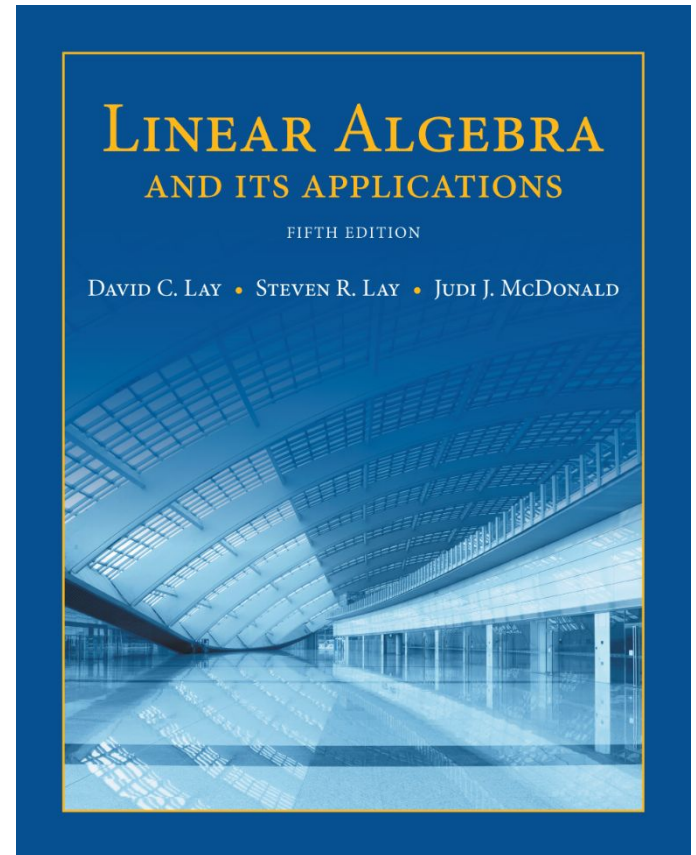


# 1

## Linear Equations in Linear Algebra

### 1.1

## SYSTEMS OF LINEAR EQUATIONS



# LINEAR EQUATION

- A **linear equation** in the variables  $x_1, \dots, x_n$  is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where  $b$  and the coefficients  $a_1, \dots, a_n$  are real or complex numbers, usually known in advance.

- A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables — say,  $x_1, \dots, x_n$ .

# LINEAR EQUATION

- A **solution** of the system is a list  $(s_1, s_2, \dots, s_n)$  of numbers that makes each equation a true statement when the values  $s_1, \dots, s_n$  are substituted for  $x_1, \dots, x_n$ , respectively.
- The set of all possible solutions is called the **solution set** of the linear system.
- Two linear systems are called **equivalent** if they have the same solution set.

# LINEAR EQUATION

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- A system of linear equations has
  1. no solution, or
  2. exactly one solution, or
  3. infinitely many solutions.
- A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions.
- A system is **inconsistent** if it has no solution.

# MATRIX NOTATION

- The essential information of a linear system can be recorded compactly in a rectangular array called a **matrix**. Given the system,

$$x_1 - 2x_2 + x_3 = 0$$

$$2x_2 - 8x_3 = 8$$

$$-4x_1 + 5x_2 + 9x_3 = -9,$$

with the coefficients of each variable aligned in columns, the matrix

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

Is called the **coefficient matrix** (or matrix of **coefficients**) of the system

# MATRIX NOTATION

- An **augmented matrix** of a system consists of the coefficient matrix with an added column containing the constants from the right sides of the equations.
- For the given system of equations,

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

is called the **augmented matrix** of the system.

# MATRIX SIZE

- The size of a matrix tells how many rows and columns it has. If  $m$  and  $n$  are positive integers, an  **$m \times n$  matrix** is a rectangular array of numbers with  $m$  rows and  $n$  columns. (The number of rows always comes first.)
- The basic strategy for solving a linear system is to *replace one system with an equivalent system (i.e., one with the same solution set) that is easier to solve.*

# SOLVING SYSTEM OF EQUATIONS

- **Example 1:** Solve the given system of equations.

$$x_1 - 2x_2 + x_3 = 0 \quad \text{----(1)}$$

$$2x_2 - 8x_3 = 8 \quad \text{----(2)}$$

$$-4x_1 + 5x_2 + 9x_3 = -9 \quad \text{----(3)}$$

- **Solution:** The elimination procedure is shown here with and without matrix notation, and the results are placed side by side for comparison.



# SOLVING SYSTEM OF EQUATIONS

$$\begin{array}{rcl} x_1 - 2x_2 + x_3 & = & 0 \\ 2x_2 - 8x_3 & = & 8 \\ -4x_1 + 5x_2 + 9x_3 & = & -9 \end{array} \quad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

- Keep  $x_1$  in the first equation and eliminate it from the other equations. To do so, add 4 times equation 1 to equation 3.

$$\begin{array}{rcl} 4x_1 - 8x_2 + 4x_3 & = & 0 \\ -4x_1 + 5x_2 + 9x_3 & = & -9 \\ \hline -3x_2 + 13x_3 & = & -9 \end{array}$$

# SOLVING SYSTEM OF EQUATIONS

- The result of this calculation is written in place of the original third equation:

$$\begin{array}{rcl} x_1 - 2x_2 + x_3 & = & 0 \\ 2x_2 - 8x_3 & = & 8 \\ -3x_2 + 13x_3 & = & -9 \end{array} \quad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

- Now, multiply equation 2 by  $1/2$  in order to obtain 1 as the coefficient for  $x_2$ .

# SOLVING SYSTEM OF EQUATIONS

$$\begin{array}{rcl} x_1 - 2x_2 + x_3 & = & 0 \\ x_2 - 4x_3 & = & 4 \\ -3x_2 + 13x_3 & = & -9 \end{array} \quad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

- Use the  $x_2$  in equation 2 to eliminate the  $-3x_2$  in equation 3.

$$\begin{array}{rcl} 3x_2 - 12x_3 & = & 12 \\ -3x_2 + 13x_3 & = & -9 \\ \hline x_3 & = & 3 \end{array}$$

# SOLVING SYSTEM OF EQUATIONS

- The new system has a *triangular* form.

$$\begin{array}{rcl} x_1 - 2x_2 + x_3 & = & 0 \\ x_2 - 4x_3 & = & 4 \\ \hline & & x_3 = 3 \end{array} \quad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

- Eventually, you want to eliminate the  $-2x_2$  term from equation 1, but it is more efficient to use the  $x_3$  term in equation 3 first to eliminate the  $-4x_3$  and  $x_3$  terms in equations 2 and 1.

# SOLVING SYSTEM OF EQUATIONS

$$4x_3 = 12$$

$$-x_3 = -3$$

$$\underline{x_2 - 4x_3 = 4}$$

$$\underline{x_1 - 2x_2 + x_3 = 0}$$

$$x_2 = 16$$

$$x_1 - 2x_2 = -3$$

- Now, combine the results of these two operations.

$$x_1 - 2x_2 = -3$$

$$x_2 = 16$$

$$x_3 = 3$$

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

# SOLVING SYSTEM OF EQUATIONS

- Move back to the  $x_2$  in equation 2, and use it to eliminate the  $-2x_2$  above it. Because of the previous work with  $x_3$ , there is now no arithmetic involving  $x_3$  terms. Add 2 times equation 2 to equation 1 and obtain the system:

$$\begin{array}{l} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3 \end{array} \quad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

# SOLVING SYSTEM OF EQUATIONS

- Thus, the only solution of the original system is  $(29, 16, 3)$ . To verify that  $(29, 16, 3)$  is a solution, substitute these values into the left side of the original system, and compute.

$$(29) - 2(16) + (3) = 29 - 32 + 3 = 0$$

$$2(16) - 8(3) = 32 - 24 = 8$$

$$-4(29) + 5(16) + 9(3) = -116 + 80 + 27 = -9$$

- The results agree with the right side of the original system, so  $(29, 16, 3)$  is a solution of the system.

# ELEMENTARY ROW OPERATIONS

- Elementary row operations include the following:
  1. (Replacement) Replace one row by the sum of itself and a multiple of another row.
  2. (Interchange) Interchange two rows.
  3. (Scaling) Multiply all entries in a row by a nonzero constant.
  
- Two matrices are called **row equivalent** if there is a sequence of elementary row operations that transforms one matrix into the other.



# ELEMENTARY ROW OPERATIONS

- It is important to note that row operations are *reversible*.
- If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.
- Two fundamental questions about a linear system are as follows:
  1. Is the system consistent; that is, does at least one solution *exist*?
  2. If a solution exists, is it the *only* one; that is, is the solution *unique*?

# EXISTENCE AND UNIQUENESS OF SYSTEM OF EQUATIONS

- **Example 3:** Determine if the following system is consistent:

$$\begin{aligned}x_2 - 4x_3 &= 8 \\2x_1 - 3x_2 + 2x_3 &= 1 \\5x_1 - 8x_2 + 7x_3 &= 1\end{aligned}\tag{5}$$

- **Solution:** The augmented matrix is

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

# EXISTENCE AND UNIQUENESS OF SYSTEM OF EQUATIONS

- To obtain an  $x_1$  in the first equation, interchange rows 1 and 2:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

- To eliminate the  $5x_1$  term in the third equation, add  $-5/2$  times row 1 to row 3.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -1/2 & 2 & -3/2 \end{bmatrix} \quad (6)$$

# EXISTENCE AND UNIQUENESS OF SYSTEM OF EQUATIONS

- Next, use the  $x_2$  term in the second equation to eliminate the  $-(1/2)x_2$  term from the third equation. Add  $1/2$  times row 2 to row 3.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix} \quad (7)$$

- The augmented matrix is now in triangular form. To interpret it correctly, go back to equation notation.

$$2x_1 - 3x_2 + 2x_3 = 1$$

$$x_2 - 4x_3 = 8 \quad (8)$$

$$0 = 5/2$$

# EXISTENCE AND UNIQUENESS OF SYSTEM OF EQUATIONS

- The equation  $0 = 5 / 2$  is a short form of  $0x_1 + 0x_2 + 0x_3 = 5 / 2$ .
- There are no values of  $x_1, x_2, x_3$  that satisfy (8) because the equation  $0 = 5 / 2$  is never true.
- Since (8) and (5) have the same solution set, the original system is inconsistent (*i.e.*, has no solution).