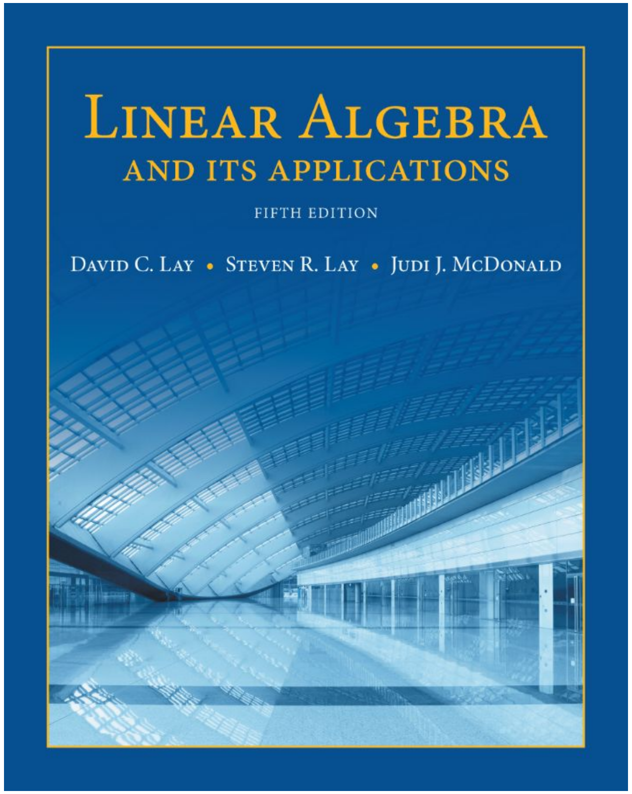
3 Determinants

3.3

CRAMER'S RULE, VOLUME, AND LINEAR TRANSFORMATIONS



CRAMER'S RULE

- **Theorem 7**: Let A be an invertible $n \times n$ matrix. For any b in $\mathbb{R}^n$, the unique solution x of Ax=b has entries given by

$$x_i = \frac{detA_i(b)}{detA}, \qquad i = 1, 2, \ldots, n \qquad (1)$$

- **Proof** Denote the columns of A by $a_1, \ldots, a_n$ and the columns of the $n \times n$ identity matrix I by $e_1, \ldots, e_n$. If Ax $= b$, the definition of matrix multiplication shows that

$$A \cdot I_i(x) = A[e_1 \ \ldots \ x \ \ldots \ e_n] = A[e_1 \ \ldots \ Ax \ \ldots \ Ae_n]$$
$$= [a_1 \ \ldots \ b \ \ldots \ a_n] = A_i(b)$$

CRAMER'S RULE

- By the multiplicative property of determinants,

$$(detA)\big(detI_i(x)\big) = detA_i(b)$$

- The second determinant on the left is simply x_i. Hence $(detA) \cdot x_i = detA_i(b)$. This proves (1) because A is invertible and det $A \neq 0$.

- **Example 1** Use Cramer's rule to solve the system

$$3x_1 - 2x_2 = 6$$
$$-5x_1 + 4x_2 = 8$$

CRAMER'S RULE

- **Solution** View the system as $A\mathrm{x} = b$. Using the notation introduced above,

$$A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}, \quad A_1(b) = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix}, \quad A_2(b) = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}$$

- Since $\det A = 2$, the system has a unique solution. By Cramer's rule,

$$x_1 = \frac{detA_1(b)}{detA} = \frac{24+16}{2} = 20$$

$$x_2 = \frac{detA_2(b)}{detA} = \frac{24+30}{2} = 27$$

A FORMULA FOR A^{-1}

- **Theorem 8**: Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

- **Example 3** Find the inverse of the matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$.

- **Solution** The nine cofactors are

$$C_{11} = +\begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} = -2, \quad C_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 3, \quad C_{13} = +\begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} = 5$$

$$C_{21} = -\begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = 14, \quad C_{22} = +\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -7, \quad C_{23} = -\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = -7$$

$$C_{31} = +\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 4, \quad C_{32} = -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 1, \quad C_{33} = +\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3$$

A FORMULA FOR A^{-1}

- The adjugate matrix is the *transpose* of the matrix of cofactors. Thus

$$adj A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}$$

- We could compute det A directly, but the following computation provides a check on the calculations above and produces det A:

$$(adj A) \cdot A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix} = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} = 14\mathrm{I}$$

A FORMULA FOR A^{-1}

- Since (adj A)$A = 14I$, Theorem 8 shows that det $A = 14$ and

$$A^{-1} = \frac{1}{14}\begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} = \begin{bmatrix} -1/7 & 1 & 2/7 \\ 3/14 & -1/2 & 1/14 \\ 5/14 & -1/2 & -3/14 \end{bmatrix}$$

DETERMINANTS AS AREA OR VOLUME

- **Theorem 9**: If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $[\det A]$. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

- **Proof** The theorem is obviously true for any 2×2 diagonal matrix:

$$\left| det \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \right| = |ad| = \begin{Bmatrix} area\ of \\ rectangle \end{Bmatrix}$$

- See Fig. 1 on the next slide.

DETERMINANTS AS AREA OR VOLUME

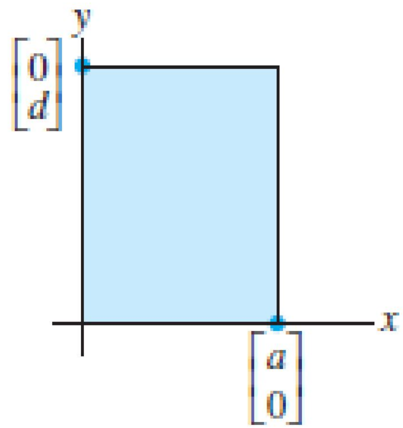
- 

FIGURE 1
Area $= |ad|$.

- It will suffice to show that any 2×2 matrix $A = [a_1 \ \ a_2]$ can be transformed into a diagonal matrix in a way that changes neither the area of the associated parallelogram nor $|\det A|$.

DETERMINANTS AS AREA OR VOLUME

- It suffices to prove the following simple geometric observation that applies to vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$:
- Let a_1 and a_2 be nonzero vectors. Then for any scalar c, the area of the parallelogram determined by a_1 and a_2 equals the area of the parallelogram determined by a_1 and a_2+ca_1.
- To prove this statement, we may assume that a_2 is not a multiple of a_1, for otherwise the two parallelograms would be degenerate and have zero area.
- If L is the line through 0 and a_1, then $a_2 + L$ is the line through a_2 parallel to L, and $a_2 + ca_1$ is on this line. See Fig. 2 on the next slide.

DETERMINANTS AS AREA OR VOLUME

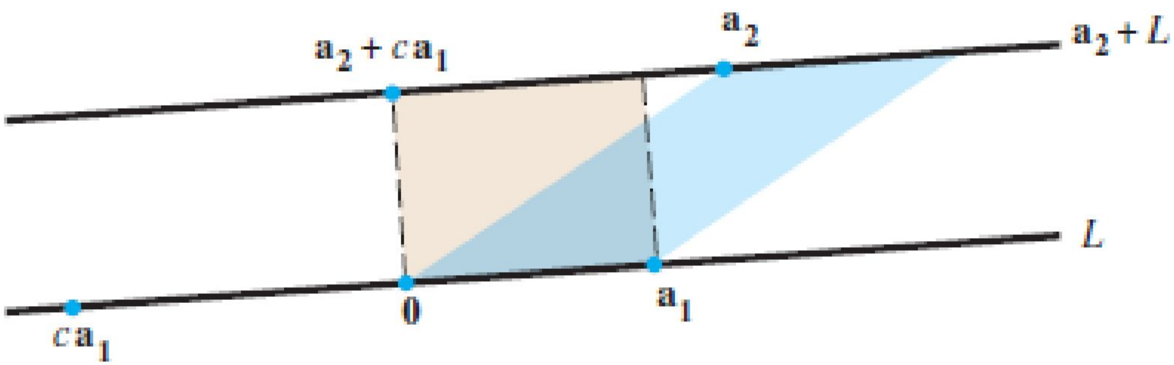


FIGURE 2 Two parallelograms of equal area.

- The points a_2 and $a_2 + ca_1$ have the same perpendicular distance to L. Hence the two parallelograms in Fig. 2 have the same area, since they share the base from 0 to a_1.

DETERMINANTS AS AREA OR VOLUME

- The proof for $\mathbb{R}^3$ is similar. The theorem is obviously true for a 3×3 diagonal matrix. See Fig. 3 below:

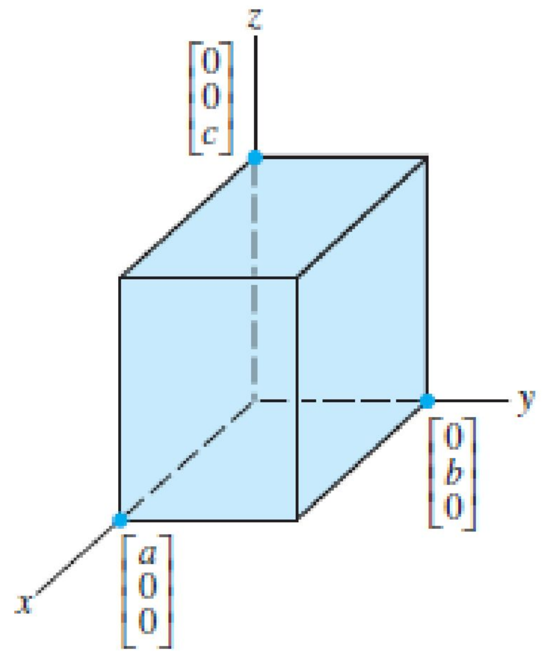


FIGURE 3
Volume $= |abc|$.

DETERMINANTS AS AREA OR VOLUME

- And any 3 × 3 matrix A can be transformed into a diagonal matrix using column operations that do not change $|\det A|$.
- A parallelepiped is shown in Fig. 4 below as a shaded box with two sloping sides.

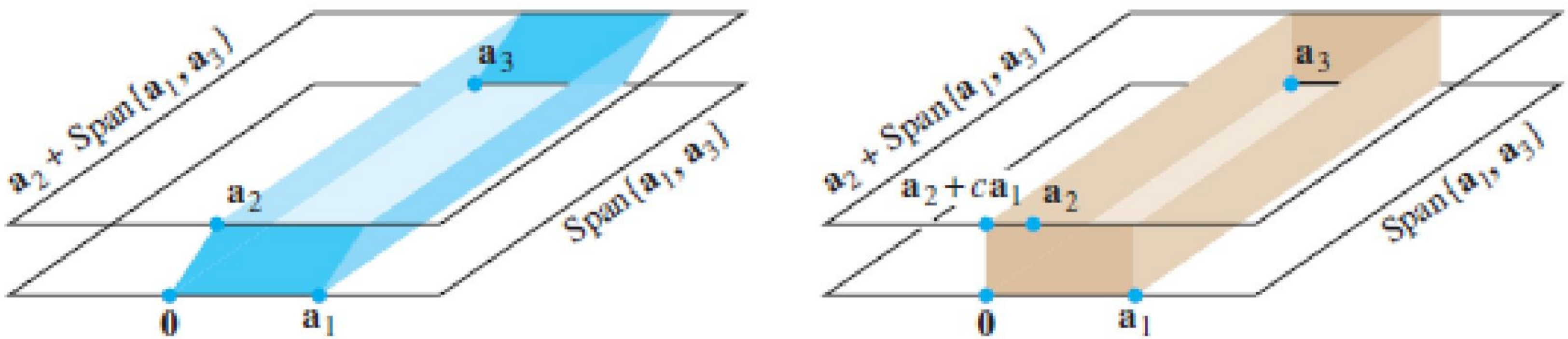


FIGURE 4 Two parallelepipeds of equal volume.

DETERMINANTS AS AREA OR VOLUME

- Its volume is the area of the base in the plane Span$\{a_1, a_3\}$ times the altitude of a_2 above Span$\{a_1, a_3\}$. Any vector $a_2 + ca_1$ lies in the plan Span$\{a_1, a_3\}$, which is parallel to Span$\{a_1, a_3\}$.
- Hence the volume of the parallelepiped is unchanged when $[a_1 \;\; a_2 \;\; a_3]$ is changed to $[a_1 \quad a_2 + ca_1 \quad a_3]$.
- Thus a column replacement operation does not affect the volume of the parallelepiped. Since the column interchanges have no effect on the volume, the proof is complete.

DETERMINANTS AS AREA OR VOLUME

- **Example 4** Calculate the area of the parallelogram determined by the points (-2, -2), (0, 3), (4, -1), and (6, 4). See Fig. 5(a) below:

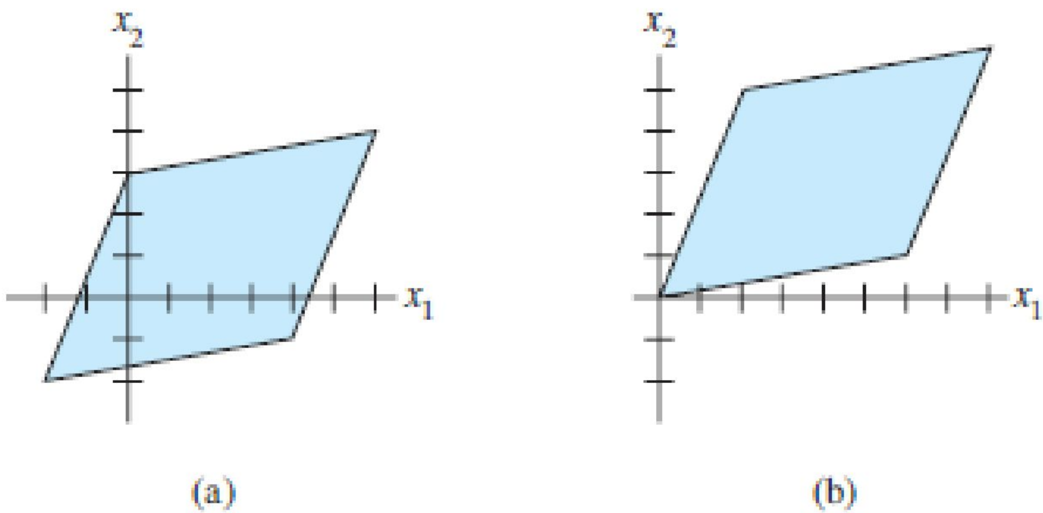


FIGURE 5 Translating a parallelogram does not change its area.

DETERMINANTS AS AREA OR VOLUME

- **Solution** First translate the parallelogram to one having the origin as a vertex. For example, subtract the vertex (-2, -2) from each of the four vertices.
- The new parallelogram has the same area, and its vertices are (0, 0), (2, 5), (6, 1), and (8, 6). See Fig. 5(b) on the previous slide.
- This parallelogram is determined by the columns of

$$A = \begin{bmatrix} 2 & 6 \\ 5 & 1 \end{bmatrix}$$

- Since $|\det A| = |-28|$, the area of the parallelogram is 28.

LINEAR TRANSFORMATIONS

- **Theorem 10**: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{area\ of\ T(S)\} = |detA| \cdot \{area\ of\ S\} \qquad (5)$$

- If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then

$$\{volume\ of\ T(S)\} = |detA| \cdot \{volume\ of\ S\} \qquad (6)$$

- **Proof** Consider the 2×2 case, with $A = [a_1\ \ a_2]$. A parallelogram at the origin in $\mathbb{R}^2$ determined by vectors b_1 and b_2 has the form

$$S = \{s_1 b_1 + s_2 b_2 \colon 0 \le s_1 \le 1,\ 0 \le s_2 \le 1\}$$

LINEAR TRANSFORMATIONS

- The image of S under T consists of points of the form

$$\begin{aligned} T(s_1 b_1 + s_2 b_2) &= s_1 T(b_1) + s_2 T(b_2) \\ &= s_1 b_1 + s_2 b_2 \end{aligned}$$

- where $0 \le s_1 \le 1,\ 0 \le s_2 \le 1$. It follows that $T(S)$ is the parallelogram determined by the columns of the matrix $[A\mathrm{b}_1 \ A\mathrm{b}_2]$. This matrix can be written as AB, where $B = [b_1 \ b_2]$.
- By Theorem 9 and the product theorem for determinants,

$$\begin{aligned} \{area\ of\ T(S)\} &= |detAB| = |detA| \cdot |detB| \\ &= |detA| \cdot \{area\ of\ S\} \qquad (7) \end{aligned}$$

LINEAR TRANSFORMATIONS

- An arbitrary parallelogram has the form $\mathbf{p} + S$, where $\mathbf{p}$ is a vector and S is a parallelogram at the origin.
- It is easy to see that T transforms $\mathbf{p} + S$ into $T(p) + T(S)$. Since translation does not affect the area of a set,

$$
\begin{aligned}
\{\text{area of } \mathrm{T}(p+S)\} &= \{\text{area of } \mathrm{T}(p) + T(S)\} \\
&= \{\text{area of } \mathrm{T}(S)\} && \text{Translation} \\
&= |detA| \cdot \{area\ of\ S\} && \text{By equation (7)} \\
&= |detA| \cdot \{area\ of\ \mathbf{p} + S\} && \text{Translation}
\end{aligned}
$$

- This shows that (5) holds for all parallelograms in $\mathbb{R}^2$. The proof of (6) for the 3×3 case is analogous.

LINEAR TRANSFORMATIONS

- **Example 5** Let a and b be positive numbers. Find the area of the region E bounded by the ellipse whose equation is

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$$

- **Solution** We claim that E is the image of the unit disk D under the linear transformation T determined by the matrix $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, because if $\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and x = A**u**, then

$$u_1 = \frac{x_1}{a} \text{ and } u_2 = \frac{x_2}{b}$$

LINEAR TRANSFORMATIONS

- It follows that **u** is in the unit disk, with $u_2^1 + u_2^2 \leq 1$, if any only if **x** is in E, with $(x_1/a)^2 + (x_2/b)^2 \leq 1$. By generalization of Theorem 10,

$$\begin{aligned} \{area\ of\ ellipse\} &= \{area\ of\ T(D)\} \\ &= |detA| \cdot \{area\ of\ D\} \\ &= ab \cdot \pi(1)^2 = \mu ab \end{aligned}$$