Exercise 2.15. *Consider the elliptic problem*

$$-u_{xx} = f, \quad \forall x \in (a,b),$$
$$u(a) = 0, \; u'(b) = \beta,$$

and its finite difference schema

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = f_i, \quad \forall i = 1, \cdots, N-1,$$
$$u_0 = 0,$$
$$\frac{u_N - u_{N-1}}{h} = \beta,$$

in an uniform mesh $\{x_i\}_{i=0}^N$, $x_i = a + ih$, $h = (b-a)/N$.
1) Derive an estimate for the truncation errors:

$$R_i^{(1)} = L_h[u(x_i)] - [Lu](x_i) \text{ for } i = 1, \cdots, N-1, \; R^{(2)} = \frac{u(x_N) - u(x_{N-1})}{h} - u'(x_N).$$

2) Rewrite the discrete problem under matrix form.
3) Establish an a priori estimate for $\|u_h\|_1$.
4) Derive an error estimate for $\|e_h\|_1$, *where* $e_i = u(x_i) - u_i$.

Appendix: Notations for Discrete Representation

Let $I = [a, b]$. We define the discrete grid points as

$$a = x_0 < x_1 < \cdots < x_N = b.$$

We introduce the following sets:

$$I_h = \{x_1, \cdots, x_{N-1}\},\ \bar{I}_h = \{x_0, x_1, \cdots, x_N\},\ I_h^+ = \{x_1, \cdots, x_N\}.$$

The grid spacing is defined as

$$h_i = x_i - x_{i-1}, \quad i = 1, \cdots, N.$$

Additionally, we define the averaged grid spacing:

$$\begin{aligned} \bar{h}_i &= \frac{1}{2}(h_i + h_{i+1}),\ i = 1, \cdots, N-1, \\ \bar{h}_0 &= \frac{1}{2}h_1, \quad \bar{h}_N = \frac{1}{2}h_N. \end{aligned}$$

A discrete function defined on $\bar{I}_h$ is denoted as

$$v_h = \{v_0, v_1, \cdots, v_N\}.$$

We define the following difference operators:

$$\begin{aligned} (v_i)_{\bar{x}} &:= v_{i,\bar{x}} := \frac{v_i - v_{i-1}}{h_i},\ i = 1, \cdots, N, \\ (v_i)_x &:= v_{i,x} := \frac{v_{i+1} - v_i}{h_{i+1}},\ i = 0, \cdots, N-1, \\ (v_i)_{\hat{x}} &:= v_{i,\hat{x}} := \frac{v_{i+1} - v_i}{\bar{h}_i},\ i = 0, \cdots, N-1. \end{aligned}$$

The discrete inner products are given by

$$(u_h, v_h)_{I_h} = \sum_{i=1}^{N-1} u_i v_i \bar{h}_i,\ (u_h, v_h)_{\bar{I}_h} = \sum_{i=0}^{N} u_i v_i \bar{h}_i,\ (u_h, v_h)_{I_h^+} = \sum_{i=1}^{N} u_i v_i h_i. \tag{1}$$

We define the discrete norms as follows:

$$\begin{aligned} \|v_h\|_c &:= \max_{\bar{I}_h} |v_i|,\ \|v_h\|_0 := (v_h, v_h)_{\bar{I}_h}^{1/2}, \\ |v_h|_1 &:= ((v_h)_{\bar{x}}, (v_h)_{\bar{x}})_{I_h^+}^{1/2},\ \|v_h\|_1^2 = \|v_h\|_0^2 + |v_h|_1^2. \end{aligned} \tag{2}$$

The discrete integral by parts:

$$\sum_{i=m+1}^{n} v_i (w_i)_{\bar{x}} h_i = -\sum_{i=m}^{n-1} (v_i)_x w_i h_{i+1} + v_n w_n - v_m w_m,\ \text{for some } 0 \leqslant m < n \leqslant N. \tag{3}$$

The discrete Green formula:

$$\sum_{i=m+1}^{n-1} ((u_i)_{\bar{x}})_{\hat{x}}\, v_i \bar{h}_i = -\sum_{i=m+1}^{n} (u_i)_{\bar{x}} (v_i)_{\bar{x}} h_i + (u_n)_{\bar{x}} v_n - (u_m)_x v_m,\ \text{for some } 0 \leqslant m < n \leqslant N. \tag{4}$$

The discrete Cauchy-Schwarz inequality states that

$$|(u_h, v_h)_{\bar{I}_h}| \leqslant (u_h, u_h)_{\bar{I}_h}^{1/2} (v_h, v_h)_{\bar{I}_h}^{1/2}. \tag{5}$$

If $v_0 = 0$ (or $v_N = 0$ or $v_0 = v_N = 0$), the discrete Poincaré inequality holds:

$$\|v_h\|_c \leqslant C|v_h|_1, \quad \|v_h\|_0 \leqslant C|v_h|_1, \tag{6}$$

where C is a constant depending only on a and b.