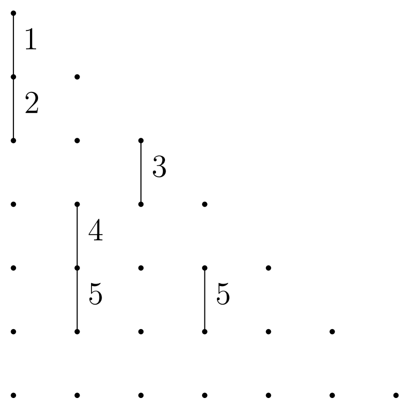

Figure 2.1 Lattice graph

For any $D \in L(w)$, the weight of each element $[i, j]$ is $a_i - b_{j-i+1}$, the weight of D is defined to be the product of the weight of all elements $[i, j] \in D$, denoted by $w(D)$. Now, the weight of the above lattice graph is

$$(a_1 - b_2)(a_1 - b_1)(a_2 - b_4)(a_2 - b_3)(a_3 - b_1)(a_4 - b_2).$$

To get all the lattice graph for a permutation, we first introduce the maximal lattice graph $L_{Max}(w)$ for $w \in S_\infty$.

$$L_{Max}(w) = \{(c, j) : c \leqslant n_j\},\ where\ n_j = \#\{i : i > j\ and\ w_i < w_j\},\ i.e.,\ n_j = c_j(w).$$

Here is L_{Max} for the permutation $[3, 1, 6, 2, 5, 4]$, its code is $[2, 0, 3, 0, 1, 0]$. Then we show moves on lattice graphs. For $D \in L(w)$, An L-move LM_{ij} is a change of the following type: Formally, $LM_{ij}(D) = D \cup \{[i-n, j-n+1]\} \setminus \{[i, j]\}$ when the following conditions are satisfied:

- $[i, j] \in D$, $[i, j + 1] \notin D$.
- $[i - n, j - n] \notin D$, $[i - n, j - n + 1] \notin D$, for some $0 < n < i$. (2.8)
- $[i - k, j - k] \in D$, $[i - k, j - k + 1] \in D$, for each $0 < k < n$.

Another kind of moves is very similar to L-move, which we call R-move. Formally, R-move $RM_{ij}(D) = D \cup \{[i + 1, j - m + 1]\} \setminus \{[i, j]\}$ when the

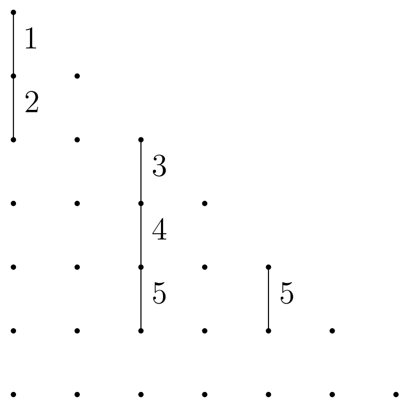

Figure 2.2 L_{Max}

following conditions are satisfied:

- $[i,j] \in D, \ [i+1, j+1] \notin D.$
- $[i, j-m] \notin D, \ [i+1, j-m+1] \notin D,$ for some $j-1 \geq j-m \geq i.$
- $[i, j-k] \in D, \ [i+1, j-k+1] \in D,$ for each $j-m+1 \leq j-k \leq j-1.$

Now, we can get some results closely related to RC-graphs in [1], but most of proofs we give are self-contained.

Lemma 2.2 *L-moves preserve the permutation associated with a lattice graph, i.e., if $D \in L(w)$ and satisfying condition (2.8), then $LM_{ij}(D) \in L(w)$.*

Proof. As shown in Figure 1, for $0 \leqslant k \leqslant n$, let u_{i-k} denote the permutation which corresponds to elements above $[i-k, j-k]$ in $(i-k)$-th column and v_{i-k} denote the permutation corresponding to elements under $[i-k, j-k+1]$ in the same column, so

$$u_{i-k} = s_{r_1} s_{r_2} \cdots s_{r_{m_1}}, \quad r_p < j-k \quad for \quad p = 1, 2, \cdots m_1. \tag{2.9}$$
$$v_{i-k} = s_{t_1} s_{t_2} \cdots s_{t_{m_2}}, \quad t_p > j-k+1 \quad for \quad p = 1, 2, \cdots m_2. \tag{2.10}$$

In fact, elements which come before $[i-n, j-n+1]$ (corresponding to permutation $w^{(1)}$) and those which come behind $[i,j]$ (corresponding to permutation $w^{(2)}$) are same in $LM_{ij}(D)$ and D, so it's enough to think about the reduced words corresponding to the elements between $[i-n, j-n+1]$ and $[i,j]$. In lattice graph D, the reduced decomposition is (we added parentheses in order to distinguish columns)

$$w' = w^{(1)}(u_{i-n})(v_{i-n+1} s_{j-n+2} s_{j-n+1} u_{i-n+1}) \ldots (v_{i-1} s_j s_{j-1} u_{i-1})(v_i s_j) w^{(2)},$$

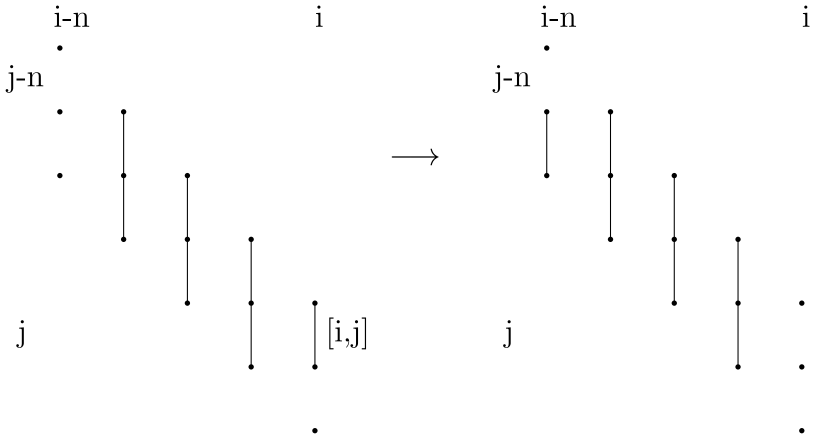

Figure 2.3 L-move

But as for $LM_{ij}(D)$, the reduced decomposition is

$$w'' = w^{(1)}(s_{j-n+1}u_{i-n})(v_{i-n+1}s_{j-n+2}s_{j-n+1}u_{i-n+1})\ldots(v_{i-1}s_js_{j-1}u_{i-1})(v_i)w^{(2)}.$$

Using (2.9), (2.10) and (1.1), we can deduce

$$\begin{aligned}
w' &= w^{(1)}(u_{i-n})(v_{i-n+1}s_{j-n+2}s_{j-n+1}u_{i-n+1})\ldots v_{i-1}s_js_{j-1}s_ju_{i-1}v_iw^{(2)}\\
&= w^{(1)}(u_{i-n})(v_{i-n+1}s_{j-n+2}s_{j-n+1}u_{i-n+1})\ldots v_{i-1}s_{j-1}s_js_{j-1}u_{i-1}v_iw^{(2)}\\
&\cdots\cdots\cdots\cdots\\
&= w^{(1)}(u_{i-n})(v_{i-n+1}s_{j-n+2}s_{j-n+1}s_{j-n+2}u_{i-n+1})\ldots(v_{i-1}s_js_{j-1}u_{i-1})(v_i)w^{(2)}\\
&= w^{(1)}(u_{i-n})(v_{i-n+1}s_{j-n+1}s_{j-n+2}s_{j-n+1}u_{i-n+1})\ldots(v_{i-1}s_js_{j-1}u_{i-1})(v_i)w^{(2)}\\
&= w^{(1)}(s_{j-n+1}u_{i-n})(v_{i-n+1}s_{j-n+2}s_{j-n+1}u_{i-n+1})\ldots(v_{i-1}s_js_{j-1}u_{i-1})(v_i)w^{(2)}\\
&= w''.
\end{aligned}$$

∎

Remark. Notice that if $n = 1$ in Figure 1, L-move from D to $LM_{ij}(D)$, i.e. from w' to w'', can be decomposed to a sequences of changes which satisfy $s_is_j = s_js_i\ for\ |i-j| > 1$. In particular, in this case if v_i and u_{i-1} are identity permutations, L-move from D to $LM_{ij}(D)$ will preserve the reduced word, only change the compatible sequence. On the other cases, one can see that L-move is a combination of a series of braid relations, so it will change the reduced words but preserve the permutation. For example $w = [3, 1, 5, 4, 2]$, the process of L-move from $L_{Max}(w)$ is

Lemma 2.3 *A lattice graph $D \in L(w)$ is the result of a L-move if and only if there is an element $[i, j] \notin D$ such that $[i, j + 1] \in D$.*

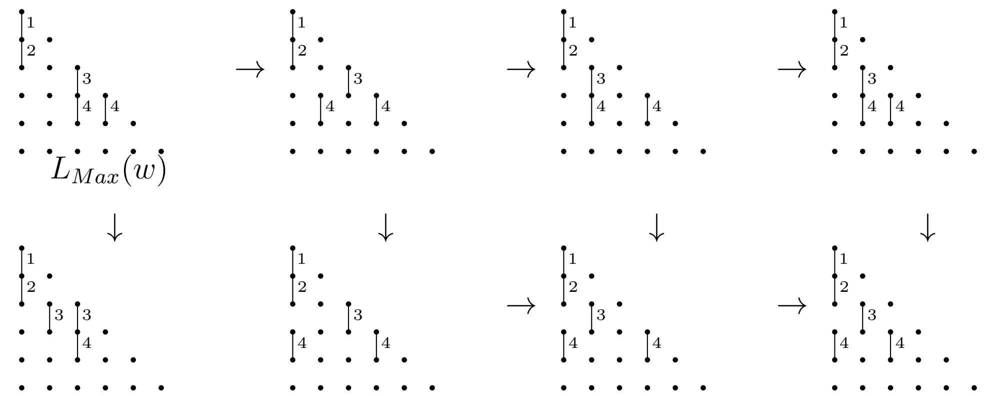

Figure 2.4

Proof. We can prove this along the lines in [1]. ∎

Geometrically, the criterion for there not being an inverse L-move is that of all the element in each column are clumped together at the top. Now we can state our main theorem :

Theorem 2.4
$$L(w) = \sum_{D \in LM(L_{Max}(w))} D,$$
where $w \in S_\infty$, $LM(L_{Max}(w))$ be the set of lattice graphs that can be obtained from the maximal lattice graph $L_{Max}(w)$ by a sequence of L-moves.

Proof. From the definition of $L_{Max}(w)$ and Lemma 2.3, we can easily get that $L_{Max}(w)$ does not admit an inverse L-move and any lattice graph of $L(w)$ other than $L_{Max}(w)$ admits an inverse L-move. Moreover, any lattice graph from $L_{Max}(w)$ by L-moves is in $L(w)$ by Lemma 2.2. In fact from the initial graph D, any inverse L-move make the element closer to the diagonal $y = x$, at last it can't admit an inverse L-move, that is $L_{Max}(w)$. Reversing this sequence and applying L-moves to $L_{Max}(w)$ we recover D. ∎

According to the Leibnitz formulas for divided differences (1.2), we can easily deduce the rule ∂_i acting on a lattice graph. In fact we only need consider the elements in the i-th and $(i+1)$-th columns, because the other columns are symmetric for a_i and a_{i+1}. Notice that $(a_i - b_{i-j+1})(a_{i+1} - b_{i-j+1})$ is symmetric in a_i and a_{i+1}, so $[i, j]$ and $[i+1, j+1]$ are symmetric, we say that such two elements are "paired", they are fixed or both vanish under ∂_i. Then, $(a_i - b_{i-j+1})^{s_i} = (a_{i+1} - b_{i-j+1})$, so the image of $[i, j]$ under s_i

is element $[i+1, j+1]$, that's a south-east move. Further, the image of element $[i+1, j+1]$ under s_i is element $[i, j]$, that's a reverse move, but $\partial_i(a_{i+1} - b_{i-j+1}) = -1$, So the move from $[i+1, j+1]$ to $[i, j]$ change the sign of the lattice graph.

Given a lattice graph D and an integer r, we give a precise description for the action of ∂_r on D. First we need to pair the segments of the lattice graph, and we say that $[r, s]$ and $[r+1, t]$ are r-paired if and only if $t = s+1$. Now let SI_m denote the unpaired segments $([r, m], \emptyset)$ or $(\emptyset, [r+1, m+1])$, that is to say, exactly one of the elements $[r, m]$ and $[r+1, m+1]$ in D, and we say SI_m is *positive* if $[r, m] \in D$, otherwise we say SI_m is *negative.* Let $SI(r, D) = (SI_{m_1}, SI_{m_2}, \ldots, SI_{m_t})$ be all such pairs, where $m_1 < m_2 < \cdots < m_t$. Now the operator ∂_r on the lattice graph D is defined by

$$\partial_r D = \bigcup_{k=1}^{t} D_k$$

where D_k have the same part as D except SI_{m_1} up to $SI_{m_{k-1}}$ change into $\overline{SI_{m_1}}$ up to $\overline{SI_{m_{k-1}}}$ ($\overline{SI_{m_j}}$ means an interchange between $[r, m_j]$ and $[r+1, m_j+1]$), and SI_{m_k} is removed from D. It's worth mentioning that D_k is assigned a sign coincident with the sign of SI_{m_k}.

Here is an example.

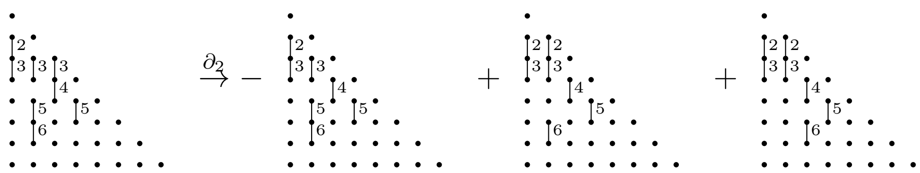

Figure 2.5

Now let $f(r,D)$ be the minimal integer such that $SI_{m_{f(r,D)}}$ is *positive* and $SI_{m_{f(r,D)+1}}$ is *negative*, if each of $SI(r,D)$ is *positive* we let $f(r,D)=t$, if each of $SI(r,D)$ is *negative* we let $f(r,D)=0$. We can classify $\partial_r D$ into the two classes $\Psi_1(D)$ and $\Psi_2(D)$ as

$$\Psi_1(D)=\bigcup_{k=1}^{f(r,D)} D_k,\ \ \Psi_2(D)=\bigcup_{k=f(r,D)+1}^{t} D_k.$$

Of course, if $SI(r,D)$ is empty then let $\partial_r D=\emptyset$. Denote a subset of $L(v)$ as

$$L_0(v)=\{D\in L(v): f(r,D)\neq 0\ and\ (r,j)\in D\ for\ r\leqslant j\leqslant m_{f(r,D)}\}.$$

Here is an example.

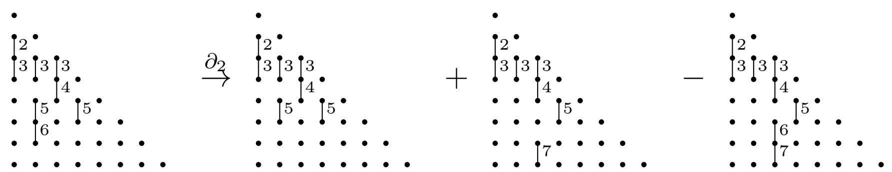

Figure 2.6

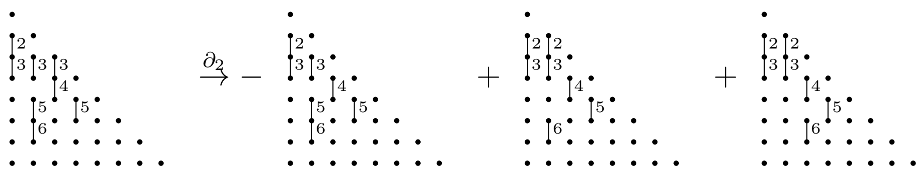

Figure 2.7

Two results are equivalent, it's determined by two equivalent description of Leibnitz formulas (1.2). As a rule, we adopt the second method which acts on elements from top to bottom.

Theorem 2.5

$$\mathbb{X}_w(\mathbb{A},\mathbb{B})=\sum_{D\in L(w)} w(D)\ , \tag{2.11}$$

where $w(D)$ denotes the weight of lattice graph D.

Proof. To prove this theorem we will proceed by reverse induction on $\ell(w)$.

If $w = w_0$ (the longest element of S_n) then (2.11) holds, because $L(w_0)$ contains only the lattice graph $D_0 = L_{Max}(w_0)$ and

$$w(D_0) = \prod_{1\leqslant i,j\leqslant n,\, i+j\leqslant n} (a_i - b_j).$$

On the other hand, from the definition of Schubert polynomials 1.2,

$$\mathbb{X}_{w_0} = \prod_{1\leqslant i,j\leqslant n,\, i+j\leqslant n} (a_i - b_j).$$

Now if $w \neq w_0$, let $v = ws_r$ and satisfy $\ell(v) = \ell(w) + 1$, from (1.3) we have $\mathbb{X}_w = \partial_r \mathbb{X}_v$. By the induction hypothesis equation (2.11) holds for $\mathbb{X}_v$. The induction step will be to "apply" the operator ∂_r to the lattice graphs in $L(v)$. Following the above analysis, we give more precise description for the action of ∂_r. Given a lattice graph D of v, let SI_m denote the pair $([r,m], \emptyset)$ or $(\emptyset, [r+1, m+1])$, that is to say, exactly one of the elements $[r,m]$ and $[r+1, m+1]$ in D, and we say SI_m is *positive* if $[r,m] \in D$, otherwise we say SI_m is *negative*. Let $SI(r,D) = (SI_{m_1}, SI_{m_2}, \ldots, SI_{m_t})$ be all such pairs. Now the operator ∂_r on the lattice graph D is defined by

$$\partial_r D = \bigcup_{k=1}^{t} D_k$$

where D_k have the same part as D except SI_{m_1} up to $SI_{m_{k-1}}$ change into $\overline{SI_{m_1}}$ up to $\overline{SI_{m_{k-1}}}$ ($\overline{SI_{m_j}}$ means an interchange between $[r, m_j]$ and $[r+1, m_{j+1}]$), and SI_{m_k} is removed from D. It's worth mentioning that D_k is assigned a sign coincident with the sign of SI_{m_k}. Now let $f(r,D)$ be the minimal integer such that $SI_{m_{f(r,D)}}$ is *positive* and $SI_{m_{f(r,D)+1}}$ is *negative*, if each of $SI(r,D)$ is *positive* we let $f(r,D) = t$, if each of $SI(r,D)$ is *negative* we let $f(r,D) = 0$. We can classify $\partial_r D$ into the two classes $\Psi_1(D)$ and $\Psi_2(D)$ as

$$\Psi_1(D) = \bigcup_{k=1}^{f(r,D)} D_k, \;\; \Psi_2(D) = \bigcup_{k=f(r,D)+1}^{t} D_k.$$

Of course, if $SI(r,D)$ is empty then let $\partial_r D = \emptyset$.

Denote a subset of $L(v)$ as

$$L_0(v) = \{D \in L(v) : f(r,D) \neq 0 \; and \; (r,j) \in D \; for \; r \leqslant j \leqslant m_{f(r,D)}\}.$$

Now we shall claim the following lemma.

Lemma 2.6

$$L(w) = \bigcup_{D\in L_0(v)} \Psi_1(D) \tag{2.12}$$

Proof. It is clear that all the sets $\partial_r D$ are disjoint from each other for $D \in L(v)$. Since all the graphs in $\Psi_1(D)$, $D \in L_0(v)$ are positive, we only need to show that all of them are lattice graphs of w, and every lattice graph of w can be obtained in this way. For one element D_k of $\Psi_1(D)$, we know that $(m_1, m_2, \ldots, m_k)$ is an ordered subsequence of $(r, r+1, \ldots, m_k)$. A little thinking leads to that we need only consider the reading words $ReadW(D)$ and $ReadW(D_k)$ which correspond to elements above point (r, m_k+1) in r-th column and above point $(r+1, m_k+2)$ in $r+1$-th column. More precisely, we suppose that

$$ReadW(D) = s_{m_k}\xi_k s_{m_{k-1}} \cdots s_{m_2}\xi_2 s_{m_1}\xi_1\xi_k^+ \cdots \xi_2^+\xi_1^+, \tag{2.13}$$

where

$$\xi_j = s_{n_{j,1}} \cdots s_{n_{j,t_j}}, \xi_j^+ = s_{n_{j,1}+1} \cdots s_{n_{j,t_j}+1}$$

and let $m_0 = r$, then for each $j = 0, 1, \ldots, k-1$

$$m_j < n_{j+1,1} < \cdots < n_{j+1,t_{j+1}} < m_{j+1}.$$

are all the integers between r and m_k. By the difference between D and D_k, we get

$$ReadW(D_k) = \xi_k \cdots \xi_2\xi_1\xi_k^+ s_{m_{k-1}+1} \cdots s_{m_2+1}\xi_2^+ s_{m_1+1}\xi_1^+. \tag{2.14}$$

It is a straight forward process of verifying that $ReadW(D) = ReadW(D_k)\cdot s_r$ by the braid relations (1.1).

On the other hand, for every lattice graph $D' \in L(w)$, let $h(r, D')$ be the minimal number such that neither of $(r, h(r, D'))$ and $(r+1, h(r, D')+1)$ is in $L(w)$, we declare that SI_{m_j} must be *negative* for every $m_j < h(r, D')$, otherwise it will be contrary to the fact $\ell(w\, s_r) = \ell(w) + 1$. From the above discussion we can easily find a lattice graph $D \in L(v)$ such that $D' \in \Psi_1(D)$ and $D \in L_0(v)$, which can be obtained from D by replacing SI_{m_j}, $m_j < h(r, D')$ with $\overline{SI_{m_j}}$, adding the element $(r, h(r, D'))$, and fixing the left part. ∎

If we let $L_1(v) = L(v) - L_0(v)$, we get

Lemma 2.7

$$\sum_{D\in L_0(v)} \Psi_2(D) + \sum_{D\in L_1(v)} \partial_r D = 0. \tag{2.15}$$

Proof. Notice that all the sets of $\partial_r D$ are disjoint. There are three classes of graphs to consider :

(1) $D' \in \Psi_2(D)$, and $D \in L_0(v)$,

(2) $D' \in \partial_r D$, and $f(r, D) = 0$,

(3) $D' \in \partial_r D$, and $f(r, D) \neq 0$, but there exists at least one j $(r \leqslant j \leqslant f(r, D))$ such that $(r, j) \notin D$.

If $D' = D_k$ and SI_{m_k} is *positive*, we can move (r, m_k) to the right through a shortest "R-move" sequence, if SI_{m_k} is *negative*, we can move $(r+1, m_k+1)$ to the left through a shortest inverse "R-move" sequence. Suppose the result lattice graph of the sequence is $D^{(1)}$ and the first move of this sequence generate an element $SI_{m_{k'}}$. We see that D_k and $D^{(1)}_{k'}$ are same but with different sign, moreover $D^{(1)}$ belongs to these three classes of graphs. That's the proof. ∎

Lemma 2.8

$$\partial_r L(v) = L(w).$$

Proof. It's a direct result of the above two lemmas. ∎

Now we complete the proof of Theorem 2.5 by combining the above three lemmas. More precisely, using the induction hypothesis, we have

$$\begin{aligned}
\mathbb{X}_w &= \partial_r \mathbb{X}_v \\
&= \textstyle\sum_{D \in L(v)} \partial_r w(D) \\
&= \textstyle\sum_{D \in L(v)} w(\partial_r D) \\
&= \textstyle\sum_{D \in L_0(v)} w(\Psi_1(D)) + \sum_{D \in L_0(v)} w(\Psi_2(D)) + \sum_{D \in L_1(v)} w(\partial_r D) \\
&= \textstyle\sum_{D \in L_0(v)} w(\Psi_1(D)) \\
&= \textstyle\sum_{D \in L(w)} w(D).
\end{aligned}$$

∎

Using the definition of weight of lattice graph, one can deduce the following theorem from (2.11) :

Theorem 2.9

$$\mathbb{X}_w(\mathbb{A}, \mathbb{B}) = \sum_{rev(\mathbf{r}) \in R(w)} \sum_{\mathbf{cp} \in C(\mathbf{r})} \prod_{j=1}^{p} (a_{cp_j} - b_{r_j - cp_j + 1}). \tag{2.16}$$

For example, for $w = [2,3,1,5,4]$, the reduced word are $124, 142$ and 412. For each of these we compute all compatible sequences as follows:

$$\begin{array}{ccc} \underline{421} & \underline{241} & \underline{214} \\ 321 & 221 & 211 \\ 421 & & \end{array}$$

Therefore

$$\begin{aligned} \mathbb{X}_{[2,3,1,5,4]} &= (a_1-b_1)(a_2-b_1)(a_4-b_1)+(a_1-b_1)(a_2-b_1)(a_3-b_2) \\ &\quad +(a_1-b_1)(a_2-b_3)(a_2-b_1)+(a_1-b_4)(a_1-b_1)(a_2-b_1). \end{aligned}$$

In fact, Theorem 2.9 is a generalization in Schubert polynomials of Theorem 2.10, which is conjectured by Stanley and first proved in [4], subsequently in [11].

Theorem 2.10 *[4, 11]*

$$X_w(\mathbb{A}) = \sum_{rev(\mathbf{r})\in R(w)} \sum_{\mathbf{cp}\in C(\mathbf{r})} a_{cp_1}a_{cp_2}\dots a_{cp_p}.$$

Remark. It states that L-move of lattice graphs is equivalent to the chute move of RC-graph. In fact the two combinatorial objects are same in essence. If we transpose the RC-graph and keep the bottom in alignment,then it's same with lattice graph. Now the relations between Kohnert's Rothe diagram configuration [16] and lattice graph is also clear by RC-graph.

The following corollaries are the generalization of the properties for simple Schubert polynomials which were proved in [1] :

Corollary 2.11 *Given permutation $u \in S_m$ and $v \in S_n$, let $u \times v = [u_1,\dots,u_m, v_1+m,\dots,v_n+m]$ and $1_m \times v = [1,\dots,m,v_1+m,\dots,v_n+m]$. Then*

$$\mathbb{X}_u(\mathbb{A},\mathbb{B})\mathbb{X}_{1_m\times v}(\mathbb{A},\mathbb{B}) = \mathbb{X}_{u\times v}(\mathbb{A},\mathbb{B}).$$

Proof. Every lattice graph D_1 in $L(u)$ satisfy that $i < m$ holds for any element $[i,j]$, and $i > m$ holds for any element of every lattice graph D_2 in $L(1_m\times v)$. No lattice graph in $L(u\times v)$ contains an element $[m,j]$. Therefore, there is a bijection between $L(u)\times L(1_m\times v)$ and $L(u\times v)$, given by sending (D_1,D_2) to $D_1\cup D_2$. ∎

If $v\in S_n$ and $v_1 = 1$, we define the operator $\downarrow$

$$\downarrow v = [v_2, v_3, \dots, v_n].$$

Notice $\downarrow v$ is undefined if $v_1 \neq 1$. Moreover, for any $\mathbb{A} = \{a_1, a_2, \ldots, a_n\}$ and $m < n$, we define

$$\mathbb{A}^+ = \{a_2, a_3, \ldots, a_n\},\ \mathbb{A}_m = \{a_1, a_2, \ldots, a_m\},\ \mathbb{A}^{-m} = \{a_{m+1}, a_{m+2}, \ldots a_n\}.$$

Corollary 2.12 *For $w \in S_\infty$, we have*

$$\mathbb{X}_w(\mathbb{A}, \mathbb{B}) = \sum_v \prod_k (a_1 - b_k) \mathbb{X}_{\downarrow vw}(\mathbb{A}^+, \mathbb{B}),$$

where $k = i_1, i_2, \ldots, i_p$, the sum is over all permutations $v \in S_\infty$ such that

$$\ell(w) = \ell(v) + \ell(vw),\ v = s_{i_1} s_{i_2} \ldots s_{i_p} \text{ with } i_1 < i_2 < \cdots < i_p, \text{ and } (vw)_1 = 1. \tag{2.17}$$

Proof. There is a bijection

$$L(w) \leftrightarrow \cup(v, L(\downarrow vw)),$$

where the union is over all permutations $v \in S_n$ with condition (2.17) satisfied. The bijection is given by sending $D \in L(w)$ to (v, D'), where the first column of D are elements $[1, k],\ k = i_1, i_2, \ldots, i_p$, and D' is the lattice graph obtained by removing the first column of D. ∎

The next result is a generalization of Corollary 2.12.

Corollary 2.13 *For any fixed positive integer m and any $w \in S_n$, we have the decomposition*

$$\mathbb{X}_w(\mathbb{A}, \mathbb{B}) = \sum d^w_{uv} \mathbb{X}_u(\mathbb{A}_m, \mathbb{B}) \mathbb{X}_v(\mathbb{A}^{-m}, \mathbb{B}),$$

where the d^w_{uv} are non-negative integers and the sum is over all permutations u,v such that

$$l(u) + l(v) = l(w),\ 1_m \times v = u^{-1} w. \tag{2.18}$$

Proof: Given a lattice graph D, we let $c_m(D) = \{[i, j] \in D :\ i \leqslant m\}$, and let $c_m(L(w)) = \{D \in L(w) :\ c_m(D) = D\}$. For each $w \in S_\infty$ and each m, there exists a bijection

$$L(w) \leftrightarrow \cup c_m(L(u)) \times L(v),$$

where the union is over all permutations u, v satisfying condition (2.18). The bijection takes D to

$$\{[i, j] \in D :\ i \leqslant m\} \times \{[i - m, j - m] :\ [i, j] \in D,\ i > m\}.$$

Therefore,

$$\mathbb{X}_w(\mathbb{A}, \mathbb{B}) = \sum (c_m \mathbb{X}_u(\mathbb{A}, \mathbb{B})) \mathbb{X}_v(\mathbb{A}^{-m}, \mathbb{B}),$$

where the sum is over all permutations u, v with condition (2.18) satisfied. ∎

3. Lattice graph and the skew Schubert polynomial

Skew Schubert polynomial is a special class of Schubert polynomial, whose specialization has been studied by many authors. Fox example, Wachs [26] and Billey, Jockusch, Stanley [4] studied the flag skew Schur function, Chen, Li and Louck [5] introduced the double Schur function. The determinant properties of these polynomials can be well expressed in terms of nonintersecting lattice path, you can find the detailed information in the forthcoming work of Chen, Yan and Yang [6].

Now we shall show that lattice graph configuration coincides with lattice path configuration for skew Schubert polynomial. A skew Schubert polynomial is indexed by a skew partition $c(w) =< J/I >$, which is the code of permutation w, here $< J/I >= (0^{i_1}, j_1-i_1, 0^{i_2-i_1}, j_2-i_2, 0^{i_3-i_2}, j_3-i_3, \ldots, j_n-i_n)$, and $J = (j_1, j_2, \ldots, j_n)$, $I = (i_1, i_2, \ldots, i_n)$ are Grassmannian with satisfying $I \leqslant J$ in componentwise order.

As usual, a lattice path in the plane is a path from an origin to a destination in which every step is either going up (vertical step) or going right (horizontal step). The weight of each step is defined as follows:

(1). For a vertical step from point $(i, j+1)$ to (i, j), the weight is $a_i - b_{j-i+1}$.

(2). For a horizontal step from (i, j) to $(i+1, j)$, the weight is 1.

(3). The weight of a path P is the product of the weights of the steps in the path, denoted by $w(P)$.

For a set of paths $P_1, P_2, \cdots, P_m$, the weight is defined to be the products of all the weights. Let $O = \{O_1, O_2, \cdots, O_m\}$ and $O' = \{O'_1, O'_2, \cdots, O'_m\}$ be two sequences of lattice points, we say that $P_1, P_2, \cdots, P_m$ is a group of nonintersecting lattice paths from O to O' if P_i's are nonintersecting and P_i is a lattice path with origin O_i and destination O'_i. Denote $w(O, O')$ as the sum of weights of all possible groups of nonintersecting paths, then Chen, Yan and Yang [6] prove that

Theorem 3.1 *[6] Let $\rho = (1, \cdots, n)$, $V = J + \rho$ and $U = I + \rho$, then the Skew Schubert Polynomials $\mathbb{Y}_{<J/I>}(A, B)$ can be evaluated by $w(O, O')$ for $O_k = (1, v_k)$ and $O'_k = (u_k, u_k)$.*

Without losing generality we can suppose $j_k > i_k$ for all $1 \leqslant k \leqslant n$, otherwise we can decrease the size n of the vector. For example $< (2,3,4)/(1,3,3) >= (0,1,0,0,0,1) =< (2,5)/(1,4) >$. Notice that the maximal lattice graph $L_{Max}(w)$ has the element set $\{[i_k+k, i_k+k], [i_k+k, i_k+k+1], \ldots, [i_k+k, j_k+k-1]\}$, $\forall k \in \{1, 2, \ldots, n\}$. Moreover,

$$[i_k+k, j_k+k] \notin L_{Max}(w) \quad \& \quad j_k+k < j_{k+1}+k+1,$$

so join up the points (i_k+k, j_k+k) and $(1, j_k+k)$ with horizontal steps, we obtain a group of non-intersecting paths which are same as those in Theorem 3.1. For example $c(w) =< (3,4,6)/(1,2,2) >= [0,2,0,2,4]$, the corresponding relation is: Since every lattice graph can be obtained by a sequence of

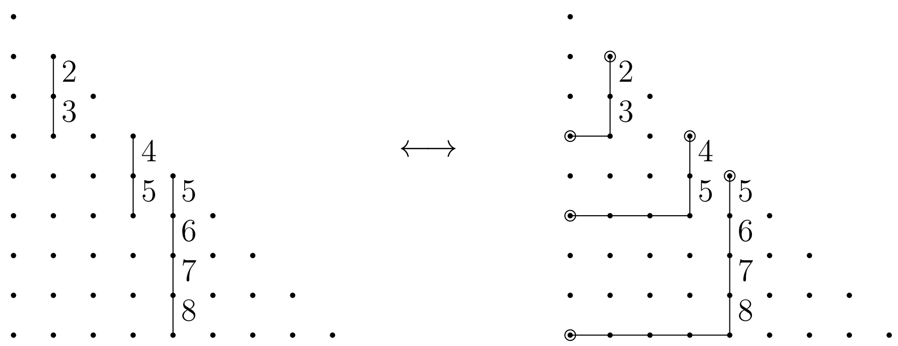

Figure 3.1

L-moves from $L_{Max}(w)$, we must show that

Theorem 3.2 *L-move transforms the non-intersecting paths graph D into the non-intersecting ones D'.*

Proof. Suppose the lattice graph which corresponds to D is D_L and $[i,j] \in D_L$. If $[i,j]$ is L-moveable, for Theorem 2.1 (b) in [4], it must satisfy the condition that $[i, j+1]$, $[i-1, j]$, $[i-1, j-1]$ are not in D_L. In this case, we get D'_L by a horizontal move replacing $[i,j]$ by $[i-1,j]$ in D_L and get D' by replacing the horizontal step from point $(i-1, j+1)$ to point $(i, j+1)$ by another horizontal step from $(i-1, j)$ to (i, j). If $[i-1, j+1] \notin D_L$, then $[i-1, j+1] \notin D'_L$ and notice $[i-1, j-1]$ also isn't in D'_L, so it's evident that D' is a non-intersecting paths graph. While $[i-1, j+1] \in D_L$, because D is a non-intersecting paths graph, so $[i-1, j+1]$ and $[i,j]$ must belong to the same path, so D' is also a non-intersecting paths graph. ∎

Remark. If we fix the starting points in the axis $x = 1$ and the ending points in the axis $y = x$, this proof also implies that if there is a lattice graph configuration can be realized by non-intersecting paths, then any graph configurations can be transformed into non-intersecting paths configuration.

Furthermore, such a Schubert polynomial must be skew, we can find J and I from the maximal lattice graph. In fact, lattice path is a very power tool which was introduced by Gessel and Viennot [12, 13].

4. NilCoxeter Algebra, NilPlactic Algebra and Key Polynomials

Lattice graph can easily explain some results in NilCoxeter algebra, we will discuss in detail following Fomin, Stanley [11]. Let $\mathbb{F}$ be any commutative ring, which contains various variables $a_1, a_2, \cdots, b_1, b_2, \cdots$. Let $n \in N$, we define the NilCoxeter $\mathbb{F}$-algebra $\Re_n$ of the symmetric group S_n is the algebra with the generators $u_1, u_2, \cdots, u_{n-1}$ and the Coxeter relations

$$u_i^2 = 0, \qquad u_i u_j = u_j u_i \; if \; |i-j| \geqslant 2, \qquad u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}. \quad (4.19)$$

Given a reduced word $\mathbf{r} = r_1, r_2, \cdots, r_p$, we identify a monomial $u_{r_1} u_{r_2} \cdots u_{r_p}$ in $\Re_n$ with the permutation $w = s_{r_1} s_{r_2} \cdots s_{r_p}$ in S_n. Some times we denote $u_w = u_{r_1} u_{r_2} \cdots u_{r_p}$ Now Coxeter relations (4.19) ensure that this notation is well defined and that $\Re_n$ has the $\mathbb{F}$-basis S_n. Write $< f, w >$ for the coefficient of $w \in S_n$ in the element f of $\Re_n$. For instance,

$$< ((1+u_1)(1+u_2))^2, 321 >= 2.$$

Define

$$A_i(z) = (I + z u_{n-1})(I + z u_{n-2}) \ldots (I + z u_i)$$

for each $i = 1, 2, \ldots, n-1$,I is the identity in $\Re_n$. Let

$$\varphi(\mathbb{A}) = A_1(a_1) A_2(a_2) \ldots A_{n-1}(a_{n-1}),$$

thus $\varphi(\mathbb{A}) \in \Re_n$. For each $w \in S_n$, we denote $\varphi_w(\mathbb{A}) =< \varphi, w >$, then

Theorem 4.1 *[11] $\varphi_w(\mathbb{A})$ is equal to Schubert polynomial $X_w(\mathbb{A})$.*

Proof. From the definition, we know if $(r_1, r_2, \ldots, r_p)$ is a reduced word, then we can identity a monomial $u_{r_1} u_{r_2} \ldots u_{r_p}$ in $\Re_n$ with the permutation $w = s_{r_1} s_{r_2} \ldots s_{r_p} \in S_n$. So the coefficient of $w \in S_n$ in the element $\varphi(\mathbb{A})$ of $\Re_n$ is equivalent to the product of weights of elements in lattice graph whose product is the permutation w. In terms of combinatorial meaning of lattice graph, $\varphi_w(\mathbb{A}) = X_w(\mathbb{A})$. ∎

From Theorem 4.1, we can deduce

$$\varphi(\mathbb{A}) = \sum_{\sigma \in S_n} X_\sigma(\mathbb{A}) u_\sigma,$$

we often call it *NilCauchy kernel*. As a generalization, we can define the double *NilCauchy kernel* to be

$$\aleph\Re_n(\mathbb{A},\mathbb{B}) = \sum_{\sigma\in S_n} \mathbb{X}_\sigma(\mathbb{A},\mathbb{B})u_\sigma, \tag{4.20}$$

Formulas involving reduced factorizations are conveniently written in $\Re_n$ by the same way with above.

Theorem 4.2 *[10, 11]*

$$\begin{aligned}&\aleph\Re_n(\mathbb{A},\mathbb{B}) =\\ &((I+(a_1-b_{n-1})u_{n-1})(I+(a_1-b_{n-2})u_{n-2})\dots(I+(a_1-b_1)u_1))\\ &((I+(a_2-b_{n-2})u_{n-1})\dots(I+(a_2-b_1)u_2))\dots((I+(a_{n-1}-b_1)u_{n-1}))\,.\end{aligned}$$

We have introduced some combinatorial descriptions of key polynomials in Section 1., but all descriptions are not enough to study the Schubert polynomials in two sets of variables, in fact Theorem 1.7 leads us to define the tableau polynomial $\mathbb{K}_T(\mathbb{A},\mathbb{B})$ indexed by the reduced column strict tableau T :

Definition 4.3 *Suppose T is a reduced column strict tableau,* $\mathbf{r}$ *and* $\mathbf{i}$ *are of length p, we have*

$$\mathbb{K}_T(\mathbb{A},\mathbb{B}) = \sum_{rev(\mathbf{r})\equiv T,\,\mathbf{i}\,is\,\mathbf{rev}(\mathbf{r})-compatible} \prod_{j=1}^{p}(a_{i_j}-b_{r_j-i_j+1}). \tag{4.21}$$

If $content(K_-^\circ(T))=\alpha$, we identify that $\mathbb{K}_T(\mathbb{A},\mathbf{0})=\kappa_\alpha(\mathbb{A})$ by Theorem 1.7. If α is the code of vexillary permutation, both the Schubert polynomial $\mathbb{Y}_\alpha(\mathbb{A},\mathbb{B})$ and the tableau polynomial $\mathbb{K}_T(\mathbb{A},\mathbb{B})$ (T is the reduced column strict tableau corresponding to the permutation whose inverse is a permutation with code α) coincide with the double key polynomial $\mathbf{DK}_\alpha(\mathbb{A},\mathbb{B})$.

Now we refine the double NilCauchy kernel by replacing the Coxeter relations by the nilplactic relations

$$\begin{aligned} y_iy_{i+1}y_i &\equiv y_{i+1}y_iy_{i+1}\\ y_jy_iy_k &\equiv y_jy_ky_i\\ y_ky_iy_j &\equiv y_iy_ky_j\end{aligned}$$

for $i<j<k$. Two reduced words which are equivalent in the nilplactic monoid gives the same permutation, when evaluated in the symmetric group

$(y_i \to s_i)$. The double NilPlactic kernel is defined by

$$\aleph\mathfrak{P}\mathfrak{R}_n(\mathbb{A},\mathbb{B}) =$$
$$((I + (a_1 - b_{n-1})y_{n-1})(I + (a_1 - b_{n-2})y_{n-2}) \ldots (I + (a_1 - b_1)y_1))$$
$$((I + (a_2 - b_{n-2})y_{n-1}) \ldots (I + (a_2 - b_1)y_2)) \ldots ((I + (a_{n-1} - b_1)y_{n-1}))\,.$$

Notice that each block expands as a sum of strictly column decreasing words in the $y_i's$. Expand the NilPlactic kernel, we get

Theorem 4.4 *The NilPlactic kernel expands as a sum of tableaux, with tableau polynomials on two sets of alphabets* $\mathbb{A}$ *and* $\mathbb{B}$ *as coefficients:*

$$\aleph\mathfrak{P}\mathfrak{R}_n \equiv \sum_T \mathbb{K}_T T, \tag{4.22}$$

sum over all reduced column strict tableaux T, whose column words appear as subwords of

$$(y_{n-1} \cdots y_1)(y_{n-1} \cdots y_2) \cdots (y_{n-1}).$$

Proof. It is an immediate result of the definition of tableau polynomial. ∎

Now projecting the NilPlactic algebra onto the NilCoxeter algebra, and comparing (4.20) and (4.22) we get that Schubert polynomials are positive sums of tableau polynomials :

Corollary 4.5

$$\mathbb{X}_w(\mathbb{A},\mathbb{B}) = \sum_{T \in P(w)} \mathbb{K}_T(\mathbb{A},\mathbb{B}), \tag{4.23}$$

where $P(w)$ are all the tableaux which are reduced decompositions of w.

Further, specialize the second alphabet $\mathbb{B}$ into $\{0, 0, \cdots\}$, we get

Corollary 4.6 *[20, 23, 25]*

$$X_w(\mathbb{A}) = \sum_{T \in P(w)} \kappa_{content(K_-^{\circ}(T^t))}(\mathbb{A}). \tag{4.24}$$

From definition 1.3, we have

$$\kappa_\alpha = \sum_T w(T), \tag{4.25}$$

where T sums over all column strict tableaux of shape $\lambda(\alpha)$ such that $K_+(T) \leq key(\alpha)$.

Combining (4.25) and (4.24), we get

Corollary 4.7 *[20, 21]*

$$X_w = \sum_{T \in \mathcal{I}(w)} w(T), \tag{4.26}$$

where $\mathcal{I}(w) = \{T | K_+(T) \leq key(content(K_-^\circ(T_1^t))), T_1 \in P(w)\}$ *is a multiset.*

Now consider the simple case of Theorem 2.5, Lascoux suggest us to find the connection between these tableaux $\mathcal{I}(w)$ and lattice graphs $L(w)$. We find the following bijection, which has been given by Lenart [23] in terms of RC-graphs.

Theorem 4.8 *Let* $\mathcal{E}(w) = \{(T, T_1) | K_+(T) \leq key(content(K_-^\circ(T_1^t))), T_1 \in P(w)\}$ *and* $L(w)$ *be all the lattice graphs. Then there is a bijection between* $\mathcal{E}(w)$ *and* $L(w)$.

Proof. For each lattice graph $D \in L(w)$, there is a unique pair $(\mathbf{r}, \mathbf{c})$ corresponding to D such that $\mathbf{c}$ is $\mathbf{r}$-compatible . Now use the Edelman-Greene correspondence [9, 4], we associate to $(\mathbf{r}, \mathbf{c})$ the pair of semistandard tableaux of conjugate shapes (T_1, T), which satisfies that $K_+(T) \leq key(content(K_-^\circ(T_1^t)))$. Now we let

$$\phi : L(w) \to \mathcal{E}(w),\ \phi(D) = (T_1, T),\ \forall D \in L(w). \tag{4.27}$$

Since every step is invertible, it is a tedious chase to check ϕ is a bijection. ∎

Now we let $\mathcal{E}'(w) = \{T | (T, T_1) \in \mathcal{E}(w)\}$, which is a multiset. In [4], one open problem is when does $\mathcal{E}'(w)$ have a simple direct description avoiding the use of Edelman-Greene correspondence? From some results in [19, 25], the following results immediately follow

Theorem 4.9 *If* w *is a vexillary permutation, then* $\mathcal{E}'(w)$ *is multiplicity free.*

Proof. Lascoux [19] first proved that a Schubert polynomial $\mathbb{X}_w$ is a key polynomial κ_α for some composition α if and only if w is vexillay and $\alpha = c(w)$. Each tableau in $\mathcal{E}'(w)$ has the shape $\lambda(\alpha)$, and Theorem 4.8 shows any two tableaux are different from each other. ∎

Remark. We have test that $\mathcal{E}'(w)$ is multiplicity free for every permutation $w \in S_6$. But Even if $\mathbb{X}_w$ is a skew Schubert polynomial, $\mathcal{E}'(w)$ is not multiplicity free. For example, Let $w = [1, 3, 2, 5, 4, 7, 6]$, $c(w) = [0, 1, 0, 1, 0, 1, 0]$, we have

$$\mathbb{X}_{[1,3,2,5,4,7,6]} = \kappa_{[0,1,0,1,0,1]} + \kappa_{[0,2,0,0,0,1]} + \kappa_{[0,1,0,2]} + \kappa_{[0,3]}. \tag{4.28}$$

One question is how to describe those permutations whose $\mathcal{E}'(w)$ are not multiplicity free.

5. Program to compute lattice configurations

We have known that the lattice graph is in one-to-one correspondence with the reduced word compatible word pair, which enable us to write a program to compute all the lattice configurations for each permutation. Here we use some functions included in ACE package [29], and the reader can compute with the following program with Maple.

```
## Better take reversed words. For every decrease of rd, there
#  must be a decrease at the same position in each word
 MajorElem:=proc(lw, pa) local i,k,mot,word,res; res:=NULL;
 if lw=[w[]] then
     RETURN (map(i->w[op(i)], ListPartIn(pa, [1$nops(pa)])))
 fi;
 for mot in lw do
    k:=op(1,mot);
    res:=res, seq(w[ op(word),op(mot)],
         word= ListPartIn(pa,[(k+1)$nops(pa)]));
 od;
 [res]
 end:
## transforms rd into a sequence of rectangular partitions
 Rd2ListPa:=proc(rd) local v,i,k,u,res; res:=NULL;
 k:=rd[1]; u:=NULL;
 v:=[op(rd),0];
 for i from 1 to nops(rd) do
    if v[i]<= v[i+1] then
     u:=u,k
    else
     res:=res, [u,k]; u:=NULL; k:=v[i+1];
    fi;
 od;
 [res]
 end:
## List all majorized words for fixed reduced word
 Rd2ListMajorizedWords:=proc(rd) local lpa,i,res;
 option remember;
 lpa:=Rd2ListPa(rd);
 res:=[w[]];
 for i from nops(lpa) by -1 to 1 do
    res:=MajorElem(res, op(i,lpa))
 od;
 res
```

```
 end:
## reading the columns of lattice graph from right to left
 WordRd2LattGraph:=proc(mot,rd) local i,lattgraph;
  lattgraph:=NULL:
  if convert(mot,`+`)=0 then RETURN(NULL) fi;
  for i from 1 to nops(mot) do
  lattgraph:=lattgraph,[op(i,mot),op( i,rd)]
  od:
  lattgraph:=[lattgraph];
  end:
## List all lattice configurations for fixed permutation
 Perm2ListLattConf:=proc(perm)local rd,pol,lw,fr;
 fr:=NULL;
 if perm=[$1..nops(perm)] then RETURN([]) fi;
 for rd in Perm2ListRd(Perm2Inv(perm)) do
 fr:=fr,op(map(WordRd2LattGraph,Rd2ListMajorizedWords(rd),rd));
 od;
 [fr];
end:
```

The function *Perm2ListLattConf* allows the reader to list all the lattice configurations of a given permutation. Some programs are supported by Lascoux, and we are very pleased to be allowed to include these programs here.

6. Acknowledgements

This work was done under the auspices of the Key Laboratory of Pure Mathematics and Combinatorics of the Ministry of Education of China, and the 973 Project on Mathematical Mechanization of the Ministry of Science and Technology of China. We thank Professor Alain Lascoux for his lecture notes on Schubert polynomials given at the Nankai University, and for many generous suggestions. We also thank Professor Axel Kohnert for paying attention to the work of Lenart [23] and Winkel [27, 28]. Special thanks are given to Professor William Y. C. Chen for his giving us a chance to study symmetric functions and Schubert polynomials.

References

[1] N. Bergeron and S. Billey, "RC-graphs and Schubert polynomials", *Experimental Math.* **2** (1993), 257-269.

[2] N. Bergeron, "A combinatorial construction of the Schubert polynomials", *J. Comb. Theory A* **60** (1992), 168-182.

[3] I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand, "Schubert celles and cohomology of the spaces G/P", *Russian Math. Surveys* **28** (1973), 1-26.

[4] S.Billey, W.Jockusch and R.Stanley, "Some combinatorial properties of Schubert polynomials", *J. Alg. Comb.* **2** (1993), 345-374.

[5] W.Y.C. Chen, Bingqin Li, and J.D. Louck, "The flagged double Schur function", *J. Alg. Comb.* **15** (2002), 7-26.

[6] W.Y.C. Chen, G.G. Yan and A.L.B. Yang, "The skew Schubert polynomials", preprint.

[7] M. Demazure, "Désingularisation des variétés de Schubert généralisées", *Ann. Sci. École Norm. Sup. (4)* **7** (1974), 53-58.

[8] M. Demazure, "Une formule des caractères", *Bull. Sc. Math.* **98** (1974), 163-172.

[9] P. Edelman and C. Greene, "Balanced tableaux", *Adv. Math.* **63** (1987), 42-99.

[10] S. Fomin and A. N. Kirillov, "Yang-Baxter equation, symmetric functions, and Schubert polynomials", submitted to the *Proceedings of the Conference on Power Series and Algebraic Combinatorics, Firenze, 1993.*

[11] S. Fomin and R. Stanley, "Schubert polynomials and the NilCoxeter algebra", *Adv. Math.* **103** (1994), 196-207.

[12] I. Gessel and G. Viennot, "Binomial determinants, paths, and hook length formulae", *Adv. Math.* **58** (1985), 300-321.

[13] I. Gessel and G. Viennot, "Determinants, paths, and plane partitions", preprint.

[14] A. Kohnert, *Thesis*, Bayreuth (1990).

[15] A. Kohnert, "Weintrauben, Polynome, Tableaux", *Bayreuth. Math. Schrift.* **38** (1991), 1-97.

[16] A. Kohnert, “Multiplication of a Schubert polynomial by a Schur polynomial”, *Ann. Comb.* **1** (1997), 367-375.

[17] A. Lascoux and M. P. Schützenberger, “Polynômes de Schubert”, *C.R. Acad. Sci. Paris* **294** (1982) 447-450.

[18] A. Lascoux, “Classes de Chern des variétés de drapeaux”, *Comptes Rendus* **295** (1982) 393-398.

[19] A. Lascoux and M. P. Schützenberger, “Schubert polynomials and the Littlewood-Richardson rule”, *Let. Math. Phys.* **10** (1985) 111-124.

[20] A. Lascoux and M. P. Schützenberger, “Tableaux and non-commutative Schubert polynomials”, *Funct. Anal. Appl.* **23** (1989), 63-64.

[21] A. Lascoux and M. P. Schützenberger, “Keys and standard bases.” in “Tableaux and Invariant Theory”, IMA Volumes in Math and its Applications (D. Stanton. Ed.). Vol 19. pp. 125-144, Southend on Sea. UK. 1990.

[22] A. Lascoux, “Double crystal graphs”, in Studies in Memory of Issai Schur, A. Joseph, A. Melnikov, and R. Rentschler, Ed., Progress in Mathematics 210, Birkhäuser, 2003, 95-114.

[23] C. Lenart, “A unified approach to combinatorial formulas for Schubert polynomials”, *J. Alg. Comb.*, to appear. preprint.

[24] I. G. Macdonald, “Notes on Schubert polynomials”, *Publications du LACIM 6*, Université du Québec, à Montréal(1991).

[25] V. Reiner and M. Shimozono, “Key polynomials and a flagged Littlewood-Richardson rule”, *J. Combin. Theory Ser. A* **70** (1995), 107-143.

[26] M. L. Wachs, “Flagged Schur functions, Schubert polynomials, and Symmetrizing Operators”, *J. Combin. Theory Ser. A* **40** (1985), 276-289.

[27] R. Winkel, “Diagram rules for the generation of Schubert polynomials”, *J.* Combin. Theory Ser. A **86** (1999), 14-48.

[28] R. Winkel, “A short proof of Kohnert’s algorithm for the generation of Schubert polynomials”, To appear in *Sem. Loth. Comb.*

[29] S. Veigneau, *ACE, an Algebraic Combinatorics Environment for the computer algebra system MAPLE*, *User’s Reference Manual*, *Version 3.0*, IGM **98–11**, Université de Marne-la-Vallée, 1998.