Problem Set 3

Problem 3.8

(a) $$p(x|\pi) = \pi^{x}(1-\pi)^{1-x} \qquad D=\{x_1, x_2 \cdots x_n\} \text{ and } s=\sum_{i=1}^{n} x_i$$

$$p(D|\pi) = \prod_{i=1}^{n} p(x_i|\pi) = \prod_{i=1}^{n}\left[\pi^{x_i}(1-\pi)^{1-x_i}\right]$$

$$= \pi^{\sum_{i=1}^{n} x_i}(1-\pi)^{\sum_{i=1}^{n}(1-x_i)} = \pi^{s}(1-\pi)^{n-s}$$

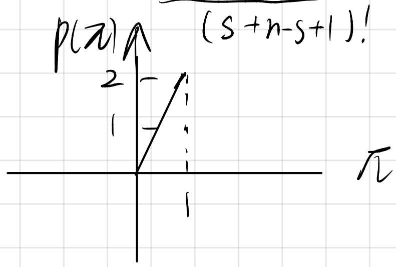
(b) Since π is the uniform prior let $p(\pi)=1$

$$p(\pi|D) = \frac{p(D|\pi)\,p(\pi)}{\int p(D|\pi)\,p(\pi)\,d\pi} = \frac{\pi^{s}(1-\pi)^{n-s}}{\int \pi^{s}(1-\pi)^{n-s}\,d\pi} = \frac{\pi^{s}(1-\pi)^{n-s}}{\frac{s!\,(n-s)!}{(s+n-s+1)!}}$$

$$= \frac{(n+1)!\,\pi^{s}(1-\pi)^{n-s}}{s!\,(n-s)!}$$

when $n=1$ $$p(\pi|D) = \frac{2!\,\pi^{s}(1-\pi)^{n-s}}{s!\,(1-s)!}$$

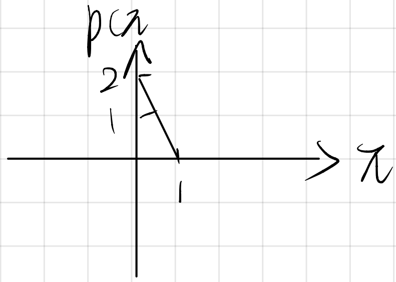
when $s=1$



when $s=1$ $$p(\pi|D) = \frac{2\pi}{1} = 2\pi$$

$s=0$ $$p(\pi|D) = \frac{2(1-\pi)}{1} = 2-2\pi$$

when $s=0$



(c) For predictive distribution

$$p(x|D) = \int p(x|\pi)\,p(\pi|D) = \int \pi^{x}(1-\pi)^{1-x}\,\frac{(n+1)!}{s!\,(n-s)!}\,\pi^{s}(1-\pi)^{n-s}\,d\pi$$

$$= \frac{(n+1)!}{s!\,(n-s)!}\int \pi^{s+x}(1-\pi)^{[n-s+(1-x)]}\,d\pi = \frac{(n+1)!}{s!\,(n-s)!}\,\frac{(s+x)!\,[n-s+(1-x)]!}{(n+2)!}$$

$$= \frac{1}{n+2} \times \frac{(s+x)!}{s!} \times \frac{[n-s+(1-x)]!}{(n-s)!}$$

when $x=1$ $p(x=0|D) = \frac{s+1}{n+2}$, when $x=0$ $p(x=1|D) = \frac{n-s+1}{n+2}$

$$p(x|D) = \prod_{k=0}^{1} p(x=k|D) = \left(\frac{s+1}{n+2}\right)^{1}\left(1-\frac{s+1}{n+2}\right)^{1-0}$$

$$= \left(\frac{s+1}{n+2}\right)^{x}\left(1-\frac{s+1}{n+2}\right)^{1-x}$$

since x with Bernolli Distribution and $P(x|D) = \left(\frac{S+1}{n+2}\right)^x \left(1-\frac{S+1}{n+2}\right)^{1-x}$

$\hat{\pi} = P(x=1) = \frac{S+1}{n+2}$ So Bayesian Estimate $\hat{\pi} = \frac{S+1}{n+2}$

the virtual sample is add one of each to let the the $\hat{\pi}_{Bays} = \frac{S+1}{n+2}$.

(d) MLE $l(\pi) = \log P(D|\pi) = s\log\pi + (n-s)\log(1-\pi)$

Let $\frac{\partial l(\pi)}{\partial \pi} = \frac{s}{\pi} - \frac{n-s}{1-\pi} = 0$ Thus $\hat{\pi}_{...} = \frac{S}{n}$

MAP : $\hat{\pi}_{MAP} = \arg\max_{} P(\pi|D)$, and $P(\pi|D) = \frac{(n+1)!}{S!(n-S)!}\pi^S(1-\pi)^{n-S}$

argmax $P(\pi|D)$: $\arg\max \log P(\pi|D) = \log\frac{(n+1)!}{S!(n-S)!} + S\log\pi + (n-S)\log(1-\pi)$

$\frac{\partial \arg\max \log P(\pi|D)}{\partial \pi} = \frac{S}{\pi} - \frac{n-S}{1-\pi}$ let it equal to 0

$\hat{\pi}_{MAP} = \frac{S}{n}$ which is also equal to $\hat{\pi}_{MLE}$

Since the uniform prior was assumed for π, which makes each Prior density same, So the requlate part is zero and Let MAP = MLE

(e) for $P_1(\pi)$ $P_1(\pi|D) = \frac{P(D|\pi)P_1(\pi)}{\int P(D|\pi)P_1(\pi)d\pi} = \frac{\pi^S(1-\pi)^{n-S}2\pi}{2\int\pi^{S+1}(1-\pi)^{n-S}}$

$= \frac{(n+2)!}{(S+1)!(n-S)!}\pi^{S+1}(1-\pi)^{n-S}$ $\hat{\pi}_{MAP} = \arg\max P_1(\pi|D)$

$= \arg\max \log P_1(\pi|D) = \arg\max \log\frac{(n+2)!}{(S+1)!(n-S)!} (S+1)\log\pi + (n-S)\log(1-\pi)$

let $\frac{\partial \arg}{\partial \pi} = \frac{S+1}{\pi} - \frac{n-S}{1-\pi} = 0$, $S - S\pi + 1 - \pi = n\pi - S\pi$

$\hat{\pi}_{MAP} = \frac{s+1}{n+1}$ when x biased to 1

For $P_0(\pi)$ $\quad P_0(\pi|D) = \frac{P(D|\pi)P_0(\pi)}{\int P(D|\pi)P_0(\pi)d\pi} = \frac{2\pi^s(1-\pi)^{n-s+1}}{2\int \pi^s(1-\pi)^{n-s+1}}$

$= \frac{(n+2)!}{s!(n-s+1)!}\pi^s(1-\pi)^{n-s+1}$, $\quad \hat{\pi}_{MAP} = \arg\max P_0(\pi|D)$

$= \arg\max \log P_0(\pi|D) \quad \frac{\partial \arg}{\partial \pi} = \frac{s}{\pi} - \frac{n-s+1}{1-\pi}$ let it $=0$

$s - s\pi = n\pi - s\pi + \pi \quad \hat{\pi}_{MAP} = \frac{s}{n+1}$

Thus, when favor 1's $\hat{\pi}_{MAP} = \frac{s+1}{n+1}$, when favor 0's $\hat{\pi}_{MAP} = \frac{s}{n+1}$

$P(x|D) = \int P(x|\pi)P(\pi|D)d\pi$ = biased 1

$= \int \pi^x(1-\pi)^{1-x}\pi^{s+1}(1-\pi)^{n-s}d\pi \cdot \frac{(n+2)!}{(s+1)!(n-s)!}$

$= \frac{(s+1+x)![(n-s)+(1-x)]!}{(n+3)!}\frac{(n+2)!}{(s+1)!(n-s)!}$

$= \frac{1}{n+3}\frac{(s+1+x)!}{(s+1)!}\frac{((n-s)+(1-x))!}{(n-s)!}$

$x=0$ or $x=1$ $\quad P(x|D) = \left(\frac{s+2}{n+3}\right)^x\left(1-\frac{s+2}{n+3}\right)^{1-x}$

$P(x|D)$ biased 0 : $\frac{(s+x)![(n-s+1)+(1-x)]!}{(n+3)!}\frac{(n+2)!}{s!(n-s+1)!}$

$= \frac{1}{n+3}\frac{(s+x)!}{s!}\frac{(n-s+1+(1-x))!}{(n-s+1)!}$, $x=0$ or $x=1$ $\quad P(x|D) = \left(\frac{s+1}{n+3}\right)^x\left(1-\frac{s+1}{n+3}\right)^{1-x}$

Above all, $\hat{\pi}_{\text{Bayse}} = \frac{S+2}{n+3}$ or $\frac{S+1}{n+3}$

Besides $P(\pi) = P(x=1)$ so $\hat{\pi}_{\text{Bays}} = \frac{S+2}{n+3}$ means virtual sample add 3 but two x is 1

$\hat{\pi}_{\text{Bays}} = \frac{S+1}{n+3}$ means viturual sample add 3 but one x is 1.

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