3 Determinants

3.2

PROPERTIES OF DETERMINANTS

LINEAR ALGEBRA
AND ITS APPLICATIONS
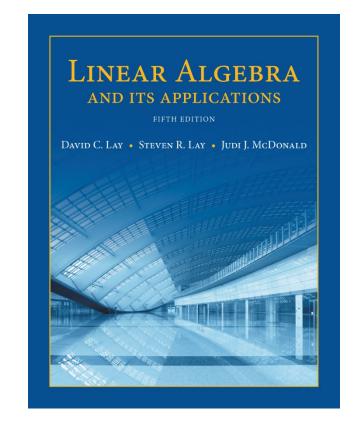
FIFTH EDITION

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PROPERTIES OF DETERMINANTS

- **Theorem 3**: Let A be a square matrix

  a) If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.

  b) If two rows of A are interchanged to produce B, then $\det B = -\det A$.

  c) If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$

PROPERTIES OF DETERMINANTS

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- ▪ **Example 1** Compute det A, where $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$

- ▪ **Solution** The strategy is to reduce A to echelon form and then to use the fact that the determinant of a triangular matrix is the product of the diagonal entries. The first two row replacements in column 1 do not change the determinant:

$$det A = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix}$$

PROPERTIES OF DETERMINANTS

- An interchange of rows 2 and 3 reverses the sign of the determinant, so

$$det A = - \begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = -(1)(3)(-5) = 15$$

PROPERTIES OF DETERMINANTS

- **Theorem 4**: A square matrix A is invertible if and only if det $A \neq 0$.

- **Example 3** Compute det A, where $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$

- **Solution** Add 2 times row 1 to row 3 to obtain

$$detA = det\begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix} = 0$$

COLUMN OPERATIONS

- **Theorem 5:** If A is a $n \times n$ matrix, then $\det A^T = \det A$.

- **Proof**: The theorem is obvious for $n = 1$. Suppose the theorem is true for $k \times k$ determinants and let $n = k + 1$.

- Then the cofactor of a_{1j} in A equals the cofactor of a_{j1} in A^T, because the cofactors involve $k \times k$ determinants.

- Hence the cofactor expansion of $\det A$ along the first row equals the cofactor expansion of $\det A^T$ down the first column. That is, A and A^T have equal determinants.

- Thus the theorem is true for $n = 1$, and the truth of the theorem for one value of n implies its truth for the next value of n. By the principle of induction, the theorem is true

DETERMINANTS AND MATRIX PRODUCTS

- **Theorem 6:** If A and B are $n \times n$ matrices, then det AB = (det A)(det A).

- **Example 5** Verify Theorem 6 for $A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$.

- Solution

$$AB = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 14 & 13 \end{bmatrix}$$

- and

$$\det AB = 25 \cdot 13 - 20 \cdot 14 = 325 - 280 = 45$$

Since det A = 9 and det B = 5,