Exercise 1. *Let* $\Omega = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| < 1/2\}$*. Define*

$$v(\mathbf{x}) = (\log |\mathbf{x}|)^k, \ \forall \mathbf{x} \in \Omega \backslash \{0\}, \ 0 < k < 1/2.$$

Prove $v \in H^1(\Omega)$*.*

Proof. It is clear that $v \in L^2(\Omega)$ and

$$\nabla v(\mathbf{x}) = k(-\log |\mathbf{x}|)^{k-1} \frac{\mathbf{x}}{|\mathbf{x}|^2},$$

which is also in $L^2(\Omega)$, i.e., $\|\nabla v\|_0 < \infty$. $\square$

Exercise 2. *1). If* K *is a rectangle,* $P(K) \hat{=} Q_1(K) = \mathrm{span}\{1, x, y, xy\}$, $\Sigma_K = \{$*mid-points of four sides*$\}$*. Prove that* $(K, p(K), \Sigma_K)$ *is not a finite element.*
2). If K *is a rectangle,* $P(K) \hat{=} Q_1^T(K) = \mathrm{span}\{1, x, y, x^2 - y^2\}$, $\Sigma_K = \{$*mid-points of four sides*$\}$*. Prove that* $(K, P(K), \Sigma_K)$ *is a finite element.*