

1. Another view of NPHSS

Given a linear mapping $F: \mathbb{D} \subset \mathbb{C}^n \rightarrow \mathbb{C}^n$, let $F(x) = Ax - b$, with $A \in \mathbb{C}^{n \times n}$ a non-Hermitian positive definite matrix and $b \in \mathbb{C}^n$ a given vector, we can construct a linear equations systems as $Ax = b$, $A \in \mathbb{C}^{n \times n}$, $x, b \in \mathbb{C}^n$.

Based on Hermitian and skew-Hermitian splitting for matrix $A \in \mathbb{C}^{n \times n}$, we could get: $A = H + S$.

where $H = \frac{1}{2}(A + A^*)$, $S = \frac{1}{2}(A - A^*)$.

Given an initial guess $x_0 \in \mathbb{C}^n$, we compute x_{k+1} for $k = 0, 1, 2, \dots$, using the following iteration scheme until sequence $\{x_k\}_{k=0}^{\infty} \subset \mathbb{C}^n$ satisfies the stopping criterion: $(\alpha P + H)x_{k+1} = (\alpha P - S)x_k + b$ where α is a given positive constant and $P \in \mathbb{C}^{n \times n}$ is a prescribed Hermitian positive definite matrix.

We can rewrite as: $x_{k+1} = T(P; \alpha)x_k + G(P; \alpha)b$, $k = 0, 1, 2, \dots$,

Keep doing this until we can use x_0 to express it: $x_{k+1} = T(P; \alpha)^{k+1}x_0 + \sum_{j=0}^k T(P; \alpha)^j G(P; \alpha)b$.

Where $T(P; \alpha) = (\alpha P + H)^{-1}(\alpha P - S)$, $G(P; \alpha) = (\alpha P + H)^{-1}$.

Here, $T(P; \alpha)$ is the iteration matrix of the NPHSS method. In fact, we can split the coefficient matrix A by: $A = B(P; \alpha) - C(P; \alpha)$, where $B(P; \alpha) = \alpha P + H$, $C(P; \alpha) = \alpha P - S$,

So, the result in (2.3) can be represented by $B(P; \alpha)$ and $C(P; \alpha)$:

$$T(P; \alpha) = B(P; \alpha)^{-1}C(P; \alpha), \quad G(P; \alpha) = B(P; \alpha)^{-1}.$$

Note that $T(P; \alpha)$ can be reformulated as: $T(P; \alpha) = I - (\alpha P + H)^{-1}A$.

Then the matrix $(\alpha P + H)$ can be viewed as a preconditioner for the coefficient matrix $A \in \mathbb{C}^{n \times n}$.

1. convergence of NPHSS iteration method

Next we show the conditions for the convergence of NPHSS method.

Since H is Hermitian and S is skew-Hermitian, we could know that all the eigenvalues of $P^{-1}H$ are real positive and all the eigenvalues of $P^{-1}S$ are imaginary.

Here, we denote::

$$\lambda_{\max} = \max_{\lambda_j \in \text{sp}(P^{-1}H)} \{\lambda_j\},$$

$$\lambda_{\min} = \min_{\lambda_j \in \text{sp}(P^{-1}H)} \{\lambda_j\},$$

$$\xi_{\max} = \max_{i\xi_j \in \text{sp}(P^{-1}S)} \{|\xi_j|\},$$

Where $\text{sp}(X)$ is the spectrum of the matrix X and $i = \sqrt{-1}$

We use the following theorem to give the convergence result of NPHSS iteration method.

Theorem 1 Let $A \in \mathbb{C}^{n \times n}$ be a positive definite matrix, let $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$ be its

Hermitian and skew-Hermitian parts, and let α be a positive constant. Let $P \in \mathbb{C}^{n \times n}$ be a Hermitian positive definite matrix. Then the spectral radius $\rho(T(P; \alpha))$ of the NPHSS iteration matrix satisfies

$$\rho(T(P; \alpha)) \leq \sigma(\alpha), \text{ where } \sigma(\alpha) = \frac{\sqrt{\alpha^2 + \xi_{\max}^2}}{\alpha + \lambda_{\min}}.$$

We can find the range of α :

(1) If $\lambda_{\min} \geq \xi_{\max}$, then $\sigma(\alpha) < 1$ for any $\alpha > 0$, which means that the NPHSS iteration method is unconditionally convergent;

(2) If $\lambda_{\min} < \xi_{\max}$, then $\sigma(\alpha) < 1$ if and only if $\alpha > \frac{\xi_{\max}^2 - \lambda_{\min}^2}{2\lambda_{\min}}$.

Which means that under the condition above, NPHSS iteration method is convergent.

Detailed proof by [1]

2. More about MN-NPHSS

$$d_{k,l_k} = -\sum_{j=0}^{l_k-1} T(P; \alpha; x_k)^j G(P; \alpha; x_k) F(x_k), \quad h_{k,m_k} = -\sum_{j=0}^{m_k-1} T(P; \alpha; x_k)^j G(P; \alpha; x_k) F(y_k),$$

Where $T(P; \alpha; x) = (\alpha P + H(x))^{-1}(\alpha P - S(x))$, $G(P; \alpha; x) = (\alpha P + H(x))^{-1}$.

So the MN-NPHSS method could be written as:

$$\begin{cases} y_k = x_k - \sum_{j=0}^{l_k-1} T(P; \alpha; x_k)^j G(P; \alpha; x_k) F(x_k) \\ x_{k+1} = y_k - \sum_{j=0}^{m_k-1} T(P; \alpha; x_k)^j G(P; \alpha; x_k) F(y_k) \end{cases}.$$

Define: $B(P; \alpha; x) = \alpha P + H(x)$, $C(P; \alpha; x) = \alpha P - S(x)$,

Then the Jacobian matrix $F'(x)$ can be written as: $F'(x) = B(P; \alpha; x) - C(P; \alpha; x)$,

From above, $T(P; \alpha; x) = I - (\alpha P + H(x))^{-1} F'(x)$, i.e., $I - T(P; \alpha; x) = (\alpha P + H(x))^{-1} F'(x)$,

Then, $F'(x)^{-1} = (I - T(P; \alpha; x))^{-1} (\alpha P + H(x))^{-1} = (I - T(P; \alpha; x))^{-1} B(P; \alpha; x)^{-1}$,

We obtain $T(P; \alpha; x) = B(P; \alpha; x)^{-1} C(P; \alpha; x)$, $G(P; \alpha; x) = B(P; \alpha; x)^{-1}$,

Hence, $F'(x)^{-1} = (I - T(P; \alpha; x))^{-1} G(P; \alpha; x)$,

We can rewrite the equation again, take the first line as example,

Because: $\sum_{j=0}^{l_k-1} T(P; \alpha; x_k)^j G(P; \alpha; x_k) F(x_k) = (I - T(P; \alpha; x_k)^{l_k})(I - T(P; \alpha; x_k))^{-1}$,

i.e., $\sum_{j=0}^{l_k-1} T(P; \alpha; x_k)^j G(P; \alpha; x_k) F(x_k) = (I - T(P; \alpha; x_k)^{l_k}) F'(x_k)^{-1} F(x_k),$

So, MN-NPHSS method could be represented as:

$$\begin{cases} y_k = x_k - (I - T(P; \alpha; x_k)^{l_k}) F'(x_k)^{-1} F(x_k) \\ x_{k+1} = y_k - (I - T(P; \alpha; x_k)^{m_k}) F'(x_k)^{-1} F(y_k) \end{cases} \quad k = 0, 1, 2, \dots$$

3. Local convergence theorem for MN-NPHSS iteration method

In this part, we will prove the local convergence for MN-NPHSS iteration method.

First, let us clarify the conditions for local convergence theorem.

Let $F: \mathbb{D} \subset \mathbb{C}^n \rightarrow \mathbb{C}^n$ be G-differentiable on an open neighborhood $\mathbb{N}_0 \subset \mathbb{D}$ of a point $x_* \in \mathbb{D}$ at which $F'(x)$ is continuous and positive definite. Also, $F(x_*) = 0$.

A mapping $F: \mathbb{D} \subset \mathbb{C}^n \rightarrow \mathbb{C}^n$ is Gateaux-(or G-) differentiable at an interior point x of \mathbb{D} , if there exists a linear operator $J \in \mathbb{C}^{n \times n}$, such that, for any $h \in \mathbb{C}^n$, $\lim_{t \rightarrow 0} \frac{1}{t} \|F(x + th) - F(x) - tJh\| = 0$,

$F: \mathbb{D} \subset \mathbb{C}^n \rightarrow \mathbb{C}^n$ is said to be G-differentiable on an open set $\mathbb{D}_0 \subset \mathbb{D}$ if it is G-differentiable at any point in \mathbb{D}_0 .

Suppose that $F'(x) = H(x) + S(x)$, where $H(x) = \frac{1}{2}(F'(x) + F'(x)^*)$, $S(x) = \frac{1}{2}(F'(x) - F'(x)^*)$

are the Hermitian and skew-Hermitian parts of the Jacobian matrix $F'(x)$. Denote with $\mathbb{N}(x_*, r)$ an open ball centered at x_* , with radius $r > 0$.

Given the conditions above, let us provide two assumptions. For all $x \in \mathbb{N}(x_*, r) \subset \mathbb{N}_0$, assume the following conditions below are hold.

Bounded Condition

There exist positive constants β and γ , such that: $\max\{\|H(x_*)\|, \|S(x_*)\|\} \leq \beta$ and $\|F'(x_*)^{-1}\| \leq \gamma$.

Lipschitz Condition

There exist nonnegative constants L_h and L_s , such that:

$$\|H(x) - H(x_*)\| \leq L_h \|x - x_*\|,$$

$$\|S(x) - S(x_*)\| \leq L_s \|x - x_*\|.$$

Based on the conditions and assumptions above, we can prove two lemmas.

Lemma 1. Given the assumption, if $r \in (0, \frac{1}{\gamma L})$, then $F'(x)^{-1}$ exists for any $x \in \mathbb{N}(x_*, r) \subset \mathbb{N}_0$. And

the following inequalities hold with $L = L_h + L_s$ for all $x, y \in \mathbb{N}(x_*, r)$:

$$\|F'(x) - F'(x_*)\| \leq L \|x - x_*\|,$$

$$\|F'(x)^{-1}\| \leq \frac{\gamma}{1 - \gamma L \|x - x_*\|},$$

$$\begin{aligned}
\|F(y)\| &\leq \frac{L}{2}\|y - x_*\|^2 + 2\beta\|y - x_*\|, \\
&\|y - x_* - F'(x)^{-1}F(y)\| \\
&\leq \frac{\gamma}{1 - \gamma L\|x - x_*\|} \left(\frac{L}{2}\|y - x_*\| + L\|x - x_*\| \right) \|y - x_*\|.
\end{aligned}$$

Proof:

The triangle inequality and the Lipschitz condition directly implies:

$$\begin{aligned}
\|F'(x) - F'(x_*)\| &= \|H(x) + S(x) - H(x_*) - S(x_*)\| \\
&\leq \|H(x) - H(x_*)\| + \|S(x) - S(x_*)\| \\
&\leq (L_h + L_s)\|x - x_*\| \\
&= L\|x - x_*\|.
\end{aligned}$$

Hence the first inequality is proved.

According to the inequality for absolute value,

$$\|F'(x_*)^{-1}(F'(x_*) - F'(x))\| \leq \|F'(x_*)^{-1}\| \|F'(x_*) - F'(x)\|,$$

Then from bounded condition and the first inequality above,

$$\|F'(x_*)^{-1}\| \|F'(x_*) - F'(x)\| \leq \gamma L\|x - x_*\|,$$

Which means

$$\|F'(x_*)^{-1}(F'(x_*) - F'(x))\| \leq \gamma L\|x - x_*\|,$$

From $x \in \mathbb{N}(x_*, r)$, i.e. $\|x - x_*\| \leq r$, and $r \in \left(0, \frac{1}{\gamma L}\right)$,

We know that $\|x - x_*\| < \frac{1}{\gamma L}$, then $\gamma L\|x - x_*\| < \gamma L \frac{1}{\gamma L} = 1$.

By making use of Banach Lemma, $F'(x)^{-1}$ exists, and

$$\begin{aligned}
\|F'(x)^{-1}\| &\leq \frac{\|F'(x_*)^{-1}\|}{1 - \|F'(x_*)^{-1}(F'(x_*) - F'(x))\|} \\
&\leq \frac{\gamma}{1 - \gamma L\|x - x_*\|}.
\end{aligned}$$

Hence the second inequality is proved.

Given $F(x_*) = 0$, we have:

$$\begin{aligned}
F(y) &= F(y) - F(x_*) - F'(x_*)(y - x_*) + F'(x_*)(y - x_*) \\
&= \int_0^1 \left(F'(x_* + t(y - x_*)) - F'(x_*) \right) dt (y - x_*) + F'(x_*)(y - x_*).
\end{aligned}$$

The triangle inequality and the bounded condition leads to:

$$\|F'(x_*)\| = \|H(x_*) + S(x_*)\| \leq \|H(x_*)\| + \|S(x_*)\| \leq 2\beta,$$

So by the first inequality and equality above, we know that:

$$\begin{aligned}
\|F(y)\| &= \left\| \int_0^1 \left(F'(x_* + t(y - x_*)) - F'(x_*) \right) dt (y - x_*) + F'(x_*)(y - x_*) \right\| \\
&\leq \left\| \int_0^1 \left(F'(x_* + t(y - x_*)) - F'(x_*) \right) dt (y - x_*) \right\| + \|F'(x_*)(y - x_*)\| \\
&\leq \int_0^1 L t \|y - x_*\| dt (y - x_*) + \|F'(x_*)(y - x_*)\| \\
&\leq \int_0^1 L t \|y - x_*\| dt (y - x_*) + 2\beta \|y - x_*\| \\
&= \frac{L}{2} \|y - x_*\|^2 + 2\beta \|y - x_*\|.
\end{aligned}$$

Hence the third inequality is proved.

Again, given $F(x_*) = 0$, it's obvious that:

$$\begin{aligned}
&y - x_* - F'(x)^{-1}F(y) \\
&= -F'(x)^{-1}(F(y) - F(x_*) - F'(x)(y - x_*)) \\
&= -F'(x)^{-1}(F(y) - F(x_*) - F'(x_*)(y - x_*)) \\
&\quad + F'(x)^{-1}(F'(x) - F'(x_*))(y - x_*) \\
&= -F'(x)^{-1} \int_0^1 \left(F'(x_* + t(y - x_*)) - F'(x_*) \right) dt (y - x_*) \\
&\quad + F'(x)^{-1}(F'(x) - F'(x_*))(y - x_*).
\end{aligned}$$

So:

$$\begin{aligned}
&\|y - x_* - F'(x)^{-1}F(y)\| \\
&= \left\| -F'(x)^{-1} \int_0^1 \left(F'(x_* + t(y - x_*)) - F'(x_*) \right) dt (y - x_*) \right. \\
&\quad \left. + F'(x)^{-1}(F'(x) - F'(x_*))(y - x_*) \right\| \\
&\leq \| -F'(x)^{-1} \| \left(\int_0^1 \|F'(x_* + t(y - x_*)) - F'(x_*)\| dt \right. \\
&\quad \left. + \|F'(x) - F'(x_*)\| \right) \|y - x_*\| \\
&\leq \frac{\gamma}{1 - \gamma L \|x - x_*\|} \left(\frac{L}{2} \|y - x_*\| + L \|x - x_*\| \right) \|y - x_*\|.
\end{aligned}$$

Hence the fourth inequality is proved.

The proof of Lemma 1 is completed.

Lemma 2. Under the assumptions of Lemma 1, suppose $r \in (0, r_0)$, define $r_0 = \min_{1 \leq j \leq 2} \{r_+^{(j)}\}$,

$$\text{Where } r_+^{(1)} = \frac{\tau\theta}{2\gamma L + 2L_h\gamma\tau\theta}, r_+^{(2)} = \frac{1 - 2\beta\gamma[(\tau+1)\theta]^u}{3\gamma L},$$

with $u = \min\{l_*, m_*\}$, $l_* = \liminf_{k \rightarrow \infty} l_k$, $m_* = \liminf_{k \rightarrow \infty} m_k$, $\tau \in \left(0, \frac{1-\theta}{\theta}\right)$ a prescribed positive

constant and $\theta \equiv \theta(\alpha; x_*) = \|T(P; \alpha; x_*)\| \leq \frac{\sqrt{\alpha^2 + \xi_{max}^2}}{\alpha + \lambda_{min}} \equiv \sigma(\alpha; x_*)$,

(Furthermore, by the convergence of NPHSS iteration method, we know that when constant α satisfies $\alpha > \frac{\xi_{max}^2 - \lambda_{min}^2}{2\lambda_{min}}$, $\sigma(\alpha; x_*) < 1$)

Then, for any $x \in \mathbb{N}(x_*, r) \subset \mathbb{N}_0$, $t \in (0, r)$ and $v > u$, it holds that :

$$\begin{aligned} \|T(P; \alpha; x)\| &\leq (\tau + 1)\theta < 1, \\ g(t; v) &= \frac{2\gamma}{1 - \gamma Lt} (Lt + \beta[(\tau + 1)\theta]^v) < g(r_0; u) < 1. \end{aligned}$$

Proof: To prove this lemma, we need to prove the two inequalities above.

The first part illustrates that for iteration matrix $T(P; \alpha; x)$ of the MN-NPHSS method, its norm $\|T(P; \alpha; x)\|$ is less than 1.

The bounded condition directly implies the bounds:

$$\|F'(x_*)\| = \|H(x_*) + S(x_*)\| \leq \|H(x_*)\| + \|S(x_*)\| \leq 2\beta.$$

And from, we know that: $B(P; \alpha; x_*)^{-1} = (I - T(P; \alpha; x_*)) F'(x_*)^{-1}$,

Hence

$$\begin{aligned} \|B(P; \alpha; x_*)^{-1}\| &= \|(I - T(P; \alpha; x_*)) F'(x_*)^{-1}\| \\ &\leq \|(I - T(P; \alpha; x_*))\| \|F'(x_*)^{-1}\| \\ &\leq (1 + \|T(P; \alpha; x_*)\|) \|F'(x_*)^{-1}\|. \end{aligned}$$

From, $\|T(P; \alpha; x_*)\| \leq \sigma(\alpha; x_*) < 1$, according to the bounded condition,

we know that: $\|B(P; \alpha; x_*)^{-1}\| \leq 2\gamma$,

Given the definition of $B(P; \alpha; x)$ and $C(P; \alpha; x)$, we have:

$$B(P; \alpha; x) - B(P; \alpha; x_*) = H(x) - H(x_*),$$

$$C(P; \alpha; x) - C(P; \alpha; x_*) = -S(x) + S(x_*),$$

So, according to the Lipschitz condition,

$$\|B(P; \alpha; x) - B(P; \alpha; x_*)\| = \|H(x) - H(x_*)\| \leq L_h \|x - x_*\|,$$

$$\|C(P; \alpha; x) - C(P; \alpha; x_*)\| = \|-S(x) + S(x_*)\| \leq L_s \|x - x_*\|,$$

Next, we introduce the perturbation lemma.

(Perturbation lemma: Let $M, N \in \mathbb{C}^{n \times n}$, and assume that M is nonsingular, with $\|M^{-1}\| \leq \varphi$. If

$\|M - N\| \leq \delta$ and $\delta \varphi < 1$, then N is also nonsingular, and $\|N^{-1}\| \leq \frac{\varphi}{1 - \delta \varphi}$)

Because $\|B(P; \alpha; x) - B(P; \alpha; x_*)\| = \|H(x) - H(x_*)\| \leq L_h \|x - x_*\|$,

And given: $\|B(P; \alpha; x_*)^{-1}\| \leq 2\gamma$,

So, for any $x \in \mathbb{N}(x_*, r)$, provided r is small enough such that x satisfies $2\gamma L_h \|x - x_*\| < 1$,

Then by perturbation lemma: $\|B(P; \alpha; x)^{-1}\| \leq \frac{2\gamma}{1 - 2\gamma L_h \|x - x_*\|}$,

Also, given the second inequality from lemma 1, we have: $\|F'(x)^{-1}\| \leq \frac{\gamma}{1 - \gamma L \|x - x_*\|}$,

So we also need to restrict r , such that $\gamma L \|x - x_*\| < 1$,

Actually, given the condition above, we have already controlled r such that: $r < \frac{\tau\theta}{2\gamma L + 2L_h \gamma \tau\theta}$,

Because $\tau \in (0, \frac{1-\theta}{\theta})$, indicating $\tau\theta < 1$, then it is obvious that: $r < \frac{1}{\gamma L}$, and $r < \frac{1}{2L_h \gamma}$,

So for any $x \in \mathbb{N}(x_*, r)$, $2\gamma L_h \|x - x_*\| < 1$ and $\gamma L \|x - x_*\| < 1$ hold.

We obtain:

$$\begin{aligned} & T(P; \alpha; x) - T(P; \alpha; x_*) \\ &= B(P; \alpha; x)^{-1} C(P; \alpha; x) - B(P; \alpha; x_*)^{-1} C(P; \alpha; x_*) \\ &= B(P; \alpha; x)^{-1} \left((C(P; \alpha; x) - C(P; \alpha; x_*)) - (B(P; \alpha; x) - B(P; \alpha; x_*)) T(P; \alpha; x_*) \right). \end{aligned}$$

Because:

$$\begin{aligned} & \|T(P; \alpha; x) - T(P; \alpha; x_*)\| \\ &\leq \|B(P; \alpha; x)^{-1}\| \|C(P; \alpha; x) - C(P; \alpha; x_*)\| \\ &\quad + \|B(P; \alpha; x) - B(P; \alpha; x_*)\| \|T(P; \alpha; x_*)\| \\ &\leq \frac{2\gamma}{1 - 2\gamma L_h \|x - x_*\|} (L_h + L_s) \|x - x_*\| \\ &= \frac{2\gamma}{1 - 2\gamma L_h \|x - x_*\|} L \|x - x_*\|, \end{aligned}$$

And we have already controlled: $r < \frac{\tau\theta}{2\gamma L + 2L_h \gamma \tau\theta}$,

Hence, $\frac{2\gamma}{1 - 2\gamma L_h \|x - x_*\|} L \|x - x_*\| < \tau\theta$, i.e. $\|T(P; \alpha; x) - T(P; \alpha; x_*)\| \leq \tau\theta$,

Which means:

$$\begin{aligned} \|T(P; \alpha; x)\| &\leq \|T(P; \alpha; x) - T(P; \alpha; x_*)\| + \|T(P; \alpha; x_*)\| \\ &\leq \tau\theta + \theta = (\tau + 1)\theta < 1. \end{aligned}$$

Thus, the normal for iteration matrix $T(P; \alpha; x)$: $\|T(P; \alpha; x)\| \leq (\tau + 1)\theta < 1$ is proved.

Another part is to prove $g(t; v) < g(r_0; u) < 1$, this is the same with proof for Theorem 3.2 in [1], please read that for reference.

Lemma 2 is proved.

Next we will prove the local convergence theorem for MN-NPHSS.

Local Convergence Theorem

Under the assumptions of Lemma 1 and Lemma 2, for any $x_0 \in \mathbb{N}(x_*, r)$ and any positive integral sequences $\{l_k\}_{k=0}^\infty$, $\{m_k\}_{k=0}^\infty$, the iteration sequence $\{x_k\}_{k=0}^\infty$ generated by the MN-NPHSS method is well-defined and converges to x_* . Furthermore, it holds that, $\lim_{k \rightarrow \infty} \|x_k - x_*\|^{\frac{1}{k}} \leq g(r_0; u)^2$.

Where $g(r_0; u) = \frac{2\gamma}{1-\gamma L r_0} (L r_0 + \beta[(\tau + 1)\theta]^u)$,

And $u = \min\{l_*, m_*\}$, $l_* = \liminf_{k \rightarrow \infty} l_k$, $m_* = \liminf_{k \rightarrow \infty} m_k$.

Proof: From the representation method for MN-NPHSS above, we obtain:

$$\begin{aligned} & \|y_k - x_*\| \\ &= \|x_k - x_* - (I - T(P; \alpha; x_k)^{l_k})F'(x_k)^{-1}F(x_k)\| \\ &\leq \|x_k - x_* - F'(x_k)^{-1}F(x_k)\| + \|T(P; \alpha; x_k)^{l_k}\| \|F'(x_k)^{-1}F(x_k)\|. \end{aligned}$$

According to the fourth inequality from Lemma 1:

$$\begin{aligned} & \|x_k - x_* - F'(x_k)^{-1}F(x_k)\| \\ &\leq \frac{\gamma}{1 - \gamma L \|x_k - x_*\|} \left(\frac{L}{2} \|x_k - x_*\| + L \|x_k - x_*\| \right) \|x_k - x_*\| \\ &= \frac{\gamma}{1 - \gamma L \|x_k - x_*\|} \left(\frac{3L}{2} \|x_k - x_*\|^2 \right). \end{aligned}$$

From the first part result in Lemma 2, $\|T(P; \alpha; x_k)^{l_k}\| \leq [(\tau + 1)\theta]^{l_k}$,

And given the first and the third inequality in Lemma 1:

$$\begin{aligned} & \|F'(x_k)^{-1}F(x_k)\| \leq \|F'(x_k)^{-1}\| \|F(x_k)\| \\ &\leq \frac{\gamma}{1 - \gamma L \|x_k - x_*\|} \left(\frac{L}{2} \|x_k - x_*\|^2 + 2\beta \|x_k - x_*\| \right). \end{aligned}$$

Hence,

$$\begin{aligned} & \|y_k - x_*\| \leq \\ & \frac{\gamma}{1 - \gamma L \|x_k - x_*\|} \left(\frac{3L}{2} \|x_k - x_*\|^2 \right) + [(\tau + 1)\theta]^{l_k} \times \frac{\gamma}{1 - \gamma L \|x_k - x_*\|} \left(\frac{L}{2} \|x_k - x_*\|^2 + 2\beta \|x_k - x_*\| \right) \\ &= \frac{(3 + [(\tau + 1)\theta]^{l_k})\gamma L}{2(1 - \gamma L \|x_k - x_*\|)} \|x_k - x_*\|^2 + \frac{2\beta\gamma [(\tau + 1)\theta]^{l_k}}{1 - \gamma L \|x_k - x_*\|} \|x_k - x_*\|. \end{aligned}$$

From the first part result in Lemma 2: $[(\tau + 1)\theta]^{l_k} < 1$,

i.e.,

$$\frac{(3 + [(\tau + 1)\theta]^{l_k})\gamma L}{2(1 - \gamma L \|x_k - x_*\|)} < \frac{2\gamma L}{1 - \gamma L \|x_k - x_*\|},$$

hence

$$\begin{aligned}\|y_k - x_*\| &= \frac{(3 + [(\tau + 1)\theta]^{l_k})\gamma L}{2(1 - \gamma L\|x_k - x_*\|)}\|x_k - x_*\|^2 + \frac{2\beta\gamma[(\tau + 1)\theta]^{l_k}}{1 - \gamma L\|x_k - x_*\|}\|x_k - x_*\| \\ &\leq \frac{2\gamma}{1 - \gamma L\|x_k - x_*\|}(L\|x_k - x_*\| + \beta[(\tau + 1)\theta]^{l_k})\|x_k - x_*\|\end{aligned}$$

Given the definition of $g(t; v)$ before, we notice that:

$$\frac{2\gamma}{1 - \gamma L\|x_k - x_*\|}(L\|x_k - x_*\| + \beta[(\tau + 1)\theta]^{l_k})\|x_k - x_*\| = g(\|x_k - x_*\|; l_k)\|x_k - x_*\|,$$

So,

$$\|y_k - x_*\| \leq g(\|x_k - x_*\|; l_k)\|x_k - x_*\| < g(r_0; u)\|x_k - x_*\| < \|x_k - x_*\|.$$

Then we analyze $\|x_{k+1} - x_*\|$, similarly, we obtain:

$$\begin{aligned}\|x_{k+1} - x_*\| &= \|y_k - x_* - (I - T(P; \alpha; x_k)^{m_k})F'(x_k)^{-1}F(y_k)\| \\ &\leq \|y_k - x_* - F'(x_k)^{-1}F(y_k)\| + \|T(P; \alpha; x_k)^{m_k}\| \|F'(x_k)^{-1}F(y_k)\|,\end{aligned}$$

and

$$\|y_k - x_* - F'(x_k)^{-1}F(y_k)\| \leq \frac{\gamma}{1 - \gamma L\|x_k - x_*\|} \left(\frac{L}{2} \|y_k - x_*\| + L\|x_k - x_*\| \right) \|y_k - x_*\|,$$

And $\|T(P; \alpha; x_k)^{m_k}\| \leq [(\tau + 1)\theta]^{m_k}$,

and

$$\begin{aligned}\|F'(x_k)^{-1}F(y_k)\| &\leq \|F'(x_k)^{-1}\| \|F(y_k)\| \\ &\leq \frac{\gamma}{1 - \gamma L\|x_k - x_*\|} \left(\frac{L}{2} \|y_k - x_*\|^2 + 2\beta\|y_k - x_*\| \right).\end{aligned}$$

so

$$\begin{aligned}\|x_{k+1} - x_*\| &\leq \frac{\gamma}{1 - \gamma L\|x_k - x_*\|} \left(\frac{L}{2} \|y_k - x_*\| + L\|x_k - x_*\| \right) \|y_k - x_*\| \\ &\quad + [(\tau + 1)\theta]^{m_k} \times \frac{\gamma}{1 - \gamma L\|x_k - x_*\|} \left(\frac{L}{2} \|y_k - x_*\|^2 + 2\beta\|y_k - x_*\| \right) \\ &= \left(\frac{\gamma L}{1 - \gamma L\|x_k - x_*\|} \left(\frac{1 + [(\tau + 1)\theta]^{m_k}}{2} \|y_k - x_*\| + \|x_k - x_*\| \right) \right. \\ &\quad \left. + \frac{2\beta\gamma[(\tau + 1)\theta]^{m_k}}{1 - \gamma L\|x_k - x_*\|} \right) \|y_k - x_*\|.\end{aligned}$$

We have already proved $\|y_k - x_*\| \leq g(\|x_k - x_*\|; l_k)\|x_k - x_*\|$, by above,

Thus,

$$\begin{aligned}\|x_{k+1} - x_*\| &\leq \left(\frac{\gamma L}{1 - \gamma L\|x_k - x_*\|} \left(\frac{1 + [(\tau + 1)\theta]^{m_k}}{2} g(\|x_k - x_*\|; l_k)\|x_k - x_*\| + \|x_k - x_*\| \right) \right. \\ &\quad \left. + \frac{2\beta\gamma[(\tau + 1)\theta]^{m_k}}{1 - \gamma L\|x_k - x_*\|} \right) g(\|x_k - x_*\|; l_k)\|x_k - x_*\|.\end{aligned}$$

Also, given the result in Lemma 2 $[(\tau + 1)\theta]^{m_k} < 1$, we have:

$$\begin{aligned}\|x_{k+1} - x_*\| &\leq \frac{2\gamma g(\|x_k - x_*\|; l_k)}{1 - \gamma L \|x_k - x_*\|} \left(\frac{1 + g(\|x_k - x_*\|; l_k)}{2} \right. \\ &\quad \left. \times L \|x_k - x_*\| + \beta[(\tau + 1)\theta]^{m_k} \|x_k - x_*\| \right).\end{aligned}$$

We also know that $g(\|x_k - x_*\|; l_k) < 1$, so

$$\|x_{k+1} - x_*\| \leq \frac{2\gamma g(\|x_k - x_*\|; l_k)}{1 - \gamma L \|x_k - x_*\|} (L \|x_k - x_*\| + \beta[(\tau + 1)\theta]^{m_k} \|x_k - x_*\|),$$

Given the definition of $g(t; v)$ before, we notice that:

$$\frac{2\gamma}{1 - \gamma L \|x_k - x_*\|} (L \|x_k - x_*\| + \beta[(\tau + 1)\theta]^{m_k} \|x_k - x_*\|) = g(\|x_k - x_*\|; m_k) \|x_k - x_*\|.$$

So:

$$\begin{aligned}\|x_{k+1} - x_*\| &\leq g(\|x_k - x_*\|; l_k) (\|x_k - x_*\|; m_k) \|x_k - x_*\| \\ &\leq g(\|x_k - x_*\|; u)^2 \|x_k - x_*\| \\ &\leq g(r_0; u)^2 \|x_k - x_*\| < \|x_k - x_*\|.\end{aligned}$$

Similar to the proof for $\|x_{k+1} - x_*\| < \|x_k - x_*\|$ above, we can get the convergent sequence $\{x_k\}_{k=0}^\infty \in \mathbb{N}(x_*, r)$ by induction. The details are shown below:

In fact, when $k = 0$, according to the condition $x_0 \in \mathbb{N}(x_*, r)$, i.e., $\|x_0 - x_*\| < r < r_0$,

We can obtain that: $\|x_1 - x_*\| < g(\|x_0 - x_*\|; u)^2 \|x_0 - x_*\| < \|x_0 - x_*\| < r$,

Furthermore, because $x_0 \in \mathbb{N}(x_*, r)$, we have $x_1 \in \mathbb{N}(x_*, r)$.

When $k = n$, assume that $x_n \in \mathbb{N}(x_*, r)$,

We can obtain that:

$$\begin{aligned}\|x_{n+1} - x_*\| &< g(\|x_n - x_*\|; u)^2 \|x_n - x_*\| \\ &< g(r_0; u)^2 \|x_n - x_*\| \\ &< g(r_0; u)^{2(n+1)} \|x_0 - x_*\| \\ &< r.\end{aligned}$$

Which means, when $k = n + 1$, $x_{n+1} \in \mathbb{N}(x_*, r)$ still holds.

As $n \rightarrow \infty$, we have $x_{n+1} \rightarrow x_*$.

So the local convergence theorem is proved.

[1]. Y.-J. Wu, X. Li, J.-Y. Yuan. A Non-Alternating Preconditioned HSS Iteration Method For Non-Hermitian Positive Definite Linear Systems[J]. Comp. Appl. Math. DOI: 10.1007/s40314-015-0231-6, 2015.