Exercise 2.3. *Determine the absolute stability region of the Crank-Nicolson schema and Leapfrog schema.*

Solution. Model problem:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \lambda u, \quad \text{where } \lambda \text{ is a constant.}$$

- Crank-Nicolson schema:

$$u^{n+1} = u^n + \frac{\lambda h(u^{n+1} + u^n)}{2} \Rightarrow u^{n+1} = \frac{2 + h\lambda}{2 - h\lambda} u^n.$$

Let $z = \lambda h$. The absolute stability region of this schema is

$$\left\{ z : \left| \frac{2+z}{2-z} \right| \leqslant 1 \right\} = \{ z : \mathrm{Re}(z) \leqslant 0 \}.$$

In fact, if we suppose that $z = a + b\mathrm{i}$, then $|2 + z| \leqslant |2 - z|$ is equivalent to

$$(a+2)^2 + b^2 \leqslant (a-2)^2 + b^2 \Leftrightarrow a \leqslant 0.$$

- Leapfrog schema:

$$u^{n+1} = 2h\lambda u^n + u^{n-1}.$$

Let $z = \lambda h$, then its characteristic polynomial is $\rho(\xi) = \xi^2 - 2z\xi - 1$. Hence roots of $\rho(\xi)$ are $\xi_{1,2} = z \pm \sqrt{z^2+1}$. The schema is absolute stable if and only if the roots satisfy the condition (see [LeVeque (2007), Definition 7.1, p. 153]):

$$\begin{aligned} &|\xi_j| \leqslant 1, \quad j = 1, 2. \\ &\text{If } \xi_1 = \xi_2, \text{ then } |\xi_1| < 1. \end{aligned}$$

We note that $|\xi_1||\xi_2| = 1$, then the only possibility to satisfy the root condition is $|\xi_1| = |\xi_2| = 1$, which leads to z is the pure imaginary number. By setting $z = b\mathrm{i}$, we obtain $b \in (-1, 1)$. Hence the absolute stability region is $\{ z = b\mathrm{i} : b \in (-1, 1) \}$. □

References

[LeVeque (2007)] LeVeque R J. Finite difference methods for ordinary and partial differential equations: steady-state and time-dependent problems[M]. Society for Industrial and Applied Mathematics, 2007.