

Exercise 1 (See Remark 1.3). *Prove*

$$\begin{aligned}\|u - u_I\|_0 &\leq Ch^2 \|u''\|_0, \\ \|u' - u'_I\|_0 &\leq Ch \|u''\|_0.\end{aligned}$$

Proof. • **Step 1.** Show that for $I = (0, 1)$ and $f \in H^2(I) \cap H_0^1(I)$, we have

$$\int_0^1 f(X)^2 dX \leq \int_0^1 f'(X)^2 dX, \text{ and } \int_0^1 f'(X)^2 dX \leq \int_0^1 f''(X)^2 dX.$$

Since $f(0) = f(1) = 0$, there exists $X_0 \in I$ such that $f'(X_0) = 0$. Thus

$$f(X) = \int_0^X f'(X) dX, \text{ and } f'(X) = \int_{X_0}^X f''(X) dX.$$

Therefore, the conclusion is clear by Cauchy-Schwarz inequality.

• **Step 2.** Make variable change to subinterval $I_{j+1} = (x_j, x_{j+1})$ by $x = x_{j-1} + X(x_j - x_{j-1})$, we have

$$\int_{x_{j-1}}^{x_j} \tilde{f}(x)^2 dx \leq (x_j - x_{j-1})^2 \int_{x_{j-1}}^{x_j} \tilde{f}'(x) dx, \text{ and } \int_{x_{j-1}}^{x_j} \tilde{f}'(x) dx \leq (x_j - x_{j-1})^2 \int_{x_{j-1}}^{x_j} \tilde{f}''(x) dx,$$

where $\tilde{f}(x) = f(X) = f(\frac{x-x_{j-1}}{x_j-x_{j-1}}) \in H^2(I_j) \cap H_0^1(I_j)$. The conclusion is obvious since

$$dx = (x_j - x_{j-1}) dX, \quad f'(X) = \tilde{f}'(x)(x_j - x_{j-1}), \quad f''(X) = \tilde{f}''(x)(x_j - x_{j-1})^2.$$

• **Step 3.** Let $\tilde{f}(x) = u(x) - u_I(x)$. It is obvious that $\tilde{f}(x_i) = 0$ for $i = 0, \dots, N+1$. Thus $(\tilde{f}|_{I_j})'' = u''$ and $\tilde{f}|_{I_j} \in H^2(I_j) \cap H_0^1(I_j)$ for $j = 1, \dots, N+1$, and we have

$$\int_{I_j} (u - u_I)^2 dx \leq h_j^2 \int_{I_j} (u' - u'_I)^2 dx, \quad \int_{I_j} (u' - u'_I)^2 dx \leq h_j^2 \int_{I_j} (u'')^2 dx,$$

both of which leads to

$$\|u - u_I\|_0 \leq h \|u' - u'_I\|_0, \text{ and } \|u' - u'_I\|_0 \leq h \|u''\|_0.$$

□

Exercise 2. Consider the mixed boundary problem

$$\begin{cases} -u'' = f, & x \in I := (0, 1), \\ u(0) = 0, \quad u'(1) = \beta, \end{cases}$$

where $\beta \in \mathbb{R}$ and $f \in L^2(I)$. Construct and analyze P_1 -FEM for this problem.

Proof. • Variational form. Let $V = \{v \in H^1(I) : v(0) = 0\}$, the bilinear form $a(u, v) = (u', v')$, and the functional $\mathcal{F}(v) = (f, v) + v(1)$. Then the variational problem reads

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ a(u, v) = \mathcal{F}(v), \quad \forall v \in V, \end{cases}$$

which is clearly equivalent to the strong problem.

• Galerkin Approximation. Let V_h be a subspace of V with finite dimension. Then the Galerkin approximation reads

$$\begin{cases} \text{Find } u_h \in V_h \text{ such that} \\ a(u_h, v_h) = \mathcal{F}(v_h), \quad \forall v_h \in V_h. \end{cases}$$

• *P1*–FEM. By construct the space of piecewise linear polynomials X_h^1 and its basis $\varphi_0, \dots, \varphi_{N+1}$, shown in the Appendix, we let the finite element space $V_h = X_h^1 \cap V$, then

$$V_h = \text{span}\{\varphi_1, \dots, \varphi_{N+1}\}.$$

• FEM Implementation. Let $u_h = \sum_{j=1}^{N+1} u_j \varphi_j(x)$, then

$$\sum_{j=1}^{N+1} u_j a(\varphi_j, \varphi_i) = \mathcal{F}(\varphi_i), \quad i = 1, \dots, N+1.$$

Let $\mathbf{A} = (a_{i,j})$ be the $(N+1) \times (N+1)$ matrix with its entries $a_{i,j} = a(\varphi_j, \varphi_i)$. Then we have

$$\begin{aligned} a_{N+1,N+1} &= \frac{1}{h_{N+1}}, \quad a_{j,j} = \frac{1}{h_j} + \frac{1}{h_{j+1}}, \quad j = 1, \dots, N, \\ a_{j,j-1} &= -\frac{1}{h_j}, \quad j = 1, \dots, N+1, \\ a_{i,j} &= 0, \quad \text{if } |i-j| \geq 2. \end{aligned}$$

Thus

$$\frac{1}{h} \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \\ u_{N+1} \end{bmatrix} = \begin{bmatrix} (f, \varphi_1) \\ (f, \varphi_2) \\ \vdots \\ (f, \varphi_N) \\ (f, \varphi_{N+1}) + \beta \end{bmatrix}.$$

• Error Estimate. We denote u_I being the interpolation of u into V_h , then it is clear that

$$\|u - u_I\|_0 \leq Ch \|u' - u'_I\|_0 \leq Ch^2 \|u''\|_0.$$

We know $a(u - u_h, v_h) = 0$ for any $v_h \in V_h$. Then

$$\|u' - u'_h\|_0^2 = a(u - u_h, u - u_h) = a(u - u_h, u - v_h) \leq \|u' - u'_h\|_0 \|u' - v'_h\|_0, \quad \forall v_h \in V_h,$$

which leads to

$$\|u' - u'_h\|_0 \leq \inf_{v_h \in V_h} \|u' - v'_h\|_0 \leq \|u' - u'_I\|_0 \leq Ch \|u''\|_0.$$

In the following, we derive the estimate for $\|u - u_h\|_0$.

Dual problem: given $r \in L^2(I)$,

$$\begin{cases} \text{Find } \varphi(r) \in V \text{ such that} \\ a(v, \varphi(r)) = (r, v), \quad \forall v \in V. \end{cases}$$

The dual problem admits a unique solution $\varphi(r)$ since $a(\cdot, \cdot)$ is continuous and coercive. Moreover, we have

$$a(v, \varphi(r)) = (r, v), \quad \forall v \in C_0^\infty(I),$$

if we suppose $\varphi(r) \in H^2(I)$, which gives $(-\varphi''(r), v) = (r, v)$, $\forall v \in C_0^\infty(I)$. Since $C_0^\infty(I)$ is dense in $L^2(I)$, we have

$$\|\varphi''(r)\|_0 = \sup_{v \in L^2(I), v \neq 0} \frac{(\varphi, v)}{\|v\|_0} = \sup_{v \in L^2(I), v \neq 0} \frac{(r, v)}{\|v\|_0} = \|r\|_0.$$

Thus we denote $\varphi_I(r)$ being the interpolation of $\varphi(r)$ into V_h and obtain

$$\begin{aligned}
\|u - u_h\|_0 &= \sup_{r \in L^2(I), r \neq 0} \frac{(r, u - u_h)}{\|r\|_0} = \sup_{r \in L^2(I), r \neq 0} \frac{a(u - u_h, \varphi(r))}{\|r\|_0} \\
&= \sup_{r \in L^2(I), r \neq 0} \frac{a(u - u_h, \varphi(r) - \varphi_I(r))}{\|r\|_0} \\
&\leq \sup_{r \in L^2(I), r \neq 0} \frac{\|u' - u'_h\|_0 \|\varphi'(r) - \varphi'_I(r)\|_0}{\|r\|_0} \\
&\leq Ch \|u' - u'_h\|_0 \sup_{r \in L^2(I), r \neq 0} \frac{\|\varphi''(r)\|_0}{\|r\|_0} \\
&\leq Ch \|u' - u'_h\|_0.
\end{aligned}$$

□

Appendix

Let $I = (0, 1)$ and $\{x_n\}_{n=0}^{N+1}$ be a grid on I such that $0 = x_0 < x_1 < \dots < x_N < x_{N+1} = 1$. Denote by each subintervals (or elements) $I_n = (x_{n-1}, x_n)$ for $1 \leq n \leq N+1$ of length $h_n = x_n - x_{n-1}$. Let $h = \max_{1 \leq n \leq N+1} h_n$.

The piecewise linear polynomials on such grid is denoted by

$$X_h^1 := \{v \in C(\bar{I}) : v|_{I_{j+1}} \in \mathbb{P}_1, j = 0, \dots, N\}.$$

We construct a nodal basis for X_h^1 , which is based on nodes in every element (how many nodes in every element depends on the degree of freedom, or the degree of polynomials required parameters to be determined).

$$\begin{aligned}
\varphi_0(x) &= \begin{cases} \frac{x_1 - x}{x_1 - x_0}, & x \in I_1, \\ 0, & \text{else,} \end{cases} & \varphi_{N+1}(x) &= \begin{cases} \frac{x - x_N}{x_{N+1} - x_N}, & x \in I_{N+1}, \\ 0, & \text{else,} \end{cases} \\
\varphi_n(x) &= \begin{cases} \frac{x - x_{n-1}}{x_n - x_{n-1}}, & x \in I_n, \\ \frac{x_{n+1} - x}{x_{n+1} - x_n}, & x \in I_{n+1}, \\ 0, & \text{else.} \end{cases}
\end{aligned}$$

Clearly, we have $X_h^1 = \text{span}\{\varphi_0, \varphi_1, \dots, \varphi_{N+1}\}$.

For any $u \in C(\bar{I})$, its interpolation into X_h^1 is denoted by $u_I(x)$. Clearly, we have $u_I(x) = \sum_{i=0}^{N+1} u(x_i) \varphi_i(x)$ and

$$u_I|_{I_{j+1}} = u(x_j) \varphi_j(x) + u(x_{j+1}) \varphi_{j+1}(x) = u(x_j) \frac{x_{j+1} - x}{x_{j+1} - x_j} + u(x_{j+1}) \frac{x - x_j}{x_{j+1} - x_j}.$$