Exercise 1 (See Remark 1.3)**.** *Prove*

$$\|u-u_I\|_0 \leqslant Ch^2\|u''\|_0,$$
$$\|u'-u_I'\|_0 \leqslant Ch\|u''\|_0.$$

Proof. • **Step 1**. Show that for $I=(0,1)$ and $f\in H^2(I)\cap H_0^1(I)$, we have

$$\int_0^1 f(X)^2\mathrm{d}X \leqslant \int_0^1 f'(X)^2\mathrm{d}X, \text{ and } \int_0^1 f'(X)^2\mathrm{d}X \leqslant \int_0^1 f''(X)^2\mathrm{d}X.$$

Since $f(0)=f(1)=0$, there exists $X_0\in I$ such that $f'(X_0)=0$. Thus

$$f(X)=\int_0^X f'(X)\mathrm{d}X, \text{ and } f'(X)=\int_{X_0}^X f''(X)\mathrm{d}X.$$

Therefore, the conclusion is clear by Cauchy-Schwarz inequality.

• **Step 2.** Make variable change to subinterval $I_{j+1}=(x_j,x_{j+1})$ by $x=x_{j-1}+X(x_j-x_{j-1})$, we have

$$\int_{x_{j-1}}^{x_j} \tilde{f}(x)^2\mathrm{d}x \leqslant (x_j-x_{j-1})^2\int_{x_{j-1}}^{x_j}\tilde{f}'(x)^2\mathrm{d}x, \text{ and } \int_{x_{j-1}}^{x_j}\tilde{f}'(x)^2\mathrm{d}x \leqslant (x_j-x_{j-1})^2\int_{x_{j-1}}^{x_j}\tilde{f}''(x)^2\mathrm{d}x,$$

where $\tilde{f}(x)=f(X)=f(\frac{x-x_{j-1}}{x_j-x_{j-1}})\in H^2(I_j)\cap H_0^1(I_j)$. The conclusion is obvious since

$$\mathrm{d}x=(x_j-x_{j-1})\mathrm{d}X,\ f'(X)=\tilde{f}'(x)(x_j-x_{j-1}),\ f''(X)=\tilde{f}''(x)(x_j-x_{j-1})^2.$$

• **Step 3.** Let $\tilde{f}(x)=u(x)-u_I(x)$. It is obvious that $\tilde{f}(x_i)=0$ for $i=0,\cdots,N+1$. Thus $\left(\tilde{f}\big|_{I_j}\right)''=u''$ and $\tilde{f}\big|_{I_j}\in H^2(I_j)\cap H_0^1(I_j)$ for $j=1,\cdots,N+1$, and we have

$$\int_{I_j}(u-u_I)^2\mathrm{d}x\leqslant h_j^2\int_{I_j}(u'-u_I')^2\mathrm{d}x,\ \int_{I_j}(u'-u_I')^2\mathrm{d}x\leqslant h_j^2\int_{I_j}(u'')^2\mathrm{d}x,$$

both of which leads to

$$\|u-u_I\|_0\leqslant h\|u'-u_I'\|_0, \text{ and } \|u'-u_I'\|_0\leqslant h\|u''\|_0.$$

□

Exercise 2. *Consider the mixed boundary problem*

$$\begin{cases} -u''=f, \quad x\in I:=(0,1),\\ u(0)=0,\ u'(1)=\beta, \end{cases}$$

where $\beta\in\mathbb{R}$ and $f\in L^2(I)$. Construct and analyze P_1-FEM for this problem.

Proof. • Variational form. Let $V=\{v\in H^1(I): v(0)=0\}$, the bilinear form $a(u,v)=(u',v')$, and the functional $\mathcal{F}(v)=(f,v)+\beta v(1)$. Then the variational problem reads

$$\begin{cases} \text{Find } u\in V \text{ such that}\\ a(u,v)=\mathcal{F}(v), \quad \forall v\in V, \end{cases}$$

which is clearly equivalent to the strong problem.

• Galerkin Approximation. Let V_h be a subspace of V with finite dimension. Then the Galerkin approximation reads

$$\begin{cases} \text{Find } u_h\in V_h \text{ such that}\\ a(u_h,v_h)=\mathcal{F}(v_h), \quad \forall v_h\in V_h. \end{cases}$$

• $P1-$FEM. By construct the space of piecewise linear polynomials X_h^1 and its basis $\varphi_0, \cdots, \varphi_{N+1}$, shown in the Appendix, we let the finite element space $V_h = X_h^1 \cap V$, then

$$V_h = \text{span}\{\varphi_1, \cdots, \varphi_{N+1}\}.$$

• FEM Implementation. Let $u_h = \sum_{j=1}^{N+1} u_j \varphi_j(x)$, then

$$\sum_{j=1}^{N+1} u_j a(\varphi_j, \varphi_i) = \mathcal{F}(\varphi_i), \quad i = 1, \cdots, N+1.$$

Let $\mathbf{A} = (a_{i,j})$ be the $(N+1) \times (N+1)$ matrix with its entries $a_{i,j} = a(\varphi_j, \varphi_i)$. Then we have

$$\begin{aligned}
&a_{N+1,N+1} = \frac{1}{h_{N+1}}, \ a_{j,j} = \frac{1}{h_j} + \frac{1}{h_{j+1}}, \quad j = 1, \cdots, N, \\
&a_{j,j-1} = -\frac{1}{h_j}, \ j = 1, \cdots, N+1, \\
&a_{i,j} = 0, \ \text{if } |i-j| \geqslant 2.
\end{aligned}$$

Thus

$$\frac{1}{h} \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \\ u_{N+1} \end{bmatrix} = \begin{bmatrix} (f, \varphi_1) \\ (f, \varphi_2) \\ \vdots \\ (f, \varphi_N) \\ (f, \varphi_{N+1}) + \beta \end{bmatrix}.$$

• Error Estimate. We denote u_I being the interpolation of u into V_h, then it is clear that

$$\|u - u_I\|_0 \leqslant Ch\|u' - u_I'\|_0 \leqslant Ch^2\|u''\|_0.$$

We know $a(u - u_h, v_h) = 0$ for any $v_h \in V_h$. Then

$$\|u' - u_h'\|_0^2 = a(u - u_h, u - u_h) = a(u - u_h, u - v_h) \leqslant \|u' - u_h'\|_0 \|u' - v_h'\|_0, \quad \forall v_h \in V_h,$$

which leads to

$$\|u' - u_h'\|_0 \leqslant \inf_{v_h \in V_h} \|u' - v_h'\|_0 \leqslant \|u' - u_I'\|_0 \leqslant Ch\|u''\|_0.$$

In the following, we derive the estimate for $\|u - u_h\|_0$.

Dual problem: given $r \in L^2(I)$,

$$\begin{cases} \text{Find } \varphi(r) \in V \text{ such that} \\ a(v, \varphi(r)) = (r, v), \quad \forall v \in V. \end{cases}$$

The dual problem admits a unique solution $\varphi(r)$ since $a(\cdot, \cdot)$ is continuous and coercive. Moreover, we have

$$a(v, \varphi(r)) = (r, v), \quad \forall v \in C_0^\infty(I),$$

if we suppose $\varphi(r) \in H^2(I)$, which gives $(-\varphi''(r), v) = (r, v)$, $\forall v \in C_0^\infty(I)$. Since $C_0^\infty(I)$ is dense in $L^2(I)$, we have

$$\|\varphi''(r)\|_0 = \sup_{v \in L^2(I), \ v \neq 0} \frac{(\varphi, v)}{\|v\|_0} = \sup_{v \in L^2(I), \ v \neq 0} \frac{(r, v)}{\|v\|_0} = \|r\|_0.$$

Thus we denote $\varphi_I(r)$ being the interpolation of $\varphi(r)$ into V_h and obtain

$$\begin{aligned}
\|u-u_h\|_0 &= \sup_{r\in L^2(I),\ r\neq 0} \frac{(r,u-u_h)}{\|r\|_0} = \sup_{r\in L^2(I),\ r\neq 0} \frac{a(u-u_h,\varphi(r))}{\|r\|_0} \\
&= \sup_{r\in L^2(I),\ r\neq 0} \frac{a(u-u_h,\varphi(r)-\varphi_I(r))}{\|r\|_0} \\
&\leqslant \sup_{r\in L^2(I),\ r\neq 0} \frac{\|u'-u_h'\|_0\|\varphi'(r)-\varphi_I'(r)\|_0}{\|r\|_0} \\
&\leqslant Ch\|u'-u_h'\|_0 \sup_{r\in L^2(I),\ r\neq 0} \frac{\|\varphi''(r)\|_0}{\|r\|_0} \\
&\leqslant Ch\|u'-u_h'\|_0.
\end{aligned}$$

□

Appendix

Let $I=(0,1)$ and $\{x_n\}_{n=0}^{N+1}$ be a grid on I such that $0=x_0<x_1<\cdots<x_N<x_{N+1}=1$. Denote by each subintervals (or elements) $I_n=(x_{n-1},x_n)$ for $1\leqslant n\leqslant N+1$ of length $h_n=x_n-x_{n-1}$. Let $h=\max_{1\leqslant n\leqslant N+1} h_n$.

The piecewise linear polynomials on such grid is denoted by

$$X_h^1 := \{v\in C(\bar I) : v\big|_{I_{j+1}} \in \mathbb{P}_1,\ j=0,\cdots,N\}.$$

We construct a nodal basis for X_h^1, which is based on nodes in every element (how many nodes in every element depends on the degree of freedom, or the degree of polynomials required parameters to be determined).

$$\varphi_0(x)=\begin{cases}\dfrac{x_1-x}{x_1-x_0}, & x\in I_1,\\ 0, & \text{else},\end{cases}\qquad \varphi_{N+1}(x)=\begin{cases}\dfrac{x-x_N}{x_{N+1}-x_N}, & x\in I_{N+1},\\ 0, & \text{else},\end{cases}$$

$$\varphi_n(x)=\begin{cases}\dfrac{x-x_{n-1}}{x_n-x_{n-1}}, & x\in I_n,\\ \dfrac{x_{n+1}-x}{x_{n+1}-x_n}, & x\in I_{n+1},\\ 0, & \text{else}.\end{cases}$$

Clearly, we have $X_h^1=\mathrm{span}\{\varphi_0,\varphi_1,\cdots,\varphi_{N+1}\}$.

For any $u\in C(\bar I)$, its interpolation into X_h^1 is denoted by $u_I(x)$. Clearly, we have $u_I(x)=\sum_{i=0}^{N+1} u(x_i)\varphi_i(x)$ and

$$u_I\big|_{I_{j+1}} = u(x_j)\varphi_j(x)+u(x_{j+1})\varphi_{j+1}(x) = u(x_j)\frac{x_{j+1}-x}{x_{j+1}-x_j}+u(x_{j+1})\frac{x-x_j}{x_{j+1}-x_j}.$$