Consider the initial value plroblem (IVP):

$$\begin{cases} u'(t) = f(t, u(t)), t \in (0, T], \\ u(0) = u_0. \end{cases} \tag{1}$$

Discretize the interval $[0, T]$ into $M+1$ equally spaced nodes, forming a grid defined by $t^n = nh$ for $n = 0, 1, \cdots, M$, where $h = T/M$.We assume that f is Lipschitz continuous in its second variable uniformly with respect to its first variable and that u is at least three times differentiable.

Exercise 2.1. *Carry out an error analysis for Backward Euler schema.*

Solution. Backward Euler schema:

$$\frac{u^n - u^{n-1}}{h} = f(t^n, u^n), \quad n = 1, 2, \cdots, M. \tag{2}$$

Its truncation error R_b^n:

$$R_b^n = \frac{u(t^n) - u(t^{n-1})}{h} - f(t^n, u(t^n)), \ n = 1, \cdots, M. \tag{3}$$

By the Tylor development:

$$u(t^{n-1}) = u(t^n) - hu'(t^n) + \frac{h^2}{2} u''(\xi^n), \text{ for some } \xi^n \in [t^{n-1}, t^n].$$

We have $R_b^n = O(h)$ as $h \to 0$. Let $e^n = u(t^n) - u^n$. Since (2) and (3), we have

$$\frac{e^n - e^{n-1}}{h} - R_b^n = f(t^n, u(t^n)) - f(t^n, u^n).$$

Let $R = \max_{1 \leqslant n \leqslant M} |R_b^n|$, and by $|f(t^n, u(t^n)) - f(t^n, u^n)| \leqslant L|e^n|$, we have

$$|e^n| \leqslant |e^{n-1}| + hL|e^n| + hR.$$

Thus if $h < L^{-1}$,

$$|e^n| \leqslant \frac{|e^{n-1}|}{1-hL} + \frac{hR}{1-hL} \leqslant \cdots \leqslant \frac{|e^0|}{(1-hL)^n} + hR \sum_{k=1}^{n} \frac{1}{(1-hL)^k} = \frac{|e^0|}{(1-hL)^n} + \frac{hR}{hL}[(1-hL)^{-n} - 1].$$

Note that $|e^0| = 0$, $(1-hL)^{-n} \leqslant (1-hL)^{-M} = (1-hL)^{-T/h}$, $(1-hL)^{-T/h}$ is non-decreasing over $h \in (0, L^{-1})$, and

$$\lim_{h \to 0} (1-hL)^{-T/h} = e^{LT}.$$

Thus $|e^n| = O(h)$ as $h \to 0$. □

Exercise 2.2. *Carry out an error analysis for Crank-Nicolson (Trapezoidal) schema.*

Solution. Crank-Nicolson schema:

$$\frac{u^{n+1} - u^n}{h} = \frac{f\left(t^{n+1}, u^{n+1}\right) + f\left(t^n, u^n\right)}{2}, \ n = 0, 1, \cdots, M-1. \tag{4}$$

Its truncation error R_c^n:

$$R_c^n = \frac{u(t^{n+1}) - u(t^n)}{h} - \frac{f(t^{n+1}, u(t^{n+1})) + f(t^n, u(t^n))}{2}. \tag{5}$$

By Tylor development:

$$u(t^{n+1}) = u(t^n) + hu'(t^n) + \frac{h^2}{2}u''(t^n) + \frac{h^3}{6}u'''(\xi^n), \qquad \text{for some } \xi^n \in [t^n, t^{n+1}],$$

and

$$u'(t^{n+1}) = u'(t^n) + hu''(t^n) + \frac{h^2}{2}u'''(\eta^n), \qquad \text{for some } \eta^n \in [t^n, t^{n+1}],$$

we have $R_c^n = O(h^2)$ as $h \to 0$. Let $e^n = u(t^n) - u^n$. Since (4) and (5), we have

$$\frac{e^{n+1} - e^n}{h} = \frac{f(t^{n+1}, u(t^{n+1})) - f(t^{n+1}, u^{n+1}) + f(t^n, u(t^n)) - f(t^n, u^n)}{2} + R_c^n.$$

Let $R = \max_{0 \leqslant n \leqslant M-1} |R_c^n|$, and by $|f(t^n, u(t^n)) - f(t^n, u^n)| \leqslant L|e^n|$, we have

$$|e^{n+1}| \leqslant |e^n| + hR + h\frac{L|e^{n+1}| + L|e^n|}{2}.$$

Hence

$$|e^{n+1}| \leqslant \frac{2+hL}{2-hL}|e^n| + hR \leqslant \cdots \leqslant \left(\frac{2+hL}{2-hL}\right)^{n+1} |e^0| + hR\frac{1 - \left(\frac{2+hL}{2-hL}\right)^{n+1}}{1 - \frac{2+hL}{2-hL}}.$$

Note that $|e^0| = 0$ and

$$\lim_{h \to 0} \frac{1 - \left(\frac{2+hL}{2-hL}\right)^{n+1}}{1 - \frac{2+hL}{2-hL}} \leqslant M = T/h.$$

Thus $|e^n| = O(h^2)$ as $h \to 0$. $\square$