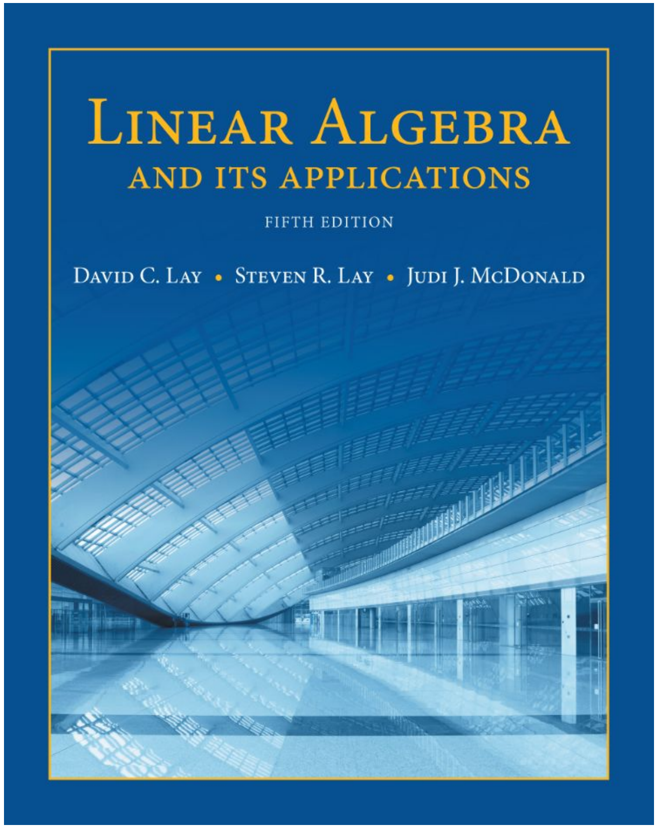
1 Linear Equations in Linear Algebra

1.8

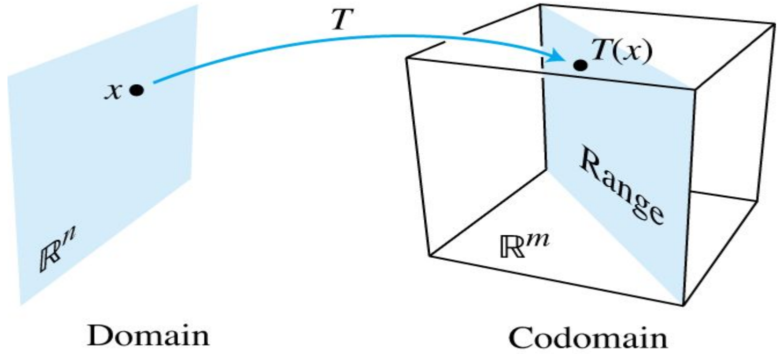
INTRODUCTION TO LINEAR TRANSFORMATIONS



LINEAR TRANSFORMATIONS

- A **transformation** (or **function** or **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.
- The set $\mathbb{R}^n$ is called **domain** of T, and $\mathbb{R}^m$ is called the **codomain** of T.
- The notation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ indicates that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$.
- For $\mathbf{x}$ in $\mathbb{R}^n$, the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is called the **image** of $\mathbf{x}$ (under the action of T).
- The set of all images $T(\mathbf{x})$ is called the **range** of T. See Fig. 2 on the next slide

MATRIX TRANSFORMATIONS



Domain, codomain, and range of $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$.

- For each $\mathbf{x}$ in $\mathbb{R}^n$, $T(\mathbf{x})$ is computed as $A\mathbf{x}$, where A is an $m \times n$ matrix.
- For simplicity, we denote such a *matrix transformation* by $\mathrm{x} \mapsto A\mathrm{x}$.
- Observe that the domain of T is $\mathbb{R}^n$ when A has n columns and the codomain of T is $\mathbb{R}^m$ when each column of A has m entries.

MATRIX TRANSFORMATIONS

- The range of T is the set of all linear combinations of the columns of A, because each image $T(\mathbf{x})$ is of the form $A\mathbf{x}$.
- **Example 1:**

Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\mathrm{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\mathrm{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$.

and define a transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ by $T(\mathrm{x}) = A\mathrm{x}$, so that

$$T(\mathrm{x}) = A\mathrm{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}.$$

MATRIX TRANSFORMATIONS

a. Find $T(\mathbf{u})$, the image of $\mathbf{u}$ under the transformation T.

b. Find an $\mathbf{x}$ in $\mathbb{R}^2$ whose image under T is $\mathbf{b}$.

c. Is there more than one $\mathbf{x}$ whose image under T is $\mathbf{b}$?

d. Determine if $\mathbf{c}$ is in the range of the transformation T.

Solution:

a. Compute

$$T(\mathrm{u}) = A\mathrm{u} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}.$$

b. Solve $T(\mathrm{x}) = \mathrm{b}$ for **x**. That is, solve $A\mathrm{x} = \mathrm{b}$, or

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \quad (1)$$

MATRIX TRANSFORMATIONS

- Row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -.5 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -.5 \\ 0 & 0 & 0 \end{bmatrix} \quad (2)$$

- Hence $x_1 = 1.5$, $x_2 = -.5$, and $\mathbf{x} = \begin{bmatrix} 1.5 \\ -.5 \end{bmatrix}$.
- The image of this **x** under T is the given vector **b**.

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c. Any **x** whose image under T is **b** must satisfy equation (1).
   - From (2), it is clear that equation (1) has a unique solution.
   - So there is exactly one **x** whose image is **b**.

d. The vector **c** is in the range of T if **c** is the image of some **x** in $\mathbb{R}^2$, that is, if $\mathbf{c} = T(\mathbf{x})$ for some **x**.
   - This is another way of asking if the system $A\mathbf{x} = \mathbf{c}$ is consistent.

MATRIX TRANSFORMATIONS

- To find the answer, row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 14 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{bmatrix}$$

- The third equation, $0 = -35$, shows that the system is inconsistent.
- So **c** is *not* in the range of T.

SHEAR TRANSFORMATION

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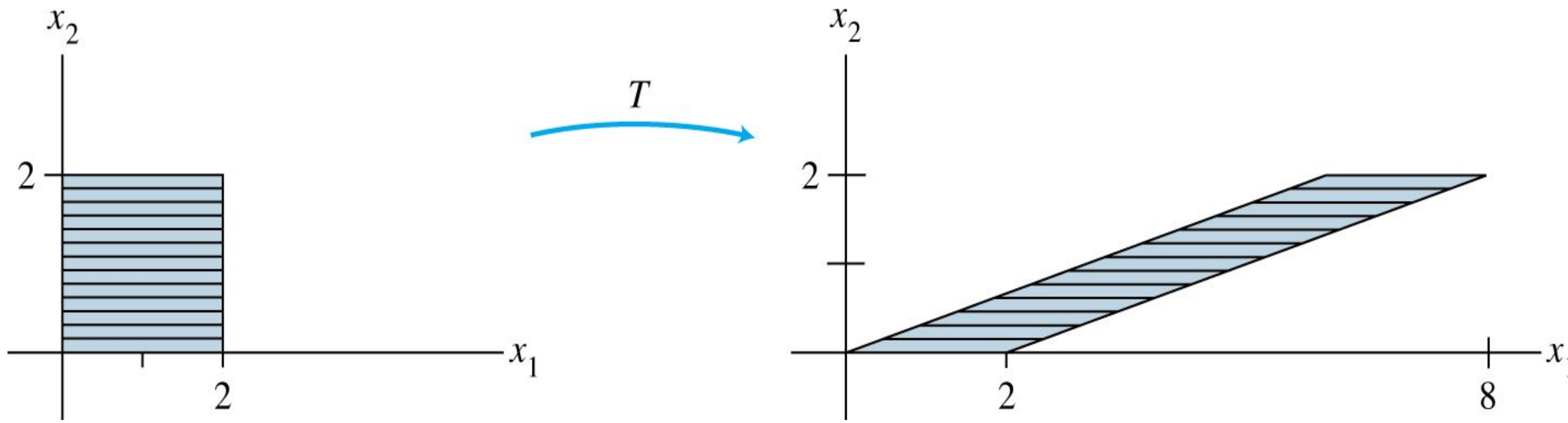
- **Example 3:** Let $A=\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}$. The transformation

$T:\mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(\mathrm{x})=A\mathrm{x}$ is called a **shear transformation**.

- It can be shown that if T acts on each point in the 2×2 square shown in Fig. 4 on the next slide, then the set of images forms the shaded parallelogram.

SHEAR TRANSFORMATION



- The key idea is to show that T maps line segments onto line segments and then to check that the corners of the square map onto the vertices of the parallelogram.
- For instance, the image of the point $\mathbf{u} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ is

$$T(\mathbf{u}) = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix},$$

LINEAR TRANSFORMATIONS

and the image of $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$.

- T deforms the square as if the top of the square were pushed to the right while the base is held fixed.
- **Definition:** A transformation (or mapping) T is **linear** if:
  i. $T(\mathrm{u}+\mathrm{v}) = T(\mathrm{u}) + T(\mathrm{v})$ for all **u**, **v** in the domain of T;
  ii. $T(c\mathrm{u}) = cT(\mathrm{u})$ for all scalars c and all **u** in the domain of T.

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- Linear transformations *preserve the operations of vector addition and scalar multiplication*.
- Property (i) says that the result $T(\mathrm{u}+\mathrm{v})$ of first adding $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and then applying T is the same as first applying T to $\mathbf{u}$ and $\mathbf{v}$ and then adding $T(\mathbf{u})$ and $T(\mathbf{v})$ in $\mathbb{R}^m$.
- These two properties lead to the following useful facts.
- If T is a linear transformation, then

$$T(0)=0 \tag{3}$$

LINEAR TRANSFORMATIONS

and $T(c\mathrm{u}+d\mathrm{v})=cT(\mathrm{u})+dT(\mathrm{v})$. (4)

for all vectors **u**, **v** in the domain of T and all scalars c, d.

- Property (3) follows from condition (ii) in the definition, because $T(0)=T(0\mathrm{u})=0T(\mathrm{u})=0$.
- Property (4) requires both (i) and (ii): $T(c\mathrm{u}+d\mathrm{v})=T(c\mathrm{u})+T(d\mathrm{v})=cT(\mathrm{u})+dT(\mathrm{v})$
- *If a transformation satisfies* (4) *for all* **u**, **v** and c, d, *it must be linear*.
- (Set $c=d=1$ for preservation of addition, and set $d=0$ for preservation of scalar multiplication.)

LINEAR TRANSFORMATIONS

- Repeated application of (4) produces a useful generalization:

$$T(c_1\mathrm{v}_1 + \ldots + c_p\mathrm{v}_p) = c_1T(\mathrm{v}_1) + \ldots + c_pT(\mathrm{v}_p) \quad (5)$$

- In engineering and physics, (5) is referred to as a *superposition principle*.

- Think of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ as signals that go into a system and $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)$ as the responses of that system to the signals.

LINEAR TRANSFORMATIONS

- The system satisfies the superposition principle if whenever an input is expressed as a linear combination of such signals, the system's response is the *same* linear combination of the responses to the individual signals.

- Given a scalar r, define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathrm{x}) = r\mathrm{x}$.

- T is called a **contraction** when $0 \leq r \leq 1$ and a **dilation** when $r > 1$.