

Exercise 1. Consider the BVP

$$\begin{cases} -u'' = f, & x \in \Lambda := (0, 1), \\ u(0) = 0, \quad u'(1) = \beta. \end{cases} \quad (1)$$

Construct a $P1$ -FEM for this problem and compare with central finite difference scheme.

Solution.

1. $P1$ -FEM.

1.1 Variational form. Let $V = \{v \in H^1(\Lambda) : v(0) = 0\}$. Then the variational form reads

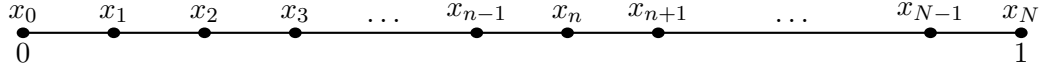
$$\begin{cases} \text{find } u \in V \text{ s.t.} \\ a(u, v) = \mathcal{F}(v), \quad \forall v \in V, \end{cases} \quad (2)$$

where $a(u, v) = (u', v')$ and $\mathcal{F}(v) = (f, v) + \beta v(1)$. Clearly, the weak problem (2) is equivalent to the strong problem (1). The well-posedness of (2) is guaranteed by Lax-Milgram theorem.

1.2 Galerkin approximation. Let $V_h \subset V$ be a finite dimensional space. Replacing V with V_h in (2) leads to the Galerkin approximation

$$\begin{cases} \text{find } u_h \in V_h \text{ s.t.} \\ a(u_h, v_h) = \mathcal{F}(v_h), \quad \forall v_h \in V_h. \end{cases} \quad (3)$$

1.3 $P1$ finite element space. Let $\{x_i\}_{i=0}^N$ be the uniform grid on Λ shown as follows:



Let $\mathcal{T}_h = \{I_i\}_{i=1}^N$, where $I_i = (x_{i-1}, x_i)$ is the element with length of $h = h_i := x_i - x_{i-1}$ for $i = 1, \dots, N$. Then the $P1$ finite element space is

$$X_h^1 = \{v \in C^0(\bar{\Lambda}) : v|_K \in \mathbb{P}_1, \quad \forall K \in \mathcal{T}_h\},$$

where \mathbb{P}_1 is the space of polynomials degree of at most 1. We construct a nodal basis for X_h^1 , defined by nodes within each element. The number of nodes per element depends on the degrees of freedom, which are determined by the required polynomial degree.

$$\varphi_0(x) = \begin{cases} \frac{x_1 - x}{x_1 - x_0}, & x \in I_1, \\ 0, & \text{else,} \end{cases} \quad \varphi_N(x) = \begin{cases} \frac{x - x_{N-1}}{x_N - x_{N-1}}, & x \in I_N, \\ 0, & \text{else,} \end{cases} \quad (4)$$

$$\varphi_n(x) = \begin{cases} \frac{x - x_{n-1}}{x_n - x_{n-1}}, & x \in I_n, \\ \frac{x_{n+1} - x}{x_{n+1} - x_n}, & x \in I_{n+1}, \\ 0, & \text{else.} \end{cases} \quad (5)$$

Clearly, we have $X_h^1 = \text{span}\{\varphi_0, \varphi_1, \dots, \varphi_N\}$. For any $u \in C(\bar{\Lambda})$, its interpolation into X_h^1 is denoted by $u_I(x)$. Clearly, we have $u_I(x) = \sum_{j=0}^N u(x_j)\varphi_j(x)$ and

$$u_I(x)|_{I_n} = u(x_{n-1})\varphi_{n-1}(x) + u(x_n)\varphi_n(x) = u(x_{n-1})\frac{x_n - x}{x_n - x_{n-1}} + u(x_n)\frac{x - x_{n-1}}{x_n - x_{n-1}}. \quad (6)$$

1.4 Implementation. We consider solving (3) by $P1$ -FEM. Let $V_h = X_h^1 \cap V$. It is known that $X_h^1 \subset H^1(I)$ (see [HW 9, Exercise 1]), then $V_h = \{v \in X_h^1 : v(0) = 0\}$, i.e.,

$$V_h = \text{span}\{\varphi_1, \dots, \varphi_N\}. \quad (7)$$

Let $u_h = \sum_{j=1}^N u_j \varphi_j(x)$, where u_j are the parameters to be determined. Then the finite element approximation reads

$$\begin{cases} \text{find } \{u_j\}_{j=1}^N \text{ s.t.} \\ \sum_{j=1}^N a(\varphi_j, \varphi_i) u_j = \mathcal{F}(\varphi_i), \quad i = 1, \dots, N. \end{cases} \quad (8)$$

By (4) and (5), it can be computed that

$$a(\varphi_j, \varphi_i) = (\varphi_j', \varphi_i') = \begin{cases} 1/h, & i = j = N, \\ 2/h, & i = j \neq N, \\ -1/h, & |i - j| = 1, \\ 0, & |i - j| > 1. \end{cases}$$

Thus (8) is equivalent to the linear system

$$\frac{1}{h} \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & & \\ 0 & \cdots & 0 & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} (f, \varphi_1) \\ (f, \varphi_2) \\ (f, \varphi_3) \\ \vdots \\ (f, \varphi_{N-1}) \\ (f, \varphi_N) + \beta \end{bmatrix}.$$

1.5 Error estimates.

- Interpolation error.

The following error estimates hold:

$$\|u' - u_I'\|_{L^\infty(\Lambda)} \leq h \|u''\|_{L^\infty(\Lambda)}, \quad \|u - u_I\|_{L^\infty(\Lambda)} \leq h^2 \|u''\|_{L^\infty(\Lambda)}, \quad (9)$$

and

$$\|u' - u_I'\|_{L^2(\Lambda)} \leq h \|u''\|_{L^\infty(\Lambda)}, \quad \|u - u_I\|_{L^2(\Lambda)} \leq h^2 \|u''\|_{L^\infty(\Lambda)}. \quad (10)$$

Both estimates (9) and (10) can be derived by applying Taylor series expansions in (6), specifically by expanding $u(x_{n-1})$ and $u(x_n)$ around a point $x \in I_n$ (see [slides part2.pdf, pp. 32-35]). There are better estimates bounded by L^2 -norm (see [HW 10, Exercise 1]):

$$\|u - u_I\|_{L^2(\Lambda)} \leq h \|u' - u_I'\|_{L^2(\Lambda)} \leq h^2 \|u''\|_{L^2(\Lambda)}. \quad (11)$$

- Equation error.

Let u be the solution of (2), and u_h the solution of (3) with V_h replaced by (7). Thus we have

$$a(u - u_h, v_h) = 0, \quad \forall v_h \in V_h.$$

Thus

$$\|u' - u_h'\|_0^2 = a(u - u_h, u - u_h) = a(u - u_h, u - v_h) \leq \|u' - u_h'\|_0 \|u' - v_h'\|_0, \quad \forall v_h \in V_h,$$

which, by (11), leads to

$$\|u' - u'_h\|_0 \leq \inf_{v_h \in V_h} \|u' - v'_h\|_0 \leq \|u' - u'_I\|_0 \leq h\|u''\|_0.$$

In the following, we derive the estimate for $\|u - u_h\|_0$ by using Aubin-Nitsche trick.

Consider the dual problem of (2): given $r \in L^2(I)$,

$$\begin{cases} \text{Find } \varphi(r) \in V \text{ such that} \\ a(v, \varphi(r)) = (r, v), \quad \forall v \in V. \end{cases}$$

The dual problem admits a unique solution $\varphi(r)$ since $a(\cdot, \cdot)$ is continuous and coercive. Moreover, we have

$$a(v, \varphi(r)) = (r, v), \quad \forall v \in C_0^\infty(I),$$

if we suppose $\varphi(r) \in H^2(I)$, which gives $(-\varphi''(r), v) = (r, v)$, $\forall v \in C_0^\infty(I)$, leading to $-\varphi''(r) = r$ in L^2 since $C_0^\infty(I)$ is dense in $L^2(I)$. Let $\varphi_I(r)$ be the interpolation of $\varphi(r)$ into V_h . By (11), we have $\|\varphi'(r) - \varphi'_I(r)\|_0 \leq h\|\varphi''(r)\|_0$. Then

$$\begin{aligned} \|u - u_h\|_0 &= \sup_{r \in L^2(\Lambda), r \neq 0} \frac{(r, u - u_h)}{\|r\|_0} = \sup_{r \in L^2(\Lambda), r \neq 0} \frac{a(u - u_h, \varphi(r))}{\|r\|_0} \\ &= \sup_{r \in L^2(\Lambda), r \neq 0} \frac{a(u - u_h, \varphi(r) - \varphi_I(r))}{\|r\|_0} \\ &\leq \sup_{r \in L^2(\Lambda), r \neq 0} \frac{\|u' - u'_h\|_0 \|\varphi'(r) - \varphi'_I(r)\|_0}{\|r\|_0} \\ &\leq h\|u' - u'_h\|_0 \sup_{r \in L^2(\Lambda), r \neq 0} \frac{\|\varphi''(r)\|_0}{\|r\|_0} \\ &\leq h\|u' - u'_h\|_0. \end{aligned}$$

Thus we have $\|u - u_h\|_0 \leq h^2\|u''\|_0$.

2. Central finite difference schema. Let $\bar{I}_h = \{x_i\}_{i=0}^N$ be the grid on Λ with uniform distance $h = h_i = x_i - x_{i-1}$. Then the central schema reads

$$\begin{cases} -\frac{\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}}{h^2} = f(x_i), \quad i = 1, \dots, N-1, \\ \bar{u}_0 = 0, \quad \frac{\bar{u}_N - \bar{u}_{N-1}}{h} = \frac{h}{2}f(x_N) + \beta. \end{cases} \quad (12)$$

2.1 Linear system. The finite difference schema (12) is equivalent to the linear system:

$$\frac{1}{h} \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ & & & \ddots & & & \\ 0 & \cdots & 0 & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \vdots \\ \bar{u}_{N-1} \\ \bar{u}_N \end{bmatrix} = \begin{bmatrix} hf(x_1) \\ hf(x_2) \\ hf(x_3) \\ \vdots \\ hf(x_{N-1}) \\ \frac{h}{2}f(x_N) + \beta \end{bmatrix}.$$

2.2 Local error estimate. Let the operator $Lu = -u_{xx}$ and the discrete operator L_h on $\{\bar{u}_i\}_{i=1}^{N-1}$ as

$$L_h \bar{u}_i = -\frac{\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}}{h^2}.$$

By the Tylor development:

$$u(x_{i+1}) = u(x_i) + hu'(x_i) + \frac{h^2}{2}u''(x_i) + \frac{h^3}{3!}u^{(3)}(x_i) + \frac{h^4}{4!}u^{(4)}(\xi_i), \text{ for some } \xi_i \in (x_i, x_{i+1}),$$

$$u(x_{i-1}) = u(x_i) - hu'(x_i) + \frac{h^2}{2}u''(x_i) - \frac{h^3}{3!}u^{(3)}(x_i) + \frac{h^4}{4!}u^{(4)}(\eta_i), \text{ for some } \eta_i \in (x_{i-1}, x_i),$$

we obtain the local error estimates $R_i^{(1)} = L_h[u(x_i)] - [Lu](x_i) = O(h^2)$, and

$$R^{(2)} = \frac{u(x_N) - u(x_{N-1})}{h} - \frac{h}{2}f(x_N) - u'(x_N) = O(h^2).$$

2.3 Stability. This proof adopts the notation for discrete functions (see [HW 4, Appendix]). Note that $L_h \bar{u}_i = -((\bar{u}_i)_{\bar{x}})_{\hat{x}}$, then multiplying both sides of the finite difference schema $L_h \bar{u}_i = f_i$ by $\bar{u}_i h_i$ yields

$$-((\bar{u}_i)_{\bar{x}})_{\hat{x}} \bar{u}_i h_i = f_i \bar{u}_i h_i, \quad \forall i = 1, \dots, N-1.$$

Summing in i from 1 to $N-1$ gives

$$-((\bar{u}_h)_{\bar{x}})_{\hat{x}} \bar{u}_h = (f_h, \bar{u}_h)_{I_h}, \quad (13)$$

where $\bar{u}_h = \{\bar{u}_0, \dots, \bar{u}_N\}$ and $f_h = \{f_0, \dots, f_N\}$, both of which are the discrete functions defined on the grid \bar{I}_h . In virtue of discrete Green formula and the fact that $\bar{u}_0 = 0$, we have

$$-((\bar{u}_h)_{\bar{x}})_{\hat{x}} \bar{u}_h = ((\bar{u}_h)_{\bar{x}}, (\bar{u}_h)_{\bar{x}})_{I_h^+} - (\bar{u}_N)_{\bar{x}} \bar{u}_N. \quad (14)$$

Thus inserting (14) into (13) we obtain

$$((\bar{u}_h)_{\bar{x}}, (\bar{u}_h)_{\bar{x}})_{I_h^+} = (f_h, \bar{u}_h)_{I_h} + (\bar{u}_N)_{\bar{x}} \bar{u}_N. \quad (15)$$

Note that $(\bar{u}_N)_{\bar{x}} = \beta + \frac{h}{2}f(x_N) \leq |\beta| + \|f_h\|_0$, and

$$\bar{u}_N = \sum_{i=1}^N (\bar{u}_i)_{\bar{x}} h \leq \left(\sum_{i=1}^N h \right)^{1/2} \left(\sum_{i=1}^N (\bar{u}_i)_{\bar{x}}^2 h \right)^{1/2} = |\bar{u}_h|_1.$$

We have

$$|\bar{u}_h|_1^2 = ((\bar{u}_h)_{\bar{x}}, (\bar{u}_h)_{\bar{x}})_{I_h^+} \leq (|\beta| + \|f_h\|_0) |\bar{u}_h|_1 + (f_h, \bar{u}_h)_{I_h} \leq (|\beta| + \|f_h\|_0) |\bar{u}_h|_1 + \|f_h\|_0 \|\bar{u}_h\|_0.$$

By discrete Poincaré inequality (see [HW 3, Exercise 2.12]): $\|\bar{u}_h\|_0 \leq |\bar{u}_h|_1$, we obtain

$$|\bar{u}_h|_1 \leq (2\|f_h\|_0 + |\beta|).$$

Thus by using discrete Poincaré again, we obtain

$$\|\bar{u}_h\|_1 = (\|\bar{u}_h\|_0^2 + |\bar{u}_h|_1^2)^{1/2} \leq \sqrt{2} |\bar{u}_h|_1 \leq \sqrt{2} (2\|f_h\|_0 + |\beta|).$$

2.4 Global error estimate. Let $e_i = u(x_i) - \bar{u}_i$ for $i = 0, 1, \dots, N$. It is obvious that

$$\begin{cases} L_h e_i = R_i^{(1)}, & \forall i = 1, 2, \dots, N-1, \\ e_0 = 0, \\ \frac{e_N - e_{N-1}}{h} = R^{(2)}. \end{cases}$$

Thus $\|e_h\|_1 \leq \sqrt{2} (2\|R_h^{(1)}\|_0 + |R^{(2)}|) = O(h^2)$ as $h \rightarrow 0$.

3. Comparison of these two methods. Let $\bar{\mu}_h(x) = \sum_{j=1}^N \bar{u}_j \varphi_j(x)$.

3.1 Show that $\bar{\mu}_h$ satisfies

$$(\bar{\mu}'_h, v'_h) = (f, v_h)_h + \beta v_h(1), \quad \forall v_h \in V_h,$$

where

$$(f, v_h)_h = \sum_{i=1}^{N-1} h f(x_i) v_h(x_i) + \frac{h}{2} f(x_0) v_h(x_0) + \frac{h}{2} f(x_N) v_h(x_N).$$

Proof. It is sufficient to show that $(\bar{\mu}'_h, \varphi'_i) = (f, \varphi_i)_h + \beta \varphi_i(1)$ holds for each $i = 1, \dots, N$. By (4) and (5), we obtain

$$(\bar{\mu}'_h, \varphi'_i) = \begin{cases} \bar{u}_1(\varphi'_1, \varphi'_1) + \bar{u}_2(\varphi'_2, \varphi'_1), & i = 1, \\ \bar{u}_{i-1}(\varphi'_{i-1}, \varphi'_i) + \bar{u}_i(\varphi'_i, \varphi'_i) + \bar{u}_{i+1}(\varphi'_{i+1}, \varphi'_i), & 1 < i < N, \\ \bar{u}_{N-1}(\varphi'_{N-1}, \varphi'_N) + \bar{u}_N(\varphi'_N, \varphi'_N), & i = N. \end{cases}$$

Thus it can be computed as

$$(\bar{\mu}'_h, \varphi'_i) = \begin{cases} -\frac{\bar{u}_2 - 2\bar{u}_1}{h}, & i = 1, \\ -\frac{\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}}{h}, & 1 < i < N, \\ \frac{\bar{u}_N - \bar{u}_{N-1}}{h}, & i = N. \end{cases}$$

On the other hand, we have

$$(f, \varphi_i)_h + \beta \varphi_i(1) = \begin{cases} h f(x_1), & i = 1, \\ h f(x_i), & 1 < i < N, \\ \frac{h}{2} f(x_N) + \beta, & i = N. \end{cases}$$

Therefore by finite difference schema (12), we have $(\bar{\mu}'_h, \varphi'_i) = (f, \varphi_i)_h + \beta \varphi_i(1)$. \square

3.2 Let $Q(w) = (w, 1)_h$. Show that

$$\left| \int_{\Lambda} w(t) dt - Q(w) \right| \leq h^2 \|w''\|_0.$$

Proof. Since

$$Q(w) = \sum_{i=1}^{N-1} h w(x_i) + \frac{h}{2} w(x_0) + \frac{h}{2} w(x_N) = \frac{h}{2} \sum_{i=1}^N (w(x_{i-1}) + w(x_i)),$$

then

$$\begin{aligned} \int_{\Lambda} w(t) dt - Q(w) &= \sum_{i=1}^N \left[\int_{x_{i-1}}^{x_i} w(t) dt - \frac{h}{2} (w(x_{i-1}) + w(x_i)) \right] \\ &= \sum_{i=1}^N \left[\int_{x_{i-1}}^{x_i} w(t) - \left(w(x_{i-1}) \frac{x_i - t}{x_i - x_{i-1}} + w(x_i) \frac{t - x_{i-1}}{x_i - x_{i-1}} \right) dt \right] \\ &= \sum_{i=1}^N \left[\int_{x_{i-1}}^{x_i} w(t) - w_I(t) dt \right] \\ &= \int_{\Lambda} w(t) - w_I(t) dt, \end{aligned}$$

where $w_I(t)$ is the interpolation of $w(t)$ into X_h^1 as shown in (6). Thus by employing (11) we have

$$\left| \int_{\Lambda} w(t) dt - Q(w) \right| = \left| \int_{\Lambda} w(t) - w_I(t) dt \right| \leq \|w - w_I\|_0 \leq h^2 \|w''\|_0.$$

□

3.3 Show that for any $v_h \in V_h$ it holds $|(\mu'_h - \bar{u}'_h, v'_h)| \leq 2h^2 (\|f'\|_0 + \|f''\|_0) (\|v_h\|_0 + \|v'_h\|_0)$.

Proof. It is evident that

$$\begin{aligned} |(u'_h - \bar{\mu}'_h, v'_h)| &= |(f, v_h) - (f, v_h)_h| \leq h^2 \|(f v_h)''\|_0 = h^2 \|f'' v_h + 2f' v'_h\|_0 \\ &\leq 2h^2 (\|f'\|_0 + \|f''\|_0) (\|v_h\|_0 + \|v'_h\|_0). \end{aligned}$$

□

3.4 Show that $\|u_h - \bar{\mu}_h\|_1 \leq ch^2 (\|f'\|_0 + \|f''\|_0)$, where c is a constant.

Proof. It is known that $u_h - \bar{\mu}_h \in V_h$, then by Poincaré inequality (see [HW 3, Exercise 2.12]), we have $\|u_h - \bar{\mu}_h\|_0 \leq \|u'_h - \bar{\mu}'_h\|_0$. Thus

$$\begin{aligned} \|u_h - \bar{\mu}_h\|_1^2 &\leq 2\|u'_h - \bar{\mu}'_h\|_0^2 = 2(u'_h - \bar{\mu}'_h, u'_h - \bar{\mu}'_h) \\ &\leq 4h^2 (\|f'\|_0 + \|f''\|_0) (\|u_h - \bar{\mu}_h\|_0 + \|u'_h - \bar{\mu}'_h\|_0) \\ &\leq 4\sqrt{2}h^2 (\|f'\|_0 + \|f''\|_0) \|u_h - \bar{\mu}_h\|_1, \end{aligned}$$

which leads to the desired result.

□

□