Exercise 1.1. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive define matrix, $b \in \mathbb{R}^n$. Prove that the following two problems have a same solution:*
1. Find $x \in \mathbb{R}^n$ such that

$$Ax = b. \tag{1}$$

2. Find $x \in \mathbb{R}^n$ such that

$$J(x) = \min_{y\in\mathbb{R}^n} J(y), \tag{2}$$

where $J(y) = \frac{1}{2}(Ax, x) - (x, b)$.

Proof. Suppose that x is the solution of (1). Then for any $y \in \mathbb{R}^n$, we set $w = y - x$. Then it is clear that $(Ax, w) = (w, b)$, $(Aw, w) \geqslant 0$, and

$$\begin{aligned}
J(y) = J(w + x) &= \frac{1}{2}(Aw + Ax, w + x) - (w + x, b) \\
&= \frac{1}{2}(Ax, x) - (x, b) + \frac{1}{2}(Aw, w) + (Ax, w) - (w, b) \\
&= J(x) + \frac{1}{2}(Aw, w) \geqslant J(x).
\end{aligned}$$

Suppose that x is the solution of (2). Then for any $y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, we have $J(x+\alpha y) \geqslant J(x)$, which is convex over α and attains its minimum at $\alpha = 0$. Thus

$$\frac{\mathrm{d}J(x + \alpha y)}{\mathrm{d}\alpha}\Big|_{\alpha=0} = 0,$$

which leads to $(Ax - b, y) = 0$ giving rise to $Ax = b$. $\square$