Exercise 2.4. *We consider the problem*

$$\begin{cases} -(\alpha u')'(x) + (\beta u')(x) + (\gamma u)(x) = f(x), & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{11}$$

where α, β, and γ are continuous functions on $[0,1]$ with $\alpha(x) \geqslant \alpha_0 > 0$ for all $x \in [0,1]$.
1) Give the weak form of the problem (11).
2) Prove the weak problem admits a unique solution under the following assumption

a. $\beta(x) = 0$, $\gamma \geqslant 0$ for all $x \in [0,1]$;

b. $-\frac{1}{2}\beta' + \gamma \geqslant 0$ for all $x \in [0,1]$;

c. see [Brezis p. 224].

3) Propose a $P1-$FEM for the numerical solution of (11).
4) Carry out an error analysis.

Proof.
1). Let $I = (0,1)$ and $V = H_0^1(I)$. The weak form reads

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ a(u,v) = \mathcal{F}(v), \ \forall v \in V, \end{cases}$$

where the bilinear form $a(u,v) = (\alpha u', v') + (\beta u', v') + (\gamma u, v)$ and $\mathcal{F}(v) = (f, v)$.
2). It is clear that $a(\cdot,\cdot)$ is a bilinear form, and $\mathcal{F}$ is a continuous functional from V to $\mathbb{R}$. By Lax-Milgram Lemma, it remains to show that $a(\cdot,\cdot)$ is continuous and coercive. Continuity is clear for all cases since the coefficients are continuous on $\bar{I}$, i.e.,

$$\begin{aligned} |a(u,v)| &\leqslant \|\alpha\|_\infty \|u'\|_0 \|v'\|_0 + \|\beta\|_\infty \|u'\|_0 \|v\|_0 + \|\gamma\|_\infty \|u\|_0 \|v\|_0 \\ &\leqslant (\|\alpha\|_\infty + \|\beta\|_\infty + \|\gamma\|_\infty) \|u\|_1 \|v\|_1, \quad \forall u, v \in V. \end{aligned}$$

For the coercivity, we consider case by case:

a. By Poincaré inequality on $H_0^1(I)$, there exists $C > 0$ such that

$$a(v,v) = (\alpha v', v') + (\gamma v, v) \geqslant \alpha_0 \|v'\|_0^2 \geqslant C\|v\|_1^2, \quad \forall v \in V.$$

b. Note that for any $v \in V$, we have $(\beta v', v) = (\beta/2, (v^2)') = (-\beta'/2, v^2)$. Thus by Poincaré inequality on $H_0^1(I)$, we have

$$\begin{aligned} a(v,v) &= (\alpha v', v') + (\beta v', v) + (\gamma v, v) = (\alpha v', v') + ((-\beta'/2 + \gamma)v, v) \\ &\geqslant \alpha_0 \|v'\|_0^2 \geqslant C\|v\|_1^2, \quad \forall v \in V. \end{aligned}$$

c. Coming soon...

3). Obvious.
4). Let u_h be the solution of $P1-$FEM. Note that $a(u - u_h, v_h) = 0$, $\forall v_h \in V_h$. Thus

$$\|u - u_h\|_1^2 \leqslant Ca(u - u_h, u - u_h) = Ca(u - u_h, u - v_h) \leqslant C\|u - u_h\|_1 \|u - v_h\|_1,$$

for any $v_h \in V_h$. It leads to $\|u - u_h\|_1 \leqslant \inf_{v_h \in V_h} \|u - v_h\|_1 \leqslant \|u - u_I\|_1$, where u_I denotes the interpolation of u into V_h. By the Poincaré inequality, we know that $C\|v\|_1 \leqslant \|v'\|_0 \leqslant C\|v\|_1$, then

$$\|u' - u_h'\|_0 \leqslant \|u' - u_I'\|_0 \leqslant Ch\|u''\|_0.$$

To obtain error estimate for $\|u-u_h\|_0$, we use duality argument. Given $r \in L^2(I)$,

$$\begin{cases} \text{Find } \varphi(r) \in V \text{ such that} \\ a(v, \varphi(r)) = (r, v), \quad \forall v \in V. \end{cases}$$

The dual problem admits a unique solution $\varphi(r)$ since $a(\cdot,\cdot)$ is continuous and coercive. If we suppose $\varphi(r) \in H^2(I)$, and there exists constant $C > 0$ such that $\|\varphi''(r)\| \leqslant C\|r\|_0$.

Thus we denote $\varphi_I(r)$ being the interpolation of $\varphi(r)$ into V_h and obtain

$$\begin{aligned} \|u-u_h\|_0 &= \sup_{r\in L^2(I),\ r\neq 0} \frac{(r, u-u_h)}{\|r\|_0} = \sup_{r\in L^2(I),\ r\neq 0} \frac{a(u-u_h, \varphi(r))}{\|r\|_0} \\ &= \sup_{r\in L^2(I),\ r\neq 0} \frac{a(u-u_h, \varphi(r)-\varphi_I(r))}{\|r\|_0} \\ &\leqslant \sup_{r\in L^2(I),\ r\neq 0} \frac{\|u-u_h\|_1\|\varphi(r)-\varphi_I(r)\|_1}{\|r\|_0} \\ &\leqslant \sup_{r\in L^2(I),\ r\neq 0} \frac{\|u'-u_h'\|_0\|\varphi'(r)-\varphi_I'(r)\|_0}{\|r\|_0} \\ &\leqslant Ch\|u'-u_h'\|_0 \sup_{r\in L^2(I),\ r\neq 0} \frac{\|\varphi''(r)\|_0}{\|r\|_0} \\ &\leqslant Ch\|u'-u_h'\|_0. \end{aligned}$$

□