Problem Set 10

10.1

a) to construct dot-product kernel. $k(x,z) = c k_1(x,z)$, $c>0$

let $k(x,z) = c < \Phi'(x), \Phi'(z) >$

$$= < \sqrt{c}\, \Phi'(x), \sqrt{c}\, \Phi'(z) >$$

to the pos-def kernel k, k_1 is valid, $c>0$

thus, k_1 is a positive def kernel, the $ck_1(x,z)$ is also

b) for $k(x,z) = k_1(x,z) + k_2(x,z) = \Phi^T(x)\Phi(z)$

let $\Phi(x) = \begin{bmatrix} \Phi^1(x) \\ \Phi^2(x) \end{bmatrix}$

Then $\Phi(x)^T \Phi(z) = \Phi^1(x)^T \Phi^1(z) + \Phi^2(x)^T \Phi^2(z)$

$$= k_1(x,z) + k_2(x,z)$$

to the pos-def kernel k, k_1, k_2 are valid,

k_1, k_2 are pos-def kernel $k_1 + k_2$ will also.

c) for $k(x,z) = k_1(x,z)\, k_2(x,z) = \Phi^T(x)\Phi(z)$

let $\Phi(x) = \Phi^1(x)\, \Phi^2(x)$

Then $\Phi(x)^T \Phi(z) = \Phi^2(x)^T \Phi^1(x)^T \Phi^1(z) \Phi^2(z)$

$$= k_1(x,z)\, k_2(x,z)$$

to the pos-def kernel k, k_1, k_2 are valid

k_1, k_2 are pos-def kernel, $k_1 k_2$ will also.

d) for $k(x,z) = f(x)\, k_1(x,z)\, f(z)$, for any $f(\cdot)$

$$= \Phi(x)^T \Phi(z)$$

let $\Phi(x) = f(x)\, \Phi'(x)$

Then $\Phi^T(x)\Phi(z) = f(x)\,\Phi'(x)^T \Phi'(z)\, f(z) = f(x)\, k_1(x,z)\, f(z)$

Thus, $k(x,z) = f(x)\, k_1(x,z)\, f(z)$ is valid kernel.

e). for $k(x,z) = k_1(x,z)^q$ where q is positive integer.

from c) can get $k(x,z)$ can be $k_1(x,z)^2$

Thus let $k(x,z)^{old} = k_1(x,z)^2$, $k(x,z) = k(x,z)^{old}\, k_1(x,z)$

$$= k_1(x,z)^3$$

by indicating, $k(x,z)$ can be $k_1(x,z)^q$

f) for $k(x,z) = \exp(k_1(x,z))$

According to Taylor expansion:

$\exp(x) = \sum_{n=0}^{\infty} \frac{a^n}{n!}$, to $\exp(k_1(x,z))$

it can be $\sum_{n=0}^{\infty} \frac{k_1(x,z)^n}{n!}$

and since
$$\begin{cases} k(x,z) = k_1(x,z)^q \text{ is valid} \\ k(x,z) = c\, k_1(x,z) \text{ is valid} \\ k(x,z) = k_1(x,z) + k_2(x,z) \text{ is valid} \end{cases}$$

Thus $1 + k_1(x,z)^1 + \frac{k_1(x,z)^2}{2!} + \cdots$ is valid kernel.

10.2

a). for $k(x,z) = \exp(-\alpha \|x-z\|^2)$ for $\alpha > 0$

$$\exp(-\alpha\|x-z\|^2) = \exp(-\alpha(x^Tx - 2x^Tz + z^Tz))$$
$$= e^{-\alpha x^Tx} \; e^{2\alpha x^Tz} \; e^{-\alpha z^Tz}$$

And $e^{2\alpha x^Tz}$ can be $e^{2\alpha k_1(x,z)}$, according to 10.1 a)

$e^{2\alpha k_1(x,z)} = e^{k(x,z)}$, according to 10.1 f)

$e^{k_1(x,z)} = k(x,z)$ is valid

Thus, $\exp(-\alpha\|x-z\|^2) = e^{-\alpha x^Tx} \, k(x,z) \, e^{-\alpha z^Tz}$

and let $f(x) = e^{-\alpha x^Tx}$

$$\exp(-\alpha\|x-z\|^2) = f(x) \, k_1(x,z) \, f(z)$$

according to 10.1 d)

$\exp(-\alpha\|x-z\|^2) = k(x,z)$ is valid kernel.

10.4.

a). for $\tilde{k}(x,z) = \dfrac{k(x,z)}{\sqrt{k(x,x)k(z,z)}}$

and $\sqrt{k(x,x)}$, $\sqrt{k(z,z)}$ can be considered as $f(x)$, $f(z)$.

Thus, $\tilde{k}(x,z) = f(x) \, k(x,z) \, f(z)$,

and according to 10.1 d)

$\tilde{k}(x,z)$ is also a valid kernel.

b) Since $\Phi(x)$ be the feature transformation associated with the kernel k.

Thus, $k(x,z) = \Phi(x)^T \Phi(z)$

$$\tilde{k}(x,z) = \frac{\Phi(x)^T \Phi(z)}{\|\Phi(x)\| \, \|\Phi(z)\|}$$

and $\cos\theta = \frac{x \cdot z}{|x||z|}$, Thus, $\tilde{k}(x,z)$ can be written as the $\cos\theta$ format.

c). Since $\tilde{k}(x,z)$ is an angle's cosine;

So $-1 \le \tilde{k}(x,z) \le 1$

which means $\begin{cases} \tilde{k}(x,z) = -1, & \text{when } \Phi(x), \Phi(z) \text{ are opposite} \\ \tilde{k}(x,z) = 1, & \text{when } \Phi(x), \Phi(z) \text{ are same direction} \\ \tilde{k}(x,z) = 0, & \text{when they are orthogonal.} \end{cases}$

this can also simpify : $\tilde{k}(x,z)^2 \le 1$

which means $\frac{[\Phi(x)^T\Phi(z)]^T(\Phi(x)^T\Phi(z))}{\Phi(x)^T\Phi(x) \quad \Phi^T(z)\Phi(z)}$

$$= \frac{k(x,z)^2}{k(x,x)\ k(z,z)} \le 1$$

$$|k(x,z)| \le \sqrt{k(x,x)\ k(z,z)}$$

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