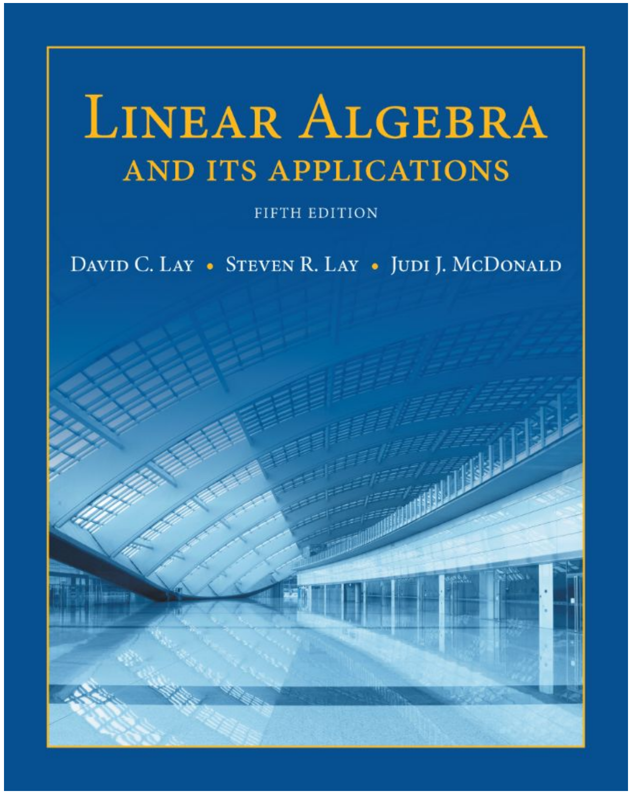
2 Matrix Algebra

2.5

MATRIX FACTORIZATIONS



MATRIX FACTORIZATIONS

- A *factorization* of a matrix A is an equation that expresses A as a product of two or more matrices.
- Whereas matrix multiplication involves a *synthesis* of data (combining the effects of two or more linear transformations into a single matrix), matrix factorization is an *analysis* of data.

THE LU FACTORIZATION

- The LU factorization, described on the next few slides, is motivated by the fairly common industrial and business problem of solving a sequence of equations, all with the same coefficient matrix:

$$Ax = b_1, \quad Ax = b_2, \ldots , \quad Ax = b_p \qquad (1)$$

- When A is invertible, one could compute A^{-1} and then compute $A^{-1}b_1$, $A^{-1}b_2$, and so on.
- However, it is more efficient to solve the first equation in the sequence (1) by row reduction and obtain the LU factorization of A at the same time. Thereafter, the remaining equations in sequence (1) are solved with the LU factorization.

THE LU FACTORIZATION

- At first, assume that A is an $m \times n$ matrix that can be row reduced to echelon form, *without row interchanges*.
- Then A can be written in the form $A = LU$, were L is an $m \times m$ lower triangular matrix with 1's on the diagonal and U is an $m \times n$ echelon form of A.
- For instance, see Fig. 1 below. Such a factorization is called an **LU factorization** of A. The matrix L is invertible and is called a unit lower triangular matrix.

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{U}$$

THE LU FACTORIZATION

- Before studying how to construct L and U, we should look at why they are so useful. When $A = LU$, the equation $Ax = b$ can be written as $L(Ux) = b$.
- Writing y for Ux, we can find x by solving the pair of equations

$$\boxed{\begin{aligned} Ly &= b \\ Ux &= y \end{aligned}}$$

- First solve $Ly = b$ for y, and then solve $Ux = y$ for x. See Fig. 2 on the next slide. Each equation is easy to solve because L and U are triangular.

THE LU FACTORIZATION

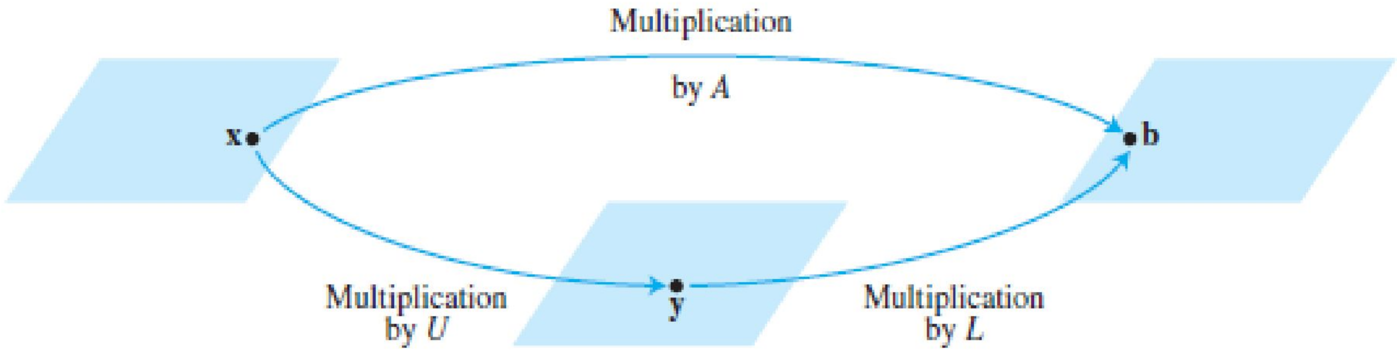
- 

FIGURE 2 Factorization of the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

- **Example 1** It can be verified that

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} = LU$$

- Use this factorization of A to solve $Ax=b$, where $\mathrm{b} = \begin{bmatrix} -9 \\ 5 \\ 7 \\ 11 \end{bmatrix}$

THE LU FACTORIZATION

- **Solution** The solution of $Ly = b$ needs only 6 multiplications and 6 additions, because the arithmetic takes place only in column 5.

$$[\,L \quad \mathbf{b}\,] = \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = [\,I \quad \mathbf{y}\,]$$

- Then, for $Ux = y$, the "backward" phase of row reduction requires 4 divisions, 6 multiplications, and 6 additions.

THE LU FACTORIZATION

- For instance, creating the zeros in column 4 of $[U \quad y]$ requires 1 division in row 4 and 3 multiplication-addition pairs to add multiples of row 4 to the rows above.

$$[U \quad \mathbf{y}] = \begin{bmatrix} 3 & -7 & -2 & 2 & -9 \\ 0 & -2 & -1 & 2 & -4 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ -6 \\ -1 \end{bmatrix}$$

To find x requires 28 arithmetic operations, or "flops" (floating point operations), excluding the cost of finding L and U. In contrast, row reduction of $[A \quad b]$ to $[I \quad x]$ takes 62 operations.

AN LU FACTORIZATION ALGORITHM

- Suppose A can be reduced to an echelon form U using only row replacements that add a multiple of one row to another below it.
- In this case, there exist unit lower triangular elementary matrices $E_1, \ldots, E_p$ such that

$$E_p \ldots E_1 A = U$$

- Then (3)

$$A = (E_p \ldots E_1)^{-1} U = LU$$

- where

$$L = (E_p \ldots E_1)^{-1} \qquad (4)$$

- It can be shown that products and inverses of unit lower triangular matrices are also unit lower triangular. Thus L is unit lower triangular.

AN LU FACTORIZATION ALGORITHM

- Note that row operations in equation (3), which reduce A to U, also reduce the L in equation (4) to I, because $E_p \ldots E_1 L = (E_p \ldots E_1)(E_p \ldots E_1)^{-1} = I$. This observation is the key to *constructing* L.

Algorithm for an LU Factorization

1. Reduce A to an echelon form U by a sequence of row replacement operations, if possible.
2. Place entries in L such that the *same sequence of row operations* reduces L to I.

AN LU FACTORIZATION ALGORITHM

- Step 1 is not always possible, but when it is, the argument above shows that an LU factorization exists.
- Example 2 on the followings slides will show how to implement step 2. By construction, L will satisfy

$$(E_p \dots E_1)L = I$$

- using the same $E_p, \dots, E_1$ as in equation (3). Thus L will be invertible, by the Invertible Matrix Theorem, with $(E_p \dots E_1) = L^{-1}$. From (3), $L^{-1}A = U$, and $A = LU$. So step 2 will produce an acceptable L.

AN LU FACTORIZATION ALGORITHM

- **Example 2** Find an LU factorization of

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

- **Solution** Since A has four rows, L should be 4×4. The first column of L is the first column of A divided by the top pivot entry:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & & 1 & 0 \\ -3 & & & 1 \end{bmatrix}$$

AN LU FACTORIZATION ALGORITHM

- Compare the first columns of A and L. *The row operations that create zeros in the first column of A will also create zeros in the first column of L.*
- To make this same correspondence of row operations on A hold for the rest of L, watch a row reduction of A to an echelon form U. That is, *highlight the entries* in each matrix that are used to determine the sequence of row operations that transform A onto U.

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix} = A_1$$

$$\sim A_2 = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = U \qquad (5)$$

AN LU FACTORIZATION ALGORITHM

- The highlighted entries above determine the row reduction of A to U. At each pivot column, divide the highlighted entries by the pivot and place the result onto L:

$$\begin{bmatrix} 2 \\ -4 \\ 2 \\ -6 \end{bmatrix} \begin{bmatrix} 3 \\ -9 \\ 12 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \begin{bmatrix} 5 \end{bmatrix}$$

$$\div 2 \quad \div 3 \quad \div 2 \quad \div 5$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$\begin{bmatrix} 1 & & & \\ -2 & 1 & & \\ 1 & -3 & 1 & \\ -3 & 4 & 2 & 1 \end{bmatrix}, \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix}$$

- An easy calculation verifies that this L and U satisfy $LU = A$.