# Problem Set 5.

5.2

a) Since the $E_{\hat{p}}[X] = \int \hat{p}(x)\, x\, dx$, and $\hat{p}(x) = \frac{1}{n}\sum_{i=1}^{n} \tilde{k}(x - x_i)$

$$E_{\hat{p}}[X] = \int \left[\frac{1}{n}\sum_{i=1}^{n} \tilde{k}(x-x_i)\right] x\, dx = \frac{1}{n}\sum_{i=1}^{n} \int \tilde{k}(x-x_i)\, x\, dx$$

Let $x - x_i = t$ $\quad E_{\hat{p}}[X] = \frac{1}{n}\sum_{i=1}^{n} \int \tilde{k}(t)(t+x_i)\, dt$

$$= \frac{1}{n}\sum_{i=1}^{n} \left[x_i \int \tilde{k}(t)\, dt + \int \tilde{k}(t)\, t\, dt\right]$$

Besides. $E_{\tilde{k}}[X] = \int \tilde{k}(x)\, x\, dx = 0$

and $\int k(x)\, dx = 1$ $\qquad \tilde{k}(x) = \frac{1}{h^d} k\left(\frac{x}{h}\right)$

The $E_{\hat{p}}(X) = \frac{1}{n}\sum_{i=1}^{n} [x_i + 0] = \frac{\sum_{i=1}^{n} x_i}{n} = \hat{u}$

b) Since $\mathrm{Cov}_{\hat{p}}[X] = \int \hat{p}(x)\, [x - E_{\hat{p}}[X]]\, [x - E_{\hat{p}}[X]]^T dx$

$$\mathrm{Cov}_{\hat{p}}[X] = \int \hat{p}(x)\, (x-\hat{u})(x-\hat{u})^T dx = \frac{1}{n}\sum_{i=1}^{n} \int \tilde{k}(x-x_i)(x-\hat{u})(x-\hat{u})^T dx$$

and $\begin{cases} \mathrm{Cov}_{\tilde{k}}[X] = \int \tilde{k}(x)\, [x - E_{\tilde{k}}[X]]\, [x - E_{\tilde{k}}[X]]^T dx = H \\ E_{\tilde{k}}[X] = 0 \end{cases}$, So $\mathrm{Cov}_{\tilde{k}}[X] = \int \tilde{k}(x)\, x\, x^T dx = H$

let $t = x - x_i$ $\quad \mathrm{Cov}_{\hat{p}}[X] = \frac{1}{n}\sum_{i=1}^{n} \int \tilde{k}(t)\, [t + (x_i - \hat{u})]\, [t + (x_i - \hat{u})]^T dt$

$$= \frac{1}{n}\sum_{i=1}^{n} \int \tilde{k}(t)\, [tt^T + 2t(x_i-\hat{u})^T + (x_i-\hat{u})(x_i-\hat{u})^T]\, dt$$

$$= \frac{1}{n}\sum_{i=1}^{n} \left[H + (x_i-\hat{u})\int \tilde{k}(t)\, t\, dt + (x_i-\hat{u})(x_i-\hat{u})^T \int \tilde{k}(t)\, dt\right]$$

and also like in a): $\begin{cases} \int k(t)\, dt = 1 \\ \int \tilde{k}(t)\, t\, dt = 0 \end{cases}$, $\quad \tilde{k}(t) = \frac{1}{h^d} k\left(\frac{t}{h}\right)$

So $\mathrm{Cov}_{\hat{p}}[x] = \frac{1}{n}\sum_{i=1}^{n}[H + (x_i-\hat{u})\times 0 + (x_i-\hat{u})(x_i-\hat{u})^T \times 1]$

$$= H + \frac{1}{n}\sum_{i=1}^{n}(x_i-\hat{u})(x_i-\hat{u})^T$$

c) To the mean of distribution that $\hat{u}$ is equal to mean of sample $\frac{\sum_{i=1}^{N} x_i}{N}$

However, $\mathrm{Cov}_{\hat{p}}[x] = H +$ covarrance of sample that $\hat{\Sigma} = H + \frac{1}{n}\sum_{i=1}^{n}(x_i-\hat{u})(x_i-\hat{u})^T$

Thus, the kernel function can large the covarinnce than sample, which can lead the estimate bias.

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