

Exercise 2.3. Determine the absolute stability region of the Crank-Nicolson schema and Leapfrog schema.

Solution. Model problem:

$$\frac{du}{dt} = \lambda u, \quad \text{where } \lambda \text{ is a constant.}$$

• Crank-Nicolson schema:

$$u^{n+1} = u^n + \frac{\lambda h(u^{n+1} + u^n)}{2} \Rightarrow u^{n+1} = \frac{2 + h\lambda}{2 - h\lambda} u^n.$$

Let $z = \lambda h$. The absolute stability region of this schema is

$$\left\{ z : \left| \frac{2+z}{2-z} \right| \leq 1 \right\} = \{ z : \operatorname{Re}(z) \leq 0 \}.$$

In fact, if we suppose that $z = a + bi$, then $|2+z| \leq |2-z|$ is equivalent to

$$(a+2)^2 + b^2 \leq (a-2)^2 + b^2 \Leftrightarrow a \leq 0.$$

• Leapfrog schema:

$$u^{n+1} = 2h\lambda u^n + u^{n-1}.$$

Let $z = \lambda h$, then its characteristic polynomial is $\rho(\xi) = \xi^2 - 2z\xi - 1$. Hence roots of $\rho(\xi)$ are $\xi_{1,2} = z \pm \sqrt{z^2 + 1}$. The schema is absolute stable if and only if the roots satisfy the condition (see [LeVeque (2007), Definition 7.1, p. 153]):

$$|\xi_j| \leq 1, \quad j = 1, 2.$$

$$\text{If } \xi_1 = \xi_2, \text{ then } |\xi_1| < 1.$$

We note that $|\xi_1||\xi_2| = 1$, then the only possibility to satisfy the root condition is $|\xi_1| = |\xi_2| = 1$, which leads to that z is the pure imaginary number. By setting $z = bi$, we obtain $b \in (-1, 1)$. Hence the absolute stability region is $\{z = bi : b \in (-1, 1)\}$. \square

References

[LeVeque (2007)] LeVeque R J. Finite difference methods for ordinary and partial differential equations: steady-state and time-dependent problems[M]. Society for Industrial and Applied Mathematics, 2007.