Exercise 2.7. *Prove that the consistency and stability leads to the convergence.*

Proof. The elliptic problem:

$$\begin{cases} Lu(x) = f(x), & \forall x \in (a,b), \\ u(a) = u(b) = 0, \end{cases}$$

where L is a linear elliptic operator. Let $\{x_i\}_{i=0}^N$ be a equispaced $(N+1)-$grid in the interval $[a,b]$. That is, $x_i = a + ih$ where $h = (b-a)/N$. Let L_h be a discrete operator to approximate L, which leads to the difference schema:

$$\begin{cases} L_h u_i = f(x_i), & i = 1, 2, \cdots, N-1 \\ u_0 = u_N = 0. \end{cases}$$

Let $\mathbf{f} = [f(x_1), \cdots, f(x_{N-1})]^{\mathrm{T}}$ and $\mathbf{u} = [u(x_1), \cdots, u(x_{N-1})]^{\mathrm{T}}$. We say the difference schema is *stable* if there exists a constant c, independend of h, such that for all h sufficiently small, it holds

$$\|\mathbf{u}\| \leqslant c\|\mathbf{f}\|,$$

where any norm is possible since they are equivalent in finite dimensional space. The truncation error is defined as $R_i = L_h[u(x_i)] - [Lu](x_i)$, where $i = 1, \cdots, N-1$. We say the difference schema is *consistent* if

$$R_i \to 0 \text{ as } h \to 0, \quad \text{for all } \ i = 1, \cdots, N-1.$$

Let $e_i = u(x_i) - u_i$, $i = 0, \cdots, N$. Note that $e_0 = e_N = 0$, and for $1 \leqslant i \leqslant N-1$ we have

$$R_i = L_h[u(x_i)] - [Lu](x_i) = L_h[u(x_i)] - f(x_i) = L_h[u(x_i)] - L_h[u_i] = L_h e_i.$$

Hence $\{e_i\}_{i=0}^N$ is the solution of the discrete problem:

$$\begin{cases} L_h e_i = R_i, & i = 1, 2, \cdots, N-1, \\ e_0 = e_N = 0. \end{cases}$$

By its consistency and stability, we have the convergence, i.e.,

$$|e_i| \leqslant \|\mathbf{e}\| \leqslant c\|\mathbf{R}\| \to 0 \text{ as } h \to 0,$$

where $\mathbf{e} = [e_1, \cdots, e_{N-1}]^{\mathrm{T}}$ and $\mathbf{R} = [R_1, \cdots, R_{N-1}]^{\mathrm{T}}$. □

Exercise 2.8. *Consider the variable coefficient equation:*

$$\begin{cases} Lu = f(x), & \forall x \in (a,b), \\ u(a) = u(b) = 0, \end{cases}$$

where $Lu = -(pu')'(x)$, $p_M \geqslant p(x) \geqslant p_0 > 0, \quad \forall x \in [a,b]$.
1) Establish an energy inequality.
2) Set $p(x) = 1$. *Consider the central schema on the non-uniform mesh* $\{x_i\}_{i=0}^N$, $h_i = x_i - x_{i-1}$*:*

$$\begin{cases} L_h u_i = f(x_i), & i = 1, 2, \cdots, N-1, \\ u_0 = u_N = 0, \end{cases}$$

where

$$L_h u_i = -\frac{\frac{u_{i+1}-u_i}{h_{i+1}} - \frac{u_i - u_{i-1}}{h_i}}{\frac{h_i + h_{i+1}}{2}}, \quad i = 1, 2, \cdots, N-1.$$

Analyze the truncation error $R_i = L_h[u(x_i)] - [Lu](x_i)$, $i = 1, 2, \cdots, N-1$ *in term of* $h = \max_{1 \leqslant i \leqslant N} |h_i|$.

Solution.
1). We know that $(u, Lu) = (u, f)$ and leverage integral by parts

$$(u, Lu) = (u, -(pu')') = (pu', u') \geqslant p_0 \|u'\|_0^2.$$

By Cauchy-Schwarz inequality and Poincaré inequality, we have

$$(f, u) \leqslant \|f\|_0 \|u\|_0 \leqslant c\|f\|_0 \|u'\|_0,$$

where c is a constant, independent of f and h. Hence it is obvious that $\|u'\|_0 \leqslant c/p_0 \|f\|_0$.
2). By Tylor development,

$$u(x_{i+1}) = u(x_i) + h_{i+1}u'(x_i) + \frac{h_{i+1}^2}{2}u''(x_i) + \frac{h_{i+1}^3}{6}u'''(x_i) + \frac{h_{i+1}^4}{24}u^{(4)}(\xi_i), \text{ for some } \xi_i \in (x_i, x_{i+1}).$$

and

$$u(x_{i-1}) = u(x_i) - h_i u'(x_i) + \frac{h_i^2}{2}u''(x_i) - \frac{h_i^3}{6}u'''(x_i) + \frac{h_i^4}{24}u^{(4)}(\xi_{i-1}), \text{ for some } \xi_{i-1} \in (x_{i-1}, x_i).$$

Hence

$$L_h[u(x_i)] = -u''(x_i) - \frac{h_{i+1}^2 - h_i^2}{3(h_i + h_{i+1})}u'''(x_i) - \frac{h_{i+1}^3 u^{(4)}(\xi_i) + h_i^3 u^{(4)}(\xi_{i-1})}{12(h_i + h_{i+1})}.$$

Hence

$$\begin{aligned} |R_i| &= |L_h[u(x_i)] - [Lu](x_i)| = |L_h[u(x_i)] + u''(x_i)| \\ &\leqslant \frac{\max_{x\in[a,b]} |u'''(x)|}{3}|h_{i+1} - h_i| + \frac{\max_{x\in[a,b]} |u^{(4)}(x)|}{12}|h_{i+1}^2 - h_i h_{i+1} + h_i^2| = O(h). \end{aligned}$$

□

Exercise 2.12. *Let v_h be a discrete function defined in $\bar{I}_h$. Prove:*
1). Discrete Poincaré inequality holds if only $v_0 = 0$ or $v_N = 0$.
2). If $v_0 = v_N = 0$, then it holds

$$\|v_h\|_0^2 \leqslant \frac{(b-a)^2}{4}|v_h|_1^2.$$

Proof.
1). If $v_0 = 0$, we have $v_i = \sum_{j=1}^{i} v_{j,\bar{x}} h_j$ for $i = 1, \cdots, N$. Then by Cauchy inequality

$$v_i^2 = \left(\sum_{j=1}^{i} v_{j,\bar{x}} h_j\right)^2 \leqslant (x_i - a)\sum_{j=1}^{i} v_{j,\bar{x}}^2 h_j \leqslant (b-a)\sum_{j=1}^{N} v_{j,\bar{x}}^2 h_j = (b-a)|v_h|_1^2.$$

If $v_N = 0$, we have $v_i = -\sum_{j=i+1}^{N} v_{j,\bar{x}} h_j$.
2). It is known that

$$v_i^2 \leq \frac{(x_i - a)(b - x_i)}{b-a}|v_h|_1^2 \leq \frac{b-a}{4}|v_h|_1^2, \quad \forall i \in \bar{I}_h.$$

Therefore

$$\|v_h\|_0^2 = \sum_{\bar{I}_h} v_i^2 h_i \leq \frac{b-a}{4}|v_h|_1^2 \sum_{\bar{I}_h} \bar{h}_i \leq \frac{(b-a)^2}{4}|v_h|_1^2.$$

□

Appendix: Notations for Discrete Representation

Let $I = [a, b]$. We define the discrete grid points as

$$a = x_0 < x_1 < \cdots < x_N = b.$$

We introduce the following sets:

$$I_h = \{x_1, \cdots, x_{N-1}\},\ \bar{I}_h = \{x_0, x_1, \cdots, x_N\},\ I_h^+ = \{x_1, \cdots, x_N\}.$$

The grid spacing is defined as

$$h_i = x_i - x_{i-1}, \quad i = 1, \cdots, N.$$

Additionally, we define the averaged grid spacing:

$$\begin{aligned} \bar{h}_i &= \frac{1}{2}(h_i + h_{i+1}),\ i = 1, \cdots, N-1, \\ \bar{h}_0 &= \frac{1}{2}h_1, \quad \bar{h}_N = \frac{1}{2}h_N. \end{aligned}$$

A discrete function defined on $\bar{I}_h$ is denoted as

$$v_h = \{v_0, v_1, \cdots, v_N\}.$$

We define the following difference operators:

$$\begin{aligned} (v_i)_{\bar{x}} &:= v_{i,\bar{x}} := \frac{v_i - v_{i-1}}{h_i},\ i = 1, \cdots, N, \\ (v_i)_x &:= v_{i,x} := \frac{v_{i+1} - v_i}{h_{i+1}},\ i = 0, \cdots, N-1, \\ (v_i)_{\hat{x}} &:= v_{i,\hat{x}} := \frac{v_{i+1} - v_i}{\bar{h}_i},\ i = 0, \cdots, N-1. \end{aligned}$$

The discrete inner products are given by

$$(u_h, v_h)_{I_h} = \sum_{i=1}^{N-1} u_i v_i \bar{h}_i,\ (u_h, v_h)_{\bar{I}_h} = \sum_{i=0}^{N} u_i v_i \bar{h}_i,\ (u_h, v_h)_{I_h^+} = \sum_{i=1}^{N} u_i v_i h_i.$$

We define the discrete norms as follows:

$$\begin{aligned} \|v_h\|_c &:= \max_{\bar{I}_h} |v_i|,\ \|v_h\|_0 := (v_h, v_h)_{\bar{I}_h}^{1/2}, \\ |v_h|_1 &:= ((v_h)_{\bar{x}}, (v_h)_{\bar{x}})_{I_h^+}^{1/2},\ \|v_h\|_1^2 = \|v_h\|_0^2 + |v_h|_1^2. \end{aligned}$$

The discrete integral by parts:

$$\sum_{i=m+1}^{n} v_i (w_i)_{\bar{x}} h_i = -\sum_{i=m}^{n-1} (v_i)_x w_i h_{i+1} + v_n w_n - v_m w_m,\ \text{for some } 0 \leqslant m < n \leqslant N.$$

The discrete Green formula:

$$\sum_{i=m+1}^{n-1} ((u_i)_{\bar{x}})_{\hat{x}}\, v_i \bar{h}_i = -\sum_{i=m+1}^{n} (u_i)_{\bar{x}} (v_i)_{\bar{x}} h_i + (u_n)_{\bar{x}} v_n - (u_m)_x v_m,\ \text{for some } 0 \leqslant m < n \leqslant N.$$

The discrete Cauchy-Schwarz inequality states that

$$|(u_h, v_h)_{\bar{I}_h}| \leqslant (u_h, u_h)_{\bar{I}_h}^{1/2} (v_h, v_h)_{\bar{I}_h}^{1/2}.$$

If $v_0 = 0$ (or $v_N = 0$ or $v_0 = v_N = 0$), the discrete Poincaré inequality holds:

$$\|v_h\|_c \leqslant C|v_h|_1, \quad \|v_h\|_0 \leqslant C|v_h|_1,$$

where C is a constant depending only on a and b.