Exercise 1. *Consider the BVP*

$$\begin{cases} -u'' = f, \quad x \in \Lambda := (0,1), \\ u(0) = 0, \;\; u'(0) = \beta. \end{cases} \tag{1}$$

Construct a $P1-$FEM for this problem and compare with central finite difference scheme.

Solution.

1. $P1-$FEM.

1.1 Variational form. Let $V = \{v \in H^1(\Lambda) : v(0) = 0\}$. Then the variational form reads

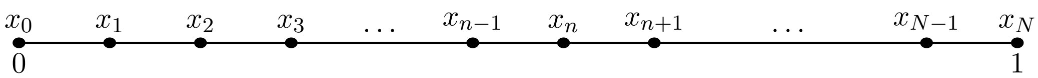
$$\begin{cases} \text{find } u \in V \text{ s.t.} \\ a(u,v) = \mathcal{F}(v), \;\; \forall v \in V, \end{cases} \tag{2}$$

where $a(u,v) = (u', v')$ and $\mathcal{F}(v) = (f,v) + \beta v(1)$. Clearly, the weak problem (2) is equivalent to the strong problem (1). The well-poseness of (2) is guaranteed by Lax-Milgram theorem.

1.2 Galerkin approximation. Let $V_h \subset V$ be a finite dimensional space. Replacing V with V_h in (2) leads to the Galerkin approximation

$$\begin{cases} \text{find } u_h \in V_h \text{ s.t.} \\ a(u_h, v_h) = \mathcal{F}(v_h), \;\; \forall v_h \in V_h. \end{cases} \tag{3}$$

1.3 $P1$ finite element space. Let $\{x_i\}_{i=0}^N$ be the uniform grid on Λ shown as follows:



Let $\mathcal{T}_h = \{I_i\}_{i=1}^N$, where $I_i = (x_{i-1}, x_i)$ is the element with length of $h = h_i := x_i - x_{i-1}$ for $i = 1, \cdots, N$. Then the $P1$ finite element space is

$$X_h^1 = \left\{ v \in C^0(\bar{\Lambda}) : v|_K \in \mathbb{P}_1, \; \forall K \in \mathcal{T}_h \right\},$$

where $\mathbb{P}_1$ is the space of polynomials degree of at most 1. We construct a nodal basis for X_h^1, defined by nodes within each element. The number of nodes per element depends on the degrees of freedom, which are determined by the required polynomial degree.

$$\varphi_0(x) = \begin{cases} \dfrac{x_1 - x}{x_1 - x_0}, & x \in I_1, \\ 0, & \text{else}, \end{cases} \qquad \varphi_N(x) = \begin{cases} \dfrac{x - x_{N-1}}{x_N - x_{N-1}}, & x \in I_N, \\ 0, & \text{else}, \end{cases} \tag{4}$$

$$\varphi_n(x) = \begin{cases} \dfrac{x - x_{n-1}}{x_n - x_{n-1}}, & x \in I_n, \\ \dfrac{x_{n+1} - x}{x_{n+1} - x_n}, & x \in I_{n+1}, \\ 0, & \text{else}. \end{cases} \tag{5}$$

Clearly, we have $X_h^1 = \text{span}\{\varphi_0, \varphi_1, \cdots, \varphi_N\}$. For any $u \in C(\bar{I})$, its interpolation into X_h^1 is denoted by $u_I(x)$. Clearly, we have $u_I(x) = \sum_{j=0}^N u(x_j)\varphi_j(x)$ and

$$u_I(x)\big|_{I_n} = u(x_{n-1})\varphi_{n-1}(x) + u(x_n)\varphi_n(x) = u(x_{n-1})\frac{x_n - x}{x_n - x_{n-1}} + u(x_n)\frac{x - x_{n-1}}{x_n - x_{n-1}}. \tag{6}$$

1.4 Implementation. We consider solving (3) by $P1-$FEM. Let $V_h = X_h^1 \cap V$. It is known that $X_h^1 \subset H^1(I)$ (see [HW 9, Exercise 1]), then $V_h = \{v \in X_h^1 : v(0) = 0\}$, i.e.,

$$V_h = \operatorname{span}\{\varphi_1, \cdots, \varphi_N\}. \tag{7}$$

Let $u_h = \sum_{j=1}^N u_j \varphi_j(x)$, where u_j are the parameters to be determined. Then the finite element approximation reads

$$\begin{cases} \text{find } \{u_j\}_{j=1}^N \text{ s.t.} \\ \displaystyle\sum_{j=1}^N a(\varphi_j, \varphi_i) u_j = \mathcal{F}(\varphi_i), \ i = 1, \cdots, N. \end{cases} \tag{8}$$

By (4) and (5), it can be computed that

$$a(\varphi_j, \varphi_i) = (\varphi_j', \varphi_i') = \begin{cases} 1/h, & i = j = N, \\ 2/h, & i = j \neq N, \\ -1/h, & |i-j| = 1, \\ 0, & |i-j| > 1. \end{cases}$$

Thus (8) is equivalent to the linear system

$$\frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ 0 & \cdots & 0 & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} (f, \varphi_1) \\ (f, \varphi_2) \\ (f, \varphi_3) \\ \vdots \\ (f, \varphi_{N-1}) \\ (f, \varphi_N) + \beta \end{bmatrix}.$$

1.5 Error estimates.

- Interpolation error.

The following error estimates hold:

$$\|u' - u_I'\|_{L^\infty(\Lambda)} \leqslant h \|u''\|_{L^\infty(\Lambda)}, \ \|u - u_I\|_{L^\infty(\Lambda)} \leqslant h^2 \|u''\|_{L^\infty(\Lambda)}, \tag{9}$$

and

$$\|u' - u_I'\|_{L^2(\Lambda)} \leqslant h \|u''\|_{L^\infty(\Lambda)}, \ \|u - u_I\|_{L^2(\Lambda)} \leqslant h^2 \|u''\|_{L^\infty(\Lambda)}. \tag{10}$$

Both estimates (9) and (10) can be derived by applying Taylor series expansions in (6), specifically by expanding $u(x_{n-1})$ and $u(x_n)$ around a point $x \in I_n$ (see [slides part2.pdf, pp. 32-35]). There are better estimates bounded by L^2-norm (see [HW 10, Exercise 1]):

$$\|u - u_I\|_{L^2(\Lambda)} \leqslant h \|u' - u_I'\|_{L^2(\Lambda)} \leqslant h^2 \|u''\|_{L^2(\Lambda)}. \tag{11}$$

- Equation error.

Let u be the solution of (2), and u_h the solution of (3) with V_h replaced by (7). Thus we have

$$a(u - u_h, v_h) = 0, \quad \forall v_h \in V_h.$$

Thus

$$\|u' - u_h'\|_0^2 = a(u - u_h, u - u_h) = a(u - u_h, u - v_h) \leqslant \|u' - u_h'\|_0 \|u' - v_h'\|_0, \quad \forall v_h \in V_h,$$

which, by (11), leads to

$$\|u' - u_h'\|_0 \leqslant \inf_{v_h \in V_h} \|u' - v_h'\|_0 \leqslant \|u' - u_I'\|_0 \leqslant h\|u''\|_0.$$

In the following, we derive the estimate for $\|u - u_h\|_0$ by using Aubin-Nitsche trick. Consider the dual problem of (2): given $r \in L^2(I)$,

$$\begin{cases} \text{Find } \varphi(r) \in V \text{ such that} \\ a(v, \varphi(r)) = (r, v), \quad \forall v \in V. \end{cases}$$

The dual problem admits a unique solution $\varphi(r)$ since $a(\cdot,\cdot)$ is continuous and coercive. Moreover, we have

$$a(v, \varphi(r)) = (r, v), \quad \forall v \in C_0^\infty(I),$$

if we suppose $\varphi(r) \in H^2(I)$, which gives $(-\varphi''(r), v) = (r, v)$, $\forall v \in C_0^\infty(I)$, leading to $-\varphi''(r) = r$ in L^2 since $C_0^\infty(I)$ is dense in $L^2(I)$. Let $\varphi_I(r)$ be the interpolation of $\varphi(r)$ into V_h. By (11), we have $\|\varphi'(r) - \varphi_I'(r)\|_0 \leqslant h\|\varphi''(r)\|_0$. Then

$$\begin{aligned} \|u - u_h\|_0 &= \sup_{r \in L^2(\Lambda),\ r \neq 0} \frac{(r, u - u_h)}{\|r\|_0} = \sup_{r \in L^2(\Lambda),\ r \neq 0} \frac{a(u - u_h, \varphi(r))}{\|r\|_0} \\ &= \sup_{r \in L^2(\Lambda),\ r \neq 0} \frac{a(u - u_h, \varphi(r) - \varphi_I(r))}{\|r\|_0} \\ &\leqslant \sup_{r \in L^2(\Lambda),\ r \neq 0} \frac{\|u' - u_h'\|_0 \|\varphi'(r) - \varphi_I'(r)\|_0}{\|r\|_0} \\ &\leqslant h\|u' - u_h'\|_0 \sup_{r \in L^2(\Lambda),\ r \neq 0} \frac{\|\varphi''(r)\|_0}{\|r\|_0} \\ &\leqslant h\|u' - u_h'\|_0. \end{aligned}$$

Thus we have $\|u - u_h\|_0 \leqslant h^2\|u''\|_0$.

2. Central finite difference schema. Let $\bar{I}_h = \{x_i\}_{i=0}^N$ be the grid on Λ with uniform distance $h = h_i = x_i - x_{i-1}$. Then the central schema reads

$$\begin{cases} -\dfrac{\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}}{h^2} = f(x_i), \ i = 1, \cdots, N-1, \\ \bar{u}_0 = 0, \quad \dfrac{\bar{u}_N - \bar{u}_{N-1}}{h} = \dfrac{h}{2} f(x_N) + \beta. \end{cases} \tag{12}$$

2.1 Linear system. The finite difference schema (12) is equivalent to the linear system:

$$\frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ 0 & \cdots & 0 & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \vdots \\ \bar{u}_{N-1} \\ \bar{u}_N \end{bmatrix} = \begin{bmatrix} hf(x_1) \\ hf(x_2) \\ hf(x_3) \\ \vdots \\ hf(x_{N-1}) \\ \frac{h}{2} f(x_N) + \beta \end{bmatrix}.$$

2.2 Local error estimate. Let the operator $Lu = -u_{xx}$ and the discrete operator L_h on $\{\bar{u}_i\}_{i=1}^{N-1}$ as

$$L_h \bar{u}_i = -\frac{\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}}{h^2}.$$

By the Tylor development:

$$u(x_{i+1}) = u(x_i) + hu'(x_i) + \frac{h^2}{2}u''(x_i) + \frac{h^3}{3!}u^{(3)}(x_i) + \frac{h^4}{4!}u^{(4)}(\xi_i), \text{ for some } \xi_i \in (x_i, x_{i+1}),$$

$$u(x_{i-1}) = u(x_i) - hu'(x_i) + \frac{h^2}{2}u''(x_i) - \frac{h^3}{3!}u^{(3)}(x_i) + \frac{h^4}{4!}u^{(4)}(\eta_i), \text{ for some } \eta_i \in (x_{i-1}, x_i),$$

we obtain the local error estimates $R_i^{(1)} = L_h[u(x_i)] - [Lu](x_i) = O(h^2)$, and

$$R^{(2)} = \frac{u(x_N) - u(x_{N-1})}{h} - \frac{h}{2}f(x_N) - u'(x_N) = O(h^2).$$

2.3 Stability. This proof adopts the notation for discrete functions (see [HW 4, Appendix]). Note that $L_h\bar{u}_i = -\left((\bar{u}_i)_{\bar{x}}\right)_{\hat{x}}$, then multiplying both sides of the finite difference schema $L_h\bar{u}_i = f_i$ by $\bar{u}_i h_i$ yields

$$-\left((\bar{u}_i)_{\bar{x}}\right)_{\hat{x}} \bar{u}_i h_i = f_i \bar{u}_i h_i, \quad \forall i = 1, \cdots, N-1.$$

Summing in i from 1 to $N-1$ gives

$$-\left(((\bar{u}_h)_{\bar{x}})_{\hat{x}}, \bar{u}_h\right)_{I_h} = (f_h, \bar{u}_h)_{I_h}, \tag{13}$$

where $\bar{u}_h = \{\bar{u}_0, \cdots, \bar{u}_N\}$ and $f_h = \{f_0, \cdots, f_N\}$, both of which are the discrete functions defined on the grid $\bar{I}_h$. In virtue of discrete Green formula and the fact that $\bar{u}_0 = 0$, we have

$$-\left(((\bar{u}_h)_{\bar{x}})_{\hat{x}}, \bar{u}_h\right)_{I_h} = ((\bar{u}_h)_{\bar{x}}, (\bar{u}_h)_{\bar{x}})_{I_h^+} - (\bar{u}_N)_{\bar{x}}\bar{u}_N. \tag{14}$$

Thus inserting (14) into (13) we obtain

$$((\bar{u}_h)_{\bar{x}}, (\bar{u}_h)_{\bar{x}})_{I_h^+} = (f_h, \bar{u}_h)_{I_h} + (\bar{u}_N)_{\bar{x}}\bar{u}_N. \tag{15}$$

Note that $(\bar{u}_N)_{\bar{x}} = \beta + \frac{h}{2}f(x_N) \leqslant |\beta| + \|f_h\|_0$, and

$$\bar{u}_N = \sum_{i=1}^{N}(\bar{u}_i)_{\bar{x}}h \leqslant \left(\sum_{i=1}^{N} h\right)^{1/2} \left(\sum_{i=1}^{N}(\bar{u}_i)_{\bar{x}}^2 h\right)^{1/2} = |\bar{u}_h|_1.$$

We have

$$|u_h|_1^2 = ((u_h)_{\bar{x}}, (u_h)_{\bar{x}})_{I_h^+} \leqslant (|\beta| + \|f_h\|_0)|\bar{u}_h|_1 + (f_h, \bar{u}_h)_{I_h} \leqslant (|\beta| + \|f_h\|_0)|u_h|_1 + \|f_h\|_0\|\bar{u}_h\|_0.$$

By discrete Poincaré inequality (see [HW 3, Exercise 2.12]): $\|\bar{u}_h\|_0 \leqslant |\bar{u}_h|_1$, we obtain

$$|\bar{u}_h|_1 \leqslant (2\|f_h\|_0 + |\beta|).$$

Thus by using discrete Poincaré again, we obtain

$$\|\bar{u}_h\|_1 = \left(\|\bar{u}_h\|_0^2 + |\bar{u}_h|_1^2\right)^{1/2} \leqslant \sqrt{2}|\bar{u}_h|_1 \leqslant \sqrt{2}(2\|f_h\|_0 + |\beta|).$$

2.4 Global error estimate. Let $e_i = u(x_i) - \bar{u}_i$ for $i = 0, 1, \cdots, N$. It is obvious that

$$\begin{cases} L_h e_i = R_i^{(1)}, \quad \forall i = 1, 2, \cdots, N-1, \\ e_0 = 0, \\ \dfrac{e_N - e_{N-1}}{h} = R^{(2)}. \end{cases}$$

Thus $\|e_h\|_1 \leqslant \sqrt{2}(2\|R_h^{(1)}\|_0 + |R^{(2)}|) = O(h^2)$ as $h \to 0$.

3. Comparison of these two methods. Let $\bar{\mu}_h(x) = \sum_{j=1}^N \bar{u}_j \varphi_j(x)$.

3.1 Show that $\bar{\mu}_h$ satisfies

$$(\bar{\mu}_h', v_h') = (f, v_h)_h + \beta v_h(1), \quad \forall v_h \in V_h,$$

where

$$(f, v_h)_h = \sum_{i=1}^{N-1} h f(x_i) v_h(x_i) + \frac{h}{2} f(x_0) v_h(x_0) + \frac{h}{2} f(x_N) v_h(x_N).$$

Proof. It is sufficient to show that $(\bar{\mu}_h', \varphi_i') = (f, \varphi_i)_h + \beta \varphi_i(1)$ holds for each $i = 1, \cdots, N$. By (4) and (5), we obtain

$$(\bar{\mu}_h', \varphi_i') = \begin{cases} \bar{u}_1(\varphi_1', \varphi_1') + \bar{u}_2(\varphi_2', \varphi_1'), & i = 1, \\ \bar{u}_{i-1}(\varphi_{i-1}', \varphi_i') + \bar{u}_i(\varphi_i', \varphi_i') + \bar{u}_{i+1}(\varphi_{i+1}', \varphi_i'), & 1 < i < N, \\ \bar{u}_{N-1}(\varphi_{N-1}', \varphi_N') + \bar{u}_N(\varphi_N', \varphi_N'), & i = N. \end{cases}$$

Thus it can be computed as

$$(\bar{\mu}_h', \varphi_i') = \begin{cases} -\dfrac{\bar{u}_2 - 2\bar{u}_1}{h}, & i = 1, \\ -\dfrac{\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}}{h}, & 1 < i < N, \\ \dfrac{\bar{u}_N - \bar{u}_{N-1}}{h}, & i = N. \end{cases}$$

On the other hand, we have

$$(f, \varphi_i)_h + \beta \varphi_i(1) = \begin{cases} h f(x_1), & i = 1, \\ h f(x_i), & 1 < i < N, \\ \dfrac{h}{2} f(x_N) + \beta, & i = N. \end{cases}$$

Therefore by finite difference shcema (12), we have $(\bar{\mu}_h', \varphi_i') = (f, \varphi_i)_h + \beta \varphi_i(1)$. $\square$

3.2 Let $Q(w) = (w, 1)_h$. Show that

$$\left| \int_\Lambda w(t) \mathrm{d}t - Q(w) \right| \leqslant h^2 \|w''\|_0.$$

Proof. Since

$$Q(w) = \sum_{i=1}^{N-1} h w(x_i) + \frac{h}{2} w(x_0) + \frac{h}{2} w(x_N) = \frac{h}{2} \sum_{i=1}^{N} \left(w(x_{i-1}) + w(x_i) \right),$$

then

$$\begin{aligned} \int_\Lambda w(t) \mathrm{d}t - Q(w) &= \sum_{i=1}^N \left[\int_{x_{i-1}}^{x_i} w(t) \mathrm{d}t - \frac{h}{2} \left(w(x_{i-1}) + w(x_i) \right) \right] \\ &= \sum_{i=1}^N \left[\int_{x_{i-1}}^{x_i} w(t) - \left(w(x_{i-1}) \frac{x_i - t}{x_i - x_{i-1}} + w(x_i) \frac{t - x_{i-1}}{x_i - x_{i-1}} \right) \mathrm{d}t \right] \\ &= \sum_{i=1}^N \left[\int_{x_{i-1}}^{x_i} w(t) - w_I(t) \mathrm{d}t \right] \\ &= \int_\Lambda w(t) - w_I(t) \mathrm{d}t, \end{aligned}$$

where $w_I(t)$ is the interpolation of $w(t)$ into X_h^1 as shown in (6). Thus by employing (11) we have

$$\left|\int_\Lambda w(t)\mathrm{d}t - Q(w)\right| = \left|\int_\Lambda w(t) - w_I(t)\mathrm{d}t\right| \leqslant \|w - w_I\|_0 \leqslant h^2\|w''\|_0.$$

□

3.3 Show that for any $v_h \in V_h$ it holds $|(\mu_h' - \bar{u}_h', v_h')| \leqslant 2h^2 (\|f'\|_0 + \|f''\|_0)(\|v_h\|_0 + \|v_h'\|_0)$.

Proof. It is evident that

$$\begin{aligned}|(u_h' - \bar{\mu}_h', v_h')| = |(f, v_h) - (f, v_h)_h| &\leqslant h^2 \left\|(f v_h)''\right\|_0 = h^2 \left\|f'' v_h + 2f' v_h'\right\|_0 \\ &\leqslant 2h^2 \left(\|f'\|_0 + \|f''\|_0\right)\left(\|v_h\|_0 + \|v_h'\|_0\right).\end{aligned}$$

□

3.4 Show that $\|u_h - \bar{\mu}_h\|_1 \leqslant ch^2 (\|f'\|_0 + \|f''\|_0)$, where c is a constant.

Proof. It is known that $u_h - \bar{\mu}_h \in V_h$, then by Poincaré inequality (see [HW 3, Exercise 2.12]), we have $\|u_h - \bar{\mu}_h\|_0 \leqslant \|u_h' - \bar{\mu}_h'\|_0$. Thus

$$\begin{aligned}\|u_h - \bar{\mu}_h\|_1^2 \leqslant 2\|u_h' - \bar{\mu}_h'\|_0^2 &= 2(u_h' - \bar{\mu}_h', u_h' - \bar{\mu}_h') \\ &\leqslant 4h^2 \left(\|f'\|_0 + \|f''\|_0\right)\left(\|u_h - \bar{\mu}_h\|_0 + \|u_h' - \bar{\mu}_h'\|_0\right) \\ &\leqslant 4\sqrt{2}h^2 \left(\|f'\|_0 + \|f''\|_0\right)\|u_h - \bar{\mu}_h\|_1,\end{aligned}$$

which leads to the desired result. □

□