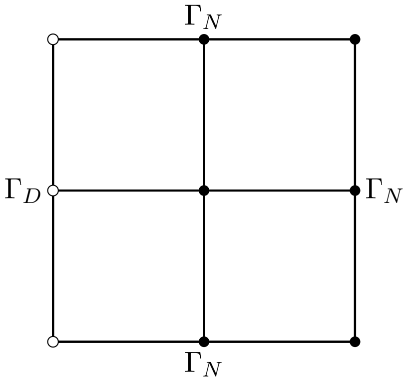
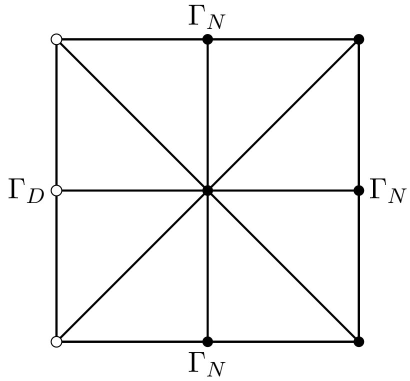
Exercise 1. *Let $\Omega = (-1,1)^2$. For $\alpha \in \mathbb{R}$, construct for the problem*

$$\begin{cases} \alpha u - \Delta u = 1, \quad \forall x \in \Omega, \\ u|_{\Gamma_D} = 0, \\ \left.\dfrac{\partial u}{\partial n}\right|_{\Gamma_N} = 2, \end{cases}$$

respectively a $Q1-$FEM based on the rectangular mesh in Fig. a. and a $P1-$FEM based on the triangular mesh in Fig. b.



a. Mesh for $Q1-$FEM.



b. Mesh for $P1-$FEM.

Solution. Let $V = \left\{ v \,\middle|\, v \in H^1(\Omega), v\big|_{\Gamma_D} = 0 \right\}$. The variational form reads

$$\begin{cases} \text{Find } u \in V \text{ s.t.} \\ a(u,v) = \mathcal{F}(v), \ \forall v \in V, \end{cases}$$

where $a(u,v) = \alpha(u,v) + (\nabla u, \nabla v)$ and $\mathcal{F}(v) = (1,v) + (2,v)_{\Gamma_N}$. Let $V_h \subset V$ be the finite dimensional space. Then the Galerkin approximation reads

$$\begin{cases} \text{Find } u_h \in V_h \text{ s.t.} \\ a(u_h,v_h) = \mathcal{F}(v_h), \ \forall v_h \in V_h. \end{cases} \tag{1}$$

In what follows, we consider the V_h as the $Q1$ and $P1$ finite element spaces respectively.

1. $Q1-$FEM.

We relabel the subelements of Fig. a. as shown in Fig. 2, and define the finite element space

$$X_h^1 = \{ v \in C(\bar{\Omega}) : v|_{K_i} \in \mathbb{Q}_1, \ i = 1,2,3,4 \}.$$

Next, we consider the basis functions for X_h^1. Let ψ_i for $i = 1, \cdots, 9$ denote the basis function associated with the node A_i. Instead of examining each basis function individually, we adopt an approach based on affine mapping from a reference element to the physical element. Thus the basis functions are constructed in a way: "reference basis" $\rightarrow$ "local basis" $\rightarrow$ "global basis". This method is commonly used in both theoretical analysis and practical implementations.

Let $T_{K_i} : \hat{K} \rightarrow K_i$ denote the affine mapping from the reference element $\hat{K}$ in Fig. A.1 to physical element K_i in Fig. 2 for $i = 1,2,3,4$. The exact expression of T_{K_i} is given in (A.2).

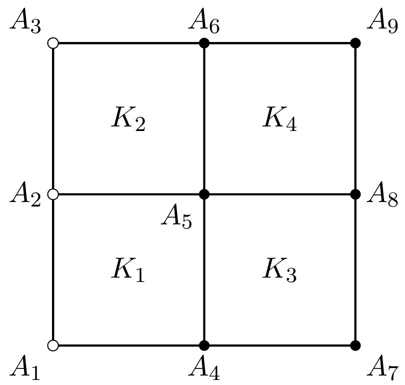


Figure 2: Mesh for $Q1-$FEM.

For each K_i, we arrange its vertices in counterclockwise order starting from the bottom-left vertex. Thus it is clear that

$$
\begin{aligned}
T_{K_1}(\hat{x},\hat{y}) &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \\
T_{K_2}(\hat{x},\hat{y}) &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \\
T_{K_3}(\hat{x},\hat{y}) &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \\
T_{K_4}(\hat{x},\hat{y}) &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.
\end{aligned}
$$

The mapping from local basis functions $\{\hat{\phi}_i\}$ in (A.1) to local part of ψ_i gives

$$
\begin{aligned}
&\psi_1|_{K_1} = (\hat{\phi}_1 \circ T_{K_1}^{-1})(x,y), \\
&\psi_2|_{K_1} = (\hat{\phi}_4 \circ T_{K_1}^{-1})(x,y),\ \ \psi_2|_{K_2} = (\hat{\phi}_1 \circ T_{K_2}^{-1})(x,y), \\
&\psi_3|_{K_2} = (\hat{\phi}_4 \circ T_{K_2}^{-1})(x,y), \\
&\psi_4|_{K_1} = (\hat{\phi}_2 \circ T_{K_1}^{-1})(x,y),\ \ \psi_4|_{K_3} = (\hat{\phi}_1 \circ T_{K_3}^{-1})(x,y), \\
&\psi_5|_{K_1} = (\hat{\psi}_3 \circ T_{K_1}^{-1})(x,y),\ \ \psi_5|_{K_2} = (\hat{\phi}_2 \circ T_{K_2}^{-1})(x,y), \\
&\psi_5|_{K_3} = (\hat{\phi}_4 \circ T_{K_3}^{-1})(x,y),\ \ \psi_5|_{K_4} = (\hat{\phi}_1 \circ T_{K_1}^{-1})(x,y), \\
&\psi_6|_{K_2} = (\hat{\phi}_3 \circ T_{K_2}^{-1})(x,y),\ \ \psi_6|_{K_4} = (\hat{\phi}_4 \circ T_{K_4}^{-1})(x,y), \\
&\psi_7|_{K_3} = (\hat{\phi}_2 \circ T_{K_3}^{-1})(x,y), \\
&\psi_8|_{K_3} = (\hat{\phi}_3 \circ T_{K_3}^{-1})(x,y),\ \ \psi_8|_{K_4} = (\hat{\phi}_2 \circ T_{K_4}^{-1})(x,y), \\
&\psi_9|_{K_4} = (\hat{\phi}_3 \circ T_{K_4}^{-1})(x,y).
\end{aligned}
$$

Thus it is clear that the nodal basis are

$$
\psi_1 = \begin{cases} xy, & (x,y) \in K_1, \\ 0, & else, \end{cases} \quad \psi_2 = \begin{cases} -x(1+y), & (x,y) \in K_1, \\ -x(1-y), & (x,y) \in K_2, \\ 0, & else, \end{cases} \quad \psi_3 = \begin{cases} -xy, & (x,y) \in K_2, \\ 0, & else, \end{cases}
$$

$$\psi_4=\begin{cases}-y(1+x), & (x,y)\in K_1,\\ -y(1-x), & (x,y)\in K_3,\\ 0, & else,\end{cases}\quad \psi_5=\begin{cases}(1+x)(1+y), & (x,y)\in K_1,\\ (1+x)(1-y), & (x,y)\in K_2,\\ (1-x)(1+y), & (x,y)\in K_3,\\ (1-x)(1-y), & (x,y)\in K_4,\\ 0, & else,\end{cases}\quad \psi_6=\begin{cases}y(1+x), & (x,y)\in K_2,\\ y(1-x), & (x,y)\in K_4,\\ 0, & else,\end{cases}$$

$$\psi_7=\begin{cases}-xy, & (x,y)\in K_3,\\ 0, & else,\end{cases}\quad \psi_8=\begin{cases}x(1+y), & (x,y)\in K_3,\\ x(1-y), & (x,y)\in K_4,\\ 0, & else,\end{cases}\quad \psi_9=\begin{cases}xy, & (x,y)\in K_4,\\ 0, & else.\end{cases}$$

Thus we have $X_h^1=\text{span}\{\psi_1,\cdots,\psi_9\}$. Let $V_h=V\cap X_h^1=\text{span}\{\psi_4,\psi_5,\psi_6,\psi_7,\psi_8,\psi_9\}$ and $u(x,y)=\sum_{j=4}^{9}u_j\psi_j(x,y)$. Inserting u into (1) gives the implementation of $Q1-$FEM

$$\sum_{i=4}^{9}a(\psi_j,\psi_i)u_j=\mathcal{F}(\psi_i),\quad i=4,5,6,7,8,9.$$

2. $P1-$FEM. We relabel the subelements of Fig. b. as shown in Fig. 3, and define the finite element space

$$X_h^1=\{v\in C(\bar{\Omega}):v|_{E_i}\in\mathbb{P}_1,\ i=1,\cdots,8\}.$$

Next, we consider the basis functions for X_h^1. Let φ_i for $i=1,\cdots,9$ denote the basis function associated with the node B_i. The basis functions are constructed in a way: "reference basis" $\to$ "local basis" $\to$ "global basis".

Let $T_{E_i}:\hat{E}\to E_i$ denote the affine mapping from the reference element $\hat{E}$ in Fig. B.1 to physical element E_i in Fig. 3 for $i=1,\cdots,8$. The exact expression of T_{E_i} is given in (B.2).

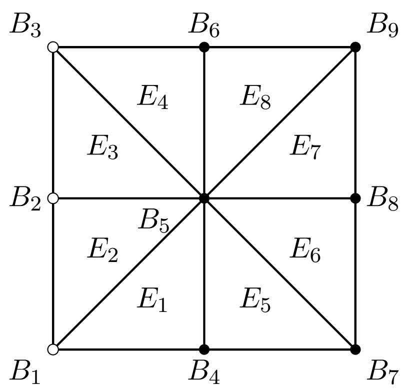
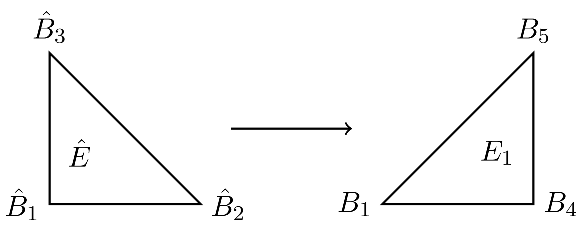


Figure 3: Mesh for $P1-$FEM.

For each element E_i, we arrange its vertices in counterclockwise order, starting from the right-angled vertex. For example, the vertices of the element E_1 are $\{B_1,B_4,B_5\}$, then we map from $\hat{E}:\Delta\hat{B}_1\hat{B}_2\hat{B}_3$ to $E_1:\Delta B_4B_5B_1$ shown as



Thus it is clear that

$$
\begin{aligned}
T_{E_1}(\hat{x},\hat{y}) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad & T_{E_2}(\hat{x},\hat{y}) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \\
T_{E_3}(\hat{x},\hat{y}) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad & T_{E_4}(\hat{x},\hat{y}) &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\
T_{E_5}(\hat{x},\hat{y}) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad & T_{E_6}(\hat{x},\hat{y}) &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
T_{E_7}(\hat{x},\hat{y}) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad & T_{E_8}(\hat{x},\hat{y}) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\end{aligned}
$$

The mapping from local basis functions $\{\hat{\eta}_i\}$ in (B.1) to local part of φ_i gives

$$
\begin{aligned}
\varphi_1|_{E_1} &= \left(\hat{\eta}_3 \circ T_{E_1}^{-1}\right)(x,y), & \varphi_1|_{E_2} &= \left(\hat{\eta}_2 \circ T_{E_2}^{-1}\right)(x,y), \\
\varphi_2|_{E_2} &= \left(\hat{\eta}_1 \circ T_{E_2}^{-1}\right)(x,y), & \varphi_2|_{E_3} &= \left(\hat{\eta}_1 \circ T_{E_3}^{-1}\right)(x,y), \\
\varphi_3|_{E_3} &= \left(\hat{\eta}_3 \circ T_{E_3}^{-1}\right)(x,y), & \varphi_3|_{E_4} &= \left(\hat{\eta}_2 \circ T_{E_4}^{-1}\right)(x,y), \\
\varphi_4|_{E_1} &= \left(\hat{\eta}_1 \circ T_{E_1}^{-1}\right)(x,y), & \varphi_4|_{E_5} &= \left(\hat{\eta}_1 \circ T_{E_5}^{-1}\right)(x,y), \\
\varphi_5|_{E_1} &= \left(\hat{\eta}_2 \circ T_{E_1}^{-1}\right)(x,y), & \varphi_5|_{E_2} &= \left(\hat{\eta}_3 \circ T_{E_2}^{-1}\right)(x,y), \\
\varphi_5|_{E_3} &= \left(\hat{\eta}_2 \circ T_{E_3}^{-1}\right)(x,y), & \varphi_5|_{E_4} &= \left(\hat{\eta}_3 \circ T_{E_4}^{-1}\right)(x,y), \\
\varphi_5|_{E_5} &= \left(\hat{\eta}_3 \circ T_{E_5}^{-1}\right)(x,y), & \varphi_5|_{E_6} &= \left(\hat{\eta}_2 \circ T_{E_6}^{-1}\right)(x,y), \\
\varphi_5|_{E_7} &= \left(\hat{\eta}_3 \circ T_{E_7}^{-1}\right)(x,y), & \varphi_5|_{E_8} &= \left(\hat{\eta}_2 \circ T_{E_8}^{-1}\right)(x,y), \\
\varphi_6|_{E_4} &= \left(\hat{\eta}_1 \circ T_{E_4}^{-1}\right)(x,y), & \varphi_6|_{E_8} &= \left(\hat{\eta}_1 \circ T_{E_8}^{-1}\right)(x,y), \\
\varphi_7|_{E_5} &= \left(\hat{\eta}_2 \circ T_{E_5}^{-1}\right)(x,y), & \varphi_7|_{E_6} &= \left(\hat{\eta}_3 \circ T_{E_6}^{-1}\right)(x,y), \\
\varphi_8|_{E_6} &= \left(\hat{\eta}_1 \circ T_{E_6}^{-1}\right)(x,y), & \varphi_8|_{E_7} &= \left(\hat{\eta}_1 \circ T_{E_7}^{-1}\right)(x,y), \\
\varphi_9|_{E_7} &= \left(\hat{\eta}_2 \circ T_{E_7}^{-1}\right)(x,y), & \varphi_9|_{E_8} &= \left(\hat{\eta}_3 \circ T_{E_8}^{-1}\right)(x,y).
\end{aligned}
$$

Thus it is clear that the nodal basis are

$$\varphi_1(x,y)=\begin{cases}-x, & (x,y)\in E_1,\\ -y, & (x,y)\in E_2,\\ 0, & else,\end{cases}\quad \varphi_2(x,y)=\begin{cases}y-x, & (x,y)\in E_2,\\ -x-y, & (x,y)\in E_3,\\ 0, & else,\end{cases}$$

$$\varphi_3(x,y)=\begin{cases}y, & (x,y)\in E_3,\\ -x, & (x,y)\in E_4,\\ 0, & else,\end{cases}\quad \varphi_4(x,y)=\begin{cases}x-y, & (x,y)\in E_1,\\ -x-y, & (x,y)\in E_5,\\ 0, & else,\end{cases}$$

$$\varphi_6(x,y)=\begin{cases}x+y, & (x,y)\in E_4,\\ y-x, & (x,y)\in E_8,\\ 0, & else,\end{cases}\quad \varphi_7(x,y)=\begin{cases}x, & (x,y)\in E_5,\\ -y, & (x,y)\in E_6,\\ 0, & else,\end{cases}$$

$$\varphi_8(x,y)=\begin{cases}x+y, & (x,y)\in E_6,\\ x-y, & (x,y)\in E_7,\\ 0, & else,\end{cases}\quad \varphi_9(x,y)=\begin{cases}y, & (x,y)\in E_7,\\ x, & (x,y)\in E_8,\\ 0, & else,\end{cases}$$

$$\varphi_5(x,y)=\begin{cases}1+y, & (x,y)\in E_1,\\ 1+x, & (x,y)\in E_2,\\ 1+x, & (x,y)\in E_3,\\ 1-y, & (x,y)\in E_4,\\ 1+y, & (x,y)\in E_5,\\ 1-x, & (x,y)\in E_6,\\ 1-x, & (x,y)\in E_7,\\ 1-y, & (x,y)\in E_8.\end{cases}$$

Thus we have $X_h^1=\text{span}\{\varphi_1,\cdots,\psi_9\}$. Let $V_h=V\cap X_h^1=\text{span}\{\varphi_4,\varphi_5,\varphi_6,\varphi_7,\varphi_8,\varphi_9\}$ and $u(x,y)=\sum_{j=4}^9 u_j\varphi_j(x,y)$. Inserting u into (1) gives the implementation of $P1-$FEM

$$\sum_{i=4}^{9} a(\varphi_j,\varphi_i)u_j=\mathcal{F}(\varphi_i),\quad i=4,5,6,7,8,9.$$

$\square$

Appendix

A Affine mapping between rectangles

Let the reference rectangle $\hat{K}$ with its vertices $\hat{A}_1 = (-1,-1)$, $\hat{A}_2 = (1,-1)$, $\hat{A}_3 = (1,1)$ and $\hat{A}_4 = (-1,1)$. For any physical rectangle K with vertices A_1, A_2, A_3 and A_4, the affine mapping from $\hat{K}$ to K is shown as follows

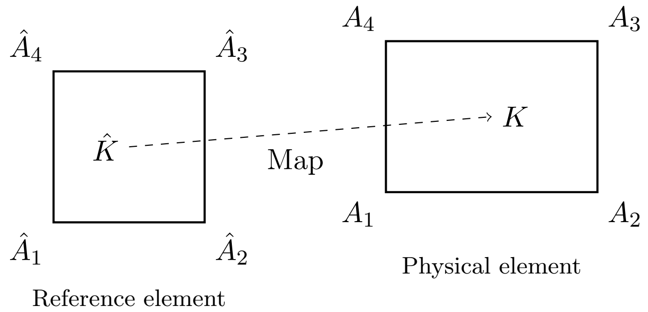


Figure A.1: Affine mapping from the reference element $\hat{K}$ to a physical element K.

$Q1-$basis on element $\hat{K} = \square\hat{A}_1\hat{A}_2\hat{A}_3\hat{A}_4$ is denoted as $\{\hat{\phi}_i\}_{i=1}^4$, each of which corresponds to the node $\hat{A}_i$, satisfying $\hat{\phi}_i(\hat{A}_j) = \delta_{ij}$. It is clear that

$$\begin{aligned} \hat{\phi}_1(\hat{x},\hat{y}) &= \frac{1-\hat{x}-\hat{y}+\hat{x}\hat{y}}{4}, \\ \hat{\phi}_2(\hat{x},\hat{y}) &= \frac{1+\hat{x}-\hat{y}-\hat{x}\hat{y}}{4}, \\ \hat{\phi}_3(\hat{x},\hat{y}) &= \frac{1+\hat{x}+\hat{y}+\hat{x}\hat{y}}{4}, \\ \hat{\phi}_4(\hat{x},\hat{y}) &= \frac{1-\hat{x}+\hat{y}-\hat{x}\hat{y}}{4}. \end{aligned} \tag{A.1}$$

Assume that $K = \square A_1A_2A_3A_4$ is an arbitrary rectangle with the coordinates

$$A_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \ i = 1,2,3,4.$$

Then the affine mapping from $\hat{K}$ to K is

$$\begin{pmatrix} x \\ y \end{pmatrix} = T_K(\hat{x},\hat{y}) = \begin{pmatrix} \frac{1}{2}h_1 & 0 \\ 0 & \frac{1}{2}h_2 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} x_1 + \frac{1}{2}h_1 \\ y_1 + \frac{1}{2}h_2 \end{pmatrix}, \tag{A.2}$$

where $h_1 = x_2 - x_1, \ h_2 = y_4 - y_1$.

B Affine mapping between triangles

Let the reference triangle $\hat{E}$ with its vertices $\hat{B}_1 = (0,0)$, $\hat{B}_2 = (1,0)$, and $\hat{B}_3 = (0,1)$. For any physical triangle E with vertices B_1, B_2, and B_3, the affine mapping from $\hat{E}$ to E is shown as follows

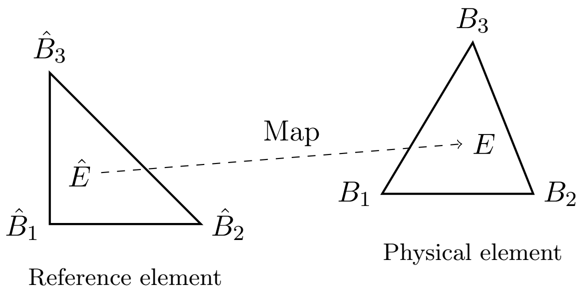


Figure B.1: Affine mapping from the reference element $\hat{E}$ to a physical element E.

$P1-$basis on element $\hat{E} = \Delta \hat{B}_1 \hat{B}_2 \hat{B}_3$ is denoted as $\{\hat{\eta}_i\}_{i=1}^3$, each of which corresponds to the node $\hat{B}_i$, satisfying $\hat{\eta}_i(\hat{B}_j) = \delta_{ij}$. It is clear that

$$\begin{aligned}
\hat{\eta}_1(\hat{x}, \hat{y}) &= -\hat{x} - \hat{y} + 1, \\
\hat{\eta}_2(\hat{x}, \hat{y}) &= \hat{x}, \\
\hat{\eta}_3(\hat{x}, \hat{y}) &= \hat{y}.
\end{aligned} \tag{B.1}$$

Assume that for an arbitrary triangle $E = \Delta B_1 B_2 B_3$ with the coordinates

$$B_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \quad i = 1, 2, 3.$$

The affine mapping from $\hat{K}$ to K is

$$\begin{pmatrix} x \\ y \end{pmatrix} = T_K(\hat{x}, \hat{y}) = \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}. \tag{B.2}$$