1. Another view of NPHSS

Given a linear mapping $F\colon\ \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$,let $F(x) = Ax - b$,with $A \in \mathbb{C}^{n\times n}$ a non-Hermitian positive definite matrix and $b \in \mathbb{C}^n$ a given vector, we can construct a linear equations systems as

$Ax = b,\ A \in \mathbb{C}^{n\times n},\ x, b \in \mathbb{C}^n.$

Based on Hermitian and skew-Hermitian splitting for matrix $\mathrm{A} \in \mathbb{C}^{n\times n}$, we could get: $A = H + S$.

where $H = \frac{1}{2}(A + A^*),\ S = \frac{1}{2}(A - A^*)$.

Given an initial guess $x_0 \in \mathbb{C}^n$,we compute x_{k+1} for k= 0,1,2 …, using the following iteration scheme until sequence $\{x_k\}_{k=0}^{\infty} \subset \mathbb{C}^n$ satisfies the stopping criterion: $(\alpha P + H)x_{k+1} = (\alpha P - S)x_k + b$ where α is a given positive constant and $P \in \mathbb{C}^{n\times n}$ is a prescribed Hermitian positive definite matrix.

We can rewrite as: $x_{k+1} = T(P;\alpha)x_k + G(P;\alpha)b,\ k = 0,1,2 \ldots,$

Keep doing this until we can use x_0 to express it：$x_{k+1} = T(P;\alpha)^{k+1}x_0 + \sum_{j=0}^{l} T(P;\alpha)^j G(P;\alpha)b.$

Where $T(P;\alpha) = (\alpha P + H)^{-1}(\alpha P - S), G(P;\alpha) = (\alpha P + H)^{-1}$.

Here, $T(P;\alpha)$ is the iteration matrix of the NPHSS method. In fact, we can split the coefficient matrix A by: $A = B(P;\alpha) - C(P;\alpha)$, where $B(P;\alpha) = \alpha P + H,\ C(P;\alpha) = \alpha P - S,$

So，the result in (2.3) can represented by $\mathrm{B}(P;\alpha)$ and $C(P;\alpha)$:

$$T(P;\alpha) = B(P;\alpha)^{-1}C(P;\alpha),\ G(P;\alpha) = B(P;\alpha)^{-1}.$$

Note that $T(P;\alpha)$ can be reformulated as: $T(P;\alpha) = I - (\alpha P + H)^{-1}A$.

Then the matrix $(\alpha P + H)$ can be viewed as a preconditioner for the coefficient matrix $A \in \mathbb{C}^{n\times n}$.

1. convergence of NPHSS iteration method

Next we show the conditions for the convergence of NPHSS method.

Since H is Hermitian and S is skew-Hermitian, we could know that all the eigenvalues of $P^{-1}H$ are real positive and all the eigenvalues of $P^{-1}S$ are imaginary.

Here, we denote:：

$$\lambda_{max} = \max_{\lambda_j \epsilon sp(P^{-1}H)}\{\lambda_j\},$$

$$\lambda_{min} = \min_{\lambda_j \epsilon sp(P^{-1}H)}\{\lambda_j\},$$

$$\xi_{max} = \max_{i\xi_j \epsilon sp(P^{-1}S)}\{|\xi_j|\},$$

Where $sp(X)$ is the spectrum of the matrix X and $i = \sqrt{-1}$

We use the following theorem to give the convergence result of NPHSS iteration method.

Theorem 1 Let $A \in \mathbb{C}^{n\times n}$ be a positive definite matrix, let $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$ be its Hermitian and skew-Hermitian parts, and let α be a positive constant. Let $P \in \mathbb{C}^{n\times n}$ be a Hermitian positive definite matri. Then the spectral radius $\rho\big(T(P;\alpha)\big)$ of the NPHSS iteration matrix satisfies $\rho\big(T(P;\alpha)\big) \le \sigma(\alpha)$, where $\sigma(\alpha) = \frac{\sqrt{\alpha^2+\xi_{max}{}^2}}{\alpha+\lambda_{min}}$.

We can find the range of α:

（1）If $\lambda_{min} \ge \xi_{max}$, then $\sigma(\alpha) < 1$ for any $\alpha > 0$, which means that the NPHSS iteration method is unconditionally convergent;

（2）If $\lambda_{min} < \xi_{max}$时， then $\sigma(\alpha) < 1$ if and only if $\alpha > \frac{\xi_{max}{}^2-\lambda_{min}{}^2}{2\lambda_{min}}$.

Which means that under the condition above, NPHSS iteration method is convergent.

Detailed proof by [1]

2. More about MN-NPHSS

$d_{k,l_k} = -\sum_{j=0}^{l_k-1} T(P;\alpha;x_k)^j\, G(P;\alpha;x_k)F(x_k),\ \ h_{k,m_k} = -\sum_{j=0}^{m_k-1} T(P;\alpha;x_k)^j\, G(P;\alpha;x_k)F(y_k),$

Where $T(P;\alpha;x) = \big(\alpha P + H(x)\big)^{-1}\big(\alpha P - S(x)\big), G(P;\alpha;x) = \big(\alpha P + H(x)\big)^{-1}$.

So the MN-NPHSS method could be written as:

$$\begin{cases} y_k = x_k - \displaystyle\sum_{j=0}^{l_k-1} T(P;\alpha;x_k)^j\, G(P;\alpha;x_k)F(x_k) \\ x_{k+1} = y_k - \displaystyle\sum_{j=0}^{m_k-1} T(P;\alpha;x_k)^j\, G(P;\alpha;x_k)F(y_k) \end{cases}.$$

Define: $B(P;\alpha;x) = \alpha P + H(x),\ C(P;\alpha;x) = \alpha P - S(x),$

Then the Jacobian matrix $F'(x)$ can be written as: $F'(x) = B(P;\alpha;x) - C(P;\alpha;x),$

From above, $T(P;\alpha;x) = I - \big(\alpha P + H(x)\big)^{-1}F'(x)$, i.e., $I - T(P;\alpha;x) = \big(\alpha P + H(x)\big)^{-1}F'(x),$

Then, $F'(x)^{-1} = \big(I - T(P;\alpha;x)\big)^{-1}\big(\alpha P + H(x)\big)^{-1} = \big(I - T(P;\alpha;x)\big)^{-1} B(P;\alpha;x)^{-1}$,

We obtain $T(P;\alpha;x) = \mathrm{B}(P;\alpha;x)^{-1}C(P;\alpha;x),\ G(P;\alpha;x) = B(P;\alpha;x)^{-1},$

Hence, $F'(x)^{-1} = \big(I - T(P;\alpha;x)\big)^{-1}G(P;\alpha;x),$

We can rewrite the equation again, take the first line as example,

Because: $\sum_{j=0}^{l_k-1} T(P;\alpha;x_k)^j\, G(P;\alpha;x_k)F(x_k) = (I - T(P;\alpha;x_k)^{l_k})\big(I - T(P;\alpha;x_k)\big)^{-1},$

i.e., $\sum_{j=0}^{l_k-1} T(P;\alpha;x_k)^j\, G(P;\alpha;x_k)F(x_k) = (I - T(P;\alpha;x_k)^{l_k})F'(x_k)^{-1}F(x_k)$,

So, MN-NPHSS method could be represented as:

$$\begin{cases} y_k = x_k - (I - T(P;\alpha;x_k)^{l_k})F'(x_k)^{-1}F(x_k) \\ x_{k+1} = y_k - (I - T(P;\alpha;x_k)^{m_k})F'(x_k)^{-1}F(y_k) \end{cases}. \quad k = 0,1,2 \ldots.$$

3. Local convergence theorem for MN-NPHSS iteration method

In this part, we will prove the local convergence for MN-NPHSS iteration method.

First, let us clarify the conditions for local convergence theorem.

Let $F: \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ be G-differentiable on an open neighborhood $\mathbb{N}_0 \subset \mathbb{D}$ of a point $x_* \in \mathbb{D}$ at which $F'(x)$ is continuous and positive definite. Also, $F(x_*) = 0$.

A mapping $F: \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ is Gateaux-(or G-) differentiable at an interior point x of $\mathbb{D}$,if there exists a linear operator $J \in \mathbb{C}^{n\times n}$, such that, for any $h \in \mathbb{C}^n$, $\lim_{t\to 0}\frac{1}{t}\|F(x+th) - F(x) - tJh\| = 0$, $F: \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ is said to be G-differentiable on an open set $\mathbb{D}_0 \subset \mathbb{D}$ if it is G-differentiable at any point in $\mathbb{D}_0$.

Suppose that $F'(x) = H(x) + S(x)$,where$H(x) = \frac{1}{2}(F'(x) + F'(x)^*)$, $S(x) = \frac{1}{2}(F'(x) - F'(x)^*)$ are the Hermitian and skew-Hermitian parts of the Jacobian matrix $F'(x)$. Denote with $\mathbb{N}(x_*, r)$ an open ball centered at x_*, with radius $r > 0$.

Given the conditions above, let us provide two assumptions. For all $x \in \mathbb{N}(x_*, r) \subset \mathbb{N}_0$, assume the following conditions below are hold.

Bounded Condition

There exist positive constants β and γ, such that:$max\{\|H(x_*)\|, \|S(x_*)\|\} \leq \beta$ and $\|F'(x_*)^{-1}\| \leq \gamma$.

Lipschitz Condition

There exist nonnegative constants L_h and L_s,such that:

$$\|H(x) - H(x_*)\| \leq L_h\|x - x_*\|,$$

$$\|S(x) - S(x_*)\| \leq L_s\|x - x_*\|.$$

Based on the conditions and assumptions above, we can prove two lemmas.

Lemma 1. Given the assumption, if $r \in \left(0, \frac{1}{\gamma L}\right)$, then $F'(x)^{-1}$ exists for any $x \in \mathbb{N}(x_*, r) \subset \mathbb{N}_0$.And the following inequalities hold with $L = L_h + L_s$ for all $x, y \in \mathbb{N}(x_*, r)$:

$$\|F'(x) - F'(x_*)\| \leq L\|x - x_*\|,$$

$$\|F'(x)^{-1}\| \leq \frac{\gamma}{1 - \gamma L\|x - x_*\|},$$

$$\|F(y)\| \leq \frac{L}{2}\|y - x_*\|^2 + 2\beta\|y - x_*\|,$$

$$\|y - x_* - F'(x)^{-1}F(y)\|$$
$$\leq \frac{\gamma}{1 - \gamma L\|x - x_*\|}\left(\frac{L}{2}\|y - x_*\| + L\|x - x_*\|\right)\|y - x_*\|.$$

Proof：

The triangle inequality and the Lipschitz condition directly implies:

$$\begin{aligned}\|F'(x) - F'(x_*)\| &= \|H(x) + S(x) - H(x_*) - S(x_*)\| \\ &\leq \|H(x) - H(x_*)\| + \|S(x) - S(x_*)\| \\ &\leq (L_h + L_s)\|x - x_*\| \\ &= L\|x - x_*\|.\end{aligned}$$

Hence the first inequality is proved.

According to the inequality for absolute value,

$$\left\|F'(x_*)^{-1}\big(F'(x_*) - F'(x)\big)\right\| \leq \|F'(x_*)^{-1}\|\|F'(x_*) - F'(x)\|,$$

Then from bounded condition and the first inequality above,

$$\|F'(x_*)^{-1}\|\|F'(x_*) - F'(x)\| \leq \gamma L\|x - x_*\|,$$

Which means

$$\left\|F'(x_*)^{-1}\big(F'(x_*) - F'(x)\big)\right\| \leq \gamma L\|x - x_*\|,$$

From $x \in \mathbb{N}(x_*, r)$, i.e. $\|x - x_*\| \leq r$, and $\mathrm{r} \in \left(0, \frac{1}{\gamma \mathrm{L}}\right)$,

We know that $\|x - x_*\| < \frac{1}{\gamma \mathrm{L}}$, then $\gamma L\|x - x_*\| < \gamma L\frac{1}{\gamma \mathrm{L}} = 1.$

By making use of Banach Lemma, $F'(x)^{-1}$ exists, and

$$\begin{aligned}\|F'(x)^{-1}\| &\leq \frac{\|F'(x_*)^{-1}\|}{1 - \left\|F'(x_*)^{-1}\big(F'(x_*) - F'(x)\big)\right\|} \\ &\leq \frac{\gamma}{1 - \gamma L\|x - x_*\|}.\end{aligned}$$

Hence the second inequality is proved.

Given $F(x_*) = 0$, we have:

$$\begin{aligned}F(y) &= F(y) - F(x_*) - F'(x_*)(y - x_*) + F'(x_*)(y - x_*) \\ &= \int_0^1 \Big(F'\big(x_* + t(y - x_*)\big) - F'(x_*)\Big)\,dt(y - x_*) + F'(x_*)(y - x_*).\end{aligned}$$

The triangle inequality and the bounded condition leads to:

$$\|F'(x_*)\| = \|H(x_*) + S(x_*)\| \leq \|H(x_*)\| + \|S(x_*)\| \leq 2\beta,$$

So by the fist inequality and equality above, we know that:

$$\begin{aligned}
\|F(y)\| &= \left\|\int_0^1 \left(F'\big(x_* + t(y-x_*)\big) - F'(x_*)\right) dt(y-x_*) + F'(x_*)(y-x_*)\right\| \\
&\le \left\|\int_0^1 \left(F'\big(x_* + t(y-x_*)\big) - F'(x_*)\right) dt(y-x_*)\right\| + \|F'(x_*)(y-x_*)\| \\
&\le \int_0^1 Lt\,\|y-x_*\| dt(y-x_*) + \|F'(x_*)(y-x_*)\| \\
&\le \int_0^1 Lt\,\|y-x_*\| dt(y-x_*) + 2\beta\|y-x_*\| \\
&= \frac{L}{2}\|y-x_*\|^2 + 2\beta\|y-x_*\|.
\end{aligned}$$

Hence the third inequality is proved.

Again, given $F(x_*) = 0$, it's obvious that:

$$\begin{aligned}
& y - x_* - F'(x)^{-1}F(y) \\
&= -F'(x)^{-1}\big(F(y) - F(x_*) - F'(x)(y-x_*)\big) \\
&= -F'(x)^{-1}\big(F(y) - F(x_*) - F'(x_*)(y-x_*)\big) \\
&\quad + F'(x)^{-1}\big(F'(x) - F'(x_*)\big)(y-x_*) \\
&= -F'(x)^{-1}\int_0^1 \left(F'\big(x_* + t(y-x_*)\big) - F'(x_*)\right) dt(y-x_*) \\
&\quad + F'(x)^{-1}\big(F'(x) - F'(x_*)\big)(y-x_*).
\end{aligned}$$

So:

$$\begin{aligned}
& \|y - x_* - F'(x)^{-1}F(y)\| \\
&= \Big\| -F'(x)^{-1}\int_0^1 \left(F'\big(x_* + t(y-x_*)\big) - F'(x_*)\right) dt(y-x_*) \\
&\quad + F'(x)^{-1}\big(F'(x) - F'(x_*)\big)(y-x_*)\Big\| \\
&\le \|-F'(x)^{-1}\| \Big(\int_0^1 \left\|F'\big(x_* + t(y-x_*)\big) - F'(x_*)\right\| dt \\
&\quad + \|F'(x) - F'(x_*)\|\Big)\|y-x_*\| \\
&\le \frac{\gamma}{1-\gamma L\|x-x_*\|}\left(\frac{L}{2}\|y-x_*\| + L\|x-x_*\|\right)\|y-x_*\|.
\end{aligned}$$

Hence the fourth inequality is proved.

The proof of Lemma 1 is completed.

Lemma 2. Under the assumptions of Lemma 1, suppose $r \in (0, r_0)$, define $r_0 = \min_{1\le j\le 2}\left\{r_+^{(j)}\right\}$,

Where $r_+^{(1)} = \frac{\tau\theta}{2\gamma L + 2L_h\gamma\tau\theta}, r_+^{(2)} = \frac{1-2\beta\gamma[(\tau+1)\theta]^u}{3\gamma L}$,

with $u = min\{l_*, m_*\}$, $l_* = \liminf_{k\to\infty} l_k$, $m_* = \liminf_{k\to\infty} m_k$, $\tau \in \left(0, \frac{1-\theta}{\theta}\right)$ a prescribed positive constant and $\theta \equiv \theta(\alpha;x_*) = \|T(P;\alpha;x_*)\| \le \frac{\sqrt{\alpha^2+{\xi_{max}}^2}}{\alpha+\lambda_{min}} \equiv \sigma(\alpha;x_*)$,

(Furthermore, by the convergence of NPHSS iteration method, we know that when constant α satisfies $\alpha > \frac{{\xi_{max}}^2-{\lambda_{min}}^2}{2\lambda_{min}}$, $\sigma(\alpha;x_*) < 1$)

Then, for any $x \in \mathbb{N}(x_*,r) \subset \mathbb{N}_0$, $t \in (0,r)$ and $v > u$, it holds that :

$$\|T(P;\alpha;x)\| \le (\tau+1)\theta < 1,$$

$$g(t;v) = \frac{2\gamma}{1-\gamma Lt}(Lt + \beta[(\tau+1)\theta]^v) < g(r_0;u) < 1.$$

Proof: To prove this lemma, we need to prove the two inequalities above.

The first part illustrates that for iteration matrix $T(P;\alpha;x)$ of the MN-NPHSS method, its norm $\|T(P;\alpha;x)\|$ is less than 1.

The bounded condition directly implies the bounds:

$$\|F'(x_*)\| = \|H(x_*) + S(x_*)\| \le \|H(x_*)\| + \|S(x_*)\| \le 2\beta.$$

And from, we know that: $B(P;\alpha;x_*)^{-1} = \left(I - T(P;\alpha;x_*)\right) F'(x_*)^{-1}$,

Hence

$$\begin{aligned}\|B(P;\alpha;x_*)^{-1}\| &= \left\|\left(I - T(P;\alpha;x_*)\right) F'(x_*)^{-1}\right\| \\ &\le \left\|\left(I - T(P;\alpha;x_*)\right)\right\| \| F'(x_*)^{-1}\| \\ &\le (1 + \|T(P;\alpha;x_*)\|)\| F'(x_*)^{-1}\|.\end{aligned}$$

From, $\|T(P;\alpha;x_*)\| \le \sigma(\alpha;x_*) < 1$,according to the bounded condition,

we know that: $\|B(P;\alpha;x_*)^{-1}\| \le 2\gamma$,

Given the definition of $B(P;\alpha;x)$ and $C(P;\alpha;x)$, we have:

$$B(P;\alpha;x) - B(P;\alpha;x_*) = H(x) - H(x_*),$$

$$C(P;\alpha;x) - C(P;\alpha;x_*) = -S(x) + S(x_*),$$

So, according to the Lipschitz condition,

$$\|B(P;\alpha;x) - B(P;\alpha;x_*)\| = \| H(x) - H(x_*)\| \le L_h\|x - x_*\|,$$

$$\|C(P;\alpha;x) - C(P;\alpha;x_*)\| = \|-S(x) + S(x_*)\| \le L_s\|x - x_*\|,$$

Next, we introduce the perturbation lemma.

(Perturbation lemma: Let $M, N \in \mathbb{C}^{n\times n}$,and assume that M is nonsingular, with $\|M^{-1}\| \le \varphi$.If $\|M-N\| \le \delta$ and$\delta\,\varphi < 1$, then N is also nonsingular, and $\|N^{-1}\| \le \frac{\varphi}{1-\delta\,\varphi}$)

Becuase $\|B(P;\alpha;x) - B(P;\alpha;x_*)\| = \| H(x) - H(x_*)\| \le L_h\|x - x_*\|$, And given: $\|B(P;\alpha;x_*)^{-1}\| \leq 2\gamma$,

So, for any $x \in \mathbb{N}(x_*, r)$, provided r is small enough such that x satisfies $2\gamma\, L_h\|x - x_*\| < 1$,

Then by perturbation lemma: $\|B(P;\alpha;x)^{-1}\| \leq \frac{2\gamma}{1-2\gamma\, L_h\|x-x_*\|}$,

Also , given the second inequality from lemma 1, we have: $\|F'(x)^{-1}\| \leq \frac{\gamma}{1-\gamma L\|x-x_*\|}$,

So we also need to restrict r, such that $\gamma L\|x - x_*\| < 1$,

Actually, given the condition above, we have already controlled r such that: $r < \frac{\tau\theta}{2\gamma L+2L_h\gamma\tau\theta}$,

Because $\tau \in \left(0, \frac{1-\theta}{\theta}\right)$, indicating $\tau\theta < 1$, then it is obvious that：$r < \frac{1}{\gamma L}$, and $r < \frac{1}{2L_h\gamma}$,

So for any $x \in \mathbb{N}(x_*, r)$， $2\gamma\, L_h\|x - x_*\| < 1$ and $\gamma L\|x - x_*\| < 1$ hold.

We obtain:

$$
\begin{aligned}
&T(P;\alpha;x) - T(P;\alpha;x_*)\\
&= \mathrm{B}(P;\alpha;x)^{-1}C(P;\alpha;x) - \mathrm{B}(P;\alpha;x_*)^{-1}C(P;\alpha;x_*)\\
&= B(P;\alpha;x)^{-1}\Big(\big(C(P;\alpha;x) - C(P;\alpha;x_*)\big) - \big(B(P;\alpha;x) - B(P;\alpha;x_*)\big)T(P;\alpha;x_*)\Big).
\end{aligned}
$$

Because:

$$
\begin{aligned}
&\|T(P;\alpha;x) - T(P;\alpha;x_*)\|\\
&\leq \|B(P;\alpha;x)^{-1}\|[\|C(P;\alpha;x) - C(P;\alpha;x_*)\|\\
&\quad +\|B(P;\alpha;x) - B(P;\alpha;x_*)\|\|T(P;\alpha;x_*)\|]\\
&\leq \frac{2\gamma}{1-2\gamma\, L_h\|x-x_*\|}(L_h + L_s)\|x - x_*\|\\
&= \frac{2\gamma}{1-2\gamma\, L_h\|x-x_*\|}L\|x - x_*\|,
\end{aligned}
$$

And we have already controlled: $r < \frac{\tau\theta}{2\gamma L+2L_h\gamma\tau\theta}$,

Hence, $\frac{2\gamma}{1-2\gamma\, L_h\|x-x_*\|}L\|x - x_*\| < \tau\theta$, i.e. $\|T(P;\alpha;x) - T(P;\alpha;x_*)\| \leq \tau\theta$,

Which means:

$$
\begin{aligned}
\|T(P;\alpha;x)\| &\leq \|T(P;\alpha;x) - T(P;\alpha;x_*)\| + \|T(P;\alpha;x_*)\|\\
&\leq \tau\theta + \theta = (\tau + 1)\theta < 1\,.
\end{aligned}
$$

Thus, the normal for iteration matrix $T(P;\alpha;x)$: $\|T(P;\alpha;x)\| \leq (\tau + 1)\theta < 1$ is proved.

Another part is to prove $g(t;v) < g(r_0;u) < 1$, this is the same with proof for Theorem 3.2 in [1], please read that for reference.

Lemma 2 is proved.

Next we will prove the local convergence theorem for MN-NPHSS.

Local Convergence Theorem

Under the assumptions of Lemma 1 and Lemma 2, for any $x_0 \in \mathbb{N}(x_*, r)$ and any positive integral sequences $\{l_k\}_{k=0}^{\infty}$, $\{m_k\}_{k=0}^{\infty}$, the iteration sequence $\{x_k\}_{k=0}^{\infty}$ generated by the MN-NPHSS method is well-defined and converges to x_*. Furthermore, it holds that, $\lim_{k\to\infty} \|x_k - x_*\|^{\frac{1}{k}} \le g(r_0; u)^2$.

Where $g(r_0; u) = \frac{2\gamma}{1-\gamma L r_0}(Lr_0 + \beta[(\tau+1)\theta]^u)$,

And $u = \min\{l_*, m_*\}$, $l_* = \lim\inf_{k\to\infty} l_k$, $m_* = \lim\inf_{k\to\infty} m_k$.

Proof: From the representation method for MN-NPHSS above, we obtain:

$$
\begin{aligned}
&\|y_k - x_*\| \\
&= \left\|x_k - x_* - (I - T(P;\alpha;x_k)^{l_k})F'(x_k)^{-1}F(x_k)\right\| \\
&\le \|x_k - x_* - F'(x_k)^{-1}F(x_k)\| + \left\|T(P;\alpha;x_k)^{l_k}\right\| \|F'(x_k)^{-1}F(x_k)\|.
\end{aligned}
$$

According to the fourth inequality from Lemma 1:

$$
\begin{aligned}
&\|x_k - x_* - F'(x_k)^{-1}F(x_k)\| \\
&\le \frac{\gamma}{1-\gamma L\|x_k - x_*\|}\left(\frac{L}{2}\|x_k - x_*\| + L\|x_k - x_*\|\right)\|x_k - x_*\| \\
&= \frac{\gamma}{1-\gamma L\|x_k - x_*\|}\left(\frac{3L}{2}\|x_k - x_*\|^2\right).
\end{aligned}
$$

From the first part result in Lemma 2, $\left\|T(P;\alpha;x_k)^{l_k}\right\| \le [(\tau+1)\theta]^{l_k}$,

And given the first and the third inequality in Lemma 1:

$$
\begin{aligned}
&\|F'(x_k)^{-1}F(x_k)\| \le \|F'(x_k)^{-1}\|\|F(x_k)\| \\
&\le \frac{\gamma}{1-\gamma L\|x_k - x_*\|}\left(\frac{L}{2}\|x_k - x_*\|^2 + 2\beta\|x_k - x_*\|\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\|y_k - x_*\| \le \\
&\frac{\gamma}{1-\gamma L\|x_k - x_*\|}\left(\frac{3L}{2}\|x_k - x_*\|^2\right) + [(\tau+1)\theta]^{l_k} \times \frac{\gamma}{1-\gamma L\|x_k - x_*\|}\left(\frac{L}{2}\|x_k - x_*\|^2 + 2\beta\|x_k - x_*\|\right) \\
&= \frac{(3 + [(\tau+1)\theta]^{l_k})\gamma L}{2(1-\gamma L\|x_k - x_*\|)}\|x_k - x_*\|^2 + \frac{2\beta\gamma[(\tau+1)\theta]^{l_k}}{1-\gamma L\|x_k - x_*\|}\|x_k - x_*\|.
\end{aligned}
$$

From the first part result in Lemma 2: $[(\tau+1)\theta]^{l_k} < 1$,

i.e.,

$$
\frac{(3 + [(\tau+1)\theta]^{l_k})\gamma L}{2(1-\gamma L\|x_k - x_*\|)} < \frac{2\gamma L}{1-\gamma L\|x_k - x_*\|},
$$

hence

$$\begin{aligned}\|y_k - x_*\| &= \frac{(3 + [(\tau+1)\theta]^{l_k})\gamma L}{2(1-\gamma L\|x_k - x_*\|)}\|x_k - x_*\|^2 + \frac{2\beta\gamma[(\tau+1)\theta]^{l_k}}{1-\gamma L\|x_k - x_*\|}\|x_k - x_*\| \\ &\leq \frac{2\gamma}{1-\gamma L\|x_k - x_*\|}(L\|x_k - x_*\| + \beta[(\tau+1)\theta]^{l_k})\|x_k - x_*\|\end{aligned}$$

Given the definition of $g(t;v)$ before, we notice that:

$$\frac{2\gamma}{1-\gamma L\|x_k - x_*\|}(L\|x_k - x_*\| + \beta[(\tau+1)\theta]^{l_k})\|x_k - x_*\| = g(\|x_k - x_*\|; l_k)\|x_k - x_*\|,$$

So,

$$\|y_k - x_*\| \leq g(\|x_k - x_*\|; l_k)\|x_k - x_*\| < g(r_0; u)\|x_k - x_*\| < \|x_k - x_*\|.$$

Then we analyze $\|x_{k+1} - x_*\|$, similarly, we obtain:

$$\begin{aligned}\|x_{k+1} - x_*\| &= \|y_k - x_* - (I - T(P;\alpha;x_k)^{m_k})F'(x_k)^{-1}F(y_k)\| \\ &\leq \|y_k - x_* - F'(x_k)^{-1}F(y_k)\| + \|T(P;\alpha;x_k)^{m_k}\|\|F'(x_k)^{-1}F(y_k)\|,\end{aligned}$$

and

$$\|y_k - x_* - F'(x_k)^{-1}F(x_k)\| \leq \frac{\gamma}{1-\gamma L\|x_k - x_*\|}\left(\frac{L}{2}\|y_k - x_*\| + L\|x_k - x_*\|\right)\|y_k - x_*\|,$$

And $\|T(P;\alpha;x_k)^{m_k}\| \leq [(\tau+1)\theta]^{m_k}$,

and

$$\begin{aligned}\|F'(x_k)^{-1}F(y_k)\| &\leq \|F'(x_k)^{-1}\|\|F(y_k)\| \\ &\leq \frac{\gamma}{1-\gamma L\|x_k - x_*\|}\left(\frac{L}{2}\|y_k - x_*\|^2 + 2\beta\|y_k - x_*\|\right).\end{aligned}$$

so

$$\begin{aligned}&\|x_{k+1} - x_*\| \\ &\leq \frac{\gamma}{1-\gamma L\|x_k - x_*\|}\left(\frac{L}{2}\|y_k - x_*\| + L\|x_k - x_*\|\right)\|y_k - x_*\| \\ &\quad + [(\tau+1)\theta]^{m_k} \times \frac{\gamma}{1-\gamma L\|x_k - x_*\|}\left(\frac{L}{2}\|y_k - x_*\|^2 + 2\beta\|y_k - x_*\|\right) \\ &= \left(\frac{\gamma L}{1-\gamma L\|x_k - x_*\|}\left(\frac{1 + [(\tau+1)\theta]^{m_k}}{2}\|y_k - x_*\| + \|x_k - x_*\|\right)\right. \\ &\quad \left. + \frac{2\beta\gamma[(\tau+1)\theta]^{m_k}}{1-\gamma L\|x_k - x_*\|}\right)\|y_k - x_*\|.\end{aligned}$$

We have already proved $\|y_k - x_*\| \leq g(\|x_k - x_*\|; l_k)\|x_k - x_*\|$, by above,

Thus,

$$\begin{aligned}\|x_{k+1} - x_*\| \leq &\left(\frac{\gamma L}{1-\gamma L\|x_k - x_*\|}\left(\frac{1 + [(\tau+1)\theta]^{m_k}}{2} g(\|x_k - x_*\|; l_k)\|x_k - x_*\| + \|x_k - x_*\|\right)\right. \\ &\left. + \frac{2\beta\gamma[(\tau+1)\theta]^{m_k}}{1-\gamma L\|x_k - x_*\|}\right) g(\|x_k - x_*\|; l_k)\|x_k - x_*\|.\end{aligned}$$

Also, given the result in Lemma 2 $[(\tau+1)\theta]^{m_k} < 1$, we have:

$$\|x_{k+1} - x_*\| \le \frac{2\gamma g(\|x_k - x_*\|; l_k)}{1 - \gamma L\|x_k - x_*\|} \Bigg(\frac{1 + g(\|x_k - x_*\|; l_k)}{2}$$

$$\times L\|x_k - x_*\| + \beta[(\tau + 1)\theta]^{m_k} \Bigg) \|x_k - x_*\|.$$

We also know that $g(\|x_k - x_*\|; l_k) < 1$, so

$$\|x_{k+1} - x_*\| \le \frac{2\gamma g(\|x_k - x_*\|; l_k)}{1 - \gamma L\|x_k - x_*\|} (L\|x_k - x_*\| + \beta[(\tau + 1)\theta]^{m_k}) \|x_k - x_*\|,$$

Given the definition of $g(t; v)$ before, we notice that:

$$\frac{2\gamma}{1 - \gamma L\|x_k - x_*\|} (L\|x_k - x_*\| + \beta[(\tau + 1)\theta]^{m_k}) \|x_k - x_*\| \quad = g(\|x_k - x_*\|; m_k) \|x_k - x_*\|.$$

So:

$$\begin{aligned} \|x_{k+1} - x_*\| &\le g(\|x_k - x_*\|; l_k)(\|x_k - x_*\|; m_k) \|x_k - x_*\| \\ &\le g(\|x_k - x_*\|; u)^2 \|x_k - x_*\| \\ &\le g(r_0; u)^2 \|x_k - x_*\| < \|x_k - x_*\|. \end{aligned}$$

Similar to the proof for $\|x_{k+1} - x_*\| < \|x_k - x_*\|$ above, we can get the convergent sequence $\{x_k\}_{k=0}^{\infty} \in \mathbb{N}(x_*, r)$ by induction. The details are shown below:

In fact, when $k = 0$, according to the condition $x_0 \in \mathbb{N}(x_*, r)$, i.e., $\|x_0 - x_*\| < r < r_0$,

We can obtain that: $\|x_1 - x_*\| < g(\|x_0 - x_*\|; u)^2 \|x_0 - x_*\| < \|x_0 - x_*\| < r$,

Furthermore, because $x_0 \in \mathbb{N}(x_*, r)$, we have $x_1 \in \mathbb{N}(x_*, r)$.

When $k = n$, assume that $x_n \in \mathbb{N}(x_*, r)$,

We can obtain that:

$$\begin{aligned} \|x_{n+1} - x_*\| &< g(\|x_n - x_*\|; u)^2 \|x_n - x_*\| \\ &< g(r_0; u)^2 \|x_n - x_*\| \\ &< g(r_0; u)^{2(n+1)} \|x_0 - x_*\| \\ &< r. \end{aligned}$$

Which means, when $k = n + 1$, $x_{n+1} \in \mathbb{N}(x_*, r)$ still holds.

As $n \to \infty$, we have $x_{n+1} \to x_*$.

So the local convergence theorem is proved.

[1]. Y.-J. Wu, X. Li, J.-Y. Yuan. A Non-Alternating Preconditioned HSS Iteration Method For Non-Hermitian Positive Definite Linear Systems[J]. Comp. Appl. Math. DOI: 10.1007/s40314-015-0231-6, 2015.