Computation & optimization for Lasso - part 2

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Overview

Coordinate Descent Algorithm

What is the Coordinate Descent (CD) Algorithm?

$$\beta_k^{t+1} = \underset{\beta_k}{\operatorname{argmin}}\ f(\beta_1^t, \beta_2^t, \ldots, \beta_k, \beta_{k+1}^t, \ldots, \beta_p^t)$$

and $\beta_j^{t+1} = \beta_j^t$ for $j \neq k$

- An iterative algorithm that updates from β^t to β^{t+1} by choosing a single coordinate, and minimizing over this coordinate.

Separability Condition

Motivation

Does CD procedure converge to the global minimum of a convex function?

- **Sufficient Condition:** the function is continuously differentiable and strictly convex in each coordinate.

$\Rightarrow$ restrictive

Separability Condition

Suppose the cost function f has the additive decomposition:

$$f(\beta_1, ..., \beta_p) = g(\beta_1, ..., \beta_p) + \sum_{j=1}^{p} h_j(\beta_j)$$

where $g : \mathbb{R}^p \to \mathbb{R}$ is differentiable and convex, and the univariate functions $h_j : \mathbb{R} \to \mathbb{R}$ are convex.

- Lasso: $g(\beta) = \frac{1}{2N}\|\mathbf{y} - \mathbf{X}\beta\|_2^2$ and $h_j(\beta_j) = \lambda|\beta_j|$ satisfies the condition

Separability Condition: Example

An Example of failure of Coordinate Descent

$$\underset{\beta \in \mathbb{R}^p}{\operatorname{argmin}} \ \frac{1}{2N}\|\mathbf{y} - \mathbf{X}\beta\|_2^2 + \ \lambda_1 \sum_{j=1}^{p} |\beta_j| + \lambda_2 \sum_{j=2}^{p} |\beta_j - \beta_{j-1}|$$

- $h(\beta)$ is not separable
- Fused Lasso: coordinate descent procedure is not guaranteed to find the global minimum

Separability Condition: Example

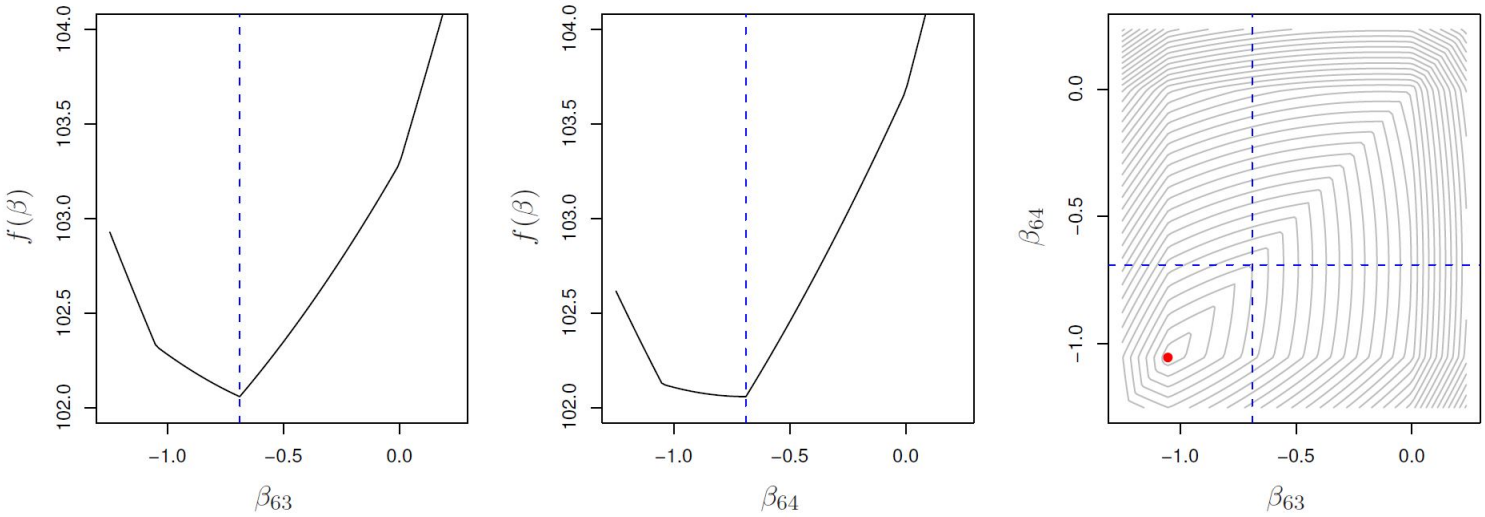


Figure: Fused Lasso: CD fails to reach the global minimum from Trevor Hastie, Robert Tibshirani, and Martin Wainwright (2015), Statistical learning with sparsity: the Lasso and generalizations, p111.

Lasso & Coordinate Descent

Optimality Condition:

$$-\frac{1}{2N}\sum_{i=1}^{N}(y_i-\beta_0-\sum_{k=1}^{p}x_{ik}\beta_k)x_{ij}+\lambda s_j=0$$

where $s_j \in sign(\beta_j)$ for $j=1,2,...,p$

- Define the **partial residual**: $r_i^{(j)}=y_i-\sum_{k\neq j}x_{ik}\hat{\beta}_k$
- Then the solution for $\hat{\beta}_j$ satisfies:

$$\hat{\beta}_j=\frac{S_\lambda(\frac{1}{N}\sum_{i=1}^{N}r_i^{(j)}x_{ij})}{\frac{1}{N}\sum_{i=1}^{N}x_{ij}^2}$$

where $S_\lambda(\theta)=sign(\theta)(|\theta|-\lambda)_+$

Lasso & Coordinate Descent

- Illustration of Coordinate Descent in R

Strategies to make the operation efficient:

Naive Updating

$$r_i^{(j)} = y_i - \sum_{k \neq j} x_{ik}\hat{\beta}_k = r_i + x_{ij}\hat{\beta}_j$$

$$\frac{1}{N}\sum_{i=1}^{N} x_{ij} r_i^{(j)} = \frac{1}{N}\sum_{i=1}^{N} x_{ij} r_i + \hat{\beta}_j$$

Covariance Updating

$$\sum_{i=1}^{N} x_{ij} r_i = \langle x_j, y \rangle - \sum_k \langle x_j, x_k \rangle \beta_{\hat{k}}$$

Warm Starts: For a decreasing sequence of values $\{\lambda_0^L\}$, $\hat{\beta}(\lambda_l)$ is typically a very good warm start for the solution $\hat{\beta}(\lambda_{l+1})$. We set $\lambda_0 = \frac{1}{N} max|\langle x_j, y \rangle|$ and $\lambda_L \approx 0$.

Lasso & Coordinate Descent

Active-set Convergence: Define the active set A and iterate the algorithm using only variables in A.

Strong-set Convergence: Define the strong set S and iterate the algorithm using only variables in S.

Sparsity: Sparsity of the design matrix X makes the operation of inner product efficient.

Details in Trevor Hastie, Robert Tibshirani, and Martin Wainwright (2015), Statistical learning with sparsity: the Lasso and generalizations, p113-114.

Elastic Net & Coordinate Descent

$$minimize_{\beta_0,\beta} \frac{1}{2}\sum_{i=1}^{N}(y_i-\beta_0-x_i^T\beta)^2+\lambda\left[\frac{1}{2}(1-\alpha)||\beta||_2^2+\alpha||\beta||_1\right]$$

- Combination of L1 and L2 penalty
- Satisfy the separability condition
- The solution satisfies:

$$\hat{\beta}_j = \frac{S_{\alpha\lambda}(\frac{1}{N}\sum_{i=1}^{N} r_i^{(j)} x_{ij})}{\frac{1}{N}\sum_{i=1}^{N} x_{ij}^2+(1-\alpha)\lambda}$$

Logistic Regression & Coordinate Descent

Background

- **Class Label G**: Take values 1 and -1

  Denote $p(x_i; \beta_0, \beta) = Pr(G = 1|x_i)$

  Define log odds: $\log \frac{Pr(G=-1|x)}{Pr(G=1|x)} = \beta_0 + x^T\beta$

- **Maximize penalized log-likelihood:**

  $$\frac{1}{N}\sum_{i=1}^{N}\{I(g_i = 1) \cdot \log(p(x_i; \beta_0, \beta)) + I(g_i = -1) \cdot \log(1 - p(x_i; \beta_0, \beta))\} - \lambda||\beta||_1$$

  Denote $y_i = I(g_i = -1)$

Explicit form of log likelihood (without penalty):

$$L(\beta_0, \beta) = \frac{1}{N}\sum_{i=1}^{N}\left[y_i \cdot (\beta_0 + x_i^T\beta) - \log(1 + e^{\beta_0 + x_i^T\beta})\right]$$

Logistic Regression & Coordinate Descent

Background

- Form a **quadratic objective function** using Taylor expansion about current estimates $(\tilde{\beta}_0, \tilde{\beta})$: Idea of Newton method, Iterated Weighted Least Square problem

$$L_Q(\beta_0, \beta) = -\frac{1}{2N} \sum_{i=1}^{N} w_i (z_i - \beta_0 - x_i^T \beta)^2 + C(\tilde{\beta}_0, \tilde{\beta})$$

- Use Coordinate Descent to solve the problem

$$minimize_{(\beta_0, \beta)} \{ - L_Q(\beta_0, \beta) + \lambda ||\beta||_1 \}$$

Logistic Regression & Coordinate Descent

Algorithm

OUTER LOOP: Decrement λ

MIDDLE LOOP: Update the **quadratic approximation** L_Q using the current parameters $(\tilde{\beta}_0, \tilde{\beta})$

INNER LOOP: Run the coordinate descent algorithm on the penalized weighted least squares problem

Least Angle Regression

Introduction

- Relates to Forward Selection method
- Relates to Lasso method
- Able to deliver the entire solution path of the Lasso problem with squared-error loss as a function of the regularization parameter λ

Least Angle Regression: Algorithm

- Start with all coefficients β_j equal to zero
- Find the predictor X_j **most correlated** with y
- **Increase** the coefficient β_j in the direction of the **sign** of its **correlation** with y
- Take **residuals** $r = y - \hat{y}$ along the way; Stop when some other predictor X_k has **as much correlation** with r as X_j has
- **Increase** β_j, β_k in their **joint least squares direction**, until some other predictor has as much correlation with the residual r

Continue until: all predictors are in the model

Least Angle Regression: Algorithm

Algorithm 5.1 LEAST ANGLE REGRESSION.

1. Standardize the predictors to have mean zero and unit ℓ_2 norm. Start with the residual $\boldsymbol{r}_0 = \mathbf{y} - \bar{\mathbf{y}}$, $\beta^0 = (\beta_1, \beta_2, \ldots, \beta_p) = \mathbf{0}$.
2. Find the predictor $\mathbf{x}_j$ most correlated with $\boldsymbol{r}_0$; i.e., with largest value for $|\langle \mathbf{x}_j, \boldsymbol{r}_0 \rangle|$. Call this value λ_0, define the active set $\mathcal{A} = \{j\}$, and $\mathbf{X}_\mathcal{A}$, the matrix consisting of this single variable.
3. For $k = 1, 2, \ldots, K = \min(N-1, p)$ do:
   (a) Define the least-squares direction $\delta = \frac{1}{\lambda_{k-1}}(\mathbf{X}_\mathcal{A}^T \mathbf{X}_\mathcal{A})^{-1} \mathbf{X}_\mathcal{A}^T \boldsymbol{r}_{k-1}$, and define the p-vector Δ such that $\Delta_\mathcal{A} = \delta$, and the remaining elements are zero.
   (b) Move the coefficients β from β^{k-1} in the direction Δ toward their least-squares solution on $\mathbf{X}_\mathcal{A}$: $\beta(\lambda) = \beta^{k-1} + (\lambda_{k-1} - \lambda)\Delta$ for $0 < \lambda \leq \lambda_{k-1}$, keeping track of the evolving residuals $\boldsymbol{r}(\lambda) = \mathbf{y} - \mathbf{X}\beta(\lambda) = \boldsymbol{r}_{k-1} - (\lambda_{k-1} - \lambda)\mathbf{X}\Delta$.
   (c) Keeping track of $|\langle \mathbf{x}_\ell, \boldsymbol{r}(\lambda) \rangle|$ for $\ell \notin \mathcal{A}$, identify the largest value of λ at which a variable "catches up" with the active set; if the variable has index j, that means $|\langle \mathbf{x}_j, \boldsymbol{r}(\lambda) \rangle| = \lambda$. This defines the next "knot" λ_k.
   (d) Set $\mathcal{A} = \mathcal{A} \cup \{j\}$, $\beta^k = \beta(\lambda_k) = \beta^{k-1} + (\lambda_{k-1} - \lambda_k)\Delta$, and $\boldsymbol{r}_k = \mathbf{y} - \mathbf{X}\beta^k$.
4. Return the sequence $\{\lambda_k, \beta^k\}_0^K$.

Figure: From Trevor Hastie, Robert Tibshirani, and Martin Wainwright (2015), Statistical learning with sparsity: the Lasso and generalizations, p119, Algorithm 5.1.

Least Angle Regression: Geometric Representation

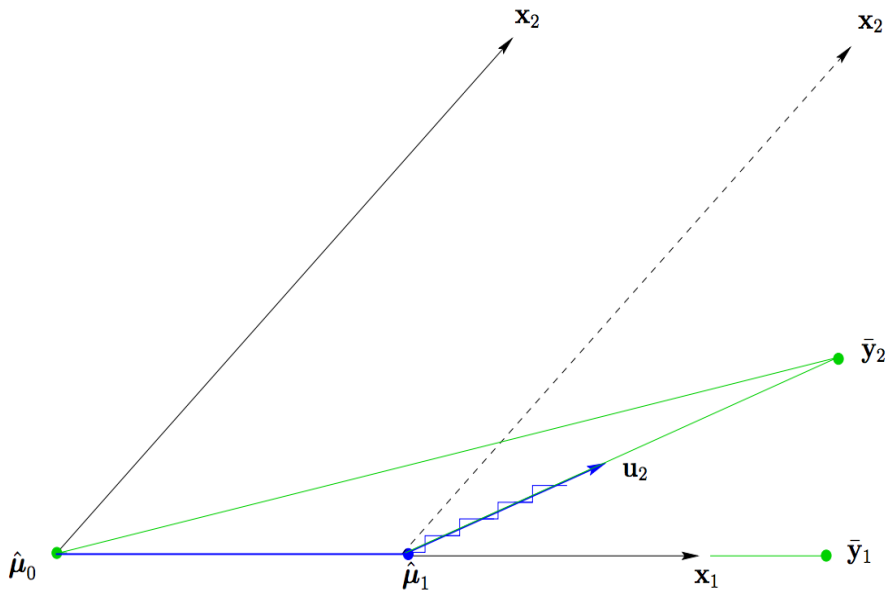


Figure: From Bradley Efron, Trevor Hastie, Iain Johnstone and Robert Tibshirani, (2004), Least Angle Regression, The Annals of Statistics, p412, Figure 2.

Connection between LAR and Lasso

LAR

Let $\mathbb{A}$ be the active set of variables for the LAR algorithm.
$\mathbf{x}_j^T(\mathbf{y} - \mathbf{X}\beta(\lambda)) = \lambda \cdot s_j,\ \forall j \in \mathbb{A}$ where s_j is the sign of inner product λ.

LASSO

Let $\mathbb{A}$ be the active set of variables in the solution for a given value of λ.
$R(\beta) = \frac{1}{2}||\mathbf{y} - \mathbf{X}\beta||_2^2 + \lambda||\beta||_1$

For differentiable $R(\beta)$, the stationary conditions give:
$\mathbf{x}_j^T(\mathbf{y} - \mathbf{X}\beta) = \lambda \cdot sign(\beta_j), \forall j \in \mathbb{A}$

If sign (β_j) matches s_j, the coefficient would be identical.

Connection between LAR and Lasso

- R Example
- LAR algorithm explains that the coefficient paths for the Lasso are **piecewise linear**
- Coefficient paths differ if $\text{sign}(\beta_j)$ is different from s_j
- **Modification** of LAR for computing Lasso solution:
  If a **nonzero** coefficient **crosses zero** before the next variable enters, **drop** it from $\mathbb{A}$ and recompute the current joint least squares direction.

Connection between LAR and Lasso

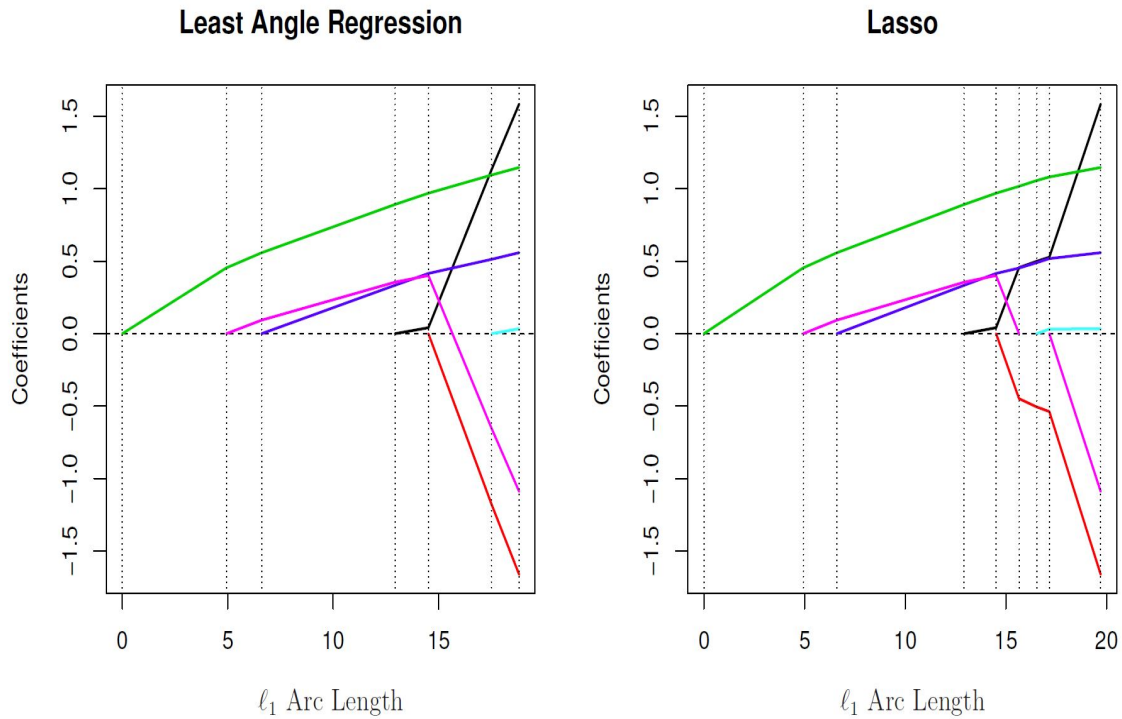


Figure: Cases where signs of λ and β disagree, from Trevor Hastie, Robert Tibshirani, and Martin Wainwright (2015), Statistical learning with sparsity: the Lasso and generalizations, p120, Figure 5.9.

Algorithm Performance

Simulation: Comparison of computation efficiency between CD and LAR

Set Up: [1]

- Generate Gaussian data with N observations and p predictors, with each pair of predictors X_j, X_k having the same population correlation ρ.
- Try different combination of N and p; Range ρ from 0 to 0.95.
  $Y = \sum_{j=1}^{p} X_j \beta_j + kZ$ where
  $\beta_j = (-1)^j exp(\frac{-2(j-1)}{20}), Z \sim N(0,1)$ and k is a constant.

[1] Jerome Friedman, Trevor Hastie, and Rob Tibshirani (2010), Regularization Paths for Generalized Linear Models via Coordinate Descent, Journal of Statistical Software, p12.

Algorithm Performance

	Linear Regression — Dense Features					
	Correlation					
	0	0.1	0.2	0.5	0.9	0.95
	$N = 1000, p = 100$					
glmnet-naive	0.05	0.06	0.06	0.09	0.08	0.07
glmnet-cov	0.02	0.02	0.02	0.02	0.02	0.02
lars	0.11	0.11	0.11	0.11	0.11	0.11
	$N = 5000, p = 100$					
glmnet-naive	0.24	0.25	0.26	0.34	0.32	0.31
glmnet-cov	0.05	0.05	0.05	0.05	0.05	0.05
lars	0.29	0.29	0.29	0.30	0.29	0.29
	$N = 100, p = 1000$					
glmnet-naive	0.04	0.05	0.04	0.05	0.04	0.03
glmnet-cov	0.07	0.08	0.07	0.08	0.04	0.03
lars	0.73	0.72	0.68	0.71	0.71	0.67
	$N = 100, p = 5000$					
glmnet-naive	0.20	0.18	0.21	0.23	0.21	0.14
glmnet-cov	0.46	0.42	0.51	0.48	0.25	0.10
lars	3.73	3.53	3.59	3.47	3.90	3.52
	$N = 100, p = 20000$					
glmnet-naive	1.00	0.99	1.06	1.29	1.17	0.97
glmnet-cov	1.86	2.26	2.34	2.59	1.24	0.79
lars	18.30	17.90	16.90	18.03	17.91	16.39
	$N = 100, p = 50000$					

Figure: Comparison of computing time, from Jerome Friedman, Trevor Hastie, and Rob Tibshirani (2010), Regularization Paths for Generalized Linear Models via Coordinate Descent, Journal of Statistical Software, Appendix 1.

Algorithm Performance

Simulation: Comparison of computation efficiency between Coordinate Descent, Proximal Gradient Descent and Nesterov Method

Set Up: [2]

- Generated an $N \times p$ predictor matrix X with standard Gaussian entries and pairwise correlation 0 or 0.5 between the features.
- $|\beta_j| = exp\big[-0.5(u(j-1))^2\big]$ and $u = \sqrt{\frac{\pi}{20}}$ and alternating signs -1,+1,-1...

[2] Trevor Hastie, Robert Tibshirani, and Martin Wainwright (2015), Statistical learning with sparsity: the Lasso and generalizations, p117.

Algorithm Performance

Table 5.1 *Lasso for linear regression: Average (standard error) of CPU times over ten realizations, for coordinate descent, generalized gradient, and Nesterov's momentum methods. In each case, time shown is the total time over a path of 20 λ values.*

	$N = 10000,\ p = 100$		$N = 200,\ p = 10000$	
Correlation	0	0.5	0	0.5
Coordinate descent	0.110 (0.001)	0.127 (0.002)	0.298 (0.003)	0.513 (0.014)
Proximal gradient	0.218 (0.008)	0.671 (0.007)	1.207 (0.026)	2.912 (0.167)
Nesterov	0.251 (0.007)	0.604 (0.011)	1.555 (0.049)	2.914 (0.119)

Figure: Comparison of computing efficiency between 3 methods, from Trevor Hastie, Robert Tibshirani, and Martin Wainwright (2015), Statistical learning with sparsity: the Lasso and generalizations, p117, Table 5.1.

Recall: Duality in optimization

Primal	
Optimize	$\min f(x)$
Constraints	$g_i(x) \leq 0, h_j(x) = 0, x \in X$
Function	$L(x, \lambda, \mu) := f(x) + \sum_i \lambda_i g_i(x) + \sum_j \mu_j h_j(x)$
Dual	
Function	$q(\lambda, \mu) = \inf_{x \in X} L(x, \lambda, \mu)$
Constraints	$\lambda \geq 0$
Optimize	$\max_{\lambda \geq 0, \mu} q(\lambda, \mu)$

Why though? - **Dual problem is always convex!**

Alternating Direction Method of Multipliers (ADMM)

Problem - decomposable !

$$\underset{\beta\in\mathbb{R}^m,\theta\in\mathbb{R}^n}{\text{minimize}}\, f(\beta)+g(\theta) \quad \text{subject to } \mathbf{A}\beta+\mathbf{B}\theta-c=0$$

Lagrangian - decomposable !

$$f(\beta)+g(\theta)+\langle\mu,\mathbf{A}\beta+\mathbf{B}\theta-c\rangle$$

Augmented Lagrangian - NOT decomposable !

$$L_\rho(\beta,\theta,\mu):=f(\beta)+g(\theta)+\langle\mu,\mathbf{A}\beta+\mathbf{B}\theta-c\rangle+\frac{\rho}{2}||\mathbf{A}\beta+\mathbf{B}\theta-c||_2^2$$

Dual Variable Update

Alternating Direction Method of Multipliers

$$\beta^{t+1} = \arg\min_{\beta \in \mathbb{R}^m} L_\rho(\beta, \theta^t, \mu^t)$$

$$\theta^{t+1} = \arg\min_{\theta \in \mathbb{R}^m} L_\rho(\beta^{t+1}, \theta, \mu^t)$$

$$\mu^{t+1} = \mu^t + \rho(\mathbf{A}\beta^{t+1} + \mathbf{B}\theta^{t+1} - c)$$

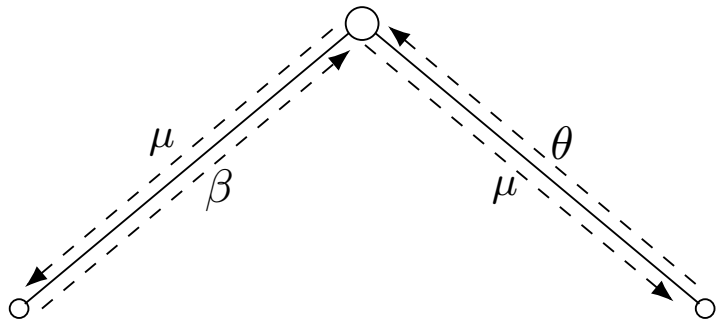


Figure: Own illustration of the dual ascent step in the ADMM algorithm utilising dual decomposition based on Geoff Gordon and Ryan Tibshirani (2012), Uses of Duality, `https://www.cs.cmu.edu/~ggordon/10725-F12/slides/18-dual-uses.pdf`.

ADMM - Why?

- convex problems with nondifferentiable constraints
- blockwise computation
  - sample blocks
  - feature blocks

ADMM for the Lasso - Problem

Problem in Lagrangian form

$$\underset{\beta \in \mathbb{R}^p, \theta \in \mathbb{R}^p}{\text{minimize}} \left\{ \frac{1}{2} \|\mathbf{y} - \mathbf{X}\beta\|_2^2 + \lambda \|\theta\|_1 \right\} \quad \text{such that } \beta - \theta = 0$$

Augmented Lagrangian

$$L_\rho(\beta, \theta, \mu) := \left\{ \frac{1}{2} \|\mathbf{y} - \mathbf{X}\beta\|_2^2 + \lambda \|\theta\|_1 \right\} + \langle \mu, \beta - \theta \rangle + \frac{\rho}{2} ||\beta - \theta||_2^2$$

ADMM for the Lasso - Update

Update

$$\beta^{t+1} = (\mathbf{X}^T\mathbf{X} + \rho\mathbf{I})^{-1}(\mathbf{X}^T\mathbf{y} + \rho\theta^t - \mu^t)$$
$$\theta^{t+1} = \mathcal{S}_{\lambda/\rho}(\beta^{t+1} + \mu^t/\rho)$$
$$\mu^{t+1} = \mu^t + \rho(\beta^{t+1} - \theta^{t+1})$$

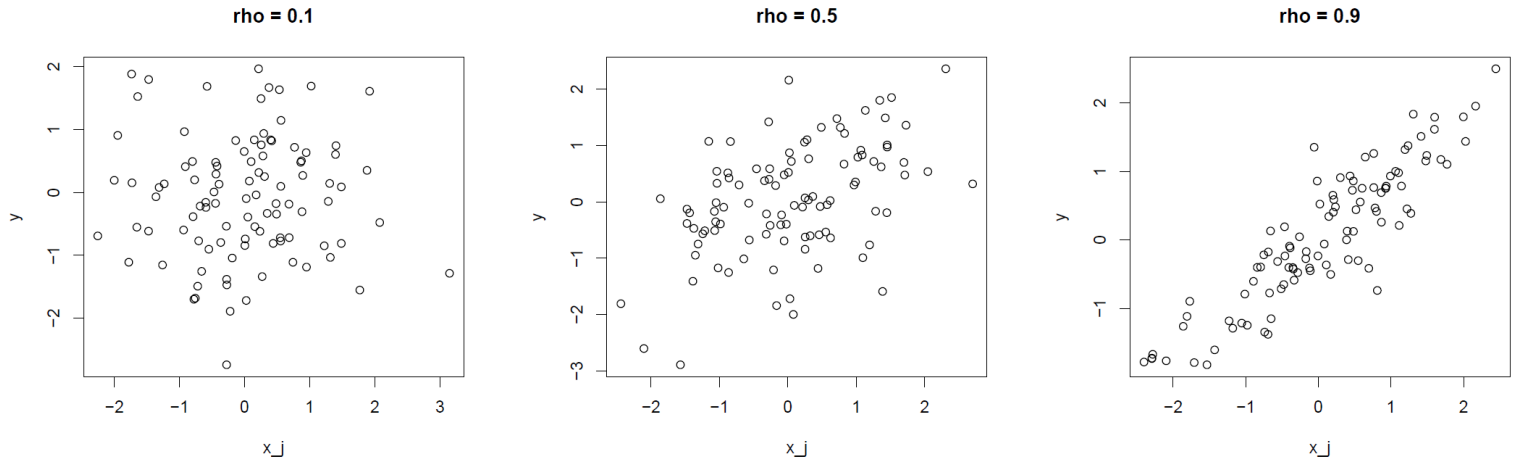
where $\mathcal{S}_{\lambda/\rho}(z) = \text{sign}(z)(|z| - \frac{\lambda}{\rho})_+$.

Screening Rules

- Pre-processing to eliminate features
- Very big data set, esp. large number of predictors
- Maybe too big to load into memory
- Screening rules eliminate predictors with minor calculation
- And very high / safe certainty (i.e. eliminated predictors would not show up in Lasso model based on full data)

They achieve a reduction in the number of variables, typically by an order of magnitude

What is a good predictor?



correlation is an inner product

high absolute correlation (= large absolute inner product)

$>$ high predictive power (compare plots)

$>$ x_j with largest inner product has highest predictive power

$>$ thus for that j we are most willing to accept some penalty from λ

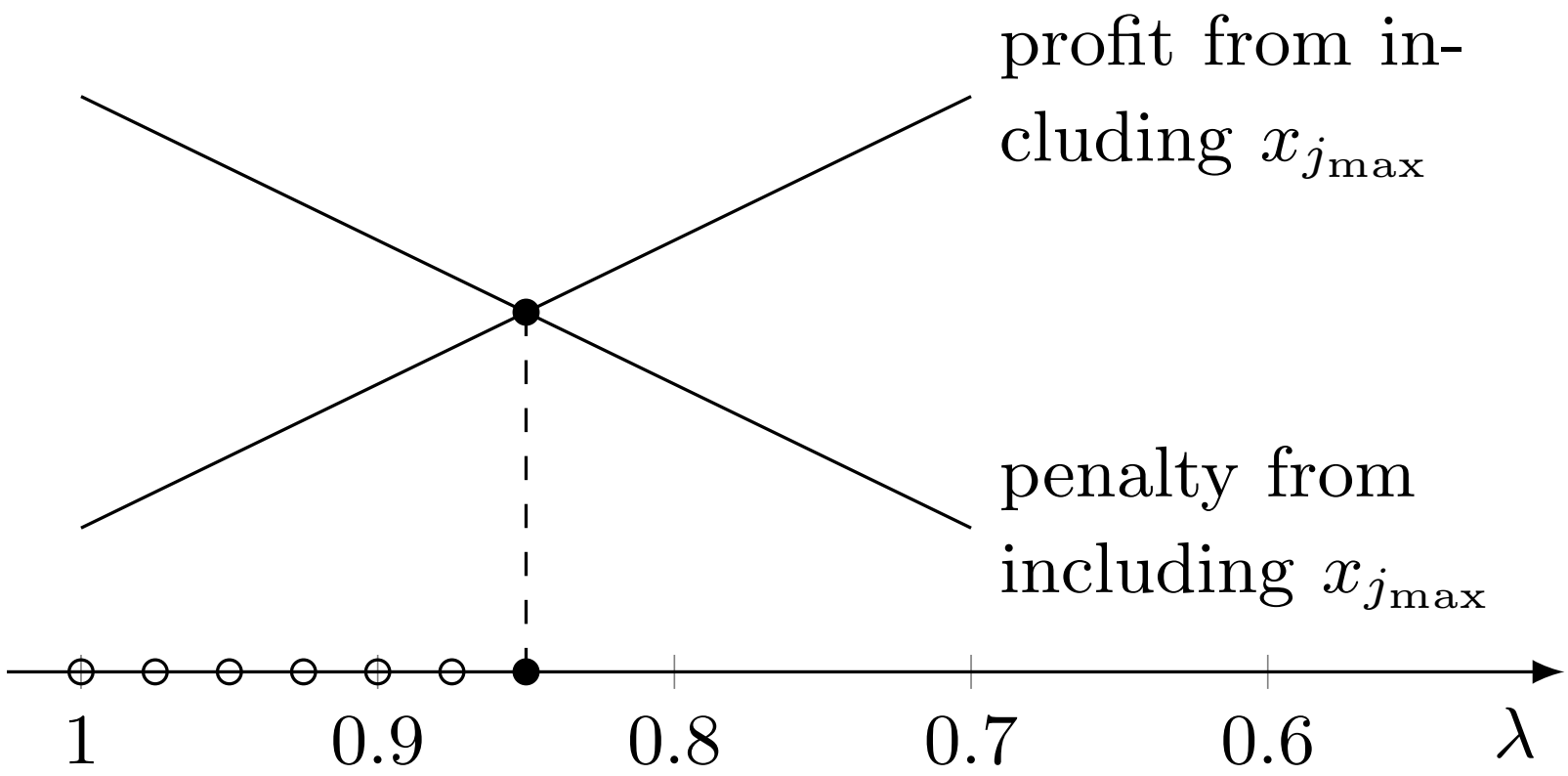
Lasso - an iterative algorithm

Let $\mathcal{A}$ be the active set of predictors. Let λ take values on a decreasing sequence.

iterate

1. order predictors x_j not in $\mathcal{A}$ by their "effectiveness" using $\left|x_j^T y\right|$ or better $\left|x_j^T (y - \hat{y}_\lambda)\right|$, call the best predictor $x_{j_{\max}}$
2. move λ such that the positive effect from the best predictor $x_{j_{\max}}$ compensates the penalty by λ
3. calculate solution for chosen λ

Lasso - a visual intepretation



Back to screening rules

Let λ take values on a decreasing sequence. Let $\lambda_{\max}$ be the λ where the first predictor has a non-zero coefficient.

$$\lambda_{\max} = \max_j \left| x_j^T y \right|$$

Let $\mathcal{A}$ be the active set of predictors.

$$\forall j \in \mathcal{A} \quad \lambda = \left| x_j^T (y - \hat{y}) \right|$$

$$\forall j \notin \mathcal{A} \quad \lambda > \left| x_j^T (y - \hat{y}) \right|$$

$$\forall j \notin \mathcal{A} \quad \lambda > \left| x_j^T (y - \hat{y}) \right|$$

R example

Global vs. Sequential

Global (one-time screening):
Suppose we want to calculate a Lasso solution at $\lambda < \lambda_{\max}$.

Sequential (iterative screening):
Suppose we have the Lasso solution $\hat{\beta}(\lambda')$ at λ' and want to screen variables for solutions at $\lambda < \lambda'$.

Dual Polytope Projection (DPP)

Global DPP Rule

Suppose we want to calculate a Lasso solution at $\lambda < \lambda_{\max}$. The DPP rule discards the j^{th} variable if

$$\left|\mathbf{x}_j^T \mathbf{y}\right| < \lambda_{\max} - \|\mathbf{x}_j\|_2 \|\mathbf{y}\|_2 \frac{\lambda_{\max} - \lambda}{\lambda}$$

Sequential DPP rule

Suppose we have the Lasso solution $\hat{\beta}(\lambda')$ at λ' and want to screen variables for solutions at $\lambda < \lambda'$. We discard the j^{th} variable if

$$\left|\mathbf{x}_j^T (\mathbf{y} - \mathbf{X}\hat{\beta}(\lambda'))\right| < \lambda' - \|\mathbf{x}_j\|_2 \|\mathbf{y}\|_2 \frac{\lambda_{\max} - \lambda}{\lambda}$$

Strong Rule

Global Strong Rule

Suppose we want to calculate a Lasso solution at $\lambda < \lambda_{\max}$. The global strong rule discards the j^{th} variable if

$$\left|\mathbf{x}_j^T \mathbf{y}\right| < \lambda - (\lambda_{\max} - \lambda) = 2\lambda - \lambda_{\max}$$

Sequential Strong Rule

Suppose we have the Lasso solution $\hat{\beta}(\lambda')$ at λ' and want to screen variables for solutions at $\lambda < \lambda'$. We discard the j^{th} variable if

$$\left|\mathbf{x}_j^T (\mathbf{y} - \mathbf{X}\hat{\beta}(\lambda'))\right| < 2\lambda - \lambda'$$

Screening Rules - Example Setup

- simulated dataset
- $N = 200, p = 5000$ uncorrelated Gaussian predictors
- $1/4$ true non-zero coefficients
- 100 decreassing lambda values equally spaced on the log-scale
- Compare Global DPP, Global Strong, Sequential DDP, Sequential Strong
- no violations for either of the strong rules

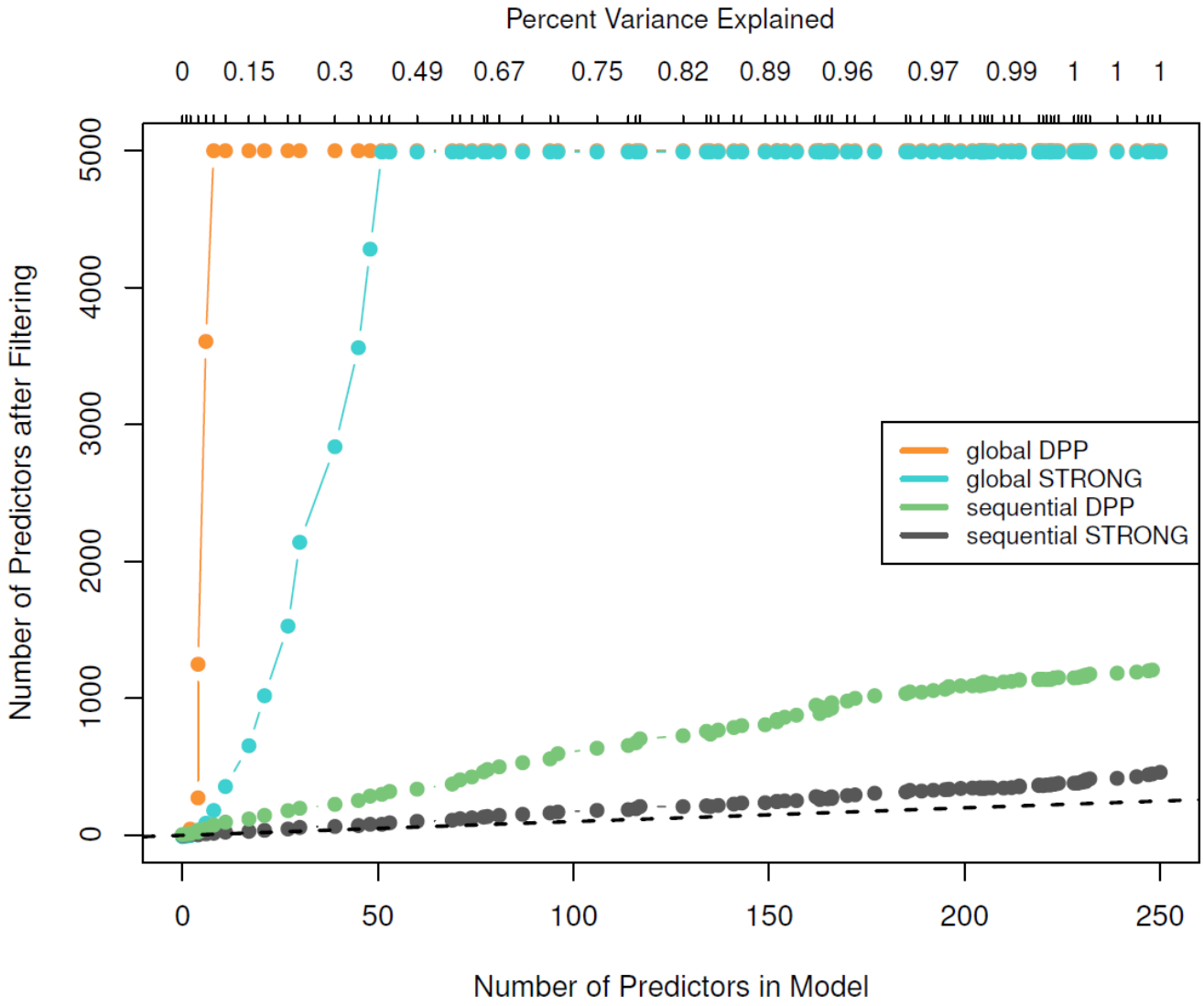


Figure: From Trevor Hastie, Robert Tibshirani, and Martin Wainwright (2015), Statistical learning with sparsity: the Lasso and generalizations, p129.

Summary I

Coordinate Descent

- An efficient algorithm implemented in glmnet but requires separability condition
- Application: Ridge, Lasso, Elastic Net, Logistic Regression, etc.; Failure: Fused Lasso

Least Angle Regression

- Similar to the idea of Forward Selection
- Computationally efficient but does not scale well to large problems

Connection between LASSO and LAR

- LAR could be modified to obtain Lasso solution
- Explains the fact that Lasso coefficient solution path is piece-wise linear

Summary II

ADMM

- Use duality to your advantage
- Limitations in speed for Lasso, but useful in more complex settings

Screening Rules

- Promising for very large p-s
- Difficult to find best rule, field in development

Additional slides: Minorization-Maximization Algorithms (MMA)

- Problem: minimize $f(\beta)$ over $\beta \in \mathbb{R}^p$ for f possibly non-convex
- Introduce additional variable θ
- Use θ to majorize (bound from above) the objective function to be minimized

Majorization-Minimization Algorithms work analoguosly.

MMA visually

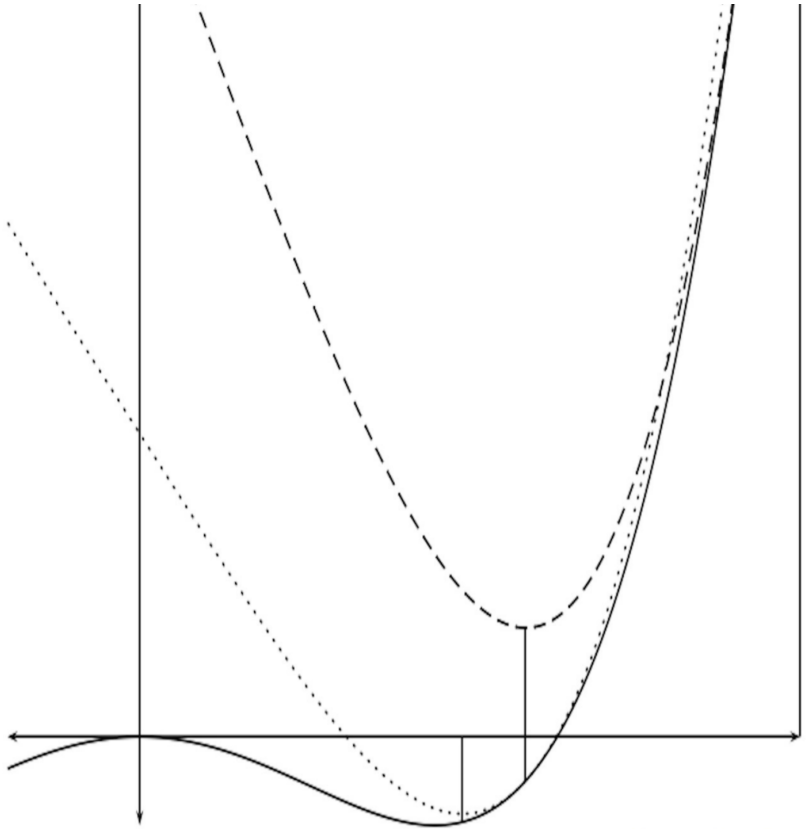


Figure: Figure from Jan De Leeuw (2015), Block Relaxation Methods in Statistics, doi.org/10.13140/RG.2.1.3101.9607

MMA analytically I

Def. $\Psi : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ majorizes f at $\beta \in \mathbb{R}^p$ if

$$\forall \theta \in \mathbb{R}^p \quad \Psi(\beta, \theta) \geq f(\beta)$$

with equality for $\theta = \beta$.

Minor-Max algorithm

- initialize β^0
- update with $\beta^{t+1} = \underset{\beta \in \mathbb{R}^p}{\arg\min}\, \Psi(\beta, \beta^t)$

MMA analytically II

This scheme generates a sequence of βs for which the cost $f(\beta^t)$ is nonincreasing, because

$$f(\beta^t) \overset{(i)}{=} \Psi(\beta^t, \beta^t) \overset{(ii)}{\geq} \Psi(\beta^{t+1}, \beta^t) \overset{(iii)}{\geq} f(\beta^{t+1})$$

where

(i) & (iii) Definiton of majorize

(ii) β^{t+1} is a minimizer of $\beta \mapsto \Psi(\beta, \beta^t)$

Biconvexity

Let's consider an example ...

$f(\alpha, \beta) = (1 - \alpha\beta)^2$

Def. A function $f(\alpha, \beta) : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ is biconvex, if for each $\alpha \in \mathbb{R}^m$ the function $\alpha \mapsto f(\alpha, \beta)$ is convex and for each $\beta \in \mathbb{R}^n$ the function $\beta \mapsto f(\alpha, \beta)$ is convex.

Alternate Convex Search

Block coordinate descent applied to α and β blocks

1. Initialize (α^0, β^0) at some point in the biconvex set to minimize over
2. For $t = 0, 1, 2, \ldots$
   (i) Fix $\beta = \beta^t$ and update $\alpha^{t+1} \in \underset{\alpha \in \mathcal{C}_{\beta^t}}{\arg\min}\, f(\alpha, \beta^t)$
   (ii) Fix $\alpha = \alpha^{t+1}$ and update $\beta^{t+1} \in \underset{\alpha \in \mathcal{C}_{\alpha^{t+1}}}{\arg\min}\, f(\alpha^{t+1}, \beta)$

For a function bounded from below, the algorithm converges to a partial optimum (i.e. as biconvexity, only optimal in one coordinate if the other coordinate is fixed).

Comments . . .

Questions . . .

Suggestions . . .

That's it.
Thanks for listening.