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Guaranteed Tensor Recovery fused Low-rankness and Smoothness

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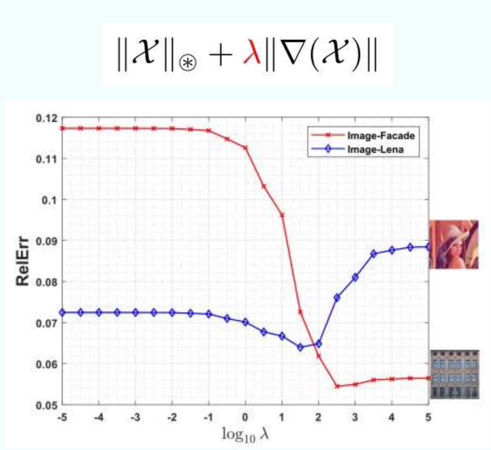
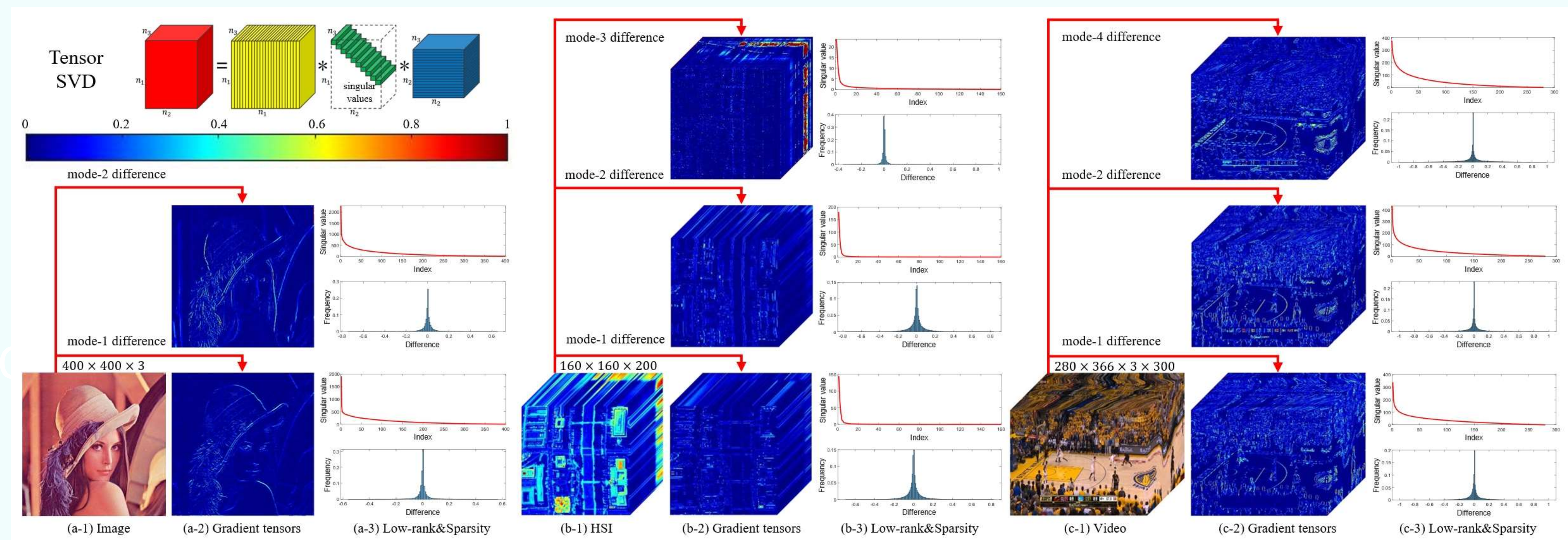
Motivations

Tensor recovery is a typical inverse problem that estimates an underlying tensor from its compressed/incompleted/corrupted observations using regularization method. Low-rankness (**L**) and smoothness (**S**) are the most two important structural information of various visual tensor data. However, almost all existing works encode these two priors separately with "**L+S**" regularized models, which is not only lack of theories but also affected heavily by the trade-off parameter.

Literature	Problem	Model	Theory
Ji et al. [23]	TC	MF + STV	✘
Li et al. [24]	TC	SNN + TV-1	✘
Ko et al. [30]	TC	TT + TV-2	✘
Yokota et al. [26]	TC	CP + TV-1/TV-2	✘
He et al. [25]	TRPCA	MF + STV	✘
Wang et al. [27]	TRPCA	SNN + SSTV	✘
Chen et al. [28]	TRPCA	TNN + HTV	✘
Zhang et al. [29]	TRPCA	NLTRD + SSTV	✘
This work	TC&TRPCA	t-CTV	✔



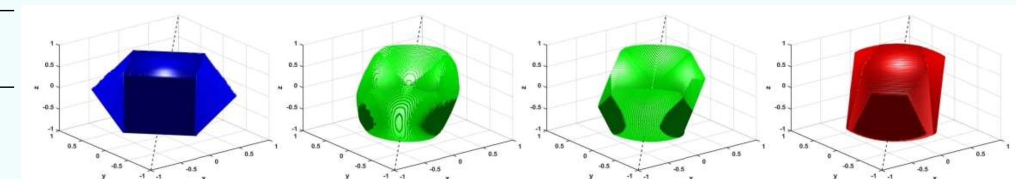
tensor Correlated Total Variation (t-CTV)



Definition

$$\|\mathcal{X}\|_{\text{t-CTV}} \triangleq \frac{1}{\gamma}\sum_{k\in\Gamma}\|\nabla_k(\mathcal{X})\|_{\circledast}$$

t-CTV can encode both low-rankness and smoothness	
Low-rankness	$\text{rank}(\mathcal{X})-1 \leq \text{rank}(\nabla(\mathcal{X})) \leq \text{rank}(\mathcal{X})$
Smoothness	$\|\nabla(\mathcal{X})\|_2 \lesssim \|\nabla(\mathcal{X})\|_{\circledast} \lesssim \|\nabla(\mathcal{X})\|_1$



(a) TNN (b) TV-1 (c) TV-2 (d) t-CTV

Theoretical Guarantees

Tensor Completion (TC)

$$\min_{\mathcal{X}} \|\mathcal{X}\|_{\text{t-CTV}}, \quad \text{s.t.} \quad \mathcal{P}_\Omega(\mathcal{X}) = \mathcal{P}_\Omega(\mathcal{X}_0) \qquad (1)$$

Theorem (Exact Recovery for TC Problem)

Suppose that $\mathcal{X}_0$ obeys the gradient tensor incoherence condition and $\Omega \sim \text{Ber}(p)$. Then, there exist universal constants $c_0, c_1, c_2 > 0$ such that $\mathcal{X}_0$ is the unique solution of model (1) with probability at least $1 - c_1\gamma(n_{(1)}n_3\cdots n_d)^{-c_2}$, provided that

$$p \geq c_0\mu R(\log(n_{(1)}\ell))^2/n_{(2)}\ell,$$

where ℓ is the specific scale factor given in t-SVD framework, $n_{(1)} := \max\{n_1, n_2\}$ and $n_{(2)} := \min\{n_1, n_2\}$.

Proposition (Worst-case Sampling Complexity)

Let $\boldsymbol{\mathcal{X}}_0 \in \mathbb{R}^{n_1\times\cdots\times n_d}$ with multi-structural prior simultaneously. Consider the following general TC model

$$\min_{\mathcal{T}} f(\boldsymbol{\mathcal{X}}) := \sum w_i\|\boldsymbol{\mathcal{X}}\|_{(i)} \ \ s.t. \ \ \mathcal{P}_\Omega(\boldsymbol{\mathcal{X}}) = \mathcal{P}_\Omega(\boldsymbol{\mathcal{X}}_0),$$

where $\|\cdot\|_{(i)}$ denotes a regularization norm (such as TNN, TV, and t-CTV norm) modeling certain prior with Lipschitz constant L_i, $w_i > 0$ is the wight parameter. Suppose $\Omega \sim \text{Ber}(p)$ and m is the number of sampling entries. Then, there exist constant $c_0, c_1 > 0$ such that $\boldsymbol{\mathcal{X}}_0$ **is not** *the unique solution of model (1) with probability at least $1 - \exp(-\frac{c_1 m}{n_1\cdots n_d\|\bar{\boldsymbol{\mathcal{X}}}_0\|_\infty^2})$, provided that $m \leq m_{\text{low}} := c_0\kappa_{\min}^2 n_1\cdots n_d$, where $\kappa_{\min} = \min\{\kappa_i = \|\bar{\boldsymbol{\mathcal{X}}}_0\|_{(i)}/L_i\}$ and $\bar{\boldsymbol{\mathcal{X}}}_0 = \boldsymbol{\mathcal{X}}_0/\|\boldsymbol{\mathcal{X}}_0\|_{\text{F}}$.*

Tensor Robust PCA (TRPCA)

$$\min_{\mathcal{X},\mathcal{S}} \|\mathcal{X}\|_{\text{t-CTV}} + \tau\|\mathcal{S}\|_1, \quad \text{s.t.} \quad \mathcal{Y} = \mathcal{X} + \mathcal{S} \qquad (2)$$

Theorem (Exact Recovery for TRPCA Problem)

Suppose that $\mathcal{X}_0$ obeys the gradient tensor incoherence condition and $\mathcal{S}_0$'s support set, denoted as Ω_0, is uniformly distributed among all sets of cardinality m. Then, there exist universal constants $c_1, c_2 > 0$ such that $(\mathcal{X}_0, \mathcal{S}_0)$ is the unique solution of model (2) when $\tau = 1/\sqrt{n_{(1)}\ell}$ with probability at least $1 - c_1\gamma(n_{(1)}n_3\cdots n_d)^{-c_2}$, provided that

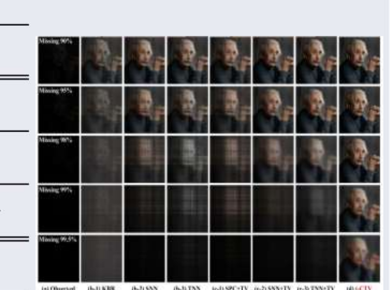
$$\text{rank}_{\text{t-SVD}}(\mathcal{X}_0) \leq \frac{\rho_r n_{(2)}\ell}{\mu\log^2(n_{(1)}\ell)} \quad \text{and} \quad m \leq \rho_s n_1\cdots n_d,$$

where $\rho_r, \rho_s > 0$ are some universal constants.

Theorem (Smaller Sampling Lower Bound)

*For order-d tensor $\mathcal{X}_0 \in \mathbb{R}^{N\times\cdots\times N}$ with low-rankness (**L**) and smoothness (**S**) priors structures simultaneously, denote its t-SVD rank as R and gradient tensor $\mathcal{G}_k$'s sparsity (number of nonzero entries) as S_k, and $S = \min_{k\in\Gamma}\{S_k\}$. Then, the corresponding lower bounds of the following **L** and/or **S** models satisfy:*

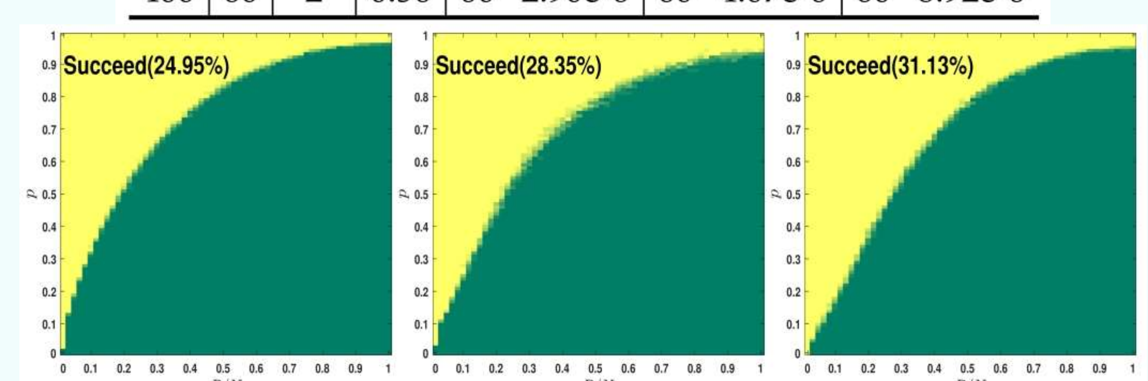
Model	$f(\mathcal{X}) =$	$m_{\text{low}} \lesssim$
L	$\|\mathcal{X}\|_{\circledast}$	$N^d\cdot\frac{R}{N}$
S	$\|\mathcal{X}\|_{\text{TV}}$	$N^d\cdot\frac{S}{N^d}$
L+S	$\|\mathcal{X}\|_{\circledast} + \lambda\|\mathcal{X}\|_{\text{TV}}$	$N^d\cdot\min\{\frac{R}{N}, \frac{S}{N^d}\}$
t-CTV	$\|\mathcal{X}\|_{\text{t-CTV}}$	$N^d\cdot\frac{R}{N}\cdot\frac{S}{N^d}$



Simulation Studies

Tensor Completion

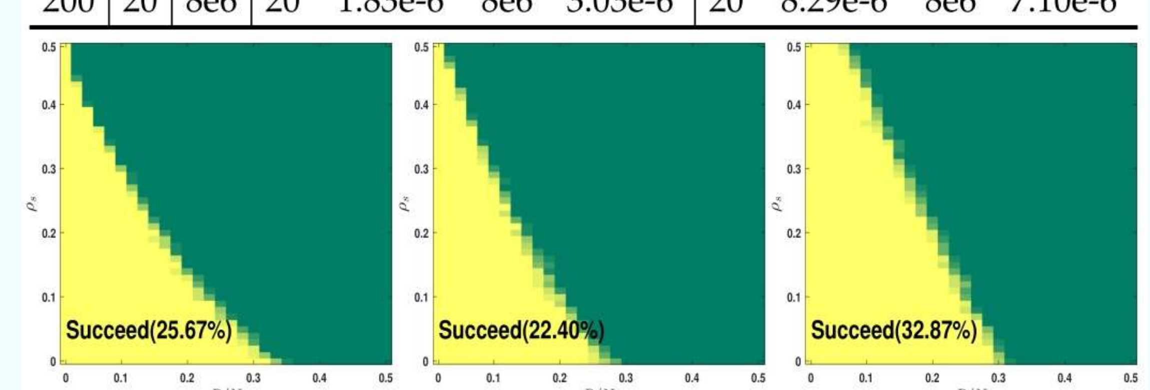
N	R	$\frac{m}{d_r}$	p	DFT		DCT		ROT	
				$\hat{R}$	RelErr	$\hat{R}$	RelErr	$\hat{R}$	RelErr
100	5	4	0.39	5	4.54e-7	5	6.01e-6	5	4.73e-7
200	20	3	0.57	20	1.83e-7	20	3.30e-6	20	1.59e-7
400	60	2	0.56	60	2.90e-6	60	4.07e-6	60	8.92e-6



(a) TNN (b) TNN+TV (c) t-CTV

Tensor Robust PCA

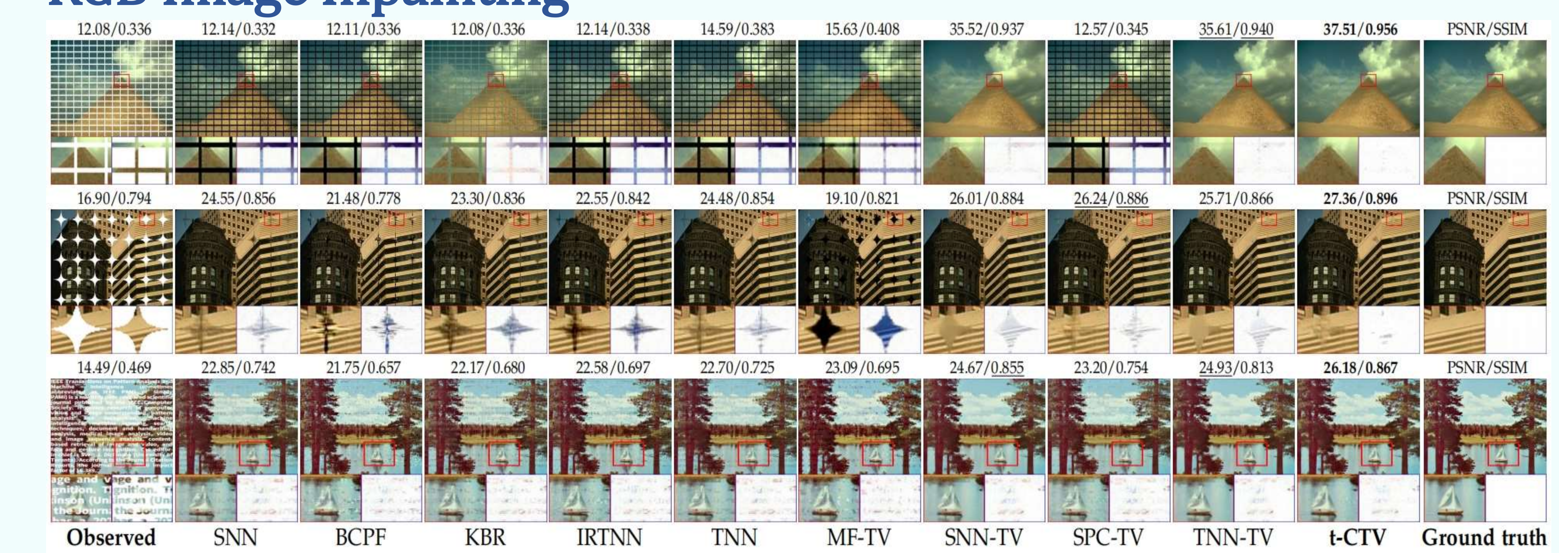
N	R	m	DFT				DCT			
			$\hat{R}$	RelErr$\mathcal{T}$	$\hat{m}$	RelErr$\mathcal{E}$	$\hat{R}$	RelErr$\mathcal{T}$	$\hat{m}$	RelErr$\mathcal{E}$
100	5	2e6	5	1.85e-7	2e6	2.46e-7	5	9.97e-6	2e6	1.14e-7
100	10	2e6	10	2.18e-6	2e6	3.96e-6	10	4.38e-7	2e6	7.27e-7
200	10	8e6	10	9.32e-7	8e6	9.84e-7	10	5.03e-7	8e6	3.64e-6
200	20	8e6	20	1.83e-6	8e6	3.03e-6	20	8.29e-6	8e6	7.10e-6



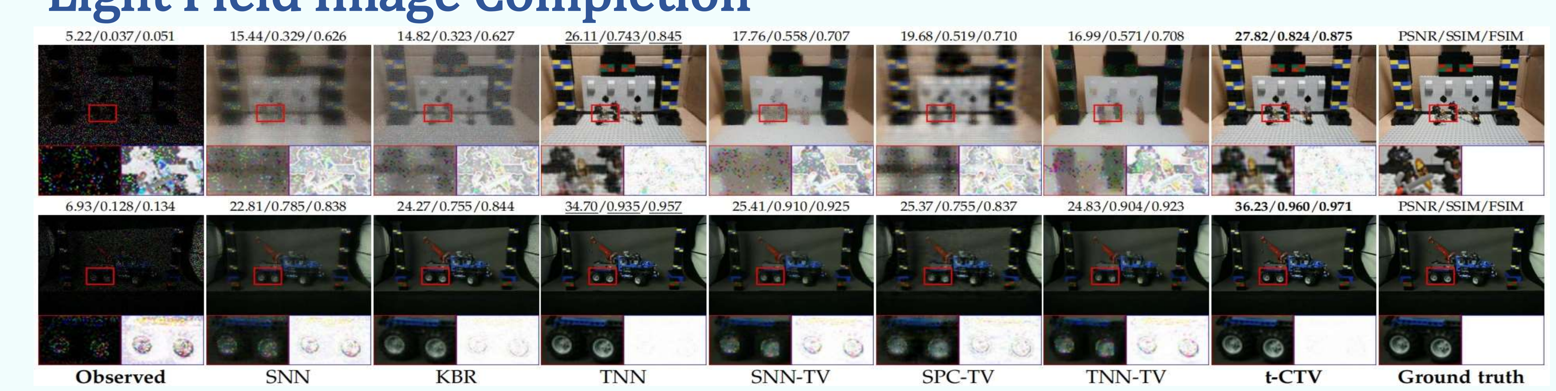
(a) TNN (b) TNN+TV (c) t-CTV

Real Applications

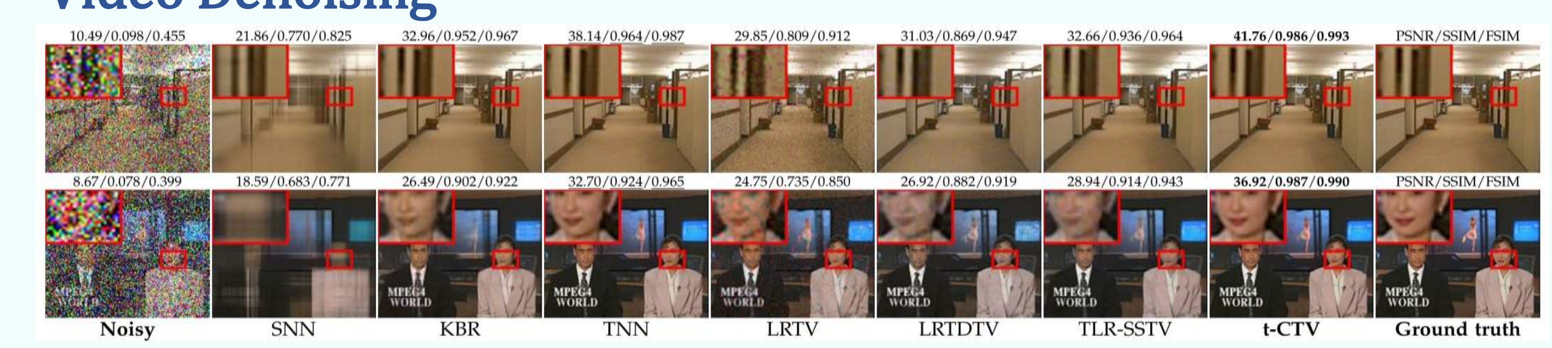
- RGB Image Inpainting



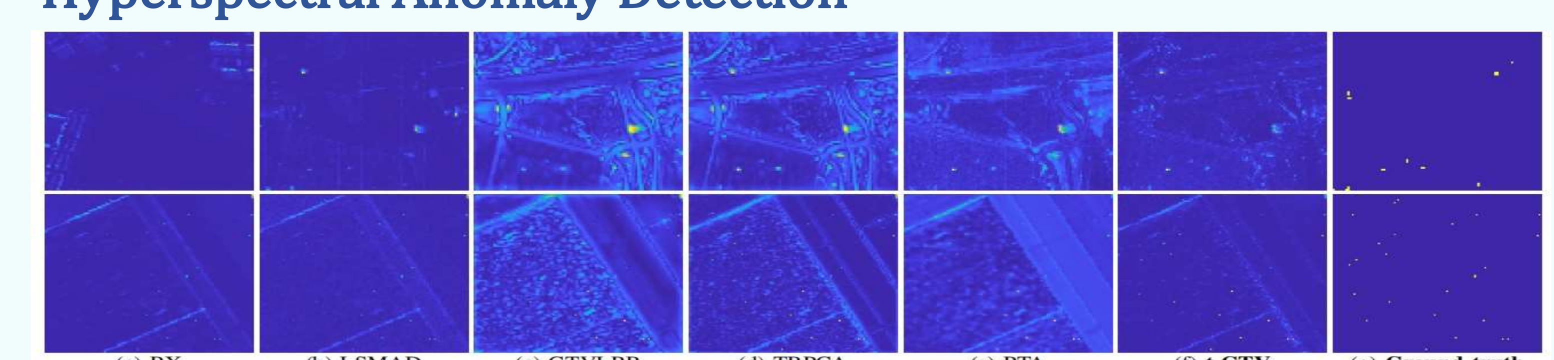
- Light Field Image Completion



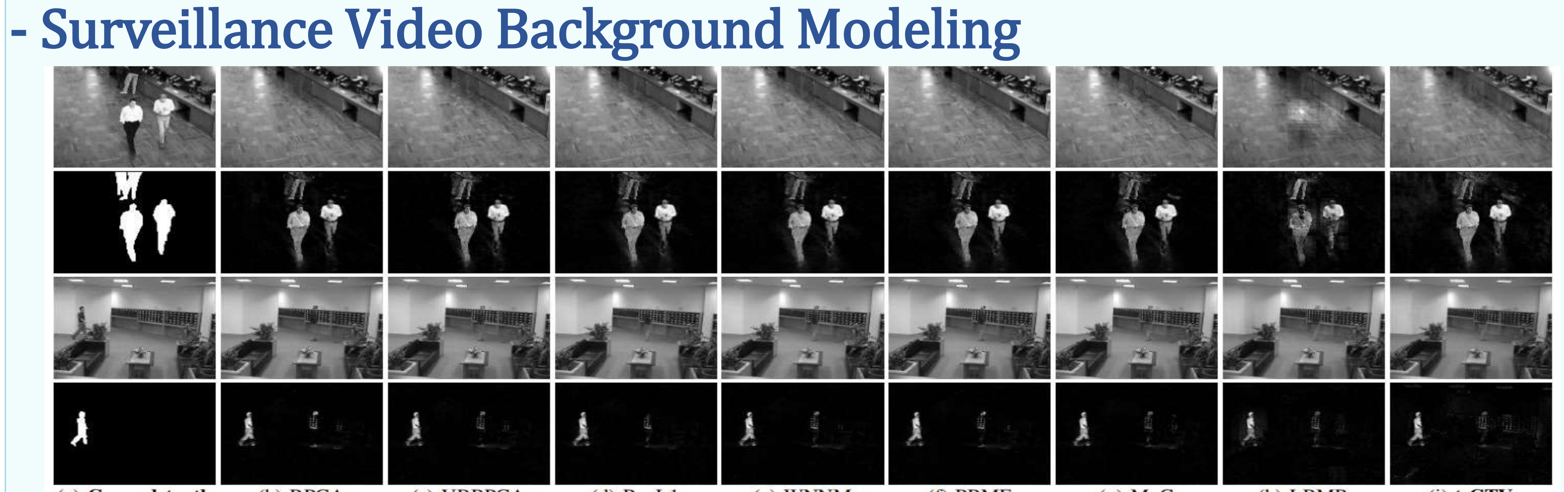
- Video Denoising



- Hyperspectral Anomaly Detection



- Surveillance Video Background Modeling



*It also performs well for CT/MRI, hyperspectral video, traffic flow data, etc

Conclusion

t-CTV is a simple, powerful and user-friendly regularizer with theoretical guarantees for low-rank&smooth tensor!