Exercises Review

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Contents

1 Week 4

Exercise 2.14. *Consider the elliptic problem*

$$\begin{cases} Lu := -u_{xx} + u_x + u = f, \quad \forall x \in (a,b), \\ u(a) = u(b) = 0, \end{cases}$$

and its finite difference schema

$$\begin{cases} L_h u_i := -\dfrac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \dfrac{u_{i+1} - u_{i-1}}{2h} + u_i = f_i, \quad \forall i = 1, \cdots, N-1, \\ \qquad\qquad\qquad\qquad\qquad\qquad u_0 = u_N = 0, \end{cases} \tag{1}$$

in an uniform mesh $\{x_i\}_{i=0}^N$, $x_i = a + ih$, $h = (b-a)/N$.
1). *Derive an estimate for the truncation error;*
2). *Establish an a priori estimate for* $\|u_h\|_1$*;*
3). *Prove the existence and uniqueness of the solution of the finite difference schema;*
4). *Derive an error estimate for* $\|e_h\|_1$*, where* $e_i = u(x_i) - u_i$.

Solution. 1). The truncation error is

$$R_i = L_h[u(x_i)] - [Lu](x_i) = -\left(\frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2} - u_{xx}(x_i)\right) + \frac{u(x_{i+1}) - u(x_{i-1})}{2h} - u_x(x_i),$$

where $i = 1, \cdots, N-1$. By the Tylor developments

$$u(x_{i+1}) = u(x_i) + hu'(x_i) + \frac{h^2}{2}u''(x_i) + \frac{h^3}{3!}u^{(3)}(x_i) + \frac{h^4}{4!}u^{(4)}(\xi_i), \text{ for some } \xi_i \in (x_i, x_{i+1}),$$

$$u(x_{i-1}) = u(x_i) - hu'(x_i) + \frac{h^2}{2}u''(x_i) - \frac{h^3}{3!}u^{(3)}(x_i) + \frac{h^4}{4!}u^{(4)}(\eta_i), \text{ for some } \eta_i \in (x_{i-1}, x_i),$$

we obtain that $R_i = O(h^2)$ as $h \to 0$ for $i = 1, \cdots, N-1$.

2). We introduce some results in discrete form first (see [slides part1.pdf, pp. 41-48]).

(i) Sets of noses:

$$I_h = \{x_1, \cdots, x_{N-1}\}, \ \bar{I}_h = \{x_0, x_1, \cdots, x_N\}, \ I_h^+ = \{x_1, \cdots, x_N\}.$$

(ii) Grid spacing: $h = h_i := x_i - x_{i-1}$, $i = 1, \cdots, N$, and

$$\bar{h}_0 = \frac{1}{2}h_1, \ \bar{h}_N = \frac{1}{2}h_N, \quad \bar{h}_i = \frac{1}{2}(h_i + h_{i+1}), \ i = 1, \cdots, N-1.$$

(iii) Discrete functions: $v_h = \{v_0, v_1, \cdots, v_N\}$ defined on $\bar{I}_h$.

(iv) Difference operators:

$$(v_i)_{\bar{x}} := v_{i,\bar{x}} := \frac{v_i - v_{i-1}}{h_i}, \ i = 1, \cdots, N,$$
$$(v_i)_x := v_{i,x} := \frac{v_{i+1} - v_i}{h_{i+1}}, \ i = 0, \cdots, N-1,$$
$$(v_i)_{\hat{x}} := v_{i,\hat{x}} := \frac{v_{i+1} - v_i}{\bar{h}_i}, \ i = 0, \cdots, N-1.$$

(v) Discrete inner products:

$$(u_h, v_h)_{I_h} = \sum_{i=1}^{N-1} u_i v_i \bar{h}_i,\ (u_h, v_h)_{\bar{I}_h} = \sum_{i=0}^{N} u_i v_i \bar{h}_i,\ (u_h, v_h)_{I_h^+} = \sum_{i=1}^{N} u_i v_i h_i. \tag{2}$$

(vi) Discrete norms:

$$\begin{aligned} &\|v_h\|_c := \max_{\bar{I}_h} |v_i|,\ \|v_h\|_0 := (v_h, v_h)_{\bar{I}_h}^{1/2}, \\ &|v_h|_1 := ((v_h)_{\bar{x}}, (v_h)_{\bar{x}})_{I_h^+}^{1/2},\ \|v_h\|_1^2 = \|v_h\|_0^2 + |v_h|_1^2. \end{aligned} \tag{3}$$

We have some conclusions:

(i) Discrete integral by parts (see [slides part1.pdf, p. 44]).

$$\sum_{i=m+1}^{n} v_i (w_i)_{\bar{x}} h_i = -\sum_{i=m}^{n-1} (v_i)_x w_i h_{i+1} + v_n w_n - v_m w_m, \quad 0 \leqslant m < n \leqslant N. \tag{4}$$

(ii) Discrete Green formula (see [slides part1.pdf, p. 45]).

$$\sum_{i=m+1}^{n-1} ((u_i)_{\bar{x}})_{\hat{x}}\, v_i \bar{h}_i = -\sum_{i=m+1}^{n} (u_i)_{\bar{x}} (v_i)_{\bar{x}} h_i + (u_n)_{\bar{x}} v_n - (u_m)_x v_m, \quad 0 \leqslant m < n \leqslant N. \tag{5}$$

(iii) Discrete Cauchy-Schwarz inequality (see [slides part1.pdf, p. 47]).

$$|(u_h, v_h)_{\bar{I}_h}| \leqslant (u_h, u_h)_{\bar{I}_h}^{1/2} (v_h, v_h)_{\bar{I}_h}^{1/2}. \tag{6}$$

(iv) Discrete Poincaré inequalities (see [slides part1.pdf, pp. 47-48] and [HW 3, Exercise 2.12]). Assume that $v_0 = 0$ or $v_N = 0$,

$$\|v_h\|_c \leqslant C|v_h|_1, \quad \|v_h\|_0 \leqslant C|v_h|_1, \tag{7}$$

where C is a constant depending only on a and b.

Note that

$$L_h u_i = -\left((u_i)_{\bar{x}}\right)_{\hat{x}} + \frac{1}{2}\left((u_i)_{\bar{x}} + (u_i)_x\right) + u_i, \quad i = 1, \cdots, N-1.$$

Multiplying both sides of the finite difference schema $L_h u_i = f_i$ by $u_i h_i$ yields

$$-\left((u_i)_{\bar{x}}\right)_{\hat{x}} u_i h_i + \frac{1}{2}\left((u_i)_{\bar{x}} + (u_i)_x\right) u_i h + u_i^2 h = f_i u_i h_i, \quad \forall i = 1, \cdots, N-1.$$

Summing in i from 1 to $N-1$ gives

$$-\left(((u_h)_{\bar{x}})_{\hat{x}}\,, u_h\right)_{I_h} + \frac{1}{2}\left((u_h)_{\bar{x}} + (u_h)_x, u_h\right)_{I_h} + (u_h, u_h)_{I_h} = (f_h, u_h)_{I_h}.$$

In virtue of discrete integral by parts (4), discrete Green formula (5) and the fact that $u_0 = u_N = 0$, we have

$$-\left(((u_h)_{\bar{x}})_{\hat{x}}\,, u_h\right)_{I_h} = ((u_h)_{\bar{x}}, (u_h)_{\bar{x}})_{I_h^+}\,, \quad ((u_h)_{\bar{x}}, u_h)_{I_h} = -\left((u_h)_x, u_h\right)_{I_h}.$$

In fact, set $m = 0$ and $n = N$ in discrete Green formula (5), we have

$$-\left(((u_h)_{\bar{x}})_{\hat{x}}\,, u_h\right)_{I_h} = -\sum_{i=1}^{N-1} (u_i)_{\bar{x}})_{\hat{x}} u_i h = \sum_{i=1}^{N-1} (u_i)_{\bar{x}} (u_i)_{\bar{x}} h - (u_N)_{\bar{x}} u_N + (u_0)_{\bar{x}} u_0 = ((u_h)_{\bar{x}}, (u_h)_{\bar{x}})_{I_h^+}\,.$$

And set $m = 0$ and $n = N$ in discrete integral by parts (4), we have

$$
\begin{aligned}
((u_h)_{\bar{x}}, u_h)_{I_h} &= \sum_{i=1}^{N-1} (u_i)_{\bar{x}} u_i h = \sum_{i=1}^{N} (u_i)_{\bar{x}} u_i h = -\sum_{i=0}^{N-1} (u_i)_x u_i h + (u_N)^2 - (u_0)^2 \\
&= -\sum_{i=1}^{N-1} (u_i)_x u_i h = -((u_h)_x, u_h)_{I_h}.
\end{aligned}
$$

Thus

$$((u_h)_{\bar{x}}, (u_h)_{\bar{x}})_{I_h^+} + (u_h, u_h)_{I_h} = (f_h, u_h)_{I_h}\,.$$

Using the fact that $u_0 = u_N = 0$, it is equivalent to

$$((u_h)_{\bar{x}}, (u_h)_{\bar{x}})_{I_h^+} + (u_h, u_h)_{\bar{I}_h} = (f_h, u_h)_{\bar{I}_h}\,.$$

By the definition of the discrete inner norm (3), the left-hand side of the above formula is $\|u_h\|_1^2$. By the discrete Cauchy-Schwarz inequality (6), and the inequality: $\|u_h\|_0 \leqslant \|u_h\|_1$, we have

$$\|u_h\|_1^2 \leqslant \|f_h\|_0 \|u_h\|_0 \leqslant \|f_h\|_0 \|u_h\|_1 \implies \|u_h\|_1 \leqslant \|f_h\|_0.$$

3). The finite difference schema is equivalent to solve the linear system:

$$\mathbf{D}\mathbf{u} = \mathbf{f},$$

where $\mathbf{u} = [u_1, \cdots, u_{N-1}]^{\mathrm{T}}$, $\mathbf{f} = [f_1, \cdots, f_{N-1}]^{\mathrm{T}}$ and

$$
\mathbf{D} = \begin{bmatrix}
1 + \frac{2}{h^2} & -\frac{1}{h^2} + \frac{1}{2h} & & & \\
-\frac{1}{h^2} - \frac{1}{2h} & 1 + \frac{2}{h^2} & -\frac{1}{h^2} + \frac{1}{2h} & & \\
 & \ddots & \ddots & \ddots & \\
 & & -\frac{1}{h^2} - \frac{1}{2h} & 1 + \frac{2}{h^2} & -\frac{1}{h^2} + \frac{1}{2h} \\
 & & & -\frac{1}{h^2} - \frac{1}{2h} & 1 + \frac{2}{h^2}
\end{bmatrix}.
$$

Note that $\mathbf{D}$ is strictly diagonally dominant, i.e.,

$$\sum_{j=1, j\neq i}^{N-1} |D_{ij}| < |D_{ii}|, \quad i = 1, \cdots, N-1.$$

Then $\mathbf{D}$ is nonsingular, which leads to the existence and uniqueness of the solution of the finite difference schema.

4). Note that $L_h e_i = L_h[u(x_i)] - L_h u_i = R_i + [Lu](x_i) - L_h u_i = R_i$ for $i = 1, \cdots, N-1$. Then

$$
\begin{cases}
L_h e_i = R_i, \ i = 1, \cdots, N-1, \\
e_0 = e_N = 0.
\end{cases}
$$

By 1) and 2) we have $\|e_h\|_1 \leqslant C \|R_h\|_0 = O(h^2)$ as $h \to 0$. $\square$

2 Week 6

Exercise 3.3. *Consider the transport-diffusion problem*

$$\begin{cases} u_t - u_{xx} + vu_x = 0, \quad \forall x \in (a,b), \; t \in (0,T), \\ u(a,t) = u(b,t) = 0, \quad t \in (0,T), \\ \qquad u(x,0) = u_0(x), \quad \forall x \in (a,b), \end{cases}$$

where v is a constant. Derive estimates for the truncation error and global error of the following schema, and prove that

$$\|u_h^n\|_0 \leqslant \|u_h^0\|_0, \quad \forall n = 0,1,\ldots,$$

- If $v \geqslant 0$,

$$\begin{aligned} \frac{u_i^{n+1} - u_i^n}{k} - \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} + v\frac{u_i^{n+1} - u_{i-1}^{n+1}}{h} &= 0, \quad \forall i = 1, \cdots, N-1, \\ u_0^{n+1} = u_N^{n+1} &= 0, \\ u^0 &= u_0, \end{aligned}$$

- if $v \leqslant 0$,

$$\begin{aligned} \frac{u_i^{n+1} - u_i^n}{k} - \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} + v\frac{u_{i+1}^{n+1} - u_i^{n+1}}{h} &= 0, \quad \forall i = 1, \cdots, N-1, \\ u_0^{n+1} = u_N^{n+1} &= 0, \\ u^0 &= u_0, \end{aligned}$$

in an uniform mesh $\{x_i\}_{i=0}^N$, $x_i = a + ih$, $h = (b-a)/N$, $\{t^n\}_{n=0}^M$, $t^n = nk$, $k = T/M$.

Solution.

• Truncation Error.

Let $Lu = u_t - u_{xx} + vu_x$ and

$$L_h u_i^{n+1} = \begin{cases} \dfrac{u_i^{n+1} - u_i^n}{k} - \dfrac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} + v\dfrac{u_i^{n+1} - u_{i-1}^{n+1}}{h}, & \text{if } v \geqslant 0, \\ \dfrac{u_i^{n+1} - u_i^n}{k} - \dfrac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} + v\dfrac{u_{i+1}^{n+1} - u_i^{n+1}}{h}, & \text{if } v \leqslant 0. \end{cases}$$

Then $R_i^{n+1} = L_h u(x_i, t^{n+1}) - [Lu](x_i, t^{n+1})$. If $v \geqslant 0$, by Tylor developments:

$$\begin{aligned} &u(x_i, t^{n+1}) - u(x_i, t^n) = ku_t(x_i, t^{n+1}) + O(k^2), \\ &u(x_{i+1}, t^{n+1}) - 2u(x_i, t^{n+1}) + u(x_{i-1}, t^{n+1}) = h^2 u_{xx}(x_i, t^{n+1}) + O(h^4), \\ &u(x_i, t^{n+1}) - u(x_{i-1}, t^{n+1}) = hu_x(x_i, t^{n+1}) + O(h^2), \end{aligned}$$

we have $R_i^{n+1} = O(k+h)$. The similar result can also be obtained for $v \leqslant 0$.

• Global Error.

Let $e_i^{n+1} = u(x_i, t^{n+1}) - u_i^{n+1}$. Then $L_h e_i^{n+1} = R_i^{n+1}$, $\forall i = 1, \cdots, N-1$.

- If $v \geqslant 0$, we have

$$\left(1 + \frac{2k}{h^2} + v\frac{k}{h}\right) e_i^{n+1} = \frac{k}{h^2} e_{i+1}^{n+1} + \left(\frac{k}{h^2} + v\frac{k}{h}\right) e_{i-1}^{n+1} + e_i^n + kR_i^{n+1}.$$

Multiplying both sides of the above formula by $e_i^{n+1}h$, summing in i from 1 to $N-1$, and using $e_0^{n+1} = e_N^{n+1} = 0$ gives

$$\left(1 + \frac{2k}{h^2} + v\frac{k}{h}\right) \|e_h^{n+1}\|_0^2 = \frac{k}{h^2} \sum_{i=0}^{N-1} e_{i+1}^{n+1} e_i^{n+1} h + \left(\frac{k}{h^2} + v\frac{k}{h}\right) \sum_{i=1}^{N} e_{i-1}^{n+1} e_i^{n+1} h + \sum_{i=0}^{N} (e_i^n + kR_i^{n+1}) e_i^{n+1} h.$$

By Cauchy-Schwarz inequality, we have

$$\left(1+\frac{2k}{h^2}+v\frac{k}{h}\right)\|e_h^{n+1}\|_0^2 \leqslant \left(\frac{2k}{h^2}+v\frac{k}{h}\right)\|e_h^{n+1}\|_0^2+(\|e_h^n\|_0+k\|R_h^{n+1}\|_0)\|e_h^{n+1}\|_0.$$

Thus

$$\|e_h^{n+1}\|_0 \leqslant \|e_h^n\|_0+k\|R_h^{n+1}\|_0 \leqslant \cdots \leqslant \|e_h^0\|_0+k\sum_{j=1}^{n+1}\|R_h^j\|_0 \leqslant T\max_j\|R_h^j\|_0 = O(k+h).$$

The similar result can be obtained for $v \leqslant 0$.

• Stability.

If $v \geqslant 0$, multiplying both sides of $L_h u_i^n = 0$ by $u_i^{n+1}h$ yields

$$\frac{u_i^{n+1}-u_i^n}{k}u_i^{n+1}h-\frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{h}u_i^{n+1}+v(u_i^{n+1}-u_{i-1}^{n+1})u_i^{n+1}=0.$$

Summing in i from 1 to $N-1$ gives

$$\frac{h}{k}\sum_{i=1}^{N-1}(u_i^{n+1}-u_i^n)u_i^{n+1}-\frac{1}{h}\sum_{i=1}^{N-1}(u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1})u_i^{n+1}+v\sum_{i=1}^{N-1}(u_i^{n+1}-u_{i-1}^{n+1})u_i^{n+1}=0.$$

The first term

$$\begin{aligned}\frac{h}{k}\sum_{i=1}^{N-1}(u_i^{n+1}-u_i^n)u_i^{n+1} &= \frac{1}{k}\left(u_h^{n+1}-u_h^n,u_h^{n+1}\right)_{I_h} = \frac{1}{2k}\left(u_h^{n+1}-u_h^n,u_h^{n+1}-u_h^n+u_h^{n+1}+u_h^n\right)_{I_h}\\ &\geqslant \frac{1}{2k}\left(u_h^{n+1}-u_h^n,u_h^{n+1}+u_h^n\right)_{I_h} = \frac{1}{2k}\left(u_h^{n+1}-u_h^n,u_h^{n+1}+u_h^n\right)_{\bar{I}_h}\\ &= \frac{1}{2k}\left(\|u_h^{n+1}\|_0^2-\|u_h^n\|_0^2\right).\end{aligned}$$

The second term (set $m=0$ and $n=N$ in discrete Green formula (5))

$$\begin{aligned}-\frac{1}{h}\sum_{i=1}^{N-1}(u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1})u_i^{n+1} &= -\left(((u_h^{n+1})_{\bar{x}})_{\hat{x}},u_h^{n+1}\right)_{I_h}\\ &= \left((u_h^{n+1})_{\bar{x}},(u_h^{n+1})_{\bar{x}}\right)_{I_h^+}-(u_N^{n+1})_{\bar{x}}u_N^{n+1}+(u_0^{n+1})_x u_0^{n+1}\\ &= \left((u_h^{n+1})_{\bar{x}},(u_h^{n+1})_{\bar{x}}\right)_{I_h^+} \geqslant 0.\end{aligned}$$

The third term

$$\begin{aligned}v\sum_{i=1}^{N-1}(u_i^{n+1}-u_{i-1}^{n+1})u_i^{n+1} &= \frac{v}{2}\sum_{i=1}^{N-1}(u_i^{n+1}-u_{i-1}^{n+1})(u_i^{n+1}-u_{i-1}^{n+1}+u_i^{n+1}+u_{i-1}^{n+1})\\ &\geqslant \frac{v}{2}\sum_{i=1}^{N-1}\left[(u_i^{n+1})^2-(u_{i-1}^{n+1})^2\right] = \frac{v}{2}(u_{N-1}^{n+1})^2 \geqslant 0.\end{aligned}$$

Thus we obtain $\|u_h^{n+1}\|_0 \leqslant \|u_h^n\|_0$, which leads to $\|u_h^n\|_0 \leqslant \|u_h^0\|_0$. For $v \leqslant 0$, a similar approach can be applied to obtain the desired result, except for the treatment of the third term:

$$\begin{aligned}v\sum_{i=1}^{N-1}(u_{i+1}^{n+1}-u_i^{n+1})u_i^{n+1} &= \frac{-v}{2}\sum_{i=1}^{N-1}(u_i^{n+1}-u_{i+1}^{n+1})(u_i^{n+1}-u_{i+1}^{n+1}+u_i^{n+1}+u_{i+1}^{n+1})\\ &\geqslant \frac{-v}{2}\sum_{i=1}^{N-1}\left[(u_i^{n+1})^2-(u_{i+1}^{n+1})^2\right] = \frac{-v}{2}(u_1^{n+1})^2 \geqslant 0.\end{aligned}$$

□

3 Week 8

Exercise 1.2. *Prove some alternative forms of the Poincaré inequality:*

$$\|v\|_{L^\infty} \leqslant c_1\|v'\|_0, \quad \forall v \in \{v \in H^1(I),\ v(0) = 0\}.$$
$$\|v\|_0 \leqslant c_2\|v'\|_0, \quad \forall v \in \{v \in H^1(I),\ v(0) = 0\}.$$

Proof. Let $V = \{v \in H^1(I),\ v(0) = 0\}$ and $U = \{v \in C^\infty(I),\ v(0) = 0\}$. Then U is dense in V with respect to $\|\cdot\|_1$, i.e., $\forall v \in V$, there exists $\{v_n\} \subset U$ such that

$$\lim_{n\to\infty} \|v_n - v\|_1 = 0.$$

Thus

$$\begin{aligned}
&\|v - v_n\|_{L^\infty} \leqslant C\|v - v_n\|_1 \to 0, \text{ as } n \to \infty,\\
&\|v - v_n\|_0 \leqslant \|v - v_n\|_1 \to 0, \text{ as } n \to \infty,\\
&\|v' - v_n'\|_0 = |v - v_n|_1 \leqslant \|v - v_n\|_1 \to 0, \text{ as } n \to \infty.
\end{aligned}$$

For the first inequality, we obtain it by employing embedding theorem: $H^1(I) \hookrightarrow C^0(\bar{I})$, or by Gagliardo-Nirenberg inequality:

$$\|u\|_{L^\infty(I)} \leq \left(\frac{1}{|I|} + 2\right)^{1/2} \|u\|^{1/2}_{L^2(I)}\|u\|^{1/2}_{H^1(I)}, \quad \forall u \in H^1(I).$$

Therefore, it is sufficient to show that the inequalities hold for any $v \in U$, which is obvious since

$$|v(x)| = \left|\int_0^x v'(x)\mathrm{d}x\right| \leqslant \left(\int_0^x 1^2 \ \mathrm{d}t\right)^{\frac{1}{2}} \left(\int_0^x \left|v_n'(t)\right|^2 \ \mathrm{d}t\right)^{\frac{1}{2}} \leqslant \|v'\|_0.$$

Since

$$\big|\|v - v_n\|_{L^\infty} - \|v\|_{L^\infty}\big| \leqslant \|v_n\|_{L^\infty} \leqslant C\|v_n'\|_0 \leqslant C\|v' - v_n'\|_0 + C\|v'\|_0$$

leads to $\|v\|_{L^\infty} \leqslant C\|v'\|_0$. □

4 Week 9

Exercise 1. *Let $\{x_n\}_{n=0}^{N+1}$ be a grid in the interval $\Lambda = (0, 1)$, i.e., $0 = x_0 < x_1 < x_2 < \cdots < x_N < x_{N+1} = 1$. Let $I_n = (x_{n-1}, x_n)$, $h_n = x_n - x_{n-1}$, and $h = \max_{1\leqslant n\leqslant N+1} h_n$. Prove*

$$\{v \in C^0(\Lambda) : v|_{I_n} \in H^1(I_n), n = 1, \cdots , N + 1\} \subset H^1(\Lambda).$$

Proof. For any $v \in \{v \in C^0(\Lambda) : v|_{I_n} \in H^1(I_n), n = 1, \cdots , N + 1\}$, it is clear that $v \in L^2(\Lambda)$ because continuity implies square integrability on the bounded domain Λ. It remains to show that the weak derivative of v also belongs to $L^2(\Lambda)$. Since $v|_{I_n} \in H^1(I_n)$, we define its piecewise derivative by

$$g|_{I_n}(x) = (v|_{I_n})'(x), \quad x \in I_n,\ n = 1, \cdots , N + 1.$$

Obviously, $g \in L^2(\Lambda)$, as each piece $(v|_{I_n})' \in L^2(I_n)$ and the intervals I_n are disjoint and cover Λ. We claim that g is the derivative of v. Indeed, for any test function $\phi(x) \in C_0^\infty(\Lambda)$, we have

$$\begin{aligned}
\int_0^1 g(x)\phi(x)\mathrm{d}x &= \sum_{n=1}^{N+1}\int_{I_n} g|_{I_n}(x)\phi(x)\mathrm{d}x = \sum_{n=1}^{N+1}\int_{I_n} (v|_{I_n})'(x)\phi(x)\mathrm{d}x\\
&= \sum_{n=1}^{N+1} [v(x)\phi(x)]\big|_{x_{n-1}}^{x_n} - \sum_{n=1}^{N+1}\int_{I_n} (v|_{I_n})(x)\phi'(x)\mathrm{d}x\\
&= \sum_{n=1}^{N+1} \left(v(x_n^-)\phi(x_n^-) - v(x_{n-1}^+)\phi(x_{n-1}^+)\right) - \sum_{n=1}^{N+1}\int_{I_n} (v|_{I_n})(x)\phi'(x)\mathrm{d}x.
\end{aligned}$$

Due to the continuity of v across element interfaces, we have $v(x_n^-) = v(x_n^+)$ for $n = 1, \cdots, N$, and since $\phi \in C_0^\infty(\Lambda)$ we have $\phi(x_0) = \phi(x_{N+1}) = 0$. Hence, the sum of boundary terms cancels out, yielding

$$\int_0^1 g(x)\phi(x)\mathrm{d}x = -\int_0^1 v(x)\phi'(x)\mathrm{d}x,$$

which confirms that g is the weak derivative of v. Therefore, $v \in H^1(\Lambda)$. $\square$

5 Week 10

Exercise 2. *Consider the mixed boundary problem*

$$\begin{cases} -u'' = f, \quad x \in I := (0,1), \\ u(0) = 0, \ u'(1) = \beta, \end{cases}$$

where $\beta \in \mathbb{R}$ and $f \in L^2(I)$. Construct and analyze P_1-FEM for this problem.

Proof. • Variational form. Let $V = \{v \in H^1(I) : v(0) = 0\}$, the bilinear form $a(u,v) = (u', v')$, and the functional $\mathcal{F}(v) = (f, v) + \beta v(1)$. Then the variational problem reads

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ a(u,v) = \mathcal{F}(v), \quad \forall v \in V, \end{cases}$$

which is clearly equivalent to the strong problem.

It is obvious that the solution of the strong problem is also the solution of the weak problem. Conversely, suppose that u is the solution of the weak problem. Then $(u', v') = (f, v) + \beta v(1), \ \forall v \in V$, which leads to $(u', v') = (f, v), \ \forall v \in C_0^\infty(I)$, and then

$$(-u'', v) = (f, v), \quad \forall v \in C_0^\infty(I),$$

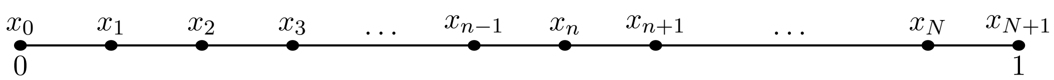
where u'' is the derivative of u' in the distribution sense. Thus $-u'' = f$ in the distribution sense (If $f \in L^2(I)$ and $u \in H^2(I)$, it is clear that $-u'' = f$ in L^2 sense; if $f \in C(I)$ and $u \in C^2(I)$, then $-u'' = f$ pointwise). For the boundary conditions, $u(0) = 0$ is obvious since $u \in V$, and $u'(1) = \beta$ follows from the integral by parts, i.e.,

$$(u', v') = (-u'', v) + u'(1)v(1) = (f, v) + \beta v(1) \Rightarrow u'(1)v(1) = \beta v(1), \ \forall v \in V.$$

• Galerkin Approximation. Let V_h be a subspace of V with finite dimension. Then the Galerkin approximation reads

$$\begin{cases} \text{Find } u_h \in V_h \text{ such that} \\ a(u_h, v_h) = \mathcal{F}(v_h), \quad \forall v_h \in V_h. \end{cases}$$

• $P1-$FEM. Let $\{x_n\}_{n=0}^{N+1}$ be a grid on I such that $0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1$. Denote by each subintervals (or elements) $I_n = (x_{n-1}, x_n)$ for $1 \leqslant n \leqslant N+1$ of length $h_n = x_n - x_{n-1}$. Let $h = \max_{1 \leqslant n \leqslant N+1} h_n$.



The piecewise linear polynomials on such grid is denoted by

$$X_h^1 := \{v \in C(\bar{I}) : v\big|_{I_{n+1}} \in \mathbb{P}_1,\ n = 0, \cdots, N\}.$$

We construct a nodal basis for X_h^1, which is based on nodes in every element (how many nodes in every element depends on the degree of freedom, or the degree of polynomials required parameters to be determined).

$$\varphi_0(x) = \begin{cases} \dfrac{x_1 - x}{x_1 - x_0}, & x \in I_1, \\ 0, & \text{else}, \end{cases} \qquad \varphi_{N+1}(x) = \begin{cases} \dfrac{x - x_N}{x_{N+1} - x_N}, & x \in I_{N+1}, \\ 0, & \text{else}, \end{cases}$$

$$\varphi_n(x) = \begin{cases} \dfrac{x - x_{n-1}}{x_n - x_{n-1}}, & x \in I_n, \\ \dfrac{x_{n+1} - x}{x_{n+1} - x_n}, & x \in I_{n+1}, \\ 0, & \text{else}. \end{cases}$$

Clearly, we have $X_h^1 = \text{span}\{\varphi_0, \varphi_1, \cdots, \varphi_{N+1}\}$. For any $u \in C(\bar{I})$, its interpolation into X_h^1 is denoted by $u_I(x)$. Clearly, we have $u_I(x) = \sum_{i=0}^{N+1} u(x_i)\varphi_i(x)$ and

$$u_I\big|_{I_{n+1}} = u(x_n)\varphi_n(x) + u(x_{n+1})\varphi_{n+1}(x) = u(x_n)\frac{x_{n+1} - x}{x_{n+1} - x_n} + u(x_{n+1})\frac{x - x_n}{x_{n+1} - x_n}.$$

Let the finite element space $V_h = X_h^1 \cap V$. It is known that $X_h^1 \subset H^1(I)$ [see HW 9, Exercise 1], then $V_h = \{v \in X_h^1 : v(0) = 0\}$, i.e.,

$$V_h = \text{span}\{\varphi_1, \cdots, \varphi_{N+1}\}.$$

• FEM Implementation. Let $u_h = \sum_{j=1}^{N+1} u_j\varphi_j(x)$, then

$$\sum_{j=1}^{N+1} u_j a(\varphi_j, \varphi_i) = \mathcal{F}(\varphi_i), \quad i = 1, \cdots, N+1.$$

Let $\mathbf{A} = (a_{i,j})$ be the $(N+1) \times (N+1)$ matrix with its entries $a_{i,j} = a(\varphi_j, \varphi_i)$. Then we have

$$\begin{aligned} &a_{N+1,N+1} = \frac{1}{h_{N+1}},\ a_{j,j} = \frac{1}{h_j} + \frac{1}{h_{j+1}}, \quad j = 1, \cdots, N, \\ &a_{j,j+1} = a_{j+1,j} = -\frac{1}{h_{j+1}},\ j = 1, \cdots, N, \\ &a_{i,j} = 0,\ \text{if } |i - j| \geqslant 2. \end{aligned}$$

Thus

$$\frac{1}{h}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \\ u_{N+1} \end{bmatrix} = \begin{bmatrix} (f, \varphi_1) \\ (f, \varphi_2) \\ \vdots \\ (f, \varphi_N) \\ (f, \varphi_{N+1}) + \beta \end{bmatrix}.$$

• Error Estimate. We denote u_I being the interpolation of u into V_h, then it is known [see HW 10, Exercise 1] that

$$\|u - u_I\|_0 \leqslant Ch\|u' - u_I'\|_0 \leqslant Ch^2\|u''\|_0.$$

We know $a(u-u_h, v_h) = 0$ for any $v_h \in V_h$. Then

$$\|u' - u_h'\|_0^2 = a(u-u_h, u-u_h) = a(u-u_h, u-v_h) \leqslant \|u'-u_h'\|_0 \|u'-v_h'\|_0, \quad \forall v_h \in V_h,$$

which leads to

$$\|u'-u_h'\|_0 \leqslant \inf_{v_h \in V_h} \|u'-v_h'\|_0 \leqslant \|u'-u_I'\|_0 \leqslant Ch\|u''\|_0.$$

In the following, we derive the estimate for $\|u-u_h\|_0$ by using Aubin-Nitsche trick.

Consider the dual problem: given $r \in L^2(I)$,

$$\begin{cases} \text{Find } \varphi(r) \in V \text{ such that} \\ a(v, \varphi(r)) = (r, v), \quad \forall v \in V. \end{cases}$$

The dual problem admits a unique solution $\varphi(r)$ since $a(\cdot,\cdot)$ is continuous and coercive. Moreover, we have

$$a(v, \varphi(r)) = (r, v), \quad \forall v \in C_0^\infty(I),$$

if we suppose $\varphi(r) \in H^2(I)$, which gives $(-\varphi''(r), v) = (r, v)$, $\forall v \in C_0^\infty(I)$, leading to $-\varphi''(r) = r$ in L^2 since $C_0^\infty(I)$ is dense in $L^2(I)$. Since for any $v \in L^2(I)$, there exists $\{v_n\} \subset C_0^\infty(I)$ such that $\lim_{n\to\infty} \|v - v_n\|_0 = 0$, then $(\varphi''(r) + r, v_n) = 0$ and $(\varphi'' + r, v) \leqslant \|\varphi'' + r\|_0 \|v - v_n\|_0 \to 0$, as $n \to \infty$. Take $v = \varphi''(r) + r$ leading to the desired result.

Let $\varphi_I(r)$ be the interpolation of $\varphi(r)$ into V_h. We have $\|\varphi'(r) - \varphi_I'(r)\|_0 \leqslant Ch\|\varphi''(r)\|_0$ and

$$\begin{aligned} \|u-u_h\|_0 &= \sup_{r\in L^2(I),\ r\neq 0} \frac{(r, u-u_h)}{\|r\|_0} = \sup_{r\in L^2(I),\ r\neq 0} \frac{a(u-u_h, \varphi(r))}{\|r\|_0} \\ &= \sup_{r\in L^2(I),\ r\neq 0} \frac{a(u-u_h, \varphi(r) - \varphi_I(r))}{\|r\|_0} \\ &\leqslant \sup_{r\in L^2(I),\ r\neq 0} \frac{\|u'-u_h'\|_0 \|\varphi'(r) - \varphi_I'(r)\|_0}{\|r\|_0} \\ &\leqslant Ch\|u'-u_h'\|_0 \sup_{r\in L^2(I),\ r\neq 0} \frac{\|\varphi''(r)\|_0}{\|r\|_0} \\ &\leqslant Ch\|u'-u_h'\|_0. \end{aligned}$$

□