Exercise 2.15. *Consider the elliptic problem*

$$
\begin{aligned}
-u_{xx} &= f, \quad \forall x \in (a,b),\\
u(a) &= 0, \;\; u'(b) = \beta,
\end{aligned}
$$

and its finite difference schema

$$
\begin{aligned}
-\frac{u_{i+1}-2u_i+u_{i-1}}{h^2} &= f_i, \quad \forall i = 1,\cdots,N-1,\\
u_0 &= 0,\\
\frac{u_N-u_{N-1}}{h} &= \beta,
\end{aligned}
$$

in an uniform mesh $\{x_i\}_{i=0}^N$, $x_i = a+ih$, $h=(b-a)/N$.
1) Derive an estimate for the truncation errors:

$$
R_i^{(1)} = L_h[u(x_i)] - [Lu](x_i) \text{ for } i = 1,\cdots,N-1, \;\; R^{(2)} = \frac{u(x_N)-u(x_{N-1})}{h} - u'(x_N).
$$

2) Rewrite the discrete problem under matrix form.
3) Establish an a priori estimate for $\|u_h\|_1$.
4) Derive an error estimate for $\|e_h\|_1$, *where* $e_i = u(x_i)-u_i$.

Solution. 1). Let the operator $Lu = -u_{xx}$ and the discrete operator L_h on $\{u_i\}_{i=1}^{N-1}$ as

$$
L_h u_i = -\frac{u_{i+1}-2u_i+u_{i-1}}{h^2}.
$$

By the Tylor development:

$$
u(x_{i+1}) = u(x_i) + hu'(x_i) + \frac{h^2}{2}u''(x_i) + \frac{h^3}{3!}u^{(3)}(x_i) + \frac{h^4}{4!}u^{(4)}(\xi_i), \text{ for some } \xi_i \in (x_i, x_{i+1}),
$$

$$
u(x_{i-1}) = u(x_i) - hu'(x_i) + \frac{h^2}{2}u''(x_i) - \frac{h^3}{3!}u^{(3)}(x_i) + \frac{h^4}{4!}u^{(4)}(\eta_i), \text{ for some } \eta_i \in (x_{i-1}, x_i),
$$

we obtain that $R_i^{(1)} = L_h[u(x_i)] - [Lu](x_i) = O(h^2)$, while $R^{(2)} = O(h)$ as $h \to 0$.
2).

$$
\begin{bmatrix}
\frac{2}{h^2} & -\frac{1}{h^2} & & & \\
-\frac{1}{h^2} & \frac{2}{h^2} & -\frac{1}{h^2} & & \\
 & \ddots & \ddots & \ddots & \\
 & & -\frac{1}{h^2} & \frac{2}{h^2} & -\frac{1}{h^2} \\
 & & & -\frac{1}{h} & \frac{1}{h}
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix}
=
\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N-1} \\ \beta \end{bmatrix},
$$

or

$$
\begin{bmatrix}
2 & -1 & & & \\
-1 & 2 & -1 & & \\
 & \ddots & \ddots & \ddots & \\
 & & -1 & 2 & -1 \\
 & & & -1 & 1
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix}
=
\begin{bmatrix} h^2 f_1 \\ h^2 f_2 \\ \vdots \\ h^2 f_{N-1} \\ h\beta \end{bmatrix}.
$$

3). Note that $L_h u_i = -\left((u_i)_{\bar{x}}\right)_{\hat{x}}$, then multiplying both sides of the finite difference schema $L_h u_i = f_i$ by $u_i h_i$ yields

$$
-\left((u_i)_{\bar{x}}\right)_{\hat{x}} u_i h_i = f_i u_i h_i, \quad \forall i = 1,\cdots,N-1.
$$

Summing in i gives

$$
-\left(\left((u_h)_{\bar{x}}\right)_{\hat{x}}, u_h\right)_{I_h} = (f_h, u_h)_{I_h}.
$$

In virtue of discrete Green formula (9) and the fact that $u_0 = 0$, we have

$$-\left(((u_h)_{\bar{x}})_{\hat{x}}, u_h\right)_{I_h} = ((u_h)_{\bar{x}}, (u_h)_{\bar{x}})_{I_h^+} - (u_N)_{\bar{x}} u_N.$$

Note that $(u_N)_{\bar{x}} = \beta$, and

$$u_N = \sum_{i=1}^{N} (u_i)_{\bar{x}} h \leqslant \left(\sum_{i=1}^{N} h\right)^{1/2} \left(\sum_{i=1}^{N} (u_i)_{\bar{x}}^2 h\right)^{1/2} = \sqrt{b-a}|u_h|_1.$$

We have

$$|u_h|_1^2 = ((u_h)_{\bar{x}}, (u_h)_{\bar{x}})_{I_h^+} = \beta u_N + (f_h, u_h)_{I_h}.$$

By discrete Cauchy-Schwarz inequality:

$$(f_h, u_h)_{I_h} \leqslant \left(\sum_{i=1}^{N-1} f_i^2 h\right)^{1/2} \left(\sum_{i=1}^{N-1} u_i^2 h\right)^{1/2} \leqslant \|f_h\|_0 \|u_h\|_0.$$

By discrete Poincaré inequality: $\|u_h\|_0 \leqslant C|u_h|_1$, we obtain

$$|u_h|_1 \leqslant C(\|f_h\|_0 + |\beta|),$$

where C represents a constant depending only on a and b, and it may different in different scenarios. Thus by using discrete Poincaré again, we obtain

$$\|u_h\|_1 = \left(\|u_h\|_0^2 + |u_h|_1^2\right)^{1/2} \leqslant C|u_h|_1 \leqslant C(\|f_h\|_0 + |\beta|).$$

4). It is obvious that

$$\begin{cases} L_h e_i = R_i, \quad \forall i = 1, 2, \cdots, N-1, \\ e_0 = 0, \\ \dfrac{e_N - e_{N-1}}{h} = R^{(2)}. \end{cases}$$

By 1) and 3) we have

$$\|e_h\|_1 \leqslant C(\|R_h^{(1)}\|_0 + |R^{(2)}|) = O(h) \text{ as } h \to 0.$$

□

Exercise 3.1. *Derive an estimate for the truncation error of the 9-point schema.*

Solution. Consider the 2D problem:

$$\begin{cases} Lu(\mathbf{x}) = f(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega, \\ \quad\quad u(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in \partial\Omega, \end{cases}$$

where $\Omega = (a, b)^2$ and $Lu = -\Delta u$. Let $\{\mathbf{x}_{i,j} : i = 0, \cdots, N_x, j = 0, \cdots, N_y\}$ be the discrete mesh in Ω equispaced of $h_x = (b-a)/N_x$ and $h_y = (b-a)/N_y$ over x and y axis respectively. Then the coordinate of $\mathbf{x}_{i,j}$ is $(a + ih_x, a + jh_y)$.

The 9-point schema is

$$\begin{cases} \bar{L}_h u_{i,j} = \bar{f}(\mathbf{x}_{i,j}), \quad \forall \mathbf{x}_{i,j} \in \Omega, \\ \quad\quad u_{i,j} = 0, \quad \forall \mathbf{x}_{i,j} \in \partial\Omega, \end{cases}$$

where

$$\begin{aligned} \bar{L}_h u_{i,j} = L_h u_{i,j} - \frac{h_x^2 + h_y^2}{12 h_x^2 h_y^2} & [u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} \\ & - 2u_{i+1,j} + 4u_{i,j} - 2u_{i-1,j} \\ & + u_{i+1,j-1} - 2u_{i,j-1} + u_{i-1,j-1}], \end{aligned} \tag{1}$$

$$L_h u_{i,j} = -\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h_x^2} - \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h_y^2}, \tag{2}$$

and $\bar{f}(\mathbf{x}_{i,j})$ is given by

$$\bar{f}(\mathbf{x}_{i,j}) = f(\mathbf{x}_{i,j}) + \frac{1}{12}\left(h_x^2 \frac{\partial^2 f}{\partial x^2} + h_y^2 \frac{\partial^2 f}{\partial y^2}\right)(\mathbf{x}_{i,j}). \tag{3}$$

The truncation error $\bar{R}_{i,j} = \bar{L}_h[u(\mathbf{x}_{i,j})] - \bar{f}(\mathbf{x}_{i,j})$. By the Tylor development, we have

$$u(\mathbf{x}_{i+1,j}) = u(\mathbf{x}_{i,j}) + h_x \frac{\partial u}{\partial x}(\mathbf{x}_{i,j}) + \frac{h_x^2}{2}\frac{\partial^2 u}{\partial x^2}(\mathbf{x}_{i,j}) + \frac{h_x^3}{3!}\frac{\partial^3 u}{\partial x^3}(\mathbf{x}_{i,j}) + \frac{h_x^4}{4!}\frac{\partial^4 u}{\partial x^4}(\mathbf{x}_{i,j}) + O(h_x^4),$$

$$u(\mathbf{x}_{i-1,j}) = u(\mathbf{x}_{i,j}) - h_x \frac{\partial u}{\partial x}(\mathbf{x}_{i,j}) + \frac{h_x^2}{2}\frac{\partial^2 u}{\partial x^2}(\mathbf{x}_{i,j}) - \frac{h_x^3}{3!}\frac{\partial^3 u}{\partial x^3}(\mathbf{x}_{i,j}) + \frac{h_x^4}{4!}\frac{\partial^4 u}{\partial x^4}(\mathbf{x}_{i,j}) + O(h_x^4),$$

$$u(\mathbf{x}_{i,j+1}) = u(\mathbf{x}_{i,j}) + h_y \frac{\partial u}{\partial y}(\mathbf{x}_{i,j}) + \frac{h_y^2}{2}\frac{\partial^2 u}{\partial y^2}(\mathbf{x}_{i,j}) + \frac{h_y^3}{3!}\frac{\partial^3 u}{\partial y^3}(\mathbf{x}_{i,j}) + \frac{h_y^4}{4!}\frac{\partial^4 u}{\partial y^4}(\mathbf{x}_{i,j}) + O(h_y^4),$$

$$u(\mathbf{x}_{i,j-1}) = u(\mathbf{x}_{i,j}) - h_y \frac{\partial u}{\partial y}(\mathbf{x}_{i,j}) + \frac{h_y^2}{2}\frac{\partial^2 u}{\partial y^2}(\mathbf{x}_{i,j}) - \frac{h_y^3}{3!}\frac{\partial^3 u}{\partial y^3}(\mathbf{x}_{i,j}) + \frac{h_y^4}{4!}\frac{\partial^4 u}{\partial y^4}(\mathbf{x}_{i,j}) + O(h_y^4).$$

Replacing $u_{i,j}$ with $u(\mathbf{x}_{i,j})$ in (2) and inserting those above formulae into it, we obtain

$$L_h[u(\mathbf{x}_{i,j})] = -\frac{\partial^2 u}{\partial x^2}(\mathbf{x}_{i,j}) - \frac{\partial^2 u}{\partial y^2}(\mathbf{x}_{i,j}) - \frac{1}{12}\left(h_x^2 \frac{\partial^4 u}{\partial x^4}(\mathbf{x}_{i,j}) + h_y^2 \frac{\partial^4 u}{\partial y^4}(\mathbf{x}_{i,j})\right) + O(h_x^2 + h_y^2). \tag{4}$$

Similarly,

$$u(\mathbf{x}_{i+1,j+1}) - 2u(\mathbf{x}_{i,j+1}) + u(\mathbf{x}_{i-1,j+1}) = h_x^2 \frac{\partial^2 u}{\partial x^2}(\mathbf{x}_{i,j+1}) + \frac{h_x^4}{12}\frac{\partial^4 u}{\partial x^4}(\mathbf{x}_{i,j+1}) + O(h_x^4),$$

$$-2u(\mathbf{x}_{i+1,j}) + 4u(\mathbf{x}_{i,j}) - 2u(\mathbf{x}_{i-1,j}) = -2h_x^2 \frac{\partial^2 u}{\partial x^2}(\mathbf{x}_{i,j}) - \frac{h_x^4}{6}\frac{\partial^4 u}{\partial x^4}(\mathbf{x}_{i,j}) + O(h_x^4),$$

$$u(\mathbf{x}_{i+1,j-1}) - 2u(\mathbf{x}_{i,j-1}) + u(\mathbf{x}_{i-1,j-1}) = h_x^2 \frac{\partial^2 u}{\partial x^2}(\mathbf{x}_{i,j-1}) + \frac{h_x^4}{12}\frac{\partial^4 u}{\partial x^4}(\mathbf{x}_{i,j-1}) + O(h_x^4).$$

By Tylor development, we have

$$\frac{\partial^2 u}{\partial x^2}(\mathbf{x}_{i,j+1}) = \frac{\partial^2 u}{\partial x^2}(\mathbf{x}_{i,j}) + h_y \frac{\partial^3 u}{\partial y \partial x^2}(\mathbf{x}_{i,j}) + \frac{h_y^2}{2}\frac{\partial^4 u}{\partial y^2 \partial x^2}(\mathbf{x}_{i,j}) + \frac{h_y^3}{3!}\frac{\partial^5 u}{\partial y^3 \partial x^2}(\mathbf{x}_{i,j}) + \frac{h_y^4}{4!}\frac{\partial^6 u}{\partial y^4 \partial x^2}(\mathbf{x}_{i,j}) + O(h_y^4),$$

$$\frac{\partial^2 u}{\partial x^2}(\mathbf{x}_{i,j-1}) = \frac{\partial^2 u}{\partial x^2}(\mathbf{x}_{i,j}) - h_y \frac{\partial^3 u}{\partial y \partial x^2}(\mathbf{x}_{i,j}) + \frac{h_y^2}{2}\frac{\partial^4 u}{\partial y^2 \partial x^2}(\mathbf{x}_{i,j}) - \frac{h_y^3}{3!}\frac{\partial^5 u}{\partial y^3 \partial x^2}(\mathbf{x}_{i,j}) + \frac{h_y^4}{4!}\frac{\partial^6 u}{\partial y^4 \partial x^2}(\mathbf{x}_{i,j}) + O(h_y^4),$$

$$\frac{\partial^4 u}{\partial x^4}(\mathbf{x}_{i,j+1}) = \frac{\partial^4 u}{\partial x^4}(\mathbf{x}_{i,j}) + h_y \frac{\partial^5 u}{\partial y \partial x^4}(\mathbf{x}_{i,j}) + \frac{h_y^2}{2}\frac{\partial^6 u}{\partial y^2 \partial x^4}(\mathbf{x}_{i,j}) + O(h_y^2),$$

$$\frac{\partial^4 u}{\partial x^4}(\mathbf{x}_{i,j-1}) = \frac{\partial^4 u}{\partial x^4}(\mathbf{x}_{i,j}) - h_y \frac{\partial^5 u}{\partial y \partial x^4}(\mathbf{x}_{i,j}) + \frac{h_y^2}{2}\frac{\partial^6 u}{\partial y^2 \partial x^4}(\mathbf{x}_{i,j}) + O(h_y^2).$$

Then

$$\begin{aligned} &u(\mathbf{x}_{i+1,j+1}) - 2u(\mathbf{x}_{i,j+1}) + u(\mathbf{x}_{i-1,j+1}) - 2u(\mathbf{x}_{i+1,j}) + 4u(\mathbf{x}_{i,j}) - 2u(\mathbf{x}_{i-1,j}) + \\ &\qquad u(\mathbf{x}_{i+1,j-1}) - 2u(\mathbf{x}_{i,j-1}) + u(\mathbf{x}_{i-1,j-1}) \\ &= h_x^2 h_y^2 \frac{\partial^4 u}{\partial y^2 \partial x^2}(\mathbf{x}_{i,j}) + \frac{1}{12} h_x^2 h_y^4 \frac{\partial^6 u}{\partial y^4 \partial x^2}(\mathbf{x}_{i,j}) + \frac{1}{12} h_x^4 h_y^2 \frac{\partial^6 u}{\partial y^2 \partial x^2}(\mathbf{x}_{i,j}) + O(h_x^2 h_y^4) + O(h_x^4 h_y^2). \end{aligned} \tag{5}$$

Combining (4) with (5), we have

$$\bar{L}_h[u(\mathbf{x}_{i,j})] = -\frac{\partial^2 u}{\partial x^2}(\mathbf{x}_{i,j}) - \frac{\partial^2 u}{\partial y^2}(\mathbf{x}_{i,j}) - \frac{1}{12}\left(h_x^2 \frac{\partial^2}{\partial x^2} + h_y^2 \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)(\mathbf{x}_{i,j}) + O((h_x^2 + h_y^2)^2).$$

It remains to determine $\bar{f}(\mathbf{x}_{i,j})$. We note that $f(\mathbf{x}_{i,j}) = -[\Delta u](\mathbf{x}_{i,j})$, and

$$\frac{1}{12}\left(h_x^2 \frac{\partial^2 f}{\partial x^2} + h_y^2 \frac{\partial^2 f}{\partial y^2}\right)(\mathbf{x}_{i,j}) = -\frac{1}{12}\left(h_x^2 \frac{\partial^2}{\partial x^2} + h_y^2 \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)(\mathbf{x}_{i,j}),$$

which leads to the desired result: $\bar{R}_{i,j} = O(h_x^4 + h_y^4)$ if $h_x = O(h_y)$. $\square$

Appendix: Notations for Discrete Representation

Let $I = [a, b]$. We define the discrete grid points as

$$a = x_0 < x_1 < \cdots < x_N = b.$$

We introduce the following sets:

$$I_h = \{x_1, \cdots, x_{N-1}\},\ \bar{I}_h = \{x_0, x_1, \cdots, x_N\},\ I_h^+ = \{x_1, \cdots, x_N\}.$$

The grid spacing is defined as

$$h_i = x_i - x_{i-1}, \quad i = 1, \cdots, N.$$

Additionally, we define the averaged grid spacing:

$$\begin{aligned} \bar{h}_i &= \frac{1}{2}(h_i + h_{i+1}),\ i = 1, \cdots, N-1, \\ \bar{h}_0 &= \frac{1}{2}h_1, \quad \bar{h}_N = \frac{1}{2}h_N. \end{aligned}$$

A discrete function defined on $\bar{I}_h$ is denoted as

$$v_h = [v_0, v_1, \cdots, v_N]^{\mathrm{T}}.$$

We define the following difference operators:

$$\begin{aligned} (v_i)_{\bar{x}} &:= v_{i,\bar{x}} := \frac{v_i - v_{i-1}}{h_i},\ i = 1, \cdots, N, \\ (v_i)_x &:= v_{i,x} := \frac{v_{i+1} - v_i}{h_{i+1}},\ i = 0, \cdots, N-1, \\ (v_i)_{\hat{x}} &:= v_{i,\hat{x}} := \frac{v_{i+1} - v_i}{\bar{h}_i},\ i = 0, \cdots, N-1. \end{aligned}$$

The discrete inner products are given by

$$(u_h, v_h)_{I_h} = \sum_{i=1}^{N-1} u_i v_i \bar{h}_i,\ (u_h, v_h)_{\bar{I}_h} = \sum_{i=0}^{N} u_i v_i \bar{h}_i,\ (u_h, v_h)_{I_h^+} = \sum_{i=1}^{N} u_i v_i h_i. \tag{6}$$

We define the discrete norms as follows:

$$\begin{aligned} &\|v_h\|_c := \max_{\bar{I}_h} |v_i|,\ \|v_h\|_0 := (v_h, v_h)_{\bar{I}_h}^{1/2}, \\ &|v_h|_1 := ((v_h)_{\bar{x}}, (v_h)_{\bar{x}})_{I_h^+}^{1/2},\ \|v_h\|_1^2 = \|v_h\|_0^2 + |v_h|_1^2. \end{aligned} \tag{7}$$

The discrete integral by parts:

$$\sum_{i=m+1}^{n} v_i (w_i)_{\bar{x}} h_i = -\sum_{i=m}^{n-1} (v_i)_x w_i h_{i+1} + v_n w_n - v_m w_m,\ \text{for some } 0 \leqslant m < n \leqslant N. \tag{8}$$

The discrete Green formula:

$$\sum_{i=m+1}^{n-1} ((u_i)_{\bar{x}})_{\hat{x}}\, v_i \bar{h}_i = -\sum_{i=m+1}^{n} (u_i)_{\bar{x}} (v_i)_{\bar{x}} h_i + (u_n)_{\bar{x}} v_n - (u_m)_x v_m,\ \text{for some } 0 \leqslant m < n \leqslant N. \tag{9}$$

The discrete Cauchy-Schwarz inequality states that

$$|(u_h, v_h)_{\bar{I}_h}| \leqslant (u_h, u_h)_{\bar{I}_h}^{1/2} (v_h, v_h)_{\bar{I}_h}^{1/2}. \tag{10}$$

If $v_0 = 0$ (or $v_N = 0$ or $v_0 = v_N = 0$), the discrete Poincaré inequality holds:

$$\|v_h\|_c \leqslant C|v_h|_1, \quad \|v_h\|_0 \leqslant C|v_h|_1, \tag{11}$$

where C is a constant depending only on a and b.