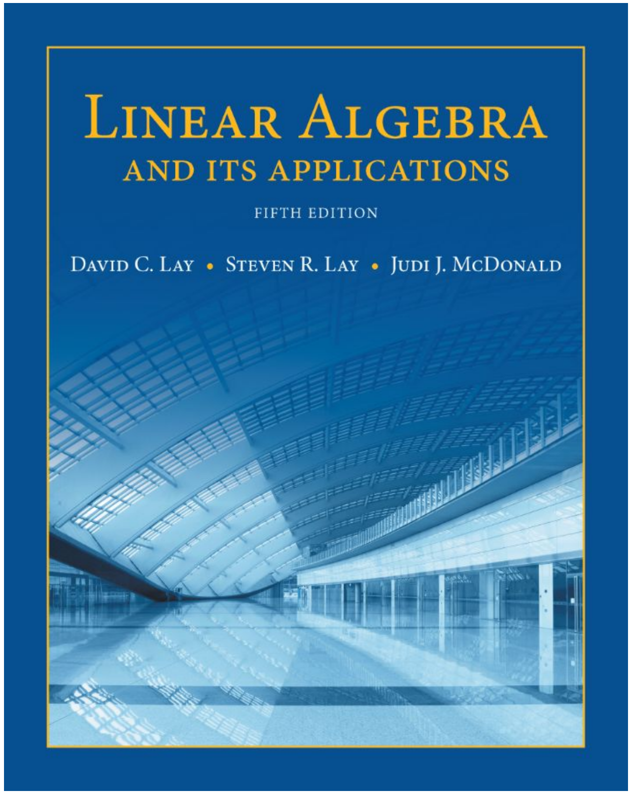
2 Matrix Algebra

2.6

THE LEONTIEF INPUT-OUTPUT MODEL



THE LEONTIEF INPUT-OUTPUT MODEL

- Linear algebra played an essential role in the Nobel prize-winning work of Wassily Leontief.
- Suppose a nation's economy is divided into *n* sectors that produce goods or services, and let $\mathbf{x}$ be a **production vector** in $\mathbb{R}^n$ that lists the output of each sector for one year.
- Also, suppose another part of the economy (called the *open sector*) does not produce goods or services but only consumes them, and let $\mathbf{d}$ be a **final demand vector** (or **bill of final demands**) that lists the values of the goods and services demanded from the various sectors by the nonproductive part of the economy.

THE LEONTIEF INPUT-OUTPUT MODEL

- As the various sectors produce goods to meet consumer demand, the producers themselves create additional intermediate demand for goods they need as inputs for their own production.
- Leontief asked if there is a production level x such that the amounts produced (or "supplied") will exactly balance the total demand for that production, so that

$$\left\{\begin{matrix} amount \\ produced \\ \boldsymbol{x} \end{matrix}\right\} = \left\{\begin{matrix} intermediate \\ demand \end{matrix}\right\} + \left\{\begin{matrix} final \\ demand \\ \boldsymbol{d} \end{matrix}\right\} \qquad (1)$$

THE LEONTIEF INPUT-OUTPUT MODEL

- The basic assumption of Leontief's input-output model is that for each sector, there is a **unit consumption vector** in $\mathbb{R}^n$ that lists the inputs needed *per unit of output* of the sector.
- As a simple example, suppose the economy consists of three sectors—manufacturing, agriculture, and services—with unit consumption vectors c_1, c_2, and c_3, as shown in the table that follows.

	Inputs Consumed per Unit of Output		
Purchased from:	**Manufacturing**	**Agriculture**	**Services**
Manufacturing	.50	.40	.20
Agriculture	.20	.30	.10
Services	.10	.10	.30
	$\uparrow$ c_1	$\uparrow$ c_2	$\uparrow$ c_3

THE LEONTIEF INPUT-OUTPUT MODEL

- **Example 1** What amounts will be consumed by the manufacturing sector if it decides to produce 100 units?

- **Solution** Compute

$$100c_1 = 100\begin{bmatrix} .50 \\ .20 \\ .10 \end{bmatrix} = \begin{bmatrix} 50 \\ 20 \\ 10 \end{bmatrix}$$

- To produce 100 units, manufacturing will order (i.e., "demand") and consume 50 units from other parts of the manufacturing sector, 20 units from agriculture, and 10 units from services.

THE LEONTIEF INPUT-OUTPUT MODEL

- If manufacturing decides to produce x_1 units of output, then x_1c_1 represents the *intermediate demands* of manufacturing, because the amounts in x_1c_1 will be consumed in the process of creating the x_1 units of output.
- Likewise, if x_2 and x_3 denote the planed outputs of the agriculture and services sectors, x_2c_2 and x_3c_3 list their corresponding intermediate demands.

THE LEONTIEF INPUT-OUTPUT MODEL

- The total intermediate demand from all three sectors is given by

$$\begin{aligned}\{\text{intermediate demand}\} &= x_1c_1 + x_2c_2 + x_3c_3 \\ &= Cx \qquad (2)\end{aligned}$$

- where C is the consumption matrix $[c_1 \quad c_2 \quad c_3]$, namely,

$$C = \begin{bmatrix} .50 & .40 & .20 \\ .20 & .30 & .10 \\ .10 & .10 & .30 \end{bmatrix} \qquad (3)$$

- Equations (1) and (2) yield Leontief's model.

$$\underset{\text{Amount produced}}{\mathbf{x}} = \underset{\text{Intermediate demand}}{C\mathbf{x}} + \underset{\text{Final Demand}}{\mathbf{d}} \qquad (4)$$

THE LEONTIEF INPUT-OUTPUT MODEL

- Equation (4) may also be written as $I\mathbf{x} - C\mathbf{x} = \mathbf{d}$, or

$$(I - C)\mathbf{x} = \mathbf{d} \tag{5}$$

- **Example 2** Consider the economy whose consumption matrix is given by (3). Suppose the final demand is 50 units for manufacturing, 30 units for agriculture, and 20 units for services. Find the production level *x* that will satisfy this demand.

- **Solution** The coefficient matrix in (5) is

$$I - C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} .5 & .4 & .2 \\ .2 & .3 & .1 \\ .1 & .1 & .3 \end{bmatrix} = \begin{bmatrix} .5 & -.4 & -.2 \\ -.2 & .7 & -.1 \\ -.1 & -.1 & .7 \end{bmatrix}$$

THE LEONTIEF INPUT-OUTPUT MODEL

- To solve (5), row reduce the augmented matrix

$$\begin{bmatrix} .5 & -.4 & -.2 & 50 \\ -.2 & .7 & -.1 & 30 \\ -.1 & -.1 & .7 & 20 \end{bmatrix} \sim \begin{bmatrix} 5 & -4 & -2 & 500 \\ -2 & 7 & -1 & 300 \\ -1 & -1 & 7 & 200 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & 226 \\ 0 & 1 & 0 & 119 \\ 0 & 0 & 1 & 78 \end{bmatrix}$$

- The last column is rounded to the nearest whole unit. Manufacturing must produce approximately 226 units, agriculture 119 units, and services only 78 units.

THE LEONTIEF INPUT-OUTPUT MODEL

- In the theorem, the term **column s**um denotes the sum of the entries in a column of a matrix. Under ordinary circumstances, the column sums of a consumption matrix are less than 1 because a sector should require less than one unit's worth of inputs to produce on unit of output.
- **Theorem 11**: Let C be the consumption matrix for an economy, and let d be the final demand. If C and d have nonnegative entries and if each column sum of C is less than 1, then $(I - C)^{-1}$ exists and the production vector

$$\mathrm{x} = (I - C)^{-1}\mathbf{d}$$

has nonnegative entries and is the unique solution of

$$\mathrm{x} = \mathrm{Cx} + \mathbf{d}$$