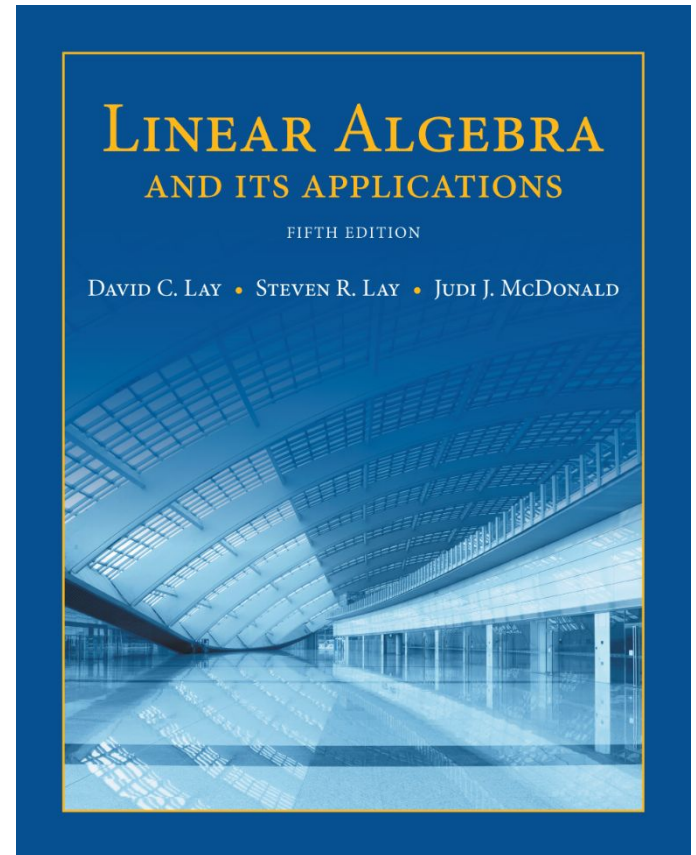


# 1

## Linear Equations in Linear Algebra

### 1.3

## VECTOR EQUATIONS



# VECTOR EQUATIONS

## Vectors in $\mathbb{R}^2$

- A matrix with only one column is called a **column vector**, or simply a **vector**.
- An example of a vector with two entries is

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$

where  $w_1$  and  $w_2$  are any real numbers.

- The set of all vectors with two entries is denoted by  $\mathbb{R}^2$  (read “r-two”).

# VECTOR EQUATIONS

- The  $\mathbb{R}$  stands for the real numbers that appear as entries in the vector, and the exponent 2 indicates that each vector contains two entries.
- Two vectors in  $\mathbb{R}^2$  are **equal** if and only if their corresponding entries are equal.
- Given two vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathbb{R}^2$ , their **sum** is the vector  $\mathbf{u} + \mathbf{v}$  obtained by adding corresponding entries of  $\mathbf{u}$  and  $\mathbf{v}$ .
- Given a vector  $\mathbf{u}$  and a real number  $c$ , the **scalar multiple** of  $\mathbf{u}$  by  $c$  is the vector  $c\mathbf{u}$  obtained by multiplying each entry in  $\mathbf{u}$  by  $c$ .

# VECTOR EQUATIONS

**Example 1:** Given  $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ , find  $4\mathbf{u}(-3)\mathbf{v}$  , and  $4\mathbf{u} + (-3)\mathbf{v}$  .

**Solution:**  $4\mathbf{u} = \begin{bmatrix} 4 \\ -8 \end{bmatrix}$  ,  $(-3)\mathbf{v} = \begin{bmatrix} -6 \\ 15 \end{bmatrix}$  and

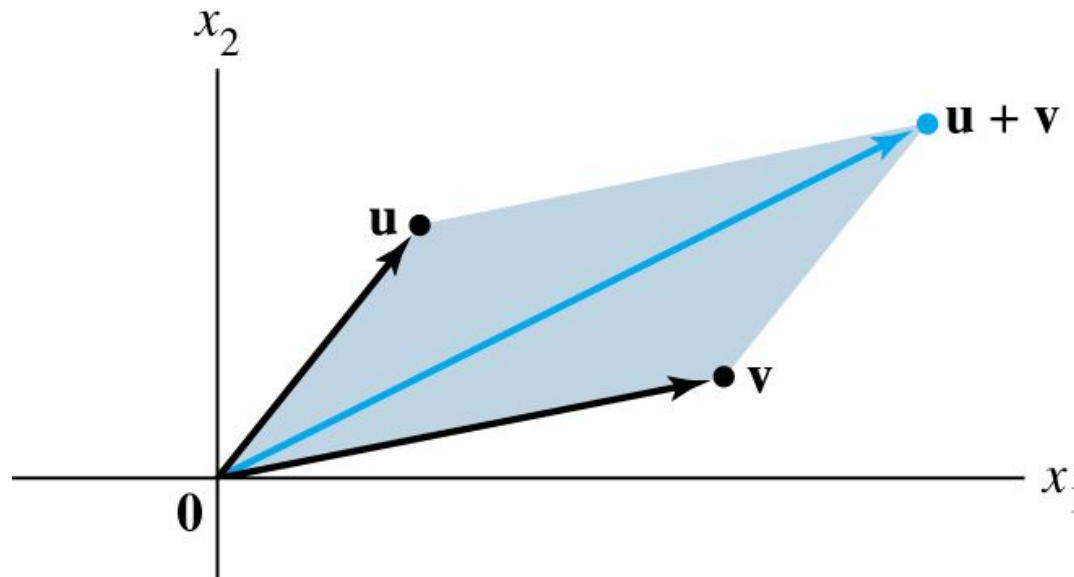
$$4\mathbf{u} + (-3)\mathbf{v} = \begin{bmatrix} 4 \\ -8 \end{bmatrix} + \begin{bmatrix} -6 \\ 15 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$$

# GEOMETRIC DESCRIPTIONS OF $\mathbb{R}^2$

- Consider a rectangular coordinate system in the plane. Because each point in the plane is determined by an ordered pair of numbers, *we can identify a geometric point  $(a, b)$  with the column vector  $\begin{bmatrix} a \\ b \end{bmatrix}$ .*
- So we may regard  $\mathbb{R}^2$  as the set of all points in the plane.

# PARALLELOGRAM RULE FOR ADDITION

- If  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathbb{R}^2$  are represented as points in the plane, then  $\mathbf{u} + \mathbf{v}$  corresponds to the fourth vertex of the parallelogram whose other vertices are  $\mathbf{u}$ ,  $\mathbf{0}$ , and  $\mathbf{v}$ . See Fig. 3 below.



# VECTORS IN $\mathbb{R}^3$ and $\mathbb{R}^n$

- Vectors in  $\mathbb{R}^3$  are  $3 \times 1$  column matrices with three entries.
- They are represented geometrically by points in a three-dimensional coordinate space, with arrows from the origin sometimes included for visual clarity.
- If  $n$  is a positive integer,  $\mathbb{R}^n$  (read “r-n”) denotes the collection of all lists (or *ordered  $n$ -tuples*) of  $n$  real numbers, usually written as  $n \times 1$  column matrices, such as

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

# ALGEBRAIC PROPERTIES OF $\mathbb{R}^n$

- The vector whose entries are all zero is called the **zero vector** and is denoted by **0**.
- For all  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  in  $\mathbb{R}^n$  and all scalars  $c$  and  $d$ :
  - (i)  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  - (ii)  $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
  - (iii)  $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
  - (iv)  $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$ ,  
where  $-\mathbf{u}$  denotes  $(-1)\mathbf{u}$
  - (v)  $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
  - (vi)  $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$



# LINEAR COMBINATIONS

- (vii)  $c(du) = (cd)(u)$

- (viii)  $1u = u$

- Given vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$  in  $\mathbb{R}^n$  and given scalars  $c_1, c_2, \dots, c_p$ , the vector  $\mathbf{y}$  defined by

$$\mathbf{y} = c_1 \mathbf{v}_1 + \dots + c_p \mathbf{v}_p$$

is called a **linear combination** of  $\mathbf{v}_1, \dots, \mathbf{v}_p$  with **weights**  $c_1, \dots, c_p$ .

- The weights in a linear combination can be any real numbers, including zero.

# LINEAR COMBINATIONS

- **Example 5:** Let  $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$ ,  $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$  and  $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$ .

Determine whether  $\mathbf{b}$  can be generated (or written) as a linear combination of  $\mathbf{a}_1$  and  $\mathbf{a}_2$ . That is, determine whether weights  $x_1$  and  $x_2$  exist such that

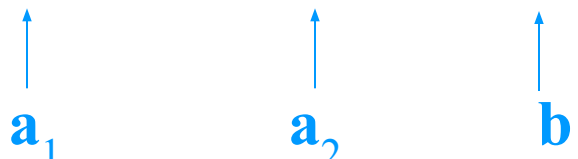
$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \quad (1)$$

If vector equation (1) has a solution, find it.

# LINEAR COMBINATIONS

**Solution:** Use the definitions of scalar multiplication and vector addition to rewrite the vector equation

$$x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix},$$

  
 $\mathbf{a}_1 \qquad \mathbf{a}_2 \qquad \mathbf{b}$

which is same as

$$\begin{bmatrix} x_1 \\ -2x_1 \\ -5x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 5x_2 \\ 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

# LINEAR COMBINATIONS

$$\text{and } \begin{bmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}. \quad (2)$$

- The vectors on the left and right sides of (2) are equal if and only if their corresponding entries are both equal. That is,  $x_1$  and  $x_2$  make the vector equation (1) true if and only if  $x_1$  and  $x_2$  satisfy the following system.

$$\begin{aligned} x_1 + 2x_2 &= 7 \\ -2x_1 + 5x_2 &= 4 \\ -5x_1 + 6x_2 &= -3 \end{aligned} \quad (3)$$

# LINEAR COMBINATIONS

- To solve this system, row reduce the augmented matrix of the system as follows:

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

- The solution of (3) is  $x_1 = 3$  and  $x_2 = 2$ . Hence  $\mathbf{b}$  is a linear combination of  $\mathbf{a}_1$  and  $\mathbf{a}_2$ , with weights  $x_1 = 3$  and  $x_2 = 2$ . That is,

$$3 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}.$$

# LINEAR COMBINATIONS

- Now, observe that the original vectors  $\mathbf{a}_1$ ,  $\mathbf{a}_2$ , and  $\mathbf{b}$  are the columns of the augmented matrix that we row reduced:

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix}$$

$\mathbf{a}_1$     $\mathbf{a}_2$     $\mathbf{b}$

- Write this matrix in a way that identifies its columns.

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{b}] \quad (4)$$

# LINEAR COMBINATIONS

- A vector equation

$$x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \dots + x_n \mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix} \quad (5)$$

- In particular,  $\mathbf{b}$  can be generated by a linear combination of  $\mathbf{a}_1, \dots, \mathbf{a}_n$  if and only if there exists a solution to the linear system corresponding to the matrix (5).

# LINEAR COMBINATIONS

- **Definition:** If  $\mathbf{v}_1, \dots, \mathbf{v}_p$  are in  $\mathbb{R}^n$ , then the set of all linear combinations of  $\mathbf{v}_1, \dots, \mathbf{v}_p$  is denoted by  $\text{Span} \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  and is called the **subset of  $\mathbb{R}^n$  spanned (or generated) by  $\mathbf{v}_1, \dots, \mathbf{v}_p$** . That is,  $\text{Span} \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is the collection of all vectors that can be written in the form

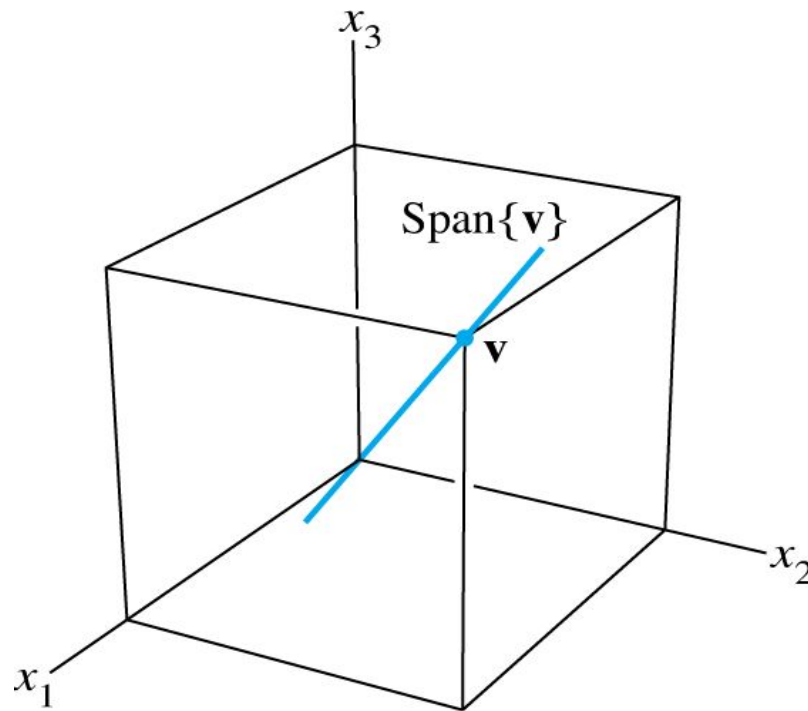
$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_p \mathbf{v}_p$$

with  $c_1, \dots, c_p$  scalars.



# A GEOMETRIC DESCRIPTION OF $\text{Span}\{\mathbf{v}\}$

- Let  $\mathbf{v}$  be a nonzero vector in  $\mathbb{R}^3$ . Then  $\text{Span}\{\mathbf{v}\}$  is the set of all scalar multiples of  $\mathbf{v}$ , which is the set of points on the line in  $\mathbb{R}^3$  through  $\mathbf{v}$  and  $\mathbf{0}$ . See Fig. 10 below:



# A GEOMETRIC DESCRIPTION OF SPAN $\{\mathbf{u}, \mathbf{v}\}$

- If  $\mathbf{u}$  and  $\mathbf{v}$  are nonzero vectors in  $\mathbb{R}^3$ , with  $\mathbf{v}$  not a multiple of  $\mathbf{u}$ , then  $\text{Span}\{\mathbf{u}, \mathbf{v}\}$  is the plane in  $\mathbb{R}^3$  that contains  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{0}$ .
- In particular,  $\text{Span}\{\mathbf{u}, \mathbf{v}\}$  contains the line in  $\mathbb{R}^3$  through  $\mathbf{u}$  and  $\mathbf{0}$  and the line through  $\mathbf{v}$  and  $\mathbf{0}$ . See Fig. 11 below:

