PREFACE

This monograph presents an introductory study of of the properties of certain Banach spaces of weakly differentiable functions of several real variables that arise in connection with numerous problems in the theory of partial differential equations, approximation theory, and many other areas of pure and applied mathematics. These spaces have become associated with the name of the late Russian mathematician S. L. Sobolev, although their origins predate his major contributions to their development in the late 1930s.

Even by 1975 when the first edition of this monograph was published, there was a great deal of material on these spaces and their close relatives, though most of it was available only in research papers published in a wide variety of journals. The monograph was written to fill a perceived need for a single source where graduate students and researchers in a wide variety of disciplines could learn the essential features of Sobolev spaces that they needed for their particular applications. No attempt was made even at that time for complete coverage. To quote from the Preface of the first edition:

> The existing mathematical literature on Sobolev spaces and their generalizations is vast, and it would be neither easy nor particularly desirable to include everything that was known about such spaces between the covers of one book. An attempt has been made in this monograph to present all the core material in sufficient generality to cover most applications, to give the reader an overview of the subject that is difficult to obtain by reading research papers, and finally ... to provide a ready reference for someone requiring a result about Sobolev spaces for use in some application.

This remains as the purpose and focus of this second edition. During the intervening twenty-seven years the research literature has grown exponentially, and there are now several other books in English that deal in whole or in part with Sobolev spaces. (For example, see [Ad2], [Bu1], [Mz1], [Tr1], [Tr3], and [Tr4].) However, there is still a need for students in other disciplines than mathematics, and in other areas of mathematics than just analysis to have available a book that describes these spaces and their core properties based only a background in mathematical analysis at the senior undergraduate level. We have tried to make this such a book.

The organization of this book is similar but not identical to that of the first edition:

Chapter 1 remains a potpourri of standard topics from real and functional analysis, included, mainly without proofs, because they provide a necessary background for what follows.

Chapter 2 on the Lebesgue Spaces $L^p(\Omega)$ is much expanded and reworked from the previous edition. It provides, in addition to standard results about these spaces, a brief treatment of mixed-norm L^p spaces, weak-L^p spaces, and the Marcinkiewicz interpolation theorem, all of which will be used in a new treatment of the Sobolev Imbedding Theorem in Chapter 4. For the most part, complete proofs are given, as they are for much of the rest of the book.

Chapter 3 provides the basic definitions and properties of the Sobolev spaces $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$. There are minor changes from the first edition.

Chapter 4 is now completely concerned with the imbedding properties of Sobolev Spaces. The first half gives a more streamlined presentation and proof of the various imbeddings of Sobolev spaces into L^p spaces, including traces on subspaces of lower dimension, and spaces of continuous and uniformly continuous functions. Because the approach to the Sobolev Imbedding Theorem has changed, the roles of Chapters 4 and 5 have switched from the first edition. The latter part of Chapter 4 deals with situations where the regularity conditions on the domain Ω that are necessary for the full Sobolev Imbedding Theorem do not apply, but some weaker imbedding results are still possible.

Chapter 5 now deals with interpolation, extension, and approximation results for Sobolev spaces. Part of it is expanded from material in Chapter 4 of the first edition with newer results and methods of proof.

Chapter 6 deals with establishing compactness of Sobolev imbeddings. It is only slightly changed from the first edition.

Chapter 7 is concerned with defining and developing properties of scales of spaces with fractional orders of smoothness, rather than the integer orders of the Sobolev spaces themselves. It is completely rewritten and bears little resemblance to the corresponding chapter in the first edition. Much emphasis is placed on real interpolation methods. The J-method and K-method are fully presented and used to develop the theory of Lorentz spaces and Besov spaces and their imbeddings, but both families of spaces are also provided with intrinsic characterizations. A key theorem identifies lower dimensional traces of functions in Sobolev spaces as constituting certain Besov spaces. Complex interpolation is used to introduce Sobolev spaces of fractional order (also called spaces of Bessel potentials) and Fourier transform methods are used to characterize and generalize these spaces to yield the Triebel Lizorkin spaces and illuminate their relationship with the Besov spaces.

Chapter 8 is very similar to its first edition counterpart. It deals with Orlicz and Orlicz-Sobolev spaces which generalize L^p and $W^{m,p}$ spaces by allowing the role of the function t^p to be assumed by a more general convex function $A(t)$. An important result identifies a certain Orlicz space as a target for an imbedding of $W^{m,p}(\Omega)$ in a limiting case where there is an imbedding into $L^p(\Omega)$ for $1 \le p < \infty$ but not into $L^\infty(\Omega)$.

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RAA & JJFF
Vancouver, August 2002

List of Spaces and Norms

Space	Norm	Paragraph
$B^{s;p,q}(\Omega)$	$\left\|\cdot\,; B^{s;p,q}(\Omega)\right\|$	7.32
$B^{s;p,q}(\mathbb{R}^n)$	$\left\|\cdot\,; B^{s;p,q}(\mathbb{R}^n)\right\|$	7.67
$\dot{B}^{s;p,q}(\mathbb{R}^n)$		7.68
$C^m(\Omega),\ C^\infty(\Omega)$		1.26
$C_0(\Omega),\ C_0^\infty(\Omega)$		1.26
$C^m(\overline{\Omega})$	$\left\|\cdot\,; C^m(\overline{\Omega})\right\|$	1.28
$C^{m,\lambda}(\overline{\Omega})$	$\left\|\cdot\,; C^{m,\lambda}(\overline{\Omega})\right\|$	1.29
$C_B^m(\Omega)$	$\left\|\cdot\,; C_B^m(\Omega)\right\|$	1.27, 4.2
$C^j(\overline{\Omega})$	$\left\|\cdot\,; C^j(\overline{\Omega})\right\|$	4.2
$C^{j,\lambda}(\overline{\Omega})$	$\left\|\cdot\,; C^{j,\lambda}(\overline{\Omega})\right\|$	4.2
$C^{j,\lambda,q}(\overline{\Omega})$	$\left\|\cdot\,; C^{j,\lambda,q}(\overline{\Omega})\right\|$	7.35
$\mathscr{D}(\Omega)$		1.56
$\mathscr{D}'(\Omega)$		1.57
$E_A(\Omega)$	$\|\cdot\|_A = \|\cdot\|_{A,\Omega}$	8.14

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PRELIMINARIES

1.1 (Introduction) Sobolev spaces are vector spaces whose elements are functions defined on domains in n-dimensional Euclidean space $\mathbb{R}^n$ and whose partial derivatives satisfy certain integrability conditions. In order to develop and elucidate the properties of these spaces and mappings between them we require some of the machinery of general topology and real and functional analysis. We assume that readers are familiar with the concept of a vector space over the real or complex scalar field, and with the related notions of dimension, subspace, linear transformation, and convex set. We also expect the reader will have some familiarity with the concept of topology on a set, at least to the extent of understanding the concepts of an open set and continuity of a function.

In this chapter we outline, mainly without any proofs, those aspects of the theories of topological vector spaces, continuity, the Lebesgue measure and integral, and Schwartz distributions that will be needed in the rest of the book. For a reader familiar with the basics of these subjects, a superficial reading to settle notations and review the main results will likely suffice.

Notation

1.2 Throughout this monograph the term *domain* and the symbol Ω will be reserved for a nonempty open set in n-dimensional real Euclidean space $\mathbb{R}^n$. We shall be concerned with the differentiability and integrability of functions defined on Ω; these functions are allowed to be complex-valued unless the contrary is explicitly stated. The complex field is denoted by $\mathbb{C}$. For $c \in \mathbb{C}$ and two functions u and v, the scalar multiple cu, the sum $u+v$, and the product uv are always defined pointwise:

$$(cu)(x) = cu(x),$$
$$(u+v)(x) = u(x) + v(x),$$
$$(uv)(x) = u(x)v(x)$$

at all points x where the right sides make sense.

A typical point in $\mathbb{R}^n$ is denoted by $x = (x_1, \ldots, x_n)$; its norm is given by $|x| = (\sum_{j=1}^n x_j^2)^{1/2}$. The inner product of two points x and y in $\mathbb{R}^n$ is $x \cdot y = \sum_{j=1}^n x_j y_j$.

If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of nonnegative integers α_j, we call α a *multi-index* and denote by x^α the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, which has degree $|\alpha| = \sum_{j=1}^n \alpha_j$. Similarly, if $D_j = \partial/\partial x_j$, then

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$$

denotes a differential operator of order $|\alpha|$. Note that $D^{(0,\ldots,0)}u = u$.

If α and β are two multi-indices, we say that $\beta \le \alpha$ provided $\beta_j \le \alpha_j$ for $1 \le j \le n$. In this case $\alpha - \beta$ is also a multi-index, and $|\alpha - \beta| + |\beta| = |\alpha|$. We also denote

$$\alpha! = \alpha_1! \cdots \alpha_n!$$

and if $\beta \le \alpha$,

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}.$$

The reader may wish to verify the Leibniz formula

$$D^\alpha(uv)(x) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta u(x) D^{\alpha-\beta} v(x)$$

valid for functions u and v that are $|\alpha|$ times continuously differentiable near x.

1.3 If $G \subset \mathbb{R}^n$ is nonempty, we denote by $\overline{G}$ the closure of G in $\mathbb{R}^n$. We shall write $G \Subset \Omega$ if $\overline{G} \subset \Omega$ and $\overline{G}$ is a compact (that is, closed and bounded) subset of $\mathbb{R}^n$. If u is a function defined on G, we define the *support* of u to be the set

$$\operatorname{supp}(u) = \overline{\{x \in G : u(x) \neq 0\}}.$$

We say that u has *compact support* in Ω if $\operatorname{supp}(u) \Subset \Omega$. We denote by "bdry G" the boundary of G in $\mathbb{R}^n$, that is, the set $\overline{G} \cap \overline{G^c}$, where G^c is the complement of G in $\mathbb{R}^n$; $G^c = \mathbb{R}^n - G = \{x \in \mathbb{R}^n : x \notin G\}$.

If $x \in \mathbb{R}^n$ and $G \subset \mathbb{R}^n$, we denote by "dist(x, G)" the distance from x to G, that is, the number $\inf_{y \in G} |x - y|$. Similarly, if $F,\ G \subset \mathbb{R}^n$ are both nonempty,

$$\operatorname{dist}(F, G) = \inf_{y \in F} \operatorname{dist}(y, G) = \inf_{\substack{x \in G \\ y \in F}} |y - x|.$$

Topological Vector Spaces

1.4 (Topological Spaces) If X is any set, a *topology* on X is a collection $\mathcal{O}$ of subsets of X which contains

(i) the whole set X and the empty set $\emptyset$,

(ii) the union of any collection of its elements, and

(iii) the intersection of any finite collection of its elements.

The pair $(X, \mathcal{O})$ is called a *topological space* and the elements of $\mathcal{O}$ are the *open sets* of that space. An open set containing a point x in X is called a *neighbourhood* of x. The complement $X - U = \{x \in X : x \notin U\}$ of any open set U is called a *closed* set. The closure $\overline{S}$ of any subset $S \subset X$ is the smallest closed subset of X that contains S.

Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two topologies on the same set X. If $\mathcal{O}_1 \subset \mathcal{O}_2$, we say that $\mathcal{O}_2$ is *stronger* than $\mathcal{O}_1$, or that $\mathcal{O}_1$ is *weaker* than $\mathcal{O}_2$.

A topological space $(X, \mathcal{O})$ is called a *Hausdorff space* if every pair of distinct points x and y in X have disjoint neighbourhoods.

The *topological product* of two topological spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is the topological space $(X \times Y, \mathcal{O})$, where $X \times Y = \{(x, y) : x \in X,\ y \in Y\}$ is the Cartesian product of the sets X and Y, and $\mathcal{O}$ consists of arbitrary unions of sets of the form $\{O_X \times O_Y : O_X \in \mathcal{O}_X,\ O_Y \in \mathcal{O}_Y\}$.

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two topological spaces. A function f from X into Y is said to be *continuous* if the preimage $f^{-1}(O) = \{x \in X : f(x) \in O\}$ belongs to $\mathcal{O}_X$ for every $O \in \mathcal{O}_Y$. Evidently the stronger the topology on X or the weaker the topology on Y, the more such continuous functions f there will be.

1.5 (Topological Vector Spaces) We assume throughout this monograph that all vectors spaces referred to are taken over the complex field unless the contrary is explicitly stated.

A *topological vector space*, hereafter abbreviated TVS, is a Hausdorff topological space that is also a vector space for which the vector space operations of addition and scalar multiplication are continuous. That is, if X is a TVS, then the mappings

$$(x, y) \to x + y \qquad \text{and} \qquad (c, x) \to cx$$

from the topological product spaces $X \times X$ and $\mathbb{C} \times X$, respectively, into X are continuous. (Here $\mathbb{C}$ has its usual topology induced by the Euclidean metric.)

X is a *locally convex* TVS if each neighbourhood of the origin in X contains a convex neighbourhood of the origin.

We outline below those aspects of the theory of topological and normed vector spaces that play a significant role in the study of Sobolev spaces. For a more thorough discussion of these topics the reader is referred to standard textbooks on functional analysis, for example [Ru1] or [Y].

1.6 (Functionals) A scalar-valued function defined on a vector space X is called a *functional*. The functional f is linear provided

$$f(ax + by) = af(x) + bf(y), \qquad x, y \in X, \quad a, b \in \mathbb{C}.$$

If X is a TVS, a functional on X is continuous if it is continuous from X into $\mathbb{C}$ where $\mathbb{C}$ has its usual topology induced by the Euclidean metric.

The set of all continuous, linear functionals on a TVS X is called the *dual* of X and is denoted by X'. Under pointwise addition and scalar multiplication X' is itself a vector space:

$$(f + g)(x) = f(x) + g(x), \quad (cf)(x) = cf(x), \qquad f, g \in X', \ x \in X, \ c \in \mathbb{C}.$$

X' will be a TVS provided a suitable topology is specified for it. One such topology is the *weak-star topology*, the weakest topology with respect to which the functional F_x, defined on X' by $F_x(f) = f(x)$ for each $f \in X'$, is continuous for each $x \in X$. This topology is used, for instance, in the space of Schwartz distributions introduced in Paragraph 1.57. The dual of a normed vector space can be given a stronger topology with respect to which it is itself a normed space. (See Paragraph 1.11.)

Normed Spaces

1.7 (Norms) A *norm* on a vector space X is a real-valued function f on X satisfying the following conditions:

(i) $f(x) \geq 0$ for all $x \in X$ and $f(x) = 0$ if and only if $x = 0$,

(ii) $f(cx) = |c| f(x)$ for every $x \in X$ and $c \in \mathbb{C}$,

(iii) $f(x + y) \leq f(x) + f(y)$ for every $x, y \in X$.

A *normed space* is a vector space X provided with a norm. The norm will be denoted $\|\cdot\,; X\|$ except where other notations are introduced.

If $r > 0$, the set

$$B_r(x) = \{y \in X : \|y - x\,; X\| < r\}$$

is called the *open ball* of radius r with center at $x \in X$. Any subset $A \subset X$ is called *open* if for every $x \in A$ there exists $r > 0$ such that $B_r(x) \subset A$. The open sets thus defined constitute a topology for X with respect to which X is a TVS. This topology is the *norm topology* on X. The closure of $B_r(x)$ in this topology is

$$\overline{B_r(x)} = \{y \in X \ : \ \|y - x\,; X\| \le r\}.$$

A TVS X is *normable* if its topology coincides with the topology induced by some norm on X. Two different norms on a vector space X are equivalent if they induce the same topology on X. This is the case if and only if there exist two positive constants a and b such that,

$$a\,\|x\|_1 \le \|x\|_2 \le b\,\|x\|_1$$

for all $x \in X$, where $\|x\|_1$ and $\|x\|_2$ are the two norms.

Let X and Y be two normed spaces. If there exists a one-to-one linear operator L mapping X onto Y having the property $\|L(x)\,; Y\| = \|x\,; X\|$ for every $x \in X$, then we call L an *isometric isomorphism* between X and Y, and we say that X and Y are *isometrically isomorphic*. Such spaces are often identified since they have identical structures and only differ in the nature of their elements.

1.8 A sequence $\{x_n\}$ in a normed space X is *convergent* to the limit x_0 if and only if $\lim_{n\to\infty} \|x_n - x_0\,; X\| = 0$ in $\mathbb{R}$. The norm topology of X is completely determined by the sequences it renders convergent.

A subset S of a normed space X is said to be *dense* in X if each $x \in X$ is the limit of a sequence of elements of S. The normed space X is called *separable* if it has a countable dense subset.

1.9 (Banach Spaces) A sequence $\{x_n\}$ in a normed space X is called a *Cauchy sequence* if and only if for every $\epsilon > 0$ there exists an integer N such that $\|x_m - x_n\,; X\| < \epsilon$ holds whenever $m, n > N$. We say that X is *complete* and a *Banach space* if every Cauchy sequence in X converges to a limit in X. Every normed space X is either a Banach space or a dense subset of a Banach space Y called the *completion* of X whose norm satisfies

$$\|x\,; Y\| = \|x\,; X\| \qquad \text{for every } x \in X.$$

1.10 (Inner Product Spaces and Hilbert Spaces) If X is a vector space, a functional $(\cdot\,,\,\cdot)_X$ defined on $X \times X$ is called an *inner product* on X provided that for every $x, y \in X$ and $a, b \in \mathbb{C}$

(i) $(x, y)_X = \overline{(y, x)}_X$, (where $\bar{c}$ denotes the complex conjugate of $c \in \mathbb{C}$)

(ii) $(ax + by, z)_X = a(x, z)_X + b(y, z)_X$,

(iii) $(x,x)_X = 0$ if and only if $x = 0$,

Equipped with such a functional, X is called an *inner product space,* and the functional

$$\|x\,;X\| = \sqrt{(x,x)_X} \tag{1}$$

is, in fact, a norm on X If X is complete (i.e. a Banach space) under this norm, it is called a *Hilbert space*. Whenever the norm on a vector space X is obtained from an inner product via (1), it satisfies the *parallelogram law*

$$\|x+y\,;X\|^2 + \|x-y\,;X\|^2 = 2\,\|x\,;X\|^2 + 2\,\|y\,;X\|^2 . \tag{2}$$

Conversely, if the norm on X satisfies (2) then it comes from an inner product as in (1).

1.11 (The Normed Dual) A norm on the dual X' of a normed space X can be defined by setting

$$\left\|x'\,;X'\right\| = \sup\left\{|x'(x)| \;:\; \|x\,;X\| \le 1\right\},$$

for each $x' \in X'$. Since $\mathbb{C}$ is complete, with the topology induced by this norm X' is a Banach space (whether or not X is) and it is called the *normed dual* of X. If X is infinite dimensional, the norm topology of X' is stronger (has more open sets) than the weak-star topology defined in Paragraph 1.6.

The following theorem shows that if X is a Hilbert space, it can be identified with its normed dual.

1.12 THEOREM (The Riesz Representation Theorem) Let X be a Hilbert space. A linear functional x' on X belongs to X' if and only if there exists $x \in X$ such that for every $y \in X$ we have

$$x'(y) = (y,x)_X,$$

and in this case $\left\|x'\,;X'\right\| = \|x\,;X\|$. Moreover, x is uniquely determined by $x' \in X'$. ∎

A vector subspace M of a normed space X is itself a normed space under the norm of X, and so normed is called a *subspace* of X. A closed subspace of a Banach space is itself a Banach space.

1.13 THEOREM (The Hahn-Banach Extension Theorem) Let M be a subspace of the normed space X. If $m' \in M'$, then there exists $x' \in X'$ such that $\left\|x'\,;X'\right\| = \left\|m'\,;M'\right\|$ and $x'(m) = m'(m)$ for every $m \in M$. ∎

1.14 (Reflexive Spaces) A natural linear injection of a normed space X into its second dual space $X'' = (X')'$ is provided by the mapping J whose value Jx at $x \in X$ is given by

$$Jx(x') = x'(x), \qquad x' \in X'.$$

Since $|Jx(x')| \le \|x' ; X'\| \, \|x ; X\|$, we have

$$\|Jx ; X''\| \le \|x ; X\| .$$

However, the Hahn-Banach Extension Theorem assures us that for any $x \in X$ we can find $x' \in X'$ such that $\|x' ; X'\| = 1$ and $x'(x) = \|x ; X\|$. Therefore J is an isometric isomorphism of X into X''.

If the range of the isomorphism J is the entire space X'', we say that the normed space X is *reflexive*. A reflexive space must be complete, and hence a Banach space.

1.15 THEOREM Let X be a normed space. X is reflexive if and only if X' is reflexive. X is separable if X' is separable. Hence if X is separable and reflexive, so is X'. ∎

1.16 (Weak Topologies and Weak Convergence) The *weak topology* on a normed space X is the weakest topology on X that still renders continuous each x' in the normed dual X' of X. Unless X is finite dimensional, the weak topology is weaker than the norm topology on X. It is a consequence of the Hahn-Banach Theorem that a closed, convex set in a normed space is also closed in the weak topology of that space.

A sequence convergent with respect to the weak topology on X is said to *converge weakly*. Thus x_n converges weakly to x in X provided $x'(x_n) \to x'(x)$ in $\mathbb{C}$ for every $x' \in X'$. We denote norm convergence of a sequence $\{x_n\}$ to x in X by $x_n \to x$, and we denote weak convergence by $x_n \rightharpoonup x$. Since we have $|x'(x_n - x)| \le \|x' ; X'\| \, \|x_n - x ; X\|$, we see that $x_n \to x$ implies $x_n \rightharpoonup x$. The converse is generally not true (unless X is finite dimensional).

1.17 (Compact Sets) A subset A of a normed space X is called *compact* if every sequence of points in A has a subsequence converging in X to an element of A. (This definition is equivalent in normed spaces to the definition of compactness in a general topological space; A is compact if whenever A is a subset of the union of a collection of open sets, it is a subset of the union of a finite subcollection of those sets.) Compact sets are closed and bounded, but closed and bounded sets need not be compact unless X is finite dimensional. A is called *precompact* in X if its closure $\overline{A}$ in the norm topology of X is compact. A is called *weakly sequentially compact* if every sequence in A has a subsequence converging weakly in X to a point in A. The reflexivity of a Banach space can be characterized in terms of this property.

1.18 THEOREM A Banach space is reflexive if and only if its closed unit ball $\overline{B_1(0)} = \{x \in X : \|x ; X\| \le 1\}$ is weakly sequentially compact. ∎

1.19 THEOREM A set A is precompact in a Banach space X if and only if for every positive number ϵ there is a finite subset N_ϵ of points of X such that

$$A \subset \bigcup_{y \in N_\epsilon} B_\epsilon(y).$$

A set N_ϵ with this property is called a *finite ϵ-net* for A. ∎

1.20 (Uniform Convexity) Any normed space is locally convex with respect to its norm topology. The norm on X is called *uniformly convex* if for every number ϵ satisfying $0 < \epsilon \leq 2$, there exists a number $\delta(\epsilon) > 0$ such that if $x, y \in X$ satisfy $\|x\,; X\| = \|y\,; X\| = 1$ and $\|x - y\,; X\| \geq \epsilon$, then $\|(x+y)/2\,; X\| \leq 1 - \delta(\epsilon)$. The normed space X itself is called "uniformly convex" in this case. It should be noted, however, that uniform convexity is a property of the norm—X may have another equivalent norm that is not uniformly convex. Any normable space is called *uniformly convex* if it possesses a uniformly convex norm. The parallelogram law (2) shows that a Hilbert space is uniformly convex.

1.21 THEOREM A uniformly convex Banach space is reflexive. ∎

The following two theorems will be used to establish the separability, reflexivity, and uniform convexity of the Sobolev spaces introduced in Chapter 3.

1.22 THEOREM Let X be a Banach space and M a subspace of X closed with respect to the norm topology of X. Then M is also a Banach space under the norm inherited from X. Furthermore

(i) M is separable if X is separable,

(ii) M is reflexive if X is reflexive,

(iii) M is uniformly convex if X is uniformly convex. ∎

The completeness, separability, and uniform convexity of M follow easily from the corresponding properties of X. The reflexivity of M is a consequence of Theorem 1.18 and the fact that M, being closed and convex, is closed in the weak topology of X.

1.23 THEOREM For $j = 1, 2, \ldots, n$ let X_j be a Banach space with norm $\|\cdot\|_j$. The Cartesian product $X = \prod_{j=1}^n X_j$, consisting of points $(x_1, \ldots, x_n)$ with $x_j \in X_j$, is a vector space under the definitions

$$x + y = (x_1 + y_1, \ldots, x_n + y_n), \qquad cx = (cx_1, \ldots, cx_n),$$

and is a Banach space with respect to any of the equivalent norms

$$\|x\|_{(p)} = \left(\sum_{j=1}^n \|x_j\|_j^p \right)^{1/p}, \qquad 1 \leq p < \infty,$$

$$\|x\|_{(\infty)} = \max_{1 \leq j \leq n} \|x_j\|_j .$$

Furthermore,

(i) if X_j is separable for $1 \le j \le n$, then X is separable,

(ii) if X_j is reflexive for $1 \le j \le n$, then X is reflexive,

(iii) if X_j is uniformly convex for $1 \le j \le n$, then X is uniformly convex. More precisely, $\|\cdot\|_{(p)}$ is a uniformly convex norm on X provided $1 < p < \infty$. ∎

The functionals $\|\cdot\|_{(p)}$, $1 \le p \le \infty$, are norms on X, and X is complete with respect to each of them. Equivalence of these norms follows from the inequalities

$$\|x\|_{(\infty)} \le \|x\|_{(p)} \le \|x\|_{(1)} \le n\,\|x\|_{(\infty)}\,.$$

The separability and uniform convexity of X are readily deduced from the corresponding properties of the spaces X_j. The reflexivity of X follows from that of X_j, $1 \le j \le n$, via Theorem 1.18 or via the natural isomorphism between X' and $\prod_{j=1}^n X_j'$.

1.24 (Operators) Since the topology of a normed space X is determined by the sequences it renders convergent, an operator f defined on X into a topological space Y is continuous if and only if $f(x_n) \to f(x)$ in Y whenever $x_n \to x$ in X. Such is also the case for any topological space X whose topology is determined by the sequences it renders convergent. (These are called *first countable spaces*.)

Let X, Y be normed spaces and f an operator from X into Y. We say that f is *compact* if $f(A)$ is precompact in Y whenever A is bounded in X. (A bounded set in a normed space is one which is contained in the ball $B_R(0)$ for some R.) If f is continuous and compact, we say that f is *completely continuous*. We say that f is *bounded* if $f(A)$ is bounded in Y whenever A is bounded in X.

Every compact operator is bounded. Every bounded linear operator is continuous. Therefore, every compact linear operator is completely continuous. The norm of a linear operator f is $\sup\{\|f(x)\,;Y\| : \|x\,;X\| \le 1\}$.

1.25 (Imbeddings) We say the normed space X is *imbedded* in the normed space Y, and we write $X \to Y$ to designate this imbedding, provided that

(i) X is a vector subspace of Y, and

(ii) the identity operator I defined on X into Y by $Ix = x$ for all $x \in X$ is continuous.

Since I is linear, (ii) is equivalent to the existence of a constant M such that

$$\|Ix\,;Y\| \le M\,\|x\,;X\|\,, \qquad x \in X.$$

Sometimes the requirement that X be a subspace of Y and I be the identity map is weakened to allow as imbeddings certain canonical transformations of X into Y. Examples are trace imbeddings of Sobolev spaces as well as imbeddings of Sobolev spaces into spaces of continuous functions. See Chapter 5.

We say that X is *compactly imbedded* in Y if the imbedding operator I is compact.

Spaces of Continuous Functions

1.26 Let Ω be a domain in $\mathbb{R}^n$. For any nonnegative integer m let $C^m(\Omega)$ denote the vector space consisting of all functions ϕ which, together with all their partial derivatives $D^\alpha \phi$ of orders $|\alpha| \le m$, are continuous on Ω. We abbreviate $C^0(\Omega) \equiv C(\Omega)$. Let $C^\infty(\Omega) = \bigcap_{m=0}^\infty C^m(\Omega)$.

The subspaces $C_0(\Omega)$ and $C_0^\infty(\Omega)$ consist of all those functions in $C(\Omega)$ and $C^\infty(\Omega)$, respectively, that have compact support in Ω.

1.27 (Spaces of Bounded, Continuous Functions) Since Ω is open, functions in $C^m(\Omega)$ need not be bounded on Ω. We define $C_B^m(\Omega)$ to consist of those functions $\phi \in C^m(\Omega)$ for which $D^\alpha u$ is bounded on Ω for $0 \le |\alpha| \le m$. $C_B^m(\Omega)$ is a Banach space with norm given by

$$\|\phi\,; C_B^m(\Omega)\| = \max_{0 \le \alpha \le m} \sup_{x \in \Omega} |D^\alpha \phi(x)|.$$

1.28 (Spaces of Bounded, Uniformly Continuous Functions) If $\phi \in C(\Omega)$ is bounded and uniformly continuous on Ω, then it possesses a unique, bounded, continuous extension to the closure $\overline{\Omega}$ of Ω. We define the vector space $C^m(\overline{\Omega})$ to consist of all those functions $\phi \in C^m(\Omega)$ for which $D^\alpha \phi$ is bounded and uniformly continuous on Ω for $0 \le |\alpha| \le m$. (This convenient abuse of notation leads to ambiguities if Ω is unbounded; e.g., $C^m(\overline{\mathbb{R}^n}) \ne C^m(\mathbb{R}^n)$ even though $\overline{\mathbb{R}^n} = \mathbb{R}^n$.) $C^m(\overline{\Omega})$ is a closed subspace of $C_B^m(\Omega)$, and therefore also a Banach space with the same norm

$$\|\phi\,; C^m(\overline{\Omega})\| = \max_{0 \le \alpha \le m} \sup_{x \in \Omega} |D^\alpha \phi(x)|.$$

1.29 (Spaces of Hölder Continuous Functions) If $0 < \lambda \le 1$, we define $C^{m,\lambda}(\overline{\Omega})$ to be the subspace of $C^m(\overline{\Omega})$ consisting of those functions ϕ for which, for $0 \le \alpha \le m$, $D^\alpha \phi$ satisfies in Ω a Hölder condition of exponent λ, that is, there exists a constant K such that

$$|D^\alpha \phi(x) - D^\alpha \phi(y)| \le K|x-y|^\lambda, \qquad x, y \in \Omega.$$

$C^{m,\lambda}(\overline{\Omega})$ is a Banach space with norm given by

$$\|\phi\,; C^{m,\lambda}(\overline{\Omega})\| = \|\phi\,; C^m(\overline{\Omega})\| + \max_{0 \le |\alpha| \le m} \sup_{\substack{x,y \in \Omega \\ x \ne y}} \frac{|D^\alpha \phi(x) - D^\alpha \phi(y)|}{|x-y|^\lambda}.$$

It should be noted that for $0 < \nu < \lambda \le 1$,

$$C^{m,\lambda}(\overline{\Omega}) \subsetneq C^{m,\nu}(\overline{\Omega}) \subsetneq C^m(\overline{\Omega}).$$

Since Lipschitz continuity (that is, Hölder continuity of exponent 1) does not imply everywhere differentiability, it is clear that $C^{m,1}(\overline{\Omega}) \not\subset C^{m+1}(\overline{\Omega})$. In general, $C^{m+1}(\overline{\Omega}) \not\subset C^{m,1}(\overline{\Omega})$ either, but the inclusion is possible for many domains Ω, for instance convex ones as can be seen by using the Mean-Value Theorem. (See Theorem 1.34.)

1.30 If Ω is bounded, the following two well-known theorems provide useful criteria for the denseness and compactness of subsets of $C(\overline{\Omega})$. If $\phi \in C(\overline{\Omega})$, we may regard ϕ as defined on $\overline{\Omega}$, that is, we identify ϕ with its unique continuous extension to the closure of Ω.

1.31 THEOREM (The Stone-Weierstrass Theorem) Let Ω be a bounded domain in $\mathbb{R}^n$. A subset $\mathscr{A}$ of $C(\overline{\Omega})$ is dense in $C(\overline{\Omega})$ if it has the following four properties:

(i) If $\phi, \psi \in \mathscr{A}$ and $c \in \mathbb{C}$, then $\phi + \psi$, $\phi\psi$, and $c\phi$ all belong to $\mathscr{A}$.

(ii) If $\phi \in \mathscr{A}$, then $\overline{\phi} \in \mathscr{A}$, where $\overline{\phi}$ is the complex conjugate of ϕ.

(iii) If $x, y \in \overline{\Omega}$ and $x \neq y$, there exists $\phi \in \mathscr{A}$ such that $\phi(x) \neq \phi(y)$.

(iv) If $x \in \overline{\Omega}$, there exists $\phi \in \mathscr{A}$ such that $\phi(x) \neq 0$. ∎

1.32 COROLLARY If Ω is bounded in $\mathbb{R}^n$, then the set P of all polynomials in $x = (x_1, \dots, x_n)$ having rational-complex coefficients is dense in $C(\overline{\Omega})$. (A *rational-complex* number is a number of the form $c_1 + ic_2$ where c_1 and c_2 are rational numbers.) Hence $C(\overline{\Omega})$ is separable.

Proof. The set of all polynomials in x is dense in $C(\overline{\Omega})$ by the Stone-Weierstrass Theorem. Any polynomial can be uniformly approximated on the compact set $\overline{\Omega}$ by elements of the countable set P, which is therefore also dense in $C(\overline{\Omega})$. ∎

1.33 THEOREM (The Ascoli-Arzela Theorem) Let Ω be a bounded domain in $\mathbb{R}^n$. A subset K of $C(\overline{\Omega})$ is precompact in $C(\overline{\Omega})$ if the following two conditions hold:

(i) There exists a constant M such that $|\phi(x)| \leq M$ holds for every $\phi \in K$ and $x \in \Omega$.

(ii) For every $\epsilon > 0$ there exists $\delta > 0$ such that if $\phi \in K$, $x, y \in \Omega$, and $|x - y| < \delta$, then $|\phi(x) - \phi(y)| < \epsilon$. ∎

The following is a straightforward imbedding theorem for the various continuous function spaces introduced above. It is a preview of the main attraction, the Sobolev imbedding theorem of Chapter 5.

1.34 THEOREM Let m be a nonnegative integer and let $0 < \nu < \lambda \leq 1$. Then the following imbeddings exist:

$$C^{m+1}(\overline{\Omega}) \to C^m(\overline{\Omega}), \tag{3}$$

$$C^{m,\nu}(\overline{\Omega}) \to C^m(\overline{\Omega}), \tag{4}$$
$$C^{m,\lambda}(\overline{\Omega}) \to C^{m,\nu}(\overline{\Omega}). \tag{5}$$

If Ω is bounded, then imbeddings (4) and (5) are compact. If Ω is convex, we have the further imbeddings

$$C^{m+1}(\overline{\Omega}) \to C^{m,1}(\overline{\Omega}), \tag{6}$$
$$C^{m+1}(\overline{\Omega}) \to C^{m,\lambda}(\overline{\Omega}). \tag{7}$$

If Ω is convex and bounded, then imbeddings (3) is compact, and so is (7) if $\lambda < 1$.

Proof. The existence of imbeddings (3) and (4) follows from the obvious inequalities

$$\left\|\phi\,;C^m(\overline{\Omega})\right\| \le \left\|\phi\,;C^{m+1}(\overline{\Omega})\right\|,$$
$$\left\|\phi\,;C^m(\overline{\Omega})\right\| \le \left\|\phi\,;C^{m,\lambda}(\overline{\Omega})\right\|.$$

To establish (5) we note that for $|\alpha| \le m$,

$$\sup_{\substack{x,y\in\Omega\\ 0<|x-y|<1}} \frac{|D^\alpha\phi(x) - D^\alpha\phi(y)|}{|x-y|^\nu} \le \sup_{x,y\in\Omega} \frac{|D^\alpha\phi(x) - D^\alpha\phi(y)|}{|x-y|^\lambda}$$

and

$$\sup_{\substack{x,y\in\Omega\\ |x-y|\ge 1}} \frac{|D^\alpha\phi(x) - D^\alpha\phi(y)|}{|x-y|^\nu} \le 2\sup_{x\in\Omega}|D^\alpha\phi(x)|,$$

from which we conclude that

$$\left\|\phi\,;C^{m,\nu}(\overline{\Omega})\right\| \le 2\left\|\phi\,;C^{m,\lambda}(\overline{\Omega})\right\|.$$

If Ω is convex and $x, y \in \Omega$, then by the Mean-Value Theorem there is a point $z \in \Omega$ on the line segment joining x and y such that $D^\alpha\phi(x) - D^\alpha\phi(y)$ is given by $(x-y)\cdot\nabla D^\alpha\phi(z)$, where $\nabla u = (D_1u, \dots, D_nu)$. Thus

$$|D^\alpha\phi(x) - D^\alpha\phi(y)| \le n|x-y|\left\|\phi\,;C^{m+1}(\overline{\Omega})\right\|, \tag{8}$$

and so

$$\left\|\phi\,;C^{m,1}(\overline{\Omega})\right\| \le n\left\|\phi\,;C^{m+1}(\overline{\Omega})\right\|.$$

Thus (6) is proved, and (7) follows from (5) and (6).

Now suppose that Ω is bounded. If A is a bounded set in $C^{0,\lambda}(\overline{\Omega})$, then there exists M such that $\left\|\phi\,;C^{0,\lambda}(\overline{\Omega})\right\| \le M$ for all $\phi \in A$. But then $|\phi(x)-\phi(y)| \le M|x-y|^\lambda$ for all $\phi \in A$ and all $x, y \in \Omega$, whence A is precompact in $C(\overline{\Omega})$ by the Ascoli-Arzela Theorem 1.33. This proves the compactness of (4) for $m = 0$. If $m \ge 1$ and

A is bounded in $C^{m,\lambda}(\overline{\Omega})$, then A is bounded in $C^{0,\lambda}(\overline{\Omega})$ and there is a sequence $\{\phi_j\} \subset A$ such that $\phi_j \to \phi$ in $C(\overline{\Omega})$. But $\{D_1\phi_j\}$ is also bounded in $C^{0,\lambda}(\overline{\Omega})$ so there exists a subsequence of $\{\phi_j\}$ which we again denote by $\{\phi_j\}$ such that $D_1\phi_j \to \psi_1$ in $C(\overline{\Omega})$. Convergence in $C(\overline{\Omega})$ being uniform convergence on Ω, we have $\psi_1 = D_1\phi$. We may continue to extract subsequences in this manner until we obtain one for which $D^\alpha\phi_j \to D^\alpha\phi$ in $C(\overline{\Omega})$ for each α satisfying $0 \le |\alpha| \le m$. This proves the compactness of (4). For (5) we argue as follows:

$$\begin{aligned}\frac{|D^\alpha\phi(x) - D^\alpha\phi(y)|}{|x-y|^\nu} &= \left(\frac{|D^\alpha\phi(x) - D^\alpha\phi(y)|}{|x-y|^\lambda}\right)^{\nu/\lambda} |D^\alpha\phi(x) - D^\alpha\phi(y)|^{1-\nu/\lambda} \\ &\le \text{const}\, |D^\alpha\phi(x) - D^\alpha\phi(y)|^{1-\nu/\lambda} \qquad (9)\end{aligned}$$

for all ϕ in a bounded subset of $C^{m,\lambda}(\overline{\Omega})$. Since (9) shows that any sequence bounded in $C^{m,\lambda}(\overline{\Omega})$ and converging in $C^m(\overline{\Omega})$ is Cauchy and so converges in $C^{m,\nu}(\overline{\Omega})$, the compactness of (5) follows from that of (4).

Finally, if Ω is both convex and bounded, the compactness of (3) and (7) follows from composing the continuous imbedding (6) with the compact imbeddings (4) and (5) for the case $\lambda = 1$. ∎

1.35 The existence of imbeddings (6) and (7), as well as the compactness of (3) and (7), can be obtained under less restrictive hypotheses than the convexity of Ω. For instance, if every pair of points $x, y \in \Omega$ can be joined by a rectifiable arc in Ω having length not exceeding some fixed multiple of $|x-y|$, then we can obtain an inequality similar to (8) and carry out the proof. We leave it to the reader to show that (6) is not compact.

The Lebesgue Measure in $\mathbb{R}^n$

1.36 Many of the vector spaces considered in this monograph consist of functions integrable in the Lebesgue sense over domains in $\mathbb{R}^n$. While we assume that most readers are familiar with Lebesgue measure and integration, we nevertheless include here a brief discussion of that theory, especially those aspects of it relevant to the study of the L^p spaces and Sobolev spaces considered hereafter. All proofs are omitted. For a more complete and systematic discussion of the Lebesgue theory, as well as more general measures and integrals, we refer the reader to any of the books [Fo], [Ro], [Ru2], and [Sx].

1.37 (Sigma Algebras) A collection Σ of subsets of $\mathbb{R}^n$ is called a *σ-algebra* if the following conditions hold:

(i) $\mathbb{R}^n \in \Sigma$.

(ii) If $A \in \Sigma$, then its complement $A^c \in \Sigma$.

(iii) If $A_j \in \Sigma$, $j = 1, 2, \ldots$, then $\bigcup_{j=1}^{\infty} \in \Sigma$.

It follows from (i)–(iii) that:

(iv) The empty set $\emptyset \in \Sigma$.

(v) If $A_j \in \Sigma$, $j = 1, 2, \ldots$, then $\bigcap_{j=1}^{\infty} \in \Sigma$.

(vi) If $A,\ B \in \Sigma$, then $A - B = A \cap B^c \in \Sigma$.

1.38 (Measures) By a *measure* μ on a σ-algebra Σ we mean a function on Σ taking values in either $\mathbb{R} \cup \{+\infty\}$ (a *positive measure*) or $\mathbb{C}$ (a *complex measure*) which is *countably additive* in the sense that

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)$$

whenever $A_j \in \Sigma$, $j = 1, 2, \ldots$ and the sets A_j are pairwise disjoint, that is, $A_j \cap A_k = \emptyset$ for $j \neq k$. For a complex measure the series on the right must converge to the same sum for all permutations of the indices in the sequence $\{A_j\}$, and so must be absolutely convergent. If μ is a positive measure and if $A,\ B \in \Sigma$ and $A \subset B$, then $\mu(A) \leq \mu(B)$. Also, if $A_j \in \Sigma$, $j = 1, 2, \ldots$ and $A_1 \subset A_2 \subset \cdots$, then $\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \lim_{j\to\infty} \mu(A_j)$.

1.39 THEOREM (Existence of Lebesgue Measure) There exists a σ-algebra Σ of subsets of $\mathbb{R}^n$ and a positive measure μ on Σ having the following properties:

(i) Every open set in $\mathbb{R}^n$ belongs to Σ.

(ii) If $A \subset B$, $B \in \Sigma$, and $\mu(B) = 0$, then $A \in \Sigma$ and $\mu(A) = 0$.

(iii) If $A = \{x \in \mathbb{R}^n : a_j \leq x_j \leq b_j,\ j = 1, 2, \ldots, n\}$, then $A \in \Sigma$ and $\mu(A) = \prod_{j=1}^{n}(b_j - a_j)$.

(iv) μ is translation invariant. This means that if $x \in \mathbb{R}^n$ and $A \in \Sigma$, then $x + A = \{x + y : y \in A\} \in \Sigma$, and $\mu(x + A) = \mu(A)$. ∎

The elements of Σ are called *(Lebesgue) measurable subsets* of $\mathbb{R}^n$, and μ is called the *(Lebesgue) measure* in $\mathbb{R}^n$. (We normally suppress the word "Lebesgue" in these terms as it is the measure on $\mathbb{R}^n$ we mainly use.) For $A \in \Sigma$ we call $\mu(A)$ the *measure of* A or the *volume of* A, since Lebesgue measure is the natural extension of volume in $\mathbb{R}^3$. While we make no formal distinction between "measure" and "volume" for sets that are easily visualized geometrically, such as balls, cubes, and domains, and we write $\mathrm{vol}(A)$ in place of $\mu(A)$ in these cases. Of course the terms *length* and *area* are more appropriate in $\mathbb{R}^1$ and $\mathbb{R}^2$.

The reader may wonder whether in fact all subsets of $\mathbb{R}^n$ are Lebesgue measurable. The answer depends on the axioms of one's set theory. Under the most common axioms the answer is no; it is possible using the Axiom of Choice to construct a nonmeasurable set. There is a version of set theory where every subset of $\mathbb{R}^n$ is measurable, but the Hahn-Banach theorem 1.13 becomes false in that version.

1.40 (Almost Everywhere) If $B \subset A \subset \mathbb{R}^n$ and $\mu(B) = 0$, then any condition that holds on the set $A - B$ is said to hold *almost everywhere* (abbreviated a.e.) in A. It is easily seen that any countable set in $\mathbb{R}^n$ has measure zero. The converse is, however, not true.

1.41 (Measurable Functions) A function f defined on a measurable set and having values in $\mathbb{R} \cup \{-\infty, +\infty\}$ is itself called *measurable* if the set

$$\{x : f(x) > a\}$$

is measurable for every real a. Some of the more important aspects of this definition are listed in the following theorem.

1.42 THEOREM (a) If f is measurable, so is $|f|$.

(b) If f and g are measurable and real-valued, so are $f + g$ and fg.

(c) If $\{f_j\}$ is a sequence of measurable functions, then $\sup_j f_j$, $\inf_j f_j$, $\limsup_{j\to\infty} f_j$, and $\liminf_{j\to\infty} f_j$ are measurable.

(d) If f is continuous and defined on a measurable set, then f is measurable.

(e) If f is continuous on $\mathbb{R}$ into $\mathbb{R}$ and g is measurable and real-valued, then the composition $f \circ g$ defined by $f \circ g(x) = f\big((g(x)\big)$ is measurable.

(f) **(Lusin's Theorem)** If f is measurable and $f(x) = 0$ for $x \in A^c$ where $\mu(A) < \infty$, and if $\epsilon > 0$, then there exists a function $g \in C_0(\mathbb{R}^n)$ such that $\sup_{x\in\mathbb{R}^n} g(x) \le \sup_{x\in\mathbb{R}^n} f(x)$ and $\mu\left(\{x \in \mathbb{R}^n : f(x) \ne g(x)\}\right) < \epsilon$. ∎

1.43 (Characteristic and Simple Functions) Let $A \subset \mathbb{R}^n$. The function χ_A defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

is called the *characteristic function* of A. A real-valued function s on $\mathbb{R}^n$ is called a *simple function* if its range is a finite set of real numbers. If for every x, we have $s(x) \in \{a_1, \dots, a_n\}$, then $s = \sum_{j=1}^m \chi_{A_j}(x)$, where $A_j = \{x \in \mathbb{R}^n : s(x) = a_j\}$, and s is measurable if and only if $A_1, A_2, \dots, A_m$ are all measurable. Because of the following approximation theorem, simple functions are a very useful tool in integration theory.

1.44 THEOREM Given a real-valued function f with domain $A \subset \mathbb{R}^n$ there is a sequence $\{s_j\}$ of simple functions converging pointwise to f on A. If f is bounded, $\{s_j\}$ may be chosen so that the convergence is uniform. If f is measurable, each s_j may be chosen measurable. If f is nonnegative-valued, the sequence $\{s_j\}$ may be chosen to be monotonically increasing at each point. ∎

The Lebesgue Integral

1.45 We are now in a position to define the *(Lebesgue) integral* of a measurable, real-valued function defined on a measurable subset $A \subset \mathbb{R}^n$. For a simple function $s = \sum_{j=1}^{m} a_j \chi_{A_j}$, where $A_j \subset A$, A_j measurable, we define

$$\int_A s(x)\,dx = \sum_{j=1}^{m} a_j \mu(A_j). \tag{10}$$

If f is measurable and nonnegative-valued on A, we define

$$\int_A f(x)\,dx = \sup \int_A s(x)\,dx, \tag{11}$$

where the supremum is taken over measurable, simple functions s vanishing outside A and satisfying $0 \le s(x) \le f(x)$ in A. If f is a nonnegative simple function, then the two definitions of $\int_A f(x)\,dx$ given by (10) and (11) coincide. Note that the integral of a nonnegative function may be $+\infty$.

If f is measurable and real-valued, we set $f = f^+ - f^-$, where $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$ are both measurable and nonnegative. We define

$$\int_A f(x)\,dx = \int_A f^+(x)\,dx - \int_A f^-(x)\,dx$$

provided at least one of the integrals on the right is finite. If *both integrals* are finite, we say that f is (Lebesgue) *integrable* on A. The class of integrable functions on A is denoted $L^1(A)$.

1.46 THEOREM Assume all of the functions and sets appearing below are measurable.

(a) If f is bounded on A and $\mu(A) < \infty$, then $f \in L^1(A)$.

(b) If $a \le f(x) \le b$ for all $x \in A$ and if $\mu(A) < \infty$, then

$$a\,\mu(A) \le \int_A f(x)\,dx \le b\,\mu(A).$$

(c) If $f(x) \le g(x)$ for all $x \in A$, and if both integrals exist, then

$$\int_A f(x)\,dx \le \int_A g(x)\,dx.$$

(d) If $f, g \in L^1(A)$, then $f + g \in L^1(A)$ and

$$\int_A (f+g)(x)\,dx = \int_A f(x)\,dx + \int_A g(x)\,dx.$$

(e) If $f \in L^1(A)$ and $c \in \mathbb{R}$, then $cf \in L^1(A)$ and

$$\int_a (cf)(x)\,dx = c\int_A f(x)\,dx.$$

(f) If $f \in L^1(A)$, then $|f| \in L^1(A)$ and

$$\left|\int_A f(x)\,dx\right| \le \int_A |f(x)|\,dx.$$

(g) If $f \in L^1(A)$ and $B \subset A$, then $f \in L^1(B)$. If, in addition, $f(x) \ge 0$ for all $x \in A$, then

$$\int_B f(x)\,dx \le \int_A f(x)\,dx.$$

(h) If $\mu(A) = 0$, then $\int_A f(x)\,dx = 0$.

(i) If $f \in L^1(A)$ and $\int_B f(x) = 0$ for every $B \subset A$, then $f(x) = 0$ a.e. on A. ∎

One consequence of part (i) and the additivity of the integral is that sets of measure zero may be ignored for purposes of integration. That is, if f and g are measurable on A and if $f(x) = g(x)$ a.e. on A, then $\int_A f(x)\,dx = \int_A g(x)\,dx$. Accordingly, two elements of $L^1(A)$ are considered identical if they are equal almost everywhere. Thus the elements of $L_1(A)$ are actually not functions but equivalence classes of functions; two functions belong to the same element of $L_1(A)$ if they are equal a.e. on A. Nevertheless, we will continue to refer (loosely) to the elements of $L_1(A)$ as functions on A.

1.47 THEOREM If f is either an element of $L^1(\mathbb{R}^n)$ or measurable and nonnegative on $\mathbb{R}^n$, then the set function λ defined by

$$\lambda(A) = \int_A f(x)\,dx$$

is countably additive, and hence a measure on the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}^n$. ∎

The following three theorems are concerned with the interchange of integration and limit processes.

1.48 THEOREM (The Monotone Convergence Theorem) Let $A \subset \mathbb{R}^n$ be measurable and let $\{f_j\}$ be a sequence of measurable functions satisfying $0 \le f_1(x) \le f_2(x) \le \cdots$ for every $x \in A$. Then

$$\lim_{j\to\infty}\int_A f_j(x)\,dx = \int_A \left(\lim_{j\to\infty} f_j(x)\right)dx. \blacksquare$$

1.49 THEOREM (Fatou's Lemma) Let $A \subset \mathbb{R}^n$ be measurable and let $\{f_j\}$ be a sequence of nonnegative measurable functions. Then

$$\int_A \left(\liminf_{j\to\infty} \right) dx \le \liminf_{j\to\infty} \int_A f_j(x)\, dx. \blacksquare$$

1.50 THEOREM (The Dominated Convergence Theorem) Let $A \subset \mathbb{R}^n$ be measurable and let $\{f_j\}$ be a sequence of measurable functions converging to a limit pointwise on A. If there exists a function $g \in L^1(A)$ such that $|f_j(x)| \le g(x)$ for every j and all $x \in A$, then

$$\lim_{j\to\infty} \int_A f_j(x)\, dx = \int_A \left(\lim_{j\to\infty} f_j(x) \right) dx. \blacksquare$$

1.51 (Integrals of Complex-Valued Functions) The integral of a complex-valued function over a measurable set $A \subset \mathbb{R}^n$ is defined as follows. Set $f = i+iv$, where u and v are real-valued and call f measurable if and only if u and v are measurable. We say f is integrable over A, and write $f \in L^1(A)$, provided $|f| = (u^2+v^2)^{1/2}$ belongs to $L^1(A)$ in the sense described in Paragraph 1.45. For $f \in L^1(A)$, and only for such f, the integral is defined by

$$\int_A f(x)\, dx = \int_A u(x)\, dx + i \int_A v(x)\, dx.$$

It is easily checked that $f \in L^1(A)$ if and only if $u, v \in L^1(A)$. Theorem 1.42(a,b,d–f), Theorem 1.46(a,d–i), Theorem 1.47 (assuming $f \in L^1(\mathbb{R}^n)$), and Theorem 1.50 all extend to cover the case of complex f.

The following theorem enables us to express certain complex measures in terms of Lebesgue measure μ. It is the converse of Theorem 1.47.

1.52 THEOREM (The Radon-Nikodym Theorem) Let λ be a complex measure defined on the σ-algebra Σ of Lebesgue measurable subsets of $\mathbb{R}^n$. Suppose that $\lambda(A) = 0$ for every $A \in \Sigma$ for which $\mu(A) = 0$. Then there exists $f \in L^1(\mathbb{R}^n)$ such that for every $A \in \Sigma$

$$\lambda(A) = \int_A f(x)\, dx.$$

The function f is uniquely determined by λ up to sets of measure zero. $\blacksquare$

1.53 If f is a function defined on a subset A of $\mathbb{R}^{n+m}$, we may regard f as depending on the pair of variables (x, y) with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. The integral of f over A is then denoted by

$$\int_A f(x, y)\, dx\, dy$$

or, if it is desired to have the integral extend over all of $\mathbb{R}^{n+m}$,

$$\int_{\mathbb{R}^{n+m}} f(x, y)\chi_A(x, y)\,dx\,dy,$$

where χ_A is the characteristic function of A. In particular, if $A \subset \mathbb{R}^n$, we may write

$$\int_A f(x)\,dx = \int_A f(x_1, \ldots, x_n)\,dx_1 \cdots dx_n.$$

1.54 THEOREM (Fubini's Theorem) Let f be a measurable function on $\mathbb{R}^{m+n}$ and suppose that at least one of the integrals

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^{n+m}} |f(x, y)|\,dx, dy, \\ I_2 &= \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} |f(x, y)|\,dx \right) dy, \\ I_3 &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} |f(x, y)|\,dy \right) dx \end{aligned} \tag{12}$$

exists and is finite. For I_2, we mean by this that there is an integrable function g on $\mathbb{R}^n$ such that $g(y)$ is equal to the inner integral for almost all y, and similarly for I_3. Then

(a) $f(\cdot, y) \in L^1(\mathbb{R}^n)$ for almost all $y \in \mathbb{R}^m$.

(b) $f(x, \cdot) \in L^1(\mathbb{R}^m)$ for almost all $x \in \mathbb{R}^n$.

(c) $\int_{R^m} f(\cdot, y)\,dy \in L^1(\mathbb{R}^n)$.

(d) $\int_{R^n} f(x, \cdot)\,dx \in L^1(\mathbb{R}^m)$.

(e) $I_1 = I_2 = I_3$.

Distributions and Weak Derivatives

1.55 We require in subsequent chapters some of the basic concepts and techniques of the Schwartz theory of distributions [Sch], and we present here a brief description of those aspects of the theory that are relevant for our purposes. Of special importance is the notion of weak or distributional derivative of an integrable function. One of the standard definitions of Sobolev spaces is phrased in terms of such derivatives. (See Paragraph 3.2.) Besides [Sch], the reader is referred to [Ru1] and [Y] for more complete treatments of the spaces $\mathscr{D}(\Omega)$ and $\mathscr{D}'(\Omega)$ introduced below, as well as useful generalizations of these spaces.

1.56 (Test Functions) Let Ω be a domain in $\mathbb{R}^n$. A sequence $\{\phi_j\}$ of functions belonging to $C_0^\infty(\Omega)$ is said to *converge in the sense of the space* $\mathscr{D}(\Omega)$ to the function $\phi \in C_0^\infty(\Omega)$ provided the following conditions are satisfied:

(i) there exists $K \Subset \Omega$ such that $\operatorname{supp}(\phi_j - \phi) \subset K$ for every j, and

(ii) $\lim_{j\to\infty} D^\alpha \phi_j(x) = D^\alpha \phi(x)$ uniformly on K for each multi-index α.

There is a locally convex topology on the vector space $C_0^\infty(\Omega)$ which respect to which a linear functional T is continuous if and only if $T(\phi_j) \to T(\phi)$ in $\mathbb{C}$ whenever $\phi_j \to \phi$ in the sense of the space $\mathscr{D}(\Omega)$. Equipped with this topology, $C_0^\infty(\Omega)$ becomes a TVS called $\mathscr{D}(\Omega)$ whose elements are called *test functions*. $\mathscr{D}(\Omega)$ is not a normable space. (We ignore the question of uniqueness of the topology asserted above. It uniquely determines the dual of $\mathscr{D}(\Omega)$ which is sufficient for our purposes.)

1.57 (Schwartz Distributions) The dual space $\mathscr{D}'(\Omega)$ of $\mathscr{D}(\Omega)$ is called the *space of (Schwartz) distributions* on Ω. $\mathscr{D}'(\Omega)$ is given the weak-star topology as the dual of $\mathscr{D}(\Omega)$, and is a locally convex TVS with that topology. We summarize the vector space and convergence operations in $\mathscr{D}'(\Omega)$ as follows: if S, T, T_j belong to $\mathscr{D}'(\Omega)$ and $c \in \mathbb{C}$, then

$$\begin{aligned} (S+T)(\phi) &= S(\phi) + T(\phi), & \phi &\in \mathscr{D}(\Omega), \\ (cT)(\phi) &= c\,T(\phi), & \phi &\in \mathscr{D}(\Omega), \end{aligned}$$

$T_j \to T$ in $\mathscr{D}'(\Omega)$ if and only if $T_j(\phi) \to T(\phi)$ in $\mathbb{C}$ for every $\phi \in \mathscr{D}(\Omega)$.

1.58 (Locally Integrable Functions) A function u defined almost everywhere on Ω is said to be *locally integrable* on Ω provided $u \in L^1(U)$ for every open $U \Subset \Omega$. In this case we write $u \in L^1_{\mathrm{loc}}(\Omega)$. Corresponding to every $u \in L^1_{\mathrm{loc}}(\Omega)$ there is a distribution $T_u \in \mathscr{D}'(\Omega)$ defined by

$$T_u(\phi) = \int_\Omega u(x)\phi(x)\,dx, \qquad \phi \in \mathscr{D}(\Omega). \tag{13}$$

Evidently T_u, thus defined, is a linear functional on $\mathscr{D}(\Omega)$. To see that it is continuous, suppose that $\phi_j \to \phi$ in $\mathscr{D}(\Omega)$. Then there exists $K \Subset \Omega$ such that $\operatorname{supp}(\phi_j - \phi) \subset K$ for all j. Thus

$$|T_u(\phi_j) - T_u(\phi)| \le \sup_{x\in K} |\phi_j(x) - \phi(x)| \int_K |u(x)|\,dx.$$

The right side of the above inequality tends to zero as $j \to \infty$ since $\phi_j \to \phi$ uniformly on K.

1.59 Not every distribution $T \in \mathscr{D}'(\Omega)$ is of the form T_u defined by (13) for some $u \in L^1_{\mathrm{loc}}(\Omega)$. Indeed, if $0 \in \Omega$, there can be no locally integrable function δ on Ω such that for every $\phi \in \mathscr{D}(\Omega)$

$$\int_\Omega \delta(x)\phi(x)\,dx = \phi(0).$$

However, the linear functional δ defined on $\mathscr{D}(\Omega)$ by

$$\delta(\phi) = \phi(0) \tag{14}$$

is easily seen to be continuous and hence a distribution on Ω. It is called a *Dirac distribution.*

1.60 (Derivatives of Distributions) Let $u \in C^1(\Omega)$ and $\phi \in \mathscr{D}(\Omega)$. Since ϕ vanishes outside some compact subset of Ω, we obtain by integration by parts in the variable x_j

$$\int_\Omega \left(\frac{\partial}{\partial x_j} u(x)\right) \phi(x)\, dx = -\int_\Omega u(x) \left(\frac{\partial}{\partial x_j}\phi(x)\right)\, dx.$$

Similarly, if $u \in C^{|\alpha|}(\Omega)$, then integration by parts $|\alpha|$ times leads to

$$\int_\Omega \big(D^\alpha u(x)\big)\phi(x)\, dx = (-1)^{|\alpha|} \int_\Omega u(x) D^\alpha \phi(x)\, dx.$$

This motivates the following definition of the derivative $D^\alpha T$ of a distribution $T \in \mathscr{D}'(\Omega)$:

$$(D^\alpha T)(\phi) = (-1)^{|\alpha|} T(D^\alpha \phi). \tag{15}$$

Since $D^\alpha \phi \in \mathscr{D}(\Omega)$ whenever $\phi \in \mathscr{D}(\Omega)$, $D^\alpha T$ is a functional on $\mathscr{D}(\Omega)$, and it is clearly linear. We show that it is continuous, and hence a distribution on Ω. To this end suppose $\phi_j \to \phi$ in $\mathscr{D}(\Omega)$. Then

$$\operatorname{supp}\big(D^\alpha(\phi_j - \phi)\big) \subset \operatorname{supp}\big(\phi_j - \phi\big) \subset K$$

for some $K \Subset \Omega$. Moreover,

$$D^\beta\big(D^\alpha(\phi_j - \phi)\big) = D^{\beta+\alpha}(\phi_j - \phi)$$

converges to zero uniformly on K as $j \to \infty$ for each multi-index β. Hence $D^\alpha \phi_j \to D^\alpha \phi$ in $\mathscr{D}(\Omega)$. Since $T \in \mathscr{D}'(\Omega)$ it follows that

$$D^\alpha T(\phi_j) = (-1)^{|\alpha|} T(D^\alpha \phi_j) \to (-1)^{|\alpha|} T(D^\alpha \phi) = D^\alpha T(\phi)$$

in $\mathbb{C}$. Thus $D^\alpha T \in \mathscr{D}'(\Omega)$.

We have shown that every distribution in $\mathscr{D}'(\Omega)$ possesses derivatives of all orders in $\mathscr{D}'(\Omega)$ in the sense of definition (15). Furthermore, the mapping D^α from $\mathscr{D}'(\Omega)$ into $\mathscr{D}'(\Omega)$ is continuous; if $T_j \to T$ in $\mathscr{D}'(\Omega)$ and $\phi \in \mathscr{D}(\Omega)$, then

$$D^\alpha T_j(\phi) = (-1)^{|\alpha|} T_j(D^\alpha \phi) \to (-1)^{|\alpha|} T(D^\alpha \phi) = D^\alpha T(\phi).$$

1.61 EXAMPLES

1. If $0 \in \Omega$ and $\delta \in \mathscr{D}'(\Omega)$ is the Dirac distribution defined by (14), then $D^\alpha \delta$ is given by
$$D^\alpha \delta(\phi) = (-1)^{|\alpha|} D^\alpha \phi(0).$$

2. If $\Omega = \mathbb{R}$ (i.e., $n = 1$) and $H \in L^1_{\mathrm{loc}}(\mathbb{R})$ is the Heaviside function defined by
$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0, \end{cases}$$
then the derivative $(T_H)'$ of the corresponding distribution T_H is δ. To see this, let $\phi \in \mathscr{D}(\mathbb{R})$ have support in the interval $[-a, a]$. Then
$$(T_H)'(\phi) = -T_H(\phi') = -\int_0^a \phi'(x)\, dx = \phi(0) = \delta(\phi).$$

1.62 (Weak Derivatives) We now define the concept of a function being the weak derivative of another function. Let $u \in L^1_{\mathrm{loc}}(\Omega)$. There may or may not exist a function $v_\alpha \in L^1_{\mathrm{loc}}(\Omega)$ such that $T_{v_\alpha} = D^\alpha T_u$ in $\mathscr{D}'(\Omega)$. If such a v_α exists, it is unique up to sets of measure zero and is called the *weak* or *distributional* partial derivative of u, and is denoted by $D^\alpha u$. Thus $D^\alpha u = v_\alpha$ in the weak (or distributional) sense provided $v_\alpha \in L^1_{\mathrm{loc}}(\Omega)$ satisfies
$$\int_\Omega u(x) D^\alpha \phi(x)\, dx = (-1)^{|\alpha|} \int_\Omega v_\alpha(x)\phi(x)\, dx$$
for every $\phi \in \mathscr{D}(\Omega)$.

If u is sufficiently smooth to have a continuous partial derivative $D^\alpha u$ in the usual (classical) sense, then $D^\alpha u$ is also a weak partial derivative of u. Of course, $D^\alpha u$ may exist in the weak sense without existing in the classical sense. We shall show in Theorem 3.17 that certain functions having weak derivatives (those in Sobolev spaces) can be suitably approximated by smooth functions.

1.63 Let us note in conclusion that distributions in Ω can be multiplied by smooth functions. If $T \in \mathscr{D}'(\Omega)$ and $\omega \in C^\infty(\Omega)$, the product $\omega T \in \mathscr{D}'(\Omega)$ is defined by
$$(\omega T)(\phi) = T(\omega\phi), \qquad \phi \in \mathscr{D}(\Omega).$$
If $T = T_u$ for some $u \in L^1_{\mathrm{loc}}(\Omega)$, then $\omega T = T_{\omega u}$. The Leibniz rule (see Paragraph 1.2) is easily checked to hold for $D^\alpha(\omega T)$.

2

THE LEBESGUE SPACES $L^p(\Omega)$

Definition and Basic Properties

2.1 (The Space $L^p(\Omega)$) Let Ω be a domain in $\mathbb{R}^n$ and let p be a positive real number. We denote by $L^p(\Omega)$ the class of all measurable functions u defined on Ω for which

$$\int_\Omega |u(x)|^p \, dx < \infty. \tag{1}$$

We identify in $L^p(\Omega)$ functions that are equal almost everywhere in Ω; the elements of $L^p(\Omega)$ are thus equivalence classes of measurable functions satisfying (1), two functions being equivalent if they are equal a.e. in Ω. For convenience, we ignore this distinction, and write $u \in L^p(\Omega)$ if u satisfies (1), and $u = 0$ in $L^p(\Omega)$ if $u(x) = 0$ a.e. in Ω. Evidently $cu \in L^p(\Omega)$ if $u \in L^p(\Omega)$ and $c \in \mathbb{C}$. To confirm that $L^p(\Omega)$ is a vector space we must show that if $u, v \in L^p(\Omega)$, then $u + v \in L^p(\Omega)$. This is an immediate consequence of the following inequality, which will also prove useful later on.

2.2 LEMMA If $1 \le p < \infty$ and $a, b \ge 0$, then

$$(a + b)^p \le 2^{p-1}(a^p + b^p). \tag{2}$$

Proof. If $p = 1$, then (2) is an obvious equality. For $p > 1$, the function t^p is convex on $[0, \infty)$; that is, its graph lies below the chord line joining the points

(a, a^p) and (b, b^p). Thus

$$\left(\frac{a+b}{2}\right)^p \le \frac{a^p+b^p}{2},$$

from which (2) follows at once. ∎

If $u, v \in L^p(\Omega)$, then integrating

$$|u(x)+v(x)|^p \le \big(|u(x)|+|v(x)|\big)^p \le 2^{p-1}\big(|u(x)|^p+|v(x)|^p\big)$$

over Ω confirms that $u+v \in L^p(\Omega)$.

2.3 (The L_p Norm) We shall verify presently that the functional $\|\cdot\|_p$ defined by

$$\|u\|_p = \left(\int_\Omega |u(x)^p\,dx\right)^{1/p}$$

is a norm on $L^p(\Omega)$ provided $1 \le p < \infty$. (It is not a norm if $0 < p < 1$.) In arguments where confusion of domains may occur, we use $\|\cdot\|_{p,\Omega}$ in place of $\|\cdot\|_p$. It is clear that $\|u\|_p \ge 0$ and $\|u\|_p = 0$ if and only if $u = 0$ in $L^p(\Omega)$. Moreover,

$$\|cu\|_p = |c|\,\|u\|_p\,, \qquad c \in \mathbb{C}.$$

Thus we will have shown that $\|\cdot\|_p$ is a norm on $L^p(\Omega)$ once we have verified the triangle inequality

$$\|u+v\|_p \le \|u\|_p + \|v\|_p\,,$$

which is known as *Minkowski's inequality.* We verify it in Paragraph 2.8 below, for which we first require Hölder's inequality.

2.4 THEOREM (Hölder's Inequality) Let $1 < p < \infty$ and let p' denote the *conjugate exponent* defined by

$$p' = \frac{p}{p-1}, \qquad \text{that is} \qquad \frac{1}{p}+\frac{1}{p'} = 1$$

which also satisfies $1 < p' < 1$. If $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$, then $uv \in L^1(\Omega)$, and

$$\int_\Omega |u(x)v(x)|\,dx \le \|u\|_p\,\|v\|_{p'}\,. \tag{3}$$

Equality holds if and only if $|u(x)|^p$ and $|v(x)|^{p'}$ are proportional a.e. in Ω.

Proof. Let $a, b > 0$ and let $A = \ln(a^p)$ and $B = \ln(b^{p'})$. Since the exponential function is strictly convex, $\exp\big((A/p)+(B/p')\big) \le (1/p)\exp A + (1/p')\exp B$, with equality only if $A = B$. Hence

$$ab \le (a^p/p) + (b^{p'}/p'),$$

with equality occurring if and only if $a^p = b^{p'}$. If either $\|u\|_p = 0$ or $\|v\|_{p'} = 0$, then $u(x)v(x) = 0$ a.e. in Ω, and (3) is satisfied. Otherwise we can substitute $a = |u(x)| / \|u\|_p$ and $b = |v(x)| / \|v\|_{p'}$ in the above inequality and integrate over Ω to obtain (3). ∎

2.5 COROLLARY If $p > 0, q > 0$ and $r > 0$ satisfy $(1/p) + (1/q) = 1/r$, and if $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^r(\Omega)$ and $\|uv\|_r \le \|u\|_p \|v\|_q$. To see this, we can apply Hölder's inequality to $|u|^r |v|^r$ with exponents p/r and $q/r = (p/r)'$. ∎

2.6 COROLLARY Hölder's inequality can be extended to products of more than two functions. Suppose $u = \prod_{j=1}^N u_j$ where $u_j \in L^{p_j}(\Omega)$, $1 \le j \le N$, where $p_j > 0$. If $\sum_{j=1}^N (1/p_j) = 1/q$, then $u \in L^q(\Omega)$ and $\|u\|_q \le \prod_{j=1}^N \|u_j\|_{p_j}$. This follows from the previous corollary by induction on N. ∎

2.7 LEMMA (A Converse of Hölder's Inequality) A measurable function u belongs to $L^p(\Omega)$ if and only if

$$\sup\left\{\int_\Omega |u(x)|v(x)\,dx \;:\; v(x) \ge 0 \text{ on } \Omega,\ \|v\|_{p'} \le 1\right\} \tag{4}$$

is finite, and then that supremum equals $\|u\|_p$.

Proof. This is obvious if $\|u\|_p = 0$. If $0 < \|u\|_p < \infty$, then for nonnegative v with $\|v\|_{p'} \le 1$ we have, by Hölder's inequality,

$$\int_\Omega |u(x)|v(x)\,dx \le \|u\|_p \|v\|_{p'} \le \|u\|_p\,,$$

and equality holds if $v = \left(|u| / \|u\|_p\right)^{p/p'}$, for which $\|v\|_{p'} = 1$.

Conversely, if $\|u\|_p = \infty$ we can find an increasing sequence s_j of nontrivial simple functions satisfying $0 \le s_j(x) \le |u(x)|$ on Ω for which $\|s_j\|_p \to \infty$. If $v_j = \left(|s_j| / \|s_j\|_p\right)^{p/p'}$, then

$$\int_\Omega |u(x)|v_j(x)\,dx \ge \int_\Omega s_j(x)v_j(x)\,dx = \|s_j\|_p$$

so the supremum (4) must be infinite. ∎

2.8 THEOREM (Minkowski's Inequality) If $1 \le p < \infty$, then

$$\|u + v\|_p \le \|u\|_p + \|v\|_p\,. \tag{5}$$

Proof. Inequality (5) certainly holds if $p = 1$ since

$$\int_\Omega |u(x) + v(x)|\, dx \le \int_\Omega |u(x)|\, dx + \int_\Omega |v(x)|\, dx.$$

For $1 < p < \infty$ observe that for $w \ge 0$, $\|w\|_{p'} \le 1$ we have, by Hölder's inequality,

$$\begin{aligned}\int_\Omega \big(|u(x)| + |v(x)|\big) w(x)\, dx &\le \int_\Omega |u(x)| w(x)\, dx + \int_\Omega |v(x)| w(x)\, dx \\ &\le \|u\|_p + \|v\|_p,\end{aligned}$$

whence $\|u + v\|_p \le \|u\|_p + \|v\|_p$ follows by Lemma 2.7. ∎

2.9 THEOREM (Minkowski's Inequality for Integrals) Let $1 \le p < \infty$. Suppose that f is measurable on $\mathbb{R}^m \times \mathbb{R}^n$, that $f(\cdot, y) \in L^p(\mathbb{R}^m)$ for almost all $y \in \mathbb{R}^n$, and that the function $y \to \|f(\cdot, y)\|_{p,\mathbb{R}^m}$ belongs to $L^1(\mathbb{R}^n)$. Then the function $x \to \int_{\mathbb{R}^n} f(x, y)\, dy$ belongs to $L^p(\mathbb{R}^m)$ and

$$\left(\int_{\mathbb{R}^m} \left|\int_{\mathbb{R}^n} f(x, y)\, dy\right|^p dx\right)^{1/p} \le \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} |f(x, y)|^p\, dx\right)^{1/p} dy.$$

That is,

$$\left\|\int_{\mathbb{R}^n} f(\cdot, y)\, dy\right\|_{p,\mathbb{R}^m} \le \int_{\mathbb{R}^n} \|f(\cdot, y)\|_{p,\mathbb{R}^m}\, dy.$$

Proof. Suppose initially that $f \ge 0$. When $p = 1$, the inequalities above become equalities given in Fubini's theorem. When $p > 1$, use a nonnegative function $\|w\|$ in the unit ball of $L^p(\Omega)$ as in Theorem 2.8. By Fubini's theorem and Hölder's inequality,

$$\begin{aligned}\int_{\mathbb{R}^m} \int_{\mathbb{R}^n} f(x, y)\, dy w(x)\, dx &= \int_{\mathbb{R}^{m+n}} f(x, y) w(x)\, dx\, dy \\ &\le \int_{\mathbb{R}^n} \|w\|_{p',\mathbb{R}^m} \|f(\cdot, y)\|_{p,\mathbb{R}^m}\, dy \\ &\le \int_{\mathbb{R}^n} \|f(\cdot, y)\|_{p,\mathbb{R}^m}\, dy.\end{aligned}$$

This case now follows by Lemma 2.7. For a general function f as above, split f into real and imaginary parts and split these as differences of nonnegative functions satisfying the hypotheses. It follows that the function mapping x to $\int_{R^n} f(x, y)\, dy$ belongs to $L^p(R^m)$. To get the norm estimate, replace f by $|f|$. ∎

2.10 (The Space $L^\infty(\Omega)$) A function u that is measurable on Ω is said to be *essentially bounded* on Ω if there is a constant K such that $|u(x)| \le K$ a.e. on Ω. The greatest lower bound of such constants K is called the essential supremum of $|u|$ on Ω, and is denoted by $\operatorname{ess\,sup}_{x\in\Omega} |u(x)|$. We denote by $L^\infty(\Omega)$ the vector space of all functions u that are essentially bounded on Ω, functions being once again identified if they are equal a.e. on Ω. It is easily checked that the functional $\|\cdot\|_\infty$ defined by

$$\|u\|_\infty = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)|$$

is a norm on $L^\infty(\Omega)$. Moreover, Hölder's inequality (3) and its corollaries extend to cover the two cases $p = 1,\ p' = \infty$ and $p = \infty,\ p' = 1$.

2.11 THEOREM (An Interpolation Inequality) Let $1 \le p < q < r$, so that

$$\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r}$$

for some θ satisfying $0 < \theta < 1$. If $u \in L^p(\Omega) \cap L^r(\Omega)$, then $u \in L^q(\Omega)$ and

$$\|u\|_q \le \|u\|_p^\theta \, \|u\|_r^{1-\theta}.$$

Proof. Let $s = p/(\theta q)$. Then $s \ge 1$ and $s' = s/(s-1) = r/((1-\theta)q)$ if $r < \infty$. In this case, by Hölder's inequality

$$\begin{aligned}\|u\|_q^q &= \int_O |u(x)|^{\theta q}|u(x)|^{(1-\theta)q}\,dx \\ &\le \left(\int_\Omega |u(x)|^{\theta q s}\,dx\right)^{1/s}\left(\int_\Omega |u(x)|^{(1-\theta)qs'}\,dx\right)^{1/s'} = \|u\|_p^{\theta q}\,\|u\|_r^{(1-\theta)q}\end{aligned}$$

and the result follows at once. The proof if $r = \infty$ is similar. ∎

The following two theorems establish reverse forms of Hölder's and Minkowski's inequalities for the case $0 < p < 1$. The latter inequality, which indicates that $\|\cdot\|_p$ is not a norm in this case, will be used to prove the Clarkson inequalities in Theorem 2.38.

2.12 THEOREM (A Reverse Hölder Inequality) Let $0 < p < 1$, so that $p' = p/(p-1) < 0$. If $f \in L^p(\Omega)$ and

$$0 < \int_\Omega |g(x)|^{p'}\,dx < \infty,$$

then

$$\int_\Omega |f(x)g(x)|\,dx \ge \left(\int_\Omega |f(x)|^p\,dx\right)^{1/p}\left(\int_\Omega |g(x)|^{p'}\,dx\right)^{1/p'}. \qquad (6)$$

Proof. We can assume $fg \in L^1(\Omega)$; otherwise the left side of (6) is infinite. Let $\phi = |g|^{-p}$ and $\psi = |fg|^p$ so that $\phi\psi = |f|^p$. Then $\psi \in L^q(\Omega)$, where $q = 1/p > 1$, and since $p' = -pq'$ where $q' = q/(q-1)$, we have $\phi \in L^{q'}(\Omega)$. By the direct form of Hölder's inequality (3) we have

$$\begin{aligned}\int_\Omega |f(x)|^p\,dx &= \int_\Omega \phi(x)\psi(x)\,dx \le \|\psi\|_q\,\|\phi\|_{q'}\\ &= \left(\int_\Omega |f(x)g(x)|\,dx\right)^p \left(\int_\Omega |g(x)|^{p'}\,dx\right)^{1-p}.\end{aligned}$$

Taking pth roots and dividing by the last factor on the right side we obtain (6). ∎

2.13 THEOREM (A Reverse Minkowski Inequality) Let $0 < p < 1$. If $u, v \in L^p(\Omega)$, then

$$\||u| + |v|\|_p \ge \|u\|_p + \|v\|_p\,. \tag{7}$$

Proof. In $u = v = 0$ in $L^p(\Omega)$, then the right side of (7) is zero. Otherwise, the left side is greater than zero and we can apply the reverse Hölder inequality (6) to obtain

$$\begin{aligned}\||u| + |v|\|_p^p &= \int_\Omega \big(|u(x)| + |v(x)|\big)^{p-1}\big(|u(x)| + |v(x)|\big)\,dx\\ &\ge \left(\int_\Omega \big(|u(x)| + |v(x)|\big)^p\,dx\right)^{1/p'} \big(\|u\|_p + \|v\|_p\big)\\ &= \||u| + |v|\|_p^{p/p'} \big(\|u\|_p + \|v\|_p\big)\end{aligned}$$

and (7) follows by cancellation. ∎

Here is a useful imbedding theorem for L^p spaces over domains with finite volume.

2.14 THEOREM (An Imbedding Theorem for L^p Spaces) Suppose that $\mathrm{vol}(\Omega) = \int_\Omega 1\,dx < \infty$ and $1 \le p \le q \le \infty$. If $u \in L^q(\Omega)$, then $u \in L^p(\Omega)$ and

$$\|u\|_p \le \big(\mathrm{vol}(\Omega)\big)^{(1/p)-(1/q)}\,\|u\|_q\,. \tag{8}$$

Hence

$$L^q(\Omega) \to L^p(\Omega). \tag{9}$$

If $u \in L^\infty(\Omega)$, then

$$\lim_{p\to\infty} \|u\|_p = \|u\|_\infty\,. \tag{10}$$

Finally, if $u \in L^p(\Omega)$ for $1 \le p < \infty$ and if there exists a constant K such that for all such p

$$\|u\|_p \le K, \tag{11}$$

then $u \in L^\infty(\Omega)$ and

$$\|u\|_\infty \le K. \tag{12}$$

Proof. If $p = q$ or $q = \infty$, (8) and (9) are trivial. If $1 \le p < q < \infty$ and $u \in L^q(\Omega)$, Hölder's inequality gives

$$\int_\Omega |u(x)|^p\,dx \le \left(\int_\Omega |u(x)|^q\,dx\right)^{p/q}\left(\int_\Omega 1\,dx\right)^{1-(p/q)}$$

from which (8) and (9) follow immediately. If $u \in L^\infty(\Omega)$, we obtain from (8)

$$\limsup_{p\to\infty} \|u\|_p \le \|u\|_\infty\,. \tag{13}$$

On the other hand, for any $\epsilon > 0$ there exists a set $A \subset \Omega$ having positive measure $\mu(A)$ such that

$$|u(x)| \ge \|u\|_\infty - \epsilon \qquad \text{if } x \in A.$$

Hence

$$\int_\Omega |u(x)^p\,dx \ge \int_A |u(x)|^p\,dx \ge \mu(A)(\|u\|_\infty - \epsilon)^p.$$

It follows that $\|u\|_p \ge \big(\mu(A)\big)^{1/p}(\|u\|_\infty - \epsilon)$, whence

$$\liminf_{p\to\infty} \|u\|_p \ge \|u\|_\infty\,. \tag{14}$$

Equation (10) now follows from (13) and (14).

Now suppose (11) holds for $1 \le p < \infty$. If $u \notin L^\infty(\Omega)$ or else if (12) does not hold, then we can find a constant $K_1 > K$ and a set $A \subset \Omega$ with $\mu(A) > 0$ such that for $x \in A$, $|u(x)| \ge K_1$. The same argument used to obtain (14) now shows that

$$\liminf_{p\to\infty} \|u\|_p \ge K_1$$

which contradicts (11). ∎

2.15 COROLLARY $L^p(\Omega) \subset L^1_{\mathrm{loc}}(\Omega)$ for $1 \le p \le \infty$ and any domain Ω.

Completeness of $L^p(\Omega)$

2.16 THEOREM $L^p(\Omega)$ is a Banach space if $1 \le p \le \infty$.

Proof. First assume $1 \le p < \infty$ and let $\{u_n\}$ be a Cauchy sequence in $L^p(\Omega)$. There is a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that

$$\left\|u_{n_{j+1}} - u_{n_j}\right\|_p \le \frac{1}{2^j}, \qquad j = 1, 2, \ldots .$$

Let $v_m(x) = \sum_{j=1}^m |u_{n_{j+1}}(x) - u_{n_j}(x)|$. Then

$$\|v_m\|_p \le \sum_{j=1}^m \frac{1}{2^j} < 1, \qquad m = 1, 2, \ldots .$$

Putting $v(x) = \lim_{m\to\infty} v_m(x)$, which may be infinite for some x, we obtain by the Monotone Convergence Theorem 1.48

$$\int_\Omega |v(x)|^p \, dx = \lim_{m\to\infty} \int_\Omega |v_m(x)|^p \, dx \le 1.$$

Hence $v(x) < \infty$ a.e. on Ω and the series

$$u_{n_1}(x) + \sum_{j=1}^\infty \big(u_{n_{j+1}}(x) - u_{n_j}(x)\big) \tag{15}$$

converges to a limit $u(x)$ a.e. on Ω by Theorem 1.50. Let $u(x) = 0$ wherever it is undefined by (15). Since (15) telescopes, we have

$$\lim_{m\to\infty} u_{n_m}(x) = u(x) \qquad \text{a.e. in } \Omega.$$

For any $\epsilon > 0$ there exists N such that if $m, n \ge N$, then $\|u_m - u_n\|_p < \epsilon$. Hence, by Fatou's lemma 1.49

$$\begin{aligned}\int_\Omega |u(x) - u_n(x)|^p \, dx &= \int_\Omega \lim_{j\to\infty} |u_{n_j}(x) - u_n(x)|^p \, dx \\ &\le \liminf_{j\to\infty} \int_\Omega |u_{n_j}(x) - u_n(x)|^p \, dx \le \epsilon^p\end{aligned}$$

if $n \ge N$. Thus $u = (u - u_n) + u_n \in L^p(\Omega)$ and $\|u - u_n\|_p \to 0$ as $n \to \infty$. Therefore $L^p(\Omega)$ is complete and so is a Banach space.

Finally, if $\{u_n\}$ is a Cauchy sequence in $L^\infty(\Omega)$, then there exists a set $A \subset \Omega$ having measure zero such that if $x \notin A$, then for every $n, m = 1, 2, \ldots$

$$|u_n(x)| \le \|u_n\|_\infty, \qquad |u_n(x) - u_m(x)| \le \|u_n - u_m\|_\infty .$$

Therefore, $\{u_n\}$ converges uniformly on $\Omega - A$ to a bounded function u. Setting $u = 0$ for $x \in A$, we have $u \in L^\infty(\Omega)$ and $\|u_n - u\|_\infty \to 0$ as $n \to \infty$. Thus $L^\infty(\Omega)$ is also complete and a Banach space. ∎

2.17 COROLLARY If $1 \le p \le \infty$, each Cauchy sequence in $L^p(\Omega)$ has a subsequence converging pointwise almost everywhere on Ω. ∎

2.18 COROLLARY $L^2(\Omega)$ is a Hilbert space with respect to the inner product

$$(u, v) = \int_\Omega u(x)\overline{v(x)}\, dx.$$

Hölder's inequality for $L^2(\Omega)$ is just the well-known Schwarz inequality

$$|(u, v)| \le \|u\|_2 \, \|v\|_2 \,. \blacksquare$$

Approximation by Continuous Functions

2.19 THEOREM $C_0(\Omega)$ is dense in $L^p(\Omega)$ if $1 \le p < \infty$.

Proof. Any $u \in L^p(\Omega)$ can be written in the form $u = u_1 - u_2 + i(u_3 - u_4)$ where, for $1 \le j \le 4$, $u_j \in L^p(\Omega)$ is real-valued and nonnegative. Thus it is sufficient to prove that if $\epsilon > 0$ and $u \in L^p(\Omega)$ is real-valued and nonnegative then there exists $\phi \in C_0(\Omega)$ such that $\|\phi - u\|_p < \epsilon$. By Theorem 1.44 for such a function u there exists a monotonically increasing sequence $\{s_n\}$ of nonnegative simple functions converging pointwise to u on Ω. Since $0 \le s_n(x) \le u(x)$, we have $s_n \in L^p(\Omega)$ and since $\big(u(x) - s_n(x)\big)^p \le \big(u(x)\big)^p$, we have $s_n \to u$ in $L^p(\Omega)$ by the Dominated Convergence Theorem 1.50. Thus there exists an $s \in \{s_n\}$ such that $\|u - s\|_p < \epsilon/2$. Since s is simple and $p < \infty$ the support of s has finite volume. We can also assume that $s(x) = 0$ if $x \in \Omega^c$. By Lusin's Theorem 1.42(f) there exists $\phi \in C_0(\mathbb{R}^n)$ such that

$$|\phi(x)| \le \|s\|_\infty \qquad \text{for all } x \in \mathbb{R}^n,$$

and

$$\operatorname{vol}(\{x \in \mathbb{R}^n \, : \, \phi(x) \ne s(x)\}) < \left(\frac{\epsilon}{4\,\|s\|_\infty}\right)^p.$$

By Theorem 2.14

$$\begin{aligned} \|s - \phi\|_p &\le \|s - \phi\|_\infty \big(\operatorname{vol}(\{x \in \mathbb{R}^n : \phi(x) \ne s(x)\})\big)^{1/p} \\ &< 2\,\|s\|_\infty \left(\frac{\epsilon}{4\,\|s\|_\infty}\right) = \frac{\epsilon}{2}. \end{aligned}$$

It follows that $\|u - \phi\|_p < \epsilon$. $\blacksquare$

2.20 The above proof shows that the set of simple functions in $L^p(\Omega)$ is dense in $L^p(\Omega)$ for $1 \le p < \infty$. That this is also true for $L^\infty(\Omega)$ is a direct consequence of Theorem 1.44.

2.21 THEOREM $L^p(\Omega)$ is separable if $1 \le p < \infty$.

Proof. For $m = 1, 2, \ldots$ let

$$\Omega_m = \{x \in \Omega : |x| \le m \quad \text{and} \quad \operatorname{dist}(x, \operatorname{bdry}(\Omega)) \ge 1/m\}.$$

Then Ω_m is a compact subset of Ω. Let P be the set of all polynomials on $\mathbb{R}^n$ having rational-complex coefficients, and let $P_m = \{\chi_m f : f \in P\}$ where χ_m is the characteristic function of Ω_m. As shown in Paragraph 1.32, P_m is dense in $C(\Omega_m)$. Moreover, $\bigcup_{m=1}^{\infty} P_m$ is countable.

If $u \in L^p(\Omega)$ and $\epsilon > 0$, there exists $\phi \in C_0(\Omega)$ such that $\|u - \phi\|_p < \epsilon/2$. If $1/m < \operatorname{dist}(\operatorname{supp}(\phi), \operatorname{bdry}(\Omega))$, then there exists f in the set P_m such that $\|\phi - f\|_\infty < (\epsilon/2)\big(\operatorname{vol}(\Omega_m)\big)^{-1/p}$. It follows that

$$\|\phi - f\|_p \le \|\phi - f\|_\infty \big(\operatorname{vol}(\Omega_m)\big)^{1/p} < \epsilon/2$$

and so $\|u - f\|_p < \epsilon$. Thus the countable set $\bigcup_{m=1}^{\infty} P_m$ is dense in $L^p(\Omega)$ and $L^p(\Omega)$ is separable. ∎

2.22 $C_B^0(\Omega)$ is a proper closed subset of $L^\infty(\Omega)$ and so is not dense in that space. Therefore, neither are $C_0(\Omega)$ or $C_0^\infty(\Omega)$. In fact, $L^\infty(\Omega)$ is not separable.

Convolutions and Young's Theorem

2.23 (The Convolution Product) It is often useful to form a non-pointwise product of two functions that smooth out irregularities of each of them to produce a function better behaved locally than either factor alone. One such product is the *convolution* $u * v$ of two functions u and v defined by

$$u * v(x) = \int_{\mathbb{R}^n} u(x - y)v(y)\, dy \tag{16}$$

when the integral exists. For instance, if $u \in L^p(\mathbb{R}^n)$ and $v \in L^{p'}(\mathbb{R}^n)$, then the integral (16) converges absolutely by Hölder's inequality, and we have $|u * v(x)| \le \|u\|_p \|v\|_{p'}$ for all values of x. Moreover, $u * v$ is uniformly continuous in these cases. To see this, observe first that if $u \in L^p(\mathbb{R}^n)$ and $v \in C_0(\mathbb{R}^n)$, then applying Hölder's inequality to the convolution of u with differences between v and translates of v shows that $u * v$ is uniformly continuous. When $1 \le p' < \infty$ a general function v in $L^{p'}(\mathbb{R}^n)$ is the $L^{p'}$-norm limit of a sequence, $\{v_j\}$ say, of functions in $C_0(\mathbb{R}^n)$; then $u * v$ is the L^∞-norm limit of the sequence $\{u * v_j\}$, and so is still uniformly continuous. In any event, the change of variable $y = x - z$ shows that $u * v = v * u$. Thus $u * v$ is also uniformly continuous when $u \in L^1(\mathbb{R}^n)$ and $v \in L^\infty(\mathbb{R}^n)$.

2.24 THEOREM (Young's Theorem) Let $p, q, r \geq 1$ and suppose that $(1/p) + (1/q) + (1/r) = 2$. Then

$$\left| \int_{\mathbb{R}^n} (u * v)(x) w(x)\, dx \right| \leq \|u\|_p \|v\|_q \|w\|_r \tag{17}$$

holds for all $u \in L^p(\mathbb{R}^n)$, $v \in L^q(\mathbb{R}^n)$, $w \in L^r(\mathbb{R}^n)$.

Proof. For now, we prove this estimate when $u \in C_0(\mathbb{R}^n)$, and we explain in the proof of the Corollary below how to deal with more general functions u. This special case is the one we use in applications of convolution. The function mapping (x, y) to $u(x - y)$ is then jointly continuous on $\mathbb{R}^n \times \mathbb{R}^n$, and hence is a measurable function on on $\mathbb{R}^n \times \mathbb{R}^n$. This justifies the use of Fubini's theorem below. First observe that

$$\frac{1}{p'} + \frac{1}{q'} + \frac{1}{r'} = 3 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r} = 1,$$

so the functions

$$\begin{aligned} U(x, y) &= |v(y))|^{q/p'} |w(x)|^{r/p'} \\ V(x, y) &= |u(x - y))|^{p/q'} |w(x)|^{r/q'} \\ W(x, y) &= |u(x - y))|^{p/r'} |v(y)|^{q/r'} \end{aligned}$$

satisfy $(UVW)(x, y) = u(x - y)v(y)w(x)$. Moreover,

$$\begin{aligned} \|V\|_{q'} &= \left(\int_{\mathbb{R}^n} |w(x)|^r\, dx \int_{\mathbb{R}^n} |u(x - y)|^p\, dy \right)^{1/q'} \\ &= \left(\int_{\mathbb{R}^n} |w(x)|^r\, dx \int_{\mathbb{R}^n} |u(z)|^p\, dz \right)^{1/q'} = \|u\|_p^{p/q'} \|w\|_r^{r/q'}, \end{aligned}$$

and similarly $\|U\|_{p'} = \|v\|_q^{q/p'} \|w\|_r^{r/p'}$ and $\|W\|_{r'} = \|u\|_p^{p/r'} \|v\|_q^{q/r'}$. Combining these results, we have, by the three-function form of Hölder's inequality,

$$\begin{aligned} \left| \int_{\mathbb{R}^n} (u * v)(x) w(x)\, dx \right| &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x - y||v(y||w(x)|\, dy\, dx \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} U(x, y) V(x, y) W(x, y)\, dy\, dx \\ &\leq \|U\|_{p'} \|V\|_{q'} \|W\|_{r'} = \|u\|_p \|v\|_q \|w\|_r. \end{aligned}$$

We remark that (17) holds with a constant $K = K(p, q, r, n) < 1$ included on the right side. The best (smallest) constant is

$$K(p, q, r, n) = \left(\frac{p^{1/p} q^{1/q} r^{1/r}}{(p')^{1/p'} (q')^{1/q'} (r')^{1/r'}} \right)^{n/2}.$$

See [LL] for a proof of this. ∎

2.25 COROLLARY If $(1/p) + (1/q) = 1 + (1/r)$, and if $u \in L^p(\mathbb{R}^n)$ and $v \in L^q(\mathbb{R}^n)$, then $u * v \in L^r(\mathbb{R}^n)$, and

$$\|u * v\|_r \le K(p, q, r', n)\, \|u\|_p\, \|v\|_q \le \|u\|_p\, \|v\|_q .$$

This is known as *Young's inequality for convolution.* It also implies Young's Theorem. When $u \in C_0(\mathbb{R}^n)$, it follows from Lemma 2.7 and the case of inequality (17) proved above, with r' in place of r.

2.26 (Proof of the General Case of Corollary 2.25 and Theorem 2.24) We remove the restriction $u \in C_0(\mathbb{R}^n)$ from the above Corollary and therefore from Young's Theorem itself. We can assume that p and q are both finite, since the only other pairs satisfying the hypotheses are $(p, q) = (1, \infty)$ and $(\infty, 1)$, and these were covered before the statement of the theorem.

Fix a simple function v in $L^q(\mathbb{R}^n)$, and regard the functions u as running through the subspace $C_0(\mathbb{R}^n)$ of $L^p(\mathbb{R}^n)$. Then convolution with v is a bounded operator, T_v say, from this dense subspace of $L^p(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$, and the norm of T_v is at most $\|v\|_q$. By the norm density of $C_0(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$, the operator T_v extends uniquely to one with the same norm mapping all of $L^p(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$.

Given u in $L^p(\mathbb{R}^n)$, find a sequence $\{u_j\}$ in $C_0(\mathbb{R}^n)$ converging in L^p norm to u. Then $T_v(u_j)$ converges in L^r norm to $T_v(u)$. Pass to a subsequence, if necessary, to also get almost-everywhere convergence of $T_v(u_j)$ to $T_v(u)$. Since the simple function v also belongs to $L^{p'}$, the integrals (16) defining $u * v$ and $u_j * v$ all converge absolutely, and

$$u * v(x) = \lim_{j\to\infty} \big(u_j * v(x)\big) \quad \text{for all } x \in R^n.$$

So $T_v(u)(x)$ agrees almost everywhere with $u * v(x)$ as given in (16), and hence $\|u * v\|_r \le \|u\|_p \|v\|_q$ when u is any function in $L^p(\mathbb{R}^n)$ and v is any simple function in $L^q(\mathbb{R}^n)$.

We complete the proof with an argument passing from simple functions v to general functions in $L^q(\mathbb{R}^n)$. For any fixed u in $L^p(\mathbb{R}^n)$ convolution with u defines an operator, S_u say, with norm at most $\|u\|_p$, from the subspace of simple functions in $L^q(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$. By the density of that subspace, the operator S_u extends uniquely to one with the same norm mapping all of $L^q(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$.

To relate this extended operator S_u to formula (16), it suffices to deal with the case where the functions u and v are both nonnegative. Pick an increasing sequence $\{v_j\}$ of nonnegative simple functions converging in L^q norm to v. Then the sequence $\{u * v_j\}$ converges in L^r norm to $S_u(v)$. Again pass to a subsequence that converges almost everywhere to $S_u(v)$. Since the function u is nonnegative, the product sequence $\{u * v_j(x)\}$ increases for each x. So it either diverges to ∞ or converges to a finite value for $u * v(x)$. From the a.e. convergence above, the latter must happen for almost all x, and $\|u * v\|_r = \|S_u(v)\|_r \le \|u\|_p \|v\|_q$ as required. ∎

2.27 (The Space ℓ^p) It is sometimes useful to classify sequences of real or complex numbers according to their degree of summability. We denote by ℓ^p the set of doubly infinite sequences $a = \{a_i\}_{i=-\infty}^{\infty}$ for which

$$\|a\,;\ell^p\| = \begin{cases} \left(\displaystyle\sum_{i=-\infty}^{\infty} |a_i|^p\right)^{1/p} & \text{if } 0 < p < \infty \\ \displaystyle\sup_{-\infty<i<\infty} |a_i| & \text{if } p = \infty \end{cases}$$

is finite. Evidently, $\|a\,;\ell^p\| = \|f\|_p$ where f is the function defined on $\mathbb{R}$ by $f(t) = a_i$ for $i \le t < i+1$, $-\infty < i < \infty$.

If $1 \le p \le \infty$, then ℓ^p is a Banach space with norm $\|\cdot\,;\ell^p\|$. Singly infinite sequences such as $\{a_i\}_{i=0}^{\infty}$ or even finite sequences such as $\{a_i\}_{i=m}^{n}$ can be regarded as defined for $-\infty < i < \infty$ with all $a_i = 0$ for i outside the appropriate interval, and as such they determine subspaces of of ℓ^p.

Hölder's inequality, Minkowski's inequality, and Young's inequality follow for the spaces ℓ^p by the same methods used for $L^p(\mathbb{R})$. Specifically, suppose that $a = \{a_i\}_{i=-\infty}^{\infty}$ and $b = \{b_i\}_{i=-\infty}^{\infty}$.

(a) If $a \in \ell^p$ and $b \in \ell^q$, then $ab = \{a_i b_i\}_{i=-\infty}^{\infty} \in \ell^r$ where r satisfies $(1/r) = (1/p) + (1/q)$, and

$$\|ab\,;\ell^r\| \le \|a\,;\ell^p\|\ \|b\,;\ell^q\|. \qquad \text{(Hölder's Inequality)}$$

(b) If $a, b \in \ell^p$, then

$$\|a+b\,;\ell^p\| \le \|a\,;\ell^p\| + \|b\,;\ell^p\|. \qquad \text{(Minkowski's Inequality)}$$

(c) If $a \in \ell^p$ and $b \in \ell^q$ where $(1/p) + (1/q) \ge 1$, then the series $(a * b)_i$ defined by

$$(a * b)_i = \sum_{j=-\infty}^{\infty} a_{i-j} b_j, \qquad (-\infty < i < \infty),$$

converges absolutely. Moreover, the sequence $a * b$, called the *convolution* of a and b, belongs to ℓ^r, where $1 + (1/r) = (1/p) + (1/q)$, and

$$\|a * b\,;\ell^r\| \le \|a\,;\ell^p\|\ \|b\,;\ell^q\|. \qquad \text{(Young's Inequality)}$$

Note, however, that the ℓ^p spaces imbed into one another in the reverse order to the imbeddings of the spaces $L^p(\Omega)$ where Ω has finite volume. (See Theorem 2.14.) If $0 < p \le q \le \infty$, then

$$\ell^p \to \ell^q, \qquad \text{and} \qquad \left\|a\,;\ell^q\right\| \le \left\|a\,;\ell^p\right\|.$$

The latter inequality is obvious if $q = \infty$ and follows for other $q \ge p$ from summing the inequality

$$|a_i|^q = |a_i|^p |a_i|^{q-p} \le |a_i|^p \left\|a\,;\ell^\infty\right\|^{q-p} \le |a_i|^p \left\|a\,;\ell^p\right\|^{q-p}.$$

Mollifiers and Approximation by Smooth Functions

2.28 (Mollifiers) Let J be a nonnegative, real-valued function belonging to $C_0^\infty(\mathbb{R}^n)$ and having the properties

(i) $J(x) = 0$ if $|x| \ge 1$, and

(ii) $\int_{\mathbb{R}^n} J(x)\,dx = 1$.

For example, we may take

$$J(x) = \begin{cases} k\exp[-1/(1-|x|^2)] & \text{if } |x| < 1 \\ 0 & \text{if } |x| \ge 1, \end{cases}$$

where $k > 0$ is chosen so that condition (ii) is satisfied. If $\epsilon > 0$, the function $J_\epsilon(x) = \epsilon^{-n} J(x/\epsilon)$ is nonnegative, belongs to $C_0^\infty(\mathbb{R}^n)$, and satisfies

(i) $J_\epsilon(x) = 0$ if $|x| \ge \epsilon$, and

(ii) $\int_{\mathbb{R}^n} J_\epsilon(x)\,dx = 1$.

J_ϵ is called a *mollifier* and the convolution

$$J_\epsilon * u(x) = \int_{\mathbb{R}^n} J_\epsilon(x-y)u(y)\,dy, \tag{18}$$

defined for functions u for which the right side of (18) makes sense, is called a *mollification* or *regularization* of u. The following theorem summarizes some properties of mollification.

2.29 THEOREM (Properties of Mollification) Let u be a function which is defined on $\mathbb{R}^n$ and vanishes identically outside Ω.

(a) If $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then $J_\epsilon * u \in C^\infty(\mathbb{R}^n)$.

(b) If $u \in L^1_{\mathrm{loc}}(\Omega)$ and $\operatorname{supp}(u) \Subset \Omega$, then $J_\epsilon * u \in C_0^\infty(\Omega)$ provided

$$\epsilon < \operatorname{dist}\big(\operatorname{supp}(u), \operatorname{bdry}(\Omega)\big).$$

(c) If $u \in L^p(\Omega)$ where $1 \le p < \infty$, then $J_\epsilon * u \in L^p(\Omega)$. Also

$$\|J_\epsilon * u\|_p \le \|u\|_p \quad \text{and} \quad \lim_{\epsilon\to 0+} \|J_\epsilon * u - u\|_p = 0.$$

(d) If $u \in C(\Omega)$ and if $G \Subset \Omega$, then $\lim_{\epsilon\to 0+} J_\epsilon * u(x) = u(x)$ uniformly on G.

(e) If $u \in C(\overline{\Omega})$, then $\lim_{\epsilon\to 0+} J_\epsilon * u(x) = u(x)$ uniformly on Ω.

Proof. Since $J_\epsilon(x-y)$ is an infinitely differentiable function of x and vanishes if $|y-x| \ge \epsilon$, and since for every multi-index α we have

$$D^\alpha(J_\epsilon * u)(x) = \int_{\mathbb{R}^n} D_x^\alpha J_\epsilon(x-y)u(y)\,dy,$$

conclusions (a) and (b) are valid.

If $u \in L^p(\Omega)$ where $1 < p < \infty$, then by Hölder's inequality (3),

$$\begin{aligned} |J_\epsilon * u(x)| &= \left|\int_{\mathbb{R}^n} J_\epsilon(x-y)u(y)\,dy\right| \\ &\le \left(\int_{\mathbb{R}^n} J_\epsilon(x-y)\,dy\right)^{1/p'} \left(\int_{\mathbb{R}^n} J_\epsilon(x-y)|u(y)|^p\,dy\right)^{1/p} \\ &= \left(\int_{\mathbb{R}^n} J_\epsilon(x-y)|u(y)|^p\,dy\right)^{1/p}. \end{aligned} \tag{19}$$

Hence by Fubini's Theorem 1.54

$$\begin{aligned} \int_\Omega |J_\epsilon * u(x)|^p\,dx &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_\epsilon(x-y)|u(y)|^p\,dy\,dx \\ &= \int_{\mathbb{R}^n} |u(y)|^p\,dy \int_{\mathbb{R}^n} J_\epsilon(x-y)\,dx = \|u\|_p^p. \end{aligned}$$

For $p = 1$ this inequality follows directly from (18).

Now let $\eta > 0$ be given. By Theorem 2.19 there exists $\phi \in C_0(\Omega)$ such that $\|u - \phi\|_p < \eta/3$. Thus also $\|J_\epsilon * u - J_\epsilon * \phi\|_p < \eta/3$. Now

$$\begin{aligned} |J_\epsilon * \phi(x) - \phi(x)| &= \left|\int_{\mathbb{R}^n} J_\epsilon(x-y)\big(\phi(y) - \phi(x)\big)\,du\right| \\ &\le \sup_{|y-x|<\epsilon} |\phi(y) - \phi(x)|. \end{aligned} \tag{20}$$

Since ϕ is uniformly continuous on Ω, the right side of (20) tends to zero as $\epsilon \to 0+$. Since supp (ϕ) is compact, we can ensure that $\|J_\epsilon * \phi - \phi\|_p < \eta/3$

by choosing ϵ sufficiently small. For such ϵ we have $\|J_\epsilon * u - u\|_p < \eta$ and (c) follows.

The proofs of (d) and (e) may be obtained by replacing ϕ by u in inequality (20). ∎

2.30 COROLLARY $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$ if $1 \le p < \infty$. ∎

This is an immediate consequence of conclusions (b) and (e) of the theorem and Theorem 2.19.

Precompact Sets in $L^p(\Omega)$

2.31 The following theorem plays a role in the study of L^p spaces similar to that played by the Arzela-Ascoli Theorem 1.33 in the study of spaces of continuous functions. If u is a function defined a.e. on $\Omega \subset \mathbb{R}^n$, let $\tilde{u}$ denote the zero extension of u outside Ω:

$$\tilde{u} = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n - \Omega. \end{cases}$$

2.32 THEOREM Let $1 \le p < \infty$. A bounded subset $K \subset L^p(\Omega)$ is precompact in $L^p(\Omega)$ if and only if for every number $\epsilon > 0$ there exists a number $\delta > 0$ and a subset $G \Subset \Omega$ such that for every $u \in K$ and $h \in \mathbb{R}^n$ with $|h| < \delta$ both of the following inequalities hold:

$$\int_\Omega |\tilde{u}(x+h) - \tilde{u}(x)|^p \, dx < \epsilon^p, \tag{21}$$

$$\int_{\Omega - \overline{G}} |u(x)|^p \, dx < \epsilon^p. \tag{22}$$

Proof. Let $T_h u$ denote the translate of u by h:

$$T_h u(x) = u(x+h).$$

First we assume that K is precompact in $L^p(\Omega)$. Let $\epsilon > 0$ be given. Since K has a finite $\epsilon/6$-net (Theorem 1.19), and since $C_0(\Omega)$ is dense in $L^p(\Omega)$ (Theorem 2.19), there exists a finite set S of continuous functions having compact support in Ω, such that for each $u \in K$ there exists $\phi \in S$ satisfying $\|u - \phi\|_p < \epsilon/3$. Let G be the union of the supports of the finitely many functions in S. Then $G \Subset \Omega$ and inequality (22) follows immediately. To prove inequality (21) choose a closed ball $\overline{B_r}$ of radius r centred at the origin and containing G. Note that $(T_h\phi - \phi)(x) = \phi(x+h) - \phi(x)$ is uniformly continuous and vanishes outside B_{r+1} provided $|h| < 1$. Hence

$$\lim_{|h| \to 0} \int_{\mathbb{R}^n} |T_h\phi(x) - \phi(x)|^p \, dx = 0,$$

the convergence being uniform for $\phi \in S$. For $|h|$ sufficiently small, we have $\|T_h\phi - \phi\|_p < \epsilon/3$. If $\phi \in S$ satisfies $\|u - \phi\|_p < \epsilon/3$, then also $\|T_h\tilde{u} - T_h\phi\|_p < \epsilon/3$. Hence we have for $|h|$ sufficiently small (independent of $u \in K$),

$$\|T_h\tilde{u} - \tilde{u}\|_p \le \|T_h\tilde{u} - T_h\phi\|_p + \|T_h\phi - \phi\|_p + \|\phi - u\|_p < \epsilon$$

and (21) follows. (This argument shows that translation is continuous in $L^p(\mathbb{R}^n)$.)

It is sufficient to prove the converse for the special case $\Omega = \mathbb{R}^n$, as it follows for general Ω from its application in this special case to the set $\tilde{K} = \{\tilde{u} : u \in K\}$.

Let $\epsilon > 0$ be given and choose $G \Subset \mathbb{R}^n$ such that for all $u \in K$

$$\int_{\mathbb{R}^n - \overline{G}} |u(x)|^p \, dx < \frac{\epsilon}{3}. \tag{23}$$

For any $\eta > 0$ the function $J_\eta * u$ defined as in (18) belongs to $C^\infty(\mathbb{R}^n)$ and in particular to $C(\overline{G})$. If $\phi \in C_0(\mathbb{R}^n)$, then by Hölder's inequality,

$$\begin{aligned} \left|J_\eta * \phi(x) - \phi(x)\right|^p &= \left|\int_{\mathbb{R}^n} J_\eta(y)\big(\phi(x-y) - \phi(x)\big)\, dy\right|^p \\ &\le \int_{B_\eta} J_\eta(y)\left|T_{-y}\phi(x) - \phi(x)\right|^p dy. \end{aligned}$$

Hence

$$\left\|J_\eta * \phi - \phi\right\|_p \le \sup_{h \in B_\eta} \|T_h\phi - \phi\|_p \,.$$

If $u \in L^p(\mathbb{R}^n)$, let $\{\phi_j\}$ be a sequence in $C_0(\mathbb{R}^n)$ converging to u in L^p norm. By 2.29(c), $\{J_\eta * \phi_j\}$ is a Cauchy sequence converging to $J_\eta * u$ in $L^p(\mathbb{R}^n)$. Since also $T_h\phi_j \to T_h u$ in $L^p(\mathbb{R}^n)$, we have

$$\left\|J_\eta * u - u\right\|_p \le \sup_{h \in B_\eta} \|T_h u - u\|_p \,.$$

Now (21) implies that $\lim_{|h|\to 0} \|T_h u - u\|_p = 0$ uniformly for $u \in K$. Hence $\lim_{\eta\to 0} \left\|J_\eta * u - u\right\|_p = 0$ uniformly for $u \in K$. Fix $\eta > 0$ so that

$$\int_{\overline{G}} |J_\eta * u(x) - u(x)|^p \, dx < \frac{\epsilon}{3 \cdot 2^{p-1}} \tag{24}$$

for all $u \in K$.

We show that $\{J_\eta * u : u \in K\}$ satisfies the conditions of the Arzela-Ascoli Theorem 1.33 on $\overline{G}$ and hence is precompact in $C(\overline{G})$. By (19) we have

$$|J_\eta * u(x)| \le \left(\sup_{y \in \mathbb{R}^n} J_\eta(y)\right)^{1/p} \|u\|_p$$

which is bounded uniformly for $x \in \mathbb{R}^n$ and $u \in K$ since K is bounded in $L^p(\mathbb{R}^n)$ and η is fixed. Similarly,

$$|J_\eta * u(x+h) - J_\eta * u(x)| \le \left(\sup_{y \in \mathbb{R}^n} J_\eta(y) \right)^{1/p} \|T_h u - u\|_p$$

and so $\lim_{|h| \to 0} J_\eta * u(x+h) = J_\eta * u(x)$ uniformly for $x \in \mathbb{R}^n$ and $u \in K$. Thus $\{J_\eta * u \;:\; u \in K\}$ is precompact in $C(\overline{G})$, and by Theorem 1.19 there exists a finite set $\{\psi_1, \dots, \psi_m\}$ of functions in $C(\overline{G})$ such that if $u \in K$, then for some j, $1 \le j \le m$, and all $x \in \overline{G}$ we have

$$|\psi_j(x) - J_\eta * u(x)| < \frac{\epsilon}{3 \cdot 2^{p-1} \cdot \operatorname{vol}(\overline{G})}.$$

This, together with (23), (24), and the inequality $(|a| + |b|)^p \le 2^{p-1}(|a|^p + |b|^p)$ of Lemma 2.2, implies that

$$\begin{aligned}
\int_{\mathbb{R}^n} |u(x) - \tilde{\psi}_j(x)|^p \, dx &= \int_{\mathbb{R}^n - \overline{G}} |u(x)|^p \, dx + \int_{\overline{G}} |u(x) - \psi_j(x)|^p \, dx \\
&< \frac{\epsilon}{3} + 2^{p-1} \int_{\overline{G}} \left(|u(x) - J_\eta * u(x)|^p + |J_\eta * u(x) - \psi_j(x)|^p \right) dx \\
&< \frac{\epsilon}{3} + 2^{p-1} \left(\frac{\epsilon}{3 \cdot 2^{p-1}} + \frac{\epsilon}{3 \cdot 2^{p-1} \cdot \operatorname{vol}(\overline{G})} \operatorname{vol}(\overline{G}) \right) = \epsilon.
\end{aligned}$$

Hence K has a finite ϵ-net in $L^p(\mathbb{R}^n)$ and is precompact there by Theorem 1.19. ∎

2.33 THEOREM Let $1 \le p < \infty$ and let $K \subset L^p(\Omega)$. Suppose there exists a sequence $\{\Omega_j\}$ of subdomains of Ω having the following properties:

(i) $\Omega_j \subset \Omega_{j+1}$ for each j.

(ii) The set of restrictions to Ω_j of the functions in K is precompact in $L^p(\Omega_j)$ for each j.

(iii) For every $\epsilon > 0$ there exists j such that

$$\int_{\Omega - \Omega_j} |u(x)|^p \, dx < \epsilon \qquad \text{for every } u \in K.$$

Then K is precompact in $L^p(\Omega)$.

Proof. Let $\{u_n\}$ be a sequence in K. By (ii) there exists a subsequence $\{u_n^{(1)}\}$ whose restrictions of Ω_1 converge in $L^p(\Omega_1)$. Having selected $\{u_n^{(1)}\}, \dots, \{u_n^{(k)}\}$, we may select a subsequence $\{u_n^{(k+1)}\}$ of $\{u_n^{(k)}\}$ whose restrictions to Ω_{k+1} converge in $L^p(\Omega_{k+1})$. The restrictions of $\{u_n^{(k+1)}\}$ to Ω_j also converge in $L^p(\Omega_j)$ for $1 \le j \le k$ by (i).

Let $v_n = u_n^{(n)}$ for $n = 1, 2, \ldots$. Clearly $\{v_n\}$ is a subsequence of $\{u_n\}$. Given $\epsilon > 0$, (iii) assures us that there exists j such that

$$\int_{\Omega-\Omega_j} |v_n(x) - v_m(x)|^p \, dx < \frac{\epsilon}{2}$$

for all $n, m = 1, 2, \ldots$. Except for the first $j-1$ terms, $\{v_n\}$ is a subsequence of $\{u_n^{(j)}\}$, so its restrictions to Ω_j form a Cauchy sequence in $L^p(\Omega_j)$. Thus for n, m sufficiently large,

$$\int_{\Omega_j} |v_n(x) - v_m(x)|^p \, dx < \frac{\epsilon}{2},$$

and

$$\int_{\Omega} |v_n(x) - v_m(x)|^p \, dx < \epsilon.$$

Thus $\{v_n\}$ is a Cauchy sequence in $L^p(\Omega)$ and so converges there. Hence K is precompact in $L^p(\Omega)$. ∎

Uniform Convexity

2.34 As noted previously, the parallelogram law in an inner product space guarantees the uniform convexity of the corresponding norm on that space. This applies to $L^2(\Omega)$. Now we will develop certain inequalities due to Clarkson [Clk] that generalize the parallelogram law and verify the uniform convexity of $L^p(\Omega)$ for $1 < p < \infty$.

We begin by preparing three technical lemmas needed for the proof.

2.35 LEMMA If $0 < s < 1$, then $f(t) = (1 - s^t)/t$ is a decreasing function of $t > 0$.

Proof. $f'(t) = (1/t^2)\big(g(s^t) - 1\big)$ where $g(r) = r - r\ln r$. Since $0 < s^t < 1$ and since $g'(r) = -\ln r \geq 0$ for $0 < r \leq 1$, it follows that $g(s^t) < g(1) = 1$ whence $f'(t) < 0$. ∎

2.36 LEMMA If $1 < p \leq 2$ and $0 \leq t \leq 1$, then

$$\left(\frac{1+t}{2}\right)^{p'} + \left(\frac{1-t}{2}\right)^{p'} \leq \left(\frac{1}{2} + \frac{1}{2}t^p\right)^{1/(p-1)}, \tag{25}$$

where $p' = p/(p-1)$ is the exponent conjugate to p.

Proof. Since equality holds in (25) if either $p = 2$ or $t = 0$ or $t = 1$, we may assume that $1 < p < 2$ and that $0 < t < 1$. Under the transformation

$t = (1-s)/(1+s)$, which maps $0 < t < 1$ onto $1 > s > 0$, (25) reduces to the equivalent form

$$\frac{1}{2}\left((1+s)^p + (1-s)^p\right) - (1+s^{p'})^{p-1} \geq 0. \tag{26}$$

The power series expansion of the left side of (26) takes the form

$$\begin{aligned}
&\frac{1}{2}\sum_{k=0}^{\infty}\binom{p}{k}s^k + \frac{1}{2}\sum_{k=0}^{\infty}\binom{p}{k}(-s)^k - \sum_{k=0}^{\infty}\binom{p-1}{k}s^{p'k} \\
&\quad = \sum_{k=0}^{\infty}\binom{p}{2k}s^{2k} - \sum_{k=0}^{\infty}\binom{p-1}{k}s^{p'k} \\
&\quad = \sum_{k=1}^{\infty}\left[\binom{p}{2k}s^{2k} - \binom{p-1}{2k-1}s^{p'(2k-1)} - \binom{p-1}{2k}s^{2p'k}\right],
\end{aligned}$$

where

$$\binom{p}{0} = 1 \qquad \text{and} \qquad \binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}, \quad k \geq 1.$$

The latter series certainly converges for $0 \leq s < 1$. We prove (26) by showing that each term of the series is positive for $0 < s < 1$. The kth term (in square brackets above) can be written in the form

$$\begin{aligned}
&\frac{p(p-1)(2-p)(3-p)\cdots(2k-1-p)}{(2k)!}s^{2k} \\
&\quad - \frac{(p-1)(2-p)\cdots(2k-1-p)}{(2k-1)!}s^{p'(2k-1)} + \frac{(p-1)(2-p)\cdots(2k-p)}{(2k)!}s^{2kp'} \\
&= \frac{(2-p)\cdots(2k-p)}{(2k-1)!}s^{2k}\left[\frac{p(p-1)}{2k(2k-p)} - \frac{p-1}{2k-p}s^{p'(2k-1)-2k} + \frac{p-1}{2k}s^{2kp'-2k}\right] \\
&= \frac{(2-p)\cdots(2k-p)}{(2k-1)!}s^{2k}\left[\frac{1-s^{(2k-p)/(p-1)}}{(2k-p)/(p-1)} - \frac{1-s^{2k/(p-1)}}{2k/(p-1)}\right].
\end{aligned}$$

The first factor is positive since $p < 2$; the factor in the square brackets is positive by Lemma 2.35 since $0 < (2k-p)/(p-1) < 2k/(p-1)$. Thus (26) and hence (25) is established. ∎

2.37 LEMMA Let $z, w \in \mathbb{C}$. If $1 < p \leq 2$ and $p' = p/(p-1)$, then

$$\left|\frac{z+w}{2}\right|^{p'} + \left|\frac{z-w}{2}\right|^{p'} \leq \left(\frac{1}{2}|z|^p + \frac{1}{2}|w|^p\right)^{1/(p-1)}. \tag{27}$$

If $2 \le p < \infty$, then

$$\left|\frac{z+w}{2}\right|^p + \left|\frac{z-w}{2}\right|^p \le \frac{1}{2}|z|^p + \frac{1}{2}|w|^p. \tag{28}$$

Proof. Since (27) obviously holds if $z = 0$ or $w = 0$ and is symmetric in z and w, we can assume that $|z| \ge |w| > 0$. If $w/z = re^{i\theta}$ where $0 \le r \le 1$ and $0 \le \theta < 2\pi$, then (27) can be rewritten in the form

$$\left|\frac{1+re^{i\theta}}{2}\right|^{p'} + \left|\frac{1-re^{i\theta}}{2}\right|^{p'} \le \left(\frac{1}{2} + \frac{1}{2}r^p\right)^{1/(p-1)}. \tag{29}$$

If $\theta = 0$, then (29) is just the result of Lemma 2.36. We complete the proof of (29) by showing that for fixed r, $0 < r \le 1$, the function

$$f(\theta) = |1 + re^{i\theta}|^{p'} + |1 - re^{i\theta}|^{p'}$$

has a maximum value for $0 \le \theta < 2\pi$ at $\theta = 0$. Since

$$f(\theta) = (1 + r^2 + 2r\cos\theta)^{p'/2} + (1 + r^2 - 2r\cos\theta)^{p'/2},$$

satisfies $f(2\pi - \theta) = f(\pi - \theta) = f(\theta)$, we need consider f only on the interval $0 \le \theta \le \pi/2$. Since $p' \ge 2$, on that interval

$$f'(\theta) = -p'r\sin\theta\big[(1 + r^2 + 2r\cos\theta)^{(p'/2)-1} - (1 + r^2 - 2r\cos\theta)^{(p'/2)-1}\big] \le 0.$$

Thus the maximum value of f does indeed occur at $\theta = 0$ and (29), and therefore also (27), is proved.

If $2 \le p < \infty$, then $1 < p' \le 2$, and we have by interchanging p and p' in (27) and using Lemma 2.2,

$$\begin{aligned}
\left|\frac{z+w}{2}\right|^p + \left|\frac{z-w}{2}\right|^p &\le \left(\frac{1}{2}|z|^{p'} + \frac{1}{2}|w|^{p'}\right)^{1/(p'-1)} \\
&= \left(\frac{1}{2}|z|^{p'} + \frac{1}{2}|w|^{p'}\right)^{p/p'} \\
&\le 2^{(p/p')-1}\left[\left(\frac{1}{2}\right)^{p/p'}|z|^p + \left(\frac{1}{2}\right)^{p/p'}|w|^p\right] \\
&= \frac{1}{2}|z|^p + \frac{1}{2}|w|^p,
\end{aligned}$$

so that (28) is also proved. ∎

2.38 THEOREM (Clarkson's Inequalities) Let $u, v \in L^p(\Omega)$. For $1 < p < \infty$ let $p' = p/(p-1)$. If $2 \le p < \infty$, then

$$\left\|\frac{u+v}{2}\right\|_p^p + \left\|\frac{u-v}{2}\right\|_p^p \le \frac{1}{2}\|u\|_p^p + \frac{1}{2}\|v\|_p^p, \tag{30}$$

$$\left\|\frac{u+v}{2}\right\|_p^{p'} + \left\|\frac{u-v}{2}\right\|_p^{p'} \ge \left(\frac{1}{2}\|u\|_p^p + \frac{1}{2}\|v\|_p^p\right)^{p'-1}. \tag{31}$$

If $1 < p \le 2$, then

$$\left\|\frac{u+v}{2}\right\|_p^{p'} + \left\|\frac{u-v}{2}\right\|_p^{p'} \le \left(\frac{1}{2}\|u\|_p^p + \frac{1}{2}\|v\|_p^p\right)^{p'-1}, \tag{32}$$

$$\left\|\frac{u+v}{2}\right\|_p^p + \left\|\frac{u-v}{2}\right\|_p^p \ge \frac{1}{2}\|u\|_p^p + \frac{1}{2}\|v\|_p^p. \tag{33}$$

Proof. For $2 \le p < \infty$, (30) is obtained by using $z = u(x)$ and $w = v(x)$ in (28) and integrating over Ω. To prove (32) for $1 < p \le 2$ we first note that $\left\| |u|^{p'} \right\|_{p-1} = \|u\|_p^{p'}$ for any $u \in L^p(\Omega)$. Using the reverse Minkowski inequality (7) corresponding to the exponent $p - 1 < 1$, and (27) with $z = u(x)$ and $w = v(x)$, we obtain

$$\begin{aligned}
\left\|\frac{u+v}{2}\right\|_p^{p'} + \left\|\frac{u-v}{2}\right\|_p^{p'} &= \left\|\left|\frac{u+v}{2}\right|^{p'}\right\|_{p-1} + \left\|\left|\frac{u-v}{2}\right|^{p'}\right\|_{p-1} \\
&\le \left[\int_\Omega \left(\left|\frac{u(x)+v(x)}{2}\right|^{p'} + \left|\frac{u(x)-v(x)}{2}\right|^{p'}\right)^{p-1} dx\right]^{1/(p-1)} \\
&\le \left[\int_\Omega \left(\frac{1}{2}|u(x)|^p + \frac{1}{2}|v(x)|^p\right) dx\right]^{p'-1} \\
&= \left(\frac{1}{2}\|u\|_p^p + \frac{1}{2}\|v\|_p^p\right)^{p'-1}
\end{aligned}$$

which is (32).

Inequality (31) is proved for $2 \le p < \infty$ by the same method used to prove (32) except that the direct Minkowski inequality (5), corresponding to $p - 1 \ge 1$, is used in place of the reverse inequality, and in place of (27) is used the inequality

$$\left(\left|\frac{\xi+\eta}{2}\right|^{p'} + \left|\frac{\xi-\eta}{2}\right|^{p'}\right)^{p-1} \ge \frac{1}{2}|\xi|^p + \frac{1}{2}|\eta|^p,$$

which is obtained from (27) by replacing p by p', z by $\xi + \eta$, and w by $\xi - \eta$. Finally, (33) can be obtained from a similar revision of (28).

We remark that if $p = 2$, all four Clarkson inequalities reduce to the parallelogram law

$$\|u + v\|_2^2 + \|u - v\|_2^2 = 2\,\|u\|_2^2 + 2\,\|v\|_2^2 . \blacksquare$$

2.39 THEOREM If $1 < p < \infty$, then $L^p(\Omega)$ is uniformly convex.

Proof. Let $u, v \in L^p(\Omega)$ satisfy $\|u\|_p = \|v\|_p = 1$ and $\|u - v\|_p \ge \epsilon$ where $0 < \epsilon < 2$. If $2 \le p < \infty$, then (30) implies that

$$\left\| \frac{u+v}{2} \right\|_p^p \le 1 - \frac{\epsilon^p}{2^p}.$$

If $1 < p \le 2$, then (32) implies that

$$\left\| \frac{u+v}{2} \right\|_p^{p'} \le 1 - \frac{\epsilon^{p'}}{2^{p'}}.$$

In either case there exists $\delta = \delta(\epsilon) > 0$ such that $\|(u + v)/2\|_p \le 1 - \delta$. $\blacksquare$

See [BKC] for sharper information on L^p geometry.

2.40 COROLLARY $L^p(\Omega)$ is reflexive if $1 < p < \infty$. $\blacksquare$

This is a consequence of uniform convexity via Theorem 1.21. We will give a direct proof after calculating the normed dual of $L^p(\Omega)$.

The Normed Dual of $L^p(\Omega)$

2.41 (Linear Functionals) Let $1 \le p \le \infty$ and let p' be the exponent conjugate to p. Each element $v \in L^{p'}(\Omega)$ defines a linear functional L_v on $L^p(\Omega)$ via

$$L_v(u) = \int_\Omega u(x)v(x)\,dx, \qquad u \in L^p(\Omega).$$

By Hölder's inequality $|L_v(u)| \le \|u\|_p \, \|v\|_{p'}$, so that $L_v \in [L^p(\Omega)]'$ and

$$\|L_v; [L^p(\Omega)]'\| \le \|v\|_{p'} .$$

Equality must hold above. If $1 < p \le \infty$, let $u(x) = |v(x)|^{p'-2}\overline{v(x)}$ if $v(x) \ne 0$ and $u(x) = 0$ otherwise. Then $u \in L^p(\Omega)$ and $L_v(u) = \|u\|_p \, \|v\|_{p'}$.

Now suppose $p = 1$ so $p' = \infty$. If $\|v\|_{p'} = 0$, let $u(x) = 0$. Otherwise let $0 < \epsilon < \|v\|_\infty$ and let A be a measurable subset of Ω such that $0 < \mu(A) < \infty$

and $|v(x)| > \|v\|_\infty - \epsilon$ on A. Let $u(x) = \overline{v(x)}/|v(x)|$ on A and $u(x) = 0$ elsewhere. Then $u \in L^1(\Omega)$ and $L_v(u) \geq \|u\|_1 (\|v\|_\infty - \epsilon)$. Thus we have shown that

$$\|L_v; [L^p(\Omega)]'\| = \|v\|_{p'}, \tag{34}$$

so that the operator $\mathcal{L}$ mapping v to L_v is an isometric isomorphism of $L^{p'}(\Omega)$ onto a subspace of $[L^p(\Omega)]'$.

2.42 It is natural to ask if the range if the isomorphism $\mathcal{L}$ is all of $[L^p(\Omega)]'$. That is, is every continuous linear functional on $L^p(\Omega)$ of the form L_v for some $v \in L^{p'}(\Omega)$? We will show that such is the case if $1 \leq p < \infty$. For $p = 2$, this is an immediate consequence of the Riesz Representation Theorem 1.12 for Hilbert spaces. For general p a direct proof can be based on the Radon-Nikodym Theorem 1.52 (see [Ru2] or Theorem 8.19). We will give a more elementary proof based on a variational argument and uniform convexity. We will use a limiting argument to obtain the case $p = 1$ from the case $p > 1$.

2.43 LEMMA Let $1 < p < \infty$. If $L \in [L^p(\Omega)]'$, and $\|L; [L^p(\Omega)]'\| = 1$, then there exists a unique $w \in L^p(\Omega)$ such that $\|w\|_p = L(w) = 1$. Dually, if $w \in L^p(\Omega)$ is given and $\|w\|_p = 1$, then there exists a unique $L \in [L^p(\Omega)]'$ such that $\|L; [L^p(\Omega)]'\| = L(w) = 1$.

Proof. First assume that $L \in [L^p(\Omega)]'$ is given and $\|L; [L^p(\Omega)]'\| = 1$. Then there exists a sequence $\{w_n\} \in L^p(\Omega)$ satisfying $\|w_n\|_p = 1$ and such that $\lim_{n\to\infty} |L(w_n)| = 1$. We may assume that $|L(w_n)| > 1/2$ for each n, and, replacing w_n by a suitable multiple of w_n by a complex number of unit modulus, that $L(w_n)$ is real and positive. Let $\epsilon > 0$. By the definition of uniform convexity, there exists a positive number $\delta > 0$ such that if u and v belong to the unit ball of $L^p(\Omega)$ and if $\|(u+v)/2\|_p > 1-\delta$, then $\|u - v\|_p < \epsilon$. On the other hand, there exist an integer N such that $L(w_n) > 1 - \delta$ for all $n > N$. When $m > N$ also, we have that $L\big((w_m + w_n)/2\big) > 1 - \delta$, and then $\|w_m - w_n\|_p < \epsilon$. Therefore $\{w_n\}$ is a Cauchy sequence in $L^p(\Omega)$ and so converges to a limit w in that space. Clearly, $\|w\|_p = 1$ and $L(w) = \lim_{n\to\infty} L(w_n) = 1$. For uniqueness, if there were two candidates v and w, then the sequence $\{v, w, v, w, \ldots\}$ would have to converge, forcing $v = w$.

Now suppose $w \in L^p(\Omega)$ is given and $\|w\|_p = 1$. As noted in Paragraph 2.41 the functional L_v defined by

$$L_v(u) = \int_\Omega u(x)v(x)\,dx, \qquad u \in L^p(\Omega), \tag{35}$$

where

$$v(x) = \begin{cases} |w(x)|^{p-2}\overline{w(x)} & \text{if } w(x) \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{36}$$

satisfies $L_v(w) = \|w\|_p^p = 1$ and $\|L_v; [L^p(\Omega)]'\| = \|v\|_{p'} = \|w\|_p^{p/p'} = 1$. It remains to be shown, therefore, that if $L_1, L_2 \in [L^p(\Omega)]'$ satisfy $\|L_1\| = \|L_2\| = 1$ and $L_1(w) = L_2(w) = 1$, then $L_1 = L_2$. Suppose not. Then there exists $u \in L^p(\Omega)$ such that $L_1(u) \neq L_2(u)$. Replacing u by a suitable multiple of u, we may assume that $L_1(u) - L_2(u) = 2$. Then replacing u by its sum with a suitable multiple of w, we can arrange that $L_1(u) = 1$ and $L_2(u) = -1$. If $t > 0$, then $L(w + tu) = 1 + t$. Since $\|L_1\| = 1$, therefore $\|w + tu\|_p \geq 1 + t$. Similarly, $L_2(w - tu) = 1 + t$ and so $\|w - tu\|_p \geq 1 + t$. If $1 < p \leq 2$, Clarkson's inequality (33) gives

$$\begin{aligned} 1 + t^p \|u\|_p^p &= \left\| \frac{(w+tu) + (w-tu)}{2} \right\|_p^p + \left\| \frac{(w+tu) - (w-tu)}{2} \right\|_p^p \\ &\geq \frac{1}{2} \|w + tu\|_p^p + \frac{1}{2} \|w - tu\|_p^p \geq (1+t)^p, \end{aligned}$$

which is not possible for all $t > 0$. Similarly, if $2 \leq p < \infty$, Clarkson's inequality (31) gives

$$\begin{aligned} 1 + t^{p'} \|u\|_p^{p'} &= \left\| \frac{(w+tu) + (w-tu)}{2} \right\|_p^{p'} + \left\| \frac{(w+tu) - (w-tu)}{2} \right\|_p^{p'} \\ &\geq \left(\frac{1}{2} \|w + tu\|_p^p + \frac{1}{2} \|w - tu\|_p^p \right)^{p'-1} \geq (1+t)^{p'}, \end{aligned}$$

which is also not possible for all $t > 0$. Thus no such u can exist, and $L_1 = L_2$. ∎

2.44 THEOREM (The Riesz Representation Theorem for $L^p(\Omega)$) Let $1 < p < \infty$ and let $L \in [L^p(\Omega)]'$. Then there exists $v \in L^{p'}(\Omega)$ such that for all $u \in L^p(\Omega)$

$$L(u) = L_v(u) = \int_\Omega u(x)v(x)\,dx.$$

Moreover, $\|v\|_{p'} = \|L; [L^p(\Omega)]'\|$. Thus $[L^p(\Omega)]' \cong L^{p'}(\Omega)$; $[L^p(\Omega)]'$ is isometrically isomorphic to $L^{p'}(\Omega)$.

Proof. If $L = 0$ we may take $v = 0$. Thus we can assume $L \neq 0$, and, without loss of generality, that $\|L; [L^p(\Omega)]'\| = 1$. By Lemma 2.43 there exists $w \in L^p(\Omega)$ with $\|w\|_p = 1$ such that $L(w) = 1$. Let v be given by (36). Then L_v, defined by (35), satisfies $\|L_v; [L^p(\Omega)]'\| = 1$ and $L_v(w) = 1$. By Lemma 2.43 again, we have $L = L_v$. Since $\|v\|_{p'} = 1$, the proof is complete. ∎

2.45 THEOREM (The Riesz Representation Theorem for $L^1(\Omega)$) Let $L \in [L^1(\Omega)]'$. Then there exists $v \in L^\infty(\Omega)$ such that for all $u \in L^1(\Omega)$

$$L(u) = \int_\Omega u(x)v(x)\,dx$$

and $\|v\|_\infty = \|L; [L^1(\Omega)]'\|$. Thus $[L^1(\Omega)]' \cong L^\infty(\Omega)$.

Proof. Once again we assume that $L \neq 0$ and $\|L; [L^1(\Omega)]'\| = 1$. Let us suppose, for the moment, that Ω has finite volume. If $1 < p < \infty$, then by Theorem 2.14 we have $L^p(\Omega) \subset L^1(\Omega)$ and

$$|L(u)| \leq \|u\|_1 \leq \big(\mathrm{vol}(\Omega)\big)^{1-(1/p)} \|u\|_p$$

for any $u \in L^p(\Omega)$. Hence $L \in [L^p(\Omega)]'$ and by Theorem 2.44 there exists $v_p \in L^{p'}(\Omega)$ such that

$$L(u) = \int_\Omega u(x)v_p(x)\,dx, \qquad u \in L^p(\Omega) \tag{37}$$

and

$$\|v_p\|_{p'} \leq \big(\mathrm{vol}(\Omega)\big)^{1-(1/p)}. \tag{38}$$

Since $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$ for $1 < p < \infty$, and since for any p, q satisfying $1 < p, q < \infty$ and any $\phi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega \phi(x)v_p(x)\,dx = L(\phi) = \int_\Omega \phi(x)v_q(x)\,dx,$$

it follows that $v_p = v_q$ a.e. on Ω. Hence we may replace v_p in (37) with a function v belonging to $L^p(\Omega)$ for each p, $1 < p < \infty$, and satisfying, following (38)

$$\|v\|_{p'} \leq \big(\mathrm{vol}(\Omega)\big)^{1-(1/p)} = \big(\mathrm{vol}(\Omega)\big)^{1/p'}.$$

It follows by Theorem 2.14 again that $v \in L^\infty(\Omega)$ and

$$\|v\|_\infty \leq \lim_{p'\to\infty} \big(\mathrm{vol}(\Omega)\big)^{1/p'} = 1. \tag{39}$$

The argument of Paragraph 2.41 shows that there must be equality in (39).

Even if Ω does not have finite volume, we can still write $\Omega = \bigcup_{j=1}^\infty G_j$, where $G_j = \{x \in \Omega : j-1 \leq |x| < j\}$ has finite volume. The sets G_j are mutually disjoint. Let χ_j be the characteristic function of G_j. If $u_j \in L^1(G_j)$, let $\tilde{u}_j$ denote the zero extension of u_j outside G_j. Let $L_j(u_j) = L(\tilde{u}_j)$. Then $L_j \in [L^1(G_j)]'$ and $\|L_j; [L^1(G_j)]'\| \leq 1$. By the finite volume case considered above, there exists $v_j \in L^\infty(G_j)$ such that $\|v_j\|_{\infty,G_j} \leq 1$ and

$$L_j(u_j) = \int_{G_j} u_j(x)v_j(x)\,dx = \int_\Omega \tilde{u}_j(x)v(x)\,dx,$$

where $v(x) = v_j(x)$ for $x \in G_j$, $j = 1, 2, \ldots$, so that $\|v\|_\infty \le 1$. If $u \in L^1(\Omega)$, we put $u = \sum_{j=1}^{\infty} \chi_j u$; the series is norm convergent in $L^1(\Omega)$ by dominated convergence. Since

$$L\left(\sum_{j=1}^{k} \chi_j u\right) = \sum_{j=1}^{k} L_j(\chi_j u) = \int_\Omega \sum_{j=1}^{k} \chi_j(x) u(x) v(x)\, dx,$$

we obtain, passing to the limit by dominated convergence,

$$L(u) = \int_\Omega u(x) v(x)\, dx.$$

It then follows, as in the finite volume case, that $\|v\|_\infty = 1$. ∎

2.46 THEOREM (Reflexivity of $L^p(\Omega)$) $L^p(\Omega)$ is reflexive if and only if $1 < p < \infty$.

Proof. Let $X = L^p(\Omega)$, where $1 < p < \infty$. Since $X' \cong L^{p'}(\Omega)$, we have

$$X'' \cong [L^{p'}(\Omega)]' \cong L^p(\Omega).$$

That is, for every element $w \in X''$ there exists $u \in L^p(\Omega) = X$ such that $w(v) = v(u) = Ju(v)$ for all $v \in X'$, where J is the natural isometric isomorphism of x into X''. (See Paragraph 1.14.) Since the range of J is therefore all of X'', X is reflexive. ∎

Since $L^1(\Omega)$ is separable while its dual, which is isometrically isomorphic to $L^\infty(\Omega)$ is not separable, neither $L^1(\Omega)$ nor $L^\infty(\Omega)$ can be reflexive.

2.47 The Riesz Representation Theorem cannot hold for the space $L^\infty(\Omega)$ in a form analogous to Theorem 2.44, for if so, then the argument of Theorem 2.46 would show that $L^1(\Omega)$ was reflexive. The dual of $L^\infty(\Omega)$ is larger than $L^1(\Omega)$. It may be identified with a space of absolutely continuous, finitely additive set functions of bounded total variation on Ω. See, for example, [Y, p 118] for details.

Mixed-Norm L^p Spaces

2.48 It is sometimes useful to consider L^p type norms of functions on $\mathbb{R}^n$ involving different exponents in different coordinate directions. Given a measurable function u on $\mathbb{R}^n$ and an index vector $\mathbf{p} = (p_1, \ldots, p_n)$ where $0 < p_i \le \infty$ for $1 \le i \le n$, we can calculate the number $\|u\|_\mathbf{p}$ by calculating first the L^{p_1}-norm of $u(x_1, x_2, \ldots, x_n)$ with respect to the variable x_1, and then the L^{p_2}-norm of the result with respect to the variable x_2, and so on, finishing with the L^{p_n}-norm with respect to x_n:

$$\|u\|_\mathbf{p} = \left\| \cdots \left\| \|u\|_{L^{p_1}(dx_1)} \right\|_{L^{p_2}(dx_2)} \cdots \right\|_{L^{p_n}(dx_n)}$$

where

$$\|f\|_{L^q(dt)} = \begin{cases} \left[\displaystyle\int_{-\infty}^{\infty} |f(\ldots,t,\ldots)|^q\,dt\right]^{1/q} & \text{if } 0 < q < \infty \\ \operatorname*{ess\,sup}_t |f(\ldots,t,\ldots)| & \text{if } q = \infty. \end{cases}$$

Of course, $\|\cdot\|_{L^q(dt)}$ is not a norm unless $q \geq 1$. For instance, if all the numbers p_i are finite, then

$$\|u\|_{\mathbf{p}} = \left[\int_{-\infty}^{\infty} \cdots \left[\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} |u(x_1,\ldots,x_n)|^{p_1}\,dx_1\right]^{p_2/p_1} dx_2\right]^{p_3/p_2} dx_3 \cdots dx_n\right]^{1/p}$$

We will denote by $L^{\mathbf{p}} = L^{\mathbf{p}}(\mathbb{R}^n)$ the set of (equivalence classes of almost everywhere equal) functions u for which $\|u\|_{\mathbf{p}} < \infty$; this is a Banach space with norm $\|\cdot\|_{\mathbf{p}}$ if all $p_i \geq 1$. The standard reference for information on these *mixed-norm* spaces is [BP]. All that we require about mixed norms in this book are two elementary results, a version of Hölder's inequality, and an inequality concerning the effect on mixed norms of permuting the order in which the L^{p_i}-norms are calculated.

2.49 (Hölder's Inequality for Mixed Norms) Let $0 < p_i \leq \infty$ and let $0 < q_i \leq \infty$ for $1 \leq i \leq n$. If $u \in L^{\mathbf{p}}$ and $v \in L^{\mathbf{q}}$, then $uv \in L^{\mathbf{r}}$ where

$$\frac{1}{r_i} = \frac{1}{p_i} + \frac{1}{q_i}, \qquad 1 \leq i \leq n, \tag{40}$$

and we have Hölder's inequality:

$$\|uv\|_{\mathbf{r}} \leq \|u\|_{\mathbf{p}}\,\|v\|_{\mathbf{q}}$$

This inequality can be proved by simply applying the (scalar) version of Hölder's inequality given in Corollary 2.5 one variable at a time. As in Corollary 2.5, p_i and q_i are allowed to be less than 1 in this form of Hölder's inequality. The n equations (40) are usually summarized with the convenient abuse of notation

$$\frac{1}{\mathbf{r}} = \frac{1}{\mathbf{p}} + \frac{1}{\mathbf{q}}.$$

The above form of Hölder's inequality can be iterated to provide a version for a product of k functions:

$$\left\|\prod_{j=1}^{k} u_j\right\|_{\mathbf{r}} \leq \prod_{j=1}^{k} \|u_j\|_{\mathbf{p}_j} \qquad \text{where} \quad \frac{1}{\mathbf{r}} = \sum_{j=1}^{k} \frac{1}{\mathbf{p}_j}.$$

2.50 (Permuted Mixed Norms) The definition of $\|u\|_{\mathbf{p}}$ requires the successive L^{p_i}-norms to be calculated in the order of appearance of the variables in the argument of u. This order can be changed by permuting the arguments and associated indices. If σ is a permutation of the set $\{1, 2, \ldots, n\}$, denote $\sigma x = \left(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}\right)$, and let $\sigma\mathbf{p}$ be defined similarly. Let σu be defined by $\sigma u(\sigma x) = u(x)$, that is, $\sigma u(x) = u(\sigma^{-1}x)$. Then $\|\sigma u\|_{\sigma\mathbf{p}}$ is called a permuted mixed norm of u. For example, if $n = 2$ and $\sigma\{1, 2\} = \{2, 1\}$, then

$$\|u\|_{\mathbf{p}} = \left[\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}|u(x_1, x_2)|^{p_1}\,dx_1\right]^{p_2/p_1} dx_2\right]^{1/p_2}$$

$$\|\sigma u\|_{\sigma\mathbf{p}} = \left[\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}|u(x_1, x_2)|^{p_2}\,dx_2\right]^{p_1/p_2} dx_1\right]^{1/p_1}.$$

Note that $\|u\|_{\mathbf{p}}$ and $\|\sigma u\|_{\sigma\mathbf{p}}$ involve the same L^{p_i}-norms with respect to the same variables; only the order of evaluation of those norms has been changed. The question of comparing the sizes of these mixed norms naturally arises.

2.51 THEOREM (The Permutation Inequality for Mixed Norms) Given an index vector $\mathbf{p}$, let σ_* and σ^* be permutations of $\{1, 2, \ldots, n\}$ having components in nondecreasing order and nonincreasing order respectively:

$$p_{\sigma_*(1)} \le p_{\sigma_*(2)} \le \cdots \le p_{\sigma_*(n)},$$
$$p_{\sigma^*(1)} \ge p_{\sigma^*(2)} \ge \cdots \ge p_{\sigma^*(n)}.$$

Then for any permutation σ of $\{1, 2, \ldots, n\}$ and any function u we have

$$\|\sigma_* u\|_{\sigma_*\mathbf{p}} \le \|\sigma u\|_{\sigma\mathbf{p}} \le \left\|\sigma^* u\right\|_{\sigma^*\mathbf{p}}.$$

Proof. Since any permutation can be decomposed into a product of special permutations each of which transposes two adjacent elements and leaves the rest unmoved, proving the inequality reduces to demonstrating the special case: if $p_1 \le p_2 < \infty$, then

$$\left[\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}|u|^{p_1}\,dx_1\right]^{p_2/p_1} dx_2\right]^{1/p_2} \le \left[\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}|u|^{p_2}\,dx_2\right]^{p_1/p_2} dx_1\right]^{1/p_1}.$$

But this is just a version of Minkowski's inequality for integrals (Theorem 2.9), namely

$$\left\|\int_{-\infty}^{\infty}|v(x_1, x_2)|\,dx_1\right\|_{L^r(dx_2)} \le \int_{-\infty}^{\infty}\|v(x_1, \cdot)\|_{L^r(dx_2)}\,dx_1$$

applied to $v = |u|^{p_1}$ with $r = p_2/p_1$. The case where $p_2 = \infty$ is easier. ∎

2.52 REMARK Similar permutation inequalities hold for mixed norm ℓ^p spaces and for hybrid mixtures of ℓ^p and L^q norms. We will use such inequalities in Chapter 7.

The Marcinkiewicz Interpolation Theorem

2.53 (Distribution Functions) Let Ω be a domain in $\mathbb{R}^n$ and u be a measurable function defined on Ω. For $t \geq 0$, let

$$\Omega_{u,t} = \{x \in \Omega : |u(x)| > t\}.$$

We define the *distribution function* of u to be

$$\delta_u(t) = \mu\left(\Omega_{u,t}\right),$$

where μ is the Lebesgue measure on $\mathbb{R}^n$. Evidently δ_u is nonincreasing for $t \geq 0$ and if $|u(x)| \leq |v(x)|$ a.e. on Ω, then $\delta_u(t) \leq \delta_v(t)$ for $t \geq 0$.

Since $|u(x)| > t$ implies $|u(x)| > t + (1/k)$ for some integer $k > 0$, we have $\Omega_{u,t} = \bigcup_{k=1}^{\infty} \Omega_{u,t+(1/k)}$ and it follows that δ_u is right continuous on the interval $[0, \infty)$. Similarly, if $|u(x)|$ is an increasing limit of $\{|u_j(x)|\}$ at each x, then $|u(x)| > t$ implies $|u_j(x)| > t$ for some j and so $\Omega_{u,t} = \bigcup_{j=1}^{\infty} \Omega_{u_j,t}$. Hence $\lim_{j\to\infty} \delta_{u_j}(t) = \delta_u(t)$.

If $|u(x) + v(x)| > t$, then either $|u(x)| > t/2$ or $|v(x)| > t/2$ (or both), so that $\Omega_{u+v,t} \subset \Omega_{u,t/2} \cup \Omega_{v,t/2}$ and hence

$$\delta_{u+v}(t) \leq \delta_u(t/2) + \delta_v(t/2). \tag{41}$$

Now suppose $u \in L^p(\Omega)$ for some p satisfying $0 < p < \infty$. For $t > 0$ we have

$$\|u\|_p^p = \int_\Omega |u(x)|^p\,dx \geq \int_{\Omega_{u,t}} |u(x)|^p\,dx \geq t^p \mu\left(\Omega_{u,t}\right),$$

from which we obtain *Chebyshev's inequality*

$$\delta_u(t) = \mu\left(\Omega_{u,t}\right) \leq t^{-p}\,\|u\|_p^p. \tag{42}$$

2.54 LEMMA If $0 < p < \infty$, then

$$\|u\|_p^p = \int_\Omega |u(x)|^p\,dx = p\int_0^\infty t^{p-1}\delta_u(t)\,dt. \tag{43}$$

Proof. First suppose $|u|$ is a simple function, say

$$|u(x)| = a_j \quad \text{on} \quad A_j \subset \Omega, \qquad 1 \le j \le k,$$

where $0 < a_1 < a_2 < \cdots < a_k$ and $A_i \cap A_j$ is empty for $i \ne j$. Then

$$\delta_u(t) = \begin{cases} \sum_{i=1}^{k} \mu(A_i) & \text{if } t < a_1 \\ \sum_{i=j}^{k} \mu(A_i) & \text{if } a_{j-1} \le t < a_j, \quad (2 \le j \le k) \\ 0 & \text{if } t \ge a_k. \end{cases}$$

Therefore,

$$\begin{aligned} p\int_0^\infty t^{p-1}\delta_u(t)\,dt &= p\left(\int_0^{a_1} + \sum_{j=2}^{k}\int_{a_{j-1}}^{a_j} + \int_{a_k}^\infty\right) t^{p-1}\delta_u(t)\,dt \\ &= a_1^p \sum_{j=1}^{k}\mu(A_j) + \sum_{j=2}^{k}\left(a_j^p - a_{j-1}^p\right)\sum_{i=j}^{k}\mu(A_i) \\ &= \sum_{j=1}^{k} a_j^p \mu(A_j) = \|u\|_p^p, \end{aligned}$$

so (43) holds for simple functions. By Theorem 1.44, if u is measurable, then $|u|$ is a limit of a monotonically increasing sequence of measurable simple functions. Equation (43) now follows by monotone convergence. ∎

2.55 (Weak L^p Spaces) If u is a measurable function on Ω, let

$$[u]_p = [u]_{p,\Omega} = \left(\sup_{t>0} t^p \delta_u(t)\right)^{1/p}.$$

We define the space weak-$L^p(\Omega)$ as follows:

$$\text{weak-}L^p(\Omega) = \big\{u : [u]_p < \infty\big\}.$$

It is easily checked that $[cu]_p = |c|[u]_p$ for complex c, but $[\cdot]_p$ is not a norm on weak-$L^p(\Omega)$ because it does not satisfy the triangle inequality. However, by (41)

$$\begin{aligned} [u+v]_p &= \left(\sup_{t>0} t^p \delta_{u+v}(t)\right)^{1/p} \\ &\le \left(2^p \sup_{t>0}\left(\frac{t}{2}\right)^p \delta_u(t/2) + 2^p \sup_{t>0}\left(\frac{t}{2}\right)^p \delta_v(t/2)\right)^{1/p} \\ &= 2\big([u]_p + [v]_p\big), \end{aligned}$$

so weak-$L^p(\Omega)$ is a vector space and the "open balls" $B_r(u) = \{v \in \text{weak-}L^p(\Omega) : [v - u]_p < r\}$ do generate a topology on weak-$L^p(\Omega)$ with respect to which weak-$L^p(\Omega)$ is a topological vector space. A functional $[\cdot]$ with the properties of a norm except that the triangle inequality is replaced with a weaker version of the form $[u + v] \le K([u] + [v])$ for some constant $K > 1$ is called a *quasi-norm*.

Chebyshev's inequality (42) shows that $[u]_p \le \|u\|_p$ so that $L^p(\Omega) \subset \text{weak-}L^p(\Omega)$. The inclusion is strict since, if $x_0 \in \Omega$ it is easily shown that $u(x) = |x - x_0|^{-n/p}$ belongs to weak-$L^p(\Omega)$ but not to $L^p(\Omega)$.

2.56 (Strong and Weak Type Operators) A operator F mapping a vector space X of measurable functions into another such space Y is called *sublinear* if, for all $u, v \in X$ and scalars c,

$$|F(u + v)| \le |F(u)| + |F(v)|, \qquad \text{and}$$
$$|T(cu)| = |c||T(u)|.$$

A linear operator from X into Y is certainly sublinear. We will be especially concerned with operators from L^p spaces on a domain Ω in $\mathbb{R}^n$ into $L^q(\Omega')$ or weak-$L^q(\Omega')$ where Ω' is a domain in $\mathbb{R}^k$ with k not necessarily equal to n.

We distinguish two important classes of sublinear operators:

(a) F is of *strong type* (p, q), where $1 \le p \le \infty$ and $1 \le q \le \infty$, if F maps $L^p(\Omega)$ into $L^q(\Omega')$ and there exists a constant K such that for all $u \in L^p(\Omega)$,

$$\|F(u)\|_{q,\Omega'} \le K\, \|u\|_{p,\Omega}\,.$$

(b) F is of *weak type* (p, q), where $1 \le p \le \infty$ and $1 \le q < \infty$, if F maps $L^p(\Omega)$ into weak-$L^q(\Omega')$ and there exists a constant K such that for all $u \in L^p(\Omega)$,

$$[F(u)]_{q,\Omega'} \le K\, \|u\|_{p,\Omega}\,.$$

We also say that F is of weak type (p, ∞) if F is of strong type (p, ∞).

Strong type (p, q) implies weak type (p, q) but not conversely unless $q = \infty$.

2.57 The following theorem has its origins in the work of Marcinkiewicz [Mk] and was further developed by Zygmund [Z]. It is valid in more general contexts than stated here, but we only need it for operators between L^p spaces on domains in $\mathbb{R}^n$ and only state it in this context. It will form one of the cornerstones on which our proof of the Sobolev imbedding theorem will rest. In that context it will only be used for linear operators.

Because the Marcinkiewicz theorem involves an operator on a vector space containing two different L^p spaces, say X and Y, (over the same domain) it is convenient to consider its domain to be the sum of those spaces, that is the vector space consisting of sums $u + v$ where $u \in X$ and $v \in Y$.

There are numerous proofs of the Marcinkiewicz theorem in the literature. See, for example, [St] and [SW]. Our proof is based on Folland [Fo].

2.58 THEOREM (The Marcinkiewicz Interpolation Theorem) Let $1 \le p_1 \le q_1 < \infty$ and $1 \le p_2 \le q_2 \le \infty$, with $q_1 < q_2$. Suppose the numbers p and q satisfy

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \qquad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2},$$

where $0 < \theta < 1$. Let Ω and Ω' be domains in $\mathbb{R}^n$ and $\mathbb{R}^k$, respectively; k may or may not be equal to n. Let F be a sublinear operator from $L^{p_1}(\Omega) + L^{p_2}(\Omega)$ into the space of measurable functions on Ω'. If F is of weak type (p_1, q_1) and also of weak type (p_2, q_2), then F is of strong type (p, q). That is, if

$$[F(u)]_{q_j,\Omega'} \le K_j \, \|u\|_{p_j,\Omega}\,, \qquad j = 1, 2,$$

then

$$\|F(u)\|_{q,\Omega'} \le K \, \|u\|_{p,\Omega}\,,$$

where the constant K depends only on p, p_1, q_1, p_2, q_2, K_1, and K_2.

Proof. First consider the case where $q_1 < q < q_2 < \infty$ so that p_1 and p_2 are necessarily both finite. The conditions satisfied by p and q imply that $(1/p, 1/q)$ is an interior point of the line segment joining (p_1^{-1}, q_1^{-1}) and (p_2^{-1}, q_2^{-1}) in the (p, q)-plane. Let c be the extended real number equal to q/p times the slope of that line segment;

$$c = \frac{p_1(q_1 - q)}{q_1(p_1 - p)} = \frac{p_2(q_2 - q)}{q_2(p_2 - p)}. \tag{44}$$

Given any $T > 0$, a measurable function u on Ω can be written as a sum of a "small" part $u_{S,T}$ and a "big" part $u_{B,T}$ defined as follows:

$$u_{S,T}(x) = \begin{cases} u(x) & \text{if } |u(x)| \le T \\ T\,\dfrac{u(x)}{|u(x)|} & \text{if } |u(x)| > T, \end{cases}$$

$$u_{B,T}(x) = u(x) - u_{S,T}(x) = \begin{cases} 0 & \text{if } |u(x)| \le T \\ u(x)\left(1 - \dfrac{T}{|u(x)|}\right) & \text{if } |u(x)| > T. \end{cases}$$

Since $|u_{S,T}(x)| \le T$ and $|u_{B,T}(x)| = \max\{0, |u(x)| - T\}$ for all $x \in \Omega$, the distribution functions of $u_{S,T}$ and $u_{B,T}$ are given by

$$\delta_{u_{S,T}}(t) = \begin{cases} \delta_u(t) & \text{if } t < T \\ 0 & \text{if } t \ge T, \end{cases}$$

$$\delta_{u_{B,T}}(t) = \delta_u(t + T).$$

It follows, using (43), that

$$\begin{aligned}
\int_\Omega |u_{S,T}(x)|^{p_2}\,dx &= p_2 \int_0^\infty t^{p_2-1}\delta_{u_{S,T}}(t)\,dt = p_2 \int_0^T t^{p_2-1}\delta_u(t)\,dt \\
\int_\Omega |u_{B,T}(x)|^{p_1}\,dx &= p_1 \int_0^\infty t^{p_1-1}\delta_{u_{B,T}}(t)\,dt = p_1 \int_0^\infty t^{p_1-1}\delta_u(t+T)\,dt \\
&= p_1 \int_T^\infty (t-T)^{p_1-1}\delta_u(t)\,dt \le p_1 \int_T^\infty t^{p_1-1}\delta_u(t)\,dt.
\end{aligned}$$

Using (43) followed by the sublinearity of F and inequality (41), we calculate

$$\begin{aligned}
\int_{\Omega'} |F(u)(y)|^q\,dy &= q \int_0^\infty t^{q-1}\delta_{F(u)}(t)\,dt \\
&= 2^q q \int_0^\infty t^{q-1}\delta_{F(u)}(2t)\,dt \\
&\le 2^q q \int_0^\infty t^{q-1}\delta_{F(u_{S,T})+F(u_{B,T})}(2t)\,dt \\
&\le 2^q q \int_0^\infty t^{q-1}\delta_{F(u_{S,T})}(t)\,dt + 2^q q \int_0^\infty t^{q-1}\delta_{F(u_{B,T})}(t)\,dt. \qquad (45)
\end{aligned}$$

This inequality holds for any $T > 0$; we can choose T to depend on t if we wish. In the following, let $T = t^c$ where c is given by (44). For positive s, the definition of $[\cdot]_s$ implies that $\delta_v(t) \le t^{-s}[v]_s^s$. Using this and the given estimate $[F(v)]_{q_2,\Omega'} \le K_2\,\|v\|_{p,\Omega}$ we obtain

$$\begin{aligned}
\int_0^\infty t^{q-1}\delta_{F(u_{S,T})}(t)\,dt &\le \int_0^\infty t^{q-1-q_2}[F(u_{S,T})]_{q_2}^{q_2}\,dt \\
&\le \int_0^\infty t^{q-1-q_2}\left(K_2\,\|u_{S,T}\|_{p_2}\right)^{q_2}\,dt \\
&\le K_2^{q_2} p_2^{q_2/p_2} \int_0^\infty t^{q-1-q_2}\left[\int_0^{t^c} \tau^{p_2-1}\delta_u(\tau)\,d\tau\right]^{q_2/p_2} dt \\
&= K_2^{q_2} p_2^{q_2/p_2} I_2.
\end{aligned}$$

Since $q_2 \ge p_2$ we can estimate the latter iterated integral I_2 using Minkowski's inequality for integrals, Theorem 2.9.

$$
\begin{aligned}
I_2 &= \int_0^\infty \left[\int_0^{t^c} t^{(q-1-q_2)(p_2/q_2)} \tau^{p_2-1} \delta_u(\tau)\, d\tau\right]^{q_2/p_2} dt \\
&\le \left[\int_0^\infty \left(\int_{\tau^{1/c}}^\infty t^{q-1-q_2} \left(\tau^{p_2-1}\delta_u(\tau)\right)^{q_2/p_2} dt\right)^{p_2/q_2} d\tau\right]^{q_2/p_2} \\
&= \left[\int_0^\infty \tau^{p_2-1}\delta_u(\tau) \left(\int_{\tau^{1/c}}^\infty t^{q-1-q_2}\, dt\right)^{p_2/q_2} d\tau\right]^{q_2/p_2} \\
&= \left[\frac{1}{q_2-q}\int_0^\infty \tau^{p_2-1+[(q-q_2)/c](p_2/q_2)} \delta_u(\tau)\, d\tau\right]^{q_2/p_2} \\
&= \left(\frac{1}{q_2-q}\int_0^\infty \tau^{p-1}\delta_u(\tau)\, d\tau\right)^{q_2/p_2} = \left(\frac{1}{p(q_2-q)} \|u\|_{p,\Omega}^p\right)^{q_2/p_2}.
\end{aligned}
$$

It follows that

$$
2^q q \int_0^\infty t^{q-1} \delta_{F(u_{S,T})}(t)\, dt \le 2^q q K_2^{q_2} \left(\frac{p_2}{p(q_2-q)} \|u\|_{p,\Omega}^p\right)^{q_2/p_2}. \tag{46}
$$

An entirely parallel argument using $q_1 < q$ instead of $q_2 > q$ shows that

$$
2^q q \int_0^\infty t^{q-1} \delta_{F(u_{B,T})}(t)\, dt \le 2^q q K_1^{q_1} \left(\frac{p_1}{p(q-q_1)} \|u\|_{p,\Omega}^p\right)^{q_1/p_1}. \tag{47}
$$

If $\|u\|_{p,\Omega} = 1$, we therefore have

$$
\|F(u)\|_{q,\Omega'} \le K = 2q^{1/q}\left[\left(\frac{p_2 K_2^{p_2}}{p(q_2-q)}\right)^{q_2/p_2} + \left(\frac{p_1 K_1^{p_1}}{p(q-q_1)}\right)^{q_1/p_1}\right]^{1/q}.
$$

By the homogeneity of F, if $u \ne 0$ in $L^p(\Omega)$, then

$$
\begin{aligned}
\|F(u)\|_{q,\Omega'} &= \left\| F\left(\|u\|_{p,\Omega} \frac{u}{\|u\|_{p,\Omega}}\right)\right\|_{q,\Omega'} \\
&= \|u\|_{p,\Omega} \left\| F\left(\frac{u}{\|u\|_{p,\Omega}}\right)\right\|_{q,\Omega'} \le K\, \|u\|_{p,\Omega}.
\end{aligned}
$$

Now we examine the case where $q_2 = \infty$. It is possible to choose T (depending on t) in the above argument to ensure that $\delta_{F(u_{S,T})}(t) = 0$ for all $t > 0$. If $p_2 = \infty$, the appropriate choice is $T = t/K_2$ for then

$$
\left\|F(u_{S,T})\right\|_{\infty,\Omega'} \le K_2 \left\|u_{S,T}\right\|_{\infty,\Omega} \le K_2 T = t,
$$

and $\delta_{F(u_{S,T})}(t) = 0$. If $p_2 < \infty$, the appropriate choice is

$$T = \left(\frac{t}{K_2\bigl(p_2 \|u\|_{p,\Omega}^p / p\bigr)^{1/p_2}} \right)^c ,$$

where $c = p_2/(p_2 - p)$, the limit as $q_2 \to \infty$ of the value of c used in the earlier part of this proof. For this choice of T,

$$\begin{aligned}
\|F(u_{S,T})\|_{\infty,\Omega'}^{p_2} &\le K_2^{p_2} \|u_{S,T}\|_{p_2}^{p_2} = K_2^{p_2} p_2 \int_0^T t^{p_2-1}\delta_{u_{S,T}}(t)\,dt \\
&\le K_2^{p_2} p_2 T^{p_2-p} \int_0^T t^{p-1}\delta_u(t)\,dt \\
&\le K_2^{p_2} p_2 T^{p_2-p} \int_0^\infty t^{p-1}\delta_u(t)\,dt \\
&= K_2^{p_2} p_2 T^{p_2-p}(1/p) \|u\|_{p,\Omega}^p = t^{p_2},
\end{aligned}$$

and again $\delta_{F(u_{S,T})}(t) = 0$. In either of these cases the first term in (45) is zero and an estimate similar to (47) holds for the second term provided $p_1 < p_2$.

If $q_2 = \infty$ and $p_2 < p_1 < \infty$ we can instead assure that the second term in (45) is zero by choosing T to force $\delta_{F(u_{B,t})}(t) = 0$ and obtain an estimate similar to (46) for the first term.

There remains one case to be considered: $q_1 < q < q_2 = \infty$, $p_1 = p = p_2 < \infty$. In this case it follows directly from the definition of $[\cdot]_s$ that

$$t^{q_1}\delta_{F(u)}(t) \le [F(u)]_{q_1}^{q_1} \le K_1^{q_1} \|u\|_{p,\Omega}^{q_1},$$

and hence $\delta_{F(u)} \le \bigl(K_1 \|u\|_{p,\Omega}/t\bigr)^{q_1}$. On the other hand, $\delta_{F(u)}(t) = 0$ if we have $t \ge T = K_2 \|u\|_{p,\Omega} \ge \|F(u)\|_{\infty,\Omega'}$. Thus

$$\begin{aligned}
\|F(u)\|_{q,\Omega'}^q &= q\int_0^\infty t^{q-1}\delta_{F(u)}(t)\,dt = q\int_0^T t^{q-1}\delta_{F(u)}(t)\,dt \\
&\le q\bigl(K_1 \|u\|_{p,\Omega}\bigr)^{q_1} \int_0^T t^{q-1-q_1}\,dt = K^q \|u\|_{p,\Omega}^{q_1},
\end{aligned}$$

where K is a finite constant because $q_1 < q$. This completes the proof. ∎

3

THE SOBOLEV SPACES $W^{m,p}(\Omega)$

In this chapter we introduce Sobolev spaces of integer order and establish some of their most important properties. These spaces are defined over an arbitrary domain $\Omega \subset \mathbb{R}^n$ and are vector subspaces of various Lebesgue spaces $L^p(\Omega)$.

Definitions and Basic Properties

3.1 (The Sobolev Norms) We define a functional $\|\cdot\|_{m,p}$, where m is a positive integer and $1 \le p \le \infty$, as follows:

$$\|u\|_{m,p} = \left(\sum_{0 \le |\alpha| \le m} \|D^\alpha u\|_p^p \right)^{1/p} \quad \text{if} \quad 1 \le p < \infty, \tag{1}$$

$$\|u\|_{m,\infty} = \max_{0 \le |\alpha| \le m} \|D^\alpha u\|_\infty \tag{2}$$

for any function u for which the right side makes sense, $\|\cdot\|_p$ being, of course, the norm in $L^p(\Omega)$. In some situations where confusion of domains may occur we will use $\|u\|_{m,p,\Omega}$ in place of $\|u\|_{m,p}$. Evidently (1) or (2) defines a norm on any vector space of functions on which the right side takes finite values provided functions are identified in the space if they are equal almost everywhere in Ω.

3.2 (Sobolev Spaces) For any positive integer m and $1 \le p \le \infty$ we consider three vector spaces on which $\|\cdot\|_{m,p}$ is a norm:

(a) $H^{m,p}(\Omega) \equiv$ the completion of $\{u \in C^m(\Omega) : \|u\|_{m,p} < \infty\}$ with respect to the norm $\|\cdot\|_{m,p}$,

(b) $W^{m,p}(\Omega) \equiv \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega)$ for $0 \le |\alpha| \le m\}$, where $D^\alpha u$ is the weak (or distributional) partial derivative of Paragraph 1.62, and

(c) $W_0^{m,p}(\Omega) \equiv$ the closure of $C_0^\infty(\Omega)$ in the space $W^{m,p}(\Omega)$.

Equipped with the appropriate norm (1) or (2) these are called *Sobolev spaces* over Ω. Clearly $W^{0,p}(\Omega) = L^p(\Omega)$, and if $1 \le p < \infty$, $W_0^{0,p}(\Omega) = L^p(\Omega)$ because $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$. (See Paragraph 2.30.) For any m, we have the obvious chain of imbeddings

$$W_0^{m,p}(\Omega) \to W^{m,p}(\Omega) \to L^p(\Omega).$$

We will show in Theorem 3.17 that $H^{m,p}(\Omega) = W^{m,p}(\Omega)$ for every domain Ω. This result, published in 1964 by Meyers and Serrin [MS] ended much confusion about the relationship of these spaces that existed in the literature before that time. It is surprising that this elementary result remained undiscovered for so long.

The spaces $W^{m,p}(\Omega)$ were introduced by Sobolev [So1,So2]. Many related spaces were being studied by other writers, in particular Morrey [Mo] and Deny and Lions [DL]. Many different symbols ($W^{m,p}$, $H^{m,p}$, $P^{m,p}$, L_p^m, etc.) have been used to denote these spaces and their variants, and before they became generally associated with the name of Sobolev they were sometimes referred to under other names, for example, as Beppo Levi spaces.

Numerous generalizations and specializions of the basic spaces $W^{m,p}(\Omega)$ have been constructed. Much of this literature originated in the Soviet Union. In particular, there are extensions that allow arbitrary real values of m (see Chapter 7) which are interpreted as corresponding to fractional orders of differentiation. There are weighted spaces that introduce weight functions into the L^p norms; see Kufner [Ku]. There are spaces of vector fields that are annihilated by differential operators like curl and divergence; see [DaL]. Other generalizations involve different orders of differentiation and different L^p norms in different coordinate directions (anisotropic spaces — see [BIN1, BIN2]), and Orlicz-Sobolev spaces (see Chapter 8) modeled on the generalizations of L^p spaces known as Orlicz spaces. Finally, there has been much work on the interaction between Sobolev spaces and differential geometry [Hb] and a flurry of recent activity on Sobolev spaces on metric spaces [Hn, HK].

We will not be able to investigate the most of these generalizations here.

3.3 THEOREM $W^{m,p}(\Omega)$ is a Banach space.

Proof. Let $\{u_n\}$ be a Cauchy sequence in $W^{m,p}(\Omega)$. Then $\{D^\alpha u\}$ is a Cauchy sequence in $L^p(\Omega)$ for $0 \le |\alpha| \le m$. Since $L^p(\Omega)$ is complete there exist functions u and u_α, $0 \le |\alpha| \le m$, such that $u_n \to u$ and $D^\alpha u_n \to u_\alpha$ in $L^p(\Omega)$ as

$n \to \infty$. Now $L^p(\Omega) \subset L^1_{\mathrm{loc}}(\Omega)$ and so u_n determines a distribution $T_{u_n} \in \mathscr{D}'(\Omega)$ as in Paragraph 1.58. For any $\phi \in \mathscr{D}(\Omega)$ we have

$$|T_{u_n}(\phi) - T_u(\phi)| \le \int_\Omega |u_n(x) - u(x)||\phi(x)|\,dx \le \|\phi\|_{p'}\,\|u_n - u\|_p$$

by Hölder's inequality, where p' is the exponent conjugate to p. Therefore $T_{u_n}(\phi) \to T_u(\phi)$ for every $\phi \in \mathscr{D}(\Omega)$ as $n \to \infty$. Similarly, $T_{D^\alpha u_n}(\phi) \to T_{u_\alpha}(\phi)$ for every $\phi \in \mathscr{D}(\Omega)$. It follows that

$$T_{u_\alpha}(\phi) = \lim_{n\to\infty} T_{D^\alpha u_n}(\phi) = \lim_{n\to\infty} (-1)^{|\alpha|} T_{u_n}(D^\alpha \phi) = (-1)^{|\alpha|} T_u(D^\alpha\phi)$$

for every $\phi \in \mathscr{D}(\Omega)$. Thus $u_\alpha = D^\alpha u$ in the distributional sense on Ω for $0 \le |\alpha| \le m$, whence $u \in W^{m,p}(\Omega)$. Since $\lim_{n\to\infty} \|u_n - u\|_{m,p} = 0$, the space $W^{m,p}(\Omega)$ is complete. ∎

3.4 COROLLARY $H^{m,p}(\Omega) \subset W^{m,p}(\Omega)$.

Proof. Distributional and classical partial derivatives coincide whenever the latter exist and are continuous on Ω. Therefore the set

$$S = \{\phi \in C^m(\Omega) : \|\phi\|_{m,p} < \infty\}$$

is contained in $W^{m,p}(\Omega)$. Since $W^{m,p}(\Omega)$ is complete, the identity operator on S extends to an isometric isomorphism between $H^{m,p}(\Omega)$, the completion of S, and the closure of S in $W^{m,p}(\Omega)$. We can identify $H^{m,p}(\Omega)$ with this closure. ∎

3.5 Several important properties of the spaces $W^{m,p}(\Omega)$ can be easily obtained by regarding $W^{m,p}(\Omega)$ as a closed subspace of an L^p space on a union of disjoint copies of Ω.

Given integers $n \ge 1$ and $m \ge 0$, let $N \equiv N(n, m)$ be the number of multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ such that $|\alpha| \le m$. For each such multi-index α let Ω_α be a copy of Ω in a different copy of $\mathbb{R}^n$, so that the N domains Ω_α are *de facto* mutually disjoint. Let $\Omega^{(m)}$ be the union of these N domains; $\Omega^{(m)} = \bigcup_{|\alpha|\le m} \Omega_\alpha$. Given a function u in $W^{m,p}(\Omega)$, let U be the function on $\Omega^{(m)}$ that coincides with $D^\alpha u$ on Ω_α. It is easy to check that the map P taking u to U is an isometry from $W^{m,p}(\Omega)$ into $L^p(\Omega^{(m)})$. Since $W^{m,p}(\Omega)$ is complete, the range W of the isometry P is a closed subspace of $L^p(\Omega^{(m)})$. It follows that W is separable if $1 \le p < \infty$, and is uniformly convex and reflexive if $1 < p < \infty$. The same conclusions must therefore hold for $W^{m,p}(\Omega) = P^{-1}(W)$.

3.6 THEOREM $W^{m,p}(\Omega)$ is separable if $1 \le p < \infty$, and is uniformly convex and reflexive if $1 < p < \infty$. In particular, $W^{m,2}(\Omega)$ is a separable Hilbert space with inner product

$$(u, v)_m = \sum_{0\le|\alpha|\le m} (D^\alpha u, D^\alpha v),$$

where $(u, v) = \int_\Omega u(x)\overline{v(x)}\,dx$ is the inner product on $L^2(\Omega)$. ∎

Duality and the Spaces $W^{-m,p'}(\Omega)$

3.7 In this section we shall take, for fixed Ω, m, and p, the number N, the spaces $L^p(\Omega^{(m)})$ and W, and the operator P to be specified as in Paragraph 3.5. We also define

$$\langle u, v\rangle = \int_\Omega u(x)v(x)\,dx$$

for any functions u, v for which the right side makes sense. For given p let us agree that p' always denotes the conjugate exponent:

$$p' = \begin{cases} \infty & \text{if } p = 1 \\ p/(p-1) & \text{if } 1 < p < \infty \\ 1 & \text{if } p = \infty. \end{cases}$$

First we extend the Riesz Representation Theorem to the space $W^{m,p}(\Omega)$. Then, we identify the dual of $W_0^{m,p}(\Omega)$ with a subspace of $\mathscr{D}'(\Omega)$. Finally, we show that if $1 < p < \infty$, the dual of $W_0^{m,p}(\Omega)$ can also be identified with the completion of $L^{p'}(\Omega)$ with respect to a norm weaker than the usual $L^{p'}$ norm.

3.8 (The Dual of $L^p(\Omega^{(m)})$) To every $L \in (L^p(\Omega^{(m)}))'$, where $1 \le p < \infty$, there corresponds a unique $v \in L^{p'}(\Omega^{(m)})$ such that for every $u \in L^p(\Omega^{(m)})$,

$$L(u) = \int_{\Omega^{(m)}} u(x)v(x)\,dx = \sum_{|\alpha|\le m}\int_{\Omega_\alpha} u_\alpha(x)v_\alpha(x)\,dx = \sum_{|\alpha|\le m}\langle u_\alpha, v_\alpha\rangle,$$

where u_α and v_α are the restrictions of u and v, respectively, to Ω_α. Moreover, $\left\|L\,;\left(L^p(\Omega^{(m)})\right)'\right\| = \left\|v\,;L^{p'}(\Omega^{(m)})\right\|$. Thus $(L^p(\Omega^{(m)}))' \equiv L^{p'}(\Omega^{(m)})$.

This is valid because $L^p(\Omega^{(m)})$ is, after all, an L^p space, albeit one defined on an unusual domain.

3.9 THEOREM (The Dual of $W^{m,p}(\Omega)$) Let $1 \le p < \infty$. For every $L \in \left(W^{m,p}(\Omega)\right)'$ there exist elements $v \in L^{p'}(\Omega^{(m)})$ such that if the restriction of v to Ω_α is v_α, we have for all $u \in W^{m,p}(\Omega)$

$$L(u) = \sum_{0\le|\alpha|\le m}\langle D^\alpha u, v_\alpha\rangle. \tag{3}$$

Moreover

$$\|L;\left(W^{m,p}(\Omega)\right)'\| = \inf\|v;L^{p'}(\Omega^{(m)})\| = \min\|v;L^{p'}(\Omega^{(m)})\|, \tag{4}$$

the infimum being taken over, and *attained on* the set of all $v \in L^{p'}(\Omega^{(m)})$ for which (3) holds for every $u \in W^{m,p}(\Omega)$.

If $1 < p < \infty$, the element $v \in L^{p'}(\Omega^{(m)})$ satisfying (3) and (4) is unique.

Proof. A linear functional L^* is defined as follows on the range W of the operator P defined in Paragraph 3.5:

$$L^*(Pu) = L(u), \qquad u \in W^{m,p}(\Omega).$$

Since P is an isometric isomorphism, $L^* \in W'$ and

$$\|L^*;\, W'\| = \left\| L\,;\, \big(W^{m,p}(\Omega)\big)' \right\|.$$

By the Hahn-Banach Theorem 1.13 there exists a norm preserving extension $\hat{L}$ of L^* to all of $L^p(\Omega^{(m)})$, and, as observed in Paragraph 3.8 there exists $v \in L^{p'}(\Omega^{(m)})$ such that if $u \in L^p(\Omega^{(m)})$, then

$$\hat{L}(u) = \sum_{0\le|\alpha|\le m} \langle u_\alpha, v_\alpha\rangle.$$

Thus, for $u \in W^{m,p}(\Omega)$ we obtain

$$L(u) = L^*(Pu) = \hat{L}(Pu) = \sum_{0\le|\alpha|\le m} \langle D^\alpha u, v_\alpha\rangle.$$

Moreover,

$$\|L;\, \big(W^{m,p}(\Omega)\big)'\| = \|L^*;\, W'\| = \|\hat{L};\, (L^p(\Omega^{(m)}))'\| = \|v;\, L^{p'}(\Omega^{(m)})\|.$$

Now (4) must hold because any element $v \in L^{p'}(\Omega^{(m)})$ for which (3) holds for every $u \in W^{m,p}(\Omega)$ corresponds to an extension L of L^* and so will have norm $\|v;\, L^{p'}(\Omega^{(m)})\|$ not less than $\|L;\, \big(W^{m,p}(\Omega)\big)'\|$.

The uniqueness of v if $1 < p < \infty$ follows from the uniform convexity of $L^p(\Omega^{(m)})$ and $L^{p'}(\Omega^{(m)})$ by an argument similar to that in Lemma 2.43. ∎

3.10 If $1 \le p < \infty$ every element L of $\big(W^{m,p}(\Omega)\big)'$ is an extension to $W^{m,p}(\Omega)$ of a distribution $T \in \mathscr{D}'(\Omega)$. To see what form this distribution takes, suppose L is given by (3) for some $v \in L^{p'}(\Omega^{(m)})$ and define T and T_{v_α} on $\mathscr{D}(\Omega)$ by

$$T = \sum_{0\le|\alpha|\le m} (-1)^{|\alpha|} D^\alpha T_{v_\alpha}, \quad T_{v_\alpha}(\phi) = \langle \phi, v_\alpha\rangle. \quad 0 \le |\alpha| \le m, \tag{5}$$

For every $\phi \in \mathscr{D}(\Omega) \subset W^{m,p}(\Omega)$ we have $T(\phi) = \sum_{0\le|\alpha|\le m} T_{v_\alpha}(D^\alpha \phi) = L(\phi)$ so that L is clearly an extension of T. Moreover, by (4)

$$\|L;\, \big(W^{m,p}(\Omega)\big)'\| = \min\big\{\|v;\, L^{p'}(\Omega^{(m)})\| \;:\; L \text{ extends } T \text{ given by (5)}\big\}.$$

These remarks also hold for $L \in \left(W_0^{m,p}(\Omega)\right)'$ since any such functional possesses a norm-preserving extension to $W^{m,p}(\Omega)$.

3.11 Now suppose T is any element of $\mathscr{D}'(\Omega)$ having the form (5) for some $v \in L^{p'}(\Omega^{(m)})$, where $1 \le p' \le \infty$. Then T possesses (possibly non-unique) continuous extensions to $W^{m,p}(\Omega)$. However, T possesses a unique continuous extension to $W_0^{m,p}(\Omega)$. To see this, for $u \in W_0^{m,p}(\Omega)$ let $\{\phi_n\}$ be a sequence in $C_0^\infty(\Omega) = \mathscr{D}(\Omega)$ converging to u in norm in $W_0^{m,p}(\Omega)$. Then

$$\begin{aligned} |T(\phi_k) - T(\phi_n)| &\le \sum_{0\le|\alpha|\le m} |T_{v_\alpha}(D^\alpha\phi_k - D^\alpha\phi_n)| \\ &\le \sum_{0\le|\alpha|\le m} \|D^\alpha(\phi_k - \phi_n)\|_p \, \|v_\alpha\|_{p'} \\ &\le \|\phi_k - \phi_n\|_{m,p} \, \|v; L^{p'}(\Omega^{(m)})\| \to 0 \quad \text{as } k, n \to \infty. \end{aligned}$$

Thus $\{T(\phi_n)\}$ is a Cauchy sequence in $\mathbb{C}$ and so converges to a limit that we can denote by $L(u)$ since it is clear that if also $\{\psi_n\} \subset \mathscr{D}(\Omega)$ and $\|\psi_n - u\|_{m,p} \to 0$, then $T(\phi_n) - T(\psi_n) \to 0$ as $n \to \infty$. The functional L thus defined is linear and belongs to $\left(W_0^{m,p}(\Omega)\right)'$, for if $u = \lim_{n\to\infty} \phi_n$ as above, then

$$|L(u)| = \lim_{n\to\infty} |T(\phi_n)| \le \lim_{n\to\infty} \|\phi_n\|_{m,p} \, \|v; L^{p'}(\Omega^{(m)})\| = \|u\|_{m,p} \, \|v; L^{p'}(\Omega^{(m)})\|.$$

We have therefore proved the following theorem.

3.12 THEOREM (The Normed Dual of $W_0^{m,p}(\Omega)$) If $1 \le p < \infty$, p' is the exponent conjugate to p, and $m \ge 1$, the dual space $\left(W_0^{m,p}(\Omega)\right)'$ is isometrically isomorphic to the Banach space $W^{-m,p'}(\Omega)$ consisting of those distributions $T \in \mathscr{D}'(\Omega)$ that satisfy (5) and having norm

$$\|T\| = \min\left\{\|v; L^{p'}(\Omega^{(m)})\| \,:\, v \text{ satisfies } (5)\right\}. \quad \blacksquare$$

The completeness of this space is a consequence of the isometric isomorphism. Evidently $W^{-m,p'}(\Omega)$ is separable and reflexive if $1 < p < \infty$.

When $W_0^{m,p}(\Omega)$ is a proper subset of $W^{m,p}(\Omega)$, continuous linear functionals on $W^{m,p}(\Omega)$ are not fully determined by their restrictions to $C_0(\Omega)$, and so are not determined by distributions T given by (5).

3.13 (The $(-m, p')$ norm on $L^{p'}(\Omega)$) There is another way of characterizing the dual of $W_0^{m,p}(\Omega)$ if $1 < p < \infty$. Each element $v \in L^{p'}(\Omega)$ determines an element L_v of $\left(W_0^{m,p}(\Omega)\right)'$ by means of $L_v(u) = \langle u, v\rangle$, because

$$|L_v(u)| = |\langle u, v\rangle| \le \|v\|_{p'} \, \|u\|_p \le \|v\|_{p'} \, \|u\|_{m,p}\,.$$

We define the $(-m, p')$-norm of $v \in L^{p'}(\Omega)$ to be the norm of L_v, that is

$$\|v\|_{-m,p'} = \|L_v; \left(W_0^{m,p}(\Omega)\right)'\| = \sup_{u\in W_0^{m,p}(\Omega),\, \|u\|_{m,p}\le 1} |\langle u, v\rangle|.$$

Clearly $\|v\|_{-m,p'} \le \|v\|_{p'}$ and for any $u \in W_0^{m,p}(\Omega)$ and $v \in L^{p'}(\Omega)$ we have

$$|\langle u, v\rangle| = \|u\|_{m,p} \left|\left\langle \frac{u}{\|u\|_{m,p}}, v\right\rangle\right| \le \|u\|_{m,p} \|v\|_{-m,p'}, \tag{6}$$

which is a generalization of Hölder's inequality.

Let $V = \{L_v : v \in L^{p'}(\Omega)\}$, which is a vector subspace of $\left(W_0^{m,p}(\Omega)\right)'$. We show that V is dense in $\left(W_0^{m,p}(\Omega)\right)'$. To this end it is sufficient to show that if $F \in \left(W_0^{m,p}(\Omega)\right)''$ satisfies $F(L_v) = 0$ for every $L_v \in V$, then $F = 0$ in $\left(W_0^{m,p}(\Omega)\right)''$. But since $W_0^{m,p}(\Omega)$ is reflexive, there exists $f \in W_0^{m,p}(\Omega)$ corresponding to $F \in \left(W_0^{m,p}(\Omega)\right)''$ such that $\langle f, v\rangle = L_v(f) = F(L_v) = 0$ for every $v \in L^{p'}(\Omega)$. But then $f(x)$ must be zero a.e. in Ω. Hence $f = 0$ in $W_0^{m,p}(\Omega)$ and $F = 0$ in $\left(W_0^{m,p}(\Omega)\right)''$.

Let $H^{-m,p'}(\Omega)$ denote the completion of $L^{p'}(\Omega)$ with respect to the norm $\|\cdot\|_{-m,p'}$. Then we have

$$H^{-m,p'}(\Omega) \equiv \left(W_0^{m,p}(\Omega)\right)' \equiv W^{-m,p'}(\Omega).$$

In particular, corresponding to each $v \in H^{-m,p'}(\Omega)$, there exists a distribution $T_v \in W^{-m,p'}(\Omega)$ such that $T_v(\phi) = \lim_{n\to\infty}\langle \phi, v_n\rangle$ for every $\phi \in \mathscr{D}(\Omega)$ and every sequence $\{v_n\} \subset L^{p'}(\Omega)$ for which $\lim_{n\to\infty} \|v_n - v\|_{-m,p'} = 0$. Conversely, any $T \in W^{-m,p'}(\Omega)$ satisfies $T = T_v$ for some such v. Moreover, by (6), $|T_v(\phi)| \le \|\phi\|_{m,p} \|v\|_{-m,p'}$.

3.14 A similar argument to that above shows that the dual space $\left(W^{m,p}(\Omega)\right)'$ can be characterized for $1 < p < \infty$ as the completion of $L^{p'}(\Omega)$ with respect to the norm

$$\|v\|^*_{-m,p'} = \sup_{u\in W^{m,p}(\Omega),\ \|u\|_{m,p}\le 1} |\langle u, v\rangle|.$$

Approximation by Smooth Functions on Ω

We wish to prove that $\{\phi \in C^\infty(\Omega) : \|\phi\|_{m,p} < \infty\}$ is dense in $W^{m,p}(\Omega)$. For this we require the following existence theorem for infinitely differentiable *partitions of unity*.

3.15 THEOREM (Partitions of Unity) Let A be an arbitrary subset of $\mathbb{R}^n$ and let $\mathscr{O}$ be a collection of open sets in $\mathbb{R}^n$ which cover A, that is, $A \subset \bigcup_{U\in\mathscr{O}} U$. Then there exists a collection Ψ of functions $\psi \in C_0^\infty(\mathbb{R}^n)$ having the following properties:

(i) For every $\psi \in \Psi$ and every $x \in \mathbb{R}^n$, $0 \le \psi(x) \le 1$.

(ii) If $K \Subset A$, all but finitely many $\psi \in \Psi$ vanish identically on K.

(iii) For every $\psi \in \Psi$ there exists $U \in \mathcal{O}$ such that supp $(\psi) \subset U$.

(iv) For every $x \in A$, we have $\sum_{\psi \in \Psi} \psi(x) = 1$.

Such a collection Ψ is called a *C^∞-partition of unity for A subordinate to $\mathcal{O}$*.

Proof. Since the proof can be found in many texts, we give only an outline of it. First suppose that A is compact. Then there is a finite collection of sets in $\mathcal{O}$ that cover A, say $A \subset \bigcup_{j=1}^N U_j$. Compact sets $K_1 \subset U_1, \dots, K_N \subset U_N$ can then be constructed so that $A \subset \bigcup_{j=1}^N K_j$. For each j a nonnegative-valued function $\phi_j \in C_0^\infty(U_j)$ can be found such that $\phi_j(x) > 0$ for $x \in K_j$. A function ϕ in $C^\infty(\mathbb{R}^n)$ can then be constructed so that $\phi(x) > 0$ on $\mathbb{R}^n$ and $\phi(x) = \sum_{j=1}^N \phi_j(x)$ for $x \in A$. Now $\Psi = \{\psi_n : \psi_j(x) = \phi_j(x)/\phi(x), 1 \le j \le N\}$ has the required properties. If A is an arbitrary open set. Then $A = \bigcup_{j=1}^\infty A_j$, where

$$A_j = \{x \in A : |x| \le j \text{ and } \operatorname{dist}(x, \operatorname{bdry} A) \ge 1/j\}$$

is compact. Taking $A_0 = A_{-1} = \emptyset$, for each $j \ge 1$ the collection

$$\mathcal{O}_j = \{U \cap (\text{interior of } A_{j+1} \cap A_{j-2}^c) : U \in \mathcal{O}\}$$

covers A_j and so there exists a finite C^∞-partition of unity Ψ_j for A_j subordinate to $\mathcal{O}_j$. The sum $\sigma(x) = \sum_{j=1}^\infty \sum_{\phi \in \Psi_j} \phi(x)$ involves only finitely many nonzero terms at each $x \in A$. The collection $\Psi = \{\psi : \psi(x) = \phi(x)/\sigma(x)$ for some ϕ in some Ψ_j if $x \in A$, $\psi(x) = 0$ if $x \notin A\}$ has the prescribed properties.

Finally, if A is arbitrary, then $A \subset B$ where B is the union of all $U \in \mathcal{O}$ and is an open set. Any partition of unity for B will do for A as well. ∎

3.16 LEMMA (Mollification in $W^{m,p}(\Omega)$) Let J_ϵ be defined as in Paragraph 2.28 and let $1 \le p < \infty$ and $u \in W^{m,p}(\Omega)$. If Ω' is a subdomain with compact closure in Ω, then $\lim_{\epsilon \to 0+} J_\epsilon * u = u$ in $W^{m,p}(\Omega')$.

Proof. Let $\epsilon < \operatorname{dist}(\Omega', \operatorname{bdry}\Omega)$ and $\tilde{u}$ be the zero extension of u outside Ω. If $\phi \in \mathcal{D}(\Omega')$,

$$\begin{aligned}
\int_{\Omega'} J_\epsilon * u(x) D^\alpha \phi(x)\, dx &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{u}(x-y) J_\epsilon(y) D^\alpha \phi(x)\, dx\, dy \\
&= (-1)^{|\alpha|} \int_{\mathbb{R}^n} \int_{\Omega'} D_x^\alpha u(x-y) J_\epsilon(y) \phi(x)\, dx\, dy \\
&= (-1)^{|\alpha|} \int_{\Omega'} J_\epsilon * D^\alpha u(x) \phi(x)\, dx.
\end{aligned}$$

Thus $D^\alpha J_\epsilon * u = J_\epsilon * D^\alpha u$ in the distributional sense in Ω'. Since $D^\alpha u \in L^p(\Omega)$ for $0 \le |\alpha| \le m$ we have by Theorem 2.29(c)

$$\lim_{\epsilon \to 0+} \|D^\alpha J_\epsilon * u - D^\alpha u\|_{p,\Omega'} = \lim_{\epsilon \to 0+} \|J_\epsilon * D^\alpha u - D^\alpha u\|_{p,\Omega'} = 0.$$

Thus $\lim_{\epsilon \to 0+} \|J_\epsilon u - u\|_{m,p,\Omega'} = 0$. ∎

3.17 THEOREM (H = W) (See [MS].) If $1 \le p < \infty$, then

$$H^{m,p}(\Omega) = W^{m,p}(\Omega).$$

Proof. By Corollary 3.4 it is sufficient to show that $W^{m,p}(\Omega) \subset H^{m,p}(\Omega)$, that is, that $\{\phi \in C^m(\Omega) : \|\phi\|_{m,p} < \infty\}$ is dense in $W^{m,p}(\Omega)$. If $u \in W^{m,p}(\Omega)$ and $\epsilon > 0$, we in fact show that there exists $\phi \in C^\infty(\Omega)$ such that $\|\phi - u\|_{m,p} < \epsilon$, so that $C^\infty(\Omega)$ is dense in $W^{m,p}(\Omega)$. For $k = 1, 2, \ldots$ let

$$\Omega_k = \{x \in \Omega : |x| < k \text{ and } \operatorname{dist}(x, \operatorname{bdry}\Omega) > 1/k,$$

and let $\Omega_0 = \Omega_{-1} = \emptyset$, the empty set. Then

$$\mathscr{O} = \{U_k : U_k = \Omega_{k+1} \cap (\overline{\Omega_{k-1}})^c,\ k = 1, 2, \ldots\}$$

is a collection of open subsets of Ω that covers Ω. Let Ψ be a C^∞-partition or unity for Ω subordinate to $\mathscr{O}$. Let ψ_k denote the sum of the finitely many functions $\psi \in \Psi$ whose supports are contained in U_k. Then $\psi_k \in C_0^\infty(U_k)$ and $\sum_{k=1}^\infty \psi_k(x) = 1$ on Ω.

If $0 < \epsilon < 1/(k+1)(k+2)$, then $J_\epsilon * (\psi_k u)$ has support in the intersection $V_k = \Omega_{k+2} \cap (\Omega_{k-2})^c \Subset \Omega$. Since $\psi_k u \in W^{m,p}(\Omega)$ we may choose ϵ_k, satisfying $0 < \epsilon_k < 1/(k+1)(k+2)$, such that

$$\left\| J_{\epsilon_k} * (\psi_k u) - \psi_k u \right\|_{m,p,\Omega} = \left\| J_{\epsilon_k} * (\psi_k u) - \psi_k u \right\|_{m,p,V_k} < \epsilon/(2^k).$$

Let $\phi = \sum_{k=1}^\infty J_{\epsilon_k} * (\psi_k u)$. On any $\Omega' \Subset \Omega$ only finitely many terms in the sum can be nonzero. Thus $\phi \in C^\infty(\Omega)$. For $x \in \Omega_k$, we have

$$u(x) = \sum_{j=1}^{k+2} \psi_j(x) u(x), \qquad \text{and} \qquad \phi(x) = \sum_{j=1}^{k+2} J_{\epsilon_j} * (\psi_j u)(x).$$

Thus

$$\|u - \phi\|_{m,p,\Omega_k} \le \sum_{j=1}^{k+2} \left\| J_{\epsilon_j} * (\psi_j u) - \psi_j u \right\|_{m,p,\Omega} < \epsilon.$$

By the monotone convergence theorem 1.48, $\|u - \phi\|_{m,p,\Omega} < \epsilon$. ∎

3.18 EXAMPLE Theorem 3.17 can not be extended to the case $p = \infty$. For instance, if $\Omega = \{x \in \mathbb{R} : -1 < x < 1$, and $u(x) = |x|$, then $u'(x) = x/|x|$ for $x \neq 0$ and so $u \in W^{1,\infty}(\Omega)$. But $u \notin H^{1,\infty}(\Omega)$. In fact, if $0 < \epsilon < 1/2$, there exists no function $\phi \in C^1(\Omega)$ such that $\left\|\phi' - u'\right\|_\infty < \epsilon$.

Approximation by Smooth Functions on $\mathbb{R}^n$

3.19 Having shown that an element of $W^{m,p}(\Omega)$ can always be approximated by functions smooth on Ω we now ask whether the approximation can in fact be done with bounded functions having bounded derivatives of all orders, or at least of all orders up to and including at least m. That is, we are asking whether, for any values of $k \ge m$, the space $C^k(\overline{\Omega})$ is dense in $W^{m,p}(\Omega)$. The following example shows that the answer may be negative.

3.20 EXAMPLE Let $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < |x| < 1,\ 0 < y < 1\}$. Then the function defined on Ω by

$$u(x, y) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

evidently belongs to $W^{1,p}(\Omega)$. However, if $\epsilon > 0$ is sufficiently small, there can exist no $\phi \in C^1(\overline{\Omega})$ such that $\|u - \phi\|_{1,p,\Omega} < \epsilon$. To see this, suppose there exists such a ϕ. If $L = \{(x, y) : -1 \le x \le 0,\ 0 \le y \le 1\}$ and $R = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le 1\}$, then $\overline{\Omega} = L \cup R$. We have $\|\phi\|_{1,L} \le \|\phi\|_{p,L} < \epsilon$ and similarly $\|1 - \phi\|_{1,R} < \epsilon$ from which we obtain $\|\phi\|_{1,R} > 1 - \epsilon$. If

$$\Phi(x) = \int_0^1 \phi(x, y)\, dy,$$

then there exist a and b with $-1 \le a < 0$ and $0 < b \le 1$ such that $\Phi(a) < \epsilon$ and $\Phi(b) > 1 - \epsilon$. If $0 < \epsilon < 1/2$, then

$$\begin{aligned} 1 - 2\epsilon < \Phi(b) - \Phi(a) = \int_a^b \Phi'(x)\, dx &\le \int_{\overline{\Omega}} |D_x\phi(x, y)|\, dx\, dy \\ &\le 2^{1/p'} \|D_x\phi\|_{p,\Omega} < 2^{1/p'}\epsilon. \end{aligned}$$

Thus $1 < \epsilon(2 + 2^{1/p'})$, which is not possible for small ϵ.

The difficulty with the domain in this example is that it lies on both sides of part of its boundary, namely the line segment $x = 0$, $0 \le y \le 1$. We now formulate a condition on a domain Ω that prevents this from happening and guarantees that for any k and m, $C^k(\overline{\Omega})$ is dense in $W^{m,p}(\Omega)$ provided $1 \le p < \infty$.

3.21 (The Segment Condition) We say that a domain Ω satisfies the *segment condition* if every $x \in \operatorname{bdry}\Omega$ has a neighbourhood U_x and a nonzero vector y_x such that if $z \in \overline{\Omega} \cap U_x$, then $z + t y_x \in \Omega$ for $0 < t < 1$.

If nonempty, the boundary of a domain satisfying this condition must be $(n-1)$-dimensional, and the domain cannot lie on both sides of any part of its boundary.

3.22 THEOREM If Ω satisfies the segment condition, then the set of restrictions to Ω of functions in $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{m,p}(\Omega)$ for $1 \le p < \infty$.

Proof. Let f be a fixed function in $C_0^\infty(\mathbb{R}^n)$ satisfying

(i) $f(x) = 1$ if $|x| \le 1$,

(ii) $f(x) = 0$ if $|x| \ge 2$,

(iii) $|D^\alpha f(x)| \le M$ (constant) for all x and $0 \le |\alpha| \le m$.

For $\epsilon > 0$ let $f_\epsilon(x) = f(\epsilon x)$. Then $f_\epsilon(x) = 1$ for $|x| \le 1/\epsilon$ and also $|D^\alpha f_\epsilon(x)| \le M\epsilon^{|\alpha|} \le M$ if $\epsilon \le 1$. If $u \in W^{m,p}(\Omega)$, then $u_\epsilon = f_\epsilon u$ belongs to $W^{m,p}(\Omega)$ and has bounded support. For $0 < \epsilon \le 1$ and $|\alpha| \le m$

$$|D^\alpha u_\epsilon(x)| = \left|\sum_{\beta\le\alpha}\binom{\alpha}{\beta} D^\beta u(x) D^{\alpha-\beta} f_\epsilon(x)\right| \le M \sum_{\beta\le\alpha}\binom{\alpha}{\beta} |D^\beta u(x)|.$$

Therefore, setting $\Omega_\epsilon = \{x \in \Omega : |x| > 1/\epsilon\}$, we have

$$\begin{aligned}\|u - u_\epsilon\|_{m,p,\Omega} &= \|u - u_\epsilon\|_{m,p,\Omega_\epsilon} \\ &\le \|u\|_{m,p,\Omega_\epsilon} + \|u_\epsilon\|_{m,p,\Omega_\epsilon} \le \text{const}\, \|u\|_{m,p,\Omega_\epsilon}.\end{aligned}$$

The right side approaches zero as $\epsilon \to 0+$. Thus any $u \in W^{m,p}(\Omega)$ can be approximated in that space by functions with bounded supports.

We now, therefore, assume that $K = \{x \in \Omega : u(x) \ne 0\}$ is bounded. The set $F = \overline{K} - \left(\bigcup_{x\in \mathrm{bdry}\,\Omega} U_x\right)$ is thus compact and contained in Ω, $\{U_x\}$ being the collection of open sets referred to in the definition of the segment condition. There exists an open set U_0 such that $F \Subset U_0 \Subset \Omega$. Since $\overline{K}$ is compact, there exists finitely many of the sets U_x, let us rename them $U_1, \dots, U_k$, such that $\overline{K} \subset U_0 \cup U_1 \cup \cdots \cup U_k$. Moreover, there are other open sets $V_0, V_1, \dots, V_k$ such that $V_j \Subset U_j$ for $0 \le j \le k$ but still $\overline{K} \subset V_0 \cup V_1 \cup \cdots \cup V_k$.

Let Ψ be a C^∞-partition of unity subordinate to $\{V_j : 0 \le j \le k\}$, and let ψ_j be the sum of the finitely many functions $\psi \in \Psi$ whose supports lie in V_j. Let $u_j = \psi_j u$. Suppose that for each j we can find $\phi_j \in C_0^\infty(\mathbb{R}^n)$ such that

$$\left\|u_j - \phi_j\right\|_{m,p,\Omega} < \epsilon/(k+1). \tag{7}$$

Then, putting $\phi = \sum_{j=0}^k \phi_j$, we would obtain

$$\|u - \phi\|_{m,p,\Omega} \le \sum_{j=0}^{k} \left\|u_j - \phi_j\right\|_{m,p,\Omega} < \epsilon.$$

A function $\phi_0 \in C_0^\infty(\mathbb{R}^n)$ satisfying (7) for $j = 0$ can be found via Lemma 3.16 since $\mathrm{supp}\,(u_0) \subset V_0 \Subset \Omega$. It remains, therefore, to find ϕ_j satisfying (7) for $1 \le j \le k$. For fixed such j we extend u_j to be identically zero outside Ω. Thus $u_j \in W^{m,p}(\mathbb{R}^n - \Gamma)$, where $\Gamma = \overline{V_j} \cap \mathrm{bdry}\,\Omega$. Let y be the nonzero vector associated with the set U_j in the definition of the segment condition. (See Fig. 1.) Let $\Gamma_t = \{x - ty : x \in \Gamma\}$, where t is so chosen that

$$0 < t < \min\{1, \mathrm{dist}(V_j, \mathbb{R}^n - U_j)/|y|\}.$$

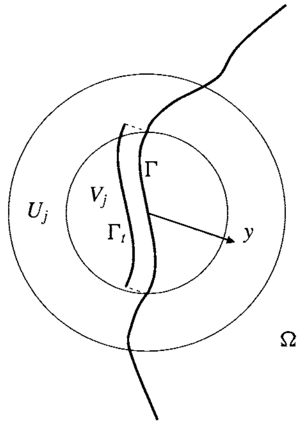


Fig. 1

Then $\Gamma_t \subset U_j$ and $\Gamma_t \cap \overline{\Omega}$ is empty by the segment condition. Let us define $u_{j,t}(x) = u_j(x + ty)$. Then $u_{j,t} \in W^{m,p}(\mathbb{R}^n - \Gamma_t)$. Translation is continuous in $L^p(\Omega)$ (see the proof of Theorem 2.32) so $D^\alpha u_{j,t} \to D^\alpha u_j$ in $L^p(\Omega)$ as $t \to 0+$ for $|\alpha| \le m$. Thus $u_{j,t} \to u_j$ in $W^{m,p}(\Omega)$ as $t \to 0+$, and so it is sufficient to find $\phi_j \in C_0^\infty(\mathbb{R}^n)$ such that $\|u_{j,t} - \phi_j\|_{m,p}$ is sufficiently small. However, $\Omega \cap U_j \Subset \mathbb{R}^n - \Gamma_t$, and so by Lemma 3.16 we can take $\phi_j = J_\delta * u_{j,t}$ for suitably small $\delta > 0$. This completes the proof. ∎

3.23 COROLLARY $W_0^{m,p}(\mathbb{R}^n) = W^{m,p}(\mathbb{R}^n)$.

Approximation by Functions in $C_0^\infty(\Omega)$

3.24 Corollary 3.23 suggests the question: For what domains Ω is it true that $W^{m,p}(\Omega) = W_0^{m,p}(\Omega)$, that is, when is $C_0^\infty(\Omega)$ dense in $W^{m,p}(\Omega)$? A partial answer to this question can be formulated in terms of the nature of the distributions belonging to $W^{-m,p'}(\mathbb{R}^n)$. The approach below is due to Lions [Lj]. Throughout this discussion we assume $1 < p < \infty$ and p' is the conjugate exponent $p' = p/(p-1)$.

3.25 ((m, p')-Polar sets) Let F be a closed subset of $\mathbb{R}^n$. A distribution $T \in \mathscr{D}'(\mathbb{R}^n)$ is said to have support in F ($\operatorname{supp}(T) \subset F$) provided that $T(\phi) = 0$ for every $\phi \in \mathscr{D}(\mathbb{R}^n - F)$. We say that the closed set F is (m, p')-polar if the only distribution $T \in W^{-m,p'}(\mathbb{R}^n)$ having support in F is the zero distribution $T = 0$.

If F has positive measure, it cannot be (m, p')-polar because the characteristic function of any compact subset of F having positive measure belongs to $L^{p'}(\mathbb{R}^n)$ and hence to $W^{-m,p'}(\mathbb{R}^n)$.

We shall show later that if $mp > n$, then $W^{m,p}(\mathbb{R}^n) \to C(\mathbb{R}^n)$ in the sense that if $u \in W^{m,p}(\mathbb{R}^n)$, then there exists $v \in C(\mathbb{R}^n)$ such that $u(x) = v(x)$ a.e. in $\mathbb{R}^n$ and

$$|v(x)| \le \text{const } \|u\|_{m,p},$$

the constant being independent of x and u. It follows that the Dirac distribution δ_x given by $\delta_x(\phi) = \phi(x)$ belongs to $\left(W^{m,p}(\mathbb{R}^n)\right)' = \left(W_0^{m,p}(\mathbb{R}^n)\right)' = W^{-m,p'}(\mathbb{R}^n)$. Hence, if $mp > n$ a set F cannot be (m, p')-polar unless it is empty.

Since $W^{m+1,p}(\Omega) \to W^{m,p}(\Omega)$ any bounded linear functional on the latter space is also bounded on the former. Thus $W^{-m,p'}(\Omega) \subset W^{-m-1,p'}(\Omega)$ and, in particular, any $(m+1, p')$-polar set is also (m, p')-polar. The converse is, of course, generally not true.

3.26 (Zero Extensions) If function u is defined on Ω let $\tilde{u}$ denote the zero extension of u to the complement Ω^c of Ω in $\mathbb{R}^n$:

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \Omega^c. \end{cases}$$

The following lemma shows that the mapping $u \mapsto \tilde{u}$ maps $W_0^{m,p}(\Omega)$ (isometrically) into $W^{m,p}(\mathbb{R}^n)$.

3.27 LEMMA Let $u \in W_0^{m,p}(\Omega)$. If $|\alpha| \le m$, then $D^\alpha \tilde{u} = \widetilde{D^\alpha u}$ in the distributional sense in $\mathbb{R}^n$. Hence $\tilde{u} \in W^{m,p}(\mathbb{R}^n)$.

Proof. Let $\{\phi_j\}$ be a sequence in $C_0^\infty(\Omega)$ converging to u in $W_0^{m,p}(\Omega)$. If $\psi \in \mathscr{D}(\mathbb{R}^n)$, then for $|\alpha| \le m$

$$\begin{aligned}
(-1)^{|\alpha|} \int_{\mathbb{R}^n} \tilde{u}(x) D^\alpha \psi(x)\, dx &= (-1)^{|\alpha|} \int_\Omega u(x) D^\alpha \psi(x)\, dx \\
&= \lim_{j\to\infty} (-1)^{|\alpha|} \int_\Omega \phi_j(x) D^\alpha \psi(x)\, dx \\
&= \lim_{j\to\infty} \int_\Omega D^\alpha \phi_j(x) \psi(x)\, dx \\
&= \int_{\mathbb{R}^n} \widetilde{D^\alpha u}(x) \psi(x)\, dx.
\end{aligned}$$

Thus $D^\alpha \tilde{u} = \widetilde{D^\alpha u}$ in the distributional sense in $\mathbb{R}^n$ and these locally integrable functions are equal a.e. in $\mathbb{R}^n$. It follows that $\|\tilde{u}\|_{m,p,\mathbb{R}^n} = \|u\|_{m,p,\Omega}$. ∎

We can now give a necessary and sufficient condition that the mapping $u \mapsto \tilde{u}$ carries $W_0^{m,p}(\Omega)$ *onto* $W^{m,p}(\mathbb{R}^n)$.

3.28 THEOREM $C_0^\infty(\Omega)$ is dense in $W^{m,p}(\mathbb{R}^n)$ if and only if the complement Ω^c of Ω is (m, p')-polar.

Proof. First suppose $C_0^\infty(\Omega)$ is dense in $W^{m,p}(\mathbb{R}^n)$. Let $T \in W^{-m,p'}(\mathbb{R}^n)$ have support in Ω^c. If $u \in W^{m,p}(\mathbb{R}^n)$, then there exists a sequence $\{\phi_j\} \subset C_0^\infty(\Omega)$ converging to u in $W^{m,p}(\mathbb{R}^n)$. Hence $T(u) = \lim_{j\to\infty} T(\phi_j) = 0$ and so $T = 0$. Thus Ω^c is (m, p')-polar.

Conversely, suppose that $C_0^\infty(\Omega)$ is not dense in $W^{m,p}(\mathbb{R}^n)$. Then there exists $u \in W^{m,p}(\mathbb{R}^n)$ and a constant $k > 0$ such that for all $\phi \in C_0^\infty(\Omega)$ we have $\|u - \phi\|_{m,p,\mathbb{R}^n} \geq k$. The Hahn-Banach theorem 1.13 can be used to show that there exists $T \in W^{-m,p'}(\mathbb{R}^n)$ such that $T(\phi) = 0$ for all $u \in C_0^\infty(\Omega)$ but $T(u) \neq 0$. Since supp $(T) \subset \Omega^c$ but $T \neq 0$, Ω^c cannot be (m, p')-polar. ∎

As a final preparation for our investigation of the possible identity of $W_0^{m,p}(\Omega)$ and $W^{m,p}(\Omega)$ we establish a distributional analog of the fact, obvious for differentiable functions, that the vanishing of first derivatives over a rectangle implies constancy on that rectangle. We extend this first to distributions (in Corollary 3.30) and then to locally integrable functions.

3.29 LEMMA Let $B = (a_1, b_1) \times \cdots \times (a_n, b_n)$ be an open rectangular box in $\mathbb{R}^n$ and let $\phi \in \mathscr{D}(B)$. If $\int_B \phi(x)\,dx = 0$, then $\phi(x) = \sum_{j=1}^n \phi_j(x)$, where $\phi_j \in \mathscr{D}(B)$ and

$$\int_{a_j}^{b_j} \phi_j(x_1, \ldots, x_j, \ldots, x_n)\,dx_j = 0 \tag{8}$$

for every fixed $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \in \mathbb{R}^{n-1}$.

Proof. For $1 \leq j \leq n$ select functions $u_j \in C_0^\infty(a_j, b_j)$ such that $\int_{a_j}^{b_j} u_j(t)\,dt = 1$. For $2 \leq j \leq n$, let

$$\begin{aligned}
B_j &= (a_j, b_j) \times (a_{j+1}, b_{j+1}) \times \cdots \times (a_n, b_n),\\
\psi_j(x_j, \ldots, x_n) &= \int_{a_1}^{b_1} dt_1 \int_{a_2}^{b_2} dt_2 \cdots \int_{a_{j-1}}^{b_{j-1}} \phi(t_1, \ldots, t_{j-1}, x_j, \ldots, x_n)\,dt_{j-1},\\
\omega_j(x) &= u_1(x_1) \cdots u_{j-1}(x_{j-1})\psi_j(x_j, \ldots, x_n).
\end{aligned}$$

Then $\psi_j \in \mathscr{D}(B_j)$ and $\omega_j \in \mathscr{D}(B)$. Moreover

$$\int_{B_j} \psi_j(x_j, \ldots, x_n)\,dx_j \cdots dx_n = \int_B \phi(x)\,dx = 0.$$

Let $\phi_1 = \phi - \omega_2$, $\phi_j = \omega_j - \omega_{j+1}$ if $2 \leq j \leq n-1$, and $\phi_n = \omega_n$. Clearly $\phi_j \in \mathscr{D}(B)$ for $1 \leq j \leq n$, and $\phi = \sum_{j=1}^n \phi_j$. Finally,

$$
\begin{aligned}
&\int_{a_1}^{b_1} \phi_1(x_1,\dots,x_n)\,dx_1 \\
&\qquad = \int_{a_1}^{b_1} \phi(x_1,\dots,x_n)\,dx_1 - \psi_2(x_2,\dots,x_n)\int_{a_1}^{b_1} u_1(x_1)\,dx_1 = 0 \\
&\int_{a_j}^{b_j} \phi_j(x_1,\dots,x_n)\,dx_j \\
&\qquad = u_1(x_1)\cdots u_{j-1}(x_{j-1}) \\
&\qquad\quad \times \left(\int_{a_j}^{b_j} \psi_j(x_1,\dots,x_n)\,dx_j - \psi_{j+1}(x_{j+1},\dots,x_n)\int_{a_j}^{b_j} u_j(x_j)\,dx_j\right) \\
&\qquad = 0, \qquad 2 \le j \le n-1, \\
&\int_{a_n}^{b_n} \phi_n(x_1,\dots,x_n)\,dx_n = u_1(x_1)\cdots u_{n-1}(x_{n-1})\int_{a_n}^{b_n} \psi_n(x_n)\,dx_n \\
&\qquad = u_1(x_1)\cdots u_{n-1}(x_{n-1})\int_B \phi(x)\,dx = 0. \quad \blacksquare
\end{aligned}
$$

3.30 COROLLARY If $T \in \mathscr{D}'(B)$ and $D_jT = 0$ for $1 \le j \le n$, then there exists a constant k such that for all $\phi \in \mathscr{D}(B)$,

$$T(\phi) = k\int_B \phi(x)\,dx.$$

Proof. First note that if $\int_B \phi(x)\,dx = 0$, then $T(\phi) = 0$, for, by the above lemma we may write $\phi = \sum_{j=1}^n \phi_j$, where $\phi_j \in \mathscr{D}(B)$ satisfies (8), and hence $\phi_j = D_j\theta_j$, where θ_j defined by

$$\theta_j(x) = \int_{a_j}^{x_j} \phi_j(x_1,\dots,x_{j-1},t,x_{j+1},\dots,x_n)\,dt$$

belongs to $\mathscr{D}(B)$. Thus $T(\phi) = \sum_{j=1}^n T(D_j\theta_j) = -\sum_{j=1}^n (D_jT)(\theta_j) = 0$.

Now suppose $T \ne 0$. Then there exists $\phi_0 \in \mathscr{D}(B)$ such that $T(\phi_0) = k_1 \ne 0$. Thus $\int_B \phi_0(x)\,dx = k_2 \ne 0$ and $T(\phi_0) = k\int_B \phi_0(x)\,dx$, where $k = k_1/k_2$. If $\phi \in \mathscr{D}(B)$ is arbitrary, let $K(\phi) = \int_B \phi(x)\,dx$. Then

$$\int_B \left(\phi(x) - \frac{K(\phi)}{k_2}\phi_0(x)\right) dx = 0$$

and so $T\big(\phi - [K(\phi)/k_2]\phi_0\big) = 0$. It follows that

$$T(\phi) = \frac{T(\phi_0)}{k_2} K(\phi) = kK(\phi) = k \int_B \phi(x)\, dx. \quad \blacksquare$$

Note that this corollary can be extended to any connected set $\Omega \in \mathbb{R}^n$ via a partition of unity for Ω subordinate to some open cover of Ω by open rectangular boxes that are contained in Ω. We do not, however, require this extension.

The following lemma shows that different locally integrable functions on an open set Ω determine different distributions on Ω.

3.31 LEMMA Let $u \in L^1_{\mathrm{loc}}(\Omega)$ satisfy $\int_\Omega u(x)\phi(x)\, dx = 0$ for every ϕ in $\mathscr{D}(\Omega)$. Then $u(x) = 0$ a.e. in Ω.

Proof. If $\psi \in C_0(\Omega)$, then for sufficiently small positive ϵ, the mollifier $J_\epsilon * \psi$ belongs to $\mathscr{D}(\Omega)$. By Lemma 2.29, $J_\epsilon * \psi \to \psi$ uniformly on Ω as $\epsilon \to 0+$. Hence $\int_\Omega u(x)\psi(x)\, dx = 0$ for every $\psi \in C_0(\Omega)$.

Let $K \Subset \Omega$ and let $\epsilon > 0$. Let χ_K be the characteristic function of K. Then $\int_K |u(x)|\, dx < \infty$. There exists $\delta > 0$ such that for any measurable set $A \subset K$ with $\mu(A) < \delta$ we have $\int_A |u(x)|\, dx < \epsilon/2$ (see, for example, [Ru2, p. 124]). By Lusin's theorem 1.42(f) there exists $\psi \in C_0(\mathbb{R}^n)$ with $|\psi(x)| \le 1$ for all x, such that

$$\mu\big(\{x \in \mathbb{R}^n : \psi(x) \ne \chi_K(x)\operatorname{sgn}\overline{u(x)}\}\big) < \delta.$$

Here

$$\operatorname{sgn} v(x) = \begin{cases} v(x)/|v(x)| & \text{if } v(x) \ne 0 \\ 0 & \text{if } v(x) = 0. \end{cases}$$

Hence

$$\begin{aligned} \int_K |u(x)|\, dx &= \int_\Omega u(x)\chi_K(x)\operatorname{sgn}\overline{u(x)}\, dx \\ &= \int_\Omega u(x)\psi(x)\, dx + \int_\Omega u(x)\big(\chi_K(x)\operatorname{sgn}\overline{u(x)} - \psi(x)\big)\, dx \\ &\le 0 + 2\int_{\{x\in\Omega:\psi(x)\ne\chi_K(x)\operatorname{sgn}\overline{u(x)}\}} |u(x)|\, dx < \epsilon. \end{aligned}$$

Since ϵ is arbitrary, $u(x) = 0$ a.e. in K for each such K, and hence a.e. in Ω. $\blacksquare$

3.32 COROLLARY If B is a rectangular box as in Lemma 3.29 and u in $L^1_{\mathrm{loc}}(B)$ possesses weak derivatives $D_j u = 0$ for $1 \le j \le n$, then for some constant k, $u(x) = k$ a.e. in B.

Proof. By Corollary 3.30, since $D_j T_u = 0$ for $1 \le j \le n$, we have

$$\int_B u(x)\phi(x)\, dx = T_u(\phi) = k \int_B \phi(x)\, dx.$$

Hence $u(x) - k = 0$ a.e. in B. $\blacksquare$

3.33 THEOREM Let $m \geq 1$.

(a) If $W^{m,p}(\Omega) = W_0^{m,p}(\Omega)$, then Ω^c is (m, p')-polar.

(b) If Ω^c is both $(1, p)$-polar and (m, p')-polar, then $W^{m,p}(\Omega) = W_0^{m,p}(\Omega)$.

Proof. (a) Assume $W^{m,p}(\Omega) = W_0^{m,p}(\Omega)$. We deduce first that Ω^c must have measure zero. If not, there would exist some finite open rectangle $B \subset \mathbb{R}^n$ which intersects both Ω and Ω^c in sets of positive measure. Let u be the restriction to Ω of a function in $C_0^\infty(\mathbb{R}^n)$ which is identically one on $B \cap \Omega$. Then $u \in W^{m,p}(\Omega)$ and so $u \in W_0^{m,p}(\Omega)$. By Lemma 3.27, the zero extension $\tilde{u}$ of u to $\mathbb{R}^n$ belongs to $W^{m,p}(\mathbb{R}^n)$ and $D_j\tilde{u} = \widetilde{D_j u}$ in the distributional sense in $\mathbb{R}^n$ for $1 \leq j \leq n$. Now $\widetilde{D_j u}$ is identically zero on B and so $D_j\tilde{u} = 0$ as a distribution on B. By Corollary 3.32, $\tilde{u}$ must have a constant value a.e. in B. Since $\tilde{u} = 1$ on $B \cap \Omega$ and $\tilde{u} = 0$ on $B \cap \Omega^c$, we have a contradiction. Thus Ω^c has measure zero.

Now if $v \in W^{m,p}(\mathbb{R}^n)$ and u is the restriction of v to Ω, then u belongs to $W^{m,p}(\Omega)$ and hence, by assumption, also to $W_0^{m,p}(\Omega)$. By Lemma 3.27, $\tilde{u} \in W^{m,p}(\mathbb{R}^n)$ and can be approximated by elements of $C_0^\infty(\Omega)$. But $v(x) = \tilde{u}(x)$ on Ω, that is, a.e. in $\mathbb{R}^n$. Hence v and $\tilde{u}$ have the same distributional derivatives, and coincide in $W^{m,p}(\mathbb{R}^n)$. Therefore $C_0^\infty(\Omega)$ is dense in $W^{m,p}(\mathbb{R}^n)$ and Ω^c is (m, p')-polar by Theorem 3.28.

(b) Now assume Ω^c is $(1, p)$-polar and (m, p')-polar. Let $u \in W^{m,p}(\Omega)$. We show that $u \in W_0^{m,p}(\Omega)$. Since $\tilde{u} \in L^p(\mathbb{R}^n)$, the distribution $T_{D_j\tilde{u}}$, corresponding to $D_j\tilde{u}$, belongs to $W^{-1,p}(\mathbb{R}^n)$. Since $\widetilde{D_j u} \in L^p(\mathbb{R}^n) \subset H^{-1,p}(\mathbb{R}^n)$ (see Paragraph 3.13), therefore $T_{\widetilde{D_j u}} \in W^{-1,p}(\mathbb{R}^n)$. Hence $T_{D_j\tilde{u}-\widetilde{D_j u}} \in W^{-1,p}(\mathbb{R}^n)$. But $D_j\tilde{u} - \widetilde{D_j u} = 0$ on Ω so $\operatorname{supp}\left(T_{D_j\tilde{u}-\widetilde{D_j u}}\right) \subset \Omega^c$. Since Ω^c is $(1, p)$-polar, $D_j\tilde{u} = \widetilde{D_j u}$ in the distributional sense on $\mathbb{R}^n$, whence $D_j\tilde{u} \in L^p(\mathbb{R}^n)$ and $\tilde{u} \in W^{m,p}(\mathbb{R}^n)$. Since Ω^c is (m, p')-polar, $C_0^\infty(\Omega)$ is dense in $W^{m,p}(\mathbb{R}^n)$, and thus $u \in W_0^{m,p}(\Omega)$. ∎

3.34 If (m, p')-polarity implies $(1, p)$-polarity, then Theorem 3.33 amounts to the assertion that (m, p')-polarity of Ω^c is necessary and sufficient for the equality of $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$. This is certainly the case if $p = 2$.

The following two lemmas develop properties of polarity. The first of these shows that it is a local property.

3.35 LEMMA $F \subset \mathbb{R}^n$ is (m, p')-polar if and only if $F \cap K$ is (m, p')-polar for every compact set $K \subset \mathbb{R}^n$.

Proof. Clearly the (m, p')-polarity of F implies that of $F \cap K$ for every compact K. We need only prove the converse.

Let $T \in W^{-m,p'}(\mathbb{R}^n)$ be given by $T = \sum_{0 \leq |\alpha| \leq m} (-1)^{|\alpha|} D^\alpha T_{v_\alpha}$, where sequence $\{v_\alpha\} \subset L^{p'}(\mathbb{R}^n)$. Suppose T has support in F. We must show that $T = 0$. Let $f \in C_0^\infty(\mathbb{R}^n)$ satisfy $f(x) = 1$ if $|x| \leq 1$ and $f(x) = 0$ if $|x| \geq 2$. For $\epsilon > 0$, let

$f_\epsilon(x) = f(\epsilon x)$ so that $D^\alpha f_\epsilon(x) = \epsilon^{|\alpha|} D^\alpha f(\epsilon x) \to 0$ uniformly in x as $\epsilon \to 0+$. Then $f_\epsilon T \in W^{-m,p'}(\mathbb{R}^n)$ by induction on m, and for any $\phi \in \mathscr{D}(\mathbb{R}^n)$ we have

$$\begin{aligned}
|T(\phi) - f_\epsilon T(\phi)| &= |T(\phi) - T(f_\epsilon \phi)| \\
&= \left| \sum_{0 \le |\alpha| \le m} \int_{\mathbb{R}^n} v_\alpha(x) D^\alpha \big[\phi(x)\big(1 - f_\epsilon(x)\big)\big]\, dx \right| \\
&= \left| \sum_{0 \le |\alpha| \le m} \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \int_{\mathbb{R}^n} v_\alpha(x) D^\beta \phi(x) D^{\alpha-\beta}\big(1 - f_\epsilon(x)\big)\, dx \right| \\
&\le \sum_{\beta \le \alpha} \int_{\mathbb{R}^n} |w_\beta(x) D^\beta \phi(x)|\, dx \le \|\phi\|_{m,p} \left\| w\, ; L^{p'}(\Omega^{(m)}) \right\|,
\end{aligned}$$

where

$$\begin{aligned}
w_\beta(x) &= \sum_{|\alpha| \le m,\, \beta \le \alpha} \binom{\alpha}{\beta} v_\alpha(x) D^{\alpha-\beta}\big(1 - f_\epsilon(x)\big) \\
&= v_\beta(x)\big(1 - f_\epsilon(x)\big) - \sum_{|\alpha| \le m,\, \beta \le \alpha, \beta \ne \alpha} \binom{\alpha}{\beta} v_\alpha(x) D^{\alpha-\beta} f_\epsilon(x).
\end{aligned}$$

Since $f_\epsilon(x) = 1$ for $|x| \le 1/\epsilon$, we have $\lim_{\epsilon \to 0+} \left\| w_\beta \right\|_{p'} = 0$. Thus $f_\epsilon T \to T$ in $W^{-m,p'}(\mathbb{R}^n)$ as $\epsilon \to 0+$. But $f_\epsilon T = 0$ by assumption since it has compact support in K. Thus $T = 0$. ∎

3.36 LEMMA If $p' < q'$ (that is, $p > q$) and $f \subset \mathbb{R}^n$ is (m, p')-polar, then F is also (m, q')-polar.

Proof. Let $K \subset \mathbb{R}^n$ be compact. By the previous lemma it is sufficient to show that $F \cap K$ is (m, q')-polar. Let G be an open, bounded set in $\mathbb{R}^n$ containing K. By Theorem 2.14, $W_0^{m,p}(G) \to W_0^{m,q}(G)$, so that $W^{-m,q'}(G) \subset W^{-m,p'}(G)$. Any distribution $T \in W^{-m,q'}(\mathbb{R}^n)$ having support in $K \cap F$ also belongs to $W^{-m,q'}(G)$ and so to $W^{-m,p'}(G)$. Since $K \cap F$ is (m, p')-polar, $T = 0$. Thus $K \cap F$ is also (m, q')-polar. ∎

3.37 THEOREM Let $m \ge 1$ and $p \ge 2$. Then $W^{m,p}(\Omega) = W_0^{m,p}(\Omega)$ if and only of Ω^c is (m, p')-polar.

Proof. Since $p' \le 2$, Ω^c is (m, p)-polar and therefore also $(1, p)$-polar. The result now follows by Theorem 3.33.

3.38 The Sobolev Imbedding Theorem 4.12 can be used to extend the previous theorem to cover certain values of $p < 2$. If $(m-1)p < n$, the imbedding theorem gives

$$W^{m,p}(\mathbb{R}^n) \to W^{1,q}(\mathbb{R}^n), \qquad q = \frac{np}{n - (m-1)p},$$

which in turn implies that $W^{-1,q'}(\mathbb{R}^n) \subset W^{-m,p'}(\mathbb{R}^n)$. If also $p \geq 2n/(n+m-1)$, then $q' \leq p$ and so by Lemma 3.36, Ω^c is $(1, p)$-polar if it is (m, p')-polar. Note that $2n/(n+m-1) < 2$ if $m > 1$. If, on the other hand, $(m-1)p \geq n$, then $mp > n$, and, as pointed out in Paragraph 3.25, Ω^c cannot be (m, p')-polar unless it is empty, in which case it is trivially $(1, p)$-polar.

The only values of p for which we do not know that the (m, p')-polarity of Ω^c implies $(1, p)$-polarity and hence is equivalent to the identity of $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$, are given by $1 \leq p \leq \min\{n/(m-1), 2n/(n+m-1)\}$.

3.39 Whenever $W_0^{m,p}(\Omega) \neq W^{m,p}(\Omega)$, the former space is a closed subspace of the latter. In the Hilbert space case, $p = 2$, we may consider the space $W_0^{\perp}$ consisting of all $v \in W^{m,2}(\Omega)$ such that $(v, \phi)_m = 0$ for all $\phi \in C_0^{\infty}(\Omega)$. Every $u \in W^{m,2}(\Omega)$ can be uniquely decomposed in the form $u = u_0 + v$, where $u_0 \in W_0^{m,2}(\Omega)$ and $v \in W_0^{\perp}$. Integration by parts shows that any $v \in W_0^{\perp}$ must satisfy

$$\sum_{0 \leq |\alpha| \leq m} (-1)^{|\alpha|} D^{2\alpha} v(x) = 0$$

in the weak sense, and hence a.e. in Ω.

Coordinate Transformations

3.40 Let Φ be a one-to-one transformation of a domain $\Omega \subset \mathbb{R}^n$ onto a domain $G \in \mathbb{R}^n$, having inverse $\Psi = \Phi^{-1}$. We say that Φ is *m-smooth* if, when we write $y = \Phi(x)$ and $x = \Psi(y)$ in the form

$$\begin{aligned}
y_1 &= \phi_1(x_1, \ldots, x_n), & x_1 &= \psi_1(y_1, \ldots, y_n), \\
y_2 &= \phi_2(x_1, \ldots, x_n), & x_2 &= \psi_2(y_1, \ldots, y_n), \\
&\vdots & &\vdots \\
y_n &= \phi_n(x_1, \ldots, x_n), & x_n &= \psi_n(y_1, \ldots, y_n),
\end{aligned}$$

then $\phi_1, \ldots, \phi_n$ belong to $C^m(\overline{\Omega})$ and $\psi_1, \ldots, \psi_n$ belong to $C^m(\overline{G})$.

If u is a measurable function on Ω, we define a measurable function Au on G by

$$Au(y) = u\big(\Psi(y)\big). \tag{9}$$

Suppose that Φ is 1-smooth so that there exist constants $0 < c < C$ such that for all $x \in \Omega$

$$c \leq |\det \Phi'(x)| \leq C, \tag{10}$$

where Φ' denotes the Jacobian matrix $\partial(y_1, \ldots, y_n)/\partial(y_1, \ldots, y_n)$. Since smooth functions are dense in L^p spaces, the operator A defined by (9) transforms $L^p(\Omega)$ boundedly onto $L^p(G)$ and has a bounded inverse; in fact, for $1 \leq p < \infty$,

$$c^{1/p} \|u\|_{p,\Omega} \leq \|Au\|_{p,G} \leq C^{1/p} \|u\|_{p,\Omega}.$$

We establish a similar result for Sobolev spaces.

3.41 THEOREM Let Φ be m-smooth, where $m \ge 1$. The operator A defined by (9) transforms $W^{m,p}(\Omega)$ boundedly onto $W^{m,p}(G)$ and has a bounded inverse.

Proof. We show that the inequality $\|Au\|_{m,p,G} \le \text{const } \|u\|_{m,p,\Omega}$ holds for every $u \in W^{m,p}(\Omega)$, the constant depending only on the transformation Φ. The reverse inequality $\|Au\|_{m,p,G} \ge \text{const } \|u\|_{m,p,\Omega}$ (with a different constant) can be established similarly, using the inverse operator A^{-1}. By Theorem 3.17 for given $u \in W^{m,p}(\Omega)$, there exists a sequence $\{u_j\} \subset C^\infty(\Omega)$ converging to u in $W^{m,p}(\Omega)$-norm. For such smooth u_j it is readily checked by induction on $|\alpha|$ that

$$D^\alpha(Au_j)(y) = \sum_{\beta \le \alpha} M_{\alpha\beta}(y) A(D^\beta u_j)(y), \tag{11}$$

where $M_{\alpha\beta}$ is a polynomial of degree not exceeding $|\beta|$ in derivatives of orders not exceeding $|\alpha|$ of the various components of Ψ. If $\theta \in \mathscr{D}(G)$ integration by parts gives

$$(-1)^{|\alpha|} \int_G (Au_j)(y) D^\alpha \theta(y)\, dy = \sum_{\beta \le \alpha} \int_G A(D^\beta u_j)(y) M_{\alpha\beta}(y)\theta(y)\, dy, \tag{12}$$

or, replacing y by $\Phi(x)$ and expressing the integrals over Ω,

$$\begin{aligned}(-1)^{|\alpha|} &\int_\Omega u_j(x)(D^\alpha\theta)\big(\Phi(x)\big)|\det \Phi'(x)|\, dx \\ &= \sum_{\beta \le \alpha} \int_\Omega D^\beta u_j(x) M_{\alpha\beta}\big(\Phi(x)\big)\theta\big(\Phi(x)\big)|\det \Phi'(x)|\, dx. \end{aligned} \tag{13}$$

Since $D^\beta u_j \to u$ in $L^p(\Omega)$ for $|\beta| \le m$, we can take the limit through (13) as $n \to \infty$ and hence obtain (12) with u replacing u_j. Thus (11) holds in the weak sense for any $u \in W^{m,p}(\Omega)$. Therefore

$$\begin{aligned}\int_G |D^\alpha(Au)(y)|^p\, dy &\le \Big(\sum_{\beta \le \alpha} 1\Big)^p \max_{|\beta| \le |\alpha|} \Big(\sup_{y \in G} |M_{\alpha\beta}| \int_G |(D^\beta u)|\big(\Psi(y)\big)|^p\, dy\Big) \\ &\le \text{const } \max_{|\beta| \le |\alpha|} \int_\Omega |D^\beta u(x)|^p\, dx,\end{aligned}$$

from which it follows that $\|Au\|_{m,p,G} \le \text{const } \|u\|_{m,p,\Omega}$. ∎

Of special importance in later chapters is the case of the above theorem corresponding to nonsingular linear transformations Φ or, more generally, affine transformations (compositions of nonsingular linear transformations and translations). For such transformations $\det \Phi'(x)$ is a nonzero constant.

4

THE SOBOLEV IMBEDDING THEOREM

4.1 The imbedding characteristics of Sobolev spaces are essential in their uses in analysis, especially in the study of differential and integral operators. The most important imbedding results for Sobolev spaces are often gathered together into a single "theorem" called *the Sobolev Imbedding Theorem* although they are of several different types and can require different methods of proof. The core results are due to Sobolev [So2] but our statement (Theorem 4.12) also includes refinements due to others, in particular Morrey [Mo] and Gagliardo [Ga1].

Most of the imbeddings hold for domains $\Omega \subset \mathbb{R}^n$ satisfying some form of "cone condition" that enables us to derive pointwise estimates for the value of a function at the vertex of a truncated cone from suitable averages of the values of the function and its derivatives over the cone. Some of the imbeddings require stronger geometric hypotheses which, roughly speaking, force Ω to have an $(n-1)$-dimensional boundary that is locally the graph of a Lipschitz continuous function and which, like the segment condition described in Paragraph 3.21, requires Ω to lie on only one side of its boundary. We will discuss these geometric properties of domains prior to the statement of the imbedding theorem itself.

4.2 (Targets of the Imbeddings) The Sobolev imbedding theorem asserts the existence of imbeddings of $W^{m,p}(\Omega)$ (or $W_0^{m,p}(\Omega)$) into Banach spaces of the following types:

(i) $W^{j,q}(\Omega)$, where $j \le m$, and in particular $L^q(\Omega)$,

(ii) $W^{j,q}(\Omega_k)$, where, for $1 \le k < n$, Ω_k is the intersection of Ω with a

k-dimensional plane in $\mathbb{R}^n$.

(iii) $C_B^j(\Omega)$, the space of functions having bounded, continuous derivatives up to order j on Ω (see Paragraph 1.27) normed by

$$\left\| u\,;\, C_B^j(\Omega) \right\| = \max_{0 \le |\alpha| \le j} \sup_{x \in \Omega} |D^\alpha u(x)|.$$

(iv) $C^j(\overline{\Omega})$, the closed subspace of $C_B^j(\Omega)$ consisting of functions having bounded, uniformly continuous derivatives up to order j on Ω (see Paragraph 1.28) with the same norm as $C_B^j(\Omega)$:

$$\left\| \phi\,;\, C^j(\overline{\Omega}) \right\| = \max_{0 \le \alpha \le j} \sup_{x \in \Omega} |D^\alpha \phi(x)|.$$

This space is smaller than $C_B^j(\Omega)$ in that its elements must be uniformly continuous on Ω. For example, the function u of Example 3.20 belongs to $C_B^1(\Omega)$ but certainly not to $C^1(\overline{\Omega})$ for the domain Ω of that example.

(v) $C^{j,\lambda}(\overline{\Omega})$, the closed subspace of $C^j(\overline{\Omega})$ consisting of functions whose derivatives up to order j satisfy Hölder conditions of exponent λ in Ω (see Paragraph 1.29). The norm on $C^{j,\lambda}(\overline{\Omega})$ is

$$\left\| \phi\,;\, C^{j,\lambda}(\overline{\Omega}) \right\| = \left\| \phi\,;\, C^j(\overline{\Omega}) \right\| + \max_{0 \le |\alpha| \le j} \sup_{\substack{x,y \in \Omega \\ x \ne y}} \frac{|D^\alpha \phi(x) - D^\alpha \phi(y)|}{|x-y|^\lambda}.$$

Since elements of $W^{m,p}(\Omega)$ are, strictly speaking, not functions defined everywhere on Ω, but rather equivalence classes of such functions defined and equal up to sets of measure zero, we must clarify what is meant by imbeddings of types (ii)–(v). What is intended for imbeddings into the continuous function spaces (types (iii)-(v)) is that the "equivalence class" $u \in W^{m,p}(\Omega)$ should contain an element that belongs to the continuous function space that is the target of the imbedding and is bounded in that space by a constant times $\|u\|_{m,p,\Omega}$. Hence, for example, existence of the imbedding

$$W^{m,p}(\Omega) \to C^j(\overline{\Omega})$$

means that each $u \in W^{m,p}(\Omega)$ when considered as a function, can be redefined on a subset of Ω having measure zero to produce a new function $u^* \in C^j(\overline{\Omega})$ such that $u^* = u$ in $W^{m,p}(\Omega)$ (i.e. u^* and u belong to the same "equivalence class" in $W^{m,p}(\Omega)$) and

$$\left\| u^*\,;\, C^j(\overline{\Omega}) \right\| \le K\, \|u\|_{m,p,\Omega}$$

with K independent of u.

Even more care is necessary in interpreting imbeddings into spaces of type (ii):

$$W^{m,p}(\Omega) \to W^{j,q}(\Omega_k)$$

where Ω_k is the intersection of Ω with a plane of dimension $k < n$. Each element of $W^{m,p}(\Omega)$ is, by Theorem 3.17, a limit in that space of a sequence $\{u_i\}$ of functions in $C^\infty(\Omega)$. The functions u_i have *traces on* Ω_k (that is, restrictions to Ω_k) that belong to $C^\infty(\Omega_k)$. The above imbedding signifies that these traces converge in $W^{j,q}(\Omega_k)$ to a function u^* that is independent of the choice of $\{u_i\}$ and satisfies

$$\left\|u^*\right\|_{j,q,\Omega_k} \le K \left\|u\right\|_{m,p,\Omega}$$

with K independent of u.

4.3 Let us note as a point of interest, though of no particular use to us later, that the imbedding $W^{m,p}(\Omega) \to W^{j,q}(\Omega)$ is equivalent to the simple containment $W^{m,p}(\Omega) \subset W^{j,q}(\Omega)$. Certainly the former implies the latter. To verify the converse, suppose $W^{m,p}(\Omega) \subset W^{j,q}(\Omega)$, and let I be the linear operator taking $W^{m,p}(\Omega)$ into $W^{j,q}(\Omega)$ defined by $Iu = u$ for $u \in W^{m,p}(\Omega)$. If $u_k \to u$ in $W^{m,p}(\Omega)$ (and hence in $L^p(\Omega)$) and $Iu_k \to v$ in $W^{j,q}(\Omega)$ (and hence in $L^q(\Omega)$), then, passing to a subsequence if necessary, we have by Corollary 2.17 that $u_k(x) \to u(x)$ a.e. on Ω, $u_k(x) = Iu_k(x) \to v(x)$ a.e. on Ω. Thus $u(x) = v(x)$ a.e. on Ω, that is, $Iu = v$, and I is continuous by the closed graph theorem of functional analysis.

Geometric Properties of Domains

4.4 (Some Definitions) Many properties of Sobolev spaces defined on a domain Ω, and in particular the imbedding properties of these spaces, depend on regularity properties of Ω. Such regularity is normally expressed in terms of geometric or analytic conditions that may or may not be satisfied by a given domain. We specify below several such conditions and consider their relationships. First we make some definitions.

Let v be a nonzero vector in $\mathbb{R}^n$, and for each $x \neq 0$ let $\angle(x, v)$ be the angle between the position vector x and v. For given such v, $\rho > 0$, and κ satisfying $0 < \kappa \le \pi$, the set

$$C = \{x \in \mathbb{R}^n : x = 0 \text{ or } 0 < |x| \le \rho,\ \angle(x, v) \le \kappa/2\}$$

is called a *finite cone* of height ρ, axis direction v and aperture angle κ with vertex at the origin. Note that $x + C = \{x + y : y \in C\}$ is a finite cone with vertex at x but the same dimensions and axis direction as C and is obtained by parallel translation of C.

Given n linearly independent vectors $y_1, \ldots, y_n \in \mathbb{R}^n$, the set

$$P = \left\{ \sum_{j=1}^{n} \lambda_j y_j \; : \; 0 \le \lambda_j \le 1, \; 1 \le j \le n \right\}$$

is a *parallelepiped* with one vertex at the origin. Similarly, $x + P$ is a parallel translate of P having one vertex at x. The *centre* of $x + P$ is the point given by $c(x + P) = x + (1/2)(y_1 + \cdots + y_n)$. Every parallelepiped with a vertex at x is contained in a finite cone with vertex at x and also contains such a cone.

An open cover $\mathcal{O}$ of a set $S \subset \mathbb{R}^n$ is said to be *locally finite* if any compact set in $\mathbb{R}^n$ can intersect at most finitely many members of $\mathcal{O}$. Such locally finite collections of sets must be countable, so their elements can be listed in sequence. If S is closed, then any open cover of S by sets with a uniform bound on their diameters possesses a locally finite subcover.

We now specify six regularity properties that a domain $\Omega \subset \mathbb{R}^n$ may possess. We denote by Ω_δ the set of points in Ω within distance δ of the boundary of Ω:

$$\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \operatorname{bdry}\Omega) < \delta\}.$$

4.5 (The Segment Condition) As defined in Paragraph 3.21, a domain Ω satisfies the *segment condition* if every $x \in \operatorname{bdry}\Omega$ has a neighbourhood U_x and a nonzero vector y_x such that if $z \in \overline{\Omega} \cap U_x$, then $z + ty_x \in \Omega$ for $0 < t < 1$. Since the boundary of Ω is necessarily closed, we can replace its open cover by the neighbourhoods U_x with a locally finite subcover $\{U_1, U_2, \ldots\}$ with corresponding vectors $y_1, y_2, \ldots$ such that if $x \in \overline{\Omega} \cap U_j$ for some j, then $x + ty_j \in \Omega$ for $0 < t < 1$.

4.6 (The Cone Condition) Ω satisfies the *cone condition* if there exists a finite cone C such that each $x \in \Omega$ is the vertex of a finite cone C_x contained in Ω and congruent to C. Note that C_x need not be obtained from C by parallel translation, but simply by rigid motion.

4.7 (The Weak Cone Condition) Given $x \in \Omega$, let $R(x)$ consist of all points $y \in \Omega$ such that the line segment from x to y lies in Ω; thus $R(x)$ is a union of rays and line segments emanating from x. Let

$$\Gamma(x) = \{y \in R(x) : |y - x| < 1\}.$$

We say that Ω satisfies the *weak cone condition* if there exists $\delta > 0$ such that

$$\mu_n\big(\Gamma(x)\big) \ge \delta \qquad \text{for all } x \in \Omega,$$

where μ_n is the Lebesgue measure in $\mathbb{R}^n$. Clearly the cone condition implies the weak cone condition, but there are many domains satisfying the weak cone condition that do not satisfy the cone condition.

4.8 (The Uniform Cone Condition) Ω satisfies the *uniform cone condition* if there exists a locally finite open cover $\{U_j\}$ of the boundary of Ω and a corresponding sequence $\{C_j\}$ of finite cones, each congruent to some fixed finite cone C, such that

(i) There exists $M < \infty$ such that every U_j has diameter less then M.

(ii) $\Omega_\delta \subset \bigcup_{j=1}^{\infty} U_j$ for some $\delta > 0$.

(iii) $Q_j \equiv \bigcup_{x \in \Omega \cap U_j} (x + C_j) \subset \Omega$ for every j.

(iv) For some finite R, every collection of $R + 1$ of the sets Q_j has empty intersection.

4.9 (The Strong Local Lipschitz Condition) Ω satisfies the *strong local Lipschitz condition* if there exist positive numbers δ and M, a locally finite open cover $\{U_j\}$ of bdry Ω, and, for each j a real-valued function f_j of $n - 1$ variables, such that the following conditions hold:

(i) For some finite R, every collection of $R + 1$ of the sets U_j has empty intersection.

(ii) For every pair of points $x, y \in \Omega_\delta$ such that $|x - y| < \delta$, there exists j such that

$$x, y \in V_j \equiv \left\{x \in U_j : \operatorname{dist}(x, \operatorname{bdry} U_j) > \delta\right\}.$$

(iii) Each function f_j satisfies a Lipschitz condition with constant M: that is, if $\xi = (\xi_1, \dots, \xi_{n-1})$ and $\rho = (\rho_1, \dots, \rho_{n-1})$ are in $\mathbb{R}^{n-1}$, then

$$|f(\xi) - f(\rho)| \le M|\xi - \rho|.$$

(iv) For some Cartesian coordinate system $(\zeta_{j,1}, \dots, \zeta_{j,n})$ in U_j, $\Omega \cap U_j$ is represented by the inequality

$$\zeta_{j,n} < f_j(\zeta_{j,1}, \dots, \zeta_{j,n-1}).$$

If Ω is bounded, the rather complicated set of conditions above reduce to the simple condition that Ω should have a locally Lipschitz boundary, that is, that each point x on the boundary of Ω should have a neighbourhood U_x whose intersection with bdry Ω should be the graph of a Lipschitz continuous function.

4.10 (The Uniform C^m-Regularity Condition) Ω satisfies the *uniform C^m-regularity condition* is there exists a locally finite open cover $\{U_j\}$ of bdry Ω, and a corresponding sequence $\{\Phi_j\}$ of m-smooth transformations (see Paragraph 3.40) with Φ_j taking U_j onto the ball $B = \{y \in \mathbb{R}^n : |y| < 1$ and having inverse $\Psi_j = \Phi_j^{-1}$, such that:

(i) For some finite R. every collection of $R + 1$ of the sets U_j has empty intersection.

(ii) For some $\delta > 0$, $\Omega_\delta \subset \bigcup_{j=1}^{\infty} \Psi_j(\{y \in \mathbb{R}^n : |y| < \frac{1}{2}\})$.

(iii) For each j, $\Phi_j(U_j \cap \Omega) = \{y \in B : y_n > 0\}$.

(iv) If $(\phi_{j,1}, \ldots, \phi_{j,n})$ and $(\psi_{j,1}, \ldots, \psi_{j,n})$ are the components of Φ_j and Ψ_j, then there is a finite constant M such that for every α with $0 < |\alpha| \le m$, every i, $1 \le i \le n$, and every j we have

$$\begin{aligned} |D^\alpha \phi_{j,i}(x)| &\le M, &\quad &\text{for } x \in U_j, \\ |D^\alpha \psi_{j,i}(y)| &\le M, &\quad &\text{for } y \in B. \end{aligned}$$

4.11 Except for the cone condition and the weak cone condition, the other conditions defined above all require that the boundary of Ω be $(n-1)$-dimensional and that Ω lie on only one side of its boundary. The domain Ω of Example 3.20 satisfies the cone condition (and therefore the weak cone condition), but none of the other four conditions. Among those four we have:

$$\begin{aligned} &\text{the uniform } C^m\text{-regularity condition } (m \ge 2) \\ &\Longrightarrow \text{ the strong local Lipschitz condition} \\ &\Longrightarrow \text{ the uniform cone condition} \\ &\Longrightarrow \text{ the segment condition.} \end{aligned}$$

Also,

$$\begin{aligned} &\text{the uniform cone condition} \\ &\Longrightarrow \text{ the cone condition} \\ &\Longrightarrow \text{ the weak cone condition} \end{aligned}$$

Typically, most of the imbeddings of $W^{m,p}(\Omega)$ have been proven for domains satisfying the cone condition. Exceptions are the imbeddings into spaces $C^j(\overline{\Omega})$ and $C^{j,\lambda}(\overline{\Omega})$ of uniformly continuous functions which, as suggested by Example 3.20, require that Ω lie on one side of its boundary. These imbeddings are usually proved for domains satisfying the strong local Lipschitz condition. It should be noted, however, that Ω need not satisfy any of these conditions for appropriate imbeddings of $W_0^{m,p}(\Omega)$ to be valid.

4.12 THEOREM (The Sobolev Imbedding Theorem) Let Ω be a domain in $\mathbb{R}^n$ and, for $1 \le k \le n$, let Ω_k be the intersection of Ω with a plane of dimension k in $\mathbb{R}^n$. (If $k = n$, then $\Omega_k = \Omega$.) Let $j \ge 0$ and $m \ge 1$ be integers and let $1 \le p < \infty$.

PART I Suppose Ω satisfies the cone condition.

Case A If either $mp > n$ or $m = n$ and $p = 1$, then

$$W^{j+m,p}(\Omega) \to C_B^j(\Omega). \tag{1}$$

Moreover, if $1 \le k \le n$, then

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega_k) \qquad \text{for } p \le q \le \infty, \tag{2}$$

and, in particular,

$$W^{m,p}(\Omega) \to L^q(\Omega) \qquad \text{for } p \le q \le \infty.$$

Case B If $1 \le k \le n$ and $mp = n$, then

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega_k), \qquad \text{for } p \le q < \infty, \tag{3}$$

and, in particular,

$$W^{m,p}(\Omega) \to L^q(\Omega), \qquad \text{for } p \le q < \infty.$$

Case C If $mp < n$ and either $n - mp < k \le n$ or $p = 1$ and $n - m \le k \le n$, then

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega_k), \qquad \text{for } p \le q \le p* = kp/(n - mp). \tag{4}$$

In particular,

$$W^{m,p}(\Omega) \to L^q(\Omega), \qquad \text{for } p \le q \le p* = np/(n - mp). \tag{5}$$

The imbedding constants for the imbeddings above depend only on n, m, p, q, j, k, and the dimensions of the cone C in the cone condition.

PART II Suppose Ω satisfies the strong local Lipschitz condition. (See Paragraph 4.9.) Then the target space $C_B^j(\Omega)$ of the imbedding (1) can be replaced with the smaller space $C^j(\overline{\Omega})$, and the imbedding can be further refined as follows:

If $mp > n > (m - 1)p$, then

$$W^{j+m,p}(\Omega) \to C^{j,\lambda}(\overline{\Omega}) \qquad \text{for } 0 < \lambda \le m - (n/p), \tag{6}$$

and if $n = (m-1)p$, then

$$W^{j+m,p}(\Omega) \to C^{j,\lambda}(\overline{\Omega}) \qquad \text{for } 0 < \lambda < 1. \tag{7}$$

Also, if $n = m - 1$ and $p = 1$, then (7) holds for $\lambda = 1$ as well.

PART III All of the imbeddings in Parts A and B are valid for *arbitrary* domains Ω if the W-space undergoing the imbedding is replaced with the corresponding W_0-space.

4.13 REMARKS

1. Imbeddings (1)–(4) are essentially due to Sobolev [So1, So2], although his original proof did not cover the all cases. Imbeddings (6)–(7) originate in the work of Morrey [Mo].
2. Imbeddings (2)–(4) involving traces of functions on planes of lower dimension can be extended in a reasonable manner to apply to traces on more general smooth manifolds. For example, see Theorem 5.36.
3. If Ω_k (or Ω) has finite volume, then imbeddings (2)–(4) also hold for $1 \le q < p$ in addition to the values of q asserted in the statement of the theorem. This follows from Theorem 2.14. It will be shown in Theorem 6.43 that no imbedding of the form $W^{m,p}(\Omega) \to L^q(\Omega)$ where $q < p$ is possible unless Ω has finite volume.
4. Part III of the theorem is an immediate consequence of Parts I and II applied to $\mathbb{R}^n$ because, by Lemma 3.27, the operator of zero extension of functions outside Ω maps $W_0^{m,p}(\Omega)$ isometrically into $W^{m,p}(\mathbb{R}^n)$.
5. More generally, suppose there exists an operator E mapping $W^{m,p}(\Omega)$ into $W^{m,p}(\mathbb{R}^n)$ such that $Eu(x) = u(x)$ a.e. in Ω and such that $\|Eu\|_{m,p,\mathbb{R}^n} \le K_1 \|u\|_{m,p,\Omega}$. (Such an operator is called an (m,p)-extension operator for Ω. If the imbedding theorem has already been proved for $\mathbb{R}^n$, then it must hold for the domain Ω as well. For example, if $W^{m,p}(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$, and $u \in W^{m,p}(\Omega)$, then

$$\|u\|_{q,\Omega} \le \|Eu\|_{q,\mathbb{R}^n} \le K_2 \|Eu\|_{m,p,\mathbb{R}^n} \le K_2 K_1 \|u\|_{m,p,\Omega}.$$

In Chapter 5 we will establish the existence of such extension operators, but only for domains satisfying conditions stronger than the cone condition, so we will not use such a technique to prove Theorem 4.12.

6. It is sufficient to prove imbeddings (1)–(4), (6)–(7) for the special case $j = 0$, as the general case follows by applying this special case to derivative $D^\alpha u$ of u for $|\alpha| \le j$. For example, if the imbedding $W^{m,p}(\Omega) \to L^q(\Omega)$ has been proven, then for any $u \in W^{j+m,p}(\Omega)$ we have $D^\alpha u \in W^{m,p}(\Omega)$

for $|\alpha| \le j$, whence $D^\alpha u \in L^q(\Omega)$. Thus $u \in W^{j,q}(\Omega)$ and

$$\|u\|_{j,q} = \left(\sum_{|\alpha| \le j} \|D^\alpha u\|_{0,q}^q \right)^{1/q}$$
$$\le K_1 \left(\sum_{|\alpha| \le j} \|D^\alpha u\|_{m,p}^p \right)^{1/p} \le K_2 \|u\|_{j+m,p} .$$

7. The authors have shown that all of Part I can be proved for domains satisfying only the weak cone condition instead of the cone condition. See [AF1].

4.14 (Strategy for Proving the Imbedding Theorem) We use two overlapping methods to prove the imbeddings in Part I of Theorem 4.12. The first, potential theoretic in nature, was used by Sobolev. It works when $p > 1$, and gives the right order of growth of imbedding constants as $q \to \infty$ when $mp = n$; this will be useful in Chapter 7. Here we use the potential method to prove Case A and the imbeddings in Cases B and C for $p > 1$. The other approach is based on a combinatorial-averaging argument due to Gagliardo [Ga1]. We will use it to establish Cases B and C for $p = 1$, though it could be adapted (with a bit more difficulty) to prove all of Part I. (See, in particular, Theorem 5.10 and the Remark following that theorem.)

Part II of the theorem follows by sharpening certain estimates used in obtaining Case A of Part I.

The entire proof of Theorem 4.12 is fairly lengthy and is broken down into several lemmas. Throughout we use K, and occasionally $K_1, K_2, \ldots$, to represent various constants that can depend on parameters of the spaces being imbedded. The values of these constants can change from line to line. While stated for the cone condition, the potential method works verbatim under the weak cone condition as well.

Imbeddings by Potential Arguments

4.15 LEMMA (A Local Estimate) Let domain $\Omega \subset \mathbb{R}^n$ satisfy the cone condition. There exists a constant K depending on m, n, and the dimensions ρ and κ of the cone C specified in the cone condition for Ω such that for every $u \in C^\infty(\Omega)$, every $x \in \Omega$, and every r satisfying $0 < r \le \rho$, we have

$$|u(x)| \le K \Bigg(\sum_{|\alpha| \le m-1} r^{|\alpha|-n} \int_{C_{x,r}} |D^\alpha u(y)|\, dy + \sum_{|\alpha| = m} \int_{C_{x,r}} |D^\alpha u(y)| |x-y|^{m-n}\, dy \Bigg), \tag{8}$$

where $C_{x,r} = \{y \in C_x : |x - y| \le r\}$. Here $C_x \subset \Omega$ is a cone congruent to C having vertex at x.

Proof. We apply Taylor's formula with integral remainder,

$$f(1) = \sum_{j=0}^{m-1} \frac{1}{j!} f^{(j)}(0) + \frac{1}{(m-1)!} \int_0^1 (1-t)^{m-1} f^{(m)}(t)\, dt$$

to the function $f(t) = u\big(tx + (1-t)y\big)$, where $x \in \Omega$ and $y \in C_{x,r}$. Noting that

$$f^{(j)}(t) = \sum_{|\alpha|=j} \frac{j!}{\alpha!} D^\alpha u\big(tx + (1-t)y\big)(x-y)^\alpha,$$

where $\alpha! = \alpha_1! \cdots \alpha_n!$ and $(x-y)^\alpha = (x_1 - y_1)^{\alpha_1} \cdots (x_n - y_n)^{\alpha_n}$, we obtain

$$\begin{aligned} |u(x)| \le &\sum_{|\alpha| \le m-1} \frac{1}{\alpha!} |D^\alpha u(y)| |x-y|^{|\alpha|} \\ &+ \sum_{|\alpha|=m} \frac{m}{\alpha!} |x-y|^m \int_0^1 (1-t)^{m-1} |D^\alpha u\big(tx + (1-t)y\big)|\, dt. \end{aligned}$$

If C has volume $c\rho^n$, then $C_{x,r}$ has volume cr^n. Integration of y over $C_{x,r}$ leads to

$$\begin{aligned} &cr^n |u(x)| \\ &\quad \le \sum_{|\alpha| \le m-1} \frac{r^{|\alpha|}}{\alpha!} \int_{C_{x,r}} |D^\alpha u(y)|\, dy \\ &\quad + \sum_{|\alpha|=m} \frac{m}{\alpha!} \int_{C_{x,r}} |x-y|^m\, dy \int_0^1 (1-t)^{m-1} |D^\alpha u\big(tx + (1-t)y\big)|\, dt. \end{aligned}$$

In the final (double) integral we first change the order of integration, then substitute $z = tx + (1-t)y$, so that $z - x = (1-t)(y-x)$ and $dz = (1-t)^n\, dy$, to obtain, for that integral,

$$\int_0^1 (1-t)^{-n-1}\, dt \int_{C_{x,(1-t)r}} |z-x|^m |D^\alpha u(z)|\, dz.$$

A second change of order of integration now gives for the above integral

$$\begin{aligned} &\int_{C_{x,r}} |x-z|^m |D^\alpha u(z)|\, dz \int_0^{1-(|z-x|/r)} (1-t)^{-n-1}\, dt \\ &\quad \le \frac{r^n}{n} \int_{C_{x,r}} |x-z|^{m-n} |D^\alpha u(z)|\, dz. \end{aligned}$$

Inequality (8) now follows immediately. ∎

4.16 (Proof of Part I, Case A of Theorem 4.12) As noted earlier, we can assume that $j = 0$. Let $u \in W^{m,p}(\Omega) \cap C^\infty(\Omega)$ and let $x \in \Omega$. We must show that

$$|u(x)| \le K \, \|u\|_{m,p} \,. \tag{9}$$

For $p = 1$ and $m = n$, this follows immediately from (8). For $p > 1$ and $mp > n$, we apply Hölder's inequality to (8) with $r = \rho$ to obtain

$$|u(x)| \le K \Bigg(\sum_{|\alpha| \le m-1} c^{1/p'} \rho^{|\alpha| - (n/p)} \, \|D^\alpha u\|_{p, C_{x,\rho}} + \sum_{|\alpha| = m} \|D^\alpha u\|_{p, C_{x,\rho}} \left[\int_{C_{x,\rho}} |x - y|^{(m-n)p'} \, dy \right]^{1/p'} \Bigg),$$

where c is the volume of $C_{x,1}$ and $p' = p/(p-1)$. The final integral is finite since $(m-n)p' > -n$ when $mp > n$. Thus

$$|u(x)| \le K \sum_{|\alpha| \le m} \|D^\alpha u\|_{p, C_{x,\rho}} \tag{10}$$

and (9) follows because $C_{x,\rho} \subset \Omega$.

Next observe that since any $u \in W^{m,p}(\Omega)$ is the limit of a Cauchy sequence of continuous functions by Theorem 3.17, and since (9) implies this Cauchy sequence converges to a continuous function on Ω, u must coincide with a continuous function a.e. on Ω. Thus $u \in C_B^0(\Omega)$ and imbedding (1) is proved.

Now let Ω_k denote the intersection of Ω with a k-dimensional plane H, let $\Omega_{k,\rho} = \{x \in \mathbb{R}^n : \operatorname{dist}(x, \Omega_k) < \rho\}$, and let u and all its derivatives be extended to be zero outside Ω. Since $C_{x,\rho} \subset B_\rho(x)$, the ball of radius ρ with centre at x, we have, using (10) and denoting by dx' the k-volume element in H,

$$\begin{aligned} \int_{\Omega_k} |u(x)|^p \, dx' &\le K \sum_{|\alpha| \le m} \int_{\Omega_k} dx' \int_{B_\rho(x)} |D^\alpha u(y)|^p dy \\ &= K \sum_{|\alpha| \le m} \int_{\Omega_{k,\rho}} |D^\alpha u(y)|^p dy \int_{H \cap B_\rho(y)} dx' \le K_1 \, \|u\|_{m,p,\Omega}^p \,, \end{aligned}$$

and $W^{m,p}(\Omega) \to L^p(\Omega_k)$. But (9) shows that $W^{m,p}(\Omega) \to L^\infty(\Omega_k)$ and so imbedding (2) follows by Theorem 2.11. ∎

Let χ_r be the characteristic function of the ball $B_r(0) = \{x \in \mathbb{R}^n : |x| < r\}$. In the following discussion we will develop estimates for convolutions of L^p functions with the kernels $\omega_m(x) = |x|^{m-n}$ and

$$\chi_r \omega_m(x) = \begin{cases} |x|^{m-n} & \text{if } |x| < r, \\ 0 & \text{if } |x| \ge r. \end{cases}$$

Observe that if $m \le n$ and $0 < r \le 1$, then

$$\chi_r(x) \le \chi_r\omega_m(x) \le \omega_m(x).$$

4.17 LEMMA Let $p \ge 1$, $1 \le k \le n$, and $n - mp < k$. There exists a constant K such that for every $r > 0$, every k-dimensional plane $H \subset \mathbb{R}^n$, and every $v \in L^p(\mathbb{R}^n)$, we have $\chi_r\omega_m * |v| \in L^p(H)$ and

$$\|\chi_r\omega_m * |v|\|_{p,H} \le K r^{m-(n-k)/p} \|v\|_{p,\mathbb{R}^n}. \tag{11}$$

In particular,

$$\|\chi_1 * |v|\|_{p,H} \le \|\chi_1\omega_m * |v|\|_{p,H} \le K \|v\|_{p,\mathbb{R}^n}.$$

Proof. If $p > 1$, then by Hölder's inequality

$$\begin{aligned}
\chi_r\omega_m * |v|(x) &= \int_{B_r(x)} |v(y)||x-y|^{-s}|x-y|^{s+m-n}\,dy \\
&\le \left(\int_{B_r(x)} |v(y)|^p|x-y|^{-sp}\,dy\right)^{1/p} \left(\int_{B_r(x)} |x-y|^{(s+m-n)p'}\,dy\right)^{1/p'} \\
&= K r^{s+m-(n/p)} \left(\int_{B_r(x)} |v(y)|^p|x-y|^{-sp}\,dy\right)^{1/p},
\end{aligned}$$

provided $s + m - (n/p) > 0$. If $p = 1$ the same estimate holds provided $s + m - n \ge 0$ without using Hölder's inequality.

Integrating the pth power of the above estimate over H (with volume element dx'), we obtain

$$\begin{aligned}
\|\chi_r\omega_m * |v|\|_{p,H}^p &= \int_H \left|\chi_r\omega_m * |v|(x)\right|^p dx' \\
&\le K r^{(s+m)p-n} \int_H dx' \int_{B_r(x)} |v(y)|^p|x-y|^{-sp}\,dy \\
&\le K r^{(s+m)p-n} r^{k-sp} \|v\|_{p,\mathbb{R}^n}^p = K r^{mp-(n-k)} \|v\|_{p,\mathbb{R}^n}^p,
\end{aligned}$$

provided $k > sp$.

Since $n - mp < k$ there exists s satisfying $(n/p) - m < s < k/p$, so both estimates above are valid and (11) holds. ∎

4.18 LEMMA Let $p > 1$, $mp < n$, $n - mp < k \le n$, and $p^* = kp/(n-mp)$. There exists a constant K such that for every k-dimensional plane H in $\mathbb{R}^n$ and every $v \in L^p(\mathbb{R}^n)$, we have $\omega_m * |v| \in L^{p^*}(H)$ and

$$\|\chi_1 * |v|\|_{p^*,H} \le \|\chi_1\omega_m * |v|\|_{p^*,H} \le \|\omega_m * |v|\|_{p^*,H} \le K \|v\|_{p,\mathbb{R}^n}. \tag{12}$$

Proof. Only the final inequality of (12) requires proof. Since $mp < n$, for each $x \in \mathbb{R}^n$ Hölder's inequality gives

$$\begin{aligned}\int_{\mathbb{R}^n - B_r(x)} |v(y)||x-y|^{m-n}\,dy &\le \|v\|_{p,\mathbb{R}^n}\left(\int_{\mathbb{R}^n - B_r(x)} |x-y|^{(m-n)p'}\,dy\right)^{1/p'}\\ &= K_1\,\|v\|_{p,\mathbb{R}^n}\left(\int_r^\infty t^{(m-n)p'+n-1}\,dt\right)^{1/p'}\\ &= K_1 r^{m-(n/p)}\,\|v\|_{p,\mathbb{R}^n}.\end{aligned}$$

If $t > 0$, choose r so that $K_1 r^{m-(n/p)}\,\|v\|_{p,\mathbb{R}^n} = t/2$. If

$$\omega_m * |v|(x) = \int_{\mathbb{R}^n} |v(y)||x-y|^{m-n}\,dy > t,$$

then

$$\chi_r\omega_m * |v|(x) = \int_{B_r(x)} |v(y)||x-y|^{m-n}\,dy > t/2.$$

Thus

$$\begin{aligned}\mu_k\big(\{x \in H : \omega_m * |v|(x) > t\}\big) &\le \mu_k\big(\{x \in H : \chi_r\omega_m * |v|(x) > t/2\}\big)\\ &\le \left(\frac{2}{t}\right)^p \|\chi_r\omega_m * |v|\|_{p,H}^p\\ &\le \left(\frac{r^{(n/p)-m}}{K_1\,\|v\|_{p,\mathbb{R}^n}}\right)^p K r^{mp-n+k}\,\|v\|_{p,\mathbb{R}^n}^p = K_2 r^k\end{aligned}$$

by inequality (11). But $r^k = \big(2K_1\,\|v\|_{p,\mathbb{R}^n}/t\big)^{p^*}$, so

$$\mu_k\big(\{x \in H : \omega_m * |v|(x) > t\}\big) \le K_2\left(\frac{2K_1}{t}\,\|v\|_{p,\mathbb{R}^n}\right)^{p^*}.$$

Thus the mapping $I : v \mapsto (\omega_m * |v|)\big|_H$ is of weak type (p, p^*).

For fixed m, n, k, the values of p satisfying the conditions of this lemma constitute an open interval, so there exist p_1 and p_2 in that interval, and a number θ satisfying $0 < \theta < 1$ such that

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2},$$

and

$$\frac{1}{p^*} = \frac{n/k}{p} - \frac{m}{k} = \frac{1-\theta}{p_1^*} + \frac{\theta}{p_2^*}.$$

Since $p^* > p$, the Marcinkiewicz interpolation theorem 2.58 assures us that I is bounded from $L^p(\mathbb{R}^n)$ into $L^{p^*}(H)$, that is, (12) holds. ∎

4.19 (Proof of Part I, Case C of Theorem 4.12 for $p > 1$) We have $mp < n$, $n - mp < k \le n$, and $p \le q \le p* = kp/(n - mp)$. Let $u \in C^\infty(\Omega)$ and extend u and all its derivatives to be zero on $\mathbb{R}^n - \Omega$. Taking $r = \rho$ in Lemma 4.15 and replacing $C_{x,r}$ with the larger ball $B_1(x)$, we obtain

$$|u(x)| \le K\left(\sum_{|\alpha|\le m-1} \chi_1 * |D^\alpha u|(x) + \sum_{|\alpha|=m} \chi_1\omega_m * |D^\alpha u|(x)\right). \tag{13}$$

If $1/q = \theta/p + (1-\theta)/p^*$ where $0 \le \theta \le 1$, then by the interpolation inequality of Theorem 2.11 and Lemmas 4.17 and 4.18

$$\begin{aligned}\|u\|_{q,\Omega_k} &\le \|u\|_{p,H}^{\theta}\,\|u\|_{p^*,H}^{1-\theta}\\ &\le K\left(\sum_{|\alpha|\le m}\|D^\alpha u\|_{p,\mathbb{R}^n}\right)^{\theta}\left(\sum_{|\alpha|\le m}\|D^\alpha u\|_{p,\mathbb{R}^n}\right)^{1-\theta}\\ &\le K\,\|u\|_{m,p,\Omega}\end{aligned}$$

as required. ∎

4.20 (Proof of Part I, Case B of Theorem 4.12 for $p > 1$) We have $mp = n$, $1 \le k \le n$, and $p \le q < \infty$. We can select numbers p_1, p_2, and θ such that $1 < p_1 < p < p_2$, $n - mp_1 < k$, $0 < \theta < 1$, and

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \qquad \frac{1}{q} = \frac{\theta}{p_1}.$$

As in the above proof of Case C for $p > 1$, the maps $v \mapsto (\chi_1 * |v|)\big|_H$ and $v \mapsto (\chi_1\omega_m * |v|)\big|_H$ are bounded from $L^{p_1}(\mathbb{R}^n)$ into $L^{p_1}\left(\mathbb{R}^k\right)$ and so are of weak type (p_1, p_1). As in the proof of Case A, these same maps are bounded from $L^{p_2}(\mathbb{R}^n)$ into $L^\infty\left(\mathbb{R}^k\right)$ and so are of weak type (p_2, ∞). By the Marcinkiewicz theorem again, they are bounded from $L^p(\mathbb{R}^n)$ into $L^q\left(\mathbb{R}^k\right)$ and

$$\|\chi_1 * |v|\|_{q,H} \le \|\chi_1\omega_m * |v|\|_{q,H} \le K\,\|v\|_{p,\mathbb{R}^n}$$

and the desired result follows by applying these estimates to the various terms of (13). ∎

Imbeddings by Averaging

4.21 We still need to prove the imbeddings for Cases B and C with $p = 1$. We first prove that $W^{1,1}(\Omega) \to L^{n/(n-1)}(\Omega)$ and deduce from this and the imbeddings already established for $p > 1$ that all but one of the remaining imbeddings in Cases B and C are valid. The remaining imbedding is the special case of C where $k = n - m$, $p = 1$, $p* = 1$ which will require a special proof.

We first show that any domain satisfying the cone condition is the union of finitely many subdomains each of which is a union of parallel translates of a fixed parallelepiped. Then we establish a special case of a combinatorial lemma estimating a function in terms of averages in coordinate directions. Both of these results are due to Gagliardo [Ga1] and constitute the foundation on which rests his proof of all of Cases B and C of Part I.

4.22 LEMMA (Decomposition of Ω) Let $\Omega \subset \mathbb{R}^n$ satisfy the cone condition. Then there exists a finite collection $\{\Omega_1, \dots, \Omega_N\}$ of open subsets of Ω such that $\Omega = \bigcup_{j=1}^{N} \Omega_j$, and such that to each Ω_j there corresponds a subset $A_j \subset \Omega_j$ and an open parallelepiped P_j with one vertex at 0 such that $\Omega_j = \bigcup_{x \in A_j} (x + P_j)$. If Ω is bounded and $\rho > 0$ is given, we can accomplish the above decomposition using sets A_j each satisfying $\operatorname{diam}(A_j) < \rho$.

Finally, if Ω is bounded and $\rho > 0$ is sufficiently small, then each Ω_j will satisfy the strong local Lipschitz condition.

Proof. Let C be the finite cone with vertex at 0 such that any $x \in \Omega$ is the vertex of a finite cone $C_x \subset \Omega$ congruent to C. We can select a finite number of finite cones $C_1, \dots, C_N$ each having vertex at 0 (and each having the same height as C but smaller aperture angle than C) such that any finite cone congruent to C and having vertex at 0 must contain one of the cones C_j. For each j, let P_j be an open parallelepiped with one vertex at the origin, contained in C_j, and having positive volume. Then for each $x \in \Omega$ there exists j, $1 \le j \le N$, such that

$$x + P_j \subset x + C_j \subset C_x \subset \Omega.$$

Since Ω is open and $\overline{x + P_j}$ is compact, $y + P_j \subset \Omega$ for any y sufficiently close to x. Hence every $x \in \Omega$ belongs to $y + P_j$ for some j and some $y \in \Omega$. Let $A_j = \{y \in \overline{\Omega} : y + P_j \subset \Omega\}$ and let $\Omega_j = \bigcup_{y \in A_j} (y + P_j)$. Then $\Omega = \bigcup_{j=1}^{N} \Omega_j$.

Now suppose Ω is bounded and $\rho > 0$ is given. If $\operatorname{diam}(A_j) \ge \rho$ we can decompose A_j into a finite union of sets A_{ji} each with diameter less than ρ and define the corresponding parallelepiped $P_{ji} = P_j$. We then rename the totality of such sets A_{ji} as a single finite family, which we again call $\{A_j\}$ and define Ω_j as above.

Figure 2 attempts to illustrate these notions for the domain in $\mathbb{R}^2$ considered in

Example 3.20:

$$\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < |x| < 1, \ 0 < y < 1\},$$
$$C = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0, x^2 + y^2 < 1/4\},$$
$$\rho < 0.98.$$

Finally, we show that if ρ is sufficiently small, then Ω_j satisfies the strong local Lipschitz condition. For simplicity of notation, let $G = \bigcup_{x \in A}(x + P)$, where $\operatorname{diam}(A) < \rho$ and P is a fixed parallelepiped. We show that G satisfies the strong local Lipschitz condition if ρ is suitably small. For each vertex v_j of P let $Q_j = \{y = v_j + \lambda(x - v_j) : x \in P, \lambda > 0\}$ be the infinite pyramid with vertex v_j generated by P. Then $P = \bigcap Q_j$, the intersection being taken over all 2^n vertices of P. Let $G_j = \bigcup_{x \in A}(x + Q_j)$. Let δ be the distance from the centre of P to the boundary of P and let B be an arbitrary ball of radius $\sigma = \delta/2$. For any fixed $x \in G$, B cannot intersect opposite faces of $x + P$ so we may pick a vertex v_j of P with the property that $x + v_j$ is common to all faces of $x + P$ that intersect B, if any such faces exist. Then $B \cap (x + P) = B \cap (x + Q_j)$. Now let $x, y \in A$ and suppose B could intersect relatively opposite faces of $x + P$ and $y + P$, that is, there exist points a and b on opposite faces of P such that $x + a \in B$ and $y + b \in B$. Then

$$\begin{aligned}\rho \geq \operatorname{dist}(x, y) &= \operatorname{dist}(x + b, y + b)\\ &\geq \operatorname{dist}(x + b, x + a) - \operatorname{dist}(x + a, y + b)\\ &\geq 2\delta - 2\sigma = \delta.\end{aligned}$$

It follows that if $\rho < \delta$, then B cannot intersect relatively opposite faces of $x + P$ and $y + P$ for any $x, y \in A$. Thus $B \cap (x + P) = B \cap (x + Q_j)$ for some fixed j independent of $x \in A$, whence $B \cap G = B \cap G_j$.

Choose coordinates $\xi = (\xi', \xi_n) = (\xi_1, \dots, \xi_{n-1}, \xi_n)$ in B so that the ξ_n-axis lies in the direction of the vector from the centre of P to the vertex v_j. Then $B \cap (x + Q_j)$ is specified in B by an inequality of the form $\xi_n < f_x(\xi')$ where f_x satisfies a Lipschitz condition with constant independent of x. Thus $B \cap G_j$, and hence $B \cap G$, is specified by $\xi_n < f(\xi')$, where $f(\xi') = \sup_{x \in A} f_x(\xi')$ is itself a Lipschitz continuous function. Since this can be done for a neighbourhood B of any point on the boundary of G, it follows that G satisfies the strong local Lipschitz condition. ∎

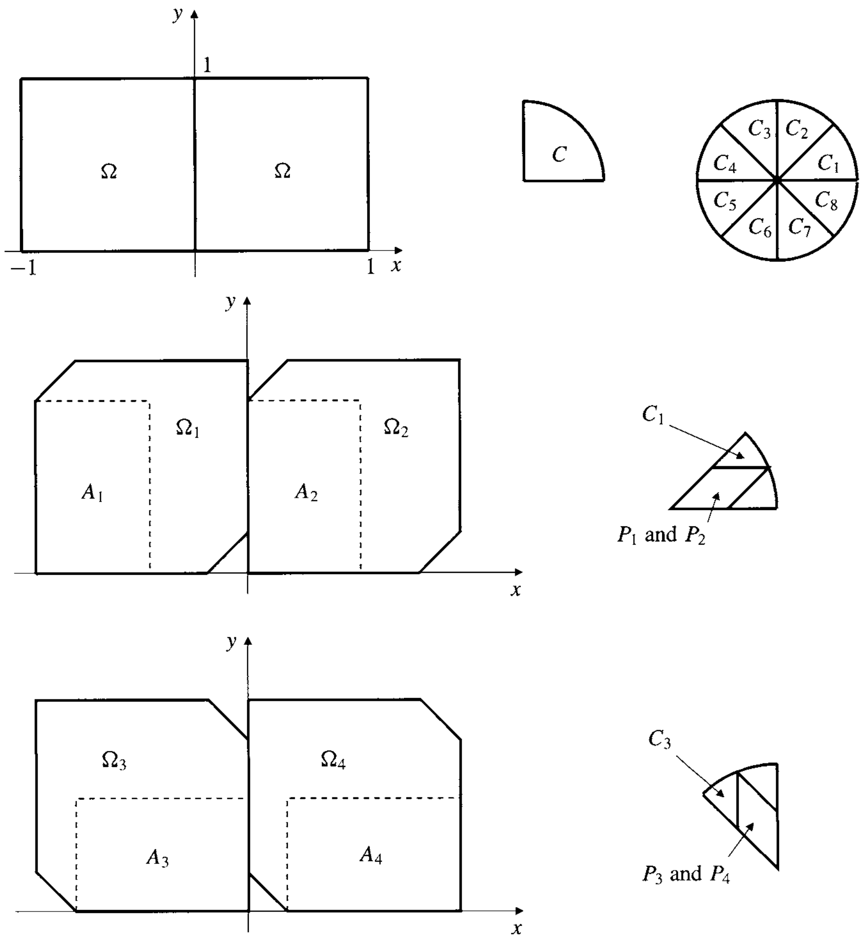


Fig. 2

4.23 LEMMA (An Averaging Lemma) Let Ω be a domain in $\mathbb{R}^n$ where $n \geq 2$. Let k be an integer satisfying $1 \leq k \leq n$, and let $\kappa = (\kappa_1, \ldots, \kappa_k)$ be a k-tuple of integers satisfying $1 \leq \kappa_1 < \kappa_2 < \cdots < \kappa_k \leq n$. Let S be the set of all $\binom{n}{k}$ such k-tuples. Given $x \in R^n$, let x_κ denote the point $(x_{\kappa_1}, \ldots, x_{\kappa_k})$ in $\mathbb{R}^k$ and let $dx_\kappa = dx_{\kappa_1} \cdots dx_{\kappa_k}$.

For $\kappa \in S$ let E_κ be the k-dimensional plane in $\mathbb{R}^n$ spanned by the coordinate axes corresponding to the components of x_κ:

$$E_\kappa = \left\{x \in \mathbb{R}^n \ : \ x_i = 0 \text{ if } i \notin \kappa\right\},$$

and let Ω_κ be the projection of Ω onto E_κ:

$$\Omega_\kappa = \{x \in E_\kappa \, : \, x_\kappa = y_\kappa \text{ for some } y \in \Omega\}.$$

Let $F_\kappa(x_\kappa)$ be a function depending only on the k components of x_κ and belonging to $L^\lambda(\Omega_\kappa)$, where $\lambda = \binom{n-1}{k-1}$. Then the function F defined by

$$F(x) = \prod_{\kappa \in S} F_\kappa(x_\kappa)$$

belongs to $L^1(\Omega)$, and $\|F\|_{1,\Omega} \le \prod_{\kappa \in S} \|F_\kappa\|_{\lambda,\Omega_\kappa}$, that is,

$$\left(\int_\Omega |F(x)|\, dx\right)^\lambda \le \prod_{\kappa \in S} \int_{\Omega_\kappa} |F_\kappa(x_\kappa)|^\lambda \, dx_\kappa. \tag{14}$$

Proof. We use the mixed-norm Hölder inequality of Paragraph 2.49 to provide the proof. For each $\kappa \in S$ let $\mathbf{p}_\kappa$ be the n-vector whose ith component is λ if $i \in \kappa$ and ∞ if $i \notin \kappa$. For each i, $1 \le i \le n$, exactly $(k/n)\binom{n}{k} = \lambda$ of the vectors $\mathbf{p}_\kappa$ have ith component equal to λ. Therefore, in the notation of Paragraph 2.49

$$\sum_{\kappa \in S} \frac{1}{\mathbf{p}_\kappa} = \frac{1}{\mathbf{w}},$$

where $\mathbf{w}$ is the n-vector $(1, 1, \ldots, 1)$.

Let $F_\kappa(x_\kappa)$ be extended to be zero for $x_\kappa \notin \Omega_\kappa$ and consider F_κ to be defined on $\mathbb{R}^n$ but independent of x_j if $j \notin \kappa$. Then F_κ is its own supremum over those x_j and

$$\|F_\kappa\|_{\lambda,\Omega_\kappa} = \|F_\kappa\|_{\mathbf{p}_\kappa,\mathbb{R}^n}.$$

From the mixed-norm Hölder inequality

$$\|F\|_{1,\Omega} \le \|F\|_{\mathbf{w},\mathbb{R}^n} \le \prod_{\kappa \in S} \|F_\kappa\|_{\mathbf{p}_\kappa,\mathbb{R}^n} = \prod_{\kappa \in S} \|F_\kappa\|_{\lambda,\Omega_\kappa}$$

as required. ∎

4.24 LEMMA If Ω satisfies the cone condition, then $W^{1,1}(\Omega) \to L^p(\Omega)$ for $1 \le p \le n/(n-1)$.

Proof. By Lemma 4.22, Ω is a finite union of subdomains each of which is a union of parallel translates of a fixed parallelepiped. It is therefore sufficient to prove the imbedding for one such subdomain. Thus we assume $\Omega = \bigcup_{x \in A}(x + P)$ where P is a parallelepiped. There is a linear transformation of $\mathbb{R}^n$ onto itself that maps P onto a cube Q of unit edge with edges parallel to the coordinate axes. By Theorem 3.41 it is therefore sufficient to prove the lemma for $\Omega = \bigcup_{x \in A}(x + Q)$. For $x \in \Omega$ let ℓ be the intersection of Ω with the line through x parallel to the x_1-axis. Evidently ℓ contains a closed interval of length 1 containing x_1, say the interval $[\xi_1, \xi_2]$. If $f \in C^1([0,1])$, then $|f(t_0)| \le |f(t)| + \left|\int_{t_0}^t f'(\tau)\, d\tau\right|$, and integrating over t over $[0,1]$ yields

$$|f(t_0)| \le \int_0^1 (|f(t)| + |f'(t)|)\, dt.$$

For $u \in C^\infty(\Omega)$ we apply this inequality to $u(t, \hat{x}_1)$ (where $\hat{x}_1 = (x_2, \dots, x_n)$) to obtain

$$\begin{aligned} |u(x)| &\le \int_{\xi_1}^{\xi_2} \big(|u(t,\hat{x}_1)| + |D_1 u(t, \hat{x}_1)|\big)\, dt \\ &\le \int_\ell \big(|u(t,\hat{x}_1)| + |D_1 u(t, \hat{x}_1)|\big)\, dt. \end{aligned}$$

Let Ω_1 be the orthogonal projection of Ω onto the hyperplane of coordinates $\hat{x}_1$, and let

$$u_1(\hat{x}_1) = \left(\int_\ell \big(|u(t,\hat{x}_1)| + |D_1 u(t,\hat{x}_1)|\big)\, dt\right)^{1/(n-1)}.$$

(Evidently $u_1(\hat{x}_1)$ is independent of x_1) We have

$$\|u_1\|_{1/(n-1),\Omega_1} = \int_{\Omega_1} |u_1(x)|^{n-1}\, d\hat{x}_1 \le \|u\|_{1,1,\Omega}.$$

Similarly, for $2 \le j \le n$ we can define u_j to be independent of x_j and to satisfy $|u(x)| \le \big(u_j(x)\big)^{1/(n-1)}$ and

$$\left\|u_j\right\|_{1/(n-1),\Omega_j} \le \|u\|_{1,1,\Omega}.$$

Since $|u(x)|^{n/(n-1)} \le \prod_{j=1}^n u_j(x)$, applying inequality (14) with $k = n-1 = \lambda$ now gives

$$\int_\Omega |u(x)|^{n/(n-1)}\, dx \le \prod_{j=1}^n \int_{\Omega_j} |u_j|(\hat{x}_j)|^{n-1}\, d\hat{x}_j \le \|u\|_{1,1,\Omega}^{n/(n-1)}.$$

For the original domain Ω, this will imply that

$$\|u\|_{n/(n-1),\Omega} \le K\, \|u\|_{1,1,\Omega}$$

where the constant K depends on n and the cone C of the cone condition. These determine the number N of subdomains needed, and the size of the determinant of the linear transformation needed to transform the parallelepipeds for each subdomain into Q. The imbedding $W^{1,1}(\Omega) \to L^p(\Omega)$ for $1 \le p \le n/(n-1)$ now follows by L^p interpolation (Theorem 2.11) ∎

4.25 (Proof of Part I, Cases B and C of Theorem 4.12 for $p = 1, k > n - m$) Let $m \le n$. By the above lemma and previously proved parts of Cases B and C for $p > 1$, we have

$$W^{m,1}(\Omega) \to W^{m-1,p}(\Omega) \qquad \text{for } 1 \le p \le n/(n-1).$$

Since $k > n - m$, therefore $k \ge n - m + 1 > n - (m-1)p$ for any $p > 1$. Therefore $W^{m-1,p}(\Omega) \to L^q(\Omega_k)$ holds for

$$1 \le q \le p^* = \frac{kp}{n-(m-1)p} = \frac{kn/(n-1)}{n-(m-1)n/(n-1)} = \frac{k}{n-m}.$$

Combining these imbeddings we get $W^{m,1}(\Omega) \to L^p(\Omega)$, $1 \le q \le k/(n-m)$. For $p = 1$, $m = n$ the imbedding $W^{n,1}(\Omega) \to L^q(\Omega_k)$, $1 \le q \le \infty$, $1 \le k \le n$ was already proved under Case A. ∎

4.26 (Proof of Part I, Case C of Theorem 4.12 for $p = 1, k = n - m$) In this case we want to show $W^{m,1}(\Omega) \to L^1(\Omega_k)$. As in the proof in Paragraph 4.24 it is sufficient to establish the imbedding for a domain Ω that is a union of parallel translates of a unit cube with edges parallel to the coordinate axes. We can also assume that $0 \in \Omega$ and that

$$\Omega_k = \{x = (x', x'') \in \Omega : x' = 0\},$$

where $x' = (x_1, \ldots x_m)$ and $x'' = (x_{m+1}, \ldots, x_n)$. For $x \in \Omega$ let Ω_x be the intersection of Ω with the m-plane of variables x' passing through x. Ω_x contains an m-cube Q_x of edge 1 containing x, and so by Case A of Theorem 4.12 applied to this cube, we have for $u \in C^\infty(\Omega)$,

$$|u(x)| \le K \sum_{|\alpha| \le m} \int_{\Omega_x} |D^\alpha u(x', x'')|\, dx'.$$

Integrating x'' over Ω_k then gives

$$\int_{\Omega_k} |u(x)|\, dx'' \le K \sum_{|\alpha| \le m} \int_{\Omega} |D^\alpha u(x)|\, dx.$$

The proof of Part I of Theorem 4.12 is now complete. ∎

Imbeddings into Lipschitz Spaces

4.27 To prove Part II of Theorem 4.12, we now assume that the domain $\Omega \subset \mathbb{R}^n$ satisfies the strong local Lipschitz condition defined in Paragraph 4.9, and that $mp > n \geq (m-1)p$. We shall show that $W^{m,p}(\Omega) \to C^{0,\lambda}(\overline{\Omega})$ where:

(i) $0 < \lambda \leq m - (n/p)$ if $n > (m-1)p$, or

(ii) $0 < \lambda < 1$ if $n = (m-1)p$ and $p > 1$, or

(iii) $0 < \lambda \leq 1$ if $n = m - 1$ and $p = 1$.

In particular, therefore, $W^{m,p}(\Omega) \to C^0(\overline{\Omega})$. The imbedding constants may depend on m, p, n, and the parameters δ and M specified in the definition of the strong local Lipschitz condition. Since that condition implies the cone condition, we already know that $W^{m,p}(\Omega) \to C^0_B(\Omega)$, so if $u \in W^{m,p}(\Omega)$, then

$$\sup_{x \in \Omega} |u(x)| \leq K_1 \|u\|_{m,p,\Omega} .$$

It is therefore sufficient to establish further that for the appropriate λ,

$$\sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|u(x) - u(y)|}{|x-y|^\lambda} \leq K_2 \|u\|_{m,p,\Omega} .$$

Since $mp > n \geq (m-1)p$, Cases B and C of Part I of Theorem 4.12 yields the imbedding $W^{m,p}(\Omega) \to W^{1,r}(\Omega)$ where:

(i) $r = np/(n-m+1)p$ and so $1 - (n/r) = m - (n/p)$ if $n > (m-1)p$, or

(ii) $p < r < \infty$ and so $0 < 1 - (n/r) < 1$ if $n > (m-1)p$, or

(iii) $r = \infty$ and so $1 - (n/r) = 1$ if $n = m - 1$ and $p = 1$.

It is therefore sufficient to establish the special case $m = 1$.

4.28 LEMMA Let Ω satisfy the strong local Lipschitz condition. If u belongs to $W^{1,p}(\Omega)$ where $n < p \leq \infty$, and if $0 \leq \lambda \leq 1 - (n/p)$, then

$$\sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|u(x) - u(y)|}{|x-y|^\lambda} \leq K \|u\|_{1,p,\Omega} . \tag{15}$$

Proof. Suppose, for the moment, that Ω is a cube having unit edge length. For $0 < t < 1$ let Q_t denote a subset of Ω that is a closed cube having edge length t and faces parallel to those of Ω. If $x, y \in \Omega$ and $|x - y| = \sigma < 1$, then there is a fixed such cube Q_σ such that $x, y \in Q_\sigma$.

Let $u \in C^\infty(\Omega)$ If $z \in Q_\sigma$, then

$$u(x) - u(z) = \int_0^1 \frac{d}{dt} u\big((x + t(z-x)\big)\, dt,$$

so that

$$|u(x) - u(z)| \le \sigma\sqrt{n}\int_0^1 |\operatorname{grad} u\big((x + t(z - x)\big)|\, dt.$$

It follows that

$$\begin{aligned}
\left| u(x) - \frac{1}{\sigma^n}\int_{Q_\sigma} u(z)\,dz \right| &= \left| \frac{1}{\sigma^n}\int_{Q_\sigma} \big(u(x) - u(z)\big)\,dz \right| \\
&\le \frac{\sqrt{n}}{\sigma^{n-1}}\int_{Q_\sigma} dz \int_0^1 |\operatorname{grad} u\big((x + t(z - x)\big)|\,dt \\
&= \frac{\sqrt{n}}{\sigma^{n-1}}\int_0^1 t^{-n}\,dt \int_{Q_{t\sigma}} |\operatorname{grad} u(\zeta)|\,d\zeta \\
&\le \frac{\sqrt{n}}{\sigma^{n-1}} \|\operatorname{grad} u\|_{0,p,\Omega} \int_0^1 \big(\operatorname{vol}(Q)_{t\sigma}\big)^{1/p'} t^{-n}\,dt \qquad (16) \\
&\le K\sigma^{1-(n/p)} \|\operatorname{grad} u\|_{0,p,\Omega},
\end{aligned}$$

where $K = K(n, p) = \sqrt{n}\int_0^1 t^{-n/p}\,dt < \infty$. A similar inequality holds with y in place of x and so

$$|u(x) - u(y)| \le 2K|x - y|^{1-(n/p)} \|\operatorname{grad} u\|_{0,p,\Omega}.$$

It follows that (15) holds for $0 < \lambda \le 1 - (n/p)$ for Ω a cube, and therefore via a nonsingular linear transformation, for Ω a parallelepiped.

Now suppose that Ω is an arbitrary domain satisfying the strong local Lipschitz condition. Let δ, M, Ω_δ, U_j and V_j be as specified in the definition of that condition in Paragraph 4.9. There exists a parallelepiped P of diameter δ whose dimensions depend only on δ and M such that to each j there corresponds a parallelepiped P_j congruent to P and having one vertex at the origin, such that for every $x \in V_j \cap \Omega$ we have $x + P_j \subset \Omega$. Furthermore, there exist constants δ_0 and δ_1 depending only on δ and P, with $\delta_0 \le \delta$, such that if $x, y \in V_j \cap \Omega$ and $|x - y| < \delta_0$, then there exists $z \in (x + P_j) \cap (y + P_j)$ with $|x - z| + |y - z| \le \delta_1|x - y|$. If follows from applications of (15) to $x + P_j$ and $y + P_j$ that if $u \in C^\infty(\Omega)$, then

$$\begin{aligned}
|u(x) - u(y)| &\le |u(x) - u(z)| + |u(y) - u(z)| \\
&\le K|x - z|^\lambda \|u\|_{1,p,\Omega} + K|y - z|^\lambda \|u\|_{1,p,\Omega} \\
&\le K_1|x - y|^\lambda \|u\|_{1,p,\Omega}. \qquad (17)
\end{aligned}$$

Now let x, y be arbitrary points in Ω. If $|x - y| < \delta_0 \le \delta$ and $x, y \in \Omega_\delta$, then $x, y \in V_j$ for some j and (17) holds. If $|x - y| < \delta_0$, $x \in \Omega_\delta$, $y \in \Omega - \Omega_\delta$, then $x \in V_j$ for some j and (17) still follows by an applications of (15) to $x + P_j$ and $y + P_j$. If $|x - y| < \delta_0$, $x, y \in \Omega - \Omega_\delta$, then (17) follows from applications of (15) to $x + P'$ and $y + P'$ where P' is any parallelepiped congruent to p and having one vertex at the origin. Finally, if $|x - y| \geq \delta_0$, then

$$|u(x) - u(y)| \leq |u(x)| + |u(y)| \leq K_1 \|u\|_{1,p,\Omega} \leq K\delta_0^{-\lambda}|x - y|^{\lambda} \|u\|_{1,p,\Omega} .$$

Thus (15) holds for all $u \in C^{\infty}(\Omega)$ and, by Theorem 3.17, for all $u \in C_B^0(\Omega)$. ∎

This completes the proof of Part II of Theorem 4.12 and therefore of the whole theorem since, as remarked earlier, Part III follows from the fact that Parts I and II hold for $\Omega = \mathbb{R}^n$.

Sobolev's Inequality

4.29 (Seminorms) For $1 \leq p < \infty$ and for integers j, $0 \leq j \leq m$, we introduce functionals $|\cdot|_{j,p}$ on $W^{m,p}(\Omega)$ as follows:

$$|u|_{j,p} = |u|_{j,p,\Omega} = \left(\sum_{|\alpha|=j} |D^{\alpha}u(x)|^p \, dx \right)^{1/p} .$$

Clearly $|u|_{0,p} = \|u\|_{0,p} = \|u\|_p$ is the norm on $L^p(\Omega)$ and

$$\|u\|_{m,p} = \left(\sum_{j=0}^{m} |u|_{j,p}^p \right)^{1/p} .$$

If $j \geq 1$, we call $|\cdot|_{j,p}$ a *seminorm*. It has all the properties of a norm except that $|u|_{j,p} = 0$ need not imply $u = 0$ in $W^{m,p}(\Omega)$. For example, u may be a nonzero constant function if Ω has finite volume. Under certain circumstances which we begin to investigate in Paragraph 6.29, $|\cdot|_{m,p}$ is a norm on $W_0^{m,p}(\Omega)$ equivalent to the usual norm $\|\cdot\|_{m,p}$. In particular, this is so if Ω is bounded.

For now we will confine our attention to these seminorms as they apply to functions in $C_0^{\infty}(\mathbb{R}^n)$.

4.30 The Sobolev imbedding theorem tells us that $W_0^{m,p}(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ for certain finite values of q depending on m, p, and n; for such q there is a finite constant K such that for all $\phi \in C_0^{\infty}(\mathbb{R}^n)$ we have

$$\|\phi\|_q \leq K \|\phi\|_{m,p} .$$

We now ask whether such an inequality can hold with $|\cdot|_{m,p}$ in place of $\|\cdot\|_{m,p}$. That is, do there exist constants $K < \infty$ and $q \geq 1$ such that for all $\phi \in C_0^{\infty}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |\phi(x)|^q \, dx \leq K^q \left(\sum_{|\alpha|=m} \int_{\mathbb{R}^n} |D^{\alpha}\phi(x)|^p \, dx \right)^{q/p} ? \qquad (18)$$

If so, for any given $\phi \in C_0^\infty(\mathbb{R}^n)$, the inequality must also hold for all dilates $\phi_t(x) = \phi(tx)$, $0 < t < \infty$, as these functions also belong to $C_0^\infty(\mathbb{R}^n)$. Since $\|\phi_t\|_q = t^{-n/q}\,\|\phi\|_q$ and $\|D^\alpha \phi_t\|_p = t^{m-(n/p)}\,\|D^\alpha \phi\|_p$ if $|\alpha| = m$, we must have

$$\int_{\mathbb{R}^n} |\phi(x)|^q\,dx \le K^q t^{n+mq-(nq/p)} \left(\sum_{|\alpha|=m} \int_{\mathbb{R}^n} |D^\alpha \phi(x)|^p\,dx \right)^{q/p}$$

This is clearly not possible for all $t > 0$ unless the exponent of t on the right side is zero, that is, unless $q = p^* = np/(n - mp)$. Thus no inequality of the form (18) is possible unless $mp < n$ and $q = p^* = np/(n - mp)$. We now show that (18) does hold if these conditions are satisfied.

4.31 THEOREM (Sobolev's Inequality) When $mp < n$, there exists a finite constant K such that (18) holds for every $\phi \in C_0^\infty(\mathbb{R}^n)$:

$$\|\phi\|_{q,\mathbb{R}^n} \le K\,|\phi|_{m,p,\mathbb{R}^n} \tag{19}$$

if and only if $q = p^* = np/(n - mp)$. This is known as *Sobolev's inequality*.

Proof. The "only if" part was demonstrated above. For the "if" part note first that it is sufficient to establish the inequality for $m = 1$ as its validity for higher m (with $mp < n$) can be confirmed by induction on m. We leave the details to the reader.

Next, it suffices to prove the case $m = 1$, $p = 1$, that is

$$\int_{\mathbb{R}^n} |\phi(x)|^{n/(n-1)}\,dx \le K \left(\sum_{j=1}^{n} \int_{\mathbb{R}^n} |D_j \phi(x)|\,dx \right)^{n/(n-1)}, \tag{20}$$

for if $1 < p < n$ and $p^* = np/(n - p)$ we can apply (20) to $|\phi(x)|^s$ where $s = (n-1)p^*/n$ and obtain, using Hölder's inequality,

$$\begin{aligned}\int_{\mathbb{R}^n} |\phi(x)|^{p^*}\,dx &\le K \left(\sum_{j=1}^{n} s|\phi(x)|^{s-1}|D_j\phi(x)|\,dx \right)^{n/(n-1)} \\ &\le K_1 \left(\sum_{j=1}^{n} \|\phi\|_{(s-1)p'}^{s-1} \left\|D_j\phi\right\|_p \right)^{n/(n-1)}.\end{aligned}$$

Since $(s-1)p' = p^*$ and $p^* - (s-1)n/(n-1) = n/(n-1)$, it follows by cancellation that

$$\|\phi\|_{p^*} \le K_2\,|\phi|_{1,p}\,.$$

It remains, therefore, to prove (20). Let $\phi \in C_0^\infty(\mathbb{R}^n)$ and for $x \in \mathbb{R}^n$ and $1 \le j \le n$ let $\hat{x}_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$. Let

$$u_j(\hat{x}_j) = \left(\sum_{i=1}^n \int_{-\infty}^{\infty} |D_j\phi(x)|\, dx_j\right)^{1/(n-1)},$$

which is evidently independent of x_j and satisfies

$$\left(\|u_j\|_{n-1,\mathbb{R}^{n-1}}\right)^{n-1} \le |u|_{1,1,\mathbb{R}^n}.$$

Since

$$\phi(x) = \int_{-\infty}^{x_1} D_1\phi(t, \hat{x}_1)\, dt$$

we have

$$|\phi(x)| \le \int_{-\infty}^{\infty} |D_1\phi(t, \hat{x}_1)|\, dt \le \left(u_1(\hat{x}_1)\right)^{n-1}.$$

Similarly, $|\phi(x)| \le \left(u_j(\hat{x}_j)\right)^{n-1}$. Applying the inequality (14) from Lemma 4.23 with $k = n - 1 = \lambda$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} |\phi(x)|^{n/(n-1)}\, dx &\le \int_{\mathbb{R}^n} \prod_{j=1}^n u_j(\hat{x}_j)\, dx \\ &\le \left(\prod_{j=1}^n \int_{\mathbb{R}^{n-1}} |u_j(\hat{x}_j)|^{n-1}\, d\hat{x}_j\right)^{1/(n-1)} \le |u|_{1,1,\mathbb{R}^n}^{n/(n-1)},\end{aligned}$$

which completes the proof of (20) and hence the theorem. ∎

4.32 (REMARK) For the case $m = 1$, $1 < p < n$, Talenti [T] and Aubin, as exposed in Section 2.6 of [Au], obtained the best constant for the equivalent form of Sobolev's inequality

$$\|\phi\|_{np/(n-p),\mathbb{R}^n} \le K\, \|\mathrm{grad}\, \phi\|_{p,\mathbb{R}^n} \tag{21}$$

by showing that the ratio

$$\frac{\|\phi\|_{np/(n-p)}}{\|\mathrm{grad}\, \phi\|_{1,p}}$$

is maximized if u is a radially symmetric function of the form

$$u(x) = \left(a + b|x|^{p/(p-1)}\right)^{1-(n/p)}$$

which, while not in $C_0^\infty(\mathbb{R}^n)$ is a limit of functions in that space. His method involved first showing that replacing an arbitrary function u vanishing at infinity with a radially symmetric, non-increasing, equimeasurable rearrangement of u decreased $\|\operatorname{grad} u\|_{p,\mathbb{R}^n}$ while, of course, leaving $\|u\|_{np/(n-p),\mathbb{R}^n}$ unchanged. Talenti's best constant for (21) is

$$K = \pi^{-1/2} n^{-1/p} \left(\frac{p-1}{n-p}\right)^{1/p'} \left(\frac{\Gamma(1+n/2)\Gamma(n)}{\Gamma(n/p)\Gamma(1+n-(n/p))}\right)^{1/n}.$$

Variations of Sobolev's Inequality

4.33 Mixed-norm L^p estimates of the type considered in Paragraphs 2.48–2.51 and used in the proof of Gagliardo's averaging lemma 4.23 can contribute to generalizations of Sobolev's inequality. We examine briefly two such generalizations:

(a) **anisotropic Sobolev inequalities**, in which different L^p norms are used for different partial derivatives on the right side of (19), and

(b) **reduced Sobolev inequalities**, in which the seminorm $|\phi|_{m,p,\mathbb{R}^n}$ on the right side of (19) is replaced with a similar seminorm involving only a subset of the partial derivatives of order m of ϕ.

Questions of this sort are discussed in [BIN1] and [BIN2]. We follow the treatment in [A3] and [A4] and most of the details will be omitted here.

4.34 (A First-Order Anisotropic Sobolev Inequality) If $p_j \geq 1$ for each j with $1 \leq j \leq n$ and $\phi \in C_0^\infty(\mathbb{R}^n)$, then an inequality of the form

$$\|\phi\|_q \leq K \sum_{j=1}^{n} \|D_j \phi\|_{p_j} \tag{22}$$

is a (first-order) *anisotropic* Sobolev inequality because different L^p norms are used to estimate the derivatives of ϕ in different coordinate directions. A dilation argument involving $\phi(\lambda_1 x_1, \dots, \lambda_n x_n)$ for $0 < \lambda_j < \infty$, $1 \leq j \leq n$ shows that no such anisotropic inequality is possible for finite q unless

$$\sum_{j=1}^{n} \frac{1}{p_j} > 1 \qquad \text{and} \qquad \frac{1}{q} = \frac{1}{n}\sum_{j=1}^{n} \frac{1}{p_j} - \frac{1}{n}.$$

If these conditions are satisfied, then (22) does hold. The proof is a generalization of that of Theorem 4.31 and uses the mixed-norm Hölder and permutation inequalities. (See [A3] for the details.)

4.35 (Higher-Order Anisotropic Sobolev Inequalities) The generalization of (22) to an mth order inequality by induction on m is somewhat more problematic.

The mth order isotropic inequality (19) follows from its special case $m = 1$ by simple induction. We can also obtain

$$\|\phi\|_q \le K \sum_{|\alpha|=m} \|D^\alpha \phi\|_{p_\alpha},$$

where

$$\frac{1}{q} = \frac{1}{n^m} \sum_{|\alpha|=m} \binom{m}{\alpha} \frac{1}{p_\alpha} - \frac{m}{n}, \qquad \binom{m}{\alpha} = \frac{m!}{\alpha_1!\alpha_2!\cdots\alpha_n!}$$

by induction from (22) under suitable restrictions on the exponents p_α, but the restriction

$$\frac{1}{n^m} \sum_{|\alpha|=m} \binom{m}{\alpha} \frac{1}{p_\alpha} > \frac{m}{n}$$

will not suffice in general for the induction even though $\sum_{|\alpha|=m} \binom{m}{\alpha} = n^m$. The conditions $mp_\alpha < n$ for each α with $|\alpha| = m$ will suffice, but are stronger than necessary.

For any multi-index β and $1 \le j \le n$, let

$$\beta[j] = (\beta_1, \ldots, \beta_{j-1}, \beta_j + 1, \beta_{j+1}, \ldots, \beta_n).$$

Evidently, $|\beta[j]| = |\beta| + 1$ and it can be verified that if the numbers p_α are defined for all α with $|\alpha| = m$, then

$$\sum_{|\beta|=m-1} \binom{m-1}{\beta} \sum_{j=1}^{n} \frac{1}{p_{\beta[j]}} = \sum_{|\alpha|=m} \binom{m}{\alpha} \frac{1}{p_\alpha}.$$

This provides the induction step necessary to verify the following theorem, for which the details can again be found in [A3].

4.36 THEOREM Let $p_\alpha \ge 1$ for all α with $|\alpha| = m$. Suppose that for every β with $|\beta| = m - 1$ we have

$$\sum_{j=1}^{n} \frac{1}{p_{\beta[j]}} > m.$$

Then there exists a constant K such that the inequality

$$\|\phi\|_q \le K \sum_{|\alpha|=m} \|D^\alpha \phi\|_{p_\alpha}$$

holds for all $\phi \in C_0^\infty(\mathbb{R}^n)$, where

$$\frac{1}{q} = \frac{1}{n^m} \sum_{|\alpha|=m} \binom{m}{\alpha} \frac{1}{p_\alpha} - \frac{m}{n}.$$

4.37 (Reduced Sobolev Inequalities) Another variation of Sobolev's inequality addresses the question of whether the number of derivatives estimated in the seminorm on the right side of (19) (or, equivalently, (18)) can be reduced without jeopardizing the validity of the inequality for all $\phi \in C_0^\infty(\mathbb{R}^n)$. If $m \geq 2$, the answer is yes; only those partial derivatives of order m that are "completely mixed" (in the sense that all m differentiations are taken with respect to different variables) need be included in the seminorm. Specifically, if we denote

$$\mathcal{M} = \mathcal{M}(n,m) = \{\alpha : |\alpha| = m, \quad \alpha_j = 0 \text{ or } \alpha_j = 1 \text{ for } 1 \leq j \leq n,$$

then the *reduced* Sobolev inequality

$$\|\phi\|_q \leq K \sum_{\alpha \in \mathcal{M}} \|D^\alpha \phi\|_p$$

holds for all $\phi \in C_0^\infty(\mathbb{R}^n)$, provided $mp < n$ and $q = np/(n - mp)$. Again the proof depends on mixed-norm estimates; it can be found in [A4] where the possibility of further reductions in the number of derivatives estimated on the right side of Sobolev's inequality is also considered. See also Section 13 in [BIN1].

$W^{m,p}(\Omega)$ as a Banach Algebra

4.38 Given u and v in $W^{m,p}(\Omega)$, where Ω is a domain in $\mathbb{R}^n$, one cannot in general expect that their pointwise product uv will belong to $W^{m,p}(\Omega)$. The imbedding theorem, however, shows that this is the case provided $mp > n$ and Ω satisfies the cone condition. (See [Sr] and [Mz2].)

4.39 THEOREM Let Ω be a domain in $\mathbb{R}^n$ satisfying the cone condition. If $mp > n$ or $p = 1$ and $m \geq n$, then there exists a constant K^* depending on m, p, n, and the cone C determining the cone condition for Ω, such that for $u, v \in W^{m,p}(\Omega)$ the product uv, defined pointwise a.e. in Ω, satisfies

$$\|uv\|_{m,p,\Omega} \leq K^* \|u\|_{m,p,\Omega} \|v\|_{m,p,\Omega}. \tag{23}$$

In particular, equipped with the equivalent norm $\|\cdot\|^*_{m,p,\Omega}$ defined by

$$\|u\|^*_{m,p,\Omega} = K^* \|u\|_{m,p,\Omega},$$

$W^{m,p}(\Omega)$ is a commutative Banach algebra with respect to pointwise multiplication in that

$$\|uv\|^*_{m,p,\Omega} \leq \|u\|^*_{m,p,\Omega} \|v\|^*_{m,p,\Omega}.$$

Proof. We assume $mp > n$; the case $p = 1$, $m = n$ is simpler. In order to establish (23) it is sufficient to show that if $|\alpha| \leq m$, then

$$\int_\Omega |D^\alpha[u(x)v(x)]|^p \leq K_\alpha \|u\|_{m,p,\Omega} \|v\|_{m,p,\Omega},$$

where $K_\alpha = K_\alpha(m, p, n, C)$. Let us assume for the moment that $u \in C^\infty(\Omega)$. By the Leibniz rule for distributional derivatives, that is,

$$D^\alpha(uv) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta u D^{\alpha-\beta} v,$$

it is sufficient to show that for any $\beta \le \alpha$, $|\alpha| \le m$, we have

$$\int_\Omega |D^\beta u(x) D^{\alpha-\beta} v(x)|^p \, dx \le K_{\alpha,\beta} \|u\|^p_{m,p,\Omega} \|v\|^p_{m,p,\Omega},$$

where $K_{\alpha,\beta} = K_{\alpha,\beta}(m, p, n, C)$. By the imbedding theorem there exists, for any β with $|\beta| \le m$, a constant $K(\beta) = K(\beta, m, p, n, C)$ such that for any $w \in W^{m,p}(\Omega)$,

$$\int_\Omega |D^\beta w(x)|^r \, dx \le K(\beta) \|w\|^r_{m,p,\Omega}, \tag{24}$$

provided $(m - |\beta|)p \le n$ and $p \le r \le np/(n - [m - |\beta|]p)$ [or $p \le r < \infty$ if $(m - |\beta|)p = n$], or alternatively

$$|D^\beta w(x)| \le K(\beta) \|w\|_{m,p,\Omega} \qquad \text{a.e. in } \Omega$$

provided $(m - |\beta|)p > n$.

Let k be the largest integer such that $(m-k)p > n$. Since $mp > n$ we have $k \ge 0$. If $|\beta| \le k$, then $(m - |\beta|)p > n$, so

$$\begin{aligned}\int_\Omega |D^\beta u(x) D^{\alpha-\beta} v(x)|^p \, dx &\le K(\beta)^p \|u\|^p_{m,p,\Omega} \left\|D^{\alpha-\beta} v\right\|^p_{0,p,\Omega} \\ &\le K(\beta)^p \|u\|^p_{m,p,\Omega} \|v\|^p_{m,p,\Omega}.\end{aligned}$$

Similarly, if $|\alpha - \beta| \le k$, then

$$\int_\Omega |D^\beta u(x) D^{\alpha-\beta} v(x)|^p \, dx \le K(\alpha-\beta)^p \|u\|^p_{m,p,\Omega} \|v\|^p_{m,p,\Omega}.$$

Now if $|\beta| > k$ and $|\alpha - \beta| > k$, then, in fact, $|\beta| \ge k+1$ and $|\alpha - \beta| \ge k+1$ so that $n \ge (m - |\beta|)p$ and $n \ge (m - |\alpha - \beta|)p$. Moreover,

$$\frac{n - (m - |\beta|)p}{n} + \frac{n - (m - |\alpha - \beta|)p}{n} = 2 - \frac{(2m - |\alpha|)p}{n} < 2 - \frac{mp}{n} < 1.$$

Hence there exist positive numbers r and r' with $(1/r) + (1/r') = 1$ such that

$$p \le rp < \frac{np}{n - (m - |\beta|)p}, \qquad p \le r'p < \frac{np}{n - (m - |\alpha - \beta|)p}.$$

Thus by Hölder's inequality and (24) we have

$$\int_\Omega |D^\beta u(x) D^{\alpha-\beta} v(x)|^p \, dx \le \left(\int_\Omega |D^\beta u(x)|^{rp} \, dx \right)^{1/r} \left(\int_\Omega |D^{\alpha-\beta} v(x)|^{r'p} \, dx \right)^{1/r'}$$
$$\le \big(K(\beta)\big)^{1/r} \big(K(\alpha-\beta)\big)^{1/r'} \|u\|^p_{m,p,\Omega} \, \|v\|^p_{m,p,\Omega} .$$

This completes the proof of (23) for $u \in C^\infty(\Omega)$, $v \in W^{m,p}(\Omega)$.

If $u \in W^{m,p}(\Omega)$ then by Theorem 3.17 there exists a sequence $\{u_j\}$ of $C^\infty(\Omega)$ functions converging to u in $W^{m,p}(\Omega)$. By the above argument, $\{u_j v\}$ is a Cauchy sequence in $W^{m,p}(\Omega)$ and so it converges to an element w of that space. Since $mp > n$, u and v may be assumed to be continuous and bounded on Ω. Thus

$$\begin{aligned}\|w - uv\|_{0,p,\Omega} &\le \|w - u_j v\|_{0,p,\Omega} + \|(u_j - u)v\|_{0,p,\Omega} \\ &\le \|w - u_j v\|_{0,p,\Omega} + \|v\|_{0,\infty,\Omega} \|u_j - u\|_{0,p,\Omega} \\ &\to 0 \qquad \text{as } j \to \infty.\end{aligned}$$

Hence $w = uv$ in $L^p(\Omega)$ and so $w = uv$ in the sense of distributions. Therefore, $w = uv$ in $W^{m,p}(\Omega)$ and

$$\|uv\|_{m,p,\Omega} = \|w\|_{m,p,\Omega} \le \limsup_{j\to\infty} \|u_j v\|_{m,p,\Omega} \le K^* \|u\|_{m,p,\Omega} \|v\|_{m,p,\Omega}$$

as was to be shown. ∎

We remark that the Banach algebra $W^{m,p}(\Omega)$ has an identity element if an only if Ω is bounded. That is, the function $e(x) = 1$ belongs to $W^{m,p}(\Omega)$ if and only if Ω has finite volume, but there are no unbounded domains of finite volume that satisfy the cone condition.

Optimality of the Imbedding Theorem

4.40 The imbeddings furnished by the Sobolev Imbedding Theorem 4.12 are "best possible" in the sense that no imbeddings of the types asserted there are possible for any domain for parameter values m, p, q, λ etc. not satisfying the restrictions imposed in the statement of the theorem. We present below a number of examples to illustrate this fact. In these examples it is the local behaviour of functions in $W^{m,p}(\Omega)$ rather than their behaviour near the boundary that prevents extending the parameter intervals for imbeddings.

There remains the possibility that a weaker version of Part I of the imbedding theorem may hold for certain domains not nice enough to satisfy the (weak) cone condition. We will examine some such possibilities later in this chapter.

4.41 EXAMPLE Let k be an integer such that $1 \le k \le n$ and suppose that $mp < n$ and $q > p^* = kp/(n-mp)$. We construct a function $u \in W^{m,p}(\Omega)$ such that $u \notin L^q(\Omega_k)$, where Ω_k is the intersection of Ω with a k-dimensional plane, thus showing that $W^{m,p}(\Omega)$ does not imbed into $L^q(\Omega_k)$.

Without loss of generality, we can assume that the origin belongs to Ω and that $\Omega_k = \{x \in \Omega : x_{k+1} = \cdots = x_n = 0\}$. For $R > 0$, let $B_R = \{x \in \mathbb{R}^n : |x| < R\}$. We fix R small enough that $\overline{B_{2R}} \subset \Omega$. Let $v(x) = |x|^\mu$; the value of μ will be set later. Evidently $v \in C^\infty(\mathbb{R}^n - \{0\})$. Let $u \in C^\infty(\mathbb{R}^n - \{0\})$ be a function satisfying $u(x) = v(x)$ in B_R and $u(x) = 0$ outside B_{2R}. The membership of u in $W^{m,p}(\Omega)$ depends only on the behaviour of v near the origin:

$$u \in W^{m,p}(\Omega) \iff v \in W^{m,p}(B_R).$$

It is easily checked by induction on $|\alpha|$ that

$$D^\alpha v(x) = P_\alpha(x)|x|^{\mu-2|\alpha|},$$

where $P_\alpha(x)$ is a polynomial homogeneous of degree $|\alpha|$ in the components of x. Thus $|D^\alpha v(x)| \le K_\alpha |x|^{\mu-|\alpha|}$ and, setting $\rho = |x|$,

$$\int_{B_R} |D^\alpha v(x)|^p \, dx \le K_n K_\alpha^P \int_0^R \rho^{(\mu-|\alpha|)p+n-1} \, d\rho,$$

where K_n is the $(n-1)$-measure of the sphere of radius 1 in $\mathbb{R}^n$. Therefore $v \in W^{m,p}(B_R)$ and $u \in W^{m,p}(\Omega)$ provided $\mu > m - (n/p)$.

On the other hand, denoting $\tilde{x}_k = (x_1, \ldots, x_k)$ and $r = |\tilde{x}_k|$, we have

$$\int_{\Omega_k} |u(x)|^q \, d\tilde{x}_k \ge \int_{(B_R)_k} |v(x)|^q \, d\tilde{x}_k = K_k \int_0^R r^{\mu q+k-1} \, dr.$$

Thus $u \notin L^q(\Omega_k)$ if $\mu \le -(k/q)$.

Since $q > kp/(n-mp)$ we can pick μ so that $m - (n/p) < \mu \le -(k/q)$, thus completing the specification of u. ∎

Note that $\mu < 0$, so u is unbounded near the origin. Hence no imbedding of the form $W^{m,p}(\Omega) \to C_B^0(\Omega)$ is possible if $mp < n$.

4.42 EXAMPLE Suppose $mp > n > (m-1)p$, and let $\lambda > m - (n/p)$. Fix μ so that $m - (n/p) < \mu < \lambda$. Then the function u constructed in Example 4.41 continues to belong to $W^{m,p}(\Omega)$. However, if $|x| < R$,

$$\frac{|u(x) - u(0)|}{|x - 0|^\lambda} = |x|^{\mu-\lambda} \to \infty \quad \text{as } |x| \to 0.$$

Thus $u \notin C^{0,\lambda}(\overline{\Omega})$, and the imbedding $W^{m,p}(\Omega) \to C^{0,\lambda}(\overline{\Omega})$ is not possible. ∎

4.43 EXAMPLE Suppose $p > 1$ and $mp = n$. We construct a function u in $W^{m,p}(\Omega)$ such that $u \notin L^\infty(\Omega)$. Hence the imbedding $W^{m,p}(\Omega) \to L^q(\Omega)$, valid for $p \le q < \infty$, cannot be extended to yield $W^{m,p}(\Omega) \to L^\infty(\Omega)$ or $W^{m,p}(\Omega) \to C^0(\overline{\Omega})$ unless $p = 1$ and $m = n$. (See, however, Theorem 8.27.)

Again we assume $0 \in \Omega$ and define $u(x)$ as in Example 4.41 except with a different function $v(x)$ defined by

$$v(x) = \log\big(\log(4R/|x|)\big).$$

Clearly v is not bounded near the origin, so $u \notin L^\infty(\Omega)$. It can be checked by induction on $|\alpha|$ that

$$D^\alpha v(x) = \sum_{j=1}^{|\alpha|} P_{\alpha,j}(x)|x|^{-2|\alpha|}\big(\log(4R/|x|)\big)^{-j},$$

where $P_{\alpha,j}(x)$ is a polynomial homogeneous of degree $|\alpha|$ in the components of x. Since $p = n/m$, we have

$$|D^\alpha v(x)|^p \le \sum_{j=1}^{|\alpha|} K_{\alpha,j}|x|^{-|\alpha|n/m}\big(\log(4R/|x|)\big)^{-jp},$$

so that, setting $\rho = |x|$,

$$\int_{B_R} |D^\alpha v(x)|^p\, dx \le K \sum_{j=1}^{|\alpha|} \int_0^R \big(\log(4R/\rho)\big)^{-jp} \rho^{-|\alpha|n/m+n-1}\, d\rho.$$

The right side of the above inequality is certainly finite if $|\alpha| < m$. If $|\alpha| = m$, we have, setting $\sigma = \log(4R/\rho)$,

$$\int_{B_R} |D^\alpha v(x)|^p\, dx \le K \sum_{j=1}^{|\alpha|} \int_{\log 4}^\infty \sigma^{-jp}\, d\sigma$$

which is finite since $p > 1$. Thus $v \in W^{m,p}(B_R)$ and $u \in W^{m,p}(\Omega)$. ∎

It is interesting that the same function v (and hence u) works for any choice of m and p with $mp = n$.

4.44 EXAMPLE Suppose $(m-1)p = n$ and $p > 1$. We construct u in $W^{m,p}(\Omega)$ such that $u \notin C^{0,1}(\overline{\Omega})$. Hence the imbedding $W^{m,p}(\Omega) \to C^{0,\lambda}(\overline{\Omega})$, valid for $0 < \lambda < 1$ whenever Ω satisfies the strong local Lipschitz condition, cannot be extended to yield $W^{m,p}(\Omega) \to C^{0,1}(\overline{\Omega})$ unless $p = 1$ and $m - 1 = n$. Here u is constructed as in the previous example except using

$$v(x) = |x| \log\bigl(\log(4R/|x|)\bigr).$$

Since $|v(x) - v(0)|/|x - 0| = \log\bigl(\log(4R/|x|)\bigr) \to \infty$ as $x \to 0$ it is clear that $v \notin C^{0,1}(\overline{B_R})$ and therefore $u \notin C^{0,1}(\overline{\Omega})$. The fact that $v \in W^{m,p}(B_R)$ and hence $u \in W^{m,p}(\Omega)$ is shown just as in the previous example. ∎

Nonimbedding Theorems for Irregular Domains

4.45 The above examples show that even for very regular domains there can exist no imbeddings of the types considered in Theorem 4.12 except those explicitly stated there. It remains to be seen whether any imbeddings of those types can exist for domains that do not satisfy the cone condition (or at least the weak cone condition). We will show below that Theorem 4.12 can be extended, with weakened conclusions, to certain types of irregular domains, but first we show that no extension is possible if the domain is "too irregular." This can happen if either the domain is unbounded and too narrow at infinity, or if it has a cusp of exponential sharpness on its boundary.

An unbounded domain $\Omega \subset \mathbb{R}^n$ may have a smooth boundary and still fail to satisfy the cone condition if it becomes narrow at infinity, that is, if

$$\lim_{\substack{|x|\to\infty \\ x\in\Omega}} \operatorname{dist}(x, \operatorname{bdry}\Omega) = 0.$$

The following theorem shows that Parts I and II of Theorem 4.12 fail completely for any unbounded Ω which has finite volume.

4.46 THEOREM Let Ω be an unbounded domain in $\mathbb{R}^n$ having finite volume, and let $q > p$. Then $W^{m,p}(\Omega)$ is not imbedded in $L^q(\Omega)$.

Proof. We construct a function $u(x)$ depending only on distance $\rho = |x|$ of x from the origin whose growth as ρ increases is rapid enough to prevent membership in $L^q(\Omega)$ but not so rapid as to prevent membership in $W^{m,p}(\Omega)$.

Without loss of generality we assume $\operatorname{vol}(\Omega) = 1$. Let $A(\rho)$ denote the surface area ($(n-1)$-measure) of the intersection of Ω with the surface $|x| = \rho$. Then

$$\int_0^\infty A(\rho)\, d\rho = 1.$$

Let $r_0 = 0$ and define r_k for $k = 1, 2, \ldots$ by

$$\int_{r_k}^\infty A(\rho)\, d\rho = \frac{1}{2^k} = \int_{r_{k-1}}^{r_k} A(\rho)\, d\rho.$$

Since Ω is unbounded, r_k increases to infinity with k. Let $\Delta r_k = r_{k+1} - r_k$ and fix ϵ such that $0 < \epsilon < [1/(mp)] - [1/(mq)]$. There must exist an increasing sequence $\{k_j\}_{j=1}^{\infty}$ such that $\Delta r_{k_j} \geq 2^{-\epsilon k_j}$, for otherwise $\Delta r_k < 2^{-\epsilon k}$ for all but possibly finitely many values of k and we would have $\sum_{k=0}^{\infty} \Delta r_k < \infty$, contradicting $\lim r_k = \infty$. For convenience we assume $k_1 \geq 1$ so $k_j \geq j$ for all j. Let $a_0 = 0$, $a_j = r_{k_j+1}$, and $b_j = r_{k_j}$. Note that $a_{j-1} \leq b_j < a_j$ and $a_j - b_j = \Delta r_{k_j} \geq 2^{-\epsilon k_j}$. Let f be an infinitely differentiable function on $\mathbb{R}$ having the properties:

(i) $0 \leq f(t) \leq 1$ for all t,

(ii) $f(t) = 0$ if $t \leq 0$ and $f(t) = 1$ if $t \geq 1$,

(iii) $|(d/dt)^{\kappa} f(t)| \leq M$ for all t if $1 \leq \kappa \leq m$.

If $x \in \Omega$ and $\rho = |x|$, set

$$u(x) = \begin{cases} 2^{k_{j-1}/q} & \text{for } a_{j-1} \leq \rho \leq b_j \\ 2^{k_{j-1}/q} + \left(2^{k_j/q} - 2^{k_{j-1}/q}\right) f\left(\dfrac{\rho - b_j}{a_j - b_j}\right) & \text{for } b_j \leq \rho \leq a_j. \end{cases}$$

Clearly $u \in C^{\infty}(\Omega)$. Denoting $\Omega_j = \{x \in \Omega : a_{j-1} \leq \rho \leq a_j\}$, we have

$$\begin{aligned} \int_{\Omega_j} |u(x)|^p \, dx &= \left(\int_{a_{j-1}}^{b_j} + \int_{b_j}^{a_j}\right) \left(u(x)\right)^p A(\rho)\, d\rho \\ &\leq 2^{k_{j-1}p/q} \int_{a_{j-1}}^{\infty} A(\rho)\, d\rho + 2^{k_j p/q} \int_{b_j}^{a_j} A(\rho)\, d\rho \\ &= \frac{2^{-k_{j-1}(1-p/q)} + 2^{-k_j(1-p/q)}}{2} \leq \frac{1}{2^{(j-1)(1-p/q)}}. \end{aligned}$$

Since $p < q$, the above inequality forces

$$\int_{\Omega} |u(x)|^p \, dx = \sum_{j=1}^{\infty} \int_{\Omega_j} |u(x)|^p \, dx < \infty.$$

Also, if $1 \leq \kappa \leq m$, we have

$$\begin{aligned} \int_{\Omega_j} \left|\frac{d^{\kappa} u}{d\rho^{\kappa}}\right|^p dx &= \int_{b_j}^{a_j} \left|\frac{d^{\kappa} u}{d\rho^{\kappa}}\right|^p A(\rho)\, d\rho \\ &\leq M^p 2^{k_j p/q} \left(a_j - b_j\right)^{-\kappa p} \int_{b_j}^{a_j} A(\rho)\, d\rho \\ &= \frac{M^p 2^{-k_j(1-p/q-\epsilon\kappa p)}}{2} \leq \frac{M^p 2^{-Cj}}{2}, \end{aligned}$$

where $C = 1 - p/q - \epsilon\kappa p > 0$ because of the choice of ϵ. Hence $D^\alpha u \in L^p(\Omega)$ for $|\alpha| \le m$, that is, $u \in W^{m,p}(\Omega)$. However, $u \notin L^q(\Omega)$ because we have for each j,

$$\begin{aligned}\int_{\Omega_j} |u(x)|^q\, dx &\ge 2^{k_{j-1}} \int_{a_{j-1}}^{a_j} A(\rho)\, d\rho \\ &= 2^{k_{j-1}}\left(2^{-k_{j-1}-1} - 2^{-k_j-1}\right) \ge \frac{1}{4}.\end{aligned}$$

Therefore $W^{m,p}(\Omega)$ cannot be imbedded in $L^q(\Omega)$. ∎

The conclusion of the above theorem can be extended to unbounded domains having infinite volume but satisfying

$$\limsup_{N\to\infty} \operatorname{vol}(\{x \in \Omega : N \le |x| \le N+1\}) = 0.$$

(See Theorem 6.41.)

4.47 Parts I and II of Theorem 4.12 also fail completely for domains with sufficiently sharp boundary cusps. If Ω is a domain in $\mathbb{R}^n$ and x_0 is a point on its boundary, let $B_r = B_r(x_0)$ denote the open ball of radius r and centre at x_0. Let $\Omega_r = B_r \cap \Omega$, let $S_r = (\operatorname{bdry} B_r) \cap \Omega$, and let $A(r, \Omega)$ be the surface area ($(n-1)$-measure) of S_r. We shall say that Ω has a *cusp of exponential sharpness* at its boundary point x_0 if for every real number k we have

$$\lim_{r\to 0+} \frac{A(r,\Omega)}{r^k} = 0. \tag{25}$$

4.48 THEOREM If Ω is a domain in $\mathbb{R}^n$ having a cusp of exponential sharpness at a point x_0 on its boundary, then $W^{m,p}(\Omega)$ is not imbedded in $L^q(\Omega)$ for any $q > p$.

Proof. We construct $u \in W^{m,p}(\Omega)$ which fails to belong to $L^q(\Omega)$ because it becomes unbounded too rapidly near x_0. Without loss of generality we may assume $x_0 = 0$, so that $r = |x|$. Let $\Omega^* = \{x/|x|^2 : x \in \Omega,\ |x| < 1\}$. Then Ω^* is unbounded and has finite volume by (25), and

$$A(r, \Omega^*) = r^{2(n-1)} A(1/r, \Omega).$$

Let t satisfy $p < t < q$. By Theorem 4.46 there exists a function $\tilde{v} \in C^m(0,\infty)$ such that

(i) $\tilde{v}(r) = 0$ if $0 < r \le 1$,

(ii) $\displaystyle\int_1^\infty |\tilde{v}^{(j)}|^t A(r,\Omega^*)\, dr < \infty$ if $0 \le j \le m$,

(iii) $\int_1^\infty |\tilde{v}(r)|^q A(r, \Omega^*)\, dr = \infty.$

[Specifically, $v(y) = \tilde{v}(|y|)$ defines $v \in W^{m,t}(\Omega^*)$ but $v \notin L^q(\Omega^*)$.] Let $x = y/|y|^2$ so that $\rho = |x| = 1/|y| = 1/r$. Set $\lambda = 2n/q$ and define

$$u(x) = \tilde{u}(\rho) = r^\lambda \tilde{v}(r) = |y|^\lambda v(y).$$

It follows for $|\alpha| = j \le m$ that

$$|D^\alpha u(x)| \le |\tilde{u}^{(j)}(\rho)| \le \sum_{i=1}^{j} c_{ij} r^{\lambda+j+i} \tilde{v}^{(i)}(r),$$

where the coefficients c_{ij} depend only on λ. Now $u(x)$ vanishes for $|x| \ge 1$ and so

$$\int_\Omega |u(x)|^q\, dx = \int_0^1 |\tilde{u}(\rho)|^q A(\rho, \Omega)\, d\rho = \int_1^\infty |\tilde{v}(r)|^q A(r, \Omega^*)\, dr = \infty.$$

On the other hand, if $0 \le |\alpha| = j \le m$, we have

$$\begin{aligned} \int_\Omega |D^\alpha u(x)|^p\, dx &\le \int_0^1 |\tilde{u}^{(j)}(\rho)|^p A(\rho, \Omega)\, d\rho \\ &\le K \sum_{i=0}^{j} \int_1^\infty |\tilde{v}^{(i)}(r)|^p r^{(\lambda+j+i)p-2n} A(r, \Omega^*)\, dr. \end{aligned}$$

If it happens that $(\lambda + 2m)p \le 2n$, then, since $p < t$ and $\operatorname{vol}(\Omega^*) < \infty$, all the integrals in the above sum are finite by Hölder's inequality, and $u \in W^{m,p}(\Omega)$. Otherwise let

$$k = \big((\lambda + 2m)p - 2n\big)\frac{t}{t-p} + 2n.$$

By (25) there exists $a \le 1$ such that if $\rho \le a$, then $A(\rho, \Omega) \le \rho^k$. It follows that if $r \ge 1/a$, then

$$r^{k-2n} A(r, \Omega^*) \le r^{k-2} \rho^k = r^{-2}.$$

Thus

$$\begin{aligned} &\int_1^\infty |\tilde{v}^{(i)}(r)|^p r^{(\lambda+j+i)p-2n} A(r, \Omega^*)\, dr \\ &\quad = \int_1^\infty |\tilde{v}^{(i)}(r)|^p r^{(k-2n)(t-p)/t} A(r, \Omega^*)\, dr \\ &\quad \le \left(\int_1^\infty |\tilde{v}^{(i)}(r)|^t A(r, \Omega^*)\, dr\right)^{p/t} \left(\int_1^\infty r^{k-2n} A(r, \Omega^*)\, dr\right)^{(t-p)/t} \end{aligned}$$

which is finite. Hence $u \in W^{m,p}(\Omega)$ and the proof is complete. ∎

Imbedding Theorems for Domains with Cusps

4.49 Having proved that Theorem 4.12 fails completely for sufficiently irregular domains, we now propose to show that certain imbeddings of the types considered in that theorem do hold for less irregular domains that nevertheless fail to satisfy even the weak cone condition. Questions of this sort have been considered by many writers. The treatment here follows that in [A1].

We consider domains Ω in $\mathbb{R}^n$ whose boundaries consist only of $(n-1)$-dimensional surfaces, and it is assumed that Ω lies on only one side of its boundary. For such domains we shall say, somewhat loosely, that Ω has a *cusp* at point x_0 on its boundary if no finite open cone of positive volume contained in Ω can have its vertex at x_0. The failure of a domain to have any cusps does not, of course, imply that it satisfies the cone condition.

We consider a family of special domains in $\mathbb{R}^n$ that we call *standard cusps* and that have cusps of power sharpness (less sharp than exponential sharpness).

4.50 **(Standard Cusps)** If $1 \le k \le n-1$ and $\lambda > 1$, let the standard cusp $Q_{k,\lambda}$ be the set of points $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ that satisfy the inequalities

$$\begin{gathered} x_1^2 + \cdots + x_k^2 < x_{k+1}^{2\lambda}, \quad x_{k+1} > 0, \ldots, x_n > 0, \\ (x_1^2 + \cdots + x_k^2)^{1/\lambda} + x_{k+1}^2 + \cdots + x_n^2 < a^2, \end{gathered} \tag{26}$$

where a is the radius of the ball of unit volume in $\mathbb{R}^n$. Note that $a < 1$. The cusp $Q_{k,\lambda}$ has axial plane spanned by the $x_k, \ldots, x_n$ axes, and verticial plane (cusp plane) spanned by $x_{k+2}, \ldots, x_n$. If $k = n-1$, the origin is the only vertex point of $Q_{k,\lambda}$. The outer boundary surface of $Q_{k,\lambda}$ corresponds to equality in (26) in order to simplify calculations later. A sphere or other suitable surface bounded and bounded away from the origin could be used instead.

Corresponding to the standard cusp $Q_{k,\lambda}$ we consider the associated *standard cone* $C_k = Q_{k,1}$ consisting of points $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$ that satisfy the inequalities

$$\begin{gathered} y_1^2 + \cdots + y_k^2 < y_{k+1}^2, \quad y_{k+1} > 0, \ldots, y_n > 0, \\ y_1^2 + \cdots + y_n^2 < a^2. \end{gathered}$$

Figure 3 illustrates the standard cusps $Q_{1,2}$ in $\mathbb{R}^2$, and $Q_{2,2}$ and $Q_{1,2}$ in $\mathbb{R}^3$, together with their associated standard cones. In $\mathbb{R}^3$ the cusp $Q_{2,2}$ has a single cusp point (vertex) at the origin, while $Q_{1,2}$ has a cusp line along the x_3-axis.

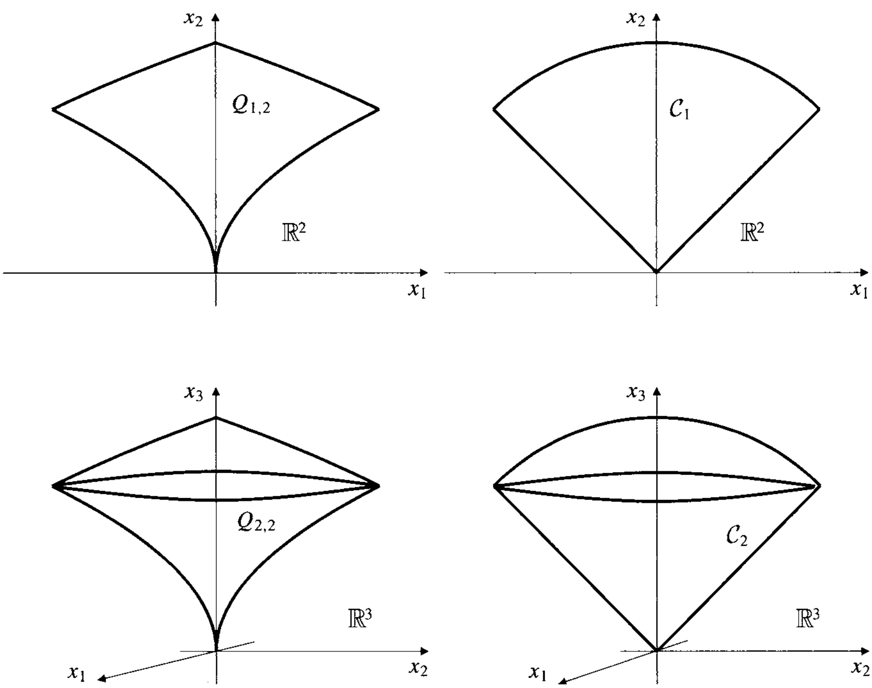
It is convenient to adopt a system of generalized "cylindrical" coordinates in $\mathbb{R}^n$, $(r_k, \phi_1, \ldots, \phi_{k-1}, y_{k+1}, \ldots, y_n)$, so that $r_k \ge 0$, $-\pi \le \phi_1 \le \pi$, $0 \le \phi_2, \ldots,$

$\phi_{k-1} \leq \pi$, and

$$\begin{aligned} y_1 &= r_k \sin\phi_1 \sin\phi_2 \cdots \sin\phi_{k-1} \\ y_2 &= r_k \cos\phi_1 \sin\phi_2 \cdots \sin\phi_{k-1} \\ y_3 &= \qquad\quad r_k \cos\phi_2 \cdots \sin\phi_{k-1} \\ &\ \ \vdots \\ y_k &= \qquad\qquad\qquad\quad r_k \cos\phi_{k-1}. \end{aligned} \tag{27}$$

In terms of these coordinates, $\mathcal{C}_k$ is represented by

$$\begin{gathered} 0 \leq r_k < y_{k+1}, \quad y_{k+1} > 0, \ldots, y_n > 0, \\ r_k^2 + y_{k+1}^2 + \cdots + y_n^2 < a^2. \end{gathered}$$



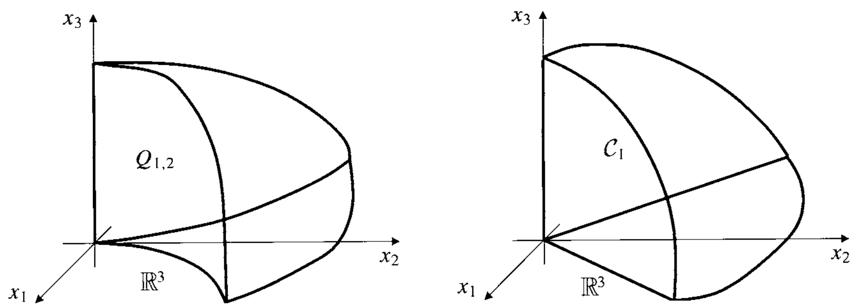


Fig. 3

The standard cusp $Q_{k,\lambda}$ may be transformed into the associated cone C_k by means of the one-to-one transformation

$$\begin{aligned}
x_1 &= r_k^\lambda \sin\phi_1 \sin\phi_2 \cdots \sin\phi_{k-1} \\
x_2 &= r_k^\lambda \cos\phi_1 \sin\phi_2 \cdots \sin\phi_{k-1} \\
x_3 &= \qquad\quad r_k^\lambda \cos\phi_2 \cdots \sin\phi_{k-1} \\
&\;\vdots \\
x_k &= \qquad\qquad\qquad r_k^\lambda \cos\phi_{k-1} \\
x_{k+1} &= y_{k+1} \\
&\;\vdots \\
x_n &= y_n,
\end{aligned} \tag{28}$$

which has Jacobian determinant

$$\left|\frac{\partial(x_1, \ldots, x_n)}{\partial(y_1, \ldots, y_n)}\right| = \lambda\, r_k^{(\lambda-1)k}. \tag{29}$$

We now state three theorems extending imbeddings of the types considered in Theorem 4.12 (except the trace imbeddings) to domains with boundary irregularities comparable to standard cusps. The proofs of these theorems will be given later in this chapter.

4.51 THEOREM Let Ω be a domain in $\mathbb{R}^n$ having the following property: There exists a family Γ of open subsets of Ω such that

(i) $\Omega = \bigcup_{G\in\Gamma} G$,

(ii) Γ has the finite intersection property, that is, there exists a positive integer N such that any $N+1$ distinct sets in Γ have empty intersection,

(iii) at most one set $G \in \Gamma$ satisfies the cone condition,

(iv) there exist positive constants ν and A such that for each $G \in \Gamma$ not satisfying the cone condition there exists a one-to-one function $\Psi = (\psi_1, \dots, \psi_n)$ mapping G onto a standard cusp $Q_{k,\lambda}$, where $(\lambda - 1)k \le \nu$, and such that for all i, j, $(1 \le i, j \le n)$, all $x \in G$, and all $y \in Q_{k,\lambda}$,

$$\left|\frac{\partial \psi_j}{\partial x_i}\right| \le A \qquad \text{and} \qquad \left|\frac{\partial (\psi^{-1})_j}{\partial y_i}\right| \le A.$$

If $\nu > mp - n$, then

$$W^{m,p}(\Omega) \to L^q(\Omega), \qquad \text{for} \quad p \le q \le \frac{(\nu + n)p}{\nu + n - mp}.$$

If $\nu = mp - n$, then the same imbedding holds for $p \le q < \infty$, and for $q = \infty$ if $p = 1$.

If $\nu < mp - n$, then the imbedding holds for $p \le q \le \infty$.

4.52 THEOREM Let Ω be a domain in $\mathbb{R}^n$ having the following property: There exist positive constants $\nu < mp - n$ and A such that for each $x \in \Omega$ there exists an open set G with $x \in G \subset \Omega$ and a one-to-one mapping $\Psi = (\psi_1, \dots, \psi_n)$ mapping G onto a standard cusp $Q_{k,\lambda}$, where $(\lambda - 1)k \le \nu$, and such that for all i, j, $(1 \le i, j \le n)$, all $x \in G$, and all $y \in Q_{k,\lambda}$,

$$\left|\frac{\partial \psi_j}{\partial x_i}\right| \le A \qquad \text{and} \qquad \left|\frac{\partial (\psi^{-1})_j}{\partial y_i}\right| \le A.$$

Then

$$W^{m,p}(\Omega) \to C_B^0(\Omega).$$

More generally, if $\nu < (m - j)p - n$ where $0 \le j \le m - 1$, then

$$W^{m,p}(\Omega) \to C_B^j(\Omega).$$

4.53 THEOREM Let Ω be a domain in $\mathbb{R}^n$ having the following property: There exist positive constants ν, δ, and A such that for each pair of points $x, y \in \Omega$ with $|x - y| \le \delta$ there exists an open set G with $x, y \in G \subset \Omega$ and a one-to-one mapping $\Psi = (\psi_1, \dots, \psi_n)$ mapping G onto a standard cusp $Q_{k,\lambda}$, where $(\lambda - 1)k \le \nu$, and such that for all i, j, $(1 \le i, j \le n)$, all $x \in G$, and all $y \in Q_{k,\lambda}$,

$$\left|\frac{\partial \psi_j}{\partial x_i}\right| \le A \qquad \text{and} \qquad \left|\frac{\partial (\psi^{-1})_j}{\partial y_i}\right| \le A.$$

Suppose that $(m-j-1)p < \nu+n < (m-j)p$ for some integer $j, (0 \le j \le m-1)$. Then

$$W^{m,p}(\Omega) \to C^{j,\mu}(\overline{\Omega}) \qquad \text{for} \quad 0 < \mu \le m - j - \frac{n+\nu}{p}.$$

If $(m-j-1)p = \nu + n$, then the same imbedding holds for $0 < \mu < 1$. In either event we have $W^{m,p}(\Omega) \to C^j(\overline{\Omega})$.

4.54 REMARKS

1. In these theorems the role played by the parameter ν is equivalent to an increase in the dimension n in Theorem 4.12, where increasing n results in weaker imbedding results for given m and p. Since $\nu \ge (\lambda - 1)k$, the sharper the cusp, the greater the equivalent increase in dimension.
2. The reader may wish to construct examples similar to those of Paragraphs 4.41–4.44 to show that the three theorems above give the best possible imbeddings for the domains and types of spaces considered.

4.55 EXAMPLE To illustrate Theorem 4.51, consider the domain

$$\Omega = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_2 > 0,\ x_2^2 < x_1 < 3x_2^2\}.$$

If $a = (4\pi/3)^{-1/3}$, the radius of the ball of unit volume in $\mathbb{R}^3$, it is readily verified that the transformation

$$y_1 = x_1 + 2x_2^2, \quad y_2 = x_2, \quad y_3 = x_3 - (k/a), \quad k = 0, \pm 1, \pm 2, \ldots$$

transforms a subdomain G_k of Ω onto the standard cusp $Q_{1,2} \subset \mathbb{R}^3$ in the manner required of the transformation Ψ in the statement of the theorem. Moreover, $\{G\}_{k=-\infty}^{\infty}$ has the finite intersection property and covers Ω up to a set satisfying the cone condition. Using $\nu = 1$, we conclude that $W^{m,p}(\Omega) \to L^q(\Omega)$ for $p \le q \le 4p/(4-mp)$ if $mp < 4$, or for $p \le q < \infty$ if $mp = 4$, or for $p \le q \le \infty$ if $mp > 4$.

Imbedding Inequalities Involving Weighted Norms

4.56 The technique of mapping a standard cusp onto its associated standard cone via (28) and (29) is central to the proof of Theorem 4.51. Such a transformation introduces into any integrals involved a weight factor in the form of the Jacobian determinant (29). Accordingly, we must obtain imbedding inequalities for such standard cones involving L^p-norms weighted by powers of distance from the axial plane of the cone. Such inequalities are also useful in extending the imbedding theorem 4.12 to more general Sobolev spaces involving weighted norms. Many authors have treated the subject of weighted Sobolev spaces. We mention, in particular, Kufner's monograph [Ku] which focuses on a different class of weights depending on distance from the boundary of Ω.

We begin with some one-dimensional inequalities for functions continuously differentiable on an open interval $(0, T)$ in $\mathbb{R}$.

4.57 LEMMA Let $\nu > 0$ and $u \in C^1(0, T)$. If $\int_0^T |u'(t)|t^\nu \, dt < \infty$, then $\lim_{t \to 0+} |u(t)|t^\nu = 0$.

Proof. Let $\epsilon > 0$ be given and fix s in $(0, T/2)$ small enough so that for any t, $0 < t < s$, we have

$$\int_t^s |u'(\tau)|\tau^\nu \, d\tau < \epsilon/3.$$

Now there exists δ in $(0, s)$ such that

$$\delta^\nu |u'(T/2)| < \epsilon/3 \qquad \text{and} \qquad (\delta/s)^\nu \int_s^{T/2} |u'(\tau)|\tau^\nu \, d\tau < \epsilon/3.$$

If $0 < t \le \delta$, then

$$|u(t)| \le |u(T/2)| + \int_t^{T/2} |u'(\tau)| \, d\tau$$

so that

$$t^\nu |u(t)| \le \delta^\nu |u(T/2)| + \int_t^s |u'(\tau)|\tau^\nu \, d\tau + (\delta/s)^\nu \int_s^{T/2} |u'(\tau)|\tau^\nu \, d\tau < \epsilon.$$

Hence $\lim_{t \to 0+} |u(t)|t^\nu = 0$. ∎

4.58 LEMMA Let $\nu > 0$, $p \ge 1$, and $u \in C^1(0, T)$. Then

$$\int_0^T |u(t)|^p t^{\nu-1} \, dt \le \frac{\nu+1}{\nu T} \int_0^T |u(t)|^p t^\nu \, dt + \frac{p}{\nu} \int_0^T |u(t)|^{p-1} |u'(t)| t^\nu \, dt. \tag{30}$$

Proof. We may assume without loss of generality that the right side of (30) is finite and that $p = 1$. Integration by parts gives

$$\int_0^T |u(t)| \left(\nu t^{\nu-1} - \frac{\nu+1}{T} t^\nu \right) dt = -\int_0^T \left(t^\nu - \frac{1}{T} t^{\nu+1} \right) \frac{d}{dt} |u(t)| \, dt,$$

the previous lemma assuring the vanishing of the integrated term at zero. Transposition and estimation of the term on the right now yields

$$\nu \int_0^T |u(t)| t^{\nu-1} \, dt \le \frac{\nu+1}{T} \int_0^T |u(t)| t^\nu \, dt + \int_0^T |u'(t)| t^\nu \, dt,$$

which is (30) for $p = 1$. ∎

4.59 LEMMA Let $\nu > 0$, $p \ge 1$, and $u \in C^1(0,T)$. Then

$$\sup_{0<t<T} |u(t)|^p \le \frac{2}{T}\int_0^T |u(t)|^p\,dt + p\int_0^T |u(t)|^{p-1}|u'(t)|\,dt, \tag{31}$$

$$\sup_{0<t<T} |u(t)|^p t^\nu \le \frac{\nu+3}{T}\int_0^T |u(t)|^p t^\nu\,dt + 2p\int_0^T |u(t)|^{p-1}|u'(t)|t^\nu\,dt. \tag{32}$$

Proof. Again the inequalities need only be proved for $p = 1$. If $0 < t \le T/2$, we obtain by integration by parts

$$\int_0^{T/2}\left|u\left(t+\frac{T}{2}-\tau\right)\right|d\tau = \frac{T}{2}|u(t)| - \int_0^{T/2}\tau\frac{d}{d\tau}\left|u\left(t+\frac{T}{2}-\tau\right)\right|d\tau$$

whence

$$|u(t)| \le \frac{2}{T}\int_0^T |u(\sigma)|\,d\sigma + \int_0^T |u'(\sigma)|\,d\sigma.$$

For $T/2 \le t < T$ the same inequality results from the partial integration of $\int_0^{T/2}|u(t+\tau-T/2)|\,d\tau$. This proves (31) for $p = 1$. Replacing $u(t)$ by $u(t)t^\nu$ in this inequality, we obtain

$$\begin{aligned}\sup_{0<t<T} |u(t)|t^\nu &\le \frac{2}{T}\int_0^T |u(t)|t^\nu\,dt + \int_0^T \left(|u'(t)|t^\nu + \nu|u(t)|t^{\nu-1}\right)dt\\ &\le \frac{2}{T}\int_0^T |u(t)|t^\nu\,dt + \int_0^T |u'(t)|t^\nu\,dt\\ &\quad + \nu\left(\frac{\nu+1}{\nu T}\int_0^T |u(t)|t^\nu\,dt + \frac{1}{\nu}\int_0^T |u'(t)|t^\nu\,dt\right),\end{aligned}$$

where (30) has been used to obtain the last inequality. This is the desired result (32) for $p = 1$. ∎

4.60 Now we return to $\mathbb{R}^n$ for $n \ge 2$. If $x \in \mathbb{R}^n$, we shall make use of the spherical polar coordinate representation

$$x = (\rho, \phi) = (\rho, \phi_1, \ldots, \phi_{n-1}),$$

where $\rho \ge 0$, $-\pi \le \phi_1 \le \pi$, $0 \le \phi_2, \ldots, \phi_{n-1} \le \pi$, and

$$\begin{aligned} x_1 &= \rho\sin\phi_1\sin\phi_2\cdots\sin\phi_{n-1},\\ x_2 &= \rho\cos\phi_1\sin\phi_2\cdots\sin\phi_{n-1},\\ x_3 &= \qquad\ \rho\cos\phi_2\cdots\sin\phi_{n-1},\\ &\ \ \vdots\\ x_n &= \qquad\qquad\quad \rho\cos\phi_{n-1}.\end{aligned}$$

The volume element is

$$dx = dx_1\, dx_2 \cdots dx_n = \rho^{n-1} \prod_{j=1}^{n-1} \sin^{j-1} \phi_j \, d\rho\, d\phi,$$

where $d\phi = d\phi_1 \cdots d\phi_{n-1}$.

We define functions $r_k = r_k(x)$ for $1 \le k \le n$ as follows:

$$r_1(x) = \rho |\sin \phi_1| \prod_{j=2}^{n-1} \sin \phi_j,$$

$$r_k(x) = \rho \prod_{j=k}^{n-1} \sin \phi_j, \qquad k = 2, 3, \ldots, n-1,$$

$$r_n(x) = \rho.$$

For $1 \le k \le n-1$, $r_k(x)$ is the distance of x from the coordinate plane spanned by the axes $x_{k+1}, \ldots, x_n$; of course $r_n(x)$ is the distance of x from the origin. In connection with the use of product symbols of the form $P = \prod_{j=k}^{m} P_j$, we follow the convention that $P = 1$ if $m < k$.

Let C be an open, conical domain in $\mathbb{R}^n$ specified by the inequalities

$$0 < \rho < a, \quad -\beta_1 < \phi_1 < \beta_1, \quad 0 \le \phi_j < \beta_j, \quad (2 \le j \le n-1), \tag{33}$$

where $0 < \beta_i \le \pi$. (Inequalities "$<$" in (33) corresponding to any $\beta_i = \pi$ are replaced by "$\le$." If all $\beta_i = \pi$, the first inequality is replaced with $0 \le \rho < a$.) Note that any standard cone C_k (introduced in section 4.50) is of the form (33) for some choice of the parameters β_i, $1 \le i \le n-1$.

4.61 LEMMA Let C be as specified by (33) and let $p \ge 1$. Suppose that either $m = k = 1$, or $2 \le m \le n$ and $1 \le k \le n$. Let $1 - k < \nu_1 \le \nu \le \nu_2 < \infty$. Then there exists a constant $K = K(m, k, n, p, \nu 1, \nu 2, \beta_1, \ldots, \beta_{n-1})$ independent of ν and a, such that for every $u \in C^1(C)$ we have

$$\begin{aligned} &\int_C |u(x)|^p [r_k(x)]^\nu [r_m(x)]^{-1}\, dx \\ &\qquad \le K \int_C |u(x)|^{p-1} \left(\frac{1}{a} |u(x)| + |\operatorname{grad} u(x)| \right) [r_k(x)]^\nu \, dx. \end{aligned} \tag{34}$$

Proof. Once again it is sufficient to establish (34) for $p = 1$. Let C_+ be the set $\{x = (\rho, \phi) : \phi_1 \ge 0\}$ and C_- the set $\{x = (\rho, \phi) : \phi_1 \le 0\}$. Then $C = C_+ \cup C_-$. We prove (34) only for C_+ (which, however, we continue to call C); a similar proof holds for C_-, so that (34) holds for the given C. Accordingly, assume $C = C_+$.

For $k \le m$ we may write (34) in the form (taking $p = 1$)

$$\int_C |u| \prod_{j=2}^{k-1} \sin^{j-1} \phi_j \prod_{j=k}^{m-1} \sin^{\nu+j-1} \phi_j \prod_{j=m}^{n-1} \sin^{\nu+j-2} \phi_j \rho^{\nu+n-2}\, d\rho\, d\phi$$
$$\le K \int_C \left(\frac{1}{a}|u| + |\operatorname{grad} u|\right) \prod_{j=2}^{k-1} \sin^{j-1} \phi_j \prod_{j=k}^{n-1} \sin^{\nu+j-1} \phi_j \rho^{\nu+n-1}\, d\rho\, d\phi.$$

For $k > m \ge 2$ we may write (34) in the form

$$\int_C |u| \prod_{j=2}^{m-1} \sin^{j-1} \phi_j \prod_{j=m}^{k-1} \sin^{j-2} \phi_j \prod_{j=k}^{n-1} \sin^{\nu+j-2} \phi_j \rho^{\nu+n-2}\, d\rho\, d\phi$$
$$\le K \int_C \left(\frac{1}{a}|u| + |\operatorname{grad} u|\right) \prod_{j=2}^{k-1} \sin^{j-1} \phi_j \prod_{j=k}^{n-1} \sin^{\nu+j-1} \phi_j \rho^{\nu+n-1}\, d\rho\, d\phi.$$

By virtue of the restrictions placed on ν, m, and k in the statement of the lemma, each of the two inequalities above is a special case of

$$\begin{aligned} &\int_C |u| \prod_{j=1}^{i-1} \sin^{\mu_j} \phi_j \prod_{j=i}^{n-1} \sin^{\mu_j - 1} \phi_j \rho^{\nu+n-2}\, d\rho\, d\phi \\ &\qquad \le K \int_C \left(\frac{1}{a}|u| + |\operatorname{grad} u|\right) \prod_{j=1}^{n-1} \sin^{\mu_j} \phi_j \rho^{\nu+n-1}\, d\rho\, d\phi, \end{aligned} \tag{35}$$

where $1 \le i \le n$, $\mu_j \ge 0$, and $\mu_j > 0$ if $j \ge i$. We prove (35) by backwards induction on i. For $i = n$, (35) is obtained by applying Lemma 4.58 to u considered as a function of ρ on $(0, a)$, and then integrating the remaining variables with the appropriate weights. Assume, therefore, that (35) has been proved for $i = k + 1$ where $1 \le k \le n - 1$. We prove it must also hold for $i = k$.

If $\beta_k < \pi$, we have

$$\sin \phi_k \le \phi_k \le K_1 \sin \phi_k, \qquad 0 \le \phi_k \le \beta_k, \tag{36}$$

where $K_1 = K_1(\beta_k)$. By Lemma 4.58, and since

$$\left|\frac{\partial u}{\partial \phi_k}\right| \le \rho |\operatorname{grad} u| \prod_{j=k+1}^{n-1} \sin \phi_j,$$

we have

$$\int_0^{\beta_k} |u(\rho, \phi)| \sin^{\mu_k - 1} \phi_k\, d\phi_k$$

$$
\begin{aligned}
&\le \int_0^{\beta_k} |u|\phi_k^{\mu_k-1}\,d\phi_k \\
&\le K_2 \int_0^{\beta_k} \left(|u| + |\operatorname{grad} u|\rho \prod_{j=k+1}^{n-1} \sin\phi_j\right) \phi_k^{\mu_k}\,d\phi_k \\
&\le K_3 \int_0^{\beta_k} \left(|u| + |\operatorname{grad} u|\rho \prod_{j=k+1}^{n-1} \sin\phi_j\right) \sin^{\mu_k}\phi_k\,d\phi_k. \qquad (37)
\end{aligned}
$$

Note that K_2, and hence K_3, depends on β_k but may be chosen independent of μ_k, and hence of ν, under the conditions of the lemma. If $\beta_k = \pi$, we obtain (37) by writing $\int_0^\pi = \int_0^{\pi/2} + \int_{\pi/2}^\pi$ and using the inequalities

$$
\begin{aligned}
&\sin\phi_k \le \phi_k \le (\pi/2)\sin\phi_k && \text{if} \quad 0 \le \phi_k \le \pi/2 \\
&\sin\phi_k \le \pi - \phi_k \le (\pi/2)\sin\phi_k && \text{if} \quad \pi/2 \le \phi_k \le \pi.
\end{aligned} \qquad (38)
$$

We now obtain, using (37) and the induction hypothesis,

$$
\begin{aligned}
&\int_C |u| \prod_{j=1}^{k-1} \sin^{\mu_j}\phi_j \prod_{j=k}^{n-1} \sin^{\mu_j-1}\phi_j\,\rho^{\nu+n-2}\,d\rho\,d\phi \\
&\quad \le \int_0^a \rho^{\nu+n-2}\,d\rho \prod_{j=1}^{k-1} \int_0^{\beta_j} \sin^{\mu_j}\phi_j\,d\phi_j \\
&\qquad \times \prod_{j=k+1}^{n-1} \int_0^{\beta_j} \sin^{\mu_j-1}\phi_j\,d\phi_j \times \int_0^{\beta_k} |u| \sin^{\mu_k-1}\phi_k\,d\phi_k \\
&\quad \le K_3 \int_C |\operatorname{grad} u| \prod_{j=1}^{n-1} \sin^{\mu_j}\phi_j\,\rho^{\nu+n-1}\,d\rho\,d\phi \\
&\qquad + K_3 \int_C |u| \prod_{j=1}^{k} \sin^{\mu_j}\phi_j \prod_{j=k+1}^{n-1} \sin^{\mu_j-1}\phi_j\,\rho^{\nu+n-2}\,d\rho\,d\phi \\
&\quad \le K \int_C \left(\frac{1}{a}|u| + |\operatorname{grad} u|\right) \prod_{j=1}^{n-1} \sin^{\mu_j}\phi_j\,\rho^{\nu+n-1}\,d\rho\,d\phi.
\end{aligned}
$$

This completes the induction establishing (35) and hence the lemma. ∎

The following lemma provides a weighted imbedding inequality for the L^q-norm of a function defined on a conical domain of the type (33) in terms of the $W^{m,p}$-norm, both norms being weighted with a power of distance r_k from a coordinate $(n-k)$-plane. It provides the core of the proof of Theorem 4.51.

4.62 LEMMA Let C be as specified by (33) and let $p \ge 1$ and $1 \le k \le n$. Suppose that $\max\{1-k, p-n\} < \nu_1 < \nu_2 < \infty$. Then there exists a constant

$K = K(k, n, p, \nu_1, \nu_2, \beta_1, \dots, \beta_{n-1})$, independent of a, such that for every ν satisfying $\nu_1 \le \nu \le \nu_2$ and every function $u \in C^1(\mathcal{C}) \cap C(\overline{\mathcal{C}})$ we have

$$\left(\int_{\mathcal{C}} |u(x)|^q [r_k(x)]^\nu \, dx \right)^{1/q} \le K \left(\int_{\mathcal{C}} \left(\frac{1}{a^p} |u(x)|^p + |\operatorname{grad} u(x)|^p \right) [r_k(x)]^\nu \, dx \right)^{1/p}, \tag{39}$$

where $q = (\nu + n)p/(\nu + n - p)$.

Proof. Let $\delta = (\nu + n - 1)p/(\nu + n - p)$, let $s = (\nu + n - 1)/\nu$, and let $s' = (\nu + n - 1)/(n - 1)$. We have by Hölder's inequality and Lemma 4.61 (the case $m = k$)

$$\begin{aligned} \int_{\mathcal{C}} |u(x)|^q [r_k(x)]^\nu \, dx &\le \left(\int_{\mathcal{C}} |u|^\delta r_k^{\nu-1} \, dx \right)^{1/s} \left(\int_{\mathcal{C}} |u|^{n\delta/(n-1)} r_k^{n\nu/(n-1)} \, dx \right)^{1/s'} \\ &\le K_1 \left(\int_{\mathcal{C}} |u|^{\delta-1} \left(\frac{1}{a}|u| + |\operatorname{grad} u| \right) r_k^\nu \, dx \right)^{1/s} \\ &\quad \times \left(\int_{\mathcal{C}} |u|^{n\delta/(n-1)} r_k^{n\nu/(n-1)} \, dx \right)^{1/s'} . \end{aligned} \tag{40}$$

In order to estimate the last integral above we adopt the notation

$$\rho^* = (\phi_1, \dots, \phi_{n-1}), \qquad \phi_j^* = (\rho, \phi_1, \dots, \hat{\phi}_j, \phi_{j+1}, \dots, \phi_{n-1}), \quad 1 \le j \le n-1,$$

where the caret denotes omission of a component. Let

$$\begin{aligned} \mathcal{C}_0^* &= \{\rho^* : (\rho, \rho^*) \in \mathcal{C} \text{ for } 0 < \rho < a\} \\ \mathcal{C}_j^* &= \{\phi_j^* : (\rho, \phi) \in \mathcal{C} \text{ for } 0 < \phi_j < \beta_j\}. \end{aligned}$$

$\mathcal{C}_0^*$ and $\mathcal{C}_j^*$, $(1 \le j \le n - 1)$, are domains in $\mathbb{R}^{n-1}$. We define functions F_0 on $\mathcal{C}_0^*$ and F_j on $\mathcal{C}_j^*$ as follows:

$$\begin{aligned} \left(F_0(\rho^*)\right)^{n-1} &= \sup_{0<\rho<a} \left(|u|^\delta \rho^{\nu+n-1}\right) \prod_{i=k}^{n-1} \sin^\nu \phi_i \prod_{i=2}^{n-1} \sin^{i-1} \phi_i, \\ \left(F_j(\phi_j^*)\right)^{n-1} &= \left(\sup_{0<\phi_j<\beta_j} \left(|u|^\delta \sin^{\nu+n-1} \phi_j\right) \right) \rho^{\nu+n-2} \\ &\quad \times \prod_{i=k}^{n-1} \sin^\nu \phi_i \prod_{i=2}^{j-1} \sin^{i-1} \phi_i \prod_{i=j+1}^{n-1} \sin^{i-2} \phi_i . \end{aligned}$$

Then we have

$$|u|^{n\delta/(n-1)} r_k^{n\nu/(n-1)} \rho^{n-1} \prod_{i=2}^{n-1} \sin^{i-1} \phi_i \le F_0(\rho^*) \prod_{j=1}^{n-1} F_j(\phi_j^*).$$

Applying the combinatorial lemma 4.23 with $k = n - 1 = \lambda$ we obtain

$$\begin{aligned} &\int_C |u|^{n\delta/(n-1)} r_k^{n\nu/(n-1)}\, dx \\ &\quad \le \int_C F_0(\rho^*) \prod_{j=1}^{n-1} F_j(\phi_j^*)\, d\rho\, d\phi \\ &\quad \le \left(\int_{C_0^*} \big(F_0(\rho^*)\big)^{n-1} d\phi \prod_{j=1}^{n-1} \int_{C_j^*} \big(F_j(\phi_j^*)\big)^{n-1} d\rho\, d\hat{\phi}_j \right)^{1/(n-1)}. \end{aligned} \tag{41}$$

Now by Lemma 4.59, and since $|\partial u/\partial\rho| \le |\operatorname{grad} u|$,

$$\sup_{0<\rho<a} |u|^\delta \rho^{\nu+n-1} \le K_2 \int_0^a |u|^{\delta-1} \left(\frac{1}{a}|u| + |\operatorname{grad} u| \right) \rho^{\nu+n-1}\, d\rho,$$

where K_2 is independent of ν for $1 - n < \nu_1 \le \nu \le \nu_2 < \infty$. It follows that

$$\int_{C_0^*} \big(F_0(\rho^*)\big)^{n-1} d\phi \le K_2 \int_C |u|^{\delta-1} \left(\frac{1}{a}|u| + |\operatorname{grad} u| \right) r_k^\nu\, dx. \tag{42}$$

Similarly, by making use of (36) or (38) as in Lemma 4.61, we obtain from Lemma 4.59

$$\begin{aligned} &\sup_{0<\phi_j<\beta_j} |u|^\delta \sin^{\nu+j-1} \phi_j \\ &\quad \le K_{2,j} \int_0^{\beta_j} |u|^{\delta-1} \left(|u| + \left| \frac{\partial u}{\partial \phi_j} \right| \right) \sin^{\nu+j-1} \phi_j\, d\phi_j \\ &\quad \le K_{2,j} \int_0^{\beta_j} |u|^{\delta-1} \left(|u| + |\operatorname{grad} u| \rho \prod_{i=j+1}^{n-1} \sin \phi_i \right) \sin^{\nu+j-1} \phi_j\, d\phi_j, \end{aligned}$$

since $|\partial u/\phi_j| \le \rho \prod_{i=j+1}^{n-1} \sin \phi_i$. Hence

$$\begin{aligned} &\int_{C_j^*} \big(F_j(\phi_j^*)\big)^{n-1} d\rho\, d\hat{\phi}_j \\ &\quad \le K_{2,j} \int_C |\operatorname{grad} u| |u|^{\delta-1} r_k^\nu\, dx + K_{2,j} \int_C |u|^\delta\, r_k^\nu\, r_{j+1}^{-1}\, dx \\ &\quad \le K_{3,j} \int_C |u|^{\delta-1} \left(\frac{1}{a}|u| + |\operatorname{grad} u| \right) r_k^\nu\, dx, \end{aligned} \tag{43}$$

where we have used Lemma 4.61 again to obtain the last inequality. Note that the constants $K_{2,j}$ and $K_{3,j}$ can be chosen independent of ν for the values of ν allowed. Substitution of (42) and (43) into (41) and then into (40) leads to

$$\begin{aligned}\int_C |u|^q r_k^\nu \, dx &\le K_4 \left(\int_C |u|^{\delta-1} \left(\frac{1}{a}|u| + |\operatorname{grad} u| \right) r_k^\nu \, dx \right)^{1/s+n/((n-1)s')} \\ &\le K_4 \left(\left[\int_C |u|^q r_k^\nu \, dx \right]^{(p-1)/p} \right. \\ &\quad \left. \times \left[2^{p-1} \int_C \left(\frac{1}{a^p}|u|^p + |\operatorname{grad} u|^p \right) r_k^\nu \, dx \right]^{1/p} \right)^{(\nu+n)/(\nu+n-1)} .\end{aligned}$$

Since $(\nu + n - 1)/(\nu + n) - (p - 1)/p = 1/q$, inequality (39) follows by cancellation for, since u is bounded on C and $\nu > 1 - n$, $\int_C |u|^q r_k^\nu \, dx$ is finite. ∎

4.63 REMARKS

1. The assumption that $u \in C(\overline{C})$ was made only to ensure that the above cancellation was justified. In fact, the lemma holds for any $u \in C^1(C)$.
2. If $1 - k < \nu_1 < \nu_2 < \infty$ and $\nu_1 \le \nu \le \nu_2$, where $p \ge \nu + n$, then (39) holds for any q satisfying $1 \le q < \infty$. It is sufficient to prove this for large q. If $q \ge (\nu + n)/(\nu + n - 1)$, then $q = (\nu + n)s/(\nu + n - s)$ for some s satisfying $1 \le s < p$. Thus

$$\begin{aligned}\left(\int_C |u|^q r_k^\nu \, dx \right)^{s/q} &\le K \int_C \left(\frac{1}{a^s}|u|^s + |\operatorname{grad} u|^s \right) r_k^\nu \, dx \\ &\le K \left(2^{(p-2)/s} \int_C \left(\frac{1}{a^p}|u|^p + |\operatorname{grad} u|^p \right) r_k^\nu \, dx \right)^{s/p} \left(\int_C r_k^\nu \, dx \right)^{(p-s)/p},\end{aligned}$$

   which yields (39) since the last factor is finite.
3. If $\nu = m$, a positive integer, then (39) can be obtained very simply as follows. Let $y = (x, z) = (x_1, \ldots, x_n, z_1, \ldots, z_m)$ denote a point in $\mathbb{R}^{n+m}$ and define $u^*(y) = u(x)$ for $x \in C$. If

$$C^* = \left\{ y \in \mathbb{R}^{n+m} \, : \, y = (x, z), \, x \in C, \, 0 < z_j < r_k(x), \, 1 \le j \le m \right\},$$

   then C^* satisfies the cone condition in $\mathbb{R}^{n+m}$, whence by Theorem 4.12 we have, putting $q = (n + m)p/(n + m - p)$,

$$\begin{aligned}\left(\int_C |u|^q r_k^m \, dx \right)^{1/q} &= \left(\int_{C^*} |u^*(y)|^q \, dy \right)^{1/q} \\ &\le K \left(\int_{C^*} \left(\frac{1}{a^p}|u^*(y)|^p + |\operatorname{grad} u^*(y)|^p \right) dy \right)^{1/p} \\ &= K \left(\int_C \left(\frac{1}{a^p}|u|^p + |\operatorname{grad} u|^p \right) r_k^m \, dx \right)^{1/p}\end{aligned}$$

since $|\mathrm{grad}\, u^*(y)| = |\mathrm{grad}\, u(x)|$, u^* being independent of z.

4. Suppose that $u \in C_0^\infty(\mathbb{R}^n)$, or, more generally, that

$$\int_{\mathbb{R}^n} |u(x)|^p \, [r_k(x)]^\nu \, dx < \infty$$

with ν as in the above lemma. If we take $\beta_i = \pi$, $1 \le i \le n-1$, and let $a \to \infty$ in (39), we obtain

$$\left(\int_{\mathbb{R}^n} |u(x)|^q \, [r_k(x)]^\nu \, dx \right)^{1/q} \le K \left(\int_{\mathbb{R}^n} |\mathrm{grad}\, u(x)|^p \, [r_k(x)]^\nu \, dx \right)^{1/p}.$$

This generalizes (the case $m = 1$ of) Sobolev's inequality, Theorem 4.31.

As final preparations for the proofs of Theorems 4.51–4.53 we need to obtain weighted analogs of the L^∞ and Hölder imbedding inequalities provided by Theorem 4.12. It is convenient here to deal with arbitrary domains satisfying the cone condition rather than the special case C considered in the lemmas above. The following elementary result will be needed.

4.64 LEMMA Let $z \in \mathbb{R}^k$ and let Ω be a domain of finite volume in $\mathbb{R}^k$. If $0 \le \nu < k$, then

$$\int_\Omega |x - z|^{-\nu} \, dx \le \frac{K}{k - \nu} (\mathrm{vol}(\Omega))^{1-\nu/k},$$

where the constant K depends on ν and k, but not on z or Ω.

Proof. Let B be the ball in $\mathbb{R}^k$ having centre z and the same volume as Ω. It is easily seen that the left side of the above inequality does not exceed $\int_B |x - z|^{-\nu} \, dx$, and that the inequality holds for $\Omega = B$. ∎

4.65 LEMMA Let $\Omega \subset \mathbb{R}^n$ satisfy the cone condition. Let $1 \le k \le n$ and let P be an $(n-k)$-dimensional plane in $\mathbb{R}^n$. Denote by $r(x)$ the distance from x to P. If $0 \le \nu < p - n$, then for all $u \in C^1(\Omega)$ we have

$$\sup_{x \in \Omega} |u(x)| \le K \left(\int_\Omega \left(|u(x)|^p + |\mathrm{grad}\, u|^p \right) [r(x)]^\nu \, dx \right)^{1/p}, \tag{44}$$

where the constant K may depend on ν, n, p, k, and the cone C determining the cone condition for Ω, but not on u.

Proof. Throughout this proof A_i and K_i will denote various constants depending on one or more of the parameters on which K is allowed to depend above. It is sufficient to prove that if C is a finite cone contained in Ω having vertex at, say, the origin, then

$$|u(0)| \le K\left(\int_C \left(|u(x)|^p + |\operatorname{grad} u|^p\right)[r(x)]^\nu\, dx\right)^{1/p}. \tag{45}$$

For $0 \le j \le n$, let A_j denote the supremum of the Lebesgue j-dimensional measure of the projection of C onto $\mathbb{R}^j$, taken over all j-dimensional subspaces $\mathbb{R}^j$ of $\mathbb{R}^n$. Writing $x = (x', x'')$ where $x' = (x_1, \dots, x_{n-k})$ and $x'' = (x_{n-k+1}, \dots, x_n)$, we may assume, without loss of generality, that P is orthogonal to the coordinate axes corresponding to the components of x''. Define

$$S = \{x' \in \mathbb{R}^{n-k} : (x', x'') \in C \text{ for some } x'' \in \mathbb{R}^k\},$$
$$R(x') = \{x'' \in \mathbb{R}^k : (x', x'') \in C\} \quad \text{for each } x' \in S.$$

For $0 \le t \le 1$ we denote by C_t the cone $\{tx : x \in C\}$ so that $C_t \subset C$ and $C_t = C$ if $t = 1$. For C_t we define the quantities $A_{t,j}$, S_t, and $R_t(x')$ analogously to the similar quantities defined for C. Clearly $A_{t,j} = t^j A_j$. If $x \in C$, we have

$$u(x) = u(0) + \int_0^1 \frac{d}{dt} u(tx)\, dt,$$

so that

$$|u(0)| \le |u(x)| + |x| \int_0^1 |\operatorname{grad} u(tx)|\, dt.$$

Setting $V = \operatorname{vol}(C)$ and $a = \sup_{x\in C} |x|$, and integrating the above inequality over C, we obtain

$$\begin{aligned} V|u(0)| &\le \int_C |u(x)|\, dx + a \int_C \int_0^1 |\operatorname{grad} u(tx)|\, dt\, dx \\ &= \int_C |u(x)|\, dx + a \int_0^1 t^{-n}\, dt \int_{C_t} |\operatorname{grad} u(x)|\, dx. \end{aligned} \tag{46}$$

Let z denote the orthogonal projection of x onto P. Then $r(x) = |x'' - z''|$. Since $0 \le \nu < p - n$, we have $p > 1$, and so by the previous lemma

$$\begin{aligned} \int_{C_t} [r(x)]^{-\nu/(p-1)}\, dx &= \int_{S_t} dx' \int_{R_t(x')} |x'' - z''|^{-\nu/(p-1)}\, dx'' \\ &\le K_1 \int_{S_t} [A_{t,k}]^{1-\nu/(k(p-1))}\, dx' \\ &\le K_1 [A_{t,k}]^{1-\nu/(k(p-1))} [A_{t,n-k}] = K_2 t^{n-\nu/(p-1)}. \end{aligned}$$

It follows that

$$\begin{aligned}
&\int_{C_t} |\operatorname{grad} u(x)|\,dx \\
&\quad \le \left(\int_{C_t} |\operatorname{grad} u(x)|^p [r(x)]^\nu\,dx\right)^{1/p} \left(\int_{C_t} [r(x)]^{-\nu/(p-1)}\,dx\right)^{1/p'} \\
&\quad \le K_3 t^{n-(\nu+n)/p} \left(\int_{C_t} |\operatorname{grad} u(x)|^p [r(x)]^\nu\,dx\right)^{1/p}. \qquad (47)
\end{aligned}$$

Hence, since $\nu < p - n$,

$$\int_0^1 t^{-n}\,dt \int_{C_t} |\operatorname{grad} u(x)|\,dx \le K_4 \left(\int_C |\operatorname{grad} u(x)|^p [r(x)]^\nu\,dx\right)^{1/p}. \qquad (48)$$

Similarly,

$$\begin{aligned}
\int_C |u(x)|\,dx &\le \left(\int_C |u(x)|^p [r(x)]^\nu\,dx\right)^{1/p} \left(\int_C [r(x)]^{-\nu/(p-1)}\,dx\right)^{1/p'} \\
&\le K_5 \left(\int_C |u(x)|^p [r(x)]^\nu\,dx\right)^{1/p}. \qquad (49)
\end{aligned}$$

Inequality (45) now follows from (46), (48), and (49). ∎

4.66 LEMMA Suppose all the conditions of the previous lemma are satisfied and, in addition, Ω satisfies the strong local Lipschitz condition. Then for all $u \in C^1(\Omega)$ we have

$$\sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\mu} \le K \left(\int_\Omega \big(|u(x)|^p + |\operatorname{grad} u(x)|^p\big)[r(x)]^\nu\,dx\right)^{1/p}, \qquad (50)$$

where $\mu = 1 - (\nu + n)/p$ satisfies $0 < \mu < 1$, and K is independent of u.

Proof. The proof is the same as that given for inequality (15) in Lemma 4.28 except that the inequality

$$\int_{\Omega_{t\sigma}} |\operatorname{grad} u(z)|\,dz \le K_1 t^{n-(\nu+n)/p} \big(|\operatorname{grad} u(z)|^p [r(z)]^\nu\,dz\big)^{1/p} \qquad (51)$$

is used in (16) in place of the special case $\nu = 0$ actually used there. Inequality (51) is obtained in the same way as (47) above. ∎

Proofs of Theorems 4.51–4.53

4.67 LEMMA Let $\bar{\nu} \geq 0$. If $\bar{\nu} > p - n$, let $1 \leq q \leq (\bar{\nu} + n)/(\bar{\nu} + n - p)$; otherwise, let $1 \leq q < \infty$. There exists a constant $K = K(n, p, \bar{\nu})$ such that for every standard cusp $Q_{k,\lambda}$ (see Paragraph 4.50) for which $(\lambda - 1)k \equiv \nu \leq \bar{\nu}$, and every $u \in C^1(Q_{k,\lambda})$, we have

$$\|u\|_{0,q,Q_{k,\lambda}} \leq K \, \|u\|_{1,p,Q_{k,\lambda}} . \tag{52}$$

Proof. Since each $Q_{k,\lambda}$ has the segment property, it suffices to prove (52) for $u \in C^1(\overline{Q_{k,\lambda}})$. We first do so for given k and λ and then show that K may be chosen to be independent of these parameters.

First suppose $\bar{\nu} > p - n$. It suffices to prove (52) for

$$q = (\bar{\nu} + n)/(\bar{\nu} + n - p).$$

For $u \in C^1(\overline{Q_{k,\lambda}})$ define $\tilde{u}(y) = u(x)$, where y is related to x by (27) and (28). Thus $\tilde{u} \in C^1(C_k) \cap C(\overline{C_k})$, where C_k is the standard cone associated with $Q_{k,\lambda}$. By Lemma 4.62, and since $q \leq (\nu + n)p/(\nu + n - p)$, we have

$$\begin{aligned} \|u\|_{0,q,Q_{k,\lambda}} &= \left(\lambda \int_{C_k} |\tilde{u}(y)|^q [r_k(y)]^\nu \, dy\right)^{1/q} \\ &\leq K_1 \left(\int_{C_k} \left(|\tilde{u}(y)|^p + |\operatorname{grad} \tilde{u}(y)|^p\right)[r_k(y)]^\nu \, dy\right)^{1/q} . \end{aligned} \tag{53}$$

Now $x_j = r_k^{\lambda-1} y_j$ if $1 \leq j \leq k$ and $x_j = y_j$ if $k + 1 \leq j \leq n$. Since $r_k^2 = y_1^2 + \cdots + y_k^2$ we have

$$\frac{\partial x_j}{\partial y_i} = \begin{cases} \delta_{ij} r_k^{\lambda-1} + (\lambda - 1) r_k^{\lambda-3} y_i y_j & \text{if } 1 \leq i, j \leq k \\ \delta_{ij} & \text{otherwise,} \end{cases}$$

where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$. Since $r_k(y) \leq 1$ on C_k it follows that

$$|\operatorname{grad} \tilde{u}(y)| \leq K_2 |\operatorname{grad} u(x)|.$$

Hence (52) follows from (53) in this case. For $\bar{\nu} \leq p - n$ and arbitrary q the proof is similar, being based on Remark 2 of Paragraph 4.63.

In order to show that the constant K in (52) can be chosen independent of k and λ provided $\nu = (\lambda - 1)k \leq \bar{\nu}$, we note that it is sufficient to prove that there is a constant $\tilde{K}$ such that for any such k, λ and all $v \in C^1(C_k) \cap C(\overline{C_k})$ we have

$$\begin{aligned} &\left(\int_{C_k} |v(y)|^q [r_k(y)]^\nu \, dy\right)^{1/q} \\ &\qquad \leq \tilde{K} \left(\int_{C_k} \left(|v(y)|^p + |\operatorname{grad} v(y)|^p\right)[r_k(y)]^\nu \, dy\right)^{1/p} . \end{aligned} \tag{54}$$

In fact, it is sufficient to establish (54) with $\tilde{K}$ depending on k as we can then use the maximum of $\tilde{K}(k)$ over the finitely many values of k allowed. We distinguish three cases.

Case I $\bar{\nu} < p - n$, $1 \le q < \infty$. By Lemma 4.65 we have for $0 \le \nu \le \bar{\nu}$,

$$\sup_{y \in C_k} |v(y)| \le K(\nu) \left(\int_{C_k} \left(|v(y)|^p + |\operatorname{grad} v(y)|^p \right) [r_k(y)]^\nu \, dy \right)^{1/p}. \tag{55}$$

Since the integral on the right decreases as ν increases, we have $K(\nu) \le K(\bar{\nu})$ and (54) now follows from (55) and the boundedness of C_k.

Case II $\bar{\nu} > p - n$. Again it is sufficient to deal with $q = (\bar{\nu}+n)p/(\bar{\nu}+n-p)$. From Lemma 4.62 we obtain

$$\left(\int_{C_k} |v|^s r_k^\nu \, dy \right)^{1/s} \le K_1 \left(\int_{C_k} \left(|v|^p + |\operatorname{grad} v|^p \right) r_k^\nu \, dy \right)^{1/p}, \tag{56}$$

where $s = (\nu+n)p/(\nu+n-p) \ge q$ and K_1 is independent of ν for $p-n < \nu_0 \le \bar{\nu}$. By Hölder's inequality, and since $r_k(y) \le 1$ on C_k, we have

$$\left(\int_{C_k} |v|^q r_k^\nu \, dy \right)^{1/q} \le \left(\int_{C_k} |v|^s r_k^\nu \, dy \right)^{1/s} \left(\operatorname{vol}(C_k) \right)^{(s-q)/sq}$$

so that if $\nu_0 \le \nu \le \bar{\nu}$, then (54) follows from (56).

If $p - n < 0$, we can take $\nu_0 = 0$ and be done. Otherwise, $p \ge n \ge 2$. Fixing $\nu_0 = (\bar{\nu} - n + p)/2$, we can find ν_1 such that $0 \le \nu_1 \le p - n$ (or $\nu_1 = 0$ if $p = n$) such that for $\nu_1 \le \nu \le \nu_0$ we have

$$1 \le t = \frac{(\nu+n)(\bar{\nu}+n)p}{(\nu+n)(\bar{\nu}+n) + (\bar{\nu}-\nu)p} \le \frac{p}{1+\epsilon_0},$$

where $\epsilon_0 > 0$ and depends only on $\bar{\nu}$, n, and p. Because of the latter inequality we may also assume $t - n < \nu_1$. Since $(\nu+n)t/(\nu+n-t) = q$ we have, again by Lemma 4.62 and Hölder's inequality,

$$\begin{aligned} \left(\int_{C_k} |v|^q r_k^\nu \, dy \right)^{1/q} &\le K_2 \left(\int_{C_k} \left(|v|^t + |\operatorname{grad} v|^t \right) r_k^\nu \, dy \right)^{1/t} \\ &\quad 2^{(p-t)/pt} K_2 \left(\int_{C_k} \left(|v|^p + |\operatorname{grad} v|^p \right) r_k^\nu \, dy \right)^{1/p} \left(\operatorname{vol}(C_k) \right)^{(p-t)/pt}, \end{aligned} \tag{57}$$

where K_2 is independent of ν for $\nu_1 \le \nu \le \nu_0$.

In the case $\nu_1 > 0$ we can obtain a similar (uniform) estimate for $0 \le \nu \le \nu_1$ by the method of Case I. Combining this with (56) and (57), we prove (54) for this case.

Case III $\bar{\nu} = p - n$, $1 \le q < \infty$. Fix $s \ge \max\{q, n/(n-1)\}$ and let $t = (\nu+n)s/(\nu+n+s)$, so $s = (\nu+n)t/(\nu+n-t)$. Then $1 \le t \le ps/(p+s) < p$ for $0 \le \nu \le \bar{\nu}$. Hence we can select $\nu_1 \ge 0$ such that $t - n < \nu_1 < p - n$. The rest of the proof is similar to Case II. This completes the proof of the lemma. ∎

4.68 (Proof of Theorem 4.51) It is sufficient to prove only the special case $m = 1$, for the general case then follows by induction on m. Let q satisfy $p \le q \le (\nu+n)p/(\nu+n-p)$ if $\nu + n > p$, or $p \le q < \infty$ otherwise. Clearly $q < np/(n-p)$ if $n > p$ so in either case we have by Theorem 4.12

$$\|u\|_{0,q,G} \le K_1 \|u\|_{1,p,G}$$

for every $u \in C^1(\Omega)$ and that element G of Γ that satisfies the cone condition (if such a G exists). If $G \in \Gamma$ does not satisfy the cone condition, and if $\Psi : G \to Q_{k,\lambda}$, where $(\lambda - 1)k \le \nu$, is the 1-smooth mapping specified in the statement of the theorem. Then by Theorem 3.41 and Lemma 4.67

$$\|u\|_{0,q,G} \le K_2 \left\|u \circ \Psi^{-1}\right\|_{0,q,Q_{k,\lambda}} \le K_3 \left\|u \circ \Psi^{-1}\right\|_{1,p,Q_{k,\lambda}} \le K_4 \|u\|_{1,p,G},$$

where K_4 is independent of G. Thus, since $q/p \ge 1$,

$$\begin{aligned}\|u\|_{0,q,\Omega}^q &\le \sum_{G\in\Gamma} \|u\|_{0,q,G}^q \le K_5 \sum_{G\in\Gamma} \left(\|u\|_{1,p,G}^p\right)^{q/p} \\ &\le K_5 \left(\sum_{G\in\Gamma} \|u\|_{1,p,G}^p\right)^{q/p} \le K_5 N^{q/p} \|u\|_{1,p,\Omega}^q,\end{aligned}$$

where we have used the finite intersection property of Γ to obtain the final inequality. The required imbedding inequality now follows by completion.

If $\nu < mp - n$, we require that $W^{m,p}(\Omega) \to L^q(\Omega)$ also holds for $q = \infty$. This is a consequence of Theorem 4.52 proved below. ∎

4.69 LEMMA Let $0 \le \bar{\nu} < mp - n$. Then there exists a constant $K = K(m, p, n, \bar{\nu})$ such that if $Q_{k,\lambda}$ is any standard cusp domain for which $(\lambda - 1)k = \nu \le \bar{\nu}$ and if $u \in C^m(Q_{k,\lambda})$, then

$$\sup_{x\in Q_{k,\lambda}} |u(x)| \le K \|u\|_{m,p,Q_{k,\lambda}}. \tag{58}$$

Proof. Again it is sufficient to prove the lemma for the case $m = 1$. If u belongs to $C^1(Q_{k,\lambda})$ where $(\lambda - 1)k = \nu \le \bar{\nu}$, then we have by Lemma 4.65 and via the method of the second paragraph of the proof of Lemma 4.67,

$$
\begin{aligned}
\sup_{x\in Q_{k,\lambda}} |u(x)| &= \sup_{y\in C_k} |\tilde{u}(y)| \\
&\le K_1 \left(\int_{C_k} \left(|\tilde{u}(y)|^p + |\operatorname{grad} \tilde{u}(y)|^p \right) [r_k(y)]^\nu \, dy \right)^{1/p} \\
&\le K_2 \left(\int_{Q_{k,\lambda}} \left(|u(x)|^p + |\operatorname{grad} u(x)|^p \right) dx \right)^{1/p} . \qquad (59)
\end{aligned}
$$

Since $r_k(y) \le 1$ for $y \in C_k$ it is evident that K_1, and hence K_2, can be chosen independent of k and λ provided $0 \le \nu = (\lambda - 1)k \le \bar{\nu}$. ∎

4.70 (Proof of Theorem 4.52) It is sufficient to prove that

$$W^{m,p}(\Omega) \to C_B^0(\Omega).$$

Let $u \in C^\infty(\Omega)$. If $x \in \Omega$, then $x \in G \subset \Omega$ for some domain G for which there exists a 1-smooth transformation $\Psi : G \to Q_{k,\lambda}$, $(\lambda - 1)k \le \nu$, as specified in the statement of the theorem. Thus

$$
\begin{aligned}
|u(x)| \le \sup_{x\in G} |u(x)| &= \sup_{y\in Q_{k,\lambda}} |u \circ \Psi^{-1}(y)| \\
&\le K_1 \left\| u \circ \Psi^{-1} \right\|_{m,p,Q_{k,\lambda}} \le K_2 \|u\|_{m,p,G} \\
&\le K_2 \|u\|_{m,p,\Omega} , \qquad (60)
\end{aligned}
$$

where K_1 and K_2 are independent of G. The rest of the proof is similar to the second paragraph of the proof in Paragraph 4.16. ∎

4.71 (Proof of Theorem 4.53) As in Lemma 4.28 it is sufficient to prove that

$$W^{1,p}(\Omega) \to C^{0,\mu}(\overline{\Omega}) \qquad \text{if} \quad 0 < \mu \le 1 - \frac{n+\nu}{p},$$

that is, that

$$\sup_{\substack{x,y\in\Omega \\ x\ne y}} \frac{|u(x) - u(y)|}{|x-y|^\mu} \le K \|u\|_{1,p,\Omega} \qquad (61)$$

holds when $\nu + n < p$ and $0 < \mu \le 1 - (\nu + n)/p$. For $x, y \in \Omega$ satisfying $|x - y| \ge \delta$, (61) holds by virtue of (60). If $|x - y| < \delta$, then there exists $G \subset \Omega$ with $x, y \in G$, and a 1-smooth transformation Ψ from G onto a standard cusp $Q_{k,\lambda}$ with $(\lambda - 1)k \le \nu$, satisfying the conditions of the theorem. Inequality (61) can then be derived from Lemma 4.66 by the same method used in the proof of Lemma 4.69. The details are left to the reader. ∎

5

INTERPOLATION, EXTENSION, AND APPROXIMATION THEOREMS

Interpolation on Order of Smoothness

5.1 We consider the problem of determining upper bounds for L^p norms of derivatives $D^\beta u$, $0 < |\beta| < m$, of functions in $W^{m,p}(\Omega)$ in terms of the L^p norms of u and its partial derivatives of order m. Such estimates are conveniently expressed in terms of the seminorms $|\cdot|_{j,p}$ defined in Paragraph 4.29. Theorem 5.2 below provides such an estimate for the seminorm $|u|_{j,p}$ in terms of $|u|_{m,p}$ and $\|u\|_p$, as well as some elementary consequences of this estimate. Such estimates arose in the work of of Ehrling [E], Nirenberg [Nr1, Nr2], Gagliardo [Ga1, Ga2], and Browder [Br1, Br2], and were frequently proved under the assumption that Ω satisfies the uniform cone condition, at least if Ω is unbounded. However, we will prove Theorem 5.2 assuming only the cone condition. In fact, even the weak cone condition is sufficient for the proof, as is shown in [AF1].

5.2 THEOREM Let Ω be a domain in $\mathbb{R}^n$ satisfying the cone condition. For each $\epsilon_0 > 0$ there exist finite constants K and K', each depending on n, m, p, ϵ_0 and the dimensions of the cone C providing the cone condition for Ω such that if $0 < \epsilon \le \epsilon_0$, $0 \le j \le m$, and $u \in W^{m,p}(\Omega)$, then

$$|u|_{j,p} \le K\big(\epsilon\,|u|_{m,p} + \epsilon^{-j/(m-j)}\,\|u\|_p\big), \tag{1}$$

$$\|u\|_{j,p} \le K'\big(\epsilon\,\|u\|_{m,p} + \epsilon^{-j/(m-j)}\,\|u\|_p\big), \tag{2}$$

$$\|u\|_{j,p} \le 2K'\,\|u\|_{m,p}^{j/m}\,\|u\|_p^{(m-j)/m}. \tag{3}$$

5.3 Inequality (2) follows from repeated applications of (1), and (3) by setting $\epsilon_0 = 1$ in (2) and choosing ϵ in (2) so that the two terms on the right side are equal. Furthermore, (1) holds when $\epsilon < \epsilon_0$ if it holds for $\epsilon < \epsilon_1$ for any specific positive ϵ_1; to see this just replace ϵ by $\epsilon\epsilon_1/\epsilon_0$ and suitably adjust K. Thus we need only prove (1), and that for just one value of ϵ_0.

We carry out the proof in three lemmas. The first develops a one-dimensional version for the case $m = 2$, $j = 1$. The second establishes (1) for $m = 2$, $j = 1$ for general Ω satisfying the cone condition. The third shows that (1) is valid for general $m \geq 2$ and $1 \leq j \leq m - 1$ whenever the case $m = 2$, $j = 1$ is known to hold.

5.4 LEMMA If $\rho > 0$, $1 \leq p < \infty$, $K_p = 2^{p-1}9^p$, and $g \in C^2([0, \rho])$, then

$$|g'(0)|^p \leq \frac{K_p}{\rho}\left(\rho^p \int_0^\rho |g''(t)|^p \, dt + \rho^{-p} \int_0^\rho |g(t)|^p \, dt\right). \tag{4}$$

Proof. Let $f \in C^2([0, 1])$, let $x \in [0, 1/3]$, and let $y \in [2/3, 1]$. By the mean-value theorem there exists $z \in (x, y)$ such that

$$|f'(z)| = \left|\frac{f(y) - f(x)}{y - x}\right| \leq 3|f(x)| + 3|f(y)|.$$

Thus

$$\begin{aligned}|f'(0)| &= \left|f'(z) - \int_0^z f''(t)\, dt\right| \\ &\leq 3|f(x)| + 3|f(y)| + \int_0^1 |f''(t)|\, dt.\end{aligned}$$

Integration of x over $[0, 1/3]$ and y over $[2/3, 1]$ yields

$$\frac{1}{9}|f'(0)| \leq \int_0^{1/3} |f(x)|\, dx + \int_{2/3}^1 |f(y)|\, dy + \frac{1}{9}\int_0^1 |f''(t)|\, dt.$$

For $p \geq 1$ we therefore have (using Hölder's inequality if $p > 1$)

$$|f'(0)|^p \leq K_p\left(\int_0^1 |f''(t)|^p \, dt + \int_0^1 |f(t)|^p \, dt\right).$$

where $K_p = 2^{p-1}9^p$.

Inequality (4) now follows by substituting $f(t) = g(\rho t)$. ∎

5.5 LEMMA If $1 \leq p < \infty$ and the domain $\Omega \subset \mathbb{R}^n$ satisfies the cone condition, then there exists a constant K depending on n, p, and the height ρ_0 and aperture angle κ of the cone C providing the cone condition for Ω such that for all ϵ, $0 < \epsilon \le \rho_0$ and all $u \in W^{2,p}(\Omega)$ we have

$$|u|_{1,p} \le K\big(\epsilon\, |u|_{2,p} + \epsilon^{-1}\, \|u\|_p\big). \tag{5}$$

Proof. Let $\Sigma = \{\sigma \in \mathbb{R}^n : |\sigma| = 1\}$ be the unit sphere in $\mathbb{R}^n$ with volume element $d\sigma$ and $(n-1)$-volume $K_0 = K_0(n) = \int_\Sigma d\sigma$. If $x \in \Omega$ let σ_x be the unit vector in the direction of the axis of a cone $C_x \subset \Omega$ congruent to C and having vertex at x, and let $\Sigma_x = \{\sigma \in \Sigma : \angle(\sigma, \sigma_x) \le \kappa/2\}$.

Let $u \in C^\infty(\Omega)$. If $x \in \Omega$, $\sigma \in \Sigma_x$, and $0 < \rho \le \rho_0$, then

$$|\sigma \cdot \operatorname{grad} u(x)|^p \le \frac{K_p}{\rho}\, I(\rho, p, u, x, \sigma),$$

where

$$I(\rho, p, u, x, \sigma) = \rho^p \int_0^\rho |D_t^2 u(x + t\sigma)|^p\, dt + \rho^{-p} \int_0^\rho |u(x + t\sigma)|^p\, dt.$$

There exists a constant $K_1 = K_1(n, p, \kappa)$ such that

$$\int_\Sigma |\sigma \cdot \operatorname{grad} u(x)|^p\, d\sigma \ge \int_{\Sigma_x} |\sigma \cdot \operatorname{grad} u(x)|^p\, d\sigma \ge K_1 |\operatorname{grad} u(x)|^p.$$

Accordingly,

$$\int_\Omega |\operatorname{grad} u(x)|^p\, dx \le \frac{K_p}{K_1 \rho} \int_\Sigma d\sigma \int_\Omega I(\rho, p, u, x, \sigma)\, dx.$$

In order to estimate the inner integral on the right, regard u and its derivatives as extended to all of $\mathbb{R}^n$ so as to be identically zero outside Ω. For simplicity, we suppose $\sigma = e_n = (0, \ldots, 0, 1)$ and write $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$. We have

$$\begin{aligned}
&\int_\Omega I(\rho, p, u, x, e_n)\, dx \\
&\quad = \int_{\mathbb{R}^{n-1}} dx' \int_{-\infty}^{\infty} dx_n \int_0^\rho \big(\rho^p |D_n^2 u(x', x_n + t)|^p + \rho^{-p} |u(x', x_n + t)|^p\big)\, dt \\
&\quad = \int_{\mathbb{R}^{n-1}} dx' \int_0^\rho dt \int_{-\infty}^{\infty} \big(\rho^p |D_n^2 u(x)|^p + \rho^{-p} |u(x)|^p\big)\, dx_n \\
&\quad \le \rho \int_\Omega \big(\rho^p |D_n^2 u(x)|^p + \rho^{-p} |u(x)|^p\big)\, dx,
\end{aligned}$$

In general, for $\sigma \in \Sigma$

$$\int_\Omega I(\rho, p, u, x, \sigma)\, dx \le \rho \int_\Omega \big(\rho^p\, |u|_{2,p}^p + \rho^{-p}\, \|u\|_p^p\big)\, dx,$$

and since $|D_j(u)| \le |\mathrm{grad}\, u|$ and the measure of Σ is K_0,

$$|u|_{1,p}^p \le \frac{nK_pK_0}{K_1}\left(\rho^p\,|u|_{2,p}^p + \rho^{-p}\,\|u\|_p^p\right).$$

Inequality (5) now follows by taking pth roots, replacing ρ with ϵ, and noting that $C^\infty(\Omega)$ is dense in $W^{2,p}(\Omega)$. ▮

5.6 LEMMA Let $m \ge 2$, let $0 < \delta_0 < \infty$, and let $\epsilon_0 = \min\{\delta_0, \delta_0^2, \dots, \delta_0^{m-1}\}$. Suppose that for given p, $1 \le p < \infty$, and given $\Omega \subset \mathbb{R}^n$ there exists a constant $K = K(\delta_0, p, \Omega)$ such that for every δ satisfying $0 < \delta \le \delta_0$ and every $u \in W^{2,p}(\Omega)$, we have

$$|u|_{1,p} \le K\delta\,|u|_{2,p} + K\delta^{-1}\,|u|_{0,p}\,. \tag{6}$$

Then there exists a constant $K = K(\epsilon_0, m, p, \Omega)$ such that for every ϵ satisfying $0 < \epsilon \le \epsilon_0$, every integer j satisfying $0 \le j \le m-1$, and every $u \in W^{m,p}(\Omega)$, we have

$$|u|_{j,p} \le K\epsilon\,|u|_{m,p} + K\epsilon^{-j/(m-j)}\,|u|_{0,p}\,. \tag{7}$$

Proof. Since (7) is trivial for $j = 0$, we consider only the case $1 \le j \le m-1$. The proof is accomplished by a double induction on m and j. The constants $K_1, K_2, \dots$ appearing in the argument may depend on δ_0 (or ϵ_0), m, p, and Ω. First we prove (7) for $j = m-1$ by induction on m, so that (6) is the special case $m = 2$. Assume, therefore, that for some k, $2 \le k \le m-1$,

$$|u|_{k-1,p} \le K_1\delta\,|u|_{k,p} + K_1\delta^{-(k-1)}\,|u|_{0,p} \tag{8}$$

holds for all δ, $0 < \delta \le \delta_0$, and all $u \in W^{k,p}(\Omega)$. If $u \in W^{k+1,p}(\Omega)$, we prove (8) with $k+1$ replacing k (and a different constant K_1). If $|\alpha| = k-1$ we obtain from (6)

$$|D^\alpha u|_{1,p} \le K_2\delta\,|D^\alpha u|_{2,p} + K_2\delta^{-1}\,|D^\alpha u|_{0,p}\,.$$

Combining this inequality with (8) we obtain, for $0 < \eta \le \delta_0$,

$$\begin{aligned}
|u|_{k,p} &\le K_3 \sum_{|\alpha|=k-1} |D^\alpha u|_{1,p} \\
&\le K_4\delta\,|u|_{k+1,p} + K_4\delta^{-1}\,|u|_{k-1,p} \\
&\le K_4\delta\,|u|_{k+1,p} + K_4K_1\delta^{-1}\eta\,|u|_{k,p} + K_4K_1\delta^{-1}\eta^{1-k}\,|u|_{0,p}\,.
\end{aligned}$$

We may assume without prejudice that $2K_1K_4 \ge 1$. Therefore, we may take $\eta = \delta/(2K_1K_4)$ and so obtain

$$\begin{aligned}
|u|_{k,p} &\le 2K_4\delta\,|u|_{k+1,p} + \left(\delta/(2K_1K_4)\right)^{-k}\,|u|_{0,p} \\
&\le K_5\delta\,|u|_{k+1,p} + K_5\delta^{-k}\,|u|_{0,p}\,.
\end{aligned}$$

This completes the induction establishing (8) for $0 < \delta \le \delta_0$ and hence (7) for $j = m-1$ and $0 < \epsilon \le \delta_0$.

We now prove by downward induction on j that

$$|u|_{j,p} \le K_6 \delta^{m-j} |u|_{m,p} + K_6 \delta^{-j} |u|_{0,p} \tag{9}$$

holds for $1 \le j \le m-1$ and $0 < \delta \le \delta_0$. Note that (8) with $k = m$ is the special case $j = m-1$ of (9). Assume, therefore, that (9) holds for some j, $2 \le j \le m-1$. We prove that it also holds with j replaced by $j-1$ (and a different constant K_6). From (8) and (9) we obtain

$$\begin{aligned}
|u|_{j-1,p} &\le K_7 \delta\, |u|_{j,p} + K_7 \delta^{1-j} |u|_{0,p} \\
&\le K_7 \delta \big(K_6 \delta^{m-j} |u|_{m,p} + K_6 \delta^{-j} |u|_{0,p} \big) + K_7 \delta^{1-j} |u|_{0,p} \\
&\le K_8 \delta^{m-(j-1)} |u|_{m,p} + K_8 \delta^{1-j} |u|_{0,p} \,.
\end{aligned}$$

Thus (9) holds, and (7) follows by setting $\delta = \epsilon^{1/(m-j)}$ in (7) and noting that $\epsilon \le \epsilon_0$ if $\delta \le \delta_0$. ▮

This completes the proof of Theorem 5.2

5.7 REMARK Careful consideration of the proofs of the previous two lemmas shows that if the height of the cone providing the cone condition for Ω is infinite, then inequalities (5) and (7) (and therefore (1) and (2)) hold for all $\epsilon > 0$, the corresponding constants K being independent of ϵ. This is the case, for example, if $\Omega = \mathbb{R}^n$ or a half-space like $\mathbb{R}^n_+$.

Interpolation on Degree of Summability

The following two interpolation theorems provide sharp estimates for L^q norms of functions in $W^{m,p}(\Omega)$. Some of these estimates follow from Theorem 4.12 while others have traditionally been obtained for regular domains from imbeddings of Sobolev spaces of fractional order. (See Chapter 7.) We obtain them here assuming only that the domain satisfies the cone condition. Again, the weak cone condition would do as well; see [AF1].

5.8 THEOREM Let Ω be a domain in $\mathbb{R}^n$ satisfying the cone condition. If $mp > n$, let $p \le q \le \infty$; if $mp = n$, let $p \le q < \infty$; if $mp < n$, let $p \le q \le p^* = np/(n-mp)$. Then there exists a constant K depending on m, n, p, q and the dimensions of the cone C providing the cone condition for Ω, such that for all $u \in W^{m,p}(\Omega)$,

$$\|u\|_q \le K \, \|u\|^{\theta}_{m,p} \, \|u\|^{1-\theta}_p \,, \tag{10}$$

where $\theta = (n/mp) - (n/mq)$.

Proof. The case $mp < n$, $p \le q \le p^*$ follows directly from Theorems 2.11 and 4.12:

$$\|u\|_q \le \|u\|_{p^*}^{\theta} \, \|u\|_p^{1-\theta} \le K \, \|u\|_{m,p}^{\theta} \, \|u\|_p^{1-\theta},$$

where $1/q = (\theta/p^*)+(1-\theta)/p$ from which it follows that $\theta = (n/mp)-(n/mq)$. For the cases $mp = n$, $p \le q < \infty$, and $mp > n$, $p \le q \le \infty$ we use the local bound obtained in Lemma 4.15. If $0 < r \le \rho$ (the height of the cone C), then

$$|u(x)| \le K_1 \left(\sum_{|\alpha| \le m-1} r^{|\alpha|-n} \chi_r * |D^\alpha u|(x) + \sum_{|\alpha|=m} (\chi_r \omega_m) * |D^\alpha u|(x) \right), \tag{11}$$

where χ_r is the characteristic function of the ball of radius r centred at the origin in $\mathbb{R}^n$, and $\omega_m(x) = |x|^{m-n}$. We estimate the L^q norms of both terms on the right side of (11) using Young's inequality from Corollary 2.25. If $(1/p)+(1/s) = 1+(1/q)$, then

$$\begin{aligned} \|\chi_r * |D^\alpha u|\|_q &\le \|\chi_r\|_s \, \|D^\alpha u\|_p = K_2 r^{n-(n/p)+(n/q)} \, \|D^\alpha u\|_p \\ \|(\chi_r \omega_m) * |D^\alpha u|\|_q &\le \|\chi_r \omega_m\|_s \, \|D^\alpha u\|_p = K_3 r^{m-(n/p)+(n/q)} \, \|D^\alpha u\|_p . \end{aligned}$$

(Note that $m - (n/p) + (n/q) > 0$ if q satisfies the above restrictions.) Hence

$$\|u\|_q \le K_4 \left(\sum_{j=0}^{m-1} r^{j-(n/p)+(n/q)} \, |u|_{j,p} + r^{m-(n/p)+(n/q)} \, |u|_{m,p} \right).$$

By Theorem 5.2,

$$|u|_{j,p} \le K_5 \big(r^{m-j} \, |u|_{m,p} + r^{-j} \, \|u\|_p \big),$$

so

$$\|u\|_q \le K_6 \big(r^{m-(n/p)+(n/q)} \, \|u\|_{m,p} + r^{-(n/p)+(n/q)} \, \|u\|_p \big).$$

Adjusting K_6 if necessary, we can assume this inequality holds for all $r \le 1$. Choosing r to make the two terms on the right side equal, we obtain (10). ∎

A special case of the above Theorem asserts that if $mp > n$, then

$$\|u\|_\infty \le K \, \|u\|_{m,p}^{n/mp} \, \|u\|_p^{1-(n/mp)} . \tag{12}$$

A similar inequality with $\|u\|_p$ replaced by a more general $\|u\|_q$ is sometimes useful.

5.9 THEOREM Let Ω be a domain in $\mathbb{R}^n$ satisfying the cone condition. Let $p > 1$ and $mp > n$. Suppose that either $1 \le q \le p$ or both $q > p$ and $mp - p < n$. Then there exists a constant K depending on m, n, p, q and the dimensions of the cone C providing the cone condition for Ω, such that for all $u \in W^{m,p}(\Omega)$,

$$\|u\|_\infty \le K\,\|u\|_{m,p}^{\theta}\,\|u\|_q^{1-\theta},$$

where $\theta = np/[np + (mp - n)q]$.

Proof. It is sufficient to show that the inequality

$$|u(x)| \le K\,\|u\|_{m,p}^{\theta}\,\|u\|_q^{1-\theta}, \qquad \theta = np/[np + (mp - n)q] \tag{13}$$

holds for all $x \in \Omega$ and all $u \in W^{m,p}(\Omega) \cap C^\infty(\Omega)$.

First we observe that (13) is a straightforward consequence of Theorems 5.8 and 2.11 if $1 \le q \le p$; since (12) holds we can substitute

$$\|u\|_p \le \|u\|_q^{q/p}\,\|u\|_\infty^{1-(q/p)}$$

and obtain (13) by cancellation.

Now suppose $q > p$, and, for the moment, that $m = 1$ and $p > n$. We reuse the local bound (11); in this case it says

$$|u(x)| \le K_1\Big(r^{-n}\chi_r * |u|(x) + \sum_{|\alpha|=1} (\chi_r\omega_1) * |D^\alpha u|(x)\Big),$$

for $0 < r \le \rho$, the height of the cone C. By Hölder's inequality,

$$\chi_r * |u|(x) \le K_2 r^{n-(n/q)}\,\|u\|_q,$$

and, for $|\alpha| = 1$,

$$(\chi_r\omega_1) * |D^\alpha u|(x) \le K_3 r^{1-(n/p)}\,\|D^\alpha u\|_p. \tag{14}$$

Since $\|u\|_q \le K_5\,\|u\|_{1,p}$ (by Part I Case A of Theorem 4.12), and since inequality (14) may be assumed to hold for all r such that $0 < r^{1-(n/p)+(n/q)} \le K_5$ provided K_4 is suitably adjusted, we can choose r to make the two upper bounds above equal. This choice yields (13) with $m = 1$.

For general m, we have $W^{m,p}(\Omega) \to W^{1,r}(\Omega)$, where $r = np/(n - mp + p)$ satisfies $n < r < \infty$ since $(m-1)p < n < mp$. Hence, if $u \in W^{m,p}(\Omega) \cap C^\infty(\Omega)$, we have

$$|u(x)| \le K_6\,\|u\|_{1,r}^{\theta}\,\|u\|_q^{1-\theta} \le K_7\,\|u\|_{m,p}^{\theta}\,\|u\|_q^{1-\theta},$$

where $\theta = nr/[nr + (r - n)q] = np/[np + (mp - n)q]$. ∎

The following theorem makes use of the above result to provide an alternate direct proof of Part I Case C of the Sobolev imbedding theorem 4.12 as well as a hybrid imbedding inequality that will prove useful for establishing compactness of some of these imbeddings in the next chapter.

5.10 THEOREM Let Ω be a domain in $\mathbb{R}^n$ satisfying the cone condition. Let m and k be positive integers and let $p > 1$. Suppose that $mp < n$ and $n - mp < k \le n$. Let ν be the largest integer less than mp, so that $n - \nu \le k$. Let Ω_k be the intersection of Ω with a k-dimensional plane in $\mathbb{R}^n$. Then there exists a constant K such that the inequality

$$\|u\|_{0,kq/n,\Omega_k} \le K \, \|u\|_{0,q,\Omega}^{1-\theta} \, \|u\|_{m,p,\Omega}^{\theta} \tag{15}$$

holds for all $u \in W^{m,p}(\Omega)$, where

$$q = p^* = \frac{np}{n - mp} \qquad \text{and} \qquad \theta = \frac{\nu p}{\nu p + (mp - \nu)q}.$$

Note that $0 < \theta < 1$.

Proof. Again it is sufficient to establish the inequality for functions in $W^{m,p}(\Omega) \cap C^\infty(\Omega)$. Without loss of generality we assume that H is a coordinate k-plane $\mathbb{R}^k$ in $\mathbb{R}^n$, and, as we did in Lemma 4.24, that Ω is a union of coordinate cubes of fixed edge length, say 2.

Let $\mu = \binom{k}{n-\nu}$, and let E^i, $1 \le i \le \mu$, denote the various coordinate planes in $\mathbb{R}^k$ having dimension $n - \nu$. Let Ω^i be the projection of Ω_k onto E^i, and for each $x \in \Omega^i$ let Ω_x^i denote the intersection of Ω with the ν-dimensional plane through x perpendicular to E^i. Then Ω_x^i contains a ν dimensional cube of unit edge length having a vertex at x, so it satisfies a cone condition with parameters independent of i and x. By Theorem 5.9

$$\|u\|_{0,\infty,\Omega_x^i} \le K_1 \, \|u\|_{0,q,\Omega_x^i}^{1-\theta} \, \|u\|_{m,p,\Omega_x^i}^{\theta} .$$

Let $s = (n - \nu)p/(n - mp)$, and let dx^i and dx_*^i denote the volume elements in E^i and its orthogonal complement (in $\mathbb{R}^n$) respectively. Since

$$s(1 - \theta) = \frac{q(mp - \nu)}{mp} \qquad \text{and} \qquad s\theta = \frac{\nu}{m},$$

we have

$$\begin{aligned}
&\int_{\Omega^i} \sup_{y \in \Omega_x^i} |u(y)|^s \, dx^i \\
&\le K_1 \int_{\Omega^i} \left[\int_{\Omega_x^i} |u(x)|^q \, dx_*^i \right]^{(mp-\nu)/mp} \left[\int_{\Omega_x^i} \sum_{|\alpha| \le m} |D^\alpha u(x)|^p \, dx_*^i \right]^{\nu/mp} \\
&\le K_1 \, \|u\|_{0,q,\Omega}^{s(1-\theta)} \, \|u\|_{m,p,\Omega}^{s\theta} ,
\end{aligned}$$

the last line being an application of Hölder's inequality.

Let dx^k denote the k-dimensional volume element in H. We apply the averaging Lemma 4.23 to the family of μ subspaces E^i of $\mathbb{R}^k$. The parameter λ for this application of the lemma is $\lambda = \binom{k-1}{n-\nu-1} = (n-\nu)\mu/k$. Since $(kq/n)(\lambda/\mu) = s$, we obtain

$$\begin{aligned}\|u\|_{0,kq/n,\Omega_k}^{kq/n} &\le K_2 \int_{\Omega_k} \prod_{i=1}^{\mu} \sup_{y\in\Omega_x^i} |u(y)|^{kq/\mu n}\, dx^k \\ &\le K_2 \prod_{i=1}^{\mu} \left[\int_{\Omega^i} \sup_{y\in\Omega_x^i} |u(y)|^s\, dx^i \right]^{1/\lambda} \\ &\le K_3 \prod_{i=1}^{\mu} \|u\|_{0,q,\Omega}^{kq(1-\theta)/\mu n}\, \|u\|_{m,p,\Omega}^{kq\theta/\mu n},\end{aligned}$$

so that

$$\|u\|_{0,kq/n,\Omega_k} \le K\, \|u\|_{0,q,\Omega}^{1-\theta}\, \|u\|_{m,p,\Omega}^{\theta}$$

as required. ∎

5.11 REMARK If we take $k = n$ in inequality (15), then the imbedding $W^{m,p}(\Omega) \to L^q(\Omega)$ follows for $q = np/(n-mp)$ by cancellation. The corresponding imbedding inequality $\|u\|_{0,q,\Omega} \le K\, \|u\|_{m,p,\Omega}$ can then be used to further estimate the right side of (15), yielding the trace imbedding $W^{m,p}(\Omega) \to L^r(\Omega_k)$ for $r = kp/(n-mp)$.

Interpolation Involving Compact Subdomains

Sometimes it is useful to have bounds for intermediate derivatives $D^\beta u$, of a function $u \in W^{m,p}(\Omega)$, where $1 \le |\beta| \le m-1$, in terms of the seminorm $|u|_{m,p,\Omega}$ and the L^p-norm of u over a compact subdomain $\Omega' \Subset \Omega$. Such inequalities are typically not possible unless Ω is bounded, but for bounded Ω they can be established under the assumption that Ω satisfies either the segment condition or the cone condition. (A bounded domain Ω satisfying the cone condition can be decomposed into a finite union of subdomains each of which satisfies the strong local Lipschitz condition, and therefore the segment condition. See Lemma 4.22.) We will prove the following hybrid interpolation theorem. (See Agmon [Ag].)

5.12 THEOREM Let Ω be a bounded domain in $\mathbb{R}^n$ satisfying the segment condition. Let $0 < \epsilon_0 < \infty$, let $1 \le p < \infty$, and let j and m be integers with $0 \le j \le m-1$. There exists a constant $K = K(\epsilon_0, m, p, \Omega)$ and for each ϵ satisfying $0 < \epsilon \le \epsilon_0$ a domain $\Omega_\epsilon \Subset \Omega$ such that for every $u \in W^{m,p}(\Omega)$

$$|u|_{j,p,\Omega} \le K\epsilon\, |u|_{m,p,\Omega} + K\epsilon^{-j/(m-j)}\, \|u\|_{p,\Omega_\epsilon}. \tag{16}$$

Note that this theorem implies Theorem 5.2 extends to bounded domains satisfying the segment condition.

As in the proof of Theorem 5.12, we begin with a one-dimensional inequality.

5.13 LEMMA Let $1 \le p < \infty$ and let $0 < l_1 < l_2 < \infty$. Then there exists a constant $K = K(p, l_1, l_2)$ and, for every $\epsilon > 0$, a number $\delta = \delta(\epsilon, l_1, l_2)$ satisfying $0 < 2\delta < l_1$ such that if (a, b) is a finite open interval in $\mathbb{R}$ whose length $b - a$ satisfies $l_1 \le b - a \le l_2$, and $g \in C^1(a, b)$, then

$$\int_a^b |g(t)|^p\,dt \le K\epsilon \int_a^b |g'(t)|^p\,dt + K \int_{a+\delta}^{b-\delta} |g(t)|^p\,dt. \tag{17}$$

Proof. If $f \in C^1(0, 1)$, $0 < t < 1$, and $1/3 < \tau < 2/3$, then

$$|f(s)| = \left| f(\tau) + \int_\tau^s f'(\xi)\,d\xi \right| \le |f(\tau| + \int_0^1 |f'(\xi)|\,d\xi.$$

Integrating τ over $(1/3, 2/3)$, applying Hölder's inequality if $p > 1$, and finally integrating s over $(0, 1)$ gives

$$\int_0^1 |f(s)|^p \le K_p \int_{1/3}^{2/3} |f(s)|^p\,ds + K_p \int_0^1 |f'(s)|^p\,ds,$$

where $K_p = 3 \cdot 2^{p-1}$. Now substitute $f(s) = g\big(a + s(b - a)\big) = g(t)$ to obtain

$$\int_a^b |g(t)|^p \le K_p(b - a)^p \int_a^b |g'(t)|^p\,dt + K_p \int_{(2a+b)/3}^{(a+2b)/3} |g(t)|^p\,dt.$$

For given $\epsilon > 0$ pick a positive integer k such that $k^{-p} \le \epsilon$. Let $a_j = a+(b-a)j/k$ for $j = 0, 1, \dots, k$ and pick δ so that $0 < \delta \le (b - a)/3k$. Then

$$\begin{aligned}
\int_a^b |g(t)|^p\,dt &= \sum_{j=1}^k \int_{a_{j-1}}^{a_j} |g(t)|^p\,dt \\
&\le K_p \sum_{j=1}^k \left[\left(\frac{b-a}{k}\right)^p \int_{a_{j-1}}^{a_j} |g'(t)|^p\,dt + \int_{a_{j-1}+\delta}^{a_j-\delta} |g(t)|^p\,dt \right] \\
&\le K_p \max\{1, (b-a)^p\} \left[\epsilon \int_a^b |g'(t)|^p\,dt + \int_{a+\delta}^{b-\delta} |g(t)|^p\,dt \right]
\end{aligned}$$

which is the desired inequality (17). ∎

5.14 LEMMA Let Ω be a bounded domain in $\mathbb{R}^n$ that satisfies the segment condition. Then there exists a constant $K = K(p, \Omega)$ and, for any positive number ϵ, a domain $\Omega_\epsilon \Subset \Omega$, such that

$$|u|_{0,p,\Omega} \le K\epsilon \, |u|_{1,p,\Omega} + K \, |u|_{0,p,\Omega_\epsilon} \tag{18}$$

holds for every $u \in W^{1,p}(\Omega)$.

Proof. Since Ω is bounded, and its boundary is therefore compact, the open cover $\{U_j\}$ of bdry Ω and corresponding set $\{y_j\}$ of nonzero vectors referred to in the definition of the segment condition (Paragraph 3.21) are both finite sets. Therefore open sets $V_j \Subset U_j$ can be found such that bdry $\Omega \subset \bigcup_j V_j$ and even, for sufficiently small δ, $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \operatorname{bdry}\Omega) < \delta\} \subset \bigcup_j V_j$. Thus $\Omega = \bigcup_j (V_j \cap \Omega) \cup \tilde{\Omega}$, where $\tilde{\Omega} \Subset \Omega$. It is thus sufficient to prove that for each j

$$|u|_{0,p,V_j \cap \Omega^p} \le K_1 \epsilon^p \, |u|^p_{1,p,\Omega} + K_1 \, |u|^p_{0,p,\Omega_{\epsilon,j}}$$

for some $\Omega_{\epsilon,j} \Subset \Omega$. For simplicity, we now drop the subscripts j.

Consider the sets Q, Q_η, $0 \le \eta < 1$, defined by

$$Q = \{x + ty : x \in U \cap \Omega, 0 < t < 1\},$$
$$Q_\eta = \{x + ty : x \in V \cap \Omega, \eta < t < 1\}.$$

If $\eta > 0$, then $Q_\eta \Subset Q$, and by the segment condition, $Q \subset \Omega$. Any line ℓ parallel to y and passing through a point in $V \cap \Omega$ intersects Q_0 in one or more intervals each having length between $|y|$ and diam Ω. By 5.13 there exists $\eta > 0$ and a constant K_1 such that for every $u \in C^\infty(\Omega)$ and any such line ℓ

$$\int_{\ell \cap Q_0} |u(x)|^p \, ds \le K_1 \epsilon^p \int_{\ell \cap Q_0} |D_y u(x)|^p \, ds + K_1 \int_{\ell \cap Q_\eta} |u(x)|^p \, ds,$$

D_y denoting differentiation in the direction of y and ds being the length element in that direction. We integrate this inequality over the projection of Q_0 on a hyperplane perpendicular to y and so obtain

$$\begin{aligned} |u|^p_{0,p,V\cap\Omega} \le |u|^p_{0,p,Q_0} &\le K_1 \epsilon^p \, |u|^p_{1,p,Q_0} + K_1 \, |u|^p_{0,p,Q_\eta} \\ &\le K_1 \epsilon^p \, |u|^p_{1,p,\Omega} + K_1 \, |u|^p_{0,p,Q_\epsilon}, \end{aligned}$$

where $\Omega_\epsilon = \Omega_\eta \Subset \Omega$. By density, this inequality holds for every $u \in W^{1,p}(\Omega)$.

5.15 (Completion of the Proof of Theorem 5.12) We apply Lemma 5.14 to derivatives $D^\beta u$, $|\beta| = m - 1$ to obtain

$$|u|_{m-1,p,\Omega} \le K\epsilon \, |u|_{m,p,\Omega} + K_1 \, |u|_{m-1,p,\Omega_\epsilon}, \tag{19}$$

where $\Omega_\epsilon \Subset \Omega$. Since $\overline{\Omega_\epsilon}$ is a compact subset of Ω, there exists a constant $\delta > 0$ such that $\operatorname{dist}(\overline{\Omega_\epsilon}, \operatorname{bdry}\Omega) > \delta$. The union Ω' of open balls of radius δ about points in $\overline{\Omega_\epsilon}$ clearly satisfies the cone condition and also $\Omega' \Subset \Omega$. We can use Ω' in place of Ω_ϵ in (19), and so we can assume Ω_ϵ satisfies the cone condition. By Theorem 5.2, for given $\epsilon_0 > 0$ the inequality

$$|u|_{m-1,p,\Omega_\epsilon} \le K_2\epsilon\,|u|_{m,p,\Omega_\epsilon} + K_2\epsilon^{-(m-1)}\,|u|_{0,p,\Omega_\epsilon}\,.$$

Combining this with inequality (19) we obtain the case $j = m - 1$ of (16).

The rest of the proof is by downward induction on j. Assuming that (16) holds for some j satisfying $1 \le j \le m-1$, and replacing ϵ with ϵ^{m-j} (with consequent alterations to K and Ω_ϵ), we obtain

$$|u|_{j,p,\Omega} \le K_3\epsilon^{m-j}\,|u|_{m,p,\Omega} + K_3\epsilon^{-j}\,|u|_{0,p,\Omega_{\epsilon,1}}\,.$$

Also, by the case already proved,

$$|u|_{j-1,p,\Omega} \le K_4\epsilon\,|u|_{j,p,\Omega} + K_4\epsilon^{-(j-1)}\,|u|_{0,p,\Omega_{\epsilon,2}}\,.$$

Combining these we get

$$|u|_{j-1,p,\Omega} \le K_5\epsilon^{m-(j-1)}\,|u|_{m,p,\Omega} + K_5\epsilon^{-(j-1)}\,|u|_{0,p,\Omega_\epsilon}\,,$$

where $K_5 = K_4(K_3 + 1)$ and $\Omega_\epsilon = \Omega_{\epsilon,1} \cup \Omega_{\epsilon,2}$. Replacing ϵ by $\epsilon^{1/(m-j+1)}$ we complete the induction. ∎

5.16 REMARK The conclusion of Theorem 5.12 is also valid for bounded domains satisfying the cone condition. Although the cone condition does not imply the segment condition, the decomposition of a domain Ω satisfying the cone condition into a finite union of subdomains each of which is a union of parallel translates of a parallelepiped (see Lemma 4.22) can be refined, for bounded Ω, so that each of the subdomains satisfies a strong local Lipschitz condition and therefore also the segment condition.

Extension Theorems

5.17 (Extension Operators) Let Ω be a domain in $\mathbb{R}^n$. For given m and p a linear operator E mapping $W^{m,p}(\Omega)$ into $W^{m,p}(\mathbb{R}^n)$ is called a *simple* (m, p)-*extension operator for* Ω if there exists a constant $K = K(m, p)$ such that for every $u \in W^{m,p}(\Omega)$ the following conditions hold:

(i) $Eu(x) = u(x)$ a.e. in Ω,

(ii) $\|Eu\|_{m,p,\mathbb{R}^n} \le K\,\|u\|_{m,p,\Omega}$.

E is called a *strong m-extension operator for* Ω if E is a linear operator mapping functions defined a.e. in Ω to functions defined a.e. in $\mathbb{R}^n$ and if, for every p, $1 \le p < \infty$, and every integer k, $0 \le k \le m$, the restriction of E to $W^{k,p}(\Omega)$ is a simple (k, p)-extension operator for Ω.

Finally, E is called a *total extension operator for* Ω if E is a strong m-extension operator for Ω for every m. Such a total extension operator necessarily extends functions in $C^m(\overline{\Omega})$ to lie in $C^m(\mathbb{R}^n)$.

5.18 The existence of even a simple (m, p)-extension operator for Ω guarantees that $W^{m,p}(\Omega)$ inherits many properties possessed by $W^{m,p}(\mathbb{R}^n)$. For instance, if an imbedding $W^{m,p}(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is known to hold, so that

$$\|u\|_{q,\mathbb{R}^n} \le K_1 \|u\|_{m,p,\mathbb{R}^n},$$

then the imbedding $W^{m,p}(\Omega) \to L^q(\Omega)$ must also hold, for if $u \in W^{m,p}(\Omega)$, then

$$\|u\|_{0,q,\Omega} \le \|Eu\|_{0,q,\mathbb{R}^n} \le K_1 \|Eu\|_{m,p,\mathbb{R}^n} \le K_1 K \|u\|_{m,p,\Omega}.$$

The reason we did not use this technique to prove the Sobolev imbedding theorem 4.12 is that extension theorems cannot be obtained for some domains satisfying such weak conditions as the cone condition or even the weak cone condition.

We will construct extension operators of each of the three types defined above. First we will use successive reflections in smooth boundaries to construct strong and total extension operators for half spaces, and strong extension operators for domains with suitably smooth boundaries. The method is attributed to Whitney [W] and later Hestenes [He] and Seeley [Se]. Stein [St] obtained a total extension operator under the minimal assumption that Ω satisfies the strong local Lipschitz condition. He used integral averaging instead of reflections. We will give only an outline of his proof here, leaving the interested reader to consult [St] for the details. See also [Ry]. The third construction, due to Calderón [Ca1] involves the use of the Calderón-Zygmund theory of singular integrals. It is less transparent than the reflection or averaging methods, and only works when $1 < p < \infty$, but requires only that the domain Ω satisfies the uniform cone condition. Unlike the other methods, it has the property that if the trivial extension $\tilde{u}$ belongs to $W^{m,p}(\mathbb{R}^n)$, then $\tilde{u}$ is the extension produced by the method. By Theorem 5.29 below, this happens if and only if $u \in W_0^{m,p}(\Omega)$. The paper [Jn] provides an extension method that works under a geometric hypothesis that is necessary and sufficient in $\mathbb{R}^2$, and is nearly optimal in higher dimensions.

Except for very simple domains all of our constructions require the use of partitions of unity subordinate to open covers of bdry Ω chosen in such a way that the functions in the partition have uniformly bounded derivatives.

To illustrate the reflection technique we begin by constructing a strong m-extension operator and a total extension operator for a half-space. Then we extend these to apply to domains that satisfy the uniform C^m-regularity condition and also have a bounded boundary.

5.19 THEOREM Let Ω be the half-space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$. Then there exists a strong m-extension operator E for Ω. Moreover, for every multi-index α satisfying $|\alpha| \le m$ there exists a strong $(m - |\alpha|)$-extension operator E_α for Ω, such that

$$D^\alpha Eu(x) = E_\alpha D^\alpha u(x).$$

Proof. For functions u defined a.e. on $\mathbb{R}^n_+$ we define Eu and $E_\alpha u$, $|\alpha| \le m$ a.e. on $\mathbb{R}^n$ via

$$Eu(x) = \begin{cases} u(x) & \text{if } x_n > 0 \\ \sum_{j=1}^{m+1} \lambda_j u(x_1, \dots, x_{n-1}, -jx_n) & \text{if } x_n < 0, \end{cases}$$

$$E_\alpha u(x) = \begin{cases} u(x) & \text{if } x_n > 0 \\ \sum_{j=1}^{m+1} (-j)^{\alpha_n} \lambda_j u(x_1, \dots, x_{n-1}, -jx_n) & \text{if } x_n < 0, \end{cases}$$

where the coefficients $\lambda_1, \dots, \lambda_{m+1}$ are the unique solutions of the $(m+1)\times(m+1)$ system of linear equations

$$\sum_{j=1}^{m+1} (-j)^k \lambda_j = 1, \qquad k = 0, \dots, m.$$

If $u \in C^m(\overline{\mathbb{R}^n_+})$, it is readily checked that $Eu \in C^m(\mathbb{R}^n)$ and

$$D^\alpha Eu(x) = E_\alpha D^\alpha u(x), \qquad |\alpha| \le m.$$

Thus

$$\begin{aligned} &\int_{\mathbb{R}^n} |D^\alpha Eu(x)|^p \, dx \\ &= \int_{\mathbb{R}^n_+} |D^\alpha u(x)|^p \, dx + \int_{\mathbb{R}^n_-} \left| \sum_{j=1}^{m+1} (-j)^{\alpha_n} \lambda_j u(x_1, \dots, x_{n-1}, -jx_n) \right|^p dx \\ &\le K(m, p, \alpha) \int_{\mathbb{R}^n_+} |D^\alpha u(x)|^p \, dx. \end{aligned}$$

By Theorem 3.22, the above inequality extends to functions $u \in W^{k,p}\left(\mathbb{R}^n_+\right)$, $m \ge k \ge |\alpha|$. Hence, E is a strong m-extension operator for $\mathbb{R}^n_+$. Since $D^\beta E_\alpha u(x) = E_{\alpha+\beta} u(x)$, a similar calculations shows that E_α is a strong $(m - |\alpha|)$-extension. ∎

The reflection technique used in the above proof can be modified to yield a total extension operator. The proof, due to Seeley [Se], is based on the following lemma.

5.20 LEMMA There exists a sequence $\{a_k\}_{k=0}^{\infty}$ such that for every nonnegative integer n we have

$$\sum_{k=0}^{\infty} 2^{nk} a_k = (-1)^n, \tag{20}$$

and

$$\sum_{k=0}^{\infty} 2^{nk} |a_k| < \infty. \tag{21}$$

Proof. For fixed N, let $a_{k,N}$, $k = 0, 1, \dots, N$ be the solution of the system of linear equations

$$\sum_{k=0}^{N} 2^{nk} a_{k,N} = (-1)^n, \qquad n = 0, 1, \dots, N. \tag{22}$$

In terms of the Vandermonde determinant

$$V(x_0, x_1, \dots, x_N) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_N \\ x_0^2 & x_1^2 & \cdots & x_N^2 \\ \vdots & \vdots & & \vdots \\ x_0^N & x_1^N & \cdots & x_N^N \end{vmatrix} = \prod_{\substack{i,j=0 \\ i<j}}^{N} (x_j - x_i),$$

$a_{k,N}$ as given by Cramer's rule is

$$\begin{aligned} a_{k,N} &= \frac{V(1, 2, \dots, 2^{k-1}, -1, 2^{k+1}, \dots, 2^N)}{V(1, 2, \dots, 2^N)} \\ &= \left[\prod_{\substack{i,j=0 \\ i,j \neq k \\ i<j}} (2^j - 2^i) \prod_{i=0}^{k-1} (-1 - 2^i) \prod_{j=k+1}^{N} (2^j + 1) \right] \cdot \left[\prod_{\substack{i,j=0 \\ i<j}}^{N} (2^j - 2^i) \right]^{-1} \\ &= A_k B_{k,N} \end{aligned}$$

where

$$A_k = \prod_{i=1}^{k-1} \frac{1 + 2^i}{2^i - 2^k}, \qquad B_{k,N} = \prod_{j=k+1}^{N} \frac{1 + 2^j}{2^j - 2^k},$$

it being understood that $\prod_{i=l}^{m} P_i = 1$ if $l > m$. Now

$$|A_k| \le \prod_{i=1}^{k-1} \frac{2^{i+1}}{2^{k-1}} \le 2^{(5k-k^2)/2}.$$

Also

$$\begin{aligned}\log B_{k,N} &= \sum_{j=k+1}^{N} \log\left(1 + \frac{1+2^k}{2^j - 2^k}\right)\\ &< \sum_{j=k+1}^{N} \frac{1+2^k}{2^j - 2^k} < (1+2^k) \sum_{j=k+1}^{N} \frac{1}{2^{j-1}} < 4,\end{aligned}$$

where we have used the inequality $\log(1+x) < x$ valid for $x > 0$. It follows that the increasing sequence $\{B_{k,N}\}_{N=0}^{\infty}$ converges to a limit $B_k \le e^4$. Let $a_k = A_k B_k$, so that

$$|a_k| \le e^4 \cdot 2^{(5k-k^2)/2}.$$

Then for any n

$$\sum_{k=0}^{\infty} 2^{nk} |a_k| \le e^4 \sum_{k=0}^{\infty} 2^{(2nk+5k-k^2)/2} < \infty.$$

Letting $n \to \infty$ in (22) completes the proof. ∎

5.21 THEOREM Let Ω be a half-space in $\mathbb{R}^n$. Then there exists a total extension operator E for Ω.

Proof. The restrictions to $\mathbb{R}^n_+$ of functions $\phi \in C_0^\infty(\mathbb{R}^n)$ being dense in $W^{m,p}\left(\mathbb{R}^n_+\right)$ for any m and p, we need only define the extension operator for such functions. Let f be a real-valued function, infinitely differentiable on $[0, \infty)$ and satisfying $f(t) = 1$ if $0 \le t \le 1/2$ and $f(t) = 0$ if $t \ge 1$, If $\phi \in C_0^\infty(\mathbb{R}^n)$, let

$$E\phi(x) = E\phi(x', x_n) = \begin{cases} \phi(x) & \text{if } x_n \ge 0,\\ \sum_{k=0}^{\infty} a_k f(-2^k x_n)\phi(x', -2^k x_n) & \text{if } x_n < 0,\end{cases}$$

where $\{a_k\}$ is the sequence constructed in the previous lemma. $E\phi$ is well-defined on $\mathbb{R}^n$ since the sum above has only finitely many nonvanishing terms for any particular $x \in \mathbb{R}^n_- = \{x \in \mathbb{R}^n : x_n < 0\}$. Moreover, $E\phi$ has compact support and belongs to $C^\infty(\overline{\mathbb{R}^n_+}) \cap C^\infty(\overline{\mathbb{R}^n_-})$. If $x \in \mathbb{R}^n_-$, we have

$$\begin{aligned}D^\alpha E\phi(x) &= \sum_{k=0}^{\infty} \sum_{j=0}^{\alpha_n} \binom{\alpha_n}{j} (-2^k)^{\alpha_n} f^{(\alpha_n - j)}(-2^k x_n) D_n^j D^{\alpha'} \phi(x', -2^k x_n)\\ &= \sum_{k=0}^{\infty} \psi_k(x).\end{aligned}$$

Since $\psi_k(x) = 0$ when $-x_n > 1/2^{k-1}$ it follows from (21) that the above series converges absolutely and uniformly as $x_n \to 0-$. Hence by (20)

$$\begin{aligned}\lim_{x_n \to 0-} D^\alpha E\phi(x) &= \sum_{k=0}^{\infty} (-2^k)^{\alpha_n} a_k D^\alpha \phi(x', 0+) \\ &= D^\alpha \phi(x', 0+) = \lim_{x_n \to 0+} D^\alpha E\phi(x) = D^\alpha E\phi(0).\end{aligned}$$

Thus $E\phi \in C_0^\infty(\mathbb{R}^n)$. Moreover, if $|\alpha| \le m$,

$$|\psi_k(x)|^p \le K_1^p |a_k|^p 2^{kmp} \sum_{|\beta| \le m} |D^\beta \phi(x', -2^k x_n)|^p,$$

where K_1 depends only on m, p, n, and f. Thus

$$\begin{aligned}\|\psi_k\|_{0,p,\mathbb{R}^n_-} &\le K_1 |a_k| 2^{km} \left(\sum_{|\beta| \le m} \int_{\mathbb{R}^n_-} |D^\beta \phi(x', -2^k x_n)|^p \, dx \right)^{1/p} \\ &= K_1 |a_k| 2^{km} \left(\frac{1}{2^k} \sum_{|\beta| \le m} \int_{\mathbb{R}^n_+} |D^\beta \phi(y)|^p \, dy \right)^{1/p} \\ &\le K_1 |a_k| 2^{km} \, \|\phi\|_{m,p,\mathbb{R}^n_+}.\end{aligned}$$

It follows from (21) that

$$\|D^\alpha E\phi\|_{0,p,\mathbb{R}^n_-} \le K_1 \, \|\phi\|_{m,p,\mathbb{R}^n_+} \sum_{k=0}^{\infty} 2^{km} |a_k| \le K_2 \, \|\phi\|_{m,p,\mathbb{R}^n_+}.$$

Combining this with a similar (trivial) estimate for $\|D^\alpha E\phi\|_{0,p,\mathbb{R}^n_+}$, we obtain

$$\|E\phi\|_{m,p,\mathbb{R}^n} \le K_3 \, \|\phi\|_{m,p,\mathbb{R}^n_+}$$

with $K_3 = K_3(m, p, n)$. This completes the proof. ∎

5.22 THEOREM Let Ω be a domain in $\mathbb{R}^n$ satisfying the uniform C^m-regularity condition and having a bounded boundary. Then there exists a strong m-extension operator E for Ω. Moreover, if α and γ are multi-indices with $|\gamma| \le |\alpha| \le m$, then there exists a linear operator $E_{\alpha\gamma}$ continuous from $W^{j,p}(\Omega)$ into $W^{j,p}(\mathbb{R}^n)$ for $1 \le j \le m - |\alpha|$, $1 \le p < \infty$, such that if $u \in W^{|\alpha|,p}(\Omega)$, then

$$D^\alpha (Eu)(x) = \sum_{|\gamma| \le |\alpha|} E_{\alpha\gamma} D^\gamma u(x). \tag{23}$$

Proof. Since Ω is uniformly C^m-regular and has a bounded boundary the open cover $\{U_j\}$ of bdry Ω and the corresponding m-smooth maps Φ_j from U_j onto B referred to in Paragraph 4.10 are finite collections, say $1 \le j \le N$. Let $Q = \{y = (y', y_n) \in \mathbb{R}^n : |y'| < 1/2,\ |y_n| < \sqrt{3}/2\}$. Then

$$\{y \in \mathbb{R}^n : |y| < 1/2\} \subset Q \subset B = \{y \in \mathbb{R}^n : |y| < 1\}.$$

By condition (ii) of Paragraph 4.10 the open sets $V_j = \Psi_j(Q)$, $1 \le j \le N$, form an open cover of $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \operatorname{bdry}\Omega) < \delta\}$ for some $\delta > 0$. There exists an open set $V_0 \subset \Omega$, bounded away from bdry Ω, such that $\Omega \subset \bigcup_{j=0}^N V_j$. By Theorem 3.15 we can find infinitely differentiable functions $\omega_0, \omega_1, \dots, \omega_N$ such that the support of ω_j is a subset of V_j and $\sum_{j=0}^N \omega_j(x) = 1$ for all $x \in \Omega$. (Note that the support of ω_0 need not be compact if Ω is unbounded.)

Since Ω is uniformly C^m-regular it satisfies the segment condition and so restrictions to Ω of functions in $C_0^\infty(\mathbb{R}^n)$ are dense in $W^{k,p}(\Omega)$. If $\phi \in C_0^\infty(\mathbb{R}^n)$, then for $x \in \Omega$, $\phi(x) = \sum_{j=0}^N \phi_j(x)$, where $\phi_j = \omega_j \cdot \phi$.

For $j \ge 1$ and $y \in B$ let $\psi_j(y) = \phi_j(\Psi_j(y))$. Then $\psi_j \in C_0^\infty(Q)$. We extend ψ_j to be identically zero outside Q. With E and E_α defined as in Theorem 5.19, we have $E\psi_j \in C_0^m(Q)$, $E\psi_j = \psi_j$ on $Q_+ = \{y \in Q : y_n > 0\}$, and

$$\|E\psi_j\|_{k,p,Q} \le K_1 \|\psi_j\|_{k,p,Q_+}, \qquad 0 \le k \le m,$$

where K_1 depends on k, m, and p. If $\theta_j(x) = E\psi_j(\Phi_j(x))$, then $\theta_j \in C_0^\infty(V_j)$ and $\theta_j(x) = \phi_j(x)$ if $x \in \Omega$. It may be checked by induction that if $|\alpha| \le m$, then

$$D^\alpha \theta_j(x) = \sum_{|\beta| \le |\alpha|} \sum_{|\gamma| \le |\alpha|} a_{j;\alpha\beta}(x)\big[E_\beta\big(b_{j;\beta\gamma} \cdot (D^\gamma \phi_j \circ \Psi_j)\big)\big]\big(\Phi_j(x)\big),$$

where $a_{j;\alpha\beta} \in C^{m-|\alpha|}(\overline{U_j})$ and $b_{j;\beta\gamma} \in C^{m-|\beta|}(\overline{B})$ depend on the transformations Φ_j and $\Psi_j = \Phi_j^{-1}$ and satisfy

$$\sum_{|\beta| \le |\alpha|} a_{j;\alpha\beta}(x) b_{j;\beta\gamma}\big(\Phi_j(x)\big) = \begin{cases} 1 & \text{if } \gamma = \alpha \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 3.41 we have for $k \le m$,

$$\|\theta_j\|_{k,p,\mathbb{R}^n} \le K_2 \|E\psi_j\|_{k,p,Q} \le K_1 K_2 \|\psi_j\|_{k,p,Q_+} \le K_3 \|\psi_j\|_{k,p,\Omega},$$

where K_3 may be chosen to be independent of j. The operator $\tilde{E}$ defined by

$$\tilde{E}\phi(x) = \phi_0(x) + \sum_{j=1}^N \theta_j(x)$$

clearly satisfies $\tilde{E}\phi(x) = \phi(x)$ if $x \in \Omega$, and

$$\left\|\tilde{E}\phi\right\|_{k,p,\mathbb{R}^n} \le \|\phi_0\|_{k,p,\Omega} + K_3 \sum_{j=1}^{N} \|\phi_j\|_{k,p,\Omega} \le K_4(1 + NK_3)\,\|\phi\|_{k,p,\Omega}\,, \qquad (24)$$

where

$$K_4 = \max_{0\le j\le N} \max_{|\alpha|\le m} \sup |D^\alpha \omega_j(x)| < \infty.$$

Thus $\tilde{E}$ is a strong m-extension operator for Ω. Also

$$D^\alpha \tilde{E}\phi(x) = \sum_{|\gamma|\le|\alpha|} (E_{\alpha\gamma} D^\gamma \phi)(x),$$

where

$$E_{\alpha\gamma} v(x) = \sum_{j=1}^{N} \sum_{|\beta|\le|\alpha|} a_{j;\alpha\beta}(x)\big[E_\beta\big(b_{j;\beta\gamma} \cdot (v \cdot \omega_j) \circ \Psi_j\big)\big]\big(\Phi_j(x)\big)$$

if $\alpha \ne \gamma$, and

$$E_{\alpha\alpha} v(x) = (v \cdot \omega_0)(x) + \sum_{j=1}^{N} \sum_{|\beta|\le|\alpha|} a_{j;\alpha\beta}(x)\big[E_\beta\big(b_{j;\beta\gamma} \cdot (v \cdot \omega_j) \circ \Psi_j\big)\big]\big(\Phi_j(x)\big).$$

We note that if $x \in \Omega$, then $E_{\alpha\gamma} v(x) = 0$ for $\alpha \ne \gamma$ and $E_{\alpha\alpha} v(x) = v(x)$. Clearly $E_{\alpha\gamma}$ is a linear operator. By the differentiability properties of $a_{j;\alpha\beta}$ and $b_{j;\beta\gamma}$, $E_{\alpha\gamma}$ is continuous on $W^{j,p}(\Omega)$ into $W^{j,p}(\mathbb{R}^n)$ for $1 \le j \le m - |\alpha|$. This completes the proof. ∎

5.23 REMARKS

1. If Ω is uniformly C^m-regular for all m, and has a bounded boundary, then we can use the total extension operator of Theorem 5.21 in place of that of Theorem 5.19 in the above proof to obtain a total extension operator for Ω.
2. The restriction that bdry Ω be bounded was imposed in Theorem 5.22 so that the open cover $\{V_j\}$ would be finite. This finiteness was used in two places in the proof, first in asserting the existence of the constant K_4, and secondly in obtaining the last inequality in (24). This latter use is, however, not essential for the proof because (24) could still be obtained from the finite intersection property (condition (i) in Paragraph 4.10) even if the cover $\{V_j\}$ were not finite. Theorem 5.22 extends to any suitably regular domain for which there exists a partition of unity $\{\omega_j\}$ subordinate to $\{V_j\}$ with $D^\alpha \omega_j$ bounded on $\mathbb{R}^n$ uniformly in j for any given α. The reader may find it

interesting to construct, by the above techniques, extension operators for domains not covered by the above theorems, for example, quadrants, strips, rectangular boxes, and smooth images of these.

3. The previous remark also applies to the Calderón Extension Theorem 5.28 given below. Although it is proved by methods quite different from the reflection methods used above, the proof still makes use of a partition of unity in the same way as does that of Theorem 5.22. Accordingly, the above considerations also apply to it. The theorem is proved under a strengthened uniform cone condition that reduces to the uniform cone condition of Paragraph 4.8 if Ω has a bounded boundary.

Clearly subsuming the extension theorems obtained above is the following theorem of Stein [St].

5.24 THEOREM (The Stein Extension Theorem) If Ω is a domain in $\mathbb{R}^n$ satisfying the strong local Lipschitz condition, then there exists a total extension operator for Ω.

We will provide here only an outline of the proof. The details can be found in Chapter 6 of [St].

5.25 (Outline of the Proof of the Stein Extension Theorem) .

1. Let $\Omega_e = \mathbb{R}^n - \overline{\Omega}$ be the open exterior of Ω. The function $\delta(x) = \operatorname{dist}(x, \overline{\Omega})$ is Lipschitz continuous on Ω_e since

$$|\delta(x) - \delta(y)| \le |x - y| \qquad \text{for} \quad x, y \in \Omega_e,$$

but might not be smooth there. However, there exists a function Δ in $C^\infty(\Omega_e)$ and positive constants c_1, c_2, and C_α for all multiindices α such that for all $x \in \Omega_e$,

$$c_1\delta(x) \le \Delta(x) \le c_2\delta(x), \qquad \text{and}$$
$$|D^\alpha \Delta(x)| \le C_\alpha\big(\delta(x)\big)^{1-|\alpha|}.$$

2. There exists a continuous function ϕ on $[1, \infty)$ for which
   (a) $\lim_{t\to\infty} t^k\phi(t) = 0$ for $k = 0, 1, 2, \ldots,$
   (b) $\int_1^\infty \phi(t)\,dt = 1$
   (c) $\int_1^\infty t^k\phi(t) = 0$ for $k = 1, 2, \ldots.$

In fact, $\phi(t) = \dfrac{e}{\pi t}\operatorname{Im}\left(e^{-w(t-1)^{1/4}}\right)$, where $w = e^{-i\pi/4}$, is such a function.

3. For the special case $\Omega = \{(x, y) : x \in \mathbb{R}^{n-1}, y \in \mathbb{R}, y > f(x)$ where f satisfies a Lipschitz condition $|\phi(x) - \phi(x')| \le M|x - x'|$, there exists a constant c such that if $(x, y) \in \Omega_e$, then $\phi(x) - y \le c\Delta(x, y)$.
4. For Ω as specified in 3, $\Delta^*(x, y) = 2c\Delta(x, y)$, and $u \in C_0^\infty(\mathbb{R}^n)$, the operator E defined by

$$E(u)(x, y) = \begin{cases} u(x, y) & \text{if } y > f(x) \\ \int_1^\infty u\big(x, y + t\Delta^*(x, y)\big)\phi(t)\,dt & \text{if } y < f(x) \end{cases}$$

satisfies, for every $m \ge 0$ and $1 \le p \le \infty$,

$$\|E(u)\|_{m,p,\mathbb{R}^n} \le K\,\|u\|_{m,p,\Omega}\,, \tag{25}$$

where $K = K(m, p, n, M)$. Since Ω satisfies the strong local Lipschitz condition it also satisfies the segment condition and so, by Theorem 3.22 the restrictions to Ω of functions in $C_0^\infty(\mathbb{R}^n)$ are dense in $W^{m,p}(\Omega)$ and so (25) holds for all $u \in W^{m,p}(\Omega)$. Thus Stein's theorem holds for this Ω.
5. The case of general Ω satisfying the strong local Lipschitz condition now follows via a partition of unity subordinate to an open cover of bdry Ω by open sets in each of which (a rotated version of) the special case 4 can be applied. ∎

5.26 The proof of the Calderón extension theorem is based on a special case, suitable for our purposes, of a well-known inequality of Calderón and Zygmund [CZ] for convolutions involving kernels with nonintegrable singularities. The proof of this inequality is rather lengthy and can be found in many sources (e.g. Stein and Weiss [SW]). It will be omitted here. Neither the inequality nor the extension theorem itself will be required hereafter in this monograph.

Let $B_R = \{x \in \mathbb{R}^n : |x| \le R\}$, let $\Sigma_R = \{x \in \mathbb{R}^n : |x| = R\}$, and let $d\sigma_R$ be the area element (Lebesgue $(n-1)$-volume element) on Σ_R. A function g is said to be *homogeneous of degree* μ on $B_R - \{0\}$ if $g(tx) = t^\mu g(x)$ for all $x \in B_R - \{0\}$ and $0 < t \le 1$.

5.27 THEOREM (The Calderón Zygmund Inequality) Let

$$g(x) = G(x)|x|^{-n},$$

where

(i) G is bounded on $\mathbb{R}^n - \{0\}$ and has compact support,

(ii) G is homogeneous of degree 0 on $B_R - \{0\}$ for some $R > 0$, and

(iii) $\int_{\Sigma_R} G(x)\,d\sigma_R = 0$.

If $1 < p < \infty$ and $u \in L^p(\mathbb{R}^n)$, then the principal-value convolution integral

$$u * g(x) = \lim_{\epsilon \to 0+} \int_{\mathbb{R}^n - B_\epsilon} u(x-y) g(y)\, dy$$

exists for almost all $x \in \mathbb{R}^n$, and there exists a constant $K = K(G, p)$ such that for all such u

$$\|u * g\|_p \le K \|u\|_p .$$

Conversely, if G satisfies (i) and (ii) and if $u * g$ exists for all $u \in C_0^\infty(\mathbb{R}^n)$, then G satisfies (iii).

5.28 THEOREM (The Calderón Extension Theorem) Let Ω be a domain in $\mathbb{R}^n$ satisfying the uniform cone condition (Paragraph 4.8) modified as follows:

(i) the open cover $\{U_j\}$ of bdry Ω is required to be finite, and

(ii) the sets U_j are not required to be bounded.

Then for any $m \in \{1, 2, \dots\}$ and any p satisfying $1 < p < \infty$, there exists a simple (m, p)-extension operator $E = E(m, p)$ for Ω.

Proof. Let $\{U_1, \dots, U_N\}$ be the open cover of bdry Ω given by the uniform cone condition, and let U_0 be an open subset of Ω bounded away from bdry Ω such that $\Omega \subset \bigcup_{j=0}^N U_j$. (Such a U_0 exists by condition (ii) of Paragraph 4.8.) Let $\omega_0, \omega_1, \dots, \omega_N$ be a C^∞ partition of unity for Ω with $\operatorname{supp}(\omega)_j \subset U_j$. For $1 \le j \le N$ we shall define operators E_j so that if $u \in W^{m,p}(\Omega)$, then $E_j u \in W^{m,p}(\mathbb{R}^n)$ and satisfies

$$E_j u = u \qquad \text{in} \quad U_j \cap \Omega,$$
$$\left\| E_j u \right\|_{m,p,\mathbb{R}^n} \le K_{m,p,j} \|u\|_{m,p,\Omega} .$$

The desired extension operator is then clearly given by

$$Eu = \omega_0 u + \sum_{j=1}^N \omega_j E_j u.$$

We shall write $x \in \mathbb{R}^n$ in the polar coordinate form $x = \rho\sigma$ where $\rho \ge 0$ and σ is a unit vector. Let C_j, the the cone associated with U_j in the description of the uniform cone condition, have vertex at 0. Let ϕ_j be a nontrivial function defined in $\mathbb{R}^n - \{0\}$ satisfying

(i) $\phi_j(x) \ge 0$ for all $x \ne 0$,

(ii) $\operatorname{supp}(\phi_j) \subset -C_j \cup \{0\}$,

(iii) $\phi_j \in C^\infty(\mathbb{R}^n - \{0\})$, and

(iv) for some $\epsilon > 0$, ϕ_j is homogeneous of degree $m-n$ in $B_\epsilon - \{0\}$.

Now $\rho^{n-1}\phi_j$ is homogeneous of degree $m-1 \ge 0$ on $B_\epsilon - \{0\}$ and so the function $\psi_j(x) = (\partial/\partial\rho)^m\big(\rho^{n-1}\phi_j(x)\big)$ vanishes on $B_\epsilon - \{0\}$. Hence ψ_j, extended to be zero at $x = 0$, belongs to $C_0^\infty(-C_j)$. Define

$$E_j u(y) = K_j \left((-1)^m \int_\Sigma \int_0^\infty \phi_j(\rho\sigma)\rho^{n-1} \left(\frac{\partial}{\partial\rho}\right)^m \tilde{u}(y-\rho\sigma)\,d\rho\,d\sigma \right.$$
$$\left. - \int_\Sigma \int_0^\infty \psi_j(\rho\sigma)\tilde{u}(y-\rho\sigma)\,d\rho\,d\sigma \right) \tag{26}$$

where $\tilde{u}$ is the zero extension of u outside Ω and where the constant K_j will be determined shortly. If $y \in U_j \cap \Omega$, then, assuming for the moment that $u \in C^\infty(\Omega)$, we have, for $\rho\sigma \in \operatorname{supp}\big(\phi_j\big)$, by condition (iii) of Paragraph 4.8, that $\tilde{u}(y-\rho\sigma) = u(y-\rho\sigma)$ is infinitely differentiable. Now integration by parts m times yields

$$\begin{aligned}
&(-1)^m \int_0^\infty \rho^{n-1}\phi_j(\rho\sigma) \left(\frac{\partial}{\partial\rho}\right)^m u(y-\rho\sigma)\,d\rho \\
&\quad = \sum_{k=0}^{m-1} (-1)^{m-k} \left(\frac{\partial}{\partial\rho}\right)^k \big(\rho^{n-1}\phi_j(\rho\sigma)\big) \left(\frac{\partial}{\partial\rho}\right)^{m-k-1} u(y-\rho\sigma)\Bigg|_{\rho=0}^{\rho=\infty} \\
&\quad\quad + \int_0^\infty \left(\frac{\partial}{\partial\rho}\right)^m \big(\rho^{n-1}\phi_j(\rho\sigma)\big) u(y-\rho\sigma)\,d\rho \\
&\quad = \left(\frac{\partial}{\partial\rho}\right)^{m-1} \big(\rho^{n-1}\phi_j(\rho\sigma)\big)\Big|_{\rho=0} u(y) + \int_0^\infty \psi_j(\rho\sigma)u(y-\rho\sigma)\,d\rho.
\end{aligned}$$

Hence

$$E_j u(y) = K_j u(y) \int_\Sigma \left(\frac{\partial}{\partial\rho}\right)^{m-1} \big(\rho^{n-1}\phi_j(\rho\sigma)\big)\Big|_{\rho=0} d\sigma.$$

Since $(\partial/\partial\rho)^{m-1}\big(\rho^{n-1}\phi_j(\rho\sigma)\big)$ is homogeneous of degree zero near 0, the above integral does not vanish if ϕ_j is not identically zero. Hence K_j can be chosen so that $E_j u(y) = u(y)$ for $y \in U_j \cap \Omega$ and all $u \in C^\infty(\Omega)$. Since $C^\infty(\Omega)$ is dense in $W^{m,p}(\Omega)$ we have $E_j u(y) = u(y)$ a.e. in $U_j \cap \Omega$ for every $u \in W^{m,p}(\Omega)$. The same argument shows that if $\tilde{u} \in W^{m,p}(\mathbb{R}^n)$, then $E_j u(y) = \tilde{u}(y)$ a.e. in $\mathbb{R}^n$.

It remains, therefore, to show that

$$\left\| D^\alpha E_j u \right\|_{0,p,\mathbb{R}^n} \le K_\alpha \, \|u\|_{m,p,\Omega}$$

holds for any α with $|\alpha| \le m$ and all $u \in C^\infty(\Omega) \cap W^{m,p}(\Omega)$. The last integral in (26) is of the form $\theta_j * \tilde{u}(y)$, where $\theta_j(x) = \psi_j(x)|x|^{1-n}$. Since $\theta_j \in L^1(\mathbb{R}^n)$ and has compact support, we obtain via Young's inequality for convolution (Corollary 2.25),

$$\left\| D^\alpha(\theta_j * \tilde{u}) \right\|_{0,p,\mathbb{R}^n} = \left\| \theta_j * (\widetilde{D^\alpha u}) \right\|_{0,p,\mathbb{R}^n} \le \left\| \theta_j \right\|_{0,1,\mathbb{R}^n} \|D^\alpha u\|_{0,p,\Omega} .$$

It now remains to be shown that the first integral in (26) defines a bounded map from $W^{m,p}(\Omega)$ into $W^{m,p}(\mathbb{R}^n)$. Since $(\partial/\partial\rho)^m = \sum_{|\alpha|=m}(m!/\alpha!)\sigma^\alpha D^\alpha$ we obtain

$$\begin{aligned}
&\int_\Sigma \int_0^\infty \phi_j(\rho\sigma)\rho^{n-1}\left(\frac{\partial}{\partial\rho}\right)^m \tilde{u}(y-\rho\sigma)\,d\rho\,d\sigma \\
&\qquad = \sum_{|\alpha|=m} \frac{m!}{\alpha!}\int_{\mathbb{R}^n} \phi_j(x)\widetilde{D_x^\alpha u}(y-x)\sigma^\alpha\,dx \\
&\qquad = \sum_{|\alpha|=m} \xi_\alpha * \widetilde{D^\alpha u},
\end{aligned}$$

where $\xi_\alpha = (-1)^{|\alpha|}(m!/\alpha!)\sigma^\alpha\phi_j$ is homogeneous of degree $m-n$ in $B_\epsilon - \{0\}$ and belongs to $C^\infty(\mathbb{R}^n - \{0\})$. It is now clearly sufficient to show that for any β satisfying $|\beta| \le m$

$$\left\| D^\beta(\xi_\alpha * v) \right\|_{0,p,\mathbb{R}^n} \le K_{\alpha,\beta}\, \|v\|_{0,p,\mathbb{R}^n} . \tag{27}$$

If $|\beta| \le m-1$, then $D^\beta\xi_\alpha$ is homogeneous of degree not exceeding $1-n$ in $B_\epsilon - \{0\}$ and so belongs to $L^1(\mathbb{R}^n)$. Inequality (27) now follows by Young's inequality for convolution. Thus we need consider only the case $|\beta| = m$, in which we write $D^\beta = (\partial/\partial x_i)D^\gamma$ for some i, $1 \le i \le n$, and some γ with $|\gamma| = m-1$. Suppose, for the moment, that $v \in C_0^\infty(\mathbb{R}^n)$. Then we may write

$$\begin{aligned}
D^\beta(\xi_\alpha * v)(x) &= (D^\gamma\xi_\alpha) * \left[\left(\frac{\partial}{\partial x_i}\right) v\right](x) = \int_{\mathbb{R}^n} D_i v(x-y) D^\gamma \xi_\alpha(y)\,dy \\
&= \lim_{\delta\to 0+} \int_{\mathbb{R}^n - B_\delta} D_i v(x-y) D^\gamma \xi_\alpha(y)\,dy.
\end{aligned}$$

We now integrate by parts in the last integral to free v and obtain $D^\beta\xi_\alpha$ under the integral. The integrated term is a surface integral over the spherical boundary Σ_δ of B_δ of the product of $v(x-\cdot)$ and a function homogeneous of degree $1-n$ near zero. This surface integral must therefore tend to $Kv(x)$ as $\delta \to 0+$, for some constant K. Noting that $D_i v(x-y) = -(\partial/\partial y_i)v(x-y)$, we now have

$$D^\beta(\xi_\alpha * v)(x) = \lim_{\delta\to 0+} \int_{\mathbb{R}^n} v(x-y) D^\beta \xi_\alpha(y)\,dy + Kv(x).$$

Now $D^\beta\xi_\alpha$ is homogeneous of degree $-n$ near the origin and so, by the last assertion of Theorem 5.27, $D^\beta\xi_\alpha$ satisfies all the conditions for the singular kernel

g of that theorem. Since $1 < p < \infty$, we have for any $v \in L^p(\Omega)$ (regarded as being identically zero outside Ω)

$$\left\| D^\beta \xi_\alpha * v \right\|_{0,p,\mathbb{R}^n} \leq K_{\alpha,\beta} \, \|v\|_{0,p,\mathbb{R}^n} .$$

This completes the proof. ∎

As observed in the proof of the above theorem, the Calderón extension of a function $u \in W^{m,p}(\Omega)$ coincides with the zero extension $\tilde{u}$ of u if $\tilde{u}$ belongs to $W^{m,p}(\mathbb{R}^n)$. The following theorem (which could have been proved in Chapter 3) shows that in this case u must belong to $W_0^{m,p}(\Omega)$.

5.29 THEOREM (Characterization of $W_0^{m,p}(\Omega)$ by Exterior Extension) Let Ω have the segment property. Then a function u on Ω belongs to $W_0^{m,p}(\Omega)$ if and only if the zero extension $\tilde{u}$ of u belongs to $W^{m,p}(R^n)$.

Proof. Lemma 3.27 shows, with no hypotheses on Ω, that if $u \in W_0^{m,p}(\Omega)$, then $\tilde{u} \in W^{m,p}(R^n)$.

Conversely, suppose that Ω has the segment property and that $\tilde{u} \in W^{m,p}(\mathbb{R}^n)$. Proceed as in the proof of Theorem 3.22, first multiplying u by a suitable smooth cutoff function f_ϵ to approximate u in $W^{m,p}(\Omega)$ by a function in that space with a bounded support. Replace u by that approximation; then $\tilde{u}$ is replaced by $f_\epsilon \tilde{u}$, and so still belongs to $W^{m,p}(\mathbb{R}^n)$. Now split this u into finitely-many pieces u_j, where $0 \leq i \leq k$, with u_j supported in a set V_j and the union of the sets V_j covering the support of u. In the context of that theorem, u_0 already belongs to $W_0^{m,p}(\Omega)$.

For the other values of j, use a translate $u_{j,t}$ of $\widetilde{u_j}$ mapping x to $\widetilde{u_j}(x - ty)$ rather than to $\widetilde{u_j}(x + ty)$ as we did in the proof of Theorem 3.22. For small enough positive values of t, using $x - ty$ shifts the support of $\widetilde{u_j}$ strictly inside the domain Ω. Then $u_{j,t}$ belongs to $W^{m,p}(R^n)$ since $\widetilde{u_j}$ does. Since $u_{j,t}$ vanishes outside a compact subset of Ω, the restriction of $u_{j,t}$ to Ω belongs to $W_0^{m,p}(\Omega)$. As $t \to 0+$, these restrictions converge to u_j in $W^{m,p}(\Omega)$. Thus each piece u_j belongs to $W_0^{m,p}(\Omega)$, and so does u. ∎

5.30 There is a close connection between the existence of extension operators and imbeddings into spaces of Hölder continuous functions. For example, it is shown in [Ko] that the imbedding $W^{m,p}(\Omega) \to C^{0,1-(n/p)}(\overline{\Omega})$ implies the existence of a simple $(1, q)$-extension operator for Ω provided $q > p$.

A short survey of extension theorems for Sobolev spaces can be found in [Bu2].

An Approximation Theorem

5.31 (The Approximation Property) The following question is involved in the matter of interpolation of Sobolev spaces on order of smoothness that will play a central role in the development of Besov spaces and Sobolev spaces of fractional order in Chapter 7:

> If $0 < k < m$ does there exist a constant C such that for every $u \in W^{k,p}(\Omega)$ and every sufficiently small ϵ there exists $u_\epsilon \in W^{m,p}(\Omega)$ satisfying
>
> $$\|u - u_\epsilon\|_p \le C\epsilon^k \|u\|_{k,p}, \qquad \text{and} \qquad \|u_\epsilon\|_{m,p} \le C\epsilon^{k-m} \|u\|_{k,p}?$$

If the answer is "yes," we will say that the domain Ω has the *approximation property*. Combined with the interpolation Theorem 5.2, this property will show that $W^{k,p}(\Omega)$ is suitably intermediate between $L^p(\Omega)$ and $W^{m,p}(\Omega)$ for purposes of interpolation. In Theorem 5.33 we prove that $\mathbb{R}^n$ itself has the approximation property. It will therefore follow that any domain Ω admitting a total extension operator will have the approximation property for any choice of k and m with $0 < k < m$. In particular, therefore, a domain satisfying the strong local Lipschitz condition has the approximation property.

There are domains with the approximation property that do not satisfy the strong local Lipschitz condition. The approximation property does not prevent a domain from lying on both sides of a boundary hypersurface. In [AF4] the authors obtain the property under the assumption that Ω satisfies the "smooth cone condition," which is essentially a cone condition with the added restriction that the cone must vary smoothly from point to point. Our proof of Theorem 5.33 is a simplified version of the proof in [AF4].

We begin by stating an elementary lemma.

5.32 LEMMA If $u \in L^p(\mathbb{R}^n)$ and $B_\epsilon(x)$ is the ball of radius ϵ about x, then

$$\int_{\mathbb{R}^n} \left(\int_{B_\epsilon(x)} |u(y)|\, dy \right)^p dx \le K_n^p \epsilon^{np} \|u\|_{p,\mathbb{R}^n}^p,$$

where K_n is the volume of the unit ball $B_1(0)$.

Proof. The proof is immediate using Hölder's inequality and a change or order of integration. ∎

5.33 THEOREM (An Approximation Theorem for $\mathbb{R}^n$) If $0 < k < m$, there exists a constant C such that for $u \in W^{k,p}(\mathbb{R}^n)$ and $0 < \epsilon \le 1$ there exists u_ϵ in $C^\infty(\mathbb{R}^n)$ such that the following seminorm inequalities hold:

$$\|u - u_\epsilon\|_p \le C\epsilon^k |u|_{k,p}, \qquad \text{and}$$

$$|u_\epsilon|_{j,p} \le \begin{cases} C\,\|u\|_{k,p} & \text{if } j \le k-1 \\ C\,\epsilon^{k-j}\,|u|_{k,p} & \text{if } j \ge k. \end{cases}$$

In particular, $\mathbb{R}^n$ has the approximation property.

Proof. It is sufficient to establish the inequalities for $u \in C_0^\infty(\mathbb{R}^n)$ which is dense in $W^{k,p}(\mathbb{R}^n)$. We apply Taylor's formula

$$f(1) = \sum_{j=0}^{k-1} \frac{1}{j!} f^{(j)}(0) + \frac{1}{(k-1)!} \int_0^1 (1-t)^{k-1} f^{(k)}(t)\,dt$$

to the function $f(t) = u\big(tx + (1-t)y\big)$ to obtain

$$\begin{aligned} u(x) = &\sum_{|\alpha| \le k-1} \frac{1}{\alpha!} D^\alpha u(y)(x-y)^\alpha \\ &+ \sum_{|\alpha|=k} \frac{k}{\alpha!}(x-y)^\alpha \int_0^1 (1-t)^{k-1} D^\alpha u\big(tx + (1-t)y\big)\,dt. \end{aligned}$$

Now let $\phi \in C_0^\infty\big(B_1(0)\big)$ satisfy $0 \le \phi(x) \le K_0$ for all x and $\int_{\mathbb{R}^n} \phi(x)\,dx = 1$. We multiply the above Taylor formula by $\epsilon^{-n}\phi\big((x-y)/\epsilon\big)$ and integrate y over $\mathbb{R}^n$ to obtain $u(x) = u_\epsilon(x) + R(x)$ where

$$\begin{aligned} u_\epsilon(x) &= \epsilon^{-n} \sum_{|\alpha| \le k-1} \frac{1}{\alpha!} \int_{\mathbb{R}^n} \phi\left(\frac{x-y}{\epsilon}\right)(x-y)^\alpha D^\alpha u(y)\,dy \\ R(x) &= \epsilon^{-n} \sum_{|\alpha|=k} \frac{k}{\alpha!} \int_{\mathbb{R}^n} \phi\left(\frac{x-y}{\epsilon}\right)(x-y)^\alpha\,dy \\ &\qquad \times \int_0^1 (1-t)^{k-1} D^\alpha u\big(tx + (1-t)y\big)\,dt. \end{aligned}$$

We can estimate $|u(x) - u_\epsilon(x)| = |R(x)|$ by reversing the order of the double integral, substituting $z = tx + (1-t)y$ (so that $z - x = (1-t)(y-x)$ and $dz = (1-t)^n\,dy$), and reversing the order of integration again:

$$\begin{aligned} |u(x) - u_\epsilon(x)| &\le K_0 \sum_{|\alpha|=k} \frac{k}{\alpha!}\epsilon^{-n} \int_0^1 (1-t)^{-1-n}\,dt \int_{B_{\epsilon(1-t)}(x)} |x-z|^k |D^\alpha u(z)|\,dz \\ &\le K_0 \sum_{|\alpha|=k} \frac{k}{\alpha!}\epsilon^{-n} \int_{B_\epsilon(x)} |x-z|^k |D^\alpha u(z)|\,dz \int_0^{1-|z-x|/\epsilon} (1-t)^{-n-1}\,dt \\ &< K_0 \sum_{|\alpha|=k} \frac{k}{\alpha!}\epsilon^{-n} \int_{B_\epsilon(x)} |x-z|^k |D^\alpha u(z)|\,dz \\ &\le K_0 \sum_{|\alpha|=k} \frac{k}{\alpha!}\epsilon^{k-n} \int_{B_\epsilon(x)} |D^\alpha u(z)|\,dz. \end{aligned}$$

Estimating the L^p-norm of the last integral above by the previous lemma, we obtain

$$\|u(x) - u_\epsilon(x)\|_p \le K_0 \sum_{|\alpha|=k} \frac{k}{\alpha!}\epsilon^k \|D^\alpha u\|_p \le C\epsilon^k |u|_{k,p}\,.$$

On the other hand, we have

$$u_\epsilon(x) = \epsilon^{-n} \int_{\mathbb{R}^n} \phi\left(\frac{x-y}{\epsilon}\right) P_{k-1}(u; x, y)\, dy,$$

where

$$P_j(u; x, y) = \sum_{i=0}^{j} T_i(u; x, y),$$

$$T_j(u; x, y) = \sum_{|\alpha|=j} \frac{1}{\alpha!} D^\alpha u(y)(x-y)^\alpha.$$

It is readily verified that

$$\frac{\partial}{\partial x_i} T_j(u; x, y) = \begin{cases} T_{j-1}(D_i u; x, y) & \text{if } j > 0 \\ 0 & \text{if } j = 0 \end{cases}$$

$$\frac{\partial}{\partial x_i} P_j(u; x, y) = \begin{cases} P_{j-1}(D_i u; x, y) & \text{if } j > 0 \\ 0 & \text{if } j = 0 \end{cases}$$

$$\frac{\partial}{\partial y_i} P_j(u; x, y) = T_j(D_i u; x, y) \qquad \text{for} \quad j \ge 0.$$

Since $\dfrac{\partial}{\partial x_i}\phi\left(\dfrac{x-y}{\epsilon}\right) = -\dfrac{\partial}{\partial y_i}\phi\left(\dfrac{x-y}{\epsilon}\right)$, integration by parts gives

$$\begin{aligned} D_i u_\epsilon(x) = \epsilon^{-n} &\int_{\mathbb{R}^n} \phi\left(\frac{x-y}{\epsilon}\right) P_{k-2}(D_i u; x, y)\, dy \\ &+ \int_{\mathbb{R}^n} \phi\left(\frac{x-y}{\epsilon}\right) T_{k-1}(D_i u; x, y)\, dy. \end{aligned}$$

By induction, if $|\beta| = j \le k$,

$$\begin{aligned} D^\beta u_\epsilon(x) = \epsilon^{-n} &\int_{\mathbb{R}^n} \phi\left(\frac{x-y}{\epsilon}\right) P_{k-1-j}(D^\beta u; x, y)\, dy \\ &+ j\epsilon^{-n} \int_{\mathbb{R}^n} \phi\left(\frac{x-y}{\epsilon}\right) T_{k-j}(D^\beta u; x, y)\, dy. \end{aligned}$$

When $j = k$ the sums P_{k-1-j} are empty, leaving only the second line above, which becomes

$$k\epsilon^{-n} \int_{\mathbb{R}^n} \phi\left(\frac{x-y}{\epsilon}\right) T_0(D^\beta u; x, y)\, dy = k\epsilon^{-n} \int_{\mathbb{R}^n} \phi\left(\frac{x-y}{\epsilon}\right) D^\beta u(y)\, dy.$$

Write any multi-index γ with $|\gamma| > k$ in the form $\beta + \delta$ with $|\beta| = k$ to get that

$$D^\gamma u_\epsilon(x) = k\epsilon^{-n-|\delta|} \int_{\mathbb{R}^n} D^\delta \phi\left(\frac{x-y}{\epsilon}\right) D^\beta u(y)\, dy$$

in these cases. Apply the previous lemma to the various terms above to get that

$$|u_\epsilon|_{j,p} \le \begin{cases} C\,\|u\|_p & \text{if } j \le k-1 \\ C\epsilon^{k-j}\,|u|_{k,p} & \text{if } j \ge k. \end{cases}$$

In deriving this when $j < k$, expand the (nonempty) sums P_{k-1-j} to see that

$$|D^\beta u_\epsilon(x)| \le K_0 \epsilon^{-n} \int_{B_\epsilon(x)} \left[\sum_{i=0}^{k-1-j} |T_i(D^\beta u; x, y)| + j|T_{k-j}(D^\beta u; x, y)| \right] dy.$$

This completes the proof. ∎

Boundary Traces

5.34 Of importance in the study of boundary value problems for differential operators defined on a domain Ω is the determination of spaces of functions defined on the boundary of Ω that contain the traces $u\big|_{\text{bdry}\,\Omega}$ of functions u in $W^{m,p}(\Omega)$. For example, if $W^{m,p}(\Omega) \to C^0(\overline{\Omega})$, then clearly $u\big|_{\text{bdry}\,\Omega}$ belongs to $C(\text{bdry}\,\Omega)$. We outline below an L^q-imbedding result for such traces which can be obtained for domains with suitably smooth boundaries as a corollary of Theorem 4.12 via the use of an extension operator.

The more interesting problem of characterizing the image of $W^{m,p}(\Omega)$ under the mapping $u \to u\big|_{\text{bdry}\,\Omega}$ will be dealt with in Chapter 7. See, in particular, Theorem 7.39. The characterization is in terms of Besov spaces which are generalized Sobolev spaces of fractional order.

5.35 Let Ω be a domain in $\mathbb{R}^n$ satisfying the uniform C^m-regularity condition of Paragraph 4.10. Thus there exists a locally finite open cover $\{U_j\}$ of bdry Ω, and corresponding m-smooth transformations Ψ_j mapping $B = \{y \in \mathbb{R}^n : |y| < 1\}$ onto U_j such that $U_j \cap \text{bdry}\,\Omega = \Psi_j(B_0)$, where $B_0 = \{y \in B : y_n = 0\}$. If f is a function having support in U_j, we may define the integral of f over bdry Ω via

$$\int_{\text{bdry}\,\Omega} f(x)\, d\sigma = \int_{U_j \cap \text{bdry}\,\Omega} f(x)\, d\sigma = \int_{B_0} f \circ \Psi_j(y', 0)\, J_j(y')\, dy',$$

where $d\sigma$ is the $(n-1)$-volume element on bdry Ω, $y' = (y_1, \dots, y_{n-1})$, and, if $x = \Psi_j(y)$, then

$$J_j(y') = \left[\sum_{k=1}^{n} \left(\frac{\partial(x_1, \dots, \hat{x}_k, \dots, x_n)}{\partial(y_1, \dots, y_{n-1})} \right)^2 \right]^{1/2} \Bigg|_{y_n=0} .$$

If f is an arbitrary function defined on $\mathbb{R}^n$, we may set

$$\int_{\text{bdry}\,\Omega} f(x)\, d\sigma = \sum_j \int_{\text{bdry}\,\Omega} f(x) v_j(x)\, d\sigma,$$

where $\{v_j\}$ is a partition of unity for bdry Ω subordinate to $\{U_j\}$.

5.36 THEOREM (A Boundary Trace Imbedding Theorem) Let Ω be a domain in $\mathbb{R}^n$ satisfying the uniform C^m-regularity condition, and suppose there exists a simple (m, p)-extension operator E for Ω. Also suppose that $mp < n$ and $p \le q \le p^* = (n-1)p/(n-mp)$. Then

$$W^{m,p}(\Omega) \to L^q(\text{bdry}\,\Omega) . \tag{28}$$

If $mp = n$, then imbedding (28) holds for $p \le q < \infty$.

Proof. Imbedding (28) should be interpreted in the following sense. If $u \in W^{m,p}(\Omega)$, then Eu has a trace on bdry Ω in the sense described in Paragraph 4.2, and $\|Eu\|_{0,q,\text{bdry}\,\Omega} \le K \|u\|_{m,p,\Omega}$ with K independent of u. Note that since $C_0(R^n)$ is dense in $W^{m,p}(\Omega)$, $\|Eu\|_{0,q,\text{bdry}\,\Omega}$ is independent of the particular extension operator E used.

We prove the special case $mp < n$, $q = p^* = (n-1)p/(n-mp)$ of the theorem; the other cases are similar. We use the notations of the previous Paragraph.

There is a constant K_1 such that for every $u \in W^{m,p}(\Omega)$,

$$\|Eu\|_{m,p,\mathbb{R}^n} \le K_1 \|u\|_{m,p,\Omega} .$$

By the uniform C^m-regularity condition (see Paragraph 4.10) there exists a constant K_2 such that for each j and every $y \in B$ we have $x = \Psi_j(y) \in U_j$,

$$|J_j(y')| \le K_2, \qquad \text{and} \qquad \left| \frac{\partial(y_1, \dots, y_n)}{\partial(x_1, \dots, x_n)} \right| \le K_2.$$

Noting that $0 \le v_j(x) \le 1$ on $\mathbb{R}^n$, and using imbedding (4) of Theorem 4.12 applied over B, we have, for $u \in W^{m,p}(\Omega)$,

$$\int_{\text{bdry}\,\Omega} |Eu(x)|^q\, d\sigma \le \sum_j \int_{U_j \cap \text{bdry}\,\Omega} |Eu(x)|^q\, d\sigma$$

$$
\begin{aligned}
&\le K_2 \sum_j \| Eu \circ \Psi_j \|_{0,q,B_0}^q \\
&\le K_3 \sum_j \Big(\| Eu \circ \Psi_j \|_{m,p,B}^p \Big)^{q/p} \\
&\le K_4 \left(\sum_j \| Eu \|_{m,p,U_j}^p \right)^{q/p} \\
&\le K_4 R \, \| Eu \|_{m,p,\mathbb{R}^n}^q \\
&\le K_5 \, \| u \|_{m,p,\Omega}^q \, .
\end{aligned}
$$

The second last inequality above makes use of the finite intersection property possessed by the cover $\{U_j\}$. The constant K_4 is independent of j because $|D^\alpha \Psi_{j,i}(y)| \le$ const for all i, j, where $\Psi_j = (\Psi_{j,1}, \dots, \Psi_{j,n})$. This completes the proof. ∎

Finally, we show that functions in $W^{m,p}(\Omega)$ belong to $W_0^{m,p}(\Omega)$ if and only if they have suitably trivial boundary traces.

5.37 THEOREM (Trivial Traces) Under the same hypotheses as Theorem 5.36, a function u in $W^{m,p}(\Omega)$ belongs to $W_0^{m,p}(\Omega)$ if and only the boundary traces of its derivatives of order less than m all coincide with the 0-function.

Proof. Every function in $C_0^\infty(\Omega)$ has trivial boundary trace, and so do all derivatives of such functions. Since the trace mapping is a continuous linear operator from $W^{m,p}(\Omega)$ to $W^{m-1,p}(\text{bdry}\ \Omega)$, all functions in $W_0^{m,p}(\Omega)$ have trivial boundary traces, and so do their derivatives of order less than m.

To prove the converse, we suppose that $u \in W^{m,p}(\Omega)$ and that u and its derivatives or order less than m have trivial boundary traces. Localization and a suitable change of variables reduces matters to the case where Ω is the half-space $\{x \in R^n : x_n > 0\}$. We then show that the zero-extension $\tilde{u}$ must belong to $W^{m,p}(\mathbb{R}^n)$, forcing u to belong to $W_0^{m,p}(\Omega)$ by Theorem 5.29.

In fact, we claim that if $u \in W^{m,p}(\Omega)$ has trivial boundary traces for u and its derivatives of order less than m, then the distributional derivatives $D^\alpha \tilde{u}$ of order at most m coincide with the zero-extensions $\widetilde{D^\alpha u}$. To verify this, approximate the integrals

$$
\int_{\mathbb{R}^n} \tilde{u}(x) D^\alpha \phi(x)\, dx \qquad \text{and} \qquad (-1)^{|\alpha|} \int_{\mathbb{R}^n} \widetilde{D^\alpha u}(x) \phi(x)\, dx \tag{29}
$$

by approximating u with functions v_j in $C^\infty(\overline{\Omega})$, without requiring that these approximations have trivial traces.

Let e_n be the unit vector $(0, \dots, 0, 1)$. Since $v_j \in C^\infty(\overline{\Omega})$, integrating by parts with respect to the other variables and then with respect to x_n shows that the difference between the integrals

$$\int_{\mathbb{R}^n} \tilde{v}_j(x) D^\alpha \phi(x)\, dx \qquad \text{and} \qquad (-1)^{|\alpha|} \int_{\mathbb{R}^n} \widetilde{D^\alpha v_j}(x) \phi(x)\, dx$$

is a finite alternating sum of integrals of the form

$$\int_{\mathbb{R}^{n-1}} D^{\alpha - k e_n} v_j(x_1, \ldots, x_{n-1}, 0) D_n^{k-1} \phi(x_1, \ldots, x_{n-1}, 0)\, dx_1 \cdots dx_{n-1} \tag{30}$$

with $k > 0$. Choose the sequence $\{v_j\}$ to converge to u in $W^{m,p}(\Omega)$. For each multi-index β with $\beta < \alpha$, the trace of $D^\beta v_j$ will converge in $L^p(\mathbb{R}^{n-1})$ to the trace of $D^\beta u$, that is to 0 in that space. Since the restriction of $D_n^{k-1}\phi$ to R^{n-1} belongs to $L^{p'}(\mathbb{R}^{n-1})$, each of the integrals in (30) tends to 0 as $j \to \infty$.

It follows that the two integrals in (29) are equal, and that $\tilde{u} \in W^{m,p}(\mathbb{R}^n)$. This completes the proof. ∎

6

COMPACT IMBEDDINGS OF SOBOLEV SPACES

The Rellich-Kondrachov Theorem

6.1 (Restricted Imbeddings) Let Ω be a domain in $\mathbb{R}^n$ and let Ω_0 be a subdomain of Ω. Let $X(\Omega)$ denote any of the possible target spaces for imbeddings of $W^{m,p}(\Omega)$, that is, $X(\Omega)$ is a space of the form $C_B^j(\Omega)$, $C^j(\overline{\Omega})$, $C^{j,\lambda}(\overline{\Omega})$, $L^q(\Omega_k)$, or $W^{j,q}(\Omega_k)$, where Ω_k, $1 \le k \le n$, is the intersection of Ω with a k-dimensional plane in $\mathbb{R}^n$. Since the linear restriction operator $i_{\Omega_0} : u \to u\big|_{\Omega_0}$ is bounded from $X(\Omega)$ into $X(\Omega_0)$ (in fact $\|i_{\Omega_0}u; X(\Omega_0)\| \le \|u; X(\Omega)\|$) any imbedding of the form

$$W^{m,p}(\Omega) \to X(\Omega) \tag{1}$$

can be composed with this restriction to yield the imbedding

$$W^{m,p}(\Omega) \to X(\Omega_0) \tag{2}$$

and (2) has imbedding constant no larger than (1).

6.2 (Compact Imbeddings) Recall that a set A in a normed space is precompact if every sequence of points in A has a subsequence converging in norm to an element of the space. An operator between normed spaces is called compact if it maps bounded sets into precompact sets, and is called completely continuous if it is continuous and compact. (See Paragraph 1.24; for linear operators compactness and complete continuity are equivalent.) In this chapter we are concerned with the compactness of imbedding operators which are continuous whenever they exist, and so are completely continuous whenever they are compact.

If Ω satisfies the hypotheses of the Sobolev imbedding Theorem 4.12 and if Ω_0 is a bounded subset of Ω, then, with the exception of certain extreme cases, all the restricted imbeddings (1) corresponding to imbeddings asserted in Theorem 4.12 are compact. The most important of these compact imbedding results originated in a lemma of Rellich [Re] and was proved specifically for Sobolev spaces by Kondrachov [K]. Such compact imbeddings have many important applications in analysis, especially to showing that linear elliptic partial differential equations defined over bounded domains have discrete spectra. See, for example, [EE] and [ET] for such applications and further refinements.

We summarize the various compact imbeddings of $W^{m,p}(\Omega)$ in the following theorem

6.3 THEOREM (The Rellich-Kondrachov Theorem) Let Ω be a domain in $\mathbb{R}^n$, let Ω_0 be a bounded subdomain of Ω, and let Ω_0^k be the intersection of Ω_0 with a k-dimensional plane in $\mathbb{R}^n$. Let $j \geq 0$ and $m \geq 1$ be integers, and let $1 \leq p < \infty$.

PART I If Ω satisfies the cone condition and $mp \leq n$, then the following imbeddings are compact:

$$W^{j+m,p}(\Omega) \to W^{j,q}\left(\Omega_0^k\right) \quad \text{if} \quad 0 < n - mp < k \leq n \text{ and } 1 \leq q < kp/(n-mp), \tag{3}$$

$$W^{j+m,p}(\Omega) \to W^{j,q}\left(\Omega_0^k\right) \quad \text{if} \quad n = mp,\ 1 \leq k \leq n \text{ and } 1 \leq q < \infty. \tag{4}$$

PART II If Ω satisfies the cone condition and $mp > n$, then the following imbeddings are compact:

$$W^{j+m,p}(\Omega) \to C_B^j(\Omega_0) \tag{5}$$

$$W^{j+m,p}(\Omega) \to W^{j,q}\left(\Omega_0^k\right) \quad \text{if} \quad 1 \leq q < \infty. \tag{6}$$

PART III If Ω satisfies the strong local Lipschitz condition, then the following imbeddings are compact:

$$W^{j+m,p}(\Omega) \to C^j(\overline{\Omega_0}) \quad \text{if} \quad mp > m, \tag{7}$$

$$W^{j+m,p}(\Omega) \to C^{j,\lambda}(\overline{\Omega_0}) \quad \text{if} \quad mp > n \geq (m-1)p \text{ and } 0 < \lambda < m - (n/p). \tag{8}$$

PART IV If Ω is an arbitrary domain in $\mathbb{R}^n$, the imbeddings (3)–(8) are compact provided $W^{j+m,p}(\Omega)$ is replaced by $W_0^{j+m,p}(\Omega)$.

6.4 REMARKS

1. Note that if Ω is bounded, we may have $\Omega_0 = \Omega$ in the statement of the theorem.
2. If X, Y, and Z are spaces for which we have the imbeddings $X \to Y$ and $Y \to Z$, and if one of these imbeddings is compact, then the composite imbedding $X \to Z$ is compact. Thus, for example, if $Y \to Z$ is compact, then any sequence $\{u_j\}$ bounded in X will be bounded in Y and will therefore have a subsequence $\{u'_j\}$ convergent in Z.
3. Since the extension operator $u \to \tilde{u}$, where $\tilde{u}(x) = u(x)$ if $x \in \Omega$ and $\tilde{u}(x) = 0$ if $x \notin \Omega$, defines an imbedding $W_0^{j+m,p}(\Omega) \to W^{j+m,p}(\mathbb{R}^n)$ by Lemma 3.27, Part IV of Theorem 6.3 follows from application of Parts I–III to $\mathbb{R}^n$.
4. In proving the compactness of any of the imbeddings (3)–(8) it is sufficient to consider only the case $j = 0$. Suppose, for example, that (3) has been proven compact if $j = 0$. For $j \geq 1$ and $\{u_i\}$ a bounded sequence in $W^{j+m,p}(\Omega)$ it is clear that $\{D^\alpha u_i\}$ is bounded in $W^{m,p}(\Omega)$ for each α such that $|\alpha| \leq j$. Hence $\{D^\alpha u_i|_{\Omega_0^k}\}$ is precompact in $L^q(\Omega_0^k)$ with q specified as in (3). It is possible, therefore, to select (by finite induction) a subsequence $\{u'_i\}$ of $\{u_i\}$ for which $\{D^\alpha u'_i|_{\Omega_0^k}\}$ converges in $L^q(\Omega_0^k)$ for each α such that $|\alpha| \leq j$. Thus $\{u'_i|_{\Omega_0^k}\}$ converges in $W_0^{j,q}(\Omega_0^k)$ and (3) is compact.
5. Since Ω_0 is bounded, $C_B^0(\Omega_0^k) \to L^q(\Omega_0^k)$ for $1 \leq q \leq \infty$; in fact $\|u\|_{0,q,\Omega_0^k} \leq \|u; C_B^0(\Omega_0^k)\|[\mathrm{vol}(\Omega_0^k)]^{1/q}$. Thus the compactness of (6) (for $j = 0$) follows from that of (5).
6. For the purpose of proving Theorem 6.3 the bounded subdomain Ω_0 of Ω may be assumed to satisfy the cone condition in Ω does. If C is a finite cone determining the cone condition for Ω, let $\tilde{\Omega}$ be the union of all finite cones congruent to C, contained in Ω and having nonempty intersection with Ω_0. Then $\Omega_0 \subset \tilde{\Omega} \subset \Omega$ and $\tilde{\Omega}$ is bounded and satisfies the cone condition. If $W^{m,p}(\Omega) \to X(\tilde{\Omega})$ is compact, then so is $W^{m,p}(\Omega) \to X(\Omega_0)$ by restriction.

6.5 (Proof of Theorem 6.3, Part III) If $mp > n \geq (m-1)p$ and if $0 < \lambda < m - (n/p)$, then there exists μ such that $\lambda < \mu < m - (n/p)$. Since Ω_0 is bounded, the imbedding $C^{0,\mu}(\overline{\Omega_0}) \to C^{0,\lambda}(\overline{\Omega_0})$ is compact by Theorem 1.34. Since $W^{m,p}(\Omega) \to C^{0,\mu}(\overline{\Omega}) \to C^{0,\mu}(\overline{\Omega_0})$ by Theorem 4.12 and restriction, imbedding (8) is compact for $j = 0$ by Remark 6.4(2).

If $mp > n$, let j^* be the nonnegative integer satisfying the inequalities $(m - j^*)p > n \geq (m - j^* - 1)p$. Then we have the imbedding chain

$$W^{m,p}(\Omega) \to W^{m-j^*,p}(\Omega) \to C^{0,\mu}(\overline{\Omega_0}) \to C(\overline{\Omega_0}) \qquad (9)$$

where $0 < \mu < m - j^* - (n/p)$. The last imbedding in (9) is compact by Theorem 1.34. Thus (7) is compact for $j = 0$. ∎

6.6 (Proof of Theorem 6.3, Part II) As noted in Remark 6.4(6), Ω_0 may be assumed to satisfy the cone condition. Since Ω_0 is bounded it can, by Lemma 4.22 be written as a finite union, $\Omega_0 = \bigcup_{k=1}^{M} \Omega_k$, where each Ω_k satisfies the strong local Lipschitz condition. If $mp > n$, then

$$W^{m,p}(\Omega) \to W^{m,p}(\Omega_k) \to C(\overline{\Omega_k}),$$

the latter imbedding being compact as proved above. If $\{u_i\}$ is a sequence bounded in $W^{m,p}(\Omega)$, we may select by finite induction on k a subsequence $\{u_i'\}$ whose restriction to Ω_k converges in $C(\overline{\Omega_k})$ for each k, $1 \le k \le M$. But this subsequence then converges in $C_B^0(\Omega_0)$, so proving that (5) is compact for $j = 0$. Therefore (6) is also compact by Remark 6.4(5). ∎

6.7 LEMMA Let Ω be a domain in $\mathbb{R}^n$, Ω_0 a subdomain of Ω, and Ω_0^k the intersection of Ω_0 with a k-dimensional plane in $\mathbb{R}^n$ $(1 \le k \le n)$. Let $1 \le q_1 < q_0$ and suppose that

$$W^{m,p}(\Omega) \to L^{q_0}\left(\Omega_0^k\right)$$

and

$$W^{m,p}(\Omega) \to L^{q_1}\left(\Omega_0^k\right) \qquad \text{compactly.}$$

If $q_1 \le q < q_0$, then

$$W^{m,p}(\Omega) \to L^{q}\left(\Omega_0^k\right) \qquad \text{compactly.}$$

Proof. Let $\lambda = q_1(q_0 - q)/q(q_0 - q_1)$ and $\mu = q_0(q - q_1)/q(q_0 - q_1)$. Then $\lambda > 0$ and $\mu \ge 0$. By Hölder's inequality there exists a constant K such that for all $u \in W^{m,p}(\Omega)$,

$$\|u\|_{0,q,\Omega_0^k} \le \|u\|_{0,q_1,\Omega_0^k}^{\lambda} \|u\|_{0,q_0,\Omega_0^k}^{\mu} \le K \|u\|_{0,q_1,\Omega_0^k}^{\lambda} \|u\|_{m,p,\Omega}^{\mu}.$$

A sequence bounded in $W^{m,p}(\Omega)$ has a subsequence which converges in $L^{q_1}(\Omega_0^k)$ and is therefore a Cauchy sequence in that space. Applying the inequality above to differences between terms of this sequence shows that it is also a Cauchy sequence in $L^q(\Omega_0^k)$, so the imbedding of $W^{m,p}(\Omega)$ into $L^q(\Omega_0^k)$ is compact. ∎

6.8 (Proof of Theorem 6.3, Part I) First we deal with (the case $j = 0$ of) imbedding (3). Assume for the moment that $k = n$ and let $q_0 = np/(n - mp)$. In order to prove that the imbedding

$$W^{m,p}(\Omega) \to L^q(\Omega_0), \qquad 1 \le q < q_0, \tag{10}$$

is compact, it sufficed, by Lemma 6.7, to do so only for $q = 1$. For $j = 1, 2, 3, \dots$ let

$$\Omega_j = \{x \in \Omega : \operatorname{dist}(x, \operatorname{bdry}\Omega) > 2/j\}.$$

Let S be a set of functions bounded in $W^{m,p}(\Omega)$. We show that S (when restricted to Ω_0) is precompact in $L^1(\Omega_0)$ by showing that S satisfies the conditions of Theorem 2.32. Accordingly, let $\epsilon > 0$ be given and for each $u \in W^{m,p}(\Omega)$ set

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega_0 \\ 0 & \text{otherwise.} \end{cases}$$

By Hölder's inequality and since $W^{m,p}(\Omega) \to L^{q_0}(\Omega_0)$, we have

$$\begin{aligned} \int_{\Omega_0 - \Omega_j} |u(x)|\, dx &\le \left(\int_{\Omega_0 - \Omega_j} |u(x)|^{q_0}\, dx \right)^{1/q_0} \left(\int_{\Omega_0 - \Omega_j} 1\, dx \right)^{1-1/q_0} \\ &\le K_1 \, \|u\|_{m,p,\Omega} \left[\operatorname{vol}(\Omega_0 - \Omega_j)\right]^{1-1/q_0}, \end{aligned}$$

with K_1 independent of u. Since $q_0 > 1$ and Ω_0 has finite volume, j may be selected large enough to ensure that for every $u \in S$,

$$\int_{\Omega_0 - \Omega_j} |u(x)|\, dx < \epsilon$$

and also, for every $h \in \mathbb{R}^n$,

$$\int_{\Omega_0 - \Omega_j} |\tilde{u}(x+h) - \tilde{u}(x)|\, dx < \frac{\epsilon}{2}.$$

Now if $|h| < 1/j$, then $x + th \in \Omega_{2j}$ provided $x \in \Omega_j$ and $0 \le t \le 1$. If $u \in C^\infty(\Omega)$, it follows that

$$\begin{aligned} \int_{\Omega_j} |u(x+h) - u(x)|\, dx &\le \int_{\Omega_j} dx \int_0^1 \left| \frac{d}{dt} u(x+th) \right| dt \\ &\le |h| \int_0^1 dt \int_{\Omega_{2j}} |\operatorname{grad} u(y)|\, dy \\ &\le |h| \; \|u\|_{1,1,\Omega_0} \le K_2 |h| \; \|u\|_{m,p,\Omega}, \end{aligned}$$

where K_2 is independent of u. Since $C^\infty(\Omega)$ is dense in $W^{m,p}(\Omega)$, this estimate holds for any $u \in W^{m,p}(\Omega)$. Hence if $|h|$ is sufficiently small, we have

$$\int_{\Omega_0} |\tilde{u}(x+h) - \tilde{u}(x)|\, dx < \epsilon.$$

Hence S is precompact in $L^1(\Omega_0)$ by Theorem 2.32 and imbedding (10) is compact.

Next suppose that $k < n$ and $p > 1$. The Sobolev Imbedding Theorem 4.12 assures us that $W^{m,p}(\Omega) \to L^{kp/(n-mp)}(\Omega_0^k)$. For any $q < kp/(n-mp)$ we can choose r such that $1 \le r < p, n-mr < k$, and $q \le kr/(n-mr) < kp/(n-mp)$. Since Ω_0 is bounded, the imbeddings

$$W^{m,p}(\Omega) \to W^{m,p}(\Omega_0) \to W^{m,r}(\Omega_0)$$

exist. By Theorem 5.10 we have

$$\begin{aligned} \|u\|_{q,\Omega_0^k} &\le K_1 \|u\|_{kr/(n-mr),\Omega_0^k} \\ &\le K_2 \|u\|_{nr/(n-mr),\Omega_0}^{1-\theta} \|u\|_{m,r,\Omega_0}^{\theta} \\ &\le K_3 \|u\|_{nr/(n-mr),\Omega_0}^{1-\theta} \|u\|_{m,p,\Omega}^{\theta}, \end{aligned}$$

where K_j and θ are constants (independent of $u \in W^{m,p}(\Omega)$) and θ satisfies $0 < \theta < 1$. Since $nr/(n-mr) < np/(n-mp)$, a sequence bounded in $W^{m,p}(\Omega)$ must have a subsequence convergent in $L^{nr/(n-mr)}(\Omega_0)$ by the earlier part of this proof. That sequence is therefore a Cauchy sequence in $L^{nr/(n-mr)}(\Omega_0)$, and by the above inequality it is therefore a Cauchy sequence in $L^q(\Omega_0^k)$, so the imbedding $W^{m,p}(\Omega) \to L^q(\Omega_0^k)$ is compact and so is $W^{m,p}(\Omega) \to L^1(\Omega_0^k)$.

If $p = 1$ and $0 \le n - m < k < n$, then necessarily $m \ge 2$. Composing the continuous imbedding $W^{m,1}(\Omega) \to W^{m-1,r}(\Omega)$, where $r = n/(n-1) > 1$, with the compact imbedding $W^{m-1,r}(\Omega) \to L^1(\Omega_0^k)$, (which is compact because $k \ge n-(m-1) > n-(m-1)r$), completes the proof of the compactness of (3).

To prove that imbedding (4) is compact we proceed as follows. If $n = mp$, $p > 1$, and $1 \le q < \infty$, then we may select r so that $1 \le r < p$, $k > n - mr > 0$, and $kr/(n-mr) > q$. Assuming again that Ω_0 satisfies the cone condition, we have

$$W^{m,p}(\Omega) \to W^{m,r}(\Omega_0) \to L^q(\Omega_0^k).$$

The latter imbedding is compact by (3). If $p = 1$ and $n = m \ge 2$, then, setting $r = n/(n-1) > 1$ so that $n = (n-1)r$, we have for $1 \le q < \infty$,

$$W^{n,1}(\Omega) \to W^{n-1,r}(\Omega) \to L^q(\Omega_0^k),$$

the latter imbedding being compact as shown immediately above. Finally, if $n = m = p = 1$, then $k = 1$ also. Letting $q_0 > 1$ be arbitrarily chosen we prove the compactness of $W^{1,1}(\Omega) \to L^1(\Omega_0)$ exactly as in the case $k = n$ considered at the beginning of this proof. Since $W^{1,1}(\Omega) \to L^q(\Omega_0)$ for $1 \le q < \infty$, all these imbeddings are compact by Lemma 6.7. ∎

Two Counterexamples

6.9 (Quasibounded Domains) We say that an unbounded domain $\Omega \subset \mathbb{R}^n$ is *quasibounded* if

$$\lim_{\substack{x\in\Omega \\ |x|\to\infty}} \operatorname{dist}(x, \operatorname{bdry}\Omega) = 0.$$

An unbounded domain is not quasibounded if and only if it contains infinitely many pairwise disjoint congruent balls.

6.10 Two obvious questions arise from consideration of the statement of the Rellich-Kondrachov Theorem 6.3. First, can the theorem be extended to cover unbounded Ω_0? Second, can the *extreme cases*

$$W^{j+m,p}(\Omega) \to W^{j,q}\left(\Omega_0^k\right), \qquad \begin{array}{l} 0 < n-, p < k \le n, \\ q = kp/(n-mp) \end{array}$$

and

$$W^{j+m,p}(\Omega) \to C^{j,\lambda}(\overline{\Omega_0}), \qquad \begin{array}{l} mp > n > (m-1)p, \\ \lambda = m - (n/p) \end{array}$$

ever be compact? The first of these questions will be investigated later in this chapter. For the moment though we show that the answer is negative if $k = n$ and Ω_0 is not quasibounded. However, the situation changes (see [Lp]) for subspaces of symmetric functions.

6.11 EXAMPLE Let Ω be an unbounded domain in $\mathbb{R}^n$ that is not quasibounded. Then there exists a sequence $\{B_i\}$ of mutually disjoint open balls contained in Ω and all having the same positive radius. Let $\phi_1 \in C_0^\infty(B_1)$ satisfy $\|\phi_1\|_{j,p,B_1} = A_{j,p} > 0$ for each $j = 0, 1, 2, \ldots$ and each $p \ge 1$. Let ϕ_i be a translate of ϕ_1 having support in B_i. Then $\{\phi_i\}$ is a bounded sequence in $W_0^{m,p}(\Omega)$ for any fixed m and p. But for any q,

$$\|\phi_i - \phi_k\|_{j,q,\Omega} = \left(\|\phi_i\|_{j,q,B_i}^q + \|\phi_k\|_{j,q,B_i}^q\right)^{1/q} = 2^{1/q} A_{j,q} > 0$$

so that $\{\phi_i\}$ cannot have a sequence converging in $W^{j,q}(()\,\Omega)$ for any $j \ge 0$. Thus no compact imbedding of the form $W_0^{j+m,p}(\Omega) \to W^{j,q}(\Omega)$ is possible. The non-compactness of the other imbeddings of Theorem 6.3 is proved similarly. ∎

Now we provide an example showing that the answer to the second question raised in Paragraph 6.10 is always negative.

6.12 EXAMPLE Let integers j, m, n be given with $j \ge 0$ and $m, n \ge 1$. Let $p \ge 1$. If $mp < n$, let k be an integer such that $n - mp < k \le n$ and let $q = kp/(n - mp)$. If $(m-1)p < n < mp$, let $\lambda = m - (n/p)$. Let Ω

be a domain in $\mathbb{R}^n$ and let Ω_0 be a nonempty bounded subdomain of Ω having nonempty intersection Ω_0^k with a k-dimensional plane H in $\mathbb{R}^n$ which, without loss of generality, we can take to be the plane $\mathbb{R}^k$ spanned by the $x_1, x_2, \dots, x_k$ coordinate axes. We show that the imbeddings

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega_0^k) \qquad \text{if} \quad mp < n \tag{11}$$
$$W^{j+m,p}(\Omega) \to C^{j,\lambda}(\overline{\Omega_0}) \qquad \text{if} \quad (m-1)p \le n < mp \tag{12}$$

cannot be compact.

Let $B_r(x)$ be the open ball of radius r in $\mathbb{R}^n$ centred at x and let ϕ be a nontrivial function in $C_0^\infty\big(B_1(0)\big)$. Let $\{a_i\}$ be a sequence of distinct points in Ω_0^k, and let $B_i = B_{r_i}(a_i)$ where the positive radii r_i satisfy $r_i \le 1$ and are chosen so that the balls B_i are pairwise disjoint and contained in Ω_0. We define a scaled, translated dilation ϕ_i of ϕ with support in B_i by

$$\phi_i(x) = r_i^{j+m-(n/p)}\phi(y), \qquad \text{where} \quad x = a_i + r_i y.$$

The functions ϕ_i have disjoint supports in Ω_0 and, since $D^\alpha \phi_i(x) = r^{-|\alpha|}D^\alpha\phi(y)$ and $dx = r_i^n\, dy$, we have, for $|\alpha| \le j+m$,

$$\int_\Omega |D^\alpha \phi_i(x)|^p\, dx = r_i^{(j+m-|\alpha|)p}\int_\Omega |\mathscr{D}^\alpha \phi(y)|^p\, dy.$$

Therefore, $\{\phi\}$ is bounded in $W^{j+m,p}(\Omega)$.

On the other hand, $dx_1 \cdots dx_k = r_i^k\, dy_1 \cdots dy_k$, so that if $|\alpha| = j$, then

$$\int_{\Omega_0^k} |D^\alpha \phi_i(x)|^q\, dx_1 \cdots dx_k = r_i^{k+q[m-(n/p)]}\int_{\mathbb{R}^k} |D^\alpha \phi(y)|^q\, dy_1 \cdots dy_k.$$

Since $k + q\big[m - (n/p)\big] = 0$, this shows that

$$\|\phi_i\|_{j,q,\Omega_0^k} \ge |\phi_i|_{j,q,\Omega_0^k} = C_1\, |\phi|_{j,q,\mathbb{R}^k} > 0$$

for all i, and $\{\phi_i\}$ is bounded away from zero in $W^{j,q}(\Omega_0^k)$. The disjointness of the supports of the functions ϕ_i now implies that $\{\phi\}$ can have no subsequence converging in $W^{j,q}(\Omega_0^k)$, so the imbedding (11) cannot be compact.

Now suppose that $(m-1)p \le n < mp$. Let a be a point in $B_1(0)$ and β be a particular multiindex satisfying $|\beta| = j$ such that $|D^\beta \phi(a)| = C_2 > 0$. Let $b_i = a_i + r_i a$ and let c_i be the point on the boundary of B_i closest to b_i. We have

$$\big|D^\beta \phi_i(b_i)\big| = r_i^{m-(n/p)}C_2 = r_i^\lambda C_2,$$

and, since $D^\beta \phi_i(c_i) = 0$,

$$\left\|\phi_i\,;C^{j,\lambda}(\overline{\Omega_0})\right\| \ge \frac{\left|D^\beta \phi_i(b_i) - D^\beta \phi_i(c_i)\right|}{|b_i - c_i|^\lambda} = C_2 > 0.$$

Again, this precludes the existence of a subsequence of of $\{\phi_i\}$ convergent in $C^{j,\lambda}(\overline{\Omega_0})$, so the imbedding (12) cannot be compact. ∎

6.13 REMARK Observe that the above examples in fact showed that no imbeddings of $W_0^{j+m,p}(\Omega)$, not just of the larger space $W^{j+m,p}(\Omega)$, into the appropriate target space can be compact. We now examine the possibility of obtaining compact imbeddings of $W_0^{m,p}(\Omega)$ for certain unbounded domains.

Unbounded Domains — Compact Imbeddings of $W_0^{m,p}(\Omega)$

6.14 Let Ω be an unbounded domain in $\mathbb{R}^n$. We shall be concerned below with determining whether the imbedding

$$W_0^{m,p}(\Omega) \to L^p(\Omega) \tag{13}$$

is compact. If it is, then it will follow by Remark 6.4(4), Lemma 6.7, and the second part of the proof in Paragraph 6.8 that the imbeddings

$$W_0^{j+m,p}(\Omega) \to W^{j,q}(\Omega_k), \qquad 0 < n - mp < k \le n, \quad p \le q < kp/(n-mp),$$

$$W_0^{j+m,p}(\Omega) \to W^{j,q}(\Omega_k), \qquad n = mp, \quad 1 \le k \le n, p \le q < \infty$$

are also compact. See Theorem 6.28 for the corresponding compactness of imbeddings into continuous function spaces.

As was shown in Example 6.11, imbedding (13) cannot be compact unless Ω is quasibounded. In Theorem 6.16 we give a geometric condition on Ω that is sufficient to guarantee the compactness of (13), and in Theorem 6.19 we give an analytic condition that is necessary and sufficient for the compactness of (13). Both theorems are from [A2].

6.15 Let Ω_r denote the set $\{x \in \Omega : |x| \ge r\}$. In the following discussion any cube H referred to will have its edges parallel to the coordinate axes. For a domain Ω, a cube H, and an integer ν satisfying $1 \le \nu \le n$, we define the quantity $\mu_{n-\nu}(H, \Omega)$ to be the maximum of the $(n-\nu)$-measure of $P(H - \Omega)$ taken over all projections P onto $(n-\nu)$-dimensional faces of H.

6.16 THEOREM Let ν be an integer such that $1 \le \nu \le n$ and $mp > \nu$ (or $m = p = \nu = 1$). Suppose that for every $\epsilon > 0$ there exist numbers h and r with $0 < h \le 1$ and $r \ge 0$ such that for every cube $H \subset \mathbb{R}^n$ having side h and nonempty intersection with Ω_r we have

$$\frac{\mu_{n-\nu}(H, \Omega)}{h^{n-\nu}} \ge \frac{h^p}{\epsilon}.$$

Then imbedding (13) is compact.

6.17 REMARKS

1. We will deduce this theorem from Theorem 6.19 later in this section.
2. The above theorem shows that for quasibounded Ω the compactness of (13) may depend in an essential way on the dimension of bdry Ω.
3. For $\nu = n$, the condition of Theorem 6.16 places only the minimum restriction of quasiboundedness on Ω; if $mp > n$ then (13) is compact for any quasibounded Ω. It can also be shown that if $p > 1$ and Ω is quasibounded with boundary having no finite accumulation points, then (13) cannot be compact unless $mp > n$.
4. If $\nu = 1$, the condition of Theorem 6.16 places no restrictions on m and p but requires that bdry Ω be "essentially $(n - 1)$-dimensional." Any quasibounded domain whose boundary consists of reasonably regular $(n - 1)$-dimensional surfaces will satisfy that condition. An example of such a domain is the "spiny urchin" of Figure 4, a domain in $\mathbb{R}^2$ obtained by deleting from the plane the union of the sets S_k, $(k = 1, 2, \ldots)$, specified in polar coordinates by

$$S_k = \left\{(r, \theta) \,:\, r \geq k,\ \theta = n\pi/2^k,\ n = 1, 2, \ldots, 2^{k+1}\right\}.$$

Note that this domain, though quasibounded, is simply connected and has empty exterior.

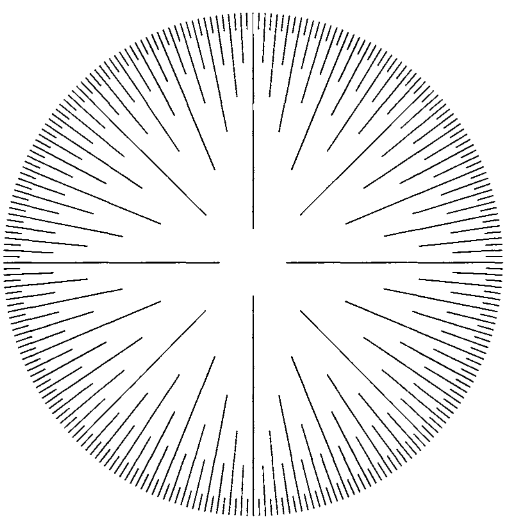


Fig. 4

5. More generally, if ν is the largest integer less than mp, the condition of Theorem 6.16 requires in a certain sense that the part of the boundary of Ω having dimension at least $n - \nu$ should bound a quasibounded domain.

6.18 (A Definition of Capacity) Let H be a cube of edge length h in $\mathbb{R}^n$ and let E be a closed subset of H. Given m and p we define a functional $I_H^{m,p}$ on $C^\infty(H)$ by

$$I_H^{m,p}(u) = \sum_{1\le j\le m} h^{jp}\,|u|_{j,p,H}^p = \sum_{1\le|\alpha|\le m} h^{|\alpha|p}\int_H |D^\alpha u(x)|^p\,dx.$$

Let $C^\infty(H,E)$ denote the set of all nontrivial functions $u \in C^\infty(H)$ that vanish identically in a neighbourhood of E. We define the (m,p)-capacity $Q^{m,p}(H,E)$ of E in H by

$$Q^{m,p}(H,E) = \inf\left\{\frac{I_H^{m,p}(u)}{\|u\|_{0,p,H}^p} : u \in C^\infty(H,E)\right\}.$$

Clearly $Q^{m,p}(H,E) \le Q^{m+1,p}(H,E)$ and, whenever $E \subset F \subset H$, we have $Q^{m,p}(H,E) \le Q^{m,p}(H,F)$.

The following theorem characterizes those domains for which imbedding (13) is compact in terms of this capacity.

6.19 THEOREM Imbedding (13) is compact if and only if Ω satisfies the following condition: For every $\epsilon > 0$ there exists $h \le 1$ and $r \ge 0$ such that the inequality

$$Q^{m,p}(H, H-\Omega) \ge h^p/\epsilon$$

holds for every n-cube H of edge length h having nonempty intersection with Ω_r. (This condition clearly implies that Ω is quasibounded.)

Prior to proving this theorem we prepare the following lemma.

6.20 LEMMA There exists a constant $K(m,p)$ such that for any n-cube H of edge length h, any measurable subset A of H with positive volume, and any $u \in C^1(H)$, we have

$$\|u\|_{0,p,H}^p \le \frac{2^{p-1}h^n}{\operatorname{vol}(A)}\|u\|_{0,p,A}^p + K\,\frac{h^{n+p}}{\operatorname{vol}(A)}\,\|\operatorname{grad} u\|_{0,p,H}^p\,.$$

Proof. Let $y \in A$ and $x = (\rho,\phi) \in H$, where (ρ,ϕ) denote spherical coordinates centred at y, in terms of which the volume element is given by $dx = \omega(\phi)\,\rho^{n-1}\,d\rho\,d\phi$. Let bdry H be specified by $\rho = f(\phi)$, $\phi \in \Sigma$. Clearly $f(\phi) \le \sqrt{n}h$. Since

$$u(x) = u(y) + \int_0^\rho \frac{d}{dr}u(r,\phi)\,dr,$$

we have by Lemma 2.2 and Hölder's inequality

$$\begin{aligned}
&\int_H |u(x)|^p\,dx \\
&\le 2^{p-1}h^n|u(y)|^p + 2^{p-1}\int_H \left|\int_0^\rho \frac{d}{dr}u(r,\phi)\,dr\right|^p dx \\
&\le 2^{p-1}h^n|u(y)|^p + 2^{p-1}\int_\Sigma \omega(\phi)\,d\phi \int_0^{f(\phi)} \rho^{n+p-2}\,d\rho \int_0^\rho |\operatorname{grad} u(r,\phi)|^p\,dr \\
&\le 2^{p-1}h^n|u(y)|^p + \frac{2^{p-1}}{n+p-1}\left(\sqrt{n}h\right)^{n+p-1}\int_H \frac{|\operatorname{grad} u(z)|^p}{|z-y|^{n-1}}\,dz.
\end{aligned}$$

Integrating y over A and using Lemma 4.64 we obtain

$$(\operatorname{vol}(A))\,\|u\|_{0,p,H}^p \le 2^{p-1}h^n\,\|u\|_{0,p,A}^p + Kh^{n+p}\,\|\operatorname{grad} u\|_{0,p,H}^p,$$

as required. ∎

6.21 (Proof of Theorem 6.19 — Necessity) Suppose that Ω does not satisfy the condition stated in the theorem. Then there exists a finite constant $K_1 = 1/\epsilon$ such that for every h with $0 < h \le 1$ there exists a sequence $\{H_j\}$ of mutually disjoint cubes of edge length h which intersect Ω and for which

$$Q^{m,p}(H_j, H_j - \Omega) < K_1 h^p.$$

By the definition of capacity, for each such cube H_j there exists a function $u_j \in C^\infty(H_j, H_j - \Omega)$ such that $\|u_j\|_{0,p,H_j}^p = h^n$, $\|\operatorname{grad} u_j\|_{0,p,H_j}^p \le K_1 h^n$, and $\|u_j\|_{m,p,H_j}^p \le K_2(h)$. Let $A_j = \{x \in H_j : |u_j(x)| < \frac{1}{2}\}$. By the previous Lemma we have

$$h^n \le \frac{2^{p-1}h^n}{\operatorname{vol}(A_j)} \cdot \frac{\operatorname{vol}(A_j)}{2^p} + \frac{KK_1}{\operatorname{vol}(A_j)} h^{2n+p}$$

from which it follows that $\operatorname{vol}(A_j) \le K_3 h^{n+p}$. Let us choose h so small that $K_3 h^p \le \frac{1}{3}$, whence $\operatorname{vol}(A_j) \le \frac{1}{3}\operatorname{vol}(H_j)$. Choose functions $w_j \in C_0^\infty(H_j)$ such that $w_j(x) = 1$ on a subset S_j of H_j having volume no less than $\frac{2}{3}\operatorname{vol}(H_j)$, and such that

$$\sup_j \max_{|\alpha|\le m} \sup_{x\in H_j} |D^\alpha w_j(x)| = K_4(h) < \infty.$$

Then $v_j = u_j w_j \in C_0^\infty(H_j \cap \Omega) \subset C_0^\infty(\Omega)$ and $|v_j(x)| \ge \frac{1}{2}$ on $S_j \cap (H_j - A_j)$, a set of volume not less than $h^n/3$. Hence $\|v_j\|_{0,p,H_j}^p \ge h^n/3 \cdot 2^p$. On the other hand

$$\int_{H_j} |D^\alpha u_j(x)|^p \cdot |D^\beta w_j(x)|^p\,dx \le K_4(h)\,K_2(h)$$

provided $|\alpha|, |\beta| \le m$. Hence $\{v_j\}$ is a bounded sequence in $W_0^{m,p}(\Omega)$. Since the supports of the functions v_j are disjoint, $\|v_i - v_j\|_{0,p,\Omega}^p \ge 2h^n/3 \cdot 2^p$ so the imbedding (13) cannot be compact. ∎

6.22 (Proof of Theorem 6.19 — Sufficiency) Suppose Ω satisfies the condition stated in the theorem. Let $\epsilon > 0$ be given and choose $r \ge 0$ and $h \le 1$ such that for every cube H of edge h intersecting Ω_r we have $Q^{m,p}(H, H - \Omega) \ge h^p/\epsilon^p$. Then for every $u \in C_0^\infty(\Omega)$ we obtain

$$\|u\|_{0,p,H}^p \le \frac{\epsilon^p}{h^p} I_H^{m,p}(u) \le \epsilon^p \|u\|_{m,p,H}^p .$$

Since a neighbourhood of Ω_r can be tessellated by such cubes H we have by summation

$$\|u\|_{0,p,\Omega_r} \le \epsilon \|u\|_{m,p,\Omega} .$$

That any bounded set S in $W_0^{m,p}(\Omega)$ is precompact in $L^p(\Omega)$ now follows from Theorems 2.33 and 6.3. ∎

6.23 LEMMA There is a constant K independent of h such that for any cube H in $\mathbb{R}^n$ having edge length h, for every q satisfying $p \le q \le np/(n - mp)$ (or $p \le q < \infty$ if $mp = n$, or $p \le q \le \infty$ if $mp > n$), and for every $u \in C^\infty(H)$ we have

$$\|u\|_{0,q,H} \le K \left(\sum_{|\alpha| \le m} h^{|\alpha|p - n + np/q} \|D^\alpha u\|_{0,p,H}^p \right)^{1/p} .$$

Proof. We may suppose H to be centred at the origin and let $\tilde{H}$ be the cube of unit edge concentric with H and having edges parallel to those of H. The stated inequality holds for $\tilde{u} \in C^\infty(\tilde{H})$ by the Sobolev imbedding theorem. It then follows for H via the dilation $u(x) = \tilde{u}(x/h)$. ∎

6.24 LEMMA If $mp > n$ (or if $m = p = n = 1$), there exists a constant $K = K(m, p, n)$ such that for every cube H of edge length h in $\mathbb{R}^n$ and every $u \in C^\infty(H)$ that vanishes in a neighbourhood of some point $y \in H$, we have

$$\|u\|_{0,p,H}^p \le K\, I_H^{m,p}(u).$$

Proof. Let (ρ, ϕ) be spherical coordinates centred at y. Then

$$u(\rho, \phi) = \int_0^\rho \frac{d}{dt} u(t, \phi)\, dt.$$

If $n > (m - 1)p$, then let $q = np/(n - mp + p)$, so that $q > n$. Otherwise let $q > \max\{n, p\}$ be an arbitrary and finite. If $(\rho, \phi) \in H$, then by Hölder's inequality

$$|u(\rho,\phi)|^q\rho^{n-1} \le \left(\sqrt{n}h\right)^{n-1}\int_0^\rho \left|\frac{d}{dt}u(t,\phi)\right|^q t^{n-1}\,dt \left(\int_0^{\sqrt{n}h} t^{-(n-1)/(q-1)}\,dt\right)^{q-1}$$
$$\le K_1 h^{q-1}\int_0^\rho \left|\frac{d}{dt}u(t,\phi)\right|^q t^{n-1}\,dt.$$

It follows, using the previous lemma with $m-1$ in place of m, that

$$\begin{aligned}\|u\|_{0,q,H}^q &\le K_2 h^q \int_H |\mathrm{grad}\, u(x)|^q\,dx \\ &\le K_2 h^q \sum_{|\alpha|=1} \|D^\alpha u\|_{0,q,H}^q \\ &\le K_3 h^q \sum_{|\alpha|=1}\left(\sum_{|\beta|\le m-1} h^{|\beta|p-n+n/q}\left\|D^{\alpha+\beta}u\right\|_{0,p,H}^p\right)^{q/p}.\end{aligned} \tag{14}$$

If $p > n$ (or $p = n = 1$) the desired result follows directly from (14) with $q = p$:

$$\|u\|_{0,p,H}^p \le K\, I_H^{1,p}(u) \le K\, I_H^{m,p}(u).$$

Otherwise, a further application of Hölder's inequality yields

$$\begin{aligned}\|u\|_{0,p,H}^p &\le \|u\|_{0,q,H}^p\,(\mathrm{vol}(H))^{(q-p)/q} \\ &\le K_2^{p/q}\sum_{1\le|\gamma|\le m} h^{|\gamma|p}\,\|D^\gamma u\|_{0,p,H}^p = K\, I_H^{m,p}(u). \quad ∎\end{aligned}$$

6.25 (Proof of Theorem 6.16) Let $mp > \nu$ (or $m = p = \nu = 1$) and let H be a cube in $\mathbb{R}^n$ for which $\mu_{n-\nu}(H,\Omega) \ge h^p/\epsilon$. Let P be the maximal projection of $H - \Omega$ onto an $(n-\nu)$-dimensional face of H and let $E = P(H - \Omega)$. Without loss of generality we may assume that the face F of H containing E is parallel to the $x_{\nu+1},\dots,x_n$ coordinate plane. For each point $x = (x', x'')$ in E, where $x' = (x_1,\dots,x_\nu)$ and $x'' = (x_{\nu+1},\dots,x_n)$ let $H_{x''}$ be the ν-dimensional cube of edge length h in which H intersects the ν-plane through x normal to F. By the definition of P there exists $y \in H_{x''} - \Omega$. If $u \in C^\infty(H, H - \Omega)$, then $u(\cdot, x'') \in C^\infty(H_{x''}, y)$. Applying the previous lemma to $u(\cdot, x'')$ we obtain

$$\int_{H_{x''}} |u(x',x'')|^p\,dx' \le K_1 \sum_{1\le|\alpha|\le m} h^{|\alpha|p}\int_{H_{x''}} |D^\alpha u(x',x'')|^p\,dx',$$

where K_1 is independent of H, x'', and u. Integrating this inequality over E and denoting $H' = \{x' : x = (x', x'') \in H \text{ for some } x''\}$, we obtain

$$\|u\|_{0,p,H'\times E}^p \le K_1 I_{H'\times E}^{m,p}(u) \le K_1 I_H^{m,p}(u).$$

Now we apply Lemma 6.20 with $A = H' \times E$ so that $\mathrm{vol}(A) = h^\nu \mu_{n-\nu}(H, \Omega)$. This yields

$$\|u\|_{0,p,H}^p \le K_2 \frac{h^{n-\nu}}{\mu_{n-\nu}(H,\Omega)} I_H^{m,p}(u),$$

where K_2 is independent of H. It follows that

$$Q^{m,p}(H, H-\Omega) \ge \frac{\mu_{n-\nu}(H,\Omega)}{K_2 h^{n-\nu}} \ge \frac{h^p}{\epsilon K_2}.$$

Hence Ω satisfies the hypothesis of Theorem 6.19 if it satisfies that of Theorem 6.16. ∎

The following two interpolation lemmas enable us to extend Theorem 6.16 to cover imbeddings into spaces of continuous functions.

6.26 LEMMA Let $1 \le p < \infty$ and $0 < \mu \le 1$. There exists a constant $K = K(n, p, \mu)$ such that for every $u \in C_0^\infty(\mathbb{R}^n)$ we have

$$\sup_{x\in\mathbb{R}^n} |u(x)| \le K \, \|u\|_{0,p,\mathbb{R}^n}^\lambda \left(\sup_{\substack{x,y\in\mathbb{R}^n \\ x\ne y}} \frac{|u(x)-u(y)|}{|x-y|^\mu} \right)^{1-\lambda}, \tag{15}$$

where $\lambda = p\mu/(n + p\mu)$.

Proof. We may assume

$$\sup_{x\in\mathbb{R}^n} |u(x)| = N > 0 \qquad \text{and} \qquad \sup_{\substack{x,y\in\mathbb{R}^n \\ x\ne y}} \frac{|u(x)-u(y)|}{|x-y|^\mu} = M < \infty.$$

Let ϵ satisfy $0 < \epsilon \le N/2$. Then there exists a point x_0 in $\mathbb{R}^n$ such that we have $|u(x_0)| \ge N - \epsilon \ge N/2$. Now $|u(x_0 - u(x)|/|x_0 - x|^\mu \le M$ for all x, so

$$|u(x)| \ge |u(x_0)| - M|x_0 - x|^\mu \ge \frac{1}{2}|u(x_0)|$$

provided $|x - x_0| \le (N/4M)^{1/\mu} = r$. Hence

$$\int_{\mathbb{R}^n} |u(x)|^p \, dx \ge \int_{B_r(x_0)} \left(\frac{|u(x_0)|}{2}\right)^p dx \ge K_1 \left(\frac{N-\epsilon}{2}\right)^p \left(\frac{N}{4M}\right)^{n/\mu}.$$

Since this holds for arbitrarily small ϵ we have

$$\|u\|_{0,p,\mathbb{R}^n} \geq \left(\frac{K_1^{1/p}}{2 \cdot 4^{n/\mu p}} \right) N^{1+(n/\mu p)} M^{-n/\mu p}$$

from which (15) follows at once. ∎

6.27 LEMMA Let Ω be an arbitrary domain in $\mathbb{R}^n$, and let $0 < \lambda < \mu \leq 1$. For every function $u \in C^{0,\mu}(\overline{\Omega})$ we have

$$\left\| u; C^{0,\lambda}(\overline{\Omega}) \right\| \leq 3^{1-\lambda/\mu} \left\| u; C(\overline{\Omega}) \right\|^{1-\lambda/\mu} \left\| u; C^{0,\mu}(\overline{\Omega}) \right\|^{\lambda/\mu}. \tag{16}$$

Proof. Let $p = \mu/\lambda$ and $p' = p/(p-1)$. Let

$$A_1 = \left\| u; C(\overline{\Omega}) \right\|^{1/p}, \qquad B_1 = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \left(\frac{|u(x)-u(y)|}{|x-y|^{\mu}} \right)^{1/p},$$

$$A_2 = \left\| u; C(\overline{\Omega}) \right\|^{1/p'}, \qquad B_2 = \sup_{\substack{x,y\in\Omega \\ x\neq y}} |u(x)-u(y)|^{1/p'}.$$

Clearly $A_1^p + B_1^p = \left\| u; C^{0,\mu}(\overline{\Omega}) \right\|$ and $B_2^{p'} \leq 2 \left\| u; C(\overline{\Omega}) \right\|$. By Hölder's inequality for sums we have

$$\begin{aligned} \left\| u; C^{0,\lambda}(\overline{\Omega}) \right\| &= \left\| u; C(\overline{\Omega}) \right\| + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{\lambda}} \\ &\leq A_1 A_2 + B_1 B_2 \\ &\leq \left(A_1^p + B_1^p \right)^{1/p} \left(A_2^{p'} + B_2^{p'} \right)^{1/p'} \\ &\leq \left\| u; C^{0,\mu}(\overline{\Omega}) \right\|^{\lambda/\mu} \left(3 \left\| u; C(\overline{\Omega}) \right\| \right)^{1-\lambda/\mu} \end{aligned}$$

as required. ∎

6.28 THEOREM Let Ω satisfy the hypotheses of Theorem 6.16. Then the following imbeddings are compact:

$$W_0^{j+m,p}(\Omega) \to C^j(\overline{\Omega}) \qquad \text{if} \quad mp > n \tag{17}$$

$$W_0^{j+m,p}(\Omega) \to C^{j,\lambda}(\overline{\Omega}) \qquad \text{if} \quad mp > n \geq (m-1)p \quad \text{and} \quad 0 < \lambda < m - (n/p). \tag{18}$$

Proof. It is sufficient to deal with the case $j = 0$. If $mp > n$, let j^* be the nonnegative integer satisfying $(m - j^*)p > n \geq (m - j^* - 1)p$. Then we have the chain of imbeddings

$$W_0^{m,p}(\Omega) \to W_0^{m-j^*,p}(\Omega) \to C^{0,\mu}(\overline{\Omega}) \to C(\overline{\Omega}),$$

where $0 < \mu < m - j^* - (n/p)$. If $\{u_i\}$ is a bounded sequence in $W_0^{m,p}(\Omega)$, then it is also bounded in $C^{0,\mu}(\overline{\Omega})$. By Theorem 6.16, $\{u_i\}$ has a subsequence $\{u_i'\}$ converging in $L^p(\Omega)$. By (15), which applies by completion to the functions u_i, this subsequence is a Cauchy sequence in $C(\overline{\Omega})$ and so converges there. Hence (17) is compact for $j = 0$. Furthermore, if $mp > n \geq (m-1)p$ (that is, if $j^* = 0$) and $0 < \lambda < \mu$, then by (16) $\{u_i'\}$ is also a Cauchy sequence in $C^{0,\lambda}(\overline{\Omega})$ whence (18) is also compact. ∎

An Equivalent Norm for $W_0^{m,p}(\Omega)$

6.29 (Domains of Finite Width) Consider the problem of determining for what domains Ω in $\mathbb{R}^n$ is the seminorm

$$|u|_{m,p,\Omega} = \left(\sum_{|\alpha|=m} \|D^\alpha u\|_{0,p,\Omega}^p \right)^{1/p}$$

actually a norm on $W_0^{m,p}(\Omega)$ equivalent to the standard norm

$$\|u\|_{m,p,\Omega} = \left(\sum_{|\alpha|\leq m} \|D^\alpha u\|_{0,p,\Omega}^p \right)^{1/p}.$$

This problem is closely related to the problem of determining for which unbounded domains Ω the imbedding $W_0^{m,p}(\Omega) \to L^p(\Omega)$ is compact because both problems depend on estimates for the L^p norm of a function in terms of L^p estimates for its derivatives.

We can easily show the equivalence of the above seminorm and norm for a domain of *finite width*, that is, a domain in $\mathbb{R}^n$ that lies between two parallel planes of dimension $(n-1)$. In particular, this is true for any bounded domain.

6.30 THEOREM (Poincaré's Inequality) If domain $\Omega \subset \mathbb{R}^n$ has finite width, then there exists a constant $K = K(p)$ such that for all $\phi \in C_0^\infty(\Omega)$

$$\|\phi\|_{0,p,\Omega} \leq K\, |\phi|_{1,p,\Omega}\,. \tag{19}$$

This inequality is known as *Poincare's Inequality.*

Proof. Without loss of generality we can assume that Ω lies between the hyperplanes $x_n = 0$ and $x_n = c > 0$. Denoting $x = (x', x_n)$, where $x' = (x_1, \ldots, x_{n-1})$, we have for any $\phi \in C_0^\infty(\Omega)$,

$$\phi(x) = \int_0^{x_n} \frac{d}{dt}\phi(x', t)\, dt$$

so that, by Hölder's inequality,

$$
\begin{aligned}
\|\phi\|_{0,p,\Omega}^p &= \int_{\mathbb{R}^{n-1}} dx' \int_0^c |\phi(x)|^p \, dx_n \\
&\le \int_{\mathbb{R}^{n-1}} dx' \int_0^c x_n^{p-1} \, dx_n \int_0^c |D_n \phi(x', t)|^p \, dt \\
&\le \frac{c^p}{p} \, |\phi|_{1,p,\Omega}^p .
\end{aligned}
$$

Inequality (19) follows with $K = c/p^{1/p}$. ∎

6.31 COROLLARY If Ω has finite width, $|\cdot|_{m,p,\Omega}$ is a norm on $W_0^{m,p}(\Omega)$ equivalent to the standard norm $\|\cdot\|_{m,p,\Omega}$.

Proof. If $\phi \in C_0^\infty(\Omega)$ then any derivative of ϕ also belongs to $C_0^\infty(\Omega)$. Now (19) implies

$$
|\phi|_{1,p,\Omega}^p \le \|\phi\|_{1,p,\Omega}^p = \|\phi\|_{0,p,\Omega}^p + |\phi|_{1,p,\Omega}^p \le (1 + K^p) \, |\phi|_{1,p,\Omega}^p ,
$$

and successive iterations of this inequality to derivatives $D^\alpha \phi$, ($|\alpha| \le m - 1$) leads to

$$
|\phi|_{m,p,\Omega}^p \le \|\phi\|_{m,p,\Omega}^p \le K_1 \, |\phi|_{m,p,\Omega}^p .
$$

By completion, this holds for all u in $W_0^{m,p}(\Omega)$. ∎

6.32 (Quasicylindrical Domains) An unbounded domain Ω in $\mathbb{R}^n$ is called *quasicylindrical* if

$$
\limsup_{x \in \Omega, \ |x| \to \infty} \operatorname{dist}(x, \operatorname{bdry} \Omega) < \infty .
$$

Every quasibounded domain is quasicylindrical, as is every (unbounded) domain of finite width. The seminorm $|\cdot|_{m,p,\Omega}$ is not equivalent to the norm $\|\cdot\|_{m,p,\Omega}$ on $W_0^{m,p}(\Omega)$ for unbounded Ω if Ω is not quasicylindrical. We leave it to the reader to construct a suitable counterexample.

The following theorem is clearly analogous to Theorem 6.16.

6.33 THEOREM Suppose there exist an integer ν and constants K, R, and h such that $1 \le \nu \le n$, $0 < K \le 1$, $0 \le R < \infty$, and $0 < h < \infty$. Suppose also that either $\nu < p$ or $\nu = p = 1$, and that for every cube H in $\mathbb{R}^n$ having edge length h and nonempty intersection with $\Omega_R = \{x \in \Omega : |x| \ge R\}$ we have

$$
\frac{\mu_{n-\nu}(H, \Omega)}{h^{n-\nu}} \ge K ,
$$

where $\mu_{n-\nu}(H, \Omega)$ is as defined prior to the statement of Theorem 6.16. Then $|\cdot|_{m,p,\Omega}$ is a norm on $W_0^{m,p}(\Omega)$ equivalent to the standard norm $\|\cdot\|_{m,p,\Omega}$.

Proof. As observed in the previous Corollary, it is again sufficient to prove that $\|u\|_{0,p,\Omega} \le K_1 \, |u|_{1,p,\Omega}$ holds for all $u \in C_0^\infty(\Omega)$. Let H be a cube of edge length h having nonempty intersection with Ω_R. Since $\nu < p$ (or $\nu = p = 1$) the proof of Theorem 6.16 shows that

$$Q^{1,p}(H, H - \Omega) \ge \frac{\mu_{n-\nu}(H, \Omega)}{K_2 h^{n-\nu}} \ge \frac{K}{K_2}$$

for all $u \in C_0^\infty(\Omega)$, K_2 being independent of u. Hence

$$\|u\|_{0,p,H}^p \le (K_2/K) I_H^{1,p} = K_3 \, |u|_{1,p,H}^p \, .$$

By summing this inequality over the cubes comprising a tessellation of some neighbourhood of Ω_R, we obtain

$$\|u\|_{0,p,\Omega_R}^p \le K_3 \, |u|_{1,p,\Omega}^p \, . \tag{20}$$

It remains to be proven that

$$\|u\|_{0,p,B_R}^p \le K_3 \, |u|_{1,p,\Omega}^p \, ,$$

where $B_R = \{x \in \mathbb{R}^n : |x| < R\}$. Let (ρ, ϕ) denote the spherical coordinates of the point $x \in \mathbb{R}^n$ ($\rho \ge 0$, $\phi \in \Sigma$) so that $dx = \rho^{n-1}\omega(\phi)\, d\rho\, d\phi$. For any $u \in C^\infty(\mathbb{R}^n)$ we have

$$u(\rho, \phi) = u(\rho + R, \phi) - \int_\rho^{R+\rho} \frac{d}{dt} u(t, \phi)\, dt$$

so that (by Lemma 2.2)

$$|u(\rho, \phi)|^p \le 2^{p-1} |u(\rho + R, \phi)|^p + 2^{p-1} R^{p-1} \rho^{1-n} \int_\rho^{R+\rho} |\operatorname{grad} u(t, \phi)|^p t^{n-1}\, dt.$$

Hence

$$\begin{aligned}
\|u\|_{0,p,B_R}^p &= \int_\Sigma \omega(\phi)\, d\phi \int_0^R |u(\rho, \phi)|^p \rho^{n-1}\, d\rho \\
&\le 2^{p-1} \int_\Sigma \omega(\phi)\, d\phi \int_0^R |u(\rho + R, \phi)|^p (\rho + R)^{n-1}\, d\rho \\
&\quad + 2^{p-1} R^p \int_\Sigma \omega(\phi)\, d\phi \int_0^{2R} |\operatorname{grad} u(t, \phi)|^p t^{n-1}\, dt.
\end{aligned}$$

Therefore, we have for $u \in C_0^\infty(\Omega)$

$$\begin{aligned}\|u\|_{0,p,B_R}^p &\le 2^{p-1}\,\|u\|_{0,p,B_{2R}-B_R}^p + 2^{p-1}R^p\,|u|_{1,p,B_{2R}}^p\\ &\le 2^{p-1}\,\|u\|_{0,p,\Omega_R}^p + 2^{p-1}R^p\,|u|_{1,p,\Omega}^p \le K_4\,|u|_{1,p,\Omega}^p\end{aligned}$$

by (20). ∎

Unbounded Domains — Decay at Infinity

6.34 The fact that elements of $W_0^{m,p}(\Omega)$ vanish in a generalized sense on the boundary of Ω played a critical role in our showing that the imbedding

$$W_0^{m,p}(\Omega) \to L^p(\Omega) \tag{21}$$

is compact for certain unbounded domains Ω. Since elements of $W^{m,p}(\Omega)$ do not have this property, there remains a question of whether an imbedding of the form

$$W^{m,p}(\Omega) \to L^p(\Omega) \tag{22}$$

can ever be compact for unbounded Ω, or even for bounded Ω which are sufficiently irregular that no imbedding of the form

$$W^{m,p}(\Omega) \to L^q(\Omega) \tag{23}$$

can exist for any $q > p$. Note that if Ω has finite volume, the existence of imbedding (23) for some $q > p$ implies the compactness of imbedding (22) by the method of the first part of the proof in Paragraph 6.8. By Theorem 4.46 imbedding (23) cannot, however, exist if $q > p$ and Ω is unbounded but has finite volume.

6.35 EXAMPLE For $j = 1, 2, \ldots$ let B_j be an open ball in $\mathbb{R}^n$ having radius r_j, and suppose that $\overline{B_j} \cap \overline{B_i}$ is empty whenever $j \neq i$. Let $\Omega = \bigcup_{j=1}^\infty B_j$. Note that Ω may be bounded or unbounded. The sequence $\{u_j\}$ defined by

$$u_j(x) = \begin{cases} (\mathrm{vol}(B_j))^{-1/p} & \text{if } x \in \overline{B_j} \\ 0 & \text{if } x \notin \overline{B_j} \end{cases}$$

is bounded in $W^{m,p}(\Omega)$ for every integer $m \ge 0$, but is not precompact in $L^p(\Omega)$ no matter how fast $r_j \to 0$ as $j \to \infty$. (Of course, imbedding (21) is compact by Theorem 6.16 provided $\lim_{j\to\infty} r_j = 0$.) Even if Ω is bounded, imbedding (23) cannot exist for any $q > p$.

6.36 Let us state at once that there do exist unbounded domains Ω for which the imbedding (22) is compact. See Example 6.53. An example of such a domain was given by the authors in [AF2] and it provided a basis for an investigation of the general problem in [AF3]. The approach of this latter paper is used in the following discussion.

First we concern ourselves with necessary conditions for the compactness of (23) for $q \geq p$. These conditions involve rapid decay at infinity for any unbounded domain (see Theorem 6.45). The techniques involved in the proof also yield a strengthened version of Theorem 4.46, namely Theorem 6.41, and a converse of the assertion [see Remark 4.13(3)] that $W^{m,p}(\Omega) \to L^q(\Omega)$ for $1 \leq q < p$ if Ω has finite volume.

A sufficient condition for the compactness of (22) is given in Theorem 6.52. It applies to many domains, bounded and unbounded, to which neither the Rellich-Kondrachov theorem nor any generalization of that theorem obtained by the same methods can be applied. (e.g. exponential cusps — see Example 6.54).

6.37 (Tessellations and λ-fat Cubes) Let T be a tessellation of $\mathbb{R}^n$ by closed n-cubes of edge length h. If H is one of the cubes in T, let $N(H)$ denote the cube of edge length $3h$ concentric with H and therefore consisting of the 3^n elements of T that intersect H. We call $N(H)$ the *neighbourhood* of H. By the *fringe* of H we shall mean the shell $F(H) = N(H) - H$.

Let Ω be a given domain in $\mathbb{R}^n$ and T a given tessellation as above. Let $\lambda > 0$. A cube $H \in T$ will be called λ-*fat* (with respect to Ω) if

$$\mu(H \cap \Omega) > \lambda\, \mu(F(H) \cap \Omega),$$

where μ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. (We use μ instead of "vol" for notational simplicity in the following discussion where the symbol must be used many times.) If H is not λ-fat then we will say it is λ-*thin.*

6.38 THEOREM Suppose there exists a compact imbedding of the form

$$W^{m,p}(\Omega) \to L^q(\Omega)$$

for some $q \geq p$. Then for every $\lambda > 0$ and every tessellation T of $\mathbb{R}^n$ by cubes of fixed size, T can have only finitely many λ-fat cubes.

Proof. Suppose, to the contrary, that for some $\lambda > 0$ there exists a tessellation T of $\mathbb{R}^n$ by cubes of edge length h containing a sequence $\{H_j\}_{j=1}^{\infty}$ of λ-fat cubes. Passing to a subsequence if necessary we may assume that $N(H_j) \cap N(H_i)$ is empty whenever $j \neq i$. For each j there exists $\phi_j \in C_0^\infty(N(H_j))$ such that

(i) $|\phi_j(x)| \leq 1$ for all $x \in \mathbb{R}^n$,

(ii) $\phi_j(x) = 1$ for $x \in H_j$, and

(iii) $|D^\alpha \phi_j(x)| \leq M$ for all j, all $x \in \mathbb{R}^n$, and all α satisfying $0 \leq |\alpha| \leq m$.

In fact, all the ϕ_j can be taken to be translates of one of them. Let $\psi_j = c_j\phi_j$, where the positive constants c_j are chosen so that

$$\|\psi_j\|_{0,q,\Omega}^q \geq c_j^q \int_{H_j\cap\Omega} |\phi_j(x)|^q\,dx = c_j^q\mu(H_j\cap\Omega) = 1.$$

But then

$$\begin{aligned}\|\psi_j\|_{m,p,\Omega}^p &= c_j^p \sum_{0\leq|\alpha|\leq m} \int_{N(H_j)\cap\Omega} |D^\alpha\phi_j(x)|^p\,dx \\ &\leq M^p c_j^p\,\mu\big(N(H_j)\cap\Omega\big) \\ &< M^p c_j^p\,\mu\big(H_j\cap\Omega\big)\left(1+\frac{1}{\lambda}\right) = M^p\left(1+\frac{1}{\lambda}\right)c_j^{p-q},\end{aligned}$$

since H_j is λ-fat. Now $\mu(H_j\cap\Omega) \leq \mu(H_j) = h^n$ so $c_j \geq h^{-n/q}$. Since $p-q \leq 0$, $\{\psi_j\}$ is bounded in $W^{m,p}(\Omega)$. But the functions ψ_j have disjoint supports, so $\{\psi_j\}$ cannot be precompact in $L^q(\Omega)$, contradicting the assumption that $W^{m,p}(\Omega) \to L^q(\Omega)$ is compact. Thus every T can possess at most finitely many λ-fat cubes. ∎

6.39 COROLLARY Suppose that $W^{m,p}(\Omega) \to L^q(\Omega)$ for some $q > p$. If T is a tessellation of $\mathbb{R}^n$ by cubes of fixed edge-length, and if $\lambda > 0$ is given, then there exists $\epsilon > 0$ such that $\mu(H\cap\Omega) \geq \epsilon$ for every λ-fat $H \in T$.

Proof. Suppose, to the contrary, that there exists a sequence $\{H_j\}$ of λ-fat cubes with $\lim_{j\to\infty}\mu(H_j\cap\Omega) = 0$. If c_j is defined as in the above proof, we have $\lim_{j\to\infty} c_j = \infty$. But then $\lim_{j\to\infty}\|\psi_j\|_{m,p,\Omega} = 0$ since $p < q$. Since $\{\psi_j\}$ is bounded away from 0 in $L^q(\Omega)$, we have contradicted the continuity of the imbedding $W^{m,p}(\Omega) \to L^q(\Omega)$. ∎

6.40 REMARK It follows from the above corollary that if there exists an imbedding

$$W^{m,p}(\Omega) \to L^q(\Omega) \tag{24}$$

for some $q > p$ then one of the following alternatives must hold:

(a) There exists $\epsilon > 0$ and a tessellation T of $\mathbb{R}^n$ consisting of cubes of fixed size such that $\mu(H\cap\Omega) \geq \epsilon$ for infinitely many cubes $H \in T$.

(b) For every $\lambda > 0$, every tessellation T of $\mathbb{R}^n$ consisting of cubes of fixed size contains only finitely many λ-fat cubes.

We will show in Theorem 6.42 that (b) implies that Ω has finite volume. By Theorem 4.46, (b) is therefore inconsistent with the existence of (24) for $q > p$. On the other hand, (a) implies that $\mu\big(\{x \in \Omega : N \leq |x| \leq N+1\}\big)$ does not approach zero as N tends to infinity. We have therefore proved the following strengthening of Theorem 4.46.

6.41 THEOREM Let Ω be an unbounded domain in $\mathbb{R}^n$ satisfying

$$\limsup_{N\to\infty} \mathrm{vol}(\{x \in \Omega : N \le |x| \le N+1\}) = 0.$$

Then there can be no imbedding of the form (24) for any $q > p$.

6.42 THEOREM Suppose that imbedding (24) is compact for some $q \ge p$. Then Ω has finite volume.

Proof. Let T be a tessellation of $\mathbb{R}^n$ by cubes of unit edge length, and let $\lambda = 1/[2(3^n - 1)]$. Let P be the union of the finitely many λ-fat cubes in T. Clearly $\mu(P \cap \Omega) \le \mu(P) < \infty$.

Let H be a λ-thin cube in T. Let H_1 be one of the $3^n - 1$ cubes in T constituting the fringe of H selected so that $\mu(H_1 \cap \Omega)$ is maximal. Thus

$$\mu(H \cap \Omega) \le \lambda\, \mu\big(F(H) \cap \Omega\big) \le \lambda(3^n - 1)\mu(H_1 \cap \Omega) = \frac{1}{2}\mu(H_1 \cap \Omega).$$

If H_1 is also λ-thin, then we may select a cube $H_2 \in T$ with $H_2 \subset F(H_1)$ such that $\mu(H_1 \cap \Omega) \le \frac{1}{2}\mu(H_2 \cap \Omega)$.

Suppose an infinite chain $\{H_1, H_2, \ldots\}$ of λ-thin cubes can be constructed in the above manner. Then

$$\mu(H \cap \Omega) \le \frac{1}{2}\mu(H_1 \cap \Omega) \le \cdots \le \frac{1}{2^j}\mu(H_j \cap \Omega) \le \frac{1}{2^j}$$

for each j since $\mu(H_j \cap \Omega) \le \mu(H_j) = 1$. Hence $\mu(H \cap \Omega) = 0$. Denoting by P_∞ the union of λ-thin cubes $H \in T$ for which such an infinite chain can be constructed, we have $\mu(P_\infty \cap \Omega) = 0$.

Let P_j denote the union of λ-thin cubes $H \in T$ for which some such chain ends on the jth step; that is, H_j is λ-fat. Any particular λ-fat cube H' can occur as the end H_j of a chain beginning at H only if H is contained in the cube of edge $2j+1$ centred on H'. Hence there are at most $(2j+1)^n$ such cubes $H \subset P_j$ having H' as chain endpoint. Thus

$$\begin{aligned}
\mu(P_j \cap \Omega) &= \sum_{H \subset P_j} \mu(H \cap \Omega) \\
&\le \frac{1}{2^j} \sum_{H \subset P_j} \mu(H_j \cap \Omega) \\
&\le \frac{(2j+1)^n}{2^j} \sum_{H' \subset P} \mu(H' \cap \Omega) = \frac{(2j+1)^n}{2^j}\mu(P \cap \Omega),
\end{aligned}$$

so that $\mu(\Omega) = \mu(P \cap \Omega) + \mu(P_\infty \cap \Omega) + \sum_{j=1}^{\infty} \mu(P_j \cap \Omega) < \infty$. ∎

Suppose $1 \le q < p$. If $\mathrm{vol}(\Omega) < \infty$, then the imbedding

$$W^{m,p}(\Omega) \to L^q(\Omega)$$

exists because $W^{m,p}(\Omega) \to L^p(\Omega)$ trivially and $L^p(\Omega) \to L^q(\Omega)$ by Theorem 2.14.

We are now in a position to prove the converse.

6.43 THEOREM If the imbedding $W^{m,p}(\Omega) \to L^q(\Omega)$ exists for some p and q satisfying $1 \le q < p$, then Ω has finite volume.

Proof. Let T, λ, and again let P denote the union of the λ-fat cubes in T. If we can show that $\mu(P \cap \Omega)$ is finite, it will follow by the same argument used in that theorem that $\mu(\Omega)$ is finite.

Accordingly, suppose that $\mu(P \cap \Omega)$ is not finite. Then there exists a sequence $\{H_j\}$ of λ-fat cubes in T such that $\sum_{j=1}^{\infty} \mu(H_j \cap \Omega) = \infty$. If L is the lattice of centres of cubes in T, we may break up L into 3^n mutually disjoint sublattices $\{L_i\}_{i=1}^{3^n}$ each having period 3 in each coordinate direction. For each i let T_i be the set of all cubes in T that have centres in L_i. For some i we must have $\sum_{\lambda-\mathrm{fat} H \in T_i} \mu(H \cap \Omega) = \infty$. Thus we may assume that the cubes of the sequence $\{H_j\}$ all belong to T_i for some i, so that $N(H_j) \cap N(H_k)$ is empty if $j \ne k$.

Choose integer j_1 so that

$$2 \le \sum_{j=1}^{j_1} \mu(H_j \cap \Omega) < 4.$$

Let ϕ_j be as in the proof of Theorem 6.38, and let

$$\psi_1(x) = 2^{-1/p} \sum_{j=1}^{j_1} \phi_j(x).$$

Since the supports of the functions ϕ_j are mutually disjoint and since the cubes H_j are λ-fat, for $|\alpha| \le m$ we have

$$\begin{aligned} \|D^\alpha \psi_1\|_{0,p,\Omega}^p &= \frac{1}{2} \sum_{j=1}^{j_1} \int_\Omega |D^\alpha \phi_j(x)|^p \, dx \\ &\le \frac{1}{2} M^p \sum_{j=1}^{j_1} \mu\big(N(H_j) \cap \Omega\big) \\ &< \frac{1}{2} M^p \left(1 + \frac{1}{\lambda}\right) \sum_{j=1}^{j_1} \mu(H_j \cap \Omega) < 2M^p \left(1 + \frac{1}{\lambda}\right). \end{aligned}$$

On the other hand,

$$\|\psi_1\|_{0,q,\Omega}^q \geq 2^{-q/p} \sum_{j=1}^{j_1} \mu(H_j \cap \Omega) \geq 2^{1-(q/p)}.$$

Having so defined j_1 and ψ_1, we can now define $j_2, j_3, \ldots$ and $\psi_2, \psi_3, \ldots$ inductively so that

$$2^k \leq \sum_{j=j_{k-1}+1}^{j_k} \mu(H_j \cap \Omega) < 2^{k+1}$$

and

$$\psi_k(x) = 2^{-k/p}\, k^{-2/p} \sum_{j=j_{k-1}+1}^{j_k} \phi_j(x).$$

As above, we have for $|\alpha| \leq m$,

$$\|D^\alpha \psi_k\|_{0,p,\Omega}^p \leq \frac{2}{k^2} M^p \left(1 + \frac{1}{\lambda}\right)$$

and

$$\|\psi_k\|_{0,q,\Omega}^q \geq 2^{k(1-q/p)} M^p \left(\frac{1}{k}\right)^{2q/p}.$$

Thus $\psi = \sum_{k=1}^{\infty} \psi_k$ belongs to $W^{m,p}(\Omega)$ but not to $L^q(\Omega)$ contradicting the imbedding $W^{m,p}(\Omega) \to L^q(\Omega)$. Hence $\mu(P \cap \Omega) < \infty$ as required. ∎

6.44 If there exists a compact imbedding of the form $W^{m,p}(\Omega) \to L^q(\Omega)$ for some $q \geq p$, then, as we have shown, Ω has finite volume. In fact, considerably more is true; $\mu(\{x \in \Omega : |x| \geq R\})$ must approach zero very rapidly as $R \to \infty$, as we show in Theorem 6.45 below.

If Q is a union of cubes H in some tessellation T of $\mathbb{R}^n$ by cubes of fixed edge length, we extend the notions of neighbourhood and fringe to Q in an obvious manner:

$$N(Q) = \bigcup_{\substack{H \in T \\ H \subset Q}} N(H), \qquad F(Q) = N(Q) - Q.$$

Given $\delta > 0$, let $\lambda = \delta/[3^n(1+\delta)]$. If all the cubes $H \in T$ satisfying $H \subset Q$ are λ-thin, then Q is itself δ-thin in the sense that

$$\mu(Q \cap \Omega) \leq \delta\mu\big(F(Q) \cap \Omega\big).$$

To see this note that as H runs through the cubes comprising Q, $F(H)$ covers $N(Q)$ at most 3^n times. Hence

$$\begin{aligned}\mu(Q \cap \Omega) &= \sum_{H \subset Q} \mu(H \cap \Omega) \leq \lambda \sum_{H \subset Q} \mu\big(F(H) \cap \Omega\big) \\ &\leq 3^n \lambda \mu\big(N(Q) \cap \Omega\big) = 3^n \lambda\big[\mu(Q \cap \Omega) + \mu\big(F(Q) \cap \Omega\big)\big]\end{aligned}$$

and the fact that Q is δ-thin follows by transposition (permissible since $\mu(\Omega) < \infty$) and since $3^n\lambda/(1-3^n\lambda) = \delta$.

For any measurable set $S \subset \mathbb{R}^n$ let Q be the union of all cubes in T whose interiors intersect S, and define $F(S) = f(Q)$. If S is at a positive distance from the finitely many λ-fat cubes in T, then Q consists of λ-thin cubes and we obtain

$$\mu(S \cap \Omega) \le \mu(Q \cap \Omega) \le \delta\mu\big(F(S) \cap \Omega\big). \tag{25}$$

6.45 THEOREM (Rapid Decay) Suppose there exists a compact imbedding of the form

$$W^{m,p}(\Omega) \to L^q(\Omega) \tag{26}$$

for some $q \ge p$. For each $r \ge 0$ let $\Omega_r = \{x \in \Omega : |x| > r\}$, let $S_r = \{x \in \Omega : |x| = r\}$, and let A_r denote the surface area (Lebesgue $(n-1)$-measure) of S_r. Then

(a) For given $\epsilon, \delta > 0$ there exists R such that if $r \ge R$, then

$$\mu(\Omega_r) \le \delta\, \mu\big(\{x \in \Omega : r - \epsilon \le |x| \le r\}\big).$$

(b) If A_r is positive and ultimately nonincreasing as $r \to \infty$, then for each $\epsilon > 0$

$$\lim_{r\to\infty} \frac{A_{r+\epsilon}}{A_r} = 0.$$

Proof. Given $\epsilon > 0$ and $\delta > 0$, let $\lambda = \delta/[3^n(1+\delta)]$ and let T be a tessellation of $\mathbb{R}^n$ by cubes of edge length $\epsilon/(2\sqrt{n})$. Let R be large enough that the finitely many λ-fat cubes in T lie in the ball of radius $R - \epsilon/2$ about the origin. If $r \ge R$ and $S = \Omega_r$, then any $H \in T$ whose interior intersects S is λ-thin. Moreover, any cube in the fringe of S can only intersect Ω at points x satisfying $r - \epsilon/2 \le |x| \le r$. By (25),

$$\mu(\Omega_r) = \mu(S \cap \Omega) \le \delta\, \mu\big(F(S) \cap \Omega\big) = \delta\, \mu\big(\{x \in \Omega : r - \epsilon \le |x| \le r\}\big),$$

which proves (a).

For (b) choose R_0 so that A_r is nonincreasing for $r \ge R_0$. Fix $\epsilon', \delta > 0$ and let $\epsilon = \epsilon'/2$. Let R be as in (a). If $r \ge \max\{R, R_0 + \epsilon'\}$, then

$$\begin{aligned} A_{r+\epsilon'} &\le \frac{1}{\epsilon}\int_{r+\epsilon}^{r+2\epsilon} A_s\, ds \le \frac{1}{\epsilon}\mu(\Omega_{r+\epsilon}) \\ &\le \frac{\delta}{\epsilon}\mu\left(\{x \in \Omega : r \le |x| \le r+\epsilon\}\right) = \frac{\delta}{\epsilon}\int_r^{r+\epsilon} A_s\, ds \le \delta A_r. \end{aligned}$$

Since ϵ' and δ are arbitrary, (b) follows. ∎

6.46 COROLLARY If there exists a compact imbedding of the form (26) for some $q \geq p$, then for every $k > 0$ we have

$$\lim_{r\to\infty} e^{kr}\mu(\Omega_r) = 0. \tag{27}$$

Proof. Fix k and let $\delta = e^{-(k+1)}$. From conclusion (a) of Theorem 6.45 there exists R such that $r \geq R$ implies $\mu(\Omega_{r+1}) \leq \delta\mu(\Omega_r)$. Thus if j is a positive integer and $0 \leq t < 1$, we have

$$\begin{aligned} e^{k(R+j+t)}\mu(\Omega_{R+j+t}) &\leq e^{k(R+j+1)}\mu(\Omega_{R+j}) \\ &\leq e^{k(R+1)}\, e^{kj}\, \delta^j\, \mu(\Omega_R) = e^{k(R+1)}\, \mu(\Omega_R)\, e^{-j}. \end{aligned}$$

The last term approaches zero as j tends to infinity. ∎

6.47 REMARKS

1. We work with Sobolev spaces defined intrinsically in domains. If instead, we had defined $W^{m,p}(\Omega)$ to consist of all restrictions to Ω of functions in $W^{m,p}(R^n)$, then the outcome for Corollary 6.46 would have been different. With that definition, it is shown in [BSc] that $W^{m,p}(\Omega)$ imbeds compactly in $L^p(\Omega)$ if and only if the volume of the intersection of Ω with cubes of fixed edge-length tends to 0 as the centres of those cubes tend to ∞. There are many domains Ω satisfying the latter condition but not satisfying (27). None of these domains can have any Sobolev extension property.
2. The argument used in the proof of Theorem 6.45(a) works for any norm ρ on $\mathbb{R}^n$ in place of the usual Euclidean norm $\rho(x) = |x|$. The same holds for Theorem 6.45(b) provided A_r is well defined (with respect to the norm ρ) and provided

$$\mu(\{x \in \Omega : r \leq \rho(x) \leq r+\epsilon\}) = \int_r^{r+\epsilon} A_s\, ds.$$

   This is true, for example, if $\rho(x) = \max_{1\leq i\leq n} |x_i|$.
3. For the proof of Theorem 6.45(b) it is sufficient that A_r have an equivalent nonincreasing majorant, that is, there should exist a positive, nonincreasing function $f(r)$ and a constant $M > 0$ such that for sufficiently large r

$$A_r \leq f(r) \leq M\, A_r.$$

4. Theorem 6.38 is sharper than Theorem 6.45, because the conclusions of the latter theorem are global whereas the compactness of (26) depends on local properties of Ω. We illustrate this by means of two examples.

6.48 EXAMPLE Let $f \in C^1\big([0,\infty)\big)$ be positive and nonincreasing with bounded derivative f'. We consider the planar domain (Figure 5)

$$\Omega = \big\{(x, y) \in \mathbb{R}^2 \,:\, x > 0,\ 0 < y < f(x)\big\}.$$

With respect to the supremum norm on $\mathbb{R}^2$, that is $\rho(x, y) = \max\{|x|, |y|\}$, we have $A_s = f(s)$ for sufficiently large s. Hence Ω satisfies conclusion (b) of Theorem 6.45 (and, since f is monotonic, conclusion (a) as well) if and only if

$$\lim_{s\to\infty} \frac{f(s+\epsilon)}{f(s)} = 0 \tag{28}$$

holds for every $\epsilon > 0$. For example, $f(x) = \exp(-x^2)$ satisfies this condition but $f(x) = \exp(-x)$ does not. We shall show in Example 6.53 that the imbedding

$$W^{m,p}(\Omega) \to L^p(\Omega) \tag{29}$$

is compact if (28) holds. Thus (28) is necessary and sufficient for compactness of the above imbedding for domains of this type. ∎

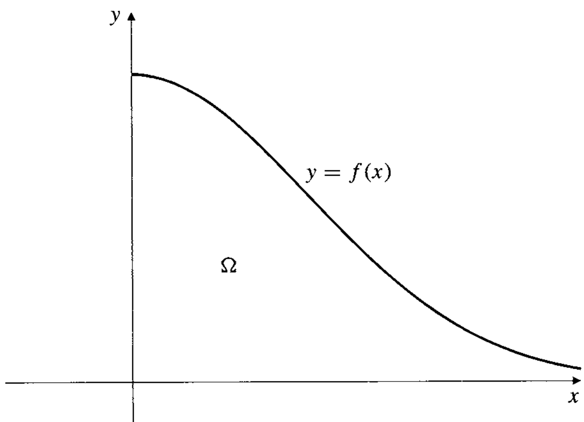


Fig. 5

6.49 EXAMPLE Let f be as in the previous example, and assume also that $f'(0) = 0$. Let g be a positive, nonincreasing function in $C^1\big([0,\infty)\big)$ satisfying

(i) $g(0) = \frac{1}{2} f(0)$, and $g'(0) = 0$,

(ii) $g(x) < f(x)$ for all $x \geq 0$,

(iii) $g(x)$ is constant on infinitely many disjoint intervals of unit length.

Let $h(x) = f(x) - g(x)$ and consider the domain (Figure 6)

$$\tilde{\Omega} = \big\{(x, y) \in \mathbb{R}^2 \,:\, 0 < y < g(x) \text{ if } x \geq 0,\ 0 < y < h(-x) \text{ if } x < 0\big\}.$$

Again we have $A_s = f(s)$ for sufficiently large s, so $\tilde{\Omega}$ satisfies the conclusions of Theorem 6.45 if (28) holds.

If, however, T is a tessellation of $\mathbb{R}^2$ by squares of edge length $\frac{1}{4}$ having edges parallel to the coordinate axes, and if one of the squares in T has centre at the origin, then T has infinitely many $\frac{1}{3}$-fat squares with centres on the positive x-axis. By Theorem 6.38 the imbedding (29) cannot be compact for the domain $\tilde{\Omega}$. ∎

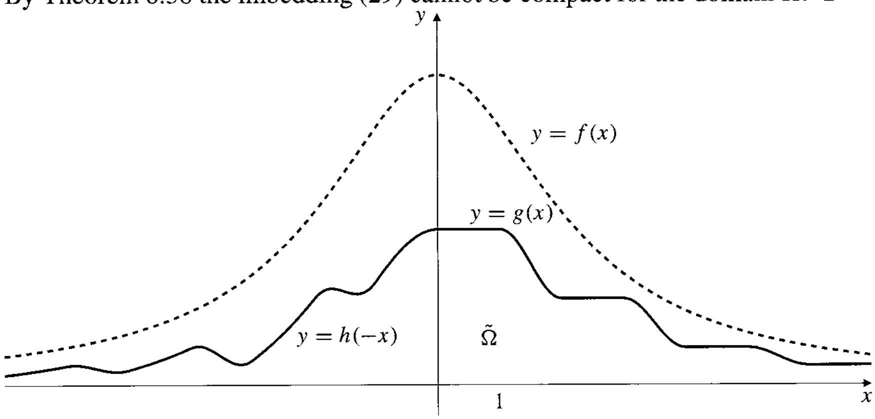


Fig. 6

Unbounded Domains — Compact Imbeddings of $W^{m,p}(\Omega)$

6.50 (Flows) The above examples suggest that any sufficient condition for the compactness of the imbedding

$$W^{m,p}(\Omega) \to L^p(\Omega)$$

for unbounded domains Ω must involve the rapid decay of volume locally in each branch of Ω_r as r tends to infinity. A convenient way to express such local decay is in terms of flows on Ω.

By a *flow* on Ω we mean a continuously differentiable map $\Phi : U \to \Omega$ where U is an open set in $\Omega \times \mathbb{R}$ containing $\Omega \times \{0\}$, and where $\Phi(x, 0) = x$ for every $x \in \Omega$.

For fixed $x \in \Omega$ the curve $t \to \Phi(x, t)$ is called a *streamline* of the flow. For fixed t the map $\Phi_t : x \to \Phi(x, t)$ sends a subset of Ω into Ω. We shall be concerned with the Jacobian of this map:

$$\det \Phi_t'(x) = \left. \frac{\partial(\Phi_1, \ldots, \Phi_n)}{\partial(x_1, \ldots, x_n)} \right|_{(x,t)} .$$

It is sometimes required of a flow Φ that $\Phi_{s+t} = \Phi_s \circ \Phi_t$ but we do not need this property and so do not assume it.

6.51 EXAMPLE Let Ω be the domain of Example 6.48. Define the flow

$$\Phi(x, y, t) = \left(x - t, \frac{f(x-t)}{f(x)} y\right), \qquad 0 < t < x.$$

The direction of the flow is towards the line $x = 0$ and the streamlines (some of which are shown in Figure 7) diverge as the domain widens. Φ_t is a local magnification for $t > 0$:

$$\det \Phi_t'(x, y) = \frac{f(x-t)}{f(x)}.$$

Note that $\lim_{x\to\infty} \det \Phi_t'(x, y) = \infty$ if f satisfies (28).
For $N = 1, 2, \ldots$ let $\Omega_N^* = \{(x, y) \in \Omega : 0 < x < N\}$. Since Ω_N^* is bounded and satisfies the cone condition, the imbedding

$$W^{1,p}\left(\Omega_N^*\right) \to L^p(\Omega_N^*)$$

is compact. This compactness, together with properties of the flow Φ are sufficient to force the compactness of $W^{m,p}(\Omega) \to L^p(\Omega)$ as we now show.

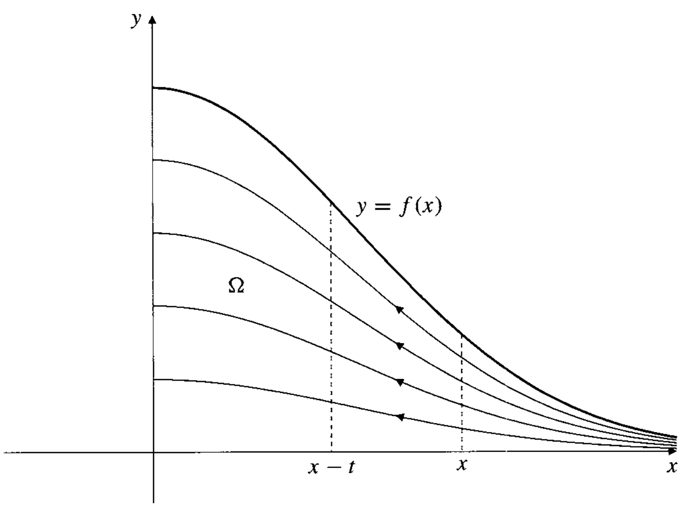


Fig. 7

6.52 THEOREM Let Ω be an open set in $\mathbb{R}^n$ having the following properties:

(a) There exists an infinite sequence $\{\Omega_N^*\}_{N=1}^\infty$ of open subsets of Ω such that $\Omega_N^* \subset \Omega_{N+1}^*$ and such that for each N the imbedding

$$W^{1,p}\left(\Omega_N^*\right) \to L^p(\Omega_N^*)$$

is compact.

(b) There exists a flow $\Phi : U \to \Omega$ such that if $\Omega_N = \Omega - \Omega_N^*$, then

(i) $\Omega_N \times [0,1] \subset U$ for each N,

(ii) Φ_t is one-to-one for all t,

(iii) $|(\partial/\partial t)\Phi(x,t)| \le M$ (constant) for all $(x,t) \in U$.

(c) The functions $d_N(t) = \sup_{x\in\Omega_N} |\det \Phi_t'(x)|^{-1}$ satisfy

(i) $\lim_{N\to\infty} d_N(1) = 0$,

(ii) $\lim_{N\to\infty} \int_0^1 d_N(t)\,dt = 0$.

Then the imbedding $W^{m,p}(\Omega) \to L^p(\Omega)$ is compact.

Proof. Since we have $W^{m,p}(\Omega) \to W^{1,p}(\Omega) \to L^p(\Omega)$ it is sufficient to prove that the latter imbedding is compact. Let $u \in C^1(\Omega)$. For each $x \in \Omega_N$ we have

$$u(x) = u\big(\Phi_1(x)\big) - \int_0^1 \frac{\partial}{\partial t} u\big(\Phi_t(x)\big)\,dt.$$

Now

$$\begin{aligned}\int_{\Omega_N} \left|u\big(\Phi_1(x)\big)\right| dx &\le d_N(1) \int_{\Omega_N} \left|u\big(\Phi_1(x)\big)\right| \left|\det \Phi_1'(x)\right| dx \\ &= d_N(1) \int_{\Phi_1(\Omega_N)} |u(y)|\,dy \\ &\le d_N(1) \int_\Omega |u(y)|\,dy.\end{aligned}$$

Also

$$\begin{aligned}\int_{\Omega_N} \left| \int_0^1 \frac{\partial}{\partial t} u\big(\Phi_t(x)\big)\,dt \right| dx &\le \int_{\Omega_N} dx \int_0^1 \left|\operatorname{grad} u\big(\Phi_t(x)\big)\right| \left|\frac{\partial}{\partial t}\Phi_t(x)\right| dt \\ &\le M \int_0^1 d_N(t)\,dt \int_{\Omega_N} \left|\operatorname{grad} u\big(\Phi_t(x)\big)\right| \left|\det \Phi_t'(x)\right| dx \\ &\le M \left(\int_0^1 d_N(t)\,dt\right) \left(\int_\Omega |\operatorname{grad} u(y)|\,dy\right).\end{aligned}$$

Putting $\delta_N = \max\left\{d_N(1), M\int_0^1 d_N(t)\,dt\right\}$, we have

$$\int_{\Omega_N} |u(x)|\,dx \le \delta_N \int_\Omega \big(|u(y)| + |\operatorname{grad} u(y)|\big)\,dy \le \delta_N \|u\|_{1,1,\Omega} \tag{30}$$

and $\lim_{N\to\infty} \delta_N = 0$.

Now suppose u is real-valued and belongs to $C^1(\Omega) \cap W^{1,p}(\Omega)$. By Hölder's inequality, the distributional derivatives of $|u|^p$

$$D_j|u|^p = p \cdot |u|^{p-1} \cdot \operatorname{sgn} u \cdot D_j u,$$

satisfy

$$\int_\Omega \left| D_j |u(x)|^p \right| dx \le p \left\| D_j u \right\|_{0,p,\Omega} \|u\|_{0,p,\Omega}^{p-1} \le p \, \|u\|_{1,p,\Omega}^p \, .$$

Thus $|u|^p \in W^{1,1}(\Omega)$ and by Theorem 3.17 there is a sequence $\{\phi_j\}$ of functions in $C^1(\Omega) \cap W^{1,1}(\Omega)$ such that $\lim_{j\to\infty} \left\| \phi_j - |u|^p \right\|_{1,1,\Omega} = 0$. Thus, by (30)

$$\begin{aligned} \int_{\Omega_N} |u(x)|^p \, dx &= \lim_{j\to\infty} \int_{\Omega_N} \phi_j(x)\, dx \le \limsup_{j\to\infty} \delta_N \left\| \phi_j \right\|_{1,1,\Omega} \\ &\le \delta_N \left\| |u|^p \right\|_{1,1,\Omega} \le K \delta_N \, \|u\|_{1,p,\Omega}^p \, , \end{aligned}$$

where $K = K(n, p)$. This inequality holds for arbitrary complex-valued function $u \in C^1(\Omega) \cap W^{1,p}(\Omega)$ by virtue of its separate applications to the real and imaginary parts of u.

If S is a bounded set in $W^{1,p}(\Omega)$ and $\epsilon > 0$, we may, by the above inequality, select N so that for all $u \in S$

$$\int_{\Omega_N} |u(x)|^p \, dx < \epsilon .$$

Since $W^{1,p}(\Omega - \Omega_N) \to L^p(\Omega - \Omega_N)$ is compact, the precompactness of S in $L^p(\Omega)$ follows by Theorem 2.33. Hence $W^{1,p}(\Omega) \to L^p(\Omega)$ is compact. ∎

6.53 EXAMPLE Consider again the domain of Examples 6.48 and 6.51 and the flow Φ given in the latter example. We have

$$d_N(t) = \sup_{x \ge N} \frac{f(x)}{f(x-t)} \le 1 \qquad \text{if} \quad 0 \le t \le 1$$

and by (28)

$$\lim_{N\to\infty} d_N(t) = 0 \qquad \text{if} \quad t > 0.$$

Thus by dominated convergence

$$\lim_{N\to\infty} \int_0^1 d_N(t)\, dt = 0.$$

The assumption that f' is bounded guarantees that the speed $|(\partial/\partial t)\Phi(x,y,t)|$ is bounded on U. Thus Ω satisfies the hypotheses of Theorem 6.52 and the imbedding $W^{m,p}(\Omega) \to L^p(\Omega)$ is compact for this domain. ∎

6.54 EXAMPLE Theorem 6.52 can also be used to show the compactness of $W^{m,p}(\Omega) \to L^p(\Omega)$ for some bounded domains to which neither the Rellich-Kondrachov theorem nor the techniques used in its proof can be applied. For example, we consider

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 : 0 < x < 2,\ 0 < y < f(x)\right\},$$

where $f \in C^1([0,2])$ is positive, nondecreasing, has bounded derivative f', and satisfies $\lim_{x\to 0+} f(x) = 0$. Let

$$U = \left\{(x,y,t) \in \mathbb{R}^3 : (x,y) \in \Omega,\ -x < t < 2 - x\right\}$$

and define the flow $\Phi : U \to \Omega$ by

$$\Phi(x,y,t) = \left(x+t,\ \frac{f(x+t)}{f(x)}\,y\right).$$

Then $\det \Phi_t'(x,y) = f(x+t)/f(x)$. If $\Omega_N^* = \{(x,y) \in \Omega : x > 1/N\}$, then

$$d_N(t) = \sup_{0<x\le 1/N} \left|\frac{f(x)}{f(x+t)}\right|$$

satisfies $d_N(t) \le 1$ for $0 \le t \le 1$, and $\lim_{N\to\infty} d_N(t) = 0$ if $t > 0$. Hence also $\lim_{N\to\infty} \int_0^1 d_N(t)\,dt = 0$ by dominated convergence. Since Ω_N^* is bounded and satisfies the cone condition, and since the boundedness of $\partial\Phi/\partial t$ is assured by that of f', we have, by Theorem 6.52 the compactness of

$$W^{m,p}(\Omega) \to L^p(\Omega). \tag{31}$$

However, suppose that $\lim_{x\to 0+} f(x)/x^k = 0$ for every k. (For example, this is true if $f(x) = e^{-1/x}$.) Then Ω has an exponential cusp at the origin and by Theorem 4.48 there exists no imbedding of the form $W^{m,p}(\Omega) \to L^q(\Omega)$ for any $q > p$ so the method of proof of the Rellich-Kondrachov theorem cannot be used to show the compactness of (31).

6.55 REMARKS

1. It is easy to imagine domains more general than those in the above examples to which Theorem 6.52 applies, although it may be difficult to specify an appropriate flow. A domain with many (perhaps infinitely many) unbounded branches can, if connected, admit a suitable flow provided volume decays sufficiently rapidly in each branch, a condition not fulfilled by the domain $\tilde{\Omega}$ in Example 6.49. For unbounded domains in which volume

decays monotonically in each branch Theorem 6.45 is essentially a converse of Theorem 6.52 in that the proof of Theorem 6.45 can be applied separately to show that the volume decays in each branch in the required way.

2. Since the only unbounded domains for which $W^{m,p}(\Omega)$ imbeds compactly into $L^p(\Omega)$ have finite volume there can be no extensions of Theorem 6.52 to give compact imbeddings into $L^q(\Omega)$ (where $q > p$), or $C_B(\Omega)$ etc.; there do not exist such imbeddings.

Hilbert-Schmidt Imbeddings

6.56 (Complete Orthonormal Systems) A *complete orthonormal system* in a separable Hilbert space X is a sequence $\{e_i\}_{i=1}^{\infty}$ of elements of X satisfying

$$(e_i, e_j)_X = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

(where $(\cdot, \cdot)_X$ is the inner product on X), and such that for each $x \in X$ we have

$$\lim_{k\to\infty} \left\| x - \sum_{i=1}^{k} (x, e_i)_X e_i \,;\, X \right\| = 0. \tag{32}$$

Thus $x = \sum_{i=1}^{\infty}(x, e_i)e_i$, the series converging with respect to the norm in X. It is well known that every separable Hilbert space possesses such a complete orthonormal system. There follows from (32) the Parseval identity

$$\|x \,;\, X\|^2 = \sum_{i=1}^{\infty} |(x, e_i)_X|^2.$$

6.57 (Hilbert-Schmidt Operators) Let X and Y be two separable Hilbert spaces and let $\{e_i\}_{i=1}^{\infty}$ and $\{f_i\}_{i=1}^{\infty}$ be given complete orthornomal systems in X and Y respectively. Let A be a bounded linear operator with domain X taking values in Y, and let A^* be the adjoint of A taking Y into X and defined by

$$(x, A^*y)_X = (Ax, y)_Y, \qquad x \in X, \quad y \in Y.$$

Define

$$|||A|||^2 = \sum_{i=1}^{\infty} \|Ae_i \,;\, Y\|^2, \qquad |||A^*|||^2 = \sum_{i=1}^{\infty} \left\|A^* f_i \,;\, X\right\|^2.$$

If $|||A|||$ is finite, A is called a *Hilbert-Schmidt operator* and we call $|||A|||$ its *Hilbert-Schmidt norm.* Recall that the operator norm of A is given by

$$\|A\| = \sup\big\{\|Ax\,;Y\| \;:\; \|x\,;X\| \le 1\big\}.$$

We must justify the definition of the Hilbert-Schmidt norm.

6.58 LEMMA The norms $|||A|||$ and $|||A^*|||$ are independent of the particular orthonormal systems $\{e_i\}$ and $\{f_i\}$ used to define them. Moreover

$$|||A||| = |||A^*||| \ge \|A\|.$$

Proof. By Parseval's identity

$$\begin{aligned} |||A|||^2 &= \sum_{i=1}^{\infty} \|Ae_i\,;Y\|^2 = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |(Ae_i, f_j)_Y|^2 \\ &= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} |(e_i, A^*f_j)_X|^2 = \sum_{j=1}^{\infty} \left\|A^*f_j\,;X\right\|^2 = |||A^*|||^2. \end{aligned}$$

Hence each expression is independent of $\{e_i\}$ and $\{f_j\}$. For any $x \in X$ we have

$$\begin{aligned} \|Ax\,;Y\|^2 &= \left\|\sum_{i=1}^{\infty}(x, e_i)_X Ae_i\,;Y\right\|^2 \le \left(\sum_{i=1}^{\infty} |(x, e_i)_X|\,\|Ae_i\,;Y\|\right)^2 \\ &\le \left(\sum_{i=1}^{\infty} |(x, e_i)_X|^2\right)\left(\sum_{j=1}^{\infty} \left\|Ae_j\,;Y\right\|^2\right) = \|x\,;X\|^2\,|||A|||^2. \end{aligned}$$

Hence $\|A\| \le |||A|||$ as required. ∎

6.59 REMARK Consider the scalars (Ae_i, f_j) for $1 \le i, j < \infty$; they are the entries in an infinite matrix representing the operator A. The lemma above shows that the Hilbert-Schmidt norm of A is the sum of the squares of the absolute values of the elements of this matrix. Similarly, the numbers (A^*f_j, e_i) are the entries in a matrix representing A^*. Since these matrices are adjoints of each other, the equality of the corresponding Hilbert-Schmidt norms of the operators is assured.

6.60 We leave to the reader the task of verifying the following assertions.

(a) If X, Y, and Z are separable Hilbert spaces and A and B are bounded linear operators from X into Y and Y into Z, respectively, then $B \circ A$, which maps X into Z, is a Hilbert-Schmidt operator if either A or B is. If A is Hilbert-Schmidt, then $|||B \circ A||| \le \|B\|\,|||A|||$.

(b) Every Hilbert-Schmidt operator is compact.

The following Theorem, due to Maurin [Mr] has far-reaching implications for eigenfunction expansions corresponding to differential operators.

6.61 THEOREM (Maurin's Theorem) Let Ω be a bounded domain in $\mathbb{R}^n$ satisfying the cone condition. Let m and k be nonnegative integers with $k > n/2$. Then the imbedding map

$$W^{m+k,2}(\Omega) \to W^{m,2}(\Omega) \tag{33}$$

is a Hilbert-Schmidt operator. Similarly the imbedding map

$$W_0^{m+k,2}(\Omega) \to W_0^{m,2}(\Omega) \tag{34}$$

is a Hilbert-Schmidt operator for any bounded domain Ω.

Proof. Given $y \in \Omega$ and α with $|\alpha| \le m$ we define a linear functional T_y^α on $W^{m+k,2}(\Omega)$ by

$$T_y^\alpha(u) = D^\alpha u(y).$$

Since $2k > m$, the Sobolev Imbedding Theorem 4.12 implies that T_y^α is bounded on $W^{m+k,2}(\Omega)$ and has norm bounded by a constant K independent of y and α:

$$|T_y^\alpha(u)| \le \max_{0\le|\alpha|\le m} \sup_{x\in\Omega} |D^\alpha u(x)| \le K\, \|u\|_{m+k,2,\Omega}\,.$$

By the Riesz representation theorem for Hilbert spaces there exists $v_y^\alpha \in W^{m+k,2}(\Omega)$ such that

$$D^\alpha u(y) = T_y^\alpha(u) = \left(u, v_y^\alpha\right)_{m+k},$$

where $(\cdot,\cdot)_{m+k}$ is the inner product on $W^{m+k,2}(\Omega)$. Moreover

$$\left\|v_y^\alpha\right\|_{m+k,2,\Omega}^2 = \left\|T_y^\alpha\,;\,\left[W^{m+k,2}(\Omega)\right]'\right\| \le K.$$

If $\{e_i\}_{i=1}^\infty$ is a complete orthonormal system in $W^{m+k,2}(\Omega)$, then

$$\left\|v_y^\alpha\right\|_{m+k,2,\Omega}^2 = \sum_{i=1}^\infty \left|\left(e_i, v_y^\alpha\right)_{m+k}\right|^2 = \sum_{i=1}^\infty |D^\alpha e_i(y)|^2.$$

Consequently,

$$\sum_{i=1}^\infty \|e_i\|_{m,2,\Omega}^2 \le \sum_{|\alpha|\le m} \int_\Omega \left\|v_y^\alpha\right\|_{m+k,2,\Omega}^2 dy \le \sum_{|\alpha|\le m} K\,\mathrm{vol}(\Omega) < \infty.$$

Hence imbedding (33) is Hilbert-Schmidt. The corresponding imbedding (34) is also Hilbert-Schmidt without the cone-condition requirement as it is not needed for the application of Theorem 4.12 in this case. ∎

The following generalization of Maurin's theorem is due to Clark [Ck].

6.62 THEOREM Let μ be a nonnegative, measurable function defined on the domain Ω in $\mathbb{R}^n$. Let $W_0^{m,2;\mu}(\Omega)$ be the Hilbert space obtained by completing $C_0^\infty(\Omega)$ with respect to the weighted norm

$$\|u\|_{m,2;\mu} = \left(\sum_{|\alpha|\le m} \int_\Omega |D^\alpha u(x)|^2 \, \mu(x)\, dx\right)^{1/2}.$$

For $y \in \Omega$ let $\tau(y) = \operatorname{dist}(y, \operatorname{bdry}\Omega)$. Suppose that

$$\int_\Omega \big(\tau(y)\big)^{2\nu} \mu(y)\, dy < \infty \tag{35}$$

for some nonnegative integer ν. If $k > \nu + n/2$, then the imbedding

$$W_0^{m+k,2}(\Omega) \to W_0^{m,2;\mu}(\Omega) \tag{36}$$

(exists and) is Hilbert-Schmidt.

Proof. The argument is parallel to that given in the proof of Maurin's theorem above. Let $\{e_i\}$, T_y^α, and v_y^α be defined as there. If $y \in \Omega$, let y_0 be chosen in $\operatorname{bdry}\Omega$ such that $\tau(y) = |y - y_0|$. If ν is a positive integer and $u \in C_0^\infty(\Omega)$, we have by Taylor's formula with remainder

$$D^\alpha u(y) = \sum_{|\beta|=\nu} \frac{1}{\beta!} D^{\alpha+\beta} u(y_\beta)(y - y_\beta)^\beta$$

for some points y_β satisfying $|y - y_\beta| \le \tau(y)$. If $|\alpha| \le m$ and $k > \nu + n/2$, we obtain from Theorem 4.12

$$|D^\alpha u(y)| \le K\, \|u\|_{m+k,2,\Omega} \big(\tau(y)\big)^\nu.$$

By completion this inequality holds for any $u \in W_0^{m+k,2}(\Omega)$. As in the proof of Maurin's theorem, it follows that

$$\left\|v_y^\alpha\right\|_{m+k,2,\Omega} = \sup_{\|u\|_{m+k,2,\Omega}=1} |D^\alpha u(y)| \le K\big(\tau(y)\big)^\nu,$$

and hence also that

$$\sum_{i=1}^{\infty} \|e_i\|_{m,2;\mu}^2 \le \sum_{|\alpha|\le m} \int_{\Omega} \|v_y^{\alpha}\|_{m+k,2,\Omega}^2 \, \mu(y)\, dy$$
$$\le K^2 \sum_{|\alpha|\le m} \int_{\Omega} \big(\tau(y)\big)^{2\nu} \mu(y)\, dy < \infty$$

by (35). Hence imbedding (36) is Hilbert-Schmidt. ∎

6.63 REMARK Various choices of μ and ν lead to generalizations of Maurin's theorem for imbeddings of the sort (34). If $\mu(x) = 1$ and $\nu = 0$ we obtain the obvious generalization to unbounded domains of finite volume. If $\mu(x) = 1$ and $\nu > 0$, Ω may be unbounded and even have infinite volume, but it must be quasibounded by (35). Of course quasiboundedness may not be sufficient to guarantee (35). If μ is the characteristic function of a bounded subdomain Ω_0 of Ω, and $\nu = 0$, we obtain the Hilbert-Schmidt imbedding

$$W_0^{m+k,2}(\Omega) \to W^{m,2}(\Omega_0), \qquad k > n/2.$$

7

FRACTIONAL ORDER SPACES

Introduction

7.1 This chapter is concerned with extending the notion of the standard Sobolev space $W^{m,p}(\Omega)$ to include spaces where m need not be an integer. There are various ways to define such *fractional order* spaces; many of them depend on using interpolation to construct scales of spaces suitably intermediate between two extreme spaces, say $L^p(\Omega)$ and $W^{m,p}(\Omega)$.

Interpolation methods themselves come in two flavours: real methods and complex methods. We have already seen an example of the real method in the Marcinkiewicz theorem of Paragraph 2.58. Although the details of the real method can be found in several sources, for example, [BB], [BL], and [BSh], we shall provide a treatment here in sufficient detail to make clear its application to the development of the Besov spaces, one of the scales of fractional order Sobolev spaces that particularly lends itself to characterizing the spaces of traces of functions in $W^{m,p}(\Omega)$ on the boundaries of smoothly bounded domains Ω; such characterizations are useful in the study of boundary-value problems. Several older interpolation methods are known [BL, pp. 70–75] to be equivalent to the now-standard real interpolation method that we use here. In the the corresponding chapter of the previous edition [A] of this book, the older method of traces was used rather than the method presented in this edition. Later in this Chapter, we prove a trace theorem (Theorem 7.39) giving an instance of that equivalence.

After that we shall describe more briefly other scales of fractional order Sobolev spaces, some obtained by complex methods and some by Fourier decompositions.

The Bochner Integral

7.2 In developing the real interpolation method below we will use the concept of the integral of a Banach-space-valued function defined on an interval on the real line $\mathbb{R}$. (For the complex method we will use the concept of analytic Banach-space-valued functions of a complex variable.) We present here a brief description of the Bochner integral, referring the reader to [Y] or [BB] for more details.

Let X be a Banach space with norm $\|\cdot\|_X$ and let f be a function defined on an interval (a, b) in $\mathbb{R}$ (which may be infinite) and having values in X. In addition, let μ be a measure on (a, b) given by $d\mu(t) = w(t)\,dt$ where w is continuous and positive on (a, b). Of special concern to us later will be the case where $a = 0$, $b = \infty$, and $w(t) = 1/t$. In this case μ is the Haar measure on $(0, \infty)$, which is invariant under scaling in the multiplicative group $(0, \infty)$: if $(c, d) \subset (0, \infty)$ and $\lambda > 0$, then $\mu(\lambda c, \lambda d) = \mu(c, d)$.

We want to define the integral of f over (a, b).

7.3 (Definition of the Bochner Integral) If $\{A_1, \dots, A_k\}$ is a finite collection of mutually disjoint subsets of (a, b) each having finite μ-measure, and if $\{x_1, \dots, x_k\}$ is a corresponding set of elements of X, we call the function f defined by

$$f(t) = \sum_{i=1}^{k} \chi_{A_i}(t) x_i, \qquad a < t < b,$$

a *simple function* on (a, b) into X. For such simple functions we define, obviously,

$$\int_a^b f(t)\,d\mu(t) = \sum_{i=1}^{k} \mu(A_i) x_i = \sum_{i=1}^{k} \left(\int_{A_i} w(t)\,dt \right) x_i.$$

Of course, a different representation of the simple function f using a different collection of subsets of (a, b) will yield the same value for the integral; the subsets in the collections need not be mutually disjoint, and given two such collections we can always form an equivalent mutually disjoint collection consisting of pairwise intersections of the elements of the two collections.

Now let f an arbitrary function defined on (a, b) into X. We say that f is *(strongly) measurable* on (a, b) if there exists a sequence $\{f_j\}$ of simple functions with supports in (a, b) such that

$$\lim_{j\to\infty} \left\| f_j(t) - f(t) \right\|_X \qquad \text{a.e. in } (a, b). \tag{1}$$

It can be shown that f is measurable if its range is separable and if, for each x' in the dual of X, the scalar-valued function $x'\big(f(\cdot)\big)$ is measurable on (a, b).

Suppose that a sequence of simple functions $\{f_j\}$ satisfying (1) can be chosen in such a way that

$$\lim_{j\to\infty} \int_a^b \|f_j(t) - f(t)\|_X \, d\mu(t) = 0.$$

Then we say that f is *Bochner integrable* on (a, b) and we define

$$\int_a^b f(t)\, d\mu(t) = \lim_{j\to\infty} \int_a^b f_j(t)\, d\mu(t).$$

Again we observe that the limit does not depend on the choice of the approximating simple functions.

A measurable function f is integrable on (a, b) if and only if the scalar-valued function $\|f(\cdot)\|_X$ is integrable on (a, b). In fact, there holds the "triangle inequality"

$$\left\| \int_a^b f(t)\, d\mu(t) \right\|_X \le \int_a^b \|f(t)\|_X \, d\mu(t).$$

7.4 (The Spaces $L^q(a, b; d\mu, X)$) If $1 \le q \le \infty$, we say that $f \in L^q(a, b; d\mu, X)$ provided $\|f\,;\, L^q(a, b; d\mu, X)\| < \infty$, where

$$\big\|f\,;\, L^q(a, b; d\mu, X)\big\| = \begin{cases} \left(\displaystyle\int_a^b \|f(t)\|_X^q \, d\mu(t) \right)^{1/q} & \text{if } 1 \le q < \infty \\ \operatorname{ess\,sup}_{a<t<b} \{\|f(t)\|_X\} & \text{if } q = \infty. \end{cases}$$

In particular, if $X = \mathbb{R}$ or $X = \mathbb{C}$, we will denote $L^q(a, b; d\mu, X)$ simply by $L^q(a, b; d\mu)$.

7.5 (The spaces L^q_*) Of much importance below is the special case where $X = \mathbb{R}$ or $\mathbb{C}$, $(a, b) = (0, \infty)$, and $d\mu = dt/t$; we will further abbreviate the notation for this , denoting $L^q(a, b; d\mu, X)$ simply L^q_*. Note that L^q_* is equivalent to $L^q(\mathbb{R})$ with Lebesgue measure via a change of variable: if $t = e^s$ and $f(t) = f(e^s) = F(s)$, then $\big\|f\,;\, L^q_*\big\| = \|F\|_{q,\mathbb{R}}$. Most of the properties of $L^q(\mathbb{R})$ transfer to properties of L^q_*. In particular Hölder's and Young's inequalities hold; we will need both of them below. It should be noted that the convolution of two functions f and g defined on $(0, \infty)$ and integrated with respect to the Haar measure dt/t is given by

$$f * g(t) = \int_0^\infty f\left(\frac{t}{s}\right) g(s)\, \frac{ds}{s},$$

and Young's inequality proclaims $\| f * g\,;L^r_*\| \le \| f\,;L^p_*\|\,\|g\,;L^q_*\|$ provided $p, q, r \ge 1$ and $1 + (1/r) = (1/p) + (1/q)$.

Intermediate Spaces and Interpolation — The Real Method

7.6 In this Section we will be discussing the construction of Banach spaces X that are suitably intermediate between two Banach spaces X_0 and X_1, each of which is (continuously) imbedded in a Hausdorff topological vector space $\mathcal{X}$, and whose intersection is nontrivial. (Such a pair of spaces $\{X_0, X_1\}$ is called an *interpolation pair* and X is called an intermediate space of the pair. In some of our later applications, we will have $X_1 \to X_0$ (for example, $X_0 = L^p(\Omega)$ and $X_1 = W^{m,p}(\Omega)$), in which case we can clearly take $\mathcal{X} = X_0$. We shall, in fact, be constructing families of such intermediate spaces $X_{\theta,q}$ between X_0 and X_1, such that if $Y_{\theta,q}$ is the corresponding intermediate space for another such interpolation pair $\{Y_0, Y_1\}$ with Y_0 and Y_1 imbedded in $\mathcal{Y}$, and if T is a linear operator from $\mathcal{X}$ into $\mathcal{Y}$ (for example an imbedding operator) such that T is bounded from X_i into Y_i, $i = 0, 1$, then T will also be bounded from $X_{\theta,q}$ into $Y_{\theta,q}$.

There are many different ways of constructing such intermediate spaces, mostly leading to the same spaces with equivalent norms. We examine here two such methods, the J-method and the K-method, (together called the real method) due to Lions and Peetre. The theory is developed in several texts, in particular [BB] and [BL]. Our approach follows [BB] and we will omit some aspects of the theory for which we have no future need.

7.7 (Intermediate Spaces) Let $\|\cdot\|_{X_i}$ denote the norm in X_i, $i = 0, 1$. The intersection $X_0 \cap X_1$ and the algebraic sum $X_0 + X_1 = \{u = u_0 + u_1 : u_0 \in X_0, u_1 \in X_1\}$ are themselves Banach spaces with respect to the norms

$$\begin{aligned}\|u\|_{X_0\cap X_1} &= \max\big\{\|u\|_{X_0}, \|u\|_{X_1}\big\}\\ \|u\|_{X_0+X_1} &= \inf\big\{\|u_0\|_{X_0} + \|u_1\|_{X_1} : u = u_0 + u_1, u_0 \in X_0, u_1 \in X_1\big\}.\end{aligned}$$

and $X_0 \cap X_1 \to X_i \to X_0 + X_1$ for $i = 0, 1$.

In general, we say that a Banach space X is *intermediate* between X_0 and X_1 if there exist the imbeddings

$$X_0 \cap X_1 \to X \to X_0 + X_1.$$

7.8 (The J and K norms) For each fixed $t > 0$ the following functionals define norms on $X_0 \cap X_1$ and $X_0 + X_1$ respectively, equivalent to the norms defined above:

$$\begin{aligned}J(t; u) &= \max\big\{\|u\|_{X_0}, t\,\|u\|_{X_1}\big\}\\ K(t; u) &= \inf\big\{\|u_0\|_{X_0} + t\,\|u_1\|_{X_1} : u = u_0 + u_1, u_0 \in X_0, u_1 \in X_1\big\}.\end{aligned}$$

Evidently $J(1; u) = \|u\|_{X_0 \cap X_1}$, $K(1; u) = \|u\|_{X_0+X_1}$, and $J(t; u)$ and $K(t; u)$ are continuous and monotonically increasing functions of t on $(0, \infty)$. Moreover

$$\min\{1, t\} \|u\|_{X_0 \cap X_1} \le J(t; u) \le \max\{1, t\} \|u\|_{X_0 \cap X_1} \tag{2}$$

$$\min\{1, t\} \|u\|_{X_0+X_1} \le K(t; u) \le \max\{1, t\} \|u\|_{X_0+X_1} . \tag{3}$$

$J(t; u)$ is a convex function of t because, if $0 < a < b$ and $0 < \theta < 1$,

$$\begin{aligned} J\big((1-\theta)a + \theta b; u\big) &= \max\big\{\|u\|_{X_0}, (1-\theta)a \|u\|_{X_1} + \theta b \|u\|_{X_1}\big\} \\ &\le (1-\theta) \max\big\{\|u\|_{X_0}, a \|u\|_{X_1}\big\} + \theta \max\big\{\|u\|_{X_0}, b \|u\|_{X_1}\big\} \\ &= (1-\theta) J(a; u) + \theta J(b; u). \end{aligned}$$

Also for such a, b, θ and any $u_0 \in X_0$ and $u_1 \in X_1$ for which $u = u_0 + u_1$ we have

$$\begin{aligned} &\|u_0\|_{X_0} + \big((1-\theta)a + \theta b\big) \|u_1\|_{X_1} \\ &\qquad = (1-\theta)\big(\|u_0\|_{X_0} + a \|u_1\|_{X_1}\big) + \theta\big(\|u_0\|_{X_0} + b \|u_1\|_{X_1}\big) \\ &\qquad \ge (1-\theta) K(a; u) + \theta K(b; u), \end{aligned}$$

so that $K\big((1-\theta)a + \theta b); u\big) \ge (1-\theta)K(a; u) + \theta K(b; u)$ and $K(t; u)$ is a concave function of t.

Finally we observe that if $u \in X_0 \cap X_1$, then for any positive t and s we have $K(t; u) \le \|u\|_{X_0} \le J(s; u)$ and $K(t; u) \le t \|u\|_{X_1} = (t/s)s \|u\|_{X_1} \le (t/s)J(s; u)$. Accordingly,

$$K(t; u) \le \min\left\{1, \frac{t}{s}\right\} J(s; u). \tag{4}$$

7.9 (The K-method) If $0 \le \theta \le 1$ and $1 \le q \le \infty$ we denote by $(X_0, X_1)_{\theta,q;K}$ the space of all $u \in X_0 + X_1$ such that the function $t \to t^{-\theta} K(t; u)$ belongs to $L^q_* = L^q(0, \infty; dt/t)$.

Of course, the zero element $u = 0$ of $X_0 + X_1$ always belongs to $(X_0, X_1)_{\theta,q;K}$. The following theorem shows that if $1 \le q < \infty$ and either $\theta = 0$ or $\theta = 1$, then $(X_0, X_1)_{\theta,q;K}$ contains only this trivial element. Otherwise $(X_0, X_1)_{\theta,q;K}$ is an intermediate space between X_0 and X_1.

7.10 THEOREM If and only if either $1 \le q < \infty$ and $0 < \theta < 1$ or $q = \infty$ and $0 \le \theta \le 1$, then the space $(X_0, X_1)_{\theta,q;K}$ is a nontrivial Banach space with norm

$$\|u\|_{\theta,q;K} = \begin{cases} \left(\displaystyle\int_0^\infty \big(t^{-\theta} K(t; u)\big)^q \frac{dt}{t}\right)^{1/q} & \text{if } 1 \le q < \infty \\ \operatorname{ess\,sup}_{0<t<\infty} \{t^{-\theta} K(t; u)\} & \text{if } q = \infty. \end{cases}$$

Furthermore,

$$\|u\|_{X_0+X_1} \le \frac{\|u\|_{\theta,q;K}}{\left\|t^{-\theta}\min\{1,t\}\,;\,L^q_*\right\|} \le \|u\|_{X_0\cap X_1} \tag{5}$$

so there hold the imbeddings

$$X_0 \cap X_1 \to (X_0, X_1)_{\theta,q;K} \to X_0 + X_1$$

and $(X_0, X_1)_{\theta,q;K}$ is an intermediate space between X_0 and X_1.
Otherwise $(X_0, X_1)_{\theta,q;K} = \{0\}$.

Proof. It is easily checked that the function $t \to t^{-\theta}\min\{1,t\}$ belongs to L^q_* if and only if θ and q satisfy the conditions of the theorem. Since (3) shows that

$$\left\|t^{-\theta}\min\{1,t\}\,;\,L^q_*\right\| \,\|u\|_{X_0+X_1} \le \left\|t^{-\theta}K(t;u)\,;\,L^q_*\right\| = \|u\|_{\theta,q;K}\,,$$

there can be no nonzero elements of $(X_0, X_1)_{\theta,q;K}$ unless those conditions are satisfied. If so, then the left inequality in (5) holds and $(X_0, X_1)_{\theta,q;K} \to X_0 + X_1$. Also, by (4) we have $K(t;u) \le \min\{1,t\}J(1;u) = \min\{1,t\}\,\|u\|_{X_0\cap X_1}$ so the right inequality in (5) holds and $X_0 \cap X_1 \to (X_0, X_1)_{\theta,q;K}$.
Verification that $\|u\|_{\theta,q;K}$ is a norm and that $(X_0, X_1)_{\theta,q;K}$ is complete under it are left as exercises for the reader. ∎

Note that $u \in X_0$ and $\theta = 0$ implies that $t^{-\theta}K(t;u) \le \|u\|_{X_0}$. Also, $u \in X_1$ and $\theta = 1$ implies that $t^{-\theta}K(t;u) \le \|u\|_{X_1}$. Thus we also have

$$X_0 \to (X_0, X_1)_{0,\infty;K} \qquad \text{and} \qquad X_1 \to (X_0, X_1)_{1,\infty;K}. \tag{6}$$

7.11 THEOREM (A Discrete Version of the K-method) For each integer i let $K_i(u) = K(2^i;u)$. Then $u \in (X_0, X_1)_{\theta,q;K}$ if and only if the sequence $\{2^{-i\theta}K_i(u)\}_{i=-\infty}^{\infty}$ belongs to the space ℓ^q (defined in Paragraph 2.27). Moreover, the ℓ^q-norm of that sequence is equivalent to $\|u\|_{\theta,q;K}$.

Proof. First write (for $1 \le q < \infty$)

$$\int_0^\infty \left(t^{-\theta}K(t;u)\right)^q \frac{dt}{t} = \sum_{i=-\infty}^{\infty} \int_{2^i}^{2^{i+1}} \left(t^{-\theta}K(t;u)\right)^q \frac{dt}{t}.$$

Since $K(t;u)$ increases and $t^{-\theta}$ decreases as t increases, we have for $2^i \le t \le 2^{i+1}$,

$$2^{-(i+1)\theta}K_i(u) \le t^{-\theta}K(t;u) \le 2^{-i\theta}K_{i+1}(u),$$

so that

$$2^{-\theta q}\ln 2\big[2^{-i\theta}K_i(u)\big]^q \le \int_{2^i}^{2^{i+1}} \big(t^{-\theta}K(t;u)\big)^q \frac{dt}{t} \le 2^{\theta q}\ln 2\big[2^{-(i+1)\theta}K_{i+1}(u)\big]^q.$$

Summing on i and taking qth roots then gives

$$2^{-\theta}(\ln 2)^{1/q}\left\|\big\{2^{-i\theta}K_i(u)\big\};\ell^q\right\| \le \|u\|_{\theta,q;K} \le 2^{\theta}(\ln 2)^{1/q}\left\|\big\{2^{-i\theta}K_i(u)\big\};\ell^q\right\|.$$

The proof for $q=\infty$ is easier and left for the reader. ∎

7.12 (The J-method) If $0\le\theta\le 1$ and $1\le q\le\infty$ we denote by $(X_0,X_1)_{\theta,q;J}$ the space of all $u\in X_0+X_1$ such that

$$u=\int_0^\infty f(t)\,\frac{dt}{t}$$

(Bochner integral) for some $f\in L^1(0,\infty;dt/t,X_0+X_1)$ having values in $X_0\cap X_1$ and such that the real-valued function $t\to t^{-\theta}J(t;f)$ belongs to L^q_*.

7.13 THEOREM If either $1<q\le\infty$ and $0<\theta<1$ or $q=1$ and $0\le\theta\le 1$, then $(X_0,X_1)_{\theta,q;J}$ is a nontrivial Banach space with norm

$$\begin{aligned}\|u\|_{\theta,q;J} &= \inf_{f\in S(u)}\left\|t^{-\theta}J\big(t;f(t)\big);L^q_*\right\|\\ &= \inf_{f\in S(u)}\left(\int_0^\infty \big[t^{-\theta}J\big(t;f(t)\big)\big]^q\frac{dt}{t}\right)^{1/q}, \quad (\text{if } q<\infty),\end{aligned}$$

where

$$S(u)=\left\{f\in L^1(0,\infty;dt/t,X_0+X_1)\ :\ u=\int_0^\infty f(t)\,\frac{dt}{t}\right\}.$$

Furthermore,

$$\|u\|_{X_0+X_1}\le\left(\left\|t^{-\theta}\min\{1,t\};L^{q'}_*\right\|\right)\|u\|_{\theta,q;J}\le\|u\|_{X_0\cap X_1} \tag{7}$$

so that

$$X_0\cap X_1\to(X_0,X_1)_{\theta,q;J}\to X_0+X_1$$

and $(X_0,X_1)_{\theta,q;J}$ is an intermediate space between X_0 and X_1.

Proof. Again we leave verification of the norm and completeness properties to the reader and we concentrate on the imbeddings.

Let $f \in S(u)$. By (3) and (4) with $t = 1$ and $s = \tau$ we have

$$\|f(\tau)\|_{X_0+X_1} \le K\big(1, f(\tau)\big) \le \min\left\{1, \frac{1}{\tau}\right\} J\big(\tau, f(\tau)\big).$$

Accordingly, If $(1/q) + (1/q') = 1$, then by Hölder's inequality

$$\begin{aligned}\|u\|_{X_0+X_1} &\le \int_0^\infty \|f(\tau)\|_{X_0+X_1} \frac{d\tau}{\tau} \le \int_0^\infty \min\left\{1, \frac{1}{\tau}\right\} J\big(\tau, f(\tau)\big) \frac{d\tau}{\tau} \\ &\le \left\| \tau^\theta \min\left\{1, \frac{1}{\tau}\right\} ; L_*^{q'} \right\| \left\| t^{-\theta} J\big(t; f(t)\big) ; L_*^q \right\|.\end{aligned}$$

The first factor in this product of norms is finite if θ and q satisfy the conditions of the theorem, and if we replace τ with $1/t$ in it, we can see that it is equal to $\left\| t^{-\theta} \min\{1, t\} ; L_*^{q'} \right\|$. Since the above inequality holds for all $f \in S(u)$, the left inequality in (7) is established and $(X_0, X_1)_{\theta,q;J} \to X_0 + X_1$.

To verify the right inequality in (7), let $u \in X_0 \cap X_1$. Let $\phi(t) \ge 0$ satisfy $\left\| t^{-\theta}\phi(t) ; L_*^q \right\| = 1$. Hölder's inequality shows that

$$\int_0^\infty \phi(\tau) \min\{1, 1/\tau\} \frac{d\tau}{\tau} < \infty.$$

If

$$f(t) = \frac{\phi(t) \min\{1, 1/t\}}{\displaystyle\int_0^\infty \phi(\tau) \min\{1, 1/\tau\} \frac{d\tau}{\tau}} u,$$

then $f \in S(u)$ and

$$\begin{aligned} J\big(t; f(t)\big) &= \frac{\phi(t) \min\{1, 1/t\}}{\displaystyle\int_0^\infty \phi(\tau) \min\{1, 1/\tau\} \frac{d\tau}{\tau}} J(t; u) \\ &\le \frac{\phi(t)}{\displaystyle\int_0^\infty \phi(\tau) \min\{1, 1/\tau\} \frac{d\tau}{\tau}} \|u\|_{X_0\cap X_1},\end{aligned}$$

the latter inequality following from (2) since $\max\{1, t\} = \big(\min\{1, 1/t\}\big)^{-1}$. Therefore,

$$\begin{aligned} &\left(\int_0^\infty \phi(\tau) \min\{1, 1/\tau\} \frac{d\tau}{\tau} \right) \|u\|_{\theta,q;J} \\ &\le \left(\int_0^\infty \phi(\tau) \min\{1, 1/\tau\} \frac{d\tau}{\tau} \right) \left(\int_0^\infty \big(t^{-\theta} J\big(t; f(t)\big)\big)^q \frac{dt}{t} \right)^{1/q} \\ &\le \left(\int_0^\infty \big(t^{-\theta}\phi(t) \|u\|_{X_0\cap X_1}\big)^q \frac{dt}{t} \right)^{1/q} = \|u\|_{X_0\cap X_1}.\end{aligned}$$

By the converse to Hölder's inequality,

$$\sup\left\{\int_0^\infty \phi(\tau)\min\{1,1/\tau\}\,\frac{d\tau}{\tau} \;:\; \left\|\tau^{-\theta}\phi(\tau)\,;\,L_*^q\right\| = 1\right\}$$
$$= \left\|\tau^{\theta}\min\{1,1/\tau\}\,;\,L_*^{q'}\right\| = \left\|t^{-\theta}\min\{1,t\}\,;\,L_*^{q'}\right\|.$$

Thus the right inequality in (7) is established and $X_0 \cap X_1 \to (X_0, X_1)_{\theta,q;J}$. ∎

7.14 Observe that if $u = \int_0^\infty f(t)\,dt/t$ where $f(t) \in X_0 \cap X_1$, then

$$\|u\|_{X_0} \le \int_0^\infty \|f(t)\|_{X_0}\,\frac{dt}{t} \le \int_0^\infty J\big(t, f(t)\big)\,\frac{dt}{t}$$
$$\|u\|_{X_1} \le \int_0^\infty \|f(t)\|_{X_1}\,\frac{dt}{t} \le \int_0^\infty t^{-1}J\big(t, f(t)\big)\,\frac{dt}{t}.$$

Each of these estimates holds for all such representations of u, so $\|u\|_{X_0} \le \|u\|_{0,1;J}$ and $\|u\|_{X_1} \le \|u\|_{1,1;J}$. Combining these with (6) we obtain

$$\begin{aligned} (X_0, X_1)_{0,1;J} &\to X_0 \to (X_0, X_1)_{0,\infty;K} \\ (X_0, X_1)_{1,1;J} &\to X_1 \to (X_0, X_1)_{1,\infty;K}. \end{aligned} \tag{8}$$

There is also a discrete version of the J-method leading to an equivalent norm for $(X_0, X_1)_{\theta,q;J}$.

7.15 THEOREM (A Discrete Version of the J-method) An element u of $X_0 + X_1$ belongs to $(X_0, X_1)_{\theta,q;J}$ if and only if $u = \sum_{i=-\infty}^{\infty} u_i$ where the series converges in $X_0 + X_1$ and the sequence $\{2^{-\theta i}J(2^i, u_i)\}$ belongs to ℓ^q. In this case

$$\inf\left\{\left\|\{2^{-\theta i}J(2^i; u_i)\}\,;\,\ell^q\right\| \;:\; u = \sum_{i=-\infty}^{\infty} u_i\right\}$$

is a norm on $(X_0, X_1)_{\theta,q;J}$ equivalent to $\|u\|_{\theta,q;J}$.

Proof. Again we will show this for $1 \le q < \infty$ and leave the easier case $q = \infty$ to the reader.

First suppose that $u \in (X_0, X_1)_{\theta,q;J}$ and let $\epsilon > 0$. Then there exists a function $f \in L^1(0,\infty; dt/t, X_0 + X_1)$ such that

$$u = \int_0^\infty f(t)\,\frac{dt}{t}$$

and

$$\int_0^\infty \left[t^{-\theta}J\big(t; f(t)\big)\right]^q \frac{dt}{t} \le (1+\epsilon)\,\|u\|_{\theta,q;J}.$$

Let the sequence $\{u_i\}_{i=-\infty}^{\infty}$ be defined by

$$u_i = \int_{2^i}^{2^{i+1}} f(t)\,\frac{dt}{t}.$$

then $\sum_{i=-\infty}^{\infty} u_i$ converges to u in $X_0 + X_1$ because the integral representation converges to u there. Moreover,

$$\begin{aligned}
2^{-i\theta} J(2^i; u_i) &\le \int_{2^i}^{2^{i+1}} 2^{-i\theta} J\big(t; f(t)\big)\,\frac{dt}{t} \\
&= 2^{\theta} \int_{2^i}^{2^{i+1}} 2^{-(i+1)\theta} J\big(t; f(t)\big)\,\frac{dt}{t} \\
&\le 2^{\theta} \int_{2^i}^{2^{i+1}} t^{-\theta} J\big(t; f(t)\big)\,\frac{dt}{t} \\
&\le 2^{\theta}\big(\ln 2\big)^{1/q'} \left(\int_{2^i}^{2^{i+1}} \big[t^{-\theta} J\big(t; f(t)\big)\big]^q\,\frac{dt}{t}\right)^{1/q},
\end{aligned}$$

where $q' = q/(q-1)$ and Hölder's inequality was used in the last line. Thus

$$\sum_{i=-\infty}^{\infty} \big[2^{-i\theta} J(2^i; u_i)\big]^q \le 2^{\theta q}\big(\ln 2\big)^{q/q'} \int_0^{\infty} \big[t^{-\theta} J\big(t; f(t)\big)\big]^q\,\frac{dt}{t}$$

and, since ϵ is arbitrary,

$$\big\| \big\{2^{-i\theta} J(2^i; u_i)\big\}\,;\, \ell^q \big\| \le 2^{\theta}\big(\ln 2\big)^{1/q'}\, \|u\|_{\theta,q;J}\,.$$

Conversely, if $u = \sum_{i=-\infty}^{\infty} u_i$ where the series converges in $X_0 + X_1$, we can define a function $f \in L^1(0, \infty; dt/t, X_0 + X_1)$ by

$$f(t) = \frac{1}{\ln 2}\, u_i, \qquad \text{for} \quad 2^i \le t < 2^{i+1}, \quad -\infty < i < \infty,$$

and we will have

$$\int_{2^i}^{2^{i+1}} f(t)\,\frac{dt}{t} = u_i \qquad \text{and} \qquad u = \int_0^{\infty} f(t)\,\frac{dt}{t}.$$

Moreover,

$$\begin{aligned}
\int_{2^i}^{2^{i+1}} \big[t^{-\theta} J\big(t; f(t)\big)\big]^q\,\frac{dt}{t} &\le \int_{2^i}^{2^{i+1}} \big[2^{-i\theta} J\big(2^{i+1}; f(t)\big)\big]^q\,\frac{dt}{t} \\
&\le \left(\frac{2}{\ln 2}\right)^q \int_{2^i}^{2^{i+1}} \big[2^{-i\theta} J(2^i; u_i)\big]^q\,\frac{dt}{t} \\
&= \frac{2^q}{(\ln 2)^{q-1}} \big[2^{-i\theta} J(2^i; u_i)\big]^q.
\end{aligned}$$

Summing on i then gives

$$\|u\|_{\theta,q;J} \le \left(\frac{2}{(\ln 2)^{1/q'}}\right) \left\| \{2^{-i\theta} J(2^i; u_i)\} ; \ell^q \right\| . \blacksquare$$

Next we prove that for $0 < \theta < 1$ the J- and K-methods generate the same intermediate spaces with equivalent norms.

7.16 THEOREM (Equivalence Theorem) If $0 < \theta < 1$ and $1 \le q \le \infty$, then

(a) $(X_0, X_1)_{\theta,q;J} \to (X_0, X_1)_{\theta,q;K}$, and

(b) $(X_0, X_1)_{\theta,q;K} \to (X_0, X_1)_{\theta,q;J}$. Therefore

(c) $(X_0, X_1)_{\theta,q;J} = (X_0, X_1)_{\theta,q;K}$, the two spaces having equivalent norms.

Proof. Conclusion (a) is a consequence of the somewhat stronger result

$$(X_0, X_1)_{\theta,p;J} \to (X_0, X_1)_{\theta,q;K}, \qquad \text{if } 1 \le p \le q \tag{9}$$

which we now prove. Let $u = \int_0^\infty f(s)\, ds/s \in (X_0, X_1)_{\theta,p;J}$. Since $K(t; \cdot)$ is a norm on $X_0 + X_1$, we have by the triangle inequality and (4)

$$\begin{aligned} t^{-\theta} K(t; u) &\le t^{-\theta} \int_0^\infty K\big(t; f(s)\big) \frac{ds}{s} \\ &\le \int_0^\infty \left(\frac{t}{s}\right)^{-\theta} \min\left\{1, \frac{t}{s}\right\} s^{-\theta} J\big(s; f(s)\big) \frac{ds}{s} \\ &= \big[t^{-\theta} \min\{1, t\}\big] * \big[t^{-\theta} J\big(t; f(t)\big)\big]. \end{aligned}$$

By Young's inequality with $1 + (1/q) = (1/r) + (1/p)$ (so $r \ge 1$)

$$\begin{aligned} \|u\|_{\theta,q;K} &= \left\|t^{-\theta} K(t; u) ; L^q_*\right\| \\ &\le \left\|t^{-\theta} \min\{1, t\} ; L^r_*\right\| \left\|t^{-\theta} J\big(t; f(t)\big) ; L^p_*\right\| \\ &\le C_{\theta,p,q} \|u\|_{\theta,p;K} , \end{aligned}$$

which confirms (9) and hence (a).

Now we prove (b) by using the discrete versions of the J and K methods. Let $u \in (X_0, X_1)_{\theta,p;K}$. By the definition of $K(t; u)$, for each integer i there exist $v_i \in X_0$ and $w_i \in X_1$ such that

$$u = v_i + w_i \qquad \text{and} \qquad \|v_i\|_{X_0} + 2^i \|w_i\|_{X_1} \le 2K(2^i; u).$$

Then the sequences $\{2^{-i\theta} \|v_i\|_{X_0}\}$ and $\{2^{i(1-\theta)} \|w_i\|_{X_1}\}$ both belong to ℓ^q and each has ℓ^q-norm bounded by a constant times $\|u\|_{\theta,q;K}$. For each index i let $u_i = v_{i+1} - v_i$. Since

$$0 = u - u = \big(v_{i+1} + w_{i+1}\big) - \big(v_i + w_i\big) = \big(v_{i+1} - v_i\big) + \big(w_{i+1} - w_i\big),$$

we have, in fact,

$$u_i = v_{i+1} - v_i = w_i - w_{i+1}.$$

The first of these representations of u_i shows that $\{2^{-i\theta} \|u_i\|_{X_0}\}$ belongs to ℓ^q; the second representations shows that $\{2^{i(1-\theta)} \|u_i\|_{X_1}\}$ also belongs to ℓ^q. Therefore $\{2^{-i\theta} J(2^i; u_i)\} \in \ell^q$ and has ℓ^q-norm bounded by a constant times $\|u\|_{\theta,q;K}$. Since $\ell^q \subset \ell^\infty$, the sequence $\left\{2^{j(1-\theta)} \left\|w_j\right\|_{X_1}\right\}$ is bounded even though $2^{j(1-\theta)} \to \infty$ as $j \to \infty$. Thus $\left\|w_j\right\|_{X_1} \to 0$ as $j \to \infty$. Since $\sum_{i=0}^{j} u_i = w_0 - w_{j+1}$, the half series $\sum_{i=0}^{\infty}$ converges to w_0 in X_1 and hence in $X_0 + X_1$. Similarly, the half-series $\sum_{i=-\infty}^{-1} u_i$ converges to v_0 in X_0, and thus in $X_0 + X_1$. Thus the full series $\sum_{i=-\infty}^{\infty} u_i$ converges to $v_0 + w_0 = u$ in $X_0 + X_1$ and we have

$$\|u\|_{\theta,q;J} \le \text{const.}\ \|u\|_{\theta,q;K}\,.$$

This completes the proof of (b) and hence (c). ∎

7.17 COROLLARY If $0 < \theta < 1$ and $1 \le p \le q \le \infty$, then

$$(X_0, X_1)_{\theta,p;K} \to (X_0, X_1)_{\theta,q;K}. \tag{10}$$

Proof. $(X_0, X_1)_{\theta,p;K} \to (X_0, X_1)_{\theta,p;J} \to (X_0, X_1)_{\theta,q;K}$ by part (b) and imbedding (9). ∎

7.18 (Classes of Intermediate Spaces) We define three classes of intermediate spaces X between X_0 and X_1 as follows:

(a) X belongs to class $\mathscr{K}(\theta; X_0, X_1)$ if for all $u \in X$

$$K(t; u) \le C_1 t^\theta \|u\|_X\,,$$

where C_1 is a constant.

(b) X belongs to class $\mathscr{J}(\theta; X_0, X_1)$ if for all $u \in X_0 \cap X_1$

$$\|u\|_X \le C_2 t^{-\theta} J(t; u),$$

where C_2 is a constant.

(c) X belongs to class $\mathscr{H}(\theta; X_0, X_1)$ if X belongs to both $\mathscr{K}(\theta; X_0, X_1)$ and $\mathscr{J}(\theta; X_0, X_1)$.

The following lemma gives necessary and sufficient conditions for membership in these classes.

7.19 LEMMA Let $0 \le \theta \le 1$ and let X be an intermediate space between X_0 and X_1.

(a) $X \in \mathscr{K}(\theta; X_0, X_1)$ if and only if $X \to (X_0, X_1)_{\theta,\infty;K}$.

(b) $X \in \mathscr{J}(\theta; X_0, X_1)$ if and only if $(X_0, X_1)_{\theta,1;J} \to X$.

(c) $X \in \mathscr{H}(\theta; X_0, X_1)$ if and only if $(X_0, X_1)_{\theta,1;J} \to X \to (X_0, X_1)_{\theta,\infty;K}$.

Proof. Conclusion (a) is immediate since $\|u\|_{\theta,\infty;K} = \sup_{0<t<\infty}\big(t^{-\theta}k(t; u)\big)$. Since (c) follows from (a) and (b), only (b) requires proof.

First suppose $X \in \mathscr{J}(\theta; X_0, X_1)$. Let $u \in (X_0, X_1)_{\theta,1;J}$. If $f(t)$ is any function on $(0, \infty)$ with values in $X_0 \cap X_1$ such that $u = \int_0^\infty f(t)\, dt/t$, then

$$\|u\|_X \le \int_0^\infty \|f(t)\|_X \frac{dt}{t} \le C_2 \int_0^\infty t^{-\theta} J\big(t; f(t)\big) \frac{dt}{t}.$$

Since this holds for all such representations of u we have

$$\|u\|_X \le C_2 (X_0, X_1)_{\theta,1;J}, \tag{11}$$

and so $(X_0, X_1)_{\theta,1;J} \to X$.

Conversely, suppose that $(X_0, X_1)_{\theta,1;J} \to X$; therefore (11) holds with some constant C_2. Let $u \in X_0 \cap X_1$, let $\lambda > 0$ and $t > 0$, and let

$$f_\lambda(s) = \begin{cases} (1/\lambda)u & \text{if } te^{-\lambda} \le s \le t \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\int_0^\infty f_\lambda(s) \frac{ds}{s} = \left(\int_{te^{-\lambda}}^t \frac{ds}{s}\right)\left(\frac{1}{\lambda}\right) u = u.$$

Since $J\big(s; (1/\lambda)u\big) = (1/\lambda) J(s; u)$ we have

$$\|u\|_{\theta,1;J} \le \int_0^\infty s^{-\theta} J\big(s; f_\lambda(s)\big) \frac{ds}{s} = \frac{1}{\lambda} \int_{te^{-\lambda}}^t s^{-\theta} J(s; u) \frac{ds}{s}.$$

Since $s^{-\theta} J(s; u)$ is continuous in s and $\int_{te^{-\lambda}}^t ds/s = \lambda$, we can let $\lambda \to 0+$ in the above inequality and obtain $\|u\|_{\theta,1;J} \le t^{-\theta} J(t; u)$. Hence

$$\|u\|_X \le C_2 (X_0, X_1)_{\theta,1;J} \le C_2 t^{-\theta} J(t; u)$$

and the proof of (b) is complete. ∎

The following corollary follows immediately, using the equivalence theorem, (10), and (8).

7.20 COROLLARY If $0 < \theta < 1$ and $1 \le q \le \infty$, then

$$(X_0, X_1)_{\theta,q;J} = (X_0, X_1)_{\theta,q;K} \in \mathscr{H}(\theta; X_0, X_1).$$

Moreover, $X_0 \in \mathscr{H}(0; X_0, X_1)$ and $X_1 \in \mathscr{H}(1; X_0, X_1)$. ∎

Next we examine the result of constructing intermediate spaces between two intermediate spaces.

7.21 THEOREM (The Reiteration Theorem) Let $0 \le \theta_0 < \theta_1 \le 1$ and let X_{θ_0} and X_{θ_1} be intermediate spaces between X_0 and X_1. For $0 \le \lambda \le 1$, let $\theta = (1-\lambda)\theta_0 + \lambda\theta_1$.

(a) If $X_{\theta_i} \in \mathscr{K}(\theta_i; X_0, X_1)$ for $i = 0, 1$, and if either $0 < \lambda < 1$ and $1 \le q < \infty$ or $0 \le \lambda \le 1$ and $q = \infty$, then

$$\left(X_{\theta_0}, X_{\theta_1}\right)_{\lambda,q;K} \to (X_0, X_1)_{\theta,q;K}\,.$$

(b) If $X_{\theta_i} \in \mathscr{J}(\theta_i; X_0, X_1)$ for $i = 0, 1$, and if either $0 < \lambda < 1$ and $1 < q \le \infty$ or $0 \le \lambda \le 1$ and $q = 1$, then

$$(X_0, X_1)_{\theta,q;J} \to \left(X_{\theta_0}, X_{\theta_1}\right)_{\lambda,q;J}\,.$$

(c) If $X_{\theta_i} \in \mathscr{H}(\theta_i; X_0, X_1)$ for $i = 0, 1$, and if $0 < \lambda < 1$ and $1 \le q \le \infty$, then

$$\left(X_{\theta_0}, X_{\theta_1}\right)_{\lambda,q;J} = \left(X_{\theta_0}, X_{\theta_1}\right)_{\lambda,q;K} = (X_0, X_1)_{\theta,q;K} = (X_0, X_1)_{\theta,q;J}\,.$$

(d) Moreover,

$$(X_0, X_1)_{\theta_0,1;J} \to \left(X_{\theta_0}, X_{\theta_1}\right)_{0,1;J} \to X_{\theta_0} \to \left(X_{\theta_0}, X_{\theta_1}\right)_{0,\infty;K} \to (X_0, X_1)_{\theta_0,\infty;K}$$
$$(X_0, X_1)_{\theta_1,1;J} \to \left(X_{\theta_0}, X_{\theta_1}\right)_{1,1;J} \to X_{\theta_1} \to \left(X_{\theta_0}, X_{\theta_1}\right)_{1,\infty;K} \to (X_0, X_1)_{\theta_1,\infty;K}\,.$$

Proof. The important conclusions here are (c) and (d) and these follow from (a) and (b) which we must prove. In both proofs we need to distinguish the function norms $K(t; u)$ and $J(t; u)$ used in the construction of the intermediate spaces between X_0 and X_1 from those used for the intermediate spaces between X_{θ_0} and X_{θ_1}. We will use K^* and J^* for the latter.

Proof of (a) If $u \in \left(X_{\theta_0}, X_{\theta_1}\right)_{\lambda,q;K}$, then $u = u_0 + u_1$ where $u_i \in X_{\theta_i}$. Since $X_{\theta_i} \in \mathscr{K}(\theta_i; X_0, X_1)$, we have

$$\begin{aligned} K(t; u) &\le K(t; u_0) + K(t; u_1) \\ &\le C_0 t^{\theta_0} \left\| u_0 ; X_{\theta_0} \right\| + C_1 t^{\theta_1} \left\| u_1 ; X_{\theta_1} \right\| \\ &\le C_0 t^{\theta_0} \left(\left\| u_0 ; X_{\theta_0} \right\| + \frac{C_1}{C_0} t^{\theta_1 - \theta_0} \left\| u_1 ; X_{\theta_1} \right\| \right). \end{aligned}$$

Since this estimate holds for all such representations of u, we have

$$K(t;u) \le C_0 t^{\theta_0} K^*\left(\frac{C_1}{C_0} t^{\theta_1-\theta_0}; u\right).$$

If $\theta = (1-\lambda)\theta_0 + \lambda\theta_1$, then $\lambda = (\theta-\theta_0)/(\theta_1-\theta_0)$, and (assuming $q < \infty$)

$$\begin{aligned}
\left\| t^{-\theta} K(t;u)\,;\, L^q_*\right\| &\le C_0 \left[\int_0^\infty \left(t^{-(\theta-\theta_0)} K^*\left(\frac{C_1}{C_0} t^{\theta_1-\theta_0}; u\right)\right)^q \frac{dt}{t}\right]^{1/q} \\
&= \frac{C_0^{1-\lambda} C_1^{\lambda}}{(\theta_1-\theta_0)^{1/q}} \left[\int_0^\infty \left(s^{-\lambda} K^*(s;u)\right)^q \frac{ds}{s}\right]^{1/q}
\end{aligned}$$

via the transformation $s = (C_1/C_0) t^{\theta_1-\theta_0}$. Hence

$$\|u\|_{\theta,q;K} \le \frac{C_0^{1-\lambda} C_1^{\lambda}}{(\theta_1-\theta_0)^{1/q}} \|u\|_{\lambda,q;K}$$

and so $\left(X_{\theta_0}, X_{\theta_1}\right)_{\lambda,q;K} \to (X_0, X_1)_{\theta,q;K}$.

Proof of (b) Let $u \in (X_0, X_1)_{\theta,q;J}$. Then $u = \int_0^\infty f(s)\,ds/s$ for some f taking values in $X_0 \cap X_1$ satisfying $s^{-\theta} J\big((s; f(s)\big) \in L^q_*$. Clearly $f(s) \in X_{\theta_0} \cap X_{\theta_1}$. Since $X_{\theta_i} \in \mathscr{J}(\theta_i; X_0, X_1)$ we have

$$\begin{aligned}
J^*\big(s; f(s)\big) &= \max\left\{\left\|f(s)\,;\, X_{\theta_0}\right\|, s\left\|f(s)\,;\, X_{\theta_1}\right\|\right\} \\
&\le \max\left\{C_0 t^{-\theta_0} J\big(t; f(s)\big), C_1 t^{-\theta_1} s J\big(t; f(s)\big)\right\} \\
&= C_0 t^{-\theta_0} \max\left\{1, \frac{C_1}{C_0} t^{-(\theta_1-\theta_0)} s\right\} J\big(t; f(s)\big).
\end{aligned}$$

This estimate holds for all $t > 0$ so we can choose t so that $t^{-(\theta_1-\theta_0)} s = C_0/C_1$ and obtain

$$J^*\big(s; f(s)\big) \le C_0 \left(\frac{C_1}{C_0} s\right)^{-\theta_0/(\theta_1-\theta_0)} J\left(\left(\frac{C_1}{C_0} s\right)^{1/(\theta_1-\theta_0)}; f(s)\right).$$

If $\theta = (1-\lambda)\theta_0 + \lambda\theta_1$, then

$$\begin{aligned}
&\left\| s^{-\lambda} J^*\big(s; f(s)\big)\,;\, L^q_*\right\| \\
&\quad \le C_0^{1-\lambda} C_1^{\lambda} \left(\int_0^\infty \left[\left(\frac{C_1}{C_0} s\right)^{-\theta/(\theta_1-\theta_0)} J\left(\left(\frac{C_1}{C_0} s\right)^{1/(\theta_1-\theta_0)}; f(s)\right)\right]^q \frac{ds}{s}\right)^{1/q} \\
&\quad \le C_0^{1-\lambda} C_1^{\lambda} (\theta_1-\theta_0)^{1/q} \left(\int_0^\infty \left[t^{-\theta} J\big(t; g(t)\big)\right]^q \frac{dt}{t}\right)^{1/q} \\
&\quad = C_0^{1-\lambda} C_1^{\lambda} (\theta_1-\theta_0)^{1/q} \left\| t^{-\theta} J\big(t; g(t)\big)\,;\, L^q_*\right\|,
\end{aligned}$$

where $g(t) = f\big((C_0/C_1)t^{\theta_1-\theta_0}\big) = f(s) \in X_0 \cap X_1$. Since

$$\int_0^\infty g(t)\,\frac{dt}{t} = \frac{1}{\theta_1-\theta_0}\int_0^\infty f(s)\,\frac{ds}{s} = \frac{1}{\theta_1-\theta_0}u,$$

we have

$$\|u\|_{\lambda,q;J} \le \frac{C_0^{1-\lambda}C_1^{\lambda}}{(\theta_1-\theta_0)^{(q-1)/q}}\,\|u\|_{\theta.q;J}$$

and so $(X_0, X_1)_{\theta,q;J} \to \big(X_{\theta_0}, X_{\theta_1}\big)_{\lambda,q;J}$. ∎

7.22 (Interpolation Spaces) Let $P = \{X_0, X_1\}$ and $Q = \{Y_0, Y_1\}$ be two interpolation pairs of Banach spaces, and let T be a bounded linear operator from $X_0 + X_1$ into $Y_0 + Y_1$ having the property that T is bounded from X_i into Y_i, with norm at most M_i, $i = 0, 1$; that is,

$$\|Tu_i\|_{Y_i} \le M_i\,\|u_i\|_{X_i}, \qquad \text{for all } u_i \in X_i), \qquad (i = 1, 2).$$

If X and Y are intermediate spaces for $\{X_0, X_1\}$ and $\{Y_0, Y_1\}$, respectively, we call X and Y *interpolation spaces of type* θ for P and Q, where $0 \le \theta \le 1$, if every such linear operator T maps X into Y with norm M satisfying

$$M \le C M_0^{1-\theta} M_1^{\theta}, \tag{12}$$

where constant $C \ge 1$ is independent of T. We say that the interpolation spaces X and Y are *exact* if inequality (12) holds with $C = 1$. If $X_0 = Y_0$, $X_1 = Y_1$, $X = Y$ and $T = I$, the identity operator on $X_0 + X_1$, then $C = 1$ for all $0 \le \theta \le 1$, so no smaller C is possible in (12).

7.23 THEOREM (An Exact Interpolation Theorem) Let $P = \{X_0, X_1\}$ and $Q = \{Y_0, Y_1\}$ be two interpolation pairs.

(a) If either $0 < \theta < 1$ and $1 \le q \le \infty$ or $0 \le \theta \le 1$ and $q = \infty$, then the intermediate spaces $(X_0, X_1)_{\theta,q;K}$ and $(Y_0, Y_1)_{\theta,q;K}$ are exact interpolation spaces of type θ for P and Q

(b) If either $0 < \theta < 1$ and $1 < q \le \infty$ or $0 \le \theta \le 1$ and $q = 1$, then the intermediate spaces $(X_0, X_1)_{\theta,q;J}$ and $(Y_0, Y_1)_{\theta,q;J}$ are exact interpolation spaces of type θ for P and Q.

Proof. Let $T : X_0 + X_1 \to Y_0 + Y_1$ satisfy $\|Tu_i\|_{Y_i} \le M_i\,\|u_i\|_{X_i}$, $i = 0, 1$. If $u \in X_0 + X_1$, then

$$\begin{aligned} K(t; Tu) &= \inf\left\{\|Tu_0\|_{Y_0} + t\,\|Tu_1\|_{Y_1} \,:\, u = u_0 + u_1,\, u_i \in X_i\right\} \\ &\le M_0 \inf_{\substack{u=u_0+u_1\\ u_i\in X_i}} \left(\|u_0\|_{X_0} + \frac{M_1}{M_0}t\,\|u_1\|_{X_1}\right) = M_0 K\big((M_1/M_0)t; u\big). \end{aligned}$$

If $u \in (X_0, X_1)_{\theta,q;K}$, then

$$\begin{aligned}\|Tu\|_{\theta,q;K} &= \left\|t^{-\theta}K(t;\,Tu)\,;\,L^q_*\right\| \le M_0\left\|t^{-\theta}K\big((M_1/M_0)t;\,u\big)\,;\,L^q_*\right\|\\ &= M_0\left(\frac{M_0}{M_1}\right)^{-\theta}\left\|s^{-\theta}K(s;\,u)\,;\,L^q_*\right\| = M_0^{1-\theta}M_1^{\theta}\,\|u\|_{\theta,q;K}\,,\end{aligned}$$

which proves (a).

If $u \in X_0 \cap X_1$, then

$$\begin{aligned}J(t;\,Tu) &= \max\left\{\|Tu\|_{Y_0}\,,\,t\,\|Tu\|_{Y_1}\right\}\\ &\le M_0\max\left\{\|u\|_{X_0}\,,\,(M_1/M_0)t\,\|u\|_{X_1}\right\} = M_0J\big((M_1/M_0)t;\,u\big).\end{aligned}$$

If $u = \int_0^\infty f(t)\,dt/t$, where $f(t) \in X_0 \cap X_1$ and $t^{-\theta}J\big(t;\,f(t)\big) \in L^q_*$, then

$$\begin{aligned}&\|Tu\|_{\theta,q;J} = \left\|t^{-\theta}J\big(t;\,Tf(t)\big)\,;\,L^q_*\right\|\\ &\quad\le M_0\left\|t^{-\theta}J\big((M_1/M_0)t;\,f(t)\,;\,L^q_*\right\| = M_0\left(\frac{M_0}{M_1}\right)^{-\theta}\left\|s^{-\theta}J\big(s;\,g(s)\big)\,;\,L^q_*\right\|,\end{aligned}$$

where $g(s) = f\big((M_0/M_1)s\big) = f(t)$. Since this estimate holds for all representations of $u = \int_0^\infty g(s)\,ds/s$, we have

$$\|Tu\|_{\theta,q;J} \le M_0^{1-\theta}M_1^{\theta}\,\|u\|_{\theta,q;J}$$

and the proof of (b) is complete. ∎

The Lorentz Spaces

7.24 (Equimeasurable Decreasing Rearrangement) Recall that, as defined in Paragraph 2.53, the distribution function δ_u corresponding to a measurable function u finite a.e. in a domain $\Omega \subset \mathbb{R}^n$ is given by

$$\delta_u(t) = \mu\{x \in \Omega \,:\, |u(x)| > t\}$$

and is nonincreasing on $[0, \infty)$. (It is also right continuous there, but that is of no relevance for integrals involving the distribution function since a nonincreasing function can have at most countably many points of discontinuity.) Moreover, if $u \in L^p(\Omega)$, then

$$\|u\|_p = \begin{cases}\left(p\displaystyle\int_0^\infty t^p\delta_u(y)\,\frac{dt}{t}\right)^{1/p} & \text{if } 1 \le p < \infty,\\ \inf\{t \,:\, \delta_u(t) = 0\} & \text{if } p = \infty.\end{cases}$$

The *equimeasurable decreasing rearrangement of u* is the function u^* defined by

$$u^*(s) = \inf\{t : \delta_u(t) \le s\}.$$

This definition and the fact that δ_u is nonincreasing imply that $u*$ is nonincreasing too. Moreover, $u^*(s) > t$ if and only if $\delta_u(t) > s$, and this latter condition is trivially equivalent to $s < \delta_u(t)$. Therefore,

$$\delta_{u^*}(t) = \mu\left\{s : u^*(s) > t\right\} = \mu\{s : 0 \le s < \delta_u(t)\} = \mu\left\{[0, \delta_u(t))\right\} = \delta_u(t).$$

This justifies our calling u^* and u equimeasurable; the size of both functions exceeds any number s on sets having the same measure. Also,

$$\delta_{u^*}(t) = \mu\{s : u^*(s) > t\} = \inf\{s : u^*(s) \le t\}$$

so that

$$\delta_u(t) = \inf\{s : u^*(s) \le t\}.$$

This further illustrates the symmetry between δ_u and u^*.

Note also that

$$u^*\big(\delta_u(t)\big) = \inf\big\{s : \delta_u(s) \le \delta_u(t)\big\} \le t.$$

If $u^*\big(\delta_u(t)\big) = s < t$, then δ_u is constant on the interval (s, t) in which case u^* has a jump discontinuity of magnitude at least $t - s$ at $\delta_u(t)$.

Similarly, $\delta_u\big(u^*(s)\big) \le s$, with equality if δ_u is continuous at $t = u^*(s)$. The relationship between δ_u and u^* is illustrated in Figure 8. Except at points where either function is discontinuous (and the other is constant on an interval), each is the inverse of the other.

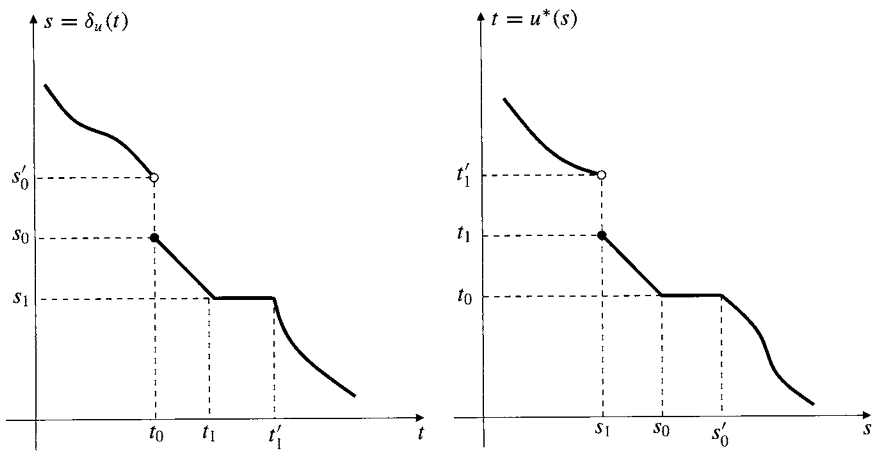


Fig. 8

If $S_t = \{x \in \Omega : |u(x)| > t\}$, then

$$\int_{S_t} |u(x)|\,dx = \int_0^{\delta_u(t)} u^*(s)\,ds, \tag{13}$$

and if $u \in L^p(\Omega)$, then

$$\|u\|_p = \begin{cases} \left(\displaystyle\int_0^\infty \big(u^*(s)\big)^p\,dx\right)^{1/p} & \text{if } 1 \le p < \infty \\ \displaystyle\sup_{0<s<\infty} u^*(s) & \text{if } p = \infty. \end{cases}$$

7.25 (The Lorentz Spaces) For u measurable on Ω let

$$u^{**}(t) = \frac{1}{t}\int_0^t u^*(s)\,ds,$$

that is, the average value of u^* over $[0, t]$. Since u^* is nonincreasing, we have $u^*(t) \le u^{**}(t)$.

For $1 \le p \le \infty$ we define the functional

$$\big\|u\,; L^{p,q}(\Omega)\big\| = \begin{cases} \left(\displaystyle\int_0^\infty \big(t^{1/p}u^{**}(t)\big)^q\,\frac{dt}{t}\right)^{1/q} & \text{if } 1 \le q < \infty \\ \displaystyle\sup_{t>0} t^{1/p}u^{**}(t) & \text{if } q = \infty. \end{cases}$$

The *Lorentz space* $L^{p,q}(\Omega)$ consists of those measurable functions u on Ω for which $\|u\,; L^{p,q}(\Omega)\| < \infty$. Theorem 7.26 below shows that if $1 < p < \infty$, then $L^{p,q}(\Omega)$ is, in fact, identical to the intermediate space $\big(L^1(\Omega), L^\infty(\Omega)\big)_{(p-1)/p,q;K}$ and $\|u\,; L^{p,q}(\Omega)\| = \|u\|_{(p-1)/p,q;K}$. Thus $L^{p,q}(\Omega)$ is a Banach space under the norm $\|u\,; L^{p,q}(\Omega)\|$. It is also a Banach space if $p = 1$ or $p = \infty$.

The second corollary to Theorem 7.26 shows that if $1 < p < \infty$, then $L^{p,q}(\Omega)$ coincides with the set of measurable u for which $\big[u; L^{p,q}(\Omega)\big] < \infty$, where

$$\big[u; L^{p,q}(\Omega)\big] = \begin{cases} \left(\displaystyle\int_0^\infty \big(t^{1/p}u^{*}(t)\big)^q\,\frac{dt}{t}\right)^{1/q} & \text{if } 1 \le q < \infty \\ \displaystyle\sup_{t>0} t^{1/p}u^{*}(t) & \text{if } q = \infty, \end{cases}$$

and that

$$\big[u; L^{p,q}(\Omega)\big] \le \big\|u\,; L^{p,q}(\Omega)\big\| \le \frac{p}{p-1}\big[u; L^{p,q}(\Omega)\big].$$

The index p in $L^{p,q}(\Omega)$ is called the principal index; q is the secondary index. Unless $q = p$, the functional $\left[\,\cdot\,; L^{p,q}(\Omega)\right]$ is not a norm since it does not satisfy the triangle inequality; it, however, is a quasi-norm since

$$\left[u+v; L^{p,q}(\Omega)\right] \le 2\left(\left[u; L^{p,q}(\Omega)\right] + \left[v; L^{p,q}(\Omega)\right]\right).$$

For $1 < p < \infty$ it is evident that $\left[\,\cdot\,; L^{p,p}(\Omega)\right] = \|\cdot\|_{p,\Omega}$, and therefore $L^{p,p}(\Omega) = L^p(\Omega)$. Moreover, if we recall the definition of the space weak-$L^p(\Omega)$ given in Paragraph 2.55 and having quasi-norm given (for $p < \infty$) by

$$[u]_p = [u]_{p,\Omega} = \left(\sup_{t>0} t^p \delta_u(t)\right)^{1/p},$$

we can show that $L^{p,\infty}(\Omega) = \text{weak-}L^p(\Omega)$. This is also clear for $p = \infty$. If $1 < p < \infty$ and $K > 0$, then for all $t > 0$ we have, putting $s = K^p t^{-p}$,

$$\delta_u(t) \le K^p t^{-p} = s \qquad \Longleftrightarrow \qquad u^*(s) \le t = K s^{-1/p}.$$

Hence $[u]_p \le K$ if and only if $\left[u; L^{p,\infty}(\Omega)\right] \le K$, and these two quasi-norms are, in fact, equal.

For $p = 1$ the situation is a little different. Observe that

$$\left\|u\,; L^{1,\infty}(\Omega)\right\| = \sup_{t>0} t u^{**}(t) = \sup_{t>0} \int_0^t u^*(s)\,ds = \int_0^\infty u^*(s)\,ds = \|u\|_1$$

so $L^1(\Omega) = L^{1,\infty}(\Omega)$ (*not* $L^{1,1}(\Omega)$ which contains only the zero function).

For $p = \infty$ we have $L^{\infty,\infty}(\Omega) = L^\infty(\Omega)$ since

$$\left\|u\,; L^{\infty,\infty}(\Omega)\right\| = \sup_{t>0} u^{**}(t) = \sup_{t>0} \frac{1}{t}\int_0^t u^*(s)\,ds = u^*(0) = \|u\|_\infty .$$

7.26 THEOREM If $u \in L^1(\Omega) + L^\infty(\Omega)$, then for $t > 0$ we have

$$K(t; u) = \int_0^t u^*(s)\,ds = t u^{**}(t). \tag{14}$$

Therefore, if $1 < p < \infty$, $1 \le q \le \infty$, and $\theta = 1 - (1/p)$,

$$L^{p,q}(\Omega) = \left(L^1(\Omega), L^\infty(\Omega)\right)_{\theta,q;K}$$

with equality of norms: $\|u\,; L^{p,q}(\Omega)\| = \|u\|_{\theta,q;K}$.

Proof. The second conclusion follows immediately from the representation (14) which we prove as follows.

Since $K(t;u) = K(t;|u|)$ we can assume that u is real-valued and nonnegative. Let $u = v + w$ where $v \in L^1(\Omega)$ and $w \in L^\infty(\Omega)$. In order to calculate

$$K(t;u) = \inf_{u=v+w}\big(\|v\|_1 + t\,\|w\|_\infty\big) \tag{15}$$

we can also assume that v and w are real-valued functions since, in any event, $u = \operatorname{Re} v + \operatorname{Re} w$ and $\|\operatorname{Re} v\|_1 \le \|v\|_1$ and $\|\operatorname{Re} w\|_\infty \le \|w\|_\infty$. We can also assume that v and w are nonnegative, for if

$$v_1(x) = \begin{cases} \min\{v(x), u(x)\} & \text{if } v(x) \ge 0 \\ 0 & \text{if } v(x) < 0 \end{cases} \qquad \text{and} \qquad w_1(x) = u(x) - v_1(x),$$

then $0 \le v_1(x) \le |v(x)|$ and $0 \le w_1(x) \le |w(x)|$. Thus the infimum in (15) does not change if we restrict to nonnegative functions v and w.

Thus we consider $u = v + w$, where $v \ge 0$, $v \in L^1(\Omega)$, $w \ge 0$, and $w \in L^\infty(\Omega)$. Let $\lambda = \|w\|_\infty$ and define $u_\lambda(x) = \min\{\lambda, u(x)\}$. Evidently $w(x) \le u_\lambda(x)$ and $u(x) - u_\lambda(x) \le u(x) - w(x) = v(x)$. Now let

$$g(t,\lambda) = \|u - u_\lambda\|_1 + t\lambda \le \|v\|_1 + t\,\|w\|_\infty .$$

Then $K(t;u) = \inf_{0<\lambda<\infty} g(t,\lambda)$. We want to show that this infimum is, in fact, a minimum and is assumed at $\lambda = \lambda_t = \inf\{\tau : \delta_u(\tau) < t\}$.

If $\lambda > \lambda_t$, then $u_\lambda(x) - u_{\lambda_t}(x) \le \lambda - \lambda_t$ if $u(x) > \lambda_t$, and $u_\lambda(x) - u_{\lambda_t}(x) = 0$ if $u(x) \le \lambda_t$. Since $\delta_u(\lambda_t) \le t$, we have

$$\begin{aligned} g(t,\lambda) - g(t,\lambda_t) &= -\int_\Omega \big(u_\lambda(x) - u_{\lambda_t}(x)\big)\,dx + t(\lambda - \lambda_t) \\ &\ge (\lambda - \lambda_t)\big(t - \delta_u(\lambda_t)\big) \ge 0. \end{aligned}$$

Thus $K(t;u) \le g(t,\lambda_t)$.

On the other hand, if $g(t,\lambda^*) < \infty$ for some $\lambda^* < \lambda_t$, then $g(t,\lambda)$ is a continuous function of λ for $\lambda \ge \lambda^*$ and so for any $\epsilon > 0$ there exists λ such that $\lambda^* \le \lambda < \lambda_t$ and

$$|g(t,\lambda) - g(t,\lambda_t)| < \epsilon.$$

Now $u_\lambda(x) - u_{\lambda^*}(x) = \lambda - \lambda^*$ if $u(x) > \lambda$, and since $\delta_u(\lambda) \ge t$ we have

$$\begin{aligned} g(t,\lambda^*) - g(t,\lambda) &= \int_\Omega \big(u_\lambda(x) - u_{\lambda^*}(x)\big)\,dx - t(\lambda - \lambda^*) \\ &\ge (\lambda - \lambda^*)\big(\delta_u(\lambda) - t\big) \ge 0. \end{aligned}$$

Thus

$$g(t,\lambda^*) - g(t,\lambda_t) \ge g(t,\lambda^*) - g(t,\lambda) - |g(t,\lambda) - g(t,\lambda_t)| \ge -\epsilon.$$

Since ϵ is arbitrary, $g(t,\lambda^*) \ge g(t,\lambda_t)$ and $K(t;u) \ge g(t,\lambda_t)$. Thus

$$K(t;u) = g(t,\lambda_t) = \left\| u - u_{\lambda_t} \right\|_1 + t\lambda_t.$$

Now $u(x) - u_{\lambda_t}(x) = 0$ except where $u(x) > \lambda_t$ and $\lambda_t = u^*(s)$ for $\delta_u(\lambda_t) \le s \le t$. Therefore, by (13),

$$\begin{aligned} K(t;u) &= \int_0^{\delta_u(\lambda_t)} \big(u^*(s) - \lambda_t\big)\,ds + t\lambda_t = \int_0^{\delta_u(\lambda_t)} u^*(s)\,ds - \lambda_t\delta_u(\lambda_t) + t\lambda_t \\ &= \int_0^{\delta_u(\lambda_t)} u^*(s)\,ds + \int_{\delta_u(\lambda_t)}^t u^*(s)\,ds = \int_0^t u^*(s)\,ds \end{aligned}$$

which completes the proof. ∎

7.27 COROLLARY If $1 \le p_1 < p < p_2 \le \infty$ and $1/p = (1-\theta)/p_1 + \theta/p_2$, then by the Reiteration Theorem 7.21, up to equivalence of norms,

$$L^{p,q}(\Omega) = \big(L^{p_1}(\Omega), L^{p_2}(\Omega)\big)_{\theta,q;K}.$$

7.28 COROLLARY For $1 < p < \infty$, $1 \le q \le \infty$, and $\theta = 1 - (1/p)$, we have

$$\big[u; L^{p,q}(\Omega)\big] \le \left\| u\,; L^{p,q}(\Omega) \right\| \le \frac{p}{p-1}\big[u; L^{p,q}(\Omega)\big].$$

Proof. Since u^* is decreasing, (14) implies that $tu^*(t) \le K(t;u)$. Thus

$$\begin{aligned} \big[u; L^{p,q}(\Omega)\big] &= \left(\int_0^\infty \big(t^{1/p}u^*(t)\big)^q \,\frac{dt}{t}\right)^{1/q} \\ &\le \left(\int_0^\infty \big(t^{-\theta}K(t;u)\big)^q \,\frac{dt}{t}\right)^{1/q} = \|u\|_{\theta,q;K} = \left\| u\,; L^{p,q}(\Omega) \right\|. \end{aligned}$$

On the other hand,

$$\begin{aligned} t^{-\theta}K(t;u) &= \int_0^t t^{-\theta}u^*(s)\,ds \\ &= \int_1^\infty \sigma^{-\theta}\left(\frac{t}{\sigma}\right)^{1-\theta} u^*\left(\frac{t}{\sigma}\right)\frac{d\sigma}{\sigma} = f * g(t), \end{aligned}$$

where

$$f(t) = t^{1-\theta}u^*(t) = t^{1/p}u^*(t), \qquad \text{and} \qquad g(t) = \begin{cases} t^{-\theta} & \text{if } t \geq 1 \\ 0 & \text{if } 0 \leq 1, \end{cases}$$

and the convolution is with respect to the measure dt/t. Since we have $\|f\,; L_*^q\| = \left[u; L^{p,q}(\Omega)\right]$ and $\|g\,; L_*^1\| = 1/\theta = p/(p-1)$, Young's inequality (see Paragraph 7.5) gives

$$\left\|u\,; L^{p,q}(\Omega)\right\| = \|u\|_{\theta,q;K} = \left\|f * g\,; L_*^q\right\| \leq \frac{p}{p-1}\left[u; L^{p,q}(\Omega)\right]. \quad \blacksquare$$

7.29 REMARK Working with Lorentz spaces and using the real interpolation method allows us to sharpen the cases of the Sobolev imbedding theorem where $p > 1$ and $mp < n$. In those cases, the proof in Chapter IV used Lemma 4.18, where convolution with the kernel ω_m was first shown to be of weak type (p, p^*) (where $p* = np/(n - mp)$) for all such indices p. Then other such indices p_1 and p_2 were chosen with $p_1 < p < p_2$, and Marcinkiewicz interpolation implied that this linear convolution operator must be of strong type (p, p^*).

We can instead apply the the Exact Interpolation Theorem 7.23 and Lorentz interpolation as in Corollary 7.27, to deduce, from the weak-type estimates above, that convolution with ω_m maps $L^p(\Omega)$ into $L^{p^*,p}(\Omega)$; this target space is strictly smaller than $L^{p^*}(\Omega)$, since $p < p^*$. It follows that $W^{m,p}(\Omega)$ imbeds in the smaller spaces $L^{p^*,p}(\Omega)$ when $p > 1$ and $mp < n$.

Recall too that convolution with ω_m is *not* of strong type $(1, 1^*)$ when $m < n$, but an averaging argument, in Lemma 4.24, showed that $W^{m,1}(\Omega) \subset L^{1^*}(\Omega)$ in that case. That argument can be refined as in Fournier [F] to show that in fact $W^{m,1}(\Omega) \subset L^{1^*,1}(\Omega)$ in these cases. This sharper endpoint imbedding had been proved earlier by Poornima [Po] using another method, and also in a dual form in Faris [Fa].

An ideal context for applying interpolation is one where there are apt endpoint estimates from which everything else follows. We illustrate that idea for convolution with ω_m. It is easy, via Fubini's theorem, to verify that if $f \in L^1(\Omega)$ then $\|f * g_0\|_\infty \leq \|f\|_1\,\|g_0\|_\infty$ and $\|f * g_1\|_1 \leq \|f\|_1\,\|g_1\|_1$ for all functions g_0 in $L^\infty(\Omega)$ and g_1 in $L^1(\Omega)$. Fixing f and interpolating between the endpoint conditions on the functions g gives that $\|f * g\,; L^{p,q}(\Omega)\| \leq C_p\,\|f\|_1\,\|g\,; L^{p,q}(\Omega)\|$ for all indices p and q in the intervals $(1, \infty)$ and $[1, \infty]$ respectively. Apply this with $g = \omega_m$, which belongs to $L^{n/(n-m),\infty}(\Omega) = L^{1^*,\infty}(\Omega) = \text{weak-}L^{1^*}(\Omega)$ to deduce that convolution with ω_m maps $L^1(\Omega)$ into $L^{1^*,\infty}(\Omega)$. On the other hand, if $f \in L^{(1^*)',1}(\Omega) = L^{n/m,1}(\Omega)$, then

$$\begin{aligned}
|\omega_m * f(x)| &\le \int_{R^n} |\omega_m(x-y)f(y)|\,dy \\
&\le \int_0^\infty (\omega_m)^*(t) f^*(t)\,dt = \int_0^\infty [t^{1/1^*}(\omega_m)^*(t)][t^{1/(1^*)'} f^*(t)]\,\frac{dt}{t} \\
&\le \left\|\omega_m\,;L^{1^*,\infty}(\Omega)\right\| \int_0^\infty \left[t^{1/(1^*)'} f^*(t)\right]\frac{dt}{t} \le C_m \left\| f\,; L^{(1^*)',1}(\Omega)\right\|.
\end{aligned}$$

That is, convolution with ω_m maps $L^1(\Omega)$ into $L^{1^*,\infty}(\Omega)$ and $L^{(1^*)',1}(\Omega)$ into $L^\infty(\Omega)$. Real interpolation then makes this convolution a bounded mapping of $L^{p,q}(\Omega)$ into $L^{p^*,q}(\Omega)$ for all indices p in the interval $(1, (1^*)') = (1, n/m)$ and all indices q in $[1, \infty]$.

These conclusions are sharper than those coming from Marcinkiewicz interpolation. On the other hand, the latter applies to mappings of weak-type $(1, 1)$, a case not covered by the K and J methods for Banach spaces, since weak L^1 is not a Banach space. The statement of the Marcinkiewicz Theorem 2.58 also applies to sublinear operators of weak-type (p, q) rather than just linear operators. It is easy, however, to extend the J and K machinery to cover sublinear operators between L^p spaces and Lorentz spaces. As above, this gives target spaces $L^{q,p}$ that are strictly smaller than L^q when $p < q$. Marcinkiewicz does not apply when $p > q$, but the J and K methods still apply, with target spaces $L^{q,p}$ that are larger than L^q in these cases.

Besov Spaces

7.30 The real interpolation method also applies to scales of spaces based on smoothness. For Sobolev spaces on sufficiently smooth domains the resulting intermediate spaces are called Besov spaces. Before defining them, we first establish the following theorem which shows that if $0 < k < m$, then $W^{k,p}(\Omega)$ is suitably intermediate between $L^p(\Omega)$ and $W^{m,p}(\Omega)$ provided Ω is sufficiently regular. Since the proof requires both Theorem 5.2, for which the cone condition suffices, and the approximation property of Paragraph 5.31 which we know holds for $\mathbb{R}^n$ and by extension for any domain satisfying the strong local Lipschitz condition, which implies the cone condition, we state the theorem for domains satisfying the strong local Lipschitz condition even though it holds for some domains which do not satisfy this condition. (See Paragraph 5.31.)

7.31 THEOREM If $\Omega \subset \mathbb{R}^n$ satisfies the strong local Lipschitz condition and if $0 < k < m$ and $1 \le p < \infty$, then

$$W^{k,p}(\Omega) \in \mathscr{H}\big(k/m;\, L^p(\Omega),\, W^{m,p}(\Omega)\big).$$

Proof. In this context we deal with the function norms

$$J(t;u) = \max\{\|u\|_p, t\,\|u\|_{m,p}\}$$
$$K(t;u) = \inf\{\|u_0\|_p + t\,\|u_1\|_{m,p} \;:\; u = u_0 + u_1, u \in L^p(\Omega), u_1 \in W^{m,p}(\Omega)\}.$$

We must show that

$$\|u\|_{k,p} \le C\,t^{-(k/m)} J(t;u) \tag{16}$$
$$K(t;u) \le C\,t^{k/m}\,\|u\|_{k,p}\,. \tag{17}$$

Now Theorem 5.2 asserts that for some constant C and all $u \in W^{m,p}(\Omega)$

$$\|u\|_{k,p} \le C\,\|u\|_p^{1-(k/m)}\,\|u\|_{m,p}^{k/m}\,.$$

The expression on the right side is C times the minimum value of

$$t^{-k/m} J(t;u) = \max\{t^{-k/m}\,\|u\|_p\,, t^{1-(k/m)}\,\|u\|_{m,p}\},$$

which occurs for $t = \|u\|_p / \|u\|_{m,p}$, the value of t making both terms in the maximum equal. This proves (16).

We show that (17) is equivalent to the approximation property. If $u \in W^{k,p}(\Omega)$, then

$$K(t;u) \le \|u\|_p + t\,\|0\|_{m,p} = \|u\|_p \le \|u\|_{k,p}\,.$$

Thus $t^{-k/m}K(t,u) \le \|u\|_p$ when $t \ge 1$, and inequality (17) holds in that case. If $t^{-(k/m)}K(t;u) \le C\,\|u\|_{k,p}$ also holds for $0 < t \le 1$, then since we can choose $u_0 \in L^p(\Omega)$ and $u_1 \in W^{m,p}(\Omega)$ with $u = u_0 + u_1$ and $\|u_0\|_p + t\,\|u_1\|_{m,p} \le 2K(t;u)$, we must have

$$\|u - u_1\|_p = \|u_0\|_p \le 2Ct^{k/m}\,\|u\|_{k,p} \qquad \text{and} \qquad \|u_1\|_{m,p} \le 2Ct^{(k/m)-1}\,\|u\|_{k,p}\,,$$

so that with $t = \epsilon^m$, $u_\epsilon = u_1$ is a solution of the approximation problem of Paragraph 5.31. Conversely, if the approximation problem has a solution, that is, if for each $\epsilon \le 1$ there exists $u_\epsilon \in W^{m,p}(\Omega)$ satisfying

$$\|u - u_\epsilon\|_p \le C\epsilon^k\,\|u\|_{k,p} \qquad \text{and} \qquad \|u_\epsilon\|_{m,p} \le C\epsilon^{k-m}\,\|u\|_{k,p}\,,$$

then, with $\epsilon = t^{1/m}$, we will have

$$t^{-(k/m)}K(t;u) \le t^{-(k/m)}\big(\|u - u_\epsilon\|_p + t\,\|u_\epsilon\|_{m,p}\big) \le C\,\|u\|_{k,p}$$

and (17) holds. This completes the proof. ∎

7.32 (The Besov Spaces) We begin with a definition of Besov spaces on general domains by interpolation.

Let $0 < s < \infty$, $1 \le p < \infty$, and $1 \le q \le \infty$. Also let m be the smallest integer larger than s. We define the *Besov space* $B^{s;p,q}(\Omega)$ to be the intermediate space between $L^p(\Omega)$ and $W^{m,p}(\Omega)$ corresponding to $\theta = s/m$, specifically:

$$B^{s;p,q}(\Omega) = \big(L^p(\Omega),\, W^{m,p}(\Omega)\big)_{s/m,q;J}.$$

It is a Banach space with norm $\|u\,;\,B^{s;p,q}(\Omega)\| = \left\|u\,;\,\big(L^p(\Omega),\, W^{m,p}(\Omega)\big)_{s/m,q;J}\right\|$ and enjoys many other properties inherited from $L^p(\Omega)$ and $W^{m,p}(\Omega)$, for example the density of the subspace $\{\phi \in C^\infty(\Omega) : \|u\|_{m,p} < \infty\}$. Also, imposing the strong local Lipschitz property on Ω guarantees the existence of an extension operator from $W^{m,p}(\Omega)$ to $W^{m,p}(\mathbb{R}^n)$ and so from $B^{s;p,q}(\Omega)$ to $B^{s;p,q}(\mathbb{R}^n)$. On $\mathbb{R}^n$, there are many equivalent definitions $B^{s;p,q}$ (see [J]), each leading to a definition of $B^{s;,p,q}(\Omega)$ by restriction. For domains with good enough extension properties, these definitions by restriction are equivalent to the definition by real interpolation. Although somewhat indirect, that definition is intrinsic. As in Remark 6.47(1), the definitions by restriction can give smaller spaces for domains without extension properties.

For domains for which the conclusion of Theorem 7.31 holds, that theorem and the Reiteration Theorem 7.21 show that, up to equivalence of norms, we get the same space $B^{s;p,q}(\Omega)$ if we use any integer $m > s$ in the definition above. In fact, if $s_1 > s$ and $1 \le q_1 \le \infty$, then

$$B^{s;p,q}(\Omega) = \big(L^p(\Omega),\, B^{s_1;p,q_1}(\Omega)\big)_{s/s_1,q;J}.$$

More generally, if $0 \le k < s < m$ and $s = (1-\theta)k + \theta m$, then

$$B^{s;p,q}(\Omega) = \big(W^{k,p}(\Omega),\, W^{m,p}(\Omega)\big)_{\theta,q;J},$$

and if $0 < s_1 < s < s_2$, $s = (1-\theta)s_1 + \theta s_2$, and $1 \le q_1, q_2 \le \infty$, then

$$B^{s;p,q}(\Omega) = \big(B^{s_1;p,q_1}(\Omega),\, B^{s_2;p,q_2}(\Omega)\big)_{\theta,q;J}.$$

7.33 Theorem 7.31 also implies that for integer m,

$$B^{m;p,1}(\Omega) \to W^{m,p}(\Omega) \to B^{m;p,\infty}(\Omega).$$

In Paragraph 7.67 we will see that

$$\begin{aligned} &B^{m;p,p}(\Omega) \to W^{m,p}(\Omega) \to B^{m;p,2}(\Omega) \qquad &&\text{for} \quad 1 < p \le 2, \\ &B^{m;p,2}(\Omega) \to W^{m,p}(\Omega) \to B^{m;p,p}(\Omega) \qquad &&\text{for} \quad 2 \le p < \infty. \end{aligned}$$

The indices here are best possible; even in the case $\Omega = \mathbb{R}^n$ it is not true that $B^{m;p,q}(\Omega) = W^{m,p}(\Omega)$ for any q unless $p = q = 2$.

The following imbedding theorem for Besov spaces requires only that Ω satisfy the cone condition (or even the weak cone condition) since it makes no use of Theorem 7.31.

7.34 THEOREM (An Imbedding Theorem for Besov Spaces) Let Ω be a domain in $\mathbb{R}^n$ satisfying the cone condition, and let $1 \le p < \infty$ and $1 \le q \le \infty$.

(a) If $sp < n$, then $B^{s;p,q}(\Omega) \to L^{r,q}(\Omega)$ for $r = np/(n - sp)$.

(b) If $sp = n$, then $B^{s;p,1}(\Omega) \to C_B^0(\Omega) \to L^\infty(\Omega)$.

(c) If $sp > n$, then $B^{s;p,q}(\Omega) \to C_B^0(\Omega)$.

Proof. Observe that part (a) follows from part (b) and the Exact Interpolation Theorem 7.23 since if $0 < s < s_1$ and $s_1 p = n$, then (b) implies

$$B^{s;p,q}(\Omega) = \left(L^p(\Omega), B^{s_1;p,1}(\Omega)\right)_{s/s_1,q;J} \to \left(L^p(\Omega), L^\infty(\Omega)\right)_{s/s_1,q;J} = L^{r,q}(\Omega),$$

where $r = [1 - (s/s_1)]/p = np/(n - sp)$.

To prove (b) let m be the smallest integer greater than $s = n/p$. Let $u \in B^{n/p;p,1}(\Omega) = \left(L^p(\Omega), W^{m,p}(\Omega)\right)_{n/(mp),1;J}$. By the discrete version of the J-method, there exist functions u_i in $W^{m,p}(\Omega)$ such that the series $\sum_{i=-\infty}^{\infty} u_i$ converges to u in $B^{n/p;p,1}(\Omega)$ and such that the sequence $\{2^{-in/mp} J(2^i; u_i)\}_{i=-\infty}^{\infty}$ belongs to ℓ^1 and has ℓ^1 norm no larger than $C \left\| u \,;\, B^{n/p;p,1}(\Omega) \right\|$. Since $mp > n$ and Ω satisfies the cone condition, Theorem 5.8 shows that

$$\|v\|_\infty \le C_1 \|v\|_p^{1-(n/mp)} \|v\|_{m,p}^{n/mp}$$

for all $v \in W^{m,p}(\Omega)$. Thus

$$\begin{aligned}
\|u\|_\infty &\le \sum_{i=-\infty}^{\infty} \|u_i\|_\infty \\
&\le C_1 \sum_{i=-\infty}^{\infty} \|u_i\|_p^{1-(n/mp)} \|u_i\|_{m,p}^{n/mp} \\
&\le C_1 \sum_{i=-\infty}^{\infty} 2^{-in/mp} J\left(2^i; u_i\right) \le C_2 \left\| u \,;\, B^{n/p;p,1}(\Omega) \right\|.
\end{aligned}$$

Thus $B^{n/p;p,1}(\Omega) \to L^\infty(\Omega)$. The continuity of u follows as in the proof of Part I, Case A of Theorem 4.12 given in Paragraph 4.16.

Part (c) follows from part (b) since $B^{s;p,q}(\Omega) \to B^{s_1;p,1}(\Omega)$ if $s > s_1$. This imbedding holds because $W^{m,p}(\Omega) \to L^p(\Omega)$. ∎

Generalized Spaces of Hölder Continuous Functions

7.35 (The Spaces $C^{j,\lambda,q}(\overline{\Omega})$) If Ω satisfies the strong local Lipschitz condition and $sp > n$, the Besov space $B^{s;p,q}(\Omega)$ also imbeds into an appropriate space of Hölder continuous functions. To formulate that imbedding we begin by generalizing the Hölder space $C^{j,\lambda}(\overline{\Omega})$ to allow for a third parameter. For this purpose we consider the *modulus of continuity* of a function u defined on Ω given by

$$\omega(u;t) = \sup\big\{|u(x) - u(y)| \,:\, x, y \in \Omega,\ |x - y| \le t\big\}, \qquad (t > 0).$$

Observe that $\omega(u;t) = \omega^*_\infty(u;t)$ in the notation of Paragraph 7.46. Also observe that if $0 < \lambda \le 1$ and $t^{-\lambda}\omega(t,u) \le k < \infty$ for all $t > 0$, then u is uniformly continuous on Ω. Since $C^j(\overline{\Omega})$ is a subspace of $W^{j,\infty}(\Omega)$ with the same norm, $C^{j,\lambda}(\overline{\Omega})$ consists of those $u \in W^{j,\infty}(\Omega)$ for which $t^{-\lambda}\omega(t, D^\alpha u)$ is bounded for all $0 < t < \infty$ and all α with $|\alpha| = j$.

We now define the generalized spaces $C^{j,\lambda,q}(\overline{\Omega})$ as follows. If $j \ge 0$, $0 < \lambda \le 1$, and $q = \infty$, then $C^{j,\lambda,\infty}(\overline{\Omega}) = C^{j,\lambda}(\overline{\Omega})$ with norm

$$\left\|u\,;C^{j,\lambda,\infty}(\overline{\Omega})\right\| = \left\|u\,;C^{j,\lambda}(\overline{\Omega})\right\| = \|u\|_{j,\infty} + \max_{|\alpha|=j}\sup_{t>0}\frac{\omega(D^\alpha u;t)}{t^\lambda}.$$

For $j \ge 0$, $0 < \lambda \le 1$, and $1 \le q < \infty$, the space $C^{j,\lambda,q}(\overline{\Omega})$ consists of those functions $u \in W^{j,\infty}(\Omega)$ for which $\left\|u\,;C^{j,\lambda,q}(\overline{\Omega})\right\| < \infty$, where

$$\left\|u\,;C^{j,\lambda,q}(\overline{\Omega})\right\| = \left\|u\,;C^{j}(\overline{\Omega})\right\| + \max_{|\alpha|=j}\left(\int_0^\infty \left(t^{-\lambda}\omega(D^\alpha u;t)\right)^q\frac{dt}{t}\right)^{1/q}.$$

$C^{j,\lambda,q}(\overline{\Omega})$ is a Banach space under the norm $\left\|\cdot\,;C^{j,\lambda,q}(\overline{\Omega})\right\|$.

7.36 LEMMA If $0 < \lambda \le 1$ and $0 < \theta < 1$, then

$$\left(L^\infty(\Omega), C^{0,\lambda}(\overline{\Omega})\right)_{\theta,q;K} \to C^{0,\theta\lambda,q}(\overline{\Omega}).$$

Proof. Let $u \in C^{0,\lambda}(\overline{\Omega})_{\theta,q;K}$. Then there exists $v \in L^\infty(\Omega)$ and $w \in C^{0,\lambda}(\overline{\Omega})$ such that $u = v + w$ and

$$\|v\|_\infty + t^\lambda\left\|w\,;C^{0,\lambda}(\overline{\Omega})\right\| \le 2K(t^\lambda;u) \qquad \text{for } t > 0.$$

If $|h| \le t$, then

$$\begin{aligned}|u(x+h) - u(x)| &\le |v(x+h)| + |v(x)| + \frac{|w(x+h) - w(x)|}{|h|^\lambda}|h|^\lambda \\ &\le 2\|v\|_\infty + \left\|w\,;C^{0,\lambda}(\overline{\Omega})\right\| t^\lambda \le 4K(t^\lambda;u).\end{aligned}$$

Thus $\omega(u;t) \le 4K(t^\lambda; u)$.

Since $\|u\|_\infty \le \left\| u\,;\,C^{0,\lambda}(\overline{\Omega})\right\|$, we have $\|u\|_\infty \le \|u\|_{\theta,q;K}$. Thus, if $1 \le q < \infty$,

$$\begin{aligned}
\left\| u\,;\,C^{0,\lambda\theta,q}(\overline{\Omega})\right\| &= \|u\|_\infty + \left(\int_0^\infty \left(t^{-\lambda\theta}\omega(u;t)\right)^q \frac{dt}{t}\right)^{1/q} \\
&\le \|u\|_{\theta,q;K} + 4\left(\int_0^\infty \left(t^{-\lambda\theta}K(t^\lambda;u)\right)^q \frac{dt}{t}\right)^{1/q} \\
&= \|u\|_{\theta,q;K} + 4\lambda^{-1/q}\left(\int_0^\infty \left(\tau^{-\theta}K(\tau;u)\right)^q \frac{d\tau}{\tau}\right)^{1/q} \\
&\le \left(1 + 4\lambda^{-1/q}\right)\|u\|_{\theta,q;K}\,.
\end{aligned}$$

Similarly, for $q = \infty$, we obtain

$$\left\| u\,;\,C^{0,\lambda\theta,\infty}(\overline{\Omega})\right\| \le \|u\|_{\theta,\infty;K} + 4\sup_t t^{-\lambda\theta}K(t^\lambda;u) \le 5\,\|u\|_{\theta,\infty;K}\,.$$

This completes the proof. ∎

7.37 THEOREM Let Ω be a domain in $\mathbb{R}^n$ satisfying the strong local Lipschitz condition. Let $m - 1 - j \le n/p < s \le m - j$ and $1 \le q \le \infty$. If $\mu = s - n/p$, then

$$B^{s;p,q}(\Omega) \to C^{j,\mu,q}(\overline{\Omega}).$$

Proof. It is sufficient to prove this for $j = 0$. By Theorem 7.34(b),

$$B^{n/p;p,1}(\Omega) \to C_B^0(\Omega) \to L^\infty(\Omega).$$

By Part II of Theorem 4.12,

$$W^{m,p}(\Omega) \to C^{0,\lambda}(\overline{\Omega}), \qquad \text{where} \quad \lambda = m - \frac{n}{p}.$$

Now $B^{s;p,q}(\Omega) = \left(B^{n/p;p,1}(\Omega),\, W^{m,p}(\Omega)\right)_{\theta,q;K}$, where

$$(1-\theta)\frac{n}{p} + \theta m = s.$$

Since $\lambda\theta = \mu$, we have by the Exact Interpolation Theorem and the previous Lemma,

$$B^{s;p,q}(\Omega) \to \left(L^\infty(\Omega),\, C^{0,\lambda}(\overline{\Omega})\right)_{\theta,q;K} \to C^{j,\mu,q}(\overline{\Omega}). \quad ∎$$

Characterization of Traces

7.38 As shown in the Sobolev imbedding theorem (Theorem 4.12) functions in $W^{m,p}(\mathbb{R}^{n+1})$ (where $mp < n+1$) have traces on $\mathbb{R}^n$ that belong to $L^q(\mathbb{R}^n)$ for $p \le q \le np/(n+1-mp)$. The following theorem asserts that these traces are exactly the functions that belong to $B^{m-(1/p);p,p}(\mathbb{R}^n)$. This is an instance of the phenomenon that passing from functions in $W^{m,p}(\Omega)$ to their traces on surfaces of codimension 1 results in a loss of smoothness corresponding to $1/p$ of a derivative.

In the following we denote points in $\mathbb{R}^{n+1}$ by (x,t) where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The trace $u(x)$ of a smooth function $U(x,t)$ defined on $\mathbb{R}^{n+1}$ is therefore given by $u(x) = U(x,0)$.

7.39 THEOREM (The Trace Theorem) If $1 < p < \infty$, the following conditions on a measurable function u on $\mathbb{R}^n$ are equivalent.

(a) There is a function U in $W^{m,p}(\mathbb{R}^{n+1})$ so that u is the trace of U.

(b) $u \in B^{m-(1/p);p,p}(\mathbb{R}^n)$. ▮

As the proof of this theorem is rather lengthy, we split it into two lemmas; (a) implies (b) and (b) implies (a).

7.40 LEMMA Let $1 < p < \infty$. If $U \in W^{m,p}(\mathbb{R}^{n+1})$, then its trace u belongs to the space $B = B^{m-(1/p);p,p}(\mathbb{R}^n)$ and

$$\|u\|_B \le K \, \|U\|_{m,p,\mathbb{R}^{n+1}}, \tag{18}$$

for some constant K independent of U.

Proof. We represent

$$B \equiv B^{m-(1/p);p,p}(\mathbb{R}^n) = \big(W^{m-1,p}(\mathbb{R}^n),\, W^{m,p}(\mathbb{R}^n)\big)_{\theta,p;J},$$

where

$$\theta = 1 - \frac{1}{p} = \frac{1}{p'}$$

and use the discrete version of the J-method; we have $u \in B^{m-(1/p);p,p}(\mathbb{R}^n)$ if and only if there exist functions u_i in $W^{m-1,p}(\mathbb{R}^n) \cap W^{m,p}(\mathbb{R}^n) = W^{m,p}(\mathbb{R}^n)$ for $-\infty < i < \infty$ such that the series $\sum_{i=-\infty}^{\infty} u_i$ converges to u in norm in the space $W^{m-1,p}(\mathbb{R}^n) + W^{m,p}(\mathbb{R}^n) = W^{m-1,p}(\mathbb{R}^n)$, and such that the sequences $\left\{2^{-i/p'} \|u_i\|_{m-1,p}\right\}$ and $\left\{2^{i/p} \|u_i\|_{m,p}\right\}$ both belong to ℓ^p. We verify (18) by splitting U into pieces U_i with traces u_i that satisfy these conditions.

Let Φ be an even function on the real line satisfying the following conditions:

(i) $\Phi(t) = 1$ if $-1 \le t \le 1$,

(ii) $\Phi(t) = 0$ if $|t| \ge 2$,

(iii) $|\Phi(t)| \le 1$ for all t,

(iv) $|\Phi^{(j)}(t)| \le C_j < \infty$ for all $j \ge 1$ and all t.

For each integer i let $\Phi_i(t) = \Phi(t/2^i)$; then Φ_i takes the value 1 on the interval $[-2^i, 2^i]$ and takes the value 0 on the intervals $[2^{i+1}, \infty)$ and $(-\infty, -2^{i+1}]$. Also, $|\Phi(t)| \le 1$ and $|\Phi_i'(t)| \le 2^{-i}C_1$ for all t.

Let $\phi_i = \Phi_{i+1} - \Phi_i$. Then $\phi_i(\tau)$ vanishes outside the open intervals $(2^i, 2^{i+2})$ and $(-2^{i+2}, -2^i)$; in particular it vanishes at the endpoints of these intervals. Also $\|\phi_i\|_\infty = 1$ and $\left\|\phi_i'\right\|_\infty \le 2^{-i}C_1$.

Now suppose that $U \in C_0^\infty(\mathbb{R}^{n+1})$. Then for each t we have

$$U(x,t) = -\int_t^\infty \frac{\partial U}{\partial \tau}(x,\tau)\,d\tau = -\int_t^\infty D^{(0,1)}U(x,\tau)\,d\tau.$$

Let

$$U_i(x,t) = -\int_t^\infty \phi_i(\tau) D^{(0,1)}U(x,\tau)\,d\tau.$$

Let $u(x) = U(x,0)$ be the trace of U on $\mathbb{R}^n$, and let u_i be the corresponding trace of U_i. Since U has compact support, the functions U_i and u_i vanish when i is sufficiently large. Moreover, $U_i(x,t) = 0$ for all i when $|x|$ is sufficiently large. Therefore the trace u vanishes except on a compact set, on which the series $\sum_{i=\infty}^{\infty} u_i(x)$ converges uniformly to $u(x)$. The terms in this series also vanish off that compact set and taking any partial derivative term-by-term gives a series that converges uniformly on that compact set to the corresponding partial derivative of u.

We use two representations of $u_i(x) = U_i(x,0)$, namely

$$u_i(x) = -\int_{2^i}^{2^{i+2}} \phi_i(\tau) D^{(0,1)}U(x,\tau)\,d\tau = \int_{2^i}^{2^{i+2}} \phi_i'(\tau) U(x,\tau)\,d\tau, \tag{19}$$

where the second expression follows from the first by integration by parts. If $|\alpha| \le m-1$ we obtain from the first representation a corresponding representations of $D^\alpha u_i(x)$:

$$D^\alpha u_i(x) = -\int_{2^i}^{2^{i+2}} \phi_i(\tau) D^{(\alpha,1)}U(x,\tau)\,d\tau,$$

so that, by Hölder's inequality,

$$|D^\alpha u_i(x)| \le (2^{i+2})^{1/p'} \left(\int_{2^i}^{2^{i+2}} \left|D^{(\alpha,1)}U(x,\tau)\right|^p d\tau \right)^{1/p}.$$

Each positive number τ lies in exactly two of the intervals $[2^i, 2^{i+1})$ over which the integrals above run. Multiplying by $2^{-i/p'}$, taking p-th powers on both sides, summing with respect to i, and integrating x over $\mathbb{R}^n$ shows that the p-th power of the ℓ^p norm of the sequence $\{2^{-i/p'}\|D^\alpha u_i\|_p\}_{i=-\infty}^{\infty}$ is no larger than

$$2^{1+2p/p'}\int_{R_+^{n+1}} \left|D^{(\alpha,1)}U(x,\tau)\right|^p \,d\tau\,dx.$$

Thus that ℓ^p norm is bounded by a constant times $\|U\|_{m,p,\mathbb{R}^{n+1}}$.

Using the second representation of u_i in (19), our bound on $\|\phi_i'\|_\infty$, and Hölder's inequality gives us a second estimate

$$|D^\alpha u_i(x)| \le 2^{-i}C_1(2^{i+2})^{1/p'}\left(\int_{2^i}^{2^{i+2}} |D^{(\alpha,0)}U(x,\tau)|^p\,d\tau\right)^{1/p},$$

this one valid for any α with $|\alpha| \le m$. Multiplying by $2^{i/p}$, taking p-th powers on both sides, and summing with respect to i shows that the p-th power of the ℓ^p norm of the sequence $\{2^{i/p}\|D^\alpha u_i\|_p\}_{i=-\infty}^{\infty}$ is no larger than

$$2^{1+2p/p'}C_1^p\int_{R_+^{n+1}} |D^{(\alpha,0)}U(x,\tau)|^p\,d\tau\,dx.$$

Thus that ℓ^p norm is also bounded by a constant times $\|U\|_{m,p,\mathbb{R}^{n+1}}$.

Together, these estimates show that the norm of u in $B^{m-(1/p);p,p}(\mathbb{R}^n)$ is bounded by a constant times the norm of U in $W^{m,p}(\mathbb{R}^{n+1})$ whenever $U \in C_0^\infty(\mathbb{R}^{n+1})$. Since the latter space is dense in $W^{m,p}(\mathbb{R}^{n+1})$, the proof is complete. ∎

7.41 LEMMA Let $1 < p < \infty$ and $B = B^{m-(1/p);p,p}(\mathbb{R}^n)$. If $u \in B$, then u is the trace of a function $U \in W^{m,p}(\mathbb{R}^{n+1})$ satisfying

$$\|U\|_{m,p,\mathbb{R}^{n+1}} \le K\,\|u\|_B \tag{20}$$

for some constant K independent of u.

Proof. In this proof it is convenient to use a characterization of B different (if $m > 1$) from the one used in the previous lemma, namely

$$B = B^{m-(1/p);p,p}(\mathbb{R}^n) = \big(L^p(\mathbb{R}^n),\, W^{m,p}(\mathbb{R}^n)\big)_{\theta,p;J},$$

where $\theta = 1 - (1/mp)$. Again we use the discrete version of the J-method. For $u \in B$ we can find $u_i \in L^p(\mathbb{R}^n) \cap W^{m,p}(\mathbb{R}^n) = W^{m,p}(\mathbb{R}^n)$ (for $-\infty < i < \infty$)

such that $\sum_{i=-\infty}^{\infty} u_i$ converges to u in $L^p(\mathbb{R}^n) + W^{m,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, and such that

$$\left\| \left\{ 2^{-\theta i} \|u_i\|_p \right\} ; \ell^p \right\| \le K_1 \|u\|_B ,$$
$$\left\| \left\{ 2^{(1-\theta) i} \|u_i\|_{m,p} \right\} ; \ell^p \right\| \le K_1 \|u\|_B .$$

These estimates imply that $\sum_{i=-\infty}^{\infty} u_i$ converges to u in B. We will construct an extension $U(x,t)$ of $u(x)$ defined on $\mathbb{R}^{n+1}$ such that (20) holds.

It is sufficient to extend the partial sums $s_k = \sum_{i=-k}^{k} u_i$ to S_k on $\mathbb{R}^{n+1}$ with control of the norms:

$$\|S_k\|_{m,p,\mathbb{R}^{n+1}} \le K_1 \left\| s_k \,;\, B^{m-(1/p);p,p}(\mathbb{R}^n) \right\| ,$$

since $\{S_k\}$ will then be a Cauchy sequence in $W^{m,p}(\mathbb{R}^{n+1})$ and so will converge there. Furthermore, we can assume that the functions u and u_i are smooth since the mollifiers $J_\epsilon * u$ and $J_\epsilon * u_i$ (as considered in Paragraphs 2.28 and 3.16) converge to u and u_i in norm in $W^{m,p}(\mathbb{R}^n)$ as $\epsilon \to 0+$. Accordingly, therefore, in the following construction we assume that the functions u and u_i are smooth and that all but finitely many of the u_i vanish identically on $\mathbb{R}^n$.

Let $\Phi(t)$ be as defined in the previous lemma. Here, however, we redefine Φ_i as follows:

$$\Phi_i(t) = \Phi\left(\frac{t}{2^{i/m}}\right), \qquad -\infty < i < \infty.$$

The derivatives of Φ_i then satisfy $|\Phi_i^{(j)}(t)| \le 2^{-ij/m} C_j$. Also, note that for $j \ge 1$, $\Phi_i^{(j)}$ is zero outside the two intervals $(-2^{(i+1)/m}, -2^{i/m})$ and $(2^{i/m}, 2^{(i+1)/m})$, which have total length not exceeding $2^{1+(i/m)}$.

We define the extension of u as follows:

$$U(x,t) = \sum_{i=-\infty}^{\infty} U_i(x,t), \qquad \text{where} \qquad U_i(x,t) = \Phi_i(t) u_i(x).$$

Note that the sum is actually a finite one under the current assumptions. In order to verify (20) it is sufficient to bound by multiples of $\|u\|_B$ the L^p-norms of U and all its mth order derivatives; the Ehrling-Nirenberg-Gagliardo interpolation theorem 5.2 then supplies similar bounds for intermediate derivatives. The mth order derivatives are of three types: $D^{(0,m)}U$, $D^{(\alpha,j)}U$ for $1 \le j \le m-1$ and $|\alpha| + j = m$, and $D^{(\alpha,0)}U$ for $|\alpha| = m$. We examine each in turn.

Since $D^{(0,m)}U_i(x,t) = 2^{-i}\Phi^{(m)}(t/2^{i/m}) u_i(x)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}} |D^{(0,m)} U_i(x,t)|^p \, dx\, dt \\
&\quad \le \left(\int_{-2^{(i+1)/m}}^{-2^{i/m}} dt + \int_{2^{i/m}}^{2^{(i+1)/m}} dt \right) \int_{\mathbb{R}^n} |D^{(0,m)} U_i(x,t)|^p \, dx \\
&\quad \le 2^{1+(i/m)} 2^{-ip} C_m^p \|u_i\|_p^p = 2 C_m^p 2^{-\theta i p} \|u_i\|_p^p .
\end{aligned}$$

Since the functions $\Phi_i^{(m)}$ have non-overlapping supports, we can sum the above inequality on i to obtain

$$\begin{aligned}\left\| D^{(0,m)} U \right\|_{p,\mathbb{R}^{n+1}}^p &\le 2C_m^p \sum_{i=-\infty}^{\infty} \left(2^{-\theta i} \|u_i\|_p\right)^p \\ &= 2C_m^p \left\| \left\{2^{-\theta i} \|u_i\|_p\right\} ; \ell^p \right\|^p \le 2C_m^p \|u\|_B^p\end{aligned}$$

and the required estimate for $D^{(0,m)}U$ is proved.

Now consider $D^{(\alpha,j)} U_i(x,t) = 2^{-ij/m} \Phi^{(j)}(t/2^{i/m}) D^\alpha u_i(x)$ for which we obtain similarly

$$\int_{\mathbb{R}^{n+1}} |D^{(\alpha,j)} U_i(x,t)|^p \, dx \, dt \le C_j^p 2^{-i(jp-1)/m} \|D^\alpha u_i\|_p^p .$$

Since $|\alpha| = m - j$, we can replace the L^p-norm of $D^\alpha u_i$ with the seminorm $|u_i|_{m-j,p}$, and again using the non-overlapping of the supports of the $\Phi_i^{(j)}$ (since $j \ge 1$) to get

$$\left\| D^{(\alpha,j)} U \right\|_{p,\mathbb{R}^{n+1}}^p \le C_j^p \sum_{i=-\infty}^{\infty} 2^{-i(jp-1)/m} |u_i|_{m-jp}^p .$$

As remarked in Paragraph 5.7, for $1 \le j \le m-1$ Theorem 5.2 assures us that there exists a constant K_2 such that for any $\epsilon > 0$ and any i

$$|u_i|_{m-j,p}^p \le K_2 \left(\epsilon^p |u_i|_{m,p}^p + \epsilon^{-(m-j)p/j} \|u\|_p^p\right).$$

Let $\epsilon = 2^{ij/m}$. Then we have

$$\begin{aligned}\left\| D^{(\alpha,j)} U \right\|_{p,\mathbb{R}^{n+1}}^p &\le C_j^p K_2 \sum_{i=-\infty}^{\infty} \left(2^{i/m} |u_i|_{m,p}^p + 2^{-ip(1-(1/mp))} \|u_i\|_p^p\right) \\ &= C_j^p K_2 \sum_{i=-\infty}^{\infty} \left(2^{(1-\theta)ip} |u_i|_{m,p}^p + 2^{-\theta ip} \|u_i\|_p^p\right) \\ &\le C_j^p K_2 \left(\left\| \left\{2^{(1-\theta)i} \|u_i\|_{m,p}\right\} ; \ell^p \right\|^p + \left\| \left\{2^{-\theta i} \|u_i\|_p\right\} ; \ell^p \right\| \right) \\ &\le 2K_1^p C_j^p K_2 \|u\|_B\end{aligned}$$

and the bound for $D^{(\alpha,j)}U$ is proved.

Finally, we consider U and $D^{(\alpha,0)}U$ together. (We allow $0 \le |\alpha| \le m$.) Unlike their derivatives, the functions Φ_i have nested rather than non-overlapping supports. We must proceed differently than in the previous cases. Consider

$D^{(\alpha,0)}U(x,t)$ on the strip $2^{j/m} < t \le 2^{(j+1)/m}$ in $\mathbb{R}^{n+1}$. Since $|\Phi_i(t)| \le 1$ and since $U_i(x,t) = 0$ on this strip if $i < j-1$, we have

$$|D^{(\alpha,0)}U(x,t)| \le \sum_{i=j-1}^{\infty} |D^{(\alpha,0)}U_i(x,t)| = \sum_{i=j-1}^{\infty} 2^{-i/mp} a_i,$$

where $a_i = 2^{i/mp}|D^\alpha u_i(x)|$. Thus,

$$\begin{aligned} b_j \equiv \left(\int_{2^{j/m}}^{2^{(j+1)/m}} |D^{(\alpha,0)}U(x,t)|^p \, dt \right)^{1/p} &\le \sum_{i=j-1}^{\infty} 2^{j/mp} 2^{-i/mp} a_i \\ &= \sum_{i=j-1}^{\infty} 2^{(j-i)/mp} a_i = (c * a)_j, \end{aligned}$$

where $c_j = 2^{j/mp}$ when $-\infty < j \le 1$ and $c_j = 0$ otherwise. Observe that $c \in \ell^1$ (say, $\|c\,;\ell^1\| = K_3$), and so by Young's inequality for sequences

$$\|b\,;\ell^p\| \le K_3 \|a\,;\ell^p\|.$$

Taking pth powers and summing on j now leads to

$$\int_0^\infty \left|D^{(\alpha,0)}U(x,t)\right|^p dt \le K_3^p \left\|\left\{2^{i/mp}|D^\alpha u_i(x)|\right\};\ell^p\right\|^p.$$

Integrating x over $\mathbb{R}^n$ and taking pth roots then gives

$$\begin{aligned} \left\|D^{(\alpha,0)}U\right\|_{0,p,\mathbb{R}^{n+1}_+} &\le K_3 \left\|\left\{2^{i/mp}\,\|D^\alpha u_i\|_p\right\};\ell^p\right\| \\ &\le K_3 \left\|\left\{2^{(1-\theta)i}\,\|u_i\|_{m,p}\right\};\ell^p\right\| \le K_1 K_3 \|u\|_B. \end{aligned}$$

A similar estimate holds for $\left\|D^{(\alpha,0)}U\right\|_{0,p,\mathbb{R}^{n+1}_-}$, so the proof is complete. ∎

7.42 We can now complete the imbedding picture for Besov spaces by proving an analog of the trace imbedding part of the Sobolev Imbedding Theorem 4.12 for Besov spaces. We will show in Lemma 7.44 below that the trace operator T defined for smooth functions U on $\mathbb{R}^{n+1}$ by

$$(TU)(x) = U(x,0)$$

is linear and bounded from $B^{1/p;p,1}(\mathbb{R}^{n+1})$ into $L^p(\mathbb{R}^n)$. Since Theorem 7.39 assures us that T is also bounded from $W^{m,p}(\mathbb{R}^{n+1})$ onto $B^{m-1/p;p;p}(\mathbb{R}^n)$ for every $m \geq 1$, by the exact interpolation theorem (Theorem 7.23), it is bounded from $B^{s;p,q}(\mathbb{R}^{n+1})$ into $B^{s-1/p;p;q}(\mathbb{R}^n)$, that is,

$$B^{s;p,q}(\mathbb{R}^{n+1}) \to B^{s-1/p;p;q}(\mathbb{R}^n),$$

for every $s > 1/p$ and $1 \leq q \leq \infty$. (Although Theorem 7.39 does not apply if $p = 1$, we already know from the Sobolev Theorem 4.12 that traces of functions in $W^{m,1}(\mathbb{R}^{n+1})$ belong to $W^{m-1,1}(\mathbb{R}^n)$.)

We can now take traces of traces. If $n - k < sp < n$ (so that $s - (n-k)/p > 0$), then

$$B^{s;p,q}(\mathbb{R}^n) \to B^{s-(n-k)/p;p;q}(\mathbb{R}^k),$$

We can combine this imbedding with Theorem 7.34 to obtain for $n - k < sp < n$ and $r = kp/(n - sp)$,

$$B^{s;p,p}(\mathbb{R}^n) \to B^{s-(n-k)/p;p;p}(\mathbb{R}^k) \to L^{r,p}(\mathbb{R}^k) \to L^r(\mathbb{R}^k).$$

More generally:

7.43 THEOREM (Trace Imbeddings for Besov Spaces on $\mathbb{R}^n$) If k is an integer satisfying $1 < k < n$, $n - k < sp < n$, and $r = kp/(n - sp)$, then

$$B^{s;p,q}(\mathbb{R}^n) \to B^{s-(n-k)/p;p;q}(\mathbb{R}^k) \to L^{r,q}(\mathbb{R}^k), \quad \text{and}$$
$$B^{s;p,q}(\mathbb{R}^n) \to L^r(\mathbb{R}^k) \quad \text{for } q \leq r.$$

To establish this theorem, we need only prove the following lemma.

7.44 LEMMA The trace operator T defined by $(TU)(x) = U(x,0)$ imbeds $B^{1/p;p,1}(\mathbb{R}^{n+1})$ into $L^p(\mathbb{R}^n)$.

Proof. Suppose that U belongs to $B \equiv B^{1/p;p,1}(\mathbb{R}^{n+1})$ and, without loss of generality, that $\|U\|_B \leq 1$. Then there exist functions U_i for $-\infty < i < \infty$ such that $U = \sum_i U_i$ and

$$\sum_i 2^{-i/p} \|U_i\|_{p,\mathbb{R}^{n+1}} \leq C \qquad \text{and} \qquad \sum_i 2^{i/p'} \|U_i\|_{1,p,\mathbb{R}^{n+1}} \leq C$$

for some constant C. As in the proof of Lemma 7.40, we can assume that only finitely many of the functions U_i have nonzero values and that they are smooth functions. For any of these functions we have, for $2^i \leq h \leq 2^{i+1}$,

$$\begin{aligned} |U_i(x,0)| &\leq \int_0^h \left|D^{(0,1)}U_i(x,t)\right| dt + |U_i(x,h)| \\ &\leq \int_0^{2^{i+1}} \left|D^{(0,1)}U_i(x,t)\right| dt + |U_i(x,h)|. \end{aligned}$$

Averaging h over $[2^i, 2^{i+1}]$ then gives the estimate

$$|U_i(x,0)| \le \int_0^{2^{i+1}} \left|D^{(0,1)}U_i(x,t)\right| dt + \frac{1}{2^i}\int_{2^i}^{2^{i+1}} |U_i(x,t)|\, dt.$$

By Hölder's inequality,

$$\begin{aligned}|U_i(x,0)| &\le 2^{(i+1)/p'}\left(\int_0^{2^{i+1}} \left|D^{(0,1)}U_i(x,t)\right|^p dt\right)^{1/p} \\ &\quad + \frac{2^{i/p'}}{2^i}\left(\int_{2^i}^{2^{i+1}} |U_i(x,t)|^p\, dt\right)^{1/p} \\ &= a_i(x) + b_i(x), \quad \text{say.}\end{aligned}$$

Then $\|a_i\|_{p,\mathbb{R}^n} \le 2(2^{i/p'})\left\|U_j\right\|_{1,p,\mathbb{R}^{n+1}}$ and $\|b_i\|_{p,\mathbb{R}^n} \le 2^{-i/p}\left\|U_j\right\|_{p,\mathbb{R}^{n+1}}$. We now have

$$\begin{aligned}\|U(\cdot,0)\|_{p,\mathbb{R}^n} &\le \sum_i \|U_i(\cdot,0)\|_{p,\mathbb{R}^n} \\ &\le 2\left(\sum_i 2^{i/p'}\left\|U_j\right\|_{1,p,\mathbb{R}^{n+1}} + \sum_i 2^{-i/p}\left\|U_j\right\|_{p,\mathbb{R}^{n+1}}\right) \le 4C.\end{aligned}$$

This completes the proof. ∎

7.45 REMARKS

1. Theorems 7.39 and 7.43 extend to traces on arbitrary planes of sufficiently high dimension, and, as a consequence of Theorem 3.41, to traces on sufficiently smooth surfaces of sufficiently high dimension.
2. Both theorems also extend to traces of functions in $B^{s;p,q}(\Omega)$ on the intersection of the domain Ω in $\mathbb{R}^n$ with planes or smooth surfaces of dimension k satisfying $k > n - sp$, provided there exists a suitable extension operator for Ω. This will be the case if, for example, Ω satisfies a strong local Lipschitz condition. (See Theorem 5.21.)
3. Before Besov spaces were fully developed, Gagliardo [Ga3] identified the trace space as a space defined by a version of the intrinsic condition (c) in the characterization of Besov spaces in Theorem 7.47 below, where $q = p$ and $s = m - (1/p)$.

Direct Characterizations of Besov Spaces

7.46 The K functional for the pair $\left(L^p(\Omega), W^{m,p}(\Omega)\right)$ measures how closely a given function u can be approximated in L^p norm by functions whose $W^{m,p}$ norm are not too large. For instance, a splitting $u = u_0 + u_1$ with $\|u_0\|_p + t\,\|u_1\|_{m,p} \le 2K(t; u)$ provides such an approximation u_1 to u; then the error $u - u_1 = u_0$ has $L^p(\Omega)$ norm at most $2K(t; u)$ and the approximation u_1 has $W^{m,p}(\Omega)$ norm at most $(2/t)K(t; u)$. So, in principle, the definition of $B^{s;p,q}(\Omega)$ by real interpolation characterizes functions in $B^{s;p,q}(\Omega)$ by the way in which they can be approximated in $L^p(\Omega)$ norm by functions in $W^{m,p}(\Omega)$.

Like many other descriptions of Besov spaces, the one above seems indirect, but it can yield useful upper bounds for Besov norms. On $\mathbb{R}^n$, more direct characterizations come from considering the *L^p-modulus of continuity* and higher-order versions of that modulus. Given a point h in $\mathbb{R}^n$ and a function u in $L^p(\mathbb{R}^n)$, let u_h be the function mapping x to $u(x - h)$, let $\Delta_h u = u - u_h$, let $\omega_p(u; h) = \|\Delta_h u\|_p$, and for positive integers m, let $\omega_p^{(m)}(u; h) = \|(\Delta_h)^m u\|_p$. When $1 \le p < \infty$, mollification shows that $\omega_p(u; h)$ tends to 0 as $h \to 0$, and the same is true for $\omega_p^{(m)}(u; h)$; as stated below, when $m > s$, the rate of the latter convergence to 0 determines whether $u \in B^{s;p,q}(\mathbb{R}^n)$. We also define functions on $\mathbb{R}_+$ by letting $\omega_p^*(u; t) = \sup\{\omega_p(u; h); |h| \le t\}$ and letting $\omega_p^{(m)*}(u; t) = \sup\{\omega_p^{(m)}(u; h); |h| \le t\}$.

7.47 THEOREM (Intrinsic Characterization of $B^{s;p,q}(\mathbb{R}^n)$) Whenever $m > s > 0$, $1 < p < \infty$, and $1 \le q < \infty$, the following conditions on a function u in $L^p(\mathbb{R}^n)$ are equivalent. If $q = \infty$ condition (a) is equivalent to the versions of conditions (b) and (c) with the integrals replaced by the suprema of the quantities inside the square brackets.

(a) $u \in B^{s;p,q}(\mathbb{R}^n)$.

(b) $\displaystyle\int_0^\infty \left[t^{-s}\omega_p^{(m)*}(u; t)\right]^q \frac{dt}{t} < \infty.$

(c) $\displaystyle\int_{R^n} \left[|h|^{-s}\omega_p^{m}(u; h)\right]^q \frac{dh}{|h|^n} < \infty.$ ∎

Before proving this theorem, we observe a few things. First, the moduli of continuity in parts (b) and (c) are never larger than $2^m\,\|u\|_p$; so we get conditions equivalent to (b) and (c) respectively if we use integrals with $t \le 1$ and $|h| \le 1$. Next, the equivalence of conditions (b) and (c) with condition (a), where m does not appear, means that if (b) or (c) holds for some $m > s$, then both conditions hold for all $m > s$.

It follows from our later discussion of Fourier decompositions that if $1 < p < \infty$, then these conditions are equivalent to requiring that the derivatives of u of order k, where k is the largest integer less than s, belong to $L^p(\mathbb{R}^n)$ and satisfy the versions of condition (b) or (c) with $m = 1$ and s replaced with $s - k$.

While we assumed $1 < p < \infty$ in the statement of the theorem, the only part of the proof that requires this is the part showing that (c) $\Rightarrow$ (a) when $m > 1$. The rest of the proof is valid for $1 \leq p \leq \infty$.

7.48 (The Proof of Theorem 7.47 for m = 1) We assume, for the moment, that $m = 1$ and $s < 1$; in the next Paragraph we will outline with rather less detail how to modify the argument for the case $m > 1$. We show that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

The first part is similar to the proof of Lemma 7.36. Suppose first that condition (a) holds and consider condition (b) with $m = 1$. Fix a positive value of the parameter t and split a nontrivial function u as $v + w$ with $\|v\|_p + t\|w\|_{1,p} \leq 2K(t; u)$. Then $\Delta_h u = \Delta_h v + \Delta_h w$, and it suffices to control the L^p norms of the these two differences. For the first term, just use the fact that $\|\Delta_h v\|_p \leq 2\|u\|_p$.

For the second term, we use mollification to replace v and w with smooth functions satisfying the same estimate on their L^p and $W^{1,p}$ norms respectively. We majorize $|w(x-h)-w(x)|$ by the integral of $|\operatorname{grad} w|$ along the line segment joining $x-h$ to x, and use Hölder's inequality to majorize that by $|h|^{1/p'}$ times the one-dimensional L^p norm of the restriction of $|\operatorname{grad} w|$ to that segment. Finally, we take p-th powers, integrate with respect to x, and take a p-th root to get that $\|\Delta_h w\|_p \leq |h||w|_{1,p}$. When $|h| \leq t$ we then obtain

$$\|\Delta_h u\|_p \leq \|\Delta_h v\|_p + \|\Delta_h w\|_p \leq 2\|v\|_p + t|w|_{1,p} \leq 4K(t; u),$$

so condition (a) implies condition (b).

Since t^{-s} decreases and $\omega_p(u; t)$ increases as t increases, condition (b) holds with $m = 1$ if and only if the sequence $\{2^{-is}\omega_p^*(u; 2^i)\}_{i=-\infty}^{\infty}$ belongs to ℓ^q. To deduce condition (c) with $m = 1$, we split the integral in (c) into dyadic pieces with $2^i < |h| \leq 2^{i+1}$. The integral of the measure $dh/|h|^n$ over each such piece is the same. In the i-th piece, $\omega_p(u; h) \leq \omega_p^*(u; 2^{i+1})$ by the definition of the latter quantity. And in that piece, $|h|^{-s} \leq 2^s 2^{-s(i+1)}$. So the integral in (c) is majorized by a constant time the q-th power of the ℓ^q norm of the sequence $\{2^{-(i+1)s}\omega_p^*(f; 2^{(i+1)})\}_{i=-\infty}^{\infty}$, and (c) follows from (b).

We now show that (c) $\Rightarrow$ (a) when $m = 1 > s > 0$. Choose a nonnegative smooth function Φ vanishing outside the ball of radius 2 centred at 0 and inside the ball of radius 1, and satisfying

$$\int_{R^n} \Phi(x)\, dx = 1.$$

For fixed $t > 0$ let $\Phi_t(x) = t^{-n}\Phi(x/t)$; this nonnegative function also integrates to 1, and it vanishes outside the ball of radius $2t$ centred at 0 and inside the ball of radius t.

For u satisfying condition (c), split $u = v + w$ where $w = u * \Phi_t$ and $v = u - w$.

The fact that the density Φ_t has mass 1 ensures that

$$v(x) = \int_{\mathbb{R}^n} \Phi_t(h)[u(x) - u(x-h)]\,dh = \int_{R^n} \Phi_t(h)\Delta_h u(x)\,dh$$
$$= \int_{t<|h|<2t} \Phi_t(h)\Delta_h u(x)\,dh.$$

The function v belongs to $L^p(\mathbb{R}^n)$, being the difference of two functions in that space. To estimate its norm, we use the converse of Hölder's inequality to linearize that norm as the supremum of $\int_{\mathbb{R}^n} |v(x)|g(x)\,dx$ over all nonnegative functions g in the unit ball of $L^{p'}(\mathbb{R}^n)$. For each such function g, we find that

$$\int_{\mathbb{R}^n} |v(x)|g(x)\,dx \le \int_{t<|h|<2t} \Phi_t(h)\left[\int_{\mathbb{R}^n} g(x)|\Delta_h u(x)|\,dx\right] dh$$
$$\le \int_{t<|h|<2t} \Phi_t(h)\|g\|_{p'}\|\Delta_h u\|_p\,dh$$
$$= \int_{t<|h|<2t} \Phi_t(h)\|\Delta_h u\|_p\,dh.$$

Since $\|\Phi_t\|_\infty \le C/t^n$, the last integral above is in turn bounded above by

$$\frac{C}{t^n}\int_{t<|h|<2t} \|\Delta_h u\|_p\,dh \le C\int_{t<|h|<2t} \|\Delta_h u\|_p \frac{dh}{|h|^n}$$
$$\le C_q\left(\int_{t<|h|<2t} \left[\|\Delta_h u\|_p\right]^q \frac{dh}{|h|^n}\right)^{1/q},$$

where the last step uses Hölder's inequality and the fact that the coronas $\{h \in R^n : t < |h| < 2t\}$ all have the same measure. Thus we have shown that

$$\|v\|_p \le C_q\left(\int_{t<|h|<2t} \left[\|\Delta_h u\|_p\right]^q \frac{dh}{|h|^n}\right)^{1/q}. \tag{21}$$

To bound $K(t;u)$ for the interpolation pair $(L^p(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n))$, we also require a bound for $\|w\|_{1,p} = \|u * \Phi_t\|_{1,p}$. Note that $\|w\|_p \le \|u\|_p\|\Phi_t\|_1 = \|u\|_p$. Moreover,

$$\operatorname{grad} w(x) = [u * \operatorname{grad}(\Phi_t)](x) = \int_{t<|h|<2t} u(x-h)\operatorname{grad}(\Phi_t)(h)\,dh$$
$$= \int_{t<|h|<2t} \big[u(x-h) - u(x)\big]\operatorname{grad}(\Phi_t)(h)\,dh$$
$$= \int_{t<|h|<2t} \Delta_h u(x)\operatorname{grad}(\Phi_t)(h)\,dh,$$

where we used the fact that the average value of $\nabla(\Phi_t)(h)$ is $\mathbf{0}$ to pass from the first line above to the second line. Linearizing as we did for v leads to an upper bound like (21) for $\|\operatorname{grad} w\|_p$, except that $\|\Phi_t\|_\infty$ is replaced by $\|\operatorname{grad}\Phi_t\|_\infty$, which is bounded by $\check{C}/t^{n+1}$ rather than by C/t^n. This division by an extra factor of t leads to the estimate

$$|w|_{1,p} \le \frac{C_q^*}{t}\left(\int_{t<|h|<2t}\left[\|(\|\Delta_h u\|_p\right]^q \frac{dh}{|h|^n}\right)^{1/q}.$$

Therefore

$$\begin{aligned} K(t;u) &\le \|v\|_p + t\|w\|_{1,p} \\ &\le \text{const.}\left(\int_{t<|h|<2t}\left[\|\Delta_h u\|_p\right]^q \frac{dh}{|h|^n}\right)^{1/q} + t\|u\|_p. \end{aligned} \tag{22}$$

We also have the cheap estimate $K(t;u) \le \|u\|_p$ from the splitting $u = u + 0$.

We use the discrete version of the K method to describe $B^{s;p,q}(\mathbb{R}^n)$. The cheap estimate suffices to make $\sum_{i=1}^{\infty}[2^{-is}K(2^i;u)]^q$ finite. When $i \le 0$ we use inequality (22) with $t = 2^i$, and we find that distinct indices i lead to disjoint coronas for the integral appearing in (22). It follows that the part of the ℓ^q norm with $i \le 0$ is bounded above by a constant times $\|u\|_p$ plus a constant times the quantity

$$\left(\int_{|h|\le 2}\left[|h|^{-s}\omega_p(u;h)\right]^q \frac{dh}{|h|^n}\right)^{1/q}.$$

This completes the proof when $m = 1$ and $1 \le q < \infty$. The proof when $m = 1$ and $q = \infty$ is similar. ∎

7.49 (The Proof of Theorem 7.47 for $m > 1$) We can easily modify some parts of the above proof for the case where $m = 1$ to work when $m > 1$. In particular, to prove that condition (b) implies condition (c) when $m > 1$, simply take the argument for $m = 1$ and replace ω_p^* by $\omega_p^{(m)*}$ and ω_p by $\omega_p^{(m)}$.

To get from (a) to (b) when $m > 1$, consider $B^{s;p,q}(\mathbb{R}^n)$ as a real interpolation space between $X_0 = L^p(\mathbb{R}^n)$ and $X_1 = W^{m,p}(\mathbb{R}^n)$ with $\theta = s/m$; since $m > s$, we have $\theta < 1$. Given a value of t, split u as $v + w$ with $\|v\|_p + t^m\|w\|_{1,p} \le 2K(t^m;u)$. Then $\|\Delta_h^m v\|_p \le 2^m\|v\|_p \le 2^{m+1}\|v\|_p$.

Again we can mollify w and then write differences of w as integrals of derivatives of w. When $m = 1$ we found that $\Delta_h w$ was an integral of a first directional derivative of w with respect to path length along the line segment from $x - h$ to x. Denote that directional derivative by $D_h w$. Then Δ_h^2 is equal to the integral along the same line segment of $\Delta_h(D_h w)$. That integrand is itself equal to an integral along a line segment of length $|h|$ with integrand $D_h^2 w$. This represents $\Delta_h^2 w(x)$ as an iterated double integral of $D_h^2 w$, with both integrations running over intervals of length $|h|$. Iteration then represents $\Delta_h^m w$ as an m-fold iterated integral of $D_h^m w$ over intervals of length $|h|$. Applying Hölder's inequality to that integral and then integrating p-th powers over $\mathbb{R}^n$ yields the estimate $\|\Delta_h^m w\|_p \leq C|h|^m |w|_{m,p}$. It follows that $\omega_p^{(m)*}(u;t) \leq \hat{C} K(t^m; u)$. Thus

$$\begin{aligned}\int_{-\infty}^{\infty} \left[t^{-s}\omega_p^{(m)*}(u;t)\right]^q \frac{dt}{t} &\leq \hat{C}^q \int_{-\infty}^{\infty} \left[t^{-s} K(t^m; u)\right]^q \frac{dt}{t} \\ &= \hat{C}^q \int_{-\infty}^{\infty} \left[(t^m)^{-s/m} K(t^m; u)\right]^q \frac{dt}{t} \\ &= \check{C} \int_{-\infty}^{\infty} [\tau^{-\theta} K(\tau; u) \frac{d\tau}{\tau},\end{aligned}$$

after the change of variable $\tau = t^m$. So condition (a) still implies condition (b) when $m > 1$.

We now give an outline of the proof that (c) implies (a). See [BB, pp. 192–194] for more details on some of what we do. Since condition (c) for any value of m implies the corresponding condition for larger values of m, we free to assume that m is even, and we do so.

Given a function u satisfying condition (c) for an even index $m > \max\{1, s\}$, and given an integer $i \leq 0$, we can split $u = v_i + w_i$, where v_i is an averaged m-fold integral of $\Delta_h^m u$; each single integral in this nest runs over an interval of length comparable to $t = 2^i$, and the averaging involves dividing by a multiple of t^m. The outcome is that we can estimate $\|v_i\|_p$ by the average of $\|\Delta_h^m u\|_p$ over a suitable h-corona. As in the case where $m = 1$, this leads to an estimate for the ℓ^q norm of the sequence $\left\{2^{-is}\|v_i\|_p\right\}_{i=-\infty}^{\infty}$ in terms of the integral in condition (c).

There is still a cheap estimate to guarantee for the pair $X_0 = L^p(\mathbb{R}^n)$ and $X_1 = W^{m,p}(\mathbb{R}^n)$ that the half-sequence $\left\{2^{-is}K(2^{im}; u)\right\}_{i=1}^{\infty}$ belongs to ℓ^q. This leaves the problem of suitably controlling the ℓ^q norm of the half-sequence $\left\{2^{i(m-s)}\|w_i\|_{1,p}\right\}_{i=-\infty}^{0}$. We can represent w_i as a sum of m terms, each involving an average, with an m-fold iterated integral, of translates of u in a fixed direction. We can use this representation to estimate the norms in $L^p(\mathbb{R}^n)$ of m-fold directional derivatives of w_i in any fixed direction. In particular, we can do this for the unmixed partial derivatives $D_j^m w_i$, in each case getting an L^p norm that we can control with the part of (c) corresponding to a suitable corona. It is known that L^p estimates for all unmixed derivatives of even order m imply similar estimate for all mixed mth-order derivatives derivatives, and thus for $|w_i|_{m,p}$. (See [St, p. 77]; this is the place where we need m to be even and $1 < p < \infty$.)

Finally, for $K(2^{im}; u)$ we also need estimates for $\|w_i\|_p$. Since w_i comes from averages of translates of u, these estimates take the form $\|w_i\|_p \leq C\|u\|_p$. For the half sequence $\{2^{-is}K(2^{im}; u)\}_{i=-\infty}^{0}$ we then need to multiply by 2^{im} and 2^{-is}; again the outcome is a finite ℓ^q norm, since $i \le 0$ and $m > s$. ∎

Other Scales of Intermediate Spaces

7.50 The Besov spaces are not the only scale of intermediate spaces that can fill the gap between Sobolev spaces of integer order. Several other such scales have been constructed, each slightly different from the others and each having properties making it useful in certain contexts. As we have seen, the Besov spaces are particularly useful for characterizing traces of functions in Sobolev spaces. However, except when $p = 2$, the Sobolev spaces do not actually belong to the scale of Besov spaces.

Two other scales we will introduce below are:

(a) the scale of fractional order Sobolev spaces (also called spaces of Bessel potentials), denoted $W^{s,p}(\Omega)$, which we will define for positive, real s by a complex interpolation method introduced below. It will turn out that if $s = m$, a positive integer and Ω is reasonable, then the space obtained coincides with the usual Sobolev space $W^{m,p}(\Omega)$.

(b) the scale of Triebel-Lizorkin spaces, $F^{s;p,q}(\mathbb{R}^n)$, which we will define only on $\mathbb{R}^n$ but which will provide a link between the Sobolev, Bessel potential, and Besov spaces, containing members of each of those scales for appropriate choices of the parameters s, p, and q.

We will use Fourier transforms to characterize both of the scales listed above, and will therefore normally work on the whole of $\mathbb{R}^n$. Some results can be extended to more general domains for which suitable extension operators exist.

For the rest of this chapter we will present only descriptive introductions to the topics considered and will eschew formal proofs, choosing to refer the reader to the available literature, e.g., [Tr1, Tr2, Tr3, Tr4], for more information. We particularly recommend the first chapter of [Tr4].

We begin by describing another interpolation method for Banach spaces; this one is based on properties of analytic functions in the complex plane.

7.51 (The Complex Interpolation Method) Let $\{X_0, X_1\}$ be an interpolation pair of complex Banach spaces defined as in Paragraph 7.7 so that $X_0 + X_1$ is a Banach space with norm

$$\|u\|_{X_0+X_1} = \inf\left\{\|u_0\|_{X_0} + \|u_1\|_{X_1} \ : \ u = u_0 + u_1, u_0 \in X_0, u_1 \in X_1\right\}.$$

Let $\mathscr{F} = \mathscr{F}(X_0, X_1)$ be the space of all functions f of the complex variable $\zeta = \theta + i\tau$ with values in $X_0 + X_1$ that satisfy the following conditions:

(a) f is continuous and bounded on the strip $0 \le \theta \le 1$ into $X_0 + X_1$.

(b) f is analytic from $0 < \theta < 1$ into $X_0 + X_1$ (i.e., the derivative $f'(\zeta)$ exists in $X_0 + X_1$ if $0 < \theta = \operatorname{Re}\zeta < 1$).

(c) f is continuous on the line $\theta = 0$ into X_0 and

$$\|f(i\tau)\|_{X_0} \to 0 \qquad \text{as} \qquad |\tau| \to \infty.$$

(d) f is continuous on the line $\theta = 1$ into X_1 and

$$\|f(1+i\tau)\|_{X_1} \to 0 \qquad \text{as} \qquad |\tau| \to \infty.$$

7.52 $\mathscr{F}$ is a Banach space with norm

$$\|f\,;\mathscr{F}\| = \max\left\{\sup_\tau \|f(i\tau)\|_{X_0}, \sup_\tau \|f(1+i\tau)\|_{X_1}\right\}.$$

Given a real number θ in the interval (0, 1), we define

$$X_\theta = [X_0, X_1]_\theta = \left\{u \in X_0 + X_1 \,:\, u = f(\theta) \quad \text{for some} \quad f \in \mathscr{F}\right\}.$$

X_θ is called a *complex interpolation space* between X_0 and X_1; it is a Banach space with norm

$$\|u\|_{X_\theta} = \|u\|_{[X_0,X_1]_\theta} = \inf\left\{\|f\,;\mathscr{F}\| \,:\, f(\theta) = u\right\}.$$

It follows from the above definitions that an analog of the Exact Interpolation Theorem (Theorem 7.23) holds for the complex interpolation method too. (See Calderón [Ca2, p. 115] and [BL, chapter 4].) If $\{X_0, X_1\}$ and $\{Y_0, Y_1\}$ are two interpolation pairs and T is a bounded linear operator from $X_0 + X_1$ into $Y_0 + Y_1$ such that T is bounded from X_0 into Y_0 with norm M_0 and from X_1 into Y_1 with norm M_1, then T is also bounded from X_θ into Y_θ with norm $M \le M_0^{1-\theta} M_1^\theta$ for each θ in the interval [0, 1].

There is also a version of the Reiteration Theorem 7.21 for complex interpolation; if $0 < \theta_0 < \theta_1 < 1$, $0 < \lambda < 1$, and $\theta = (1-\lambda)\theta_0 + \lambda\theta_1$, then

$$\left[[X_0, X_1]_{\theta_0}, [X_0, X_1]_{\theta_1}\right]_\lambda = [X_0, X_1]_\theta$$

with equivalent norms. This was originally proved under the assumption that $X_0 \cap X_1$ is dense in $[X_0, X_1]_{\theta_0} \cap [X_0, X_1]_{\theta_1}$, but this restriction was removed by Cwikel [Cw].

7.53 (Banach Lattices on Ω) Most of the Banach spaces considered in this book are spaces of (equivalence classes of almost everywhere equal) real-valued or complex-valued functions defined in a domain Ω in $\mathbb{R}^n$. Such a Banach space

B is called a *Banach lattice on* Ω if, whenever $u \in B$ and v is a measurable, real- or complex-valued function on Ω satisfying $|v(x)| \le |u(x)|$ a.e. on Ω, then $v \in B$ and $\|v\|_B \le \|u\|_B$. Evidently only function spaces whose norms depend only on the size of the function involved can be Banach lattices. The Lebesgue spaces $L^p(\Omega)$ and Lorentz spaces $L^{p,q}(\Omega)$ are Banach lattices on Ω, but Sobolev spaces $W^{m,p}(\Omega)$ (where $m \ge 1$) are not, since their norms also depend on the size of derivatives of their member functions.

We say that a Banach lattice B on Ω has the *dominated convergence property* if, whenever $u \in B$, $u_j \in B$ for $1 \le j < \infty$, and $|u_j(x)| \le |u(x)|$ a.e. in Ω, then

$$\lim_{j\to\infty} u_j(x) = 0 \text{ a.e.} \quad \Longrightarrow \quad \lim_{j\to\infty} \|u_j\|_B = 0.$$

The Lebesgue spaces $L^p(\Omega)$ and Lorentz spaces $L^{p,q}(\Omega)$ have this property provided both p and q are finite, but $L^\infty(\Omega)$, $L^{p,\infty}(\Omega)$, and $L^{\infty,q}(\Omega)$ do not. (As a counterexample for L^∞, consider a sequence of translates with non-overlapping supports of dilates of a nontrivial bounded function with bounded support.)

7.54 (The spaces $X_0^{1-\theta}X_1^\theta$) Now suppose that X_0 and X_1 are two Banach lattices on Ω and let $0 < \theta < 1$. We denote by $X_0^{1-\theta}X_1^\theta$ the collection of measurable functions u on Ω for each of which there exists a positive number λ and non-negative real-valued functions $u_0 \in X_0$ and $u_1 \in X_1$ such that $\|u_0\|_{X_0} = 1$, $\|u_1\|_{X_1} = 1$ and

$$|u(x)| \le \lambda u_0(x)^{1-\theta} u_1(x)^\theta. \tag{23}$$

Then $X_0^{1-\theta}X_1^\theta$ is a Banach lattice on Ω with respect to the norm

$$\|u\,;X_0^{1-\theta}X_1^\theta\| = \inf\{\lambda \,:\, \text{inequality (23) holds}\}.$$

The key result concerning the complex interpolation of Banach lattices is the following theorem of Calderón [Ca2, p.125] which characterizes the intermediate spaces.

7.55 THEOREM Let X_0 and X_1 be Banach lattices at least one of which has the dominated convergence property. If $0 < \theta < 1$, then

$$[X_0, X_1]_\theta = X_0^{1-\theta}X_1^\theta$$

with equality of norms. ∎

7.56 EXAMPLE It follows by factorization and Hölder's inequality that if $1 \le p_i \le \infty$ for $i = 0, 1$, $p_1 \ne p_2$, and $0 < \theta < 1$, then

$$\left[L^{p_0}(\Omega), L^{p_1}(\Omega)\right]_\theta = L^p(\Omega),$$

with equality of norms, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Moreover, if also $1 \le q_i \le \infty$ and at least one of the pairs (p_0, q_0) and (p_1, q_1) has finite components, then

$$\left[L^{p_0,q_0}(\Omega), L^{p_1,q_1}(\Omega)\right]_\theta = L^{p,q}(\Omega),$$

with equivalence of norms, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \text{and} \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

7.57 (Fractional Order Sobolev Spaces) We can define a scale of fractional order spaces by complex interpolation between L^p and Sobolev spaces. Specifically, if $s > 0$ and m is the smallest integer greater than s and Ω is a domain in $\mathbb{R}^n$, we define the space $W^{s,p}(\Omega)$ as

$$W^{s,p}(\Omega) = \left[L^p(\Omega), W^{m,p}(\Omega)\right]_{s/m}.$$

Again, as for Besov spaces, we can use the Reiteration Theorem to replace m with a larger integer and also observe that $W^{s,p}(\Omega)$ is an appropriate complex interpolation space between $W^{s_0,p}(\Omega)$ and $W^{s_1,p}(\Omega)$ if $0 < s_0 < s < s_1$. We will see later that if s is a positive integer and Ω has a suitable extension property, then $W^{s,p}(\Omega)$ coincides with the usual Sobolev space with the same name.

Because $W^{m,p}(\Omega)$ is not a Banach lattice on Ω we cannot use Theorem 7.55 to characterize $W^{s,p}(\Omega)$. Instead we will use properties of the Fourier transform on $\mathbb{R}^n$ for this purpose. Therefore, as we did for Besov spaces, we will normally work only with $W^{s,p}(\mathbb{R}^n)$, and rely on extension theorems to supply results for domains $\Omega \subset \mathbb{R}^n$.

We begin by reviewing some basic aspects of the Fourier transform.

7.58 (The Fourier Transform) The *Fourier transform* of a function u belonging to $L^1(\mathbb{R}^n)$ is the function $\hat{u}$ defined on $\mathbb{R}^n$ by

$$\hat{u}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot y} u(x)\, dx.$$

By dominated convergence the function $\hat{u}$ is continuous; moreover, we have $\|\hat{u}\|_\infty \le (2\pi)^{-n/2} \|u\|_1$. If $u \in C^1(\mathbb{R}^n)$ and both u and $D_j u$ belong to $L^1(\mathbb{R}^n)$, then $\widehat{D_j u}(y) = i y_j \hat{u}(y)$ by integration by parts. Similarly, if both u and the function mapping x to $|x|u(x)$ belong to $L^1(\mathbb{R}^n)$, then $\hat{u} \in C^1(\mathbb{R}^n)$; in this case $D_j\hat{u}(y)$ is the value at y of the Fourier transform of the function mapping x to $-ix_j u(x)$.

7.59 (The Space of Rapidly Decreasing Functions) Let $\mathscr{S} = \mathscr{S}(\mathbb{R}^n)$ denote the space of all functions u in $C^\infty(\mathbb{R}^n)$ such that for all multi-indices $\alpha \geq 0$ and $\beta \geq 0$ the function mapping x to $x^\alpha D^\beta u(x)$ is bounded on $\mathbb{R}^n$. Unlike functions in $\mathscr{D}(\mathbb{R}^n)$, functions in $\mathscr{S}$ need not have compact support; nevertheless, they must approach 0 at infinity faster than any rational function of x. For this reason the elements of $\mathscr{S}$ are usually called *rapidly decreasing functions.*

The properties of the Fourier transform mentioned above extend to verify the assertion that the Fourier transform of an element of $\mathscr{S}$ also belongs to $\mathscr{S}$.

The *inverse Fourier transform* $\check{u}$ of an element u of $L^1(\mathbb{R}^n)$ is defined for $x \in \mathbb{R}^n$ by

$$\check{u}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot y} u(y)\, dy.$$

The *Fourier inversion theorem* [RS, chapter 9] asserts that if $u \in \mathscr{S}$, then the inverse Fourier transform of $\hat{u}$ is u ($\check{\hat{u}} = u$), and, moreover, that the same conclusion holds under the weaker assumptions that $u \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ and $\hat{u} \in L^1(\mathbb{R}^n)$. One advantage of considering the Fourier transform on $\mathscr{S}$ is that $u \in \mathscr{S}$ guarantees that $\hat{u} \in L^1(\mathbb{R}^n)$, and the same is true for the function mapping y to $y^\alpha \hat{u}(y)$ for any multi-index $\alpha \geq 0$. In fact, the inverse Fourier transforms of functions in $\mathscr{S}$ also belong to $\mathscr{S}$ and the transform of the inverse transform also returns the original function. Thus the Fourier transform is a one-to-one mapping of $\mathscr{S}$ onto itself.

7.60 (The Space of Tempered Distributions) Given a linear functional F on the space $\mathscr{S}$, we can define another such functional $\hat{F}$ by requiring $\hat{F}(u) = F(\hat{u})$ for all $u \in \mathscr{S}$. Fubini's theorem shows that if F operates by integrating functions in $\mathscr{S}$ against a fixed integrable function f, then $\hat{F}$ operates by integrating against the transform $\hat{f}$:

$$\begin{aligned} F(u) &= \int_{\mathbb{R}^n} f(x)u(x)\, dx, \qquad f \in L^1(\mathbb{R}^n), \\ \Longrightarrow \qquad \hat{F}(v) &= \int_{\mathbb{R}^n} \hat{f}(y)v(y)\, dy. \end{aligned} \tag{24}$$

There exists a locally convex topology on $\mathscr{S}$ such that the mapping $F \to \hat{F}$ maps the dual space $\mathscr{S}' = \mathscr{S}'(\mathbb{R}^n)$ in a one-to-one way onto itself. The elements of this dual space $\mathscr{S}'$ are called *tempered distributions.* As was the case for $\mathscr{D}'(\Omega)$, not all tempered distributions can be represented by integration against functions.

7.61 (The Plancherel Theorem) An easy calculation shows that if u and v belong to $L^1(\mathbb{R}^n)$, then $\widehat{u * v} = (2\pi)^{n/2}\hat{u}\hat{v}$; Fourier transformation converts convolution products into pointwise products. If $u \in L^1(\mathbb{R}^n)$, let $\tilde{u}(x) = \overline{u(-x)}$. Then $\widehat{\tilde{u}} = \overline{\hat{u}}$, and $\widehat{u * \tilde{u}} = (2\pi)^{n/2}|\hat{u}|^2$. If $u \in \mathscr{S}$, then both $u * \tilde{u}$ and $|\hat{u}|^2$ also belong to $\mathscr{S}$. Applying the Fourier inversion theorem to $u * \tilde{u}$ at $x = 0$ then gives the following result, known as *Plancherel's Theorem.*

$$u \in \mathscr{S} \qquad \Longrightarrow \qquad \|\hat{u}\|_2^2 = \|u\|_2^2 .$$

That is, the Fourier transform maps the space $\mathscr{S}$ equipped with the L^2-norm isometrically onto itself. Since $\mathscr{S}$ is dense in $L^2(\mathbb{R}^n)$, the isometry extends to one mapping $L^2(\mathbb{R}^n)$ onto itself. Also, $L^2(\mathbb{R}^n) \subset \mathscr{S}'$ and the distributional Fourier transforms of an L^2 function is the same L^2 function as defined by the above isometry. (That is, the Fourier transform of an element of $\mathscr{S}'$ that operates by integration against L^2 functions as in (24) does itself operate in that way.)

7.62 (Characterization of $W^{s,2}(\mathbb{R}^n)$) Given $u \in L^2(\mathbb{R}^n)$ and any positive integer m, let

$$u_m(y) = (1 + |y|^2)^{m/2}\hat{u}(y). \tag{25}$$

It is easy to verify that $u \in W^{m,2}(\mathbb{R}^n)$ if and only if u_m belongs to $L^2(\mathbb{R}^n)$, and the L^2-norm of u_m is equivalent to the $W^{m,2}$-norm of u. So the Fourier transform identifies $W^{m,2}(\mathbb{R}^n)$ with the Banach lattice of functions w for which $(1 + |\cdot|^2)^{m/2}\hat{w}(\cdot)$ belongs to $L^2(\mathbb{R}^n)$. For each positive integer m that lattice has the dominated convergence property. It follows that $u \in W^{s,2}(\mathbb{R}^n)$ if and only if $(1 + |\cdot|^2)^{s/2}\hat{u}(\cdot)$ belongs to $L^2(\mathbb{R}^n)$.

7.63 (Characterization of $W^{s,p}(\mathbb{R}^n)$) The description of $W^{s,p}(\mathbb{R}^n)$ when $1 < p < 2$ or $2 < p < \infty$ is more complicated. If $u \in L^p(\mathbb{R}^n)$ with $1 < p < 2$, then $u \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$; this guarantees that $\hat{u} \in L^\infty(\mathbb{R}^n) + L^2(\mathbb{R}^n)$, and in particular that the distribution $\hat{u}$ is a function. Moreover, it follows by complex interpolation that $\hat{u} \in L^{p'}(\mathbb{R}^n)$ and by real interpolation that $\hat{u} \in L^{p',p}(\mathbb{R}^n)$. But the set of such transforms of L^p functions is not a lattice when $1 < p < 2$. This follows from the fact (see [FG]) that every set of positive measure contains a subset E of positive measure so that if the Fourier transform of an L^p function, where $1 < p < 2$, vanishes off E, then the function must be 0. If $u \in L^p(\mathbb{R}^n)$ and E is such a subset on which $\hat{u}(y) \neq 0$, then the function that equals $\hat{u}$ on E and 0 off E is not trivial but would have to be trivial if the set of Fourier transforms of L^p functions were a lattice.

A duality argument shows that the set of (distributional) Fourier transforms of functions in $L^p(\mathbb{R}^n)$ for $p > 2$ cannot be a Banach lattice either. Moreover (see [Sz]), there are functions in $L^p(\mathbb{R}^n)$ whose transforms are not even functions.

Nevertheless, the product of any tempered distribution and any sufficiently smooth function that has at most polynomial growth is always defined. For any distribution $u \in \mathscr{S}'$ we can then define the distribution u_m by analogy with formula (25); we multiply the tempered distribution $\hat{u}$ by the smooth function $(1 + |\cdot|^2)^{m/2}$. When $1 < p < 2$ or $2 < p < \infty$, the theory of singular integrals [St, p. 138] then shows that $u \in W^{m,p}(\mathbb{R}^n)$ if and only if the function u_m is the Fourier transform of some function in $L^p(\mathbb{R}^n)$. Again the space $W^{s,p}(\mathbb{R}^n)$ is characterized by the version of this condition with m replaced by s. In particular, if $s = m$ we obtain the usual Sobolev space $W^{m,p}(\mathbb{R}^n)$ up to equivalence of norms, when $1 < p < \infty$. The fractional order Sobolev spaces spaces are natural generalizations of the Sobolev spaces that allow for fractional orders of smoothness.

One can pass between spaces $W^{s,p}(\mathbb{R}^n)$ having the same index p but different orders of smoothness s by multiplying or dividing Fourier transforms by factors of the form $(1 + |\cdot|^2)^{-r/2}$. When $r > 0$ these radial factors are constant multiples of Fourier transforms of certain Bessel functions; for this reason the spaces $W^{s,p}(\mathbb{R}^n)$ are often called *spaces of Bessel potentials.* (See [AMS].)

In order to show the relationship between the fractional order Sobolev spaces and the Besov spaces, it is, however, more useful to refine the scale of spaces $W^{s,p}(\mathbb{R}^n)$ using a dyadic splitting of the Fourier transform.

7.64 (An Alternate Characterization of $W^{s,p}(\mathbb{R}^n)$) In proving the Trace Theorem 7.39 we used a splitting of a function in $W^{m,p}(\mathbb{R}^{n+1})$ into dyadic pieces supported in slabs parallel to the subspace $\mathbb{R}^n$ of the traces. Here we are going to use a similar splitting of the Fourier transform of an L^p function into dyadic pieces supported between concentric spheres.

Recall the C^∞ function ϕ_i defined in the proof of Lemma 7.40 and having support in the interval $(2^i, 2^{i+2})$. For each integer i and y in $\mathbb{R}^n$, let $\psi_i(y) = \phi_i(|y|)$. Each of these radially symmetric functions belongs to $\mathscr{S}$ and so has an inverse transform, Ψ_i say, that also belongs to $\mathscr{S}$.

Fix an index p in the interval $(1, \infty)$ and let $u \in L^p(\mathbb{R}^n)$. For each integer i let $T_i u$ be the convolution of u with $(2\pi)^{-n/2}\Psi_i$; thus $\widehat{T_i u}(y) = \psi_i(y) \cdot \hat{u}(y)$. One can regard the functions $T_i u$ as dyadic parts of u with nearly disjoint frequencies. Littlewood-Paley theory [FJW] shows that the L^p-norm of u is equivalent to the L^p-norm of the function mapping x to $[\sum_{i=-\infty}^{\infty} |T_i u(x)|^2]^{1/2}$. That is

$$\|u\|_p \approx \left(\int_{R^n} \left[\sum_{i=-\infty}^{\infty} |T_i u(x)|^2 \right]^{p/2} dx \right)^{1/p}.$$

To estimate the norm of u in $W^{s,p}(\mathbb{R}^n)$ we should replace each term $T_i u$ by the function obtained by not only multiplying $\hat{u}$ by ψ_i, but also multiplying the transform by the function mapping y to $(1 + |y|^2)^{s/2}$.

On the support of ψ_i the values of that second Fourier multiplier are all roughly equal to $1 + 2^{si}$. It turns out that $u \in W^{s,p}(\mathbb{R}^n)$ if and only if

$$\|u\|_{s,p} = \left\| u\,;\, W^{s,p}(\mathbb{R}^n)\right\| \approx \left(\int_{R^n} \left[\sum_{i=-\infty}^{\infty} (1+2^{si})^2 |T_i u(x)|^2 \right]^{p/2} dx \right)^{1/p} < \infty.$$

This is a complicated but intrinsic characterization of the space $W^{s,p}(\mathbb{R}^n)$. That is, the following steps provide a recipe for determining whether an L^p function u belongs to $W^{s,p}(\mathbb{R}^n)$:

(a) Split u into the the pieces $T_i u$ by convolving with the functions Ψ_i or by multiplying the distribution $\hat{u}$ by ψ_i and then taking the inverse transform. For each point x in $\mathbb{R}^n$ this gives a sequence $\{T_i u(x)\}$.

(b) Multiply the i-th term in that sequence by $(1+2^{si})$ and compute the ℓ^2-norm of the result. This gives a function of x.

(c) Compute the L^p-norm of that function.

The steps in this recipe can be modified to produce other scales of spaces.

7.65 (The Triebel-Lizorkin Spaces) Define $F^{s;p,q}(\mathbb{R}^n)$ to be the space obtained by using steps (a) to (c) above but taking an ℓ^q-norm rather than an ℓ^2-norm in step (b). This gives the family of Triebel-Lizorkin spaces; if $1 \le q < \infty$,

$$\left\| u\,;\, F^{s;p,q}(\mathbb{R}^n)\right\| \approx \left(\int_{R^n} \left[\sum_{i=-\infty}^{\infty} (1+2^{si})^q |T_i u(x)|^q \right]^{p/q} dx \right)^{1/p} < \infty.$$

Note that $F^{m;p,2}(\mathbb{R}^n)$ coincides with $W^{m,p}(\mathbb{R}^n)$ when m is a positive integer, and $F^{s;p,2}(\mathbb{R}^n)$ coincides with $W^{s,p}(\mathbb{R}^n)$ when s is positive.

7.66 REMARKS

1. The space $F^{0;p,2}(\mathbb{R}^n)$ coincides with $L^p(\mathbb{R}^n)$ when $1 < p < \infty$.
2. The definitions of $W^{s,p}(\mathbb{R}^n)$ and $F^{s;p,q}(\mathbb{R}^n)$ also make sense if $s < 0$, and even if $0 < p, q < 1$. However they may contain distributions that are not functions if $s < 0$, and they will not be Banach spaces unless $p \ge 1$ and $1 \le q < \infty$.
3. If $s > 0$, the recipes for characterizing $W^{s,p}(\mathbb{R}^n)$ and $F^{s;p,q}(\mathbb{R}^n)$ given above can be modified to replace the multiplier $(1+2^{si})$ by 2^{si} and restricting the summations in the ℓ^2 or ℓ^p norm expressions to $i \ge 0$, provided we also explicitly require $u \in L^p(\mathbb{R}^n)$. Thus, for example, $u \in F^{s;p,q}(\mathbb{R}^n)$ if and only if

$$\|u\|_p + \left(\int_{R^n} \left[\sum_{i=0}^{\infty} 2^{siq} |T_i u(x)|^q \right]^{p/q} dx \right)^{1/p} < \infty.$$

4. If $s > 0$ and we modify the recipe for $F^{s;p,q}(\mathbb{R}^n)$ by replacing the multiplier $(1+2^{si})$ by 2^{si} but continuing to take the summation over all integers i, then we obtain the so-called *homogeneous Triebel-Lizorkin space* $\dot{F}^{s;p,q}(\mathbb{R}^n)$ which contain equivalence classes of distributions modulo polynomials of low enough degree. Only smoothness and not size determines whether a function belongs to this homogeneous space.

7.67 (An Alternate Definition of the Besov Spaces) It turns out that the Besov spaces $B^{s;p,q}(\mathbb{R}^n)$ arises from the variant of the recipe given in Paragraph 7.64 where the last two steps are modified as follows.

(b') Multiply the i-th term in the sequence $\{T_i u(x)\}$ by $(1 + 2^{si})$ and compute the L^p-norm of the result. This gives a sequence of nonnegative numbers.

(c') Compute the ℓ^q-norm of that sequence.

$$\left\| u \,;\, B^{s;p,q}(\mathbb{R}^n) \right\| \approx \left[\sum_{i=-\infty}^{\infty} \left(\int_{R^n} (1 + 2^{si})^p |T_i u(x)|^p \, dx \right)^{q/p} \right]^{1/q}.$$

This amounts to reversing the order in which the two norms are computed. That order does not matter when $q = p$; thus $B^{s;p,p}(\mathbb{R}^n) = F^{s;p,p}(\mathbb{R}^n)$ with equivalent norms. When $q \neq p$, Minkowski's inequality for sums and integrals reveals that in comparing the outcomes of steps (b) and (c), the larger norm and the smaller space of functions arises when the larger of the indices p and q is used first. That is,

$$\begin{cases} F^{s;p,q}(\mathbb{R}^n) \subset B^{s;p,q}(\mathbb{R}^n) & \text{if } q < p \\ B^{s;p,q}(\mathbb{R}^n) \subset F^{s;p,q}(\mathbb{R}^n) & \text{if } q > p. \end{cases}$$

For fixed s and p the inclusions between the Besov spaces $B^{s;p,q}(\mathbb{R}^n)$ are the same as those between ℓ^q spaces, and the same is true for the Triebel-Lizorkin spaces $F^{s;p,q}(\mathbb{R}^n)$.

Finally, the only link with the scale of fractional order Sobolev spaces and in particular with the Sobolev spaces occurs through the Triebel-Lizorkin scale with $q = 2$. We have

$$\begin{cases} W^{s,p}(\mathbb{R}^n) = F^{s;p,2}(\mathbb{R}^n) \subset F^{s;p,q}(\mathbb{R}^n) & \text{if } q \geq 2 \\ F^{s;p,q}(\mathbb{R}^n) \subset F^{s;p,2}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n) & \text{if } q \leq 2. \end{cases}$$

As another example, the trace of $W^{m,p}(\mathbb{R}^{n+1})$ on $\mathbb{R}^n$ is exactly the space $B^{m-1/p;p,p}(\mathbb{R}^n) = F^{m-1/p;p,p}(\mathbb{R}^n)$. When $p \leq 2$, this trace space is included in the space $F^{m-1/p;p,2}(\mathbb{R}^n)$ and thus in the space $W^{m-1/p,p}(\mathbb{R}^n)$. When $p \geq 2$, this inclusion is reversed.

7.68 REMARKS

1. Appropriate versions of Remarks 7.66 for the Triebel-Lizorkin spaces apply to the above characterization of the Besov spaces too. In particular, modifying recipe item (b') to use the multiplier 2^{si} instead of $1 + 2^{si}$ results in a *homogeneous Besov space* $\dot{B}^{s;p,q}(\mathbb{R}^n)$ of equivalence classes of distributions modulo certain polynomials. Again membership in this space depends only on smoothness and not on size.
2. The K-version of the definition of $B^{s;p,q}(\mathbb{R}^n)$ as an intermediate space obtained by the real method is a condition on how well $u \in L^p(\mathbb{R}^n)$ can be approximated by functions in $W^{m,p}(\mathbb{R}^n)$ for some integer $m > s$. But the J-form of the definition requires a splitting of u into pieces u_i with suitable control on the norms of the functions u_i in the spaces $L^p(\mathbb{R}^n)$ and $W^{m,p}(\mathbb{R}^n)$. The Fourier splitting also gives us pieces $T_i u$ for which we can control those two norms, and these can serve as the pieces u_i. Conversely, if we have pieces u_i with suitable control on appropriate norms, and it we apply Fourier decomposition to each piece, we would find that the norms $\|T_j u_i\|_p$ are negligible when $|j-i|$ is large, leading to appropriate estimates for the norms $\|T_j u\|_p$.

7.69 (Extensions for General Domains) Many of the properties of the scales of Besov spaces, spaces of Bessel potentials, and Triebel-Lizorkin spaces on $\mathbb{R}^n$ can be extended to more general domains Ω via the use of an extension operator. Rychkov [Ry] has constructed a linear total extension operator $\mathscr{E}$ that simultaneously and boundedly extends functions in $F^{s;p,q}(\Omega)$ to $F^{s;p,q}(\mathbb{R}^n)$ and functions in $B^{s;p,q}(\Omega)$ to $B^{s;p,q}(\mathbb{R}^n)$ provided Ω satisfies a strong local Lipschitz condition. The same operator $\mathscr{E}$ works for both scales, all real s, and all $p > 0$, $q > 0$; it is an extension operator in the sense that $\mathscr{E}u|_\Omega = u$ in $\mathscr{D}'(\Omega)$ for every u in any of the Besov or Triebel-Lizorkin spaces defined on Ω as restrictions in the sense of $\mathscr{D}'(\Omega)$ of functions in the corresponding spaces on $\mathbb{R}^n$.

The existence of this operator provides, for example, an intrinsic characterization of $B^{s;p,q}(\Omega)$ in terms of that for $B^{s;p,q}(\mathbb{R}^n)$ obtained in Theorem 7.47.

Wavelet Characterizations

7.70 We have seen above how membership of a function u in a space $B^{s;p,q}(\mathbb{R}^n)$ can be determined by the size of the sequence of norms $\|T_i u\|_p$, while its membership in the space $F^{s;p,q}(\mathbb{R}^n)$ requires pointwise information about the sizes of the functions $T_i u$ on $\mathbb{R}^n$. Both characterizations use the functions $T_i u$ of a dyadic decomposition of u defined as inverse Fourier transforms of products of $\hat{u}$ with dilates of a suitable smooth function ϕ. We conclude this chapter by describing how further refining these decompositions to the level of wavelets reduces questions about membership of u in these smoothness classes to questions about the sizes of the (scalar) coefficients of u in such decompositions. These coefficients do form a Banach lattice.

This contrasts dramatically with the situation for Fourier transforms of L^p functions with $1 < p < 2$, where these transforms fail to form a lattice.

7.71 (Wavelet Analysis) An *analyzing wavelet* is a nontrivial function on $\mathbb{R}^n$ satisfying some decay conditions, some cancellation conditions, and some smoothness conditions. Different versions of these conditions are appropriate in different contexts. Two classical examples of wavelets on $\mathbb{R}$ are the following:

(a) The basic *Haar* function h given by

$$h(x) = \begin{cases} 1 & \text{if} \le x < 1/2 \\ -1 & \text{if } 1/2 \le x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(b) A basic Shannon wavelet S defined as the inverse Fourier transform of the function $\hat{S}$ satisfying

$$\hat{S}(y) = \begin{cases} 1 & \text{if } \pi \le |y| < 2\pi \\ 0 & \text{otherwise.} \end{cases}$$

The Haar wavelet has compact support, and *a fortiori* decays extremely rapidly. The only cancellation condition it satisfies is that $\int_{\mathbb{R}} ch(x)\, dx = 0$ for all constants c. It fails to be smooth, but compensates for that by taking only two nonzero values and thus being simple to use numerically.

The Shannon wavelet does *not* have compact support; instead it decays like $1/|x|$, that is, at a fairly slow rate. However, it has very good cancellation properties, since

$$\int_{\mathbb{R}} x^m S(x)\, dx = 0 \quad \text{for all nonnegative integers } m.$$

(These integrals are equal to constants times the values at $y = 0$ of derivatives of $\hat{S}(y)$. Since $\hat{S}$ vanishes in a neighbourhood of 0, those derivatives all vanish at 0.) Also, $S \in C^\infty(\mathbb{R})$ and even extends to an entire function on the complex plane.

To get a better balance between these conditions, we will invert the roles of function and Fourier transform from the previous section, and use below a wavelet ϕ defined on $\mathbb{R}^n$ as the inverse Fourier transform of a nontrivial smooth function that vanishes outside the annulus where $1/2 < |y| < 2$. Then ϕ has all the cancellation properties of the Shannon wavelet, for the same reasons. Also ϕ decays very rapidly because $\hat{\phi}$ is smooth, and ϕ is smooth because $\hat{\phi}$ decays rapidly. Again the compact support of $\hat{\phi}$ makes ϕ the restriction of an entire function.

Given an analyzing wavelet, w say, we consider some or all of its translates mapping x to $w(x - h)$ and some or all of its (translated) dilates, mapping x to

$w(2^r x - h)$. These too are often called wavelets. Translation preserves L^p norms; dilation does *not* do so, except when $p = \infty$; however, we will use the multiple $2^{rn/2} w(2^r x - h)$ to preserve L^2 norms.

If we apply the same operations to the complex exponential that maps x to e^{ixy} on $\mathbb{R}$, we find that dilation produces other such exponentials, but that translation just multiplies the exponential by a complex constant and so does not produce anything really new. In contrast, the translates of the basic Haar wavelet by integer amounts have disjoint supports and so are orthogonal in $L^2(\mathbb{R})$. A less obvious fact is that translating the Shannon wavelet by integers yields orthogonal functions, this time without disjoint supports.

In both cases, dilating by factors 2^i, where i is an integer, yields other wavelets that are orthogonal to their translates by 2^i times integers, and these wavelets are orthogonal to those in the same family at other dyadic scales. Moreover, in both examples, this gives an orthogonal basis for $L^2(\mathbb{R})$.

Less of this orthogonality persists for wavelets like the one we called ϕ above. But it can still pay to consider the *wavelet transform* of a given function u which maps positive numbers a and vectors h in $\mathbb{R}^n$ to

$$\frac{1}{\sqrt{a^n}} \int_{\mathbb{R}^n} u(x) \phi\left(\frac{x-h}{a}\right) dx.$$

For our purposes it will suffice to consider only those dilations and translates mapping x to $\phi_{i,k}(x) = 2^{in/2}\phi(2^i x - k)$, where i runs through the set of integers, and k runs through the integer lattice in $\mathbb{R}^n$. Integrating u against such wavelets yields *wavelet coefficients* that we can index by the pairs (i, k) and use to characterize membership of u in various spaces.

For much more on wavelets, see [Db].

7.72 (Wavelet Characterization of Besov Spaces) Let ϕ be a function in $\mathscr{S}$ whose Fourier transform $\hat{\phi}$ satisfies the following two conditions:

(i) $\hat{\phi}(y) = 0$ if $|y| < 1/2$ or $|y| > 2$.

(ii) $|\hat{\phi}(y)| > c > 0$ if $3/5 < |y| < 5/3$.

Note that the conditions on $\hat{\phi}$ imply that

$$\int_{\mathbb{R}^n} P(x)\phi(x)\, dx = 0$$

for any polynomial P.

Also, it can be shown (see [FJW, p. 54]) that there exists a dual function $\psi \in \mathscr{S}$ satisfying the same conditions (i) and (ii) and such that

$$\sum_{i=-\infty}^{\infty} \overline{\hat{\phi}(2^{-i}y)}\, \hat{\psi}(2^{-i}y)\, dy = 1 \quad \text{for all } y \neq 0.$$

Let $\mathbb{Z}$ denote the set of all integers. For each $i \in \mathbb{Z}$ and each n-tuple $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ we define two wavelet families by using dyadic dilates and translates of ϕ and ψ:

$$\phi_{i,k}(x) = 2^{-in/2}\phi(2^{-i}x - k) \qquad \text{and} \qquad \psi_{i,k}(x) = 2^{in/2}\psi(2^{i}x - k).$$

Note that the dilations in these two families are in opposite directions and that $\phi_{i,k}$ and $\psi_{i,k}$ have the same L^2 norms as do ϕ and ψ respectively. Moreover, for any polynomial P,

$$\int_{\mathbb{R}^n} P(x)\phi_{i,k}(x)\,dx = 0.$$

Let I denote the set of all indices (i, k) such that $i \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, and let $\mathscr{F}$ denote the wavelet family $\{\phi_{i,k} : (i, k) \in I\}$. Given a locally integrable function u, we define its wavelet coefficients $c_{i,k}(u)$ with respect to the family $\mathscr{F}$ by

$$c_{i,k}(u) = \int_{\mathbb{R}^n} u(x)\overline{\phi_{i,k}(x)}\,dx,$$

and consider the wavelet series representation

$$u = \sum_{(i,k)\in I} c_{i,k}(u)\psi_{i,k}. \tag{26}$$

The series represents u modulo polynomials as all its terms vanish if u is a polynomial.

It turns out that u belongs to the homogeneous Besov space $\dot{B}^{s;p,q}(\mathbb{R}^n)$ if and only if its coefficients $\{c_{i,k}(u) : (i, k) \in I\}$ belong to the Banach lattice on I having norm

$$\left(\sum_{i=-\infty}^{\infty}\left[2^{i(s+n[1/2-1/p])}\sum_{k\in\mathbb{Z}^n}|c_{i,k}|^p\right]^{q/p}\right)^{1/q}. \tag{27}$$

The condition for membership in the ordinary Besov space $B^{s;p,q}(\mathbb{R}^n)$ is a bit more complicated. We use only the part of the wavelet series (26) with $i \geq 0$ and replace the rest with a new series

$$\sum_{k\in\mathbb{Z}^n} c_k(u)\Psi_k,$$

where $\Psi_k(x) = \Psi(x - k)$ and Ψ is a function in $\mathscr{S}$ satisfying the conditions $\hat{\Psi}(y) = 0$ if $|y| \geq 1$ and $|\hat{\Psi}(y)| > c > 0$ if $|y| \leq 5/6$. Again there is a dual such function Φ with the same properties such that the coefficients $c_k(u)$ are given by

$$c_k(u) = \int_{\mathbb{R}^n} u(x)\overline{\Phi_k(x)}\,dx.$$

We have $u \in B^{s;p,q}(\mathbb{R}^n)$ if and only if the expression

$$\left(\sum_{k\in\mathbb{Z}^n} |c_k|^p\right)^{1/p} + \left(\sum_{i=0}^{\infty}\left[2^{i(s+n[1/2-1/p])}\sum_{k\in\mathbb{Z}^n}|c_{i,k}|^p\right]^{q/p}\right)^{1/q} \tag{28}$$

is finite, and this expression provides an equivalent norm for $B^{s;p,q}(\mathbb{R}^n)$.

Note that, in expressions (27) and (28) the part of the recipe in item 7.66 involving the computation of an L^p-norm seems to have disappeared. In fact, however, for any fixed value of the index i, the wavelet "coefficients" $c_{i,k}$ are actually values of the convolution $u * \phi_{i,0}$ taken at points in the discrete lattice $\{2^i k\}$, where the index i is fixed but k varies. This lattice turns out to be fine enough that the L^p-norm of $u * \phi_{i,0}$ is equivalent to the ℓ^p-norm over this lattice of the values of $u * \phi_{i,0}$.

7.73 (Wavelet Characterization of Triebel-Lizorkin Spaces) Membership in the homogeneous Triebel-Lizorkin space $\dot{F}^{s;p,q}(\mathbb{R}^n)$ is also characterized by a condition where only the sizes of the coefficients $c_{i,k}$ matter, namely the finiteness of

$$\left\|\left(\sum_{i=-\infty}^{\infty}\left[2^{i(s+n/2)}\sum_{k\in\mathbb{Z}}|c_{i,k}|\chi_{i,k}\right]^q\right)^{1/q}\right\|_{p,\mathbb{R}^n}.$$

where $\chi_{i,k}$ is the characteristic function of the cube $2^i k_j \le x_j < 2^i(k_j+)$, $(1 \le j \le n)$. At any point x in $\mathbb{R}^n$ the inner sum above collapses as follows. For each value of the index i the point x belongs to the cube corresponding to i and k for only one value of k, say $k_i(x)$. This reduces matters to the finiteness of

$$\left\|\left(\sum_{i=-\infty}^{\infty}\left[2^{i(s+n/2)}\,|c_{i,k_i(\cdot)}|\right]^q\right)^{1/q}\right\|_{p,\mathbb{R}^n}.$$

We refer to section 12 in [FJ] for information on how to deal in a similar way with the inhomogeneous space $F^{s;p,q}(\mathbb{R}^n)$.

Recall that in the discrete version of the J-method, the pieces u_i in suitable splittings of u are not unique. This flexibility sometimes simplified our analysis, for instance in the proofs of (trace) imbeddings for Besov spaces. The same is true for the related idea of *atomic decomposition*, for which we refer to [FJW] and [FJ] for sharper results and much more information.

8

ORLICZ SPACES AND ORLICZ-SOBOLEV SPACES

Introduction

8.1 In this final chapter we present results on generalizations of Lebesgue and Sobolev spaces in which the role usually played by the convex function t^p is assumed by a more general convex function $A(t)$. The spaces $L_A(\Omega)$, called *Orlicz spaces*, are studied in depth in the monograph by Krasnosel'skii and Rutickii [KR] and also in the doctoral thesis by Luxemburg [Lu], to either of which the reader is referred for a more complete development of the material outlined below. The former also contains examples of applications of Orlicz spaces to certain problems in nonlinear analysis.

It is of some interest to note that a gap in the Sobolev imbedding theorem (Theorem 4.12) can be filled by an Orlicz space. Specifically, if $mp = n$ and $p > 1$, then for suitably regular Ω we have

$$W^{m,p}(\Omega) \to L^q(\Omega), \quad p \le q < \infty, \qquad \text{but} \qquad W^{m,p}(\Omega) \not\to L^\infty(\Omega);$$

there is no *best*, (i.e., smallest) target L^p-space for the imbedding. In Theorem 8.27 below we will provide an optimal imbedding of $W^{m,p}(\Omega)$ into a certain Orlicz space. This result is due to Trudinger [Td], with precedents in [Ju] and [Pz]. There has been much further work, for instance [Ms] and [Ad1].

Following [KR], we use the class of "N-functions" as defining functions A for Orlicz spaces. This class is not as wide as the class of Young's functions used in

[Lu]; for instance, it excludes $L^1(\Omega)$ and $L^\infty(\Omega)$ from the class of Orlicz spaces. However, N-functions are simpler to deal with, and are adequate for our purposes. Only once, in Theorem 8.39 below, do we need to refer to a more general Young's function.

If the role played by $L^p(\Omega)$ in the definition of the Sobolev space $W^{m,p}(\Omega)$ is assigned instead to the Orlicz space $L_A(\Omega)$, the resulting space is denoted by $W^m L_A(\Omega)$ and is called an *Orlicz-Sobolev space*. Many properties of Sobolev spaces have been extended to Orlicz-Sobolev spaces by Donaldson and Trudinger [DT]. We present some of these results in this chapter.

N-Functions

8.2 (Definition of an N-Function) Let a be a real-valued function defined on $[0, \infty)$ and having the following properties:

(a) $a(0) = 0$, $a(t) > 0$ if $t > 0$, $\lim_{t\to\infty} a(t) = \infty$;

(b) a is nondecreasing, that is, $s > t$ implies $a(s) \geq a(t)$;

(c) a is right continuous, that is, if $t \geq 0$, then $\lim_{s\to t+} a(s) = a(t)$.

Then the real-valued function A defined on $[0, \infty)$ by

$$A(t) = \int_0^t a(\tau)\, d\tau \tag{1}$$

is called an *N-function.*

It is not difficult to verify that any such N-function A has the following properties:

(i) A is continuous on $[0, \infty)$;

(ii) A is strictly increasing that is, $s > t \geq 0$ implies $A(s) > A(t)$;

(iii) A is convex, that is, if $s, t \geq 0$ and $o < \lambda < 1$, then

$$A\big(\lambda s + (1-\lambda)t\big) \leq \lambda A(s) + (1-\lambda)A(t);$$

(iv) $\lim_{t\to 0} A(t)/t = 0$, and $\lim_{t\to\infty} A(t)/t = \infty$;

(v) if $s > t > 0$, then $A(s)/s > A(t)/t$.

Properties (i), (iii), and (iv) could have been used to define an N-function since they imply the existence of a representation of A in the form (1) with a having the properties (a)–(c).

The following are examples of N-functions:

$$\begin{aligned}
&A(t) = t^p, && 1 < p < \infty,\\
&A(t) = e^t - t - 1, && \\
&A(t) = e^{(t^p)} - 1, && 1 < p < \infty,\\
&A(t) = (1+t)\log(1+t) - t. &&
\end{aligned}$$

Evidently, $A(t)$ is represented by the area under the graph $\sigma = a(\tau)$ from $\tau = 0$ to $\tau = t$ as shown in Figure 9. Rectilinear segments in the graph of A correspond to intervals on which a is constant, and angular points on the graph of A correspond to discontinuities (i.e., vertical jumps) in the graph of a.

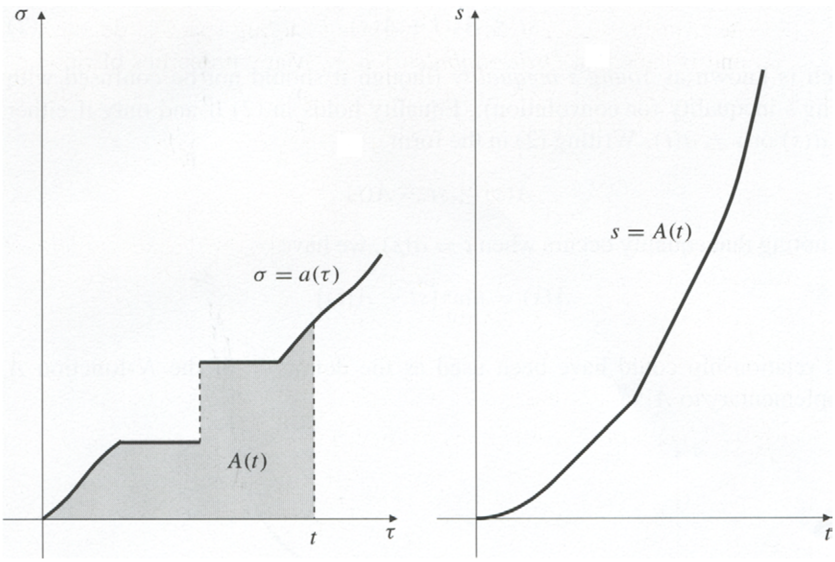


Fig. 9

8.3 (Complementary N-Functions) Given a function a satisfying conditions (a)–(c) of the previous Paragraph, we define

$$\tilde{a}(s) = \sup_{a(t) \le s} t.$$

It is readily checked that the function $\tilde{a}$ so defined also satisfies (a)–(c) and that a can be recovered from $\tilde{a}$ via

$$a(t) = \sup_{\tilde{a}(s) \le t} s.$$

If a is strictly increasing then $\tilde{a} = a^{-1}$. The N-functions A and $\tilde{A}$ given by

$$A(t) = \int_0^t a(\tau)\, d\tau, \qquad \tilde{A}(s) = \int_0^s \tilde{a}(\sigma)\, d\sigma$$

are said to be *complementary*; each is the *complement* of the other. Examples of such complementary pairs are:

$$A(t) = \frac{t^p}{p}, \qquad \tilde{A}(s) = \frac{s^{p'}}{p'}, \qquad 1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

and

$$A(t) = e^t - t - 1, \qquad \tilde{A}(s) = (1+s)\log(1+s) - s.$$

$\tilde{A}(s)$ is represented by the area to the left of the graph $\sigma = a(\tau)$ (or, more correctly, $\tau = \tilde{a}(\sigma)$) from $\sigma = 0$ to $\sigma = s$ as shown in Figure 10. Evidently, we have

$$st \le A(t) + \tilde{A}(s), \tag{2}$$

which is known as *Young's inequality* (though it should not be confused with Young's inequality for convolution). Equality holds in (2) if and only if either $t = \tilde{a}(s)$ or $s = a(t)$. Writing (2) in the form

$$\tilde{A}(s) \ge st - A(t)$$

and noting that equality occurs when $t = \tilde{a}(s)$, we have

$$\tilde{A}(s) = \max_{t \ge 0}\big(st - A(t)\big).$$

This relationship could have been used as the definition of the N-function $\tilde{A}$ complementary to A.

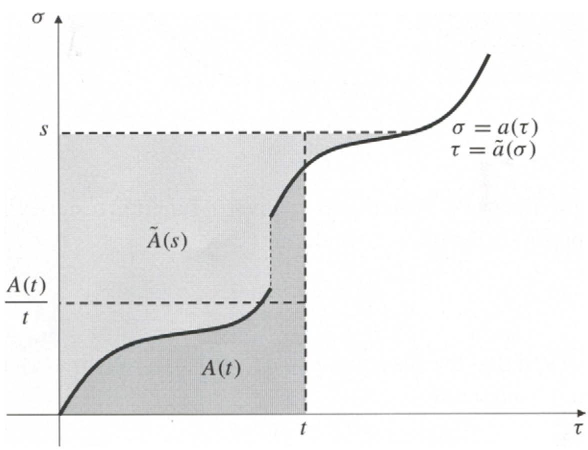


Fig. 10

Since A and $\tilde{A}$ are strictly increasing, they have inverses and (2) implies that for every $t \ge 0$

$$A^{-1}(t)\tilde{A}^{-1}(t) \le A\big(A^{-1}(t)\big) + \tilde{A}\big(\tilde{A}^{-1}(t)\big) = 2t.$$

On the other hand, $A(t) \le ta(t)$, so that, considering Figure 10 again, we have for every $t > 0$,

$$\tilde{A}\left(\frac{A(t)}{t}\right) \le \frac{A(t)}{t}\,t = A(t). \tag{3}$$

Replacing $A(t)$ by t in inequality (3), we obtain

$$\tilde{A}\left(\frac{t}{A^{-1}(t)}\right) < t.$$

Therefore, for any $t > 0$,

$$t < A^{-1}(t)\tilde{A}^{-1}(t) \le 2t. \tag{4}$$

8.4 (Dominance and Equivalence of N-Functions) We shall require certain partial ordering relationships among N-functions. If A and B are two N-functions, we say that *B dominates A globally* if there exists a positive constant k such that

$$A(t) \le B(kt) \tag{5}$$

holds for all $t \ge 0$. Similarly, *B dominates A near infinity* if there exist positive constants t_0 and k such that (5) holds for all $t \ge t_0$. The two N-functions A and B are *equivalent globally* (or *near infinity*) if each dominates the other globally (or near infinity). Thus A and B are equivalent near infinity if there exist positive constants t_0, k_1, and k_2, such that if $t \ge t_0$, then $B(k_1 t) \le A(t) \le B(k_2 t)$. Such will certainly be the case if

$$0 < \lim_{t\to\infty} \frac{B(t)}{A(t)} < \infty.$$

If A and B have respective complementary N-functions $\tilde{A}$ and $\tilde{B}$, then B dominates A globally (or near infinity) if and only if $\tilde{A}$ dominates $\tilde{B}$ globally (or near infinity). Similarly, A and B are equivalent if and only if $\tilde{A}$ and $\tilde{B}$ are.

8.5 If B dominates A near infinity and A and B are not equivalent near infinity, then we say that *A increases essentially more slowly than B* near infinity. This is the case if and only if for every positive constant k

$$\lim_{t\to\infty} \frac{A(kt)}{B(t)} = 0.$$

The reader may verify that this limit is equivalent to

$$\lim_{t\to\infty} \frac{B^{-1}(t)}{A^{-1}(t)} = 0.$$

Let $1 < p < \infty$ and let A_p denote the N-function

$$A_p(t) = \frac{t^p}{p}, \qquad 0 \le t < \infty.$$

If $1 < p < q < \infty$, then A_p increases essentially more slowly than A_q near infinity. However, A_q does not dominate A_p globally.

8.6 (The Δ_2 Condition) An N-function is said to satisfy a *global* Δ_2-*condition* if there exists a positive constant k such that for every $t \geq 0$,

$$A(2t) \leq kA(t). \tag{6}$$

This is the case if and only if for every $r > 1$ there exists a positive constant $k = k(r)$ such that for all $t \geq 0$,

$$A(rt) \leq kA(t). \tag{7}$$

Similarly, A satisfies a Δ_2 *condition near infinity* if there exists $t_0 > 0$ such that (6) (or equivalently (7) with $r > 1$) holds for all $t \geq t_0$. Evidently, t_0 may be replaced with any smaller positive number t_1, for if $t_1 \leq t \leq t_0$, then

$$A(rt) \leq \frac{A(rt_0)}{A(t_1)} A(t).$$

If A satisfies a Δ_2-condition globally (or near infinity) and if B is equivalent to A globally (or near infinity), then B also satisfies such a Δ_2-condition. Clearly the N-function $A_p(t) = t^p/p$, $(1 < p < \infty)$, satisfies a global Δ_2-condition. It can be verified that A satisfies a Δ_2-condition globally (or near infinity) if and only if there exists a positive, finite constant c such that

$$\frac{1}{c}\, t\, a(t) \leq A(t) \leq t\, a(t)$$

holds for all $t \geq 0$ (or for all $t \geq t_0 > 0$), where A is given by (1).

Orlicz Spaces

8.7 (The Orlicz Class $K_A(\Omega)$) Let Ω be a domain in $\mathbb{R}^n$ and let A be an N-function. The *Orlicz class* $K_A(\Omega)$ is the set of all (equivalence classes modulo equality a.e. in Ω of) measurable functions u defined on Ω that satisfy

$$\int_\Omega A(|u(x)|)\, dx < \infty.$$

Since A is convex $K_A(\Omega)$ is always a convex set of functions but it may not be a vector space; for instance, there may exist $u \in K_A(\Omega)$ and $\lambda > 0$ such that $\lambda u \notin K_A(\Omega)$.

We say that the pair (A, Ω) is Δ-*regular* if *either*

(a) A satisfies a global Δ_2-condition, *or*

(b) A satisfies a Δ_2-condition near infinity and Ω has finite volume.

8.8 LEMMA $K_A(\Omega)$ is a vector space (under pointwise addition and scalar multiplication) if and only if (A, Ω) is Δ-regular.

Proof. Since A is convex we have:

(i) $\lambda u \in K_A(\Omega)$ provided $u \in K_A(\Omega)$ and $|\lambda| \le 1$, and

(ii) if $u \in K_A(\Omega)$ implies that $\lambda u \in K_A(\Omega)$ for every complex λ, then $u, v \in K_A(\Omega)$ implies $u + v \in K_A(\Omega)$.

It follows that $K_A(\Omega)$ is a vector space if and only if $\lambda u \in K_A(\Omega)$ whenever $u \in K_A(\Omega)$ and $|\lambda| > 1$.

If A satisfies a global Δ_2-condition and $|\lambda| > 1$, then we have by (7) for $u \in K_A(\Omega)$

$$\int_\Omega A(|\lambda u(x)|)\,dx \le k(|\lambda|)\int_\Omega A(|u(x)|)\,dx < \infty.$$

Similarly, if A satisfies a Δ_2-condition near infinity and $\text{vol}(\Omega) < \infty$, we have for $u \in K_A(\Omega)$, $|\lambda| > 1$, and some $t_0 > 0$,

$$\begin{aligned}\int_\Omega A(|\lambda u(x)|)\,dx &= \left(\int_{\{x\in\Omega:|u(x)|\ge t_0\}} + \int_{\{x\in\Omega:|u(x)|<t_0\}}\right) A(|\lambda u(x)|)\,dx \\ &\le k(|\lambda|)\int_\Omega A(|\lambda u(x)|)\,dx + A(|\lambda| t_0)\text{vol}(\Omega) < \infty.\end{aligned}$$

In either case $K_A(\Omega)$ is seen to be a vector space.

Now suppose that (A, Ω) is not Δ-regular and, if $\text{vol}(\Omega) < \infty$, that $t_0 > 0$ is given. There exists a sequence $\{t_j\}$ of positive numbers such that

(i) $A(2t_j) \ge 2^j A(t_j)$, and

(ii) $t_j \ge t_0 > 0$ if $\text{vol}(\Omega) < \infty$.

Let $\{\Omega_j\}$ be a sequence of mutually disjoint, measurable subsets of Ω such that

$$\text{vol}(\Omega)_j = \begin{cases} 1/[2^j A(t_j)] & \text{if } \text{vol}(\Omega) = \infty \\ A(t_0)\text{vol}(\Omega)/[2^j A(t_j)] & \text{if } \text{vol}(\Omega) < \infty. \end{cases}$$

Let

$$u(x) = \begin{cases} t_j & \text{if } x \in \Omega_j \\ 0 & \text{if } x \in \Omega - \left(\bigcup_{j=1}^\infty \Omega_j\right). \end{cases}$$

Then

$$\int_\Omega A(|u(x)|)\,dx = \sum_{j=1}^{\infty} A(t_j)\mathrm{vol}(\Omega)_j$$

$$= \begin{cases} 1 & \text{if } \mathrm{vol}(\Omega) = \infty \\ A(t_0)\mathrm{vol}(\Omega) & \text{if } \mathrm{vol}(\Omega) < \infty. \end{cases}$$

But

$$\int_\Omega A(|2u(x)|)\,dx \geq \sum_{j=1}^{\infty} 2^j A(t_j)\mathrm{vol}(\Omega)_j = \infty.$$

Thus $K_A(\Omega)$ is not a vector space. ∎

8.9 (The Orlicz Space $L_A(\Omega)$) The *Orlicz space* $L_A(\Omega)$ is the linear hull of the Orlicz class $K_A(\Omega)$, that is, the smallest vector space (under pointwise addition and scalar multiplication) that contains $K_A(\Omega)$. Evidently, $L_A(\Omega)$ contains all scalar multiples λu of elements $u \in K_A(\Omega)$. Thus $K_A(\Omega) \subset L_A(\Omega)$, these sets being equal if and only if (A, Ω) is Δ-regular.

The reader may verify that the functional

$$\|u\|_A = \|u\|_{A,\Omega} = \inf\left\{k > 0 : \int_\Omega A\left(\frac{|u(x)|}{k}\right)dx \leq 1\right\}$$

is a norm on $L_A(\Omega)$. It is called the Luxemburg norm. The infimum is attained. In fact, if k decreases towards $\|u\|_A$ in the inequality

$$\int_\Omega A\left(\frac{|u(x)|}{k}\right)dx \leq 1, \tag{8}$$

we obtain by monotone convergence

$$\int_\Omega A\left(\frac{|u(x)|}{\|u\|_A}\right)dx \leq 1. \tag{9}$$

Equality may fail to hold in (9) but if equality holds in (8), then $k = \|u\|_A$.

8.10 THEOREM $L_A(\Omega)$ is a Banach space with respect to the Luxemburg norm.

The completeness proof is similar to that for the L^p spaces given in Theorem 2.16. The details are left to the reader. We remark that if $1 < p < \infty$ and $A_p(t) = t^p/p$, then

$$L^p(\Omega) = L_{A_p}(\Omega) = K_{A_p}(\Omega).$$

Moreover, $\|u\|_{A_p,\Omega} = p^{-1/p}\,\|u\|_{p,\Omega}$.

8.11 (A Generalized Hölder Inequality) If A and $\tilde{A}$ are complementary N-functions, a generalized version of Hölder's inequality

$$\left| \int_\Omega u(x)v(x)\,dx \right| \le 2\,\|u\|_{A,\Omega}\,\|v\|_{\tilde{A},\Omega} \tag{10}$$

can be obtained by applying Young's inequality (2) to $|u(x)|/\,\|u\|_A$ and $|v(x)|/\,\|v\|_{\tilde{A}}$ and integrating over Ω.

The following elementary imbedding theorem is an analog for Orlicz spaces of Theorem 2.14 for L^p spaces.

8.12 THEOREM (An Imbedding Theorem for Orlicz Spaces) The imbedding

$$L_B(\Omega) \to L_A(\Omega)$$

holds if and only if *either*

(a) B dominates A globally, *or*

(b) B dominates A near infinity and $\mathrm{vol}(\Omega) < \infty$.

Proof. If $A(t) \le B(kt)$ for all $t \ge 0$, and if $u \in L_B(\Omega)$, then

$$\int_\Omega A\left(\frac{|u(x)|}{k\,\|u\|_B}\right) dx \le \int_\Omega B\left(\frac{|u(x)|}{\|u\|_B}\right) dx \le 1.$$

Thus $u \in L_A(\Omega)$ and $\|u\|_A \le k\,\|u\|_B$.

If $\mathrm{vol}(\Omega) < \infty$, let $t_1 = A^{-1}\big((2\mathrm{vol}(\Omega))^{-1}\big)$. If B dominates A near infinity, then there exists positive numbers t_0 and k such that $A(t) \le B(kt)$ for $t \ge t_0$. Evidently, for $t \ge t_1$ we have

$$A(t) \le \max\left\{1, \frac{A(t_0)}{B(kt_1)}\right\} B(kt) = k_1 B(kt).$$

If $u \in L_B(\Omega)$ is given, let $\Omega'(u) = \{x \in \Omega : |u(x)|/[2k_1k\,\|u\|_B] < t_1\}$ and $\Omega''(u) = \Omega - \Omega'(u)$. Then

$$\begin{aligned}
\int_\Omega A\left(\frac{|u(x)|}{2k_1k\,\|u\|_B}\right) dx &= \left(\int_{\Omega'(u)} + \int_{\Omega''(u)}\right) A\left(\frac{|u(x)|}{2k_1k\,\|u\|_B}\right) dx \\
&\le \frac{1}{2\mathrm{vol}(\Omega)} \int_{\Omega'(u)} dx + k_1 \int_{\Omega''(u)} B\left(\frac{|u(x)|}{2k_1\,\|u\|_B}\right) dx \\
&\le \frac{1}{2} + \frac{1}{2}\int_\Omega B\left(\frac{|u(x)|}{\|u\|_B}\right) dx \le 1.
\end{aligned}$$

Thus $u \in L_A(\Omega)$ and $\|u\|_A \le 2k_1k\,\|u\|_B$.

Conversely, suppose that neither of the hypotheses (a) and (b) holds. Then there exist numbers $t_j > 0$ such that

$$A(t_j) \geq B(jt_j), \qquad j = 1, 2, \ldots.$$

If $\operatorname{vol}(\Omega) < \infty$, we may assume, in addition, that

$$t_j \geq \frac{1}{j} B^{-1}\left(\frac{1}{\operatorname{vol}(\Omega)}\right).$$

Let Ω_j be a subdomain of Ω having volume $1/B(jt_j)$, and let

$$u_j(x) = \begin{cases} jt_j & \text{if } x \in \Omega_j \\ 0 & \text{if } x \in \Omega - \Omega_j. \end{cases}$$

Then

$$\int_\Omega A\left(\frac{|u_j(x)|}{j}\right) dx \geq \int_\Omega B(|u_j(x)|)\, dx = 1$$

so that $\|u_j\|_B = 1$ but $\|u_j\|_A \geq j$. Thus $L_B(\Omega)$ is not imbedded in $L_A(\Omega)$. ∎

8.13 (Convergence in Mean) A sequence $\{u_j\}$ of functions in $L_A(\Omega)$ is said to *converge in mean* to $u \in L_A(\Omega)$ if

$$\lim_{j\to\infty} \int_\Omega A\big(|u_j(x) - u(x)|\big)\, dx = 0.$$

The convexity of A implies that for $0 < \epsilon \leq 1$ we have

$$\int_\Omega A\big(|u_j(x) - u(x)|\big)\, dx \leq \epsilon \int_\Omega A\left(\frac{|u_j(x) - u(x)|}{\epsilon}\right) dx$$

from which it follows that norm convergence in $L_A(\Omega)$ implies mean convergence. The converse holds, that is, mean convergence implies norm convergence, if and only if (A, Ω) is Δ-regular. The proof is similar to that of Lemma 8.8 and is left to the reader.

8.14 (The Space $E_A(\Omega)$) Let $E_A(\Omega)$ denote the closure in $L_A(\Omega)$ of the space of functions u which are bounded on Ω and have bounded support in $\overline{\Omega}$. If $u \in K_A(\Omega)$, the sequence $\{u_j\}$ defined by

$$u_j(x) = \begin{cases} u(x) & \text{if } |u(x)| \leq j \text{ and } |x| \leq j, \quad x \in \Omega \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

converges a.e. on Ω to u. Since $A(|u(x) - u_j(x)|) \leq A(|u(x)|)$, we have by dominated convergence that u_j converges to u in mean in $L_A(\Omega)$. Therefore, if

(A, Ω) is Δ-regular, then $E_A(\Omega) = K_A(\Omega) = L_A(\Omega)$. If (A, Ω) is not Δ-regular, then we have

$$E_A(\Omega) \subset K_A(\Omega) \subsetneqq L_A(\Omega)$$

so that $E_A(\Omega)$ is a proper closed subspace of $L_A(\Omega)$ in this case. To verify the first inclusion above let $u \in E_A(\Omega)$ be given. Let v be a bounded function with bounded support such that $0 < \|u - v\|_A < 1/2$. Using the convexity of A and (9), we obtain

$$\frac{1}{\|2u - 2v\|_A} \int_\Omega A\big(|2u(x) - 2v(x)|\big)\, dx \le \int_\Omega A\left(\frac{|2u(x) - 2v(x)|}{\|2u - 2v\|_A}\right) dx \le 1,$$

whence $2u - 2v \in K_A(\Omega)$. Since $2v$ clearly belongs to $K_A(\Omega)$ and $K_A(\Omega)$ is convex, we have $u = (1/2)(2u - 2v) + (1/2)(2v) \in K_A(\Omega)$.

8.15 LEMMA $E_A(\Omega)$ is the maximal linear subspace of $K_A(\Omega)$.

Proof. Let S be a linear subspace of $K_A(\Omega)$ and let $u \in S$. Then $\lambda u \in K_A(\Omega)$ for every scalar λ. If $\epsilon > 0$ and u_j is given by (11), then u_j/ϵ converges to u/ϵ in mean in $L_A(\Omega)$ as noted in Paragraph 8.14. Hence, for sufficiently large values of j we have

$$\int_\Omega A\left(\frac{|u_j(x) - u(x)|}{\epsilon}\right) dx \le 1$$

and therefore u_j converges to u in norm in $L_A(\Omega)$. Thus $S \subset E_A(\Omega)$. ∎

8.16 THEOREM Let Ω have finite volume, and suppose that the N-function A increases essentially more slowly than the N-function B near infinity. Then

$$L_B(\Omega) \to E_A(\Omega).$$

Proof. Since $L_B(\Omega) \to L_A(\Omega)$ is already established we need only show that $L_B(\Omega) \subset E_A(\Omega)$. Since $L_B(\Omega)$ is the linear hull of $K_B(\Omega)$ and $E_A(\Omega)$ is the maximal linear subspace of $K_A(\Omega)$, it is sufficient to show that $\lambda u \in K_A(\Omega)$ whenever $u \in K_B(\Omega)$ and λ is a scalar. But there exists a positive number t_0 such that $A(|\lambda|t) \le B(t)$ for all $t \ge t_0$. Thus

$$\begin{aligned}\int_\Omega A\big(|\lambda u(x)|\big)\, dx &= \left(\int_{\{x\in\Omega:|u(x)\le t_0\}} + \int_{\{x\in\Omega:|u(x)>t_0\}}\right) A\big(|\lambda u(x)|\big)\, dx \\ &\le A\big(|\lambda| t_0\big)\mathrm{vol}(\Omega) + \int_\Omega B\big(|u(x)|\big)\, dx < \infty\end{aligned}$$

whence the theorem follows. ∎

Duality in Orlicz Spaces

8.17 LEMMA Given $v \in L_{\tilde{A}}(\Omega)$, the linear functional F_v defined by

$$F_v(u) = \int_\Omega u(x)v(x)\,dx \tag{12}$$

belongs to the dual space $[L_A(\Omega)]'$ and its norm $\|F_v\|$ in that space satisfies

$$\|v\|_{\tilde{A}} \le \|F_v\| \le 2\,\|v\|_{\tilde{A}}\,. \tag{13}$$

Proof. It follows by Hölder's inequality (10) that

$$|F_v(u)| \le 2\,\|u\|_A\,\|v\|_{\tilde{A}}$$

holds for all $u \in L_A(\Omega)$, confirming the second inequality in (13).
To establish the other half of (13) we may assume that $v \ne 0$ in $L_{\tilde{A}}(\Omega)$ so that $\|F_v\| = K > 0$. Let

$$u(x) = \begin{cases} \tilde{A}\left(\dfrac{|v(x)|}{K}\right)\Big/\dfrac{|v(x)|}{K} & \text{if } v(x) \ne 0 \\ 0 & \text{if } v(x) = 0. \end{cases}$$

If $\|u\|_A > 1$, then for $0 < \epsilon \le \|u\|_A - 1$ we have

$$\frac{1}{\|u\|_A - \epsilon}\int_\Omega A\big(|u(x)|\big)\,dx \ge \int_\Omega A\left(\frac{|u(x)|}{\|u\|_A - \epsilon}\right)dx > 1.$$

Letting $\epsilon \to 0+$ we obtain, using (3),

$$\begin{aligned} \|u\|_A &\le \int_\Omega A\big(|u(x)|\big)\,dx = \int_\Omega A\left(\tilde{A}\left(\frac{|v(x)|}{K}\right)\Big/\frac{|v(x)|}{K}\right)dx \\ &< \int_\Omega \tilde{A}\left(\frac{|v(x)|}{K}\right)dx = \frac{1}{\|F_v\|}\int_\Omega u(x)v(x)\,dx \le \|u\|_A\,. \end{aligned}$$

This contradiction shows that $\|u\|_A \le 1$. Now

$$\|F_v\| = \sup_{\|u\|_A \le 1} |F_v(u)| \ge \|F_v\|\left|\int_\Omega \tilde{A}\left(\frac{|v(x)|}{\|F_v\|}\right)dx\right|$$

so that

$$\int_\Omega \tilde{A}\left(\frac{|v(x)|}{\|F_v\|}\right)dx \le 1. \tag{14}$$

Thus, $\|v\|_{\tilde{A}} \le \|F_v\|$. ∎

8.18 REMARK The above lemma also holds when F_v is restricted to act on $E_A(\Omega)$. To obtain the first inequality of (13) in this case take $\|F_u\|$ to be the norm of F_v in $[E_A(\Omega)]'$ and replace u in the above proof by $\chi_n u$ where χ_n is the characteristic function of $\Omega_n = \{x \in \Omega : |x| \le n \text{ and } |u(x)| \le n\}$. Evidently, $\chi_n u$ belongs to $E_A(\Omega)$, $\|\chi_n u\|_A \le 1$, and (14) becomes

$$\int_\Omega \chi_n(x)\tilde{A}\left(\frac{|v(x)|}{\|F_v\|}\right) dx \le 1.$$

Since $\chi_n(x)$ increases to unity a.e. on Ω as $n \to \infty$, we obtain (14) again, and $\|v\|_{\tilde{A}} \le \|F_v\|$ as before.

8.19 THEOREM (The Dual of $E_A(\Omega)$) The dual space of $E_A(\Omega)$ is isomorphic and homeomorphic to $L_{\tilde{A}}(\Omega)$.

Proof. We have already shown that any element $v \in L_{\tilde{A}}(\Omega)$ determines a bounded linear functional F_v via (12) on $L_A(\Omega)$ and also on $E_A(\Omega)$, and that in either case the norm of this functional differs from $\|v\|_{\tilde{A}}$ by at most a factor of 2. It remains to be shown that every bounded linear functional on $E_A(\Omega)$ is of the form F_v for some such v.

Let $F \in [E_A(\Omega)]'$ be given. We define a complex measure λ on the measurable subsets S of Ω having finite volume by setting

$$\lambda(S) = F(\chi_S),$$

χ_S being the characteristic function of S. Since

$$\int_\Omega A\left(|\chi_S(x)|A^{-1}\left[\frac{1}{\operatorname{vol}(S)}\right]\right) dx = \int_S \frac{1}{\operatorname{vol}(S)}\, dx = 1 \tag{15}$$

we have

$$|\lambda(S)| \le \|F\|\,\|\chi_S\|_A = \frac{\|F\|}{A^{-1}\big(1/\operatorname{vol}(S)\big)}.$$

Since the right side tends to zero with vol(S), the measure λ is absolutely continuous with respect to Lebesgue measure, and so by the Radon-Nikodym Theorem 1.52, λ can be expressed in the form

$$\lambda(S) = \int_S v(x)\, dx,$$

for some v that is integrable on Ω. Thus

$$F(u) = \int_\Omega u(x)v(x)\, dx$$

holds for measurable, simple functions u.

If $u \in E_A(\Omega)$, a sequence of measurable, simple functions u_j can be found that converges a.e. to u and satisfies $|u_j(x)| \le |u(x)|$ on Ω. Since $|u_j(x)v(x)|$ converges a.e. to $|u(x)v(x)|$, Fatou's Lemma 1.49 yields

$$\begin{aligned}\left|\int_\Omega u(x)v(x)\,dx\right| &\le \sup_j \int_\Omega |u_j(x)v(x)|\,dx = \sup_j \left|F\big(|u_j|\operatorname{sgn} v\big)\right| \\ &\le \|F\| \sup_j \left\|u_j\right\|_A \le \|F\|\,\|u\|_A .\end{aligned}$$

It follows that the linear functional

$$F_v(u) = \int_\Omega u(x)v(x)\,dx$$

is bounded on $E_A(\Omega)$ whence $v \in L_{\tilde{A}}(\Omega)$ by Remark 8.18. Since F_v and F assume the same values on the measurable, simple functions, a set that is dense in $E_A(\Omega)$ (see Theorem 8.21 below), they agree on $E_A(\Omega)$ and the theorem is proved. ∎

A simple application of the Hahn-Banach Theorem shows that if $E_A(\Omega)$ is a proper subspace of $L_A(\Omega)$ (that is, if (A, Ω) is *not* Δ-regular), then there exists a bounded linear functional F on $L_A(\Omega)$ that is not given by (12) for any $v \in L_{\tilde{A}}(\Omega)$. As an immediate consequence of this fact we have the following theorem.

8.20 THEOREM (Reflexivity of Orlicz Spaces) $L_A(\Omega)$ is reflexive if and only if both (A, Ω) and $(\tilde{A}, \Omega)$ are Δ-regular.

We omit any discussion of uniform convexity of Orlicz spaces. This subject is treated in Luxemburg's thesis [Lu].

Separability and Compactness Theorems

We next generalize to Orlicz spaces the L^p approximation Theorems 2.19, 2.21, and 2.30.

8.21 THEOREM (Approximation of Functions in $E_A(\Omega)$)

(a) $C_0(\Omega)$ is dense in $E_A(\Omega)$.

(b) $E_A(\Omega)$ is separable.

(c) If J_ϵ is the mollifier of Paragraph 2.28, then for each $u \in E_A(\Omega)$ we have $\lim_{\epsilon\to 0+} J_\epsilon * u = u$ in norm in $E_A(\Omega)$.

(d) $C_0^\infty(\Omega)$ is dense in $E_A(\Omega)$.

Proof. Part (a) is proved by the same method used in Theorem 2.19. In approximating $u \in E_A(\Omega)$ first by simple functions we can assume that u is bounded on

Ω and has bounded support. Then a dominated convergence argument shows that the simple functions converge in norm to u in $E_A(\Omega)$. (The details are left to the reader.)

Part (b) follows from part (a) by the same proof given for Theorem 2.21.

Consider part (c). If $u \in E_A(\Omega)$, let u be extended to $\mathbb{R}^n$ so as to vanish identically outside Ω. Let $v \in L_{\tilde{A}}(\Omega)$. Then

$$\begin{aligned}\left|\int_\Omega \big(J_\epsilon * u(x) - u(x)\big)v(x)\,dx\right| &\le \int_{\mathbb{R}^n} J(y)\,dy \int_\Omega |u(x-\epsilon y) - u(x)||v(x)|\,dx \\ &\le 2\,\|v\|_{\tilde{A},\Omega} \int_{|y|\le 1} J(y)\,\left\|u_{\epsilon y} - u\right\|_{A,\Omega}\,dy\end{aligned}$$

by Hölder's inequality (10), where $u_{\epsilon y}(x) = u(x-\epsilon y)$. Thus by (13) and Theorem 8.19,

$$\begin{aligned}\|J_\epsilon * u - u\|_{A,\Omega} &= \sup_{\|v\|_{\tilde{A},\Omega}\le 1} \left|\int_\Omega \big(J_\epsilon * u(x) - u(x)\big)v(x)\,dx\right| \\ &\le 2\int_{|y|\le 1} J(y)\,\left\|u_{\epsilon y} - u\right\|_{A,\Omega}\,dy.\end{aligned}$$

Given $\delta > 0$ we can find $\tilde{u} \in C_0(\Omega)$ such that $\|u - \tilde{u}\|_{A,\Omega} < \delta/6$. Clearly, $\left\|u_{\epsilon y} - \tilde{u}_{\epsilon y}\right\|_{A,\Omega} < \delta/6$ and for sufficiently small ϵ, $\left\|\tilde{u}_{\epsilon y} - \tilde{u}\right\|_{A,\Omega} < \delta/6$ for every y with $|y| \le 1$. Thus $\|J_\epsilon * u - u\|_{A,\Omega} < \delta$ and (c) is established.

Part (d) is an immediate consequence of parts (a) and (c). ∎

8.22 REMARK $L_A(\Omega)$ is not separable unless $L_A(\Omega) = E_A(\Omega)$, that is, unless (A, Ω) is Δ-regular. A proof of this fact may be found in [KR] (Chapter II, Theorem 10.2).

8.23 (Convergence in Measure) A sequence $\{u_j\}$ of measurable functions is said to *converge in measure* on Ω to the function u provided that for each $\epsilon > 0$ and $\delta > 0$ there exists an integer M such that if $j > M$, then

$$\operatorname{vol}(\{x \in \Omega : |u_j(x) - u(x)| > \epsilon\}) \le \delta.$$

Clearly, in this case there also exists an integer N such that if $j, k \ge N$, then

$$\operatorname{vol}(\{x \in \Omega : |u_j(x) - u_k(x)| \ge \epsilon\}) \le \delta.$$

8.24 THEOREM Let Ω have finite volume and suppose that the N-function B increases essentially more slowly than A near infinity. If the sequence $\{u_j\}$ is bounded in $L_A(\Omega)$ and convergent in measure on Ω, then it is convergent in norm in $L_B(\Omega)$.

Proof. Fix $\epsilon > 0$ and let $v_{j,k}(x) = \big(u_j(x) - u_k(x)\big)/\epsilon$. Clearly $\{v_{j,k}\}$ is bounded in $L_A(\Omega)$; say $\|v_{j,k}\|_{A,\Omega} \le K$. Now there exists a positive number t_0 such that if $t > t_0$, then

$$B(t) \le \frac{1}{4} A\left(\frac{t}{K}\right).$$

Let $\delta = 1/[4B(t_0)]$ and set

$$\Omega_{j,k} = \left\{x \in \Omega \,:\, |v_{j,k}(x)| \ge B^{-1}\left(\frac{1}{2\mathrm{vol}(\Omega)}\right)\right\}.$$

Since $\{u_j\}$ converges in measure, there exists an integer N such that if $j, k \ge N$, then $\mathrm{vol}(\Omega)_{j,k} \le \delta$. Set

$$\Omega'_{j,k} = \{x \in \Omega_{j,k} \,:\, |v_{j,k}(x)| \ge t_0\}, \qquad \Omega''_{j,k} = \Omega_{j,k} - \Omega'_{j,k}.$$

For $j, k \ge N$ we have

$$\begin{aligned}\int_\Omega B(|v_{j,k}(x)|)\,dx &= \left(\int_{\Omega-\Omega_{j,k}} + \int_{\Omega'_{j,k}} + \int_{\Omega''_{j,k}}\right) B(|v_{j,k}(x)|)\,dx \\ &\le \frac{\mathrm{vol}(\Omega)}{2\mathrm{vol}(\Omega)} + \frac{1}{4}\int_{\Omega'_{j,k}} A\left(\frac{|v_{j,k}(x)|}{K}\right) dx + \delta B(t_0) \le 1.\end{aligned}$$

Hence $\|u_j - u_k\|_{B,\Omega} \le \epsilon$ and so $\{u_j\}$ converges in $L_B(\Omega)$. ∎

The following theorem will be useful when we wish to extend the Rellich-Kondrachov Theorem 6.3 to imbeddings of Orlicz-Sobolev spaces.

8.25 THEOREM (Precompact Sets in Orlicz Spaces) Let Ω have finite volume and suppose that the N-function B increases essentially more slowly than A near infinity. Then any bounded subset S of $L_A(\Omega)$ which is precompact in $L^1(\Omega)$ is also precompact in $L_B(\Omega)$.

Proof. Evidently $L_A(\Omega) \to L^1(\Omega)$ since Ω has finite volume. If $\{u_j^*\}$ is a sequence in S, then it has a subsequence $\{u_j\}$ that converges in $L^1(\Omega)$; say $u_j \to u$ in $L^1(\Omega)$. Let $\epsilon, \delta > 0$. Then there exists an integer N such that if $j \ge N$, then $\|u_j - u\|_{1,\Omega} \le \epsilon\delta$. If follows that

$$\mathrm{vol}(\{x \in \Omega \,:\, |u_j(x) - u(x)| \ge \epsilon\}) \le \delta.$$

Thus $\{u_j\}$ converges to u in measure on Ω and hence also in $L_B(\Omega)$. ∎

A Limiting Case of the Sobolev Imbedding Theorem

8.26 If $mp = n$ and $p > 1$, the Sobolev Imbedding Theorem 4.12 provides no best (i.e., smallest) target space into which $W^{m,p}(\Omega)$ can be imbedded. In this case, for suitably regular Ω,

$$W^{m,p}(\Omega) \to L^q(\Omega), \qquad p \le q < \infty,$$

but (see Example 4.43)

$$W^{m,p}(\Omega) \not\subset L^\infty(\Omega).$$

If the class of target spaces for the imbedding is enlarged to contain Orlicz spaces, then a best such target space can be found.

We first consider the case of bounded Ω and later extend our consideration to unbounded domains. The following theorem was established by Trudinger [Td]. For other proofs see [B+] and [Ta]; for refinements going beyond Orlicz spaces see [BW] and [MP].

8.27 THEOREM (Trudinger's Theorem) Let Ω be a bounded domain in $\mathbb{R}^n$ satisfying the cone condition. Let $mp = n$ and $p > 1$. Set

$$A(t) = \exp\left(t^{n/(n-m)}\right) - 1 = \exp\left(t^{p/(p-1)}\right) - 1. \tag{16}$$

Then there exists the imbedding

$$W^{m,p}(\Omega) \to L_A(\Omega).$$

Proof. If $m > 1$ and $mp = n$, then $W^{m,p}(\Omega) \to W^{1,n}(\Omega)$. Therefore it is sufficient to prove the theorem for $m = 1$, $p = n > 1$. Let $u \in C^1(\Omega) \cap W^{1,n}(\Omega)$ (a set that is dense in $W^{1,n}(\Omega)$) and let $x \in \Omega$. By the special case $m = 1$ of Lemma 4.15 we have, denoting by C a cone contained in Ω, having vertex at x, and congruent to the cone specifying the cone condition for Ω,

$$\begin{aligned} |u(x)| &\le K_1 \left(\|u\|_{1,C} + \sum_{j=1}^n \int_C |D_j u(x)||x-y|^{1-n}\,dy \right) \\ &\le K_1 \left(\|u\|_{1,\Omega} + \sum_{j=1}^n \int_\Omega |D_j u(y)||x-y|^{1-n}\,dy \right). \end{aligned}$$

We want to estimate the L^s-norm $\|u\|_s$ for arbitrary $s > 1$. If $v \in L^{s'}(\Omega)$ (where

$(1/s) + (1/s') = 1$), then

$$\int_\Omega |u(x)v(x)|\,dx \le K_1 \left(\|u\|_1 \int_\Omega |v(x)|\,dx + \sum_{j=1}^n \int_\Omega \int_\Omega \frac{|D_j u(y)||v(x)|}{|x-y|^{n-1}}\,dy\,dx \right)$$
$$\le K_1 \|u\|_1 \|v\|_{s'} \big(\mathrm{vol}(\Omega)\big)^{1/s}$$
$$+ K_1 \sum_{j=1}^n \left(\int_\Omega \int_\Omega \frac{|v(x)|}{|x-y|^{n-(1/s)}}\,dy\,dx \right)^{(n-1)/n}$$
$$\times \left(\int_\Omega \int_\Omega \frac{|D_j u(y)|^n |v(x)|}{|x-y|^{(n-1)/s}}\,dy\,dx \right)^{1/n}.$$

By Lemma 4.64, if $0 \le \nu < n$,

$$\int_\Omega \frac{1}{|x-y|^\nu}\,dy \le \frac{K_2}{n-\nu}\big(\mathrm{vol}(\Omega)\big)^{1-(\nu/n)}.$$

Hence

$$\int_\Omega \int_\Omega \frac{|v(x)|}{|x-y|^{n-(1/s)}}\,dy\,dx \le K_2 s \big(\mathrm{vol}(\Omega)\big)^{1/(sn)} \int_\Omega |v(x)|\,dx$$
$$\le K_3 s \big(\mathrm{vol}(\Omega)\big)^{1/(sn)+1/s} \|v\|_{s'}.$$

Also,

$$\int_\Omega \int_\Omega \frac{|D_j u(y)|^n |v(x)|}{|x-y|^{(n-1)/s}}\,dy\,dx \le \int_\Omega |D_j u(y)|^n\,dy \;\|v\|_{s'} \left(\int_\Omega \frac{1}{|x-y|^{n-1}}\,dx \right)^{1/s}$$
$$\le \left\|D_j u\right\|_p^p \|v\|_{s'} \left(K_2 \big(\mathrm{vol}(\Omega)\big)^{1/n} \right)^{1/s}$$
$$= K_4 \left\|D_j u\right\|_n^n \|v\|_{s'} \big(\mathrm{vol}(\Omega)\big)^{1/(ns)}.$$

It follows from these estimates that

$$\int_\Omega |u(x)v(x)|\,dx \le K_1 \|u\|_1 \|v\|_{s'} \big(\mathrm{vol}(\Omega)\big)^{1/s}$$
$$+ K_4 \sum_{j=1}^n s^{(n-1)/n} \left\|D_j u\right\|_n \|v\|_{s'} \big(\mathrm{vol}(\Omega)\big)^{1/s}.$$

Since $s^{(n-1)/n} > 1$ and since $W^{1,n}(\Omega) \to L^1(\Omega)$, we now have

$$\|u\|_s = \sup_{v \in L^{s'}(\Omega)} \frac{1}{\|v\|_{s'}} \int_\Omega |u(x)v(x)|\,dx \le K_5 s^{(n-1)/n} \big(\mathrm{vol}(\Omega)\big)^{1/s} \|u\|_{1,n}.$$

The constant K_5 depends only on n and the cone determining the cone condition for Ω. Setting $s = nk/(n-1)$, we obtain

$$\begin{aligned}\int_\Omega |u(x)|^{nk/(n-1)}\,dx &\le \operatorname{vol}(\Omega)\left(\frac{nk}{n-1}\right)^k \left(K_5\,\|u\|_{1,n}\right)^{nk/(n-1)}\\ &= \operatorname{vol}(\Omega)\left(\frac{k}{e^{n/(n-1)}}\right)^k \left(eK_5\left[\frac{n}{n-1}\right]^{(n-1)/n} \|u\|_{1,n}\right)^{nk/(n-1)}\end{aligned}$$

Since $e^{n/(n-1)} > e$, the series $\sum_{k=1}^\infty (1/k!)\big(k/e^{n/(n-1)}\big)^k$ converges to a finite sum $K6$. Let $K_7 = \max\{1, K_6\operatorname{vol}(\Omega)\}$ and put

$$K_8 = eK_7K_5\left(\frac{n}{n-1}\right)^{(n-1)/n} \|u\|_{1,n} = K_9\,\|u\|_{1,n}\,.$$

Then

$$\int_\Omega \left(\frac{|u(x)|}{K_8}\right)^{nk/(n-1)} dx \le \frac{\operatorname{vol}(\Omega)}{K_7^{nk/(n-1)}}\left(\frac{k}{e^{n/(n-1)}}\right)^k < \frac{\operatorname{vol}(\Omega)}{K_7}\left(\frac{k}{e^{n/(n-1)}}\right)^k$$

since $K_7 \ge 1$ and $nk/(n-1) > 1$. Expanding $A(t)$ in a power series, we now obtain

$$\begin{aligned}\int_\Omega A\left(\frac{|u(x)|}{K_8}\right) dx &= \sum_{k=1}^\infty \frac{1}{k!}\int_\Omega \left(\frac{|u(x)|}{K_8}\right)^{nk/(n-1)} dx\\ &< \frac{\operatorname{vol}(\Omega)}{K_7}\sum_{k=1}^\infty \frac{1}{k!}\left(\frac{k}{e^{n/(n-1)}}\right)^k \le 1.\end{aligned}$$

Hence $u \in L_A(\Omega)$ and

$$\|u\|_A \le K_8 = K_9\,\|u\|_{m,p}\,,$$

where K_9 depends on n, $\operatorname{vol}(\Omega)$, and the cone C determining the cone condition for Ω. ∎

8.28 REMARK The imbedding established in Theorem 8.27 is "best possible" in the sense that if there exist an imbedding of the form

$$W_0^{m,p}(\Omega) \to L_B(\Omega),$$

then A dominates B near infinity. A proof of this fact for the case $m = 1$, $p = n > 1$ can be found in [HMT]. The general case is left to the reader as an exercise.

Trudinger's theorem can be generalized to fractional-order spaces. For results in this direction the reader is referred to [Gr] and [P].

Recent efforts have identified non-Orlicz function spaces that are smaller than Trudinger's space into which $W^{m,p}(\Omega)$ can be imbedded in the limiting case $mp = n$. See [MP] in this regard.

8.29 (Extension to Unbounded Domains) If Ω is unbounded and so (satisfying the cone condition) has infinite volume, then the N-function A given by (16) may not decrease rapidly enough at zero to to allow membership in $L_A(\Omega)$ of every $u \in W^{m,p}(\Omega)$ (where $mp = n$). Let k_0 be the smallest integer such that $k_0 \geq p - 1$ and define a modified N-function A_0 by

$$A_0(t) = \exp\left(t^{p/(p-1)}\right) - \sum_{j=0}^{k_0-1} \frac{1}{j!} t^{jp/(p-1)}.$$

Evidently A_0 is equivalent to A near infinity so for any domain Ω having finite volume, $L_A(\Omega)$ and $L_{A_0}(\Omega)$ coincide and have equivalent norms. However, A_0 enjoys the further property that for $0 < r \leq 1$,

$$A_0(rt) \leq r^{k_0 p/(p-1)} A_0(t) \leq r^p A_0(t). \tag{17}$$

We show that if $mp = n$, $p > 1$, and Ω satisfies the cone condition (but may be unbounded), then

$$W^{m,p}(\Omega) \to L_{A_0}(\Omega).$$

Lemma 4.22 implies that even an unbounded domain Ω satisfying the cone condition can be written as a union of countably many subdomains Ω_j each satisfying the cone condition specified by a cone independent of j, each having volume satisfying

$$0 < K_1 \leq \operatorname{vol}(\Omega_j) \leq K_2$$

with K_1 and K_2 independent of j, and such that any $M + 1$ of the subdomains have empty intersection. It follows from Trudinger's theorem that

$$\|u\|_{A_0,\Omega_j} \leq K_3 \|u\|_{m,p,\Omega_j}$$

with K_3 independent of j. Using (17) with $r = M^{1/p} \|u\|_{m,p,\Omega_j}^{-1} \|u\|_{m,p,\Omega}$ and the finite intersection property of the domains Ω_j, we have

$$\begin{aligned} \int_\Omega A_0\left(\frac{|u(x)|}{M^{1/p} K_3 \|u\|_{m,p,\Omega}}\right) dx &\leq \sum_{j=1}^{\infty} \int_{\Omega_j} A_0\left(\frac{|u(x)|}{M^{1/p} K_3 \|u\|_{m,p,\Omega}}\right) dx \\ &\leq \sum_{j=1}^{\infty} \frac{\|u\|_{m,p,\Omega_j}^p}{M \|u\|_{m,p,\Omega}^p} \leq 1. \end{aligned}$$

Hence $\|u\|_{A_0,\Omega} \le M^{1/p} K_3 \|u\|_{m,p,\Omega}$ as required.

We remark that if $k_0 > p - 1$, the above result can be improved slightly by using in place of A_0 the N-function $\max\{t^p, A_0(t)\}$.

Orlicz-Sobolev Spaces

8.30 (Definitions) For a given domain Ω in $\mathbb{R}^n$ and a given N-function A the *Orlicz-Sobolev space* $W^m L_A(\Omega)$ consists of those (equivalence classes of) functions u in $L_A(\Omega)$ whose distributional derivatives $D^\alpha u$ also belong to $L_A(\Omega)$ for all α with $|\alpha| \le m$. The space $W^m E_A(\Omega)$ is defined in an analogous fashion. It may be checked by the same method used for ordinary Sobolev spaces in Chapter 3 that $W^m L_A(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{m,A} = \|u\|_{m,A,\Omega} = \max_{0\le|\alpha|\le m} \|D^\alpha u\|_{A,\Omega},$$

and that $W^m E_A(\Omega)$ is a closed subspace of $W^m L_A(\Omega)$ and hence also a Banach space with the same norm. It should be kept in mind that $W^m E_A(\Omega)$ coincides with $W^m L_A(\Omega)$ if and only if (A, Ω) is Δ-regular. If $1 < p < \infty$ and $A_p(t) = t^p$, then $W^m L_{A_p}(\Omega) = W^m E_{A_p}(\Omega) = W^{m,p}(\Omega)$, the latter space having norm equivalent to those of the former two spaces.

As in the case of ordinary Sobolev spaces, $W_0^m L_A(\Omega)$ is taken to be the closure of $C_0^\infty(\Omega)$ in $W^m L_A(\Omega)$. (An analogous definition for $W_0^m E_A(\Omega)$ clearly leads to the same spaces in all cases.)

Many properties of Orlicz-Sobolev spaces are obtained by very straightforward generalization of the proofs of the same properties for ordinary Sobolev spaces. We summarize these in the following theorem and refer the reader to the corresponding results in Chapter 3 for the method of proof. The details can also be found in the article by Donaldson and Trudinger [DT].

8.31 THEOREM (Basic Properties of Orlicz-Sobolev Spaces)

(a) $W^m E_A(\Omega)$ is separable (Theorem 3.6).

(b) If (A, Ω) and $(\tilde{A}, \Omega)$ are Δ-regular, then $W^m E_A(\Omega) = W^m L_A(\Omega)$ is reflexive (Theorem 3.6).

(c) Each element F of the dual space $[W^m E_A(\Omega)]'$ is given by

$$F(u) = \sum_{0\le|\alpha|\le m} \int_\Omega D^\alpha u(x)\, v_\alpha(x)\, ds$$

for some functions $v_\alpha \in L_{\tilde{A}}(\Omega)$, $0 \le |\alpha| \le m$ (Theorem 3.9).

(d) $C^\infty(\Omega) \cap W^m E_A(\Omega)$ is dense in $W^m E_A(\Omega)$ (Theorem 3.17).

(e) If Ω satisfies the segment condition, then $C^\infty(\overline{\Omega})$ is dense in $W^m E_A(\Omega)$ (Theorem 3.22).

(f) $C_0^\infty(\mathbb{R}^n)$ is dense in $W^m E_A(\mathbb{R}^n)$. Thus $W_0^m L_A(\mathbb{R}^n) = W^m E_A(\mathbb{R}^n)$ (Theorem 3.22).

Imbedding Theorems for Orlicz-Sobolev Spaces

8.32 Imbedding results analogous to those obtained for the spaces $W^{m,p}(\Omega)$ in Chapters 4 and 6 can be formulated for the Orlicz-Sobolev spaces $W^m L_A(\Omega)$ and $W^m E_A(\Omega)$. The first results in this direction were obtained by Dankert [Da]. A fairly general imbedding theorem along the lines of Theorems 4.12 and 6.3 was presented by Donaldson and Trudinger [DT] and we develop it below.

As was the case with ordinary Sobolev spaces, most of these imbedding results are obtained for domains satisfying the cone condition. Exceptions are those yielding (generalized) Hölder continuity estimates; these require the strong local Lipschitz condition. Some results below are proved only for bounded domains. The method used in extending the analogous results for ordinary Sobolev spaces to unbounded domains does not seem to extend in a straightforward manner when general Orlicz spaces are involved. In this sense the imbedding picture we present here is incomplete. Best possible Orlicz-Sobolev imbeddings, involving a careful study of rearrangements, have been found recently by Cianchi [Ci]. We settle here for results that follow by methods we used earlier for imbeddings of $W^{m,p}(\Omega)$ and for weighted spaces; that is also how we proved Trudinger's theorem.

8.33 (A Sobolev Conjugate) We concern ourselves for the time being with imbeddings of $W^1 L_A(\Omega)$; the imbeddings of $W^m L_A(\Omega)$ are summarized in Theorem 8.43. As usual, Ω is assumed to be a domain in $\mathbb{R}^n$.

Let A be an N-function. We shall always suppose that

$$\int_0^1 \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt < \infty, \tag{18}$$

replacing, if necessary, A by another N-function equivalent to A near infinity. (If Ω has finite volume, (18) places no restrictions on A from the point of view of imbedding theory since N-functions equivalent near infinity determine identical Orlicz spaces in that case.)

Suppose also that

$$\int_1^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt = \infty. \tag{19}$$

For instance, if $A(t) = A_p(t) = t^p$, $p > 1$, then (19) holds precisely when $p \le n$. With (19) satisfied, we define the *Sobolev conjugate N-function* A_* of A by setting

$$A_*^{-1}(t) = \int_0^t \frac{A^{-1}(\tau)}{\tau^{(n+1)/n}}\,d\tau, \qquad t \ge 0. \tag{20}$$

It may readily be checked that A_* is an N-function. If $1 < p < n$, we have, setting $q = np/(n-p)$ (the normal Sobolev conjugate exponent for p),

$$A_{p*}(t) = q^{1-q} p^{-q/p} A_q(t).$$

It is also readily seen for the case $p = n$ that $A_{n*}(t)$ is equivalent near infinity to the N-function $e^t - t - 1$. In [Ci] a different Sobolev conjugate is used; it is equivalent when $p = n$ to the N-function in Trudinger's theorem.

8.34 LEMMA Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and let f satisfy a Lipschitz condition on $\mathbb{R}$. If $g(x) = f(|u(x)|)$, then $g \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and

$$D_j g(x) = f'(|u(x)|) \operatorname{sgn} u(x) \cdot D_j u(x).$$

Proof. Since $|u| \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and $D_j|u(x)| = \operatorname{sgn} u(x) \cdot D_j u(x)$ it is sufficient to establish the lemma for positive, real-valued functions u so that $g(x) = f\big(u(x)\big)$. Let $\phi \in \mathscr{D}(\Omega)$ and let $\{e_j\}_{j=1}^n$ be the standard basis in $\mathbb{R}^n$. Then

$$\begin{aligned}
-\int_\Omega f\big(u(x)\big) D_j\phi(x)\,dx &= -\lim_{h\to 0}\int_\Omega f\big(u(x)\big)\,\frac{\phi(x)-\phi(x-he_j)}{h}\,dx \\
&= \lim_{h\to 0}\int_\Omega \frac{f\big(u(x+he_j)\big)-f\big(u(x)\big)}{h}\,\phi(x)\,dx \\
&= \lim_{h\to 0}\int_\Omega Q(x,h)\,\frac{u(x+he_j)-u(x)}{h}\,\phi(x)\,dx,
\end{aligned}$$

where, since f satisfies a Lipschitz condition, for each h the function $Q(\cdot, h)$ is defined a.e. on Ω by

$$Q(x,h) = \begin{cases} \dfrac{f\big(u(x+he_j)\big)-f\big(u(x)\big)}{u(x+he_j)-u(x)} & \text{if } u(x+he_j) \neq u(x) \\ f'\big(u(x)\big) & \text{otherwise.} \end{cases}$$

Moreover, $\|Q(\cdot,h)\|_{\infty,\Omega} \le K$ for some constant K independent of h. A well-known theorem in functional analysis tells us that for some sequence of values of h tending to zero, $Q(\cdot,h)$ converges to $f'\big(u(\cdot)\big)$ in the weak-star topology of $L^\infty(\Omega)$. On the other hand, since $u \in W^{1,1}(\operatorname{supp}(\phi))$ we have

$$\lim_{h\to 0}\frac{u(x+he_j)-u(x)}{h}\,\phi(x) = D_j u(x)\cdot\phi(x)$$

in $L^1(\operatorname{supp}(\phi))$. It follows that

$$-\int_\Omega f\big(u(x)\big) D_j\phi(x)\,dx = \int_\Omega f'\big(u(x)\big)\,D_j u(x)\,\phi(x)\,dx,$$

which implies the lemma. ∎

8.35 THEOREM (Imbedding Into an Orlicz Space) Let Ω be bounded and satisfying the cone condition in $\mathbb{R}^n$. If (18) and (19) hold, then

$$W^1L_A(\Omega) \to L_{A_*}(\Omega),$$

where A_* is given by (20). Moreover, if B is any N-function increasing essentially more slowly than A_* near infinity, then the imbedding

$$W^1L_A(\Omega) \to L_B(\Omega)$$

(exists and) is compact.

Proof. The function $s = A_*(t)$ satisfies the differential equation

$$A^{-1}(s)\,\frac{ds}{dt} = s^{(n+1)/n}, \tag{21}$$

and hence, since $s < A^{-1}(s)\tilde{A}^{-1}(s)$ (see (4)),

$$\frac{ds}{dt} \le s^{1/n}\tilde{A}^{-1}(s).$$

Therefore $\sigma(t) = \big(A_*(t)\big)^{(n-1)/n}$ satisfies the differential inequality

$$\frac{d\sigma}{dt} \le \frac{n-1}{n}\tilde{A}^{-1}\Big(\big(\sigma(t)\big)^{n/(n-1)}\Big). \tag{22}$$

Let $u \in W^1L_A(\Omega)$ and suppose, for the moment, that u is bounded on Ω and is not zero in $L_A(\Omega)$. Then $\int_\Omega A_*(|u(x)|/\lambda)\,dx$ decreases continuously from infinity to zero as λ increases from zero to infinity, and, accordingly, assumes the value unity for some positive value K of λ. Thus

$$\int_\Omega A_*\left(\frac{|u(x)|}{K}\right)dx = 1, \qquad K = \|u\|_{A_*}. \tag{23}$$

Let $f(x) = \sigma\big(|u(x)|/K\big)$. Evidently, $u \in W^{1,1}(\Omega)$ and σ is Lipschitz on the range of $|u|/K$ so that, by the previous lemma, f belongs to $W^{1,1}(\Omega)$. By Theorem 4.12 we have $W^{1,1}(\Omega) \to L^{n/(n-1)}(\Omega)$ and so

$$\begin{aligned}\|f\|_{n/(n-1)} &\le K_1\left(\sum_{j=1}^n \big\|D_jU\big\|_1 + \|f\|_1\right)\\ &= K_1\left[\sum_{j=1}^n \frac{1}{K}\int_\Omega \sigma'\left(\frac{|u(x)|}{K}\right)|D_ju(x)|\,dx + \int_\Omega \sigma\left(\frac{|u(x)|}{K}\right)dx\right].\end{aligned} \tag{24}$$

By (23) and Hölder's inequality (10), we obtain

$$1 = \left(\int_\Omega A_*\left(\frac{|u(x)|}{K}\right) dx\right)^{(n-1)/n} = \|f\|_{n/(n-1)}$$
$$\leq \frac{2K_1}{K}\sum_{j=1}^{n}\left\|\sigma'\left(\frac{|u|}{K}\right)\right\|_{\tilde{A}} \left\|D_j u\right\|_A + K_1\int_\Omega \sigma'\left(\frac{|u(x)|}{K}\right) dx. \tag{25}$$

Making use of (22), we have

$$\left\|\sigma'\left(\frac{|u|}{K}\right)\right\|_{\tilde{A}} \leq \frac{n-1}{n}\left\|\tilde{A}^{-1}\left(\left(\sigma\left(\frac{|u|}{K}\right)\right)^{n/(n-1)}\right)\right\|_{\tilde{A}}$$
$$= \frac{n-1}{n}\inf\left\{\lambda > 0 : \int_\Omega \tilde{A}\left(\frac{\tilde{A}^{-1}\left(A_*(|u(x)|/K)\right)}{\lambda}\right) dx \leq 1\right\}.$$

Suppose $\lambda > 1$. Then

$$\int_\Omega \tilde{A}\left(\frac{\tilde{A}^{-1}\left(A_*(|u(x)|/K)\right)}{\lambda}\right) dx \leq \frac{1}{\lambda}\int_\Omega A_*\left(\frac{|u(x)|}{K}\right) dx = \frac{1}{\lambda} < 1.$$

Thus

$$\left\|\sigma'\left(\frac{|u|}{K}\right)\right\|_{\tilde{A}} \leq \frac{n-1}{n}. \tag{26}$$

Let $g(t) = A_*(t)/t$ and $h(t) = \sigma(t)/t$. It is readily checked that h is bounded on finite intervals and $\lim_{t\to\infty} g(t)/h(t) = \infty$. Thus there exists a constant t_0 such that $h(t) \leq g(t)/(2K)$ if $t \geq t_0$. Putting $K_2 = K_2 \sup_{0\leq t\leq t_0} h(t)$, we have, for all $t \geq 0$,

$$\sigma(t) \leq \frac{1}{2K_1} A_*(t) + \frac{K_2}{K_1} t.$$

Hence

$$K_1\int_\Omega \sigma\left(\frac{|u(x)|}{K}\right) dx \leq \frac{1}{2}\int_\Omega A_*\left(\frac{|u(x)|}{K}\right) dx + \frac{K_2}{K_1}\int_\Omega |u(x)|\, dx$$
$$\leq \frac{1}{2} + \frac{K_3}{K}\|u\|_A, \tag{27}$$

where $K_3 = 2K_2\|1\|_{\tilde{A}} < \infty$ since Ω has finite volume.

Combining (25)–(27), we obtain

$$1 \leq \frac{2K_1}{K}(n-1)\|u\|_{1,A} + \frac{1}{2} + \frac{K_3}{K}\|u\|_A,$$

so that

$$\|u\|_{A_*} = K \le K_4 \|u\|_{1,A}, \tag{28}$$

where K_4 depends only on n, A, $\mathrm{vol}(\Omega)$, and the cone determining the cone condition for Ω.

To extend (28) to arbitrary $u \in W^1L_A(\Omega)$ let

$$u_k(x) = \begin{cases} |u(x)| & \text{if } |u(x)| \le k \\ k \operatorname{sgn} u(x) & \text{if } |u(x)| > k. \end{cases} \tag{29}$$

Clearly u_k is bounded and it belongs to $W^1L_A(\Omega)$ by the previous lemma. Moreover, $\|u_k\|_{A_*}$ increases with k but is bounded by $K_4 \|u\|_A$. Therefore, $\lim_{k\to\infty} \|u_k\|_{A_*} = K$ exists and $K \le K_4 \|u\|_{1,A}$. By Fatou's lemma 1.49

$$\int_\Omega A_*\left(\frac{|u(x)|}{K}\right) dx \le \lim_{k\to\infty} \int_\Omega A_*\left(\frac{|u_k(x)|}{K}\right) dx \le 1$$

whence $u \in L_{A_*}(\Omega)$ and (28) holds.

Since Ω has finite volume we have

$$W^1L_A(\Omega) \to W^{1,1}(\Omega) \to L^1(\Omega),$$

the latter imbedding being compact by Theorem 6.3. A bounded subset of $W^1L_A(\Omega)$ is bounded in $L_{A_*}(\Omega)$ and precompact in $L^1(\Omega)$, and hence precompact in $L_B(\Omega)$ by Theorem 8.25 whenever B increases essentially more slowly than A_* near infinity. ∎

Theorem 8.35 extends to arbitrary (even unbounded) domains Ω provided W is replaced by W_0.

8.36 THEOREM Let Ω be an arbitrary domain in $\mathbb{R}^n$. If the N function A satisfies (18) and (19), then

$$W_0^m L_A(\Omega) \to L_{A_*}(\Omega).$$

Moreover, if Ω_0 is a bounded subdomain of Ω, then the imbedding

$$W_0^m L_A(\Omega) \to L_B(\Omega_0)$$

exists and is compact for any N-function B increasing essentially more slowly that A_* near infinity.

Proof. If $u \in W_0^m L_A(\Omega)$, then the function f in the proof of Theorem 8.35 can be approximated in $W^{1,1}(\Omega)$ by elements of $C_0^\infty(\Omega)$. By Sobolev's inequality

(Theorem 4.31), (24) holds with the term $\|f\|_1$ absent from the right side. Therefore (27) is not needed and the proof does not require that Ω have finite volume. The cone condition is not required either, since Sobolev's inequality holds for all $u \in C_0^\infty(\mathbb{R}^n)$. The compactness arguments are similar to those above. ∎

8.37 REMARK Theorem 8.35 is not optimal in the sense that for some A, L_{A_*} is not necessarily the smallest Orlicz space in which $W^1L_A(\Omega)$ can be imbedded. For instance, if $A(t) = A_n(t) = t^n/n$, then, as noted earlier, $A_*(t)$ is equivalent near infinity to $e^t - t - 1$, an N-function that increases essentially more slowly near infinity than does $\exp(t^{n/(n-1)}) - 1$. Thus Theorem 8.27 gives a sharper result than Theorem 8.35 in this case. In [DT] Donaldson and Trudinger state that Theorem 8.35 can be improved by the methods of Theorem 8.27 provided A dominates near infinity every t^p with $p < n$, but that Theorem 8.35 gives optimal results if for some $p < n$, t^p dominates A near infinity. The former cases are those where [Ci] improves on Theorem 8.35.

There are also some unbounded domains [Ch] for which some Orlicz-Sobolev imbeddings are compact.

The following theorem generalizes (the case $m = 1$ of) the part of Theorem 4.12 dealing with traces on lower dimensional hyperplanes.

8.38 THEOREM (Traces on Planes) Let Ω be a bounded domain in $\mathbb{R}^n$ satisfying the cone condition, and let Ω_k denote the intersection of Ω with a k-dimensional plane in $\mathbb{R}^n$. Let A be an N-function for which (18) and (19) hold, and let A_* be given by (20). Let $1 \le p < n$ where p is such that the function B defined by $B(t) = A(t^{1/p})$ is an N-function. If either $n - p < k \le n$ or $p = 1$ and $n - 1 \le k \le n$, then

$$W^1L_A(\Omega) \to L_{A_*^{k/n}}(\Omega_k),$$

where $A_*^{k/n}(t) = [A_*(t)]^{k/n}$.

Moreover, if $p > 1$ and C is an N-function increasing essentially more slowly than $A_*^{k/n}$ near infinity, then the imbedding

$$W^1L_A(\Omega) \to L_C(\Omega_k) \tag{30}$$

is compact.

Proof. The problem of verifying that $A_*^{k/n}$ is an N-function is left to the reader. Let $u \in W^1L_A(\Omega)$ be a bounded function. Then

$$\int_{\Omega_k} A_*^{k/n}\left(\frac{|u(y)|}{K}\right) dy = 1, \qquad K = \|u\|_{A_*^{k/n},\Omega_k}. \tag{31}$$

We wish to show that

$$K \le K_1 \|u\|_{1,A,\Omega} \tag{32}$$

with K_1 independent of u. Since this inequality is known to hold for the special case $k = n$ (Theorem 8.35) we may assume without loss of generality that

$$K \ge \|u\|_{A_*,\Omega} = \|u\|_{A_*^{n/n},\Omega_n}. \tag{33}$$

Let $\omega(t) = [A_*(t)]^{1/q}$ where $q = np/(n-p)$. By (case $m = 1$ of) Theorem 4.12 we have

$$\begin{aligned}
\left\|\omega\left(\frac{|u|}{K}\right)\right\|^p_{kp/(n-p),\Omega_k} &\le K_2\left[\sum_{j=1}^n \left\|D_j\omega\left(\frac{|u|}{K}\right)\right\|^p_{p,\Omega} + \left\|\omega\left(\frac{|u|}{K}\right)\right\|^p_{p,\Omega}\right] \\
&= K_2\left[\frac{1}{K^p}\sum_{j=1}^n \int_\Omega \left|\omega'\left(\frac{|u(x)|}{K}\right)\right|^p |D_ju(x)|^p\,dx \right.\\
&\qquad \left. + \int_\Omega \left|\omega\left(\frac{|u(x)|}{K}\right)\right|^p dx\right].
\end{aligned}$$

Using (31) and noting that $\||v|^p\|_{B,\Omega} \le \|v\|^p_{A,\Omega}$, we obtain

$$\begin{aligned}
1 &= \left[\int_{\Omega_k}\left(A_*\left(\frac{|u(y)|}{K}\right)\right)^{k/n} dy\right]^{(n-p)/k} = \left\|\omega\left(\frac{|u|}{K}\right)\right\|^p_{kp/(n-p),\Omega_k} \\
&\le \frac{2K_2}{K^p}\sum_{j=1}^n \left\|\left(\omega'\left(\frac{|u|}{K}\right)\right)^p\right\|_{\tilde{B},\Omega} \||D_ju|^p\|_{B,\Omega} + K_2\left\|\omega\left(\frac{|u|}{K}\right)\right\|^p_{p,\Omega} \\
&\le \frac{2nK_2}{K^p}\left\|\left(\omega'\left(\frac{|u|}{K}\right)\right)^p\right\|_{\tilde{B},\Omega} \|u\|^p_{1,A,\Omega} + K_2\left\|\omega\left(\frac{|u|}{K}\right)\right\|^p_{p,\Omega}.
\end{aligned} \tag{34}$$

Now $B^{-1}(t) = \left(A^{-1}(t)\right)^p$ and so, using (21) and (4), we have

$$\begin{aligned}
\left(\omega'(t)\right)^p &= \frac{1}{q^p}\left(A_*(t)\right)^{p(1-q)/q}\left(A_*'(t)\right)^p \\
&= \frac{1}{q^p}A_*(t)\frac{1}{B^{-1}\left(A_*(t)\right)} \le \frac{1}{q^p}\tilde{B}^{-1}\left(A_*(t)\right).
\end{aligned}$$

It follows by (33) that

$$\int_\Omega \tilde{B}\left(\left(\frac{\omega'(|u(x)|/K)}{1/q}\right)^p\right) dx \le \int_\Omega A_*\left(\frac{|u(x)|}{K}\right) dx \le 1.$$

So

$$\left\| \left(\omega' \left(\frac{|u|}{K} \right) \right)^p \right\|_{\tilde{B},\Omega} \le \frac{1}{q^p}. \tag{35}$$

Now set $g(t) = A_*(t)/t^p$ and $h(t) = \big(\omega(t)/t\big)^p$. It is readily checked that $\lim_{t\to\infty} g(t)/h(t) = \infty$. In order to see that $h(t)$ is bounded near zero let $s = A_*(t)$ and consider

$$\big(h(t)\big)^{1/p} = \frac{\big(A_*(t)\big)^{1/q}}{t} = \frac{s^{(1/p)-(1/n)}}{\displaystyle\int_0^s \frac{A^{-1}(\tau)}{\tau^{(n+1)/n}}\, d\tau} \le \frac{s^{1/p}}{\displaystyle\int_0^s \frac{\big(B^{-1}(\tau)\big)^{1/p}}{\tau}} d\tau.$$

Since B is an N-function $\lim_{\tau\to\infty} B^{-1}(\tau)/\tau = \infty$. Hence, for sufficiently small values of t we have

$$\big(h(t)\big)^{1/p} \le \frac{s^{1/p}}{\displaystyle\int_0^s \tau^{-1+(1/p)}\, d\tau} = \frac{1}{p}.$$

Therefore, there exists a constant K_3 such that for $t \ge 0$

$$\big(\omega(t)\big)^p \le \frac{1}{2K_2} A_*(t) + K_3 t^p.$$

Using (33) we now obtain

$$\begin{aligned} \left\| \omega\left(\frac{|u|}{K}\right) \right\|_{p,\Omega}^p &\le \frac{1}{2K_2} \int_\Omega A_*\left(\frac{|u(x)|}{K}\right) dx + \frac{K_3}{K^p} \int_\Omega |u(x)|^p\, dx \\ &\le \frac{1}{2K_2} + \frac{2K_3}{K^p} \big\| |u|^p \big\|_{B,\Omega} \|1\|_{\tilde{B},\Omega} \\ &\le \frac{1}{2K_2} + \frac{K_4}{K^p} \|u\|_{A,\Omega}^p . \end{aligned} \tag{36}$$

From (34)–(36) there follows the inequality

$$1 \le \frac{2nK_2}{K^p} \cdot \frac{1}{q^p} \|u\|_{1,A,\Omega}^p + \frac{1}{2} + \frac{K_4 K_2}{K^p} \|u\|_{A,\Omega}^p$$

and hence (32). The extension of (32) to arbitrary $u \in W^1 L_A(\Omega)$ now follows as in the proof of Theorem 8.35.

Since $B(t) = A(t^{1/p})$ is an N-function and Ω is bounded, we have

$$W^1 L_A(\Omega) \to W^{1,p}(\Omega) \to L^1(\Omega_k),$$

the latter imbedding being compact by Theorem 6.3 provided $p > 1$. The compactness of (30) now follows from Theorem 8.25. ∎

8.39 THEOREM (Imbedding Into a Space of Continuous Functions) Let Ω satisfy the cone condition in $\mathbb{R}^n$. Let A be an N-function for which

$$\int_1^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt < \infty. \tag{37}$$

Then

$$W^1L_A(\Omega) \to C_B^0(\Omega) = C(\Omega) \cap L^\infty(\Omega).$$

Proof. Let C be a finite cone contained in Ω. We shall show that there exists a constant K_1 depending on n, A, and the dimensions of C such that

$$\|u\|_{\infty,C} \le K_1 \|u\|_{1,A,C}\,. \tag{38}$$

In doing so, we may assume without loss of generality that A satisfies (18), for if not, and if B is an N-function satisfying (18) and equivalent to A near infinity, then $W^1L_A(C) \to W^1L_B(C)$ with imbedding constant depending on A, B, and $\operatorname{vol}(C)$ by Theorem 8.12. Since B satisfies (37) we would have

$$\|u\|_{\infty,C} \le K_2 \|u\|_{1,B,C} \le K_3 \|u\|_{1,A,C}\,.$$

Now Ω is a union of congruent copies of some such finite cone C so that (38) clearly implies

$$\|u\|_{\infty,\Omega} \le K_1 \|u\|_{1,A,\Omega}\,. \tag{39}$$

Since A is assumed to satisfy (18) and (37) we have

$$\int_0^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt = K_4 < \infty.$$

Let

$$\Lambda^{-1}(t) = \int_0^t \frac{A^{-1}(\tau)}{\tau^{(n+1)/n}}\,d\tau.$$

The Λ^{-1} maps $[0,\infty)$ in a one-to-one manner onto $[0, K_4)$ and has a convex inverse Λ. We extend the domain of definition of Λ to $[0,\infty)$ by defining $\Lambda(t) = \infty$ for $t \ge K_4$. The function Λ is a *Young's function.* (See Luxemburg [Lu] or O'Neill [O].) Although it is not an N-function in the sense defined early in this chapter, nevertheless the Luxemburg norm

$$\|u\|_{\Lambda,C} = \inf\left\{k > 0 : \int_C \Lambda\left(\frac{|u(x)|}{k}\right) dx \le 1\right\}$$

is easily seen to be a norm on $L^\infty(C)$ equivalent to the usual norm; in fact,

$$\frac{1}{K_4}\,\|u\|_{\infty,C} \le \|u\|_{\Lambda,C} \le \frac{1}{\Lambda^{-1}(1/\mathrm{vol}(C))}\,\|u\|_{\infty,C}\,.$$

Moreover, $s = \Lambda(t)$ satisfies the differential equation (21), so that the proof of Theorem 8.35 can be carried over in this case to yield, for $u \in W^1L_A(C)$,

$$\|u\|_{\Lambda,C} \le K_5\,\|u\|_{1,A,C}$$

and inequality (38) follows.

By Theorem 8.31(d) an element $u \in W^mE_A(\Omega)$ can be approximated in norm by functions continuous on Ω. It follows from (39) that u must coincide a.e. on Ω with a continuous function. (See Paragraph 4.16.)

Suppose that an N-function B can be constructed such that the following conditions are satisfied:

(a) $B(t) = A(t)$ near zero.

(b) B increases essentially more slowly than A near infinity.

(c) B satisfies

$$\int_1^\infty \frac{B^{-1}(t)}{t^{(n+1)/n}}\,dt \le 2\int_1^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt < \infty.$$

Then, by Theorem 8.16, $u \in W^1L_A(C)$ implies $u \in W^1E_B(C)$ so that we have $W^1L_A(\Omega) \subset C(\Omega)$ as required.

It remains, therefore, to construct an N-function B having the properties (a)–(c). Let $1 < t_1 < t_2 < \cdots$ be such that

$$\int_{t_k}^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt = \frac{1}{2^{2k}}\int_1^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt.$$

We define a sequence $\{s_k\}$ with $s_k \ge t_k$, and the function $B^{-1}(t)$, inductively as follows.

Let $s_1 = t_1$ and $B^{-1}(t) = A^{-1}(t)$ for $0 \le t \le s_1$. Having chosen $s_1, s_2, \dots, s_k$ and defined $B^{-1}(t)$ for $0 \le t \le s_{k-1}$, we continue $B^{-1}(t)$ to the right of s_{k-1} along a straight line with slope $(A^{-1})'(s_{k-1}-)$ (which always exists since A^{-1} is concave) until a point t_k' is reached where $B^{-1}(t_k') = 2^{k-1}A^{-1}(t_k')$. Such t_k' exists because $\lim_{t\to\infty} A^{-1}(t)/t = 0$. If $t_k' \ge t_k$, let $s_k = t_k'$. Otherwise let $s_k = t_k$ and extend B^{-1} from t_k' to s_k by setting $B^{-1}(t) = 2^{k-1}A^{-1}(t)$. Evidently B^{-1} is concave and B is an N-function. Moreover, $B(t) = A(t)$ near zero and since

$$\lim_{t\to\infty}\frac{B^{-1}(t)}{A^{-1}(t)} = \infty,$$

B increases essentially more slowly than A near infinity. Finally,

$$\begin{aligned}\int_1^\infty \frac{B^{-1}(t)}{t^{(n+1)/n}}\,dt &\le \int_1^{s_1} \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt + \sum_{k=2}^{\infty} \int_{s_{k-1}}^{s_k} \frac{2^{k-1}A^{-1}(t)}{t^{(n+1)/n}}\,dt \\ &\le \int_1^{s_1} \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt + \sum_{k=2}^{\infty} 2^{k-1} \int_{t_{k-1}}^{\infty} \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt \\ &= 2\int_1^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt,\end{aligned}$$

as required. ∎

8.40 THEOREM (Uniform Continuity) Let Ω be a domain in $\mathbb{R}^n$ satisfying the strong local Lipschitz condition. If the N-function A satisfies

$$\int_1^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt < \infty, \tag{40}$$

then there exists a constant K such that for any $u \in W^1L_A(\Omega)$ (which may be assumed continuous by the previous theorem) and all $x, y \in \Omega$ we have

$$|u(x) - u(y)| \le K\,\|u\|_{1,A,\Omega} \int_{|x-y|^{-n}}^{\infty} \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt. \tag{41}$$

Proof. We establish (41) for the case where Ω is a cube of unit edge; the extension to more general strongly Lipschitz domains can then be carried out just as in the proof of Lemma 4.28. As in that lemma we let Ω_σ denote a parallel subcube of Ω having edge σ and obtain for $x \in \overline{\Omega_\sigma}$

$$\left| u(x) - \frac{1}{\sigma^n}\int_{\Omega_\sigma} u(z)\,dz \right| \le \frac{\sqrt{n}}{\sigma^{n-1}} \int_0^1 t^{-n}\,dt \int_{\Omega_{t\sigma}} |\mathrm{grad}\,u|\,dz.$$

By (15), $\|1\|_{\tilde{A},\Omega_{t\sigma}} = 1/\tilde{A}^{-1}(t^{-n}\sigma^{-n})$. It follows by Hölder's inequality and (4) that

$$\begin{aligned}\int_{\Omega_{t\sigma}} |\mathrm{grad}\,u|\,dz &\le 2\,\|\mathrm{grad}\,u\|_{A,\Omega_{t\sigma}}\,\|1\|_{\tilde{A},\Omega_{t\sigma}} \\ &\le \frac{2}{\tilde{A}^{-1}(t^{-n}\sigma^{-n})}\,\|u\|_{1,A,\Omega} \\ &\le 2\sigma^n t^n A^{-1}(t^{-n}\sigma^{-n})\,\|u\|_{1,A,\Omega}\,.\end{aligned}$$

Hence

$$\begin{aligned}\left| u(x) - \frac{1}{\sigma^n}\int_{\Omega_\sigma} u(z)\,dz \right| &\le 2\sqrt{n}\sigma\,\|u\|_{1,A,\Omega} \int_0^1 A^{-1}\left(\frac{1}{t^n\sigma^n}\right)dt \\ &= \frac{2}{\sqrt{n}}\,\|u\|_{1,A,\Omega} \int_{\sigma^{-n}}^{\infty} \frac{A^{-1}(\tau)}{\tau^{(n+1)/n}}\,d\tau.\end{aligned}$$

If $x, y \in \Omega$ and $\sigma = |x-y| < 1$, then there exists a subcube Ω_σ with $x, y \in \overline{\Omega_\sigma}$, and it follows from the above inequality applied to both x and y that

$$|u(x) - u(y)| \le \frac{4}{\sqrt{n}} \|u\|_{1,A,\Omega} \int_{|x-y|^{-n}}^{\infty} \frac{A^{-1}(t)}{t^{(n+1)/n}}\, dt.$$

For $|x-y| \ge 1$, (41) follows directly from (39) and (40). ∎

8.41 (Generalization of Hölder Continuity) Let M denote the class of positive, continuous, increasing functions of $t > 0$. If $\mu \in M$, the space $C_\mu(\overline{\Omega})$, consisting of those functions $u \in C(\overline{\Omega})$ for which the norm

$$\left\| u\,;\, C_\mu(\overline{\Omega}) \right\| = \left\| u\,;\, C(\overline{\Omega}) \right\| + \sup_{\substack{x,y\in\Omega \\ x\ne y}} \frac{|u(x)-u(y)|}{\mu(|x-y|)}$$

is finite, is a Banach space under that norm. The theorem above asserts that if (40) holds, then

$$W^1 L_A(\Omega) \to C_\mu(\overline{\Omega}), \qquad \text{where} \quad \mu(t) = \int_{|x-y|^{-n}}^{\infty} \frac{A^{-1}(t)}{t^{(n+1)/n}}\, dt. \tag{42}$$

If $\mu, \nu \in M$ are such that $\mu/\nu \in M$, then for bounded Ω we have, as in Theorem 1.34, that the imbedding

$$C_\mu(\overline{\Omega}) \to C_\nu(\overline{\Omega})$$

exists and is compact. Hence the imbedding

$$W^1 L_A(\Omega) \to C_\nu(\overline{\Omega})$$

is also compact if μ is given as in (42).

8.42 (Generalization to Higher Orders of Smoothness) We now prepare to state the general Orlicz-Sobolev imbedding theorem of Donaldson and Trudinger [DT] by generalizing the framework used for imbeddings of $W^1 L_A(\Omega)$ considered above so that we can formulate imbeddings of $W^m L_A(\Omega)$.

For a given N-function A we define a sequence of N-functions $B_0, B_1, B_2, \ldots$ as follows:

$$B_0(t) = A(t)$$

$$(B_k)^{-1}(t) = \int_0^t \frac{(B_{k-1})^{-1}(\tau)}{\tau^{(n+1)/n}}\, d\tau, \qquad k = 1, 2, \ldots.$$

(Observe that $B_1 = A_*$.) At each stage we assume that

$$\int_0^1 \frac{(B_k)^{-1}(t)}{t^{(n+1)/n}}\, dt < \infty, \tag{43}$$

replacing B_k, if necessary, with another N-function equivalent to it near infinity and satisfying (43).

Let $J = J(A)$ be the smallest nonnegative integer such that

$$\int_1^\infty \frac{(B_J)^{-1}(t)}{t^{(n+1)/n}}\,dt < \infty.$$

Evidently, $J(A) \le n$. If μ belongs to the class M defined in the previous Paragraph, we define the space $C_\mu^m(\overline{\Omega})$ to consist of those functions $u \in C(\overline{\Omega})$ for which $D^\alpha u \in C_\mu(\overline{\Omega})$ whenever $|\alpha| \le m$. The space $C_\mu^m(\overline{\Omega})$ is a Banach space with respect to the norm

$$\left\| u\,;C_\mu^m(\overline{\Omega})\right\| = \max_{|\alpha|\le m} \left\| D^\alpha u\,; C_\mu(\overline{\Omega})\right\|.$$

8.43 THEOREM (A General Orlicz-Sobolev Imbedding Theorem) Let Ω be a bounded domain in $\mathbb{R}^n$ satisfying the cone condition. Let A be an N-function.

(a) If $m \le J(A)$, then $W^m L_A(\Omega) \to L_{B_m}(\Omega)$. Moreover, if B is an N-function increasing essentially more slowly than B_m near infinity, then the imbedding $W^m L_A(\Omega) \to L_B(\Omega)$ exists and is compact.

(b) If $m > J(A)$, then $W^m L_A(\Omega) \to C_B^0(\Omega) = C^0(\Omega) \cap L^\infty(\Omega)$.

(c) If $m > J(A)$ and Ω satisfies the strong local Lipschitz condition, then $W^m L_A(\Omega) \to C_\mu^{m-J-1}(\overline{\Omega})$ where

$$\mu(t) = \int_{t^{-n}}^\infty \frac{(B_J)^{-1}(\tau)}{\tau^{(n+1)/n}}\,d\tau.$$

Moreover, the imbedding $W^m L_A(\Omega) \to C^{m-J-1}(\overline{\Omega})$ is compact and so is $W^m L_A(\Omega) \to C_\nu^{m-J-1}(\overline{\Omega})$ provided $\nu \in M$ and $\mu/\nu \in M$.

8.44 REMARK Theorem 8.43 follows in a straightforward way from the special cases with $m = 1$ provided earlier. Also, if we replace L_A by E_A in part (a) we get $W^m E_A(\Omega) \to E_{B_m}(\Omega)$ since the sequence $\{u_k\}$ defined by (29) converges to u if $u \in W^1 E_A(\Omega)$. Theorem 8.43 holds without any restrictions on Ω if $W^m L_A(\Omega)$ is replaced with $W_0^m L_A(\Omega)$.

8.45 REMARK Since Theorem 8.43 implies that $W^m L_A(\Omega) \to W^1 L_{B_{m-1}}(\Omega)$, we will also have $W^m L_A(\Omega) \to L_{[(B_m)^{k/n}]}(\Omega_k)$, where Ω_k is the intersection of Ω with a k-dimensional plane in $\mathbb{R}^n$, provided that (using Theorem 8.38) there exists p satisfying $1 \le p < n$ for which $n - p < k \le n$ (or $n - 1 \le k \le n$ if $p = 1$) and $B(t) = B_m(t^{1/p})$ is an N-function.

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SOBOLEV SPACES

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Contents

Preface

This monograph is devoted to a study of properties of certain Banach spaces of weakly differentiable functions of several real variables which arise in connection with numerous problems in the theory of partial differential equations and related areas of mathematical analysis, and which have become an essential tool in those disciplines. These spaces are now most often associated with the name of the Soviet mathematician S. L. Sobolev, though their origins predate his major contributions to their development in the late 1930s.

Sobolev spaces are very interesting mathematical structures in their own right, but their principal significance lies in the central role they, and their numerous generalizations, now play in partial differential equations. Accordingly, most of this book concentrates on those aspects of the theory of Sobolev spaces that have proven most useful in applications. Although no specific applications to problems in partial differential equations are discussed (these are to be found in almost any modern textbook on partial differential equations), this monograph is nevertheless intended mainly to serve as a textbook and reference on Sobolev spaces for graduate students and researchers in differential equations. Some of the material in Chapters III–VI has grown out of lecture notes [18] for a graduate course and seminar given by Professor Colin Clark at the University of British Columbia in 1967–1968.

The material is organized into eight chapters. Chapter I is a potpourri of standard topics from real and functional analysis, included, mainly without proofs, because they form a necessary background for what follows. Chapter II is also largely "background" but concentrates on a specific topic, the Lebesgue spaces $L^p(\Omega)$, of which Sobolev spaces are special subspaces. For completeness, proofs are included here. Most of the material in these first two chapters will be quite familiar to the reader and may be omitted, or simply given a superficial reading to settle questions of notation and such. (Possible exceptions are Sections 1.25–1.27, 1.31, and 2.21–2.22 which may be less familiar.) The inclusion of these elementary chapters makes the book fairly self-contained. Only a solid undergraduate background in mathematical analysis is assumed of the reader.

Chapters III–VI may be described as the heart of the book. These develop all the basic properties of Sobolev spaces of positive integral order and culminate in the very important Sobolev imbedding theorem (Theorem 5.4) and the corresponding compact imbedding theorem (Theorem 6.2). Sections 5.33–5.54 and 6.12–6.50 consist of refinements and generalizations of these basic imbedding theorems, and could be omitted from a first reading.

Chapter VII is concerned with generalization of ordinary Sobolev spaces to allow fractional orders of differentiation. Such spaces are often involved in research into nonlinear partial differential equations, for instance the Navier–Stokes equations of fluid mechanics. Several approaches to defining fractional-order spaces can be taken. We concentrate in Chapter VII on the trace-interpolation approach of J. L. Lions and E. Magenes, and discuss other approaches more briefly at the end of the chapter (Sections 7.59–7.74). It is necessary to develop a reasonable body of abstract functional analysis (the trace-interpolation theory) before introducing the fractional-order spaces. Most readers will find that a reading of this material (in Sections 7.2–7.34, possibly omitting proofs) is essential for an understanding of the discussion of fractional-order spaces that begins in Section 7.35.

Chapter VIII concerns Orlicz–Sobolev spaces and, for the sake of completeness, necessarily begins with a self-contained introduction to the theory of Orlicz spaces. These spaces are finding increasingly important applications in applied analysis. The main results of Chapter VIII are the theorem of N. S. Trudinger (Theorem 8.25) establishing a limiting case of the Sobolev imbedding theorem, and the imbedding theorems of Trudinger and T. K. Donaldson for Orlicz–Sobolev spaces given in Sections 8.29–8.40.

The existing mathematical literature on Sobolev spaces and their generalizations is vast, and it would be neither easy nor particularly desirable to include everything that was known about such spaces between the covers of one book. An attempt has been made in this monograph to present all the core material in sufficient generality to cover most applications, to give the reader an overview of the subject that is difficult to obtain by reading research papers, and finally, as mentioned above, to provide a ready reference for someone requiring a result about Sobolev spaces for use in some application. Complete proofs are given for most theorems, but some assertions are left for the interested reader to verify as exercises. Literature references are given in square brackets, equation numbers in parentheses, and sections are numbered in the form *m.n* with *m* denoting the chapter.

Acknowledgments

We acknowledge with deep gratitude the considerable assistance we have received from Professor John Fournier in the preparation of this monograph. Also much appreciated are the helpful comments received from Professor Bui An Ton and the encouragement of Professor Colin Clark who originally suggested that this book be written. Thanks are also due to Mrs. Yit-Sin Choo for a superb job of typing a difficult manuscript. Finally, of course, we accept all responsibility for error or obscurity and welcome comments, or corrections, from readers.

List of Spaces and Norms

The numbers at the right indicate the sections in which the symbols are introduced. In some cases the notations are not those used in other areas of analysis.

B	$\|\cdot\|_B$	7.2
B_1+B_2	$\|\cdot\,; B_1+B_2\|$	7.11
$B^{s,p}(\mathbb{R}^n)$	$\|\cdot\,; B^{s,p}(\mathbb{R}^n)\|$	7.67
$B^{s,p}(\Omega)$	$\|\cdot\,; B^{s,p}(\Omega)\|$	7.72
$\mathbb{C}$	$\lvert\cdot\rvert$	1.1
$C(\Omega)$		1.25
$C^m(\Omega)$		1.25
$C^\infty(\Omega)$		1.25
$C_0(\Omega)$		1.25
$C_0^\infty(\Omega)$		1.25
$C^m(\overline{\Omega})$	$\|\cdot\,; C^m(\overline{\Omega})\|$	1.26
$C^{m,\lambda}(\overline{\Omega})$	$\|\cdot\,; C^{m,\lambda}(\overline{\Omega})\|$	1.27
$C_B^j(\Omega)$	$\|\cdot\,; C_B^j(\Omega)\|$	5.2
$C_\mu(\overline{\Omega})$	$\|\cdot\,; C_\mu(\overline{\Omega})\|$	8.37
$\mathscr{D}(\Omega)$		1.51
$\mathscr{D}'(\Omega)$		1.52
$D(\Lambda)$	$\|\cdot\,; D(\Lambda)\|$	7.7, 7.9

I

Introductory Topics

Notation

1.1 Throughout this monograph the term *domain* and the symbol Ω shall be reserved for an open set in n-dimensional, real Euclidean space $\mathbb{R}^n$. We shall be concerned with differentiability and integrability of functions defined on Ω—these functions are allowed to be complex valued unless the contrary is stated explicitly. The complex field is denoted by $\mathbb{C}$. For $c \in \mathbb{C}$ and two functions u and v the scalar multiple cu, the sum $u+v$, and the product uv are always taken to be defined pointwise as

$$(cu)(x) = cu(x),$$
$$(u+v)(x) = u(x) + v(x),$$
$$(uv)(x) = u(x)\,v(x),$$

at all points x where the right sides make sense.

A typical point in $\mathbb{R}^n$ is denoted by $x = (x_1, \ldots, x_n)$; its norm $|x| = (\sum_{j=1}^n x_j^2)^{1/2}$. The inner product of x and y is $x \cdot y = \sum_{j=1}^n x_j y_j$.

If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of nonnegative integers α_j, we call α a *multi-index* and denote by x^α the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, which has degree $|\alpha| = \sum_{j=1}^n \alpha_j$. Similarly, if $D_j = \partial/\partial x_j$ for $1 \le j \le n$, then

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$$

denotes a differential operator of order $|\alpha|$. $D^{(0,\ldots,0)}u = u$.

If α and β are two multi-indices, we say $\beta \le \alpha$ provided $\beta_j \le \alpha_j$ for $1 \le j \le n$. In this case $\alpha - \beta$ is also a multi-index and $|\alpha - \beta| + |\beta| = |\alpha|$. We also denote

$$\alpha! = \alpha_1! \cdots \alpha_n!$$

and if $\beta \le \alpha$,

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}.$$

The reader may wish to verify the Leibniz formula

$$D^\alpha(uv)(x) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta u(x)\, D^{\alpha-\beta} v(x)$$

valid for functions u and v that are $|\alpha|$ times continuously differentiable near x.

1.2 If $G \subset \mathbb{R}^n$, we denote by $\bar{G}$ the closure of G in $\mathbb{R}^n$. We shall write $G \subset\subset \Omega$ provided $\bar{G} \subset \Omega$ and $\bar{G}$ is a compact (i.e., closed and bounded) subset of $\mathbb{R}^n$. If u is a function defined on G, we define the *support* of u as

$$\operatorname{supp} u = \overline{\{x \in G : u(x) \ne 0\}}.$$

We say that u has compact support in Ω if $\operatorname{supp} u \subset\subset \Omega$. We shall denote by "bdry G" the boundary of G in $\mathbb{R}^n$, that is, the set $\bar{G} \cap \overline{G^c}$ where $G^c = \mathbb{R}^n \sim G = \{x \in \mathbb{R}^n : x \notin G\}$ is the complement of G.

If $x \in \mathbb{R}^n$ and $G \subset \mathbb{R}^n$, we denote by "dist(x, G)" the distance from x to G, that is, the number $\inf_{y \in G} |x - y|$. Similarly, if $F, G \subset \mathbb{R}^n$,

$$\operatorname{dist}(F, G) = \inf_{y \in F} \operatorname{dist}(y, G) = \inf_{\substack{x \in G \\ y \in F}} |x - y|.$$

Topological Vector Spaces

1.3 We assume that the reader is familiar with the concept of a vector space over the real or complex scalar field, and with the related notions of dimension, subspace, linear transformation, and convex set. We also assume familiarity with the basic concepts of general topology, Hausdorff topological spaces, weaker and stronger topologies, continuous functions, convergent sequences, topological product spaces, subspaces, and relative topology.

Let it be assumed throughout this monograph that all vector spaces referred to are taken over the complex field unless the contrary is explicitly stated.

1.4 A *topological vector space*, hereafter abbreviated TVS, is a Hausdorff topological space that is also a vector space for which the vector space operations of addition and scalar multiplication are continuous. That is, if X is a TVS, then the mappings

$$(x,y) \to x+y \qquad \text{and} \qquad (c,x) \to cx$$

from the topological product spaces $X \times X$ and $\mathbb{C} \times X$, respectively, into X are continuous. X is a *locally convex* TVS if each neighborhood of the origin in X contains a convex neighborhood.

We shall outline below, mainly omitting proofs and details, those aspects of the theory of topological and normed vector spaces that play a significant role in the study of Sobolev spaces. For a more thorough discussion of these topics the reader is referred to standard textbooks on functional analysis, for example, those by Yosida [69] or Rudin [59].

1.5 By a *functional* on a vector space X we mean a scalar-valued function f defined on X. The functional f is linear provided

$$f(ax+by) = af(x) + bf(y), \qquad x,y \in X, \quad a,b \in \mathbb{C}.$$

If X is a TVS, a functional on X is continuous if it is continuous from X into $\mathbb{C}$ where $\mathbb{C}$ has its usual topology, induced by the Euclidean metric.

The set of all continuous, linear functionals on X is called the *dual of* X and is denoted by X'. Under pointwise addition and scalar multiplication X' is a vector space:

$$(f+g)(x) = f(x) + g(x), \qquad (cf)(x) = cf(x), \qquad f,g \in X', \quad x \in X, \quad c \in \mathbb{C}.$$

X' will be a TVS provided a suitable topology is specified for it. One such topology is the *weak-star topology*, the weakest topology with respect to which the functional F_x defined on X' by $F_x(f) = f(x)$ for each $f \in X'$ is continuous for each $x \in X$. This topology is used, for instance, in the space of Schwartz distributions introduced in Section 1.52. The dual of a normed space can be given a stronger topology with respect to which it is a normed space itself (Section 1.10).

Normed Spaces

1.6 A *norm* on a vector space X is a real-valued functional f on X which satisfies

(i) $f(x) \geq 0$ for all $x \in X$ with equality if and only if $x = 0$,
(ii) $f(cx) = |c| f(x)$ for every $x \in X$ and $c \in \mathbb{C}$,
(iii) $f(x+y) \leq f(x) + f(y)$ for every $x, y \in X$.

A *normed space* is a vector space X which is provided with a norm. The norm will be denoted by $\|\cdot;X\|$ except where simpler notations are introduced. If $r > 0$, the set

$$B_r(x) = \{y \in X : \|y-x;X\| < r\}$$

is called the *open ball* of radius r with center $x \in X$. Any subset A of X is called *open* if for every $x \in A$ there exists $r > 0$ such that $B_r(x) \subset A$. The open sets thus defined constitute a topology for X with respect to which X is a TVS. This topology is called the *norm topology* on X. The closure of $B_r(x)$ in this topology is

$$\overline{B_r(x)} = \{y \in X : \|y-x;X\| \le r\}.$$

A TVS X is *normable* if its topology coincides with the topology induced by some norm on X. Two different norms on a vector space are *equivalent* if they induce the same topology on X, that is, if for some constant $c > 0$

$$c\|x\|_1 \le \|x\|_2 \le (1/c)\|x\|_1$$

for all $x \in X$, $\|\cdot\|_1$ and $\|\cdot\|_2$ being the two norms.

If X and Y are two normed spaces and if there exists a one-to-one linear operator L mapping X onto Y and having the property $\|L(x);Y\| = \|x;X\|$ for every $x \in X$, then L is called an *isometric isomorphism* between X and Y, and X and Y are said to be *isometrically isomorphic*; we write $X \cong Y$. Such spaces are often identified since they have identical structures and differ only in the nature of their elements.

1.7 A sequence $\{x_n\}$ in a normed space X is convergent to the limit x_0 if and only if $\lim_{n\to\infty}\|x_n - x_0;X\| = 0$ in $\mathbb{R}$. The norm topology of X is completely determined by the sequences it renders convergent.

A subset S of a normed space X is said to be *dense* in X if each $x \in X$ is the limit of a sequence of elements of S. The normed space X is called *separable* if it has a countable dense subset.

1.8 A sequence $\{x_n\}$ in a normed space X is called a *Cauchy sequence* if and only if $\lim_{m,n\to\infty}\|x_m - x_n;X\| = 0$. If every Cauchy sequence in X converges to a limit in X ,then X is *complete* and a *Banach space*. Every normed space X is either a Banach space or a dense subset of a Banach space Y whose norm satisfies

$$\|x;Y\| = \|x;X\| \qquad \text{for every} \quad x \in X.$$

In this latter case Y is called the *completion* of X.

1.9 If X is a vector space, a functional $(\cdot,\cdot)_X$ defined on $X \times X$ is called an *inner product* on X provided that for every $x, y, z \in X$ and $a, b \in \mathbb{C}$

(i) $(x,y)_X = \overline{(y,x)_X}$,
(ii) $(ax+by,z)_X = a(x,z)_X + b(y,z)_X$,
(iii) $(x,x)_X = 0$ if and only if $x = 0$,

where $\bar{c}$ denotes the complex conjugate of $c \in \mathbb{C}$. Given such an inner product, a norm on X can be defined by

$$\|x;X\| = (x,x)_X^{1/2}. \tag{1}$$

If X is a Banach space under this norm, it is called a *Hilbert space*. The parallelogram law

$$\|x+y;X\|^2 + \|x-y;X\|^2 = 2\|x;X\|^2 + 2\|y;X\|^2 \tag{2}$$

holds in any normed space whose norm is obtained from an inner product via (1).

The Normed Dual

1.10 A norm on the dual X' of a normed space X can be defined by setting, for $x' \in X'$,

$$\|x';X'\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{|x'(x)|}{\|x;X\|}.$$

Since $\mathbb{C}$ is complete, X', with the topology induced by this norm, is a Banach space (whether or not X is) and is called the *normed dual* of X. If X is infinite dimensional, the norm topology of X' is stronger (i.e., has more open sets) than the weak-star topology defined in Section 1.5.

If X is a Hilbert space, it can be identified with its normed dual X' in a natural way.

1.11 THEOREM (*Riesz representation theorem*) Let X be a Hilbert space. A linear functional x' on X belongs to X' if and only if there exists $x \in X$ such that for every $y \in X$ we have

$$x'(y) = (y,x)_X,$$

and in this case $\|x';X'\| = \|x;X\|$. Moreover, x is uniquely determined by $x' \in X'$.

A vector subspace M of a normed space X is itself a normed space under the norm of X, and so normed is called a *subspace* of X. A closed subspace of a Banach space is a Banach space.

1.12 THEOREM (*Hahn–Banach extension theorem*) Let M be a subspace of the normed space X. If $m' \in M'$, there exists $x' \in X'$ such that $\|x'; X'\| = \|m'; M'\|$ and $x'(m) = m'(m)$ for every $m \in M$.

1.13 A natural linear injection of a normed space X into its second dual space $X'' = (X')'$ is provided by the mapping J_X whose value at $x \in X$ is given by

$$J_X x(x') = x'(x), \qquad x' \in X'.$$

Since $|J_X x(x')| \le \|x'; X'\| \|x; X\|$ we have

$$\|J_X x; X''\| \le \|x; X\|.$$

On the other hand, the Hahn–Banach theorem assures us that for any $x \in X$ we can find an $x' \in X'$ such that $\|x'; X'\| = 1$ and $x'(x) = \|x; X\|$. Hence

$$\|J_X x; X''\| = \|x; X\|,$$

and J_X is an isometric isomorphism from X into X''.

If the range of the isomorphism is the entire space X'', we say that the normed space X is *reflexive*. A reflexive space must be complete and hence a Banach space.

1.14 THEOREM Let X be a normed space. X is reflexive if and only if X' is reflexive. X is separable if X' is separable. Hence if X is separable and reflexive, so is X'.

Weak Topology and Weak Convergence

1.15 The *weak topology* of a normed space X is the weakest topology on X that still renders continuous each $x' \in X'$. Unless X is finite dimensional the weak topology is weaker than the norm topology on X. It is a consequence of the Hahn–Banach theorem that a closed, convex set in a normed space is also closed in the weak topology of that space. A sequence convergent with respect to the weak topology on X is said to *converge weakly*. Thus x_n converges weakly to x in X provided $x'(x_n) \to x'(x)$ in $\mathbb{C}$ for every $x' \in X'$. We denote norm convergence of a sequence $\{x_n\}$ to x in X by $x_n \to x$; weak convergence by $x_n \rightharpoonup x$. Since $|x'(x_n - x)| \le \|x'; X'\| \|x_n - x; X\|$ we see that $x_n \to x$ implies $x_n \rightharpoonup x$. The converse is generally not true.

Compact Sets

1.16 A subset A of a normed space X is called *compact* if every sequence of points in A has a subsequence converging in X to an element of A. Compact sets are closed and bounded, but closed and bounded sets need not be compact unless X is finite dimensional. A is called *precompact* if its closure $\bar{A}$ (in the norm topology) is compact. A is called *weakly sequentially compact* if every sequence in A has a subsequence converging weakly in X to a point in A. The reflexivity of a Banach space can be characterized in terms of this property.

1.17 THEOREM A Banach space X is reflexive if and only if its closed unit ball $\overline{B_1(0)} = \{x \in X : \|x; X\| \le 1\}$ is weakly sequentially compact.

1.18 THEOREM A set A is precompact in a Banach space X if and only if for every positive number ε there is a finite subset N_ε of points of X with the property

$$A \subset \bigcup_{y \in N_\varepsilon} B_\varepsilon(y).$$

A set N_ε with this property is called a *finite ε-net* for A.

Uniform Convexity

1.19 Any normed space is locally convex with respect to its norm topology. The norm on X is called *uniformly convex* if for every number ε satisfying $0 < \varepsilon \le 2$ there exists a number $\delta(\varepsilon) > 0$ such that if $x, y \in X$ satisfy $\|x; X\| = \|y; X\| = 1$ and $\|x - y; X\| \ge \varepsilon$, then $\|(x+y)/2; X\| \le 1 - \delta(\varepsilon)$. The normed space X is itself called "uniformly convex" in this case. It should, however, be noted that uniform convexity is a property of the norm—X may possess an equivalent norm that is not uniformly convex. Any normable space is called "uniformly convex" if it possesses a uniformly convex norm. The parallelogram law (2) shows that a Hilbert space is uniformly convex.

1.20 THEOREM A uniformly convex Banach space is reflexive.

The following two theorems will be used to establish the separability, reflexivity, and uniform convexity of the Sobolev spaces introduced in Chapter III.

1.21 THEOREM Let X be a Banach space and M a subspace of X closed with respect to the norm topology on X. Then M is itself a Banach space under the norm inherited from X. Furthermore,

(i) M is separable if X is separable,
(ii) M is reflexive if X is reflexive,
(iii) M is uniformly convex if X is uniformly convex.

The completeness, separability, and uniform convexity of M follow easily from the corresponding properties for X. The reflexivity of M is a consequence of Theorem 1.17 and the fact that M, being closed and convex, is closed in the weak topology of X.

1.22 THEOREM For $j = 1, 2, \ldots, n$ let X_j be a Banach space with norm $\|\cdot\|_j$. The Cartesian product $X = \prod_{j=1}^n X_j$, consisting of points $x = (x_1, \ldots, x_n)$ with $x_j \in X_j$, is a vector space under the definitions

$$x + y = (x_1 + y_1, \ldots, x_n + y_n), \qquad cx = (cx_1, \ldots, cx_n),$$

and is a Banach space with respect to any of the equivalent norms

$$\|x\|_{(p)} = \left(\sum_{j=1}^{n} \|x_j\|_j^p\right)^{1/p}, \qquad 1 \le p < \infty,$$

$$\|x\|_{(\infty)} = \max_{1 \le j \le n} \|x_j\|_j.$$

Furthermore,

(i) if X_j is separable for $1 \le j \le n$, then X is separable;
(ii) if X_j is reflexive for $1 \le j \le n$, then X is reflexive;
(iii) if X_j is uniformly convex for $1 \le j \le n$, then X is uniformly convex. More precisely, $\|\cdot\|_{(p)}$ is a uniformly convex norm on X provided $1 < p < \infty$.

The reader may verify that the functionals $\|\cdot\|_{(p)}$, $1 \le p \le \infty$, are in fact norms on X and that X is complete with respect to each of them. Equivalance of the norms follows from the inequalities

$$\|x\|_{(\infty)} \le \|x\|_{(p)} \le \|x\|_{(1)} \le n\|x\|_{(\infty)}.$$

The separability and uniform convexity of X are readily deduced from the corresponding properties of the spaces X_j. The reflexivity of X follows from that of X_j, $1 \le j \le n$, via Theorem 1.17 or via a natural isomorphism between X' and $\prod_{j=1}^n X_j'$ (see, for example, Lemma 3.7).

Operators and Imbeddings

1.23 Since the topology of a normed space X is determined by its convergent sequences, an operator f defined on X into a topological space Y is continuous if and only if $f(x_n) \to f(x)$ in Y whenever $x_n \to x$ in X. Such is also the case for any topological space X whose topology is determined by the sequences it renders convergent (first countable spaces).

Let X, Y be normed spaces and f an operator from X into Y. The operator f is called *compact* if $f(A)$ is precompact in Y whenever A is bounded in X. [A bounded set in a normed space is one which is contained in the ball $B_R(0)$ for some R.] f is *completely continuous* if it is continuous and compact. f is *bounded* if $f(A)$ is bounded in Y whenever A is bounded in X.

Every compact operator is bounded. Every bounded linear operator is continuous. Hence every compact linear operator is completely continuous.

1.24 We say that the normed space X is *imbedded* in the normed space Y, and write $X \to Y$ to designate this imbedding, provided

(i) X is a vector subspace of Y, and
(ii) the identity operator I defined on X into Y by $Ix = x$ for all $x \in X$ is continuous.

Since I is linear, (ii) is equivalent to the existence of a constant M such that

$$\|Ix; Y\| \leq M \|x; X\|, \qquad x \in X.$$

In some circumstances the requirement that X be a subspace of Y and I be the identity map is weakened to allow as imbeddings certain canonical linear transformations of X into Y. (Examples are trace imbeddings of Sobolev spaces as well as imbeddings of these spaces into spaces of continuous functions. See Chapter V.)

We say that X is *compactly imbedded* in Y if the imbedding operator I is compact.

Spaces of Continuous Functions

1.25 Let Ω be a domain in $\mathbb{R}^n$. For any nonnegative integer m let $C^m(\Omega)$ be the vector space consisting of all functions ϕ which, together with all their partial derivatives $D^\alpha\phi$ of orders $|\alpha| \leq m$, are continuous on Ω. We abbreviate $C^0(\Omega) \equiv C(\Omega)$. Let $C^\infty(\Omega) = \bigcap_{m=0}^{\infty} C^m(\Omega)$. The subspaces $C_0(\Omega)$ and $C_0^\infty(\Omega)$ consist of all those functions in $C(\Omega)$ and $C^\infty(\Omega)$, respectively, which have compact support in Ω.

1.26 Since Ω is open, functions in $C^m(\Omega)$ need not be bounded on Ω. If $\phi \in C(\Omega)$ is bounded and uniformly continuous on Ω, then it possesses a unique, bounded, continuous extension to the closure $\overline{\Omega}$ of Ω. Accordingly, we define the vector space $C^m(\overline{\Omega})$ to consist of all those functions $\phi \in C^m(\Omega)$ for which $D^\alpha\phi$ is bounded and uniformly continuous on Ω for $0 \leq |\alpha| \leq m$. [Note that $C^m(\overline{\mathbb{R}^n}) \neq C^m(\mathbb{R}^n)$.] $C^m(\overline{\Omega})$ is a Banach space with norm given by

$$\|\phi; C^m(\overline{\Omega})\| = \max_{0 \leq |\alpha| \leq m} \sup_{x \in \Omega} |D^\alpha\phi(x)|.$$

1.27 If $0 < \lambda \le 1$, we define $C^{m,\lambda}(\overline{\Omega})$ to be the subspace of $C^m(\overline{\Omega})$ consisting of those functions ϕ for which, for $0 \le |\alpha| \le m$, $D^\alpha\phi$ satisfies in Ω a Hölder condition of exponent λ, that is, there exists a constant K such that

$$|D^\alpha\phi(x) - D^\alpha\phi(y)| \le K|x-y|^\lambda, \qquad x, y \in \Omega.$$

$C^{m,\lambda}(\overline{\Omega})$ is a Banach space with norm given by

$$\|\phi; C^{m,\lambda}(\overline{\Omega})\| = \|\phi; C^m(\overline{\Omega})\| + \max_{0\le|\alpha|\le m} \sup_{\substack{x,y\in\Omega \\ x\ne y}} \frac{|D^\alpha\phi(x) - D^\alpha\phi(y)|}{|x-y|^\lambda}.$$

It should be noted that for $0 < \nu < \lambda \le 1$,

$$C^{m,\lambda}(\overline{\Omega}) \subsetneqq C^{m,\nu}(\overline{\Omega}) \subsetneqq C^m(\overline{\Omega}).$$

It is also clear that $C^{m,1}(\overline{\Omega}) \not\subset C^{m+1}(\overline{\Omega})$. In general $C^{m+1}(\overline{\Omega}) \not\subset C^{m,1}(\overline{\Omega})$ either, but the inclusion is possible for some domains Ω, for instance convex ones as can be seen by appealing to the mean value theorem (see Theorem 1.31).

If Ω is bounded, the following two well-known theorems provide useful criteria for denseness and compactness of subsets of $C(\overline{\Omega})$. If $\phi \in C(\overline{\Omega})$, we may regard ϕ as defined on $\overline{\Omega}$, that is, we identify ϕ with its unique continuous extension to the closure of Ω.

1.28 THEOREM (*Stone–Weierstrass theorem*) Let Ω be a bounded domain in $\mathbb{R}^n$. A subset $\mathscr{A}$ of $C(\overline{\Omega})$ is dense in $C(\overline{\Omega})$ if it has the following four properties:

(i) If $\phi, \psi \in \mathscr{A}$ and $c \in \mathbb{C}$, then $\phi+\psi$, $\phi\psi$, and $c\phi$ all belong to $\mathscr{A}$.
(ii) If $\phi \in \mathscr{A}$, then $\bar{\phi} \in \mathscr{A}$, where $\bar{\phi}$ is the complex conjugate of ϕ.
(iii) If $x, y \in \overline{\Omega}$, $x \ne y$, there exists $\phi \in \mathscr{A}$ such that $\phi(x) \ne \phi(y)$.
(iv) If $x \in \overline{\Omega}$, there exists $\phi \in \mathscr{A}$ such that $\phi(x) \ne 0$.

1.29 COROLLARY If Ω is bounded in $\mathbb{R}^n$, then the set P of all polynomials in $x = (x_1, \ldots, x_n)$ having rational-complex coefficients is dense in $C(\overline{\Omega})$. ($c \in \mathbb{C}$ is rational complex if $c = c_1 + ic_2$, where c_1 and c_2 are rational numbers.) Hence $C(\overline{\Omega})$ is separable.

PROOF The set of all polynomials in x is dense in $C(\overline{\Omega})$ by the Stone–Weierstrass theorem. Any polynomial can be uniformly approximated on the compact set $\overline{\Omega}$ by elements of the countable set P, which is therefore also dense in $C(\overline{\Omega})$. ∎

1.30 THEOREM (*Ascoli–Arzela theorem*) Let Ω be a bounded domain in $\mathbb{R}^n$. A subset K of $C(\overline{\Omega})$ is precompact in $C(\overline{\Omega})$ providing the following two conditions hold:

(i) There exists a constant M such that for every $\phi \in K$ and $x \in \Omega$, $|\phi(x)| \le M$.

(ii) For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\phi \in K$, $x, y \in \Omega$, and $|x-y| < \delta$, then $|\phi(x) - \phi(y)| < \varepsilon$.

The following is a straightforward imbedding theorem for the spaces introduced above.

1.31 THEOREM Let m be a nonnegative integer and let $0 < \nu < \lambda \le 1$. Then the following imbeddings exist:

$$C^{m+1}(\overline{\Omega}) \to C^m(\overline{\Omega}), \tag{3}$$

$$C^{m,\lambda}(\overline{\Omega}) \to C^m(\overline{\Omega}), \tag{4}$$

$$C^{m,\lambda}(\overline{\Omega}) \to C^{m,\nu}(\overline{\Omega}). \tag{5}$$

If Ω is bounded, then imbeddings (4) and (5) are compact. If Ω is convex, we have the further imbeddings

$$C^{m+1}(\overline{\Omega}) \to C^{m,1}(\overline{\Omega}), \tag{6}$$

$$C^{m+1}(\overline{\Omega}) \to C^{m,\nu}(\overline{\Omega}). \tag{7}$$

If Ω is convex and bounded, then imbeddings (3) and (7) are compact.

PROOF The existence of imbeddings (3) and (4) follows from the obvious inequalities

$$\|\phi; C^m(\overline{\Omega})\| \le \|\phi; C^{m+1}(\overline{\Omega})\|,$$

$$\|\phi; C^m(\overline{\Omega})\| \le \|\phi; C^{m,\lambda}(\overline{\Omega})\|.$$

To establish (5) we note that for $|\alpha| \le m$,

$$\sup_{\substack{x,y \in \Omega \\ 0<|x-y|<1}} \frac{|D^\alpha\phi(x) - D^\alpha\phi(y)|}{|x-y|^\nu} \le \sup_{x,y \in \Omega} \frac{|D^\alpha\phi(x) - D^\alpha\phi(y)|}{|x-y|^\lambda}$$

and

$$\sup_{\substack{x,y \in \Omega \\ |x-y|\ge 1}} \frac{|D^\alpha\phi(x) - D^\alpha\phi(y)|}{|x-y|^\nu} \le 2 \sup_{x \in \Omega} |D^\alpha\phi(x)|,$$

from which we conclude that

$$\|\phi; C^{m,\nu}(\overline{\Omega})\| \le 3\,\|\phi; C^{m,\lambda}(\overline{\Omega})\|.$$

If Ω is convex and $x, y \in \Omega$, then by the mean value theorem there is a point $z \in \Omega$ on the line segment joining x and y such that $D^\alpha\phi(x) - D^\alpha\phi(y) =$

$(x-y)\cdot\nabla D^\alpha\phi(z)$, where $\nabla u = (D_1 u, D_2 u, \ldots, D_n u)$. Thus

$$|D^\alpha\phi(x)-D^\alpha\phi(y)| \le n|x-y|\,\|\phi; C^{m+1}(\overline{\Omega})\|, \tag{8}$$

and so

$$\|\phi; C^{m,1}(\overline{\Omega})\| \le n\,\|\phi; C^{m+1}(\overline{\Omega})\|.$$

Thus (6) is proved and (7) follows from (5) and (6).

Now suppose that Ω is bounded. If A is a bounded set in $C^{0,\lambda}(\overline{\Omega})$, then there exists M such that $\|\phi; C^{0,\lambda}(\overline{\Omega})\| \le M$ for all $\phi \in A$. But then $|\phi(x)-\phi(y)| \le M|x-y|^\lambda$ for all $\phi \in A$ and all $x, y \in \Omega$, whence A is precompact in $C(\overline{\Omega})$ by Theorem 1.30. This proves the compactness of (4) for $m = 0$. If $m \ge 1$ and A is bounded in $C^{m,\lambda}(\overline{\Omega})$, then A is bounded in $C^{0,\lambda}(\overline{\Omega})$ and there is a sequence $\{\phi_j\} \subset A$ such that $\phi_j \to \phi$ in $C(\overline{\Omega})$. But $\{D_1\phi_j\}$ is also bounded in $C^{0,\lambda}(\overline{\Omega})$ so there exists a subsequence of $\{\phi_j\}$ which we again denote by $\{\phi_j\}$ such that $D_1\phi_j \to \psi_1$ in $C(\overline{\Omega})$. Convergence in $C(\overline{\Omega})$ being uniform convergence on Ω, we have $\psi_1 = D_1\phi$. We may continue to extract subsequences in this manner until we obtain one for which $D^\alpha\phi_j \to D^\alpha\phi$ in $C(\overline{\Omega})$ for each α, $0 \le |\alpha| \le m$. This proves the compactness of (4). For (5) we argue as follows:

$$\begin{aligned}\frac{|D^\alpha\phi(x)-D^\alpha\phi(y)|}{|x-y|^\nu} &= \left(\frac{|D^\alpha\phi(x)-D^\alpha\phi(y)|}{|x-y|^\lambda}\right)^{\nu/\lambda} |D^\alpha\phi(x)-D^\alpha\phi(y)|^{1-\nu/\lambda} \\ &\le \text{const}\,|D^\alpha\phi(x)-D^\alpha\phi(y)|^{1-\nu/\lambda} \end{aligned} \tag{9}$$

for all ϕ in a bounded subset of $C^{m,\lambda}(\overline{\Omega})$. Since (9) shows that any sequence bounded in $C^{m,\lambda}(\overline{\Omega})$ and converging in $C^m(\overline{\Omega})$ also converges in $C^{m,\nu}(\overline{\Omega})$, the compactness of (5) follows from that of (4).

Finally, if Ω is convex and bounded, the compactness of (3) and (7) follows from composing the continuous imbedding (6) with the compact imbeddings (4) and (5) for the case $\lambda = 1$. ∎

The existence of imbeddings (6) and (7), as well as the compactness of (3) and (7), can be obtained under less restrictive hypotheses than the convexity of Ω. For instance, if every pair of points $x, y \in \Omega$ can be joined by a rectifiable arc in Ω having length not exceeding some fixed multiple of $|x-y|$, then we can obtain an inequality similar to (8) and carry out the proof. We leave it to the reader to show that (6) is not compact.

The Lebesgue Measure in $\mathbb{R}^n$

1.32 Many of the spaces of functions considered in this monograph consist of functions integrable in the Lebesgue sense over domains in $\mathbb{R}^n$. While we assume that most readers are familiar with Lebesgue measure and integration, we nevertheless include here a brief discussion of that theory, especially those aspects of it relevant to the study of the L^p-spaces and Sobolev spaces considered hereafter. All proofs are omitted. For a more complete and systematic discussion of the Lebesgue theory, as well as more general measures and integrals, the reader is referred to any of the standard works on integration theory, for example, the book by Munroe [48].

1.33 A collection Σ of subsets of $\mathbb{R}^n$ is called a *σ-algebra* if the following conditions hold:

(i) $\mathbb{R}^n \in \Sigma$.
(ii) If $A \in \Sigma$, then $A^c = \{x \in \mathbb{R}^n : x \notin A\} \in \Sigma$.
(iii) If $A_j \in \Sigma$, $j = 1, 2, \ldots$, then $\bigcup_{j=1}^{\infty} A_j \in \Sigma$.

It follows from (i)–(iii) that:

(iv) $\varnothing \in \Sigma$.
(v) If $A_j \in \Sigma$, $j = 1, 2, \ldots$, then $\bigcap_{j=1}^{\infty} A_j \in \Sigma$.
(vi) If $A, B \in \Sigma$, then $A - B = A \cap B^c \in \Sigma$.

1.34 By a *measure* μ on Σ we mean a function on Σ taking values in either $\mathbb{R} \cup \{+\infty\}$ (a *positive measure*) or $\mathbb{C}$ (a *complex measure*) which is *countably additive* in the sense that

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)$$

whenever $A_j \in \Sigma$, $j = 1, 2, \ldots$, and $A_j \cap A_k = \varnothing$ for $j \neq k$. (For a complex measure the series on the right, being convergent for any such sequence $\{A_j\}$, is absolutely convergent.) If μ is a positive measure and if $A, B \in \Sigma$ and $A \subset B$, then $\mu(A) \leq \mu(B)$. Also, if $A_j \in \Sigma$, $j = 1, 2, \ldots$ and $A_1 \subset A_2 \subset \cdots$, then $\mu(\bigcup_{j=1}^{\infty} A_j) = \lim_{j\to\infty} \mu(A_j)$.

1.35 THEOREM There exists a σ-algebra Σ of subsets of $\mathbb{R}^n$ and a positive measure μ on Σ having the following properties:

(i) Every open set in $\mathbb{R}^n$ belongs to Σ.
(ii) If $A \subset B$, $B \in \Sigma$, and $\mu(B) = 0$, then $A \in \Sigma$ and $\mu(A) = 0$.
(iii) If $A = \{x \in \mathbb{R}^n : a_j \leq x_j \leq b_j, j = 1, 2, \ldots, n\}$, then $A \in \Sigma$ and $\mu(A) = \prod_{j=1}^{n} (b_j - a_j)$.
(iv) μ is translation invariant, that is, if $x \in \mathbb{R}^n$ and $A \in \Sigma$, then $x + A = \{x + y : y \in A\} \in \Sigma$ and $\mu(x + A) = \mu(A)$.

The elements of Σ are called *(Lebesgue) measurable subsets* of $\mathbb{R}^n$; μ is called the (*Lebesgue*) *measure* in $\mathbb{R}^n$. (We shall normally suppress the word "Lebesgue" in these terms as it is the only measure on $\mathbb{R}^n$ which we shall require for our purposes.) For $A \in \Sigma$ we call $\mu(A)$ the *measure of* A or the *volume of* A, since Lebesgue measure is a natural generalization of the concept of volume in $\mathbb{R}^3$. While we make no formal distinction between "measure" and "volume" we shall often prefer the latter term for sets that are easily visualized geometrically (balls, cubes, domains) and shall write vol A in place of $\mu(A)$ in these cases. In $\mathbb{R}^1$ and $\mathbb{R}^2$ the terms *length* and *area* are more appropriate than *volume*.

1.36 If $B \subset A \subset \mathbb{R}^n$ and $\mu(B) = 0$, then any condition that holds at every point of the set $A - B$ is said to hold *almost everywhere* (a.e.) in A. It is easily seen that every countable set in $\mathbb{R}^n$ has measure zero. The converse is, however, not true.

A function f defined on a measurable set and having values in $\mathbb{R} \cup \{+\infty, -\infty\}$ is itself called *measurable* if the set

$$\{x : f(x) > a\}$$

is measurable for every real a. Some of the more important aspects of this definition are listed in the following theorem.

1.37 THEOREM (a) If f is measurable, so is $|f|$.

(b) If f and g are measurable and real valued, then so are $f+g$ and fg.

(c) If $\{f_n\}$ is a sequence of measurable functions, then $\sup_n f_n$, $\inf_n f_n$, $\limsup_{n\to\infty} f_n$, and $\liminf_{n\to\infty} f_n$ are measurable.

(d) If f is continuous and defined on a measurable set, then f is measurable.

(e) If f is continuous on $\mathbb{R}$ into $\mathbb{R}$ and g is measurable and real valued, then the composite function $f \circ g$ defined by $f \circ g(x) = f(g(x))$ is measurable.

(f) (*Lusin's theorem*) If f is measurable and $f(x) = 0$ for $x \in A^c$ where $\mu(A) < \infty$, and if $\varepsilon > 0$, then there exists a function $g \in C_0(A)$ such that $\sup_{x \in \mathbb{R}^n} |g(x)| \le \sup_{x \in \mathbb{R}^n} |f(x)|$ and $\mu\{x \in \mathbb{R}^n : f(x) \ne g(x)\} < \varepsilon$.

1.38 Let $A \subset \mathbb{R}^n$. The function χ_A defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

is called the *characteristic function* of A. A real-valued function s on $\mathbb{R}^n$ is called a *simple function* if its range is a finite set of real numbers. If for every x, $s(x) \in \{a_1, \ldots, a_m\}$, then clearly $s = \sum_{j=1}^m a_j \chi_{A_j}$, where $A_j = \{x \in \mathbb{R}^n : s(x) = a_j\}$, and s is measurable if and only if $A_1, A_2, \ldots, A_m$ are measurable. Because of the following approximation theorem simple functions are a very useful tool in integration theory.

1.39 THEOREM Given a real-valued function f with domain $A \subset \mathbb{R}^n$ there is a sequence $\{s_n\}$ of simple functions converging pointwise to f on A. If f is bounded, $\{s_n\}$ may be chosen so that the convergence is uniform. If f is measurable, each s_n may be chosen measurable. If f is nonnegative valued, the sequence $\{s_n\}$ may be chosen to be monotonically increasing at each point.

The Lebesgue Integral

1.40 We are now in a position to define the (*Lebesgue*) *integral* of a measurable, real-valued function defined on a measurable set $A \subset \mathbb{R}^n$. For a simple function $s = \sum_{j=1}^{m} a_j \chi_{A_j}$, where $A_j \subset A$, A_j is measurable, we define

$$\int_A s(x)\,dx = \sum_{j=1}^{m} a_j \mu(A_j). \tag{10}$$

If f is measurable and nonnegative valued, we define

$$\int_A f(x)\,dx = \sup \int_A s(x)\,dx, \tag{11}$$

the supremum being taken over measurable simple functions s vanishing outside A and satisfying $0 \le s(x) \le f(x)$ in A. If f is a nonnegative simple function, then the two definitions of $\int_A f(x)\,dx$ given by (10) and (11) coincide. Note that the integral of a nonnegative function may be $+\infty$.

If f is measurable and real valued, we set $f = f^+ - f^-$, where $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$ are both measurable and nonnegative. We define

$$\int_A f(x)\,dx = \int_A f^+(x)\,dx - \int_A f^-(x)\,dx$$

provided at least one of the integrals on the right is finite. If *both integrals are finite*, we say that f is (Lebesgue) *integrable on* A. The class of integrable functions on A is denoted $L^1(A)$.

1.41 THEOREM Assume all of the functions and sets appearing below are measurable.

(a) If f is bounded on A and $\mu(A) < \infty$, then $f \in L^1(A)$.

(b) If $a \le f(x) \le b$ for all $x \in A$ and if $\mu(A) < \infty$, then

$$a\mu(A) \le \int_A f(x)\,dx \le b\mu(A).$$

(c) If $f(x) \le g(x)$ for all $x \in A$, and if both integrals exist, then

$$\int_A f(x)\,dx \le \int_A g(x)\,dx.$$

(d) If $f, g \in L^1(A)$, then $f+g \in L^1(A)$ and

$$\int_A (f+g)(x)\,dx = \int_A f(x)\,dx + \int_A g(x)\,dx.$$

(e) If $f \in L^1(A)$ and $c \in \mathbb{R}$, then $cf \in L^1(A)$ and

$$\int_A (cf)(x)\,dx = c\int_A f(x)\,dx.$$

(f) If $f \in L^1(A)$, then $|f| \in L^1(A)$ and

$$\left|\int_A f(x)\,dx\right| \le \int_A |f(x)|\,dx.$$

(g) If $f \in L^1(A)$ and $B \subset A$, then $f \in L^1(B)$; if in addition $f(x) \ge 0$ for all $x \in A$, then

$$\int_B f(x)\,dx \le \int_A f(x)\,dx.$$

(h) If $\mu(A) = 0$, then $\int_A f(x)\,dx = 0$.

(i) If $f \in L^1(A)$ and if $\int_B f(x)\,dx = 0$ for every $B \subset A$, then $f(x) = 0$ a.e. on A.

1.42 THEOREM If f is either an element of $L^1(\mathbb{R}^n)$ or measurable and nonnegative on $\mathbb{R}^n$, then the set function λ defined by

$$\lambda(A) = \int_A f(x)\,dx$$

is countably additive, and hence a measure, on the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}^n$.

One consequence of this additivity of the integral is that sets of measure zero may be ignored for purposes of integration, that is, if f and g are measurable on A and if $f(x) = g(x)$ a.e. on A, then $\int_A f(x)\,dx = \int_A g(x)\,dx$. Accordingly, two elements of $L^1(A)$ are considered identical if they are equal almost everywhere.

The following three theorems are concerned with the interchange of integration and limit processes.

1.43 THEOREM (*Monotone convergence theorem*) Let $A \subset \mathbb{R}^n$ be measurable, and $\{f_n\}$ a sequence of measurable functions satisfying $0 \le f_1(x) \le f_2(x) \le \cdots$ for every $x \in A$. Then

$$\lim_{n\to\infty} \int_A f_n(x)\,dx = \int_A \left(\lim_{n\to\infty} f_n(x)\right) dx.$$

1.44 THEOREM (*Fatou's lemma*) Let $A \subset \mathbb{R}^n$ be measurable and let $\{f_n\}$ be a sequence of nonnegative, measurable functions. Then

$$\int_A \left(\liminf_{n\to\infty} f_n(x)\right) dx \le \liminf_{n\to\infty} \int_A f_n(x)\, dx.$$

1.45 THEOREM (*Dominated convergence theorem*) Let $A \subset \mathbb{R}^n$ be measurable and let $\{f_n\}$ be a sequence of measurable functions converging to a limit pointwise on A. If there is a function $g \in L^1(A)$ such that $|f_n(x)| \le g(x)$ for every n and all $x \in A$, then

$$\lim_{n\to\infty} \int_A f_n(x)\, dx = \int_A \left(\lim_{n\to\infty} f_n(x)\right) dx.$$

1.46 The integral of a complex-valued function over a measurable set $A \subset \mathbb{R}^n$ is defined as follows. Set $f = u + iv$, where u and v are real valued and call f measurable if and only if u and v are measurable. We shall say that f is integrable over A, and write $f \in L^1(A)$, provided $|f| = (u^2+v^2)^{1/2}$ belongs to $L^1(A)$ in the sense described in Section 1.40. For $f \in L^1(A)$, and only for such f, the integral is defined by

$$\int_A f(x)\, dx = \int_A u(x)\, dx + i \int_A v(x)\, dx.$$

It is easily checked that $f \in L^1(A)$ if and only if $u, v \in L^1(A)$. Theorem 1.37(a, b, d–f), Theorem 1.41(a, d–i), Theorem 1.42 [assuming $f \in L^1(\mathbb{R}^n)$], and Theorem 1.45 all extend to cover the case of complex f.

The following theorem enables us to express certain complex measures in terms of Lebesgue measure μ. It is the converse of Theorem 1.42.

1.47 THEOREM (*The Radon–Nikodym theorem*) Let λ be a complex measure defined on the σ-algebra Σ of Lebesgue measurable subsets of $\mathbb{R}^n$. Suppose that $\lambda(A) = 0$ for every $A \in \Sigma$ for which $\mu(A) = 0$. Then there exists $f \in L^1(\mathbb{R}^n)$ such that for every $A \in \Sigma$

$$\lambda(A) = \int_A f(x)\, dx.$$

The function f is uniquely determined by λ up to sets of measure zero.

1.48 If f is a function defined on a subset A of $\mathbb{R}^{n+m}$, we may regard f as depending on the pair of variables (x, y) with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. The integral of f over A is then commonly denoted by

$$\int_A f(x,y)\,dx\,dy$$

or, if it is desired to have the integral extend over all of $\mathbb{R}^{n+m}$,

$$\int_{\mathbb{R}^{n+m}} f(x,y)\,\chi_A(x,y)\,dx\,dy,$$

where χ_A is the characteristic function of A. In particular, if $A \subset \mathbb{R}^n$, we may write

$$\int_A f(x)\,dx = \int_A f(x_1,\ldots,x_n)\,dx_1 \cdots dx_n.$$

1.49 THEOREM (*Fubini's theorem*) Let f be a measurable function on $\mathbb{R}^{n+m}$ and suppose that at least one of the integrals

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^{n+m}} |f(x,y)|\,dx\,dy, \\ I_2 &= \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} |f(x,y)|\,dx \right) dy, \\ I_3 &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} |f(x,y)|\,dy \right) dx \end{aligned} \tag{12}$$

exists and is finite. Then

(a) $f(\cdot, y) \in L^1(\mathbb{R}^n)$ for almost all $y \in \mathbb{R}^m$,
(b) $f(x, \cdot) \in L^1(\mathbb{R}^m)$ for almost all $x \in \mathbb{R}^n$,
(c) $\int_{\mathbb{R}^n} f(x, \cdot)\,dx \in L^1(\mathbb{R}^m)$,
(d) $\int_{\mathbb{R}^m} f(\cdot, y)\,dy \in L^1(\mathbb{R}^n)$, and
(e) $I_1 = I_2 = I_3$.

Distributions and Weak Derivatives

1.50 We shall require in subsequent chapters some of the basic concepts and techniques of the Schwartz theory of distributions [60], and we present here a brief description of those aspects of the theory that are relevant for our purposes. Of special importance is the notion of weak or distributional derivative of an integrable function. One of the standard definitions of Sobolev spaces is phrased in terms of such derivatives (Section 3.1). In addition to Ref. [60] the reader is referred to Rudin [59] and Yosida [69] for more complete treatments of the spaces $\mathscr{D}(\Omega)$ and $\mathscr{D}'(\Omega)$ introduced below, as well as useful generalizations of these spaces.

1.51 Let Ω be a domain in $\mathbb{R}^n$. A sequence $\{\phi_n\}$ of functions belonging to $C_0^\infty(\Omega)$ is said to *converge in the sense of the space* $\mathscr{D}(\Omega)$ to the function $\phi \in C_0^\infty(\Omega)$ provided the following conditions are satisfied:

(i) there exists $K \subset\subset \Omega$ such that $\operatorname{supp}(\phi_n - \phi) \subset K$ for every n, and
(ii) $\lim_{n\to\infty} D^\alpha \phi_n(x) = D^\alpha \phi(x)$ uniformly on K for each multi-index α.

There exists a locally convex topology on the vector space $C_0^\infty(\Omega)$ with respect to which a linear functional T is continuous if and only if $T(\phi_n) \to T(\phi)$ in $\mathbb{C}$ whenever $\phi_n \to \phi$ in the sense of the space $\mathscr{D}(\Omega)$. This TVS is called $\mathscr{D}(\Omega)$ and its elements *testing functions*. $\mathscr{D}(\Omega)$ is not a normable space. (We ignore the question of uniqueness of the topology asserted above. It uniquely determines the dual of $\mathscr{D}(\Omega)$ which is sufficient for our purposes.)

1.52 The dual space $\mathscr{D}'(\Omega)$ of $\mathscr{D}(\Omega)$ is called the *space of* (Schwartz) *distributions*. $\mathscr{D}'(\Omega)$ is given the weak-star topology as dual of $\mathscr{D}(\Omega)$, and is a locally convex TVS with that topology. We summarize the vector space and convergence operations in $\mathscr{D}'(\Omega)$ as follows: if $S, T, T_n \in \mathscr{D}'(\Omega)$ and $c \in \mathbb{C}$, then

$$(S+T)(\phi) = S(\phi) + T(\phi), \qquad \phi \in \mathscr{D}(\Omega),$$
$$(cT)(\phi) = cT(\phi), \qquad \phi \in \mathscr{D}(\Omega),$$

$T_n \to T$ in $\mathscr{D}'(\Omega)$ if and only if $T_n(\phi) \to T(\phi)$ in $\mathbb{C}$ for every $\phi \in \mathscr{D}(\Omega)$.

1.53 A function u defined almost everywhere on Ω is said to be *locally integrable* on Ω provided $u \in L^1(A)$ for every measurable $A \subset\subset \Omega$. In this case we write $u \in L^1_{\mathrm{loc}}(\Omega)$. Corresponding to every $u \in L^1_{\mathrm{loc}}(\Omega)$ there is a distribution $T_u \in \mathscr{D}'(\Omega)$ defined by

$$T_u(\phi) = \int_\Omega u(x)\,\phi(x)\,dx, \qquad \phi \in \mathscr{D}(\Omega). \tag{13}$$

It is clear that T_u, thus defined, is a linear functional on $\mathscr{D}(\Omega)$. To see that it is continuous suppose that $\phi_n \to \phi$ in $\mathscr{D}(\Omega)$. Then there exists $K \subset\subset \Omega$ such that $\operatorname{supp}(\phi_n - \phi) \subset K$ for $n = 1, 2, 3, \ldots$. Thus

$$|T_u(\phi_n) - T_u(\phi)| \le \sup_{x\in K} |\phi_n(x) - \phi(x)| \int_K |u(x)|\,dx.$$

The right side of the inequality above tends to zero as $n \to \infty$ since $\phi_n \to \phi$ uniformly on K.

1.54 Not every distribution $T \in \mathscr{D}'(\Omega)$ is of the form $T = T_u$ [defined by (13)] for some $u \in L^1_{\mathrm{loc}}(\Omega)$. Indeed, assuming $0 \in \Omega$, the reader may wish to convince himself that there can be no locally integrable function δ on Ω such that for every $\phi \in \mathscr{D}(\Omega)$.

$$\int_\Omega \delta(x)\phi(x)\,dx = \phi(0).$$

The linear functional δ defined on $\mathscr{D}(\Omega)$ by

$$\delta(\phi) = \phi(0) \tag{14}$$

is, however, easily seen to be continuous, and hence a distribution on Ω.

1.55 Let $u \in C^1(\Omega)$ and $\phi \in \mathscr{D}(\Omega)$. Since ϕ vanishes identically outside some compact subset of Ω, we obtain by integration by parts in the variable x_j

$$\int_\Omega \left(\frac{\partial}{\partial x_j} u(x)\right)\phi(x)\,dx = -\int_\Omega u(x)\left(\frac{\partial}{\partial x_j}\phi(x)\right)dx.$$

Similarly, integration by parts $|\alpha|$ times leads to

$$\int_\Omega (D^\alpha u(x))\,\phi(x)\,dx = (-1)^{|\alpha|}\int_\Omega u(x)\,D^\alpha\phi(x)\,dx$$

if $u \in C^{|\alpha|}(\Omega)$. This motivates the following definition of the derivative $D^\alpha T$ of a distribution $T \in \mathscr{D}'(\Omega)$:

$$(D^\alpha T)(\phi) = (-1)^{|\alpha|}T(D^\alpha\phi). \tag{15}$$

Since $D^\alpha\phi \in \mathscr{D}(\Omega)$ whenever $\phi \in \mathscr{D}(\Omega)$, $D^\alpha T$ is a functional on $\mathscr{D}(\Omega)$. Clearly $D^\alpha T$ is linear on $\mathscr{D}(\Omega)$. We show that $D^\alpha T$ is continuous, and hence a distribution on Ω. To this end suppose $\phi_n \to \phi$ in $\mathscr{D}(\Omega)$. Then

$$\operatorname{supp}(D^\alpha(\phi_n - \phi)) \subset \operatorname{supp}(\phi_n - \phi) \subset K$$

for some $K \subset\subset \Omega$. Moreover,

$$D^\beta[D^\alpha(\phi_n - \phi)] = D^{\beta+\alpha}(\phi_n - \phi)$$

converges to zero uniformly on K as $n \to \infty$ for each multi-index β. Hence $D^\alpha\phi_n \to D^\alpha\phi$ in $\mathscr{D}(\Omega)$. Since $T \in \mathscr{D}'(\Omega)$ it follows that

$$D^\alpha T(\phi_n) = (-1)^{|\alpha|}T(D^\alpha\phi_n) \to (-1)^{|\alpha|}T(D^\alpha\phi) = D^\alpha T(\phi)$$

in $\mathbb{C}$. Thus $D^\alpha T \in \mathscr{D}'(\Omega)$.

We have shown that every distribution in $\mathscr{D}'(\Omega)$ possesses derivatives of arbitrary orders in $\mathscr{D}'(\Omega)$ in the sense of the definition (15). Furthermore, the mapping D^α from $\mathscr{D}'(\Omega)$ into $\mathscr{D}'(\Omega)$ is continuous. If $T_n \to T$ in $\mathscr{D}'(\Omega)$ and if $\phi \in \mathscr{D}(\Omega)$, then

$$D^\alpha T_n(\phi) = (-1)^{|\alpha|}T_n(D^\alpha\phi) \to (-1)^{|\alpha|}T(D^\alpha\phi) = D^\alpha T(\phi).$$

1.56 EXAMPLES (1) If $0 \in \Omega$ and $\delta \in \mathscr{D}'(\Omega)$ is defined by (14), then $D^\alpha \delta$ is given by

$$D^\alpha \delta(\phi) = (-1)^{|\alpha|} D^\alpha \phi(0).$$

(2) If $\Omega = \mathbb{R}$ (i.e., $n = 1$) and $H \in L^1_{\text{loc}}(\mathbb{R})$ is defined by

$$H(x) = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0, \end{cases}$$

then the derivative $(T_H)'$ is δ, for if $\phi \in \mathscr{D}(\mathbb{R})$ has support in the interval $[-a, a]$, then

$$(T_H)'(\phi) = -T_H(\phi') = -\int_0^a \phi'(x)\, dx = \phi(0) = \delta(\phi).$$

1.57 We now define the concept of weak derivative of a locally integrable function. Let $u \in L^1_{\text{loc}}(\Omega)$. There may or may not exist a function $v_\alpha \in L^1_{\text{loc}}(\Omega)$ such that $T_{v_\alpha} = D^\alpha(T_u)$ in $\mathscr{D}'(\Omega)$. If such a v_α exists, it is unique up to sets of measure zero and it is called the *weak* or *distributional partial derivative* of u and is denoted by $D^\alpha u$. Thus $D^\alpha u = v_\alpha$ in the weak (distributional) sense provided $v_\alpha \in L^1_{\text{loc}}(\Omega)$ satisfies

$$\int_\Omega u(x)\, D^\alpha \phi(x)\, dx = (-1)^{|\alpha|} \int_\Omega v_\alpha(x)\, \phi(x)\, dx$$

for every $\phi \in \mathscr{D}(\Omega)$.

If u is sufficiently smooth to have a continuous partial derivative $D^\alpha u$ in the usual (classical) sense, then $D^\alpha u$ is also a distributional partial derivative of u. Of course $D^\alpha u$ may exist in the distributional sense without existing in the classical sense. For example a function u, continuous on $\mathbb{R}$, which has a bounded derivative u' except at finitely many points, has a derivative in the distributional sense. We shall show in Theorem 3.16 that functions having weak derivatives can be suitably approximated by smooth functions.

1.58 Let us note in conclusion that distributions on Ω can be multiplied by smooth functions. If $T \in \mathscr{D}'(\Omega)$ and $\omega \in C^\infty(\Omega)$, the product $\omega T \in \mathscr{D}'(\Omega)$ is defined by

$$(\omega T)(\phi) = T(\omega\phi), \qquad \phi \in \mathscr{D}(\Omega).$$

If $T = T_u$ for some $u \in L^1_{\text{loc}}(\Omega)$, then $\omega T = T_{\omega u}$. The Leibniz rule (see Section 1.1) is easily checked to hold for $D^\alpha(\omega T)$.

II

The Spaces $L^p(\Omega)$

Definition and Basic Properties

2.1 Let Ω be a domain in $\mathbb{R}^n$ and let p be a positive real number. We denote by $L^p(\Omega)$ the class of all measurable functions u, defined on Ω, for which

$$\int_\Omega |u(x)|^p \, dx < \infty. \tag{1}$$

We identify in $L^p(\Omega)$ functions that are equal almost everywhere on Ω. The elements of $L^p(\Omega)$ are thus actually equivalence classes of measurable functions satisfying (1), two functions being equivalent if they are equal a.e. in Ω. For convenience, however, we ignore this distinction and write $u \in L^p(\Omega)$ if u satisfies (1), and $u = 0$ in $L^p(\Omega)$ if $u(x) = 0$ a.e. in Ω. It is clear that if $u \in L^p(\Omega)$ and $c \in \mathbb{C}$, then $cu \in L^p(\Omega)$. Moreover, if $u, v \in L^p(\Omega)$, then since

$$|u(x) + v(x)|^p \leq (|u(x)| + |v(x)|)^p \leq 2^p(|u(x)|^p + |v(x)|^p),$$

$u + v \in L^p(\Omega)$, so $L^p(\Omega)$ is a vector space.

2.2 We shall verify presently that the functional $\|\cdot\|_p$ defined by

$$\|u\|_p = \left\{ \int_\Omega |u(x)|^p \, dx \right\}^{1/p}$$

is a norm on $L^p(\Omega)$ provided $1 \leq p < \infty$. (It is not a norm if $0 < p < 1$.) In arguments where confusion of domains might occur we shall use $\|\cdot\|_{p,\Omega}$ in place of $\|\cdot\|_p$. It is clear that $\|u\|_p \ge 0$ and equality occurs if and only if $u = 0$ in $L^p(\Omega)$. Moreover,

$$\|cu\|_p = |c|\,\|u\|_p, \qquad c \in \mathbb{C}.$$

It remains to be shown, then, that if $1 \le p < \infty$,

$$\|u+v\|_p \le \|u\|_p + \|v\|_p, \tag{2}$$

which is known as *Minkowski's inequality*. Condition (2) certainly holds for $p = 1$ since

$$\int_\Omega |u(x)+v(x)|\,dx \le \int_\Omega |u(x)|\,dx + \int_\Omega |v(x)|\,dx.$$

If $1 < p < \infty$, we denote by p' the number $p/(p-1)$ so that $1 < p' < \infty$ and

$$(1/p) + (1/p') = 1.$$

p' is called the *exponent conjugate to p*.

2.3 THEOREM (*Hölder's inequality*) If $1 < p < \infty$ and $u \in L^p(\Omega)$, $v \in L^{p'}(\Omega)$, then $uv \in L^1(\Omega)$ and

$$\int_\Omega |u(x)v(x)|\,dx \le \|u\|_p\,\|v\|_{p'}. \tag{3}$$

PROOF The function $f(t) = (t^p/p) + (1/p') - t$ has, for $t \ge 0$, the minimum value zero, and this minimum is attained only at $t = 1$. Setting $t = ab^{-p'/p}$, we conclude, for nonnegative numbers a and b, that

$$ab \le (a^p/p) + (b^{p'}/p') \tag{4}$$

with equality occurring if and only if $a^p = b^{p'}$. If either $\|u\|_p = 0$ or $\|v\|_{p'} = 0$, then $u(x)v(x) = 0$ a.e. in Ω so (3) is satisfied. Otherwise we obtain (3) by setting $a = |u(x)|/\|u\|_p$ and $b = |v(x)|/\|v\|_{p'}$ in (4) and integrating over Ω. Equality occurs in (3) if and only if $|u(x)|^p$ and $|v(x)|^{p'}$ are proportional a.e. in Ω. ∎

We remark that a form of Hölder's inequality for finite or infinite sums,

$$\sum |a_k b_k| \le \{\sum |a_k|^p\}^{1/p}\{\sum |b_k|^{p'}\}^{1/p'},$$

can be proved in the same manner.

2.4 THEOREM (*Minkowski's inequality*) If $1 \le p < \infty$, then

$$\|u+v\|_p \le \|u\|_p + \|v\|_p. \tag{5}$$

PROOF We have already done the case in which $p=1$ so we assume $1<p<\infty$. We may also assume that $u, v \in L^p(\Omega)$, for otherwise the right side of (5) is infinite. Now

$$\int_\Omega |u(x)+v(x)|^p\,dx \le \int_\Omega |u(x)+v(x)|^{p-1}(|u(x)|+|v(x)|)\,dx$$
$$\le \left\{\int_\Omega |u(x)+v(x)|^p\,dx\right\}^{1/p'}(\|u\|_p+\|v\|_p)$$

by separate applications of Hölder's inequality. Inequality (5) follows by cancellation, which is valid since $\|u+v\|_p<\infty$. ∎

2.5 A function u, measurable on Ω, is said to be *essentially bounded* on Ω provided there exists a constant K for which $|u(x)|\le K$ a.e. on Ω. The greatest lower bound of such constants K is called the *essential supremum* of $|u|$ on Ω and is denoted by $\operatorname{ess\,sup}_{x\in\Omega}|u(x)|$. We denote by $L^\infty(\Omega)$ the vector space consisting of all functions u that are essentially bounded on Ω, functions being once again identified if they are equal a.e. on Ω. It is easily verified that the functional $\|\cdot\|_\infty$ defined by

$$\|u\|_\infty = \operatorname*{ess\,sup}_{x\in\Omega}|u(x)|$$

is a norm on $L^\infty(\Omega)$. Moreover, Hölder's inequality (3) clearly extends to cover the two cases $p=1$, $p'=\infty$, and $p=\infty$, $p'=1$.

The following pair of theorems establishes reverse forms of Hölder's and Minkowski's inequalities for the case $0<p<1$. The latter inequality will be used later in establishing the uniform convexity of certain L^p-spaces.

2.6 THEOREM Let $0<p<1$ so that $p'=p/(p-1)<0$. Suppose $f\in L^p(\Omega)$ and

$$0<\int_\Omega |g(x)|^{p'}\,dx<\infty.$$

Then

$$\int_\Omega |f(x)g(x)|\,dx \ge \left\{\int_\Omega |f(x)|^p\,dx\right\}^{1/p}\left\{\int_\Omega |g(x)|^{p'}\,dx\right\}^{1/p'}. \tag{6}$$

PROOF We may assume $fg\in L^1(\Omega)$; otherwise the left side of (6) is infinite. Set $\phi=|g|^{-p}$ and $\psi=|fg|^p$ so that $\phi\psi=|f|^p$. Then $\psi\in L^q(\Omega)$ where $q=1/p>1$, and since $p'=-pq'$ where $q'=q/(q-1)$ we have $\phi\in L^{q'}(\Omega)$. By the direct version of Hölder's inequality (3) we have

$$\int_\Omega |f(x)|^p \, dx = \int_\Omega \phi(x)\psi(x)\, dx \leq \|\psi\|_q \|\phi\|_{q'}$$
$$= \left\{\int_\Omega |f(x)g(x)|\, dx\right\}^p \left\{\int_\Omega |g(x)|^{p'}\, dx\right\}^{1-p}.$$

Taking pth roots and dividing by the last factor on the right side we obtain (6). ∎

2.7 THEOREM Let $0 < p < 1$. If $u, v \in L^p(\Omega)$, then

$$\|\, |u| + |v| \,\|_p \geq \|u\|_p + \|v\|_p. \tag{7}$$

PROOF If $u = v = 0$ in $L^p(\Omega)$, then (7) is trivial. Otherwise the left-hand side is greater than zero. Applying the reverse Hölder's inequality (6), we obtain

$$\|\, |u| + |v| \,\|_p^p = \int_\Omega (|u(x)| + |v(x)|)^{p-1}(|u(x)| + |v(x)|)\, dx$$
$$\geq \left\{\int_\Omega (|u(x)| + |v(x)|)^{(p-1)p'}\, dx\right\}^{1/p'} (\|u\|_p + \|v\|_p)$$
$$= \|\, |u| + |v| \,\|_p^{p/p'}(\|u\|_p + \|v\|_p)$$

and (7) follows by cancellation. ∎

The following theorem gives a useful imbedding result for L^p-spaces over domains with finite volume, and some consequences of this imbedding.

2.8 THEOREM Suppose $\operatorname{vol}\Omega = \int_\Omega 1\, dx < \infty$ and $1 \leq p \leq q \leq \infty$. If $u \in L^q(\Omega)$, then $u \in L^p(\Omega)$ and

$$\|u\|_p \leq (\operatorname{vol}\Omega)^{(1/p)-(1/q)} \|u\|_q. \tag{8}$$

Hence

$$L^q(\Omega) \to L^p(\Omega). \tag{9}$$

If $u \in L^\infty(\Omega)$, then

$$\lim_{p\to\infty} \|u\|_p = \|u\|_\infty. \tag{10}$$

Finally, if $u \in L^p(\Omega)$ for $1 \leq p < \infty$ and if there is a constant K such that for all such p

$$\|u\|_p \leq K, \tag{11}$$

then $u \in L^\infty(\Omega)$ and

$$\|u\|_\infty \le K. \tag{12}$$

PROOF If $p = q$, (8) and (9) are trivial. If $1 \le p < q \le \infty$ and $u \in L^q(\Omega)$, Hölder's inequality gives

$$\int_\Omega |u(x)|^p\,dx \le \left\{\int_\Omega |u(x)|^q\,dx\right\}^{p/q}\left\{\int_\Omega 1\,dx\right\}^{1-(p/q)}$$

from which (8) and (9) follow immediately. If $u \in L^\infty(\Omega)$, we obtain from (8)

$$\limsup_{p\to\infty} \|u\|_p \le \|u\|_\infty. \tag{13}$$

On the other hand, for any $\varepsilon > 0$ there exists a set $A \subset \Omega$ having positive measure $\mu(A)$ such that

$$|u(x)| \ge \|u\|_\infty - \varepsilon \qquad \text{if} \quad x \in A.$$

Hence

$$\int_\Omega |u(x)|^p\,dx \ge \int_A |u(x)|^p\,dx \ge \mu(A)(\|u\|_\infty - \varepsilon)^p.$$

It follows that $\|u\|_p \ge (\mu(A))^{1/p}(\|u\|_\infty - \varepsilon)$, whence

$$\liminf_{p\to\infty} \|u\|_p \ge \|u\|_\infty. \tag{14}$$

Equation (10) now follows from (13) and (14).

Now suppose (11) holds for $1 \le p < \infty$. If $u \notin L^\infty(\Omega)$ or else if (12) does not hold, then we can find a constant $K_1 > K$ and a set $A \subset \Omega$ with $\mu(A) > 0$ such that for $x \in A$, $|u(x)| \ge K_1$. The same argument used to obtain (14) now shows that

$$\liminf_{p\to\infty} \|u\|_p \ge K_1,$$

which contradicts (11). ∎

2.9 COROLLARY $L^p(\Omega) \subset L^1_{\text{loc}}(\Omega)$ for $1 \le p \le \infty$ and any domain Ω.

Completeness of $L^p(\Omega)$

2.10 THEOREM $L^p(\Omega)$ is a Banach space if $1 \le p \le \infty$.

PROOF First assume $1 \le p < \infty$ and let $\{u_n\}$ be a Cauchy sequence in $L^p(\Omega)$.

There is a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that

$$\|u_{n_{j+1}} - u_{n_j}\|_p \le 1/2^j, \qquad j = 1, 2, \ldots.$$

Let $v_m(x) = \sum_{j=1}^{m} |u_{n_{j+1}}(x) - u_{n_j}(x)|$. Then

$$\|v_m\|_p \le \sum_{j=1}^{m} (1/2^j) < 1, \qquad m = 1, 2, \ldots.$$

Putting $v(x) = \lim_{m\to\infty} v_m(x)$, which may be infinite for some x, we obtain by Fatou's lemma 1.44

$$\int_\Omega |v(x)|^p\, dx \le \liminf_{m\to\infty} \int_\Omega |v_m(x)|^p\, dx \le 1.$$

Hence $v(x) < \infty$ a.e. in Ω and the series

$$u_{n_1}(x) + \sum_{j=1}^{\infty} (u_{n_{j+1}}(x) - u_{n_j}(x)) \tag{15}$$

converges to a limit $u(x)$ a.e. in Ω. Let $u(x) = 0$ whenever it is undefined as the limit of (15). Since (15) telescopes we have

$$\lim_{m\to\infty} u_{n_m}(x) = u(x) \qquad \text{a.e. in } \Omega.$$

For any $\varepsilon > 0$ there exists N such that if $m, n \ge N$, then $\|u_m - u_n\|_p < \varepsilon$. Hence, by Fatou's lemma again

$$\int_\Omega |u(x) - u_n(x)|^p\, dx = \int_\Omega \lim_{j\to\infty} |u_{n_j}(x) - u_n(x)|^p\, dx$$

$$\le \liminf_{j\to\infty} \int_\Omega |u_{n_j}(x) - u_n(x)|^p\, dx \le \varepsilon^p$$

if $n \ge M$. Thus $u = (u - u_n) + u_n \in L^p(\Omega)$ and $\|u - u_n\|_p \to 0$ as $n \to \infty$. Therefore $L^p(\Omega)$ is complete.

Finally, if $\{u_n\}$ is a Cauchy sequence in $L^\infty(\Omega)$, then there exists a set $A \subset \Omega$ having measure zero such that if $x \notin A$, then for every $n, m = 1, 2, \ldots$

$$|u_n(x)| \le \|u_n\|_\infty, \qquad |u_n(x) - u_m(x)| \le \|u_n - u_m\|_\infty.$$

Since $\{\|u_n\|_\infty\}$ is bounded in $\mathbb{R}$, u_n converges uniformly on $\Omega \sim A$ to a bounded function u. Setting $u(x) = 0$ for $x \in A$, we have $u \in L^\infty(\Omega)$ and $\|u_n - u\|_\infty \to 0$ as $n \to \infty$. Thus $L^\infty(\Omega)$ is complete. ∎

2.11 COROLLARY If $1 \le p \le \infty$, a Cauchy sequence in $L^p(\Omega)$ has a subsequence converging pointwise almost everywhere on Ω.

2.12 COROLLARY $L^2(\Omega)$ is a Hilbert space with respect to the inner product

$$(u, v) = \int_\Omega u(x)\overline{v(x)}\,dx.$$

Hölder's inequality for $L^2(\Omega)$ is actually just the well-known Schwarz inequality

$$|(u, v)| \le \|u\|_2\,\|v\|_2.$$

Approximation by Continuous Functions, Separability

2.13 THEOREM $C_0(\Omega)$ is dense in $L^p(\Omega)$ if $1 \le p < \infty$.

PROOF Let $u \in L^p(\Omega)$ and let $\varepsilon > 0$. We show that there exists a function $\phi \in C_0(\Omega)$ such that $\|u-\phi\|_p < \varepsilon$. Setting $u = u_1 - u_2 + i(u_3 - u_4)$ where each u_j, $1 \le j \le 4$, is real valued and nonnegative, we find $\phi_j \in C_0(\Omega)$ such that $\|\phi_j - u_j\|_p < \varepsilon/4$, $1 \le j \le 4$. Then $\|u - \phi_1 + \phi_2 - i(\phi_3 - \phi_4)\|_p < \varepsilon$. We assume without loss of generality, therefore, that u is real valued and nonnegative. By Theorem 1.39 there exists a monotonically increasing sequence $\{s_n\}$ of non-negative simple functions converging pointwise to u on Ω. Since $0 \le s_n(x) \le u(x)$ we have $s_n \in L^p(\Omega)$. Since $(u(x)-s_n(x))^p \le (u(x))^p$ we have $s_n \to u$ in $L^p(\Omega)$ by the dominated convergence theorem 1.45. We may thus pick $s \in \{s_n\}$ such that $\|u-s\|_p < \varepsilon/2$. Since s is simple and $p < \infty$ the support of s must have finite volume. We may also assume that $s(x) = 0$ for all $x \in \Omega^c$. Applying Lusin's theorem 1.37(f) we obtain a function $\phi \in C_0(\Omega)$ such that

$$|\phi(x)| \le \|s\|_\infty \qquad \text{for all} \quad x \in \Omega,$$

and

$$\operatorname{vol}\{x \in \Omega : s(x) \ne \phi(x)\} < (\varepsilon/4\,\|s\|_\infty)^p.$$

Hence by Theorem 2.8 we have

$$\begin{aligned}\|s-\phi\|_p &\le \|s-\phi\|_\infty(\operatorname{vol}\{x \in \Omega : s(x) \ne \phi(x)\})^{1/p}\\ &< 2\,\|s\|_\infty(\varepsilon/4\,\|s\|_\infty) = \varepsilon/2.\end{aligned}$$

It follows that $\|u-\phi\|_p < \varepsilon$. ∎

2.14 The above proof shows that in fact the set of simple functions in $L^p(\Omega)$ is dense in $L^p(\Omega)$ for $1 \le p < \infty$. That this is also true for $L^\infty(\Omega)$ is a direct consequence of Theorem 1.39.

2.15 THEOREM $L^p(\Omega)$ is separable if $1 \le p < \infty$.

PROOF For $m = 1, 2, \ldots$ let

$$\overline{\Omega}_m = \{x \in \Omega : \operatorname{dist}(x, \operatorname{bdry}\Omega) \ge 1/m \text{ and } |x| \le m\}.$$

Thus $\bar{\Omega}_m$ is a compact subset of Ω. Let P be the set of all polynomials on $\mathbb{R}^n$ having rational-complex coefficients. Let $P_m = \{\chi_{\bar{\Omega}_m} f : f \in P\}$ where $\chi_{\bar{\Omega}_m}$ is the characteristic function of $\bar{\Omega}_m$. By Corollary 1.29, P_m is dense in $C(\bar{\Omega}_m)$. Moreover, $\bigcup_{m=1}^{\infty} P_m$ is countable.

If $u \in L^p(\Omega)$ and $\varepsilon > 0$, there exists $\phi \in C_0(\Omega)$ such that $\|u-\phi\|_p < \varepsilon/2$. If $1/m < \operatorname{dist}(\operatorname{supp}\phi, \operatorname{bdry}\Omega)$, there exists $f \in P_m$ such that $\|\phi - f\|_\infty < (\varepsilon/2)(\operatorname{vol}\bar{\Omega}_m)^{-1/p}$. It follows that

$$\|\phi-f\|_p \le \|\phi-f\|_\infty (\operatorname{vol}\bar{\Omega}_m)^{1/p} < \varepsilon/2$$

and so $\|u-f\|_p < \varepsilon$. Thus the countable set $\bigcup_{m=1}^{\infty} P_m$ is dense in $L^p(\Omega)$ and $L^p(\Omega)$ is separable. ▮

2.16 $C(\Omega)$, being a proper closed subspace of $L^\infty(\Omega)$, is not dense in that space. Thus neither is $C_0(\Omega)$ nor $C_0^\infty(\Omega)$, and $L^\infty(\Omega)$ is not separable.

Mollifiers, Approximation by Smooth Functions

2.17 Let J be a nonnegative, real-valued function belonging to $C_0^\infty(\mathbb{R}^n)$ and having the properties

(i) $J(x) = 0$ if $|x| \ge 1$, and
(ii) $\int_{\mathbb{R}^n} J(x)\,dx = 1$.

For example, we may take

$$J(x) = \begin{cases} k \exp[-1/(1-|x|^2)] & \text{if } |x| < 1 \\ 0 & \text{if } |x| \ge 1, \end{cases}$$

where $k > 0$ is so chosen that condition (ii) is satisfied. If $\varepsilon > 0$, the function $J_\varepsilon(x) = \varepsilon^{-n} J(x/\varepsilon)$ is nonnegative, belongs to $C_0^\infty(\mathbb{R}^n)$, and satisfies

(i) $J_\varepsilon(x) = 0$ if $|x| \ge \varepsilon$, and
(ii) $\int_{\mathbb{R}^n} J_\varepsilon(x)\,dx = 1$.

J_ε is called a *mollifier*, and the convolution

$$J_\varepsilon * u(x) = \int_{\mathbb{R}^n} J_\varepsilon(x-y)\,u(y)\,dy, \tag{16}$$

defined for functions u for which the right side of (16) makes sense, is called a *mollification* or *regularization* of u. We summarize some properties of mollification in the following lemma.

2.18 LEMMA Let u be a function which is defined on $\mathbb{R}^n$ and vanishes identically outside the domain Ω.

(a) If $u \in L^1_{\mathrm{loc}}(\overline{\Omega})$, then $J_\varepsilon * u \in C^\infty(\mathbb{R}^n)$.
(b) If also $\operatorname{supp} u \subset\subset \Omega$, then $J_\varepsilon * u \in C_0^\infty(\Omega)$ provided

$$\varepsilon < \operatorname{dist}(\operatorname{supp} u, \operatorname{bdry} \Omega).$$

(c) If $u \in L^p(\Omega)$ where $1 \le p < \infty$, then $J_\varepsilon * u \in L^p(\Omega)$. Moreover,

$$\|J_\varepsilon * u\|_p \le \|u\|_p \qquad \text{and} \qquad \lim_{\varepsilon \to 0+} \|J_\varepsilon * u - u\|_p = 0.$$

(d) If $u \in C(\Omega)$ and $G \subset\subset \Omega$, then $\lim_{\varepsilon \to 0+} J_\varepsilon u(x) = u(x)$ uniformly on G.
(e) If $u \in C(\overline{\Omega})$, then $\lim_{\varepsilon \to 0+} J_\varepsilon u(x) = u(x)$ uniformly on Ω.

z.B.: $u(x) = 1$ auf $\Omega = \{x \in \mathbb{R}^n : x_1 > 0\}$ $\Rightarrow$ $J_\varepsilon u(x) = \frac{1}{2}$ für $x \in \partial\Omega$ für alle ε

PROOF Since $J_\varepsilon(x-y)$ is an infinitely differentiable function of x and vanishes if $|y-x| \ge \varepsilon$, and since for every multi-index α and every function u that is integrable on compact sets in $\mathbb{R}^n$ we have

$$D^\alpha(J_\varepsilon * u)(x) = \int_{\mathbb{R}^n} D_x^{\alpha} J_\varepsilon(x-y)\, u(y)\, dy,$$

it follows that conclusions (a) and (b) are valid.

Suppose $u \in L^p(\Omega)$. If $1 < p < \infty$, we let $p' = p/(p-1)$ and obtain by Hölder's inequality

$$\begin{aligned} |J_\varepsilon * u(x)| &= \left| \int_{\mathbb{R}^n} J_\varepsilon(x-y)\, u(y)\, dy \right| \\ &\le \left\{ \int_{\mathbb{R}^n} J_\varepsilon(x-y)\, dy \right\}^{1/p'} \left\{ \int_{\mathbb{R}^n} J_\varepsilon(x-y)\, |u(y)|^p\, dy \right\}^{1/p} \\ &= \left\{ \int_{\mathbb{R}^n} J_\varepsilon(x-y)\, |u(y)|^p\, dy \right\}^{1/p}. \end{aligned} \tag{17}$$

Hence by Fubini's theorem

$$\begin{aligned} \int_\Omega |J_\varepsilon * u(x)|^p\, dx &\le \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} J_\varepsilon(x-y)\, |u(y)|^p\, dy\, dx \\ &= \int_{\mathbb{R}^n} |u(y)|^p\, dy \int_{\mathbb{R}^n} J_\varepsilon(x-y)\, dx = \|u\|_p^p. \end{aligned} \tag{18}$$

Let $\eta > 0$. By Theorem 2.13 there exists $\phi \in C_0(\Omega)$ such that $\|u - \phi\|_p < \eta/3$. Thus by (18), $\|J_\varepsilon * u - J_\varepsilon * \phi\|_p < \eta/3$. Now

$$\begin{aligned} |J_\varepsilon * \phi(x) - \phi(x)| &= \left| \int_{\mathbb{R}^n} J_\varepsilon(x-y)(\phi(y) - \phi(x))\, dy \right| \\ &\le \sup_{|y-x| < \varepsilon} |\phi(y) - \phi(x)|. \end{aligned} \tag{19}$$

Since ϕ is uniformly continuous on Ω the right side of (19) tends to zero as

$\varepsilon \to 0+$. Since $\operatorname{supp} \phi$ is compact we may therefore arrange to have $\|J_\varepsilon * \phi - \phi\|_p < \eta/3$ by choosing ε sufficiently small. For such ε we therefore have $\|J_\varepsilon * u - u\|_p < \eta$ and (c) follows. If $p = 1$, (18) follows directly from (16) without use of Hölder's inequality, and the rest of the proof of (c) is the same as above. The proofs of (d) and (e) may be obtained by replacing ϕ by u in (19). ∎

2.19 THEOREM $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$ if $1 \le p < \infty$.

The proof is an immediate consequence of Theorem 2.13 and Lemma 2.18(b, e).

Precompact Sets in $L^p(\Omega)$

2.20 The following theorem plays a role in the study of L^p-spaces similar to that played by the Ascoli–Arzela theorem 1.30 in the study of spaces of continuous functions. If u is a function defined (a.e.) on a domain $\Omega \subset \mathbb{R}^n$, we denote by $\tilde{u}$ the zero extension of u outside Ω, that is,

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if} \quad x \in \Omega \\ 0 & \text{if} \quad x \in \mathbb{R}^n \sim \Omega. \end{cases}$$

2.21 THEOREM Let $1 \le p < \infty$. A bounded subset $K \subset L^p(\Omega)$ is precompact in $L^p(\Omega)$ if and only if for every number $\varepsilon > 0$ there exists a number $\delta > 0$ and a subset $G \subset\subset \Omega$ such that for every $u \in K$ and every $h \in \mathbb{R}^n$ with $|h| < \delta$

$$\int_\Omega |\tilde{u}(x+h) - \tilde{u}(x)|^p \, dx < \varepsilon^p \tag{20}$$

and

$$\int_{\Omega \sim \bar{G}} |u(x)|^p \, dx < \varepsilon^p. \tag{21}$$

Proof It is sufficient to prove the theorem for the special case $\Omega = \mathbb{R}^n$, as the theorem follows for general Ω from its application in this special case to the set $\tilde{K} = \{\tilde{u} : u \in K\}$.

Let us assume first that K is precompact in $L^p(\mathbb{R}^n)$. Let $\varepsilon > 0$ be given. Since K has a finite $(\varepsilon/6)$-net (Theorem 1.18), and since $C_0(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ (Theorem 2.13), there exists a finite set S of continuous functions having compact support, such that for each $u \in K$ there exists $\phi \in S$ satisfying $\|u - \phi\|_p < \varepsilon/3$. Since S is finite there exists $r > 0$ such that $\operatorname{supp} \phi \subset \bar{B}_r$ for every $\phi \in S$, where B_r is the ball $\{x \in \mathbb{R}^n : |x| < r\}$. Setting $G = B_r$, we obtain (21). Also, $\phi(x+h) - \phi(x)$ is uniformly continuous for all x and vanishes identically outside B_{r+1} provided $|h| < 1$. Hence

$$\lim_{|h| \to 0} \int_{\mathbb{R}^n} |\phi(x+h) - \phi(x)|^p \, dx = 0. \tag{22}$$

Since S is finite, (22) is uniform for $\phi \in S$. For $u \in K$ let $T_h u$ be the translate of u by h:

$$T_h u(x) = u(x+h). \tag{23}$$

If $\phi \in S$ satisfies $\|u - \phi\|_p < \varepsilon/3$, then also $\|T_h u - T_h \phi\|_p < \varepsilon/3$. Hence by (22) we have for $|h|$ sufficiently small (independent of $u \in K$),

$$\begin{aligned} \|T_h u - u\|_p &\le \|T_h u - T_h \phi\|_p + \|T_h \phi - \phi\|_p + \|\phi - u\|_p \\ &< (2\varepsilon/3) + \|T_h \phi - \phi\|_p < \varepsilon, \end{aligned}$$

and (20) follows. [This argument shows translation is continuous in $L^p(\Omega)$.]

To prove the converse let $\varepsilon > 0$ be given and choose $G \subset\subset \mathbb{R}^n$ such that for all $u \in K$

$$\int_{\mathbb{R}^n \sim \bar{G}} |u(x)|^p \, dx < \varepsilon/3. \tag{24}$$

For any $\eta > 0$ the function $J_\eta * u$ defined as in (16) belongs to $C^\infty(\mathbb{R}^n)$ and in particular to $C(\bar{G})$. If $\phi \in C_0(\mathbb{R}^n)$, then by Hölder's inequality

$$\begin{aligned} |J_\eta * \phi(x) - \phi(x)|^p &= \left| \int_{\mathbb{R}^n} J_\eta(y)(\phi(x-y) - \phi(x)) \, dy \right|^p \\ &\le \int_{B_\eta} J_\eta(y) |T_{-y} \phi(x) - \phi(x)|^p \, dy \end{aligned}$$

where $T_h \phi$ is given as in (23). Hence

$$\|J_\eta * \phi - \phi\|_p \le \sup_{h \in B_\eta} \|T_h \phi - \phi\|_p. \tag{25}$$

If $u \in L^p(\mathbb{R}^n)$, let $\{\phi_n\}$ be a sequence in $C_0(\mathbb{R}^n)$ converging to u in L^p-norm. By Lemma 2.18(c), $\{J_\eta * \phi_n\}$ is a Cauchy sequence converging to $J_\eta * u$ in $L^p(\mathbb{R}^n)$. Since also $T_h \phi_n \to T_h u$ in $L^p(\mathbb{R}^n)$, (25) extends to all $u \in L^p(\mathbb{R}^n)$:

$$\|J_\eta * u - u\|_p \le \sup_{h \in B_\eta} \|T_h u - u\|_p.$$

Now (20) implies that $\lim_{|h| \to 0} \|T_h u - u\|_p = 0$ uniformly for $u \in K$. Hence

$\lim_{\eta\to 0}\|J_\eta * u - u\|_p = 0$ uniformly for $u \in K$. We now fix $\eta > 0$ so that

$$\int_{\bar{G}} |J_\eta * u(x) - u(x)|^p \, dx < \frac{\varepsilon}{3\cdot 2^p} \tag{26}$$

for all $u \in K$.

We show that $\{J_\eta * u : u \in K\}$ satisfies the conditions of the Ascoli–Arzela theorem 1.30 on $\bar{G}$ and hence is precompact in $C(\bar{G})$. By (12) we have

$$|J_\eta * u(x)| \le \left(\sup_{x\in\mathbb{R}^n} J_\eta(x)\right)^{1/p} \|u\|_p$$

which is bounded uniformly for $x \in \mathbb{R}^n$ and $u \in K$ since K is a bounded set in $L^p(\Omega)$ and η is fixed. Similarly

$$|J_\eta * u(x+h) - J_\eta * u(x)| \le \left(\sup_{x\in\mathbb{R}^n} J_\eta(x)\right)^{1/p} \|T_h u - u\|_p$$

and so $\lim_{|h|\to 0} J_\eta * u(x+h) = J_\eta * u(x)$ uniformly for $x \in \mathbb{R}^n$ and $u \in K$.

Thus $\{J_\eta * u : u \in K\}$ is precompact in $C(\bar{G})$ and by Theorem 1.18 there exists a finite set $\{\psi_1, \ldots, \psi_m\}$ of functions in $C(\bar{G})$ such that if $u \in K$, then for some j, $1 \le j \le m$, and all $x \in \bar{G}$ we have

$$|\psi_j(x) - J_\eta * u(x)|^p < \frac{\varepsilon}{3\cdot 2^p \cdot \operatorname{vol}\bar{G}}. \tag{27}$$

Denoting by $\tilde{\psi}_j$ the zero extension of ψ_j outside $\bar{G}$, we obtain from (24), (26), (27), and the inequality $(|a|+|b|)^p \le 2^p(|a|^p + |b|^p)$

$$\begin{aligned}
\int_{\mathbb{R}^n} |u(x) - \tilde{\psi}_j(x)|^p \, dx &= \int_{\mathbb{R}^n\sim\bar{G}} |u(x)|^p \, dx + \int_{\bar{G}} |u(x) - \psi_j(x)|^p \, dx \\
&< \frac{\varepsilon}{3} + 2^p \int_{\bar{G}} \left(|u(x) - J_\eta * u(x)|^p + |J_\eta * u(x) - \psi_j(x)|^p\right) dx \\
&< \frac{\varepsilon}{3} + 2^p\left(\frac{\varepsilon}{3\cdot 2^p} + \frac{\varepsilon}{3\cdot 2^p\cdot\operatorname{vol}\bar{G}}\operatorname{vol}\bar{G}\right) = \varepsilon.
\end{aligned}$$

Hence K has a finite ε-net in $L^p(\mathbb{R}^n)$, namely $\{\tilde{\psi}_j : 1 \le j \le m\}$, and so is precompact by Theorem 1.18. ▌

2.22 THEOREM Let $1 \le p < \infty$ and let $K \subset L^p(\Omega)$. Suppose there exists a sequence $\{\Omega_j\}$ of subdomains of Ω having the following properties:

(a) for each j, $\Omega_j \subset \Omega_{j+1}$;

(b) for each j the set of restrictions to Ω_j of the functions in K is precompact in $L^p(\Omega_j)$;

(c) for every $\varepsilon > 0$, there exists j such that

$$\int_{\Omega \sim \Omega_j} |u(x)|^p \, dx < \varepsilon \qquad \text{for every} \quad u \in K.$$

Then K is precompact in $L^p(\Omega)$.

PROOF Let $\{u_n\}$ be a sequence in K. Then by (b) there exists a subsequence $\{u_n^{(1)}\}$ such that the restrictions $\{u_n^{(1)}|_{\Omega_1}\}$ converge in $L^p(\Omega_1)$. Having selected $\{u_n^{(1)}\}, \dots, \{u_n^{(k)}\}$, we may select a subsequence $\{u_n^{(k+1)}\}$ of $\{u_n^{(k)}\}$ such that $\{u_n^{(k+1)}|_{\Omega_{k+1}}\}$ converges in $L^p(\Omega_{k+1})$. Hence also $\{u_n^{(k+1)}|_{\Omega_j}\}$ converges in $L^p(\Omega_j)$ for $1 \le j \le k+1$ by (a).

Let $v_n = u_n^{(n)}$ for $n = 1, 2, \dots$. Clearly $\{v_n\}$ is a subsequence of $\{u_n\}$. Given $\varepsilon > 0$, there exists j [by (c)] such that

$$\int_{\Omega \sim \Omega_j} |v_n(x) - v_m(x)|^p \, dx < \varepsilon/2 \tag{28}$$

for all $n, m = 1, 2, \dots$. Except for the first $j-1$ terms, $\{v_n\}$ is a subsequence of $\{u_n^{(j)}\}$ and so $\{v_n|_{\Omega_j}\}$ is a Cauchy sequence in $L^p(\Omega_j)$. Thus for n, m sufficiently large we have

$$\int_{\Omega_j} |v_n(x) - v_m(x)|^p \, dx < \varepsilon/2. \tag{29}$$

Combining (28) and (29) we see that $\{v_n\}$ is a Cauchy sequence in $L^p(\Omega)$ and so converges there. Hence K is precompact in $L^p(\Omega)$. ∎

We remark that Theorem 2.22 is just a setting, suitable for our purposes, of a well-known theorem stating that the operator-norm limit of a sequence of compact operators is compact.

The Uniform Convexity of $L^p(\Omega)$

2.23 For $1 < p < \infty$ the space $L^p(\Omega)$ is uniformly convex, its norm $\|\cdot\|_p$ satisfying the condition prescribed in Section 1.19. This result, due to Clarkson [19], is obtained via a set of inequalities for $L^p(\Omega)$ that generalizes the parallelogram law in $L^2(\Omega)$. These inequalities are given in Theorem 2.28, for the proof of which we prepare the following lemmas.

2.24 LEMMA If $1 \le p < \infty$ and $a \ge 0$, $b \ge 0$, then

$$(a+b)^p \le 2^{p-1}(a^p + b^p). \tag{30}$$

Proof If $a = 0$, (30) clearly holds. If $a > 0$, (30) may be rewritten in the form

$$(1+x)^p \le 2^{p-1}(1+x^p) \tag{31}$$

where $0 \le x = b/a$. The function $f(x) = (1+x)^p/(1+x^p)$ satisfies $f(0) = 1 = \lim_{x\to\infty} f(x)$ and $f(x) > 1$ if $0 < x < \infty$. Hence f has its maximum for $x \ge 0$ at its only critical point, $x = 1$. Since $f(1) = 2^{p-1}$, (31) follows. ∎

2.25 LEMMA If $0 < s < 1$, the function $f(x) = (1-s^x)/x$ is a decreasing function of $x > 0$.

Proof $f'(x) = (1/x^2)(g(s^x)-1)$ where $g(t) = t - t\ln t$. Since $0 < s^x < 1$ and since $g'(t) = -\ln t \ge 0$ for $0 < t \le 1$, it follows that $g(s^x) < g(1) = 1$ whence $f'(x) < 0$. ∎

2.26 LEMMA If $1 < p \le 2$ and $0 \le t \le 1$, then

$$\left|\frac{1+t}{2}\right|^{p'} + \left|\frac{1-t}{2}\right|^{p'} \le \left(\frac{1}{2} + \frac{1}{2}t^p\right)^{1/(p-1)}, \tag{32}$$

where $p' = p/(p-1)$ is the exponent conjugate to p.

Proof Since equality clearly holds in (32) if either $p = 2$ or $t = 0$ or $t = 1$, we may assume that $1 < p < 2$ and $0 < t < 1$. Under the transformation $t = (1-s)/(1+s)$, which maps the interval $0 < t < 1$ onto the interval $1 > s > 0$, (32) reduces to the equivalent form

$$\tfrac{1}{2}[(1+s)^p + (1-s)^p] - (1+s^{p'})^{p-1} \ge 0. \tag{33}$$

If we denote

$$\binom{p}{0} = 1 \qquad \text{and} \qquad \binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}, \qquad k \ge 1,$$

the power series expansion of the left side of (33) takes the form

$$\begin{aligned}
&\frac{1}{2}\sum_{k=0}^{\infty}\binom{p}{k}s^k + \frac{1}{2}\sum_{k=0}^{\infty}\binom{p}{k}(-s)^k - \sum_{k=0}^{\infty}\binom{p-1}{k}s^{p'k} \\
&= \sum_{k=0}^{\infty}\binom{p}{2k}s^{2k} - \sum_{k=0}^{\infty}\binom{p-1}{k}s^{p'k} \\
&= \sum_{k=1}^{\infty}\left\{\binom{p}{2k}s^{2k} - \binom{p-1}{2k-1}s^{p'(2k-1)} - \binom{p-1}{2k}s^{2p'k}\right\}.
\end{aligned}$$

The latter series is convergent for $0 \le s \le 1$. We prove (33) by showing that each term in the series is positive for $0 < s < 1$. The kth term can be written in the form

$$\frac{p(p-1)(2-p)(3-p)\cdots(2k-1-p)}{(2k)!}s^{2k}$$

$$-\frac{(p-1)(2-p)(3-p)\cdots(2k-1-p)}{(2k-1)!}s^{p'(2k-1)}+\frac{(p-1)(2-p)\cdots(2k-p)}{(2k)!}s^{2kp'}$$

$$=\frac{(2-p)(3-p)\cdots(2k-p)}{(2k-1)!}s^{2k}\left[\frac{p(p-1)}{2k(2k-p)}-\frac{p-1}{2k-p}s^{p'(2k-1)-2k}+\frac{p-1}{2k}s^{2kp'-2k}\right]$$

$$=\frac{(2-p)(3-p)\cdots(2k-p)}{(2k-1)!}s^{2k}\left[\frac{1-s^{(2k-p)/(p-1)}}{(2k-p)/(p-1)}-\frac{1-s^{2k/(p-1)}}{2k/(p-1)}\right].$$

The first factor above is positive since $p<2$; the factor in brackets is positive by Lemma 2.25 since $0<(2k-p)/(p-1)<2k/(p-1)$. Thus (33) and hence (32) is established. ∎

2.27 LEMMA Let $z, w \in \mathbb{C}$. If $1<p\le 2$, then

$$\left|\frac{z+w}{2}\right|^{p'}+\left|\frac{z-w}{2}\right|^{p'}\le\left(\frac{1}{2}|z|^p+\frac{1}{2}|w|^p\right)^{1/(p-1)},\tag{34}$$

where $p'=p/(p-1)$. If $2\le p<\infty$, then

$$\left|\frac{z+w}{2}\right|^{p}+\left|\frac{z-w}{2}\right|^{p}\le\frac{1}{2}|z|^p+\frac{1}{2}|w|^p.\tag{35}$$

PROOF Since (34) obviously holds if $z=0$ or $w=0$ and is symmetric in z and w, we may assume that $|z|\ge|w|>0$. In this case (34) can be rewritten in the form

$$\left|\frac{1+re^{i\theta}}{2}\right|^{p'}+\left|\frac{1-re^{i\theta}}{2}\right|^{p'}\le\left(\frac{1}{2}+\frac{1}{2}r^p\right)^{1/(p-1)},\tag{36}$$

where $w/z=re^{i\theta}$, $r\ge 0$, $0\le\theta<2\pi$. If $\theta=0$, (36) is already proved in Lemma 2.26. We complete the proof of (36) by showing that for fixed r the function

$$f(\theta)=|1+re^{i\theta}|^{p'}+|1-re^{i\theta}|^{p'}$$

has a maximum value for $0\le\theta<2\pi$ at $\theta=0$. Since

$$f(\theta)=(1+r^2+2r\cos\theta)^{p'/2}+(1+r^2-2r\cos\theta)^{p'/2},$$

it is clear that $f(2\pi-\theta)=f(\pi-\theta)=f(\theta)$, so that we need consider f only on the interval $0\le\theta\le\pi/2$. Since $p'\ge 2$, we have on that interval

$$f'(\theta)=-p'r\sin\theta[(1+r^2+2r\cos\theta)^{(p'/2)-1}-(1+r^2-2r\cos\theta)^{(p'/2)-1}]\le 0.$$

If $2 \le p < \infty$, then $1 < p' \le 2$ and we have, interchanging p and p' in (34) and using Lemma 2.24,

$$\left|\frac{z+w}{2}\right|^p + \left|\frac{z-w}{2}\right|^p \le \left(\frac{1}{2}|z|^{p'} + \frac{1}{2}|w|^{p'}\right)^{1/(p'-1)}$$
$$= \left(\frac{1}{2}|z|^{p'} + \frac{1}{2}|w|^{p'}\right)^{p/p'}$$
$$\le 2^{(p/p')-1}\left(\left(\frac{1}{2}\right)^{p/p'}|z|^p + \left(\frac{1}{2}\right)^{p/p'}|w|^p\right)$$
$$= \frac{1}{2}|z|^p + \frac{1}{2}|w|^p,$$

so that (35) is also proved. ∎

2.28 THEOREM (*Clarkson's inequalities*) Let $u, v \in L^p(\Omega)$. For $1 < p < \infty$ let $p' = p/(p-1)$. If $2 \le p < \infty$, then

$$\left\|\frac{u+v}{2}\right\|_p^p + \left\|\frac{u-v}{2}\right\|_p^p \le \frac{1}{2}\|u\|_p^p + \frac{1}{2}\|v\|_p^p, \tag{37}$$

$$\left\|\frac{u+v}{2}\right\|_p^{p'} + \left\|\frac{u-v}{2}\right\|_p^{p'} \ge \left(\frac{1}{2}\|u\|_p^p + \frac{1}{2}\|v\|_p^p\right)^{p'-1}. \tag{38}$$

If $1 < p \le 2$, then

$$\left\|\frac{u+v}{2}\right\|_p^{p'} + \left\|\frac{u-v}{2}\right\|_p^{p'} \le \left(\frac{1}{2}\|u\|_p^p + \frac{1}{2}\|v\|_p^p\right)^{p'-1}, \tag{39}$$

$$\left\|\frac{u+v}{2}\right\|_p^p + \left\|\frac{u-v}{2}\right\|_p^p \ge \frac{1}{2}\|u\|_p^p + \frac{1}{2}\|v\|_p^p. \tag{40}$$

PROOF For $2 \le p < \infty$, (37) is obtained by taking $z = u(x)$ and $w = v(x)$ in (35) and integrating over Ω. To prove (39) for $1 < p \le 2$ we first note that $\| |u|^{p'} \|_{p-1} = \|u\|_p^{p'}$ for any $u \in L^p(\Omega)$. Using the reverse Minkowski inequality (7) corresponding to the exponent $p-1 < 1$, and (34) with $z = u(x)$, $w = v(x)$, we obtain

$$\left\|\frac{u+v}{2}\right\|_p^{p'} + \left\|\frac{u-v}{2}\right\|_p^{p'} = \left\|\left|\frac{u+v}{2}\right|^{p'}\right\|_{p-1} + \left\|\left|\frac{u-v}{2}\right|^{p'}\right\|_{p-1}$$
$$\le \left(\int_\Omega \left(\left|\frac{u(x)+v(x)}{2}\right|^{p'} + \left|\frac{u(x)-v(x)}{2}\right|^{p'}\right)^{p-1} dx\right)^{1/(p-1)}$$

equation continues

$$\leq \left(\int_\Omega \left(\frac{1}{2}|u(x)|^p + \frac{1}{2}|v(x)|^p\right) dx\right)^{p'-1}$$

$$= \left(\frac{1}{2}\|u\|_p^p + \frac{1}{2}\|v\|_p^p\right)^{p'-1}$$

which is (39).

Inequality (38) can be proved for $2 \leq p < \infty$ by the same method used to prove (39), except that the direct Minkowski inequality (5), corresponding to $p-1 \geq 1$, is used in place of the reverse inequality, and in place of (34) is used the inequality

$$\left(\left|\frac{\xi+\eta}{2}\right|^{p'} + \left|\frac{\xi-\eta}{2}\right|^{p'}\right)^{p-1} \geq \frac{1}{2}|\xi|^p + \frac{1}{2}|\eta|^p,$$

which is obtained from (34) by replacing p by p', z by $\xi+\eta$, and w by $\xi-\eta$. Finally, (40) can be obtained from a similar revision of (35). We remark that all four of Clarkson's inequalities reduce to the parallelogram law

$$\|u+v\|_2^2 + \|u-v\|_2^2 = 2\|u\|_2^2 + 2\|v\|_2^2$$

in the case $p = 2$. ∎

2.29 COROLLARY If $1 < p < \infty$, $L^p(\Omega)$ is uniformly convex.

Proof Let $u, v \in L^p(\Omega)$ satisfy $\|u\|_p = \|v\|_p = 1$ and $\|u-v\|_p \geq \varepsilon > 0$. If $2 \leq p < \infty$, we have from (37)

$$\left\|\frac{u+v}{2}\right\|_p^p \leq 1 - \frac{\varepsilon^p}{2^p}.$$

If $1 < p \leq 2$, we have from (39)

$$\left\|\frac{u+v}{2}\right\|_p^{p'} \leq 1 - \frac{\varepsilon^{p'}}{2^{p'}}.$$

In either case there exists $\delta = \delta(\varepsilon) > 0$ such that $\|(u+v)/2\|_p \leq 1-\delta$. ∎

Being uniformly convex, $L^p(\Omega)$ is reflexive for $1 < p < \infty$ by Theorem 1.20. We shall give a direct proof of this reflexivity after computing the dual of $L^p(\Omega)$.

The Normed Dual of $L^p(\Omega)$

2.30 Let $1 \leq p \leq \infty$ and let p' denote the exponent conjugate to p. For each element $v \in L^{p'}(\Omega)$ we can define a linear functional L_v on $L^p(\Omega)$ via

$$L_v(u) = \int_\Omega u(x)v(x)\, dx, \qquad u \in L^p(\Omega).$$

By Hölder's inequality $|L_v(u)| \le \|u\|_p \|v\|_{p'}$ so that $L_v \in [L^p(\Omega)]'$ and

$$\|L_v; [L^p(\Omega)]'\| \le \|v\|_{p'}. \tag{41}$$

We show that equality must hold in (41). If $1 < p \le \infty$, let $u(x) = |v(x)|^{p'-2}\overline{v(x)}$ if $v(x) \ne 0$ and $u(x) = 0$ otherwise. Then $u \in L^p(\Omega)$ and $L_v(u) = \|u\|_p \|v\|_{p'}$. Now suppose $p = 1$ so $p' = \infty$. If $\|v\|_{p'} = 0$, let $u(x) = 0$. Otherwise let $0 < \varepsilon < \|v\|_\infty$ and let A be a measurable subset of Ω such that $0 < \mu(A) < \infty$ and $|v(x)| \ge \|v\|_\infty - \varepsilon$ on A. Let $u(x) = |v(x)|^{-1}\overline{v(x)}$ for $x \in A$; $u(x) = 0$ otherwise. Then $u \in L^1(\Omega)$ and $L_v(u) \ge \|u\|_1(\|v\|_\infty - \varepsilon)$. Thus we have shown that

$$\|L_v; [L^p(\Omega)]'\| = \|v\|_{p'} \tag{42}$$

so that the operator L mapping v to L_v is an isometric isomorphism of $L^{p'}(\Omega)$ onto a subspace of $[L^p(\Omega)]'$.

2.31 It is natural to ask whether the range of the isomorphism L is all of $[L^p(\Omega)]'$, that is, whether every continuous linear functional on $L^p(\Omega)$ is of the form L_v for some $v \in L^{p'}(\Omega)$. We shall show that such is the case provided $1 \le p < \infty$. For $p = 2$ this is an immediate consequence of the Riesz representation theorem for Hilbert spaces. For general p a direct proof can be given using the Radon–Nikodym theorem (see Rudin [58] or Theorem 8.18). We shall give a more elementary proof based on the uniform convexity of $L^p(\Omega)$ and a variational argument. This method of proof is also used by Hewitt and Stromberg [32]. Finally we shall use a limiting argument to obtain the case $p = 1$ from the case $p > 1$.

2.32 LEMMA Let $1 < p < \infty$. If $L \in [L^p(\Omega)]'$ and $\|L; [L^p(\Omega)]'\| = 1$, then there exists unique $w \in L^p(\Omega)$ such that $\|w\|_p = L(w) = 1$. Dually, if $w \in L^p(\Omega)$ is given and $\|w\|_p = 1$, then there exists unique $L \in [L^p(\Omega)]'$ such that $\|L; [L^p(\Omega)]'\| = L(w) = 1$.

PROOF First assume that $L \in [L^p(\Omega)]'$ is given and $\|L\| = 1$. There exists a sequence $\{w_n\} \in L^p(\Omega)$ such that $\|w_n\| = 1$ and $\lim_{n\to\infty} |L(w_n)| = 1$. We may assume that $|L(w_n)| > \frac{1}{2}$ for each n, and, replacing w_n by a suitable multiple of w_n by a complex number of unit modulus, that $L(w_n) > 0$. Suppose the sequence $\{w_n\}$ is not a Cauchy sequence in $L^p(\Omega)$. Then there exists $\varepsilon > 0$ such that $\|w_n - w_m\|_p \ge \varepsilon$ for some arbitrarily large values of m and n, so that by uniform convexity we have $\|\frac{1}{2}(w_n + w_m)\|_p \le 1 - \delta$, where δ is a fixed positive number. Thus

$$\begin{aligned} 1 \ge L\left(\frac{w_n + w_m}{\|w_n + w_m\|_p}\right) &= \left\|\frac{w_n + w_m}{2}\right\|_p^{-1} L\left(\frac{w_n + w_m}{2}\right) \\ &\ge \frac{1}{1-\delta} \cdot \frac{1}{2}[L(w_n) + L(w_m)]. \end{aligned} \tag{43}$$

Since the last expression approaches $1/(1-\delta)$ as $n, m \to \infty$, we have a contradiction. Thus $\{w_n\}$ is a Cauchy sequence in $L^p(\Omega)$ and so converges to an element w of that space. Clearly $\|w\|_p = 1$ and $L(w) = \lim_{n\to\infty} L(w_n) = 1$. The uniqueness of w follows from (43) applied to two distinct candidates.

Now suppose $w \in L^p(\Omega)$ is given and $\|w\|_p = 1$. As noted in Section 2.30 the functional L_v defined by

$$L_v(u) = \int_\Omega u(x)\, v(x)\, dx, \qquad u \in L^p(\Omega), \tag{44}$$

where

$$v(x) = \begin{cases} |w(x)|^{p-2}\overline{w(x)} & \text{if } \; w(x) \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{45}$$

satisfies $L_v(w) = \|w\|_p^p = 1$ and $\|L_v; [L^p(\Omega)]'\| = \|v\|_{p'} = \|w\|_p^{p/p'} = 1$. It remains to be shown, therefore, that if $L_1, L_2 \in [L^p(\Omega)]'$ satisfy $\|L_1\| = \|L_2\| = L_1(w) = L_2(w) = 1$, then $L_1 = L_2$. Suppose not. Then there exists $u \in L^p(\Omega)$ such that $L_1(u) \neq L_2(u)$. Replacing u by a suitable multiple of u, we may assume that $L_1(u) - L_2(u) = 2$. Then replacing u by its sum with a suitable multiple of w, we can arrange that $L_1(u) = 1$ and $L_2(u) = -1$. If $t > 0$, then $L_1(w+tu) = 1+t$; since $\|L_1\| = 1$, therefore $\|w+tu\|_p \geq 1+t$. Similarly, $L_2(w-tu) = 1+t$ so $\|w-tu\|_p \geq 1+t$. If $1 < p \leq 2$, Clarkson's inequality (40) gives

$$\begin{aligned} 1 + t^p \|u\|_p^p &= \left\| \frac{(w+tu)+(w-tu)}{2} \right\|_p^p + \left\| \frac{(w+tu)-(w-tu)}{2} \right\|_p^p \\ &\geq \frac{1}{2}\|w+tu\|_p^p + \frac{1}{2}\|w-tu\|_p^p \geq (1+t)^p. \end{aligned} \tag{46}$$

If $2 \leq p < \infty$, Clarkson's inequality (38) gives

$$\begin{aligned} 1 + t^{p'} \|u\|_p^{p'} &= \left\| \frac{(w+tu)+(w-tu)}{2} \right\|_p^{p'} + \left\| \frac{(w+tu)-(w-tu)}{2} \right\|_p^{p'} \\ &\geq \left(\frac{1}{2}\|w+tu\|_p^p + \frac{1}{2}\|w-tu\|_p^p \right)^{p'-1} \geq (1+t)^{p'}. \end{aligned} \tag{47}$$

Equations (46) and (47) are not possible for all $t > 0$ unless $\|u\|_p = 0$ which is impossible. Thus $L_1 = L_2$. ∎

2.33 THEOREM (*The Riesz representation theorem for* $L^p(\Omega)$) Let $1 < p < \infty$ and let $L \in [L^p(\Omega)]'$. Then there exists $v \in L^{p'}(\Omega)$ such that for all

$u \in L^p(\Omega)$

$$L(u) = \int_\Omega u(x)v(x)\,dx.$$

Moreover, $\|v\|_{p'} = \|L;[L^p(\Omega)]'\|$. Thus $[L^p(\Omega)]' \cong L^{p'}(\Omega)$.

PROOF Lf $L = 0$, we may take $v = 0$. Accordingly, we assume $L \neq 0$ and, without loss of generality, that $\|L;[L^p(\Omega)]'\| = 1$. By Lemma 2.32 there exists $w \in L^p(\Omega)$ with $\|w\|_p = 1$ such that $L(w) = 1$. Let v be given by (45). Then L_v, defined by (44), satisfies $\|L_v;[L^p(\Omega)]'\| = 1$ and $L_v(w) = 1$. By Lemma 2.32, again we have $L = L_v$. Since $\|v\|_{p'} = 1$, the proof is complete. ▮

2.34 THEOREM (*Riesz representation theorem for $L^1(\Omega)$*) Let $L \in [L^1(\Omega)]'$. Then there exists $v \in L^\infty(\Omega)$ such that for all $u \in L^1(\Omega)$

$$L(u) = \int_\Omega u(x)v(x)\,dx$$

and $\|v\|_\infty = \|L;[L^1(\Omega)]'\|$. Thus $[L^1(\Omega)]' \cong L^\infty(\Omega)$.

PROOF Once again we may assume that $L \neq 0$ and $\|L;[L^1(\Omega)]'\| = 1$. Let us suppose, for the moment, that Ω has finite volume. Then by Theorem 2.8 if $1 < p < \infty$, we have $L^p(\Omega) \subset L^1(\Omega)$ and

$$|L(u)| \le \|u\|_1 \le (\operatorname{vol}\Omega)^{1-(1/p)}\|u\|_p$$

for any $u \in L^p(\Omega)$. Hence $L \in [L^p(\Omega)]'$ and by Theorem 2.33 there exists $v_p \in L^{p'}(\Omega)$ such that

$$\|v_p\|_{p'} \le (\operatorname{vol}\Omega)^{1-(1/p)} \tag{48}$$

and for every $u \in L^p(\Omega)$

$$L(u) = \int_\Omega u(x)v_p(x)\,dx. \tag{49}$$

Since $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$ for $1 < p < \infty$, and since for any p, q satisfying $1 < p, q < \infty$ and any $\phi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega \phi(x)v_p(x)\,dx = L(\phi) = \int_\Omega \phi(x)v_q(x)\,dx,$$

it follows that $v_p = v_q$ a.e. on Ω. Hence we may replace v_p in (49) bv a function v belonging to $L^p(\Omega)$ for each p, $1 < p < \infty$, and satisfying, following (48),

$$\|v\|_{p'} \le (\operatorname{vol}\Omega)^{1-(1/p)}.$$

It follows by Theorem 2.8 again that $v \in L^\infty(\Omega)$ and

$$\|v\|_\infty \le \lim_{p'\to\infty} (\operatorname{vol}\Omega)^{1-(1/p)} = 1. \tag{50}$$

The argument at the beginning of Section 2.30 shows that equality must prevail in (50).

If Ω does not have finite volume, we may nevertheless write $\Omega = \bigcup_{j=1}^{\infty} G_j$, where $G_j = \{x \in \Omega : j-1 \le |x| < j\}$ has finite volume. The sets G_j are mutually disjoint. Let $\chi_j(x)$ be the characteristic function of G_j. If $u_j \in L^1(G_j)$, let $\tilde{u}_j$ denote the zero extension of u_j outside G_j, that is, $\tilde{u}_j(x) = u_j(x)$ if $x \in G_j$, $\tilde{u}_j(x) = 0$ otherwise. Let $L_j(u_j) = L(\tilde{u}_j)$. Then $L_j \in [L^1(G_j)]'$ and $\|L_j; [L^1(G_j)]'\| \le 1$. By the finite volume case considered above there exists $v_j \in L^\infty(G_j)$ such that $\|v_j\|_{\infty, G_j} \le 1$ and

$$L_j(u_j) = \int_{G_j} u_j(x) v_j(x)\, dx = \int_\Omega \tilde{u}_j(x) v(x)\, dx,$$

where $v(x) = v_j(x)$ for $x \in G_j$ $(j = 1, 2, \ldots)$, so that $\|v\|_\infty \le 1$. If $u \in L^1(\Omega)$, we put $u = \sum_{j=1}^{\infty} \chi_j u$; the series converging in norm in $L^1(\Omega)$ by dominated convergence. Since

$$L\left(\sum_{j=1}^{k} \chi_j u\right) = \sum_{j=1}^{k} L_j(\chi_j u) = \int_\Omega \sum_{j=1}^{k} \chi_j(x) u(x) v(x)\, dx,$$

we obtain, passing to the limit by dominated convergence,

$$L(u) = \int_\Omega u(x) v(x)\, dx.$$

It then follows, as in the finite volume case, that $\|v\|_\infty = 1$. ∎

2.35 THEOREM $L^p(\Omega)$ is reflexive if and only if $1 < p < \infty$.

Proof Let $X = L^p(\Omega)$, where $1 < p < \infty$. Since $X' \cong L^{p'}(\Omega)$, to any $w \in X''$ there corresponds $\tilde{w} \in [L^{p'}(\Omega)]'$ such that $w(v) = \tilde{w}(\tilde{v})$, where

$$v(u) = \int_\Omega \tilde{v}(x) u(x)\, dx, \qquad u \in X.$$

Similarly, corresponding to $\tilde{w} \in [L^{p'}(\Omega)]'$ there exists $u \in X$ such that

$$\tilde{w}(\tilde{v}) = \int_\Omega \tilde{v}(x) u(x)\, dx, \qquad \tilde{v} \in L^{p'}(\Omega).$$

It follows that

$$w(v) = \tilde{w}(\tilde{v}) = \int_\Omega \tilde{v}(x) u(x)\, dx = v(u) = J_X u(v)$$

for all $v \in X'$, J_X being the natural isometric isomorphism (see Section 1.13) of X into X''. This shows that J_X maps X onto X'' so that $X = L^p(\Omega)$ is reflexive.

Since $L^1(\Omega)$ is separable while its dual, which is isometrically isomorphic to $L^\infty(\Omega)$, is not separable, neither $L^1(\Omega)$ nor $L^\infty(\Omega)$ can be reflexive. ∎

2.36 The Riesz representation theorem cannot hold for the space $L^\infty(\Omega)$ in a form analogous to Theorem 2.33, for if so, then the argument of Theorem 2.35 would show that $L^1(\Omega)$ was reflexive. The dual of $L^\infty(\Omega)$ is larger than $L^1(\Omega)$. It may be identified with a space of absolutely continuous, finitely additive set functions of bounded total variation on Ω. The reader is referred to Yosida [69, p. 118] for details.

III

The Spaces $W^{m,p}(\Omega)$

Definitions and Basic Properties

In this chapter we introduce Sobolev spaces of integer order and establish some of their basic properties. These spaces are defined over an arbitrary domain $\Omega \subset \mathbb{R}^n$ and are vector subspaces of various spaces $L^p(\Omega)$.

3.1 We define a functional $\|\cdot\|_{m,p}$, where m is a nonnegative integer and $1 \le p \le \infty$, as follows:

$$\|u\|_{m,p} = \left\{ \sum_{0 \le |\alpha| \le m} \|D^\alpha u\|_p^p \right\}^{1/p} \qquad \text{if} \qquad 1 \le p < \infty, \tag{1}$$

$$\|u\|_{m,\infty} = \max_{0 \le |\alpha| \le m} \|D^\alpha u\|_\infty \tag{2}$$

for any function u for which the right side makes sense, $\|\cdot\|_p$ being, of course, the $L^p(\Omega)$-norm. (In situations where confusion of domains may occur we shall write $\|u\|_{m,p,\Omega}$ in place of $\|u\|_{m,p}$.) It is clear that (1) or (2) defines a norm on any vector space of functions on which the right side takes finite values provided functions are identified in the space if they are equal almost everywhere in Ω. We consider three such spaces corresponding to any given values of m and p:

$H^{m,p}(\Omega) \equiv$ the completion of $\{u \in C^m(\Omega) : \|u\|_{m,p} < \infty\}$ with respect to the norm $\|\cdot\|_{m,p}$,

$W^{m,p}(\Omega) \equiv \{u \in L^p(\Omega): D^\alpha u \in L^p(\Omega)$ for $0 \le |\alpha| \le m$, where $D^\alpha u$ is the weak (or distributional) partial derivative of Section 1.57$\}$, and

$W_0^{m,p}(\Omega) \equiv$ the closure of $C_0^\infty(\Omega)$ in the space $W^{m,p}(\Omega)$.

Equipped with the appropriate norm (1) or (2), these are called *Sobolev spaces* over Ω. Clearly $W^{0,p}(\Omega) = L^p(\Omega)$, and if $1 \le p < \infty$, $W_0^{0,p}(\Omega) = L^p(\Omega)$ by Theorem 2.19. For any m the chain of imbeddings

$$W_0^{m,p}(\Omega) \to W^{m,p}(\Omega) \to L^p(\Omega)$$

is also clear. We shall show in Theorem 3.16 that $H^{m,p}(\Omega) = W^{m,p}(\Omega)$ for every domain Ω. This result, published in 1964 by Meyers and Serrin [46] dispelled considerable confusion about the relationship of these spaces that had existed in the literature before that time. It is surprising that this elementary result remained undiscovered for so long.

The spaces $W^{m,p}(\Omega)$ were introduced by Sobolev [62, 63] with many related spaces being studied by other writers, in particular Morrey [47] and Deny and Lions [21]. Many different symbols ($W^{m,p}, H^{m,p}, P^{m,p}, L_m{}^p$, etc.) have been (and are being) used to denote these spaces and their variants, and before they became generally associated with the name of Sobolev they were sometimes referred to under other names, for example, as "Beppo Levi spaces."

Numerous generalizations and extensions of the basic spaces $W^{m,p}(\Omega)$ have been made in recent times, the great bulk of the literature originating in the Soviet Union. In particular we mention extensions that allow arbitrary real values of m (see Chapter VII) and are interpreted as corresponding to fractional orders of differentiation, weighted spaces that introduce weight functions into the L^p-norms, spaces $W^{\mathbf{m},\mathbf{p}}$ that involve different orders of differentiation and different L^p-norms in the various coordinate directions (anisotropic spaces), and Orlicz–Sobolev spaces (Chapter VIII) modeled on the generalizations of L^p-spaces known as "Orlicz spaces."

It will not be possible to investigate the complete spectrum of possible generalizations in this monograph.

3.2 THEOREM $W^{m,p}(\Omega)$ is a Banach space.

PROOF Let $\{u_n\}$ be a Cauchy sequence in $W^{m,p}(\Omega)$. Then $\{D^\alpha u_n\}$ is a Cauchy sequence in $L^p(\Omega)$ for $0 \le |\alpha| \le m$. Since $L^p(\Omega)$ is complete there exist functions u and u_α, $0 < |\alpha| \le m$, in $L^p(\Omega)$ such that $u_n \to u$ and $D^\alpha u_n \to u_\alpha$ in $L^p(\Omega)$ as $n \to \infty$. Now $L^p(\Omega) \subset L^1_{\text{loc}}(\Omega)$ and so u_n determines a distribution $T_{u_n} \in \mathscr{D}'(\Omega)$ as in Section 1.53. For any $\phi \in \mathscr{D}(\Omega)$ we have

$$|T_{u_n}(\phi) - T_u(\phi)| \le \int_\Omega |u_n(x) - u(x)|\,|\phi(x)|\,dx \le \|\phi\|_{p'} \|u_n - u\|_p$$

by Hölder's inequality, where $p' = p/(p-1)$ (or $p' = \infty$ if $p = 1$, $p' = 1$ if $p = \infty$). Hence $T_{u_n}(\phi) \to T_u(\phi)$ for every $\phi \in \mathscr{D}(\Omega)$ as $n \to \infty$. Similarly, $T_{D^\alpha u_n}(\phi) \to T_{u_\alpha}(\phi)$ for every $\phi \in \mathscr{D}(\Omega)$. It follows that

$$T_{u_\alpha}(\phi) = \lim_{n\to\infty} T_{D^\alpha u_n}(\phi) = \lim_{n\to\infty} (-1)^{|\alpha|} T_{u_n}(D^\alpha \phi) = (-1)^{|\alpha|} T_u(D^\alpha \phi)$$

for every $\phi \in \mathscr{D}(\Omega)$. Thus $u_\alpha = D^\alpha u$ in the ditributional sense on Ω for $0 \le |\alpha| \le m$, whence $u \in W^{m,p}(\Omega)$. Since $\lim_{n\to\infty} \|u_n - u\|_{m,p} = 0$, $W^{m,p}(\Omega)$ is complete. ▮

Distributional and classical partial derivatives coincide when the latter exist and are continuous; thus it is clear that the set

$$S = \{\phi \in C^m(\Omega) : \|\phi\|_{m,p} < \infty\}$$

is contained in $W^{m,p}(\Omega)$. Since $W^{m,p}(\Omega)$ is complete, the identity operator in S extends to an isometric isomorphism between $H^{m,p}(\Omega)$, the completion of S, and the closure of S in $W^{m,p}(\Omega)$. It is thus natural to identify $H^{m,p}(\Omega)$ with this closure and so obtain the following corollary.

3.3 COROLLARY $H^{m,p}(\Omega) \subset W^{m,p}(\Omega)$.

3.4 Several important properties of the spaces $W^{m,p}(\Omega)$ are most easily obtained by regarding $W^{m,p}(\Omega)$ as a closed subspace of a Cartesian product of spaces $L^p(\Omega)$. Let $N \equiv N(n,m) = \sum_{0 \le |\alpha| \le m} 1$ be the number of multi-indices α satisfying $0 \le |\alpha| \le m$. For $1 \le p \le \infty$ let $L_N{}^p = \prod_{j=1}^N L^p(\Omega)$, the norm of $u = (u_1, \dots, u_N)$ in $L_N{}^p$ being given by

$$\|u; L_N{}^p\| = \begin{cases} \left(\sum_{j=1}^N \|u_j\|_p^p \right)^{1/p} & \text{if} \quad 1 \le p < \infty \\ \max_{1 \le j \le N} \|u_j\|_\infty & \text{if} \quad p = \infty. \end{cases}$$

By Theorems 1.22, 2.10, 2.17, 2.25, and 2.31, $L_N{}^p$ s a Banach space that is separable if $1 \le p < \infty$ and reflexive and uniformly convex if $1 < p < \infty$.

Let us suppose that the N multi-indices α satisfying $0 \le |\alpha| \le m$ are linearly ordered in some convenient fashion so that to each $u \in W^{m,p}(\Omega)$ we may associate the well-defined vector Pu in $L_N{}^p$ given by

$$Pu = (D^\alpha u)_{0 \le |\alpha| \le m}. \tag{3}$$

Since $\|Pu; L_N{}^p\| = \|u\|_{m,p}$, P is an isometric isomorphism of $W^{m,p}(\Omega)$ onto a subspace $W \subset L_N{}^p$. Since $W^{m,p}(\Omega)$ is complete, W is a closed subspace of $L_N{}^p$. By Theorem 1.21, W is separable if $1 \le p < \infty$ and reflexive and uniformly convex if $1 < p < \infty$. The same conclusions must therefore hold for $W^{m,p}(\Omega) = P^{-1}(W)$.

3.5 THEOREM $W^{m,p}(\Omega)$ is separable if $1 \le p < \infty$, and is reflexive and uniformly convex if $1 < p < \infty$. In particular, therefore, $W^{m,2}(\Omega)$ is a separable Hilbert space with inner product

$$(u, v)_m = \sum_{0 \le |\alpha| \le m} (D^\alpha u, D^\alpha v),$$

where $(u, v) = \int_\Omega u(x)\overline{v(x)}\,dx$ is the inner product in $L^2(\Omega)$.

Duality, the Spaces $W^{-m,p'}(\Omega)$

3.6 In the following sections we shall take, for fixed Ω, m, and p, the number N, the spaces $L_N{}^p$ and W, and the operator P to be as specified in Section 3.4. We also define

$$\langle u, v\rangle = \int_\Omega u(x)v(x)\,dx$$

for any functions u, v for which the right side makes sense. For given p, p' shall always designate the conjugate exponent:

$$p' = \begin{cases} \infty & \text{if } \ p = 1 \\ p/(p-1) & \text{if } \ 1 < p < \infty \\ 1 & \text{if } \ p = \infty. \end{cases}$$

3.7 LEMMA Let $1 \le p < \infty$. To every $L \in (L_N{}^p)'$ there corresponds unique $v \in L_N^{p'}$ such that for every $u \in L_N{}^p$

$$L(u) = \sum_{j=1}^{N} \langle u_j, v_j\rangle.$$

Moreover,

$$\|L;(L_N{}^p)'\| = \|v; L_N^{p'}\|. \tag{4}$$

Thus $(L_N{}^p)' \cong L_N^{p'}$.

PROOF If $1 \le j \le N$ and $w \in L^p(\Omega)$, let $w_{(j)} = (0, \dots, 0, w, 0, \dots, 0)$ be that element of $L_N{}^p$ whose jth component is w, all other components being zero. Setting $L_j(w) = L(w_{(j)})$, we see that $L_j \in (L^p(\Omega))'$ and so by Theorems 2.33 and 2.34 there exists (unique) $v_j \in L^{p'}(\Omega)$ such that for every $w \in L^p(\Omega)$

$$L(w_{(j)}) = L_j(w) = \langle w, v_j\rangle.$$

If $u \in L_N{}^p$, then

$$L(u) = L\left(\sum_{j=1}^{N} u_{j(j)}\right) = \sum_{j=1}^{N} L(u_{j(j)}) = \sum_{j=1}^{N} \langle u_j, v_j\rangle.$$

By Hölder's inequality (for functions and for finite sums), we have

$$|L(u)| \le \sum_{j=1}^{N} \|u_j\|_p \|v_j\|_{p'} \le \|u; L_N{}^p\| \, \|v; L_N^{p'}\|$$

so that $\|L; (L_N{}^p)'\| \le \|v; L_N^{p'}\|$. We show that these norms are in fact equal as follows: if $1 < p < \infty$ and $1 \le j \le N$, let

$$u_j(x) = \begin{cases} |v_j(x)|^{p'-2}\overline{v_j(x)} & \text{if } \; v_j(x) \ne 0 \\ 0 & \text{if } \; v_j(x) = 0. \end{cases}$$

It is easily checked that $|L(u_1, \dots, u_N)| = \|v; L_N^{p'}\|^{p'} = \|u; L_N{}^p\| \, \|v; L_N^{p'}\|$. If $p = 1$, we choose k so that $\|v_k\|_\infty = \max_{1 \le j \le N} \|v_j\|_\infty$. For any $\varepsilon > 0$ there is a measurable subset $A \subset \Omega$ having finite, nonzero volume, such that $|v_k(x)| \ge \|v_k\|_\infty - \varepsilon$ for $x \in A$. Set

$$u(x) = \begin{cases} \overline{v_k(x)}/v_k(x) & \text{if } \; x \in A \; \text{ and } \; v_k(x) \ne 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} L(u_{(k)}) = \langle u, v_k \rangle = \int_A |v_k(x)| \, dx &\ge (\|v_k\|_\infty - \varepsilon) \, \|u\|_1 \\ &= (\|v; L_N{}^\infty\| - \varepsilon) \, \|u_{(k)}; L_N{}^1\|. \end{aligned}$$

Since ε is arbitrary, (4) must follow in this case also. ▮

3.8 THEOREM Let $1 \le p < \infty$. For every $L \in (W^{m,p}(\Omega))'$ there exists an element $v \in L_N^{p'}$ such that, writing the vector v in the form $(v_\alpha)_{0 \le |\alpha| \le m}$, we have for all $u \in W^{m,p}(\Omega)$

$$L(u) = \sum_{0 \le |\alpha| \le m} \langle D^\alpha u, v_\alpha \rangle. \tag{5}$$

Moreover,

$$\|L; (W^{m,p}(\Omega))'\| = \inf \|v; L_N^{p'}\| = \min \|v; L_N^{p'}\|, \tag{6}$$

the infimum being taken over, and *attained on* the set of all $v \in L_N^{p'}$ for which (5) holds for every $u \in W^{m,p}(\Omega)$.

PROOF A linear functional L^* is defined as follows on the range W of the operator P defined by (3):

$$L^*(Pu) = L(u), \qquad u \in W^{m,p}(\Omega).$$

Since P is an isometric isomorphism, $L^* \in W'$ and

$$\|L^*; W'\| = \|L; (W^{m,p}(\Omega))'\|.$$

By the Hahn–Banach theorem there exists a norm preserving extension $\tilde{L}$ of L^* to all of L_N^p, and by Lemma 3.7 there exists $v \in L_N^{p'}$ such that if $u = (u_\alpha)_{0 \le |\alpha| \le m} \in L_N^p$, then

$$\tilde{L}(u) = \sum_{0 \le |\alpha| \le m} \langle u_\alpha, v_\alpha \rangle.$$

Thus for $u \in W^{m,p}(\Omega)$ we obtain

$$L(u) = L^*(Pu) = \tilde{L}(Pu) = \sum_{0 \le |\alpha| \le m} \langle D^\alpha u, v_\alpha \rangle.$$

Moreover,

$$\|L; (W^{m,p}(\Omega))'\| = \|L^*; W'\| = \|\tilde{L}; (L_N^p)'\| = \|v; L_N^{p'}\|. \tag{7}$$

Now any element $v \in L_N^{p'}$ for which (5) holds for every $u \in W^{m,p}(\Omega)$ corresponds to an extension L of L^* and so will have norm $\|v; L_N^{p'}\|$ not less than $\|L; (W^{m,p}(\Omega))'\|$. Combining this with (7), we obtain (6). ∎

We remark that, at least if $1 < p < \infty$, the element $v \in L_N^{p'}$ satisfying (5) and (6) is unique. Since L_N^p and $L_N^{p'}$ are uniformly convex it follows by an argument similar to that of Lemma 2.32 that linear functionals defined on closed subspaces of L_N^p have unique norm preserving extensions to L_N^p.

3.9 For $1 \le p < \infty$ every element L of the space $(W^{m,p}(\Omega))'$ is an extension to $W^{m,p}(\Omega)$ of a distribution $T \in \mathscr{D}'(\Omega)$. To see this suppose L is given by (5) for some $v \in L_N^{p'}$ and define T_{v_α}, $T \in \mathscr{D}'(\Omega)$, by

$$T_{v_\alpha}(\phi) = \langle \phi, v_\alpha \rangle, \qquad \phi \in \mathscr{D}(\Omega), \quad 0 \le |\alpha| \le m,$$

$$T = \sum_{0 \le |\alpha| \le m} (-1)^{|\alpha|} D^\alpha T_{v_\alpha}. \tag{8}$$

For every $\phi \in \mathscr{D}(\Omega) \subset W^{m,p}(\Omega)$ we have

$$T(\phi) = \sum_{0 \le |\alpha| \le m} T_{v_\alpha}(D^\alpha \phi) = L(\phi)$$

so that L is clearly an extension of T. Moreover, we have, following (6),

$$\|L; (W^{m,p}(\Omega))'\| = \min\{\|v; L_N^{p'}\| : L \text{ extends } T \text{ given by (8)}\}.$$

The above remarks also hold for $L \in (W_0^{m,p}(\Omega))'$ since any such functional possesses a norm-preserving extension to $W^{m,p}(\Omega)$.

Now suppose T is any element of $\mathscr{D}'(\Omega)$ having the form (8) for some $v \in L_N^{p'}$, where $1 \le p' \le \infty$. Then T possesses possibly nonunique continuous extensions to $W^{m,p}(\Omega)$. We show, however, that T does possess a unique such extension to $W_0^{m,p}(\Omega)$. If $u \in W_0^{m,p}(\Omega)$, let $\{\phi_n\}$ be a sequence in $C_0^\infty(\Omega) =$

$\mathscr{D}(\Omega)$ such that $\|\phi_n - u\|_{m,p} \to 0$ as $n \to \infty$. Then

$$\begin{aligned} |T(\phi_k) - T(\phi_n)| &\le \sum_{0 \le |\alpha| \le m} |T_{v_\alpha}(D^\alpha \phi_k - D^\alpha \phi_n)| \\ &\le \sum_{0 \le |\alpha| \le m} \|D^\alpha(\phi_k - \phi_n)\|_p \|v_\alpha\|_{p'} \\ &\le \|\phi_k - \phi_n\|_{m,p} \|v; L_N^{p'}\| \to 0 \quad , \quad \text{as} \quad k, n \to \infty. \end{aligned}$$

Therefore $\{T(\phi_n)\}$ is a Cauchy sequence in $\mathbb{C}$ and so converges to a limit that we may denote by $L(u)$ since it is clear that if also $\{\psi_n\} \subset \mathscr{D}(\Omega)$ and $\|\psi_n - u\|_{m,p} \to 0$, then $T(\phi_n) - T(\psi_n) \to 0$ as $n \to \infty$. The functional L thus defined is linear and belongs to $(W_0^{m,p}(\Omega))'$, for if $u = \lim_{n\to\infty} \phi_n$ as above, then

$$|L(u)| = \lim_{n\to\infty} |T(\phi_n)| \le \lim_{n\to\infty} \|\phi_n\|_{m,p} \|v; L_N^{p'}\| = \|u\|_{m,p} \|v; L_N^{p'}\|.$$

We have therefore proved the following theorem.

3.10 THEOREM Let $1 \le p < \infty$. The dual space $(W_0^{m,p}(\Omega))'$ is isometrically isomorphic to the Banach space consisting of those distributions $T \in \mathscr{D}'(\Omega)$ satisfying (8) for some $v \in L_N^{p'}$, normed by

$$\|T\| = \inf\{\|v; L_N^{p'}\| : v \text{ satisfies (8)}\}.$$

In general one cannot expect any such simple characterization of $(W^{m,p}(\Omega))'$ if $W_0^{m,p}(\Omega)$ is a proper subspace of $W^{m,p}(\Omega)$.

3.11 If $m = 1, 2, \ldots$ and $1 \le p < \infty$, let p' denote the exponent conjugate to p and denote by $W^{-m,p'}(\Omega)$ the Banach space of distributions on Ω referred to in the above theorem. (The completeness of this space is a consequence of the isometric isomorphism asserted there.) Evidently $W^{-m,p'}(\Omega)$ is separable and reflexive if $1 < p < \infty$.

3.12 Let $1 < p < \infty$. Each element $v \in L^{p'}(\Omega)$ determines an element L_v of $(W_0^{m,p}(\Omega))'$ by means of $L_v(u) = \langle u, v\rangle$ for

$$|L_v(u)| = |\langle u, v\rangle| \le \|v\|_{p'} \|u\|_p \le \|v\|_{p'} \|u\|_{m,p}.$$

We define the $(-m, p')$-norm of $v \in L^{p'}(\Omega)$ to be the norm of L_v, that is,

$$\|v\|_{-m,p'} = \|L_v; (W_0^{m,p}(\Omega))'\| = \sup_{\substack{u \in W \\ \|u\|_{m,p} \le 1}} |\langle u, v\rangle|,$$

where we have written W to represent $W_0^{m,p}(\Omega)$ on the right side, a practice we continue below, for simplicity. Clearly, for any $u \in W$ and $v \in L^{p'}(\Omega)$ we have

$$\|v\|_{-m,p'} \le \|v\|_{p'}$$

$$|\langle u, v\rangle| = \|u\|_{m,p} |\langle u/\|u\|_{m,p}, v\rangle| \le \|u\|_{m,p} \|v\|_{-m,p'}. \qquad (9)$$

The latter formula is a generalized Hölder's inequality.

Let $V = \{L_v : v \in L^{p'}(\Omega)\}$. Thus V is a vector subspace of $W' = (W_0^{m,p}(\Omega))'$. We show that V is in fact dense in W'. This is easily seen to be equivalent to showing that if $F \in W''$ satisfies $F(L_v) = 0$ for every $L_v \in V$, then $F = 0$ in W''. Since W is reflexive, there exists $f \in W$ corresponding to $F \in W''$ such that $\langle f, v \rangle = L_v(f) = F(L_v) = 0$ for every $v \in L^{p'}(\Omega)$. Since $f \in L^p(\Omega)$, it follows that $f(x) = 0$ a.e. in Ω. Hence $f = 0$ in W and $F = 0$ in W''.

Let $H^{-m,p'}(\Omega)$ denote the completion of $L^{p'}(\Omega)$ with respect to the norm $\|\cdot\|_{-m,p'}$. Then we have

$$H^{-m,p'}(\Omega) \cong (W_0^{m,p}(\Omega))' \cong W^{-m,p'}(\Omega).$$

In particular, corresponding to each $v \in H^{-m,p'}(\Omega)$, there exists $T_v \in W^{-m,p'}(\Omega)$ such that $T_v(\Omega) = \lim_{n\to\infty} \langle \phi, v_n \rangle$ for every $\phi \in \mathscr{D}(\Omega)$ and every sequence $\{v_n\} \subset L^{p'}(\Omega)$ such that $\lim_{n\to\infty} \|v_n - v\|_{-m,p'} = 0$ and conversely any $T \in H^{-m,p'}(\Omega)$ satisfies $T = T_v$ for some such v. Moreover, by (9), $|T_v(\phi)| \leq \|\phi\|_{m,p} \|v\|_{-m,p'}$.

3.13 By an argument similar to that in Section 3.12 the dual space $(W^{m,p}(\Omega))'$ can be characterized for $1 < p < \infty$ as the completion of $L^{p'}(\Omega)$ with respect to the norm

$$\|v\|^{\sim}_{-m,p'} = \sup_{\substack{u \in W^{m,p}(\Omega) \\ \|u\|_{m,p} \leq 1}} |\langle u, v \rangle|.$$

Approximation by Smooth Functions on Ω

We wish to prove that $\{\phi \in C^\infty(\Omega) : \|\phi\|_{m,p} < \infty\}$ is dense in $W^{m,p}(\Omega)$. To this end we require the following standard existence theorem for infinitely differentiable *partitions of unity*.

3.14 THEOREM Let A be an arbitrary subset of $\mathbb{R}^n$ and let $\mathcal{O}$ be a collection of open sets in $\mathbb{R}^n$ which cover A, that is, such that $A \subset \bigcup_{U \in \mathcal{O}} U$. Then there exists a collection Ψ of functions $\psi \in C_0^\infty(\mathbb{R}^n)$ having the following properties:

(i) For every $\psi \in \Psi$ and every $x \in \mathbb{R}^n$, $0 \leq \psi(x) \leq 1$.

(ii) If $K \subset\subset A$, all but possibly finitely many $\psi \in \Psi$ vanish identically on K.

(iii) For every $\psi \in \Psi$ there exists $U \in \mathcal{O}$ such that $\operatorname{supp} \psi \subset U$.

(iv) For every $x \in A$, $\sum_{\psi \in \Psi} \psi(x) = 1$.

Such a collection Ψ is called a *C^∞-partition of unity for A subordinate to $\mathcal{O}$.*

PROOF Since the proof can be found in many texts we give only an outline of it here, leaving the details to the reader. Suppose first that A is compact so that $A \subset \bigcup_{j=1}^{N} U_j$, where $U_1, \ldots, U_N \in \mathcal{O}$. Compact sets $K_1 \subset U_1, \ldots, K_N \subset U_N$ can be constructed so that $A \subset \bigcup_{j=1}^{N} K_j$. For $1 \le j \le N$ there exists a nonnegative-valued function $\phi_j \in C_0^\infty(U_j)$ such that $\phi_j(x) > 0$ for $x \in K_j$. A function ϕ can then be constructed so as to be infinitely differentiable and positive on $\mathbb{R}^n$ and to satisfy $\phi(x) = \sum_{j=1}^{N} \phi_j(x)$ for $x \in A$. Now $\Psi = \{\psi_j : \psi_j(x) = \phi_j(x)/\phi(x),\ 1 \le j \le N\}$ has the desired properties. Now suppose A is open. Then $A = \bigcup_{j=1}^{\infty} A_j$, where

$$A_j = \{x \in A : |x| \le j \text{ and } \operatorname{dist}(x, \operatorname{bdry} A) \ge 1/j\}$$

is compact. For each j the collection

$$\mathcal{O}_j = \{U \cap (\operatorname{interior} A_{j+1} \cap A_{j-2}^c) : U \in \mathcal{O}\}$$

covers A_j and so there exists a finite C^∞-partition of unity Ψ_j for A_j subordinate to $\mathcal{O}_j$. The sum $\sigma(x) = \sum_{j=1}^{\infty} \sum_{\phi \in \Psi_j} \phi(x)$ involves only finitely many nonzero terms at each point, and is positive at each $x \in A$. The collection $\Psi = \{\psi : \psi(x) = \phi(x)/\sigma(x)$ for some ϕ in some Ψ_j if $x \in A$, $\psi(x) = 0$ if $x \notin A\}$ has the prescribed properties. Finally, if A is arbitrary, then $A \subset B = \bigcup_{U \in \mathcal{O}} U$, where B is open. Any partition of unity for B will do for A as well. ∎

3.15 LEMMA Let J_ε be defined as in Section 2.17 and let $1 \le p < \infty$ and $u \in W^{m,p}(\Omega)$. If $\Omega' \subset\subset \Omega$, then $\lim_{\varepsilon \to 0+} J_\varepsilon * u = u$ in $W^{m,p}(\Omega')$.

PROOF Let $\varepsilon < \operatorname{dist}(\Omega', \operatorname{bdry}\Omega)$. For any $\phi \in \mathcal{D}(\Omega')$ we have

$$\begin{aligned}\int_{\Omega'} J_\varepsilon * u(x)\, D^\alpha \phi(x)\, dx &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{u}(x-y) J_\varepsilon(y)\, D^\alpha \phi(x)\, dx\, dy \\ &= (-1)^{|\alpha|} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} D_x^\alpha \tilde{u}(x-y) J_\varepsilon(y)\, \phi(x)\, dx\, dy \\ &= (-1)^{|\alpha|} \int_{\Omega'} J_\varepsilon * D^\alpha u(x)\, \phi(x)\, dx,\end{aligned}$$

where $\tilde{u}$ is the zero extension of u outside Ω. Thus $D^\alpha J_\varepsilon * u = J_\varepsilon * D^\alpha u$ in the distributional sense in Ω'. Since $D^\alpha u \in L^p(\Omega)$ for $0 \le |\alpha| \le m$ we have by Lemma 2.18(c)

$$\lim_{\varepsilon \to 0+} \|D^\alpha J_\varepsilon * u - D^\alpha u\|_{p,\Omega'} = \lim_{\varepsilon \to 0+} \|J_\varepsilon * D^\alpha u - D^\alpha u\|_{p,\Omega'} = 0.$$

Thus $\lim_{\varepsilon \to 0+} \|J_\varepsilon * u - u\|_{m,p,\Omega'} = 0$. ∎

3.16 THEOREM (*Meyers and Serrin* [46]) If $1 \le p < \infty$, then

$$H^{m,p}(\Omega) = W^{m,p}(\Omega).$$

PROOF By virtue of Corollary 3.3 it is sufficient to show that $W^{m,p}(\Omega) \subset H^{m,p}(\Omega)$, that is, that $\{\phi \in C^m(\Omega) : \|\phi\|_{m,p} < \infty\}$ is dense in $W^{m,p}(\Omega)$. If

$u \in W^{m,p}(\Omega)$ and $\varepsilon > 0$, we in fact show that there exists $\phi \in C^\infty(\Omega)$ such that $\|u-\phi\|_{m,p} < \varepsilon$. For $k = 1, 2, \ldots$ let

$$\Omega_k = \{x \in \Omega : |x| < k \text{ and } \operatorname{dist}(x, \operatorname{bdry}\Omega) > 1/k\},$$

and let $\Omega_0 = \Omega_{-1} = \varnothing$, the empty set. Then

$$\mathcal{O} = \{U_k : U_k = \Omega_{k+1} \cap (\overline{\Omega_{k-1}})^c,\ k = 1, 2, \ldots\}$$

is a collection of open subsets of Ω that covers Ω. Let Ψ be a C^∞-partition of unity for Ω subordinate to $\mathcal{O}$. Let ψ_k denote the sum of the finitely many functions $\psi \in \Psi$ whose supports are contained in U_k. Then $\psi_k \in C_0^\infty(U_k)$ and $\sum_{k=1}^\infty \psi_k(x) = 1$ on Ω.

If $0 < \varepsilon < 1/(k+1)(k+2)$, then $J_\varepsilon * (\psi_k u)$ has support in $\Omega_{k+2} \cap (\Omega_{k-2})^c = \mathcal{V}_k \subset\subset \Omega$. Since $\psi_k u \in W^{m,p}(\Omega)$ we may choose ε_k, $0 < \varepsilon_k < 1/(k+1)(k+2)$, such that

$$\|J_{\varepsilon_k} * (\psi_k u) - \psi_k u\|_{m,p,\Omega} = \|J_{\varepsilon_k} * (\psi_k u) - \psi_k u\|_{m,p,\mathcal{V}_k} < \varepsilon/2^k.$$

Let $\phi = \sum_{k=1}^\infty J_{\varepsilon_k} * (\psi_k u)$. On any $\Omega' \subset\subset \Omega$ only finitely many terms in the sum can fail to vanish. Thus $\phi \in C^\infty(\Omega)$. For $x \in \Omega_k$ we have

$$u(x) = \sum_{j=1}^{k+2} \psi_j(x) u(x), \qquad \phi(x) = \sum_{j=1}^{k+2} J_{\varepsilon_j} * (\psi_j u)(x).$$

Thus

$$\|u - \phi\|_{m,p,\Omega_k} \le \sum_{j=1}^{k+2} \|J_{\varepsilon_j} * (\psi_j u) - \psi_j u\|_{m,p,\Omega} < \varepsilon.$$

By the monotone convergence theorem 1.43, $\|u-\phi\|_{m,p,\Omega} < \varepsilon$. ∎

We remark that the theorem does not extend to the case $p = \infty$. For instance, if $\Omega = \{x \in \mathbb{R} : -1 < x < 1\}$ and $u(x) = |x|$, then $u \in W^{1,\infty}(\Omega)$ but $u \notin H^{1,\infty}(\Omega)$; in fact, if $\varepsilon < \frac{1}{2}$, there exists no function $\phi \in C^1(\Omega)$ such that $\|\phi' - u'\|_\infty < \varepsilon$.

Approximation by Smooth Functions on $\mathbb{R}^n$

3.17 Having shown that an element of $W^{m,p}(\Omega)$ can always be approximated by functions smooth on Ω we now ask whether or not the approximation can in fact be done with bounded functions having bounded derivatives of all orders, or, say, of all orders up to m. That is, we are asking whether for any values of $k \ge m$ the space $C^k(\overline{\Omega})$ is dense in $W^{m,p}(\Omega)$. The answer may be negative as the following example shows:

Let $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < |x| < 1,\ 0 < y < 1\}$. Then the function u specified by

$$u(x,y) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

evidently belongs to $W^{1,p}(\Omega)$. The reader may verify, however, that for sufficiently small $\varepsilon > 0$ no function $\phi \in C^1(\bar{\Omega})$ can satisfy $\|u-\phi\|_{1,p} < \varepsilon$. The difficulty with this domain is that it lies on both sides of part of its boundary (the segment $x = 0$, $0 < y < 1$).

We shall say that a domain Ω has the *segment property* if for every $x \in \operatorname{bdry}\Omega$ there exists an open set U_x and a nonzero vector y_x such that $x \in U_x$ and if $z \in \bar{\Omega} \cap U_x$, then $z + ty_x \in \Omega$ for $0 < t < 1$. A domain having this property must have $(n-1)$-dimensional boundary and cannot simultaneously lie on both sides of any given part of its boundary.

The following theorem shows that this property is sufficient to guarantee that $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{m,p}(\Omega)$, and hence in particular that $C^k(\bar{\Omega})$ is dense in $W^{m,p}(\Omega)$ for any m.

3.18 THEOREM If Ω has the segment property, then the set of restrictions to Ω of functions in $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{m,p}(\Omega)$ for $1 \le p < \infty$.

PROOF Let f be a fixed function in $C_0^\infty(\mathbb{R}^n)$ satisfying

(i) $f(x) = 1$ if $|x| \le 1$,
(ii) $f(x) = 0$ if $|x| \ge 2$,
(iii) $|D^\alpha f(x)| \le M$ (constant) for all x and $0 \le |\alpha| \le m$.

Let $f_\varepsilon(x) = f(\varepsilon x)$ for $\varepsilon > 0$. Then $f_\varepsilon(x) = 1$ if $|x| \le 1/\varepsilon$ and $|D^\alpha f_\varepsilon(x)| \le M\varepsilon^{|\alpha|} \le M$ if $\varepsilon \le 1$. If $u \in W^{m,p}(\Omega)$, then $u_\varepsilon = f_\varepsilon \cdot u$ belongs to $W^{m,p}(\Omega)$ and has bounded support. Since, for $0 < \varepsilon \le 1$ and $|\alpha| \le m$,

$$|D^\alpha u_\varepsilon(x)| = \left| \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta u(x)\, D^{\alpha-\beta} f_\varepsilon(x) \right| \le M \sum_{\beta \le \alpha} \binom{\alpha}{\beta} |D^\beta u(x)|,$$

we have, setting $\Omega_\varepsilon = \{x \in \Omega : |x| > 1/\varepsilon\}$,

$$\begin{aligned} \|u - u_\varepsilon\|_{m,p,\Omega} &= \|u - u_\varepsilon\|_{m,p,\Omega_\varepsilon} \\ &\le \|u\|_{m,p,\Omega_\varepsilon} + \|u_\varepsilon\|_{m,p,\Omega_\varepsilon} \le \text{const}\, \|u\|_{m,p,\Omega_\varepsilon}. \end{aligned}$$

The right side tends to zero as ε tends to 0. Thus any $u \in W^{m,p}(\Omega)$ can be approximated in that space by functions with bounded supports.

We may now, therefore, assume $K = \{x \in \Omega : u(x) \ne 0\}$ is bounded. The set $F = \bar{K} \sim (\bigcup_{x \in \operatorname{bdry}\Omega} U_x)$ is thus compact and contained in Ω, $\{U_x\}$ being the collection of open sets referred to in the definition of the segment property.

There exists an open set U_0 such that $F \subset\subset U_0 \subset\subset \Omega$. Since $\bar{K}$ is compact, there exist finitely many of the sets U_x; let us rename them $U_1, \ldots, U_k$, such that $\bar{K} \subset U_0 \cup U_1 \cup \cdots \cup U_k$. Moreover, we may find other open sets $\tilde{U}_0, \tilde{U}_1, \ldots, \tilde{U}_k$ such that $\tilde{U}_j \subset\subset U_j$, $0 \le j \le k$, but still $\bar{K} \subset \tilde{U}_0 \cup \tilde{U}_1 \cup \cdots \cup \tilde{U}_k$.

Let Ψ be a C^∞-partition of unity subordinate to $\{\tilde{U}_j : 0 \le j \le k\}$, and let ψ_j be the sum of the finitely many functions $\psi \in \Psi$ whose supports lie in $\tilde{U}_j$. Let $u_j = \psi_j u$. Suppose that for each j we can find $\phi_j \in C_0^\infty(\mathbb{R}^n)$ such that

$$\|u_j - \phi_j\|_{m,p,\Omega} < \varepsilon/(k+1). \tag{10}$$

Then putting $\phi = \sum_{j=0}^k \phi_j$, we would obtain

$$\|\phi - u\|_{m,p,\Omega} \le \sum_{j=0}^{k} \|\phi_j - u_j\|_{m,p,\Omega} < \varepsilon.$$

A function $\phi_0 \in C_0^\infty(\mathbb{R}^n)$ satisfying (10) for $j = 0$ can be found via Lemma 3.15 since $\operatorname{supp} u_0 \subset \tilde{U}_0 \subset\subset \Omega$. It remains, therefore, to find ϕ_j satisfying (10) for $1 \le j \le k$. For fixed such j we extend u_j to be identically zero outside Ω. Thus $u_j \in W^{m,p}(\mathbb{R}^n \sim \Gamma)$, where $\Gamma = \bar{\tilde{U}}_j \cap \operatorname{bdry}\Omega$. Let y be the nonzero vector associated with the set U_j in the definition of the segment property (Fig. 1). Let $\Gamma_t = \Gamma - ty$, where t is so chosen that

$$0 < t < \min(1, \operatorname{dist}(\tilde{U}_j, \mathbb{R}^n \sim U_j)/|y|).$$

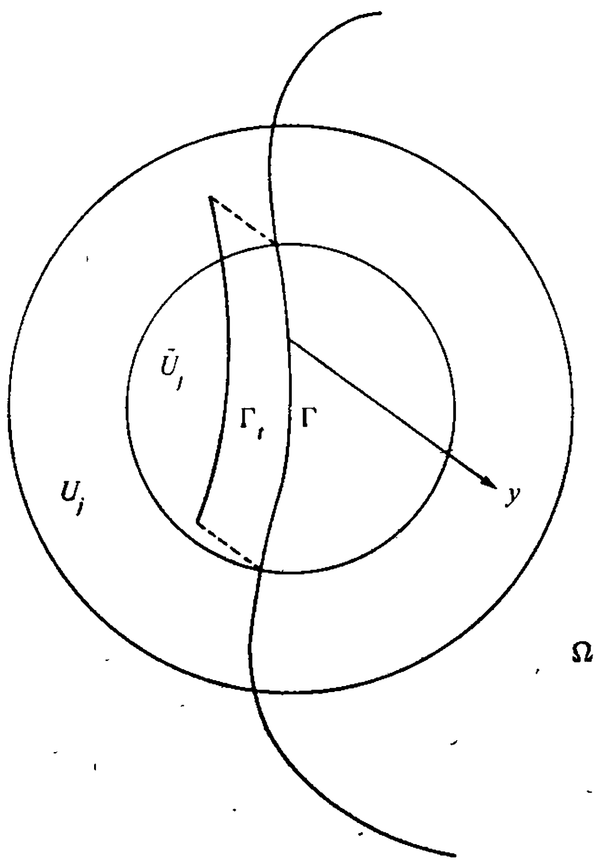


FIG. 1

Then $\Gamma_t \subset U_j$ and $\Gamma_t \cap \bar{\Omega}$ is empty by the segment property. Let $U_{j,t}(x) = u_j(x+ty)$. Then $u_{j,t} \in W^{m,p}(\mathbb{R}^n \sim \Gamma_t)$. Translation is continuous in $L^p(\Omega)$ so $D^\alpha u_{j,t} \to D^\alpha u_j$ in $L^p(\Omega)$ as $t \to 0+$, $|\alpha| \le m$. Thus $u_{j,t} \to u_j$ in $W^{m,p}(\Omega)$ as $t \to 0+$ and so it is sufficient to find $\phi_j \in C_0^\infty(\mathbb{R}^n)$ such that $\|u_{j,t} - \phi_j\|_{m,p}$ is sufficiently small. However, $\Omega \cap U_j \subset\subset \mathbb{R}^n \sim \Gamma_t$ and so by Lemma 3.15 we may take $\phi_j = J_\delta * u_{j,t}$ for suitably small $\delta > 0$. This completes the proof. ∎

3.19 COROLLARY $W_0^{m,p}(\mathbb{R}^n) = W^{m,p}(\mathbb{R}^n)$.

Approximation by Functions in $C_0^\infty(\Omega)$; (m,p')-Polar Sets

Corollary 3.19 suggests the question: For what domains Ω is it true that $W^{m,p}(\Omega) = W_0^{m,p}(\Omega)$, that is, when is $C_0^\infty(\Omega)$ dense in $W^{m,p}(\Omega)$? A partial answer to this problem can be formulated in terms of the nature of the distributions belonging to $W^{-m,p'}(\mathbb{R}^n)$. The approach given below is due to Lions [39].

3.20 Throughout the following discussion we assume that $1 < p < \infty$ and p' is conjugate to p. Let F be a closed subset of $\mathbb{R}^n$. A distribution $T \in \mathscr{D}'(\mathbb{R}^n)$ has support in F ($\operatorname{supp} T \subset F$) provided that $T(\phi) = 0$ for every $\phi \in \mathscr{D}(\mathbb{R}^n)$ vanishing identically on F. We say that the closed set F is *(m,p')-polar* if the only distribution T in $W^{-m,p'}(\mathbb{R}^n)$ having support in F is the zero distribution $T = 0$.

If F has positive measure, it cannot be (m,p')-polar for the characteristic function of any compact subset of F having positive measure belongs to $L^{p'}(\mathbb{R}^n)$ and hence to $W^{-m,p'}(\mathbb{R}^n)$.

We shall show later that if $mp > n$, then $W^{m,p}(\mathbb{R}^n) \to C(\mathbb{R}^n)$ (see Theorem 5.4) in the sense that if $u \in W^{m,p}(\mathbb{R}^n)$, then there exists $u_0 \in C(\mathbb{R}^n)$ such that $u(x) = u_0(x)$ a.e. and

$$|u_0(x)| \le \text{const}\|u\|_{m,p},$$

the constant being independent of x and u. It follows that the Dirac distribution δ_x given by $\delta_x(\phi) = \phi(x)$ belongs to $(W^{m,p}(\mathbb{R}^n))' = (W_0^{m,p}(\mathbb{R}^n))' \cong W^{-m,p'}(\mathbb{R}^n)$. Hence, if $mp > n$, a set F cannot be (m,p')-polar unless it is empty.

3.21 Since $W^{m+1,p}(\mathbb{R}^n) \to W^{m,p}(\mathbb{R}^n)$ any bounded linear functional on the latter space is bounded on the former as well, that is, $W^{-m,p'}(\Omega) \subset W^{-m-1,p'}(\Omega)$. Hence any $(m+1,p')$-polar set is also (m,p')-polar. The converse is, of course, generally not true.

Let the mapping $u \to \tilde{u}$ denote zero extension of u outside a domain

$\Omega \subset \mathbb{R}^n$:

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \Omega^c. \end{cases} \tag{11}$$

The following lemma shows that this mapping takes $W_0^{m,p}(\Omega)$ (isometrically) into $W^{m,p}(\mathbb{R}^n)$.

3.22 LEMMA Let $u \in W_0^{m,p}(\Omega)$. If $|\alpha| \le m$, then $D^\alpha \tilde{u} = (D^\alpha u)^{\sim}$ in the distributional sense in $\mathbb{R}^n$. Hence $\tilde{u} \in W^{m,p}(\mathbb{R}^n)$.

PROOF Let $\{\phi_n\}$ be a sequence in $C_0^\infty(\Omega)$ converging to u in the space $W_0^{m,p}(\Omega)$. If $\psi \in \mathscr{D}(\mathbb{R}^n)$, we have for $|\alpha| \le m$

$$\begin{aligned}(-1)^{|\alpha|}\int_{\mathbb{R}^n} \tilde{u}(x)\, D^\alpha\psi(x)\, dx &= (-1)^{|\alpha|}\int_\Omega u(x)\, D^\alpha\psi(x)\, dx \\ &= \lim_{n\to\infty} (-1)^{|\alpha|}\int_\Omega \phi_n(x)\, D^\alpha\psi(x)\, dx \\ &= \lim_{n\to\infty} \int_\Omega D^\alpha\phi_n(x)\,\psi(x)\, dx \\ &= \int_{\mathbb{R}^n} (D^\alpha u)^{\sim}(x)\,\psi(x)\, dx.\end{aligned}$$

Thus $D^\alpha \tilde{u} = (D^\alpha u)^{\sim}$ in the distributional sense on $\mathbb{R}^n$ and hence these locally integrable functions are equal a.e. in $\mathbb{R}^n$. It follows that $\|\tilde{u}\|_{m,p,\mathbb{R}^n} = \|u\|_{m,p,\Omega}$. ∎

We now give a necessary and sufficient condition on Ω that mapping (11) carry $W_0^{m,p}(\Omega)$ isometrically *onto* $W^{m,p}(\mathbb{R}^n)$.

3.23 THEOREM $C_0^\infty(\Omega)$ is dense in $W^{m,p}(\mathbb{R}^n)$ if and only if the complement Ω^c of Ω is (m,p')-polar.

PROOF First suppose $C_0^\infty(\Omega)$ is dense in $W^{m,p}(\mathbb{R}^n)$. Let $T \in W^{-m,p'}(\mathbb{R}^n)$ have support contained in Ω^c. If $u \in W^{m,p}(\mathbb{R}^n)$, then there exists a sequence $\{\phi_n\} \subset C_0^\infty(\Omega)$ converging in $W^{m,p}(\mathbb{R}^n)$ to u. Hence $T(u) = \lim_{n\to\infty} T(\phi_n) = 0$ and so $T = 0$ and Ω^c is (m,p')-polar.

Conversely, if $C_0^\infty(\Omega)$ is not dense in $W^{m,p}(\mathbb{R}^n)$, then there exists $u \in W^{m,p}(\mathbb{R}^n)$ such that $\|u-\phi\|_{m,p,\mathbb{R}^n} \ge k > 0$ for every $\phi \in C_0^\infty(\Omega)$, k being independent of ϕ. By the Hahn–Banach extension theorem there exists $T \in W^{-m,p'}(\mathbb{R}^n)$ such that $T(\phi) = 0$ for all $\phi \in C_0^\infty(\Omega)$ but $T(u) \ne 0$. Since $\operatorname{supp} T \subset \Omega^c$ but $T \ne 0$, Ω^c cannot be (m,p')-polar. ∎

As a final preparation for our investigation of the possible identity of $W_0^{m,p}(\Omega)$ and $W^{m,p}(\Omega)$ we establish a distributional analog of the fact, obvious for differentiable functions, that identical vanishing of first derivatives over a rectangle implies constancy on that rectangle. We extend this first to distributions and then to locally integrable functions.

3.24 LEMMA Let $B = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$ be an open rectangular box in $\mathbb{R}^n$. Let $\phi \in \mathscr{D}(B)$. If $\int_B \phi(x)\,dx = 0$, then $\phi(x) = \sum_{j=1}^n \phi_j(x)$, where $\phi_j \in \mathscr{D}(B)$ and

$$\int_{a_j}^{b_j} \phi_j(x_1, \ldots, x_j, \ldots, x_n)\,dx_j = 0 \tag{12}$$

for every fixed $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \in \mathbb{R}^{n-1}$.

PROOF For $1 \le j \le n$ select functions $u_j \in C_0^\infty(a_j, b_j)$ such that $\int_{a_j}^{b_j} u_j(t)\,dt = 1$. Let

$$B_j = (a_j, b_j) \times (a_{j+1}, b_{j+1}) \times \cdots \times (a_n, b_n),$$

$$\psi_j(x_j, \ldots, x_n) = \int_{a_1}^{b_1} dt_1 \int_{a_2}^{b_2} dt_2 \cdots \int_{a_{j-1}}^{b_{j-1}} \phi(t_1, \ldots, t_{j-1}, x_j, \ldots, x_n)\,dt_{j-1},$$

$$\omega_j(x) = u_1(x_1) \cdots u_{j-1}(x_{j-1})\,\psi_j(x_j, \ldots, x_n).$$

Then $\psi_j \in \mathscr{D}(B_j)$ and $\omega_j \in \mathscr{D}(B)$. Moreover,

$$\int_{B_j} \psi_j(x_j, \ldots, x_n)\,dx_j \cdots dx_n = \int_B \phi(x)\,dx = 0.$$

Let $\phi_1 = \phi - \omega_2$, $\phi_j = \omega_j - \omega_{j+1}$ $(2 \le j \le n-1)$, $\phi_n = \omega_n$. Clearly, $\phi_j \in \mathscr{D}(B)$ for $1 \le j \le n$, and $\phi = \sum_{j=1}^n \phi_j$. Finally,

$$\begin{aligned}
&\int_{a_1}^{b_1} \phi_1(x_1, \ldots, x_n)\,dx_1 \\
&\qquad = \int_{a_1}^{b_1} \phi(x_1, \ldots, x_n)\,dx_1 - \psi_2(x_2, \ldots, x_n) \int_{a_1}^{b_1} u_1(x_1)\,dx_1 = 0,
\end{aligned}$$

$$\begin{aligned}
&\int_{a_j}^{b_j} \phi_j(x_1, \ldots, x_n)\,dx_j \\
&\qquad = u_1(x_1) \cdots u_{j-1}(x_{j-1}) \\
&\qquad\quad \times \left(\int_{a_j}^{b_j} \psi_j(x_j, \ldots, x_n)\,dx_j - \psi_{j+1}(x_{j+1}, \ldots, x_n) \int_{a_j}^{b_j} u_j(x_j)\,dx_j \right) \\
&\qquad = 0, \qquad 2 \le j \le n-1,
\end{aligned}$$

$$\int_{a_n}^{b_n} \phi_n(x_1,\dots,x_n)\,dx_n = u_1(x_1)\cdots u_{n-1}(x_{n-1}) \int_{a_n}^{b_n} \psi_n(x_n)\,dx_n$$

$$= u_1(x_1)\cdots u_{n-1}(x_{n-1}) \int_B \phi(x)\,dx = 0. \quad \blacksquare$$

3.25 COROLLARY If $T \in \mathscr{D}'(B)$ and $D_j T = 0$ for $1 \le j \le n$, then there exists a constant k such that for all $\phi \in \mathscr{D}(B)$

$$T(\phi) = k \int_B \phi(x)\,dx.$$

PROOF First note that if $\int_B \phi(x)\,dx = 0$, then $T(\phi) = 0$, for, by the above lemma we may write $\phi = \sum_{j=1}^n \phi_j$, where $\phi_j \in \mathscr{D}(B)$ satisfies (12), and hence $\phi_j = D_j \theta_j$, where $\theta_j \in \mathscr{D}(B)$ is defined by

$$\theta_j(x) = \int_{a_j}^{x_j} \phi_j(x_1,\dots,x_{j-1},t,x_{j+1},\dots,x_n)\,dt.$$

Thus $T(\phi) = \sum_{j=1}^n T(D_j\theta_j) = -\sum_{j=1}^n D_j T(\theta_j) = 0$.

Now suppose $T \ne 0$. Then there exists $\phi_0 \in \mathscr{D}(B)$ such that $T(\phi_0) = k_1 \ne 0$. Hence $\int_B \phi_0(x)\,dx = k_2 \ne 0$ and $T(\phi_0) = k\int_B \phi_0(x)\,dx$, where $k = k_1/k_2$. If $\phi \in \mathscr{D}(B)$ is arbitrary, let $K(\phi) = \int_B \phi(x)\,dx$. Then

$$\int_B \left(\phi(x) - \frac{K(\phi)}{k_2}\phi_0(x)\right) dx = 0$$

and so $T(\phi - [K(\phi)/k_2]\phi_0) = 0$. It follows that

$$T(\phi) = \frac{K(\phi)}{k_2} T(\phi_0) = kK(\phi) = k\int_B \phi(x)\,dx. \quad \blacksquare$$

It should be remarked that this corollary can be extended to any open, connected set B in $\mathbb{R}^n$ via a partition of unity for Ω subordinate to some open cover of Ω by open rectangular boxes that are contained in Ω. We shall not, however, require this extension.

The following lemma shows that different locally integrable functions on an open set Ω determine different distributions on Ω.

3.26 LEMMA Let $u \in L^1_{\text{loc}}(\Omega)$ satisfy $\int_\Omega u(x)\phi(x)\,dx = 0$ for every $\phi \in \mathscr{D}(\Omega)$. Then $u(x) = 0$ a.e. in Ω.

PROOF If $\psi \in C_0(\Omega)$, then for sufficiently small positive ε, the mollifier $J_\varepsilon * \psi$ belongs to $\mathscr{D}(\Omega)$. By Lemma 2.18, $J_\varepsilon * \psi \to \psi$ uniformly on Ω as $\varepsilon \to 0+$. Hence $\int_\Omega u(x)\psi(x)\,dx = 0$ for every $\psi \in C_0(\Omega)$.

Let $K \subset\subset \Omega$ and let $\varepsilon > 0$. Let χ_K be the characteristic function of K. Then $\int_K |u(x)|\,dx < \infty$. There exists $\delta > 0$ such that for any measurable set $A \subset K$ with $\mu(A) < \delta$ we have $\int_A |u(x)|\,dx < \varepsilon/2$ (see, e.g., the book by Munroe [48, p. 136]).

By Lusin's theorem 1.37(f) there exists $\psi \in C_0(\Omega)$ with $\operatorname{supp}\psi \subset K$ and $|\psi(x)| \le 1$ for all x, such that

$$\mu(\{x \in \Omega : \psi(x) \ne \chi_K(x) \operatorname{sgn} \overline{u(x)}\}) < \delta.$$

Here

$$\operatorname{sgn} v(x) = \begin{cases} v(x)/|v(x)| & \text{if} \quad v(x) \ne 0 \\ 0 & \text{if} \quad v(x) = 0. \end{cases}$$

Hence

$$\begin{aligned} \int_K |u(x)|\,dx &= \int_\Omega u(x)\chi_K(x) \operatorname{sgn}\overline{u(x)}\,dx \\ &\quad \int_\Omega u(x)\psi(x)\,dx + \int_\Omega u(x)[\chi_K(x)\operatorname{sgn}\overline{u(x)} - \psi(x)]\,dx \\ &\le 2\int_{\{x \in \Omega : \psi(x) \ne \chi_K(x) \operatorname{sgn}\overline{u(x)}\}} |u(x)|\,dx < \varepsilon. \end{aligned}$$

Since ε is arbitrary, $u(x) = 0$ a.e. in K and hence a.e. in Ω. ∎

3.27 COROLLARY If B is a rectangular box as in Lemma 3.24 and $u \in L^1_{\mathrm{loc}}(B)$ possesses weak derivatives $D_j u = 0$ for $1 \le j \le n$, then for some constant k, $u(x) = k$ a.e. in B.

PROOF By Corollary 3.25 we have, since $D_j T_u = 0$, $1 \le j \le n$,

$$\int_B u(x)\phi(x)\,dx = T_u(\phi) = k\int_B \phi(x)\,dx.$$

Hence $u(x) - k = 0$ a.e. in B. ∎

3.28 THEOREM (1) If $W^{m,p}(\Omega) = W_0^{m,p}(\Omega)$, then Ω^c is (m, p')-polar.
(2) If Ω^c is both $(1,p)$-polar and (m,p')-polar, then $W^{m,p}(\Omega) = W_0^{m,p}(\Omega)$.

PROOF (1) Assume $W^{m,p}(\Omega) = W_0^{m,p}(\Omega)$. We deduce first that Ω^c must have measure zero. If not, there would exist some open rectangle $B \subset \mathbb{R}^n$ which intersects both Ω and Ω^c in sets of positive measure. Let u be the restriction to Ω of a function in $C_0^\infty(\mathbb{R}^n)$ which is identically one on $B \cap \Omega$. Then $u \in W^{m,p}(\Omega)$ and so $u \in W_0^{m,p}(\Omega)$. By Lemma 3.22, $\tilde{u} \in W^{m,p}(\mathbb{R}^n)$ and $D_j\tilde{u} = (D_j u)^\sim$ in the distributional sense in $\mathbb{R}^n$, for $1 \le j \le n$. Now $(D_j u)^\sim$ vanishes identically on B whence so does $D_j\tilde{u}$ as a distribution on B. By Corollary 3.27, $\tilde{u}$ must have a constant value a.e. in B. Since $\tilde{u}(x) = 1$ on

$B \cap \Omega$ and $\tilde{u}(x) = 0$ on $B \cap \Omega^c$, we have a contradiction. Thus Ω^c has measure zero.

Now if $v \in W^{m,p}(\mathbb{R}^n)$ and u is the restriction of v to Ω, then u belongs to $W^{m,p}(\Omega)$ and hence, by assumption, to $W_0^{m,p}(\Omega)$. By Lemma 3.22, $\tilde{u} \in W^{m,p}(\mathbb{R}^n)$ and can be approximated by elements of $C_0^\infty(\Omega)$. But $v(x) = \tilde{u}(x)$ on Ω, that is, a.e. in $\mathbb{R}^n$. Hence v and $\tilde{u}$ have the same distributional derivatives and so coincide in $W^{m,p}(\mathbb{R}^n)$. Therefore $C_0^\infty(\Omega)$ is dense in $W^{m,p}(\mathbb{R}^n)$ and Ω^c is (m,p')-polar by Theorem 3.23.

(2) Now assume Ω^c is $(1,p)$-polar and (m,p')-polar. Let $u \in W^{m,p}(\Omega)$. We show that $u \in W_0^{m,p}(\Omega)$. Since $\tilde{u} \in L^p(\mathbb{R}^n)$, the distribution $T_{D_j\tilde{u}}$, corresponding to $D_j\tilde{u}$, belongs to $W^{-1,p}(\mathbb{R}^n)$. Since $(D_j u)^\sim \in L^p(\mathbb{R}^n) \subset H^{-1,p}(\mathbb{R}^n)$, therefore $T_{(D_j u)^\sim} \in W^{-1,p}(\mathbb{R}^n)$. Hence $T_{D_j\tilde{u}-(D_j u)^\sim} \in W^{-1,p}(\mathbb{R}^n)$. But $D_j\tilde{u}-(D_j u)^\sim$ vanishes on Ω so $\operatorname{supp} T_{D_j\tilde{u}-(D_j u)^\sim} \subset \Omega^c$. Since Ω_c is $(1,p)$-polar $D_j\tilde{u} = (D_j u)^\sim$ in the distributional sense on $\mathbb{R}^n$. By induction on $|\alpha|$ we can show similarly that $D^\alpha\tilde{u} = (D^\alpha u)^\sim$ in the distributional sense, for $|\alpha| \le m$. Therefore $\tilde{u} \in W^{m,p}(\mathbb{R}^n)$ whence, by Theorem 3.23, u, the restriction of $\tilde{u}$ to Ω, belongs to $W_0^{m,p}(\Omega)$, Ω^c being (m,p')-polar. ∎

If (m,p')-polarity implies $(1,p)$-polarity, then Theorem 3.28 amounts to the assertion that (m,p')-polarity of Ω^c is necessary and sufficient for the equality of $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$. We now examine this possibility, establishing first two lemmas containing important properties of polarity. The first of these shows that (m,p')-polarity is a local property.

3.29 LEMMA $F \subset \mathbb{R}^n$ is (m,p')-polar if and only if $F \cap K$ is (m,p')-polar for every compact set $K \subset \mathbb{R}^n$.

PROOF Clearly the (m,p')-polarity of F implies that of $F \cap K$ for every compact K. We prove the converse. Let $T \in W^{-m,p'}(\mathbb{R}^n)$ be given by (8) and have support in F. We must show that $T = 0$. Let $f \in C_0^\infty(\mathbb{R}^n)$ satisfy $f(x) = 1$ if $|x| \le 1$ and $f(x) = 0$ if $|x| \ge 2$. For $\varepsilon > 0$ let $f_\varepsilon(x) = f(\varepsilon x)$ so that $D^\alpha f_\varepsilon(x) = \varepsilon^{|\alpha|} D^\alpha f(\varepsilon x) \to 0$ uniformly in x as $\varepsilon \to 0+$. Then $f_\varepsilon T \in W^{-m,p'}(\mathbb{R}^n)$ and for any $\phi \in \mathscr{D}(\mathbb{R}^n)$ we have

$$\begin{aligned}
|T(\phi) - f_\varepsilon T(\phi)| &= |T(\phi) - T(f_\varepsilon \phi)| \\
&= \left| \sum_{0 \le |\alpha| \le m} \int_{\mathbb{R}^n} v_\alpha(x)\, D^\alpha[\phi(x)(1 - f_\varepsilon(x))]\, dx \right| \\
&= \left| \sum_{0 \le |\alpha| \le m} \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \int_{\mathbb{R}^n} v_\alpha(x)\, D^\beta \phi(x)\, D^{\alpha-\beta}(1 - f_\varepsilon(x))\, dx \right| \\
&\le \sum_{0 \le |\beta| \le m} \int_{\mathbb{R}^n} |w_\beta(x)\, D^\beta \phi(x)|\, dx \le \|\phi\|_{m,p} \|w; L_N^{p'}\|
\end{aligned}$$

where

$$w_\beta(x) = \sum_{\substack{|\alpha|\le m\\ \alpha\ge\beta}} \binom{\alpha}{\beta} v_\alpha(x)\, D^{\alpha-\beta}(1-f_\varepsilon(x))$$

$$= v_\beta(x)(1-f_\varepsilon(x)) - \sum_{\substack{|\alpha|\le m\\ \alpha\ge\beta,\,\alpha\ne\beta}} \binom{\alpha}{\beta} v_\alpha(x)\, D^{\alpha-\beta} f_\varepsilon(x):$$

Since $f_\varepsilon(x) = 1$ for $|x| \le 1/\varepsilon$, we have $\lim_{\varepsilon\to 0+}\|w_\beta\|_p = 0$. Thus $f_\varepsilon T \to T$ in $W^{-m,p'}(\mathbb{R}^n)$ as $\varepsilon \to 0+$. Since $f_\varepsilon T$ has compact support in K, it vanishes by assumption. Thus $T = 0$. ∎

3.30 LEMMA If $p' \le q'$ and $F \subset \mathbb{R}^n$ is (m,p')-polar, then F is also (m,q')-polar.

PROOF Let $K \subset \mathbb{R}^n$ be compact. By Lemma 3.29 it is sufficient to show that $F \cap K$ is (m,q')-polar. Let G be an open, bounded set in $\mathbb{R}^n$ containing K. By Lemma 2.8, $W_0^{m,p}(G) \to W_0^{m,q}(G)$ so that $W^{-m,q'}(G) \subset W^{-m,p'}(G)$. Any distribution $T \in W^{-m,q'}(\mathbb{R}^n)$ having support in $K \cap F$ also belongs to $W^{-m,q'}(G)$ and hence to $W^{-m,p'}(G)$. Since $K \cap F$ is (m,p')-polar, $T = 0$. Thus $K \cap F$ is (m,q')-polar. ∎

3.31 THEOREM Let $p \ge 2$. Then $W^{m,p}(\Omega) = W_0^{m,p}(\Omega)$ if and only if Ω^c is (m,p')-polar.

PROOF Since $p' \le p$, Ω is (m,p)-polar, and hence $(1,p)$-polar, if it is (m,p')-polar. The result now follows by Theorem 3.28. ∎

3.32 The Sobolev imbedding theorem (Theorem 5.4) can be used to extend Theorem 3.31 to cover certain values of $p < 2$. If $(m-1)p < n$, the imbedding theorem gives

$$W^{m,p}(\mathbb{R}^n) \to W^{1,q}(\mathbb{R}^n), \qquad q = np/[n-(m-1)p],$$

which in turn implies that $W^{-1,q'}(\mathbb{R}^n) \subset W^{-1,p'}(\mathbb{R}^n)$. If also $p \le 2n/(n+m-1)$, then $q' \le p$, and so by Lemma 3.30, Ω^c is $(1,p)$-polar if it is (m,p')-polar. Note that $2n/(n+m-1) < 2$ provided $m > 1$. If, on the other hand, $(m-1)p \ge n$, then $mp > n$ and, as pointed out in Section 3.20, Ω^c cannot be (m,p')-polar unless it is empty, in which case it is $(1,p)$-polar trivially.

Thus, the only values of p for which the (m,p')-polarity of Ω^c is not known to imply $(1,p)$-polarity and hence be equivalent to the identity of $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ are given by $1 \le p \le \min(n/(m-1), 2n/(n+m-1))$.

3.33 Whenever $W_0^{m,p}(\Omega) \neq W^{m,p}(\Omega)$, the former space is a closed subspace of the latter. In the Hilbert space case, $p = 2$, we may consider the space $W_0^{\perp}$ consisting of all $v \in W^{m,p}(\Omega)$ such that $(v, \phi)_m = 0$ for every $\phi \in C_0^{\infty}(\Omega)$. Every $u \in W^{m,p}(\Omega)$ can be uniquely decomposed in the form $u = u_0 + v$, where $u_0 \in W_0^{m,p}(\Omega)$ and $v \in W_0^{\perp}$. Integration by parts shows that any $v \in W_0^{\perp}$ must satisfy

$$\sum_{0 \le |\alpha| \le m} (-1)^{|\alpha|} D^{2\alpha} v(x) = 0$$

in the weak sense, and hence a.e. in Ω.

Transformation of Coordinates

3.34 Let Φ be a one-to-one transformation of a domain $\Omega \subset \mathbb{R}^n$ onto a domain $G \subset \mathbb{R}^n$, having inverse $\Psi = \Phi^{-1}$. We call Φ m-smooth if, writing $y = \Phi(x)$ and

$$\begin{aligned}
y_1 &= \phi_1(x_1, \dots, x_n), & x_1 &= \psi_1(y_1, \dots, y_n) \\
y_2 &= \phi_2(x_1, \dots, x_n), & x_2 &= \psi_2(y_1, \dots, y_n) \\
&\vdots & &\vdots \\
y_n &= \phi_n(x_1, \dots, x_n), & x_n &= \psi_n(y_1, \dots, y_n),
\end{aligned}$$

the functions $\phi_1, \dots, \phi_n$ belong to $C^m(\overline{\Omega})$ and the functions $\psi_1, \dots, \psi_n$ belong to $C^m(\overline{G})$.

If u is a measurable function defined on Ω, we can define a measurable function on G by

$$Au(y) = u(\Psi(y)). \tag{13}$$

Suppose that Φ is 1-smooth so that for all $x \in \Omega$

$$c \le |\det \Phi'(x)| \le C \tag{14}$$

for certain constants $c, C, 0 < c \le C$. [Here, of course, $\Phi'(x)$ denotes the Jacobian matrix $\partial(y_1, \dots, y_n)/\partial(x_1, \dots, x_n)$.] It is readily seen that the operator A defined by (13) transforms $L^p(\Omega)$ boundedly onto $L^p(G)$ and has a bounded inverse; in fact (for $1 \le p < \infty$),

$$c^{1/p} \|u\|_{p,\Omega} \le \|Au\|_{p,G} \le C^{1/p} \|u\|_{p,\Omega}.$$

We establish a similar result for Sobolev spaces.

3.35 THEOREM Let Φ be m-smooth, where $m \ge 1$. Then A transforms $W^{m,p}(\Omega)$ boundedly onto $W^{m,p}(G)$ and has a bounded inverse.

PROOF We show that the inequality $\|Au\|_{m,p,G} \le \text{const}\|u\|_{m,p,\Omega}$ holds for any $u \in W^{m,p}(\Omega)$, the constant depending only on the transformation Φ. The reverse inequality $\|Au\|_{m,p,G} \ge \text{const}\|u\|_{m,p,\Omega}$ can be established in a similar manner using the operator A^{-1} taking functions defined on G into functions defined on Ω.

By Theorem 3.16 there exists for any $u \in W^{m,p}(\Omega)$ a sequence $\{u_n\}$ of functions in $C^\infty(\Omega)$ converging to u in $W^{m,p}(\Omega)$-norm. For such smooth u_n it is easily checked by induction that

$$D^\alpha(Au_n)(y) = \sum_{|\beta| \le |\alpha|} M_{\alpha\beta}(y)[A(D^\beta u_n)](y), \tag{15}$$

where $M_{\alpha\beta}$ is a polynomial of degree not exceeding $|\beta|$ in derivatives, of orders not exceeding $|\alpha|$, of the various components of Ψ. If $\phi \in \mathcal{D}(G)$, we obtain from (15) and integration by parts

$$(-1)^{|\alpha|}\int_G (Au_n)(y)\, D^\alpha\phi(y)\, dy = \sum_{|\beta| \le |\alpha|} \int_G [A(D^\beta u_n)](y)\, M_{\alpha\beta}(y)\, dy, \tag{16}$$

or, replacing y by $\Phi(x)$ and expressing the integrals over Ω,

$$(-1)^{|\alpha|}\int_\Omega u_n(x)\,(D^\alpha\phi)(\Phi(x))\,|\det \Phi'(x)|\, dx$$
$$= \sum_{|\beta| \le |\alpha|} \int_\Omega D^\beta u_n(x)\, M_{\alpha\beta}(\Phi(x))\,|\det \Phi'(x)|\, dx. \tag{17}$$

Since $D^\beta u_n \to u$ in $L^p(\Omega)$ for $|\beta| \le m$, we may take the limit through (17) as $n \to \infty$ and hence obtain (16) with u replacing u_n. Thus (15) holds in the weak sense for any $u \in W^{m,p}(\Omega)$. We now obtain from (15) and (14)

$$\int_G |D^\alpha(Au)(y)|^p\, dy$$
$$\le \Bigl(\sum_{|\beta| \le |\alpha|} 1\Bigr)^p \max_{|\beta| \le |\alpha|} \Bigl(\sup_{y \in G}|M_{\alpha\beta}(y)|^p \int_G |(D^\beta u)(\Psi(y))|^p\, dy\Bigr)$$
$$\le \text{const} \max_{|\beta| \le |\alpha|} \int_\Omega |D^\beta u(x)|^p\, dx$$

whence it follows that $\|Au\|_{m,p,G} \le \text{const}\|u\|_{m,p,\Omega}$. ∎

Of special importance in later chapters is the case of the above theorem corresponding to nonsingular linear transformations Φ or, more generally, affine transformations (compositions of nonsingular linear transformations and translations). For such transformations $\det \Phi'(x)$ is a nonzero constant.

IV

Interpolation and Extension Theorems

Geometrical Properties of Domains

4.1 Many properties of Sobolev spaces defined on a domain Ω, and in particular the imbedding properties of these spaces, depend on regularity properties of Ω. Such regularity is normally expressed in terms of geometrical conditions that may or may not be satisfied by any given domain. We specify below five such geometric conditions, including the segment property already encountered in Section 3.17, and consider their interrelationships. First, however, we standardize some geometrical concepts and notations that will prove useful.

Given a point $x \in \mathbb{R}^n$, an open ball B_1 with center x, and an open ball B_2 not containing x, the set $C_x = B_1 \cap \{x+\lambda(y-x) : y \in B_2, \lambda > 0\}$ is called a *finite cone* in $\mathbb{R}^n$ having vertex at x. We also denote by $x + C_0 = \{x+y : y \in C_0\}$ the finite cone with vertex at x obtained by parallel translation of a finite cone C_0 with vertex at 0.

Given linearly independent vectors $y_1, y_2, \ldots, y_n \in \mathbb{R}^n$, the set $P = \{\sum_{j=1}^{n} \lambda_j y_j : 0 < \lambda_j < 1, 1 \le j \le n\}$ is a *parallelepiped* with one vertex at the origin. Similarly, $x+P$ is a parallel translate of P having one vertex at x. By the center of $x+P$ we mean, of course, the point $c(x+P) = x + \frac{1}{2}(y_1 + \cdots + y_n)$. Every parallelepiped with one vertex at x contains a finite cone with vertex at x, and conversely is also contained in such a cone.

An open cover $\mathcal{O}$ of a set $S \subset \mathbb{R}^n$ is said to be *locally finite* if any compact set in $\mathbb{R}^n$ can intersect at most finitely many elements of $\mathcal{O}$. Such locally finite collections of sets must be countable, so their elements can be listed in sequence. If S is closed, then any open cover of S possesses a locally finite subcover.

We now define five regularity properties which an open domain $\Omega \subset \mathbb{R}^n$ may possess.

4.2 Ω has the *segment property* if there exists a locally finite open cover $\{U_j\}$ of $\operatorname{bdry}\Omega$ and a corresponding sequence $\{y_j\}$ of nonzero vectors such that if $x \in \overline{\Omega} \cap U_j$ for some j, then $x + ty_j \in \Omega$ for $0 < t < 1$.

4.3 Ω has the *cone property* if there exists a finite cone C such that each point $x \in \Omega$ is the vertex of a finite cone C_x contained in Ω and congruent to C. (Note that C_x need not be obtained from C by parallel translation, just by rigid motion.)

4.4 Ω has the *uniform cone property* if there exists a locally finite open cover $\{U_j\}$ of $\operatorname{bdry}\Omega$, and a corresponding sequence $\{C_j\}$ of finite cones, each congruent to some fixed finite cone C, such that:

(i) For some finite M, every U_j has diameter less than M.
(ii) For some $\delta > 0$, $\bigcup_{j=1}^{\infty} U_j \supset \Omega_\delta \equiv \{x \in \Omega : \operatorname{dist}(x, \operatorname{bdry}\Omega) < \delta\}$.
(iii) For every j, $\bigcup_{x \in \Omega \cap U_j} (x + C_j) \equiv Q_j \subset \Omega$.
(iv) For some finite R, every collection of $R+1$ of the sets Q_j has empty intersection.

4.5 Ω has the *strong local Lipschitz property* provided there exist positive numbers δ and M, a locally finite open cover $\{U_j\}$ of $\operatorname{bdry}\Omega$, and for each U_j a real-valued function f_j of $n-1$ real variables, such that the following conditions hold:

(i) For some finite R, every collection of $R+1$ of the sets U_j has empty intersection.

(ii) For every pair of points $x, y \in \Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \operatorname{bdry}\Omega) < \delta\}$ such that $|x-y| < \delta$ there exists j such that

$$x, y \in \mathscr{V}_j = \{x \in U_j : \operatorname{dist}(x, \operatorname{bdry} U_j) > \delta\}.$$

(iii) Each function f_j satisfies a Lipschitz condition with constant M:

$$|f(\xi_1, \ldots, \xi_{n-1}) - f(\eta_1, \ldots, \eta_{n-1})| \le M|(\xi_1 - \eta_1, \ldots, \xi_{n-1} - \eta_{n-1})|.$$

(iv) For some Cartesian coordinate system $(\xi_{j,1}, \ldots, \xi_{j,n})$ in U_j the set $\Omega \cap U_j$ is represented by the inequality

$$\xi_{j,n} < f_j(\xi_{j,1}, \ldots, \xi_{j,n-1}).$$

We remark that if Ω is bounded, the rather complicated conditions above reduce to the simple condition that Ω have a locally Lipschitz boundary, that is, that each point x on the boundary of Ω should have a neighborhood U_x such that $\operatorname{bdry}\Omega \cap U_x$ is the graph of a Lipschitz continuous function.

4.6 Ω has the *uniform C^m-regularity property* if there exists a locally finite open cover $\{U_j\}$ of $\operatorname{bdry}\Omega$, and a corresponding sequence $\{\Phi_j\}$ of m-smooth one-to-one transformations (see Section 3.34) with Φ_j taking U_j onto $B = \{y \in \mathbb{R}^n : |y| < 1\}$, such that:

(i) For some $\delta > 0$, $\bigcup_{j=1}^{\infty} \Psi_j(\{y \in \mathbb{R}^n : |y| < \frac{1}{2}\}) \supset \Omega_\delta$, where $\Psi_j = \Phi_j^{-1}$.

(ii) For some finite R, every collection of $R+1$ of the sets U_j has empty intersection.

(iii) For each j, $\Phi_j(U_j \cap \Omega) = \{y \in B : y_n > 0\}$.

(iv) If $(\phi_{j,1}, \ldots, \phi_{j,n})$ and $(\psi_{j,1}, \ldots, \psi_{j,n})$ denote the components of Φ_j and Ψ_j, respectively, then there exists a finite M such that for all α, $|\alpha| \le m$, for every i, $1 \le i \le n$, and for every j, we have

$$|D^\alpha \phi_{j,i}(x)| \le M, \qquad x \in U_j$$

$$|D^\alpha \psi_{j,i}(y)| \le M, \qquad y \in B.$$

4.7 With the exception of the cone property, all the other properties above require Ω to lie on only one side of its boundary. The two-dimensional domain Ω mentioned in Section 3.17, that is,

$$\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < |x| < 1,\ 0 < y < 1\}$$

has the cone property but none of the other four. The reader may wish to convince himself that

$$\text{uniform } C^m\text{-regularity property } (m \ge 1)$$

$$\Rightarrow \text{strong local Lipschitz property}$$

$$\Rightarrow \text{uniform cone property}$$

$$\Rightarrow \text{segment property}$$

for any domain Ω.

Most of the important imbedding results of Chapter V require only the cone property though one requires the strong local Lipschitz property. Although the cone property implies none of the other above properties it "almost" implies the strong local Lipschitz property for bounded domains, in a sense made precise in the following theorem of Gagliardo [24].

4.8 THEOREM (*Gagliardo* [24]) Let Ω be a bounded domain in $\mathbb{R}^n$ having the cone property. For each $\rho > 0$ there exists a finite collection $\{\Omega_1, \Omega_2, \ldots, \Omega_m\}$ of open subsets of Ω such that $\Omega = \bigcup_{j=1}^m \Omega_j$, and such that to each Ω_j there corresponds a subset A_j of $\overline{\Omega}_j$ having diameter not exceeding ρ, and an open parallelepiped P_j with one vertex at 0, such that $\Omega_j = \bigcup_{x \in A_j}(x + P_j)$. Moreover, if ρ is sufficiently small, then each Ω_j has the strong local Lipschitz property.

PROOF Let C_0 be a finite cone with vertex at 0 such that any $x \in \Omega$ is the vertex of a finite cone $C_x \subset \Omega$ congruent to C_0. It is clearly possible to select a finite number of finite cones $C_1, \ldots, C_k$ each having vertex at 0 (and each having aperture angle smaller than that of C_0) such that any finite cone congruent to C_0 and having vertex at 0 must contain one of the cones C_j, $1 \le j \le k$. For each C_j let P_j be an open parallelepiped with one vertex at the origin and such that $P_j \subset C_j$. Then for each $x \in \Omega$ there exists j, $1 \le j \le k$, such that

$$x + P_j \subset x + C_j \subset C_x \subset \Omega.$$

Since Ω is open and $\overline{x + P_j}$ is compact, it follows that $y + P_j \subset \Omega$ for all y sufficiently close to x. Hence for every $x \in \Omega$ we can find $y \in \Omega$ such that $x \in y + P_j \subset \Omega$ for some j, $1 \le j \le k$. (Any domain with the cone property can therefore be expressed as a union of translates of finitely many parallelepipeds.)

Let $\tilde{A}_j = \{x \in \overline{\Omega} : x + P_j \subset \Omega\}$. If $\operatorname{diam} \tilde{A}_j \le \rho$ for each j, we take $m = k$, set $A_j = \tilde{A}_j$ and $\Omega_j = \bigcup_{x \in A_j}(x + P_j)$, and note that the first part of the theorem is proved. Otherwise we decompose $\tilde{A}_j$ into a finite union of sets A_{ji} such that $\operatorname{diam} A_{ji} \le \rho$, set corresponding $P_{ji} = P_j$, rearrange the totality of sets A_{ji} into a finite sequence $A_1, \ldots, A_m$, rename the corresponding P_{ji}'s as $P_1, \ldots, P_m$, and finally set $\Omega_j = \bigcup_{x \in A_j}(x + P_j)$ to achieve the same end. (Figure 2 attempts to illustrate these notions for the case

$$\begin{aligned}
\Omega &= \{(x,y) \in \mathbb{R}^2 : 0 < |x| < 1,\ 0 < y < 1\}, \\
C_0 &= \{(x,y) \in \mathbb{R}^2 : x > 0,\ y > 0,\ x^2 + y^2 < \tfrac{1}{4}\}, \\
\rho &= 13/16.
\end{aligned}$$

In this case Ω can be covered by as few as four open subsets Ω_j corresponding to only two distinct parallelepipeds.)

It remains to be shown that if ρ is sufficiently small, then each Ω_j has the strong local Lipschitz property. For simplicity of notation we suppose, therefore, that $\Omega = \bigcup_{x \in A}(x + P)$, where $\operatorname{diam} A \le \rho$ and P is a fixed parallelepiped, and we show that Ω has the strong local Lipschitz property.

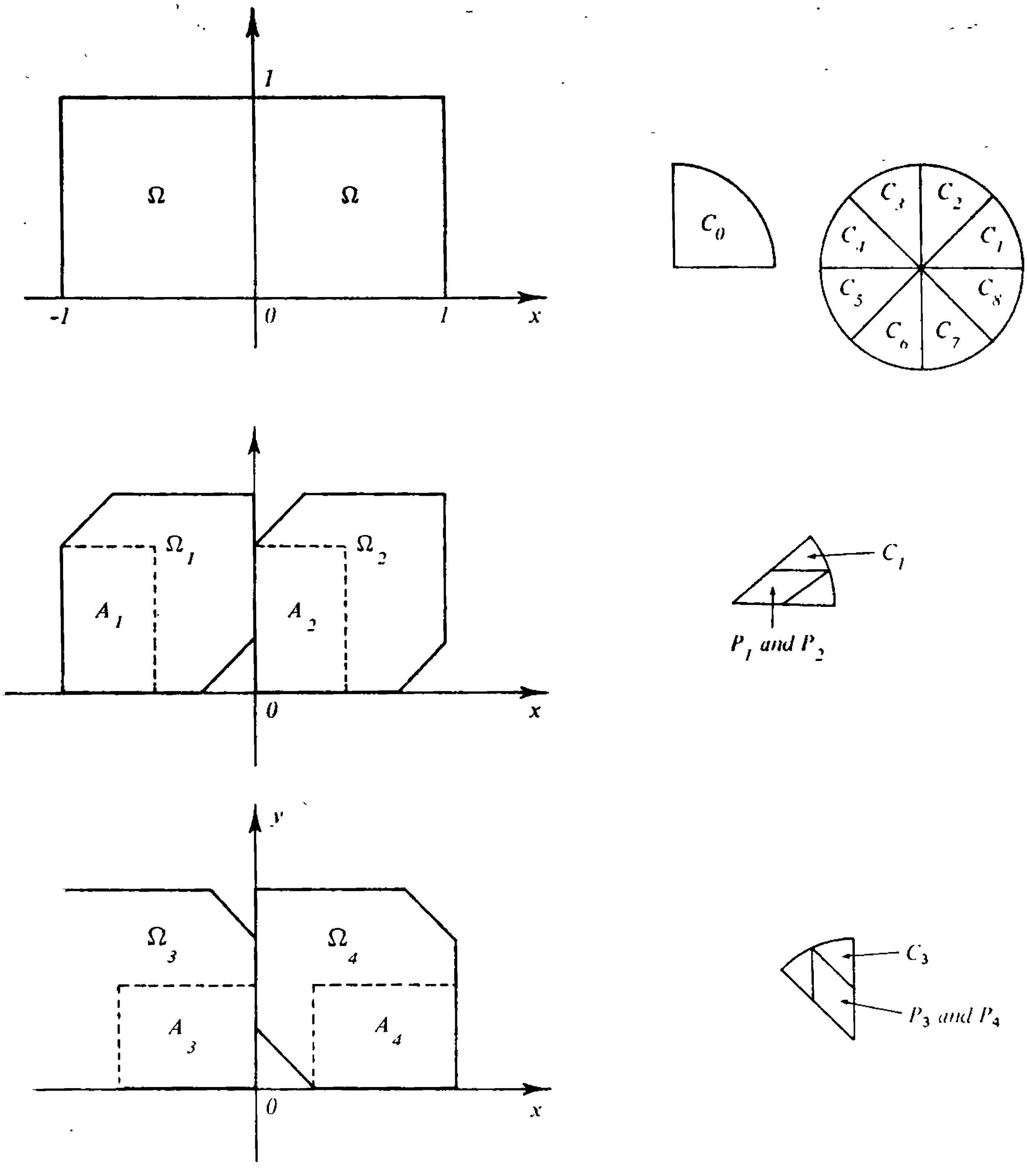


FIG. 2

For each vertex v_j of P let $Q_j = \{y = v_j + \lambda(x - v_j) : x \in P, \lambda > 0\}$ be the nfinite pyramid with vertex v_j generated by P. Then $P = \bigcap_j Q_j$, the inter-ection being taken over all 2^n vertices of P. Let $\Omega_{(j)} = \bigcup_{x \in A}(x + Q_j)$. Let $\delta = \text{dist}(\text{center of } P, \text{bdry } P)$ and let B be an arbitrary ball of radius $\sigma = \delta/2$. For any fixed $x \in \Omega$, B cannot intersect opposite faces of $x + P$ so we may pick a vertex v_j of P with the property that $x + v_j$ is common to all faces of $x + P$ that meet B, if any such faces exist. Then $B \cap (x + P) = B \cap (x + Q_j)$. Now let $x, y \in A$ and suppose B could intersect relatively opposite faces of $x + P$ and

$y+P$, that is, there exist points a and b on opposite faces of $\bar{P}$ such that $x+a \in B$ and $y+b \in B$. Then

$$\begin{aligned} \rho \geq \operatorname{dist}(x,y) &= \operatorname{dist}(x+b,\, y+b) \\ &\geq \operatorname{dist}(x+b,\, x+a) - \operatorname{dist}(x+a,\, y+b) \\ &\geq 2\delta - 2\sigma = \delta. \end{aligned}$$

It follows that if $\rho < \delta$, then B cannot meet relatively opposite faces of $x+P$ and $y+P$ for any $x, y \in A$. Thus $B \cap (x+P) = B \cap (x+Q_j)$ for some fixed j independent of $x \in A$, whence $B \cap \Omega = B \cap \Omega_{(j)}$.

Choose coordinates $\xi = (\xi', \xi_n) = (\xi_1, \ldots, \xi_{n-1}, \xi_n)$ in B so that the ξ_n axis lies in the direction of the vector from the center of P to the point v_j. Then $(x+Q_j) \cap B$ is specified in B by an inequality of the form $\xi_n < f_x(\xi')$, where f_x satisfies a Lipschitz condition with constant independent of x. Thus $\Omega_{(j)} \cap B$, and hence $\Omega \cap B$, is specified by $\xi_n < f(\xi')$ where $f(\xi') = \sup_{x \in A} f_x(\xi')$ is itself a Lipschitz continuous function. Since this can be done for a neighborhood B of any point on $\operatorname{bdry}\Omega$ it follows that Ω has the strong local Lipschitz property. ∎

Interpolation Inequalities for Intermediate Derivatives

4.9 We consider the problem of determining upper bounds for L^p-norms of derivatives $D^\beta u$, $|\beta| \leq m$, of functions $u \in W^{m,p}(\Omega)$, in terms of the L^p-norms of u and its derivatives $D^\alpha u$ of order $|\alpha| = m$. Such interpolation inequalities have been obtained by many writers including Ehrling [23], Nirenberg [53, 54], Browder [11, 12], and Gagliardo [24, 25], and are amenable to numerous generalizations. Extensions of the definition of $W^{m,p}(\Omega)$ to cover the case of nonintegral values of m can be carried out (see Chapter VII) via suitable interpolation arguments.

It is convenient to begin with a straightforward one-dimensional interpolation inequality which nevertheless typifies and provides a basis for the proof of the more general theorems which follow.

4.10 LEMMA **Let $-\infty \leq a < b \leq \infty$, let $1 \leq p < \infty$, and let $0 < \varepsilon_0 < \infty$. There exists a finite constant $K = K(\varepsilon_0, p, b-a)$, depending continuously on $b-a$ for $0 < b-a \leq \infty$, such that for every ε satisfying $0 < \varepsilon \leq \varepsilon_0$, and for every function f twice continuously differentiable on the open interval (a,b)**

$$\int_a^b |f'(t)|^p\,dt \leq K\varepsilon \int_a^b |f''(t)|^p\,dt + K\varepsilon^{-1} \int_a^b |f(t)|^p\,dt. \tag{1}$$

Moreover, if $b-a = \infty$, then $K = K(p)$ can be found so that (1) holds for every positive number ε.

Proof It is sufficient to prove (1) for real-valued functions f, for, assuming this done and writing arbitrary f in the form $f = u + iv$ with u, v real valued, we obtain

$$\begin{aligned}\int_a^b |f'(t)|^p\,dt &= \int_a^b [u'(t)^2 + v'(t)^2]^{p/2}\,dt \\ &\le \max(1, 2^{(p-2)/2}) \int_a^b [|u'(t)|^p + |v'(t)|^p]\,dt \\ &\le 2K \max(1, 2^{(p-2)/2}) \left\{ \varepsilon \int_a^b |f''(t)|^p\,dt + \varepsilon^{-1} \int_a^b |f(t)|^p\,dt \right\}.\end{aligned}$$

We may also assume, without loss of generality, that $\varepsilon_0 = 1$, for, assuming the lemma proved in this case, we obtain from (1), since $0 < \varepsilon/\varepsilon_0 \le 1$,

$$\int_a^b |f'(t)|^p\,dt \le K\cdot(\varepsilon/\varepsilon_0) \int_a^b |f''(t)|^p\,dt + K\cdot(\varepsilon_0/\varepsilon) \int_a^b |f(t)|^p\,dt.$$

This, in turn, implies (1) with $K = K(\varepsilon_0, p, b-a) = K(1, p, b-a)\max(\varepsilon_0, \varepsilon_0^{-1})$.

We assume, therefore, that f is real valued and $\varepsilon_0 = 1$. For the moment we suppose also that $a = 0$ and $b = 1$. If $0 < \xi < \frac{1}{3}$ and $\frac{2}{3} < \eta < 1$, then there exists $\lambda \in (\xi, \eta)$ such that

$$|f'(\lambda)| = \left| \frac{f(\eta) - f(\xi)}{\eta - \xi} \right| \le 3|f(\xi)| + 3|f(\eta)|.$$

It follows that for any $x \in (0, 1)$

$$\begin{aligned}|f'(x)| &= \left|f'(\lambda) + \int_\lambda^x f''(t)\,dt\right| \\ &\le 3|f(\xi)| + 3|f(\eta)| + \int_0^1 |f''(t)|\,dt.\end{aligned}$$

Integrating the above inequality with respect to ξ over $(0, \frac{1}{3})$ and with respect to η over $(\frac{2}{3}, 1)$, we obtain

$$\begin{aligned}\tfrac{1}{9}|f'(x)| &\le \int_0^{1/3} |f(\xi)|\,d\xi + \int_{2/3}^1 |f(\eta)|\,d\eta + \tfrac{1}{9} \int_0^1 |f''(t)|\,dt \\ &\le \int_0^1 |f(t)|\,dt + \tfrac{1}{9} \int_0^1 |f''(t)|\,dt,\end{aligned}$$

whence, by Hölder's inequality,

$$|f'(x)|^p \le 2^{p-1}\cdot 9^p \int_0^1 |f(t)|^p\,dt + 2^{p-1} \int_0^1 |f''(t)|^p\,dt.$$

Hence,

$$\int_0^1 |f'(t)|^p\,dt \le K_p \int_0^1 |f''(t)|^p\,dt + K_p \int_0^1 |f(t)|^p\,dt,$$

where $K_p = 2^{p-1}\cdot 9^p$. It follows by a change of variable that for any finite interval (a,b)

$$\int_a^b |f'(t)|^p\,dt \le K_p(b-a)^p \int_a^b |f''(t)|^p\,dt + K_p(b-a)^{-p} \int_a^b |f(t)|^p\,dt. \quad (2)$$

Since $0 < \varepsilon \le 1$ there exists a positive integer n such that

$$\tfrac{1}{2}\varepsilon^{1/p} \le 1/n \le \varepsilon^{1/p}.$$

Setting $a_j = a+(b-a)j/n$ for $j = 0, 1, \dots, n$, we obtain from (2), noting that $a_j - a_{j-1} = (b-a)/n$,

$$\begin{aligned}
\int_a^b |f'(t)|^p\,dt &= \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} |f'(t)|^p\,dt \\
&\le K_p \sum_{j=1}^{n} \left\{ \left(\frac{b-a}{n}\right)^p \int_{a_{j-1}}^{a_j} |f''(t)|^p\,dt + \left(\frac{n}{b-a}\right)^p \int_{a_{j-1}}^{a_j} |f(t)|^p\,dt \right\} \\
&\le \tilde{K}(p, b-a) \left\{ \varepsilon \int_a^b |f''(t)|^p\,dt + \varepsilon^{-1} \int_a^b |f(t)|^p\,dt \right\}, \qquad (3)
\end{aligned}$$

where $\tilde{K}(p, b-a) = K_p \max[(b-a)^p, 2^p(b-a)^{-p}]$.

Now let

$$K(1,p,b-a) = \begin{cases} \max\limits_{1 \le s \le 2} \tilde{K}(p,s) & \text{if } b-a \ge 1 \\ \max\limits_{b-a \le s \le 2} \tilde{K}(p,s) & \text{if } 0 < b-a < 1. \end{cases}$$

Then $K(1,p,b-a)$ is finite for $0 < b-a \le \infty$ and depends continuously on $b-a$. For $b-a < 1$, (1) follows directly from (3). For $1 \le b-a \le \infty$, the interval (a,b) may be partitioned into (possibly infinitely many) subintervals each of length between 1 and 2, whence (1) follows upon summing (3) applied to each of these subintervals.

Finally, suppose that $b-a = \infty$. To be specific we assume a is finite and $b = \infty$, the other possibilities being similar. For given $\varepsilon > 0$ let $a_j = a + j\varepsilon^{1/p}$, $j = 0, 1, 2, \dots$. Then $a_j - a_{j-1} = \varepsilon^{1/p}$ and we have, using (2),

$$\begin{aligned}
\int_a^\infty |f'(t)|^p\,dt &= \sum_{j=1}^{\infty} \int_{a_{j-1}}^{a_j} |f'(t)|^p\,dt \\
&\le K_p \varepsilon \sum_{j=1}^{\infty} \int_{a_{j-1}}^{a_j} |f''(t)|^p\,dt + K_p \varepsilon^{-1} \sum_{j=1}^{\infty} \int_{a_{j-1}}^{a_j} |f(t)|^p\,dt
\end{aligned}$$

which is (1) with $K = K_p$ depending only on p. ∎

4.11 For $1 \le p < \infty$ and for integers j, $0 \le j \le m$, we introduce functionals $|\cdot|_{j,p}$ on $W^{m,p}(\Omega)$ as follows:

$$|u|_{j,p} = |u|_{j,p,\Omega} = \left\{ \sum_{|\alpha|=j} \int_\Omega |D^\alpha u(x)|^p \, dx \right\}^{1/p}.$$

Clearly, $|u|_{0,p} = \|u\|_{0,p} = \|u\|_p$ is the norm of u in $L^p(\Omega)$ and

$$\|u\|_{m,p} = \left\{ \sum_{0 \le j \le m} |u|_{j,p}^p \right\}^{1/p}.$$

If $j \ge 1$, $|\cdot|_{j,p}$ is a seminorm—it has all the properties of a norm except that $|u|_{j,p} = 0$ does not imply that u vanishes in $W^{m,p}(\Omega)$; for instance, u may be a nonzero constant on a domain Ω having finite volume. Under certain circumstances which we investigate later, $|\cdot|_{m,p}$ is an equivalent norm for the space $W_0^{m,p}(\Omega)$. In particular this is so if Ω is bounded.

At the moment we are concerned with establishing interpolation inequalities of the form

$$|u|_{j,p} \le K\varepsilon |u|_{m,p} + K\varepsilon^{-j/(m-j)} |u|_{0,p}, \tag{4}$$

where $0 \le j \le m-1$. The following lemma shows that in general we need only establish (4) for the special case $j = 1$, $m = 2$, a reduction that will be used in the three interpolation theorems that follow.

4.12 LEMMA Let $0 < \delta_0 < \infty$, let $m \ge 2$, and let

$$\varepsilon_0 = \min(\delta_0, \delta_0^2, \ldots, \delta_0^{m-1}).$$

Suppose that for given p, $1 \le p < \infty$, and given $\Omega \subset \mathbb{R}^n$ there exists a constant $K = K(\delta_0, p, \Omega)$ such that for every finite δ, $0 < \delta \le \delta_0$, and for every $u \in W^{2,p}(\Omega)$, we have

$$|u|_{1,p} \le K\delta |u|_{2,p} + K\delta^{-1} |u|_{0,p}. \tag{5}$$

Then there exists a constant $K = K(\varepsilon_0, m, p, \Omega)$ such that for every finite ε, $0 < \varepsilon \le \varepsilon_0$, every integer j, $0 \le j \le m-1$, and every $u \in W^{m,p}(\Omega)$, we have

$$|u|_{j,p} \le K\varepsilon |u|_{m,p} + K\varepsilon^{-j/(m-j)} |u|_{0,p}. \tag{6}$$

PROOF Since (6) is obvious for $j = 0$ we consider only the case $1 \le j \le m-1$. The proof is accomplished by double induction on m and j. The constants $K_1, K_2, \ldots$ appearing in the argument may depend on δ_0 (or ε_0), m, p, and Ω. We first prove (6) for $j = m-1$ by induction on m, so that (5) is the special case $m = 2$. Assume, therefore, that for some k, $2 \le k \le m-1$,

$$|u|_{k-1,p} \le K_1 \delta |u|_{k,p} + K_1 \delta^{-(k-1)} |u|_{0,p} \tag{7}$$

holds for all δ, $0 < \delta \le \delta_0$, and all $u \in W^{k,p}(\Omega)$. If $u \in W^{k+1,p}(\Omega)$, we prove that (7) also holds with $k+1$ replacing k (and a different constant K_1'). If $|\alpha| = k-1$, we obtain from (5)

$$|D^\alpha u|_{1,p} \le K_2 \delta |D^\alpha u|_{2,p} + K_2 \delta^{-1} |D^\alpha u|_{0,p}.$$

Combining this inequality with (7), we obtain, for $0 < \eta \le \delta_0$,

$$\begin{aligned} |u|_{k,p} &\le K_3 \sum_{|\alpha| = k-1} |D^\alpha u|_{1,p} \\ &\le K_4 \delta |u|_{k+1,p} + K_4 \delta^{-1} |u|_{k-1,p} \\ &\le K_4 \delta |u|_{k+1,p} + K_4 K_1 \delta^{-1} \eta |u|_{k,p} + K_4 K_1 \delta^{-1} \eta^{1-k} |u|_{0,p}. \end{aligned}$$

We may assume without prejudice that $2K_1 K_4 \ge 1$. Hence we may take $\eta = \delta/2K_1 K_4$ and so obtain

$$\begin{aligned} |u|_{k,p} &\le 2K_4 \delta |u|_{k+1,p} + (\delta/2K_1 K_4)^{-k} |u|_{0,p} \\ &\le K_5 \delta |u|_{k+1,p} + K_5 \delta^{-k} |u|_{0,p}. \end{aligned}$$

This completes the induction establishing (7) for $0 < \delta \le \delta_0$ and hence (6) for $j = m-1$ and $0 < \varepsilon \le \delta_0$.

We now prove by downward induction on j that

$$|u|_{j,p} \le K_6 \delta^{m-j} |u|_{m,p} + K_6 \delta^{-j} |u|_{0,p} \tag{8}$$

holds for $1 \le j \le m-1$ and $0 < \delta \le \delta_0$. Note that (7) with $k = m$ is the special case $j = m-1$ of (8). Assume, therefore, that (8) holds for some j, $2 \le j \le m-1$. We prove that it also holds with j replaced by $j-1$ (and a different constant K_6). From (7) and (8) we obtain

$$\begin{aligned} |u|_{j-1,p} &\le K_7 \delta |u|_{j,p} + K_7 \delta^{1-j} |u|_{0,p} \\ &\le K_7 \delta \{K_6 \delta^{m-j} |u|_{m,p} + K_6 \delta^{-j} |u|_{0,p}\} + K_7 \delta^{1-j} |u|_{0,p} \\ &\le K_8 \delta^{m-(j-1)} |u|_{m,p} + K_8 \delta^{-(j-1)} |u|_{0,p}. \end{aligned}$$

Thus (8) holds, and (6) follows by setting $\delta = \varepsilon^{1/(m-j)}$ in (8) and noting that $\varepsilon \le \varepsilon_0$ if $\delta \le \delta_0$. ∎

4.13 THEOREM There exists a constant $K = K(m,p,n)$ such that for any $\Omega \subset \mathbb{R}^n$, any $\varepsilon > 0$, any integer j, $0 \le j \le m-1$, and any $u \in W_0^{m,p}(\Omega)$,

$$|u|_{j,p} \le K\varepsilon |u|_{m,p} + K\varepsilon^{-j/(m-j)} |u|_{0,p}. \tag{9}$$

PROOF By Lemma 3.22 the operator of zero extension outside Ω is an isometric isomorphism of $W_0^{m,p}(\Omega)$ into $W^{m,p}(\mathbb{R}^n)$. Thus it is sufficient to establish (9) for $\Omega = \mathbb{R}^n$. Also, by Lemma 4.12 we need consider only the case $j = 1$, $m = 2$. (The case $j = 0$ $m = 1$ is trivial.) For any $\varepsilon > 0$ and any $\phi \in C_0^\infty(\mathbb{R}^n)$

we obtain from Lemma 4.10

$$\int_{-\infty}^{\infty} \left|\frac{\partial}{\partial x_j}\phi(x)\right|^p dx_j \le K\varepsilon^p \int_{-\infty}^{\infty} \left|\frac{\partial^2}{\partial x_j^2}\phi(x)\right|^p dx_j + K\varepsilon^{-p}\int_{-\infty}^{\infty} |\phi(x)|^p\, dx_j.$$

Integrating the remaining components of x, we are led to

$$\|D_j\phi\|_p^p \le K\varepsilon^p \|D_j^2\phi\|_p^p + K\varepsilon^{-p}\|\phi\|_p^p,$$

whence

$$|\phi|_{1,p}^p \le K\varepsilon^p \sum_{j=1}^n \|D_j^2\phi\|_p^p + nK\varepsilon^{-p}|\phi|_{0,p}^p \le K\varepsilon^p|\phi|_{2,p}^p + nK\varepsilon^{-p}|\phi|_{0,p}^p.$$

The case $j = 1$, $m = 2$ of (9) now follows by taking pth roots and noting that $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{m,p}(\mathbb{R}^n)$. ∎

4.14 THEOREM (*Ehrling* [23], *Nirenberg* [53], *Gagliardo* [24]) Let $\Omega \subset \mathbb{R}^n$ have the uniform cone property (Section 4.4), and let ε_0 be a finite, positive number. Then there exists a constant $K = K(\varepsilon_0, m, p, \Omega)$ such that for any ε, $0 < \varepsilon \le \varepsilon_0$, any integer j, $0 \le j \le m-1$, and any $u \in W^{m,p}(\Omega)$

$$|u|_{j,p} \le K\varepsilon|u|_{m,p} + K\varepsilon^{-j/(m-j)}|u|_{0,p}. \tag{10}$$

PROOF The case $m = 1$ is trivial; again Lemma 4.12 shows that it is sufficient to establish (10) for $j = 1$, $m = 2$. In addition, the argument used in the second paragraph of the proof of Lemma 4.10 shows that we may assume $\varepsilon_0 = 1$.

In this proof we make constant use of the notations of Section 4.4 describing the uniform cone property possessed by Ω. If δ is the constant of condition (ii) of that section and if $\lambda = (\lambda_1, \ldots, \lambda_n)$ is an n-tuple of integers, we consider the cube

$$H_\lambda = \{x \in \mathbb{R}^n : \lambda_k\delta/2\sqrt{n} \le x_k \le (\lambda_k+1)\delta/2\sqrt{n}\}.$$

Then $\mathbb{R}^n = \bigcup_\lambda H_\lambda$ and $\operatorname{diam} H_\lambda = \delta/2$. Let $\Omega_0 = \bigcup_{H_\lambda \subset \Omega} H_\lambda$. Thus $\Omega \sim \Omega_\delta \subset \Omega_0 \subset \Omega$. If the sets $U_1, U_2, \ldots$ and $Q_1, Q_2, \ldots$ are as in Section 4.4, then

$$\Omega = \bigcup_{j=1}^\infty (U_j \cap \Omega) \cup \Omega_0 = \bigcup_{j=1}^\infty Q_j \cup \Omega_0.$$

We shall prove that for any $u \in W^{2,p}(\Omega)$

$$|u|_{1,p,\Omega_0}^p \le K_1\varepsilon^p|u|_{2,p,\Omega_0}^p + K_1\varepsilon^{-p}|u|_{0,p,\Omega_0}^p \tag{11}$$

and for $j = 1, 2, 3, \ldots$

$$|u|_{1,p,U_j\cap\Omega}^p \le K_2\varepsilon^p|u|_{2,p,Q_j}^p + K_2\varepsilon^{-p}|u|_{0,p,Q_j}^p, \tag{12}$$

where K_2 is independent of j. Since any $R+2$ of the sets $\Omega_0, Q_1, Q_2, \ldots$ have empty intersection, (11) and (12) imply that

$$
\begin{aligned}
|u|_{1,p,\Omega}^p &\le |u|_{1,p,\Omega_0}^p + \sum_{j=1}^{\infty} |u|_{1,p,U_j\cap\Omega}^p \\
&\le \max(K_1, K_2)\Big\{\varepsilon^p |u|_{2,p,\Omega_0}^p + \varepsilon^p \sum_{j=1}^{\infty} |u|_{2,p,Q_j}^p \\
&\qquad + \varepsilon^{-p} |u|_{0,p,\Omega_0}^p + \varepsilon^{-p} \sum_{j=1}^{\infty} |u|_{0,p,Q_j}^p\Big\} \\
&\le (R+1)\max(K_1, K_2)\{\varepsilon^p |u|_{2,p,\Omega}^p + \varepsilon^{-p} |u|_{0,p,\Omega}^p\}
\end{aligned}
$$

and this inequality yields (10) (case $j = 1$, $m = 2$) on taking pth roots. It remains therefore to verify the validity of (11) and (12).

If $u \in C^\infty(\Omega) \cap W^{2,p}(\Omega)$, we apply Lemma 4.10 to u considered as a function of x_k on the interval from $\lambda_k \delta/2\sqrt{n}$ to $(\lambda_k + 1)\delta/2\sqrt{n}$, and then integrate the remaining variables over similar intervals to obtain, for any $H_\lambda \subset \Omega$,

$$\int_{H_\lambda} |D_k u(x)|^p \, dx \le K_3 \varepsilon^p \int_{H_\lambda} |D_k{}^2 u(x)|^p \, dx + K_3 \varepsilon^{-p} \int_{H_\lambda} |u(x)|^p \, dx, \tag{13}$$

where K_3 depends only on p and the length of a side of H_λ (which in turn depends on Ω via δ and n). Summing (13) for $1 \le k \le n$, we obtain

$$|u|_{1,p,H_\lambda}^p \le K_3 \varepsilon^p |u|_{2,p,H_\lambda}^p + nK_3 \varepsilon^{-p} |u|_{0,p,H_\lambda}^p. \tag{14}$$

Since the cubes H do not overlap, we sum (14) for all cubes $H_\lambda \subset \Omega$ and obtain (11) with $K_1 = nK_3$. Since $C^\infty(\Omega) \cap W^{2,p}(\Omega)$ is dense in $W^{2,p}(\Omega)$, (11) holds for all $u \in W^{2,p}(\Omega)$.

The constant K_2 in (12) will turn out to depend only on p, M and the dimensions of the cone C_j (see Section 4.4). Anticipating this, and noting that these dimensions are specified by the single cone C to which all cones C_j are congruent, we drop, for simplicity, all subscripts j in considering (12). Let ξ be a unit vector in a direction in C, and let $\Omega_\xi = \{y + t\xi : y \in \Omega \cap U, 0 \le t \le h\}$, where h is the height of the cone C (Fig. 3). Thus $(\Omega \cap U) \subset \Omega_\xi \subset Q$ by condition (iii) of the uniform cone property. Any line L parallel to ξ either has empty intersection with Ω_ξ or else intersects Ω_ξ in an interval of length ρ, where $h \le \rho \le h + \operatorname{diam} U \le h + M$ by condition (i) of the uniform cone property. By Lemma 4.10, if $u \in C^\infty(\Omega) \cap W^{2,p}(\Omega)$,

$$\int_{L\cap\Omega_\xi} |D_\xi u|^p \, ds \le K_4 \varepsilon^p \int_{L\cap\Omega_\xi} |D_\xi{}^2 u|^p \, ds + K_4 \varepsilon^{-p} \int_{L\cap\Omega_\xi} |u|^p \, ds, \tag{15}$$

where D_ξ denotes differentiation in the direction of ξ and where K_4 can be chosen to depend only on p, h, and M, that is, on p and Ω. We now integrate

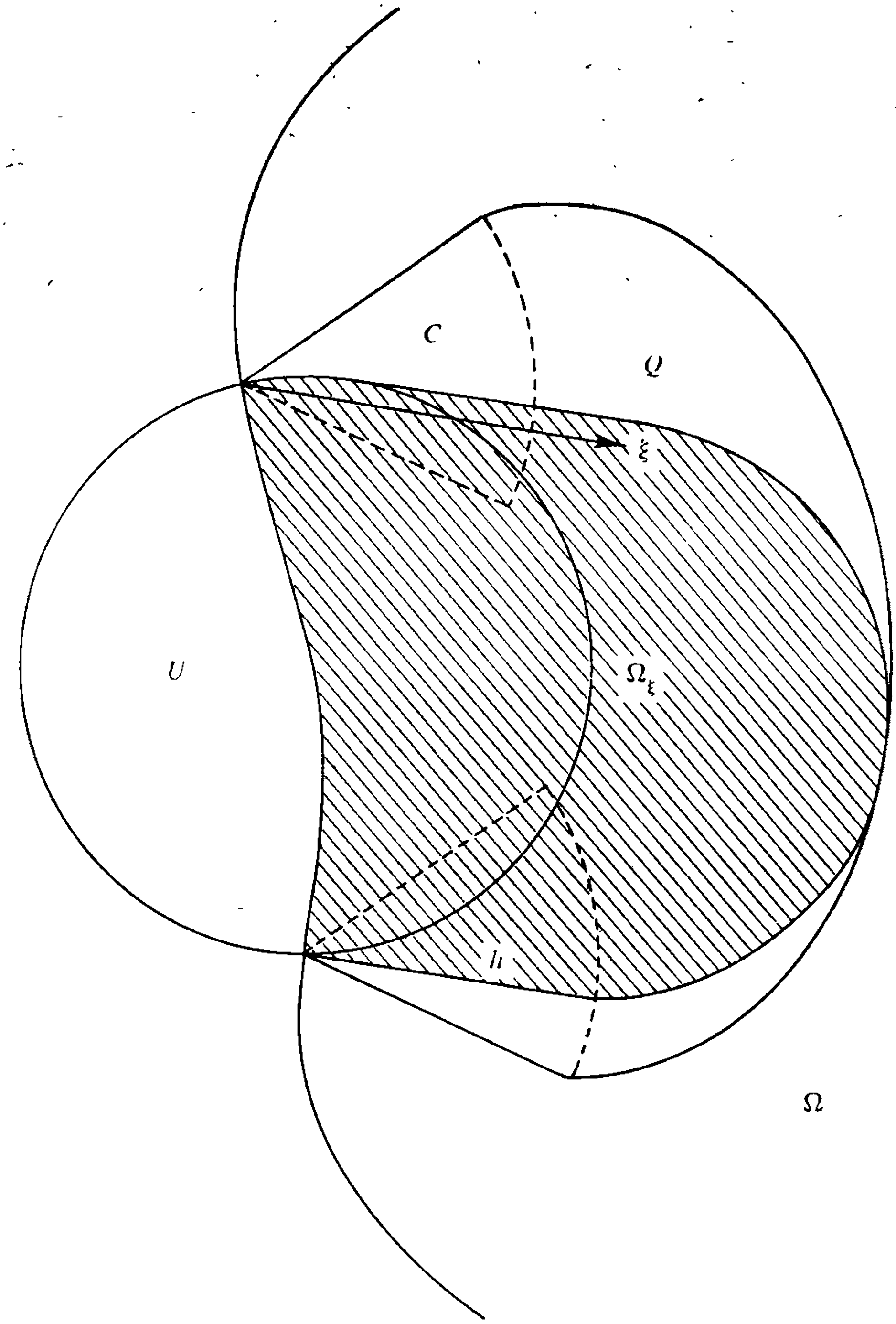


FIG. 3

(15) over the projection of Ω_ξ on a hyperplane perpendicular to ξ and so obtain

$$\begin{aligned}\int_{\Omega\cap U} |D_\xi u(x)|^p\,dx &\le \int_{\Omega_\xi} |D_\xi u(x)|^p\,dx \\ &\le K_4\varepsilon^p \int_{\Omega_\xi} |D_\xi^{\,2} u(x)|^p\,dx + K_4\varepsilon^{-p}\int_{\Omega_\xi}|u(x)|^p\,dx \\ &\le K_4\varepsilon^p \int_{Q} |D_\xi^{\,2} u(x)|^p\,dx + K_4\varepsilon^{-p}\int_{Q}|u(x)|^p\,dx. \qquad (16)\end{aligned}$$

Now let $\xi_1, \ldots, \xi_n$ be a basis of unit vectors in $\mathbb{R}^n$ each lying in a direction contained in the cone C. For $1 \le k \le n$, $D_k u(x) = \sum_{j=1}^n a_j D_{\xi_j} u(x)$ where the constants a_j satisfy $|a_j| \le 1/V$, $1 \le j \le n$, V being the volume of the parallelepiped spanned by $\xi_1, \ldots, \xi_n$. (The reader might verify this assertion. It is a simple exercise in linear algebra.) A lower bound for V can be specified in terms of the aperture angle of the cone C—that is to say, the basis $\xi_1, \ldots, \xi_n$, which varies with the covering patch U, may always be chosen in such a way that V is independent of U. It now follows from (16) that

$$\begin{aligned}\int_{\Omega \cap U} |D_k u(x)|^p\,dx &\le K_5 \sum_{j=1}^n \int_{\Omega \cap U} |D_{\xi_j} u(x)|^p\,dx \\ &\le K_5 \sum_{j=1}^n \left\{ K_4 \varepsilon^p \int_Q |D_{\xi_j}^2 u(x)|^p\,dx + K_4 \varepsilon^{-p} \int_Q |u(x)|^p\,dx \right\} \\ &\le K_6 \varepsilon^p |u|_{2,p,Q}^p + K_6 \varepsilon^{-p} |u|_{0,p,Q}^p .\end{aligned}$$

The desired inequality (12) now follows by summing on k and using the density of $C^\infty(\Omega) \cap W^{2,p}(\Omega)$ in $W^{2,p}(\Omega)$. ∎

If Ω is bounded, the above theorem can be proved under weaker hypotheses.

4.15 THEOREM Let Ω be a bounded domain in $\mathbb{R}^n$ having the cone property. Then the conclusion of Theorem 4.14 holds for Ω.

PROOF By Theorem 4.8 there exists a finite collection $\{\Omega_1, \ldots, \Omega_k\}$ of open subsets of Ω such that $\Omega = \bigcup_{j=1}^k \Omega_j$ and such that each Ω_j is a union of translates of some open parallelepiped. It is clearly sufficient to prove an inequality analogous to (11) for each Ω_j. We may therefore assume without loss of generality that $\Omega = \bigcup_{x \in A}(x+P)$, where A is a bounded set in $\mathbb{R}^n$ and P is an open parallelepiped with one vertex at the origin. Let $\xi_1, \ldots, \xi_n$ be unit vectors in the directions of the n edges of P that concur in the origin, and let l be the minimum length of these edges. Then the intersection with Ω of any line L parallel to one of the vectors ξ_j is either empty or a finite collection of segments each having length between l and $\operatorname{diam}\Omega$. It follows as in (16) that

$$\int_\Omega |D_{\xi_j} u(x)|^p\,dx \le K_1 \varepsilon^p \int_\Omega |D_{\xi_j}^2 u(x)|^p\,dx + K\varepsilon^{-p} \int_\Omega |u(x)|^p\,dx$$

for smooth functions u. Since $\{\xi_1, \xi_2, \ldots, \xi_n\}$ is a basis for $\mathbb{R}^n$ one can now show, by an argument similar to that following (16), that

$$|u|_{1,p,\Omega}^p \le K_2 \varepsilon^p |u|_{2,p,\Omega}^p + K_2 \varepsilon^{-p} |u|_{0,p,\Omega}^p$$

holds for all $u \in W^{2,p}(\Omega)$. ∎

4.16 COROLLARY The functional $((\cdot))_{m,p,\Omega}$ defined by

$$((u))_{m,p,\Omega} = \{|u|^p_{m,p,\Omega} + |u|^p_{0,p,\Omega}\}^{1/p}$$

is a norm, equivalent to the usual norm $\|\cdot\|_{m,p,\Omega}$, on each of the following spaces:

(i) $W_0^{m,p}(\Omega)$ for any domain Ω,
(ii) $W^{m,p}(\Omega)$ for any domain Ω having the uniform cone property,
(iii) $W^{m,p}(\Omega)$ for any bounded domain Ω having the cone property.

4.17 THEOREM (*Ehrling* [23], *Browder* [12]) If $\Omega \subset \mathbb{R}^n$ has the uniform cone property or if it is bounded and has the cone property, and if $1 \leq p < \infty$, then there exists a constant $K = K(m,p,\Omega)$ such that for $0 \leq j \leq m$ and any $u \in W^{m,p}(\Omega)$,

$$\|u\|_{j,p} \leq K \|u\|^{j/m}_{m,p} \|u\|^{(m-j)/m}_{0,p}. \tag{17}$$

In addition, (17) is valid for all $u \in W_0^{m,p}(\Omega)$ with a constant $K = K(m,p,n)$ independent of Ω.

PROOF Inequality (17) is obvious if either $j = 0$ or $j = m$. For $0 < j < m$ we can obtain from successive applications of (10) that

$$\|u\|_{j,p} \leq K_1 \varepsilon \|u\|_{m,p} + K_1 \varepsilon^{-j/(m-j)} \|u\|_{0,p} \tag{18}$$

holds for all ε, $0 < \varepsilon \leq 1$, and all $u \in W^{m,p}(\Omega)$ with K_1 depending only on m, p, and Ω. (By Theorem 4.13 the same inequality holds for all $u \in W_0^{m,p}(\Omega)$ with K_1 depending only on m, p, and n.) Inequality (17) now follows for $u \neq 0$ if we set $\varepsilon = (\|u\|_{0,p}/\|u\|_{m,p})^{(m-j)/m}$ in (18). ∎

We remark that (17) also implies (18) algebraically: specifically, set $p = m/j$, $p' = m/(m-j)$, $a = (\varepsilon \|u\|_{m,p})^{j/m}$, and $b = \varepsilon^{-j/m} \|u\|^{(m-j)/m}_{0,p}$ in inequality (4) of Chapter II,

$$ab \leq (a^p/p) + (b^{p'}/p'), \qquad (1/p) + (1/p') = 1,$$

to show that the right side of (17) does not exceed the right side of (18).

Interpolation Inequalities Involving Compact Subdomains

4.18 Upper bounds for the $L^p(\Omega)$-norms of intermediate derivatives $D^\beta u$, $|\beta| \leq m-1$, of a function $u \in W^{m,p}(\Omega)$ can be expressed in terms of the seminorm $|u|_{m,p,\Omega}$ and the L^p-norm of u over a suitable subdomain whose closure is a compact subset of the bounded domain Ω. We establish some hybrid interpolation inequalities of this sort by methods that follow somewhat the same lines as those used for the interpolation inequalities derived above. (See, for instance, the work of Agmon [6].)

4.19 LEMMA Let (a, b) be a finite open interval in $\mathbb{R}$ and let $1 \le p < \infty$. There exists a finite constant $K = K(p, b-a)$ and, for every positive number ε, a number $\delta = \delta(\varepsilon, b-a)$ satisfying $0 < 2\delta < b-a$ such that every continuously differentiable function f on (a, b) satisfies

$$\int_a^b |f(t)|^p\,dt \le K\varepsilon \int_a^b |f'(t)|^p\,dt + K \int_{a+\delta}^{b-\delta} |f(t)|^p\,dt. \tag{19}$$

Moreover, fixed values of K and δ, independent of $b-a$, can be chosen so that (19) holds for all intervals (a, b) whose lengths lie between fixed positive bounds: $0 < l_1 \le b-a \le l_2 < \infty$.

PROOF The proof is similar to that of Lemma 4.10. Suppose for the moment that $a = 0$ and $b = 1$, and let $\frac{1}{3} < \eta < \frac{2}{3}$. If $0 < x < 1$, then

$$|f(x)| = \left| f(\eta) + \int_\eta^x f'(t)\,dt \right| \le |f(\eta)| + \int_0^1 |f'(t)|\,dt.$$

Integrating η over $(\frac{1}{3}, \frac{2}{3})$, we are led to

$$|f(x)| \le 3 \int_{1/3}^{2/3} |f(\eta)|\,d\eta + \int_0^1 |f'(t)|\,dt,$$

so that by Hölder's inequality if $p > 1$,

$$|f(x)|^p \le 3 \cdot 2^{p-1} \int_{1/3}^{2/3} |f(t)|^p\,dt + 2^{p-1} \int_0^1 |f'(t)|^p\,dt.$$

Integrating x over $(0, 1)$, we obtain

$$\int_0^1 |f(t)|^p\,dt \le K_p \int_0^1 |f'(t)|^p\,dt + K_p \int_{1/3}^{2/3} |f(t)|^p\,dt,$$

where $K_p = 3 \cdot 2^{p-1}$. The change of variable $a + t(b-a) \to t$ now yields, for any finite interval (a, b),

$$\int_a^b |f(t)|^p\,dt \le K_p (b-a)^p \int_a^b |f'(t)|^p\,dt + K_p \int_{a+(b-a)/3}^{b-(b-a)/3} |f(t)|^p\,dt.$$

For given $\varepsilon > 0$ pick a positive integer n such that $n^{-p} \le \varepsilon$. Let $a_j = a + (b-a)j/n$ for $j = 0, 1, \ldots, n$ and pick δ so that $0 < \delta \le (b-a)/3n$. Then

$$\int_a^b |f(t)|^p\,dt = \sum_{j=1}^n \int_{a_{j-1}}^{a_j} |f(t)|^p\,dt$$

equation continues

$$\le K_p \sum_{j=1}^{n} \left\{ \left(\frac{b-a}{n}\right)^p \int_{a_{j-1}}^{a_j} |f'(t)|^p \, dt + \int_{a_{j-1}+\delta}^{a_j-\delta} |f(t)|^p \, dt \right\}$$

$$\le K_p \max(1,(b-a)^p) \left\{ \varepsilon \int_a^b |f'(t)|^p \, dt + \int_{a+\delta}^{b-\delta} |f(t)|^p \, dt \right\}$$

which is the desired inequality. ∎

The reader may convince himself that Lemma 4.19, unlike Lemma 4.10, does not extend to infinite intervals (a,b) if the intention is that the second integral on the right side of (19) should be taken over a compact subinterval.

4.20 THEOREM Let Ω be a bounded domain in $\mathbb{R}^n$ that has the segment property. Then there exists a constant $K = K(p,\Omega)$ such that to any positive number ε there corresponds a domain $\Omega_\varepsilon \subset\subset \Omega$ such that

$$|u|_{0,p,\Omega} \le K\varepsilon |u|_{1,p,\Omega} + K|u|_{0,p,\Omega_\varepsilon} \tag{20}$$

holds for every $u \in W^{1,p}(\Omega)$.

PROOF The proof is similar to that of Theorem 4.14. Since Ω is bounded the locally finite open cover $\{U_j\}$ of $\operatorname{bdry}\Omega$ and the corresponding set $\{y_j\}$ of nonzero vectors referred to in the description of the segment property (Section 4.2) are both finite sets. Therefore open sets $\mathscr{V}_j \subset\subset U_j$ can be found so that $\operatorname{bdry}\Omega \subset \bigcup_j \mathscr{V}_j$. (See the first part of the proof of Theorem 3.14.) Moreover, for some $\delta > 0$, $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \operatorname{bdry}\Omega) < \delta\} \subset \bigcup_j \mathscr{V}_j$ so that we may write $\Omega = \bigcup_j (\mathscr{V}_j \cap \Omega) \cup \tilde{\Omega}$, where $\tilde{\Omega} \subset\subset \Omega$. It is thus sufficient to prove that for each j

$$|u|^p_{0,p,\mathscr{V}_j\cap\Omega} \le K_1 \varepsilon^p |u|^p_{1,p,\Omega} + K_1 |u|^p_{0,p,\Omega_{\varepsilon,j}}$$

for some $\Omega_{\varepsilon,j} \subset\subset \Omega$. For simplicity we drop all subscripts j.

Consider the sets $Q, Q_\eta, 0 \le \eta < 1$, defined by

$$Q = \{x + ty : x \in U \cap \Omega,\ 0 < t < 1\},$$

$$Q_\eta = \{x + ty : x \in \mathscr{V} \cap \Omega,\ \eta < t < 1\}.$$

If $\eta > 0$, $Q_\eta \subset\subset Q$. By the segment property, $Q \subset \Omega$ and any line L parallel to y and passing through a point of $\mathscr{V} \cap \Omega$ intersects Q_0 in one or more intervals each having length between $|y|$ and $\operatorname{diam}\Omega$. By Lemma 4.19 there exists $\eta > 0$ and a constant K such that for any $u \in C^\infty(\Omega)$ and any such line L

$$\int_{L\cap Q_0} |u(x)|^p \, ds \le K_1 \varepsilon^p \int_{L\cap Q_0} |D_y u(x)|^p \, ds + K_1 \int_{L\cap Q_\eta} |u(x)|^p \, ds,$$

D_y denoting differentiation in the direction of y. We integrate this inequality over the projection of Q_0 on a hyperplane perpendicular to y and so obtain

$$\begin{aligned}|u|_{0,p,\Omega\cap V}^p \le |u|_{0,p,Q_0}^p &\le K_1\varepsilon^p|u|_{1,p,Q_0}^p + K_1|u|_{0,p,Q_\eta}^p \\ &\le K_1\varepsilon^p|u|_{1,p,\Omega}^p + K_1|u|_{0,p,\Omega_\varepsilon}^p,\end{aligned}$$

where $\Omega_\varepsilon = \Omega_\eta \subset\subset \Omega$. By density this inequality holds for any $u \in W^{1,p}(\Omega)$. ∎

4.21 COROLLARY The conclusion of Theorem 4.20 is also valid if Ω is bounded and has the cone property.

PROOF As remarked earlier, a domain Ω with the cone property need not have the segment property. By Theorem 4.8, however, Ω is a finite union of domains having the strong local Lipschitz property. We leave it to the reader to show that a bounded domain with the strong local Lipschitz property has the segment property, and thus complete the proof. ∎

4.22 LEMMA Let Ω_0, Ω be domains in $\mathbb{R}^n$ with $\Omega_0 \subset\subset \Omega$. Then there exists a domain Ω' having the cone property such that $\Omega_0 \subset \Omega' \subset\subset \Omega$.

PROOF Since $\overline{\Omega}_0$ is a compact subset of Ω there exists $\delta > 0$ such that $\operatorname{dist}(\overline{\Omega}_0, \operatorname{bdry}\Omega) > \delta$. The domain $\Omega' = \{y \in \mathbb{R}^n : |y-x| < \delta \text{ for some } x \in \Omega_0\}$ clearly has the desired properties. ∎

4.23 THEROEM Let Ω be a bounded domain in $\mathbb{R}^n$ having either the segment property or the cone property. Let $0 < \varepsilon_0 < \infty$, let $1 \le p < \infty$, and let j and m be integers with $0 \le j \le m-1$. Then there exists a constant $K = K(\varepsilon_0, m, p, \Omega)$, and for each ε, $0 < \varepsilon \le \varepsilon_0$, a domain Ω_ε such that $\Omega_\varepsilon \subset\subset \Omega$ and such that for every $u \in W^{m,p}(\Omega)$

$$|u|_{j,p,\Omega} \le K\varepsilon|u|_{m,p,\Omega} + K\varepsilon^{-j/(m-j)}|u|_{0,p,\Omega_\varepsilon}. \tag{21}$$

PROOF We apply Theorem 4.20 or its corollary to derivatives $D^\beta u$, $|\beta| = m-1$, to obtain

$$|u|_{m-1,p,\Omega} \le K_1\varepsilon|u|_{m,p,\Omega} + K_1|u|_{m-1,p,\Omega_\varepsilon}, \tag{22}$$

where $\Omega_\varepsilon \subset\subset \Omega$. By Lemma 4.22 we may assume that Ω_ε has the cone property. For $0 < \varepsilon \le \varepsilon_0$, we have by Theorem 4.15

$$|u|_{m-1,p,\Omega_\varepsilon} \le K_2\varepsilon|u|_{m,p,\Omega_\varepsilon} + K_2\varepsilon^{-(m-1)}|u|_{0,p,\Omega_\varepsilon}. \tag{23}$$

Combining (22) and (23), we get the case $j = m-1$ of (21). We complete the proof by downward induction on j. Assuming (21) holds for some $j \ge 1$ and replacing ε by ε^{m-j} (with consequent alteration of K and Ω_ε), we obtain

$$|u|_{j,p,\Omega} \le K_3\varepsilon^{m-j}|u|_{m,p,\Omega} + K_3\varepsilon^{-j}|u|_{0,p,\Omega_\varepsilon}. \tag{24}$$

Also by (21) with j and m replaced by $j-1$ and j, respectively (the case already proved), we have

$$|u|_{j-1,p,\Omega} \le K_4 \varepsilon |u|_{j,p,\Omega} + K_4 \varepsilon^{-(j-1)} |u|_{0,p,\Omega_\varepsilon''}. \tag{25}$$

Combining (24) and (25), we get

$$|u|_{j-1,p,\Omega} \le K_5 \varepsilon^{m-(j-1)} |u|_{m,p,\Omega} + K_5 \varepsilon^{-(j-1)} |u|_{0,p,\Omega_\varepsilon},$$

where $K_5 = K_4(K_3+1)$ and $\Omega_\varepsilon = \Omega_\varepsilon' \cup \Omega_\varepsilon''$. Replacing ε by $\varepsilon^{1/(m-j+1)}$, we complete the induction. ▌

Extension Theorems

4.24 Let Ω be a domain in $\mathbb{R}^n$. For given m and p a linear operator E mapping $W^{m,p}(\Omega)$ into $W^{m,p}(\mathbb{R}^n)$ is called a *simple (m,p)-extension operator for Ω* provided there exists a constant $K = K(m,p)$ such that for every $u \in W^{m,p}(\Omega)$ the following conditions hold:

(i) $Eu(x) = u(x)$ a.e. in Ω,
(ii) $\|Eu\|_{m,p,\mathbb{R}^n} \le K \|u\|_{m,p,\Omega}$.

E is called a *strong m-extension operator* for Ω if E is a linear operator mapping functions defined a.e. in Ω into functions defined a.e. in $\mathbb{R}^n$ and if for every p, $1 \le p < \infty$, and every k, $0 \le k \le m$, the restriction of E to $W^{k,p}(\Omega)$ is a simple (k,p)-extension operator for Ω. Finally, E is called a *total extension operator* for Ω provided E is a strong m-extension operator for Ω, for every m.

4.25 The existence of even a simple (m,p)-extension operator for a domain Ω guarantees that $W^{m,p}(\Omega)$ inherits many properties possessed by $W^{m,p}(\mathbb{R}^n)$. For instance, if the imbedding $W^{m,p}(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is known to hold, then the imbedding $W^{m,p}(\Omega) \to L^q(\Omega)$ follows via the chain of inequalities

$$\|u\|_{0,q,\Omega} \le \|Eu\|_{0,q,\mathbb{R}^n} \le K_1 \|Eu\|_{m,p,\mathbb{R}^n} \le K_1 K \|u\|_{m,p,\Omega}.$$

We shall not, however, use this technique to prove the Sobolev imbedding theorem in Chapter V as we shall prove that theorem under rather weaker hypothesis on Ω than are needed to guarantee the existence of an (m,p)-extension operator.

We shall construct extension operators of each of the three types defined above. The method used in two of these constructions is based on successive reflections in smooth boundaries. It is attributed to Lichenstein [35] and later Hestenes [31] and Seeley [61]. The third construction, due to Calderón [14] involves the use of the Calderón–Zygmund theory of singular integrals. It is less transparent than the reflections method and yields a weaker result, but requires much less regularity of the domain Ω. Except for very simple domains all of the constructions require the use of partitions of unity subordinate to open covers of $\operatorname{bdry}\Omega$ chosen in such a way that the functions in the partition have uniformly bounded derivatives. Because of this, domains with bounded boundaries (both exterior domains and bounded domains) are more likely to be easily seen to satisfy the conditions of our extension theorems. Exceptions are half-spaces, quadrants, etc., and smooth images of these.

4.26 THEOREM Let Ω be either

(i) a half-space in $\mathbb{R}^n$, or
(ii) a domain in $\mathbb{R}^n$ having the uniform C^m-regularity property, and also having a bounded boundary.

For any positive integer m there exists a strong m-extension operator E for Ω. Moreover, if α and γ are multi-indices with $|\gamma| \le |\alpha| \le m$, there exists a linear operator $E_{\alpha\gamma}$ continuous from $W^{j,p}(\Omega)$ into $W^{j,p}(\mathbb{R}^n)$ for $1 \le j \le m - |\alpha|$ such that if $u \in W^{|\alpha|,p}(\Omega)$, then

$$D^\alpha Eu(x) = \sum_{|\gamma| \le |\alpha|} E_{\alpha\gamma} D^\gamma u(x) \qquad \text{a.e. in } \mathbb{R}^n. \tag{26}$$

PROOF First let Ω be the half-space $\mathbb{R}_+{}^n = \{x \in \mathbb{R}^n : x_n > 0\}$. For functions u defined a.e. on $\mathbb{R}_+{}^n$ we define extensions Eu and $E_\alpha u$, $|\alpha| \le m$, a.e. on $\mathbb{R}^n$ via

$$\begin{aligned} Eu(x) &= \begin{cases} u(x) & x_n > 0 \\ \sum_{j=1}^{m+1} \lambda_j u(x_1, \dots, x_{n-1}, -jx_n) & \text{if } \ x_n \le 0, \end{cases} \\ E_\alpha u(x) &= \begin{cases} u(x) & \text{if } \ x_n > 0 \\ \sum_{j=1}^{m+1} (-j)^{\alpha_n} \lambda_j u(x_1, \dots, x_{n-1}, -jx_n) & \text{if } \ x_n \le 0, \end{cases} \end{aligned} \tag{27}$$

where the coefficients $\lambda_1, \dots, \lambda_{m+1}$ are the unique solutions of the $(m+1) \times (m+1)$ system of linear equations

$$\sum_{j=1}^{m+1} (-j)^k \lambda_j = 1, \qquad k = 0, 1, \dots, m.$$

If $u \in C^m(\overline{\mathbb{R}_+{}^n})$, then it is readily checked that $Eu \in C^m(\mathbb{R}^n)$ and

$$D^\alpha Eu(x) = E_\alpha D^\alpha u(x), \qquad |\alpha| \le m.$$

Thus

$$\int_{\mathbb{R}^n} |D^\alpha Eu(x)|^p\,dx = \int_{\mathbb{R}_+{}^n} |D^\alpha u(x)|^p\,dx$$
$$+ \int_{\mathbb{R}_-{}^n} \left| \sum_{j=1}^{m+1} (-j)^{\alpha_n}\lambda_j\, D^\alpha u(x_1,\dots,x_{n-1},-jx_n)\right|^p dx$$
$$\le K(m,p,\alpha)\int_{\mathbb{R}_+{}^n} |D^\alpha u(x)|^p\,dx.$$

The above inequality extends, by virtue of Theorem 3.18, to functions $u \in W^{k,p}(\mathbb{R}_+{}^n)$, $m \ge k \ge |\alpha|$. Hence E is a strong m-extension operator for $\mathbb{R}_+{}^n$. Since $D^\beta E_\alpha u(x) = E_{\alpha+\beta}u(x)$, a similar calculation shows that E_α is a strong $(m-|\alpha|)$-extension. Thus the theorem is proved for half-spaces (with $E_{\alpha\alpha} = E_\alpha$, $E_{\alpha\gamma} = 0$ otherwise).

Now suppose Ω is uniformly C^m-regular and has bounded boundary. Then the open cover $\{U_j\}$ of bdry Ω, and the corresponding m-smooth maps Φ_j from U_j onto B, referred to in Section 4.6, are finite collections, say $1 \le j \le N$. Let $Q = \{y \in \mathbb{R}^n : |y'| = (\sum_{j=1}^{n-1} y_j{}^2)^{1/2} < \frac{1}{2}, |y_n| < \sqrt{3}/2\}$. Then

$$\{y \in \mathbb{R}^n : |y| < \tfrac{1}{2}\} \subset Q \subset B = \{y \in \mathbb{R}^n : |y| < 1\}.$$

By condition (i) of Section 4.6 the open sets $\mathscr{V}_j = \Psi_j(Q)$, $1 \le j \le N$, form an open cover of $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \text{bdry}\,\Omega) < \delta\}$ for some $\delta > 0$. There exists an open set $\mathscr{V}_0$ of Ω, bounded away from bdry Ω, such that $\Omega \subset \bigcup_{j=0}^N \mathscr{V}_j$. By Theorem 3.14 we may find infinitely differentiable functions $\omega_0, \omega_1, \dots, \omega_N$ such that $\operatorname{supp}\omega_j \subset \mathscr{V}_j$ and $\sum_{j=0}^N \omega_j(x) = 1$ for all $x \in \Omega$. (Note that $\operatorname{supp}\omega_0$ need not be compact if Ω is unbounded.)

Since Ω is uniformly C^m-regular it has the segment property and so restrictions to Ω of functions in $C_0{}^\infty(\mathbb{R}^n)$ are dense in $W^{k,p}(\Omega)$. If $\phi \in C_0{}^\infty(\mathbb{R}^n)$, then ϕ agrees on Ω with the function $\sum_{j=0}^N \phi_j$, where $\phi_j = \omega_j \cdot \phi \in C_0{}^\infty(\mathscr{V}_j)$.

For $j \ge 1$ and $y \in B$ let $\psi_j(y) = \phi_j(\Psi_j(y))$. Then $\psi_j \in C_0{}^\infty(Q)$. We extend ψ_j to be identically zero outside Q. With E (and E_α) defined as in (27), we have $E\psi_j \in C_0{}^m(Q)$, $E\psi_j = \psi_j$ on $Q_+ = \{y \in Q : y_n > 0\}$, and

$$\|E\psi_j\|_{k,p,Q} \le K_1 \|\psi_j\|_{k,p,Q_+}, \qquad 0 \le k \le m,$$

where K_1 depends on k, m, and p. If $\theta_j(x) = E\psi_j(\Phi_j(x))$, then $\theta_j \in C_0{}^m(\mathscr{V}_j)$ and $\theta_j(x) = \phi_j(x)$ if $x \in \Omega$. It may be checked by induction that if $|\alpha| \le m$, then

$$D^\alpha\theta_j(x) = \sum_{|\beta|\le|\alpha|}\ \sum_{|\gamma|\le|\alpha|} a_{j;\alpha\beta}(x)\left[E_\beta(b_{j;\beta\gamma}\cdot(D^\gamma\phi_j \circ \Psi_j))\right](\Phi_j(x)),$$

where $a_{j;\alpha\beta} \in C^{m-|\alpha|}(\overline{U}_j)$ and $b_{j;\beta\gamma} \in C^{m-|\beta|}(\overline{B})$ depend on the transformations Φ_j and $\Psi_j = \Phi_j^{-1}$ and satisfy

$$\sum_{|\beta|\le|\alpha|} a_{j;\alpha\beta}(x)\,b_{j;\beta\gamma}(\Phi_j(x)) = \begin{cases} 1 & \text{if } \gamma = \alpha \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 3.35 we have for $k \le m$,

$$\|\theta_j\|_{k,p,\mathbb{R}^n} \le K_2 \|E\psi_j\|_{k,p,Q} \le K_1 K_2 \|\psi_j\|_{k,p,Q_+} \le K_3 \|\psi_j\|_{k,p,\Omega},$$

where K_3 may be chosen to be independent of j. The operator $\tilde{E}$ defined by

$$\tilde{E}\phi(x) = \phi_0(x) + \sum_{j=1}^{N} \theta_j(x)$$

clearly satisfies $\tilde{E}\phi(x) = \phi(x)$ if $x \in \Omega$, and

$$\begin{aligned}\|\tilde{E}\phi\|_{k,p,\mathbb{R}^n} &\le \|\phi_0\|_{k,p,\Omega} + K_3 \sum_{j=1}^{N} \|\phi_j\|_{k,p,\Omega} \\ &\le K_4(1+NK_3)\|\phi\|_{k,p,\Omega},\end{aligned} \tag{28}$$

where

$$K_4 = \max_{0\le j\le N} \max_{|\alpha|\le m} \sup_{x\in\mathbb{R}^n} |D^\alpha \omega_j(x)| < \infty.$$

Thus $\tilde{E}$ is a strong m-extension operator for Ω. Also

$$D^\alpha \tilde{E}\phi(x) = \sum_{|\gamma|\le|\alpha|} (E_{\alpha\gamma} D^\gamma \phi)(x),$$

where

$$E_{\alpha\gamma} v(x) = \sum_{j=1}^{N} \sum_{|\beta|\le|\alpha|} a_{j;\alpha\beta}(x)[E_\beta(b_{j;\beta\gamma}\cdot(v\cdot\omega_j)\circ\Psi_j)](\Phi_j(x))$$

if $\alpha \ne \gamma$, and

$$E_{\alpha\alpha} v(x) = (v\cdot\omega_0)(x) + \sum_{j=1}^{N} \sum_{|\beta|\le|\alpha|} a_{j;\alpha\beta}(x)[E_\beta(b_{j;\beta\alpha}\cdot(v\cdot\omega_j)\circ\Psi_j)](\Phi_j(x)).$$

We note that if $x \in \Omega$, $E_{\alpha\gamma} v(x) = 0$ for $\alpha \ne \gamma$ and $E_{\alpha\alpha} v(x) = v(x)$. Clearly $E_{\alpha\gamma}$ is a linear operator. By the differentiability properties of $a_{j;\alpha\beta}$ and $b_{j;\beta\gamma}$, $E_{\alpha\gamma}$ is continuous on $W^{j,p}(\Omega)$ into $W^{j,p}(\mathbb{R}^n)$ for $1 \le j \le m - |\alpha|$. This completes the proof. ∎

The representation (26) for derivatives of extended functions was included in the above theorem because it will be needed when we study fractional-order spaces in Chapter VII.

The reflection technique used in the above proof can be modified to yield a total extension operator for smoothly bounded domains. The proof, due to Seeley [61], is based on the following lemma.

4.27 LEMMA There exists a real sequence $\{a_k\}_{k=0}^{\infty}$ such that for every nonnegative integer n we have

$$\sum_{k=0}^{\infty} 2^{nk} a_k = (-1)^n \tag{29}$$

and

$$\sum_{k=0}^{\infty} 2^{nk} |a_k| < \infty. \tag{30}$$

PROOF For fixed N, let $a_{k,N}$, $k = 0, 1, 2, \dots, N$, be the solution of the system of linear equations

$$\sum_{k=0}^{N} 2^{nk} a_{k,N} = (-1)^n, \qquad n = 0, 1, 2, \dots, N. \tag{31}$$

In terms of the Vandermonde determinant

$$V(x_0, x_1, \dots, x_N) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_N \\ x_0^2 & x_1^2 & \cdots & x_N^2 \\ \vdots & \vdots & & \vdots \\ x_0^N & x_1^N & \cdots & x_N^N \end{vmatrix} = \prod_{\substack{i,j=0 \\ i<j}}^{N} (x_j - x_i)$$

the solution of (31), given by Cramer's rule, is

$$\begin{aligned} a_{k,N} &= \frac{V(1, 2, \dots, 2^{k-1}, -1, 2^{k+1}, \dots, 2^N)}{V(1, 2, \dots, 2^N)} \\ &= \left\{ \prod_{\substack{i,j=0 \\ i,j\neq k \\ i<j}}^{N} (2^j - 2^i) \prod_{i=0}^{k-1} (-1 - 2^i) \prod_{j=k+1}^{N} (2^j + 1) \right\} \left\{ \prod_{\substack{i,j=0 \\ i<j}}^{N} (2^j - 2^i) \right\}^{-1} \\ &= A_k B_{k,N} \end{aligned}$$

where

$$A_k = \prod_{i=0}^{k-1} \frac{1+2^i}{2^i - 2^k}, \qquad B_{k,N} = \prod_{j=k+1}^{N} \frac{1+2^j}{2^j - 2^k},$$

it being understood that $\prod_{i=l}^{m} P_i = 1$ if $l > m$. Now

$$|A_k| \leq \prod_{i=1}^{k-1} \frac{2^{i+1}}{2^{k-1}} = 2^{(3k-k^2)/2}.$$

Also

$$\begin{aligned} \log B_{k,N} &= \sum_{j=k+1}^{N} \log\left(1 + \frac{1+2^k}{2^j - 2^k}\right) \\ &< \sum_{j=k+1}^{N} \frac{1+2^k}{2^j - 2^k} < (1+2^k) \sum_{j=k+1}^{N} \frac{1}{2^{j-1}} < 4, \end{aligned}$$

where we have used the inequality $\log(1+x) < x$ valid for $x > 0$. It follows that the increasing sequence $\{B_{k,N}\}_{N=0}^{\infty}$ converges to a limit $B_k \leq e^4$. Let

$a_k = A_k B_k$ so that

$$|a_k| \le e^4 2^{(3k-k^2)/2}.$$

Then for any n

$$\sum_{k=0}^{\infty} 2^{nk} |a_k| \le e^4 \sum_{k=0}^{\infty} 2^{(2nk+3k-k^2)/2} < \infty.$$

Letting N tend to infinity in (31), we complete the proof. ∎

4.28 THEOREM Let Ω be either

(i) a half-space in $\mathbb{R}^n$, or

(ii) a domain in $\mathbb{R}^n$ having the uniform C^m-regularity property for every m, and also having a bounded boundary.

Then there exists a total extension operator for Ω.

PROOF It is sufficient to prove the theorem for the half-space $\mathbb{R}_+{}^n$; the proof for Ω satisfying (ii) then follows just as in Theorem 4.26.

The restrictions to $\mathbb{R}_+{}^n$ of functions $\phi \in C_0{}^\infty(\mathbb{R}^n)$ being dense in $W^{m,p}(\mathbb{R}_+{}^n)$ for any m and p, we define the extension operator only on such functions. Let f be a real-valued function, infinitely differentiable on $[0, \infty)$ and satisfying $f(t) = 1$ if $0 \le t \le \frac{1}{2}$, $f(t) = 0$ if $t \ge 1$. If $\phi \in C_0{}^\infty(\mathbb{R}^n)$, let

$$E\phi(x) = \begin{cases} \phi(x) & \text{if} \quad x \in \overline{\mathbb{R}_+{}^n} \\ \displaystyle\sum_{k=0}^{\infty} a_k f(-2^k x_n)\,\phi(x', -2^k x_n) & \text{if} \quad x \in \mathbb{R}_-{}^n, \end{cases} \tag{32}$$

where $\{a_k\}$ is the sequence constructed in the above lemma, and $x' = (x_1, \ldots, x_{n-1})$. Then clearly $E\phi$ is well defined on $\mathbb{R}^n$ since the sum in (32) has only finitely many nonvanishing terms for any particular $x \in \mathbb{R}_-{}^n = \{x \in \mathbb{R}^n : x_n < 0\}$. Moreover, $E\phi$ has compact support and belongs to $C^\infty(\overline{\mathbb{R}_+{}^n}) \cap C^\infty(\overline{\mathbb{R}_-{}^n})$. If $x \in \mathbb{R}_-{}^n$, we have

$$\begin{aligned} D^\alpha E\phi(x) &= \sum_{k=0}^{\infty} a_k \sum_{j=0}^{\alpha_n} \binom{\alpha_n}{j} (-2^k)^{\alpha_n} f^{(\alpha_n - j)}(-2^k x_n)\, D_n{}^j D^{\alpha'} \phi(x', -2^k x_n) \\ &= \sum_{k=0}^{\infty} \psi_k(x). \end{aligned}$$

Since $\psi_k(x) = 0$ when $-x_n > 1/2^{k-1}$ it follows from (30) that the above series converges absolutely and uniformly as x_n tends to zero from below. Hence by (29)

$$\begin{aligned} \lim_{x_n \to 0-} D^\alpha E\phi(x) &= \sum_{k=0}^{\infty} (-2^k)^{\alpha_n} a_k D^\alpha \phi(x', 0+) \\ &= D^\alpha \phi(x', 0+) = \lim_{x_n \to 0+} D^\alpha E\phi(x) = D^\alpha E\phi(0). \end{aligned}$$

Thus $E\phi \in C_0^\infty(\mathbb{R}^n)$. Moreover, if $|\alpha| \le m$,

$$|\psi_k(x)|^p \le K_1{}^p |a_k|^p 2^{kmp} \sum_{|\beta| \le m} |D^\beta \phi(x', -2^k x_n)|^p,$$

where K_1 depends on m, p, n, and f. Thus

$$\begin{aligned}\|\psi_k\|_{0,p,\mathbb{R}_-{}^n} &\le K_1 |a_k| 2^{km} \left\{ \sum_{|\beta| \le m} \int_{\mathbb{R}_-{}^n} |D^\beta \phi(x', -2^k x_n)|^p \, dx \right\}^{1/p} \\ &= K_1 |a_k| 2^{km} \left\{ (1/2^k) \sum_{|\beta| \le m} \int_{\mathbb{R}_+{}^n} |D^\beta \phi(y)|^p \, dy \right\}^{1/p} \\ &\le K_1 |a_k| 2^{km} \|\phi\|_{m,p,\mathbb{R}_+{}^n}.\end{aligned}$$

It follows by (30) that

$$\|D^\alpha E\phi\|_{0,p,\mathbb{R}_-{}^n} \le K_1 \|\phi\|_{m,p,\mathbb{R}_+{}^n} \sum_{k=0}^{\infty} 2^{km} |a_k| \le K_2 \|\phi\|_{m,p,\mathbb{R}_+{}^n}.$$

Combining this with a similar (trivial) inequality for $\|D^\alpha E\phi\|_{0,p,\mathbb{R}_+{}^n}$, we obtain

$$\|E\phi\|_{m,p,\mathbb{R}^n} \le K_3 \|\phi\|_{m,p,\mathbb{R}_+{}^n}$$

with $K_3 = K_3(m,p,n)$. This completes the proof. ▌

4.29 The restriction that bdry Ω be bounded was imposed in Theorem 4.26 (and similarly in Theorem 4.28) so that the cover $\{\mathscr{V}_j\}$ would be finite. This finiteness was used in two places in the proof, first in asserting the existence of the constant K_4, and secondly in obtaining the last inequality in (28). This latter use is, however, not essential for the proof, for, were the cover $\{\mathscr{V}_j\}$ not finite, (28) could still be obtained via the finite intersection property [Section 4.6, conidition (ii)]. Theorems 4.26 and 4.28 extend to any suitably regular domains for which there exists a partition of unity $\{\omega_j\}$ subordinate to the cover $\{\mathscr{V}_j\}$ with $D^\alpha \omega_j$ bounded on $\mathbb{R}^n$ uniformly in j for any given α. The reader may find it interesting to construct, by the above techniques, extension operators for domains not covered by the above theorems, for example, quadrants, strips, rectangular boxes, and smooth images of these.

It might be noted here that although the Calderón extension theorem (Theorem 4.32) is proved by methods quite different from the reflection approach used above, nevertheless the proof does make use of a partition of unity in the same way as does that of Theorem 4.26. Accordingly, the above considerations also apply to it. The theorem is proved under a strengthened form of uniform cone condition that reduces to the uniform cone condition of Section 4.4 if Ω has bounded boundary.

Before proceeding to Calderón's theorem we present two well-known results on convolution operators that will be needed for the proof. The first is a special case of a theorem of W. H. Young.

4.30 THEOREM (*Young*) Let $1 \le p < \infty$ and suppose that $u \in L^1(\mathbb{R}^n)$ and $v \in L^p(\mathbb{R}^n)$. Then the convolution products

$$u * v(x) = \int_{\mathbb{R}^n} u(x-y)\,v(y)\,dy, \qquad v * u(x) = \int_{\mathbb{R}^n} v(x-y)\,u(y)\,dy$$

are well defined and equal for almost all $x \in \mathbb{R}^n$. Moreover, $u * v \in L^p(\mathbb{R}^n)$ and

$$\|u * v\|_p \le \|u\|_1 \, \|v\|_p. \tag{33}$$

PROOF The proof is a simple consequence of Fubini's theorem if $p = 1$, so we assume $1 < p < \infty$. Let $w \in L^{p'}(\Omega)$. Then

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} w(x) \int_{\mathbb{R}^n} u(x-y)\,v(y)\,dy\,dx \right| \\
&\quad = \left| \int_{\mathbb{R}^n} w(x) \int_{\mathbb{R}^n} u(y)\,v(x-y)\,dy\,dx \right| \\
&\quad \le \int_{\mathbb{R}^n} |u(y)|\,dy \int_{\mathbb{R}^n} |v(x-y)|\,|w(x)|\,dx \\
&\quad \le \int_{\mathbb{R}^n} |u(y)|\,dy \left\{ \int_{\mathbb{R}^n} |v(x-y)|^p\,dx \right\}^{1/p} \left\{ \int_{\mathbb{R}^n} |w(x)|^{p'}\,dx \right\}^{1/p'} \\
&\quad = \|u\|_1 \, \|v\|_p \, \|w\|_{p'}.
\end{aligned}$$

Since w may be chosen so as to vanish nowhere, it follows that $u * v(x)$ and $v * u(x)$ must be finite a.e. Moreover, the functional

$$F_{u*v}(w) = \int_{\mathbb{R}^n} u * v(x)\,w(x)\,dx$$

belongs to $[L^{p'}(\mathbb{R}^n)]'$ and so by Theorem 2.33 there exists $\lambda \in L^p(\mathbb{R}^n)$ with $\|\lambda\|_p \le \|u\|_1 \, \|v\|_p$ such that

$$\int_{\mathbb{R}^n} \lambda(x)\,w(x)\,dx = \int_{\mathbb{R}^n} u * v(x)\,w(x)\,dx$$

for every $w \in L^{p'}(\mathbb{R}^n)$. Hence $\lambda = u * v \in L^p(\mathbb{R}^n)$ and (33) is proved. The equality of $u * v$ and $v * u$ is elementary. ∎

The following theorem is a special case, suitable for our purposes, of a well-known inequality of Calderón and Zygmund [16] for convolutions involving kernels with nonintegrable singularities. The proof, which is rather lengthy and may be found in many sources (e.g., Stein and Weiss [65]), is omitted here. Neither the inequality nor the extension theorem based on it will be required hereafter in this monograph.

Let $B_R = \{x \in \mathbb{R}^n : |x| \le R\}$, $S_R = \{x \in \mathbb{R}^n : |x| = R\}$, and let $d\sigma_R$ be the area element [Lebesgue $(n-1)$-measure] on S_R. A function g is said to be *homogeneous of degree* μ on $B_R \sim \{0\}$ if $g(tx) = t^\mu g(x)$ for all $x \in B_R \sim \{0\}$ and $0 < t \le 1$.

4.31 THEOREM (*The Calderón–Zygmund inequality*) Let

$$g(x) = G(x)|x|^{-n},$$

where

(i) G is bounded on $\mathbb{R}^n \sim \{0\}$ and has compact support,
(ii) G is homogeneous of degree 0 on $B_R \sim \{0\}$ for some $R > 0$, and
(iii) $\int_{S_R} G(x)\, d\sigma_R = 0$.

If $1 < p < \infty$ and $u \in L^p(\mathbb{R}^n)$, then the principal value convolution integral

$$u * g(x) = \lim_{\varepsilon \to 0+} \int_{\mathbb{R}^n \sim B_\varepsilon} u(x-y)g(y)\, dy$$

exists for almost all $x \in \mathbb{R}^n$, and there exists a constant $K = K(G,p)$ such that for all such u

$$\|u * g\|_p \le K\|u\|_p.$$

Conversely, if G satisfies (i) and (ii) and if $u * g$ exists for all $u \in C_0^\infty(\mathbb{R}^n)$, then G satisfies (iii).

4.32 THEOREM (*The Calderón extension theorem*) Let Ω be a domain in $\mathbb{R}^n$ having the uniform cone property (Section 4.4) modified as follows:

(i) the open cover $\{U_j\}$ of bdry Ω is required to be finite ,and
(ii) the sets U_j are not required to be bounded.

Then for any $m \in \{1, 2, \ldots\}$ and any p, $1 < p < \infty$, there exists a simple (m,p)-extension operator $E = E(m,p)$ for Ω.

PROOF Let $\{U_1, U_2, \ldots, U_N\}$ be the open cover of bdry Ω given by the uniform cone property, and let U_0 be an open subset of Ω bounded away from bdry Ω such that $\Omega \subset \bigcup_{j=0}^N U_j$. [Such U_0 exists by condition (ii), Section 4.4.] Let $\omega_0, \omega_1, \ldots, \omega_N$ be a C^∞-partition of unity for Ω with $\operatorname{supp} \omega_j \subset U_j$. For $1 \le j \le N$ we shall define operators E_j so that if $u \in W^{m,p}(\Omega)$, then $E_j u \in W^{m,p}(\mathbb{R}^n)$ and satisfies

$$E_j u = u \quad \text{in} \quad U_j \cap \Omega,$$

$$\|E_j u\|_{m,p,\mathbb{R}^n} \le K_{m,p,j} \|u\|_{m,p,\Omega}. \tag{34}$$

The desired extension operator is then clearly given by

$$Eu = \omega_0 u + \sum_{j=1}^{N} \omega_j E_j u.$$

We shall write $x \in \mathbb{R}^n$ in the polar coordinate form $x = \rho\sigma$ where $\rho \geq 0$ and σ is a unit vector. Let C_j, the cone associated with U_j in the description of the uniform cone property, have vertex at 0. Let ϕ_j be a function defined in $\mathbb{R}^n \sim \{0\}$ and satisfying

(i) $\phi_j(x) \geq 0$ for all $x \neq 0$,
(ii) $\operatorname{supp} \phi_j \subset -C_j \cup \{0\}$,
(iii) $\phi_j \in C^\infty(\mathbb{R}^n \sim \{0\})$,
(iv) for some $\varepsilon > 0$, ϕ_j is homogeneous of degree $m-n$ in $B_\varepsilon \sim \{0\}$.

Now $\rho^{n-1}\phi_j$ is homogeneous of degree $m-1 \geq 0$ on $B_\varepsilon \sim \{0\}$ and so the function $\psi_j(x) = (\partial/\partial\rho)^m[\rho^{n-1}\phi_j(x)]$ vanishes on $B_\varepsilon \sim \{0\}$. Hence ψ_j, extended to be zero at $x = 0$, belongs to $C_0^\infty(-C_j)$. Define

$$\begin{aligned} E_j u(y) = K_j\bigg\{(-1)^m \int_S \int_0^\infty \phi_j(\rho\sigma)\rho^{n-1}\left(\frac{\partial}{\partial\rho}\right)^m u(y-\rho\sigma)\, d\rho\, d\sigma \\ - \int_S \int_0^\infty \psi_j(\rho\sigma) u(y-\rho\sigma)\, d\rho\, d\sigma\bigg\} \end{aligned} \tag{35}$$

where $\int_S \cdot\, d\sigma$ denotes integration over the unit sphere, and the constant K_j will be determined shortly. If $y \in U_j \cap \Omega$, then, assuming for the moment that $u \in C^\infty(\Omega)$, we have by condition (iii) of Section 4.4 that $u(y-\rho\sigma)$ is infinitely differentiable for $\rho\sigma \in \operatorname{supp}\phi_j$. Now integration by parts m times yields

$$\begin{aligned} &(-1)^m \int_0^\infty \rho^{n-1}\phi_j(\rho\sigma)\left(\frac{\partial}{\partial\rho}\right)^m u(y-\rho\sigma)\, d\rho \\ &= \sum_{k=0}^{m-1} (-1)^{m-k}\left(\frac{\partial}{\partial\rho}\right)^k [\rho^{n-1}\phi_j(\rho\sigma)]\left(\frac{\partial}{\partial\rho}\right)^{m-k-1} u(y-\rho\sigma)\bigg|_{\rho=0}^{\rho=\infty} \\ &\quad + \int_0^\infty \left(\frac{\partial}{\partial\rho}\right)^m [\rho^{n-1}\phi_j(\rho\sigma)]\, u(y-\rho\sigma)\, d\rho \\ &= \left(\frac{\partial}{\partial\rho}\right)^{m-1}[\rho^{n-1}\phi_j(\rho\sigma)]\bigg|_{\rho=0} u(y) + \int_0^\infty \psi_j(\rho\sigma) u(y-\rho\sigma)\, d\rho. \end{aligned}$$

Hence

$$E_j u(y) = K_j u(y) \int_S \left(\frac{\partial}{\partial\rho}\right)^{m-1}[\rho^{n-1}\phi_j(\rho\sigma)]\bigg|_{\rho=0} d\sigma.$$

Since $(\partial/\partial\rho)^{m-1}[\rho^{n-1}\phi_j(\rho\sigma)]$ is homogeneous of degree zero near 0, the above integral does not vanish if ϕ_j is not identically zero. Hence K_j can be chosen so that $E_j u(y) = u(y)$ for $y \in U_j \cap \Omega$ and all $u \in C^\infty(\Omega)$. Since $C^\infty(\Omega)$ is dense in $W^{m,p}(\Omega)$ we have $E_j u(y) = u(y)$ a.e. in $U_j \cap \Omega$ for every $u \in W^{m,p}(\Omega)$. It remains, therefore, to show that (34) holds, that is, that

$$\|D^\alpha E_j u\|_{0,p,\mathbb{R}^n} \le K_\alpha \|u\|_{m,p,\Omega}$$

for any α, $|\alpha| \le m$.

The last integral in (35) is of the form $\theta_j * u(y)$, where $\theta_j(x) = \psi_j(x)|x|^{1-n}$. Since $\theta_j \in L^1(\mathbb{R}^n)$ and has compact support we obtain via Young's theorem 4.30 and a suitable approximation of u by smooth functions,

$$\|D^\alpha(\theta_j * u)\|_{0,p,\mathbb{R}^n} = \|\theta_j * (D^\alpha u)\|_{0,p,\mathbb{R}^n} \le \|\theta_j\|_{0,1,\mathbb{R}^n} \|D^\alpha u\|_{0,p,\Omega}.$$

It now remains to be shown that the first integral in (35) defines a bounded map from $W^{m,p}(\Omega)$ into $W^{m,p}(\mathbb{R}^n)$. Since $(\partial/\partial\rho)^m = \sum_{|\alpha|=m}(m!/\alpha!)\sigma^\alpha D^\alpha$ we obtain

$$\begin{aligned}\int_S \int_0^\infty \phi_j(\rho\sigma)\,\rho^{n-1}\left(\frac{\partial}{\partial\rho}\right)^m u(y-\rho\sigma)\,d\rho\,d\sigma &= \sum_{|\alpha|=m} \frac{m!}{\alpha!}\int_{\mathbb{R}^n} \phi_j(x)\,D_x^\alpha u(y-x)\,\sigma^\alpha\,dx \\ &= \sum_{|\alpha|=m} \xi_\alpha * D^\alpha u,\end{aligned}$$

where $\xi_\alpha = (-1)^{|\alpha|}(m!/\alpha!)\sigma^\alpha\phi_j$ is homogeneous of degree $m-n$ in $B_\varepsilon \sim \{0\}$ and belongs to $C^\infty(\mathbb{R}^n \sim \{0\})$. It is now clearly sufficient to show that for any β, $|\beta| \le m$

$$\|D^\beta(\xi_\alpha * v)\|_{0,p,\mathbb{R}^n} \le K_{\alpha,\beta}\|v\|_{0,p,\Omega}. \tag{36}$$

If $|\beta| \le m-1$, then $D^\beta\xi_\alpha$ is homogeneous of degree not exceeding $1-n$ in $B_\varepsilon \sim \{0\}$ and so belong to $L^1(\mathbb{R}^n)$. Inequality (36) now follows by a Young's theorem argument. Thus we need consider only the case $|\beta| = m$, in which we write $D^\beta = (\partial/\partial x_i)D^\gamma$ for some γ, $|\gamma| = m-1$, and some i, $1 \le i \le n$. Suppose, for the moment, that $v \in C_0^\infty(\Omega)$. Then we may write

$$\begin{aligned}D^\beta(\xi_\alpha * v)(x) = [D^\gamma\xi_\alpha] * \left[\left(\frac{\partial}{\partial x_i}\right)v\right](x) &= \int_{\mathbb{R}^n} D_i v(x-y)\,D^\gamma\xi_\alpha(y)\,dy \\ &= \lim_{\delta\to 0+}\int_{\mathbb{R}^n \sim B_\delta} D_i v(x-y)\,D^\gamma\xi_\alpha(y)\,dy.\end{aligned}$$

We now integrate by parts in the last integral to free v and obtain $D^\beta\xi_\alpha$ under the integral. The integrated term is a surface integral over the sphere S_δ of the product of $v(x-\cdot)$ and a function homogeneous of degree $1-n$ near zero.

This surface integral must therefore tend to $Kv(x)$ as $\delta \to 0+$, for some constant K. Noting that $D_i v(x-y) = -(\partial/\partial y_i)\, v(x-y)$, we now have

$$D^\beta(\xi_\alpha * v)(x) = \lim_{\delta \to 0+} \int_{\mathbb{R}^n} v(x-y)\, D^\beta \xi_\alpha(y)\, dy + Kv(x).$$

Now $D^\beta \xi_\alpha$ is homogeneous of degree $-n$ near the origin and so, by the last assertion of Theorem 4.31, $D^\beta \xi_\alpha$ satisfies all the conditions for the singular kernel g of that theorem. Since $p > 1$ we have for any $v \in L^p(\Omega)$ (regarded as being identically zero outside Ω)

$$\|D^\beta \xi_\alpha * v\|_{0,p,\mathbb{R}^n} \le K_{\alpha,\beta} \|v\|_{0,p,\Omega}.$$

This completes the proof. ∎

V

Imbeddings of $W^{m,p}(\Omega)$

The Sobolev Imbedding Theorem

5.1 It is primarily the imbedding characteristics of Sobolev spaces that render these spaces so useful in analysis, especially in the study of differential and integral operators. The most important of the imbedding properties of the spaces $W^{m,p}(\Omega)$ are usually lumped together in a single theorem referred to as the *Sobolev imbedding theorem*. The core results are due to Sobolev [63] but our statement (Theorem 5.4) includes refinements due to others, in particular to Morrey [47] and Gagliardo [24].

Most of the imbedding results hold for domains Ω in $\mathbb{R}^n$ having the cone property but otherwise unrestricted; some imbeddings however require the strong local Lipschitz property. Specifically no imbedding of $W^{m,p}(\Omega)$ into a space of uniformly continuous functions on Ω is possible under only the cone property, as can be seen by considering the example given in the second paragraph of Section 3.17.

5.2 The Sobolev imbedding theorem asserts the existence of imbeddings of $W^{m,p}(\Omega)$ into spaces of the following types:

(i) $W^{j,q}(\Omega)$, $j \le m$, and in particular $L^q(\Omega)$.

(ii) $C_B{}^j(\Omega) = \{u \in C^j(\Omega) : D^\alpha u \text{ is bounded on } \Omega \text{ for } |\alpha| \le j\}$. This space is larger than $C^j(\overline{\Omega})$ in that its elements need not be uniformly continuous on

Ω. However, $C_B{}^j(\Omega)$ is a Banach space under the norm

$$\|u; C_B{}^j(\Omega)\| = \max_{0 \le |\alpha| \le j} \sup_{x \in \Omega} |D^\alpha u(x)|.$$

(iii) $C^{j,\lambda}(\bar{\Omega})$ (see Section 1.27) and in particular $C^j(\bar{\Omega})$.

(iv) $W^{j,q}(\Omega^k)$, and in particular $L^q(\Omega^k)$. Here Ω^k denotes the intersection of Ω with a k-dimensional plane in $\mathbb{R}^n$, considered as a domain in $\mathbb{R}^k$.

Since elements of $W^{m,p}(\Omega)$ are, strictly speaking, not functions defined everywhere on Ω but rather equivalence classes of such functions defined and equal up to sets of measure zero, we must clarify what is meant by an imbedding of $W^{m,p}(\Omega)$ into a space of type (ii)–(iv). In the case of (ii) or (iii) what is intended is that the "equivalence class" $u \in W^{m,p}(\Omega)$ should contain an element belonging to the continuous function space that is target of the imbedding, and bounded in that space by a constant times $\|u\|_{m,p,\Omega}$. Hence, for example, $W^{m,p}(\Omega) \to C^j(\bar{\Omega})$ means that each $u \in W^{m,p}(\Omega)$ can, when considered as a function, be redefined on a set of zero measure in Ω in such a way that the modified function $\tilde{u}$ [which equals u in $W^{m,p}(\Omega)$] belongs to $C^j(\bar{\Omega})$ and satisfies $\|\tilde{u}; C^j(\bar{\Omega})\| \le K\|u\|_{m,p,\Omega}$ with K independent of u.

Even more care is necessary in interpreting the imbedding $W^{m,p}(\Omega) \to W^{j,q}(\Omega^k)$ where $k < n$. Each element $u \in W^{m,p}(\Omega)$ is, by Theorem 3.16, a limit in that space of a sequence $\{u_n\}$ of functions in $C^\infty(\Omega)$. The functions u_n have traces on Ω^k belonging to $C^\infty(\Omega^k)$. The above imbedding signifies that these traces converge in $W^{j,q}(\Omega^k)$ to a function $\tilde{u}$ satisfying $\|\tilde{u}\|_{j,q,\Omega^k} \le K\|u\|_{m,p,\Omega}$ with K independent of u.

Let us note as a point of interest (though of no use to us later) that the imbedding $W^{m,p}(\Omega) \to W^{j,q}(\Omega)$ is equivalent to the simple containment $W^{m,p}(\Omega) \subset W^{j,q}(\Omega)$. Certainly the former implies the latter. To see the converse suppose $W^{m,p}(\Omega) \subset W^{j,q}(\Omega)$ and let I be the linear operator defined on $W^{m,p}(\Omega)$ into $W^{j,q}(\Omega)$ by $Iu = u$. If $u_n \to u$ in $W^{m,p}(\Omega)$ [and hence in $L^p(\Omega)$] and $Iu_n \to v$ in $W^{j,q}(\Omega)$ [and hence in $L^q(\Omega)$], then, passing to a subsequence if necessary, we have by Corollary 2.11 that $u_n(x) \to u(x)$ a.e. in Ω and $u_n(x) = Iu_n(x) \to v(x)$ a.e. in Ω. Thus $u(x) = v(x)$ a.e. in Ω, that is, $Iu = v$, and I is continuous by the closed graph theorem of functional analysis.

5.3 Let Ω be a domain in $\mathbb{R}^n$ having the cone property specified by a certain finite cone C (see Section 4.3). C may be regarded as the intersection of an infinite cone C^* having the same vertex as C with a ball B centered at that vertex. By the *height* of C we mean the radius of B. By the *opening* of C we mean the surface area [$(n-1)$-measure] of the intersection of C^* with the sphere of unit radius having center at the vertex of C. These geometric parameters are clearly invariant under rigid transformations of C.

In asserting that an imbedding of the form

$$W^{m,p}(\Omega) \to X \tag{1}$$

(where X is a Banach space of functions defined over Ω) holds for Ω having the cone property, it is intended that an *imbedding constant* for (1), that is, a constant K for which the inequality

$$\|u; X\| \leq K\|u\|_{m,p,\Omega}$$

is satisfied for all $u \in W^{m,p}(\Omega)$, can be chosen to depend on Ω only through the dimension n and various such parameters of the cone C which are invariant under rigid motions of C.

5.4 THEOREM (*The Sobolev imbedding theorem*) Let Ω be a domain in $\mathbb{R}^n$ and let Ω^k be the k-dimensional domain obtained by intersecting Ω with a k-dimensional plane in $\mathbb{R}^n$, $1 \leq k \leq n$. (Thus $\Omega^n \equiv \Omega$.) Let j and m be non-negative integers and let p satisfy $1 \leq p < \infty$.

PART I If Ω has the cone property, then there exist the following imbeddings:

CASE A Suppose $mp < n$ and $n-mp < k \leq n$. Then

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega^k), \qquad p \leq q \leq kp/(n-mp), \tag{2}$$

and in particular,

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega), \qquad p \leq q \leq np/(n-mp), \tag{3}$$

or

$$W^{m,p}(\Omega) \to L^q(\Omega), \qquad p \leq q \leq np/(n-mp). \tag{4}$$

Moreover, if $p = 1$, so that $m < n$, imbedding (2) also exists for $k = n-m$.

CASE B Suppose $mp = n$. Then for each k, $1 \leq k \leq n$,

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega^k), \qquad p \leq q < \infty, \tag{5}$$

so that in particular

$$W^{m,p}(\Omega) \to L^q(\Omega), \qquad p \leq q < \infty. \tag{6}$$

Moreover, if $p = 1$ so that $m = n$, imbeddings (5) and (6) exist with $q = \infty$ as well; in fact,

$$W^{j+n,1}(\Omega) \to C_B^j(\Omega). \tag{7}$$

CASE C Suppose $mp > n$. Then

$$W^{j+m,p}(\Omega) \to C_B^j(\Omega). \tag{8}$$

PART II If Ω has the strong local Lipschitz property, then Case C of Part I can be refined as follows:

CASE C′ Suppose $mp > n > (m-1)p$. Then

$$W^{j+m,p}(\Omega) \to C^{j,\lambda}(\overline{\Omega}), \qquad 0 < \lambda \le m - (n/p). \tag{9}$$

CASE C″ Suppose $n = (m-1)p$. Then

$$W^{j+m,p}(\Omega) \to C^{j,\lambda}(\overline{\Omega}), \qquad 0 < \lambda < 1. \tag{10}$$

Also, if $n = m-1$ and $p = 1$, then (10) holds for $\lambda = 1$ as well.

PART III All the conclusions of Parts I and II are valid for *arbitrary* domains provided the W-spaces undergoing imbedding are replaced with the corresponding W_0-spaces.

5.5 REMARKS (1) Imbeddings (2)–(8) are essentially due to Sobolev [62, 63] whose original proof did not, however, cover the cases $q = kp/(n-mp)$ in (2), or $q = np/(n-mp)$ in (3) and (4). Imbeddings (9) and (10) find their origins in the work of Morrey [47].

(2) Imbeddings of type (2) and (5) involving traces of functions on planes of lower dimension can be extended in a reasonable manner to apply to traces on more general smooth manifolds. For example, see Theorem 5.22.

(3) Part III of the theorem is an immediate consequence of Parts I and II applied to $\mathbb{R}^n$ because, by Lemma 3.22, the operator of zero extension of functions outside Ω maps $W_0^{m,p}(\Omega)$ isometrically into $W^{m,p}(\mathbb{R}^n)$.

(4) Suppose that all the conclusions of the imbedding theorem have been proven for $\Omega = \mathbb{R}^n$. It then follows that they must also hold for any domain Ω satisfying the requirements of the Calderón extension theorem 4.32. For example, if $W^{m,p}(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$, and if E is an (m, p)-extension operator for Ω, then for any $u \in W^{m,p}(\Omega)$ we have

$$\|u\|_{0,q,\Omega} \le \|Eu\|_{0,q,\mathbb{R}^n} \le K_1 \|Eu\|_{m,p,\mathbb{R}^n} \le K_1 K_2 \|u\|_{m,p,\Omega}$$

with K_1 and K_2 independent of u. We shall not, however, prove the imbedding theorem by such extension arguments.

(5) It is sufficient to establish each of the imbeddings (2), (3), (5), (7)–(10) for the special case $j = 0$. For example, if $W^{m,p}(\Omega) \to L^q(\Omega)$ has been established, then for any $u \in W^{j+m,p}(\Omega)$ we have $D^\alpha u \in W^{m,p}(\Omega)$ for $|\alpha| \le j$, whence $D^\alpha u \in L^q(\Omega)$; thus $u \in W^{j,q}(\Omega)$; and

$$\begin{aligned}\|u\|_{j,q} &= \Bigl(\sum_{|\alpha|\le j} \|D^\alpha u\|_{0,q}^q\Bigr)^{1/q} \\ &\le K_1 \Bigl(\sum_{|\alpha|\le j} \|D^\alpha u\|_{m,p}^p\Bigr)^{1/p} \le K_2 \|u\|_{j+m,p}.\end{aligned}$$

Accordingly, we will always specialize $j = 0$ in the proofs.

(6) If Ω^k (or Ω) has finite volume, it follows by Theorem 2.8 that imbeddings (2)–(6) hold for $1 \le q < p$ in addition to the values of q asserted in the theorem. It will be shown later (Section 6.38) that no imbedding of the form $W^{m,p}(\Omega) \to L^q(\Omega)$ where $q < p$ is possible unless Ω has finite volume.

Proof of the Imbedding Theorem

5.6 The proof given here is due to Gagliardo [24]. Though it is rather lengthy, the techniques involved are quite elementary, being based on little more than simple calculus combined with astute applications of Hölder's inequality. Moreover, Gagliardo's proof establishes the imbedding theorem in the greatest possible generality and is capable of generalization to produce imbedding results for some domains not having the cone property (see Theorems 5.35–5.37).

The proof is carried out in a chain of auxiliary lemmas. In each such lemma constants $K_1, K_2, \ldots$ appearing in the proof are allowed to depend on the same parameters as the constant K referred to in the statement of the lemma.

5.7 LEMMA Let

$$R = \{x \in \mathbb{R}^n : a_i < x_i < b_i;\ 1 \le i \le n\}$$

and

$$R' = \{x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} : a_i < x_i < b_i;\ 1 \le i \le n-1\}$$

be bounded open rectangles in $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$, respectively. If $a_n < \zeta < b_n$ and $p \ge 1$, then for every $u \in C^\infty(R) \cap W^{1,p}(R)$ we have

$$\|u(\cdot,\zeta)\|_{0,p,R'} \le K\|u\|_{1,p,R} \tag{11}$$

where $K = K(p, b_n - a_n)$. Thus the trace mapping $u \to u(\cdot,\zeta)$ extends to an imbedding of $W^{1,p}(R)$ into $L^p(R_\zeta^{n-1})$, where $R_\zeta^{n-1} = R \cap \{x \in \mathbb{R}^n : x_n = \zeta\}$.

PROOF By Theorem 3.18, $C^\infty(\bar{R})$ is dense in $W^{1,p}(R)$ so we may assume $u \in C^\infty(\bar{R})$. Thus $\int_{R'} |u(x',\cdot)|^p\,dx'$ belongs to $C^\infty([a_n, b_n])$ and by the mean value theorem for integrals we have

$$\|u\|_{0,p,R}^p = \int_{a_n}^{b_n}\left(\int_{R'} |u(x',x_n)|^p\,dx'\right)dx_n = (b_n - a_n)\int_{R'} |u(x',\sigma)|^p\,dx'$$

for some $\sigma \in [a_n, b_n]$. Now

$$\begin{aligned} |u(x',\zeta)|^p &= \left|u(x',\sigma) + \int_\sigma^\zeta D_n u(x',t)\,dt\right|^p \\ &\le 2^{p-1}\left[|u(x',\sigma)|^p + |\zeta-\sigma|^{p-1}\int_\sigma^\zeta |D_n u(x',t)|^p\,dt\right] \end{aligned}$$

by Hölder's inequality. Integration over R' leads to

$$\begin{aligned}\|u(\cdot,\zeta)\|_{0,p,R'}^p &\le 2^{p-1}[\|u(\cdot,\sigma)\|_{0,p,R'}^p + (b_n-a_n)^{p-1}\|D_n u\|_{0,p,R}^p] \\ &\le 2^{p-1}[(b_n-a_n)^{-1}\|u\|_{0,p,R}^p + (b_n-a_n)^{p-1}\|D_n u\|_{0,p,R}^p]\end{aligned}$$

which yields (11) with $K = [2^{p-1}\max((b_n-a_n)^{-1},(b_n-a_n)^{p-1})]^{1/p}$. We note that K depends continuously on b_n-a_n but may tend to infinity if b_n-a_n tends to zero or infinity. ∎

5.8 LEMMA Let R be as in the previous lemma. Then

$$W^{n,1}(R) \to C(\bar{R}).$$

The imbedding constant depends only on n and the dimensions of R.

PROOF Let x be any point of R, and let R' be as in the previous lemma. If $u \in C^\infty(\bar{R})$ and $|\alpha| \le n-1$, we have by that lemma that

$$\|D^\alpha u(\cdot,x_n)\|_{0,1\,R'} \le K_1\|D^\alpha u\|_{1,1,R}.$$

Thus

$$\|u(\cdot,x_n)\|_{n-1,1,R'} \le K_2\|u\|_{n,1,R}$$

with K_2 depending on b_n-a_n. Iteration of this argument over successively lower-diemnsional rectangles leads to

$$\|u(\cdot,x_2,x_3,\ldots,x_n)\|_{1,1,(a_1,b_1)} \le K_3\|u\|_{n,1,R}$$

with K_3 depending on b_j-a_j, $2 \le j \le n$. By the mean value theorem for integrals there exists $\sigma \in [a_1,b_1]$ such that

$$\|u(\cdot,x_2,\ldots,x_n)\|_{0,1,(a_1,b_1)} = (b_1-a_1)|u(\sigma,x_2,\ldots,x_n)|.$$

Hence

$$\begin{aligned}|u(x)| &\le |u(\sigma,x_2,\ldots,x_n)| + \int_\sigma^{x_1}|D_1 u(t,x_2,\ldots,x_n)|\,dt \\ &\le [1/(b_1-a_1)]\|u(\cdot,x_2,\ldots,x_n)\|_{0,1,(a_1,b_1)} \\ &\quad + \|D_1 u(\cdot,x_2,\ldots,x_n)\|_{0,1,(a_1,b_1)} \\ &\le K\|u\|_{n,1,R}. \qquad (12)\end{aligned}$$

Now suppose $u \in W^{n,1}(R)$. By Theorem 3.18, u is the limit in $W^{n,1}(R)$ of a sequence of functions belonging to $C^\infty(\bar{R})$. It follows from (12) that this sequence converges uniformly on $\bar{R}$ to a function $\tilde{u} \in C(\bar{R})$. Since $\tilde{u}(x) = u(x)$ a.e. in R, the lemma is proved. ∎

We now turn our attention to more general domains. The following lemma of Gagliardo, which is essentially combinatorial in nature, is the foundation on which his proof of the imbedding theorem rests.

5.9 LEMMA Let Ω be a domain in $\mathbb{R}^n$ where $n \geq 2$. Let k be an integer satisfying $1 \leq k \leq n$, and let $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n)$ denote a k-tuple of integers satisfying $1 \leq \kappa_1 < \kappa_2 < \cdots < \kappa_k \leq n$. Let S be the set of all $\binom{n}{k}$ such k-tuples. Also, given $x \in \mathbb{R}^n$, let x_κ denote the point $(x_{\kappa_1}, \ldots, x_{\kappa_k}) \in \mathbb{R}^k$; $dx_\kappa = dx_{\kappa_1} \cdots dx_{\kappa_k}$.

For given $\kappa \in S$ let E_κ be the k-dimensional plane in $\mathbb{R}^n$ spanned by the coordinate axes corresponding to the components of x_κ:

$$E_\kappa = \{x \in \mathbb{R}^n : x_i = 0 \text{ if } i \notin \kappa\};$$

and for any set $G \subset \mathbb{R}^n$ let G_κ be the projection of G onto E_κ; in particular

$$\Omega_\kappa = \{x \in E_\kappa : \exists y \in \Omega \text{ such that } y_\kappa = x_\kappa\}.$$

Let F_κ be a function depending on the k components of x_κ and belonging to $L^\lambda(\Omega_\kappa)$, where $\lambda = \binom{n-1}{k-1}$. Then the function F defined on Ω by

$$F(x) = \prod_{\kappa \in S} F_\kappa(x_\kappa)$$

belongs to $L^1(\Omega)$, and $\|F\|_{1,\Omega} \leq \prod_{\kappa \in S} \|F_\kappa\|_{\lambda,\Omega_\kappa}$, that is,

$$\left[\int_\Omega |F(x)|\, dx\right]^\lambda \leq \prod_{\kappa \in S} \int_{\Omega_\kappa} |F_\kappa(x_\kappa)|^\lambda\, dx_\kappa. \tag{13}$$

PROOF For $\kappa \in S$ and $\xi_\kappa \in \mathbb{R}^k$ let $\Omega(\xi_\kappa)$ denote the k-dimensional plane section of Ω by the plane $x_\kappa = \xi_\kappa$:

$$\Omega(\xi_\kappa) = \{x \in \Omega : x_\kappa = \xi_\kappa\}.$$

We establish (13) by induction on n, and so consider first the case $n = 2$. We may also suppose that $k = 1$ since, for any n, the subcase $k = n$ of (13) is trivial. For $n = 2$, $k = 1$, we have $\lambda = 1$ and S has only two elements, $\kappa = 1$ and $\kappa = 2$. Hence

$$\begin{aligned}
\int_\Omega |F_1(x_1) F_2(x_2)|\, dx_1\, dx_2 &= \int_{\Omega_1} dx_1 \int_{\Omega(x_1)} |F_1(x_1) F_2(x_2)|\, dx_2 \\
&= \int_{\Omega_1} |F_1(x_1)|\, dx_1 \int_{(\Omega(x_1))_2} |F_2(x_2)|\, dx_2 \\
&\leq \int_{\Omega_1} |F_1(x_1)|\, dx_1 \int_{\Omega_2} |F_2(x_2)|\, dx_2
\end{aligned}$$

since clearly $(\Omega(x_1))_2 \subset \Omega_2$ for any x_1. This is (13) for the case being considered. (A similar calculation will yield (13) for arbitrary n and $k = 1$.)

Now we assume that (13) has been established for $n = N-1$. We consider the case $n = N$ and, as noted above, may assume $2 \le k \le N-1$. Thus $\lambda = \binom{N-1}{k-1}$. Let $\mu = \binom{N-2}{k-1}$ and $\nu = \binom{N-2}{k-2}$. The integrand on the left side of (13) is a product of $\binom{N}{k}$ factors $|F_\kappa|$ each belonging to the corresponding space $L^\lambda(\Omega_\kappa)$. Exactly $\binom{N-1}{k}$ of these factors, say those corresponding to $\kappa \in A \subset S$, are independent of x_n. It follows from applying the induction hypothesis over the $(N-1)$-dimensional domain $\Omega(x_N)$ and noting that $(\Omega(x_N))_\kappa \subset \Omega_\kappa$ that

$$\int_{\Omega(x_N)} \prod_{\kappa \in A} |F_\kappa(x_\kappa)|^{\lambda/\mu}\, dx_1 \cdots dx_{N-1} \le \prod_{\kappa \in A} \left[\int_{(\Omega(x_N))_\kappa} |F_\kappa(x_\kappa)|^\lambda\, dx_\kappa \right]^{1/\mu}$$
$$\le \prod_{\kappa \in A} \left[\int_{\Omega_\kappa} |F_\kappa(x_\kappa)|^\lambda\, dx_\kappa \right]^{1/\mu}. \tag{14}$$

The remaining $\binom{N}{k} - \binom{N-1}{k} = \lambda$ factors $|F_\kappa|$ depend on x_N, and so when restricted to $\Omega(x_N)$ depend on only $k-1$ variables. Applying the induction hypothesis over $\Omega(x_N)$ again, but this time with $k-1$ in place of k, we obtain

$$\int_{\Omega(x_N)} \prod_{\kappa \in S \sim A} |F_\kappa(x_\kappa)|^{\lambda/\nu}\, dx_1 \cdots dx_{N-1}$$
$$\le \prod_{\kappa \in S \sim A} \left[\int_{(\Omega(x_N))_\kappa} |F_\kappa(x_\kappa)|^\lambda\, dx_{\kappa_1} \cdots dx_{\kappa_{k-1}} \right]^{1/\nu}. \tag{15}$$

Now $\mu + \nu = \lambda$ and so by Hölder's inequality, and (14) and (15),

$$\int_{\Omega(x_N)} \prod_{\kappa \in S} |F_\kappa(x_\kappa)|\, dx_1 \cdots dx_{N-1}$$
$$\le \prod_{\kappa \in A} \left[\int_{\Omega_\kappa} |F_\kappa(x_\kappa)|^\lambda\, dx_\kappa \right]^{1/\lambda}$$
$$\times \prod_{\kappa \in S \sim A} \left[\int_{(\Omega(x_N))_\kappa} |F_\kappa(x_\kappa)|^\lambda\, dx_{\kappa_1} \cdots dx_{\kappa_{k-1}} \right]^{1/\lambda}. \tag{16}$$

Since $S \sim A$ contains λ elements we obtain by (the several function form of) Hölder's inequality that

$$\int_{\Omega_N} \prod_{\kappa \in S \sim A} \left[\int_{(\Omega(x_N))_\kappa} |F_\kappa(x_\kappa)|^\lambda\, dx_{\kappa_1} \cdots dx_{\kappa_{k-1}} \right]^{1/\lambda} dx_N$$
$$\le \prod_{\kappa \in S \sim A} \left[\int_{\Omega_N} \int_{(\Omega(x_N))_\kappa} |F_\kappa(x_\kappa)|^\lambda\, dx_\kappa \right]^{1/\lambda}$$
$$\le \prod_{\kappa \in S \sim A} \left[\int_{\Omega_\kappa} |F_\kappa(x_\kappa)|^\lambda\, dx_\kappa \right]^{1/\lambda}. \tag{17}$$

It follows by insertion of (17) into (16) that

$$\int_\Omega \prod_{\kappa\in S} |F_\kappa(x_\kappa)|\,dx = \int_{\Omega_N} dx_N \int_{\Omega(x_N)} \prod_{\kappa\in S} |F_\kappa(x_\kappa)|\,dx_1\cdots dx_{N-1}$$
$$\le \prod_{\kappa\in S}\left[\int_{\Omega_\kappa} |F_\kappa(x_\kappa)|^\lambda\,dx_\kappa\right]^{1/\lambda},$$

which completes the induction and the proof of (13). ∎

5.10 LEMMA Let Ω be a bounded domain in $\mathbb{R}^n$ having the cone property. If $1 \le p < n$, then $W^{1,p}(\Omega) \to L^q(\Omega)$, where $q = np/(n-p)$. The imbedding constant may be chosen to depend only on m, p, n, and the cone C determining the cone property for Ω.

PROOF We must show that for any $u \in W^{1,p}(\Omega)$,

$$\|u\|_{0,q,\Omega} \le K\|u\|_{1,p,\Omega}, \tag{18}$$

with $K = K(m,p,n,C)$. By Theorem 4.8, Ω may be expressed as a union of finitely many subdomains each of which has the strong local Lipschitz property (and therefore the segment property), and each of which is itself a union of parallel translates of a corresponding parallelepiped. A review of the proof of that theorem shows that the number of subdomains and the dimensions of the corresponding parallelepipeds depend on n and C. It is therefore sufficient to establish (18) for one of these subdomains.

By Theorem 3.35 and a suitable nonsingular linear transformation we may assume that the parallelepiped involved is, in fact, a cube Q having edge length 2 units, and having edges parallel to the coordinate axes. Accordingly we assume hereafter that $\Omega = \bigcup_{x\in A}(x+Q)$ with $A \subset \Omega$, and that Ω has the segment property. By Theorem 3.18 it is sufficient to establish (18) for $u \in C^\infty(\overline{\Omega})$.

For $x \in \Omega$ let $w_i(x)$ denote the intersection of Ω with the straight line through x parallel to the x_i coordinate axis. Clearly, $w_i(x)$ contains a segment of unit length with one endpoint at x, say the segment $x + te_i$, $0 \le t < 1$, where e_i is a unit vector along the x_i-axis.

Let $\gamma = (np-p)/(n-p)$ so that $\gamma \ge 1$. Integration by parts gives, for $u \in C^\infty(\overline{\Omega})$,

$$\int_0^1 |u(x+(1-t)e_i)|^\gamma\,dt$$
$$= |u(x)|^\gamma - \gamma\int_0^1 t\,|u(x+(1-t)e_i)|^{\gamma-1}\frac{d}{dt}|u(x+(1-t)e_i)|\,dt. \tag{19}$$

Let $\hat{x}_i = (x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$ and set

$$F_i(\hat{x}_i) = \sup_{y\in w_i(x)} |u(y)|^{p/(n-p)}.$$

Then (19) gives

$$|F_i(\hat{x}_i)|^{n-1} \le \int_{w_i(x)} |u(x)|^\gamma \, dx_i + \gamma \int_{w_i(x)} |u(x)|^{\gamma-1} |D_i u(x)| \, dx_i. \tag{20}$$

Integration over Ω_i, the projection of Ω onto the plane $x_i = 0$, now leads to

$$\int_{\Omega_i} |F_i(\hat{x}_i)|^{n-1} \, d\hat{x}_i \le \int_\Omega |u(x)|^\gamma \, dx + \gamma \int_\Omega |u(x)|^{\gamma-1} |D_i u(x)| \, dx.$$

If $p > 1$, then $\gamma > 1$ and an application of Hölder's inequality gives

$$\begin{aligned} \|F_i\|_{0,n-1,\Omega_i}^{n-1} &\le \gamma \left[\int_\Omega (|u(x)| + |D_i u(x)|)^p \, dx \right]^{1/p} \left[\int_\Omega |u(x)|^{(\gamma-1)p'} \, dx \right]^{1/p'} \\ &\le 2^{(p-1)/p} \gamma \, \|u\|_{1,p,\Omega} \|u\|_{0,q,\Omega}^{q/p'} \end{aligned}$$

since $(\gamma - 1)p' = q$.

We now apply Lemma 5.9 to the functions F_i, $1 \le i \le n$, noting that $k = n-1$ so that the exponent λ of that lemma is itself $n-1$:

$$\begin{aligned} \|u\|_{0,q,\Omega}^q = \int_\Omega |u(x)|^{np/(n-p)} \, dx &\le \int_\Omega \prod_{i=1}^n F_i(\hat{x}_i) \, dx \le \prod_{i=1}^n \|F_i\|_{0,n-1,\Omega_i} \\ &\le (2^{(p-1)/p} \gamma \, \|u\|_{1,p,\Omega} \|u\|_{0,q,\Omega}^{q/p'})^{n/(n-1)}. \end{aligned}$$

Since $(n-1)q/n - q/p' = 1$, (18) follows by cancellation. The cancellation is justified, for since $u \in C^\infty(\overline{\Omega})$ and Ω is bounded, $\|u\|_{0,q,\Omega}$ is finite. Since $C^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$, (18) extends by continuity to all of $W^{1,p}(\Omega)$. ▌

5.11 REMARK Let $u \in C_0(\mathbb{R}^n)$ and let q, r be as in the above proof. From the identity

$$\int_0^\infty \frac{d}{dt} |u(x + te_i)|^\gamma \, dt = -|u(x)|^\gamma,$$

we obtain

$$\sup_{y \in w_i(x)} |u(y)|^\gamma \le \gamma \int_{-\infty}^\infty |u(x)|^{\gamma-1} |D_i u(x)| \, dx_i,$$

where $w_i(x)$ is the line through x parallel to the x_i-axis. Comparing this with (20), we see that the computations of the above proof can be reproduced to yield in this case

$$\|u\|_{0,q,\mathbb{R}^n} \le K |u|_{1,p,\mathbb{R}^n}, \tag{21}$$

where the seminorm $|\cdot|_{1,p}$ is defined in Section 4.11. Inequality (21) is known as *Sobolev's inequality*.

5.12 LEMMA Let Ω be a bounded domain in $\mathbb{R}^n$ having the cone property. If $mp < n$, then $W^{m,p}(\Omega) \to L^q(\Omega)$ for $p \le q \le np/(n-mp)$. The imbedding constant may be chosen to depend only on m, p, n, q, and the cone C determining the cone property for Ω.

PROOF Let $q_0 = np/(n-mp)$. We first prove by induction on m that $W^{m,p}(\Omega) \to L^{q_0}(\Omega)$. Note that Lemma 5.10 establishes the case $m = 1$.

Assume, therefore, that $W^{m-1,p}(\Omega) \to L^r(\Omega)$ for $r = np/(n-mp+p)$ whenever $n > (m-1)p$. If $u \in W^{m,p}(\Omega)$, where $n > mp$, then u and $D_j u$ $(1 \le j \le n)$ belong to $W^{m-1,p}(\Omega)$. It follows that $u \in W^{1,r}(\Omega)$ and

$$\|u\|_{1,r,\Omega} \le K_1 \|u\|_{m,p,\Omega}.$$

Since $mp < n$, we have $r < n$ and so by Lemma 5.10 we have $W^{1,r}(\Omega) \to L^{q_0}(\Omega)$ where $q_0 = nr/(n-r) = np/(n-mp)$ and

$$\|u\|_{0,q_0,\Omega} \le K_2 \|u\|_{1,r,\Omega} \le K_3 \|u\|_{m,p,\Omega}. \tag{22}$$

This completes the induction.

Now suppose $p \le q \le q_0$. We set

$$s = (q_0-q)p/(q_0-p) \qquad \text{and} \qquad t = p/s = (q_0-p)/(q_0-q)$$

and obtain by Hölder's inequality

$$\begin{aligned}
\|u\|_{0,q,\Omega}^q &= \int_\Omega |u(x)|^s |u(x)|^{q-s}\, dx \\
&\le \left[\int_\Omega |u(x)|^{st}\, dx\right]^{1/t} \left[\int_\Omega |u(x)|^{(q-s)t'}\, dx\right]^{1/t'} \\
&= \|u\|_{0,p,\Omega}^{p/t} \|u\|_{0,q_0,\Omega}^{q_0/t'} \le K_3^{q_0/t'} \|u\|_{m,p,\Omega}^q
\end{aligned} \tag{23}$$

by (22). ∎

5.13 COROLLARY If $mp = n$, then $W^{m,p}(\Omega) \to L^q(\Omega)$ for $p \le q < \infty$. The imbedding constant here may also depend on $\operatorname{vol}\Omega$.

PROOF If $q \ge p' = p/(p-1)$, then $q = ns/(n-ms)$, where $s = pq/(p+q)$ satisfies $1 \le s < p$. By Theorem 2.8, $W^{m,p}(\Omega) \to W^{m,s}(\Omega)$ with imbedding constant dependent on $\operatorname{vol}\Omega$. Since $ms < n$, $W^{m,s}(\Omega) \to L^q(\Omega)$ by Lemma 5.12. If $p \le q \le p'$ the desired imbedding follows by interpolation between $W^{m,p}(\Omega) \to L^p(\Omega)$ and $W^{m,p}(\Omega) \to L^{p'}(\Omega)$ as in (23). ∎

For $mp = n$ and $q \ge p$ the dependence of the imbedding constant on $\operatorname{vol}\Omega$ may be removed as we show in the following lemma which removes the restriction of boundedness of Ω from Lemma 5.12 and Corollary 5.13.

5.14 LEMMA Let Ω be an arbitrary domain in $\mathbb{R}^n$ having the cone property. If $mp < n$, then $W^{m,p}(\Omega) \to L^q(\Omega)$ for $p \le q \le np/(n-mp)$. If $mp = n$, then $W^{m,p}(\Omega) \to L^q(\Omega)$ for $p \le q < \infty$. If $p = 1$ and $m = n$, then $W^{m,p}(\Omega) \to C_B{}^0(\Omega)$. The constants for these imbeddings may depend on m, p, n, q, and the cone C determining the cone property for Ω.

PROOF We tesselate $\mathbb{R}^n$ by cubes of unit side. If $\lambda = (\lambda_1, \ldots, \lambda_n)$ is an n-tuple of integers, let $H = \{x \in \mathbb{R}^n : \lambda_i \le x_i \le \lambda_i + 1;\ 1 \le i \le n\}$. Then $\mathbb{R}^n = \bigcup_\lambda H_\lambda$.

As remarked in the first paragraph of the proof of Theorem 4.8, even an unbounded domain Ω with the cone property can be expressed as a union of finitely many subdomains, say $\Omega = \bigcup_{j=1}^N \Omega_j$, such that $\Omega_j = \bigcup_{x \in A_j}(x + P_j)$, where $A_j \subset \Omega$ and P_j is a parallelepiped with one vertex at the origin. The number N and the dimensions of the parallelepipeds P_j depend on n and the cone C determining the cone property for Ω. For each λ and for $1 \le j \le N$ let

$$\Omega_{\lambda,j} = \bigcup_{x \in A_j \cap H_\lambda} (x + P_j).$$

The domains $\Omega_{\lambda,j}$ evidently possess the following properties:

(i) $\Omega = \bigcup_{\lambda,j} \Omega_{\lambda,j}$;

(ii) $\Omega_{\lambda,j}$ is bounded;

(iii) there exists a finite cone C' depending only on $P_1, \ldots, P_N$ (and hence only on n and C) such that each $\Omega_{\lambda,j}$ has the cone property determined by C';

(iv) there exists a positive integer R depending on n and C such that any $R+1$ of the domains $\Omega_{\lambda,j}$ have empty intersection;

(v) there exist constants K' and K'' depending on n and C such that for each $\Omega_{\lambda,j}$,

$$K' \le \operatorname{vol} \Omega_{\lambda,j} \le K''.$$

Suppose $mp < n$ and let $u \in W^{m,p}(\Omega)$. If $p \le q \le np/(n-mp)$, then by (ii), (iii), and Lemma 5.12, we have

$$\|u\|_{0,q,\Omega_{\lambda,j}} \le K \|u\|_{m,p,\Omega_{\lambda,j}}, \tag{24}$$

where $K = K(m,p,n,q,C)$ is independent of λ and j. Hence by (i) and (iv) and since $q \ge p$

$$\begin{aligned}\|u\|_{0,q,\Omega}^q &\le \sum_{\lambda,j} \|u\|_{0,q,\Omega_{\lambda,j}}^q \le K^q \sum_{\lambda,j} [\|u\|_{m,p,\Omega_{\lambda,j}}^p]^{q/p} \\ &\le K^q \left[\sum_{\lambda,j} \|u\|_{m,p,\Omega_{\lambda,j}}^p\right]^{q/p} \le K^q R^{q/p} \|u\|_{m,p,\Omega}^q.\end{aligned}$$

Thus $W^{m,p}(\Omega) \to L^q(\Omega)$ with imbedding constant $KR^{1/p}$.

If $mp = n$, (24) holds for any q such that $p \le q < \infty$ by virtue of Corollary 5.13, and the constant K can be chosen independent of λ and j thanks to (v). The rest of the above proof then carries over to this case.

Finally, if $p = 1$ and $m = n$, we have by Lemma 5.8 and a nonsingular linear transformation that $W^{n,1}(P) \to C^0(\bar{P})$ for any parallelepiped $P \subset \Omega$, the imbedding constant depending only on n and the dimensions of P. Hence $W^{n,1}(\Omega) \to C_B{}^0(\Omega)$ by virtue of the decomposition $\Omega = \bigcup \Omega_{\lambda,j}$. ∎

We have now proved Part I, Cases A and B of Theorem 5.1 for the case $k = n$. Before completing these cases by considering the trace imbeddinge ($k < n$), we establish the continuous function space imbeddings, Part I, Case C, and Part II.

5.15 LEMMA Let Ω be a domain in $\mathbb{R}^n$ having the cone property. If $mp > n$, then $W^{m,p}(\Omega) \to C_B{}^0(\Omega)$, the imbedding constant depending only on m, p, n, and the cone C determining the cone property for Ω.

PROOF Suppose that we can prove that for any $\phi \in C^\infty(\Omega)$,

$$\sup_{x \in \Omega} |\phi(x)| \leq K \|\phi\|_{m,p,\Omega}, \tag{25}$$

where $K = K(m,p,n,C)$. If $u \in W^{m,p}(\Omega)$, then by Theorem 3.16 there exists a sequence $\{\phi_n\}$ in $C^\infty(\Omega)$ converging to u in norm in $W^{m,p}(\Omega)$. Since $\{\phi_n\}$ is a Cauchy sequence in $W^{m,p}(\Omega)$, (25) implies that $\{\phi_n\}$ converges to a continuous function on Ω. Thus u must coincide a.e. with an element of $C_B{}^0(\Omega)$. It is therefore sufficient to establish (25).

First suppose $m = 1$ so that $p > n$. Let $x \in \Omega$ and let $C_x \subset \Omega$ be a finite cone congruent to C and having vertex at x. Let h be the height of C. Let (r, θ) denote spherical polar coordinates in $\mathbb{R}^n$ with origin at x so that C_x is specified by $0 < r < h$, $\theta \in A$. The volume element in this system is denoted by $r^{n-1}\omega(\theta)\,dr\,d\theta$. We have

$$\phi(x) = \phi(0,\theta) = \phi(r,\theta) - \int_0^r \frac{d}{dt}\phi(t,\theta)\,dt,$$

from which we conclude, for $0 < r < h$,

$$|\phi(x)| \leq |\phi(r,\theta)| + \int_0^h |\operatorname{grad} \phi(t,\theta)|\,dt.$$

Multiplying this inequality by $r^{n-1}\omega(\theta)$ and integrating r over $(0,h)$ and θ over A, we obtain

$$\begin{aligned}
(\operatorname{vol} C_x)|\phi(x)| &\leq \int_{C_x} |\phi(y)|\,dy + \frac{h^n}{n}\int_{C_x} \frac{|\operatorname{grad}\phi(y)|}{|x-y|^{n-1}}\,dy \\
&\leq (\operatorname{vol} C_x)^{1/p'} \|\phi\|_{0,p,C_x} \\
&\quad + \frac{h^n}{n}\|\operatorname{grad}\phi\|_{0,p,C_x} \left| \int_{C_x} |x-y|^{-(n-1)p'}\,dy \right|^{1/p'},
\end{aligned}$$

the last inequality following from two applications of Hölder's inequality. Since $p > n$ we have $(n-1)(1-p') > -1$ and so

$$\int_{C_x} |x-y|^{-(n-1)p'}\,dy = \int_A \omega(\theta)\,d\theta \int_0^h r^{(n-1)(1-p')}\,dr < \infty.$$

Hence

$$|\phi(x)| \le K\|\phi\|_{1,p,C_x} \le K\|\phi\|_{1,p,\Omega}$$

with $K = K(m,p,n,C_x) = K(m,p,n,C)$. Thus (25) is proved for $m = 1$.

If $m > 1$ but $p > n$, we still have

$$|\phi(x)| \le K\|\phi\|_{1,p,C_x} \le K\|\phi\|_{m,p,C_x} \le K\|\phi\|_{m,p,\Omega}.$$

If $p \le n < mp$, there exists an integer j satisfying $1 \le j \le m-1$ such that $jp \le n < (j+1)p$. If $jp < n$, set $r = np/(n-jp)$; if $jp = n$, choose $r > \max(n,p)$. In either case we have by the result proved above and by Lemma 5.14 that

$$|\phi(x)| \le K_1\|\phi\|_{1,r,C_x} \le K_1\|\phi\|_{m-j,r,C_x} \le K\|\phi\|_{m,p,C_x} \le K\|\phi\|_{m,p,\Omega},$$

the constants depending only on m, p, n, and C. This completes the proof. ∎

5.16 COROLLARY If $mp > n$, then $W^{m,p}(\Omega) \to L^q(\Omega)$ for $p \le q \le \infty$. The imbedding constants depend only on m, p, n, q, and the cone C.

PROOF We have already established that

$$\|u\|_{0,\infty,\Omega} = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)| \le K\|u\|_{m,p,\Omega}$$

for all $u \in W^{m,p}(\Omega)$. If $p \le q < \infty$, we have

$$\begin{aligned}\|u\|_{0,q,\Omega}^q &= \int_\Omega |u(x)|^p\,|u(x)|^{q-p}\,dx \\ &\le K^{q-p}\|u\|_{m,p,\Omega}^{q-p}\|u\|_{0,p,\Omega}^p \le K^{q-p}\|u\|_{m,p,\Omega}^q. \quad ∎\end{aligned}$$

5.17 LEMMA Let Ω be a domain in $\mathbb{R}^n$ having the strong local Lipschitz property, and suppose that $mp > n \ge (m-1)p$. Then $W^{m,p}(\Omega) \to C^{0,\lambda}(\bar{\Omega})$ for:

(i) $0 < \lambda \le m - n/p$ if $n > (m-1)p$, or
(ii) $0 < \lambda < 1$ if $n = (m-1)p$, or
(iii) $0 < \lambda \le 1$ if $p = 1$, $n = m-1$.

In particular $W^{m,p}(\Omega) \to C^0(\bar{\Omega})$. The imbedding constants depend on m, p, n and the parameters δ, M specified in the description of the strong local Lipschitz property for Ω (see Section 4.5).

Proof Let $u \in W^{m,p}(\Omega)$. The strong local Lipschitz property implies the cone property so by Lemma 5.15 we may assume that u is continuous on Ω and satisfies

$$\sup_{x\in\Omega}|u(x)| \le K_1\|u\|_{m,p,\Omega}. \tag{26}$$

It is therefore sufficient to establish further that for suitable λ,

$$\sup_{\substack{x,y\in\Omega\\ x\ne y}} \frac{|u(x)-u(y)|}{|x-y|^\lambda} \le K_2\|u\|_{m,p,\Omega}. \tag{27}$$

Since $mp > n \ge (m-1)p$ we have by Lemma 5.14 that $W^{m,p}(\Omega) \to W^{1,r}(\Omega)$ where:

(i) $r = np/(n-mp+p)$ and $1-(n/r) = m-(n/p)$ if $n > (m-1)p$, or
(ii) r is arbitrary, $p < r < \infty$ and $0 < 1-(n/r) < 1$ if $n = (m-1)p$, or
(iii) $r = \infty$, $1-(n/r) = m-(n/p) = 1$ if $p = 1$ and $n = m-1$.

It is therefore sufficient to eastablish (27) for $m = 1$; that is, we wish to prove that if $n < p \le \infty$ and $0 < \lambda \le 1-(n/p)$, then

$$\sup_{\substack{x,y\in\Omega\\ x\ne y}} \frac{|u(x)-u(y)|}{|x-y|^\lambda} \le K_3\|u\|_{1,p,\Omega}. \tag{28}$$

Suppose, for the moment, that Ω is a cube, which we may also assume without loss of generality to have unit edge. For $0 < t < 1$, Ω_t will denote a cube of edge t with faces parallel to those of Ω and such that $\overline{\Omega}_t \subset \Omega$. Let $u \in C^\infty(\Omega)$.

Let $x, y \in \Omega$, $|x-y| = \sigma < 1$. Then there exists a fixed cube Ω_σ with $x, y \in \overline{\Omega}_\sigma \subset \Omega$. If $z \in \Omega_\sigma$, then

$$u(x) = u(z) - \int_0^1 \frac{d}{dt} u(x + t(z-x))\,dt,$$

so that

$$|u(x)-u(z)| \le \sqrt{n}\,\sigma \int_0^1 |\operatorname{grad} u(x + t(z-x))|\,dt.$$

Hence

$$\begin{aligned}\left|u(x) - \frac{1}{\sigma^n}\int_{\Omega_\sigma} u(z)\,dz\right| &\le \left|\frac{1}{\sigma^n}\int_{\Omega_\sigma}(u(x)-u(z))\,dz\right| \\ &\le \frac{\sqrt{n}}{\sigma^{n-1}}\int_{\Omega_\sigma} dz \int_0^1 |\operatorname{grad} u(x + t(z-x))|\,dt\end{aligned}$$

equation continues

$$= \frac{\sqrt{n}}{\sigma^{n-1}} \int_0^1 t^{-n}\, dt \int_{\Omega_{t\sigma}} |\operatorname{grad} u(z)|\, dz$$

$$\le \frac{\sqrt{n}}{\sigma^{n-1}} \|\operatorname{grad} u\|_{0,p,\Omega} \int_0^1 (\operatorname{vol} \Omega_{t\sigma})^{1/p'} t^{-n}\, dt \tag{29}$$

$$\le K_4 \sigma^{1-(n/p)} \|\operatorname{grad} u\|_{0,p,\Omega},$$

where $K_4 = K_4(n,p) = \sqrt{n} \int_0^1 t^{-n/p}\, dt < \infty$. A similar inequality holds with y in place of x and so

$$|u(x) - u(y)| \le 2K_4 |x-y|^{1-(n/p)} \|\operatorname{grad} u\|_{0,p,\Omega}.$$

It follows for $0 < \lambda \le 1-(n/p)$ that (28) holds for Ω a cube, and so, via a nonsingular linear transformation, for Ω a parallelepiped.

Now suppose that Ω has the strong local Lipschitz property. Let δ, M, Ω_δ, U_j, and $\mathscr{V}_j$ be as specified in Section 4.5. There exists a parallelepiped P of diameter δ whose dimensions depend only on δ and M such that to each j there corresponds a parallelepiped P_j congruent to P and having one vertex at the origin, such that for every $x \in \mathscr{V}_j \cap \Omega$ we have $x + P_j \subset \Omega$. Furthermore there exist constants δ_0 and δ_1 depending only on δ and P, with $\delta_0 \le \delta$, such that if $x, y \in \mathscr{V}_j \cap \Omega$ and $|x-y| < \delta_0$, then there exists $z \in (x+P_j) \cap (y+P_j)$ with $|x-z| + |y-z| \le \delta_1 |x-y|$. It follows from application of (28) to $x+P_j$ and $y+P_j$ that if $u \in C^\infty(\Omega)$, then

$$\begin{aligned} |u(x)-u(y)| &\le |u(x)-u(z)| + |u(y)-u(z)| \\ &\le K_5 |x-z|^\lambda \|u\|_{1,p,\Omega} + K_5 |y-z|^\lambda \|u\|_{1,p,\Omega} \\ &\le 2^{1-\lambda} K_5 \delta_1{}^\lambda |x-y|^\lambda \|u\|_{1,p,\Omega}. \end{aligned} \tag{30}$$

Now let $x, y \in \Omega$ be arbitrary. If $|x-y| < \delta_0 \le \delta$ and $x, y \in \Omega_\delta$, then $x, y \in \mathscr{V}_j$ for some j and estimate (30) holds. If $|x-y| < \delta_0$, $x \in \Omega_\delta$, $y \in \Omega \sim \Omega_\delta$, then $x \in \mathscr{V}_j$ for some j and (30) follows by application of (28) to $x+P_j$ and $y+P_j$ again. If $|x-y| < \delta_0$ and $x, y \in \Omega \sim \Omega_\delta$, then (30) follows from application of (28) to $x+P'$, $y+P'$, where P' is any parallelepiped congruent to P and having one vertex at the origin. Finally, if $|x-y| \ge \delta_0$, then we have

$$|u(x)-u(y)| \le |u(x)| + |u(y)| \le K_6 \|u\|_{1,p,\Omega} \le K_6 \delta_0^{-\lambda} |x-y|^\lambda \|u\|_{1,p,\Omega}.$$

This completes the proof of (28) for $u \in C^\infty(\Omega)$, and so by Theorem 3.16, for all continuous u. ∎

We have now completed the proof of all parts of the imbedding theorem 5.4 except the trace imbeddings of Cases A and B (corresponding to $k < n$). For the proof of these we will need the following interpolation result.

5.18 LEMMA Let Q be a cube of edge length l, having edges parallel to the coordinate axes in $\mathbb{R}^n$. If $p > 1$, $q \geq 1$ and $mp - p < n < mp$, then there exists a constant $K = K(p, q, m, n, l)$ such that for every $u \in W^{m,p}(Q)$ we have (a.e. in Q)

$$|u(x)| \leq K \|u\|_{0,q,Q}^{s} \|u\|_{m,p,Q}^{1-s}, \tag{31}$$

where $s = (mp-n)q/[np+(mp-n)q]$.

PROOF It is sufficient to establish (31) for $u \in C^\infty(\bar{Q})$. Since each point of $\bar{Q}$ is a corner point of a cube contained in $\bar{Q}$, having edges parallel to those of Q, and having edge length $l/2$, we may assume without loss of generality that x is itself a corner point of Q, say $Q = \{y \in \mathbb{R}^n : x_i < y_i < x_i + l;\ 1 \leq i \leq n\}$.

By Lemma 5.17 we have for $y \in Q$,

$$|u(x)| - |u(y)| \leq |u(x) - u(y)| \leq K_1 |x-y|^{m-(n/p)} \|u\|_{m,p,Q}. \tag{32}$$

Let $U = \|u\|_{m,p,Q}$, which we may assume to be positive; let $\rho = |x-y|$ and $\zeta = [|u(x)|/K_1 U]^{p/(mp-n)}$. Suppose for the moment that $\zeta \leq l$. We have for $\rho \leq \zeta$,

$$|u(y)| \geq |u(x)| - K_1 U \rho^{m-(n/p)} \geq 0.$$

Raising the above inequality to the power q and integrating y over Q, we obtain

$$\begin{aligned}\int_\Omega |u(y)|^q\,dy &\geq K_2 \int_0^\zeta (|u(x)| - K_1 U \rho^{m-(n/p)})^q \rho^{n-1}\,d\rho \\ &= K_2 \zeta^n |u(x)|^q \int_0^1 (1-\sigma^{m-(n/p)})\,\sigma^{n-1}\,d\sigma \\ &= K_3 |u(x)|^{q+(np/(mp-n))} U^{-np/(mp-n)},\end{aligned}$$

from which (31) follows at once.

If, on the other hand, $\zeta > l$, then from (32) we obtain

$$\begin{aligned}|u(y)| \geq |u(x)| - K_1 U \rho^{m-(n/p)} &\geq |u(x)| - |u(x)|\,(\rho/l)^{m-(n/p)} \\ &\geq 0 \qquad \text{if} \quad \rho \leq l.\end{aligned}$$

If $t > 0$, then

$$\int_Q |u(y)|^t\,dy \geq K_2 \int_0^l |u(x)|^t (1-(\rho/l)^{m-(n/p)})^t \rho^{n-1}\,d\rho = K_4 |u(x)|^t.$$

Set $t = [(mp-n)q+np]/mp$. Then

$$\begin{aligned}|u(x)|^{[(mp-n)q+np]/mp} &\leq (1/K_4) \int_\Omega [|u(y)|^q]^{(mp-n)/mp} [|u(y)|^p]^{n/mp}\,dy \\ &\leq (1/K_4) \|u\|_{0,q,\Omega}^{q(mp-n)/mp} \|u\|_{0,p,\Omega}^{n/m},\end{aligned}$$

by an application of Hölder's inequality. Since $\|u\|_{0,p,\Omega} \le \|u\|_{m,p,\Omega}$, (31) follows at once. ∎

We remark that the above lemma also holds for the case $p = 1$, $m = n$. In this case we have from Lemma 5.14 that $W^{n,1}(\Omega) \to L^{\infty}(\Omega)$ so that $|u(x)| \le K\|u\|_{n,1,Q}$ a.e. in Q, which is (31) in this case.

5.19 LEMMA Let Ω be a domain in $\mathbb{R}^n$ having the cone property, and let Ω^k denote the intersection of Ω with some k-dimensional plane, where $1 \le k \le n$ ($\Omega^n \equiv \Omega$). If $n \ge mp$ and $n - mp < k \le n$, then

$$W^{m,p}(\Omega) \to L^q(\Omega^k) \tag{33}$$

for $p \le q \le kp/(n-mp)$ if $n > mp$, or for $p \le q < \infty$ if $n = mp$. If $p = 1$, $n > m$ and $n - m \le k \le n$, then (33) holds for $1 \le q \le k/(n-m)$.

The imbedding constants depend only on m, p, k, n, q, and the cone C determining the cone property for Ω.

PROOF It is sufficient to establish the above conclusions for Ω bounded, $n > mp$, and $q = kp/(n-mp)$, as extension to the other cases can be carried out in the same manner as was described for the case $k = n$ in Corollary 5.13 and Lemma 5.14. We may also assume, as in Lemma 5.10, that Ω is a union of coordinate cubes of edge 2 units.

Let $\mathbb{R}_0{}^k$ be a k-dimensional coordinate subspace of $\mathbb{R}^n$ on which Ω^k has a one-to-one projection $\Omega_0{}^k$. Suppose, for the moment, that $p > 1$. Let ν be the largest integer less than mp. Then $mp - p < \nu < mp$ and since $n - mp < k$ we have $n - \nu \le k$. (Note that if $p = 1$, the same conclusion holds with $k = n - m$, $\nu = m$.) Let $\mu = \binom{k}{n-\nu}$ and let E_i $(1 \le i \le \mu)$ denote the various coordinate subspaces of $\mathbb{R}_0{}^k$ having dimension $n - \nu$. Let Ω_i denote the projection of $\Omega_0{}^k$ (and hence of Ω^k) onto E_i. Also, for each $x \in \Omega_i$ let $\Omega_{i,x}$ denote the intersection of Ω with the ν-dimensional plane through x perpendicular to E_i. Then $\Omega_{i,x}$ contains a ν-dimensional coordinate cube of unit edge with one vertex at x. By Lemma 5.18, with $q = q_0 = np/(n-mp)$, we have for $u \in C^{\infty}(\Omega)$

$$\sup_{y \in \Omega_{i,x}} |u(y)|^{(n-\nu)p/(n-mp)} \le K_1 \|u\|_{0,q_0,\Omega_{i,x}}^{(mp-\nu)q_0/mp} \|u\|_{m,p,\Omega_{i,x}}^{\nu/m}. \tag{34}$$

Let dx^i and $dx_*{}^i$ denote the volume elements in E_i and the orthogonal complement of E_i, respectively. Integration of (34) over Ω_i leads to

$$\int_{\Omega_i} \sup_{y \in \Omega_{i,x}} |u(y)|^{(n-\nu)p/(n-mp)} \, dx^i$$
$$\le K_1 \int_{\Omega_i} \left[\int_{\Omega_{i,x}} |u(x)|^{q_0} \, dx_*{}^i \right]^{(mp-\nu)/mp}$$

equation continues

$$\times \left[\int_{\Omega_{i,x}} \sum_{|\alpha| \le m} |D^\alpha u(x)|^p \, dx_*^{\,i}\right]^{\nu/mp} dx^i$$

$$\le K_1 \left[\int_\Omega |u(x)|^{q_0} \, dx\right]^{(mp-\nu)/mp} \left[\int_\Omega \sum_{|\alpha| \le m} |D^\alpha u(x)|^p \, dx\right]^{\nu/mp}$$

$$= K_1 \|u\|_{0,q_0,\Omega}^{q_0(mp-\nu)/mp} \|u\|_{m,p,\Omega}^{\nu/m} \tag{35}$$

by Hölder's inequality.

Finally, we apply Lemma 5.9 to the subspaces E_i of $\mathbb{R}_0^{\,k}$. Note that the constant λ of that lemma is here equal to $\binom{k-1}{n-\nu-1}$. Letting $dx^{(k)}$ denote the volume element in $\mathbb{R}_0^{\,k}$ and setting $q = kp/(n-mp)$, we obtain

$$\|u\|_{0,q,\Omega^k}^q \le K_2 \int_{\Omega_0^{\,k}} \prod_{i=1}^{\mu} \sup_{y \in \Omega_{i,x}} |u(y)|^{q/\mu} \, dx^{(k)}$$

$$\le K_2 \prod_{i=1}^{\mu} \left[\int_{\Omega_i} \sup_{y \in \Omega_{i,x}} |u(y)|^{q\lambda/\mu} \, dx^i\right]^{1/\lambda}. \tag{36}$$

Since $q\lambda/\mu = (n-\nu)p/(n-mp)$, it follows from (35) and (36) and from Lemma 5.14 that

$$\|u\|_{0,q,\Omega^k} \le K_3 \prod_{i=1}^{\mu} \|u\|_{0,q_0,\Omega}^{q_0(mp-\nu)/mp\lambda q} \|u\|_{m,p,\Omega}^{\nu/m\lambda q}$$

$$\le K_4 [\|u\|_{m,p,\Omega}^{q_0(mp-\nu)/mp} \|u\|_{m,p,\Omega}^{\nu/m}]^{\mu/\lambda q} = K_4 \|u\|_{m,p,\Omega}.$$

This establishes the desired imbedding. ∎

We have now completed the proof of Theorem 5.4.

Traces of Functions in $W^{m,p}(\Omega)$ on the Boundary of Ω

5.20 Of importance in the study of boundary value problems for differential operators defined on a domain Ω is the determination of spaces of functions defined on the boundary of Ω containing the traces $u|_{\mathrm{bdry}\,\Omega}$ of functions u in $W^{m,p}(\Omega)$. For example, if $W^{m,p}(\Omega) \to C(\bar{\Omega})$, then clearly $u|_{\mathrm{bdry}\,\Omega}$ belongs to $C(\mathrm{bdry}\,\Omega)$. We outline below an L^q-imbedding theorem for such traces which can be obtained as a corollary of Theorem 5.4.

The problem of characterizing the image of $W^{m,p}(\Omega)$ under the operator $u \to u|_{\mathrm{bdry}\,\Omega}$ has been extensively studied by many authors. The solution, which involves Sobolev spaces of fractional order m will be given in Chapter VII (see in particular Theorem 7.53). The approach used in that chapter is due to Lions [37, 38].

5.21 Let Ω be a domain in $\mathbb{R}^n$ having the uniform C^m-regularity property. Thus there exists a locally finite open cover $\{U_j\}$ of bdry Ω, and corresponding m-smooth transformations Ψ_j mapping $B = \{y \in \mathbb{R}^n : |y| < 1\}$ onto U_j such that $U_j \cap \text{bdry}\,\Omega = \Psi_j(B_0)$; $B_0 = \{y \in B : y_n = 0\}$. If f is a function having support in U_j, we may define the integral of f over bdry Ω via

$$\int_{\text{bdry}\,\Omega} f(x)\,d\sigma = \int_{U_j \cap \text{bdry}\,\Omega} f(x)\,d\sigma = \int_{B_0} f \circ \Psi_j(y', 0) J_j(y')\,dy',$$

where $y' = (y_1, \ldots, y_{n-1})$ and if $x = \Psi_j(y)$, then

$$J_j(y') = \left\{ \sum_{k=1}^{n} \left(\frac{\partial(x_1, \ldots, \hat{x}_k, \ldots, x_n)}{\partial(y_1, \ldots, y_{n-1})} \right)^2 \right\}^{1/2} \Bigg|_{y_n = 0}.$$

If f is an arbitrary function defined on bdry Ω, we may set

$$\int_{\text{bdry}\,\Omega} f(x)\,d\sigma = \sum_j \int_{\text{bdry}\,\Omega} f(x)\, v_j(x)\,d\sigma,$$

where $\{v_j\}$ is a partition of unity for bdry Ω subordinate to $\{U_j\}$.

5.22 THEOREM Let Ω be a domain in $\mathbb{R}^n$ having the uniform C^m-regularity property, and suppose there exists a simple (m,p)-extension operator E for Ω. If $mp < n$ and $p \le q \le (n-1)p/(n-mp)$, then

$$W^{m,p}(\Omega) \to L^q(\text{bdry}\,\Omega). \tag{37}$$

If $mp = n$, then (37) holds for $p \le q < \infty$.

PROOF Imbedding (37) should be interpreted in the following sense: If $u \in W^{m,p}(\Omega)$, then Eu has a trace on bdry Ω in the sense described in the final paragraph of Section 5.2, and $\|Eu\|_{0,q,\text{bdry}\,\Omega} \le K \|u\|_{m,p,\Omega}$ with K independent of u. [Note that since $C_0(\mathbb{R}^n)$ is dense in $W^{m,p}(\Omega)$, $\|Eu\|_{0,q,\text{bdry}\,\Omega}$ is independent of the particular extension operator E used.]

It is sufficient to prove the theorem for $mp < n$ and $q = (n-1)p/(n-mp)$. There exists a constant K_1 such that for every $u \in W^{m,p}(\Omega)$

$$\|Eu\|_{m,p,\mathbb{R}^n} \le K_1 \|u\|_{m,p,\Omega}.$$

By the conditions of the uniform C^m-regularity property (Section 4.6), there exists a constant K_2 such that for each j and every $y \in B$, $x = \Psi_j(y) \in U_j$

$$|J_j(y')| \le K_2 \quad \text{and} \quad \left| \frac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots, x_n)} \right| \le K_2.$$

Noting that $0 \le v_j(x) \le 1$ on $\mathbb{R}^n$, and using imbedding (2) of Theorem 5.4

applied over B, we have for $u \in W^{m,p}(\Omega)$,

$$\begin{aligned}\int_{\operatorname{bdry}\Omega} |Eu(x)|^q \, d\sigma &\le \sum_j \int_{U_j \cap \operatorname{bdry}\Omega} |Eu(x)|^q \, d\sigma \\ &\le K_2 \sum_j \|Eu \circ \Psi_j\|_{0,q,B_0}^q \\ &\le K_3 \sum_j \left(\|Eu \circ \Psi_j\|_{m,p,B}^p \right)^{q/p} \\ &\le K_4 \left(\sum_j \|Eu\|_{m,p,U_j}^p \right)^{q/p} \\ &\le K_4 R \|Eu\|_{m,p,\mathbb{R}^n}^q \\ &\le K_5 \|u\|_{m,p,\Omega}^q.\end{aligned}$$

The second last inequality above makes use of the finite intersection property possessed by the cover $\{U_j\}$. The constant K_4 is independent of j since $|D^\alpha \Psi_{j,i}(y)| \le \text{const}$ for all i, j, where $\Psi_j = (\Psi_{j,1}, \dots, \Psi_{j,n})$. This completes the proof. ∎

$W^{m,p}(\Omega)$ as a Banach Algebra

Given functions u and v in $W^{m,p}(\Omega)$, where Ω is a domain in $\mathbb{R}^n$, one cannot in general expect their pointwise product uv to belong to $W^{m,p}(\Omega)$. [$(uv)(x) = u(x)v(x)$ a.e. in Ω.] It is, however, a straightforward application of the Sobolev imbedding theorem to show that this is the case provided $mp > n$ and Ω has the cone property.

5.23 THEOREM Let Ω be a domain in $\mathbb{R}^n$ having the cone property. If $mp > n$, then there exists a constant K^* depending on m, p, n, and the finite cone C determining the cone property for Ω, such that for all $u, v \in W^{m,p}(\Omega)$ the product uv, defined pointwise a.e. in Ω, belongs to $W^{m,p}(\Omega)$ and satisfies

$$\|uv\|_{m,p,\Omega} \le K^* \|u\|_{m,p,\Omega} \|v\|_{m,p,\Omega}. \tag{38}$$

In particular, $W^{m,p}(\Omega)$ is a commutative Banach algebra with respect to pointwise multiplication and the equivalent norm

$$\|u\|_{m,p,\Omega}^* = K^* \|u\|_{m,p,\Omega}.$$

PROOF In order to establish (38) it is sufficient to show that if $|\alpha| \le m$, then

$$\int_\Omega |D^\alpha[u(x)v(x)]|^p \, dx \le K_\alpha \|u\|_{m,p,\Omega}^p \|v\|_{m,p,\Omega}^p,$$

where $K = K(m,p,n,C)$. Let us assume for the moment that $u \in C^\infty(\Omega)$. By Leibniz's rule for distributional derivatives, that is,

$$D^\alpha[uv] = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta u D^{\alpha-\beta} v,$$

it is sufficient to show that for any $\beta \le \alpha$, $|\alpha| \le m$, we have

$$\int_\Omega |D^\beta u(x)\, D^{\alpha-\beta} v(x)|^p\, dx \le K_{\alpha,\beta} \|u\|^p_{m,p,\Omega} \|v\|^p_{m,p,\Omega},$$

where $K_{\alpha,\beta} = K_{\alpha,\beta}(m,p,n,C)$. By the imbedding theorem there exists, for each β with $|\beta| \le m$, a constant $K(\beta) = K(\beta,m,p,n,C)$ such that for any $w \in W^{m,p}(\Omega)$

$$\int_\Omega |D^\beta w(x)|^r\, dx \le K(\beta) \|w\|^r_{m,p,\Omega} \tag{39}$$

provided $(m - |\beta|)p \le n$ and $p \le r \le np/(n - (m - |\beta|)p)$ [or $p \le r < \infty$ if $(m - |\beta|)p = n$], or

$$|D^\beta w(x)| \le K(\beta) \|w\|_{m,p,\Omega} \qquad \text{a.e. in } \Omega$$

provided $(m - |\beta|)p > n$.

Let k be the largest integer such that $(m-k)p > n$. Since $mp > n$ we have that $k \ge 0$. If $|\beta| \le k$, then $(m - |\beta|)p > n$, so

$$\begin{aligned}\int_\Omega |D^\beta u(x)\, D^{\alpha-\beta} v(x)|^p\, dx &\le K(\beta)^p \|u\|^p_{m,p,\Omega} \|D^{\alpha-\beta} v\|^p_{0,p,\Omega} \\ &\le K(\beta)^p \|u\|^p_{m,p,\Omega} \|v\|^p_{m,p,\Omega}.\end{aligned}$$

Similarly, if $|\alpha - \beta| \le k$, then

$$\int_\Omega |D^\beta u(x)\, D^{\alpha-\beta} v(x)|^p\, dx \le K(\alpha-\beta)^p \|u\|^p_{m,p,\Omega} \|v\|^p_{m,p,\Omega}.$$

Now if $|\beta| > k$ and $|\alpha - \beta| > k$, then, in fact, $|\beta| \ge k+1$ and $|\alpha-\beta| \ge k+1$ so that $n \ge (m - |\beta|)p$ and $n \ge (m - |\alpha-\beta|)p$. Moreover,

$$\frac{n-(m-|\beta|)p}{n} + \frac{n-(m-|\alpha-\beta|)p}{n} = 2 - \frac{(2m-|\alpha|)p}{n} < 2 - \frac{mp}{n} < 1.$$

Hence there exist positive numbers r, r' with $(1/r)+(1/r') = 1$ such that

$$p \le rp < \frac{np}{n-(m-|\beta|)p}, \qquad p \le r'p < \frac{np}{n-(m-|\alpha-\beta|)p}.$$

Thus by Hölder's inequality and (39) we have

$$\int_\Omega |D^\beta u(x)\, D^{\alpha-\beta}v(x)|^p\, dx \le \left[\int_\Omega |D^\beta u(x)|^{rp}\, dx\right]^{1/r}\left[\int_\Omega |D^{\alpha-\beta}v(x)|^{r'p}\, dx\right]^{1/r'}$$
$$\le [K(\beta)]^{1/r}[K(\alpha-\beta)]^{1/r'}\|u\|^p_{m,p,\Omega}\|v\|^p_{m,p,\Omega}.$$

This completes the proof of (38) for $u \in C^\infty(\Omega)$, $v \in W^{m,p}(\Omega)$.

If $u \in W^{m,p}(\Omega)$, then by Theorem 3.16 there exists a sequence $\{u_n\}$ of $C^\infty(\Omega)$ functions converging to u in $W^{m,p}(\Omega)$. Then by the above argument $\{u_n v\}$ is a Cauchy sequence in $W^{m,p}(\Omega)$ so converges to an element w of that space. Since $mp > n$, u and v may be assumed continuous and bounded on Ω. Thus

$$\begin{aligned}\|w-uv\|_{0,p,\Omega} &\le \|w-u_n v\|_{0,p,\Omega} + \|(u_n-u)v\|_{0,p,\Omega}\\ &\le \|w-u_n v\|_{0,p,\Omega} + \|v\|_{0,\infty,\Omega}\|u_n-u\|_{0,p,\Omega}\\ &\to 0 \qquad \text{as} \quad n\to\infty.\end{aligned}$$

Hence $w = uv$ in $L^p(\Omega)$ and so $w = uv$ in the sense of distributions. Therefore $w = uv$ in $W^{m,p}(\Omega)$ and

$$\|uv\|_{m,p,\Omega} = \|w\|_{m,p,\Omega} \le \limsup_{n\to\infty}\|u_n v\|_{m,p,\Omega} = \|u\|_{m,p,\Omega}\|v\|_{m,p,\Omega}.$$

This completes the proof of the theorem. ▮

We remark that Banach algebra $W^{m,p}(\Omega)$ has an identity if and only if Ω is bounded, that is, the function $e(x) \equiv 1$ belongs to $W^{m,p}(\Omega)$ if and only if $\operatorname{vol}\Omega < \infty$, but there are no unbounded domains with finite volume having the cone property.

Counterexamples and Nonimbedding Theorems

5.24 Consideration of the statement of the Sobolev imbedding Theorem 5.4 may lead the reader to speculate on several directions of possible generalization. Before exploring the possibility of proving imbedding theorems for domains not satisfying the conditions of Theorem 5.4, we first construct examples showing that in certain respects that theorem gives "best possible" imbedding results for the domains considered, and indeed for any domain.

Let Ω be an arbitrary domain in $\mathbb{R}^n$ and assume, without loss of generality, that the origin belongs to Ω. Let $R > 0$ be such that the closed ball $\overline{B_{2R}}$ is contained in Ω. (Here $B_R = \{x \in \mathbb{R}^n : |x| < R\}$.) In each of the following examples we construct a function $u \in C^\infty(B_{3R} \sim \{0\})$ depending only on $\rho = |x|$. If $f \in C^\infty(0, \infty)$ satisfies $f(t) = 1$ if $t \le R$ while $f(t) = 0$ if $t \ge 2R$, then the function w defined by

$$w(x) = \begin{cases} 0 & \text{if } \rho = |x| \geq 3R \\ f(\rho)u(\rho) & \text{if } 0 < \rho < 3R \end{cases}$$

has compact support in Ω and belongs to $W^{m,p}(\Omega)$ if and only if $u \in W^{m,p}(B_R)$.

5.25 EXAMPLE Let k be an integer such that $1 \leq k \leq n$ and suppose that $mp < n$ and $q > kp/(n-mp)$. We construct u so that $u \in W^{m,p}(B_R)$ but $u \notin L^q(B_R{}^k)$, where $B_R{}^k = \{x \in B_R : x_{k+1} = \cdots = x_n = 0\}$. Hence no imbedding of the type $W^{m,p}(\Omega) \to L^q(\Omega^k)$ is possible if $q > kp/(n-mp)$.

Let $u(x) = \rho^\lambda$, where $\rho = |x|$ and the exponent λ will be specified below. It is readily checked by induction on $|\alpha|$ that

$$D^\alpha u(x) = P_\alpha(x)\rho^{\lambda - 2|\alpha|} \tag{40}$$

where P_α is a polynomial homogeneous of degree $|\alpha|$ in the components of x. Hence $|D^\alpha u(x)| \leq K_\alpha \rho^{\lambda - |\alpha|}$ and

$$\int_{B_R} |D^\alpha u(x)|^p \, dx \leq \text{const} \int_0^R \rho^{(\lambda - |\alpha|)p+n-1} \, dx.$$

Therefore u belongs to $W^{m,p}(B_R)$ provided

$$(\lambda - m)p + n > 0. \tag{41}$$

On the other hand, if $\sigma = (x_1{}^2 + \cdots + x_k{}^2)^{1/2}$, then

$$\int_{B_R{}^k} |u(x)|^q \, dx_1 \cdots dx_k = \text{const} \int_0^R \sigma^{\lambda q + k - 1} \, d\sigma$$

so that $u \notin L^q(B_R{}^k)$ if

$$\lambda q + k < 0. \tag{42}$$

Since $q > kp/(n-mp)$, it is possible to select λ to satisfy both (41) and (42) as required. ∎

Since the function u constructed above is unbounded near 0, no imbedding of the form $W^{m,p}(\Omega) \to C_B{}^0(\Omega)$ is possible if $mp < n$.

5.26 EXAMPLE Suppose $p > 1$ and $mp = n$. We construct $u \in W^{m,p}(B_R)$ so that $u \notin L^\infty(B_R)$. Hence the imbeddings $W^{m,p}(\Omega) \to L^q(\Omega)$, valid for $p \leq q < \infty$ if $mp = n$ and Ω has the cone property, cannot be extended to yield $W^{m,p}(\Omega) \to L^\infty(\Omega)$ or $W^{m,p}(\Omega) \to C_B{}^0(\Omega)$ unless $p = 1$ and $n = m$ (see, however, Theorem 8.25).

Let $u(x) = \log(\log 4R/\rho)$, where $\rho = |x|$. Clearly $u \notin L^\infty(B_R)$. Again it is easily checked by induction that

$$D^\alpha u(x) = \sum_{j=1}^{|\alpha|} P_{\alpha,j}(x)\rho^{-2|\alpha|}[\log(4R/\rho)]^{-j}, \tag{43}$$

where $P_{\alpha,j}$ is a polynomial homogeneous of degree $|\alpha|$ in the components of x. Since $p = n/m$ we have

$$|D^\alpha u(x)|^p \le \sum_{j=1}^{|\alpha|} K_{\alpha,j}\rho^{-|\alpha|n/m}[\log(4R/\rho)]^{-jp},$$

so that

$$\int_{B_R} |D^\alpha u(x)|^p\,dx \le \text{const} \sum_{j=1}^{|\alpha|} \int_0^R [\log(4R/\rho)]^{-jp}\rho^{-|\alpha|n/m+n-1}\,d\rho.$$

The right side of the above inequality is certainly finite if $|\alpha| < m$. If $|\alpha| = m$, we have, setting $\sigma = \log(4R/\rho)$,

$$\int_{B_R} |D^\alpha u(x)|^p\,dx \le \text{const} \sum_{j=1}^{|\alpha|} \int_{\log 4}^{\infty} \sigma^{-jp}\,d\sigma$$

which is finite since $p > 1$. Thus $u \in W^{m,p}(B_R)$. ▌

It is interesting that the function u above is independent of the choice of m and p with $mp = n$.

5.27 EXAMPLE Suppose $mp > n > (m-1)p$, and let $\lambda > m-(n/p)$. We construct $u \in W^{m,p}(B_R)$ such that $u \notin C^{0,\lambda}(\overline{B_R})$. Hence no imbedding of the form $W^{m,p}(\Omega) \to C^{0,\lambda}(\overline{\Omega})$ is possible if $mp > n > (m-1)p$ and $\lambda > m-(n/p)$.

As in Example 5.25 we take $u(x) = \rho^\mu$, $\rho = |x|$. From (41) we have $u \in W^{m,p}(B_R)$ provided $\mu > m-(n/p)$. Now $|u(x)-u(0)|/|x-0|^\lambda = \rho^{\mu-\lambda}$ so that $u \notin C^{0,\lambda}(\overline{B_R})$ when $\mu < \lambda$. Thus u has the required properties if we choose μ to satisfy $m-(n/p) < \mu < \lambda$. ▌

5.28 EXAMPLE Suppose $(m-1)p = n$ and $p > 1$. We construct $u \in W^{m,p}(B_R)$ such that $u \notin C^{0,1}(\overline{B_R})$. Hence the imbedding $W^{m,p}(\Omega) \to C^{0,\lambda}(\overline{\Omega})$, valid for $0 < \lambda < 1$ whenever Ω has the strong local Lipschitz property, cannot be extended to $\lambda = 1$ unless $p = 1$, $m-1 = n$.

Let $u(x) = \rho \log(\log 4R/\rho)$ where $\rho = |x|$. Since

$$|u(x)-u(0)|/|x-0| = \log(\log 4R/\rho) \to \infty \qquad \text{as} \quad x \to 0$$

it is clear that $u \notin C^{0,1}(\overline{B_R})$. Following (40) and (43) we have

$$D^\alpha u(x) = \sum_{j=1}^{|\alpha|} P_{\alpha,j}(x)\rho^{1-2|\alpha|}[\log(4R/\rho)]^{-j},$$

where $P_{\alpha,j}$ is a polynomial homogeneous of degree $|\alpha|$. Hence

$$|D^\alpha u(x)|^p \le \sum_{j=1}^{|\alpha|} K_{\alpha,j} \rho^{p(1-|\alpha|)} [\log(4R/\rho)]^{-jp}.$$

It then follows as in Example 5.26 that $u \in W^{m,p}(B_R)$. ∎

5.29 The above examples show that even for very regular domains there can exist no imbeddings of the types considered in Theorem 5.4, except those explicitly stated there. It remains to be seen whether any imbeddings of these types can exist for irregular domains not having the cone property. We shall show that Theorem 5.4 can be extended, with weakened conclusions, to certain types of such irregular domains, but we first show that no extension is possible if the domain is "too irregular."

An unbounded domain Ω in $\mathbb{R}^n$ may have a smooth boundary and still fail to have the cone property if it becomes narrow at infinity, that is, if

$$\lim_{\substack{|x|\to\infty \\ x \in \Omega}} \operatorname{dist}(x, \operatorname{bdry}\Omega) = 0.$$

The following theorem shows that Parts I and II of Theorem 5.4 fail completely for any such unbounded Ω which has finite volume.

5.30 THEOREM Let Ω be an unbounded domain in $\mathbb{R}^n$ having finite volume, and let $q > p$. Then $W^{m,p}(\Omega)$ is not imbedded in $L^q(\Omega)$.

PROOF We construct a function $u(x)$, depending only on the distance $\rho = |x|$ of x from the origin, whose growth as ρ increases is rapid enough to preclude membership of u in $L^q(\Omega)$ but not so rapid as to prevent $u \in W^{m,p}(\Omega)$.

Without loss of generality we assume $\operatorname{vol}\Omega = 1$. Let $A(\rho)$ denote the surface area [$(n-1)$-measure] of the intersection of Ω with the spherical surface of radius r centered at the origin. Then

$$\int_0^\infty A(\rho)\, d\rho = 1.$$

Let $r_0 = 0$ and define r_n for $n = 1, 2, \ldots$ by

$$\int_{r_n}^\infty A(\rho)\, d\rho = 1/2^n = \int_{r_{n-1}}^{r_n} A(\rho)\, d\rho.$$

Clearly r_n increases to infinity with n. Let $\Delta r_n = r_{n+1} - r_n$ and fix ε such that $0 < \varepsilon < [1/(mp)] - [1/(mq)]$. There must exist an increasing sequence $\{n_j\}_{j=1}^\infty$ such that $\Delta r_{n_j} \ge 2^{-\varepsilon n_j}$, for otherwise $\Delta r_n < 2^{-\varepsilon n}$ for all but possibly finitely many values of n whence we would have $\sum_{n=0}^\infty \Delta r_n < \infty$, a contradiction. For convenience we assume $n_1 \ge 1$ so that $n_j \ge j$ for all j. Let $a_0 = 0$ $a_j = r_{n_j+1}$, and $b_j = r_{n_j}$. Note that $a_{j-1} \le b_j < a_j$ and $a_j - b_j = \Delta r_{n_j} \ge 2^{-\varepsilon n_j}$.

Let f be a nonnegative, infinitely differentiable function on $\mathbb{R}$ having the properties:

(i) $0 \le f(t) \le 1$ for all t,
(ii) $f(t) = 0$ if $t \le 0, f(t) = 1$ if $t \ge 1$,
(iii) $|(d/dt)^k f(t)| \le M$ for all t if $1 \le k \le m$.

If $x \in \Omega$ and $\rho = |x|$, set

$$u(x) = \begin{cases} 2^{n_{j-1}/q} & \text{for } a_{j-1} \le \rho \le b_j \\ 2^{n_{j-1}/q} + (2^{n_j/q} - 2^{n_{j-1}/q}) f\left(\dfrac{\rho - b_j}{a_j - b_j}\right) & \text{for } b_j \le \rho \le a_j. \end{cases}$$

Clearly $u \in C^\infty(\Omega)$. Denoting $\Omega_j = \{x \in \Omega : a_{j-1} \le \rho \le a_j\}$, we have

$$\begin{aligned} \int_{\Omega_j} |u(x)|^p \, dx &= \left\{ \int_{a_{j-1}}^{b_j} + \int_{b_j}^{a_j} \right\} [u(x)]^p A(\rho) \, d\rho \\ &\le 2^{n_{j-1}p/q} \int_{a_{j-1}}^{\infty} A(\rho) \, d\rho + 2^{n_j p/q} \int_{b_j}^{a_j} A(\rho) \, d\rho \\ &= \tfrac{1}{2}[2^{-n_{j-1}(1-p/q)} + 2^{-n_j(1-p/q)}] \le 2^{-(j-1)(1-p/q)}. \end{aligned}$$

Since $p < q$ the above inequality forces

$$\int_\Omega |u(x)|^p \, dx = \sum_{j=1}^{\infty} \int_{\Omega_j} |u(x)|^p \, dt < \infty.$$

Also, if $1 \le k \le m$, we have

$$\begin{aligned} \int_{\Omega_j} \left| \frac{d^k u}{d\rho^k} \right|^p dx &= \int_{b_j}^{a_j} \left| \frac{d^k u}{d\rho^k} \right|^p A(\rho) \, d\rho \\ &\le M^p 2^{n_j p/q} [a_j - b_j]^{-kp} \int_{b_j}^{a_j} A(\rho) \, d\rho \\ &= \tfrac{1}{2} M^p 2^{-n_j(1 - p/q - \varepsilon kp)} \le \tfrac{1}{2} M^p 2^{-Cj}, \end{aligned}$$

where $C = 1 - p/q - \varepsilon kp > 0$ since $\varepsilon < [1/(mp)] - [1/(mq)]$. Hence $D^\alpha u \in L^p(\Omega)$ for $|\alpha| \le m$, that is, $u \in W^{m,p}(\Omega)$. However, $u \notin L^q(\Omega)$, for we have for each j,

$$\begin{aligned} \int_{\Omega_j} |u(x)|^q \, dx &\ge 2^{n_{j-1}} \int_{a_{j-1}}^{a_j} A(\rho) \, d\rho \\ &= 2^{n_{j-1}} [2^{-n_{j-1}-1} - 2^{-n_j - 1}] \ge \tfrac{1}{4}. \end{aligned}$$

Therefore $W^{m,p}(\Omega)$ cannot be imbedded in $L^q(\Omega)$. ∎

The conclusions of the above theorem can be extended (see Section 6.35) to unbounded domains Ω having infinite volume but satisfying

$$\limsup_{N\to\infty} \operatorname{vol}\{x \in \Omega : N \le |x| \le N+1\} = 0.$$

5.31 Parts I and II of Theorem 5.4 also fail completely for domains with sufficiently sharp boundary cusps. If Ω is a domain in $\mathbb{R}^n$ and x_0 is a point on the boundary of Ω, let $B_r = B_r(x_0)$ denote the open ball of radius r about x_0, let $\Omega_r = B_r \cap \Omega$, let $S_r = (\operatorname{bdry} B_r) \cap \Omega$, and let $A(r,\Omega)$ be the surface area [$(n-1)$-measure] of S_r. We shall say that Ω has an *exponential cusp* at $x_0 \in \operatorname{bdry}\Omega$ if for every real number k, we have

$$\lim_{r\to 0+} \frac{A(r,\Omega)}{r^k} = 0. \tag{44}$$

5.32 THEOREM If Ω is a domain in $\mathbb{R}^n$ having an exponential cusp at the point x_0 on $\operatorname{bdry}\Omega$, and if $q > p$, then $W^{m,p}(\Omega)$ is not imbedded in $L^q(\Omega)$.

PROOF We construct $u \in W^{m,p}(\Omega)$ which fails to belong to $L^q(\Omega)$ because it becomes unbounded too rapidly near x_0. Without loss of generality we may assume $x_0 = 0$ so that $r = |x|$. Let $\Omega^* = \{y = x/|x|^2 : x \in \Omega, |x| < 1\}$. It is easily seen that Ω^* is unbounded and has finite volume, and that

$$A(r,\Omega^*) = r^{2(n-1)}A(1/r,\Omega).$$

Let t satisfy $p < t < q$. By Theorem 5.30 there exists a function $\tilde{v} \in C^m(0,\infty)$ such that

(i) $\tilde{v}(r) = 0$ if $0 < r \le 1$,

(ii) $\displaystyle\int_1^\infty |v^{(j)}(r)|^t A(r,\Omega^*)\,dr < \infty$ if $0 \le j \le m$,

(iii) $\displaystyle\int_1^\infty |\tilde{v}(r)|^q A(r,\Omega^*)\,dr = \infty$.

[If $r = |y|$, then $v(y) = \tilde{v}(r)$ defines $v \in W^{m,p}(\Omega^*)$ but $v \notin L^q(\Omega^*)$.] Let $x = y/|y|^2$ so that $\rho = |x| = 1/|y| = 1/r$. Set $\lambda = 2n/q$ and define $u(x) = \tilde{u}(\rho) = r^\lambda \tilde{v}(r) = |y|^\lambda v(y)$. It follows for $|\alpha| = j \le m$ that

$$|D^\alpha u(x)| \le |\tilde{u}^{(j)}(\rho)| \le \sum_{i=1}^{j} c_{ij} r^{\lambda+j+i}\tilde{v}^{(i)}(r),$$

where the coefficients c_{ij} depend only on λ. Now $u(x)$ vanishes for $|x| \ge 1$ and so

$$\int_\Omega |u(x)|^q\,dx = \int_0^1 |\tilde{u}(\rho)|^q A(\rho,\Omega)\,d\rho = \int_1^\infty |\tilde{v}(r)|^q A(r,\Omega^*)\,dr = \infty.$$

On the other hand, if $0 \le |\alpha| = j \le m$, we have

$$\int_\Omega |D^\alpha u(x)|^p\,dx \le \int_0^1 |\tilde{u}^{(j)}(\rho)|^p A(\rho,\Omega)\,d\rho$$

$$\le \text{const} \sum_{i=0}^{j} \int_1^\infty |\tilde{v}^{(i)}(r)|^p r^{(\lambda+j+i)p-2n} A(r,\Omega^*)\,dr. \tag{45}$$

If it happens that $(\lambda+2m)p \le 2n$, then, since $p < t$ and $\operatorname{vol}\Omega^* < \infty$, all the integrals in (45) are finite by Hölder's inequality and so $u \in W^{m,p}(\Omega)$. Otherwise let

$$k = [(\lambda+2m)p - 2n][t/(t-p)] + 2n.$$

By (44) there exists $a \le 1$ such that if $\rho \le a$, then $A(\rho,\Omega) \le \rho^k$. It follows that if $r \ge 1/a$, then

$$r^{k-2n} A(r,\Omega^*) \le r^{k-2}\rho^k = r^{-2}.$$

Thus

$$\int_1^\infty |\tilde{v}^{(i)}(r)|^p r^{(\lambda+j+i)p-2n} A(r,\Omega^*)\,dr$$

$$= \int_1^\infty |\tilde{v}^{(i)}(r)|^p r^{(k-2n)(t-p)/t} A(r,\Omega^*)\,dr$$

$$\le \left\{\int_1^\infty |\tilde{v}^{(i)}(r)|^t A(r,\Omega^*)\,dr\right\}^{p/t} \left\{\int_1^\infty r^{k-2n} A(r,\Omega^*)\,dr\right\}^{(t-p)/t}$$

which is finite. Hence $u \in W^{m,p}(\Omega)$ and the proof is complete. ▌

Imbedding Theorems for Domains with Cusps

5.33 Having proved that Theorem 5.4 fails completely for domains that are sufficiently irregular we now show that certain imbeddings of the types considered in that theorem do hold for less irregular domains that however fail to have the cone property. Questions of this sort have been considered by several writers—see, for instance, the work of Globenko [26, 27] and Maz'ja [44, 45]. The treatment given below follows that given in one of the author's papers [1].

We consider domains $\Omega \subset \mathbb{R}^n$ whose boundaries consist only of $(n-1)$-dimensional surfaces, and it is assumed that Ω lies on only one side of its boundary. Ω is said to have a *cusp* at the point $x_0 \in \operatorname{bdry}\Omega$ if no finite open cone of positive volume contained in Ω can have vertex at x_0. The failure of a domain Ω to have any cusps does not, of course, imply that Ω has the cone property. We consider, for the moment, a family of special domains that we call *standard cusps* and that have cusps of power sharpness (less sharp than exponential cusps).

5.34 If $1 \le k \le n-1$ and $\lambda > 1$, let $Q_{k,\lambda}$ denote the standard cusp in E_n specified by the inequalities

$$\begin{gathered} {x_1}^2 + \cdots + {x_k}^2 < x_{k+1}^{2\lambda}, \quad x_{k+1} > 0, \ldots, x_n > 0, \\ ({x_1}^2 + \cdots + {x_k}^2)^{1/\lambda} + x_{k+1}^2 + \cdots + {x_n}^2 < a^2, \end{gathered} \tag{46}$$

where a is the radius of the ball of unit volume in $\mathbb{R}^n$. We note that $a < 1$, $Q_{k,\lambda}$ has axial plane spanned by the $x_{k+1}, \ldots, x_n$ axes, and verticial plane (cusp plane) spanned by $x_{k+2}, \ldots, x_n$. If $k = n-1$, the origin is the only vertex point of $Q_{k,\lambda}$. The outer boundary surface of $Q_{k,\lambda}$ is taken to be of the form (46) in order to simplify calculations. A sphere, or other suitable surface bounded away from the origin, could be used instead.

Corresponding to the standard cusp $Q_{k,\lambda}$ we consider the associated *standard cone* $Q_k = Q_{k,1}$ specified in terms of Cartesian coordinates $y_1, \ldots, y_n$ by

$$\begin{gathered} {y_1}^2 + \cdots + {y_k}^2 < y_{k+1}^2, \quad y_{k+1} > 0, \ldots, y_n > 0, \\ {y_1}^2 + \cdots + {y_n}^2 < a^2. \end{gathered}$$

Figure 4 illustrates standard cusps and their associated standard cones in $\mathbb{R}^2$ and $\mathbb{R}^3$. In $\mathbb{R}^3$ the cusp $Q_{2,2}$ has a single cusp point at the origin, while $Q_{1,2}$ has a cusp line along the x_3-axis.

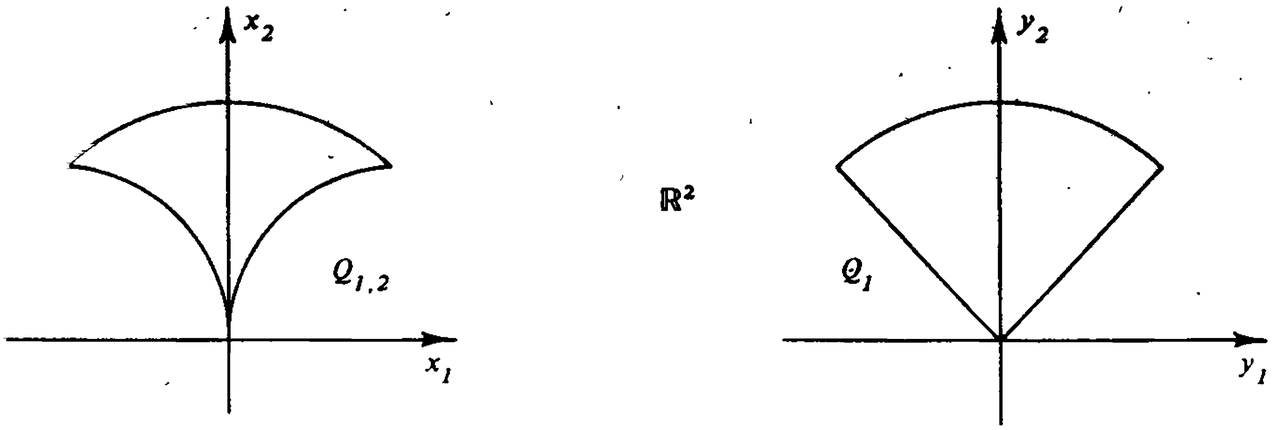
It is convenient to adopt generalized "cylindrical" coordinates $(r_k, \phi_1, \ldots, \phi_{k-1}, y_{k+1}, \ldots, y_n)$ in E_n so that $r_k \ge 0$, $-\pi \le \phi_1 \le \pi$, $0 \le \phi_2, \ldots, \phi_{k-1} \le \pi$, and

$$\begin{aligned} y_1 &= r_k \sin\phi_1 \sin\phi_2 \cdots \sin\phi_{k-1}, \\ y_2 &= r_k \cos\phi_1 \sin\phi_2 \cdots \sin\phi_{k-1}, \\ y_3 &= \qquad\quad r_k \cos\phi_2 \cdots \sin\phi_{k-1}, \\ &\vdots \\ y_k &= \qquad\qquad\qquad r_k \cos\phi_{k-1}. \end{aligned} \tag{47}$$

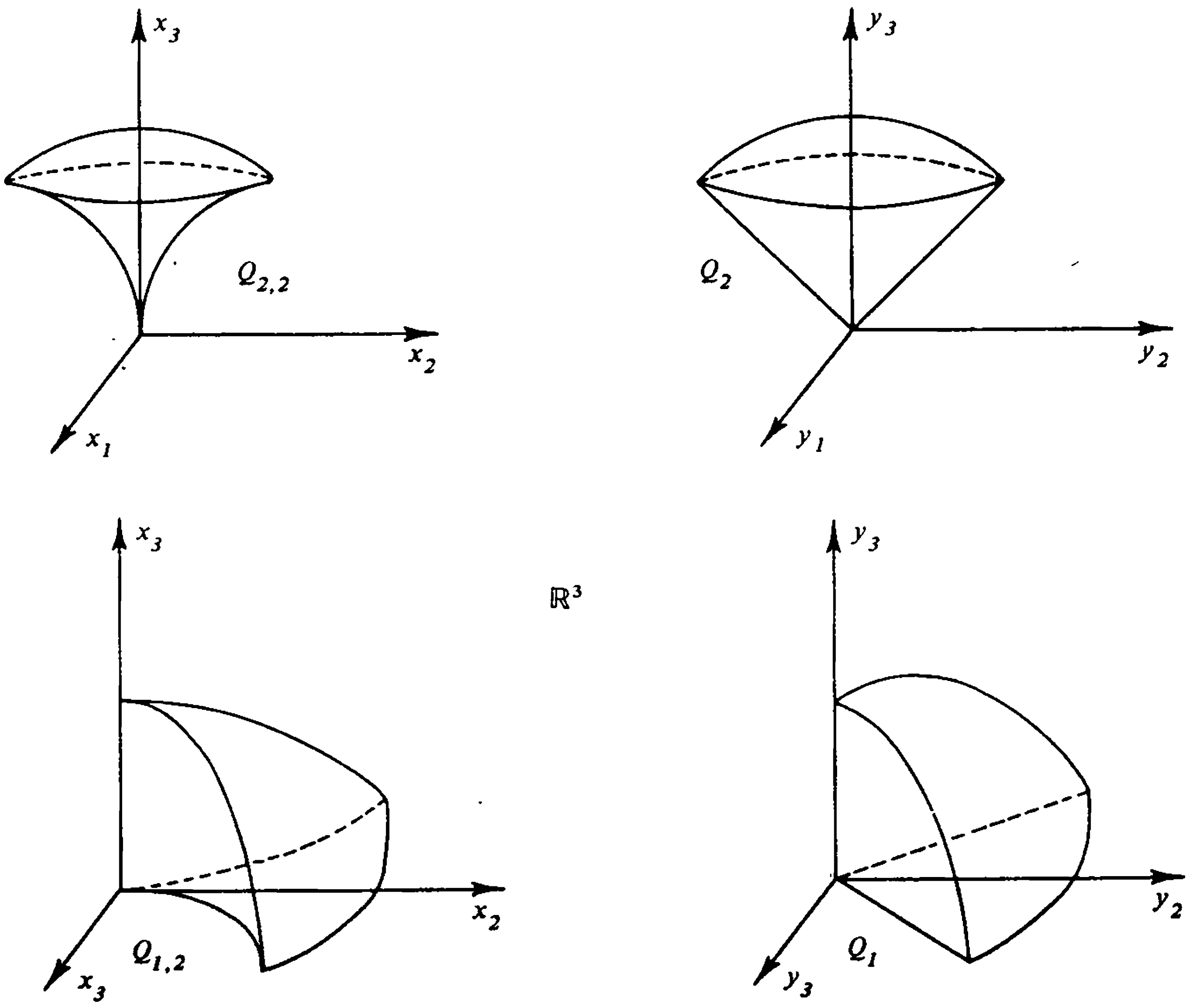
In terms of these coordinates Q_k is represented by

$$\begin{gathered} 0 \le r_k < y_{k+1}, \quad y_{k+1} > 0, \ldots, y_n > 0, \\ {r_k}^2 + y_{k+1}^2 + \cdots + {y_n}^2 < a^2. \end{gathered}$$

The standard cusp $Q_{k,\lambda}$ may be transformed into the associated cone Q_k by



(a)



(b)

FIG. 4 Standard cusps and cones in (a) $\mathbb{R}^2$ and (b) $\mathbb{R}^3$.

means of the one-to-one transformation

$$\begin{aligned}
x_1 &= r_k^{\lambda} \sin\phi_1 \sin\phi_2 \cdots \sin\phi_{k-1}, \\
x_2 &= r_k^{\lambda} \cos\phi_1 \sin\phi_2 \cdots \sin\phi_{k-1}, \\
x_3 &= \quad r_k^{\lambda} \cos\phi_2 \cdots \sin\phi_{k-1}, \\
&\vdots \\
x_k &= \qquad r_k^{\lambda} \cos\phi_{k-1}, \\
x_{k+1} &= y_{k+1}, \\
&\vdots \\
x_n &= y_n,
\end{aligned} \tag{48}$$

which has Jacobian determinant

$$\left| \frac{\partial(x_1, \ldots, x_n)}{\partial(y_1, \ldots, y_n)} \right| = \lambda r_k^{(\lambda-1)k}. \tag{49}$$

We now state three theorems extending imbeddings of the types considered in Theorem 5.4 (except trace imbeddings) to domains with boundary irregularities comparable to standard cusps. The proof of these theorems will be given later in this chapter.

5.35 THEOREM Let Ω be a domain in $\mathbb{R}^n$ having the following property: There exists a family Γ of open subsets of Ω such that

(i) $\Omega = \bigcup_{G \in \Gamma} G$;

(ii) Γ has the finite intersection property, that is, there exists a positive integer N such that any $N+1$ distinct sets in Γ have empty intersection;

(iii) at most one set $G \in \Gamma$ has the cone property;

(iv) there exist positive constants $\nu > mp-n$ and A such that for any $G \in \Gamma$ not having the cone property there exists a one to one function ψ mapping G onto a standard cusp $Q_{k,\lambda}$, where $(\lambda-1)k \le \nu$ and such that for all i, j $(1 \le i, j \le n)$, all $x \in G$, and all $y \in Q_{k,\lambda}$,

$$\left| \frac{\partial \psi_j}{\partial x_i} \right| \le A \qquad \text{and} \qquad \left| \frac{\partial (\psi^{-1})_j}{\partial y_i} \right| \le A.$$

Then

$$W^{m,p}(\Omega) \to L^q(\Omega), \qquad p \le q \le \frac{(\nu+n)p}{\nu+n-mp}. \tag{50}$$

[If $\nu = mp-n$, (50) holds for $p \le q < \infty$ (and $q = \infty$ if $p = 1$). If $\nu < mp-n$, (50) holds for $p \le q \le \infty$.]

5.36 THEOREM Let Ω be a domain in $\mathbb{R}^n$ having the following property: There exist positive constants $\nu < mp-n$ and A such that for each $x \in \Omega$ there exists an open set G with $x \in G \subset \Omega$ and a one-to-one mapping ψ of G onto a standard cusp $Q_{k,\lambda}$ with $(\lambda-1)k \leq \nu$ and such that for all i,j $(1 \leq i,j \leq n)$, all $x \in G$, and all $y \in Q_{k,\lambda}$,

$$\left|\frac{\partial \psi_j}{\partial x_i}\right| \leq A \qquad \text{and} \qquad \left|\frac{\partial (\psi^{-1})_j}{\partial y_i}\right| \leq A.$$

Then

$$W^{m,p}(\Omega) \to C_B^{\,0}(\Omega). \tag{51}$$

More generally, if $\nu < (m-j)p-n$, where $0 \leq j \leq m-1$, then

$$W^{m,p}(\Omega) \to C_B^{\,j}(\Omega).$$

5.37 THEOREM Let Ω be a domain in $\mathbb{R}^n$ with the following property: There exist positive constants ν, δ, and A such that for each pair of points $x, y \in \Omega$ with $|x-y| \leq \delta$ there exists an open set G with $x, y \in G \subset \Omega$, and a one to one mapping ψ of G onto some standard cusp $Q_{k,\lambda}$ with $(\lambda-1)k \leq \nu$, and such that for all i,j $(1 \leq i,j \leq n)$, all $x \in G$, and all $y \in Q_{k,\lambda}$

$$\left|\frac{\partial \psi_j}{\partial x_i}\right| \leq A \qquad \text{and} \qquad \left|\frac{\partial (\psi^{-1})_j}{\partial y_i}\right| \leq A.$$

Suppose that for some j with $0 \leq j \leq m-1$ we have $(m-j-1)p < \nu+n < (m-j)p$. Then

$$W^{m,p}(\Omega) \to C^{j,\mu}(\overline{\Omega}), \qquad 0 < \mu \leq m-j-[(n+\nu)/p]. \tag{52}$$

If $(m-j-1)p = \nu+n$, then (52) holds with $0 < \mu < 1$. In either event we have $W^{m,p}(\Omega) \to C^j(\overline{\Omega})$.

5.38 REMARKS (1) The reader may wish to construct examples similar to those of Sections 5.25–5.28 to show that the three theorems above give the best possible imbeddings for the domains considered.

(2) The following example may help to illustrate Theorem 5.35: Let $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 > 0,\ x_2^{\,2} < x_1 < 3x_2^{\,2}\}$. Setting $a = (3/4\pi)^{1/3}$, the radius of the ball of unit volume in $\mathbb{R}^3$, we may readily verify that the transformation

$$y_1 = x_1 + 2x_2^{\,2}, \qquad y_2 = x_2, \qquad y_3 = x_3 - (k/a), \qquad k = 0, \pm 1, \pm 2, \ldots,$$

transforms a subdomain G_k of Ω onto the standard cusp $\Omega_{1,2}$ in the manner required of the functions ψ in the statement of Theorem 5.35. Moreover, $\{G_k\}_{k=-\infty}^{\infty}$ has the finite intersection property and covers Ω up to a set with the cone property. Hence $W^{m,p}(\Omega) \to L^q(\Omega)$ for $p \le q \le 4p/(4-mp)$ if $mp < 4$, or for $p \le q < \infty$ if $mp = 4$, or $p \le q \le \infty$ if $mp > 4$.

Imbedding Inequalities Involving Weighted Norms

5.39 The technique of mapping a standard cusp onto its associated standard cone via (47) and (48) is central to the proof of Theorem 5.35. Such a transformation introduces into any integrals involved a weight factor in the form of the Jacobian determinant (49). Accordingly, we must obtain imbedding inequalities for such standard cones, corresponding to L^p-norms weighted by powers of the distance from the axial plane of the cone. Such inequalities are also useful in extending the imbedding theorem 5.4 to more general Sobolev spaces involving weighted norms.

We begin with some one-dimensional inequalities for functions continuously differentiable on a fixed open interval $(0, T)$ in $\mathbb{R}$.

5.40 LEMMA If $\nu > 0$ and $u \in C^1(0, T)$, and if $\int_0^T |u'(t)| t^\nu \, dt < \infty$, then $\lim_{t \to 0+} |u(t)| t^\nu = 0$.

PROOF Let $\varepsilon > 0$ be given and fix s, $0 < s < T/2$, small enough so that for any t, $0 < t < s$, we have

$$\int_t^s |u'(\tau)| \tau^\nu \, d\tau < \varepsilon/3.$$

Now there exists δ, $0 < \delta < s$, such that

$$\delta^\nu |u'(T/2)| < \varepsilon/3 \qquad \text{and} \qquad (\delta/s)^\nu \int_s^{T/2} |u'(\tau)| \tau^\nu \, d\tau < \varepsilon/3.$$

If $0 < t \le \delta$, we have

$$|u(t)| \le |u(T/2)| + \int_t^{T/2} |u'(\tau)| \, d\tau$$

so that

$$t^\nu |u(t)| \le \delta^\nu |u(T/2)| + \int_t^s |u'(\tau)| \tau^\nu \, d\tau + (\delta/s)^\nu \int_s^{T/2} |u'(\tau)| \tau^\nu \, d\tau < \varepsilon.$$

Hence $\lim_{t \to 0+} t^\nu |u(t)| = 0$. ∎

5.41 LEMMA If $\nu > 0$, $p \ge 1$, and $u \in C^1(0, T)$, then

$$\int_0^T |u(t)|^p t^{\nu-1} \, dt \le \frac{\nu+1}{\nu T} \int_0^T |\dot{u}(t)|^p t^\nu \, dt + \frac{p}{\nu} \int_0^T |u(t)|^{p-1} |u'(t)| t^\nu \, dt. \quad (53)$$

PROOF We may assume without loss of generality that the right side of (53) is finite, and that $p = 1$. Integration by parts gives

$$\int_0^T |u(t)|\left[\nu t^{\nu-1} - \frac{\nu+1}{T} t^\nu\right] dt = -\int_0^T \left[t^\nu - \frac{1}{T} t^{\nu+1}\right] \frac{d}{dt}|u(t)|\, dt;$$

Lemma 5.40 assures the vanishing of the integrated term at zero. Transposition and estimation of the term on the right now yields

$$\nu \int_0^T |u(t)|\, t^{\nu-1}\, dt \le \frac{\nu+1}{T} \int_0^T |u(t)|\, t^\nu\, dt + \int_0^T |u'(t)|\, t^\nu\, dt,$$

which is (53) for $p = 1$. ▌

5.42 LEMMA If $\nu > 0$, $p \ge 1$, and $u \in C^1(0, T)$, we have the following pair of inequalities:

$$\sup_{0<t<T} |u(t)|^p \le \frac{2}{T} \int_0^T |u(t)|^p\, dt + p \int_0^T |u(t)|^{p-1}\, |u'(t)|\, dt, \tag{54}$$

$$\sup_{0<t<T} |u(t)|^p t^\nu \le \frac{\nu+3}{T} \int_0^T |u(t)|^p t^\nu\, dt + 2p \int_0^T |u(t)|^{p-1}\, |u'(t)|\, t^\nu\, dt. \tag{55}$$

PROOF Again the inequalities need only be proved for $p = 1$. If $0 < t \le T/2$, we obtain by integration by parts

$$\int_0^{T/2} \left|u\left(t + \frac{T}{2} - \tau\right)\right| d\tau = \frac{T}{2}|u(t)| - \int_0^{T/2} \tau \frac{d}{d\tau}\left|u\left(t + \frac{T}{2} - \tau\right)\right| d\tau$$

whence

$$|u(t)| \le \frac{2}{T} \int_0^T |u(\sigma)|\, d\sigma + \int_0^T |u'(\sigma)|\, d\sigma.$$

For $T/2 \le t < T$ the same inequality results from partial integration of $\int_0^{T/2} |u(t+\tau-T/2)|\, d\tau$. This proves (54) for $p = 1$. Replacing $u(t)$ by $u(t)t^\nu$ in this inequality, we obtain

$$\begin{aligned}
\sup_{0<t<T} |u(t)|\, t^\nu &\le \frac{2}{T} \int_0^T |u(t)|\, t^\nu\, dt + \int_0^T [|u'(t)|\, t^\nu + \nu\, |u(t)|\, t^{\nu-1}]\, dt \\
&\le \frac{2}{T} \int_0^T |u(t)|\, t^\nu\, dt + \int_0^T |u'(t)|\, t^\nu\, dt \\
&\quad + \nu\left\{\frac{\nu+1}{\nu T} \int_0^T |u(t)|\, t^\nu\, dt + \frac{1}{\nu} \int_0^T |u'(t)|\, t^\nu\, dt\right\},
\end{aligned}$$

where (53) has been used to obtain the last inequality. This is the desired result (55) for $p = 1$. ▌

5.43 Now we turn to $\mathbb{R}^n$, $n \geq 2$. If $x \in \mathbb{R}^n$, we shall make use of the spherical polar coordinate representation

$$x = (\rho, \phi) = (\rho, \phi_1, \ldots, \phi_{n-1}),$$

where $\rho \geq 0$, $-\pi \leq \phi_1 \leq \pi$, $0 \leq \phi_2, \ldots, \phi_{n-1} \leq \pi$, and

$$\begin{aligned} x_1 &= \rho \sin\phi_1 \sin\phi_2 \cdots \sin\phi_{n-1}, \\ x_2 &= \rho \cos\phi_1 \sin\phi_2 \cdots \sin\phi_{n-1}, \\ x_3 &= \qquad \rho \cos\phi_2 \cdots \sin\phi_{n-1}, \\ &\vdots \\ x_n &= \qquad\qquad \rho \cos\phi_{n-1}. \end{aligned}$$

The volume element is

$$dx = dx_1\, dx_2 \cdots dx_n = \rho^{n-1} \prod_{j=1}^{n-1} \sin^{j-1}\phi_j \, d\rho \, d\phi,$$

where $d\phi = d\phi_1 \cdots d\phi_{n-1}$.

We define functions $r_k = r_k(x)$ for $1 \leq k \leq n$ as follows:

$$\begin{aligned} r_1(x) &= \rho |\sin\phi_1| \prod_{j=2}^{n-1} \sin\phi_j, \\ r_k(x) &= \rho \prod_{j=k}^{n-1} \sin\phi_j, \qquad k = 2, 3, \ldots, n-1, \\ r_n(x) &= \rho. \end{aligned}$$

For $1 \leq k \leq n-1$, $r_k(x)$ is the distance of x from the coordinate plane spanned by the axes $x_{k+1}, \ldots, x_n$; $r_n(x)$ being just the distance of x from the origin. In connection with the use of product symbols of the form $P = \prod_{j=k}^{m} P_j$, be it agreed hereafter that $P = 1$ if $m < k$.

Let Q be an open, conical domain in $\mathbb{R}^n$ specified by the inequalities

$$0 < \rho < a, \qquad -\beta_1 < \phi_1 < \beta_1, \qquad 0 \leq \phi_j < \beta_j, \qquad j = 2, 3, \ldots, n-1, \tag{56}$$

where $0 < \beta_i \leq \pi$. [Inequalities "$<$" in (56), corresponding to any $\beta_i = \pi$, are replaced by "$\leq$." If all $\beta_i = \pi$, the first inequality is replaced by $0 \leq \rho < a$.] It should be noted that any standard cone Q_k (introduced in Section 5.34) is of the form (56) for some choice of the parameters β_i, $1 \leq i \leq n-1$.

The following lemma generalizes Lemma 5.41 in a manner suitable for our purposes.

5.44 LEMMA Let Q be specified by (56) and let $p \geq 1$. Suppose that either $m = k = 1$, or $2 \leq m \leq n$ and $1 \leq k \leq n$. Suppose also that $1 - k < \nu_1 \leq \nu \leq$

$\nu_2 < \infty$. Then there exists a constant $K = K(m, k, n, p, \nu_1, \nu_2, \beta_1, \ldots, \beta_{n-1})$ independent of ν and a, such that for every function $u \in C^1(Q)$ we have

$$\int_Q |u(x)|^p [r_k(x)]^\nu [r_m(x)]^{-1}\, dx$$
$$\le K \int_Q |u(x)|^{p-1} [(1/a)|u(x)| + |\operatorname{grad} u(x)|] [r_k(x)]^\nu\, dx. \tag{57}$$

PROOF Once again it is sufficient to establish (57) for $p = 1$. Let $Q_+ = \{x = (\rho, \phi) \in Q : \phi_1 \ge 0\}$, $Q_- = \{x \in Q : \phi_1 \le 0\}$. Then $Q = Q_+ \cup Q_-$. We shall prove (57) only for Q_+ (which, however, we continue to call Q); a similar proof holds for Q_- so that (57) holds for the given Q. Accordingly, assume $Q = Q_+$.

For $k \le m$ we may write (57) in the form (taking $p = 1$)

$$\int_Q |u| \prod_{j=2}^{k-1} \sin^{j-1} \phi_j \prod_{j=k}^{m-1} \sin^{\nu+j-1} \phi_j \prod_{j=m}^{n-1} \sin^{\nu+j-2} \phi_j\, \rho^{\nu+n-2}\, d\rho\, d\phi$$
$$\le K \int_Q [(1/a)|u| + |\operatorname{grad} u|] \prod_{j=2}^{k-1} \sin^{j-1} \phi_j \prod_{j=k}^{n-1} \sin^{\nu+j-1} \phi_j\, \rho^{\nu+n-1}\, d\rho\, d\phi. \tag{58}$$

For $k > m \ge 2$ we may write (57) in the form

$$\int_Q |u| \prod_{j=2}^{m-1} \sin^{j-1} \phi_j \prod_{j=m}^{k-1} \sin^{j-2} \phi_j \prod_{j=k}^{n-1} \sin^{\nu+j-2} \phi_j\, \rho^{\nu+n-2}\, d\rho\, d\phi$$
$$\le K \int_Q [(1/a)|u| + |\operatorname{grad} u|] \prod_{j=2}^{k-1} \sin^{j-1} \phi_j \prod_{j=k}^{n-1} \sin^{\nu+j-1} \phi_j\, \rho^{\nu+n-1}\, d\rho\, d\phi. \tag{59}$$

By virtue of the restrictions placed on ν, m, and k in the statement of the lemma, (58) and (59) are both special cases of

$$\int_Q |u| \prod_{j=1}^{i-1} \sin^{\mu_j} \phi_j \prod_{j=i}^{n-1} \sin^{\mu_j - 1} \phi_j\, \rho^{\nu+n-2}\, d\rho\, d\phi$$
$$\le K \int_Q [(1/a)|u| + |\operatorname{grad} u|] \prod_{j=1}^{n-1} \sin^{\mu_j} \phi_j\, \rho^{\nu+n-1}\, d\rho\, d\phi, \tag{60}$$

where $1 \le i \le n$, $\mu_j \ge 0$, and $0 < \mu_j{}^* \le \mu_j$ if $j \ge i$. We prove (60) by backwards induction on i. For $i = n$, (60) is obtained by applying Lemma 5.41 to u considered as a function of ρ on $(0, a)$ and then integrating the remaining variables with the appropriate weights. Assume, therefore, that (60) has been proved for $i = k+1$ where $1 \le k \le n-1$. We prove it now holds for $i = k$.

If $\beta_k < \pi$, we have

$$\sin \phi_k \le \phi_k \le K_1 \sin \phi_k, \qquad 0 \le \phi_k \le \beta_k, \tag{61}$$

where $K_1 = K_1(\beta_k)$. By Lemma 5.41, and since

$$|\partial u/\partial\phi_k| \le \rho\,|\mathrm{grad}\,u| \prod_{j=k+1}^{n-1} \sin\phi_j,$$

we have

$$\begin{aligned}
&\int_0^{\beta_k} |u(\rho,\phi)| \sin^{\mu_k-1}\phi_k\,d\phi_k \\
&\qquad \le \int_0^{\beta_k} |u|\,\phi_k^{\mu_k-1}\,d\phi_k \\
&\qquad \le K_2 \int_0^{\beta_k} \Bigl[\,|u| + |\mathrm{grad}\,u|\,\rho \prod_{j=k+1}^{n-1} \sin\phi_j\Bigr] \phi_k^{\mu_k}\,d\phi_k \\
&\qquad \le K_3 \int_0^{\beta_k} \Bigl[\,|u| + |\mathrm{grad}\,u|\,\rho \prod_{j=k+1}^{n-1} \sin\phi_j\Bigr] \sin^{\mu_k}\phi_k\,d\phi_k. \qquad (62)
\end{aligned}$$

Note that K_2, and hence K_3, depend on β_k but may be chosen independent of μ_k, and hence of v, under the conditions of the lemma. If $\beta_k = \pi$, we obtain (62) by writing $\int_0^\pi = \int_0^{\pi/2} + \int_{\pi/2}^\pi$ and using, in place of (61) ,the inequalities

$$\begin{aligned}
\sin\phi_k \le \phi_k \le (\pi/2)\sin\phi_k \qquad &\text{if} \quad 0 \le \phi_k \le \pi/2 \\
\sin\phi_k \le \pi - \phi_k \le (\pi/2)\sin\phi_k \qquad &\text{if} \quad \pi/2 \le \phi_k \le \pi.
\end{aligned} \qquad (63)$$

We now have, using (62) and the induction hypothesis,

$$\begin{aligned}
&\int_Q |u| \prod_{j=1}^{k-1} \sin^{\mu_j}\phi_j \prod_{j=k}^{n-1} \sin^{\mu_j-1}\phi_j\,\rho^{v+n-2}\,d\rho\,d\phi \\
&\quad \le \int_0^a \rho^{v+n-2}\,d\rho \prod_{j=1}^{k-1} \int_0^{\beta_j} \sin^{\mu_j}\phi_j\,d\phi_j \\
&\qquad \times \prod_{j=k+1}^{n-1} \int_0^{\beta_j} \sin^{\mu_j-1}\phi_j\,d\phi_j \int_0^{\beta_k} |u| \sin^{\mu_k-1}\phi_k\,d\phi_k \\
&\quad \le K_3 \int_Q |\mathrm{grad}\,u| \prod_{j=1}^{n-1} \sin^{\mu_j}\phi_j\,\rho^{v+n-1}\,d\rho\,d\phi \\
&\qquad + K_3 \int_Q |u| \prod_{j=1}^{k} \sin^{\mu_j}\phi_j \prod_{j=k+1}^{n-1} \sin^{\mu_j-1}\phi_j\,\rho^{v+n-2}\,d\rho\,d\phi \\
&\quad \le K \int_Q [(1/a)|u| + |\mathrm{grad}\,u|] \prod_{j=1}^{n-1} \sin^{\mu_j}\phi_j\,\rho^{v+n-1}\,d\rho\,d\phi.
\end{aligned}$$

This completes the induction establishing (60) and hence the lemma. ∎

In the following lemma we obtain an imbedding inequality similar to that of Lemma 5.10 for the domain Q and appropriately weighted L^p-norms.

5.45 LEMMA Let Q be as specified by (56) and let $p \geq 1$ and $1 \leq k \leq n$. Suppose that $\max(1-k, p-n) < \nu_1 < \nu_2 < \infty$. Then there exists a constant $K = K(k, n, p, \nu_1, \nu_2, \beta_1, \ldots, \beta_{n-1})$, independent of a, such that for every ν satisfying $\nu_1 \leq \nu \leq \nu_2$, and every function $u \in C^1(Q) \cap C(\bar{Q})$ we have

$$\left\{\int_Q |u(x)|^q [r_k(x)]^\nu \, dx\right\}^{1/q}$$
$$\leq K\left\{\int_Q [(1/a^p)|u(x)|^p + |\operatorname{grad} u(x)|^p][r_k(x)]^\nu \, dx\right\}^{1/p}, \tag{64}$$

where $q = (\nu+n)p/(\nu+n-p)$.

PROOF Let $\delta = (\nu+n-1)p/(\nu+n-p)$, $s = (\nu+n-1)/\nu$, $s' = (\nu+n-1)/(n-1)$. We have by Hölder's inequality and Lemma 5.44 (the case $m = k$)

$$\begin{aligned}\int_Q |u(x)|^q [r_k(x)]^\nu \, dx &\leq \left\{\int_Q |u|^\delta r_k^{\nu-1} \, dx\right\}^{1/s} \left\{\int_Q |u|^{n\delta/(n-1)} r_k^{n\nu/(n-1)} \, dx\right\}^{1/s'} \\ &\leq K_1\left\{\int_Q |u|^{\delta-1}[(1/a)|u| + |\operatorname{grad} u|] r_k^{\nu} \, dx\right\}^{1/s} \\ &\quad \times \left\{\int_Q |u|^{n\delta/(n-1)} r_k^{n\nu/(n-1)} \, dx\right\}^{1/s'}.\end{aligned} \tag{65}$$

In order to estimate the last integral above we adopt the notation

$$\rho^* = (\phi_1, \ldots, \phi_{n-1}), \qquad \phi_j^* = (\rho, \phi_1, \ldots, \hat{\phi}_j, \ldots, \phi_{n-1}), \qquad j = 1, 2, \ldots, n-1,$$

where the caret denotes omission of a component. Let

$$\begin{aligned} Q_0^* &= \{\rho^* : (\rho, \rho^*) \in Q \text{ for } 0 < \rho < a\} \\ Q_j^* &= \{\phi_j^* : (\rho, \phi) \in Q \text{ for } 0 < \phi_j < \beta_j\}. \end{aligned}$$

Q_0^* and Q_j^* $(1 \leq j \leq n-1)$ are domains in $\mathbb{R}^{n-1}$. We define functions F_0 and F_j on Q_0^* and Q_j^*, respectively, as follows:

$$\begin{aligned} [F_0(\rho^*)]^{n-1} &= [F_0(\phi_1, \ldots, \phi_{n-1})]^{n-1} \\ &= \sup_{0<\rho<a} [|u|^\delta \rho^{\nu+n-1}] \prod_{i=k}^{n-1} \sin^\nu \phi_i \prod_{i=2}^{n-1} \sin^{i-1} \phi_i, \\ [F_j(\phi_j^*)]^{n-1} &= [F_j(\rho, \phi_1, \ldots, \hat{\phi}_j, \ldots, \phi_{n-1})]^{n-1} \\ &= \sup_{0<\phi_j<\beta_j} [|u|^\delta \sin^{\nu+j-1} \phi_j] \rho^{\nu+n-2} \\ &\quad \times \prod_{i=k}^{n-1} \sin^\nu \phi_i \prod_{i=2}^{j-1} \sin^{i-1} \phi_i \prod_{i=j+1}^{n-1} \sin^{i-2} \phi_i. \end{aligned}$$

Then we have

$$|u|^{n\delta/(n-1)} r_k^{n\nu/(n-1)} \rho^{n-1} \prod_{i=2}^{n-1} \sin^{i-1}\phi_i \le F_0(\rho^*) \prod_{j=1}^{n-1} F_j(\phi_j{}^*).$$

Applying the combinatorial Lemma 5.9, we obtain

$$\begin{aligned} \int_Q |u|^{n\delta/(n-1)} r_k^{n\nu/(n-1)}\,dx &\le \int_Q F_0(\rho^*) \prod_{j=1}^{n-1} F_j(\phi_j{}^*)\,d\rho\,d\phi \\ &\le \left\{ \int_{Q_0{}^*} [F_0(\rho^*)]^{n-1}\,d\phi \prod_{j=1}^{n-1} \int_{Q_j{}^*} [F_j(\phi_j{}^*)]^{n-1}\,d\rho\,d\hat{\phi}_j \right\}^{1/(n-1)} \qquad (66) \end{aligned}$$

Now by Lemma 5.42, and since $|\partial u/\partial\rho| \le |\operatorname{grad} u|$,

$$\sup_{0<\rho<a} |u|^\delta \rho^{\nu+n-1} \le K_2 \int_0^a |u|^{\delta-1} [(1/a)|u| + |\operatorname{grad} u|] \rho^{\nu+n-1}\,d\rho,$$

where K_2 is independent of ν for $1-n < \nu_1 \le \nu \le \nu_2 < \infty$. It follows that

$$\int_{Q_0{}^*} [F_0(\rho^*)]^{n-1}\,d\phi \le K_2 \int_Q |u|^{\delta-1} [(1/a)|u| + |\operatorname{grad} u|] r_k{}^\nu\,dx. \qquad (67)$$

Similarly, by making use of inequality (61) or (63) as in Lemma 5.44, we obtain from Lemma 5.42

$$\begin{aligned} \sup_{0<\phi_j<\beta_j} & |u|^\delta \sin^{\nu+j-1}\phi_j \\ &\le K_{2,j} \int_0^{\beta_j} |u|^{\delta-1} \left[|u| + \left| \frac{\partial u}{\partial\phi_j} \right| \right] \sin^{\nu+j-1}\phi_j\,d\phi_j \\ &\le K_{2,j} \int_0^{\beta_j} |u|^{\delta-1} \left[|u| + |\operatorname{grad} u|\,\rho \prod_{i=j+1}^{n-1} \sin\phi_i \right] \sin^{\nu+j-1}\phi_j\,d\phi_j \end{aligned}$$

since $|\partial u/\partial\phi_j| \le \rho \prod_{i=j+1}^{n-1} \sin\phi_i$. Hence

$$\begin{aligned} \int_{Q_j{}^*} & [F_j(\phi_j{}^*)]^{n-1}\,d\rho\,d\hat{\phi}_j \\ &\le K_{2,j} \int_Q |\operatorname{grad} u|\,|u|^{\delta-1} r_k{}^\nu\,dx + K_{2,j} \int_Q |u|^\delta r_k{}^\nu r_{j+1}^{-1}\,dx \\ &\le K_{3,j} \int_Q |u|^{\delta-1} [(1/a)|u| + |\operatorname{grad} u|] r_k{}^\nu\,dx \qquad (68) \end{aligned}$$

where we have used Lemma 5.44 again to obtain the last inequality. Note that the constants $K_{2,j}$ and $K_{3,j}$ can be chosen independent of v for the values of v allowed. Substitution of (67) and (68) into (66) and then into (65) leads to

$$\begin{aligned}
&\int_Q |u|^q r_k^{\,v}\,dx \\
&\qquad \le K_4\left\{\int_Q |u|^{\delta-1}[(1/a)|u| + |\operatorname{grad} u|]\, r_k^{\,v}\,dx\right\}^{1/s+n/(n-1)s'} \\
&\qquad \le K_4\left\{\left\{\int_Q |u|^q r_k^{\,v}\,dx\right\}^{(p-1)/p}\right. \\
&\qquad\qquad \times\left.\left\{2^{p-1}\int_Q [(1/a^p)|u|^p + |\operatorname{grad} u|^p]\, r_k^{\,v}\,dx\right\}^{1/p}\right\}^{(v+n)/(v+n-1)}.
\end{aligned}$$

Since $(v+n-1)/(v+n)-(p-1)/p = 1/q$, inequality (64) follows by cancellation for, since u is bounded on Q and $v > 1-n$, $\int_Q |u|^q r_k^{\,v}\,dx$ is finite. ▌

5.46 REMARKS (1) The assumption that $u \in C(\bar{Q})$ was made only to ensure that the above cancellation was justified. In fact the lemma holds for any $u \in C^1(Q)$.

(2) If $1-k < v_1 < v_2 < \infty$ and $v_1 \le v \le v_2$, where $v \le p-n$, then (64) holds for any q satisfying $1 \le q < \infty$. It is sufficient to prove this for large q. If $q \ge (v+n)/(v+n-1)$, then $q = (v+n)s/(v+n-s)$ for some s satisfying $1 \le s < p$. Thus

$$\begin{aligned}
&\left\{\int_Q |u|^q r_k^{\,v}\,dx\right\}^{s/q} \\
&\qquad \le K\int_Q [(1/a^s)|u|^s + |\operatorname{grad} u|^s]\, r_k^{\,v}\,dx \\
&\qquad \le K\left\{2^{(p-s)/s}\int_Q [(1/a^p)|u|^p + |\operatorname{grad} u|^p]\, r_k^{\,v}\,dx\right\}^{s/p}\left\{\int_Q r_k^{\,v}\,dx\right\}^{(p-s)/p},
\end{aligned}$$

which yields (64) since the last factor on the right is finite.

(3) If $v = m$, a positive integer, then (64) can be obtained very simply as follows. Let $y = (x,z) = (x_1,\dots,x_n, z_1,\dots,z_m)$ denote a point in $\mathbb{R}^{n+m}$ and define $u^*(y) = u(x)$ for $x \in Q$. If

$$Q^* = \{y \in \mathbb{R}^{n+m} : y = (x,z),\, x \in Q,\, 0 < z_j < r_k(x),\, 1 \le j \le m\},$$

then Q^* has the cone property in $\mathbb{R}^{n+m}$, whence by Theorem 5.4 we have, putting $q=(n+m)p/(n+m-p)$,

$$\left\{\int_Q |u|^q r_k^m\,dx\right\}^{1/q} = \left\{\int_{Q^*} |u^*(y)|^q\,dy\right\}^{1/q}$$
$$\le K\left\{\int_{Q^*} [(1/a^p)|u^*(y)|^p + |\operatorname{grad} u^*(y)|^p]\,dy\right\}^{1/p}$$
$$= K\left\{\int_Q [(1/a^p)|u|^p + |\operatorname{grad} u|^p] r_k^m\,dx\right\}^{1/p}$$

since $|\operatorname{grad} u^*(y)| = |\operatorname{grad} u(x)|$, u^* being independent of z.

(4) Suppose that $u \in C_0^\infty(\mathbb{R}^n)$, or, more generally, that

$$\int_{\mathbb{R}^n} |u(x)|^p [r_k(x)]^\nu\,dx < \infty$$

with ν as in the above lemma. If we take $\beta_i = \pi$, $1 \le i \le n-1$, and let $a \to \infty$ in (64), we obtain

$$\left\{\int_{\mathbb{R}^n} |u(x)|^q [r_k(x)]^\nu\,dx\right\}^{1/q} \le K\left\{\int_{\mathbb{R}^n} |\operatorname{grad} u(x)|^p [r_k(x)]^\nu\,dx\right\}^{1/p}.$$

This generalizes Sobolev's inequality as given in Section 5.11.

We now generalize Lemma 5.15 to allow for weighted norms. It is convenient to deal here with arbitrary domains having the cone property, rather than the special case Q considered above. The following elementary result will be needed.

5.47 LEMMA Let $z \in \mathbb{R}^k$ and let Ω be a domain of finite volume in $\mathbb{R}^k$. If $0 \le \nu < k$, then

$$\int_\Omega |x-z|^{-\nu}\,dx \le K(\operatorname{vol}\Omega)^{1-\nu/k}, \tag{69}$$

where the constant K depends on ν and k but not on z or Ω.

PROOF Let B be the ball in $\mathbb{R}^k$ having center z and the same volume as Ω. It is readily verified that the left side of (69) does not exceed $\int_B |x-z|^{-\nu}\,dx$. But (69) clearly holds for $\Omega = B$. ∎

5.48 LEMMA Let Ω be a domain with the cone property in $\mathbb{R}^n$. Let $1 \le k \le n$ and let P be an $(n-k)$-dimensional plane in $\mathbb{R}^n$. Denote by $r(x)$ the distance from x to P. If $0 \le \nu < p-n$, then for all $u \in C^1(\Omega)$ we have

$$\sup_{x\in\Omega} |u(x)| \le K\left\{\int_\Omega [|u(x)|^p + |\operatorname{grad} u(x)|^p][r(x)]^\nu\,dx\right\}^{1/p}, \tag{70}$$

where the constant K may depend on v, n, p, k and the cone C determining the cone property for Ω, but not on u.

Proof Throughout this proof A_i and K_i will denote various constants depending on one or more of the parameters on which K is allowed to depend in (70). It is sufficient to prove that if C is a finite cone contained in Ω having vertex at, say, the origin, then

$$|u(0)| \leq K\left\{\int_C [|u(x)|^p + |\operatorname{grad} u(x)|^p][r(x)]^v\,dx\right\}^{1/p}. \tag{71}$$

For $0 \leq j \leq n$ let A_j denote the supremum of the Lebesgue j-dimensional measure of the projection of C onto $\mathbb{R}^j$, taken over all j-dimensional subspaces $\mathbb{R}^j$ of $\mathbb{R}^n$. Writing $x = (x', x'')$ where $x' = (x_1, \ldots, x_{n-k})$ and $x'' = (x_{n-k+1}, \ldots, x_n)$, we may assume, without loss of generality, that P is orthogonal to the coordinate axes corresponding to the components of x''. Define

$$S = \{x' \in \mathbb{R}^{n-k} : (x', x'') \in C \text{ for some } x'' \in \mathbb{R}^k\},$$
$$R(x') = \{x'' \in \mathbb{R}^k : (x', x'') \in C\} \text{ for each } x' \in S.$$

For $0 \leq t \leq 1$ we denote by C_t the cone $\{tx : x \in C\}$ so that $C_t \subset C$ and $C_t = C$ if $t = 1$. For C_t we define the quantities $A_{t,j}$, S_t, and $R_t(x')$ analogously to the similar quantities defined above for C. Clearly $A_{t,j} = t^j A_j$. If $x \in C$, we have

$$u(x) = u(0) + \int_0^1 \frac{d}{dt} u(tx)\,dt,$$

so that

$$|u(0)| \leq |u(x)| + |x| \int_0^1 |\operatorname{grad} u(tx)|\,dt.$$

Setting $V = \operatorname{vol} C$ and $a = \sup_{x \in C} |x|$, and integrating the above inequality over C, we obtain

$$\begin{aligned} V|u(0)| &\leq \int_C |u(x)|\,dx + a \int_C \int_0^1 |\operatorname{grad} u(tx)|\,dt \\ &= \int_C |u(x)|\,dx + a \int_0^1 t^{-n}\,dt \int_{C_t} |\operatorname{grad} u(x)|\,dx. \end{aligned} \tag{72}$$

Let z denote the orthogonal projection of x onto P. Then $r(x) = |x'' - z''|$. Since $0 \leq v < p - n$ we have $p > 1$, and so by Lemma 5.47

$$\begin{aligned} \int_{C_t} [r(x)]^{-v/(p-1)}\,dx &= \int_{S_t} dx' \int_{R_t(x')} |x'' - z''|^{-v/(p-1)}\,dx'' \\ &\leq K_1 \int_{S_t} [A_{t,k}]^{1 - v/k(p-1)}\,dx' \\ &\leq K_1 [A_{t,k}]^{1 - v/k(p-1)} A_{t,n-k} = K_2 t^{n - v/(p-1)}. \end{aligned}$$

It follows that

$$\int_{C_t} |\operatorname{grad} u(x)|\, dx \le \left\{\int_{C_t} |\operatorname{grad} u(x)|^p [r(x)]^\nu\, dx\right\}^{1/p} \times \left\{\int_{C_t} [r(x)]^{-\nu/(p-1)}\, dx\right\}^{1/p'}$$
$$\le K_3 t^{n-(\nu+n)/p} \left\{\int_C |\operatorname{grad} u(x)|^p [r(x)]^\nu\, dx\right\}^{1/p}. \tag{73}$$

Hence, since $\nu < p-n$,

$$\int_0^1 t^{-n}\, dt \int_{C_t} |\operatorname{grad} u(x)|\, dx \le K_4 \left\{\int_C |\operatorname{grad} u(x)|^p [r(x)]^\nu\, dx\right\}^{1/p}. \tag{74}$$

Similarly,

$$\int_C |u(x)|\, dx \le \left\{\int_C |u(x)|^p [r(x)]^\nu\, dx\right\}^{1/p} \left\{\int_C [r(x)]^{-\nu/(p-1)}\, dx\right\}^{1/p'}$$
$$\le K_5 \left\{\int_C |u(x)|^p [r(x)]^\nu\, dx\right\}^{1/p}. \tag{75}$$

Inequality (71) now follows from (72), (74), and (75). ∎

5.49 LEMMA Suppose all the conditions of Lemma 5.48 are satisfied and, in addition, Ω has the strong local Lipschitz property. Then for all $u \in C^1(\Omega)$ we have

$$\sup_{\substack{x,y\in\Omega\\ x\ne y}} \frac{|u(x)-u(y)|}{|x-y|^\mu} \le K \left\{\int_\Omega [|u(x)|^p + |\operatorname{grad} u(x)|^p][r(x)]^\nu\, dx\right\}^{1/p}, \tag{76}$$

where $\mu = 1-(\nu+n)/p$ satisfies $0 < \mu < 1$ and K is independent of u.

PROOF The proof is the same as that given for inequality (28) in Lemma 5.17, except that the inequality

$$\int_{\Omega_{t\sigma}} |\operatorname{grad} u(z)|\, dz \le K_1 t^{n-(\nu+n)/p} \left\{\int_\Omega |\operatorname{grad} u(z)|^p [r(z)]^\nu\, dz\right\}^{1/p} \tag{77}$$

is used in (29) in place of the special case $\nu = 0$ actually used there. Inequality (77) is obtained in the same way as (73) above. ∎

Proofs of Theorems 5.35–5.37

5.50 LEMMA Let $\bar\nu \ge 0$. If $\bar\nu > p-n$, let $1 \le q \le (\bar\nu+n)p/(\bar\nu+n-p)$; otherwise let $1 \le q < \infty$. There exists a constant $K = K(n,p,\bar\nu)$ such that for every standard cusp domain $Q_{k,\lambda}$ (see Section 5.34) for which $(\lambda-1)k \equiv \nu \le \bar{\nu}$, and every $u \in C^1(Q_{k,\lambda})$, we have

$$\|u\|_{0,q,Q_{k,\lambda}} \le K\|u\|_{1,p,Q_{k,\lambda}}. \tag{78}$$

Proof Since each $Q_{k,\lambda}$ has the segment property, it suffices to prove (78) for $u \in C^1(\overline{Q_{k,\lambda}})$. We first do so for given k and λ and then show that K may be chosen so as to be independent of these parameters.

First suppose $\bar{\nu} > p-n$. It is sufficient to prove (78) for

$$q = (\bar{\nu}+n)/(\bar{\nu}+n-p).$$

For $u \in C^1(\overline{Q_{k,\lambda}})$ define $\tilde{u}(y) = u(x)$, where y is related to x by (47) and (48). Thus $\tilde{u} \in C^1(Q_k) \cap C(Q_k)$, where Q_k is the standard cone associated with $Q_{k,\lambda}$. By Lemma 5.45, and since $q \le (\nu+n)p/(\nu+n-p)$ we have

$$\begin{aligned}\|u\|_{0,q,Q_{k,\lambda}} &= \left\{\lambda \int_{Q_k} |\tilde{u}(y)|^q [r_k(y)]^\nu \, dy\right\}^{1/q} \\ &\le K_1 \left\{\int_{Q_k} [|\tilde{u}(y)|^p + |\operatorname{grad} \tilde{u}(y)|^p][r_k(y)]^\nu \, dy\right\}^{1/q}. \end{aligned} \tag{79}$$

Now $x_j = r_k^{\lambda-1} y_j$ if $1 \le j \le k$; $x_j = y_j$ if $k+1 \le j \le n$. Since $r_k{}^2 = y_1{}^2 + \cdots + y_k{}^2$ we have

$$\frac{\partial x_j}{\partial y_i} = \begin{cases} \delta_{ij} r_k^{\lambda-1} + (\lambda-1) r_k^{\lambda-3} y_i y_j & \text{if } 1 \le i,\ j \le k \\ \delta_{ij} & \text{otherwise.} \end{cases}$$

Since $r_k(y) \le 1$ on Q_k it follows that

$$|\operatorname{grad} \tilde{u}(y)| \le K_2 |\operatorname{grad} u(x)|.$$

Hence (78) follows from (79) in this case. For $\bar{\nu} \le p-n$ and arbitrary q the proof is similar, being based on Remark 5.46(2).

In order to show that the constant K in (78) can be chosen independent of k and λ provided $\nu = (\lambda-1)k \le \bar{\nu}$, we note that it is sufficient to prove that there is a constant $\tilde{K}$ such that for any such k, λ and all $v \in C^1(Q_k) \cap C(\bar{Q}_k)$ we have

$$\begin{aligned}&\left\{\int_{Q_k} |v(y)|^q [r_k(y)]^\nu \, dy\right\}^{1/q} \\ &\qquad \le \tilde{K} \left\{\int_{Q_k} [|v(y)|^p + |\operatorname{grad} v(y)|^p][r_k(y)]^\nu \, dy\right\}^{1/p}. \end{aligned} \tag{80}$$

In fact, it is sufficient to establish (80) with $\tilde{K}$ depending on k as we can then use the maximum of $\tilde{K}(k)$ over the finitely many values of k allowed. We distinguish three cases.

CASE I $\bar{\nu} < p-n$, $1 \le q < \infty$. By Lemma 5.48 we have for $0 \le \nu \le \bar{\nu}$,

$$\sup_{x \in Q_k} |v(x)| \le K(\nu)\left\{\int_{Q_k} [|v(y)|^p + |\operatorname{grad} v(y)|^p][r_k(y)]^\nu \, dy\right\}^{1/p}. \tag{81}$$

Since the integral on the right decreases as ν increases we have $K(\nu) \le K(\bar{\nu})$ and (80) now follows from (81) and the boundedness of Q_k.

CASE II $\bar{\nu} > p-n$ Again it is enough to deal with $q = (\bar{\nu}+n)p/(\bar{\nu}+n-p)$. From Lemma 5.45 we obtain

$$\left\{\int_{Q_k} |v|^s r_k{}^\nu \, dy\right\}^{1/s} \le K_1 \left\{\int_{Q_k} [|v|^p + |\operatorname{grad} v|^p] r_k{}^\nu \, dy\right\}^{1/p}, \tag{82}$$

where $s = (\nu+n)p/(\nu+n-p) \ge q$ and K_1 is independent of ν for $p-n < \nu_0 \le \nu \le \bar{\nu}$. By Hölder's inequality, and since $r_k(y) \le 1$ on Q_k, we have

$$\left\{\int_{Q_k} |v|^q r_k{}^\nu \, dy\right\}^{1/q} \le \left\{\int_{Q_k} |v|^s r_k{}^\nu \, dy\right\}^{1/s} [\operatorname{vol} Q_k]^{(s-q)/sq}$$

so that if $\nu_0 \le \nu \le \bar{\nu}$, then (80) follows from (82).

If $p-n < 0$, we can take $\nu_0 = 0$ and be done. Otherwise $p \ge n \ge 2$. Fixing $\nu_0 = (\bar{\nu}-n+p)/2$, we can find ν_1 such that $0 \le \nu_1 \le p-n$ (or $\nu_1 = 0$ if $p = n$) such that for $\nu_1 \le \nu \le \nu_0$ we have

$$1 \le t = \frac{(\nu+n)(\bar{\nu}+n)p}{(\nu+n)(\bar{\nu}+n) + (\bar{\nu}-\nu)p} \le \frac{p}{1+\varepsilon_0},$$

where $\varepsilon_0 > 0$ and depends only on $\bar{\nu}$, n, and p. Because of the latter inequality we may also assume $t-n < \nu_1$. Since $(\nu+n)t/(\nu+n-t) = q$ we have, again by Lemma 5.45 and Hölder's inequality,

$$\begin{aligned}
&\left\{\int_{Q_k} |v|^q r_k{}^\nu \, dy\right\}^{1/q} \\
&\quad \le K_2 \left\{\int_{Q_k} [|v|^t + |\operatorname{grad} v|^t] r_k{}^\nu \, dy\right\}^{1/t} \\
&\quad \le 2^{(p-t)/pt} K_2 \left\{\int_{Q_k} [|v|^p + |\operatorname{grad} v|^p] r_k{}^\nu \, dy\right\}^{1/p} [\operatorname{vol} Q_k]^{(p-t)/pt},
\end{aligned} \tag{83}$$

where K_2 is independent of ν for $\nu_1 \le \nu \le \nu_0$.

In the case $\nu_1 > 0$ we can obtain a similar (uniform) estimate for $0 \le \nu \le \nu_1$ by the method of Case I. Combining this with (82) and (83), we prove (80) for this case.

CASE III $\bar{\nu} = p-n$, $1 \le q < \infty$ Fix $s \ge \max(q, n/(n-1))$ and let $t = (\nu+n)s/(\nu+n+s)$ so that $s = (\nu+n)t/(\nu+n-t)$. Then $1 \le t \le ps/(p+s) < p$

for $0 \le \nu \le \bar{\nu}$. Hence we can select $\nu_1 \ge 0$ such that $t-n < \nu_1 < p-n$. The rest of the proof is similar to Case II. This completes the proof. ∎

5.51 PROOF OF THEOREM 5.35 By the same argument used in the proof of Lemma 5.12 it is sufficient to consider here only the special case $m = 1$. Let q satisfy $p \le q \le (\nu+n)p/(\nu+n-p)$ if $\nu+n > p$, or $p \le q < \infty$ otherwise. Clearly $q < np/(n-p)$ if $n > p$ so in either case we have by Theorem 5.4

$$\|u\|_{0,q,G} \le K_1 \|u\|_{1,p,G}$$

for every $u \in C^1(\Omega)$ and that element G of Γ which has the cone property (if such G exists.) If $G \in \Gamma$ does not have the cone property, and if $\psi: G \to Q_{k,\lambda}$, where $(\lambda-1)k \le \nu$, is the 1-smooth mapping specified in the statement of the theorem, then by Theorem 3.35 and Lemma 5.50

$$\|u\|_{0,q,G} \le K_2 \|u \circ \psi^{-1}\|_{0,q,Q_{k,\lambda}} \le K_3 \|u \circ \psi^{-1}\|_{1,p,Q_{k,\lambda}} \le K_4 \|u\|_{1,p,G},$$

where K_4 is independent of G. We have, therefore, noting that $q/p \ge 1$,

$$\|u\|_{0,q,\Omega}^q \le \sum_{G \in \Gamma} \|u\|_{0,q,G}^q \le K_5 \sum_{G \in \Gamma} \left(\|u\|_{1,p,G}^p \right)^{q/p}$$
$$\le K_5 \left(\sum_{G \in \Gamma} \|u\|_{1,p,G}^p \right)^{q/p} \le K_5 N^{q/p} \|u\|_{1,p,\Omega}^q,$$

where we have used the finite intersection property of Γ to obtain the final inequality. Imbedding (50) now follows by completion. [If $\nu < mp-n$, we require that (50) hold for $q = \infty$. This is a consequence of Theorem 5.36 proved below.] ∎

5.52 LEMMA Let $0 \le \bar{\nu} < mp-n$. There exists a constant $K = K(m,p,n,\bar{\nu})$ such that if $Q_{k,\lambda}$ is any standard cusp domain for which $(\lambda-1)k = \nu \le \bar{\nu}$ and if $u \in C^m(Q_{k,\lambda})$, then

$$\sup_{x \in Q_{k,\lambda}} |u(x)| \le K \|u\|_{m,p,Q_{k,\lambda}}. \tag{84}$$

PROOF It is sufficient to prove the lemma for the case $m = 1$; the proof for general m then follows by the same type of argument used in the last paragraph of the proof of Lemma 5.15.

If $u \in C^1(Q_{k,\lambda})$, $(\lambda-1)k = \nu \le \bar{\nu}$, we have by Lemma 5.48 and via the method of the second paragraph of the proof of Lemma 5.50,

$$\sup_{x \in Q_{k,\lambda}} |u(x)| = \sup_{y \in Q_k} |\tilde{u}(y)|$$
$$\le K_1 \left\{ \int_{Q_k} [|\tilde{u}(y)|^p + |\operatorname{grad} \tilde{u}(y)|^p] [r_k(y)]^\nu \, dy \right\}^{1/p}$$
$$\le K_2 \left\{ \int_{Q_{k,\lambda}} [|u(x)|^p + |\operatorname{grad} u(x)|^p] \, dx \right\}^{1/p}. \tag{85}$$

Since $r_k(y) \leq 1$ for $y \in Q_k$ it is evident that K_1, and hence K_2, can be chosen independent of k, λ provided $0 \leq \nu = (\lambda - 1)k \leq \bar{\nu}$. ▮

5.53 PROOF OF THEOREM 5.36 It is sufficient to prove (51). Let $u \in C^m(\Omega)$. If $x \in \Omega$, then $x \in G \subset \Omega$ for some domain G for which there exists a 1-smooth transformation $\psi: G \to Q_{k,\lambda}$, $(\lambda-1)k \leq \nu$, as specified in the statement of the theorem. Thus

$$\begin{aligned} |u(x)| &\leq \sup_{x \in G} |u(x)| = \sup_{y \in Q_{k,\lambda}} |u \circ \psi^{-1}(y)| \\ &\leq K_1 \|u \circ \psi^{-1}\|_{m,p,Q_{k,\lambda}} \leq K_2 \|u\|_{m,p,G} \\ &\leq K_2 \|u\|_{m,p,\Omega}, \end{aligned} \tag{86}$$

where K_1 and K_2 are independent of G. The rest of the proof is similar to the first paragraph of the proof of Lemma 5.15. ▮

5.54 PROOF OF THEOREM 5.37 As in Lemma 5.17 it is sufficient to prove that (52) holds when $j = 0$ and $m = 1$, that is, that

$$\sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|u(x)-u(y)|}{|x-y|^\mu} \leq K \|u\|_{1,p,\Omega} \tag{87}$$

holds when $\nu + n < p$ and $0 < \mu \leq 1 - (\nu+n)/p$. For $x, y \in \Omega$, $|x-y| > \delta$, (87) holds by virtue of (86). If $|x-y| < \delta$, there exists G with $x, y \in G \subset \Omega$, and a 1-smooth transformation ψ from G onto a standard cusp $Q_{k,\lambda}$ with $(\lambda-1)k \leq \nu$, satisfying the conditions of the theorem. Inequality (87) can then be derived from Lemma 5.49 by the same method used in the proof of Lemma 5.52. The details are left to the reader. ▮

VI

Compact Imbeddings of $W^{m,p}(\Omega)$

The Rellich–Kondrachov Theorem

6.1 Let Ω be a domain in $\mathbb{R}^n$ and let Ω_0 be a subdomain of Ω. Let $X(\Omega)$ denote any of the possible target spaces for imbeddings of $W^{m,p}(\Omega)$, that is, $X(\Omega)$ is a space of the form $C_B^j(\Omega)$, $C^{j,\lambda}(\overline{\Omega})$, $L^q(\Omega^k)$, or $W^{j,q}(\Omega^k)$, where Ω^k, $1 \le k \le n$, is the intersection of Ω with a k-dimensional plane in $\mathbb{R}^n$. Since the linear restriction operator $i_{\Omega_0}: u \to u|_{\Omega_0}$ is bounded from $X(\Omega)$ into $X(\Omega_0)$ [in fact $\|i_{\Omega_0}u; X(\Omega_0)\| \le \|u; X(\Omega)\|$] any imbedding of the form

$$W^{m,p}(\Omega) \to X(\Omega) \tag{1}$$

can be composed with this restriction to yield the imbedding

$$W^{m,p}(\Omega) \to X(\Omega_0) \tag{2}$$

and (2) has imbedding constant no larger than (1).

If Ω satisfies the hypotheses of the Sobolov imbedding Theorem 5.4 and if Ω_0 is bounded, then, with the exception of certain extreme cases, all imbeddings (2) (corresponding to imbeddings asserted in Theorem 5.4) are compact. The most important of these compact imbedding results originated in a lemma of Rellich [57] and was proved specifically for Sobolev spaces by Kondrachov [33]. Such compact imbeddings have many important applications in analysis, especially to showing the discreteness of the spectra of linear elliptic partial differential operators defined over bounded domains.

We summarize the various compact imbeddings of $W^{m,p}(\Omega)$ in the following theorem.

6.2 THEOREM (*The Rellich–Kondrachov theorem*) Let Ω be a domain in $\mathbb{R}^n$, Ω_0 a bounded subdomain of Ω, and $\Omega_0{}^k$ the intersection of Ω_0 with a k-dimensional plane in $\mathbb{R}^n$. Let j, m be integers, $j \geq 0$, $m \geq 1$, and let $1 \leq p < \infty$.

PART I If Ω has the cone property and $mp \leq n$, then the following imbeddings are compact:

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega_0{}^k) \quad \text{if} \quad 0 < n - mp < k \leq n \quad \text{and}$$

$$1 \leq q < kp/(n-mp), \tag{3}$$

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega_0{}^k) \quad \text{if} \quad n = mp, \quad 1 \leq k \leq n \quad \text{and}$$

$$1 \leq q < \infty. \tag{4}$$

PART II If Ω has the cone property and $mp > n$, then the following imbeddings are compact:

$$W^{j+m,p}(\Omega) \to C_B{}^j(\Omega), \tag{5}$$

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega_0{}^k) \quad \text{if} \quad 1 \leq q \leq \infty. \tag{6}$$

PART III If Ω has the strong local Lipschitz property, then the following imbeddings are compact:

$$W^{j+m,p}(\Omega) \to C^j(\overline{\Omega}_0) \quad \text{if} \quad mp > n, \tag{7}$$

$$W^{j+m,p}(\Omega) \to C^{j,\lambda}(\overline{\Omega}_0) \quad \text{if} \quad mp > n \geq (m-1)p \quad \text{and}$$

$$0 < \lambda < m-(n/p). \tag{8}$$

PART IV If Ω is an arbitrary domain in $\mathbb{R}^n$, all imbeddings (3)–(8) are compact provided $W^{j+m,p}(\Omega)$ is replaced by $W_0^{j+m,p}(\Omega)$.

6.3 REMARKS (1) If X, Y, and Z are spaces for which we have the imbeddings $X \to Y$ and $Y \to Z$ and if one of these imbeddings is compact, then the composite imbedding $X \to Z$ is compact. Thus, for example, if $Y \to Z$ is compact, then any sequence $\{u_i\}$ bounded in X will be bounded in Y and therefore have a subsequence $\{u_i'\}$ convergent in Z.

Since the extension operator $u \to \tilde{u}$ where $\tilde{u}(x) = u(x)$ if $x \in \Omega$, $\tilde{u}(x) = 0$ if $x \notin \Omega$ defines an imbedding $W_0^{j+m,p}(\Omega) \to W^{j+m,p}(\mathbb{R}^n)$ by Lemma 3.22, Part IV of Theorem 6.2 follows from the application of Parts I–III to $\mathbb{R}^n$.

(2) In proving the compactness of any of the imbeddings (3)–(8) it is sufficient to consider only the case $j = 0$. Suppose, for example, that (3) has been proven compact if $j = 0$. For $j \geq 1$ and $\{u_i\}$ a bounded sequence in $W^{j+m,p}(\Omega)$ it is clear that $\{D^\alpha u_i\}$ is bounded in $W^{m,p}(\Omega)$ for each α such that $|\alpha| \leq j$. Hence $\{D^\alpha u_i\}$ is precompact in $L^q(\Omega_0{}^k)$ with q specified as in (3). It is possible, therefore, to select (by finite induction) a subsequence $\{u_i'\}$ of $\{u_i\}$ for which $\{D^\alpha u_i'\}$ converges in $L^q(\Omega_0{}^k)$ for each α such that $|\alpha| \leq j$. Thus $\{u_i'\}$ converges in $W^{j,q}(\Omega_0{}^k)$ and (3) is compact.

(3) Since Ω_0 is bounded, $C_B{}^0(\Omega_0{}^k) \to L^q(\Omega_0{}^k)$ for $1 \leq q \leq \infty$; in fact, $\|u\|_{0,q,\Omega_0{}^k} \leq \|u; C_B{}^0(\Omega_0{}^k)\|[\mathrm{vol}_k \Omega_0{}^k]^{1/q}$. Thus the compactness of (6) (for $j = 0$) follows from that of (5).

(4) For the purpose of proving Theorem 6.2 the bounded subdomain Ω_0 of Ω may always be assumed to have the cone property if Ω does. If C is a finite cone determining the cone property for Ω, let $\tilde{\Omega}$ be the union of all finite cones congruent to C, contained in Ω and having nonempty intersection with Ω_0. Then $\Omega_0 \subset \tilde{\Omega} \subset \Omega$ and $\tilde{\Omega}$ is bounded and has the cone property. If $W^{m,p}(\Omega) \to X(\tilde{\Omega})$ is compact, then so is $W^{m,p}(\Omega) \to X(\Omega_0)$ by restriction.

Note that if Ω is bounded, we may have $\Omega_0 = \Omega$ in the statement of the theorem.

6.4 Proof of Theorem 6.2, Part III If $mp > n \geq (m-1)p$ and $0 < \lambda < (m-n)/p$, then there exists μ such that $\lambda < \mu < m-(n/p)$. Since Ω_0 is bounded, the imbedding $C^{0,\mu}(\overline{\Omega}_0) \to C^{0,\lambda}(\overline{\Omega}_0)$ is compact by Theorem 1.31. Since $W^{m,p}(\Omega) \to C^{0,\mu}(\overline{\Omega}) \to C^{0,\mu}(\overline{\Omega}_0)$ by Theorem 5.4 and restriction, imbedding (8) is compact for $j = 0$ by Remark 6.3(1).

If $mp > n$, let j^* be the nonnegative integer satisfying $(m-j^*)p > n \geq (m-j^*-1)p$. Then we have the imbedding chain

$$W^{m,p}(\Omega) \to W^{m-j^*,p}(\Omega) \to C^{0,\mu}(\overline{\Omega}_0) \to C(\overline{\Omega}_0) \tag{9}$$

where $0 < \mu < m-j^*-(n/p)$. The last imbedding in (9) is compact by Theorem 1.31. Thus (7) is compact for $j = 0$. ▮

6.5 Proof of Theorem 6.2, Part II As noted in Remark 6.3(4), Ω_0 may be assumed to have the cone property. Since Ω_0 is also bounded it can, by Theorem 4.8, be written as a finite union, $\Omega_0 = \bigcup_{k=1}^M \Omega_k$, where each Ω_k has the strong local Lipschitz property. If $mp > n$, then $W^{m,p}(\Omega) \to W^{m,p}(\Omega_k) \to C(\overline{\Omega}_k)$, the latter imbedding being compact as proved above. If $\{u_i\}$ is a sequence bounded in $W^{m,p}(\Omega)$, we may select (by finite induction on k) a subsequence $\{u_i'\}$ whose restriction to Ω_k converges in $C(\overline{\Omega}_k)$ for each k, $1 \leq k \leq M$. But then $\{u_i'\}$ converges in $C_B{}^0(\Omega_0)$ proving that (5) is compact for $j = 0$. Therefore (6) is also compact by Remark 6.3(3). ▮

6.6 LEMMA Let Ω be a domain in $\mathbb{R}^n$, Ω_0 a subdomain of Ω, and $\Omega_0{}^k$ the intersection of Ω_0 with a k-dimensional plane in $\mathbb{R}^n$ $(1 \le k \le n)$. Let $1 \le q_1 < q_0$ and suppose

$$W^{m,p}(\Omega) \to L^{q_0}(\Omega_0{}^k), \tag{10}$$

$$W^{m,p}(\Omega) \to L^{q_1}(\Omega_0{}^k). \tag{11}$$

Suppose also that (11) is compact. If $q_1 \le q < q_0$, then the imbedding

$$W^{m,p}(\Omega) \to L^{q}(\Omega_0{}^k) \tag{12}$$

(exists and) is compact.

PROOF Let $\lambda = q_1(q_0-q)/q(q_0-q_1)$ and $\mu = q_0(q-q_1)/q(q_0-q_1)$. Clearly $\lambda > 0$ and $\mu \ge 0$. By Hölder's inequality and (10) there exists a constant K such that for all $u \in W^{m,p}(\Omega)$,

$$\begin{aligned} \|u\|_{0,q,\Omega_0{}^k} &\le \|u\|^{\lambda}_{0,q_1,\Omega_0{}^k} \|u\|^{\mu}_{0,q_0,\Omega_0{}^k} \\ &\le K\|u\|^{\lambda}_{0,q_1,\Omega_0{}^k} \|u\|^{\mu}_{m,p,\Omega}. \end{aligned} \tag{13}$$

Let $\{u_i\}$ be a sequence bounded in $W^{m,p}(\Omega)$. Since (11) is compact there exists a subsequence $\{u_i'\}$ that converges, and is therefore a Cauchy sequence in $L^{q_1}(\Omega_0{}^k)$. By (13), $\{u_i'\}$ is a Cauchy sequence in $L^{q}(\Omega_0{}^k)$ as well. Hence (12) is compact. ∎

6.7 PROOF OF THEOREM 6.2, PART I First we deal with the case $j = 0$ of imbeddings (3). Assume, for the moment, that $k = n$ and let $q_0 = np/(n-mp)$. In order to prove that the imbeddings

$$W^{m,p}(\Omega) \to L^{q}(\Omega_0), \qquad 1 \le q < q_0, \tag{14}$$

are compact it suffices, by Lemma 6.6, to do so only for $q = 1$. For $j = 1, 2, 3, \ldots$ let

$$\Omega_j = \{x \in \Omega_0 : \operatorname{dist}(x, \operatorname{bdry}\Omega) > 2/j\}.$$

Let S be a set of functions bounded in $W^{m,p}(\Omega)$. We show that S (when restricted to Ω_0) is precompact in $L^1(\Omega_0)$ by showing that S satisfies the conditions of Theorem 2.21. Accordingly, let $\varepsilon > 0$ be given and for each $u \in W^{m,p}(\Omega)$ set

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega_0 \\ 0 & \text{otherwise.} \end{cases}$$

By Hölder's inequality and since $W^{m,p}(\Omega) \to L^{q_0}(\Omega_0)$, we have

$$\begin{aligned} \int_{\Omega_0 \sim \Omega_j} |u(x)|\,dx &\le \left\{\int_{\Omega_0 \sim \Omega_j} |u(x)|^{q_0}\,dx\right\}^{1/q_0} \left\{\int_{\Omega_0 \sim \Omega_j} 1\,dx\right\}^{1-1/q_0} \\ &\le K_1 \|u\|_{m,p,\Omega} [\operatorname{vol}(\Omega_0 \sim \Omega_j)]^{1-1/q_0}, \end{aligned}$$

with K_1 independent of u. Since Ω_0 has finite volume, j may be selected large enough to ensure that for every $u \in S$,

$$\int_{\Omega_0 \sim \Omega_j} |u(x)|\,dx < \varepsilon$$

and also, for every $h \in \mathbb{R}^n$,

$$\int_{\Omega_0 \sim \Omega_j} |\tilde{u}(x+h) - \tilde{u}(x)|\,dx < \varepsilon/2. \tag{15}$$

Now if $|h| < 1/j$, then $x + th \in \Omega_{2j}$ provided $x \in \Omega_j$ and $0 \le t \le 1$. If $u \in C^\infty(\Omega)$, it follows that

$$\begin{aligned}\int_{\Omega_j} |u(x+h) - u(x)|\,dx &\le \int_{\Omega_j} dx \int_0^1 \left|\frac{d}{dt} u(x+th)\right| dt \\ &\le |h| \int_0^1 dt \int_{\Omega_{2j}} |\operatorname{grad} u(y)|\,dy \\ &\le |h|\,\|u\|_{1,1,\Omega_0} \le K_2 |h|\,\|u\|_{m,p,\Omega},\end{aligned} \tag{16}$$

where K_2 is independent of u. Since $C^\infty(\Omega)$ is dense in $W^{m,p}(\Omega)$, (16) holds for any $u \in W^{m,p}(\Omega)$. Hence if $|h|$ is sufficiently small, we have from (15) and (16) that

$$\int_{\Omega_0} |\tilde{u}(x+h) - \tilde{u}(x)| < \varepsilon.$$

Hence S is precompact in $L^1(\Omega_0)$ by Theorem 2.21, and imbeddings (14) are compact.

Next suppose $k < n$ but $p > 1$. Let r be chosen so that $1 < r < p$ and $n - mr < k$. Let ν be the largest integer less than mr; let $s = kr/(n-mr)$, and let $q = nr/(n-mr)$. Assuming, as we may, that Ω_0 has the cone property we obtain from inequalities (35) and (36) in the proof of Lemma 5.19,

$$\begin{aligned}\|u\|_{0,1,\Omega_0^k} &\le K_3 \|u\|_{0,s,\Omega_0^k} \\ &\le K_4 \|u\|_{0,q,\Omega_0}^{\lambda} \|u\|_{m,r,\Omega_0}^{1-\lambda} \\ &\le K_5 \|u\|_{0,q,\Omega_0}^{\lambda} \|u\|_{m,p,\Omega}^{1-\lambda},\end{aligned} \tag{17}$$

where $\lambda = n(mr-\nu)/mr(n-\nu)$ satisfies $0 < \lambda < 1$, and where K_3, K_4, and K_5 are independent of u. Note that $1 < q < q_0$. If $\{u_i\}$ is a sequence bounded in $W^{m,p}(\Omega)$, we have shown that it must have a subsequence $\{u_i'\}$ which converges in $L^q(\Omega_0)$. From (17), $\{u_i'\}$ must be a Cauchy sequence in $L^1(\Omega_0^k)$ so that $W^{m,p}(\Omega) \to L^1(\Omega_0^k)$ is compact. By Lemma 6.6 so are the imbeddings $W^{m,p}(\Omega) \to L^q(\Omega_0^k)$ for $1 \le q < kp/(n-mp)$.

Finally, suppose $p = 1$ and $0 \le n-m < k < n$. Then clearly $n-m+1 \le k < n$ so that $2 \le m \le n$. By Theorem 5.4, $W^{m,1}(\Omega) \to W^{m-1,r}(\Omega)$ where $r = n/(n-1) > 1$. Also, $k \ge n-(m-1) > n-(m-1)r$ so the imbedding $W^{m-1,r}(\Omega) \to L^1(\Omega_0{}^k)$ is compact as proved above. This is sufficient to complete the proof of the compactness of (3).

To show that (4) is compact we proceed as follows. If $n = mp$, $p > 1$, and $1 \le q < \infty$, we may select r such that $1 \le r < p$, $k > n-mr > 0$, and $kr/(n-mr) > q$. Assuming again that Ω_0 has the cone property, we have

$$W^{m,p}(\Omega) \to W^{m,r}(\Omega_0) \to L^q(\Omega_0{}^k). \tag{18}$$

The latter imbedding in (18) is compact by (3). If $p = 1$ and $n = m \ge 2$, then, setting $r = n/(n-1) > 1$ so that $n = (n-1)r$, we have for $1 \le q < \infty$,

$$W^{n,1}(\Omega) \to W^{n-1,r}(\Omega) \to L^q(\Omega_0{}^k),$$

the latter imbedding being compact as proved in (18). Finally, if $n = m = p = 1$, then of necessity $k = 1$. Letting $q_0 > 1$ be arbitrary chosen, we prove the compactness of $W^{1,1}(\Omega) \to L^1(\Omega_0)$ exactly as in the case $k = n$ of (3) considered above. Since $W^{1,1}(\Omega) \to L^q(\Omega_0)$ for $1 \le q < \infty$ all these imbeddings are compact by Lemma 6.6. ▌

6.8 The reader may find it instructive to carry out the obvious generalization of Theorem 6.2 to the imbeddings supplied by Theorems 5.35–5.37.

Two Counterexamples

6.9 Two obvious questions arise from consideration of the statement of the Rellich–Kondrachov Theorem 6.2. First, can that theorem be extended to cover unbounded Ω_0? Second, can the "extreme cases"

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega_0{}^k), \qquad 0 < n-mp < k \le n, \quad q = kp/(n-mp) \tag{19}$$

and

$$W^{j+m,p}(\Omega) \to C^{j,\lambda}(\overline{\Omega}_0), \qquad mp > n > (m-1)p, \quad \lambda = m-(n/p) \tag{20}$$

ever be compact?

The first of these questions will be investigated later in this chapter. For the moment we show that, at least for $k = n$, the answer is certainly "no" unless Ω_0 is *quasibounded*, that is, unless

$$\lim_{\substack{x \in \Omega_0 \\ |x| \to \infty}} \operatorname{dist}(x, \operatorname{bdry} \Omega_0) = 0.$$

6.10 EXAMPLE Let Ω be an unbounded domain in $\mathbb{R}^n$ which is not quasibounded. Then there exists a sequence $\{B_i\}$ of mutually disjoint open balls contained in Ω and all having the same positive radius. Let $\phi_1 \in C_0^\infty(B_1)$ and suppose $\|\phi_1\|_{k,p,B_1} = A_{k,p} > 0$ for each $k = 0, 1, 2, \ldots$ and $p \geq 1$. Let ϕ_i be a translate of ϕ_1 having support in B_i. Then clearly $\{\phi_i\}$ is a bounded sequence in $W_0^{j+m,p}(\Omega)$ for any fixed j, m, p. But for any q,

$$\|\phi_i - \phi_k\|_{j,q,\Omega} = [\|\phi_i\|_{j,q,B_i}^q + \|\phi_k\|_{j,q,B_k}^q]^{1/q} = 2^{1/q} A_{j,q} > 0$$

so that $\{\phi_i\}$ cannot have a subsequence converging in $W^{j,q}(\Omega)$. Thus no imbedding of the form $W_0^{j+m,p}(\Omega) \to W^{j,q}(\Omega)$ can be compact. The noncompactness of the other imbeddings of Theorem 6.2 is proved similarly. ∎

We now show that the second question raised in Section 6.9 always has a negative answer.

6.11 EXAMPLE Let Ω be any domain in $\mathbb{R}^n$ and Ω_0 any bounded subdomain of Ω. Let Ω_0^k be the intersection of Ω_0 with a k-dimensional plane in $\mathbb{R}^n$, say (without loss of generality) the plane spanned by the $x_1, \ldots, x_k$ coordinate axes. Let $\{a_1, a_2, \ldots\}$ be a sequence of distinct points in Ω_0^k, and $\{r_1, r_2, \ldots\}$ a sequence of numbers such that $0 < r_i \leq 1$, such that $B_{r_i}(a_i) = \{x \in \mathbb{R}^n : |x - a_i| < r_i\} \subset \Omega_0$, and such that all the balls $B_{r_i}(a_i)$ are mutually disjoint.

Let $\phi \in C_0^\infty(B_1(0))$ satisfy the following conditions:

(i) For each nonnegative integer h, each real $q \geq 1$, and each k, $1 \leq k \leq n$, we have

$$|\phi|_{h,q,\mathbb{R}^k} = |\phi|_{h,q,\mathbb{R}^k \cap B_1(0)}$$
$$= \left\{ \sum_{\substack{|\alpha| = h \\ \alpha_{k+1} = \cdots = \alpha_n = 0}} \|D^\alpha \phi\|_{0,q,\mathbb{R}^k \cap B_1(0)}^q \right\}^{1/q} = A_{h,q,k} > 0.$$

(ii) There exists $a \in B_1(0)$, $a \neq 0$, such that for each nonnegative integer h,

$$|D_1^h \phi(a)| = B_h > 0. \tag{21}$$

Fix $p \geq 1$ and integers $j \geq 0$ and $m \geq 1$. For each i let

$$\phi_i(x) = r_i^{j+m-n/p} \phi((x - a_i)/r_i).$$

Then clearly $\phi_i \in C_0^\infty(B_{r_i}(a_i))$ and a simple computation shows that

$$|\phi_i|_{h,q,\mathbb{R}^k} = r_i^{j+m-n/p-h+k/q} A_{h,q,k}. \tag{22}$$

If $h \leq j+m$, it follows from (22) and $r_i \leq 1$ that

$$|\phi_i|_{h,p,\mathbb{R}^n} \leq A_{h,p,n}$$

so $\{\phi_i\}$ is a bounded sequence in $W^{j+m,p}(\Omega)$.

Suppose $mp < n$ and $n - mp < k \le n$. Taking $q = kp/(n-mp)$, we obtain from (22)

$$\|\phi_i\|_{j,q,\Omega_0^k} \ge |\phi_i|_{j,q,\mathbb{R}^k} = A_{j,q,k}.$$

Since the functions ϕ_i have disjoint supports, we have

$$\|\phi_i - \phi_h\|_{j,q,\Omega_0^k} \ge 2^{1/q} A_{j,q,k} > 0$$

and so no subsequence of $\{\phi_i\}$ can converge in $W^{j,q}(\Omega_0{}^k)$. Thus imbedding (19) cannot be compact.

On the other hand, suppose $mp > n > (m-1)p$ and let $\lambda = m - (n/p)$. Letting $b_i = a_i + r_i a$, we obtain from (21)

$$|D_1{}^j \phi_i(b_i)| = r^{m-(n/p)} |D_1{}^j \phi(a)| = r^\lambda B_j > 0.$$

Let $c_i = a_i + a r_i/|a|$ so that $c_i \in \operatorname{bdry} B_{r_i}(a_i)$ and $|b_i - c_i| = (1 - |a|) r_i$. Again since ϕ_i have disjoint supports,

$$\begin{aligned}\|\phi_i - \phi_h; C^{j,\lambda}(\overline{\Omega}_0)\| &\ge \max_{|\alpha|=j} \sup_{\substack{x,y\in\Omega_0\\ x\ne y}} \frac{|D^\alpha(\phi_i(x)-\phi_h(x)) - D^\alpha(\phi_i(y)-\phi_h(y))|}{|x-y|^\lambda}\\ &\ge \frac{|D_1{}^j\phi_i(b_i) - D_1{}^j\phi_h(b_i) - D_1{}^j\phi_i(c_i) + D_1{}^j\phi_h(c_i)|}{|b_i - c_i|^\lambda}\\ &= \frac{B_j}{(1-|a|)^\lambda} > 0.\end{aligned}$$

Thus no subsequence of $\{\phi_i\}$ can converge in $C^{j,\lambda}(\overline{\Omega}_0)$ and imbedding (20) cannot be compact. ▮

Unbounded Domains—Compact Imbeddings of $W_0^{m,p}(\Omega)$

6.12 Let Ω be an unbounded domain in $\mathbb{R}^n$. We shall be concerned below with determining whether the imbedding

$$W_0^{m,p}(\Omega) \to L^p(\Omega) \tag{23}$$

is compact. If (23) is compact, it will follow as in Remark 6.3(2) and the second paragraph of Section 6.7 that the imbeddings

$$W_0^{j+m,p}(\Omega) \to W^{j,q}(\Omega^k), \qquad 0 < n - mp < k \le n, \quad p \le q < kp/(n-mp),$$

and

$$W_0^{j+m,p}(\Omega) \to W^{j,q}(\Omega^k), \qquad n = mp, \quad 1 \le k \le n, \quad p \le q < \infty$$

are also compact.

As was shown in Example 6.10, imbedding (23) cannot be compact unless Ω is quasibounded. In Theorem 6.13 we give a geometric condition on Ω that is sufficient to guarantee the compactness of (23), and in Theorem 6.16 we give an analytic condition that is necessary and sufficient for the compactness of (23). Both theorems are from the work of Adams [2].

Let Ω_r denote the set $\{x \in \Omega : |x| \geq r\}$. Be it agreed that in the following discussion any cube H referred to will have its faces parallel to the coordinate planes.

6.13 THEOREM Let ν be an integer such that $1 \leq \nu \leq n$ and $mp > \nu$ (or $p = m = \nu = 1$). Suppose that for every $\varepsilon > 0$ there exist numbers h and r with $0 < h \leq 1$ and $r \geq 0$ such that for every cube $H \subset \mathbb{R}^n$ having edge length h and having nonempty intersection with Ω_r we have

$$\mu_{n-\nu}(H,\Omega)/h^{-\nu} \geq h^p/\varepsilon,$$

where $\mu_{n-\nu}(H,\Omega)$ is the maximum, taken over all projections P onto $(n-\nu)$-dimensional faces of H, of the area [that is, $(n-\nu)$-measure] of $P(H \sim \Omega)$. Then imbedding (23) is compact.

6.14 The above theorem shows that for given quasibounded Ω the compactness of (23) may depend in an essential way on the dimension of bdry Ω. Let us consider the two extreme cases $\nu = 1$ and $\nu = n$. For $\nu = n$ the condition of the theorem places on Ω only the minimal restriction of quasiboundedness. Thus if $mp > n$, then (23) is compact for any quasibounded Ω. It can also be shown that if $p > 1$ and Ω is quasibounded and has boundary consisting entirely of isolated points with no finite accumulation point, then (23) cannot be compact unless $mp > n$.

If $\nu = 1$, the conditions of Theorem 6.13 make no requirement of m and p but do require that bdry Ω be "essentially $(n-1)$ dimensional." Any quasibounded domain whose boundary consists of reasonably regular $(n-1)$-dimensional surfaces will satisfy these conditions. An example of such a domain is the "spiny urchin" (Fig. 5), a domain in $\mathbb{R}^2$ obtained by deleting from the plane the union of all the sets S_k $(k = 1,2,\ldots)$ specified in polar coordinates by

$$S_k = \{(r,\theta) : r \geq k,\ \theta = n\pi/2^k,\ n = 1,2,\ldots,2^{k+1}\}.$$

Note that this domain, though quasibounded, is simply connected and has empty exterior.

More generally, if ν is the largest integer less than mp the conditions of Theorem 6.13 require that in a certain sense the part of the boundary of Ω having dimension at least $n-\nu$ should bound a quasibounded domain.

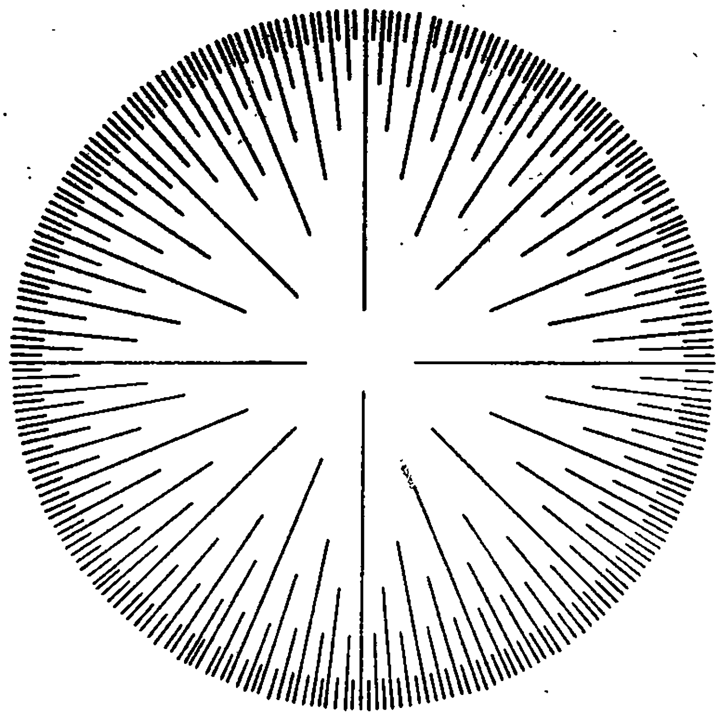


FIG. 5 A "spiny urchin."

6.15 Let H be a cube of side h in $\mathbb{R}^n$ and E a closed subset of H. Given m and p we define a functional $I_H^{m,p}$ on $C^\infty(H)$ by

$$I_H^{m,p}(u) = \sum_{1 \le j \le m} h^{jp} |u|_{j,p,H}^p = \sum_{1 \le |\alpha| \le m} h^{|\alpha|p} \int_H |D^\alpha u(x)|^p \, dx.$$

We denote by $C^{m,p}(H,E)$ the "(m,p)-capacity" of E in H defined by

$$C^{m,p}(H,E) = \inf_{u \in C^\infty(H,E)} \frac{I_H^{m,p}(u)}{\|u\|_{0,p,H}^p},$$

where $C^\infty(H,E)$ is the set of all functions $u \in C^\infty(H)$ that vanish identically in a neighborhood of E. Clearly $C^{m,p}(H,E) \le C^{m+1,p}(H,E)$ and for $E \subset F \subset H$, $C^{m,p}(H,E) \le C^{m,p}(H,F)$. Theorem 6.13 will be deduced from the following theorem characterizing in terms of the above capacity those domains for which (23) is compact.

6.16 THEOREM Imbedding (23) is compact if and only if Ω satisfies the following condition: For every $\varepsilon > 0$ there exists $h \le 1$ and $r \ge 0$ such that the inequality

$$C^{m,p}(H, H \sim \Omega) \ge h^p/\varepsilon$$

holds for every n-cube H having edge length h and having nonempty intersection with Ω_r. (This condition clearly implies that Ω is quasibounded.)

6.17 LEMMA There exists a constant $K = K(n,p)$ such that for any n-cube H of edge length h, any measurable subset A of H with positive volume, and any $u \in C^1(H)$, we have

$$\|u\|_{0,p,H}^p \le \frac{2^{p-1}h^n}{\operatorname{vol} A}\|u\|_{0,p,A}^p + K\frac{h^{n+p}}{\operatorname{vol} A}\|\operatorname{grad} u\|_{0,p,H}^p. \tag{24}$$

PROOF Let $y \in A$ and $x = (\rho, \phi) \in H$, where (ρ, ϕ) are spherical polar coordinates centered at y, in terms of which the volume element is given by $dx = \omega(\phi)\rho^{n-1}\,d\rho\,d\phi$. Let bdry H be specified by $\rho = f(\phi)$, $\phi \in \Sigma$. Clearly $f(\phi) \le \sqrt{n}\,h$. Since

$$u(x) = u(y) + \int_0^\rho \frac{d}{dr}u(r,\phi)\,dr,$$

we have by Hölder's inequality,

$$\begin{aligned}
&\int_H |u(x)|^p\,dx \\
&\quad\le 2^{p-1}|u(y)|^p h^n + 2^{p-1}\int_H \left|\int_0^\rho \frac{d}{dr}u(r,\phi)\,dr\right|^p dx \\
&\quad\le 2^{p-1}h^n|u(y)|^p + 2^{p-1}\int_\Sigma \omega(\phi)\,d\phi \int_0^{f(\phi)} \rho^{n+p-2}\,d\rho \int_0^\rho |\operatorname{grad} u(r,\phi)|^p\,dr \\
&\quad\le 2^{p-1}h^n|u(y)|^p + \frac{2^{p-1}}{n+p-1}(\sqrt{n}\,h)^{n+p-1}\int_H \frac{|\operatorname{grad} u(z)|^p}{|z-y|^{n-1}}\,dz.
\end{aligned}$$

Integration of y over A using Lemma 5.47 leads to

$$(\operatorname{vol} A)\|u\|_{0,p,H}^p \le 2^{p-1}h^n\|u\|_{0,p,A}^p + Kh^{n+p}\int_H |\operatorname{grad} u(x)|^p\,dx$$

from which (24) follows at once. ∎

6.18 PROOF OF THEOREM 6.16 (*Necessity*) Suppose Ω does not satisfy the condition stated in the theorem. Then there exists a finite constant $K_1 = 1/\varepsilon$ such that for every h with $0 < h \le 1$ there exists a sequence $\{H_j\}$ of mutually disjoint cubes of edge length h which intersect Ω and for which

$$C^{m,p}(H_j, H_j \sim \Omega) < K_1 h^p.$$

By the definition of capacity, for each such cube H_j there exists a function $u_j \in C^\infty(H_j, H_j \sim \Omega)$ such that $\|u_j\|_{0,p,H_j}^p = h^n$, $\|\operatorname{grad} u_j\|_{0,p,H_j}^p \le K_1 h^n$, and $\|u_j\|_{m,p,H_j}^p \le K_2(h)$. Let $A_j = \{x \in H_j : |u_j(x)| < \frac{1}{2}\}$. By Lemma 6.17 we have

$$h^n \le \frac{2^{p-1}h^n}{\operatorname{vol} A_j}\cdot\frac{\operatorname{vol} A_j}{2^p} + \frac{KK_1}{\operatorname{vol} A_j}h^{2n+p}$$

from which it follows that $\operatorname{vol} A_j \le K_3 h^{n+p}$. Let us choose h so small that $K_3 h^p \le \frac{1}{3}$, whence $\operatorname{vol} A_j \le \frac{1}{3} \operatorname{vol} H_j$. Choose functions $w_j \in C_0^\infty(H_j)$ such that $w_j(x) = 1$ on a subset S_j of H_j having volume no less than $\frac{2}{3} \operatorname{vol} H_j$, and such that

$$\sup_j \max_{|\alpha| \le m} \sup_{x \in H_j} |D^\alpha w_j(x)| = K_4 < \infty.$$

Then $v_j = u_j w_j \in C_0^\infty(H_j \cap \Omega) \subset C_0^\infty(\Omega)$ and $|v_j(x)| \ge \frac{1}{2}$ on $S_j \cap (H_j \sim A_j)$, a set of volume not less than $h^n/3$. Hence $\|v_j\|_{0,p,H_j}^p \ge h^n/3 \cdot 2^p$. On the other hand,

$$\int_{H_j} |D^\alpha u_j(x)|^p \cdot |D^\beta w_j(x)|^p \, dx \le K_4{}^p K_2(h)$$

provided $|\alpha|, |\beta| \le m$. Hence $\{v_j\}$ is a bounded sequence in $W_0^{m,p}(\Omega)$. Since the functions v_j have disjoint supports, $\|v_j - v_k\|_{0,p,\Omega}^p \ge 2h^n/3 \cdot 2^p$ so imbedding (23) cannot be compact.

(*Sufficiency*) Now suppose Ω satisfies the condition in the statement of the theorem. Let $\varepsilon > 0$ be given and choose $r \ge 0$ and $h \le 1$ such that for every cube H of edge h meeting Ω_r we have $C^{m,p}(H, H \sim \Omega) \ge h^p/\varepsilon^p$. Then for every $u \in C_0^\infty(\Omega)$ we obtain

$$\|u\|_{0,p,H}^p \le (\varepsilon^p/h^p) I_H^{m,p}(u) \le \varepsilon^p \|u\|_{m,p,H}^p.$$

Since a neighborhood of Ω_r can be tesselated by such cubes H we have by summation

$$\|u\|_{0,p,\Omega_r} \le \varepsilon \|u\|_{m,p,\Omega}.$$

That any bounded S in $W_0^{m,p}(\Omega)$ is precompact in $L^p(\Omega)$ now follows at once from Theorems 2.22 and 6.2. ∎

6.19 LEMMA There is a constant K independent of h such that for any cube H in $\mathbb{R}^n$ having edge length h, for every q satisfying $p \le q \le np/(n - mp)$ (or $p \le q < \infty$ if $n = mp$, or $p \le q \le \infty$ if $n < mp$), and for every $u \in C^\infty(H)$ we have

$$\|u\|_{0,q,H} \le K \left\{ \sum_{|\alpha| \le m} h^{|\alpha|p - n + np/q} \|D^\alpha u\|_{0,p,H}^p \right\}^{1/p}. \tag{25}$$

Proof We may suppose H to be centered at the origin and let $\tilde{H}$ be the cube of unit edge concentric with H. Inequality (25) holds for $\tilde{H}$ by the Sobolev imbedding theorem. For given $u \in C^\infty(H)$ we define corresponding $\tilde{u} \in C^\infty(\tilde{H})$ by $\tilde{u}(y) = u(x)$ where $x = hy$. Since

$$\left\{ \int_{\tilde{H}} |\tilde{u}(y)|^q \, dy \right\}^{1/q} \le K \left\{ \sum_{|\alpha| \le m} \int_{\tilde{H}} |D_y{}^\alpha u(y)|^p \, dy \right\}^{1/p}$$

it follows by change of variable that

$$h^{-n/q}\left\{\int_H |u(x)|^q\,dx\right\}^{1/q} \le K\left\{\sum_{|\alpha|\le m} h^{|\alpha|p-n}\int_H |D_x^\alpha u(x)|^p\,dx\right\}^{1/p},$$

whence (25) follows. ∎

6.20 LEMMA If $mp > n$ (or $m = p = n = 1$), there exists a constant $K = K(m,p,n)$ such that for every cube H of edge length h in $\mathbb{R}^n$ and every $u \in C^\infty(H)$ vanishing in a neighborhood of some point $y \in H$, we have

$$\|u\|_{0,p,H}^p \le K I_H^{m,p}(u).$$

PROOF The proof is somewhat similar to that of Lemma 5.15. First suppose $p \le n < mp$. Let (ρ, ϕ) denote polar coordinates centered at y. Then

$$u(\rho,\phi) = \int_0^\rho \frac{d}{dt}u(t,\phi)\,dt.$$

If $n > (m-1)p$, let $q = np/(n-mp+p)$ so that $q > n$. Otherwise let $q > n$ be arbitrary. If $(\rho,\phi) \in H$, then by Hölder's inequality

$$\begin{aligned} |u(\rho,\phi)|^q\rho^{n-1} &\le (\sqrt{n}\,h)^{n-1}\int_0^\rho \left|\frac{d}{dt}u(t,\phi)\right|^q t^{n-1}\,dt\left\{\int_0^{\sqrt{n}h} t^{-(n-1)/(q-1)}\,dt\right\}^{q-1} \\ &\le K_1 h^{q-1}\int_0^\rho \left|\frac{d}{dt}u(t,\phi)\right|^q t^{n-1}\,dt. \end{aligned}$$

It follows, using Lemma 6.19 with $m-1$ replacing m, that

$$\begin{aligned} \|u\|_{0,q,H}^q &\le K_2 h^q \int_H |\operatorname{grad} u(x)|^q\,dx \\ &\le K_2 h^q \sum_{|\alpha|=1} \|D^\alpha u\|_{0,q,H}^q \\ &\le K_3 h^q \sum_{|\alpha|=1}\left\{\sum_{|\beta|\le m-1} h^{|\beta|p-n+np/q}\|D^{\alpha+\beta}u\|_{0,p,H}^p\right\}^{q/p}. \end{aligned} \tag{26}$$

A further application of Hölder's inequality yields

$$\begin{aligned} \|u\|_{0,p,H}^p &\le \|u\|_{0,q,H}^p(\operatorname{vol} H)^{(q-p)/q} \\ &\le K_3^{p/q}\sum_{1\le|\gamma|\le m} h^{|\gamma|p}\|D^\gamma u\|_{0,p,H}^p = K I_H^{m,p}(u). \end{aligned}$$

If $p > n$ (or $p = n = 1$), the result follows directly from (26) with $q = p$,

$$\|u\|_{0,p,H}^p \le K I_H^{1,p}(u) \le K I_H^{m,p}(u).$$

6.21 Proof of Theorem 6.13 Let H be a cube for which, for $mp > \nu$ (or $m = p = \nu = 1$) and $\mu_{n-\nu}(H,\Omega)/h^{n-\nu} \geq h^p/\varepsilon$. Let P be the maximal projection referred to in the statement of the theorem, and let $E = P(H \sim \Omega)$. Without loss of generality we assume that the $(n-\nu)$ dimensional face F of H containing E is parallel to the $x_{\nu+1}, \dots, x_n$ coordinate plane. For each point $x = (x', x'')$ in E, where $x' = (x_1, \dots, x_\nu)$ and $x'' = (x_{\nu+1}, \dots, x_n)$, let $H_{x''}$ be the ν-dimensional cube of edge h in which H intersects the ν-plane through x normal to F. By definition of P there exists $y \in H_{x''} \sim \Omega$. If $u \in C^\infty(H, H \sim \Omega)$, then $u(\cdot, x'') \in C^\infty(H_{x''}, y)$. Applying Lemma 6.20 to $u(\cdot, x'')$, we obtain

$$\int_{H_{x''}} |u(x', x'')|^p \, dx' \leq K_1 \sum_{1 \leq |\alpha| \leq m} h^{|\alpha| p} \int_{H_{x''}} |D^\alpha u(x', x'')|^p \, dx',$$

where K_1 is independent of h, x'', and u. Integrating this inequality over E and denoting $H' = \{x' : x = (x', x'') \in H \text{ for some } x''\}$, we obtain

$$\|u\|_{0,p,H' \times E}^p \leq K_1 I_{H' \times E}^{m,p}(u) \leq K_1 I_H^{m,p}(u).$$

Now we apply Lemma 6.17 with $A = H' \times E$ so that vol $A = h^\nu \mu_{n-\nu}(H, \Omega)$. This yields

$$\|u\|_{0,p,H}^p \leq K_2 \frac{h^{n-\nu}}{\mu_{n-\nu}(H,\Omega)} I_H^{m,p}(u),$$

where K_2 is independent of h. It follows that

$$C^{m,p}(H, H \sim \Omega) \geq \frac{\mu_{n-\nu}(H,\Omega)}{K_2 h^{n-\nu}} \geq \frac{h^p}{\varepsilon K_2}.$$

Hence Ω satisfies the hypothesis of Theorem 6.16 if it satisfies that of Theorem 6.13. ∎

The following two interpolation lemmas enable us to extend Theorem 6.13 to cover imbeddings of $W_0^{m,p}(\Omega)$ into continuous function spaces.

6.22 LEMMA Let $1 \leq p < \infty$ and $0 < \mu \leq 1$. There exists a constant $K = K(n, p, \mu)$ such that for every $u \in C_0^\infty(\mathbb{R}^n)$ we have

$$\sup_{x \in \mathbb{R}^n} |u(x)| \leq K \|u\|_{0,p,\mathbb{R}^n}^\lambda \left\{ \sup_{\substack{x, y \in \mathbb{R}^n \\ x \neq y}} \frac{|u(x) - u(y)|}{|x-y|^\mu} \right\}^{1-\lambda}, \qquad (27)$$

where $\lambda = p\mu/(n + p\mu)$.

PROOF We may assume

$$\sup_{x \in \mathbb{R}^n} |u(x)| = N > 0 \qquad \text{and} \qquad \sup_{x, y \in \mathbb{R}^n} \frac{|u(x) - u(y)|}{|x-y|^\mu} = M < \infty.$$

Let ε satisfy $0 < \varepsilon \le N/2$. There exists $x_0 \in \mathbb{R}^n$ such that $|u(x_0)| \ge N-\varepsilon \ge N/2$. Now $|u(x_0)-u(x)|/|x_0-x|^\mu \le M$ for all x, so

$$|u(x)| \ge |u(x_0)| - M|x_0-x|^\mu \ge \tfrac{1}{2}|u(x_0)|$$

provided $|x-x_0| \le (N/4M)^{1/\mu} = r$. Hence

$$\int_{\mathbb{R}^n} |u(x)|^p\,dx \ge \int_{B_r(x_0)} \left(\frac{|u(x_0)|}{2}\right)^p dx \ge K_1\left(\frac{N-\varepsilon}{2}\right)^p\left(\frac{N}{4M}\right)^{n/\mu}.$$

Since this holds for arbitrarily small ε we have

$$\|u\|_{0,p,\mathbb{R}^n} \ge (K_1^{1/p}/2\cdot 4^{n/\mu p})\,N^{1+n/p\mu}M^{-n/p\mu}$$

from which (27) follows at once. ∎

6.23 LEMMA Let Ω be an arbitrary domain in $\mathbb{R}^n$, and let $0 < \lambda < \mu \le 1$. For every function $u \in C^{0,\mu}(\bar{\Omega})$ we have

$$\|u; C^{0,\lambda}(\bar{\Omega})\| \le 3^{1-\lambda/\mu}\|u; C(\bar{\Omega})\|^{1-\lambda/\mu}\|u; C^{0,\mu}(\bar{\Omega})\|^{\lambda/\mu}. \tag{28}$$

PROOF Let $p = \mu/\lambda$, $p' = p/(p-1)$, and let

$$A_1 = \|u; C(\bar{\Omega})\|^{1/p}, \qquad B_1 = \sup_{\substack{x,y\in\Omega\\ x\ne y}}\left\{\frac{|u(x)-u(y)|}{|x-y|^\mu}\right\}^{1/p},$$

$$A_2 = \|u; C(\bar{\Omega})\|^{1/p'}, \qquad B_2 = \sup_{\substack{x,y\in\Omega\\ x\ne y}} |u(x)-u(y)|^{1/p'}.$$

Clearly $A_1{}^p + B_1{}^p = \|u; C^{0,\mu}(\bar{\Omega})\|$ and $B_2^{p'} \le 2\|u; C(\bar{\Omega})\|$. We have

$$\begin{aligned}\|u; C^{0,\lambda}(\bar{\Omega})\| &= \|u; C(\bar{\Omega})\| + \sup_{\substack{x,y\in\Omega\\ x\ne y}}\frac{|u(x)-u(y)|}{|x-y|^\lambda}\\ &\le A_1A_2 + B_1B_2\\ &\le \{A_1{}^p + B_1{}^p\}^{1/p}\{A_2^{p'}+B_2^{p'}\}^{1/p'}\\ &\le \|u; C^{0,\mu}(\bar{\Omega})\|^{\lambda/\mu}(3\|u; C(\bar{\Omega})\|)^{1-\lambda/\mu}\end{aligned}$$

as required. ∎

6.24 THEOREM Let Ω satisfy the hypothesis of Theorem 6.13. Then the following imbeddings are compact:

$$W_0^{j+m,p}(\Omega) \to C^j(\bar{\Omega}) \qquad \text{if } mp > n \tag{29}$$

$$W_0^{j+m,p}(\Omega) \to C^{j,\lambda}(\bar{\Omega}) \qquad \text{if } mp > n \ge (m-1)p \text{ and } 0 < \lambda < m-(n/p). \tag{30}$$

Proof It is sufficient to deal with the case $j = 0$. If $mp > n$, let j^* be the non-negative integer satisfying $(m-j^*)p > n \ge (m-j^*-1)p$. Then we have the chain

$$W_0^{m,p}(\Omega) \to W_0^{m-j^*,p}(\Omega) \to C^{0,\mu}(\overline{\Omega}) \to C(\overline{\Omega}),$$

where $0 < \mu < m - j^* - (n/p)$. If $\{u_i\}$ is a sequence bounded in $W_0^{m,p}(\Omega)$, then $\{u_i\}$ is also bounded in $C^{0,\mu}(\overline{\Omega})$. By Theorem 6.13, $\{u_i\}$ has a subsequence $\{u_i'\}$ converging in $L^p(\Omega)$. By (27), which applies by completion to the functions u_i, $\{u_i'\}$ is a Cauchy sequence in $C(\overline{\Omega})$ and so converges there. Hence (29) is compact for $j = 0$. Furthermore, if $mp > n \ge (m-1)p$ (that is, $j^* = 0$) and $0 < \lambda < \mu$, then by (28), $\{u_i'\}$ is also a Cauchy sequence in $C^{0,\lambda}(\overline{\Omega})$ whence (30) is also compact. ∎

An Equivalent Norm for $W_0^{m,p}(\Omega)$

6.25 Closely related to the problem of determining for which unbounded domains Ω the imbedding $W_0^{m,p}(\Omega) \to L^p(\Omega)$ is compact, is that concerned with determining for which domains Ω the seminorm $|\cdot|_{m,p,\Omega}$ defined by

$$|u|_{m,p,\Omega} = \left\{ \sum_{|\alpha|=m} \|D^\alpha u\|_{0,p,\Omega}^p \right\}^{1/p}$$

is actually a norm on $W_0^{m,p}(\Omega)$, equivalent to the given norm $\|\cdot\|_{m,p,\Omega}$. Such is certainly the case for any bounded domain as we now show.

6.26 A domain $\Omega \subset \mathbb{R}^n$ is said to have *finite width* if it lies between two parallel hyperplanes. Let Ω be such a domain and suppose, without loss of generality, that Ω lies between the hyperplanes $x_n = 0$ and $x_n = d$. Letting $x = (x', x_n)$ where $x' = (x_1, \dots, x_{n-1})$, we have for any $\phi \in C_0^\infty(\Omega)$

$$\phi(x) = \int_0^{x_n} \frac{d}{dt} \phi(x', t)\, dt$$

so that

$$\begin{aligned} \|\phi\|_{0,p,\Omega}^p &= \int_{\mathbb{R}^{n-1}} dx' \int_0^d |\phi(x)|^p\, dx_n \\ &\le \int_{\mathbb{R}^{n-1}} dx' \int_0^d x_n^{p-1}\, dx_n \int_0^d |D_n \phi(x', t)|^p\, dt \\ &\le (d^p/p) |\phi|_{1,p,\Omega}^p \end{aligned} \tag{31}$$

and

$$|\phi|_{1,p,\Omega}^p \le \|\phi\|_{1,p,\Omega}^p = \|\phi\|_{0,p,\Omega}^p + |\phi|_{1,p,\Omega}^p \le (1 + (d^p/p)) |\phi|_{1,p,\Omega}^p.$$

Successive application of the above inequality to derivatives $D^\alpha\phi$, $|\alpha| \le m-1$, then yields

$$|\phi|_{m,p,\Omega} \le \|\phi\|_{m,p,\Omega} \le K|\phi|_{m,p,\Omega} \tag{32}$$

and by completion (32) holds for all $u \in W_0^{m,p}(\Omega)$. Inequality (31) is often called *Poincaré's inequality*.

6.27 An unbounded domain Ω in $\mathbb{R}^n$ is called *quasicylindrical* provided

$$\limsup_{x\in\Omega,\,|x|\to\infty} \operatorname{dist}(x, \operatorname{bdry}\Omega) < \infty.$$

Evidently every quasibounded domain is quasicylindrical, as is every (unbounded) domain of finite width. We leave to the reader the construction of a suitable counterexample to show that if Ω is not quasicylindrical, then $|\cdot|_{m,p,\Omega}$ is not an equivalent norm to $\|\cdot\|_{m,p,\Omega}$ on $W_0^{m,p}(\Omega)$.

The following theorem is clearly an analog of Theorem 6.13.

6.28 THEOREM Suppose there exist constants K, R, h, and ν with $0 < K \le 1$, $0 \le R < \infty$, $0 < h < \infty$, and $1 \le \nu \le n$, ν an integer, such that either $\nu < p$ or $\nu = p = 1$, and such that for every cube H in $\mathbb{R}^n$, having edge length h and having nonempty intersection with $\Omega_R = \{x \in \Omega : |x| \ge R\}$ we have

$$\mu_{n-\nu}(H,\Omega)/h^{n-\nu} \ge K,$$

where $\mu_{n-\nu}(H,\Omega)$ is as defined in the statement of Theorem 6.13. Then $|\cdot|_{m,p,\Omega}$ and $\|\cdot\|_{m,p,\Omega}$ are equivalent norms for $W_0^{m,p}(\Omega)$.

PROOF As noted in Section 6.26 it is sufficient to prove that $\|u\|_{0,p,\Omega} \le K_1|u|_{1,p,\Omega}$ for $u \in C_0^\infty(\Omega)$. Let H be a cube of edge length h having nonempty intersection with Ω_R. Since $\nu < p$ (or $\nu = p = 1$) the proof of Theorem 6.13 (Section 6.21) shows that

$$C^{1,p}(H, H \sim \Omega) \ge \mu_{n-\nu}(H,\Omega)/K_2 h^{n-\nu} \ge K/K_2$$

for all $u \in C_0^\infty(H)$, K_2 being independent of u. Hence

$$\|u\|_{0,p,H}^p \le (K_2/K) I_H^{1,p}(u) = K_3 |u|_{1,p,H}^p. \tag{33}$$

By summing (33) over the cubes H comprising a tesselation of some neighborhood of Ω_R, we obtain

$$\|u\|_{0,p,\Omega_R}^p \le K_3 |u|_{1,p,\Omega}^p. \tag{34}$$

It remains to be proven that

$$\|u\|_{0,p,B_R}^p \le K_4 |u|_{1,p,\Omega}^p,$$

where $B_R = \{x \in \mathbb{R}^n : |x| < R\}$. Let (ρ, ϕ) denote spherical polar coordinates of the point x in $\mathbb{R}^n$ $(\rho \geq 0, \phi \in \Sigma)$ and denote the volume element by $dx = \rho^{n-1}\omega(\phi)\, d\rho\, d\phi$. For any $u \in C^\infty(\mathbb{R}^n)$ we have

$$u(\rho, \phi) = u(\rho + R, \phi) - \int_\rho^{R+\rho} \frac{d}{dt} u(t, \phi)\, dt$$

so that

$$|u(\rho, \phi)|^p \leq 2^{p-1}|u(\rho+R, \phi)|^p + 2^{p-1}R^{p-1}\rho^{1-n} \int_\rho^{R+\rho} |\operatorname{grad} u(t, \phi)|^p t^{n-1}\, dt.$$

Hence

$$\begin{aligned}
\|u\|_{0,p,B_R}^p &= \int_\Sigma \omega(\phi)\, d\phi \int_0^R |u(\rho, \phi)|^p \rho^{n-1}\, d\rho \\
&\leq 2^{p-1} \int_\Sigma \omega(\phi)\, d\phi \int_0^R |u(\rho+R, \phi)|^p (\rho+R)^{n-1}\, d\rho \\
&\quad + 2^{p-1}R^p \int_\Sigma \omega(\phi)\, d\phi \int_0^{2R} |\operatorname{grad} u(t, \phi)|^p t^{n-1}\, dt.
\end{aligned}$$

Therefore we have for $u \in C_0^\infty(\Omega)$

$$\begin{aligned}
\|u\|_{0,p,B_R}^p &\leq 2^{p-1}\|u\|_{0,p,B_{2R}\sim B_R}^p + 2^{p-1}R^p|u|_{1,p,B_{2R}}^p \\
&\leq 2^{p-1}\|u\|_{0,p,\Omega_R}^p + 2^{p-1}R^p|u|_{1,p,\Omega}^p \leq K_4|u|_{1,p,\Omega}^p
\end{aligned}$$

by (34). ∎

Unbounded Domains—Decay at Infinity

6.29 The vanishing, in a generalized sense, on the boundary of Ω of elements of $W_0^{m,p}(\Omega)$ played a critical role in our earlier establishment of the compactness of the imbedding

$$W_0^{m,p}(\Omega) \to L^p(\Omega) \tag{35}$$

for certain unbounded domains. For elements of $W^{m,p}(\Omega)$ we no longer have this vanishing and the question remains: when, if ever, is the imbedding

$$W^{m,p}(\Omega) \to L^p(\Omega) \tag{36}$$

compact for unbounded Ω, or even for bounded Ω which are sufficiently irregular that no imbedding of the form

$$W^{m,p}(\Omega) \to L^q(\Omega) \tag{37}$$

exists for any $q > p$? Note that if Ω has finite volume, the existence of (37) for some $q > p$ implies the compactness of (36) by the method of the first part of Section 6.7. By Theorem 5.30 imbedding (37) cannot, however, exist if $q > p$ and Ω is unbounded but has finite volume.

6.30 EXAMPLE For $j = 1, 2, \ldots$ let B_j be an open ball in $\mathbb{R}^n$ having radius r_j, and suppose $\bar{B}_j \cap \bar{B}_k$ is empty if $j \neq k$. Let $\Omega = \bigcup_{j=1}^{\infty} B_j$; Ω may be bounded or unbounded. The sequence $\{u_j\}$ defined by

$$u_j(x) = \begin{cases} (\operatorname{vol} B_j)^{-1/p} & \text{if} \quad x \in \bar{B}_j \\ 0 & \text{if} \quad x \notin \bar{B}_j \end{cases}$$

is clearly bounded in $W^{m,p}(\Omega)$ but not precompact in $L^p(\Omega)$ no matter how rapidly $r_j \to 0$ as j tends to infinity. Hence (36) is not compact. [Note that (35) is compact by Theorem 6.13 provided $\lim_{j\to\infty} r_j = 0$.] If Ω is bounded, imbedding (37) cannot exist for any $q > p$.

6.31 There do exist unbounded domains Ω for which imbedding (36) is compact (see Section 6.48). An example of such a domain was given by Adams and Fournier [3] and it provided a basis for an investigation of the general problem by the same authors [4]. The approach of this latter paper is used in the following sections. First we concern ourselves with necessary conditions for the compactness of (37) ($q \geq p$). These conditions involve rapid decay at infinity for any unbounded domain (see Theorem 6.40). The techniques involved in the proof also yield a strengthened version of Theorem 5.30 (viz. Theorem 6.36) and a converse of the assertion [see Remark 5.5(6)] that $W^{m,p}(\Omega) \to L^p(\Omega)$ for $1 \leq q < p$ if Ω has finite volume.

A sufficient condition for the compactness of (36) is given in Theorem 6.47. It applies to many domains, bounded and unbounded, to which neither the Rellich–Kondrachov theorem, nor any generalizations of that theorem obtained by the same techniques can be applied (e.g., exponential cusps—see Example 6.49).

6.32 Let T be a tesselation of $\mathbb{R}^n$ by closed n-cubes of edge length h. If H is one of the cubes in T, let $N(H)$ denote the cube of side $3h$ concentric with H and having faces parallel to those of H. The $N(H)$ will be called the *neighborhood* of H. Clearly $N(H)$ is the union of the 3^n cubes in T which intersect H. By the *fringe* of H we shall mean the shell $F(H) = N(H) \sim H$.

Let Ω be a given domain in $\mathbb{R}^n$ and T a given tesselation as above. Let $\lambda > 0$. A cube $H \in T$ will be called *λ-fat* (with respect to Ω) if

$$\mu(H \cap \Omega) > \lambda\mu(F(H) \cap \Omega),$$

where μ denotes n-dimensional Lebesgue measure in $\mathbb{R}^n$. (We use "μ" instead of "vol" for notational simplicity in the following discussion where the symbol must be used many times.) If H is not λ-fat, it is called *λ-thin*.

6.33 THEOREM Suppose that there exists a compact imbedding of the form

$$W^{m,p}(\Omega) \to L^q(\Omega) \tag{38}$$

for some $q \geq p$. Then for every $\lambda > 0$ and every tesselation T of $\mathbb{R}^n$ by cubes of fixed size, T has only finitely many λ-fat cubes.

PROOF Suppose, to the contrary, that for some $\lambda > 0$ there exists a tesselation T of $\mathbb{R}^n$ by cubes of edge length h containing a sequence $\{H_j\}_{j=1}^{\infty}$ of λ-fat cubes. Passing to a subsequence if necessary we may assume that $N(H_j) \cap N(H_k) = \emptyset$ if $j \neq k$. For each j there exists $\phi_j \in C_0^\infty(N(H_j))$ such that

(i) $|\phi_j(x)| \leq 1$ for all $x \in \mathbb{R}^n$,
(ii) $\phi_j(x) = 1$ if $x \in H_j$,
(iii) $|D^\alpha \phi_j(x)| \leq M$ for all $x \in \mathbb{R}^n$ and $0 \leq |\alpha| \leq m$,

where $M = M(n, m, h)$ is independent of j. Let $\psi_j = c_j \phi_j$, where the positive constant c_j is so chosen that

$$\|\psi_j\|_{0,q,\Omega}^q \geq c_j^q \int_{H_j \cap \Omega} |\phi_j(x)|^q \, dx = c_j^q \mu(H_j \cap \Omega) = 1.$$

But then

$$\begin{aligned}\|\psi_j\|_{m,p,\Omega}^p &= c_j^p \sum_{0 \leq |\alpha| \leq m} \int_{N(H_j) \cap \Omega} |D^\alpha \phi_j(x)|^p \, dx \\ &\leq M^p c_j^p \mu(N(H_j) \cap \Omega) \\ &< M^p c_j^p \mu(H_j \cap \Omega)[1 + (1/\lambda)] = M^p[1 + (1/\lambda)] c_j^{p-q},\end{aligned}$$

since H_j is λ-fat. Now $\mu(H_j \cap \Omega) \leq \mu(H_j) = h^n$ so $c_j \geq h^{-n/q}$. Since $p - q \leq 0$, $\{\psi_j\}$ is bounded in $W^{m,p}(\Omega)$. Since the functions ψ_j have disjoint supports, $\{\psi_j\}$ cannot be precompact in $L^q(\Omega)$, contradicting the compactness of (38). Thus T can possess only finitely many λ-fat cubes. ∎

6.34 COROLLARY Suppose there exists an imbedding (38) for some $q > p$. If T is a tesselation of $\mathbb{R}^n$ by cubes of fixed edge length h, and if $\lambda > 0$ is given, then there exists $\varepsilon > 0$ such that $\mu(H \cap \Omega) \geq \varepsilon$ for every λ-fat $H \in T$.

PROOF Suppose, to the contrary, there exists a sequence $\{H_j\}$ of λ-fat cubes with $\lim_{j\to\infty} \mu(H_j \cap \Omega) = 0$. If c_j is defined as in the above proof, we have $\lim_{j\to\infty} c_j = \infty$. But then $\lim_{j\to\infty} \|\psi_j\|_{m,p,\Omega} = 0$ since $p < q$. Since $\{\psi_j\}$ is bounded away from 0 in $L^q(\Omega)$ we have contradicted the continuity of imbedding (38). ∎

6.35 Let us consider the implications of the above corollary. If imbedding (38) exists for some $q > p$, then one of the following alternatives must hold:

(a) There exists $\varepsilon > 0$ and a tesselation T of $\mathbb{R}^n$ consisting of cubes of fixed size such that $\mu(H \cap \Omega) \geq \varepsilon$ for infinitely many cubes $H \in T$.

(b) For every $\lambda > 0$ and every tesselation T of $\mathbb{R}^n$ by cubes of fixed size, T contains only finitely many λ-fat cubes.

We shall show in Theorem 6.37 that (b) implies that Ω has finite volume. By Theorem 5.30, (b) is therefore inconsistent with the existence of (38) for $q > p$. On the other hand, (a) implies that $\mu(\{x \in \Omega : N \leq |x| \leq N+1\})$ does not tend to zero as N tends to infinity. We have therefore proved the following theorem strengthening Theorem 5.30.

6.36 THEOREM Let Ω be an unbounded domain in $\mathbb{R}^n$ satisfying

$$\limsup_{N \to \infty} \operatorname{vol}\{x \in \Omega : N \leq |x| \leq N+1\} = 0.$$

Then there can be no imbedding of the form (38) for any $q > p$.

6.37 THEOREM Suppose that imbedding (38) is compact for some $q \geq p$. Then Ω has finite volume.

Proof Let T be a tesselation of $\mathbb{R}^n$ by cubes of unit edge, and let $\lambda = 1/[2(3^n - 1)]$. Let P be the union of the finitely many λ-fat cubes in T. Clearly $\mu(P \cap \Omega) \leq \mu(P) < \infty$. Let H be a λ-thin cube. Let H_1 be one of the $(3^n - 1)$ cubes of T contained in the fringe $F(H)$ and selected so that $\mu(H_1 \cap G)$ is maximal. Thus

$$\mu(H \cap G) \leq \lambda\mu(F(H) \cap G) \leq \lambda(3^n - 1)\mu(H_1 \cap G) = \tfrac{1}{2}\mu(H_1 \cap G).$$

If H_1 is also λ-thin, we may select $H_2 \in T$, $H_2 \subset F(H_1)$ such that $\mu(H_1 \cap \Omega) \leq \frac{1}{2}\mu(H_2 \cap \Omega)$.

Suppose an infinite chain $\{H, H_1, H_2, \ldots\}$ of λ-thin cubes can be constructed in the above manner. Then

$$\mu(H \cap \Omega) \leq \tfrac{1}{2}\mu(H_1 \cap \Omega) \leq \cdots \leq (1/2^j)\mu(H_j \cap \Omega) \leq 1/2^j$$

for each j, since $\mu(H_j \cap \Omega) \leq \mu(H_j) = 1$. Hence $\mu(H \cap \Omega) = 0$. Denoting by P_∞ the union of λ-thin cubes $H \in T$ for which such an infinite chain can be constructed, we have $\mu(P_\infty \cap \Omega) = 0$.

Let P_j denote the union of the λ-thin cubes $H \in T$ for which some such chain ends on the jth step (that is, H_j is λ-fat). Any particular λ-fat cube H' can occur as the end H_j of a chain beginning at H only if H is contained in the cube of edge $2j+1$ centered on H'. Hence there are at most $(2j+1)^n$ such cubes

$H \subset P_j$ having H' as chain endpoint. Thus

$$\begin{aligned}\mu(P_j \cap \Omega) &= \sum_{H \subset P_j} \mu(H \cap \Omega) \\ &\leq (1/2^j) \sum_{H \subset P_j} \mu(H_j \cap \Omega) \\ &\leq [(2j+1)^n/2^j] \sum_{H' \subset P} \mu(H' \cap \Omega) = [(2j+1)^n/2^j]\, \mu(P \cap \Omega),\end{aligned}$$

so that $\sum_{j=1}^{\infty} \mu(P_j \cap \Omega) < \infty$. Since $\mathbb{R}^n = P \cup P_\infty \cup P_1 \cup P_2 \cup \cdots$ we have $\mu(\Omega) < \infty$. ∎

Suppose $1 \leq q < p$. By Theorem 2.8 the imbedding

$$W^{m,p}(\Omega) \to L^q(\Omega) \tag{39}$$

exists if $\operatorname{vol} \Omega < \infty$. We are now in a position to prove the converse.

6.38 THEOREM Suppose imbedding (39) exists for some p, q such that $1 \leq q < p$. Then Ω has finite volume.

PROOF Let T, λ be as in the proof of the previous theorem. Once again let P denote the union of the λ-fat cubes in T. If we can show that $\mu(P \cap \Omega)$ is finite, it will follow by the same argument used in the above theorem that vol Ω is finite.

Accordingly, suppose $\mu(P \cap \Omega)$ is not finite. Then there exists a sequence $\{H_j\}_{j=1}^{\infty}$ of λ-fat cubes in T such that $\sum_{j=1}^{\infty} \mu(H_j \cap \Omega) = \infty$. If L is the lattice of centers of the cubes in T, we may break up L into 3^n mutually disjoint sublattices $\{L_i\}_{i=1}^{3^n}$ each having period 3 in each coordinate direction. For each i let T_i be the set of all cubes in T with centers in L_i. For some i we must have $\sum_{\lambda\text{-fat},\, H \in T_i} \mu(H \cap \Omega) = \infty$. Thus we may assume the cubes of the sequence $\{H_j\}$ all belong to T_i for some fixed i, so that $N(H_j) \cap N(H_k)$ do not overlap.

Let the integer j_1 be chosen so that

$$2 \leq \sum_{j=1}^{j_1} \mu(H_j \cap \Omega) < 4.$$

Let ϕ_j be as in the proof of Theorem 6.33, and let

$$\psi_1(x) = 2^{-1/p} \sum_{j=1}^{j_1} \phi_j(x).$$

We have, since the supports of the functions ϕ_j are disjoint, and since the cubes H_j are λ-fat, for $|\alpha| \le m$,

$$\begin{aligned}\|D^\alpha \psi_1\|_{0,p,\Omega}^p &= \tfrac{1}{2} \sum_{j=1}^{j_1} \int_\Omega |D^\alpha \phi_j(x)|^p \, dx \\ &\le \tfrac{1}{2} M^p \sum_{j=1}^{j_1} \mu(N(H_j) \cap \Omega) \\ &< \tfrac{1}{2} M^p (1 + (1/\lambda)) \sum_{j=1}^{j_1} \mu(H_j \cap \Omega) < 2M^p(1 + (1/\lambda)).\end{aligned}$$

On the other hand,

$$\|\psi_1\|_{0,q,\Omega}^q \ge 2^{-q/p} \sum_{j=1}^{j_1} \mu(H_j \cap \Omega) \ge 2^{1-q/p}.$$

Having so defined j_1 and ψ_1, we may now define $j_2, j_3, \dots$ and $\psi_2, \psi_3, \dots$ inductively so that

$$2^k \le \sum_{j=j_{k-1}}^{j_k} \mu(H_j \cap \Omega) < 2^{k+1}$$

and

$$\psi_k(x) = 2^{-k/p} k^{-2/p} \sum_{j=j_{k-1}}^{j_k} \phi_j(x).$$

As above we have for $|\alpha| \le m$,

$$\|D^\alpha \psi_k\|_{0,p,\Omega}^p < (2/k^2) M^p (1 + (1/\lambda))$$

and

$$\|\psi_k\|_{0,q,\Omega}^q \ge 2^{k(1-q/p)} (1/k)^{2q/p}.$$

Thus $\psi = \sum_{k=1}^\infty \psi_k$ belongs to $W^{m,p}(\Omega)$ but not $L^q(\Omega)$, contradicting (39). Hence $\mu(P \cap \Omega) < \infty$ as required. ▌

6.39 If there exists a compact imbedding of the form (38) for some $q \ge p$, then, as we have shown, Ω has finite volume. In fact, considerably more is true: $\mu(\{x \in \Omega : |x| \ge R\})$ must tend very rapidly to zero as $R \to \infty$, as we now show.

If Q is a union of cubes H in some tesselation of $\mathbb{R}^n$, we extend the notions of neighborhood and fringe to Q in an obvious manner:

$$N(Q) = \bigcup_{H \subset Q} N(H), \qquad F(Q) = N(Q) \sim Q.$$

Given $\delta > 0$, let $\lambda = \delta/3^n(1 + \delta)$. If all of the cubes $H \subset Q$ are λ-thin, then Q is itself δ-thin in the sense that

$$\mu(Q \cap \Omega) \le \delta \mu(F(Q) \cap \Omega). \tag{40}$$

To see this note that as H runs through the cubes comprising Q, $F(H)$ covers

$N(Q)$ at most 3^n times. Hence

$$\mu(Q \cap \Omega) = \sum_{H \subset Q} \mu(H \cap \Omega) \le \lambda \sum_{H \subset Q} \mu(F(H) \cap \Omega) \le 3^n \lambda \mu(N(Q) \cap \Omega)$$
$$= 3^n \lambda [\mu(Q \cap \Omega) + \mu(F(Q) \cap \Omega)]$$

from which (40) follows by transposition (permissible since $\mu(\Omega) < \infty$) and since $3^n\lambda/(1-3^n\lambda) = \delta$.

For any measurable set $S \subset \mathbb{R}^n$ let Q be the union of all cubes H of our tesselation whose interiors intersect S, and define $F(S) = F(Q)$. If S is at a positive distance from the finitely many λ-fat cubes in the tesselation, then Q consists of λ-thin cubes and we obtain from (40),

$$\mu(S \cap \Omega) \le \mu(Q \cap \Omega) \le \delta \mu(F(S) \cap \Omega). \tag{41}$$

6.40 THEOREM Suppose there exists a compact imbedding of the form (38) for some $q \ge p$. For each $r \ge 0$ let $\Omega_r = \{x \in \Omega : |x| > r\}$, let $S_r = \{x \in \Omega : |x| = r\}$, and let A_r denote the surface area [$(n-1)$-measure] of S_r. Then:

(a) For given ε, $\delta > 0$ there exists R such that if $r \ge R$, then

$$\mu(\Omega_r) \le \delta \mu(\{x \in \Omega : r-\varepsilon \le |x| \le r\}).$$

(b) If A_r is positive and ultimately nonincreasing as r tends to infinity, then for each $\varepsilon > 0$

$$\lim_{r \to \infty} \frac{A_{r+\varepsilon}}{A_r} = 0.$$

PROOF Given $\varepsilon > 0$ let T be a tesselation of $\mathbb{R}^n$ by cubes of edge length $\varepsilon/2\sqrt{n}$. Then any cube $H \in T$ whose interior intersects Ω_r is contained in $\Omega_{r-\varepsilon/2}$ and

$$F(\Omega_r) \subset \{x \in \Omega : r-\varepsilon \le |x| \le r\}.$$

For given $\delta > 0$ let λ be as given in Section 6.39. Let R be large enough that the finitely many λ-fat cubes in T are all contained in the ball of radius $R-\varepsilon/2$ centered at the origin. Then for any $r \ge R$ all the cubes in T whose interiors intersect Ω_r are λ-thin and (a) follows from (41).

For (b) choose R_0 so that A_r is nonincreasing on $[R_0, \infty)$. Fix $\varepsilon', \delta > 0$, and let $\varepsilon = \varepsilon'/2$. Let R be as in (a). If $r \ge \max(R_0+\varepsilon', R)$, then

$$A_{r+\varepsilon'} \le (1/\varepsilon) \int_{r+\varepsilon}^{r+2\varepsilon} A_s\, ds \le (1/\varepsilon)\mu(\Omega_{r+\varepsilon}) \le (\delta/\varepsilon)\mu\{x \in \Omega : r \le |x| \le r+\varepsilon\}$$
$$= (\delta/\varepsilon) \int_r^{r+\varepsilon} A_s\, ds \le \delta A_r.$$

Since ε' and δ are arbitrary, (b) follows. ∎

6.41 COROLLARY If there exists a compact imbedding of the form (38) for some $q \geq p$, then for every k we have

$$\lim_{r\to\infty} e^{kr}\mu(\Omega_r) = 0.$$

PROOF Fix k and let $\delta = e^{-(k+1)}$. From conclusion (a) of Theorem 6.40 there exists R such that $r \geq R$ implies $\mu(\Omega_{r+1}) \leq \delta\mu(\Omega_r)$. Thus if j is a positive integer and $0 \leq t < 1$, we have

$$\begin{aligned} e^{k(R+j+t)}\mu(\Omega_{R+j+t}) &< e^{k(R+j+1)}\mu(\Omega_{R+j}) \leq e^{k(R+1)}e^{kj}\delta^j\mu(\Omega_R) \\ &= e^{k(R+1)}\mu(\Omega_R)e^{-j}. \end{aligned}$$

The last term tends to zero as j tends to infinity. ∎

6.42 REMARKS (1) The argument used in the proof of Theorem 6.40(a) works for any norm ρ on $\mathbb{R}^n$ in place of the usual norm $\rho(x) = |x|$. The same holds for (b) provided A_r is well defined (with respect to the norm ρ) and provided

$$\mu(\{x \in \Omega : r \leq \rho(x) \leq r+\varepsilon\}) = \int_r^{r+\varepsilon} A_s\, ds.$$

This is true, for example, if $\rho(x) = \max_{1\leq i\leq n} |x_i|$.

(2) For the proof of (b) it is sufficient that A_r have an equivalent, positive, nonincreasing majorant, that is, there should exist a positive, nonincreasing function $f(r)$ and a constant $M > 0$ such that for all sufficiently large r,

$$A_r \leq f(r) \leq MA_r.$$

(3) Theorem 6.33 is sharper than Theorem 6.40, because the conclusions of the latter theorem are global whereas the compactness of (38) evidently depends on local properties of Ω. We illustrate this by means of two examples.

6.43 EXAMPLE Let $f \in C^1([0,\infty))$ be positive and nonincreasing with bounded derivative f'. We consider the planar domain (Fig. 6a)

$$\Omega = \{(x,y) \in \mathbb{R}^2 : x > 0,\ 0 < y < f(x)\}. \tag{42}$$

With respect to the supremum norm on $\mathbb{R}^2$, that is, $\rho(x,y) = \max(|x|,|y|)$, we have $A_s = f(s)$ for sufficiently large s. Hence Ω satisfies conclusion (b) of Theorem 6.40 [and since f is monotonic conclusion (a) as well] if and only if

$$\lim_{s\to\infty} \frac{f(s+\varepsilon)}{f(s)} = 0 \tag{43}$$

holds for every $\varepsilon > 0$. For example, $f(x) = \exp(-x^2)$ satisfies (43) but

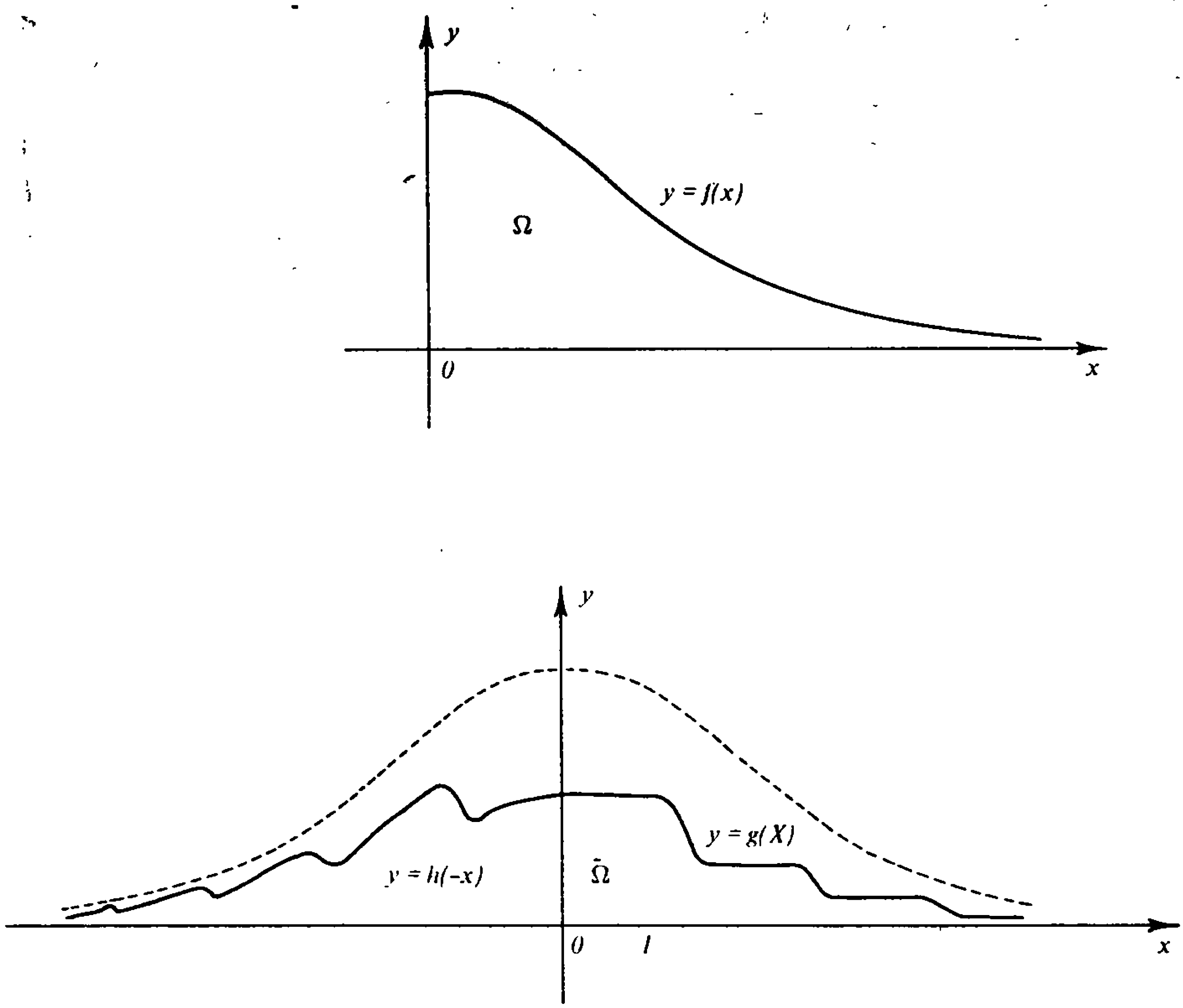


FIG. 6 Domains: (a) Ω of Example 6.43 and (b) $\tilde{\Omega}$ of Example 6.44

$f(x) = e^{-x}$ does not. We shall see (Section 6.48) that the imbedding

$$W^{m,p}(\Omega) \to L^p(\Omega) \tag{44}$$

is compact if (43) holds. Thus (43) is necessary and sufficient for the compactness of (38) for domains of the type (42).

6.44 EXAMPLE Let f be as in Example 6.43 and assume also that $f'(0) = 0$. Let g be a positive, nonincreasing function in $C^1([0, \infty))$ satisfying

(i) $g(0) = \frac{1}{2}f(0)$, $g'(0) = 0$,
(ii) $g(x) < f(x)$ for all $x \geq 0$,
(iii) $g(x)$ is constant on infinitely many disjoint intervals of unit length.

Let $h(x) = f(x) - g(x)$ and consider the domain (Fig. 6b)

$$\tilde{\Omega} = \{(x, y) \in \mathbb{R}^2 : 0 < y < g(x) \text{ if } x \geq 0,\ 0 < y < h(-x) \text{ if } x < 0\}.$$

Again we have $A_s = f(s)$ for sufficiently large s so $\tilde{\Omega}$ satisfies the conclusions of Theorem 6.40 if (43) holds.

If, however, T is a tesselation of $\mathbb{R}^2$ by squares of edge $\frac{1}{4}$ having edges parallel to the coordinate axes, and if one square of T has center at the origin, then T has infinitely many $\frac{1}{4}$-fat squares with centers on the positive x-axis. By Theorem 6.33, imbedding (44) cannot be compact.

Unbounded Domains—Compact Imbeddings of $W^{m,p}(\Omega)$

6.45 The above examples suggests that any sufficient condition for the compactness of the imbedding

$$W^{m,p}(\Omega) \to L^p(\Omega) \tag{45}$$

for unbounded domains Ω must involve the rapid decay of volume locally in each branch of Ω_r as r tends to infinity. A convenient way to express such local decay is in terms of flows on Ω.

By a *flow* on Ω we mean a continuously differentiable map $\Phi: U \to \Omega$, where U is an open set in $\Omega \times \mathbb{R}$ containing $\Omega \times \{0\}$, and where $\Phi(x,0) = x$ for every $x \in \Omega$.

For fixed $x \in \Omega$ the curve $t \to \Phi(x,t)$ is called a *streamline* of the flow. For fixed t the map $\Phi_t: x \to \Phi(x,t)$ sends a subset of Ω into Ω. We shall be concerned with the Jacobian of this map:

$$\det \Phi_t'(x) = \left. \frac{\partial(\Phi_1, \ldots, \Phi_n)}{\partial(x_1, \ldots, x_n)} \right|_{(x,t)}.$$

It is sometimes required of a flow Φ that $\Phi_{s+t} = \Phi_s \circ \Phi_t$ but we do not need this property and so do not assume it.

6.46 EXAMPLE Let Ω be the domain given by (42). Define the flow

$$\Phi(x,y,t) = (x-t, [f(x-t)/f(x)]\,y), \qquad 0 < t < x.$$

The flow is toward the line $x = 0$ and the streamlines diverge as the domain widens (see Fig. 7). Φ_t is a local magnification for $t > 0$:

$$\det \Phi_t'(x,y) = f(x-t)/f(x).$$

Note that $\lim_{x\to\infty} \det \Phi_t'(x,y) = \infty$ if f satisfies (43).

For $N = 1, 2, \ldots$ let $\Omega_N^* = \{(x,y) \in \Omega : 0 < x < N\}$. Ω_N^* is bounded and has the cone property, so the imbedding

$$W^{1,p}(\Omega_N^*) \to L^p(\Omega_N^*)$$

is compact. This compactness, together with properties of the flow Φ are sufficient to force the compactness of (45) as we now show.

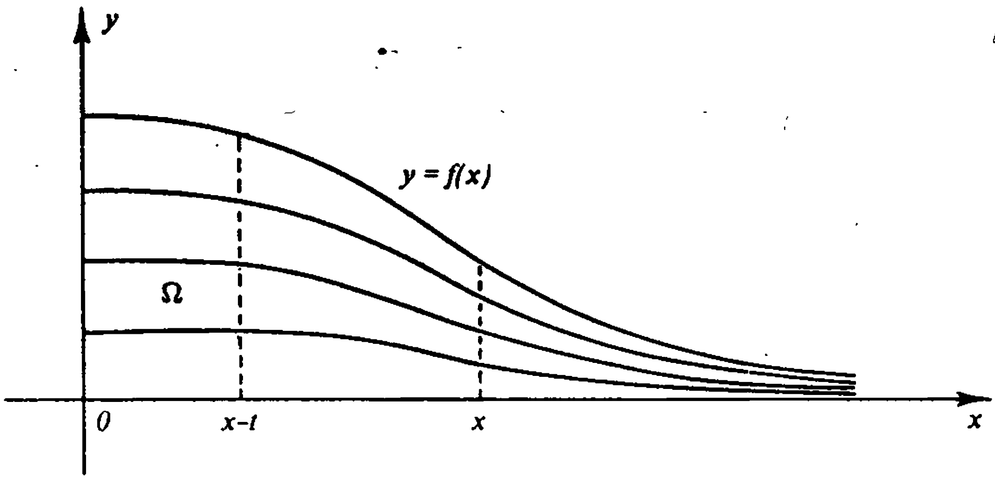


FIG. 7 Streamlines of the flow Φ given in Example 6.46

6.47 THEOREM Let Ω be an open set in $\mathbb{R}^n$ having the following properties:

(a) There exists a sequence $\{\Omega_N{}^*\}_{N=1}^{\infty}$ of open subsets of Ω such that $\Omega_N{}^* \subset \Omega_{N+1}^*$ and such that for each N the imbedding

$$W^{1,p}(\Omega_N{}^*) \to L^p(\Omega_N{}^*)$$

is compact.

(b) There exists a flow $\Phi\colon U \to \Omega$, such that if $\Omega_N = \Omega \sim \Omega_N{}^*$, then

(i) $\Omega_N \times [0,1] \subset U$ for each N,
(ii) Φ_t is one-to-one for all t,
(iii) $|(\partial/\partial t)\Phi(x,t)| \le M$ (const) for all $(x,t) \in U$.

(c) The functions $d_N(t) = \sup_{x \in \Omega_N} |\det \Phi_t'(x)|^{-1}$ satisfy

(i) $\lim_{N\to\infty} d_N(1) = 0$,
(ii) $\lim_{N\to\infty} \int_0^1 d_N(t)\, dt = 0$.

Then imbedding (45) is compact.

PROOF Since we have $W^{m,p}(\Omega) \to W^{1,p}(\Omega) \to L^p(\Omega)$ it is sufficient to prove that the latter imbedding is compact. Let $u \in C^1(\Omega)$. For each $x \in \Omega_N$ we have

$$u(x) = u(\Phi_1(x)) - \int_0^1 \frac{\partial}{\partial t} u(\Phi_t(x))\, dt.$$

Now

$$\begin{aligned}\int_{\Omega_N} |u(\Phi_1(x))|\, dx &\le d_N(1) \int_{\Omega_N} |u(\Phi_1(x))|\,|\det \Phi_1'(x)|\, dx \\ &= d_N(1) \int_{\Phi_1(\Omega_N)} |u(y)|\, dy \\ &\le d_N(1) \int_{\Omega} |u(y)|\, dy.\end{aligned}$$

Also

$$\begin{aligned}\int_{\Omega_N}\left|\int_0^1 \frac{\partial}{\partial t}u(\Phi_t(x))\,dt\right|dx &\le \int_{\Omega_N} dx \int_0^1 |\operatorname{grad} u(\Phi_t(x))|\left|\frac{\partial}{\partial t}\Phi_t(x)\right|dt\\ &\le M\int_0^1 d_N(t)\,dt \int_{\Omega_N} |\operatorname{grad} u(\Phi_t(x))||\det \Phi_t'(x)|\,dx\\ &\le M\left\{\int_0^1 d_N(t)\,dt\right\}\left\{\int_\Omega |\operatorname{grad} u(y)|\,dy\right\}.\end{aligned}$$

Putting $\delta_N = \max(d_N(1), M\int_0^1 d_N(t)\,dt)$, we have

$$\int_{\Omega_N} |u(x)|\,dx \le \delta_N \int_\Omega (|u(y)| + |\operatorname{grad} u(y)|)\,dy \le \delta_N \|u\|_{1,1,\Omega} \tag{46}$$

and $\lim_{N\to\infty}\delta_N = 0$.

Now suppose u is real valued and belongs to $C^1(\Omega)\cap W^{1,p}(\Omega)$. By Hölder's inequality the distributional derivatives of $|u|^p$,

$$D_j(|u|^p) = p\cdot|u|^{p-1}\cdot\operatorname{sgn} u\cdot D_j u,$$

satisfy

$$\int_\Omega |D_j(|u(x)|^p)|\,dx \le p\|D_j u\|_{0,p,\Omega}\|u\|_{0,p,\Omega}^{p-1} \le p\|u\|_{1,p,\Omega}^p.$$

Thus $|u|^p \in W^{1,1}(\Omega)$ and by Theorem 3.16 there is a sequence ϕ_j of functions in $C^1(\Omega)\cap W^{1,1}(\Omega)$ such that $\lim_{j\to\infty}\|\phi_j - |u|^p\|_{1,1,\Omega} = 0$. Thus by (46)

$$\begin{aligned}\int_{\Omega_N} |u(x)|^p\,dx &= \lim_{j\to\infty}\int_{\Omega_N}\phi_j(x)\,dx \le \limsup_{j\to\infty}\delta_N\|\phi_j\|_{1,1,\Omega}\\ &= \delta_N\||u|^p\|_{1,1,\Omega} \le K\delta_N\|u\|_{1,p,\Omega}^p,\end{aligned} \tag{47}$$

where $K = K(n,p)$. Inequality (47) holds for arbitrary (complex-valued) $u\in C^1(\Omega)\cap W^{1,p}(\Omega)$ by virtue of its separate application to the real and imaginary parts of u. (The constant K may be changed.)

If S is a bounded set in $W^{1,p}(\Omega)$ and $\varepsilon > 0$, we may, by (47), select N so that for all $u\in S$

$$\int_{\Omega_N}|u(x)|^p\,dx < \varepsilon.$$

Since $W^{1,p}(\Omega\sim\Omega_N)\to L^p(\Omega\sim\Omega_N)$ is compact, the precompactness of S in $L^p(\Omega)$ follows by Theorem 2.22. Hence $W^{1,p}(\Omega)\to L^p(\Omega)$ is compact. ∎

6.48 EXAMPLE Consider again the domain Ω of Examples 6.43 and 6.46 and the flow Φ given in the latter example. We have

$$d_N(t) = \sup_{x\ge N}\frac{f(x)}{f(x-t)} \le 1 \quad \text{if} \quad 0\le t\le 1$$

and by (43)

$$\lim_{N\to\infty} d_N(t) = 0 \qquad \text{if} \quad t > 0.$$

Thus by dominated convergence

$$\lim_{N\to\infty} \int_0^1 d_N(t)\,dt = 0.$$

The assumption that f' is bounded guarantees that the speed $|(\partial/\partial t)\Phi(x,y,t)|$ is bounded on U. Thus Ω satisfies the hypotheses of Theorem 6.47 and (45) is compact for this domain.

6.49 EXAMPLE Theorem 6.47 can also be used to show the compactness of (45) for some bounded domains to which neither the Rellich–Kondrachov theorem nor the techniques used in its proof can be applied. For example, we consider

$$\Omega = \{(x,y) \in \mathbb{R}^2 : 0 < x < 2,\ 0 < y < f(x)\},$$

where $f \in C^1(0,2)$ is positive, nondecreasing, has bounded derivative f', and satisfies $\lim_{x\to 0+} f(x) = 0$. Let $U = \{(x,y,t) \in \mathbb{R}^3 : (x,y) \in \Omega,\ -x < t < 2-x\}$ and define the flow $\Phi\colon U \to \Omega$ by

$$\Phi(x,y,t) = \left(x+t,\ \frac{f(x+t)}{f(x)}\,y\right).$$

Thus $\det \Phi_t'(x,y) = f(x+t)/f(x)$. If $\Omega_N{}^* = \{(x,y) \in \Omega : x > 1/N\}$, then

$$d_N(t) = \sup_{0<x\le 1/N} \left|\frac{f(x)}{f(x+t)}\right|$$

satisfies $d_N(t) \le 1$ for $0 \le t \le 1$, and $\lim_{N\to\infty} d_N(t) = 0$ if $t > 0$. Hence also $\lim_{N\to\infty} \int_0^1 d_N(t)\,dt = 0$ by dominated convergence. Since $\Omega_N{}^*$ is bounded and has the cone property, and since the boundedness of $\partial\Phi/\partial t$ is assured by that of f', we have by Theorem 6.47 the compactness of

$$W^{m,p}(\Omega) \to L^p(\Omega). \tag{48}$$

Suppose $\lim_{x\to 0+} f(x)/x^k = 0$ for every k. [For example, let $f(x) = e^{-1/x}$.] Then Ω has an exponential cusp at the origin, and so by Theorem 5.32 there exists no imbedding of the form

$$W^{m,p}(\Omega) \to L^q(\Omega)$$

for any $q > p$. Hence the method of Section 6.7 cannot be used to show the compactness of (48).

6.50 REMARKS (1) It is easy to imagine domains more general than those in the above examples to which Theorem 6.47 can be applied, although it may be difficult to specify a suitable flow. A domain with many (perhaps infinitely many) unbounded branches can, if connected, admit a suitable flow provided the volume decays sufficiently rapidly and regularly in each branch, a condition not fulfilled by the domain $\tilde{\Omega}$ of Example 6.44. For unbounded domains in which the volume decays monotonically in each branch Theorem 6.40 is essentially a converse of Theorem 6.47 in that the proof of Theorem 6.40 can be applied separately to show the volume decays in each branch in the required way.

(2) Since the only unbounded domains Ω for which $W^{m,p}(\Omega)$ imbeds compactly into $L^p(\Omega)$ have finite volume there can be no extension of Theorem 6.47 to give compact imbeddings into $L^q(\Omega)$ $(q > p)$, $C_B(\Omega)$, etc.—there do not exist such imbeddings.

Hilbert–Schmidt Imbeddings

6.51 A *complete orthonormal system* in a separable Hilbert space X is a sequence $\{e_i\}_{i=1}^{\infty}$ of elements satisfying

$$(e_i, e_j)_X = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

[where $(\cdot, \cdot)_X$ is the inner product in X], and such that for each $x \in X$ we have

$$\lim_{k \to \infty} \|x - \sum_{i=1}^{k} (x, e_i)_X e_i; X\| = 0. \tag{49}$$

Thus $x = \sum_{i=1}^{\infty} (x, e_i)_X e_i$, the series converging with respect to the norm in X. It is well known that every separable Hilbert space possesses such complete orthonormal systems. There follows from (49) the Parseval identity

$$\|x; X\|^2 = \sum_{i=1}^{\infty} |(x, e_i)_X|^2.$$

Let X and Y be two separable Hilbert spaces and let $\{e_i\}_{i=1}^{\infty}$ and $\{f_i\}_{i=1}^{\infty}$ be given complete orthonormal systems in X and Y, respectively. Let A be a bounded linear operator with domain X taking values in Y, and let A^* be the adjoint of A taking Y into X and defined by

$$(x, A^*y)_X = (Ax, y)_Y, \qquad x \in X, \quad y \in Y.$$

Define

$$|||A|||^2 = \sum_{i=1}^{\infty} \|Ae_i; Y\|^2, \qquad |||A^*|||^2 = \sum_{i=1}^{\infty} \|A^*f_i; X\|^2.$$

If $|||A|||$ is finite, A is called a *Hilbert–Schmidt operator* and we call $|||A|||$ its *Hilbert–Schmidt norm*. (Recall that the operator norm of A is given by

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax;Y\|}{\|x;X\|}.)$$

We must justify the above definition.

6.52 LEMMA The norms $|||A|||$ and $|||A^*|||$ are independent of the particular orthonormal systems $\{e_i\}$, $\{f_i\}$ used. Moreover,

$$|||A||| = |||A^*||| \geq \|A\|.$$

PROOF By Parseval's identity

$$\begin{aligned} |||A|||^2 &= \sum_{i=1}^{\infty} \|Ae_i;Y\|^2 = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |(Ae_i,f_j)_Y|^2 \\ &= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} |(e_i,A^*f_j)_X|^2 = \sum_{j=1}^{\infty} \|A^*f_j;X\|^2 = |||A^*|||. \end{aligned}$$

Hence each expression is independent of $\{e_i\}$ and $\{f_j\}$. For any $x \in X$ we have

$$\begin{aligned} \|Ax;Y\|^2 &= \left\|\sum_{i=1}^{\infty} (x,e_i)_X Ae_i;Y\right\|^2 \leq \left(\sum_{i=1}^{\infty} |(x,e_i)_X|\,\|Ae_i;Y\|\right)^2 \\ &\leq \left(\sum_{i=1}^{\infty} |(x,e_i)_X|^2\right)\left(\sum_{j=1}^{\infty} \|Ae_j;Y\|^2\right) = \|x;X\|^2\,|||A|||^2. \end{aligned}$$

Hence $\|A\| \leq |||A|||$ as required. ∎

We leave to the reader the task of verifying the following assertions:

(a) If X, Y, and Z are separable Hilbert spaces and A, B bounded linear operators from X into Y and Y into Z, respectively, then $B \circ A$, which takes X into Z, is a Hilbert–Schmidt operator if either A or B is. (If A is Hilbert–Schmidt, then $|||B \circ A||| \leq \|B\|\,|||A|||$.)

(b) Every Hilbert–Schmidt operator is compact.

The following theorem, due to Maurin [43] has far-reaching implications for eigenfunction expansions corresponding to differential operators.

6.53 THEOREM Let Ω be a bounded set having the cone property in $\mathbb{R}^n$. Let m, k be nonnegative integers with $k > n/2$. Then the imbedding mappings

$$W^{m+k,2}(\Omega) \to W^{m,2}(\Omega) \tag{50}$$

are Hilbert–Schmidt operators. Similarly, the imbeddings

$$W_0^{m+k,2}(\Omega) \to W_0^{m,2}(\Omega) \tag{51}$$

are Hilbert–Schmidt operators for any bounded domain Ω.

PROOF Given $y \in \Omega$ and α with $|\alpha| \le m$ we define a linear functional T_y^α on $W^{m+k,2}(\Omega)$ by

$$T_y^\alpha(u) = D^\alpha u(y).$$

Since $k > n/2$ the Sobolev imbedding Theorem 5.4 implies that T_y^α is bounded on $W^{m+k,2}(\Omega)$ and has norm bounded by a constant K independent of y and α:

$$|T_y^\alpha(u)| \le \max_{0 \le |\alpha| \le m} \sup_{x \in \Omega} |D^\alpha u(x)| \le K \|u\|_{m+k,2,\Omega}. \tag{52}$$

By the Riesz representation theorem for Hilbert spaces there exists $v_y^\alpha \in W^{m+k,2}(\Omega)$ such that

$$D^\alpha u(y) = T_y^\alpha(u) = (u, v_y^\alpha)_{m+k}, \tag{53}$$

where $(\cdot, \cdot)_{m+k}$ is the inner product in $W^{m+k,2}(\Omega)$, and moreover

$$\|v_y^\alpha\|_{m+k,2,\Omega} = \|T_y^\alpha; [W^{m+k,2}(\Omega)]'\| \le K. \tag{54}$$

If $\{e_i\}_{i=1}^\infty$ is a complete orthonormal system in $W^{m+k,2}(\Omega)$, then

$$\|v_y^\alpha\|_{m+k,2,\Omega}^2 = \sum_{i=1}^\infty |(e_i, v_y^\alpha)_{m+k}|^2 = \sum_{i=1}^\infty |D^\alpha e_i(y)|^2.$$

Consequently,

$$\sum_{i=1}^\infty \|e_i\|_{m,2,\Omega}^2 \le \sum_{|\alpha| \le m} \int_\Omega \|v_y^\alpha\|_{m+k,2,\Omega}^2 \, dy \le \sum_{|\alpha| \le m} K \operatorname{vol} \Omega < \infty. \tag{55}$$

Hence imbedding (50) is Hilbert–Schmidt. The same proof holds for (51) without regularity needed for the application of Theorem 5.4. ∎

The following generalization of Maurin's theorem is due to Clark [17].

6.54 THEOREM Let μ be a nonnegative, measurable function defined on the domain Ω in $\mathbb{R}^n$. Let $W_0^{m,2;\mu}(\Omega)$ be the Hilbert space obtained by completing $C_0^\infty(\Omega)$ with respect to the weighted norm

$$\|u\|_{m,2;\mu} = \left\{ \sum_{|\alpha| \le m} \int_\Omega |D^\alpha u(x)|^2 \mu(x) \, dx \right\}^{1/2}.$$

For $y \in \Omega$ let $\tau(y) = \operatorname{dist}(y, \operatorname{bdry} \Omega)$. Let ν be a nonnegative integer such that

$$\int_\Omega [\tau(y)]^{2\nu} \mu(y) \, dy < \infty. \tag{56}$$

If $k > \nu+n/2$, then the imbedding

$$W_0^{m+k,2}(\Omega) \to W_0^{m,2;\mu}(\Omega) \tag{57}$$

(exists and) is Hilbert–Schmidt.

PROOF Let $\{e_i\}$, T_y^α, and v_y^α be defined as in the previous theorem. If $y \in \Omega$, let y_0 be chosen in bdry$\,\Omega$ such that $\tau(y) = |y-y_0|$. If ν is a positive integer and $u \in C_0^\infty(\Omega)$, we have by Taylor's formula with remainder

$$D^\alpha u(y) = \sum_{|\beta|=\nu} (1/\beta!)\, D^{\alpha+\beta} u(y_\beta)(y-y_\beta)^\beta$$

for some points y_β satisfying $|y-y_\beta| \le \tau(y)$. If $|\alpha| \le m$ and $k > \nu+n/2$, we obtain from the Sobolev imbedding theorem [as in (52)]

$$|D^\alpha u(y)| \le K\|u\|_{m+k,2}[\tau(y)]^\nu. \tag{58}$$

Inequality (58) holds, by completion, for any $u \in W_0^{m+k,2}(\Omega)$, and also holds if $\nu = 0$, directly from (52). Hence by (53) and (54)

$$\|v_y^\alpha\|_{m+k,2} = \sup_{\|u\|_{m+k,2}=1} |D^\alpha u(y)| \le K[\tau(y)]^\nu.$$

It finally follows as in (55) that

$$\begin{aligned}\sum_{i=1}^\infty \|e_i\|_{m,2;\mu}^2 &\le \sum_{|\alpha|\le m} \int_\Omega \|v_y^\alpha\|_{m+k,2}^2 \mu(y)\,dy \\ &\le K^2 \sum_{|\alpha|\le m} \int_\Omega [\tau(y)]^{2\nu}\mu(y)\,dy < \infty\end{aligned}$$

by (56). Hence imbedding (57) is Hilbert–Schmidt. ▌

6.55 REMARK Various choices of μ and ν lead to generalizations of Maurin's theorem for imbeddings of the sort (51). If $\mu(x) = 1$ and $\nu = 0$, we obtain the obvious generalization to unbounded domains of finite volume. If $\mu(x) \equiv 1$ and $\nu > 0$, Ω may be unbounded and even have infinite volume, but by (56) it must be quasibounded. [Of course quasiboundedness may not be sufficient to guarantee (56).] If μ is the characteristic function of a bounded subdomain Ω_0 of Ω, and $\nu = 0$, we obtain the Hilbert–Schmidt imbedding

$$W_0^{m+k,2}(\Omega) \to W^{m,2}(\Omega_0), \qquad k > n/2.$$

VII

Fractional Order Spaces

Outline

7.1 In this chapter we are concerned with the problem of extending the notion of Sobolev space to allow for nonintegral values of m. There is no unique method for doing this, and different approaches may or may not lead to different families of spaces. The major families of spaces that arise are the following:

(i) The spaces $W^{s,p}(\Omega)$—which can be defined by a "real interpolation" method but can also be characterized in terms of an intrinsically defined norm involving first-order differences of the highest-order derivatives involved.

(ii) The spaces $L^{s,p}(\Omega)$—which can be defined by a "complex interpolation" method but, if $\Omega = \mathbb{R}^n$, can also be characterized in terms of Fourier transforms.

(iii) The Besov spaces $B^{s,p}(\Omega)$—defined in terms of an intrinsic norm similar to that of $W^{s,p}(\Omega)$ but involving second rather than first differences.

(iv) The Nikol'skii spaces $H^{s,p}(\Omega)$—having norm involving Hölder conditions in the L^p-metric.

Only the spaces $W^{s,p}(\Omega)$ and $L^{s,p}(\Omega)$ coincide with $W^{m,p}(\Omega)$ when $s = m$, an integer. $B^{s,p}(\Omega)$ coincides with $W^{s,p}(\Omega)$ for all s if $p = 2$ but otherwise only for nonintegral s. The space $H^{s,p}(\Omega)$ is always larger than (but close to)

$W^{s,p}(\Omega)$. From the point of view of imbeddings, the simplest and most complete results obtain for the spaces $B^{s,p}$ and $H^{s,p}$ (see Theorem 7.70 and Section 7.73). However, it is in terms of the $W^{s,p}$ spaces that the problem, mentioned in Section 5.20, of characterizing the traces on smooth manifolds of functions in $W^{m,p}(\Omega)$ has its solution (Theorem 7.53). For this reason we concentrate our effort in this chapter on elucidating the properties of the spaces $W^{s,p}(\Omega)$ and give only brief descriptions of the other classes.

About half of the chapter is concerned with developing the "trace interpolation method" of Lions, on which we base our study of the spaces $W^{s,p}(\Omega)$. These latter spaces are introduced in Section 7.36. The trace interpolation method is one of several essentially equivalent real interpolation methods for Banach spaces concerning which there is now a considerable literature. Descriptions of these methods may be found in the work of Butzer and Berens [13] and Stein and Weiss [65], and the interested reader is referred to the work of Peetre [56] and Grisvard [28] for some applications in the direction of fractional order Sobolev spaces. A treatment of these spaces is also given in Stein [64a]. Most of the material in this chapter follows that of Lions [37, 38] and Lions and Magenes [40].

The Bochner Integral

7.2 In this chapter we shall need to make frequent use of the notion of integral of a Banach space-valued function f defined on an interval of $\mathbb{R}$. We begin therefore with a brief discussion of the Bochner integral, referring the reader to the text by Yosida [69], for instance, for further details and proofs of our assertions.

Let B be a Banach space with norm denoted by $\|\cdot\|_B$. Let $\{A_1, A_2, \ldots, A_m\}$ be a finite collection of mutually disjoint, measurable subsets of $\mathbb{R}$, each having finite measure, and let $\{b_1, b_2, \ldots, b_m\}$ be a corresponding collection of points of B. The function f on $\mathbb{R}$ into B defined by

$$f(t) = \sum_{j=1}^{m} \chi_{A_j}(t) b_j,$$

χ_A being the characteristic function of A, is called a *simple function*. For simple functions we define, obviously,

$$\int_{\mathbb{R}} f(t)\, dt = \sum_{j=1}^{m} \mu(A_j) b_j,$$

where $\mu(A)$ denotes the (Lebesgue) measure of A.

Let A be a measurable set in $\mathbb{R}$ and f an arbitrary function defined a.e. on A into B. The function f is called (*strongly*) *measurable* on A if there exists a sequence $\{f_n\}$ of simple functions with supports in A such that

$$\lim_{n\to\infty} \|f_n(t)-f(t)\|_B = 0 \qquad \text{a.e. in } A. \tag{1}$$

Let $\langle \cdot, \cdot \rangle$ denote the pairing between B and its dual space B' (i.e., $b'(b) = \langle b, b' \rangle$, $b \in B$, $b' \in B'$). It can be shown that any function f whose range is separable is measurable provided the scalar-valued function $\langle f(\cdot), b' \rangle$ is measurable on A for each $b \in B'$.

Suppose that a sequence of simple functions f_n satisfying (1) can be chosen in such a way that

$$\lim_{n\to\infty} \int_A \|f_n(t)-f(t)\|_B \, dt = 0.$$

Then f is called *(Bochner) integrable* on A and we define

$$\int_A f(t)\, dt = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(t)\, dt. \tag{2}$$

[The integrals on the right side of (2) do converge in (the norm topology of) B to a limit which is independent of the choice of approximating sequence $\{f_n\}$.]

A measurable function f is integrable on A if and only if $\|f(\cdot)\|_B$ is (Lebesgue) integrable on A. In fact,

$$\left\| \int_A f(t)\, dt \right\|_B \le \int_A \|f(t)\|_B \, dt.$$

7.3 Let $-\infty \le a < b \le \infty$. We denote by $L^p(a,b;B)$ the vector space of (equivalence classes of) functions f measurable on (a,b) into B such that $\|f(\cdot)\|_B \in L^p(a,b)$. The space $L^p(a,b;B)$ is a Banach space with respect to the norm

$$\|f; L^p(a,b;B)\| = \begin{cases} \left\{ \int_a^b \|f(t)\|_B^p \, dt \right\}^{1/p} & \text{if} \quad 1 \le p < \infty \\ \operatorname*{ess\,sup}_{t\in(a,b)} \|f(t)\|_B & \text{if} \quad p = \infty. \end{cases}$$

Similarly, if $f \in L^p(c,d;B)$ for every c, d with $a < c < d < b$, then we write $f \in L^p_{\mathrm{loc}}(a,b;B)$, and, in case $p = 1$, call f locally integrable.

A locally integrable function g on (a,b) is called the jth distributional derivative of the locally integrable function f provided

$$\int_a^b \phi^{(j)}(t) f(t)\, dt = (-1)^j \int_a^b \phi(t) g(t)\, dt$$

for every (scalar-valued) testing function $\phi \in \mathscr{D}(a,b) = C_0^\infty(a,b)$.

Semigroups of Operators and the Abstract Cauchy Problem

7.4 The following sections present a discussion of those aspects of the theory of semigroups of operators in Banach spaces which we shall require in the subsequent development of "trace spaces" and the fractional order spaces $W^{s,p}(\Omega)$. In this treatment we follow the work of Zaidman [70].

7.5 Let B be a Banach space, and $L(B)$ the Banach space of bounded linear operators with domain B and range in B. We denote the norms in B and $L(B)$ by $\|\cdot\|_B$ and $\|\cdot\|_{L(B)}$, respectively.

A function G with domain the interval $[0, \infty)$ and range in $L(B)$ is called a (*strongly*) *continuous semigroup* on B provided

(i) $G(0) = I$, the identity operator on B,
(ii) $G(s)G(t) = G(s+t)$ for all $s, t \geq 0$,
(iii) for each $b \in B$ the function $G(\cdot)b$ is continuous from $[0, \infty)$ to (the norm topology of) B.

We note that (ii) implies that the operators $G(s)$ and $G(t)$ commute. Also, (iii) implies that for each $t_0 \geq 0$ the set $\{t : \|G(t)\|_{L(B)} > t_0\}$ is open in $\mathbb{R}$ and hence measurable. If $0 \leq t_0 < t_1 < \infty$, then for each $b \in B$, $G(\cdot)b$ is uniformly continuous on $[t_0, t_1]$ and hence there exists a constant K_b such that $\|G(t)b\|_B \leq K_b$ if $t_0 \leq t \leq t_1$. By the uniform boundedness principle of functional analysis, there exists a constant K such that $\|G(t)\|_{L(B)} \leq K$ for all $b \in B$ and all $t \in [t_0, t_1]$. Thus $\|G(\cdot)\|_{L(B)} \in L^\infty_{\mathrm{loc}}(0, \infty)$. We amplify this result in the following lemma.

7.6 LEMMA (a) The limit

$$\lim_{t\to\infty} (1/t) \log \|G(t)\|_{L(B)} = \delta_0$$

exists and is finite.

(b) For each $\delta > \delta_0$ there exists a constant M_δ such that for every $t \geq 0$

$$\|G(t)\|_{L(B)} \leq M_\delta e^{\delta t}.$$

Proof Let $N(t) = \log \|G(t)\|_{L(B)}$. Since

$$\|G(s+t)\|_{L(B)} \leq \|G(s)\|_{L(B)} \|G(t)\|_{L(B)}$$

we have that N is subadditive;

$$N(s+t) \leq N(s) + N(t).$$

Let $\delta_0 = \inf_{t>0} N(t)/t$. ~~Clearly $0 \leq \delta_0 < \infty$.~~ Let $\varepsilon > 0$ be given and choose $r > 0$ so that $N(r)/r < \delta_0 + \varepsilon$. If $t \geq 2r$, let k be the integer such that

$(k+1)r \le t < (k+2)r$. Then

$$\delta_0 \le \frac{N(t)}{t} \le \frac{N(kr)+N(t-kr)}{t} \le \frac{k}{t}N(r) + \frac{1}{t}N(t-kr).$$

Now $t-kr \in [r,2r)$ so, as noted above, $N(t-kr)$ is bounded, say by K. Thus

$$\delta_0 \le \frac{N(t)}{t} \le \frac{kr}{t}(\delta_0+\varepsilon) + \frac{K}{t} \le \left(1-\frac{r}{t}\right)(\delta_0+\varepsilon) + \frac{K}{t}.$$

The right side tends to $\delta_0+\varepsilon$ as $t \to \infty$, and (a) follows since ε is arbitrary.

If $\delta > \delta_0$, there exists t_δ such that if $t \ge t_\delta$, then $N(t) \le \delta t$, or equivalently $\|G(t)\|_{L(B)} \le e^{\delta t}$. Conclusion (b) now follows with

$$M_\delta = \max(1, \sup_{0 \le t \le t_\delta} \|G(t)\|_{L(B)}). \quad \blacksquare$$

7.7 For given $b \in B$ the quotient $(G(t)b-b)/t$ may or may not converge (strongly) in B as $t \to 0+$. Let $D(\Lambda)$ be the set of all those elements b for which the limit exists, and for $b \in D(\Lambda)$ set

$$\Lambda b = \lim_{t\to 0+} \frac{G(t)b-b}{t} = \lim_{t\to 0+} \frac{G(t)b-G(0)b}{t}.$$

Clearly $D(\Lambda)$ is a linear subspace of B and Λ a linear operator from $D(\Lambda)$ into B. We call Λ the *infinitesimal generator* of the semigroup G. Note that Λ commutes with $G(t)$ $(t \ge 0)$ on $D(\Lambda)$.

7.8 LEMMA (a) For each $b \in B$

$$\lim_{t\to 0+} (1/t)\int_0^t G(\tau)b\,d\tau = b.$$

(b) For each $b \in B$ and $t > 0$ we have

$$\int_0^t G(\tau)b\,d\tau \in D(\Lambda) \qquad \text{and} \qquad \Lambda\int_0^t G(\tau)b\,d\tau = G(t)b - b.$$

(c) For each $b \in D(\Lambda)$ and $t > 0$ we have

$$\int_0^t G(\tau)\Lambda b\,d\tau = G(t)b - b.$$

(d) $D(\Lambda)$ is dense in B.

(e) Λ is a closed operator in B, that is, the graph $\{(b, \Lambda b) : b \in D(\Lambda)\}$ is a closed subspace of $B \times B$.

PROOF Let $b \in B$. By the continuity of $G(\cdot)b$ we have $\lim_{\tau\to 0+}\|G(\tau)b-b\|_B = \lim_{\tau\to 0+}\|G(\tau)b-G(0)b\|_B = 0$. Conclusion (a) follows since $b = (1/t)\int_0^t b\,d\tau$.

For fixed t we have

$$\begin{aligned}
\lim_{s\to 0+} \frac{G(s)-G(0)}{s} \int_0^t G(\tau)b\,d\tau &= \lim_{s\to 0+} \frac{1}{s}\int_0^t [G(s+\tau)-G(\tau)]\,b\,d\tau \\
&= \lim_{s\to 0+} \left(\frac{1}{s}\int_s^{s+t} G(\tau)b\,d\tau - \frac{1}{s}\int_0^t G(\tau)b\,d\tau\right) \\
&= \lim_{s\to 0+} \left(\frac{1}{s}\int_t^{s+t} G(\tau)b\,d\tau - \frac{1}{s}\int_0^s G(\tau)b\,d\tau\right) \\
&= \lim_{s\to 0+} \frac{1}{s}\int_0^s G(\tau)\,G(t)b\,d\tau - b = G(t)b - b.
\end{aligned}$$

This proves (b). If $b \in D(\Lambda)$, then

$$\left\|\int_0^t G(\tau)\left(\frac{G(s)b-b}{s} - \Lambda b\right)d\tau\right\|_B \le t \sup_{0<\tau<t} \|G(\tau)\|_{L(B)} \left\|\frac{G(s)b-b}{s} - \Lambda b\right\|_B$$
$$\to 0 \qquad \text{as} \quad s \to 0+.$$

Thus

$$\Lambda \int_0^t G(\tau)b\,d\tau = \lim_{s\to 0+} \int_0^t G(\tau)\frac{G(s)b-b}{s}\,d\tau = \int_0^t G(\tau)\,\Lambda b\,d\tau,$$

which proves (c). Part (d) is an immediate consequence of (a) and (b).

If $b_n \in D(\Lambda)$, $b_n \to b$ and $\Lambda b_n \to b_0$ in B, then by (c)

$$G(t)b_n - b_n = \int_0^t G(\tau)\,\Lambda b_n\,d\tau.$$

We may let $n \to \infty$, justifying the interchange of limit and integral in the same way as was done in the proof of (c), and obtain

$$G(t)b - b = \int_0^t G(\tau)b_0\,d\tau.$$

Thus by (a)

$$\lim_{t\to 0+} \frac{G(t)b-b}{t} = \lim_{t\to 0+} \frac{1}{t}\int_0^t G(\tau)b_0\,d\tau = b_0,$$

whence $b \in D(\Lambda)$ and $\Lambda b = b_0$. This proves (e). ∎

7.9 REMARK It follows from the closedness of Λ that $D(\Lambda)$ is a Banach space with respect to the norm $\|b; D(\Lambda)\| = \|b\|_B + \|\Lambda b\|_B$. We have, moreover, the obvious imbedding $D(\Lambda) \to B$.

The following theorem is concerned with an abstract setting of a typical Cauchy initial value problem for a first-order differential operator.

7.10 THEOREM Let Λ be the infinitesimal generator of a continuous semigroup G on the Banach space B. Let $a \in D(\Lambda)$ and let f be a continuously differentiable function on $[0, \infty)$ into B. Then there exists a unique function u, continuous on $[0, \infty)$ into $D(\Lambda)$, and having derivative u' continuous on $[0, \infty)$ into B, such that

$$\begin{aligned} u'(t) - \Lambda u(t) &= f(t), \qquad t \geq 0 \\ u(0) &= a. \end{aligned} \tag{3}$$

In fact, u is given by

$$u(t) = G(t)a + \int_0^t G(t-\tau) f(\tau)\, d\tau. \tag{4}$$

PROOF (*Uniqueness*) We must show that the only solution of (3) for $f(t) \equiv 0$ and $a = 0$ is $u(t) \equiv 0$. If u is any solution and $t > \tau$, we have

$$\begin{aligned} \frac{\partial}{\partial \tau} G(t-\tau) u(\tau) &= \lim_{s \to 0} \frac{G(t-\tau-s) u(\tau+s) - G(t-\tau) u(\tau)}{s} \\ &= \lim_{s \to 0} \frac{G(t-\tau-s) - G(t-\tau)}{s} \cancel{u(s)}\, u(\tau) \\ &\quad + \lim_{s \to 0} G(t-\tau-s) \frac{u(\tau+s) - u(\tau)}{s} \\ &= -G(t-\tau)\Lambda u(\tau) + G(t-\tau) u'(\tau) = 0. \end{aligned}$$

Thus $G(t-\tau) u(\tau) = G(t) u(0) = 0$ for all $t > \tau$. Letting $t \to \tau+$, we obtain $u(\tau) = G(0) u(\tau) = 0$ for all $\tau \geq 0$.

(*Existence*) We verify that u given by (4) satisfies (3). First note that $u(0) = a$ and $(d/dt) G(t)a = \Lambda G(t)a$. Hence it is sufficient to show that the function

$$g(t) = \int_0^t G(t-\tau) f(\tau)\, d\tau$$

is continuously differentiable on $[0, \infty)$ into B, takes values in $D(\Lambda)$, and satisfies $g'(t) = \Lambda g(t) + f(t)$.

Now

$$\begin{aligned}\frac{g(t+s)-g(t)}{s} &= \frac{1}{s}\int_0^{t+s} G(t+s-\tau)f(\tau)\,d\tau - \frac{1}{s}\int_0^t G(t-\tau)f(\tau)\,d\tau \\ &= \frac{1}{s}\int_{-s}^{t} G(t-\tau)f(\tau+s)\,d\tau - \frac{1}{s}\int_0^t G(t-\tau)f(\tau)\,d\tau \\ &= \int_0^t G(t-\tau)\frac{f(\tau+s)-f(\tau)}{s} + \frac{1}{s}\int_t^{t+s} G(\tau)f(t+s-\tau)\,d\tau.\end{aligned}$$

Since f is continuously differentiable on $[0,\infty)$ into B we have that

$$g'(t) = \int_0^t G(t-\tau)f'(\tau)\,d\tau + G(t)f(0)$$

exists and is continuous on $[0,\infty)$ into B. On the other hand,

$$\begin{aligned}\frac{g(t+s)-g(t)}{s} &= \int_0^t \frac{G(s)-G(0)}{s} G(t-\tau)f(\tau)\,d\tau + \frac{1}{s}\int_t^{t+s} G(t+s-\tau)f(\tau)\,d\tau \\ &= \frac{G(s)-G(0)}{s} g(t) + \frac{1}{s}\int_0^s G(s-\tau)f(t+\tau)\,d\tau.\end{aligned}$$

By Lemma 7.8(a) and the continuity of f the latter integral converges to $f(t)$ as $s \to 0+$. This, together with the existence of $g'(t)$ guarantees that $\lim_{s\to 0+}[G(s)-G(0)]g(t)/s$ exists, that is, $g(t) \in D(\Lambda)$. Thus $g'(t) = \Lambda g(t)+f(t)$ as required. ∎

The Trace Spaces of Lions

7.11 Let B_1 and B_2 be two Banach spaces with norms $\|\cdot\|_{B_1}$ and $\|\cdot\|_{B_2}$, respectively, and let X be a topological vector space in which B_1 and B_2 are each imbedded continuously (i.e., $B_i \cap U$ is open in B_i, $i = 1, 2$, for every U open in X). Then the vector sum of B_1 and B_2

$$B_1 + B_2 = \{b_1+b_2 \in X : b_1 \in B_1,\ b_2 \in B_2\}$$

is itself a Banach space with respect to the norm:

$$\|u; B_1+B_2\| = \inf_{\substack{b_i \in B_i \\ b_1+b_2=u}} (\|b_1\|_{B_1} + \|b_2\|_{B_2}).$$

Let $1 \le p \le \infty$, and for each real number ν let t^ν denote the real-valued function defined on $[0,\infty)$ by $t^\nu(t) = t^\nu$, $0 \le t < \infty$.

We designate by $W(p,\nu;B_1,B_2)$, or when confusion is not likely to occur simply by W, the vector space of (equivalence classes of) measurable functions

f on $[0, \infty)$ into $B_1 + B_2$ such that

$$t^\nu f \in L^p(0, \infty; B_1) \qquad \text{and} \qquad t^\nu f' \in L^p(0, \infty; B_2),$$

f' denoting the distributional derivative of f. The space W is a Banach space with respect to the norm

$$\begin{aligned} \|f\|_W &= \|f; W(p, \infty; B_1, B_2)\| \\ &= \max(\|t^\nu f; L^p(0, \infty; B_1)\|, \|t^\nu f'; L^p(0, \infty; B_2)\|). \end{aligned}$$

As an example of this construction the reader may verify that $W(p, 0; W^{1,p}(\mathbb{R}^n), L^p(\mathbb{R}^n))$ is isomorphic to the Sobolev space $W^{1,p}(\Omega)$ where $\Omega = \{(x, t) = (x_1, \ldots, x_n, t) \in \mathbb{R}^{n+1} : t > 0\}$.

We shall show that for certain values of p and ν, functions f in W possess "traces" $f(0)$ in $B_1 + B_2$.

7.12 LEMMA Let $f \in W$. There exists $b \in B_1 + B_2$ such that

$$f(t) = b + \int_1^t f'(\tau)\, d\tau \qquad \text{a.e. in } (0, \infty). \tag{5}$$

Hence f is almost everywhere equal to a continuous function on $(0, \infty)$ into $B_1 + B_2$.

PROOF Since $t^\nu f \in L^p(0, \infty; B_1)$, therefore $f \in L^p_{\text{loc}}(0, \infty; B_1)$. Similarly $f' \in L^p_{\text{loc}}(0, \infty; B_2)$, and the function v defined a.e. on $(0, \infty)$ into $B_1 + B_2$ by

$$v(t) = f(t) - \int_1^t f'(\tau)\, d\tau$$

belongs to $L^p_{\text{loc}}(0, \infty; B_1 + B_2)$. It follows that the scalar-valued function $\langle v(\cdot), b' \rangle$ belongs to $L^p_{\text{loc}}(0, \infty)$ for each $b' \in (B_1 + B_2)'$. Thus for every $\phi \in C_0^\infty(0, \infty)$ we have

$$\begin{aligned} \int_0^\infty \frac{d}{dt} \langle v(t), b' \rangle \phi(t)\, dt &= -\int_0^\infty \langle v(t), b' \rangle \phi'(t)\, dt \\ &= -\left\langle \int_0^\infty v(t) \phi'(t)\, dt, b' \right\rangle \\ &= -\left\langle \int_0^\infty f(t) \phi'(t)\, dt - \int_0^\infty \phi'(t)\, dt \int_1^t f'(\tau)\, d\tau, b' \right\rangle \\ &= \left\langle \int_0^\infty f'(t) \phi(t)\, dt - \int_0^\infty f'(t) \phi(t)\, dt, b' \right\rangle = 0. \end{aligned}$$

(The change of order of integration and pairing $\langle \cdot, \cdot \rangle$ above is justified since v is integrable on the support of ϕ and so can be approximated there by simple functions for which the interchange is clearly valid.) By Corollary 3.27, $\langle v(t), b' \rangle$ is constant a.e. on $(0, \infty)$ for each $b' \in (B_1 + B_2)'$. Thus $v(t) = b$, a fixed vector in $B_1 + B_2$ a.e. on $(0, \infty)$ and (5) follows at once. Clearly the integral in (5) is continuous on $(0, \infty)$ into B_2, and hence into $B_1 + B_2$. ∎

7.13 LEMMA Suppose $(1/p) + \nu < 1$. Then the right side of (5) converges in $B_1 + B_2$ as $t \to 0+$. The limit is defined to be the trace, $f(0)$, of f at $t = 0$.

PROOF If $0 < s < t$, we have, for $1 < p < \infty$,

$$\left\| \int_s^t f'(\tau)\, d\tau \right\|_{B_2} \le \int_s^t \|\tau^\nu f'(\tau)\|_{B_2} \tau^{-\nu}\, d\tau$$
$$\le \|t^\nu f'; L^p(0, \infty; B_2)\| \left(\int_0^t \tau^{-\nu p/(p-1)}\, d\tau \right)^{(p-1)/p}.$$

The last factor tends to zero as $t \to 0+$ since $(1/p) + \nu < 1$. (The same argument works with the obvious modifications if $p = 1$ or $p = \infty$.) Thus $\int_1^t f'(\tau)\, d\tau$ converges in B_2 as $t \to 0+$, which proves the lemma. ∎

7.14 Given real p and ν with $1 \le p \le \infty$ and $\theta = (1/p) + \nu < 1$ we denote by $T(p, \nu; B_1, B_2)$, or simply by T, the space consisting of all traces $f(0)$ of functions f in $W = W(p, \nu; B_1, B_2)$. Called the *trace space* of W, T is a Banach space with respect to the norm

$$\|u\|_T = \inf_{\substack{u = f(0) \\ f \in W}} \|f\|_W.$$

T is a subspace of $B_1 + B_2$ and lies topologically "between" B_1 and B_2 in a sense which will become more apparent later.

Before developing some properties of the trace space T we prepare one further lemma concerning W which will be needed later.

7.15 LEMMA The subspace of W consisting of those functions $f \in W$ which are infinitely differentiable on $(0, \infty)$ into B_1 is dense in W if $1 \le p < \infty$.

PROOF Under the transformation

$$t = e^\tau, \qquad f(e^\tau) = \tilde{f}(\tau),$$

we have that $f \in W$ if and only if

$$\int_{-\infty}^{\infty} \left(e^{\theta p \tau} \|\tilde{f}(\tau)\|_{B_1}^p + e^{(\theta - 1)p\tau} \|\tilde{f}'(\tau)\|_{B_2}^p \right) d\tau < \infty,$$

where $\theta = (1/p) + \nu$. If J_ε is the mollifier of Section 2.17, then, just as in Lemma

2.18, $J_\varepsilon * \tilde{f}$ is infinitely differentiable on $\mathbb{R}$ into B_1 and

$$\lim_{\varepsilon\to 0+} \int_{-\infty}^{\infty} \left(e^{\theta p\tau} \|J_\varepsilon * \tilde{f}(\tau) - \tilde{f}(\tau)\|_{B_1}^p + e^{(\theta-1)p\tau} \|(J_\varepsilon * \tilde{f})'(\tau) - \tilde{f}'(\tau)\|_{B_2}^p\right) dt = 0.$$

Thus $f_n(t) = J_{1/n} * \tilde{f}(\log t)$ is infinitely differentiable for $0 < t < \infty$ with values in B_1. Since $f_n \to f$ in W the lemma is proved. ∎

We now investigate interpolation properties of the space T.

7.16 LEMMA Let $\theta = (1/p) + \nu$ satisfy $0 < \theta < 1$. Then

(a) every $u \in T$ satisfies

$$\|u\|_T = \inf_{\substack{f \in W \\ f(0) = u}} \|t^\nu f; L^p(0,\infty; B_1)\|^{1-\theta} \|t^\nu f'; L^p(0,\infty; B_2)\|^\theta; \tag{6}$$

(b) if $u \in B_1 \cap B_2$, then $u \in T$ and

$$\|u\|_T \le K \|u\|_{B_1}^{1-\theta} \|u\|_{B_2}^\theta, \tag{7}$$

where K is a constant independent of u.

PROOF (a) Fix $u \in T$ and $\varepsilon > 0$ and let $f \in W$ satisfy $f(0) = u$ and $\|f\|_W \le \|u\|_T + \varepsilon$. Let

$$R = \|t^\nu f; L^p(0,\infty; B_1)\|, \qquad S = \|t^\nu f'; L^p(0,\infty; B_2)\|.$$

For $\lambda > 0$ the function f_λ defined by $f_\lambda(t) = f(\lambda t)$ also belongs to W and satisfies $f_\lambda(0) = u$. Moreover,

$$\|t^\nu f_\lambda; L^p(0,\infty; B_1)\| = \lambda^{-\theta} R, \qquad \|t^\nu (f_\lambda)'; L^p(0,\infty; B_2)\| = \lambda^{1-\theta} S.$$

These two expressions are both equal to $R^{1-\theta} S^\theta$ provided we choose $\lambda = R/S$. Hence

$$\begin{aligned} \max(R,S) &= \|f\|_W \le \|u\|_T + \varepsilon \\ &\le \inf_{\lambda>0} \|f_\lambda\|_W + \varepsilon \\ &\le \inf_{\lambda>0} \max(\lambda^{-\theta} R, \lambda^{1-\theta} S) + \varepsilon \le R^{1-\theta} S^\theta + \varepsilon \le \max(R,S) + \varepsilon. \end{aligned}$$

Since ε is arbitrary, (6) follows at once.

(b) Let $\phi \in C^\infty([0,\infty))$ satisfy $\phi(0) = 1$, $\phi(t) = 0$ if $t \ge 1$, $|\phi(t)| \le 1$, and $|\phi'(t)| \le K_1$ for all $t \ge 0$. If $u \in B_1 \cap B_2$, let $f(t) = \phi(t) u$ so that $u = f(0)$. Now

$$\|t^\nu f; L^p(0,\infty; B_1)\| \le K_2 \|u\|_{B_1},$$

where $K_2 = \{\int_0^1 t^{\nu p}\,dt\}^{1/p}$. Similarly,

$$\|t^\nu f'; L^p(0,\infty; B_2)\| \le K_1 K_2 \|u\|_{B_2}.$$

Hence $f \in W$ and (7) follows from (6). ∎

The above lemma suggests the sense in which T lies "between" B_1 and B_2 provided $0 < \theta < 1$. The spaces T corresponding to all such values of θ are sometimes described as constituting a *scale* of Banach spaces interpolated between B_1 and B_2. Many properties of T can be deduced from corresponding properties of B_1 and B_2 via the following interpolation theorem.

7.17 THEOREM Suppose that $B_1 \cap B_2$ is dense in B_1 and B_2. Let $\tilde{B}_1$, $\tilde{B}_2$, and $\tilde{X}$ be three spaces having the same properties specified for B_1, B_2, and X in Section 7.11. Let L be a linear operator defined on $B_1 \cap B_2$ into $\tilde{B}_1 \cap \tilde{B}_2$ and suppose that for every $v \in B_1 \cap B_2$ we have

$$\|Lv\|_{\tilde{B}_1} \le K_1 \|v\|_{B_1}, \tag{8}$$

$$\|Lv\|_{\tilde{B}_2} \le K_2 \|v\|_{B_2}. \tag{9}$$

Thus L possesses unique continuous extensions (also denoted L) to B_1 and B_2 satisfying (8) and (9), respectively, and hence to $B_1 + B_2$ satisfying

$$\|Lu\|_{\tilde{B}_1+\tilde{B}_2} \le \max(K_1, K_2)\,\|u\|_{B_1+B_2}.$$

If $0 < \theta = (1/p) + \nu < 1$, then for every $u \in T = T(p,\nu; B_1, B_2)$ we have $Lu \in \tilde{T} = T(p,\nu; \tilde{B}_1, \tilde{B}_2)$ and

$$\|Lu\|_{\tilde{T}} \le K_1^{1-\theta} K_2^{\theta} \|u\|_T. \tag{10}$$

PROOF L is defined on $B_1 + B_2$ and hence on T. By (6) we have for $u \in T$

$$\|Lu\|_{\tilde{T}} = \inf_{\substack{\tilde{f} \in \tilde{W} \\ \tilde{f}(0) = Lu}} \|t^\nu \tilde{f}; L^p(0,\infty; \tilde{B}_1)\|^{1-\theta} \|t^\nu \tilde{f}'; L^p(0,\infty; \tilde{B}_2\|^{\theta}.$$

Also, there exists, for any given $\varepsilon > 0$, an element $f \in W$ with $f(0) = u$ such that

$$\|t^\nu f; L^p(0,\infty; B_1)\|^{1-\theta} \|t^\nu f'; L^p(0,\infty; B_2)\|^{\theta} < \|u\|_T + \varepsilon.$$

For $t \ge 0$ let $\tilde{f}(t) = Lf(t)$ so that $\tilde{f}(0) = Lu$ and

$$\|t^\nu \tilde{f}; L^p(0,\infty; \tilde{B}_1)\|^{1-\theta} \|t^\nu \tilde{f}'; L^p(0,\infty; \tilde{B}_2)\|^{\theta}$$
$$\le K_1^{1-\theta} \|t^\nu f; L^p(0,\infty; B_1)\|^{1-\theta} K_2^{\theta} \|t^\nu f'; L^p(0,\infty; B_2)\|^{\theta}.$$

Hence

$$\|Lu\|_{\tilde{T}} < K_1^{1-\theta} K_2^{\theta}(\|u\|_T + \varepsilon).$$

Since ε is arbitrary (10) follows. ∎

If we denote by $\|L\|_{L(S,\tilde{S})}$ the norm of L as an element of the Banach space $L(S,\tilde{S})$ of continuous linear operators on S into $\tilde{S}$ we may write (10) in the form

$$\|L\|_{L(T,\tilde{T})} \le \|L\|_{L(B_1,\tilde{B}_1)}^{1-\theta} \|L\|_{L(B_2,\tilde{B}_2)}^{\theta}.$$

7.18 LEMMA Suppose that $B_1 \cap B_2$ is dense in B_1 and B_2, and that there exists a sequence $\{P_j\}_{j=1}^{\infty}$ of linear operators belonging simultaneously to $L(B_1)$ and $L(B_2)$ and having range in $B_1 \cap B_2$. Suppose also that for each $b_i \in B_i$, $i = 1, 2$,

$$\lim_{j\to\infty} \|P_j b_i - b_i\|_{B_i} = 0.$$

Then for every $u \in T$ we have

$$\lim_{j\to\infty} \|P_j u - u\|_T = 0.$$

In particular, $B_1 \cap B_2$ is dense in T.

PROOF Fix $u \in T$ and choose $f \in W$ such that $f(0) = u$. Let $f_j(t) = P_j f(t)$. If $b_i \in B_i$, $i = 1, 2$, there exists an integer $j_0 = j_0(b_i)$ such that if $j \ge j_0$, then

$$\|P_j b_i - b_i\|_{B_i} \le 1.$$

Hence $\{P_j b_i\}$ is bounded in B_i, $i = 1, 2$, independently of j. By the uniform boundedness principle there exist constants K_1 and K_2 such that for every j

$$\|P_j\|_{L(B_i)} \le K_i.$$

It follows that

$$\|f_j(t)\|_{B_1} \le K_1 \|f(t)\|_{B_1}, \qquad \|f_j'(t)\|_{B_2} \le K_2 \|f'(t)\|_{B_2}.$$

Since for almost all $t > 0$, $f_j(t) \to f(t)$ in B_1 and $f_j'(t) \to f'(t)$ in B_2 as $j \to \infty$, we have by dominated convergence that $t^\nu f_j \to t^\nu f$ in $L^p(0,\infty; B_1)$ and $t^\nu f_j' \to t^\nu f'$ in $L^p(0,\infty; B_2)$. Hence $f_j \to f$ in W and so $P_j u = P_j f(0) \to f(0) = u$ in T. Since $t^\nu f_j$ and $t^\nu f_j'$ take values in $B_1 \cap B_2$, $P_j u$ belongs to $B_1 \cap B_2$. ∎

We quote now a theorem characterizing the dual of a trace space as another trace space. The proof is rather long and will not be given here—the interested reader may find it in the work of Lions [37] where trace spaces somewhat more general than those introduced above are studied. (The following theorem is a special case of Theorem 1.1 of Ref. [37, Chap. II].)

7.19 THEOREM Suppose that B_1 and B_2 are reflexive and also satisfy the conditions of Lemma 7.18. If $1 < p < \infty$ and $(1/p) + \nu = \theta$ satisfies $0 < \theta < 1$, then $(1/p')-\nu = 1-(1/p)-\nu = 1-\theta$ and

$$[T(p,\nu;B_1,B_2)]' \cong T(p',-\nu;B_2',B_1').$$

In particular, $T(p,\nu;B_1,B_2)$ is reflexive.

We now prove an imbedding theorem for trace spaces between two L^p-spaces which will play a vital role in extending certain aspects of the Sobolev imbedding theorem to fractional order spaces (see Theorem 7.57). If Ω is a domain in $\mathbb{R}^n$, then $B_1 = L^q(\Omega)$ and $B_2 = L^p(\Omega)$ are both continuously imbedded in the topological vector space $X = L^1_{\text{loc}}(\Omega)$. (A subset $U \subset L^1_{\text{loc}}(\Omega)$ is open if for every $u \in U$ there exists $\varepsilon > 0$ and $K \subset\subset \Omega$ such that $\|v-u\|_{0,1,K} < \varepsilon$, $v \in L^1_{\text{loc}}(\Omega)$ implies $v \in U$.)

7.20 THEOREM Let p,q,θ satisfy $1 \le p \le q < \infty$, $0<\theta<1$, $\theta = (1/p)+\nu$. Then

$$T(p,\nu;L^q(\Omega),L^p(\Omega)) \to L^r(\Omega), \tag{11}$$

where

$$1/r = [(1-\theta)/q] + (\theta/p).$$

PROOF Suppose that $f \in C^\infty([0,\infty))$. From the identity

$$f(0) = f(t) - \int_0^t f'(\tau)\,d\tau$$

we may readily obtain

$$\begin{aligned}|f(0)| &\le \int_0^1 |f(t)|\,dt + \int_0^1 |f'(t)|\,dt \\ &\le \left\{\left(\int_0^\infty t^{\nu p}|f(t)|^p\,dt\right)^{1/p} + \left(\int_0^\infty t^{\nu p}|f'(t)|^p\,dt\right)^{1/p}\right\}\left(\int_0^1 t^{-\nu p'}\,dt\right)^{1/p'} \\ &= K_1(\|t^\nu f\|_{0,p,(0,\infty)} + \|t^\nu f'\|_{0,p,(0,\infty)}),\end{aligned}$$

where $K_1 < \infty$ since $\theta = (1/p)+\nu < 1$. By a homogeneity argument similar to that used in the proof of Lemma 7.16(a), we may now obtain

$$|f(0)| \le 2K_1 \|t^\nu f\|^{1-\theta}_{0,p,(0,\infty)} \|t^\nu f'\|^{\theta}_{0,p,(0,\infty)}. \tag{12}$$

Now suppose that $f \in W(p,\nu;L^q(\Omega),L^p(\Omega))$ and, for the moment, that f is infinitely differentiable on $(0,\infty)$ into $L^q(\Omega)$. Let $\tilde{f}(x,t) = f(t)(x)$ for $0 \le t < \infty$, $x \in \Omega$. From (12) we have that $\tilde{f}(x,0) = \lim_{t\to 0+} \tilde{f}(x,t)$ satisfies

$$|\tilde{f}(x,0)|^r \le K_2\left(\int_0^\infty t^{\nu p}|\tilde{f}(x,t)|^p\,dt\right)^{(1-\theta)r/p}\left(\int_0^\infty t^{\nu p}\left|\frac{\partial}{\partial t}\tilde{f}(x,t)\right|^p dt\right)^{\theta r/p}$$

for almost all $x \in \Omega$. Thus, by Hölder's inequality,

$$\int_\Omega |\tilde{f}(x,0)|^r\,dx \le K_2\left(\int_\Omega\left(\int_0^\infty t^{\nu p}|\tilde{f}(x,t)|^p\,dt\right)^{(1-\theta)rs/p}dx\right)^{1/s}$$
$$\times\left(\int_\Omega\left(\int_0^\infty t^{\nu p}\left|\frac{\partial}{\partial t}\tilde{f}(x,t)\right|^p dt\right)^{\theta rs'/p}dx\right)^{1/s'},$$

where $(1/s)+(1/s')=1$. If we choose s so that $(1-\theta)rs=q$, then also $\theta rs'=p$ and we have

$$\|\tilde{f}(\cdot,0)\|_{0,r,\Omega} \le K_3\left(\int_\Omega\left(\int_0^\infty t^{\nu p}|\tilde{f}(x,t)|^p\,dt\right)^{q/p}dx\right)^{(1-\theta)/q}$$
$$\times\left(\int_\Omega\int_0^\infty t^{\nu p}\left|\frac{\partial}{\partial t}\tilde{f}(x,t)\right|^p dt\,dx\right)^{\theta/p}$$
$$= K_3\left\|\int_0^\infty t^{\nu p}|f(t)|^p\,dt\right\|_{0,q/p,\Omega}^{(1-\theta)/p}\|t^\nu f';L^p(0,\infty;L^p(\Omega))\|^\theta$$
$$\le K_3\left(\int_0^\infty t^{\nu p}\,\||f(t)|^p\|_{0,q/p,\Omega}\,dt\right)^{(1-\theta)/p}\|t^\nu f';L^p(0,\infty;L^p(\Omega))\|^\theta$$
$$= K_3\,\|t^\nu f;L^p(0,\infty;L^q(\Omega))\|^{1-\theta}\,\|t^\nu f';L^p(0,\infty;L^p(\Omega))\|^\theta.$$

By the density of infinitely differentiable functions in W (Lemma 7.15) the above inequality holds for any $f \in W$.

It now follows that if $u \in T(p,\nu;L^q(\Omega),L^p(\Omega))$, then

$$\|u\|_{0,r,\Omega} \le \inf_{\substack{f\in W\\ f(0)=u}} K_3\,\|t^\nu f;L^p(0,\infty;L^q(\Omega))\|^{1-\theta}\,\|t^\nu f';L^p(0,\infty;L^p(\Omega))\|^\theta$$
$$= K_3\,\|u\|_T$$

by Lemma 7.16(a). This establishes imbedding (11). ∎

7.21 REMARK With minor modifications in the proof, the above theorem extends to cover $q=\infty$ provided we use in place of $L^q(\Omega)$ the closure of $L^p(\Omega)\cap L^\infty(\Omega)$ in $L^\infty(\Omega)$.

Semigroup Characterization of Trace Spaces

7.22 Let B be a Banach space and G a continuous semigroup on B which is uniformly bounded, that is, for which there exists a constant M such that

$$\|G(t)\|_{L(B)} \le M, \qquad 0 \le t < \infty.$$

Let Λ be the infinitesimal generator of G so that (see Remark 7.9) $D(\Lambda)$, the domain of Λ in B, is a Banach space with respect to the graph-norm

$$\|u; D(\Lambda)\| = \|u\|_B + \|\Lambda u\|_B,$$

and is also a dense vector subspace of B. The spaces $B_1 = D(\Lambda)$ and $B_2 = X = B$ satisfy the conditions of Section 7.11, and we may accordingly construct the trace space $T = T(p, \nu; D(\Lambda), B)$ provided $0 = (1/p) + \nu < 1$ Theorem 7.24 characterizes T in terms of an explicit norm involving the semigroup G. First, however, we obtain an inequality of Hardy, Littlewood, and Pólya [28] which will be needed.

7.23 LEMMA Let f be a scalar-valued function defined a.e. on $[0, \infty)$ and let

$$g(t) = (1/t) \int_0^t f(\xi)\, d\xi.$$

If $1 \le p < \infty$ and $(1/p) + \nu = \theta < 1$, then

$$\int_0^\infty t^{\nu p} |g(t)|^p\, dt \le [1/(1-\theta)^p] \int_0^\infty t^{\nu p} |f(t)|^p\, dt. \tag{13}$$

PROOF We may certainly suppose that the right side of (13) is finite. Under the transformation $t = e^\tau$, $f(e^\tau) = \tilde{f}(\tau)$, $\xi = e^\sigma$, $g(e^\tau) = \tilde{g}(\tau)$, (13) becomes

$$\int_{-\infty}^\infty e^{\theta p \tau} |\tilde{g}(\tau)|^p\, d\tau \le [1/(1-\theta)^p] \int_{-\infty}^\infty e^{\theta p \tau} |\tilde{f}(\tau)|^p\, d\tau. \tag{14}$$

Note that

$$\tilde{g}(\tau) = e^{-\tau} \int_{-\infty}^\tau \tilde{f}(\sigma) e^\sigma\, d\sigma.$$

Let $E(\tau) = e^{\theta\tau}$ and

$$F(\tau) = \begin{cases} e^{(\theta-1)\tau} & \text{if } \tau > 0 \\ 0 & \text{if } \tau \le 0. \end{cases}$$

Then $E \cdot \tilde{g} = F * (E \cdot \tilde{f})$, and so by Young's theorem 4.30,

$$\|E \cdot \tilde{g}\|_{0,p,\mathbb{R}} \le \|F\|_{0,1,\mathbb{R}} \|E\tilde{f}\|_{0,p,\mathbb{R}}.$$

This inequality is precisely (14) since $\int_{-\infty}^\infty |F(\tau)|\, d\tau = 1/(1-\theta)$. ∎

7.24 THEOREM Let Λ be the infinitesimal generator of a uniformly bounded, continuous semigroup G on the Banach space B. If $1 \le p < \infty$ and $0 < (1/p) + \nu < 1$, then $T = T(p, \nu; D(\Lambda), B)$ coincides with the space T^0 of all $u \in B$ for which the norm

$$\|u\|_{T^0} = \left(\|u\|_B^p + \int_0^\infty t^{(\nu-1)p} \|G(t)u - u\|_B^p \, dt\right)^{1/p} \tag{15}$$

is finite. The norms $\|\cdot\|_T$ and $\|\cdot\|_{T^0}$ are equivalent.

PROOF First let $u \in T$ and choose $f \in W$ such that $f(0) = u$. For the moment let us assume that f is infinitely differentiable on $(0, \infty)$ into $D(\Lambda)$. Let $f'(t) - \Lambda f(t) = h(t)$. If $t \geq \varepsilon > 0$, we obtain by Theorem 7.10,

$$f(t) = G(t-\varepsilon) f(\varepsilon) + \int_\varepsilon^t G(t-\tau) h(\tau) \, d\tau.$$

Hence

$$G(t-\varepsilon) f(\varepsilon) - f(\varepsilon) = \int_\varepsilon^t f'(\tau) \, d\tau - \int_\varepsilon^t G(t-\tau) h(\tau) \, d\tau.$$

Letting $\varepsilon \to 0+$ in this identity and noting that $f(\varepsilon) \to f(0)$ by definition, we obtain

$$G(t) f(0) - f(0) = \int_0^t f'(\tau) \, d\tau - \int_0^t G(t-\tau) h(\tau) \, d\tau. \tag{16}$$

Now (16) holds for any $f \in W$ since, by Lemma 7.15, f is the limit in W of a sequence $\{f_n\}$ of infinitely differentiable functions on $(0, \infty)$ with values in $D(\Lambda)$. Hence for $u \in T$ and any $f \in W$ with $f(0) = u$, we have

$$G(t)u - u = \int_0^t f'(\tau) \, d\tau - \int_0^t G(t-\tau) h(\tau) \, d\tau, \qquad h(\tau) = f'(\tau) - \Lambda f(\tau).$$

Thus

$$\left\| \frac{G(t)u - u}{t} \right\|_B \leq \frac{1}{t} \int_0^t \|f'(\tau)\|_B \, d\tau + \frac{M}{t} \int_0^t \|h(\tau)\|_B \, d\tau,$$

where we have used the uniform boundedness of G. Applying Lemma 7.23, we obtain, putting $\theta = (1/p) + \nu$,

$$\begin{aligned}
&\int_0^\infty t^{(\nu-1)p} \|G(t)u - u\|_B^p \, dt \\
&\leq \frac{1}{(1-\theta)^p} \int_0^\infty t^{\nu p} (\|f'(t)\|_B + M \|h(t)\|_B)^p \, dt \\
&\leq \frac{2^{p-1}(M+1)^p}{(1-\theta)^p} (\|t^\nu f'; L^p(0, \infty; B)\|^p + \|t^\nu \Lambda f; L^p(0, \infty; B)\|^p) \\
&\leq \frac{2^p (M+1)^p}{(1-\theta)^p} \|f\|_W^p.
\end{aligned}$$

Since this holds for any $f \in W$ with $f(0) = u$ we have

$$\int_0^\infty t^{(\nu-1)p} \|G(t)u-u\|_B^p\, dt \le \left(\frac{2M+2}{1-\theta}\right)^p \|u\|_T^p.$$

Also, the identity operator on B provides imbeddings of B into B (trivially) and $D(\Lambda)$ into B, each imbedding having unit imbedding constant. By Theorem 7.17 we have as well, $T \to B$ and $\|u\|_B \le \|u\|_T$. Hence $u \in T$ implies $u \in T^0$ and

$$\|u\|_{T^0} \le \left(1 + \frac{2M+2}{1-\theta}\right)\|u\|_T.$$

Conversely, suppose that $u \in T^0$. Let $\phi \in C^\infty([0,\infty))$ satisfy $\phi(0) = 1$, $\phi(t) = 0$ for $t \ge 1$, $|\phi(t)| \le 1$ and $|\phi'(t)| \le K_1$ for $t \ge 0$. Let

$$f(t) = \phi(t)g(t),$$

where

$$g(t) = (1/t)\int_0^t G(\tau)u\, d\tau, \qquad t > 0. \tag{17}$$

In order to show that $u \in T$ and $\|u\|_T \le K_2 \|u\|_{T^0}$, it is sufficient to prove that $f \in W$ and

$$\|f\|_W \le K_2 \|u\|_{T^0} \tag{18}$$

[since $f(0) = \lim_{t\to 0+} \phi(t)g(t) = u$ by Lemma 7.8(a)] and this can be done by showing that $t^\nu g \in L^p(0,1;D(\Lambda))$ and $t^\nu g' \in L^p(0,1;B)$ with appropriate norms bounded by $K_3 \|u\|_{T^0}$.

By Lemma 7.8(b), $\int_0^t G(\tau)u\, d\tau \in D(\Lambda)$ and

$$\Lambda \int_0^t G(\tau)u\, d\tau = G(t)u - u.$$

Thus

$$\begin{aligned}
&\int_0^1 t^{\nu p} \|g(t); D(\Lambda)\|^p\, dt \\
&\quad = \int_0^1 t^{(\nu-1)p}\left(\left\|\int_0^t G(\tau)u\, d\tau\right\|_B + \left\|\Lambda\int_0^t G(\tau)u\, d\tau\right\|_B\right)^p dt \\
&\quad \le 2^{p-1}M^p\|u\|_B^p \int_0^1 t^{\nu p}\, dt + 2^{p-1}\int_0^\infty t^{(\nu-1)p}\|G(t)u-u\|_B^p\, dt \\
&\quad \le 2^{p-1}\max(M^p\theta/p, 1)\|u\|_{T^0}^p.
\end{aligned}$$

Since

$$\begin{aligned}
g'(t) &= (1/t)G(t)u - (1/t^2)\int_0^t G(\tau)u\, d\tau \\
&= (1/t)(G(t)u-u) - (1/t^2)\int_0^t (G(\tau)u-u)\, d\tau,
\end{aligned}$$

and since

$$\int_0^1 t^{\nu p} \left\| \frac{G(t)u-u}{t} \right\|_B^p dt \le \|u\|_{T^0}^p$$

and, by Lemma 7.23 with $\nu-1$ replacing ν,

$$\int_0^1 t^{\nu p} \left\| (1/t^2) \int_0^t (G(\tau)u-u)\, d\tau \right\|_B^p dt \le [1/(2-\theta)^p] \int_0^\infty t^{(\nu-1)p} \|G(t)u-u\|_B^p\, dt$$
$$\le [1/(2-\theta)^p]\, \|u\|_{T^0}^p$$

we therefore have

$$\int_0^1 t^{\nu p} \|g'(t)\|_B^p\, dt \le K_4 \|u\|_{T^0}^p.$$

Thus $g \in W$, (18) holds, and the proof is complete. ∎

7.25 For our purposes we require a slight generalization of Theorem 7.24. Let $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ be a finite family of infinitesimal generators of commuting, uniformly bounded, continuous semigroups $G_1, G_2, \ldots, G_n$ on B. Thus

$$\|G_j(t)\|_{L(B)} \le M_j; \qquad 1 \le j \le n, \quad t \ge 0;$$
$$G_j(s)G_k(t) = G_k(t)G_j(s); \qquad 1 \le j, k \le n, \quad s, t \ge 0.$$

Let B^n denote the product space $B \times B \times \cdots \times B$ (n factors) which is a Banach space with norm

$$\|(b_1, b_2, \ldots, b_n)\|_{B^n} = \sum_{j=1}^n \|b_j\|_B.$$

The operator Λ is defined on $D(\Lambda) = \bigcap_{j=1}^n D(\Lambda_j)$ into B^n by

$$\Lambda u = (\Lambda_1 u, \Lambda_2 u, \ldots, \Lambda_n u).$$

We leave it to the reader to generalize Lemma 7.8 to show that $D(\Lambda)$ is dense in B, that Λ is a closed operator, and hence that $D(\Lambda)$ is a Banach space with respect to the norm

$$\|u; D(\Lambda)\| = \|u\|_B + \|\Lambda u\|_{B^n} = \|u\|_B + \sum_{j=1}^n \|\Lambda_j u\|_B.$$

7.26 THEOREM If $0 < (1/p)+\nu < 1$, $1 \le p < \infty$, then $T = T(p, \nu; D(\Lambda), B)$ coincides with the space T^0 of all $u \in B$ for which the norm

$$\|u\|_{T^0} = \left(\|u\|_B^p + \sum_{j=1}^n \int_0^\infty t^{(\nu-1)p} \|G_j(t)u-u\|_B^p\, dt \right)^{1/p}$$

is finite. The norms $\|\cdot\|_T$ and $\|\cdot\|_{T^0}$ are equivalent.

Proof The proof is nearly identical to that of Theorem 7.24 except that in place of the function $g(t)$ given by (17) we use

$$g(t) = (1/t^n) \int_0^t \int_0^t \cdots \int_0^t G_1(\tau_1) G_2(\tau_2) \cdots G_n(\tau_n) u \, d\tau_1 \, d\tau_2 \cdots d\tau_n.$$

The details are left to the reader. ∎

7.27 EXAMPLE Let $B = L^p(\mathbb{R}^n)$, $1 \le p < \infty$, and for $u \in B$ set

$$(G_j(t)u)(x) = u(x_1, \dots, x_j + t, \dots, x_n); \qquad j = 1, 2, \dots, n.$$

Clearly the G_j are commuting, uniformly bounded ($M_j = 1$), continuous semigroups on $L^p(\mathbb{R}^n)$. (In fact they are groups if we allow $t < 0$.) The corresponding infinitesimal generators satisfy

$$D(\Lambda_j) = \{u \in L^p(\mathbb{R}^n) : D_j u \in L^p(\mathbb{R}^n)\},$$

$$\Lambda_j u = D_j u, \qquad u \in D(\Lambda_j).$$

Accordingly, $D(\Lambda) = \bigcap_{j=1}^n D(\Lambda_j) = W^{1,p}(\mathbb{R}^n)$. By Theorem 7.26 the norm

$$\left(\|u\|_{0,p,\mathbb{R}^n}^p + \sum_{j=1}^n \int_0^\infty t^{(\nu-1)p} \int_{\mathbb{R}^n} |u(x_1, \dots, x_j + t, \dots, x_n) - u(x_1, \dots, x_n)|^p \, dx \, dt \right)^{1/p}$$

is equivalent to $\|u\|_T$ on the space $T = T(p, \nu; W^{1,p}(\mathbb{R}^n), L^p(\mathbb{R}^n))$, $0 < (1/p) + \nu < 1$.

Higher-Order Traces

7.28 Up to this point we have considered only traces $f(0)$ of functions satisfying, with their first derivatives f', suitable integrability conditions on $[0, \infty)$ into various Banach spaces. We now extend the notion of trace to obtain values for $f^{(j)}(0)$, $0 \le j \le m-1$, provided $f, f', \dots, f^{(m)}$ satisfy such integrability conditions. As a result of this extension we will be able later to characterize the traces on the boundary of a regular domain Ω, of functions belonging to $W^{m,p}(\Omega)$.

7.29 Let B be a Banach space and $\Lambda_1, \dots, \Lambda_n$ infinitesimal generators of commuting, uniformly bounded, continuous semigroups $G_1, \dots, G_n$ on B. For each multi-index α we define a subspace $D(\Lambda^\alpha)$ of B and a corresponding linear operator Λ^α on $D(\Lambda^\alpha)$ into B by induction on $|\alpha|$ as follows.

If $\alpha = (0, 0, \dots, 0)$, then $D(\Lambda^\alpha) = B$ and $\Lambda^\alpha = I$, the identity on B.

If $\alpha = (0, \dots, 1, \dots, 0)$ (1 in the jth place), then $D(\Lambda^\alpha) = D(\Lambda_j)$ and $\Lambda^\alpha = \Lambda_j$.

If $D(\Lambda^\beta)$ and Λ^β have been defined for all β with $|\beta| \le r$, and if $|\alpha| = r + 1$, then

$$D(\Lambda^\alpha) = \{u : u \in D(\Lambda^\beta) \text{ and } \Lambda^{\alpha-\beta}u \in D(\Lambda^\beta) \text{ for all } \beta < \alpha\},$$
$$\Lambda^\alpha = \Lambda_1^{\alpha_1} \cdots \Lambda_n^{\alpha_n}.$$

If k is a positive integer, let $D(\Lambda^k) = \bigcap_{|\alpha| \le k} D(\Lambda^\alpha)$. Once again we leave to the reader the task of verifying (say by induction on k) that $D(\Lambda^k)$ is dense in B, that $\Lambda^k = (\Lambda^\alpha)_{|\alpha| \le k}$ is a closed operator on $D(\Lambda^k)$ into $\prod_{|\alpha| \le k} B$, and hence that $D(\Lambda^k)$ is a Banach space with respect to the norm

$$\|u; D(\Lambda^k)\| = \sum_{|\alpha| \le k} \|\Lambda^\alpha u\|_B.$$

7.30 For positive integers m and real p, $1 \le p \le \infty$, let $W^m = W^m(p, v; \Lambda; B)$ denote the space of (equivalence classes of) measurable functions f on $[0, \infty)$ into B such that

$$t^v f^{(k)} \in L^p(0, \infty; D(\Lambda^{m-k})) \qquad 0 \le k \le m,$$

$f^{(k)}$ being the distributional derivative $d^k f/dt^k$. The space W^m is a Banach space with respect to the norm

$$\|f\|_{W^m} = \max_{0 \le k \le m} \|t^v f^{(k)}; L^p(0, \infty; D(\Lambda^{m-k}))\|.$$

Note that $W^1 = W(p, v; D(\Lambda), B)$ with $D(\Lambda)$ as in Section 7.25.

7.31 LEMMA Let $f \in W^m$, $m \ge 1$. If $0 \le k \le m-1$ and α is a multi-index such that $|\alpha| + k \le m-1$, then the function $f_{\alpha k} = \Lambda^\alpha f^{(k)} \in W^1$ and

$$\|f_{\alpha k}\|_{W^1} \le \|f\|_{W^m}.$$

PROOF We have (for $1 \le p < \infty$)

$$\begin{aligned}\|t^v f_{\alpha k}; L^p(0, \infty; D(\Lambda))\|^p &= \int_0^\infty t^{vp} \left(\|\Lambda^\alpha f^{(k)}(t)\|_B + \sum_{j=1}^n \|\Lambda_j \Lambda^\alpha f^{(k)}(t)\|_B \right)^p dt \\ &\le \int_0^\infty t^{vp} \left(\sum_{|\beta| \le m-k} \|\Lambda^\beta f^{(k)}(t)\|_B \right)^p dt \\ &= \|t^v f^{(k)}; L^p(0, \infty; D(\Lambda^{m-k}))\|^p \le \|f\|_{W^m}^p.\end{aligned}$$

Also,

$$\begin{aligned}\|t^v f'_{\alpha k}; L^p(0, \infty; B)\|^p &= \int_0^\infty t^{vp} \|\Lambda^\alpha f^{(k+1)}(t)\|_B^p \, dt \\ &\le \int_0^\infty t^{vp} \left(\sum_{|\beta| \le m-k-1} \|\Lambda^\beta f^{(k+1)}(t)\|_B \right)^p dt \\ &= \|t^v f^{(k+1)}; L^p(0, \infty; D(\Lambda^{m-k-1}))\|^p \le \|f\|_{W^m}^p,\end{aligned}$$

whence the lemma follows. ∎

7.32 Let us assume hereafter that $0 < (1/p)+\nu < 1$ and denote by T^0 the trace space $T(p,\nu;D(\Lambda),B)$ corresponding to W^1. By Theorem 7.26 we may take the norm in T^0 to be

$$\|u\|_{T^0} = (\|u\|_B^p + \|u\|_G^p)^{1/p},$$

where

$$\|u\|_G = \left(\sum_{j=1}^n \int_0^\infty t^{(\nu-1)p} \|G_j(t)u-u\|_B^p\, dt\right)^{1/p}.$$

Higher-order trace spaces may now be defined as follows. For $k = 0,1,2,\ldots$ we define $T^k = T^k(p,\nu;\Lambda;B)$ to be the space consisting of those elements $u \in D(\Lambda^k)$ for which $\Lambda^\alpha u \in T^0$ whenever $|\alpha| \le k$. The space T^k is a Banach space with respect to the norm

$$\|u\|_{T^k} = \left(\|u;D(\Lambda^k)\|^p + \sum_{|\alpha|=k} \|\Lambda^\alpha u\|_G^p\right)^{1/p}.$$

It follows from Lemmas 7.31 and 7.13 that if $f \in W^m$ and $|\alpha| \le m-k-1$, then $\Lambda^\alpha f^{(k)}(0)$ exists in T^0 and

$$\|\Lambda^\alpha f^{(k)}(0)\|_{T^0} \le K_{\alpha k}\|f\|_{W^m},$$

where $K_{\alpha k}$ is a constant depending on α and k. Hence $f^{(k)}(0) \in T^{m-k-1}$ and

$$\begin{aligned}\|f^{(k)}(0)\|_{T^{m-k-1}} &= \left(\|f^{(k)}(0);D(\Lambda^{m-k-1})\|^p + \sum_{|\beta|=m-k-1} \|\Lambda^\beta f^{(k)}(0)\|_G^p\right)^{1/p}\\ &\le K_k\|f\|_{W^m}.\end{aligned}$$

It follows that the linear mapping

$$f \to (f(0), f'(0), \ldots, f^{(m-1)}(0)) \tag{19}$$

is continuous on W^m into $T^{m-1}\times T^{m-2}\times\cdots\times T^0 = \prod_{k=0}^{m-1} T^{m-k-1}$; that is,

$$\sum_{k=0}^{m-1} \|f^{(k)}(0)\|_{T^{m-k-1}} \le K\|f\|_{W^m}.$$

We shall prove that this mapping is onto (see Lions [38]).

7.33 THEOREM The range of the mapping (19) is $\prod_{k=0}^{m-1} T^{m-k-1}$. If $(u_0, u_1, \ldots, u_{m-1}) \in \prod_{k=0}^{m-1} T^{m-k-1}$, there exists $f \in W^m$ such that $f^{(k)}(0) = u_k$, $0 \le k \le m-1$, and

$$\|f\|_{W^m} \le K_0 \sum_{k=0}^{m-1} \|u_k\|_{T^{m-k-1}}.$$

Proof The proof is similar to the second part of that of Theorem 7.24, but is rather more complicated. To achieve some simplification we shall deal with the special case $n = 1$ so that Λ_j becomes just Λ and Λ^α becomes Λ^k ($|\alpha| = k$).

Suppose we have constructed for each k, $0 \le k \le m-1$, a function $f_k \in W^m$ such that $f_k^{(k)}(0) = u_k$ and

$$\|f_k\|_{W^m} \le K_k \|u_k\|_{T^{m-k-1}}.$$

Let $\lambda_{r,k}$, $0 \le r \le m-1$, satisfy the nonsingular system of equations

$$\sum_{r=0}^{m-1} r^j \lambda_{r,k} = \begin{cases} 1 & \text{if } \; j = k \\ 0 & \text{if } \; 0 \le j \le m-1, \quad j \ne k. \end{cases}$$

Then the function

$$g_k(t) = \sum_{r=0}^{m-1} \lambda_{r,k} f_k(rt)$$

satisfies

$$g_k^{(k)}(0) = u_k, \qquad g_k^{(j)}(0) = 0, \qquad 0 \le j \le m-1, \quad j \ne k.$$

Moreover, it is easily checked that

$$\|g_k\|_{W^m} \le \tilde{K}_k \|f_k\|_{W^m}.$$

Hence the function $f(t) = \sum_{k=0}^{m-1} g_k(t)$ has the properties required in the statement of the theorem. Thus we need only construct f_k.

In the rest of the proof we shall make extensive use of the convolution product of operator-valued functions on $[0, \infty)$. If for $t \ge 0$, $F_1(t)$ and $F_2(t)$ belong to $L(B)$, we define $F_1 * F_2$ on $[0, \infty)$ into $L(B)$ by

$$F_1 * F_2(t)b = \int_0^t F_1(t-\tau) F_2(\tau) b \, d\tau, \qquad b \in B.$$

(All the operators we use will commute.) If F_1 is continuously differentiable on $[0, \infty)$ into $L(B)$ so that $F_1'(t) \in L(B)$, we have evidently

$$(F_1 * F_2)'(t) = F_1' * F_2(t).$$

We denote by $F^{((m))}$ the convolution product $F * F * \cdots * F$ having m factors; this is well defined since $*$ is associative for mutually commuting factors. If $I(t) = I$ denotes the identity on B, we have clearly

$$I^{((m))}(t) = [t^{m-1}/(m-1)!] I.$$

If G is the continuous semigroup whose infinitesimal generator is Λ, then by Lemma 7.8 we have

$$\Lambda(I * G) = G - I$$

or, when both sides are restricted to elements of $D(\Lambda)$,

$$I * \Lambda G = G - I.$$

Given $u = u_k \in T^{m-k-1}$ we define

$$f_k(t) = [(m+k)!/k!]\,\phi(t)g(t)$$

where $\phi \in C^{\infty}([0,\infty))$ satisfies $\phi(t) = 1$ if $t \le \frac{1}{2}$, $\phi(t) = 0$ if $t \ge 1$, and $|\phi^{(j)}(t)| \le K_1$, $0 \le j \le m$, and where

$$g(t) = t^{-m} I^{((k+1))} * G^{((m))}(t)u. \tag{20}$$

[Note that this is the same function g as defined in (17) in the proof of Theorem 7.24 if $m = 1$ and $k = 0$.] We must verify that

$$f_k^{(k)}(0) = u \tag{21}$$

and

$$\|f_k\|_{W^m} \le K_2 \|u\|_{T^{m-k-1}}. \tag{22}$$

In view of the constancy of ϕ for $t \le \frac{1}{2}$, (21) will follow if we can show that

$$g^{(k)}(0) = [k!/(m+k)!]\,u.$$

However,

$$\begin{aligned} g(t) &= t^{-m} I^{((k+1))} * (G-I+I)^{((m))}(t)u \\ &= t^{-m} \sum_{j=0}^{m} \binom{m}{j} I^{((k+1+m-j))} * (G-I)^{((j))}(t)u. \end{aligned}$$

Since $t^{-m} I^{((k+1+m-j))}(t) = (t^{k-j}/(k+m-j)!)I$ has vanishing kth derivative if $j > 0$ we have

$$g^{(k)}(0) = \left(\frac{d}{dt}\right)^k \frac{t^k}{(k+m)!}\bigg|_{t=0} I * (G-I)^{((0))}(0)u = \frac{k!}{(k+m)!}u.$$

In order to establish (22) it is clearly sufficient to show that

$$\int_0^1 t^{\nu p} \|\Lambda^i g^{(j)}(t)\|_B^p\, dt \le K_3 \|u\|_{T^{m-k-1}}^p \tag{23}$$

holds for every i, j such that $0 \le j \le m$ and $0 \le i \le m-j$. We distinguish three cases.

Case 1 Suppose $0 \le j \le k$ and $m-k \le i \le m-j$. Let $w = \Lambda^{m-k-1}u$. Thus $w \in B$. Let $l = k+1+i-m$ so that $l > 1$ and $k+1-l \ge j$. Now

$$\Lambda^i g(t) = t^{-m} I^{((k+1-l))} * \Lambda^l I^{((l))} * G^{((l))} * G^{((m-l))}(t)w.$$

Since $\Lambda(I * G) = G - I$ we have

$$\Lambda^l I^{((l))} * G^{((l))} = (G-I)^{((l))},$$

and so

$$\Lambda^i g(t) = t^{-m} I^{((k+l-1))} * G^{((m-l))} * (G-I)^{((l))}(t)\, w.$$

Since $k+1-l \geq j$ we have, for $t > 0$,

$$\Lambda^i g^{(j)}(t) = \sum_{r=0}^{j} w_r(t),$$

where

$$w_r(t) = \tilde{K}_r t^{-m-r} I^{((k+1-l-j+r))} * G^{((m-l))} * (G-I)^{((l))}(t)\, w.$$

Now

$$\begin{aligned}\|I^{((k+1-l-j+r))} * G^{((m-l))} * G^{((l-1))}\|_{L(B)} &\leq K_4 t^{(k+1-l-j+r)+(m-l)+(l-1)} \\ &= K_4 t^{2m-i-j+r-2}\end{aligned}$$

Hence

$$\begin{aligned}\|w_r(t)\|_B &\leq K_5 t^{-m-r} \int_0^t (t-\tau)^{2m-i-j+r-2} \|G(\tau)w - w\|_B \, d\tau \\ &\leq K_5 t^{m-i-j-2} \int_0^t \|G(\tau)w - w\|_B \, d\tau.\end{aligned}$$

Since $i \leq m-j$ we therefore obtain for $0 < t \leq 1$

$$\|\Lambda^i g^{(j)}(t)\|_B \leq K_6 t^{-2} \int_0^t \|G(\tau)w - w\|_B \, d\tau.$$

It follows by Lemma 7.23 (with $\nu - 1$ in place of ν) that

$$\begin{aligned}\int_0^1 t^{\nu p} \|\Lambda^i g^{(j)}(t)\|_B^p \, dt &\leq K_6 \int_0^1 t^{(\nu-1)p} \left((1/t) \int_0^t \|G(\tau)w - w\|_B \, d\tau \right)^p dt \\ &\leq K_7 \int_0^\infty t^{(\nu-1)p} \|G(t)w - w\|_B^p \, dt \\ &\leq K_7 \|w\|_{T^0}^p \leq K_7 \|u\|_{T^{m-k-l}}.\end{aligned}$$

CASE 2 Suppose $0 \leq j \leq k$ and $0 \leq i \leq m-k-1$. Then $w = \Lambda^i u \in B$ and

$$\Lambda^i g(t) = t^{-m} I^{((k+1))} * G^{((m))}(t)\, w.$$

Hence

$$\Lambda^i g^{(j)}(t) = \sum_{r=0}^{j} w_r(t) = \sum_{r=0}^{j} \tilde{K}_r t^{-m-r} I^{((k+1-j+r))} * G^{((m))}(t)\, w.$$

Now

$$\|w_r(t)\|_B \leq K_8 t^{-m-r+(k+1-j+r)-1+m} \|w\|_B = K_8 t^{k-j} \|w\|_B.$$

Thus

$$\|\Lambda^i g^{(j)}(t)\|_B \le K_9 t^{k-j} \|w\|_B$$

and

$$\int_0^1 t^{\nu p} \|\Lambda^i g^{(j)}(t)\|_B^p \, dt \le K_9{}^p \|w\|_B^p = K_9{}^p \|\Lambda^i u\|_B \le K_9{}^p \|u\|_{T^{m-k-1}}.$$

CASE 3 Suppose $k+1 \le j \le m$ and $1 \le i \le m-j$. Then $i \le m-k-1$ and $\tilde{u} = \Lambda^i u \in T^{m-k-1-i}$ with

$$\|\tilde{u}\|_{T^{m-k-1-i}} \le \|u\|_{T^{m-k-1}}.$$

Let $h(t) = \Lambda^i g(t)$. Thus

$$h(t) = t^{-m} I^{((k+1))} * G^{((m))}(t)\tilde{u}. \tag{24}$$

In order to prove (23) in this case it is sufficient to show that

$$\int_0^1 t^{\nu p} \|h^{(j)}(t)\|_B^p \, dt \le K_{10} \|\tilde{u}\|_{T^{m-k-1-i}}^p. \tag{25}$$

Now $G = \Lambda(I * G) + I$ so that

$$h(t) = t^{-m} I^{((k+2))} * G^{((m))}(t)\Lambda\tilde{u} + t^{-m} I^{((k+2))} * G^{((m-1))}(t)\tilde{u}.$$

Another $m-1$ repetitions of this argument yields

$$h(t) = \sum_{l=0}^{m-1} t^{-m} I^{((k+2+l))} * G^{((m-l))}(t)\Lambda\tilde{u} + t^{-m} I^{((k+1+m))}(t)\tilde{u}.$$

For purposes of proving (25) we may omit the term

$$t^{-m} I^{((k+1+m))}(t)\tilde{u} = [t^k/(k+m)!]\tilde{u}$$

since the jth derivative of this term vanishes for $t > 0$. Accordingly we consider

$$h(t) \sim \sum_{l=0}^{m-1} t^{-m} I^{((k+2+l))} * G^{((m-l))}(t)\Lambda\tilde{u}. \tag{26}$$

We repeat $m-k-i-2$ more times the preceding argument used in deriving (26) from (24), each time discarding terms which are polynomials of degree $\le j-1$ and so contribute nothing to $h^{(j)}$. This leads to

$$h(t) \sim \sum_{l=0}^{m-1} t^{-m} I^{((k+2+(m-k-i-2)+l))} * G^{((m-l))}(t)\Lambda^{m-k-1-i}\tilde{u}.$$

Let $w = \Lambda^{m-k-1-i}\tilde{u} = \Lambda^{m-k-1}u$. The terms of the above sum are of the form

$$w_l(t) = t^{-m} I^{((m+l-i))} * G^{((m-l))}(t)w.$$

where $0 \leq l \leq m-1$. Note that $m+l-i \geq j$. In order to prove (25) it is sufficient to show that

$$\int_0^1 t^{\nu p} \|w_l^{(j)}(t)\|_B^p \, dt \leq K_{11} \|w\|_{T^0}^p. \tag{27}$$

At this point we must distinguish two subcases, $i \leq m-j-1$ and $i = m-j$.

If $i \leq m-j-1$, then $w_l^{(j)}$ is a linear combination of terms of the form

$$t^{-m-r} I^{((m+l-i-j+r))} * G^{((m-l))}(t) w$$

whose norms in B are bounded by

$$K_{12} t^{-m-r+(m+l-i-j+r-1)+m-l} \|w\|_B \leq K_{12} t^{m-j-i-1} \|w\|_{T^0}$$

and (27) follows at once.

If $i = m-j$, we have

$$\begin{aligned} w_l(t) &= t^{-m} I^{((j+l))} * G^{((m-l-1))} * (G-I+I)(t) w \\ &= t^{-m} I^{((j+l))} * G^{((m-l-1))} * (G-I)(t) w \\ &\quad + t^{-m} I^{((j+l+1))} * G^{((m-l-1))}(t) w. \end{aligned}$$

We repeat this procedure on the last term $m-l-1$ more times to obtain

$$w_l(t) = \sum_{s=0}^{m-l-1} t^{-m} I^{((j+l+s))} * G^{((m-l-s-1))} * (G-I)(t) w + t^{-m} I^{((j+m))}(t) w.$$

Again we may discard the last term which makes no contribution to (27). It is therefore sufficient to establish (27) with w_l replaced by

$$w_{ls}(t) = t^{-m} I^{((j+l+s))} * G^{((m-l-s-1))} * (G-I)(t) w.$$

However, $w_{ls}^{(j)}(t)$ is a linear combination of terms of the form

$$t^{-m-r} I^{((l+s+r))} * G^{((m-l-s-1))} * (G-I)(t) w$$

for $0 \leq r \leq j$. It follows just as in Case 1 that for $0 < t \leq 1$,

$$\|w_{ls}^{(j)}(t)\|_B \leq K_{13} t^{-2} \int_0^t \|G(\tau) w - w\|_B \, d\tau$$

and hence, using Lemma 7.23 again, that (27) is satisfied. This completes the proof. ∎

We remark that the proof for general n is essentially similar to that given for $n = 1$ above. In place of (20) one uses (a suitable multiple of)

$$g(t) = t^{-mn} I^{((k+1))} * G_1^{((m))} * \cdots * G_n^{((m))}(t) u.$$

7.34 EXAMPLE Let $B = L^p(\mathbb{R}^n)$ and G_j, $1 \le j \le n$, be as in Example 7.27 so that $\Lambda_j = D_j$. Evidently

$$D(\Lambda^k) = \{u \in L^p(\mathbb{R}^n) : D^\alpha u \in L^p(\mathbb{R}^n), |\alpha| \le k\} = W^{k,p}(\mathbb{R}^n).$$

For each $u \in L^p(\mathbb{R}^{n+1}_+)$ define $\tilde{u}$ a.e. on $[0, \infty)$ into $L^p(\mathbb{R}^n)$ by

$$\tilde{u}(t)(x_1, \dots, x_n) = u(x_1, \dots, x_n, t).$$

Then $u \in W^{m,p}(\mathbb{R}^{n+1}_+)$ provided $\tilde{u}^{(k)} \in L^p(0, \infty; W^{m-k,p}(\mathbb{R}^n))$, $0 \le k \le m$. Accordingly,

$$W^{m,p}(\mathbb{R}^{n+1}_+) \cong W^m(p, 0; \Lambda; L^p(\mathbb{R}^n))$$

with $\Lambda = (D_1, \dots, D_n)$, If $1 < p < \infty$, the mapping γ

$$\gamma : u \to (u(\cdot, \dots, \cdot, 0), D_{n+1}u(\cdot, \dots, \cdot, 0), \dots, D^{m-1}_{n+1}u(\cdot, \dots, \cdot, 0))$$

is an isomorphism and a homeomorphism of $W^{m,p}(\mathbb{R}^{n+1}_+)/\ker\gamma$ onto the product $\prod_{k=0}^{m-1} T^{m-k-1}$ where

$$T^k = T^k(p, 0; \Lambda, L^p(\mathbb{R}^n)) = \{v \in W^{k,p}(\mathbb{R}^n) : D^\alpha v \in T^0, |\alpha| \le k\}$$

and

$$\|v\|_{T^k} = \Bigg\{ \sum_{|\alpha| \le k} \|D^\alpha v\|^p_{0,p,\mathbb{R}^n} + \sum_{|\alpha| = k} \sum_{j=1}^{n} \int_0^\infty t^{-p} \int_{\mathbb{R}^n} |D^\alpha v(x_1, \dots, x_j + t, \dots, x_n) - D^\alpha v(x_1, \dots, x_n)|^p \, dx \, dt \Bigg\}^{1/p}.$$

The Spaces $W^{s,p}(\Omega)$

7.35 We now define spaces $W^{s,p}(\Omega)$ for arbitrary domains Ω in $\mathbb{R}^n$, arbitrary values of s, and $1 < p < \infty$. These spaces coincide for integer values of s with the spaces $W^{m,p}(\Omega)$ and $W^{-m,p}(\Omega)$ defined in Chapter III. For $s \ge 0$ the definitions can be extended to $p = 1$ and $p = \infty$, but for the time being we ignore these limiting values.

The spaces $B_1 = W^{1,p}(\Omega)$ and $B_2 = X = L^p(\Omega)$ clearly satisfy the conditions laid down in Section 7.11. For $0 < \theta < 1$ let

$$T^{\theta,p}(\Omega) = T(p, \nu; W^{1,p}(\Omega), L^p(\Omega)),$$

where $\nu + (1/p) = \theta$. Denoting $W = W(p, \nu; W^{1,p}(\Omega), L^p(\Omega))$, we write the norm of u in $T^{\theta,p}(\Omega)$ as

$$\|u; T^{\theta,p}(\Omega)\| = \inf_{\substack{f\in W\\ u=f(0)}} \max\left\{\left(\int_0^\infty t^{\nu p}\|f(t)\|_{1,p,\Omega}^p\,dt\right)^{1/p},\right.$$

$$\left.\left(\int_0^\infty t^{\nu p}\|f'(t)\|_{0,p,\Omega}^p\,dt\right)^{1/p}\right\}. \tag{28}$$

7.36 Let $s \geq 0$ be arbitrary. If $s = m$, an integer, we define $W^{s,p}(\Omega) = W^{m,p}(\Omega)$. If s is not an integer, we write $s = m+\sigma$ where m is an integer and $0 < \sigma < 1$. The space $W^{s,p}(\Omega)$ is, in this case, defined to consist of those (equivalence classes of) functions $u \in W^{m,p}(\Omega)$ whose distributional derivatives $D^\alpha u$, $|\alpha| = m$, belong to $T^{1-\sigma,p}(\Omega)$. Then $W^{s,p}(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{s,p,\Omega} = \left\{\|u\|_{m,p,\Omega}^p + \sum_{|\alpha|=m}\|D^\alpha u; T^{1-\sigma,p}(\Omega)\|^p\right\}^{1/p}. \tag{29}$$

7.37 The operator P given by

$$Pu = (u, (D^\alpha u)_{|\alpha|=m})$$

(the multi-indices α with $|\alpha| = m$ being ordered in some conveient way) is an isometric isomorphism of $W^{s,p}(\Omega)$ onto a closed subspace of the (product) Banach space

$$S = W^{m,p}(\Omega) \times \prod_{|\alpha|=m} T^{1-\sigma,p}(\Omega)$$

having norm

$$\|(u, (v_\alpha)_{|\alpha|=m}); S\| = \left\{\|u\|_{m,p,\Omega}^p + \sum_{|\alpha|=m}\|v_\alpha; T^{1-\sigma,p}(\Omega)\|^p\right\}^{1/p}.$$

Since $W^{m,p}(\Omega)$ and $T^{1-\sigma,p}(\Omega)$ are reflexive (Theorems 3.5 and 7.19) it follows by Theorems 1.21 and 1.22 that $W^{s,p}(\Omega)$ is reflexive.

7.38 THEOREM For any $s \geq 0$, $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{s,p}(\mathbb{R}^n)$.

PROOF This result has already been proved for $s = 0, 1, 2, \ldots$ (Theorems 2.19 and 3.18) and in particular $W^{1,p}(\mathbb{R}^n)$ is a dense subset of $L^p(\mathbb{R}^n)$ [i.e., dense with respect to the topology of $L^p(\mathbb{R}^n)$]. If $s = m+\sigma > 0$, where m is an integer and $0 < \sigma < 1$, the theorem may be proved as follows.

Let $\psi \in C^\infty(\mathbb{R})$ satisfy $\psi(t) = 1$ if $t \leq 0$ and $\psi(t) = 0$ if $t \geq 1$. For $j = 1, 2, 3, \ldots$ let $\psi_j \in C_0^\infty(\mathbb{R}^n)$ be defined by

$$\psi_j(x) = \psi(|x| - j).$$

Let J_ε be the mollifier introduced in Section 2.17. If u is a function defined (a.e.) on $\mathbb{R}^n$, set

$$P_j u = J_{1/j} * (\psi_j \cdot u), \qquad n = 1, 2, \ldots.$$

Evidently P_j is a bounded linear operator on $W^{m,p}(\mathbb{R}^n)$ into $W^{m,p}(\mathbb{R}^n)$ for $m = 0, 1, 2, \ldots$, and has range in $C_0^\infty(\mathbb{R}^n)$ in each case. We can deduce from Lemmas 2.18 and 3.15 that if $u \in W^{m,p}(\mathbb{R}^n)$, then

$$\lim_{j\to\infty} \|P_j u - u\|_{m,p,\mathbb{R}^n} = 0.$$

It follows by Lemma 7.18 that if $0 < \theta < 1$ and $u \in T^{\theta,p}(\mathbb{R}^n)$, then

$$\lim_{j\to\infty} \|P_j u - u; T^{\theta,p}(\mathbb{R}^n)\| = 0.$$

Since

$$D^\alpha P_j u = J_{1/j} * D^\alpha(\psi_j \cdot u) = P_j D^\alpha u + J_{1/j} * \omega_j,$$

where

$$\omega_j = \sum_{\beta < \alpha} \binom{\alpha}{\beta} D^{\alpha-\beta}\psi_j \, D^\beta u,$$

and since

$$\lim_{j\to\infty} \|\omega_j\|_{1,p,\mathbb{R}^n} = 0$$

provided $u \in W^{|\alpha|,p}(\mathbb{R}^n)$, it follows that for any $u \in W^{s,p}(\mathbb{R}^n)$ we have, taking $|\alpha| = m$,

$$\lim_{j\to\infty} \|D^\alpha P_j u - D^\alpha u; T^{1-\sigma,p}(\mathbb{R}^n)\| = 0.$$

Hence

$$\lim_{j\to\infty} \|P_j u - u\|_{s,p,\Omega} = 0$$

and the proof is complete. ∎

7.39 Let $W_0^{s,p}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in the space $W^{s,p}(\Omega)$ $(s \ge 0)$. By the above theorem $W_0^{s,p}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)$. For $s < 0$ we define

$$W^{s,p}(\Omega) = [W_0^{-s,p'}(\Omega)]', \qquad (1/p) + (1/p') = 1.$$

It follows by reflexivity that for every real s

$$[W^{s,p}(\mathbb{R}^n)]' \cong W^{-s,p'}(\mathbb{R}^n).$$

We shall not comment further on the structure of the spaces $W^{s,p}(\Omega)$ for $s < 0$ except to note that, being duals of spaces having $\mathscr{D}(\Omega) = C_0^\infty(\Omega)$ as dense subsets, they are spaces of distributions on Ω.

Many properties of the spaces $W^{s,p}(\Omega)$ are conveniently proven only for $\Omega = \mathbb{R}^n$, and must then be deduced for more general Ω with the aid of an extension operator extending functions defined on Ω to $\mathbb{R}^n$ with preservation of differential properties (see Section 4.24). For fractional $s = m+\sigma$ suitable extensions are obtained by interpolation. Thus the existence of a strong $(m+1)$-extension operator for Ω will normally be required. Such is, for example, assured if Ω satisfies the hypotheses of Theorem 4.26 (see also Section 4.29).

7.40 THEOREM If $s = m+\sigma$ where m is an integer and $0 < \sigma < 1$, and if there exists a strong $(m+1)$-extension operator E for Ω, a domain in $\mathbb{R}^n$, then the set of restrictions to Ω of functions in $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{s,p}(\Omega)$.

PROOF (Recall that the conclusion holds for $W^{m,p}(\Omega)$ under the assumption only that Ω has the segment property.) The proof follows the same lines as that of Theorem 7.38 except that in place of the operator P_j we use the operator $\tilde{P}_j$ defined by

$$\tilde{P}_j u = R_\Omega P_j Eu, \qquad u \text{ defined on } \Omega,$$

where R_Ω is the operator restricting to Ω functions defined on $\mathbb{R}^n$. The details are left to the reader. ▮

The following localization theorem requires, in addition to the existence of a strong $(m+1)$-extension operator E for Ω, a representation for the derivatives $D^\alpha Eu(x)$ in terms of the derivatives of u such as is provided by Theorem 4.26. Thus the hypotheses of the theorem below will certainly be satisfied by any domain satisfying the hypotheses of Theorem 4.26.

7.41 THEOREM Let Ω be a domain in $\mathbb{R}^n$ for which there exists a strong $(m+1)$-extension operator E and, for $|\gamma| \le |\alpha| = m$, linear operators $E_{\alpha\gamma}$ continuous from $W^{1,p}(\Omega)$ into $W^{1,p}(\mathbb{R}^n)$ and from $L^p(\Omega)$ into $L^p(\mathbb{R}^n)$ such that if $u \in W^{m,p}(\Omega)$, then

$$D^\alpha Eu(x) = \sum_{|\gamma| \le m} E_{\alpha\gamma} D^\gamma u(x) \qquad \text{a.e. in } \mathbb{R}^n. \tag{30}$$

If $s = m+\sigma > 0$, $0 \le \sigma < 1$, then $W^{s,p}(\Omega)$ coincides with the set of restrictions to Ω of funcitons in $W^{s,p}(\mathbb{R}^n)$.

PROOF If $\sigma = 0$, the result is an immediate consequence of the existence of a strong m-extension operator for Ω. Suppose $0 < \sigma < 1$. If $u \in W^{s,p}(\Omega)$, then $u \in W^{m,p}(\Omega)$ and $Eu \in W^{m,p}(\mathbb{R}_n)$; also

$$\|Eu\|_{m,p,\mathbb{R}^n} \le K_1 \|u\|_{m,p,\Omega} \le K_1 \|u\|_{s,p,\Omega}. \tag{31}$$

If $|\gamma| \le m$, then $D^\gamma u \in T^{1-\sigma,p}(\Omega)$ and

$$\|D^\gamma u; T^{1-\sigma,p}(\Omega)\| \le K_2 \|u\|_{s,p,\Omega}. \tag{32}$$

(This holds by definition if $|\gamma| = m$ and via Lemma 7.16 if $|\gamma| < m$.) Since $E_{\alpha\gamma}$ is linear and continuous on $W^{1,p}(\Omega)$ into $W^{1,p}(\mathbb{R}^n)$ and on $L^p(\Omega)$ into $L^p(\mathbb{R}^n)$, by Theorem 7.17 it is also continuous on $T^{1-\sigma,p}(\Omega)$ into $T^{1-\sigma,p}(\mathbb{R}^n)$. By (30) and (32) we have for $|\alpha| = m$

$$\|D^\alpha Eu; T^{1-\sigma}(\mathbb{R}^n)\| \le K_3 \|u\|_{s,p,\Omega}.$$

Combining this with (31), we obtain

$$\|Eu\|_{s,p,\mathbb{R}^n} \le K_4 \|u\|_{s,p,\Omega}$$

and u is the restriction to Ω of $Eu \in W^{s,p}(\mathbb{R}^n)$.

Conversely, the operator R_Ω of restriction to Ω, being continuous from $W^{m,p}(\mathbb{R}^n)$ into $W^{m,p}(\Omega)$ for any m, is also continuous by Theorem 7.17 from $W^{s,p}(\mathbb{R}^n)$ into $W^{s,p}(\Omega)$ so the restriction $R_\Omega u$ of $u \in W^{s,p}(\mathbb{R}^n)$ belongs to $W^{s,p}(\Omega)$. ∎

We remark that under the conditions of the theorem the extension operator E is continuous from $W^{s,p}(\Omega)$ into $W^{s,p}(\mathbb{R}^n)$ for any s, $0 \le s \le m+1$.

An Intrinsic Norm for $W^{s,p}(\Omega)$

7.42 We now investigate the possibility of constructing a new norm for $W^{s,p}(\Omega)$, $s \ge 0$, which is equivalent to the "trace norm" (29) (s not an integer) but which is expressed "intrinsically" in terms of properties of the element involved. In view of Example 7.27 it is most convenient to begin with the case $\Omega = \mathbb{R}^n$. Following Lions and Magenes [34] we define new spaces $\tilde{W}^{s,p}(\Omega)$ with intrinsically defined norm and then show, at least for suitably regular domains Ω, $\tilde{W}^{s,p}(\Omega)$ coincides with $W^{s,p}(\Omega)$.

7.43 For $0 < \theta < 1$ and $1 \le p < \infty$ let $\tilde{T}^{\theta,p}(\Omega)$ denote the space of (equivalence classes of) functions $u \in L^p(\Omega)$ for which the norm

$$\|u; \tilde{T}^{\theta,p}(\Omega)\| = \left\{ \|u\|_{0,p,\Omega}^p + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n-1+(1-\nu)p}}\, dx\, dy \right\}^{1/p} \tag{33}$$

is finite, where $\nu + (1/p) = \theta$.

7.44 LEMMA The space $\tilde{T}^{\theta,p}(\mathbb{R}^n)$ coincides with the Banach space $T^{\theta,p}(\mathbb{R}^n)$, the norms in the two spaces being equivalent.

PROOF The norm of an element u in $T^{\theta,p}(\mathbb{R}^n)$ was defined [in (28)] to be its norm in the trace space $T(p, \nu; W^{1,p}(\mathbb{R}^n), L^p(\mathbb{R}^n))$. By Example 7.27 we may take the norm to be

$$\|u\|_T \doteq \left\{ \|u\|_{0,p,\mathbb{R}^n}^p + \sum_{j=1}^n \int_0^t t^{(\nu-1)p} \int_{\mathbb{R}^n} |u(x_1,\ldots,x_j+t,\ldots,x_n) - u(x_1,\ldots,x_n)|^p \, dx \, dt \right\}^{1/p}.$$

Let us denote the norm given by (33) as $\|u\|_{T}$.

Let $u \in T^{\theta,p}(\mathbb{R}^n)$. Putting $\lambda = \frac{1}{2}[n-1+(1-\nu)p]$ and writing $u(x)-u(y)$ in the form

$$\sum_{j=1}^n [u(y_1,\ldots,y_{j-1},x_j,x_{j+1},\ldots,x_n) - u(y_1,\ldots,y_{j-1},y_j,x_{j+1},\ldots,x_n)],$$

we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n-1+(1-\nu)p}} \, dx \, dy \le K_1 \sum_{j=1}^n Q_j,$$

where

$$Q_j = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(y_1,\ldots,y_{j-1},x_j,\ldots,x_n) - u(y_1,\ldots,y_j,x_{j+1},\ldots,x_n)|^p}{(\sum_{k=1}^n (x_k-y_k)^2)^\lambda} \, dx \, dy.$$

Thus

$$\begin{aligned} Q_j = \int_{\mathbb{R}^j} & dy_1 \cdots dy_j \\ & \times \int_{\mathbb{R}^{n+1-j}} dx_j \cdots dx_n \, |u(y_1,\ldots,y_{j-1},x_j,\ldots,x_n) \\ & - u(y_1,\ldots,y_j,x_{j+1},\ldots,x_n)|^p R_j \end{aligned} \tag{34}$$

where

$$R_j = \int_{\mathbb{R}^{n-j}}\int_{\mathbb{R}^{j-1}} \frac{dx_1 \cdots dx_{j-1} \, dy_{j+1} \cdots dy_n}{(\sum_{k=1}^n (x_k-y_k)^2)^\lambda}.$$

Let $\rho^2 = (x_1-y_1)^2 + \cdots + (x_{j-1}-y_{j-1})^2 + (x_{j+1}-y_{j+1})^2 + \cdots + (x_n-y_n)^2$.

Then

$$\begin{aligned} R_j &= K_2 \left(\int_0^{|x_j-y_j|} + \int_{|x_j-y_j|}^\infty \right) \frac{\rho^{n-2}}{[\rho^2+(x_j-y_j)^2]^\lambda} d\rho \\ &\le \frac{K_2}{|x_j-y_j|^{2\lambda}} \int_0^{|x_j-y_j|} \rho^{n-2} \, d\rho + K_2 \int_{|x_j-y_j|}^\infty \rho^{n-2-2\lambda} \, d\rho \\ &= K_3 |x_j-y_j|^{(\nu-1)p} \end{aligned} \tag{35}$$

since $\lambda > 0$ and $n-1-2\lambda < 0$. Setting

$$y_i = z_i, \quad 1 \le i \le j, \qquad x_j = t + y_j, \qquad x_i = z_i, \quad j+1 \le i \le n$$

in (35), we obtain, using (34),

$$Q_j \le 2K_3 \int_0^\infty t^{(\nu-1)p}\, dt \int_{\mathbb{R}^n} |u(z_1,\ldots,z_j+t,\ldots,z_n) - u(z_1,\ldots,z_n)|^p\, dz.$$

Thus $u \in \tilde{T}^{\theta,p}(\mathbb{R}^n)$ and $\|u\|_{\tilde{T}} \le K_4 \|u\|_T$.

Conversely, suppose $u \in \tilde{T}^{\theta,p}(\mathbb{R}^n)$. Let $x' = (x_2,\ldots,x_n)$ and $z' = (z_2,\ldots,z_n)$ and integrate the inequality

$$\begin{aligned} |u(x_1+t,x') - u(x,x')|^p \le K_5(&|u(x_1+t,x') - u(x_1+\tfrac{1}{2}t,z')|^p \\ &+ |u(x_1+\tfrac{1}{2}t,z') - u(x,x')|^p) \end{aligned}$$

with respect to z' over the disk $D(t,x')$ centered at $x' \in \mathbb{R}^{n-1}$ and having radius $\frac{1}{2}t$, thus obtaining

$$|u(x_1+t,x') - u(x_1,x')|^p \le (K_6/t^{n-1})[I_t(t,x) + I_t(0,x)],$$

where

$$I_t(s,x) = \int_{D(t,x')} |u(x_1+s,x') - u(x_1+\tfrac{1}{2}t,z')|^p\, dz'$$

for $s = t$ or $s = 0$. Now

$$\begin{aligned}
&\int_0^\infty t^{(\nu-1)p}\, dt \int_{\mathbb{R}^n} \frac{1}{t^{n-1}} I_t(t,x)\, dx \\
&\quad = \int_{\mathbb{R}^{n-1}} dx' \int_0^\infty \frac{1}{t^{2\lambda}}\, dt \int_{D(t,x')} dz' \int_{-\infty}^\infty |u(x_1+t,x') - u(x_1+\tfrac{1}{2}t,z')|^p\, dx_1 \\
&\quad = \int_{\mathbb{R}^{n-1}} dx' \int_0^\infty \frac{1}{t^{2\lambda}}\, dt \int_{D(t,x')} dz' \int_{-\infty}^\infty |u(x_1,x') - u(x_1-\tfrac{1}{2}t,z')|^p\, dx_1 \\
&\quad = \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^{n-1}} dz' \int_{2|z'-x'|}^\infty \frac{|u(x_1,x') - u(x_1-\tfrac{1}{2}t,z')|^p}{t^{2\lambda}}\, dt \\
&\quad = \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^{n-1}} dz' \int_{-\infty}^{x_1-|z'-x'|} \frac{|u(x)-u(z)|^p}{[2(x_1-z_1)]^{2\lambda}}\, dz_1 \\
&\quad \le \frac{1}{2^\lambda} \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} \frac{|u(x)-u(z)|^p}{|x-z|^{\lambda 2}}\, dz,
\end{aligned}$$

where we have put $z_1 = x_1 - \frac{1}{2}t$, $dz = -\frac{1}{2}\, dt$ and used the fact that in the inner integral in the second last line $|x_1 - z_1| \ge |x'-z'|$ so that $|x_1 - z_1| \ge |x-z|/\sqrt{2}$.

A similar inequality holds for $I_t(0,x)$. Thus

$$\int_0^\infty t^{(\nu-1)p}\int_{\mathbb{R}^n}|u(x_1+t,x_2,\dots,x_n)-u(x_1,\dots,x_n)|^p\,dx\,dt$$
$$\leq K_7\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(z)|^p}{|x-z|^{n-1+(1-\nu)p}}\,dx\,dz.$$

Similar inequalities hold for differences in the other variables $x_2,\dots,x_n$ and combining these we obtain $\|u\|_T \leq K_8\|u\|_{\mathcal{T}}$. ∎

In order to extend the above lemma to more general domains Ω we require the following extension lemma.

7.45 LEMMA Let Ω be a half-space in $\mathbb{R}^n$ or a domain in $\mathbb{R}^n$ having the uniform 1-smooth regularity property and a bounded boundary. Then there exists a linear operator E mapping $L^p(\Omega)$ into $L^p(\mathbb{R}^n)$ such that if $u \in L^p(\Omega)$, then

$$Eu(x) = u(x) \qquad \text{a.e. in } \Omega,$$

and if $0<\theta<1$ and $u \in \mathcal{T}^{\theta,p}(\Omega)$, then $Eu \in \mathcal{T}^{\theta,p}(\mathbb{R}^n)$ and

$$\|Eu;\mathcal{T}^{\theta,p}(\mathbb{R}^n)\| \leq K\|u;\mathcal{T}^{\theta,p}(\Omega)\|$$

with K independent of u.

PROOF The proof is quite similar to that of Theorem 4.26. We begin with the case $\Omega = \mathbb{R}_+{}^n = \{x\in\mathbb{R}^n : x_n > 0\}$. Let us denote by $x' = (x_1,\dots,x_{n-1})$ and for $u \in L^p(\Omega)$ set

$$Eu(x) = \begin{cases} u(x) & \text{a.e. in } \mathbb{R}_+{}^n \\ u(x',-x_n) & \text{a.e. in } \mathbb{R}^n \sim \mathbb{R}_+{}^n. \end{cases}$$

Then

$$\|Eu\|^p_{0,p,\mathbb{R}^n} = \int_{\mathbb{R}^{n-1}} dx'\left\{\int_0^\infty |u(x)|^p\,dx_n + \int_{-\infty}^0 |u(x',-x_n)|^p\,dx_n\right\}$$
$$= 2\|u\|^p_{0,p,\mathbb{R}^n_+}.$$

Also, setting $2\lambda = n-1+(1-\nu)p = n+(1-\theta)p > 0$, we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|Eu(x)-Eu(y)|^p}{|x-y|^{2\lambda}}\,dx\,dy = I_{++}+I_{+-}+I_{-+}+I_{--},$$

where

$$I_{++} = \int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n} \frac{|u(x)-u(y)|^p}{|x-y|^{2\lambda}}\,dx\,dy,$$

$$\begin{aligned} I_{+-} &= \int_{\mathbb{R}^{n-1}} dx' \int_{\mathbb{R}^{n-1}} dy' \int_0^\infty dx_n \int_{-\infty}^0 \frac{|u(x)-u(y',-y_n)|^p}{[|x'-y'|^2+(x_n-y_n)^2]^\lambda}\,dy_n \\ &= \int_{\mathbb{R}^{n-1}} dx' \int_{\mathbb{R}^{n-1}} dy' \int_0^\infty dx_n \int_0^\infty \frac{|u(x)-u(y)|^p}{[|x'-y'|^2+(x_n+y_n)^2]^\lambda}\,dy_n \\ &\le \int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n} \frac{|u(x)-u(y)|^p}{|x-y|^{2\lambda}}\,dx\,dy \end{aligned}$$

[since $(x_n+y_n)^2 \ge (x_n-y_n)^2$ when $x_n \ge 0$ and $y_n \ge 0$], and similar inequalities hold for I_{-+} and I_{--}. Thus

$$\|Eu; \mathcal{T}^{\theta,p}(\mathbb{R}^n)\| \le 4^{1/p}\|u; \mathcal{T}^{\theta,p}(\mathbb{R}_+^n)\|.$$

Now suppose that Ω is uniformly C^1-regular and has a bounded boundary. Then the open cover $\{U_j\}$ of $\operatorname{bdry}\Omega$ and the corresponding collection $\{\Phi_j\}$ of 1-smooth maps of U_j onto $B = \{y \in \mathbb{R}^n : |y| < 1\}$ referred to in Section 4.6 are both finite collections, say $1 \le j \le N$. We may also assume that the sets U_j are bounded. Let U_0 be an open subset of Ω, bounded away from $\operatorname{bdry}\Omega$, such that $\Omega \subset \bigcup_{j=0}^N U_j$. Let $\{\omega_j\}_{j=0}^\infty$ be a C^∞-partition of unity for Ω subordinate to $\{U_j\}$. Given $u \in L^p(\Omega)$ let u_j be defined a.e. in Ω by $u_j(x) = \omega_j(x)u(x)$. Clearly $u_j \in L^p(\Omega)$ and $\|u_j\|_{0,p,\Omega} \le \|u\|_{0,p,\Omega}$. If $u \in \mathcal{T}^{\theta,p}(\Omega)$, then for $1 \le j \le N$

$$\begin{aligned} \int_\Omega\int_\Omega \frac{|u_j(x)-u_j(y)|^p}{|x-y|^{2\lambda}}\,dx\,dy &\le K_1 \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{2\lambda}}\,dx\,dy \\ &\quad + K_1 \int_{\Omega\cap U_j} |u(y)|^p\,dy \int_{U_j} \frac{|\omega_j(x)-\omega_j(y)|^p}{|x-y|^{2\lambda}}\,dx. \end{aligned}$$

But since U_j is bounded we have for $y \in \Omega \cap U_j$ by Lemma 5.47,

$$\int_{U_j} \frac{|\omega_j(x)-\omega_j(y)|^p}{|x-y|^{2\lambda}}\,dx \le K_2 \int_{U_j} |x-y|^{p+1-n}\,dx \le K_3,$$

and K_3 may be chosen independent of the finitely many values of j involved. Thus $u_j \in \mathcal{T}^{\theta,p}(\Omega)$ and

$$\|u_j; \mathcal{T}^{\theta,p}(\Omega)\| \le K_4\|u; \mathcal{T}^{\theta,p}(\Omega)\|.$$

Since $\omega_0(x) = 1$ for all $x \in \Omega$ lying outside the bounded set $\bigcup_{j=1}^N U_j$, the above inequality also holds for u_0.

For $1 \le j \le N$ let v_j be defined on $\mathbb{R}_+^n$ by

$$v_j(y) = \begin{cases} u_j \circ \Psi_j(y) & \text{if } \ y \in B \cap \mathbb{R}_+^n \\ 0 & \text{if } \ y \in \mathbb{R}_+^n \sim B, \end{cases}$$

where $\Psi_j = \Phi_j^{-1}$. Then $v_j \in \tilde{T}^{\theta,p}(\mathbb{R}_+^n)$. In fact, putting $y = \Phi_j(x)$, $\eta = \Phi_j(\xi)$, we have

$$\begin{aligned} \|v_j; \tilde{T}^{\theta,p}(\mathbb{R}_+^n)\|^p &= \int_{\mathbb{R}_+^n \cap B} |u_j(\Psi_j(y))|^p \, dy \\ &\quad + \int_{\mathbb{R}_+^n \cap B} \int_{\mathbb{R}_+^n \cap B} \frac{|u_j(\Psi_j(y)) - u_j(\Psi_j(\eta))|^p}{|y-\eta|^{2\lambda}} \, dy \, d\eta \\ &= \int_\Omega |u_j(x)|^p \, |\det \Phi_j'(x)| \, dx \\ &\quad + \int_\Omega \int_\Omega \frac{|u_j(x) - u_j(\xi)|^p}{|\Phi_j(x) - \Phi_j(y)|^{2\lambda}} |\det \Phi_j'(x)| \, |\det \Phi_j'(\xi)| \, dx \, d\xi \\ &\le K_5 \, \|u_j; \tilde{T}^{\theta,p}(\Omega)\|^p \end{aligned} \tag{36}$$

since $|\det \Phi_j'|$ is bounded and since, Ψ_j being 1-smooth on B,

$$|x - \xi| = |\Psi_j(y) - \Psi_j(\eta)| \le K_6 \, |y - \eta| = K_6 \, |\Phi_j(x) - \Phi_j(\xi)|.$$

Now $Ev_j \in \tilde{T}^{\theta,p}(\mathbb{R}^n)$ and

$$\|Ev_j; \tilde{T}^{\theta,p}(\mathbb{R}^n)\| \le K_7 \, \|v_j; \tilde{T}^{\theta,p}(\mathbb{R}_+^n)\|.$$

Also $\operatorname{supp} Ev_j \subset\subset B$. We define w_j a.e. on $\mathbb{R}^n$ by

$$w_j(x) = \begin{cases} Ev_j(\Phi(x)) & \text{if } \ x \in U_j \\ 0 & \text{if } \ x \in \mathbb{R}^n \sim U_j. \end{cases}$$

Then clearly $w_j(x) = u_j(x)$ a.e. in Ω, $\operatorname{supp} w_j \subset\subset U_j$, and by a calculation similar to the one carried on in (36),

$$\|w_j; \tilde{T}^{\theta,p}(\mathbb{R}^n)\| \le K^8 \, \|Ev_j; \tilde{T}^{\theta,p}(\mathbb{R}^n)\|.$$

Finally, we set

$$E^* u(x) = u_0(x) + \sum_{j=1}^{N} w_j(x).$$

It is clear that E^* has all the properties required of u in the statement of the lemma. ∎

It should be remarked that the comments made in Section 4.29 concerning weakening the hypotheses for the extension theorems 4.26 and 4.28 apply as well to the above lemma.

7.46 COROLLARY Under the conditions of Lemma 7.45 the spaces $\tilde{T}^{\theta,p}(\Omega)$ and $T^{\theta,p}(\Omega)$ coincide, and their norms are equivalent.

PROOF The coincidence of the two vector spaces follows from the fact that they coincide with restrictions to Ω of functions in the coincident spaces $\tilde{T}^{\theta,p}(\mathbb{R}^n)$ and $T^{\theta,p}(\mathbb{R}^n)$. If $u \in \tilde{T}^{\theta,p}(\Omega)$ and E is the extension operator constructed in the above lemma, we have

$$\|u; T^{\theta,p}(\Omega)\| \le \|Eu; T^{\theta,p}(\mathbb{R}^n)\| \le K_1 \|Eu; \tilde{T}^{\theta,p}(\mathbb{R}^n)\| \le K_2 \|u; \tilde{T}^{\theta,p}(\Omega)\|.$$

The reverse inequality follows in the same way, using instead of E the strong 1-extension operator constructed in Theorem 4.6 (the case $m = 1$) which, as is implicit in the proof of Theorem 7.41, is an extension operator for $T^{\theta,p}(\Omega)$. ∎

7.47 For $s \ge 0$ let $\tilde{W}^{s,p}(\Omega)$ be the space constructed in exactly the same way that $W^{s,p}(\Omega)$ was constructed in Section 7.36 except using the spaces $\tilde{T}^{1-\sigma,p}(\Omega)$ in place of $T^{1-\sigma,p}(\Omega)$. In view of Corollary 7.46 we have proved the following theorem.

7.48 THEOREM Let Ω be $\mathbb{R}^n$, or a half-space in $\mathbb{R}^n$, or a domain in $\mathbb{R}^n$ which is uniformly C^1-regular and has a bounded boundary. Then the spaces $\tilde{W}^{s,p}(\Omega)$ and $W^{s,p}(\Omega)$ coincide algebraically and topologically for each $s \ge 0$. In particular, if $s = m+\sigma$ where m is an integer and $0 < \sigma < 1$, then the norm $\|\cdot\|^{\sim}_{s,p,\Omega}$ given by

$$\|u\|^{\sim}_{s,p,\Omega} = \left\{ \|u\|^p_{m,p,\Omega} + \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{n+\sigma p}}\, dx\, dy \right\}^{1/p}$$

is equivalent to the original norm $\|\cdot\|_{s,p,\Omega}$ on $W^{s,p}(\Omega)$.

7.49 REMARK It is the spaces which we have above denoted $\tilde{W}^{s,p}(\Omega)$ which one encounters most frequently in the literature, and which are usually designated $W^{s,p}(\Omega)$. The space $\tilde{W}^{s,\infty}(\Omega)$ may obviously be defined in an analagous way. It consists of those $u \in W^{m,\infty}(\Omega)$ for which the norm

$$\|u\|_{s,\infty,\Omega} = \max\left(\|u\|_{m,\infty,\Omega},\ \max_{|\alpha|=m} \operatorname*{ess\,sup}_{\substack{x,y\in\Omega\\ x\ne y}} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\sigma} \right)$$

is finite.

Imbedding Theorems

7.50 As we have already seen (in Example 7.34), if $1 < p < \infty$, the linear mapping

$$u \to \gamma u = (\gamma_0 u, \ldots, \gamma_{m-1} u); \qquad \gamma_j u = D_n{}^j u(\cdot, \ldots, \cdot, 0)$$

establishes an isomorphism and a homeomorphism of $W^{m,p}(\mathbb{R}_+{}^n)/\ker\gamma$ onto $\prod_{k=0}^{m-1} T^{m-k-1}(p,0;\Lambda;L^p(\mathbb{R}^{n-1}))$, where $\Lambda = (D_1,\ldots,D_{n-1})$. Since $D(\Lambda^k) = W^{k,p}(\mathbb{R}^{n-1})$ and since $T^0(p,0;\Lambda;L^p(\mathbb{R}^{n-1})) = T^{1/p,p}(\mathbb{R}^{n-1})$, we have

$$T^k(p,0;\Lambda;L^p(\mathbb{R}^{n-1})) = W^{k+1-1/p,p}(\mathbb{R}^{n-1}).$$

Thus γ is in fact an isomorphism and homeomorphism of $W^{m,p}(\mathbb{R}_+{}^n)/\ker\gamma$ onto $\prod_{k=0}^{m-1} W^{m-k-1/p,p}(\mathbb{R}^{n-1})$. In particular, the traces on $\mathbb{R}^{n-1} = \text{bdry}\,\mathbb{R}_+{}^n$ of functions in $W^{m,p}(\mathbb{R}_+{}^n)$ belong to, and constitute the whole of the space $W^{m-1/p,p}(\text{bdry}\,\mathbb{R}_+{}^n)$. [This phenomenon is sometimes described as the loss of $(1/p)$th of a derivative on the boundary.] The result can be extended to smoothly bounded domains.

7.51 If Ω is a domain in $\mathbb{R}^n$ having the uniform C^m-regularity property and a bounded boundary, then the open cover $\{U_j\}$ of bdry Ω and the associated collection $\{\Psi_j\}$ of m-smooth maps from $B = \{y \in \mathbb{R}^n : |y| < 1\}$ onto the sets U_j (referred to in Section 4.6) are finite collections, say $1 \le j \le r$. If $\{\omega_j\}$ is a partition of unity for bdry Ω subordinate to $\{U_j\}$ and if u is a function defined on bdry Ω, we define $\theta_j u$ on $\mathbb{R}^{n-1}$ by

$$\theta_j u(y') = \begin{cases} (\omega_j u)(\Psi_j(y',0)) & \text{if } \ |y'| < 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $y' = (y_1,\ldots,y_{n-1})$.

For $s \ge 0$ and $1 < p < \infty$ we define $W^{s,p}(\text{bdry}\,\Omega)$ to be the class of functions $u \in L^p(\text{bdry}\,\Omega)$ (see Section 5.21) such that $\theta_j u$ belongs to $W^{s,p}(\mathbb{R}^{n-1})$ for $1 \le j \le r$. The space $W^{s,p}(\text{bdry}\,\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{s,p,\text{bdry}\,\Omega} = \left\{ \sum_{j=1}^{r} \|\theta_j u\|_{s,p,\mathbb{R}^{n-1}}^p \right\}^{1/p}.$$

As defined above, the space $W^{s,p}(\text{bdry}\,\Omega)$ appears to depend on the particular cover $\{U_j\}$, the mappings $\{\Psi_j\}$, and the partition of unity $\{\omega_j\}$ used in the definition. It can be checked that the same space, with an equivalent norm, results if we carry out the construction for a different collection $\{\tilde{U}_j\}$, $\{\tilde{\Psi}_j\}$, and $\{\tilde{\omega}_j\}$. (We omit the details; see Lions and Magenes [40].) It can also be checked that $C^\infty(\text{bdry}\,\Omega)$ is dense in $W^{s,p}(\text{bdry}\,\Omega)$.

7.52 Let $u \in C_0^\infty(\mathbb{R}^n)$. [The restrictions of such functions to Ω are dense in $W^{m,p}(\Omega)$.] Let γ denote the linear mapping

$$u \to \gamma u = (\gamma_0 u,\ldots,\gamma_{m-1} u); \qquad \gamma_j u = \left.\frac{\partial^j u}{\partial n^j}\right|_{\text{bdry}\,\Omega}, \tag{37}$$

where $\partial^j/\partial n^j$ denotes the jth directional derivative in the direction of the inward normal to bdry$\,\Omega$. Using a partition of unity for a neighborhood of bdry$\,\Omega$ subordinate to the open cover $\{u_j\}$, we can prove the following generalization (to Ω) of the result of Section 7.50. (The details are left to the reader.)

7.53 THEOREM Let $1 < p < \infty$ and let Ω satisfy the conditions prescribed above. Then the mapping γ given by (37) extends by continuity to an isomorphism and homeomorphism of $W^{m,p}(\Omega)/\ker\gamma$ onto

$$\prod_{k=0}^{m-1} W^{m-k-1/p,p}(\text{bdry}\,\Omega).$$

7.54 It is an immediate consequence of the following theorem that the kernel $\ker\gamma$ of the mapping γ, that is, the class of $u \in W^{m,p}(\Omega)$ for which $\gamma u = 0$, is precisely the space $W_0^{m,p}(\Omega)$. We adopt again the notations of Section 7.30. Let $W_0{}^m$ denote the closure in $W^m = W^m(p, \nu; \Lambda; B)$ of the set of functions $f \in W^m$, each of which vanishes identically on an interval $[0, \varepsilon)$ for some positive ε (which may depend on f).

7.55 THEOREM If $f \in W^m$ and if $f^{(k)}(0) = 0$ for $0 \le k \le m-1$, then $f \in W_0{}^m$. Thus $W_0{}^m$ is the kernel of the mapping

$$f \to (f(0), f'(0), \ldots, f^{(m-1)}(0))$$

of W^m onto $\prod_{k=0}^{m-1} T^{m-k-1}$.

PROOF Let $f \in W^m$ satisfy $f(0) = \cdots = f^{(m-1)}(0) = 0$. Let $\psi \in C^\infty(\mathbb{R})$ satisfy $\psi(t) = 0$ for $t \le 1$, $\psi(t) = 1$ for $t \ge 2$, $0 \le \psi(t) \le 1$ and $|\psi^{(k)}(t)| \le K_1$ for all t, $0 \le k \le m$. Let $f_n(t) = \psi(nt) f(t)$. Clearly $f_n \in W_0{}^m$. We must show that $f(t) - f_n(t) = (1 - \psi(nt)) f(t) \to 0$ in W^m as $n \to \infty$, that is, we must show that for each k, $0 \le k \le m$, and each multi-index α, $|\alpha| \le m-k$, we have

$$\int_0^\infty t^{\nu p} \|\Lambda^\alpha (f - f_n)^{(k)}(t)\|_B^p \, dt \to 0 \qquad \text{as} \quad n \to \infty.$$

Now

$$\int_0^\infty t^{\nu p} \|(1 - \psi(nt)) \Lambda^\alpha f^{(k)}(t)\|_B^p \, dt \le \int_0^{2/n} t^{\nu p} \|\Lambda^\alpha f^{(k)}(t)\|_B^p \, dt \to 0$$

as $n \to \infty$ since $f \in W^m$. Hence we need only show that if $1 \le j \le k$, then

$$\int_0^\infty t^{\nu p} \left[\left(\frac{d}{dt}\right)^j (1 - \psi(nt))\right]^p \|\Lambda^\alpha f^{(k-j)}(t)\|_B^p \, dt \to 0 \tag{38}$$

as $n \to \infty$. But the left side of (38) does not exceed a constant times

$$n^{jp} \int_{1/n}^{2/n} t^{\nu p} \|\Lambda^\alpha f^{(k-j)}(t)\|_B^p \, dt. \tag{39}$$

Since $f(0) = f'(0) = \cdots = f^{(m-1)}(0) = 0$, and since $k-j \le m-1$ we have [where $p^{-1} + (p')^{-1} = 1$]

$$\begin{aligned}
\|\Lambda^\alpha f^{(k-j)}(t)\|_B^p &\le \left\{ \frac{1}{(j-1)!} \int_0^t (t-\tau)^{j-1} \|\Lambda^\alpha f^{(k)}(\tau)\|_B \, d\tau \right\}^p \\
&\le \frac{t^{(j-1)p}}{[(j-1)!]^p} \int_0^t \tau^{\nu p} \|\Lambda^\alpha f^{(k)}(\tau)\|_B^p \, d\tau \left\{ \int_0^t \tau^{-\nu p'} \, d\tau \right\}^{p/p'} \\
&\le K_2 t^{jp-\nu p-1} \int_0^t \tau^{\nu p} \|\Lambda^\alpha f^{(k)}(\tau)\|_B^p \, d\tau.
\end{aligned}$$

It follows that (39) does not exceed a constant times

$$n^{jp} \int_{1/n}^{2/n} t^{jp-1} \, dt \int_0^t \tau^{\nu p} \|\Lambda^\alpha f^{(k)}(\tau)\|_B^p \, d\tau \le (2^{jp}/jp) \int_0^{2/n} \tau^{\nu p} \|\Lambda^\alpha f^{(k)}(\tau)\|_B^p \, d\tau \to 0$$

as $n \to \infty$ since $f \in W^m$. ∎

7.56 The characterization of traces on bdry$\,\Omega$ of functions in $W^{m,p}(\Omega)$ has important applications in the study of nonhomogeneous boundary value problems for differential operators defined on Ω. Theorem 7.53 contains both "direct" and "converse" imbedding theorems for $W^{m,p}(\Omega)$ in the following sense: If $u \in W^{m,p}(\Omega)$, then the trace $v = u|_{\mathrm{bdry}\,\Omega}$ belongs to $W^{m-1/p,p}(\mathrm{bdry}\,\Omega)$ and

$$\|v\|_{m-1/p,p,\mathrm{bdry}\,\Omega} \le K_1 \|u\|_{m,p,\Omega};$$

and conversely, if $v \in W^{m-1/p,p}(\mathrm{bdry}\,\Omega)$, then there exists $u \in W^{m,p}(\Omega)$ with $v = u|_{\mathrm{bdry}\,\Omega}$ and

$$\|u\|_{m,p,\Omega} \le K_2 \|v\|_{m-1/p,p,\mathrm{bdry}\,\Omega}.$$

Before stating a very general imbedding theorem for the spaces $W^{s,p}(\Omega)$ we show how some (but not all) imbeddings for these spaces can be obtained from known cases for integral s by the interpolation Theorem 7.17.

7.57 THEOREM Let Ω be a domain having the cone property in $\mathbb{R}^n$. Let $s > 0$ and $1 < p < n$.

(a) If $n > sp$, then $W^{s,p}(\Omega) \to L^r(\Omega)$ for $p \le r \le np/(n-sp)$.
(b) If $n = sp$, then $W^{s,p}(\Omega) \to L^r(\Omega)$ for $p \le r < \infty$.
(c) If $n < (s-j)p$ for some nonnegative integer j, then $W^{s,p}(\Omega) \to C_B^j(\Omega)$.

PROOF The results are already known for integer s so we may assume s is not an integer and write $s = m+\sigma$ where m is an integer and $0 < \sigma < 1$. First let us suppose $m = 0$. Then

$$\begin{aligned} W^{\sigma,p}(\Omega) &= L^p(\Omega) \cap T^{1-\sigma,p}(\Omega) \\ &= L^p(\Omega) \cap T(p, 1-\sigma-(1/p); W^{1,p}(\Omega), L^p(\Omega)). \end{aligned}$$

Now the identity operator is continuous from $W^{1,p}(\Omega)$ into $L^{np/(n-p)}(\Omega)$ and (trivially) from $L^p(\Omega)$ into $L^p(\Omega)$. By Theorem 7.17 it is also continuous from $T(p, 1-\sigma-(1/p); W^{1,p}(\Omega), L^p(\Omega))$ into $T(p, 1-\sigma-(1/p); L^{np/(n-p)}(\Omega), L^p(\Omega))$. By Theorem 7.20 this latter space is imbedded in $L^{np/(n-\sigma p)}$ provided $n > \sigma p$. Hence

$$W^{\sigma,p}(\Omega) \to L^{np/(n-\sigma p)}(\Omega).$$

For general m, we argue as follows. Let $u \in W^{m+\sigma,p}(\Omega)$. If $|\alpha| = m$, then $D^\alpha u \in W^{\sigma,p}(\Omega) \to L^{np/(n-\sigma p)}(\Omega)$. If $|\alpha| \le m-1$, then $D^\alpha u \in W^{1,p}(\Omega) \to L^{np/(n-\sigma p)}(\Omega)$. Hence $W^{m+\sigma,p}(\Omega) \to W^{m,np/(n-\sigma p)}(\Omega)$. If $n > sp$, we have by Theorem 5.4 that $W^{m,np/(n-\sigma p)}(\Omega) \to L^{np/(n-sp)}(\Omega)$. Hence (a) is proved. If $n = sp$, then $W^{m,np/(n-\sigma p)}(\Omega) \to L^r(\Omega)$ for any r such that $p \le r < \infty$, so (b) is proved. If $(s-j)p > n$, then $(m-j)np/(n-\sigma p) > n$ and so $W^{m,np/(n-\sigma p)}(\Omega) \to C_B{}^j(\Omega)$ and (c) is proved. ▌

The restriction $p < n$ in the above theorem is unnatural and was placed only for the purpose of achieving a very simple proof.

The following theorem contains all the imbedding results cited above as special cases. It comprises results obtained by several writers, in particular, Besov [9, 10], Uspenskii [67, 68], and Lizorkin [41]. The theorem is stated for $\mathbb{R}^n$ but can obviously be extended to domains with sufficient regularily, such as those satisfying the conditions of Theorem 7.41. We shall not attempt any proof.

7.58 THEOREM Let $s > 0$, $1 < p \le q < \infty$, and $1 \le k \le n$. Let $\chi = s-(n/p)+(k/q)$. If

(i) $\chi \ge 0$ and $p < q$, or
(ii) $\chi > 0$ and χ is not an integer, or
(iii) $\chi \ge 0$ and $1 < p \le 2$,

then (direct imbedding theorem)

$$W^{s,p}(\mathbb{R}^n) \to W^{\chi,q}(\mathbb{R}^k). \tag{40}$$

Imbedding (40) does not necessarily hold for $p = q > 2$ and χ a nonnegative integer. (In particular, one cannot in general strengthen Case A of Part I of Theorem 5.4 to allow $k = n-mp$.)

Conversely, if $p = q$ and if either

(iv) $\chi = s-(n-k)/p > 0$ and is not an integer, or
(v) $\chi \geq 0$ and $p \geq 2$,

then we have the reverse imbedding

$$W^{\chi,p}(\mathbb{R}^k) \to W^{s,p}(\mathbb{R}^n)$$

in the sense that each $u \in W^{\chi,p}(\mathbb{R}^k)$ is the trace on $\mathbb{R}^k$ (i.e., $u = w|_{\mathbb{R}^k}$) of a function $w \in W^{s,p}(\mathbb{R}^n)$ satisfying

$$\|w\|_{s,p,\mathbb{R}^n} \leq K\|u\|_{\chi,p,\mathbb{R}^k}$$

with K independent of u. (The trace is understood in the sense of Section 5.2.)

Bessel Potentials—The Spaces $L^{s,p}(\Omega)$

7.59 We shall outline here, without proofs, another method of constructing fractional order spaces which originates in studies of Bessel potentials by Aronszajn and Smith [7] (and their collaborators—Adams *et al.* [5] and Aronszajn *et al.* [8]) and which is presented by Calderón [13] and Lions and Magenes [40]. The resulting spaces, denoted $L^{s,p}(\Omega)$ (or $H^{s,p}(\Omega)$ by Lions and Magenes—but not to be confused with the H-spaces of Nikol'skii defined in Section 7.73) coincide with the spaces $W^{s,p}(\Omega)$ for integer values of s if $1 < p < \infty$, and for all s when $p = 2$.

The space $L^{s,p}(\mathbb{R}^n)$ is constructed directly in terms of Fourier transforms of tempered distributions. It is shown then that for $1 < p < \infty$, $L^{s,p}(\mathbb{R}^n)$ and $W^{s,p}(\mathbb{R}^n)$ are isomorphic and homeomorphic when s is an integer. For any values of s_1, s, s_2 with $s_1 \leq s \leq s_2$, the space $L^{s,p}(\mathbb{R}^n)$ can be identified as an intermediate space interpolated between $L^{s_1,p}(\mathbb{R}^n)$ and $L^{s_2,p}(\mathbb{R}^n)$ by a "complex" interpolation method (see Calderón [15] or Lions [36]) which is not identical to the trace method of Lions described earlier. This interpolation method then provides a means of defining $L^{s,p}(\Omega)$ for domain $\Omega \subset \mathbb{R}^n$ as an intermediate space between spaces of the form $W^{m,p}(\Omega)$ for integer values of m.

Proofs of assertions made in the discussion of the spaces $L^{s,p}(\Omega)$ and their relationship to the spaces $W^{s,p}(\Omega)$ can be found in one or another of the papers by Calderón, Lions, and Lions and Magenes cited above.

7.60 First we introduce the notion of tempered distribution. We denote by $\mathscr{S}(\mathbb{R}^n)$ the space of rapidly decreasing functions in $\mathbb{R}^n$, that is, functions ϕ satisfying

$$\sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta \phi(x)| < \infty$$

for all multi-indices α and β. The space $\mathscr{S}(\mathbb{R}^n)$ carries a locally convex topology characterized by the following notion of convergence: The sequence $\{\phi_j\}$ converges to 0 in $\mathscr{S}(\mathbb{R}^n)$ if for all α and β

$$\lim_{j\to\infty} x^\alpha D^\beta \phi_j(x) = 0 \quad \text{uniformly on } \mathbb{R}^n.$$

It may be readily verified that the Fourier transform

$$\mathscr{F}\phi(y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot y}\phi(x)\,dx$$

and the inverse Fourier transform

$$\mathscr{F}^{-1}\phi(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot y}\phi(y)\,dy$$

are each continuous on $\mathscr{S}(\mathbb{R}^n)$ into $\mathscr{S}(\mathbb{R}^n)$, and, since $\mathscr{F}^{-1}\mathscr{F}\phi = \mathscr{F}\mathscr{F}^{-1}\phi = \phi$, each is in fact an isomorphism and a homeomorphism of $\mathscr{S}(\mathbb{R}^n)$ onto $\mathscr{S}(\mathbb{R}^n)$.

It is clear from the definition of $\mathscr{S}(\mathbb{R}^n)$ that $\mathscr{D}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$. Hence the dual space $\mathscr{S}'(\mathbb{R}^n)$ consists of those distributions $T \in \mathscr{D}'(\mathbb{R}^n)$ which possess continuous extensions to $\mathscr{S}(\mathbb{R}^n)$. For instance, if $1 \le p \le \infty$ and $f \in L^p(\mathbb{R}^n)$, then

$$T_f(\phi) = \int_{\mathbb{R}^n} f(x)\,\phi(x)\,dx$$

defines $T_f \in \mathscr{S}'(\mathbb{R}^n)$. The same holds for any function f of "slow growth" at infinity, that is, for which for some finite k we have $|f(x)| \le \text{const}\,|x|^k$ a.e. in some neighborhood of infinity. The elements of $\mathscr{S}'(\mathbb{R}^n)$ are therefore called *tempered distributions*. $\mathscr{S}'(\mathbb{R}^n)$ is given the weak-star topology as dual of $\mathscr{S}(\mathbb{R}^n)$ and is a locally convex topological vector space with this topology.

The direct and inverse Fourier transformations are extended to $\mathscr{S}'(\mathbb{R}^n)$ by

$$\mathscr{F}T(\phi) = T(\mathscr{F}\phi), \qquad \mathscr{F}^{-1}T(\phi) = T(\mathscr{F}^{-1}\phi).$$

Once again, each is an isomorphism and a homeomorphism of $\mathscr{S}'(\mathbb{R}^n)$ onto $\mathscr{S}'(\mathbb{R}^n)$ and $\mathscr{F}^{-1}\mathscr{F}T = \mathscr{F}\mathscr{F}^{-1}T = T$.

7.61 Given a tempered distribution u on $\mathbb{R}^n$ and a complex number z the Bessel potential of order z of u is denoted $J^z u$ and defined by

$$J^z u = \mathscr{F}^{-1}((1 + |\cdot|^2)^{-z/2}\mathscr{F}u).$$

Evidently J^z is one-to-one on $\mathscr{S}'(\mathbb{R}^n)$ into $\mathscr{S}'(\mathbb{R}^n)$. If $\operatorname{Re} z > 0$ and $1 \le p \le \infty$, or if $\operatorname{Re} z \geq 0$ and $1 < p < \infty$, then J^z transforms $L^p(\mathbb{R}^n)$ continuously into $L^p(\mathbb{R}^n)$, and $D^\alpha J^{z+|\alpha|}$ does likewise.

7.62 For real s and $1 \leq p \leq \infty$ let $L^{s,p}(\mathbb{R}^n)$ denote the image of $L^p(\mathbb{R}^n)$ under the linear mapping J^s. Thus $L^{s,p}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ for every s, and $L^{s,p}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for $s \geq 0$. If $u \in L^{s,p}(\mathbb{R}^n)$, then there exists unique $\tilde{u} \in L^p(\mathbb{R}^n)$ with $u = J^s\tilde{u}$. We define

$$\|u; L^{s,p}(\mathbb{R}^n)\| = \|\tilde{u}\|_{0,p,\mathbb{R}^n}.$$

With respect to this norm, $L^{s,p}(\mathbb{R}^n)$ is a Banach space. We summarize some of its properties.

7.63 THEOREM (a) If $s \geq 0$ and $1 \leq p < \infty$, then $\mathscr{D}(\mathbb{R}^n)$ is dense in $L^{s,p}(\mathbb{R}^n)$.

(b) If $1 < p < \infty$ and $p' = p/(p-1)$, then $[L^{s,p}(\mathbb{R}^n)]' \cong L^{-s,p'}(\mathbb{R}^n)$.

(c) It $t < s$, then $L^{s,p}(\mathbb{R}^n) \to L^{t,p}(\mathbb{R}^n)$.

(d) If $t \leq s$ and if either $1 < p \leq q \leq np/[n-(s-t)p] < \infty$ or $p = 1$ and $1 \leq q < n/(n-s+t)$, then $L^{s,p}(\mathbb{R}^n) \to L^{t,q}(\mathbb{R}^n)$.

(e) If $0 \leq \mu \leq s-(n/p) < 1$, then $L^{s,p}(\mathbb{R}^n) \to C^{0,\mu}(\mathbb{R}^n)$.

(f) If s is a nonnegative integer and $1 < p < \infty$, then $L^{s,p}(\mathbb{R}^n)$ coincides with $W^{s,p}(\mathbb{R}^n)$, the norms in the two spaces being equivalent. This conclusion also holds for any s if $p = 2$.

(g) If $1 < p < \infty$ and $\varepsilon > 0$, then for every s we have

$$L^{s+\varepsilon,p}(\mathbb{R}^n) \to W^{s,p}(\mathbb{R}^n) \to L^{s-\varepsilon,p}(\mathbb{R}^n).$$

7.64 We now describe a complex interpolation method of Calderón [15] and Lions [36] in which setting the spaces $L^{s,p}(\mathbb{R}^n)$ can also arise.

Let B_0 and B_1 be Banach spaces both imbedded in a topological vector space X as in Section 7.11, and let the Banach space $B_0 + B_1$ be defined as in that section. We denote by $F(B_0, B_1)$ the space of functions f of a complex variable $z = \sigma + i\tau$ taking values in $B_0 + B_1$ and satisfying

(i) f is holomorphic on the strip $0 < \sigma < 1$,

(ii) f is continuous and bounded on the strip $0 \leq \sigma \leq 1$,

(iii) $f(i\tau) \in B_0$ for $\tau \in \mathbb{R}$, the map $\tau \to f(i\tau)$ is continuous on $\mathbb{R}$ into B_0, and $\lim_{|\tau|\to\infty} f(i\tau) = 0$, and

(iv) $f(1+i\tau) \in B_1$ for $\tau \in \mathbb{R}$, the map $\tau \to f(1+i\tau)$ is continuous on $\mathbb{R}$ into B_1, and $\lim_{|\tau|\to\infty} f(1+i\tau) = 0$.

$F(B_0, B_1)$ is a Banach space with respect to the norm

$$\|f; F(B_0, B_1)\| = \max\left\{\sup_{\tau\in\mathbb{R}} \|f(i\tau)\|_{B_0}, \sup_{\tau\in\mathbb{R}} \|f(1+i\tau)\|_{B_1}\right\}.$$

For $0 \le \sigma \le 1$ set

$$B_\sigma = [B_0; B_1]_\sigma = \{u \in B_0 + B_1 : u = f(\sigma) \text{ for some } f \in F(B_0, B_1)\}.$$

With respect to the norm

$$\|u\|_{B_\sigma} = \|u; [B_0; B_1]_\sigma\| = \inf_{\substack{f \in F(B_0, B_1)\\ f(\sigma) = u}} \|f; F(B_0, B_1)\|,$$

B_σ is a Banach space imbedded in $B_0 + B_1$.

The intermediate spaces B_σ possess interpolation characteristics similar to those of the trace spaces of Lions. If C_0, C_1, and Y are spaces having properties similar to those specified for B_0, B_1, and X, and if L is a linear mapping from $B_0 + B_1$ into $C_0 + C_1$ satisfying

$$\|Lu\|_{C_0} \le K_0 \|u\|_{B_0}, \qquad \|Lu\|_{C_1} \le K_1 \|u\|_{B_1},$$

then for each $u \in B_\sigma$ we have $Lu \in C_\sigma$ and

$$\|Lu\|_{C_\sigma} \le K_0^{1-\sigma} K_1^{\sigma} \|u\|_{B_\sigma}.$$

The following theorem may be found in the papers by Calderón [15] or Lions [36].

7.65 THEOREM For any real s_0 and s_1, and for $0 \le \sigma \le 1$ we have

$$[L^{s_0,p}(\mathbb{R}^n); L^{s_1,p}(\mathbb{R}^n)]_\sigma = L^{(1-\sigma)s_0 + \sigma s_1, p}(\mathbb{R}^n).$$

We remark that the corresponding statement for intermediate spaces between $W^{s_0,p}(\mathbb{R}^n)$ and $W^{s_1,p}(\mathbb{R}^n)$ obtained by trace interpolation is not valid for all s_0 and s_1 though it is for certain values, in particular if s_0 and s_1 are consecutive nonnegative integers.

7.66 The above theorem suggests how the spaces $L^{s,p}(\Omega)$ may be defined for arbitrary domains $\Omega \subset \mathbb{R}^n$. If $s \ge 0$, let m be the integer satisfying $s \le m < s+1$ and define $L^{s,p}(\Omega) = [W^{m,p}(\Omega); L^p(\Omega)]_{(m-s)/m}$. If Ω is sufficiently regular to possess a strong m-extension operator, then an interpolation argument shows that $L^{s,p}(\Omega)$ coincides with the space of restrictions to Ω of functions in $L^{s,p}(\mathbb{R}^n)$. Also, Theorem 7.65 is valid for the spaces $L^{s,p}(\Omega)$ provided $0 \le s_0, s_1 \le m$.

The definition of $L^{s,p}(\Omega)$ for negative s is carried out in the same manner as for the spaces $W^{s,p}(\Omega)$. One denotes by $L_0^{s,p}(\Omega)$ (where $s > 0$) the closure of $\mathcal{D}(\Omega)$ in $L^{s,p}(\Omega)$ and defines, for $1 < p < \infty$ and $s < 0$, the space $L^{s,p}(\Omega)$ to be $[L_0^{-s,p'}(\Omega)]'$, where $(1/p) + (1/p') = 1$.

All the properties stated for $L^{s,p}(\mathbb{R}^n)$ in Theorem 7.63 possess analogs for $L^{s,p}(\Omega)$ provided Ω is suitably regular.

Other Fractional Order Spaces

7.67 Certain gaps in the general imbedding theorem for the spaces $W^{s,p}(\mathbb{R}^n)$ (see Theorem 7.58) led to the construction by Besov [9, 10] of a family of spaces $B^{s,p}(\mathbb{R}^n)$ which differ from $W^{s,p}(\mathbb{R}^n)$ when s is a positive integer and which naturally supplement these latter spaces in a sense to be made precise below.

$B^{s,p}(\mathbb{R}^n)$ is defined for $s > 0$ and $1 \leq p \leq \infty$ as follows. Let $s = m + \sigma$ where m is a nonnegative integer and $0 < \sigma \leq 1$. The space $B^{s,p}(\mathbb{R}^n)$ consists of those functions u in $W^{m,p}(\mathbb{R}^n)$ for which the norm $\|u; B^{s,p}(\mathbb{R}^n)\|$ is finite. If $1 \leq p < \infty$,

$$\|u; B^{s,p}(\mathbb{R}^n)\|$$

$$= \left\{ \|u\|_{m,p,\mathbb{R}^n}^p + \sum_{|\alpha|=m} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|D^\alpha u(x) - 2D^\alpha u((x+y)/2) + D^\alpha u(y)|^p}{|x-y|^{n+\sigma p}} \, dx \, dy \right\}^{1/p}.$$

If $p = \infty$,

$$\|u; B^{s,\infty}(\mathbb{R}^n)\|$$

$$= \max \left\{ \|u\|_{m,\infty,\mathbb{R}^n}, \max_{|\alpha|=m} \operatorname*{ess\,sup}_{\substack{x,y \in \mathbb{R}^n \\ x \neq y}} \frac{|D^\alpha u(x) - 2D^\alpha u((x+y)/2) + D^\alpha u(y)|}{|x-y|^\sigma} \right\}.$$

$B^{s,p}(\mathbb{R}^n)$ is a Banach space with respect to the above norm. If $1 \leq p < \infty$, $C_0^\infty(\mathbb{R}^n)$ is dense in $B^{s,p}(\mathbb{R}^n)$.

7.68 LEMMA If $1 \leq p < \infty$ and $s > 0$ is not an integer, then the spaces $W^{s,p}(\mathbb{R}^n)$ and $B^{s,p}(\mathbb{R}^n)$ coincide, and have equivalent norms.

PROOF For functions u defined on $\mathbb{R}^n$ we define the difference operator Δ_z by

$$\Delta_z u(x) = u(x+z) - u(x).$$

The second difference operator Δ_z^2 is then given by

$$\Delta_z^2 u(x) = \Delta_z \Delta_z u(x) = u(x+2z) - 2u(x+z) + u(x).$$

The identity

$$\Delta_z u = (1/2^k) \Delta_{2^k z} u - \tfrac{1}{2} \sum_{j=0}^{k-1} (1/2^j) \Delta_{2^j z}^2 u \tag{41}$$

may readily be verified by expanding the sum on the right side.

Evidently, the norm of a function u in $B^{s,p}(\mathbb{R}^n)$ is equivalent to

$$\left\{ \|u\|_{m,p,\mathbb{R}^n}^p + \sum \int |z|^{-n-\sigma p} \, dz \int |\Delta_z^2 D^\alpha u(x)|^p \, dx \right\}^{1/p}. \tag{42}$$

while by Theorem 7.48 the norm of u in $W^{s,p}(\mathbb{R}^n)$ can be expressed in the form

$$\left\{\|u\|_{m,p,\mathbb{R}^n}^p + \sum_{|\alpha|=m} \int_{\mathbb{R}^n} |z|^{-n-\sigma p}\, dz \int_{\mathbb{R}^n} |\Delta_z D^\alpha u(x)|^p\, dx\right\}^{1/p}. \tag{43}$$

It is clear that (42) is bounded by a constant times (43); we must prove the reverse assertion.

Suppose, therefore, that $u \in C_0^\infty(\mathbb{R}^n)$. We have, using (41),

$$\begin{aligned}
&\left\{\int_{\mathbb{R}^n} |z|^{-n-\sigma p}\, dz \int_{\mathbb{R}^n} |\Delta_z D^\alpha u(x)|^p\, dx\right\}^{1/p} \\
&\quad\le (1/2^k)\left\{\int_{\mathbb{R}^n} |z|^{-n-\sigma p}\, dz \int_{\mathbb{R}^n} |\Delta_{2^k z} D^\alpha u(x)|^p\, dx\right\}^{1/p} \\
&\qquad + \tfrac{1}{2}\sum_{j=0}^{k-1} (1/2^j)\left\{\int_{\mathbb{R}^n} |z|^{-n-\sigma p}\, dz \int_{\mathbb{R}^n} |\Delta_{2^j z}^2 D^\alpha u(x)|^p\, dx\right\}^{1/p} \\
&\quad= (1/2^{k(1-\sigma)})\left\{\int_{\mathbb{R}^n} |\rho|^{-n-\sigma p}\, d\rho \int_{\mathbb{R}^n} |\Delta_\rho D^\alpha u(x)|^p\, dx\right\}^{1/p} \\
&\qquad + \tfrac{1}{2}\sum_{j=0}^{k-1} (1/2^{j(1-\sigma)})\left\{\int_{\mathbb{R}^n} |\rho|^{-n-\sigma p}\, d\rho \int_{\mathbb{R}^n} |\Delta_\rho{}^2 D^\alpha u(x)|^p\, dx\right\}^{1/p}.
\end{aligned}$$

(We have substituted $\rho = 2^k z$ in the first integral, $\rho = 2^j z$ in the second.) Since s is not an integer, we have $\sigma < 1$ and so k may be chosen large enough that $k(1-\sigma) \ge 1$. It then follows that (43) is bounded by a constant times (42). Since $C_0^\infty(\mathbb{R}^n)$ is dense in $B^{s,p}(\mathbb{R}^n)$ the lemma follows. ∎

7.69 If s is a positive integer and $p = 2$, the spaces $W^{s,2}(\mathbb{R}^n)$ and $B^{s,2}(\mathbb{R}^n)$ coincide. For $p \ne 2$, s an integer they are distinct but for any $\varepsilon > 0$ we have

$$\begin{aligned}
&W^{s+\varepsilon,p}(\mathbb{R}^n) \to B^{s,p}(\mathbb{R}^n) \to W^{s,p}(\mathbb{R}^n) \qquad \text{if}\quad 1 < p \le 2 \\
&B^{s+\varepsilon,p}(\mathbb{R}^n) \to W^{s,p}(\mathbb{R}^n) \to B^{s,p}(\mathbb{R}^n) \qquad \text{if}\quad p \ge 2.
\end{aligned}$$

The spaces $B^{s,p}(\mathbb{R}^n)$ are of interest primarily for their imbedding characteristics. They possess a "closed system" of imbeddings and at the same time fill gaps in the system of imbeddings for the spaces $W^{s,p}(\mathbb{R}^n)$.

7.70 THEOREM Let $s > 0$, $1 \le p \le q \le \infty$, and $1 \le k \le n$, k an integer. Suppose

$$r = s - (n/p) + (k/q) > 0.$$

Then

$$B^{s,p}(\mathbb{R}^n) \to B^{r,q}(\mathbb{R}^k).$$

Conversely, if $p = q$ and $r = [s-(n-k)]/p > 0$, then the reverse imbedding

$$B^{r,p}(\mathbb{R}^k) \to B^{s,p}(\mathbb{R}^n)$$

holds, in the sense that each element u in $B^{r,p}(\mathbb{R}^k)$ is the trace $u = v|_{\mathbb{R}^k}$ of some element v in $B^{s,p}(\mathbb{R}^n)$ satisfying

$$\|v; B^{s,p}(\mathbb{R}^n)\| \le K\|u; B^{r,p}(\mathbb{R}^k)\|,$$

where K is independent of u.

7.71 THEOREM If $s > 0$, $1 \le p \le \infty$, and $1 \le k \le n$, and if $r = [s-(n-k)]/p$, then

$$W^{s,p}(\mathbb{R}^n) \to B^{r,p}(\mathbb{R}^n)$$

and conversely

$$B^{r,p}(\mathbb{R}^k) \to W^{s,p}(\mathbb{R}^n).$$

7.72 The definitions and theorems above can be extended to suitable domains $\Omega \subset \mathbb{R}^n$ and smooth manifolds Ω^k of dimension k contained in $\bar{\Omega}$. For $1 \le p < \infty$ the norm in $B^{s,p}(\Omega)$ is

$$\|u; B^{s,p}(\Omega)\|$$
$$= \left\{\|u\|_{m,p,\Omega}^p + \sum_{|\alpha|=m} \int_\Omega \int_{\Omega_x} \frac{|D^\alpha u(x) - 2D^\alpha u((x+y)/2) + D^\alpha u(y)|^p}{|x-y|^{n+\sigma p}}\, dy\, dx\right\}^{1/p},$$

where $\Omega_x = \{y \in \Omega : (x+y)/2 \in \Omega\}$.

7.73 A different class of spaces having imbedding properties similar to the Besov spaces are the spaces $H^{s,p}(\Omega)$ introduced by Nikol'skii [49–51]. These spaces, having norms involving Hölder conditions in the L^p-metric, were studied earlier than the (fractional order) W- or B-spaces and provided impetus for the latter.

Again we set $s = m+\sigma$ where $m \ge 0$ is an integer and $0 < \sigma \le 1$. For $1 \le p < \infty$ and $\Omega \subset \mathbb{R}^n$ a function u belongs to $H^{s,p}(\Omega)$ provided the norm

$$\|u; H^{s,p}(\Omega)\| = \left\{\|u\|_{0,p,\Omega}^p + \sum_{|\alpha|=m} \sup_{\substack{h \in \mathbb{R}^n \\ \eta > 0 \\ 0 < |h| < \eta}} \int_{\Omega_\eta} \frac{|\Delta_h{}^2 D^\alpha f(x)|^p}{|h|^{\sigma p}}\, dx\right\}^{1/p}$$

is finite, where $\Omega_\eta = \{x \in \Omega; \operatorname{dist}(x, \operatorname{bdry}\Omega) \ge 2\eta\}$. The obvious modification is made for $p = \infty$ so that in fact $H^{s,\infty}(\Omega) = B^{s,\infty}(\Omega)$. An argument similar to that of Lemma 7.68 shows that if s is not an integer, the second difference Δ^2

in the norm of $H^{s,p}(\Omega)$ can be replaced by the first difference Δ without changing the space.

The spaces $H^{s,p}(\Omega)$ are larger than the corresponding spaces $W^{s,p}(\Omega)$; but if $\varepsilon > 0$, we have

$$H^{s+\varepsilon,p}(\Omega) \to W^{s,p}(\Omega) \to H^{s,p}(\Omega).$$

The spaces $H^{s,p}(\mathbb{R}^n)$ possess a closed system of imbeddings identical to those of the Besov spaces, that is, Theorem 7.70 holds with B everywhere replaced by H. Strong extension theorems can be proved for H-spaces over smoothly bounded domains so that the imbedding theorem can be extended to such domains and traces on smooth manifolds in them.

The imbedding theorems for the spaces $H^{s,p}(\mathbb{R}^n)$ and $B^{s,p}(\mathbb{R}^n)$ are proved by a technique involving approximation of functions in these spaces by entire functions of exponential type in several complex variables (see Nikol'skii [49], for example).

7.74 Numerous generalizations of the above spaces have been made, partly for their own sake and partly to facilitate the solution of other problems in analysis. We mention two such directions of generalization. The first involves replacing ordinary L^p-norms by weighted norms. The second involves the use of different values of s and p in terms of the norm involving integration in different coordinate directions (anisotropic spaces). The interested reader is referred to two excellent survey articles (Nikol'skii [52] and Sobolev and Nikol'skii [64]), and their bibliographies for further information on the whole spectrum of spaces of differentiable functions of several real variables.

VIII

Orlicz and Orlicz–Sobolev Spaces

Introduction

8.1 In this final chapter we present some recent results involving replacement of the spaces $L^p(\Omega)$ with more general spaces $L_A(\Omega)$ in which the role usually played by the convex function t^p is assumed by more general convex functions $A(t)$. The spaces $L_A(\Omega)$, called *Orlicz spaces*, are studied in depth in the monograph by Krasnosel'skii and Rutickii [34] and also in the doctoral thesis by Luxemburg [42] to either of which the reader is referred for a more complete development of the material outlined below. The former also contains examples of applications of Orlicz spaces to certain problems in nonlinear analysis.

Following Krasnosel'skii and Rutickii [34] we use the class of "N-functions" as defining functions A for Orlicz spaces. This class is not as wide as the class of Young's functions used by Luxemburg [42] (see also O'Neill [55]); for instance, it excludes $L^1(\Omega)$ and $L^\infty(\Omega)$ from the class of Orlicz spaces However, N-functions are simpler to deal with and are adequate for our purposes. Only once, in the proof of Theorem 8.35, is it necessary to refer to a more general Young's function.

If the role played by $L^p(\Omega)$ in the definition of the Sobolev space $W^{m,p}(\Omega)$ is assigned instead to an Orlicz space $L_A(\Omega)$, the resulting space is denoted by $W^mL_A(\Omega)$ and called an *Orlicz–Sobolev space*. Many properties of Sobolev spaces have been extended to Orlicz–Sobolev spaces, mainly by Donaldson and Trudinger [22]. We present some of these results in this chapter.

It is also of some interest to note that a gap in the Sobolev imbedding theorem 5.4 can be filled by consideration of Orlicz spaces. Specifically, Case B of that theorem provides no "best" target space for imbeddings of $W^{m,p}(\Omega)$ with Ω a "regular" domain in $\mathbb{R}^n$ and $mp = n$. We have $W^{m,p}(\Omega) \to L^q(\Omega)$ for $p \le q < \infty$ but $W^{m,p}(\Omega) \not\to L^\infty(\Omega)$. In Theorem 8.25 an optimal imbedding of $W^{m,p}(\Omega)$ into an Orlicz space is constructed. This result is due to Trudinger [66].

N-Functions

8.2 Let a be a real valued function defined on $[0, \infty)$ and having the following properties:

(a) $a(0) = 0$, $a(t) > 0$ if $t > 0$, $\lim_{t\to\infty} a(t) = \infty$;
(b) a is nondecreasing, that is, $s > t \ge 0$ implies $a(s) \ge a(t)$;
(c) a is right continuous, that is, if $t \ge 0$, then $\lim_{s\to t+} a(s) = a(t)$.

Then the real valued function A defined on $[0, \infty)$ by

$$A(t) = \int_0^t a(\tau)\, d\tau \tag{1}$$

is called an *N-function*.

It is not difficult to verify that any such N-function A has the following properties:

(i) A is continuous on $[0, \infty)$;
(ii) A is strictly increasing, that is, $s > t \ge 0$ implies $A(s) > A(t)$;
(iii) A is convex, that is, if s, $t \ge 0$ and $0 < \lambda < 1$, then

$$A(\lambda s + (1-\lambda)t) \le \lambda A(s) + (1-\lambda) A(t);$$

(iv) $\lim_{t\to 0+} A(t)/t = 0$, $\lim_{t\to\infty} A(t)/t = \infty$;
(v) if $s > t > 0$, then $A(s)/s > A(t)/t$.

Properties (i), (iii), and (iv) could have been used to define N-function since they imply the existence of a representation of A in the form (1) with a having the required properties (a)–(c).

The following are examples of N-functions:

$$A(t) = t^p, \qquad 1 < p < \infty,$$
$$A(t) = e^t - t - 1,$$
$$A(t) = e^{(t^p)} - 1, \qquad 1 < p < \infty,$$
$$A(t) = (1+t)\log(1+t) - t.$$

Evidently $A(t)$ is represented by the area under the graph $\sigma = a(\tau)$ from $\tau = 0$ to $\tau = t$ as shown (Fig. 8). Rectilinear segments in the graph of A correspond to intervals of constancy of a, and angular points in the graph of A correspond to discontinuities (i.e., vertical jumps) in the graph of a.

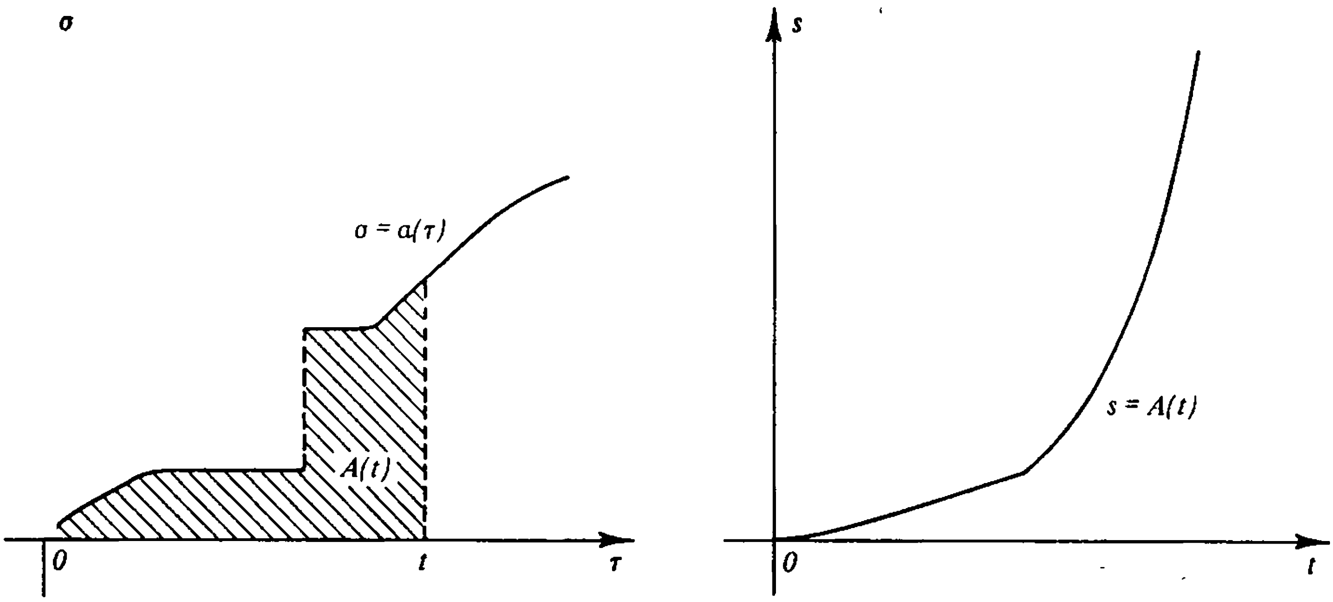


FIG. 8

8.3 Given a satisfying (a)–(c), we define

$$\tilde{a}(s) = \sup_{a(t) \le s} t. \tag{2}$$

It is readily checked that the function a so defined also satisfies (a)–(c) and that a can be recovered from $\tilde{a}$ via

$$a(t) = \sup_{\tilde{a}(s) \le t} s. \tag{3}$$

(If a is strictly increasing, then $\tilde{a} = a^{-1}$.) The N-functions A and $\tilde{A}$ given by

$$A(t) = \int_0^t a(\tau)\, d\tau, \qquad \tilde{A}(s) = \int_0^s \tilde{a}(\sigma)\, d\sigma \tag{4}$$

are said to be *complementary*; each is the complement of the other. Examples of such complementary pairs are:

$$A(t) = t^p/p, \qquad \tilde{A}(s) = s^{p'}/p', \qquad 1 < p < \infty, \quad (1/p) + (1/p') = 1;$$

$$A(t) = e^t - t - 1, \qquad \tilde{A}(s) = (1+s)\log(1+s) - s.$$

$\tilde{A}(s)$ is represented by the area to the left of the graph $\sigma = a(\tau)$ [or more correctly $\tau = \tilde{a}(\sigma)$] from $\sigma = 0$ to $\sigma = s$ as shown in Fig. 9. Evidently we have

$$st \le A(t) + \tilde{A}(s), \tag{5}$$

which is known as *Young's inequality*. Equality holds in (5) if and only if either $t = \tilde{a}(s)$ or $s = a(t)$. Writing (5) in the form

$$\tilde{A}(s) \geq st - A(t)$$

and noting that equality occurs when $t = \tilde{a}(s)$, we have

$$\tilde{A}(s) = \max_{t \geq 0}(st - A(t)).$$

This relationship could have been used as the definition of the N-function $\tilde{A}$ complementary to A.

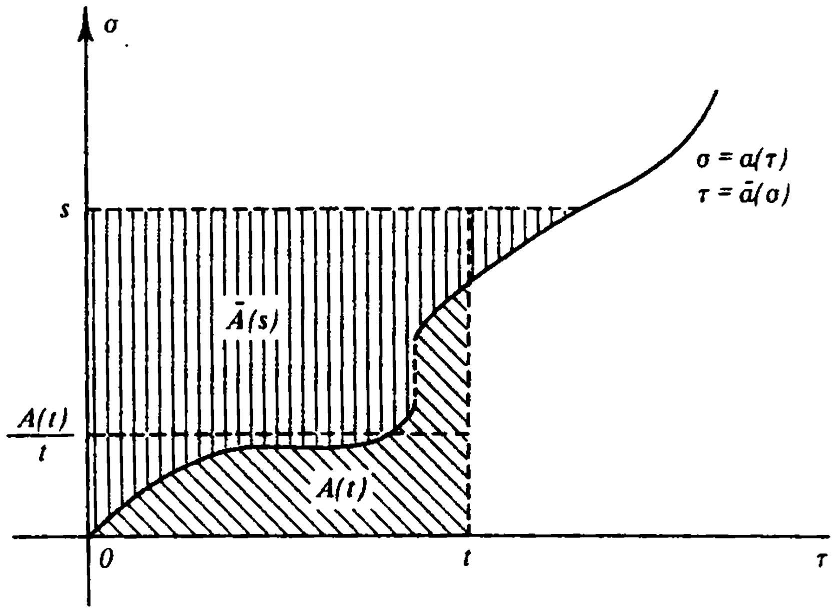


FIG. 9

Since A and $\tilde{A}$ are strictly increasing they have inverses and (5) implies that for every $t \geq 0$

$$A^{-1}(t)\tilde{A}^{-1}(t) \leq A(A^{-1}(t)) + \tilde{A}(\tilde{A}^{-1}(t)) = 2t.$$

On the other hand, $A(t) \leq ta(t)$ so that, considering Fig. 9 again, we have for $t > 0$

$$\tilde{A}(A(t)/t) < (A(t)/t)t = A(t). \tag{6}$$

Replacing $A(t)$ by t in (6), we obtain

$$\tilde{A}(t/A^{-1}(t)) < t.$$

Hence, for any $t > 0$ we have

$$t < A^{-1}(t)\,\tilde{A}^{-1}(t) \leq 2t. \tag{7}$$

8.4 We shall require certain partial-ordering relationships among N-functions. If A and B are two N-functions, we say that *B dominates A globally* provided there exists a positive constant k such that

$$A(t) \leq B(kt) \tag{8}$$

holds for all $t \geq 0$. Similarly *B dominates A near infinity* if there exist positive constants k and t_0 such that (8) holds for all $t \geq t_0$. The two N-functions A and B are *equivalent globally* (or *near infinity*) if each dominates the other globally (or near infinity). Thus A and B are equivalent near infinity if and only if there exist positive constants k_1, k_2, and t_0 such that if $t \geq t_0$, then $B(k_1 t) \leq A(t) \leq B(k_2 t)$. Such will certainly be the case if

$$0 < \lim_{t \to \infty} \frac{B(t)}{A(t)} < \infty.$$

If A and B have respective complementary N-functions $\tilde{A}$ and $\tilde{B}$, then B dominates A globally (or near infinity) if and only if $\tilde{A}$ dominates $\tilde{B}$ globally (or near infinity). Similarly A and B are equivalent if and only if $\tilde{A}$ and $\tilde{B}$ are.

8.5 If B dominates A near infinity and A and B are not equivalent near infinity, then we say *A increases essentially more slowly than B* near infinity. This is the case if and only if for every $k > 0$

$$\lim_{t \to \infty} \frac{A(kt)}{B(t)} = 0. \tag{9}$$

The reader may verify that (9) is in turn equivalent to the condition

$$\lim_{t \to \infty} \frac{B^{-1}(t)}{A^{-1}(t)} = 0.$$

Let $1 < p < \infty$. We shall hereafter denote by A_p the N-function

$$A_p(t) = t^p/p, \qquad 0 \leq t < \infty. \tag{10}$$

If $1 < p < q < \infty$, then A_p increases essentially more slowly than A_q near infinity. However, A_q does not dominate A_p globally.

8.6 An N-function A is said to satisfy a *global Δ_2-condition* if there exists a positive constant k such that for every $t \geq 0$,

$$A(2t) \leq kA(t). \tag{11}$$

It is readily seen that this will be the case if and only if for every $r > 1$ there exists a positive constant $k = k(r)$ such that for all $t \geq 0$

$$A(rt) \leq kA(t) \tag{12}$$

Similarly A is said to satisfy a Δ_2-*condition near infinity* if there exists $t_0 > 0$ such that (11) [or equivalently (12) with $r > 1$] holds for all $t \geq t_0$. Evidently t_0 may be replaced by any smaller positive number t_1, for if $t_1 \leq t \leq t_0$, then

$$A(rt) \leq [A(rt_0)/A(t_1)]\,A(t).$$

If A satisfies a Δ_2-condition globally (or near infinity) and if B is equivalent to A globally (or near infinity), then B also satisfies such a Δ_2-condition. Clearly the N-function $A_p(t) = t^p/p$, $1 < p < \infty$, satisfies a global Δ_2-condition. It may be verified that A satisfies a Δ_2-condition globally (or near infinity) if and only if there exists a finite constant c such that

$$(1/c)\,t a(t) \leq A(t) \leq t a(t)$$

holds for all $t \geq 0$ (or for all $t \geq t_0 > 0$) where A is given by (1).

Orlicz Spaces

8.7 Let Ω be a domain in $\mathbb{R}^n$ and let A be an N-function. The *Orlicz class* $K_A(\Omega)$ is the set of all (equivalence classes modulo equality a.e. in Ω of) measurable functions u defined on Ω and satisfying

$$\int_\Omega A(|u(x)|)\,dx < \infty.$$

Since A is convex $K_A(\Omega)$ is always a convex set of functions but it may not be a vector space; for instance there may exist $u \in K_A(\Omega)$ and $\lambda > 0$ such that $\lambda u \notin K_A(\Omega)$.

We call the pair (A, Ω) Δ-*regular* if either

(a) A satisfies a global Δ_2-condition, or
(b) A satisfies a Δ_2-condition near infinity and Ω has finite volume.

8.8 LEMMA $K_A(\Omega)$ is a vector space (under pointwise addition and scalar multiplication) if and only if (A, Ω) is Δ-regular.

PROOF Since A is convex we have:

(i) $\lambda u \in K_A(\Omega)$ provided $u \in K_A(\Omega)$ and $|\lambda| \leq 1$, and
(ii) if $u \in K_A(\Omega)$ implies $\lambda u \in K_A(\Omega)$ for every complex λ, then $u, v \in K_A(\Omega)$ implies $u + v \in K_A(\Omega)$.

It follows that $K_A(\Omega)$ is a vector space if and only if $u \in K_A(\Omega)$ and $|\lambda| > 1$ implies $\lambda u \in K_A(\Omega)$.

If A satisfies a global Δ_2-condition and $|\lambda| > 1$, then we have by (12) for

$u \in K_A(\Omega)$

$$\int_\Omega A(|\lambda u(x)|)\,dx \le k(|\lambda|)\int_\Omega A(|u(x)|) < \infty.$$

Similarly, if A satisfies a Δ_2-condition near infinity and $\operatorname{vol}\Omega < \infty$, we have for $|\lambda| > 1$, $u \in K_A(\Omega)$, and some $t_0 > 0$

$$\int_\Omega A(|\lambda u(x)|)\,dx = \left(\int_{\{x:|u(x)|\ge t_0\}} + \int_{\{x:|u(x)|<t_0\}}\right)(|\lambda u(x)|)\,dx$$
$$\le k(|\lambda|)\int_\Omega A(|u(x)|)\,dx + A(|\lambda|\,t_0)\operatorname{vol}\Omega < \infty.$$

In either case $K_A(\Omega)$ is seen to be a vector space.

Now suppose that (A, Ω) is not Δ-regular and, if $\operatorname{vol}\Omega < \infty$, that $t_0 > 0$ is given. There exists a sequence $\{t_j\}$ of positive numbers such that

(i) $A(2t_j) \ge 2^j A(t_j)$, and
(ii) $t_j \ge t_0 > 0$ if $\operatorname{vol}\Omega < \infty$.

Let $\{\Omega_j\}$ be a sequence of mutually disjoint, measurable subsets of Ω such that

$$\operatorname{vol}\Omega_j = \begin{cases} 1/2^j A(t_j) & \text{if } \operatorname{vol}\Omega = \infty \\ A(t_0)\operatorname{vol}\Omega/2^j A(t_j) & \text{if } \operatorname{vol}\Omega < \infty. \end{cases}$$

Let

$$u(x) = \begin{cases} t_j & \text{if } x \in \Omega_j \\ 0 & \text{if } x \in \Omega \sim \left(\bigcup_{j=1}^{\infty}\Omega_j\right). \end{cases}$$

Then

$$\int_\Omega A(|u(x)|)\,dx = \sum_{j=1}^{\infty} A(t_j)\operatorname{vol}\Omega_j$$
$$= \begin{cases} 1 & \text{if } \operatorname{vol}\Omega = \infty \\ A(t_0)\operatorname{vol}\Omega & \text{if } \operatorname{vol}\Omega < \infty. \end{cases}$$

But

$$\int_\Omega A(|2u(x)|) \ge \sum_{j=1}^{\infty} 2^j A(t_j)\operatorname{vol}\Omega_j = \infty.$$

Thus $K_A(\Omega)$ is not a vector space. ∎

8.9 The *Orlicz space* $L_A(\Omega)$ is defined to be the linear hull of the Orlicz class $K_A(\Omega)$, that is, the smallest vector space (under pointwise addition and scalar multiplication) containing $K_A(\Omega)$. Evidently $L_A(\Omega)$ consists of all scalar multiples λu of elements $u \in K_A(\Omega)$. Thus $K_A(\Omega) \subset L_A(\Omega)$; these sets being equal if and only if (A, Ω) is Δ-regular.

The reader may verify that the functional

$$\|u\|_A = \|u\|_{A,\Omega} = \inf\left\{k > 0 : \int_\Omega A\left(\frac{|u(x)|}{k}\right) dx \le 1\right\} \tag{13}$$

is a norm on $L_A(\Omega)$. (This norm is due to Luxemburg [42].) For $\|u\|_A > 0$ the infimum in (13) is attained; in fact, letting k decrease toward $\|u\|_A$ in the inequality

$$\int_\Omega A\left(\frac{|u(x)|}{k}\right) dx \le 1, \tag{14}$$

we obtain by monotone convergence

$$\int_\Omega A\left(\frac{|u(x)|}{\|u\|_A}\right) dx \le 1. \tag{15}$$

Equality may fail to hold in (15) but if equality holds in (14), then $k = \|u\|_A$.

8.10 THEOREM $L_A(\Omega)$ is a Banach space with respect to the norm (13).

The completeness proof is quite similar to that for the space $L^p(\Omega)$ given in Theorem 2.10. The details are left to the reader. [We remark that if $1 < p < \infty$ and A_p is given by (10), then

$$L^p(\Omega) = L_{A_p}(\Omega) = K_{A_p}(\Omega).$$

Moreover, $\|u\|_{A_p,\Omega} = p^{-1/p}\|u\|_{p,\Omega}$.]

8.11 If A and $\tilde{A}$ are complementary N-functions, a generalized version of Hölder's inequality

$$\left|\int_\Omega u(x)\,v(x)\,dx\right| \le 2\|u\|_{A,\Omega}\|v\|_{\tilde{A},\Omega} \tag{16}$$

can be obtained by applying Young's inequality (5) to $|u(x)|/\|u\|_A$ and $|v(x)|/\|v\|_{\tilde{A}}$ and integrating over Ω.

The following elementary imbedding theorem is an analog for Orlicz spaces of Lemma 2.8.

8.12 THEOREM The imbedding $L_B(\Omega) \to L_A(\Omega)$ holds if and only if either

(a) B dominates A globally, or
(b) B dominates A near infinity and $\operatorname{vol}\Omega < \infty$.

Proof If $A(t) \le B(kt)$ for all $t \ge 0$, and if $u \in L_B(\Omega)$, then

$$\int_\Omega A\left(\frac{|u(x)|}{k\,\|u\|_B}\right) dx \le \int_\Omega B\left(\frac{|u(x)|}{\|u\|_B}\right) dx \le 1.$$

Thus $u \in L_A(\Omega)$ and $\|u\|_A \le k\,\|u\|_B$.

If $\operatorname{vol}\Omega < \infty$, let $t_1 = A^{-1}((2\operatorname{vol}\Omega)^{-1})$. If B dominates A near infinity, then there exist positive t_0 and k such that $A(t) \le B(kt)$ for $t \ge t_0$. Evidently we have for $t \ge t_1$

$$A(t) \le K_1 B(kt),$$

where $K_1 = \max(1, A(t_0)/B(kt_1))$. If $u \in L_B(\Omega)$ is given, let $\Omega'(u) = \{x \in \Omega : |u(x)/2K_1 k\,\|u\|_B < t_1\}$ and $\Omega''(u) = \Omega \sim \Omega'(u)$. Then

$$\begin{aligned}\int_\Omega A\left(\frac{|u(x)|}{2K_1 k\,\|u\|_B}\right) dx &= \left(\int_{\Omega'(u)} + \int_{\Omega''(u)}\right) A\left(\frac{|u(x)|}{2K_1 k\,\|u\|_B}\right) dx \\ &\le \frac{1}{2\operatorname{vol}\Omega}\int_{\Omega'(u)} dx + K_1 \int_{\Omega''(u)} B\left(\frac{|u(x)|}{2K_1\|u\|_B}\right) dx \\ &\le \frac{1}{2} + \frac{1}{2}\int_\Omega B\left(\frac{|u(x)|}{\|u\|_B}\right) \le 1.\end{aligned}$$

Thus $u \in L_A(\Omega)$ and $\|u\|_A \le 2kK_1\,\|u\|_B$.

Conversely, suppose neither hypothesis (a) nor (b) holds. Then there exist points $t_j > 0$ such that

$$A(t_j) \ge B(jt_j), \qquad j = 1, 2, \ldots.$$

If $\operatorname{vol}\Omega < \infty$, we may assume in addition that

$$t_j \ge (1/j)\,B^{-1}(1/\operatorname{vol}\Omega).$$

Let Ω_j be a subdomain of Ω having volume $1/B(jt_j)$, and let

$$u_j(x) = \begin{cases} jt_j & \text{if } x \in \Omega_j \\ 0 & \text{if } x \in \Omega \sim \Omega_j. \end{cases}$$

Then

$$\int_\Omega A(|u_j(x)|/j)\,dx \ge \int_\Omega B(|u_j(x)|)\,dx = 1$$

so that $\|u_j\|_B = 1$ but $\|u_j\|_A \ge j$. Thus $L_B(\Omega)$ is not imbedded in $L_A(\Omega)$. ∎

8.13 A sequence $\{u_j\}$ of functions in $L_A(\Omega)$ is said to *converge in mean* to $u \in L_A(\Omega)$ if

$$\lim_{j\to\infty} \int_\Omega A(|u_j(x) - u(x)|)\,dx = 0.$$

Convexity of A implies that for $0 < \varepsilon \le 1$ we have

$$\int_\Omega A(|u_j(x)-u(x)|)\,dx \le \varepsilon \int_\Omega A(|u_j(x)-u(x)|/\varepsilon)\,dx$$

from which it is clear that norm convergence in $L_A(\Omega)$ implies mean convergence. The converse holds, that is, mean convergence implies norm convergence if and only if the pair (A, Ω) is Δ-regular. The proof is similar to that of Lemma 8.8 and is left to the reader.

8.14 Let $E_A(\Omega)$ denote the closure in $L_A(\Omega)$ of the space of functions u which are bounded on Ω and have bounded support in $\bar{\Omega}$. If $u \in K_A(\Omega)$, the sequence $\{u_j\}$ given by

$$u_j(x) = \begin{cases} u(x) & \text{if } \ |u(x)| \le j \ \text{ and } \ |x| \le j, \ \ x \in \Omega \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

converges a.e. on Ω to u. Since $A(|u(x)-u_j(x)|) \le A(|u(x)|)$, we have by dominated convergence that u_j converges to u in mean in $L_A(\Omega)$. Therefore if (A, Ω) is Δ-regular, then $E_A(\Omega) = K_A(\Omega) = L_A(\Omega)$. If (A, Ω) is not Δ-regular, we have

$$E_A(\Omega) \subset K_A(\Omega) \subsetneqq L_A(\Omega) \tag{18}$$

so that $E_A(\Omega)$ is a proper closed subspace of $L_A(\Omega)$ in this case. To verify the first inclusion of (18) let $u \in E_A(\Omega)$ be given. Let v be a bounded function with bounded support such that $\|u-v\|_A < \frac{1}{2}$. Using the convexity of A and (15), we obtain

$$\frac{1}{\|2u-2v\|_A}\int_\Omega A(|2u(x)-2v(x)|)\,dx \le \int_\Omega A\left(\frac{|2u(x)-2v(x)|}{\|2u-2v\|_A}\right)dx \le 1,$$

whence $2u-2v \in K_A(\Omega)$. Since $2v$ clearly belongs to $K_A(\Omega)$ and $K_A(\Omega)$ is convex we have $u = \frac{1}{2}(2u-2v)+\frac{1}{2}(2v)$ belongs to $K_A(\Omega)$.

8.15 LEMMA $E_A(\Omega)$ is the maximal linear subspace of $K_A(\Omega)$.

PROOF Let S be a linear subspace of $K_A(\Omega)$ and let $u \in S$. Then $\lambda u \in K_A(\Omega)$ for every scalar λ. If $\varepsilon > 0$ and u_j is given by (17), then u_j/ε converges to u/ε in mean in $L_A(\Omega)$ as noted in Section 8.14. Hence for sufficiently large values of j

$$\int_\Omega A\,(|u_j(x)-u(x)|/\varepsilon)\,dx \le 1$$

and therefore u_j converges to u in $L_A(\Omega)$. Thus $S \subset E_A(\Omega)$. ∎

8.16 THEOREM Let Ω have finite volume and suppose that the N-function A increases essentially more slowly than B near infinity. Then

$$L_B(\Omega) \to E_A(\Omega).$$

PROOF Since $L_B(\Omega) \to L_A(\Omega)$ is already established we need only show that $L_B(\Omega) \subset E_A(\Omega)$. Since $L_B(\Omega)$ is the linear hull of $K_B(\Omega)$ and $E_A(\Omega)$ is the maximal linear subspace of $K_A(\Omega)$, it is sufficient to show that $\lambda u \in K_A(\Omega)$ whenever $u \in K_B(\Omega)$ and λ is a scalar. But there exists a positive number t_0 such that $A(|\lambda| t) \le B(t)$ for all $t \ge t_0$. Thus

$$\begin{aligned}\int_\Omega A(|\lambda u(x)|)\, dx &= \left\{\int_{\{x:\,|u(x)| \le t_0\}} + \int_{\{x:\,|u(x)| > t_0\}}\right\} A(|\lambda u(x)|)\, dx \\ &\le A(|\lambda| t_0) \operatorname{vol}\Omega + \int_\Omega B(|u(x)|)\, dx < \infty\end{aligned}$$

whence the theorem follows. ∎

Duality in Orlicz Spaces

8.17 LEMMA For fixed $v \in L_{\tilde{A}}(\Omega)$ the linear functional L_v defined by

$$L_v(u) = \int_\Omega u(x)\, v(x)\, dx \tag{19}$$

belongs to $[L_A(\Omega)]'$. Denoting by $\|L_v\|$ its norm in that space, we have

$$\|v\|_{\tilde{A}} \le \|L_v\| \le 2\|v\|_{\tilde{A}}. \tag{20}$$

PROOF It follows by Hölder's inequality (16) that if $u \in L_A(\Omega)$, then

$$|L_v(u)| \le 2\|u\|_A \|v\|_{\tilde{A}}.$$

Thus L_v is bounded on $L_A(\Omega)$ and the second inequality of (20) holds.

To establish the first inequality we may assume that $v \ne 0$ in $L_{\tilde{A}}(\Omega)$ so that $\|L_v\| = K > 0$. Let

$$u(x) = \begin{cases} \tilde{A}\left(\dfrac{|v(x)|}{K}\right) \Big/ \dfrac{v(x)}{K} & \text{if } v(x) \ne 0 \\ 0 & \text{if } v(x) = 0. \end{cases}$$

If $\|u\|_A > 1$, then for sufficiently small $\varepsilon > 0$ we have

$$\frac{1}{\|u\|_A - \varepsilon} \int_\Omega A(|u(x)|)\, dx \ge \int_\Omega A\left(\frac{|u(x)|}{\|u\|_A - \varepsilon}\right) > 1.$$

Letting $\varepsilon \to 0+$ we obtain, using inequality (6),

$$\|u\|_A \le \int_\Omega A(|u(x)|)\,dx = \int_\Omega A\left(\tilde{A}\left(\frac{|v(x)|}{K}\right)\Big/\frac{|v(x)|}{K}\right)dx$$
$$< \int_\Omega \tilde{A}\left(\frac{v(x)}{K}\right)dx = \frac{1}{\|L_v\|}\int_\Omega u(x)v(x)\,dx \le \|u\|_A.$$

This contradiction shows that $\|u\|_A \le 1$. Now

$$\|L_v\| = \sup_{\|u\|_A \le 1} |L_v(u)| \ge \|L_v\|\left|\int_\Omega \tilde{A}\left(\frac{|v(x)|}{\|L_v\|}\right)dx\right|$$

so that

$$\int_\Omega \tilde{A}\left(\frac{|v(x)|}{\|L_v\|}\right)dx \le 1. \tag{21}$$

Thus $\|v\|_{\tilde{A}} \le \|L_v\|$. ∎

We remark that the lemma holds also when L_v is restricted to act on $E_A(\Omega)$. To obtain the first inequality of (20) in this case take $\|L_v\|$ to be the norm of L_v in $[E_A(\Omega)]'$ and replace u in the above proof by $\chi_n u$ where χ_n is the characteristic function of $\Omega_n = \{x \in \Omega : |x| \le n \text{ and } |u(x)| \le n\}$. Evidently $\chi_n u$ belongs to $E_A(\Omega)$, $\|\chi_n u\|_A \le 1$, and (21) becomes

$$\int_\Omega \chi_n(x)\,\tilde{A}\left(\frac{|v(x)|}{\|L_v\|}\right)dx \le 1.$$

Since $\chi_n(x)$ increases to unity a.e. in Ω we obtain by monotone convergence

$$\int_\Omega \tilde{A}\left(\frac{|v(x)|}{\|L_v\|}\right)dx \le 1$$

so that $\|v\|_{\tilde{A}} \le \|L_v\|$ as before.

8.18 THEOREM The dual space $[E_A(\Omega)]'$ of $E_A(\Omega)$ is isomorphic and homeomorphic to $L_{\tilde{A}}(\Omega)$.

PROOF We have already shown that any element $v \in L_{\tilde{A}}(\Omega)$ determines via (19) a bounded linear functional on $L_A(\Omega)$, and so also on $E_A(\Omega)$, having in either case norm differing from $\|v\|_{\tilde{A}}$ by at most a factor of 2. It remains to be shown that every bounded linear functional on $E_A(\Omega)$ is of the form L_v for some such v.

Let $L \in [E_A(\Omega)]'$ be given. We define a complex measure λ on the measurable subsets of Ω having finite volume by setting

$$\lambda(S) = L(\chi_S),$$

χ_S being the characteristic function of S. Since

$$\int_\Omega A(|\chi_S(x)|A^{-1}(1/\operatorname{vol} S))\,dx = \int_S (1/\operatorname{vol} S)\,dx = 1 \tag{22}$$

we have

$$|\lambda(S)| \le \|L\|\,\|\chi_S\|_A = \|L\|\,[1/A^{-1}(1/\operatorname{vol} S)].$$

Since the right side tends to zero with $\operatorname{vol} S$, the measure λ is absolutely continuous with respect to Lebesgue measure and by the Radon–Nikodym theorem 1.47, λ may be expressed in the form

$$\lambda(S) = \int_S v(x)\,dx,$$

where v is integrable on Ω. Thus

$$L(u) = \int_\Omega u(x)\,v(x)\,dx$$

holds for measurable, simple functions u.

If $u \in E_A(\Omega)$ a sequence of measurable, simple functions u_j converging a.e. to u can be found such that $|u_j(x)| \le |u(x)|$ on Ω. Since $|u_j(x)\,v(x)|$ converges a.e. to $|u(x)\,v(x)|$, Fatou's Lemma 1.44 yields

$$\begin{aligned}\left|\int_\Omega u(x)\,v(x)\,dx\right| &\le \sup_j \int_\Omega |u_j(x)\,v(x)|\,dx = \sup_j |L(|u_j|\operatorname{sgn} v)| \\ &\le \|L\| \sup_j \|u_j\|_A \le \|L\|\,\|u\|_A.\end{aligned}$$

It follows that the linear functional

$$L_v(u) = \int_\Omega u(x)\,v(x)\,dx$$

is bounded on $E_A(\Omega)$ whence $v \in L_{\tilde{A}}(\Omega)$ by the remark following Lemma 8.17. Since L_v and L assume the same values on the measurable simple functions, a set which [see the proof of Theorem 8.20(a)] is dense in $E_A(\Omega)$, they agree on $E_A(\Omega)$ and the theorem is proved. ∎

A simple application of the Hahn–Banach extension theorem shows that if $E_A(\Omega)$ is a proper subspace of $L_A(\Omega)$ [that is, if (A, Ω) is not Δ-regular], then there exists a bounded linear functional L on $L_A(\Omega)$ not given by (19) for any $v \in L_{\tilde{A}}(\Omega)$. We have as an immediate consequence the following theorem.

8.19 THEOREM $L_A(\Omega)$ **is reflexive if and only if both** (A,Ω) **and** $(\tilde{A},\Omega)$ **are Δ-regular.**

We omit any discussion of uniform convexity for Orlicz spaces. This subject is treated in Luxemburg's thesis [42].

Separability and Compactness Theorems

We next generalize the approximation Theorems 2.13, 2.15, and 2.19.

8.20 THEOREM (a) $C_0(\Omega)$ is dense in $E_A(\Omega)$.

(b) $E_A(\Omega)$ is separable.

(c) If J_ε is the mollifier introduced in Section 2.17, then for each $u \in E_A(\Omega)$ we have $\lim_{\varepsilon\to 0+} J_\varepsilon * u = u$ in $E_A(\Omega)$.

(d) $C_0^\infty(\Omega)$ is dense in $E_A(\Omega)$.

PROOF Part (a) is proved by the same method used in Theorem 2.13. In approximating $u \in E_A(\Omega)$ first by simple functions we may assume that u is bounded on Ω and has bounded support. This is required for the dominated convergence argument used to show that the simple functions converge in norm to u in $E_A(\Omega)$. (The details are left to the reader.)

Part (b) follows from part (a) by the same proof as given for Theorem 2.15.

Consider (c). If $u \in E_A(\Omega)$, let u be extended to $\mathbb{R}^n$ so as to vanish identically outside Ω. Let $v \in L_{\tilde{A}}(\Omega)$. Then

$$\left|\int_\Omega (J_\varepsilon * u(x) - u(x))v(x)\,dx\right| \le \int_{\mathbb{R}^n} J(y)\,dy \int_\Omega |u(x-\varepsilon y) - u(x)||v(x)|\,dx$$
$$\le 2\|v\|_{\tilde{A},\Omega} \int_{|y|\le 1} J(y)\,\|u_{\varepsilon y} - u\|_{A,\Omega}\,dy$$

by Hölder's inequality (16), where $u_{\varepsilon y}(x) = u(x-\varepsilon y)$. Thus by (20) and Theorem 8.18,

$$\|J_\varepsilon * u - u\|_{A,\Omega} = \sup_{\|v\|_{\tilde{A},\Omega}\le 1} \left|\int_\Omega (J_\varepsilon * u(x) - u(x))v(x)\,dx\right|$$
$$\le 2\int_{|y|\le 1} J(y)\,\|u_{\varepsilon y} - u\|_{A,\Omega}\,dy.$$

Given $\delta > 0$ we can find $\tilde{u} \in C_0(\Omega)$ such that $\|u-\tilde{u}\|_{A,\Omega} < \delta/6$. Evidently $\|u_{\varepsilon y} - \tilde{u}_{\varepsilon y}\|_{A,\Omega} < \delta/6$ and for sufficiently small ε, $\|\tilde{u}_{\varepsilon y} - \tilde{u}\|_{A,\Omega} < \delta/6$ for every y with $|y| \le 1$. Thus $\|J_\varepsilon * u - u\|_{A,\Omega} < \delta$ and (c) is established.

Part (d) is an immediate consequence of (a) and (c). ∎

We remark that $L_A(\Omega)$ is not separable unless $L_A(\Omega) = E_A(\Omega)$, that is, unless (A, Ω) is Δ-regular. A proof of this fact may be found in the work of Krasnosel'skii and Rutickii [34, Chapter II, Theorem 10.2].

8.21 A sequence u_j of measurable functions is said to *converge in measure* on Ω to the function u provided that for each $\varepsilon > 0$ and $\delta > 0$ there exists an integer M such that if $j > M$, then

$$\operatorname{vol}\{x \in \Omega : |u_j(x) - u(x)| > \varepsilon\} \leq \delta.$$

Clearly, in this case there also exists an integer N such that if j, $k \geq N$, then

$$\operatorname{vol}\{x \in \Omega : |u_j(x) - u_k(x)| \geq \varepsilon\} \leq \delta.$$

8.22 THEOREM Let Ω have finite volume and suppose that the N-function B increases essentially more slowly than A near infinity. If the sequence $\{u_j\}$ is bounded in $L_A(\Omega)$ and convergent in measure on Ω, then it is convergent in $L_B(\Omega)$.

PROOF Fix $\varepsilon > 0$ and let $v_{j,k}(x) = [u_j(x) - u_k(x)]/\varepsilon$. Clearly $\{v_{j,k}\}$ is bounded in $L_A(\Omega)$; say $\|v_{j,k}\|_{A,\Omega} \leq K$. Now there exists a positive number t_0 such that if $t > t_0$, then

$$B(t) \leq \tfrac{1}{4} A(t/K).$$

Let $\delta = 1/4B(t_0)$ and set

$$\Omega_{j,k} = \{x \in \Omega : |v_{j,k}(x)| \geq B^{-1}(1/2 \operatorname{vol} \Omega)\}.$$

Since $\{u_j\}$ converges in measure there exists an integer N such that if j, $k \geq N$, then $\operatorname{vol} \Omega_{j,k} \leq \delta$. Set

$$\Omega'_{j,k} = \{x \in \Omega_{j,k} : |v_{j,k}(x)| \geq t_0\}, \qquad \Omega''_{j,k} = \Omega_{j,k} \sim \Omega'_{j,k}.$$

For j, $k \geq M$ we have

$$\begin{aligned}\int_\Omega B(|v_{j,k}(x)|)\,dx &= \left(\int_{\Omega \sim \Omega_{j,k}} + \int_{\Omega'_{j,k}} + \int_{\Omega''_{j,k}}\right) B(|v_{j,k}(x)|)\,dx \\ &\leq \frac{\operatorname{vol}\Omega}{2\operatorname{vol}\Omega} + \frac{1}{4}\int_{\Omega'_{j,k}} A\left(\frac{|v_{j,k}(x)|}{K}\right) dx + \delta B(t_0) \leq 1.\end{aligned}$$

Hence $\|u_j - u_k\|_{B,\Omega} \leq \varepsilon$ and so $\{u_j\}$ converges in $L_B(\Omega)$. ∎

The following theorem will prove useful when we wish to extend the Rellich–Kondrachov Theorem 6.2 to imbeddings of Orlicz–Sobolev spaces.

8.23 THEOREM Let Ω have finite volume and suppose that the N-function B increases essentially more slowly than A near infinity. Then any bounded subset S of $L_A(\Omega)$ which is precompact in $L^1(\Omega)$ is also precompact in $L_B(\Omega)$.

Proof Evidently $L_A(\Omega) \to L^1(\Omega)$ since Ω has finite volume. If $\{u_j^*\}$ is a sequence in S, it has a subsequence $\{u_j\}$ convergent in $L^1(\Omega)$; say $u_j \to u$ in $L^1(\Omega)$. Let $\varepsilon, \delta > 0$. Then there exists an integer N such that if $j \ge N$, then $\|u_j - u\|_{1,\Omega} \le \varepsilon\delta$. It follows that $\operatorname{vol}\{x \in \Omega : |u_j(x) - u(x)| \ge \varepsilon\} \le \delta$ so $\{u_j\}$ converges in measure on Ω and hence also in $L_B(\Omega)$. ∎

A Limiting Case of the Sobolev Imbedding Theorem

8.24 If $mp = n$ and $p > 1$, the Sobolev imbedding Theorem 5.4 provides no best (i.e., smallest) target space into which $W^{m,p}(\Omega)$ can be imbedded. In fact we have in this case, for suitably regular Ω,

$$W^{m,p}(\Omega) \to L^q(\Omega), \qquad p \le q < \infty,$$

but (see Example 5.26)

$$W^{m,p}(\Omega) \not\to L^\infty(\Omega).$$

If the class of target spaces for the imbedding is enlarged to include Orlicz spaces, then a best target space can be found. We consider first bounded domains Ω. The case $m = 1$ of the following theorem was established by Trudinger [66].

8.25 THEOREM Let Ω be a bounded domain in $\mathbb{R}^n$ having the cone property. Let $mp = n$ and $p > 1$. Set

$$A(t) = \exp[t^{n/(n-m)}] - 1 = \exp[t^{p/(p-1)}] - 1. \qquad (23)$$

Then there exists the imbedding

$$W^{m,p}(\Omega) \to L_A(\Omega).$$

Proof Let $x \in \Omega$ and let C be a finite cone contained in Ω having vertex at x. Let $u \in C^m(\bar{C})$. Applying Taylor's formula

$$f(1) = \sum_{j=0}^{m-1} \frac{f^{(j)}(0)}{j!} + \frac{1}{(m-1)!} \int_0^1 (1-t)^{m-1} f^{(m)}(t)\, dt$$

to the function $f(t) = u(y + t(x-y))$, and noting that

$$f^{(j)}(t) = \sum_{|\alpha| = j} \frac{j!}{\alpha!} D^\alpha u(y + t(x-y))(x-y)^\alpha,$$

we obtain

$$|u(x)| \le \sum_{|\alpha|\le m-1} \frac{1}{\alpha!}|D^\alpha u(y)||x-y|^{|\alpha|}$$
$$+ \sum_{|\alpha|=m} \frac{m}{\alpha!}|x-y|^m \int_0^1 (1-t)^{m-1}|D^\alpha u(y+t(x-y))|\,dt.$$

Let V be the volume and h the height of C. Let (ρ,θ) denote spherical polar coordinates of $y \in C$ referred to x as origin so that C is specified by $0<\rho<h$, $\theta \in \Sigma$, and the volume element dy can be written in the form $\rho^{n-1}\omega(\theta)\,d\rho\,d\theta$. Then

$$|u(x)| = \frac{1}{V}\int_C |u(x)|\,dy \le \frac{1}{V}\sum_{|\alpha|\le m-1}\frac{h^{|\alpha|}}{\alpha!}\int_C |D^\alpha u(y)|\,dy$$
$$+\frac{1}{V}\sum_{|\alpha|=m}\frac{m}{\alpha!}\int_\Sigma \omega(\theta)\,d\theta\int_0^h \rho^{n+m-1}\,d\rho\int_0^1 (1-t)^{m-1}|D^\alpha u((1-t)\rho,\theta)|\,dt$$
$$\le K_1\left\{\|u\|_{m-1,1,C} + \sum_{|\alpha|=m}\int_\Sigma \omega(\theta)\,d\theta\int_0^h \rho^{n-1}\,d\rho\int_0^\rho \sigma^{m-1}|D^\alpha u(\sigma,\theta)|\,d\sigma\right\}$$
$$= K_1\left\{\|u\|_{m-1,1,C} + \sum_{|\alpha|=m}\int_\Sigma \omega(\theta)\,d\theta\int_0^h \sigma^{m-1}|D^\alpha u(\sigma,\theta)|\,d\sigma\int_0^h \rho^{n-1}\,d\rho\right\}$$
$$\le K_2\left\{\|u\|_{m-1,1,C} + \sum_{|\alpha|=m}\int_C \frac{|D^\alpha u(z)|}{|z-x|^{n-m}}\,dz\right\}.$$

By density the above inequality holds for all $u \in W^{m,1}(C)$. In particular, for any $u \in W^{m,p}(\Omega)$, and for almost all $x \in \Omega$, we have

$$|u(x)| \le K_2\left\{\|u\|_{m-1,1,\Omega} + \sum_{|\alpha|=m}\int_\Omega \frac{|D^\alpha u(y)|}{|x-y|^{n-m}}\,dy\right\},$$

where K_2 depends on m, n and the height h and volume V of the cone determining the cone property for Ω.

We wish to estimate $\|u\|_{0,s}$ for arbitrary $s>1$. Accordingly, if $v \in L^{s'}(\Omega)$ where $s' = s/(s-1)$, then

$$\int_\Omega |u(x)v(x)|\,dx$$
$$\le K_2\|u\|_{m-1,1}\int_\Omega |v(x)|\,dx + K_2\sum_{|\alpha|=m}\int_\Omega\int_\Omega \frac{|D^\alpha u(y)||v(x)|}{|x-y|^{n-m}}\,dy\,dx$$
$$\le K_2\|u\|_{m-1,1}\|v\|_{0,s'}(\operatorname{vol}\Omega)^{1/s}$$

$$+ K_2 \sum_{|\alpha|=m} \left\{ \int_\Omega \int_\Omega \frac{|v(x)|}{|x-y|^{n-(m/s)}} \, dy \, dx \right\}^{1-(1/p)}$$
$$\times \left\{ \int_\Omega \int_\Omega \frac{|D^\alpha u(y)|^p |v(x)|}{|x-y|^{(n-m)/s}} \, dy \, dx \right\}^{1/p}.$$

Now we have from Lemma 5.47 that if $0 \le \nu < n$, then

$$\int_\Omega \frac{1}{|x-y|^\nu} \, dx \le K_3(\nu, n)(\operatorname{vol}\Omega)^{1-(\nu/n)}.$$

In fact a review of the proof of that lemma shows that $K_3(\nu, n) = K_4/(n-\nu)$ with K_4 depending only on n. Hence

$$\begin{aligned}\int_\Omega \int_\Omega \frac{|v(x)|}{|x-y|^{n-(m/s)}} \, dy \, dx &\le K_4 \frac{s}{m} (\operatorname{vol}\Omega)^{m/sn} \int_\Omega |v(x)| \, dx \\ &\le K_5 s (\operatorname{vol}\Omega)^{(1/sp)+(1/s)} \|v\|_{0,s'}.\end{aligned}$$

Also

$$\begin{aligned}\int_\Omega \int_\Omega \frac{|D^\alpha u(y)|^p |v(x)|}{|x-y|^{(n-m)/s}} \, dy \, dx &\le \int_\Omega |D^\alpha u(y)|^p \, dy \cdot \|v\|_{0,s'} \cdot \left\{ \int_\Omega \frac{1}{|x-y|^{n-m}} \, dx \right\}^{1/s} \\ &\le \|D^\alpha u\|_{0,p}^p \|v\|_{0,s'} (K_5 (\operatorname{vol}\Omega)^{1/p})^{1/s}.\end{aligned}$$

Hence

$$\begin{aligned}\int_\Omega |u(x)\,v(x)| \, dx &\le K_2 \|u\|_{m-1,1} \|v\|_{0,s'} (\operatorname{vol}\Omega)^{1/s} \\ &\quad + K_6 \sum_{|\alpha|=m} s^{(p-1)/p} \|D^\alpha u\|_{0,p} \|v\|_{0,s'} (\operatorname{vol}\Omega)^{1/s}.\end{aligned}$$

Since $s^{(p-1)/p} > 1$ and since $W^{m-1,1}(\Omega) \to W^{m,p}(\Omega)$ it follows that

$$\|u\|_{0,s} = \sup_{v \in L^{s'}(\Omega)} \frac{\int_\Omega |u(x)\,v(x)| \, dx}{\|v\|_{0,s'}} \le K_7 s^{(p-1)/p} (\operatorname{vol}\Omega)^{1/s} \|u\|_{m,p}.$$

The constant K_7 depends only on m, n and the cone determining the cone property for Ω. Setting $s = nk/(n-m) = pk/(p-1)$, we obtain

$$\begin{aligned}\int_\Omega |u(x)|^{pk/(p-1)} \, dx &\le \operatorname{vol}\Omega \cdot \left\{ \frac{pk}{p-1} \right\}^k \{K_7 \|u\|_{m,p}\}^{pk/(p-1)} \\ &= \operatorname{vol}\Omega \cdot \left\{ \frac{k}{e^{p/(p-1)}} \right\}^k \left\{ eK_7 \left(\frac{p}{p-1} \right)^{(p-1)/p} \|u\|_{m,p} \right\}^{pk/(p-1)}\end{aligned}$$

Since $e^{p/(p-1)} > e$, the series $\sum_{k=1}^{\infty} (1/k!)(k/e^{p/(p-1)})^k$ converges to a finite sum, say K_8. Let $K_9 = \max(1, K_8 \operatorname{vol}\Omega)$ and put

$$K_{10} = eK_9 K_7 [p/(p-1)]^{(p-1)/p} \|u\|_{m,p} = K_{11} \|u\|_{m,p}.$$

Then

$$\int_\Omega \left(\frac{|u(x)|}{K_{10}}\right)^{pk/(p-1)} dx \le \frac{\operatorname{vol}\Omega}{K_9^{pk/(p-1)}} \left(\frac{k}{e^{p/(p-1)}}\right)^k < \frac{\operatorname{vol}\Omega}{K_9} \left(\frac{k}{e^{p/(p-1)}}\right)^k$$

since $K_9 \ge 1$ and $pk/(p-1) > 1$. Expanding $A(t)$ in power series, we now obtain

$$\begin{aligned} \int_\Omega A\left(\frac{|u(x)|}{K_{10}}\right) dx &= \sum_{k=1}^{\infty} \frac{1}{k!} \int_\Omega \left(\frac{|u(x)|}{K_{10}}\right)^{pk/(p-1)} dx \\ &< \frac{\operatorname{vol}\Omega}{K_9} \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{k}{e^{p/(p-1)}}\right)^k \le 1. \end{aligned}$$

Hence $u \in L_A(\Omega)$ and

$$\|u\|_A \le K_{10} = K_{11} \|u\|_{m,p},$$

where K_{11} depends on $n, m, \operatorname{vol}\Omega$, and the cone determining the cone property for Ω. ∎

The imbedding established in the above theorem is "best possible" in the sense that if there exists any imbedding of the form

$$W_0^{m,p}(\Omega) \to L_B(\Omega),$$

then A dominates B near infinity. A proof of this fact for the case $m = 1$, $n = p > 1$ can be found in the notes of Hempel and co-workers [30]. The general case is left to the reader as an exercise.

Theorem 8.25 can be generalized to fractional-order spaces. For results in this direction the reader is referred to Grisvard [28] and Peetre [56].

8.26 If Ω is unbounded and so (having the cone property) has infinite volume, then the N-function A given by (23) may not decrease rapidly enough at zero to allow membership in $L_A(\Omega)$ of every $u \in W^{m,p}(\Omega)$ (where $mp = n$). Let k_0 be the smallest integer such that $k_0 \ge p-1$ and define a modified N-function A_0 by

$$A_0(t) = \exp(t^{p/(p-1)}) - \sum_{j=0}^{k_0-1} (1/j!)\, t^{jp/(p-1)}.$$

Evidently A_0 is equivalent to A near infinity so for any domain Ω having finite volume, $L_A(\Omega)$ and $L_{A_0}(\Omega)$ coincide and have equivalent norms. However, A_0 enjoys the further property that for $0 < r \le 1$,

$$A_0(rt) \le r^{k_0 p/(p-1)} A_0(t) \le r^p A_0(t). \tag{24}$$

We show that if $mp = n$ and Ω has the cone property (but may be unbounded), then

$$W^{m,p}(\Omega) \to L_{A_0}(\Omega).$$

As in the proof of Lemma 5.14 we may write Ω as a union of countably many subdomains Ω_j each having the cone property determined by some fixed cone independent of j, satisfying for some constants K_1 and K_2

$$0 < K_1 \le \operatorname{vol} \Omega_j \le K_2,$$

and finally such that for some positive integer R any $R+1$ of the subdomains Ω_j have empty intersection. It follows by Theorem 8.25 that if $u \in W^{m,p}(\Omega)$, then

$$\|u\|_{A_0, \Omega_j} \le K_3 \|u\|_{m,p,\Omega_j},$$

where K_3 does not depend on j. Using (24) with $r = R^{1/p} \|u\|_{m,p,\Omega_j}^{-1} \|u\|_{m,p,\Omega}$ and the finite intersection property of the domains Ω_j, we have

$$\int_\Omega A_0\left(\frac{|u(x)|}{R^{1/p} K_3 \|u\|_{m,p,\Omega}}\right) dx \le \sum_{j=1}^{\infty} \int_{\Omega_j} A_0\left(\frac{|u(x)|}{R^{1/p} K_3 \|u\|_{m,p,\Omega}}\right) dx$$

$$\le \sum_{j=1}^{\infty} \frac{\|u\|_{m,p,\Omega_j}^p}{R \|u\|_{m,p,\Omega}^p} \le 1.$$

Hence $\|u\|_{A_0,\Omega} \le R^{1/p} K_3 \|u\|_{m,p,\Omega}$ as required.

We remark that if $k_0 > p-1$, the above result can be improved slightly by using in place of A_0 the N-function $\max(t^p, A_0(t))$.

Orlicz–Sobolev Spaces

8.27 For a given domain Ω in $\mathbb{R}^n$ and a given defining N-function A the *Orlicz–Sobolev space* $W^m L_A(\Omega)$ consists of those (equivalence classes of) functions u in $L_A(\Omega)$ whose distributional derivatives $D^\alpha u$ also belong to $L_A(\Omega)$ for all α with $|\alpha| \le m$. The space $W^m E_A(\Omega)$ is defined in analogous fashion. It may be checked by the same method used in the proof of Theorem 3.2 that $W^m L_A(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{m,A} = \|u\|_{m,A,\Omega} = \max_{0 \le |\alpha| \le m} \|D^\alpha u\|_{A,\Omega}, \tag{25}$$

and that $W^m E_A(\Omega)$ is a closed subspace of $W^m L_A(\Omega)$ and hence also a Banach space under (25). It should be kept in mind that $W^m E_A(\Omega)$ coincides with $W^m L_A(\Omega)$ if and only if (A, Ω) is Δ-regular. If $1 < p < \infty$ and $A_p(t) = t^p$, then $W^m L_{A_p}(\Omega) = W^m E_{A_p}(\Omega) = W^{m,p}(\Omega)$, the latter space having norm equivalent to that of the former.

As in the case of ordinary Sobolev spaces, $W_0{}^m L_A(\Omega)$ is taken to be the closure of $C_0{}^\infty(\Omega)$ in $W^m L_A(\Omega)$. [An analogous definition for $W_0{}^m E_A(\Omega)$ clearly leads to the same space in all cases.]

Many properties of Orlicz–Sobolev spaces are obtained by very straightforward generalization of the proofs of the same properties for ordinary Sobolev spaces. We summarize some of these in the following theorem and refer the reader to the corresponding results in Chapter III for method of proof. The details can also be found in the article by Donaldson and Trudinger [22].

8.28 THEOREM (a) $W^m E_A(\Omega)$ is separable (Theorem 3.5).

(b) If (A, Ω) and $(\tilde{A}, \Omega)$ are Δ-regular, then $W^m E_A(\Omega) = W^m L_A(\Omega)$ is reflexive (Theorem 3.5).

(c) Each element L of the dual space $[W^m E_A(\Omega)]'$ is given by

$$L(u) = \sum_{0 \le |\alpha| \le m} \int_\Omega D^\alpha u(x)\, v_\alpha(x)\, dx$$

for some functions $v_\alpha \in L_{\tilde{A}}(\Omega)$, $0 \le |\alpha| \le m$ (Theorem 3.8).

(d) $C^\infty(\Omega) \cap W^m E_A(\Omega)$ is dense in $W^m E_A(\Omega)$ (Theorem 3.16).

(e) If Ω has the segment property, then $C^\infty(\bar{\Omega})$ is dense in $W^m E_A(\Omega)$ (Theorem 3.18).

(f) $C_0{}^\infty(\mathbb{R}^n)$ is dense in $W^m E_A(\mathbb{R}^n)$. Thus $W_0{}^m L_A(\mathbb{R}^n) = W^m E_A(\mathbb{R}^n)$ (Theorem 3.18).

Imbedding Theorems for Orlicz–Sobolev Spaces

8.29 Imbedding results analogous to those obtained for the spaces $W^{m,p}(\Omega)$ in Chapters V and VI can also be formulated for the Orlicz–Sobolev spaces $W^m L_A(\Omega)$ and $W^m E_A(\Omega)$. The first results in this direction were obtained by Dankert [20] and by Donaldson. A fairly general imbedding theorem along the lines of Theorems 5.4 and 6.2 was presented by Donaldson and Trudinger [22] and we develop it below.

As was the case with ordinary Sobolev spaces, most of the imbedding results are obtained for domains having the cone property. Exceptions are those yielding (generalized) Hölder continuity estimates; these require the strong local Lipschitz property. Some results below are obtained only for bounded domains. The method used in extending the analogous results for ordinary Sobolev spaces to unbounded domains (see Lemma 5.14) does not appear to extend in a straightforawrd manner when general Orlicz spaces are involved. In this sense the imbedding picture is still incomplete.

8.30 We concern ourselves for the time being with imbeddings of $W^1L_A(\Omega)$; the imbeddings of $W^mL_A(\Omega)$ are summarized in Theorem 8.40. In the following Ω is always understood to be a domain in $\mathbb{R}^n$.

Let A be a given N-function. We shall always suppose that

$$\int_0^1 \frac{A^{-1}(t)}{t^{(n+1)/n}}\, dt < \infty, \tag{26}$$

replacing, if necessary, A by another N-function equivalent to A near infinity. [If Ω has finite volume, (26) places no restrictions on A from the point of view of imbedding theory since N-functions equivalent near infinity determine identical Orlicz spaces.]

Suppose also that

$$\int_1^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}\, dt = \infty. \tag{27}$$

For instance, if $A = A_p$ given by (10), then (27) holds precisely when $p \le n$. With (27) satisfied we define the *Sobolev conjugate* A_* of A by setting

$$A_*^{-1}(t) = \int_0^t \frac{A^{-1}(\tau)}{\tau^{(n+1)/n}}\, d\tau, \qquad t \ge 0. \tag{28}$$

It may readily be checked that A_* is an N-function. If $1 < p < n$, we have, setting $q = np/(n-p)$,

$$A_{p*}(t) = q^{1-q}p^{-q/p}A_q(t).$$

It is also readily seen for the case $p = n$ that $A_{n*}(t)$ is equivalent near infinity to the N-function $e^t - t - 1$.

Before stating the first imbedding theorem we prepare the following technical lemma that will be needed in the proof.

8.31 LEMMA Let $u \in W^{1,1}_{\text{loc}}(\Omega)$ and let f satisfy a Lipschitz condition on $\mathbb{R}$. Then $g \in W^{1,1}_{\text{loc}}(\Omega)$ where $g(x) = f(|u(x)|)$, and

$$D_j g(x) = f'(|u(x)|)\,\operatorname{sgn} u(x) \cdot D_j u(x).$$

PROOF Since $|u| \in W^{1,1}_{\text{loc}}(\Omega)$ and $D_j|u(x)| = \operatorname{sgn} u(x) \cdot D_j u(x)$ it is sufficient to establish the lemma for positive, real-valued functions u so that $g(x) = f(u(x))$. Let $\phi \in \mathscr{D}(\Omega)$. Letting $\{e_j\}_{j=1}^n$ be the usual basis in $\mathbb{R}^n$, we obtain

$$-\int_\Omega f(u(x))\, D_j\phi(x)\, dx = -\lim_{h\to 0}\int_\Omega f(u(x))\frac{\phi(x)-\phi(x-he_j)}{h}\, dx$$
$$= \lim_{h\to 0}\int_\Omega \frac{f(u(x+he_j))-f(u(x))}{h}\phi(x)\, dx$$
$$= \lim_{h\to 0}\int_\Omega Q(x,h)\frac{u(x+he_j)-u(x)}{h}\phi(x)\, dx,$$

where, since f is Lipschitz, for each h the function $Q(\cdot,h)$ is defined a.e. on Ω by

$$Q(x,h) = \begin{cases} \dfrac{f(u(x+he_j))-f(u(x))}{u(x+he_j)-u(x)} & \text{if} \quad u(x+he_j) \neq u(x) \\ f'(u(x)) & \text{otherwise.} \end{cases}$$

Moreover, $\|Q(\cdot,h)\|_{\infty,\Omega} \le K$ for some constant K independent of h. A well-known theorem in functional analysis assures us that for some sequence of values of h tending to zero, $Q(\cdot,h)$ converges to $f'(u(\cdot))$ in the weak-star topology of $L^\infty(\Omega)$. On the other hand, since $u \in W^{1,1}(\operatorname{supp}\phi)$ we have

$$\lim_{h\to 0}\frac{u(x+he_j)-u(x)}{h}\phi(x) = D_j u(x)\cdot\phi(x)$$

in $L^1(\operatorname{supp}\phi)$. It follows that

$$-\int_\Omega f(u(x))\, D_j\phi(x) = \int_\Omega f'(u(x))\, D_j(x)\,\phi(x)\, dx,$$

which evidently implies the lemma. ∎

8.32 THEOREM Let Ω be bounded and have the cone property in $\mathbb{R}^n$. If (26) and (27) hold, then

$$W^1L_A(\Omega) \to L_{A_*}(\Omega).$$

Moreover, if B is any N-function increasing essentially more slowly than A_* near infinity, then the imbedding

$$W^1L_A(\Omega) \to L_B(\Omega)$$

is compact.

PROOF The function $s = A_*(t)$ as defined by (28) satisfies the differential equation

$$A^{-1}(s)\frac{ds}{dt} = s^{(n+1)/n}, \tag{29}$$

and hence, by virtue of the left inequality of (7),

$$\frac{ds}{dt} \leqslant s^{1/n}\tilde{A}^{-1}(s).$$

Therefore $\sigma(t) = [A_*(t)]^{(n-1)/n}$ satisfies the differential inequality

$$\frac{d\sigma}{dt} \leqslant \frac{n-1}{n}\tilde{A}^{-1}((\sigma(t))^{n/(n-1)}). \tag{30}$$

Let $u \in W^1L_A(\Omega)$ and suppose for the moment that u is bounded on Ω and is not zero in $L_A(\Omega)$. Then $\int_\Omega A_*(|u(x)|/\lambda)\,dx$ decreases continuously from infinity to zero as λ increases from zero to infinity, and accordingly assumes the value unity for some positive value of λ. Thus

$$\int_\Omega A_*\left(\frac{|u(x)|}{K}\right)dx = 1, \qquad K = \|u\|_{A_*}. \tag{31}$$

Let $f(x) = \sigma(|u(x)|/K)$. Evidently $u \in W^{1,1}(\Omega)$ and σ is Lipschitz on the range of $|u|/K$ so that, by Lemma 8.31, f belongs to $W^{1,1}(\Omega)$. By Theorem 5.4 we have $W^{1,1}(\Omega) \to L^{n/(n-1)}(\Omega)$ and so

$$\begin{aligned}\|f\|_{0,n/(n-1)} &\leqslant K_1\left\{\sum_{j=1}^{n}\|D_jf\|_{0,1} + \|f\|_{0,1}\right\}\\ &= K_1\left\{\sum_{j=1}^{\infty}\frac{1}{K}\int_\Omega\sigma'\left(\frac{|u(x)|}{K}\right)|D_ju(x)|\,dx + \int_\Omega\sigma\left(\frac{|u(x)|}{K}\right)dx\right\}.\end{aligned} \tag{32}$$

By (31) and Hölder's inequality (16), we obtain

$$\begin{aligned}1 &= \left\{\int_\Omega A_*\left(\frac{|u(x)|}{K}\right)dx\right\}^{(n-1)/n} = \|f\|_{0,n/(n-1)}\\ &\leqslant \frac{2K_1}{K}\sum_{j=1}^{n}\left\|\sigma'\left(\frac{|u|}{K}\right)\right\|_{\tilde{A}}\|D_ju\|_A + K_1\int_\Omega\sigma\left(\frac{|u(x)|}{K}\right)dx.\end{aligned} \tag{33}$$

Making use of (30), we have

$$\begin{aligned}\left\|\sigma'\left(\frac{|u|}{K}\right)\right\|_{\tilde{A}} &\leqslant \frac{n-1}{n}\left\|\tilde{A}^{-1}\left(\left(\sigma\left(\frac{|u|}{K}\right)\right)^{n/(n-1)}\right)\right\|_{\tilde{A}}\\ &= \frac{n-1}{n}\inf\left\{\lambda > 0 : \int_\Omega\tilde{A}\left(\frac{\tilde{A}^{-1}(A_*(|u(x)|/K))}{\lambda}\right)dx \leqslant 1\right\}.\end{aligned}$$

Suppose $\lambda > 1$. Then

$$\int_\Omega\tilde{A}\left(\frac{\tilde{A}^{-1}(A_*(|u(x)|/K))}{\lambda}\right)dx \leqslant \frac{1}{\lambda}\int_\Omega A_*\left(\frac{|u(x)|}{K}\right)dx = \frac{1}{\lambda} < 1.$$

Thus

$$\left\|\sigma'\left(\frac{|u|}{K}\right)\right\|_{\tilde{A}} \leqslant \frac{n-1}{n}. \tag{34}$$

Let $g(t) = A_*(t)/t$ and $h(t) = \sigma(t)/t$. It is readily checked that h is bounded on finite intervals and $\lim_{t\to\infty} g(t)/h(t) = \infty$. Thus there exists a constant t_0 such that if $t \geqslant t_0$, then $h(t) \leqslant g(t)/2K_1$. Putting $K_2 = K_1 \sup_{0\leqslant t\leqslant t_0} h(t)$, we have, for all $t \geqslant 0$,

$$\sigma(t) \leqslant (1/2K_1)\, A_*(t) + (K_2/K_1)\, t.$$

Hence

$$\begin{aligned} K_1 \int_\Omega \sigma\left(\frac{|u(x)|}{K}\right) dx &\leqslant \frac{1}{2}\int_\Omega A_*\left(\frac{|u(x)|}{K}\right) dx + \frac{K_2}{K}\int_\Omega \|u(x)\|\, dx \\ &\leqslant \frac{1}{2} + \frac{K_3}{K}\|u\|_A, \end{aligned} \tag{35}$$

where $K_3 = 2K_2\|1\|_{\tilde{A}} < \infty$ since Ω has finite volume.

Combining (33)–(35), we obtain

$$1 \leqslant (2K_1/K)(n-1)\|u\|_{1,A} + \tfrac{1}{2} + (K_3/K)\|u\|_A,$$

so that

$$\|u\|_{A_*} = K \leqslant K_4 \|u\|_{1,A}. \tag{36}$$

We note that K_4 can depend on $n, A, \operatorname{vol}\Omega$, and the cone determining the cone property for Ω.

To extend (36) to arbitrary $u \in W^1L_A(\Omega)$ let

$$u_k(x) = \begin{cases} u(x) & \text{if } \ |u(x)| \leqslant k \\ k \operatorname{sgn} u(x) & \text{if } \ |u(x)| > k. \end{cases} \tag{37}$$

Evidently u_k is bounded and belongs to $W^1L_A(\Omega)$ by Lemma 8.31. Moreover, $\|u_k\|_{A_*}$ increases with k but is bounded by $K_4\|u\|_{1,A}$. Thus $\lim_{k\to\infty}\|u_k\|_{A_*} = K$ exists and $K \leqslant K_4\|u\|_{1,A}$. By Fatou's lemma

$$\int_\Omega A_*\left(\frac{|u(x)|}{K}\right) dx \leqslant \lim_{k\to\infty}\int_\Omega A_*\left(\frac{|u_k(x)|}{K}\right) dx \leqslant 1$$

whence $u \in L_{A_*}(\Omega)$ and (36) holds.

Since Ω has finite volume we have

$$W^1L_A(\Omega) \to W^{1,1}(\Omega) \to L^1(\Omega),$$

the latter imbedding being compact by Theorem 6.2. A bounded subset S of $W^1L_A(\Omega)$ is bounded in $L_{A_*}(\Omega)$ and precompact in $L^1(\Omega)$, and hence precompact in $L_B(\Omega)$ by Theorem 8.23 whenever B increases essentially more slowly than A near infinity. ∎

Theorem 8.32 extends to arbitrary (even unbounded) domains Ω provided W is replaced by W_0.

8.33 THEOREM Let Ω be any domain in $\mathbb{R}^n$. If the N-function A satisfies (26) and (27), then

$$W_0{}^1L_A(\Omega) \to L_{A_*}(\Omega).$$

Moreover, if Ω_0 is a bounded subdomain of Ω, then the imbeddings

$$W_0{}^1L_A(\Omega) \to L_B(\Omega_0)$$

exist and are compact for any N-function B increasing essentially more slowly than A near infinity.

PROOF If $u \in W_0{}^1L_A(\Omega)$, then the function f in the above proof can be approximated in $W^{1,1}(\Omega)$ by elements of $C_0{}^\infty(\Omega)$. By Sobolev's inequality (Section 5.11), (32) holds with the term $\|f\|_{0,1}$ absent from the right side. Therefore (35) is not needed and the proof does not require that Ω has finite volume. The cone property is not required either since Sobolev's inequality holds for all $u \in C_0{}^\infty(\mathbb{R}^n)$. The compactness arguments are similar to those above. ∎

8.34 REMARK Theorem 8.32 is not optimal in the sense that for some A, L_{A_*} is not necessarily the smallest Orlicz space into which $W^1L_A(\Omega)$ can be imbedded. For instance if $A(t) = A_n(t) = t^n/n$, then, as noted earlier, $A_*(t)$ is equivalent near infinity to $e^t - t - 1$. However this N-function increases essentially more slowly near infinity than does $\exp(t^{n/(n-1)}) - 1$ so that Theorem 8.25 gives a sharper result than does Theorem 8.32. Donaldson and Trudinger [22] assert that Theorem 8.32 can be improved by the methods of Theorem 8.25 provided A dominates near infinity every A_p with $p < n$, but that Theorem 8.32 gives optimal results if for some $p < n$, A_p dominates A near infinity.

8.35 THEOREM Let Ω have the cone property in $\mathbb{R}^n$. Let A be an N-function for which

$$\int_1^\infty \frac{A^{-1}(\tau)}{\tau^{(n+1)/n}}\, d\tau < \infty \tag{38}$$

Then

$$W^1L_A(\Omega) \to C_B(\Omega) = C(\Omega) \cap L^\infty(\Omega).$$

Proof Let C be a finite cone contained in Ω. We shall show that there exists a constant K_1 depending on n, A, and the dimensions of C such that

$$\|u\|_{\infty,C} \leqslant K_1 \|u\|_{1,A,C}. \tag{39}$$

In so doing, we may assume without loss of generality that A satisfies (26), for if not, and if B is an N-function satisfying (26) and equivalent to A near infinity, then $W^1L_A(C) \to W^1L_B(C)$ with imbedding constant depending on A, B, and vol C by Theorem 8.12. Since B satisfies (38) we would have

$$\|u\|_{\infty,C} \leqslant K_2 \|u\|_{1,B,C} \leqslant K_3 \|u\|_{1,A,C}.$$

Now Ω can be expressed as a union of congruent copies of some such finite cone C so that (39) clearly implies

$$\|u\|_{\infty,\Omega} \leqslant K_1 \|u\|_{1,A,\Omega}. \tag{40}$$

Since A is assumed to satisfy (26) and (38) we have

$$\int_0^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}\, dt = K_4 < \infty.$$

Let

$$\Lambda^{-1}(t) = \int_0^t \frac{A^{-1}(\tau)}{\tau^{(n+1)/n}}\, d\tau.$$

Then Λ^{-1} maps $[0, \infty)$ in a one-to-one way onto $[0, K_4)$ and has convex inverse Λ. We extend the domain of definition of Λ to $[0, \infty)$ by setting $\Lambda(t) = \infty$ for $t \geqslant K_4$. The function Λ, which is a Young's function (see Luxemburg [42] or O'Neill [55]), is not an N-function as defined in Section 8.2 but nevertheless the Luxemburg norm

$$\|u\|_{\Lambda,C} = \inf\left\{k > 0 : \int_\Omega \Lambda(|u(x)|/k)\, dx \leqslant 1\right\}$$

is easily seen to be a norm on $L^\infty(C)$ equivalent to the usual norm; in fact,

$$(1/K_4)\|u\|_{\infty,C} \leqslant \|u\|_{\Lambda,C} \leqslant [1/\Lambda^{-1}(1/\mathrm{vol}\, C)]\, \|u\|_{\infty,C}. \tag{41}$$

Moreover, $s = \Lambda(t)$ satisfies the differential equation (29) so that the proof of Theorem 8.32 can be carried over in this case to yield, for $u \in W^1L_A(C)$,

$$\|u\|_{\Lambda,C} \leqslant K_5 \|u\|_{1,A,C}. \tag{42}$$

Inequality (39) now follows from (41) and (42).

By Theorem 8.28(d) an element $u \in W^1E_A(\Omega)$ can be approximated in norm by functions continuous on Ω. It follows from (40) that u must coincide a.e. in Ω with a continuous function. (See the first paragraph of the proof of Lemma 5.15.)

Suppose that an N-function B can be constructed so that $B(t) = A(t)$ near zero, B increases essentially more slowly than A near infinity, and

$$\int_1^\infty \frac{B^{-1}(t)}{t^{(n+1)/n}}\,dt \leqslant 2\int_1^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt < \infty.$$

Then by Theorem 8.16, $u \in W^1L_A(C)$ implies $u \in W^1E_B(C)$ so that $W^1L_A(\Omega) \subset C(\Omega)$ as required.

It remains, therefore, to construct such an N-function B. Let $1 < t_1 < t_2 < \cdots$ be such that

$$\int_{t_k}^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt = \frac{1}{2^{2k}}\int_1^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt.$$

We define a sequence $\{s_k\}$ with $s_k \geqslant t_k$, and the function $B^{-1}(t)$, inductively as follows.

Let $s_1 = t_1$ and $B^{-1}(t) = A^{-1}(t)$ for $0 \leqslant t \leqslant s_1$. Having chosen $s_1, s_2, \ldots, s_{k-1}$ and defined $B^{-1}(t)$ for $0 \leqslant t \leqslant s_{k-1}$, we continue $B^{-1}(t)$ to the right of s_{k-1} along a straight line with slope $(A^{-1})'(s_{k-1}-)$ (which always exists since A^{-1} is concave) until a point t_k' is reached where $B^{-1}(t_k') = 2^{k-1}A^{-1}(t_k')$. Such t_k' exists because $\lim_{t\to\infty} A^{-1}(t)/t = 0$. If $t_k' \geqslant t_k$, let $s_k = t_k'$. Otherwise let $s_k = t_k$ and extend B^{-1} from t_k' to s_k by setting $B^{-1}(t) = 2^{k-1}A^{-1}(t)$. Evidently B^{-1} is concave and B is an N-function. Moreover, $B(t) = A(t)$ near zero and since

$$\lim_{t\to\infty} \frac{B^{-1}(t)}{A^{-1}(t)} = \infty,$$

B increases essentially more slowly than A near infinity. Finally,

$$\begin{aligned}
\int_1^\infty \frac{B^{-1}(t)}{t^{(n+1)/n}}\,dt &\leqslant \int_1^{s_1} \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt + \sum_{k=2}^{\infty}\int_{s_{k-1}}^{s_k} \frac{2^{k-1}A^{-1}(t)}{t^{(n+1)/n}}\,dt \\
&\leqslant \int_1^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt + \sum_{k=2}^{\infty} 2^{k-1}\int_{t_{k-1}}^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt \\
&= 2\int_1^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt.
\end{aligned}$$

as required. ∎

8.36 THEOREM Let Ω be a domain in $\mathbb{R}^n$ having the strong local Lipschitz property. If the N-function A satisfies

$$\int_1^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}\,dt < \infty, \tag{43}$$

then there exists a constant K such that for any $u \in W^1L_A(\Omega)$ (which may be assumed continuous by the previous theorem) and all $x, y \in \Omega$ we have

$$|u(x)-u(y)| \leqslant K\|u\|_{1,A,\Omega} \int_{|x-y|^{-n}}^{\infty} \frac{A^{-1}(t)}{t^{(n+1)/n}}\, dt. \tag{44}$$

PROOF We establish (44) for the case when Ω is a cube of unit edge; the extension to more general strongly Lipschitz domains can be carried out just as in the proof of Lemma 5.17. As in that lemma we let Ω_σ denote a parallel subcube of Ω and obtain for $x \in \bar{\Omega}_\sigma$

$$\left|u(x) - \frac{1}{\sigma^n}\int_{\Omega_\sigma} u(z)\, dz\right| \leqslant \frac{\sqrt{n}}{\sigma^{n-1}} \int_0^1 t^{-n}\, dt \int_{\Omega_{t\sigma}} |\operatorname{grad} u(z)|\, dz.$$

By (22), $\|1\|_{\tilde{A},\Omega_{t\sigma}} = 1/\tilde{A}^{-1}(t^{-n}\sigma^{-n})$. It follows by Hölder's inequality and (7) that

$$\begin{aligned}\int_{\Omega_{t\sigma}} |\operatorname{grad} u(z)|\, dz &\leqslant 2\|\operatorname{grad} u\|_{A,\Omega_{t\sigma}} \|1\|_{\tilde{A},\Omega_{t\sigma}} \\ &\leqslant 2\|u\|_{1,A,\Omega}/\tilde{A}^{-1}(t^{-n}\sigma^{-n}) \\ &\leqslant 2\sigma^n t^n \|u\|_{1,A,\Omega} A^{-1}(t^{-n}\sigma^{-n}).\end{aligned}$$

Hence

$$\begin{aligned}\left|u(x) - \frac{1}{\sigma^n}\int_{\Omega_\sigma} u(z)\, dz\right| &\leqslant 2\sqrt{n}\,\sigma \|u\|_{1,A,\Omega} \int_0^1 A^{-1}\left(\frac{1}{t^n\sigma^n}\right) dt \\ &= \frac{2}{\sqrt{n}} \|u\|_{1,A,\Omega} \int_{\sigma^{-n}}^{\infty} \frac{A^{-1}(\tau)}{\tau^{(n+1)/n}}\, d\tau. \end{aligned} \tag{45}$$

If $x, y \in \Omega$ and $\sigma = |x-y| < 1$, there exists such a subcube Ω_σ with $x, y \in \bar{\Omega}_\sigma \subset \Omega$. Using (45) for x and y, we obtain

$$|u(x)-u(y)| \leqslant \frac{4}{\sqrt{n}} \|u\|_{1,p,\Omega} \int_{|x-y|^{-n}}^{\infty} \frac{A^{-1}(t)}{t^{(n+1)/n}}\, dt.$$

For $|x-y| \geqslant 1$, (44) follows from (40) and (43). ∎

8.37 Let M denote the class of positive, continuous, increasing functions of $t > 0$ which tend to zero as t decreases to zero. If $\mu \in M$, the space $C_\mu(\bar{\Omega})$, consisting of functions $u \in C(\bar{\Omega})$ for which the norm

$$\|u; C_\mu(\bar{\Omega})\| = \|u; C(\bar{\Omega})\| + \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|u(x)-u(y)|}{\mu(|x-y|)}$$

is finite, is a Banach space under that norm. The theorem above asserts that if (43) holds, then

$$W^1L_A(\Omega) \to C_\mu(\overline{\Omega}) \qquad \text{with} \quad \mu(t) = \int_{t^{-n}}^{\infty} \frac{A^{-1}(\tau)}{\tau^{(n+1)/n}}\, d\tau. \tag{46}$$

If $\mu, \nu \in M$ are such that $\mu/\nu \in M$, then for bounded Ω we have, as in Theorem 1.31, that the imbedding

$$C_\mu(\overline{\Omega}) \to C_\nu(\overline{\Omega})$$

is compact. Hence so also is

$$W^1L_A(\Omega) \to C_\nu(\overline{\Omega}),$$

if μ is given by (46).

The following is a trace imbedding theorem which generalizes (the case $m = 1$ of) Lemma 5.19.

8.38 THEOREM Let Ω be a bounded domain in $\mathbb{R}^n$ having the cone property, and let Ω^k denote the intersection of Ω with a k-dimensional plane in $\mathbb{R}^n$. Let A be an N-function for which (26) and (27) hold, and let A_* be given by (28). Let $1 \leqslant p < n$ where p is such that the function B given by $B(t) = A(t^{1/p})$ is an N-function. If either $n-p < k \leqslant n$, or $p = 1$, and $n-1 \leqslant k \leqslant n$, then

$$W^1L_A(\Omega) \to L_{A_*^{k/n}}(\Omega^k),$$

where $A_*^{k/n}(t) = [A_*(t)]^{k/n}$.

Moreover, if $p > 1$ and C is an N-function increasing essentially more slowly than $A_*^{k/n}$ near infinity, then the imbedding

$$W^1L_A(\Omega) \to L_C(\Omega^k) \tag{47}$$

is compact.

PROOF The problem of verifying that $A_*^{k/n}$ is an N-function is left to the reader. Let $u \in W^1L_A(\Omega)$ be a bounded function. Then

$$\int_{\Omega^k} A_*^{k/n}\left(\frac{|u(y)|}{K}\right) dy = 1, \qquad K = \|u\|_{A_*^{k/n}, \Omega^k}. \tag{48}$$

We wish to show that

$$K \leqslant K_1 \|u\|_{1,A,\Omega} \tag{49}$$

with K_1 independent of u. Since (49) is known to hold for the special case

$k = n$ (Theorem 8.32) we may assume without loss of generality that

$$K \geqslant \|u\|_{A_*,\Omega} = \|u\|_{A_*^{n/n},\Omega^n}. \tag{50}$$

Let $\omega(t) = [A_*(t)]^{1/q}$ where $q = np/(n-p)$. By Lemma 5.19 (the case $m = 1$) we have

$$\begin{aligned}\left\|\omega\left(\frac{|u|}{K}\right)\right\|^p_{kp/(n-p),\Omega^k} &\leqslant K_2\left\{\sum_{j=1}^n \left\|D_j\omega\left(\frac{|u|}{K}\right)\right\|^p_{p,\Omega} + \left\|\omega\left(\frac{|u|}{K}\right)\right\|^p_{p,\Omega}\right\}\\ &= K_2\left\{\frac{1}{K^p}\sum_{j=1}^n \int_\Omega \left|\omega'\left(\frac{|u(x)|}{K}\right)\right|^p |D_j u(x)|^p\,dx\right.\\ &\qquad \left. + \int_\Omega \left|\omega\left(\frac{|u(x)|}{K}\right)\right|^p dx\right\}.\end{aligned}$$

Using (48) and noting that $\||v|^p\|_{B,\Omega} \leqslant \|v\|^p_{A,\Omega}$, we obtain

$$\begin{aligned}1 &= \left(\int_{\Omega^k}\left(A_*\left(\frac{|u(y)|}{K}\right)\right)^{k/n} dy\right)^{(n-p)/k} = \left\|\omega\left(\frac{|u|}{K}\right)\right\|^p_{kp/(n-p),\Omega^k}\\ &\leqslant \frac{2K_2}{K^p}\sum_{j=1}^n \left\|\left(\omega'\left(\frac{|u|}{K}\right)\right)^p\right\|_{B,\Omega} \||D_j u|^p\|_{B,\Omega} + K_2\left\|\omega\left(\frac{|u|}{K}\right)\right\|^p_{p,\Omega}\\ &\leqslant \frac{2nK_2}{K^p}\left\|\left(\omega'\left(\frac{|u|}{K}\right)\right)^p\right\|_{B,\Omega}\|u\|^p_{1,A,\Omega} + K_2\left\|\omega\left(\frac{|u|}{K}\right)\right\|^p_{p,\Omega}.\end{aligned} \tag{51}$$

Now $B^{-1}(t) = [A^{-1}(t)]^p$ and so, using (29) and (7), we have

$$\begin{aligned}[\omega'(t)]^p &= (1/q^p)[A_*(t)]^{p(1-q)/q}[A_*'(t)]^p\\ &= (1/q^p)A_*(t)[1/B^{-1}(A_*(t))] \leqslant (1/q^p)\tilde{B}^{-1}(A_*(t)).\end{aligned}$$

It follows by (50) that

$$\int_\Omega \tilde{B}\left(\left(\frac{\omega'(|u(x)|/K)}{1/q}\right)^p\right) dx \leqslant \int_\Omega A_*\left(\frac{|u(x)|}{K}\right) dx \leqslant 1.$$

So

$$\|(\omega'(|u|/K))^p\|_{B,\Omega} \leqslant 1/q^p. \tag{52}$$

Now set $g(t) = A_*(t)/t^p$ and $h(t) = (\omega(t)/t)^p$. It is readily checked that $\lim_{t\to\infty} g(t)/h(t) = \infty$. In order to see that $h(t)$ is bounded near zero let $s = A_*(t)$ and consider

$$(h(t))^{1/p} = \frac{(A_*(t))^{1/q}}{t} = \frac{s^{(1/p)-(1/n)}}{\displaystyle\int_0^s \frac{A^{-1}(\tau)}{\tau^{(n+1)/n}}\,d\tau} \leqslant \frac{s^{1/p}}{\displaystyle\int_0^s \frac{[B^{-1}(\tau)]^{1/p}}{\tau}\,d\tau}.$$

Since B is an N-function $\lim_{\tau\to\infty+} B^{-1}(\tau)/\tau = \infty$. Hence for sufficiently small values of t we have

$$(h(t))^{1/p} \leqslant \frac{s^{1/p}}{\int_0^s \tau^{-1+1/p}\,d\tau} = \frac{1}{p}.$$

Therefore there exists a constant K_3 such that for $t \geqslant 0$

$$(\omega(t))^p \leqslant (1/2K_2)\,A_*(t) + K_3\,t^p.$$

Using (50), we now obtain

$$\begin{aligned}\left\|\omega\left(\frac{|u|}{K}\right)\right\|_{p,\Omega}^p &\leqslant \frac{1}{2K_2}\int_\Omega A_*\left(\frac{|u(x)|}{K}\right)dx + \frac{K_3}{K^p}\int_\Omega |u(x)|^p\,dx \\ &\leqslant \frac{1}{2K_2} + \frac{2K_3}{K^p}\|\,|u|^p\|_{B,\Omega}\|1\|_{B,\Omega} \\ &\leqslant \frac{1}{2K_2} + \frac{K_4}{K^p}\|u\|_{A,\Omega}^p. \end{aligned} \tag{53}$$

From (51)–(53) there follows the inequality

$$1 \leqslant \frac{2nK_2}{K^p}\cdot\frac{1}{q^p}\|u\|_{1,A,\Omega}^p + \frac{1}{2} + \frac{K_4 K_2}{K^p}\|u\|_{A,\Omega}^p$$

and hence (49). The extension of (49) to arbitrary $u \in W^1L_A(\Omega)$ now follows as in the proof of Theorem 8.32.

Since $B(t) = A(t^{1/p})$ is an N-function and Ω is bounded we have $W^1L_A(\Omega) \to W^{1,p}(\Omega) \to L^1(\Omega^k)$, the latter imbedding being compact by Theorem 6.2 (provided $p > 1$). The compactness of (47) now follows by Theorem 8.23. ▌

8.39 We conclude this chapter with the general Orlicz–Sobolev imbedding theorem of Donaldson and Trudinger [22]. For a given N-function A we define a sequence of N-functions $B_0, B_1, B_2, \ldots$ as follows:

$$\begin{aligned} B_0(t) &= A(t) \\ (B_k)^{-1}(t) &= \int_0^t \frac{(B_{k-1})^{-1}(\tau)}{\tau^{(n+1)/n}}\,d\tau, \qquad k = 1,2,\ldots. \end{aligned}$$

At each stage we assume that

$$\int_0^1 \frac{(B_k)^{-1}(\tau)}{\tau^{(n+1)/n}}\,d\tau < \infty, \tag{54}$$

replacing B_k, if necessary, by another N-function equivalent to it near infinity and satisfying (54). Let $J = J(A)$ be the smallest nonnegative integer such that

$$\int_1^\infty \frac{(B_J)^{-1}(\tau)}{\tau^{(n+1)/n}}\, d\tau < \infty.$$

Evidently $J(A) \leqslant n$.

If μ belongs to the class M defined in Section 8.37, we define the space $C_\mu{}^m(\overline{\Omega})$ to consist of those functions $u \in C^m(\overline{\Omega})$ for which $D^\alpha u \in C_\mu(\overline{\Omega})$. The space $C_\mu{}^m(\overline{\Omega})$ is a Banach space with respect to the norm

$$\|u; C_\mu{}^m(\overline{\Omega})\| = \max_{|\alpha| \leq m} \|D^\alpha u; C_\mu(\overline{\Omega})\|.$$

8.40 THEOREM Let Ω be a bounded domain in $\mathbb{R}^n$ having the cone property. Let A be an N-function.

(a) If $m \leqslant J(A)$, then $W^m L_A(\Omega) \to L_{B_m}(\Omega)$ and the imbedding $W^m L_A(\Omega) \to L_C(\Omega)$ is compact for any N-function C increasing essentially more slowly than B_m near infinity.

(b) If $m > J(A)$, then $W^m L_A(\Omega) \to C_B(\Omega) = C(\Omega) \cap L^\infty(\Omega)$.

(c) If also Ω has the strong local Lipschitz property and if $m > J = J(A)$, then $W^m L_A(\Omega) \to C_\mu^{m-J-1}(\overline{\Omega})$ where

$$\mu(t) = \int_{t^{-n}}^\infty \frac{(B_J)^{-1}(\tau)}{\tau^{(n+1)/n}}\, d\tau.$$

Moreover, the imbeddings $W^m L_A(\Omega) \to C^{m-J-1}(\overline{\Omega})$ and $W^m L_A(\Omega) \to C_\nu^{m-J-1}(\overline{\Omega})$ are compact provided $\nu \in M$ and $\mu/\nu \in M$.

8.41 REMARK The above theorem follows in a straightforward way from Theorems 8.32, 8.35, and 8.36. Moreover, if we replace L_A by E_A in part (a), we get $W^m E_A(\Omega) \to E_{B_m}(\Omega)$ since the sequence $\{u_k\}$ defined by (37) in the proof of Theorem 8.32 converges to u if $u \in W^1 E_A(\Omega)$. Theorem 8.40 holds without restriction on Ω if $W^m L_A(\Omega)$ is everywhere replaced by $W_0{}^m L_A(\Omega)$.

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Index

目　录

第一章　预备知识

记　号

1.1　贯穿本专著的始终，区域这个术语和符号Ω专门用来表示实n维欧氏空间$\mathbf{R}^n$中的开集. 我们将论及定义在Ω上的函数的可微性和可积性，这些函数都允许是复值函数，除非另有相反的声明. $\mathbf{C}$表示复数域. 对于$c\in\mathbf{C}$和函数u, v, 数乘cu, 和$u+v$, 积uv总是按照

$$(cu)(x)=cu(x),$$
$$(u+v)(x)=u(x)+v(x),$$
$$(uv)(x)=u(x)v(x)$$

在一切使右端有意义的点上逐点定义的.

$x=(x_1,\cdots,x_n)$表示$\mathbf{R}^n$中一个点；它的范数是 $|x|=\left(\sum_{j=1}^{n}x_j^2\right)^{\frac{1}{2}}$, x和y的内积是$x\cdot y=\sum_{j=1}^{n}x_jy_j$.

如果$\alpha=(\alpha_1, \cdots, \alpha_n)$是非负整数$\alpha_j$的一个$n$重组，我们把$x$叫做一个多重指标(multi-index)且用$x^\alpha$来表示次数为$|\alpha|=\sum_{j=1}^{n}\alpha_j$的单项式$x_1^{\alpha_1}\cdots x_n^{\alpha_n}$. 类似地，如果对于$1\leqslant j\leqslant n$, $D_j=\dfrac{\partial}{\partial x_j}$, 则

$$D^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$$

表示一个阶数为$|\alpha|$的微分算子. $D^{(0,\cdots,0)}u=u$.

如果α和β是两个多重指标，假如对$1\leqslant j\leqslant n$, $\beta_j\leqslant\alpha_j$我们就说$\beta\leqslant\alpha$, 这时$\alpha-\beta$也是一个多重指标而且$|\alpha-\beta|+|\beta|=|\alpha|$.

我们还记

$$\alpha! = \alpha_1! \cdots \alpha_n!.$$

而且，如果 $\beta \leqslant \alpha$ 则

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!} = \binom{\alpha^1}{\beta_1} \cdots \binom{\alpha^n}{\beta_n}.$$

对在 x 附近 $|\alpha|$ 次连续可微的函数 u 和 v，读者可以验证 Leibniz 公式

$$D^\alpha(uv)(x) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} D^\beta u(x) D^{\alpha-\beta} v(x)$$

是成立的.

1.2 如果 $G \subset \mathbf{R}^n$，我们用 $\bar{G}$ 来表示 G 在 $\mathbf{R}^n$ 中的闭包，假如 $\bar{G} \subset \Omega$ 而且 $\bar{G}$ 是 $\mathbf{R}^n$ 的紧子集(即有界闭集)，那么记作 $G \subset\subset \Omega$. 如果 u 是定义在 G 上的函数，我们把

$$\operatorname{supp} u = \overline{\{x \in G : u(x) \neq 0\}}$$

定义为 u 的支集. 我们说 u 在 Ω 中具有紧支集，如果 $\operatorname{supp} u \subset\subset \Omega$. 我们将用"bdry$G$"来表示 G 在 $\mathbf{R}^n$ 中的边界，即，集合 $\bar{G} \cap \bar{G^c}$，其中 $G^c = \mathbf{R}^n \sim G = \{x \in \mathbf{R}^n : x \notin G\}$ 是 G 的余集.

如果 $x \in \mathbf{R}^n$ 且 $G \subset \mathbf{R}^n$，我们用 "dist(x, G)"来表示 x 到 G 的距离，也就是数 $\inf\limits_{y \in G} |x-y|$. 类似地，如果 $F, G \subset \mathbf{R}^n$，

$$\operatorname{dist}(F, G) = \inf_{y \in F} \operatorname{dist}(y, G) = \inf_{\substack{x \in G \\ y \in F}} |x-y|.$$

拓扑向量空间

1.3 我们假定读者熟悉实或复数域上向量空间的概念，以及与之有关的维数、子空间、线性变换和凸集的概念. 我们还假定读者熟悉一般拓扑学，Hausdorff 拓扑空间，较弱和较强的拓扑，连续函数，收敛序列，拓扑积空间，子空间和相对拓扑的基本概念.

除非有相反的声明，贯穿本专著的始终一切向量空间都被认为是复数域上的向量空间.

1.4 *拓扑向量空间*，以后缩写为 TVS，是一个 Hausdorff 拓扑空间，也是一个向量空间，对此空间向量空间的加法和数乘运算都是连续的，即，如果 X 是一个 TVS，则分别从拓扑积空间 $X\times X$ 和 $\mathbf{C}\times X$ 到 X 中的映射

$$(x, y)\to x+y \text{ 和 } (c, x)\to cx$$

是连续的，如果 X 的原点的每一个邻域包含一个凸的邻域，则 X 是*局部凸的* TVS.

下面我们将概略地叙述一下在 Sobolev 空间研究中起重要作用的拓扑空间和赋范空间理论的一些方面，大部分结果都将略去证明和细节. 对于这些课题的更完全的讨论，介绍读者去看泛函分析的标准教科书，例如，Yosida[69]或 Rudin[59].

1.5 我们把定义在 X 上的数值函数 f 叫做向量空间 X 上的*泛函*. 泛函 f 是线性的，假如

$$f(ax+by)=af(x)+bf(y),\quad x, y\in X, a, b\in \mathbf{C}.$$

当 X 是一个 TVS 时，X 上的泛函叫做连续的，如果这个泛函从 X 到 $\mathbf{C}$ 是连续的，其中 $\mathbf{C}$ 具有由 Euclid 度量导出的通常的拓扑.

X 上一切连续线性泛函组成的集合叫做 X 的*对偶空间*，并用 X' 来表示，在逐点的加法和数乘的意义下，X' 是一个向量空间:

$$(f+g)(x)=f(x)+g(x), (cf)(x)=cf(x),$$
$$f, g\in X', x\in X, c\in \mathbf{C}.$$

假如对 X' 规定一个适当的拓扑，则 X' 是一个TVS. 这种拓扑之一就是*弱-星拓扑*，也就是使得对每个 $f\in X'$，由 $F_x(f)=f(x)$ 定义的 X' 上的泛函 F_x 对于每个 $x\in X$ 是连续的最弱的拓扑，例如，在 1.52 节中介绍的Schwartz广义函数空间中就用了这种拓扑，对于

赋范空间的对偶空间能够给出一个更强的拓扑，关于这个拓扑对偶空间本身就是一个赋范空间(1.10 节).

赋 范 空 间

1.6 向量空间 X 上的一个范数是 X 上的一个实值泛函，它满足

(i) 对一切 $x\in X$, $f(x)\geqslant 0$, 等号当且仅当 $x=0$ 时成立,

(ii) 对每个 $x\in X$ 和 $c\in\mathbf{R}$, $f(cx)=|c|f(x)$,

(iii) 对于 $x, y\in X$, $f(x+y)\leqslant f(x)+f(y)$.

赋范空间是已经规定了范数的向量空间. 除在有些地方引入更简单的记号外，范数将用 $\|\cdot;\ X\|$ 来表示. 当 $r>0$ 时，集合

$$B_r(x)=\{y\in X: \|y-x; X\|<r\}$$

叫做中心在 $x\in X$ 半径为 r 的开球. X 的子集合 A 叫做开集，如果对每个 $x\in A$ 存在 $r>0$ 使得 $B_r(x)\subset A$, 这样定义的开集构成 X 的一个拓扑. 对这个拓扑而言 X 是一个 TVS. 这个拓扑叫做 X 上的范数拓扑，在这个拓扑下 $B_r(x)$ 的闭包是

$$\overline{B_r(x)}=\{y\in X: \|y-x; X\|\leqslant r\}.$$

一个 TVS X 叫做可赋范的，如果 X 的拓扑和由 X 上的某个范数导出的拓扑一致. 向量空间上两个不同的范数叫做等价的，如果它们导出 X 上的相同的拓扑，即，如果对某个常数 $c>0$, 对一切 $x\in X$

$$c\|x\|_1\leqslant\|x\|_2\leqslant\left(\frac{1}{c}\right)\|x\|_1,$$

$\|\cdot\|_1$ 和 $\|\cdot\|_2$ 是 X 上的两个范数。

如果 X 和 Y 是两个赋范空间，又如果存在一个把 X 映到 Y 上的一对一的线性算子 L，而且对每个 $x\in X$ 具有性质 $\|L(x); Y\|=\|x; X\|$，则 L 叫做 X 和 Y 间的一个等距同构算子，而 X 和 Y 叫做等距同构的；记作 $X\cong Y$. 互相等距同构的空间常常看成是一样

的，因为它们具有同样的结构而仅有的差别只是它们的元素的性质不同而已.

1.7 赋范空间X中的序列$\{x_n\}$收敛到x_0当且仅当在$\mathbf{R}$中$\lim\limits_{n\to\infty}\|x_n-x_0;X\|=0$, X的范数拓扑由收敛的序列完全确定.

赋范空间X的子集合S叫做在X中稠密，如果每个$x\in X$是S中的元素构成的序列的极限. 赋范空间X叫做可分的，如果X有可数的稠密子集.

1.8 赋范空间X中的序列$\{x_n\}$叫做Cauchy序列当且仅当$\lim\limits_{n,m\to\infty}\|x_m-x_n;X\|=0$. 如果$X$中每个Cauchy序列收敛到$X$中的一个极限，则$X$是完备的而且是一个Banach空间. 每一个赋范空间X或者是一个Banach空间或者是一个Banach空间Y的一个稠密子集，它们的范数满足

$$\|x;Y\|=\|x;X\| \qquad \text{对一切} \quad x\in X$$

如果是后面一种情况，Y叫做X的完备化.

1.9 如果X是一个向量空间，定义在$X\times X$上的泛函$(\cdot,\cdot)_X$叫做X上的内积，假如对一切$x, y, z\in X$和$a, b\in\mathbf{C}$

(i) $(x, y)_X=\overline{(y, x)_X}$,

(ii) $(ax+by, z)_X=a(x, z)_X+b(y, z)_X$,

(iii) $(x, x)_X=0$ 当且仅当 $x=0$,

其中$\bar{c}$表示$c\in\mathbf{C}$的共轭复数. 给定了这样一种内积后，X上的一种范数就能够用

$$\|x;X\|=(x, x)_X^{1/2} \tag{1}$$

来定义. 如果在这个范数下X是一个Banach空间，则X叫做Hilbert空间. 在范数是从内积经由(1)得到的任何赋范空间中，平行四边形定律

$$\|x+y;X\|^2+\|x-y;X\|^2=2\|x;X\|^2+2\|y;X\|^2 \tag{2}$$

是成立的.

赋范对偶

1.10 能够用如下方式来定义赋范空间X的对偶空间X'上的一种范数:

$$\|x';X'\|=\sup_{\substack{x\in X\\ x\neq 0}}\frac{|x'(x)|}{\|x;X\|},\qquad x'\in X'.$$

由于$\mathbf{C}$是完备的,具有由这种范数导出的拓扑的X'是一个Banach空间(不论X是不是Banach空间)而且叫做X的*赋范对偶*. 如果X是无穷维的,则X'的范数拓扑是比1.5节中定义的弱-星拓扑更强的拓扑(即具有更多的开集).

如果X是一个Hilbert空间,就能够用一种自然的方式把X和X的赋范对偶X'看成是一样的.

1.11 **定理**(Riesz *表示定理*)设X是一个Hilbert空间,X上的线性泛函x'属于X'当且仅当存在$x\in X$使得对一切$y\in X$有

$$x'(y)=(y,x)_X$$

这时$\|x';X'\|=\|x;X\|$. 而且x是由$x'\in X'$唯一确定的.

赋范空间X的向量子空间M本身在X的范数下是一个赋范空间,而且这样赋范的M叫做X的一个*子空间*. Banach空间的闭子空间是Banach空间.

1.12 **定理** (Hahn-Banach *延拓定理*)设M是赋范空间X的子空间. 如果$m'\in M'$,则存在$x'\in X'$,使得对一切$m\in M$, $\|x';X'\|=\|m';M'\|$而且$x'(m)=m'(m)$.

1.13 赋范空间X到其二次对偶空间$X''=(X')'$的一个自然线性内射是由映射J_X给出的,J_X在$x\in X$的值由

$$J_Xx(x')=x'(x),\quad x'\in X'$$

给出. 由于$|J_Xx(x')|\leqslant\|x';X'\|\|x;X\|$,我们有

$$\|J_Xx;X''\|\leqslant\|x;X\|.$$

另一方面, Hahn-Banach 定理保证了, 对任何 $x\in X$ 能够找到一个 $x'\in X'$ 使得 $\|x';X'\|=1$, 而且 $x'(x)=\|x;X\|$. 因此

$$\|J_X x;X''\|=\|x;X\|,$$

从而 J_X 就是 X 到 X'' 的一个等距同构.

如果同构的值域是全空间 X'', 我们说赋范空间 X 是*自反*的. 自反空间必定是完备的, 因此是 Banach 空间.

1.14 定理 设 X 是赋范空间, X 是自反的当且仅当 X' 是自反的. X 是可分的当且仅当 X' 是可分的. 因此, 如果 X 是可分且自反的, 则 X' 也是可分且自反的.

弱拓扑和弱收敛

1.15 赋范空间 X 的*弱拓扑*是使每个 $x'\in X'$ 仍然连续的 X 上的最弱的拓扑. 除非 X 是有限维的情形, 弱拓扑比 X 上的范数拓扑更弱. Hahn-Banach 定理的一个推论就是赋范空间中的一个闭凸集在该空间的弱拓扑下也是闭的. 关于 X 上的弱拓扑收敛的序列叫做*弱收敛*. 于是, 假如对一切 $x'\in X'$ 在 $\mathbf{C}$ 中 $x'(x_n)\to x'(x)$ 那么在 X 中 x_n 弱收敛到 x. 我们用 $x_n\to x$ 表示一个序列 $\{x_n\}$ 在 X 中依范数收敛到 x; 用 $x_n\rightharpoonup x$ 来表示弱收敛. 由于 $|x'(x_n-x)|\leqslant\|x';X'\|\|x_n-x;X\|$ 我们知道 $x_n\to x$ 蕴涵 $x_n\rightharpoonup x$. 一般说弱收敛不一定依范数收敛.

紧 集

1.16 赋范空间 X 的子集 A 叫做*紧的*, 如果 A 中的每个点列包含一个子序列, 该子序列在 X 中收敛到 A 中的一个元素. 紧集是闭有界集, 但闭有界集不一定是紧的, 除非 X 是有限维的. A 叫做*准紧*(Precompact)*的*, 如果其闭包 $\bar{A}$ (在范数拓扑下)是紧的. A 叫做*弱序列紧的*(Weakly Sequentially Compact), 如果 A 中的每个

序列包含一个子序列，该子序列在X中弱收敛到A中的一点. Banach 空间的自反性能借助于这个性质表征出来.

1.17 定理 Banach 空间X是自反的当且仅当X的闭单位球$\overline{B_1(0)}=\{x\in X;\|x;X\|\leqslant 1\}$是弱序列紧的.

1.18 定理 集合A在 Banach 空间X中是准紧的当且仅当对于每个正数$\varepsilon>0$存在一个由X中的点构成的有限子集N_ε，具有如下性质:

$$A\subset\bigcup_{y\in N_\varepsilon}B_\varepsilon(y).$$

具有这种性质的集合N_ε叫做A的一个有限ε-网.

一致凸性

1.19 任何赋范空间关于其范数拓扑是局部凸的. X上的范数叫做一致凸的，如果对每个满足$0<\varepsilon\leqslant 2$的数ε，存在一个数$\delta(\varepsilon)>0$，使得如果$x, y\in X$满足$\|x;X\|=\|y;X\|=1$和$\|x-y;X\|\geqslant\varepsilon$，则$\|(x+y)/2;X\|\leqslant 1-\delta(\varepsilon)$. 这时赋范空间$X$本身叫做“一致凸的”. 但是要注意一致凸性是范数的性质——X可以有一个非一致凸的等价范数. 任何可赋范的空间叫做“一致凸的”如果该空间具有一个一致凸的范数. 平行四边形定律 (2) 表明 Hilbert 空间是一致凸的.

1.20 定理 一致凸的 Banach 空间是自反的.

将用下面两个定理来建立在第三章中引入的 Sobolev 空间的可分性，自反性和一致凸性.

1.21 定理 设X是 Banach 空间而且X的子空间M关于X上的范数拓扑是闭的. 则在X的范数下M本身是一个 Banach 空间. 而且

(i) 如果X是可分的，则M是可分的，

(ii) 如果X是自反的，则M是自反的，

(iii) 如果X是一致凸的，则M是一致凸的.

M的完备性，可分性和一致凸性容易从X的相应的性质得到M的自反性是定理 1.17 和闭而且凸的M在X的弱拓扑下是闭的这一事实的推论.

1.22 定理 对于 $j=1, 2, \cdots, n$, 设 X_j 是范数为 $\|\cdot\|_j$ 的 Banach 空间. 由点 $x=(x_1, \cdots, x_n)$, $x_j \in X_j$, 组成的笛卡尔积空间

$$X=\prod_{j=1}^{n} X_j$$

在定义

$$x+y=(x_1+y_1, \cdots, x_n+y_n), \quad cx=(cx_1, \cdots, cx_n)$$

下是一个向量空间而且关于以下互相等价的范数

$$\|x\|_{(p)}=\left(\sum_{j=1}^{n}\|x_j\|_j^p\right)^{\frac{1}{p}}, \quad 1 \leqslant p<\infty,$$

$$\|x\|_{(\infty)}=\max_{1 \leqslant j \leqslant n}\|x_j\|_j$$

中的任何一个而言是一个 Banach 空间. 而且

(i) 如果对 $1 \leqslant j \leqslant n$, X_j 是可分的，则X是可分的;

(ii) 如果对 $1 \leqslant j \leqslant n$, X_j 是自反的，则X是自反的;

(iii) 如果对 $1 \leqslant j \leqslant n$, X_j 是一致凸的，则X是一致凸的. 更确切地说，假如 $1<p<\infty$ 则$\|\cdot\|_{(p)}$是X上一致凸的范数.

读者可以验证泛函$\|\cdot\|_{(p)}$, $1 \leqslant p \leqslant \infty$，实际上是$X$上的范数而且关于这些范数中的每一个$X$都是完备的. 从不等式$\|x\|_{(\infty)} \leqslant\|x\|_{(p)} \leqslant\|x\|_{(1)} \leqslant n\|x\|_{(\infty)}$可得出这些范数的等价性. X的可分性与一致凸性容易从空间X_j的相应性质得出. X的自反性可通过定理1.17 或者通过X'和$\prod_{j=1}^{n} X_j'$间的一个自然同构(例如见引理3.7)从X_j, $1 \leqslant j \leqslant n$, 的自反性得到.

算子和嵌入

1.23 因为赋范空间X的拓扑是由它的收敛序列决定的，定义在X上到拓扑空间Y中的算子f是连续的当且仅当在X中$x_n \to x$时在Y中就有$f(x_n) \to f(x)$. 对于任何其拓扑是由收敛序列来决定的拓扑空间X(第一可数空间)而言，也是这样的情形.

设X, Y是赋范空间而f是从X到Y中的一个算子. 算子f叫做*紧的*，如果当A在X中有界时$f(A)$是准紧的，[赋范空间中的有界集是包含在对于某个$R>0$的球$B_R(0)$中的集合]. 如果f是连续而且紧的算子，那么f是*完全连续的*. 如果当A在X中有界时$f(A)$在Y中有界，那么f是*有界的*.

每个紧算子是有界算子. 每个有界线性算子是连续算子，因此每个紧线性算子是完全连续的.

1.24 我们说赋范空间X*嵌入*到赋范空间Y，并且记作$X \to Y$来表示这种嵌入，假如

(i) X是Y的向量子空间，而且

(ii) 对一切$x \in X$由$Ix = x$定义的X到Y中的恒同算子是连续的.

因为I是线性的，(ii)等价于存在一个常数M使得

$$\|Ix; Y\| \leqslant M \|x; X\|, \qquad x \in X.$$

在某些情形X是Y的子空间和I是恒同映射的要求可以减弱为允许把X的某些典型线性变换嵌入到Y中去. (Sobolev 空间的迹嵌入以及 Sobolev 空间嵌入到连续函数空间中去都是例子，见第五章.)

如果嵌入算子I是紧的，我们就说X*紧嵌入*到Y中.

连续函数空间

1.25 设Ω是$\mathbf{R}^n$中一个区域. 对任何非负整数m, 设$C^m(\Omega)$是由一切在Ω上连续的函数ϕ且其偏导数$D^\alpha\phi$, $|\alpha|\leqslant m$, 也在Ω上连续的函数组成的向量空间. 我们简写作$C^0(\Omega)\equiv C(\Omega)$. 令$C^\infty(\Omega)=\bigcap_{m=0}^{\infty}C^m(\Omega)$. 子空间$C_0(\Omega)$和$C_0^\infty(\Omega)$分别由$C(\Omega)$和$C^\infty(\Omega)$中有紧支集的函数组成.

1.26 因为Ω是开集, $C^m(\Omega)$中的函数不必在Ω上有界. 如果$\phi\in C(\Omega)$在Ω上有界且一致连续, 则ϕ有一个到Ω的闭包$\overline{\Omega}$的唯一的,有界且连续的延拓. 因此, 我们定义向量空间$C^m(\overline{\Omega})$是由$\phi\in C^m(\Omega)$, 对$0\leqslant|\alpha|\leqslant m$, $D^\alpha\phi$在Ω上有界且一致连续的函数组成的. [注意$C^m(\overline{\mathbf{R}^n})\neq C^m(\mathbf{R}^n)$.] $C^m(\overline{\Omega})$是一个Banach空间,其范数由

$$\|\phi;C^m(\overline{\Omega})\|=\max_{0\leqslant|\alpha|\leqslant m}\sup_{x\in\Omega}|D^\alpha\phi(x)|$$

给出.

1.27 如果$0<\lambda\leqslant1$,我们定义$C^{m,\lambda}(\overline{\Omega})$是$C^m(\overline{\Omega})$的子空间,是由$C^m(\overline{\Omega})$中的那样一些函数$\phi$组成,对于$0\leqslant|\alpha|\leqslant m$; $D^\alpha\phi$在Ω中满足指数为λ的Hölder条件,即存在常数K使得

$$|D^\alpha\phi(x)-D^\alpha\phi(y)|\leqslant K|x-y|^\lambda,\quad x,y\in\Omega.$$

$C^{m,\lambda}(\overline{\Omega})$是一个Banach空间,其范数由

$$\|\phi;C^{m,\lambda}(\overline{\Omega})\|=\|\phi;C^m(\overline{\Omega})\|+\max_{0\leqslant|\alpha|\leqslant m}x,\ \sup_{\substack{x,y\in\Omega\\x\neq y}}\frac{|D^\alpha\phi(x)-D^\alpha\phi(y)|}{|x-y|^\lambda}.$$

给出. 要注意,对于$0<\nu<\lambda\leqslant1$,

$$C^{m,\lambda}(\overline{\Omega})\subsetneqq C^{m,\nu}(\overline{\Omega})\subsetneqq C^m(\overline{\Omega}).$$

$C^{m,1}(\overline{\Omega})\not\subset C^{m+1}(\overline{\Omega})$ 也是显然的. 一般说也有 $C^{m+1}(\overline{\Omega})\not\subset$

$C^{m,1}(\bar{\Omega})$，但是对某些区域 $C^{m+1}(\bar{\Omega})\subset C^{m,1}(\bar{\Omega})$是可能的，例如平均值定理所要求的凸区域就是这样一种区域(见定理 1.31)。

如果Ω是有界区域，下面两个众所周知的定理对确定$C(\bar{\Omega})$的子集合的稠密性和紧性提供了有效的判别准则. 如果 $\phi\in C(\bar{\Omega})$，我们可以认为ϕ是定义在$\bar{\Omega}$上的，即把ϕ和ϕ在Ω的闭包上的唯一的连续延拓看成是一样的.

1.28 定理 (Stone-Weierstrass 定理) 设Ω是 $\mathbf{R}^n$ 中的有界区域，$C(\bar{\Omega})$的子集合 $\mathbf{R}$ 在$C(\bar{\Omega})$中稠密，若 A 具有下面四条性质:

(i) 如果 ϕ, $\psi\in A$ 而 $c\in\mathbf{C}$，则 $\phi+\psi$, $\phi\psi$ 和$c\phi$ 都属于 A;

(ii) 如果 $\phi\in A$，则 $\overline{\Phi}\in A$，其中 $\overline{\Phi}$ 是ϕ 的复共轭;

(iii) 如果 $x, y\in\bar{\Omega}$, $x\neq y$，则存在 $\phi\in A$ 使得 $\phi(x)\neq\phi(y)$;

(iv) 如果 $x\in\bar{\Omega}$，则存在 $\phi\in A$ 使得 $\phi(x)\neq 0$.

1.29 推论 如果Ω在 $\mathbf{R}^n$ 中有界，则 $x=(x_1,\cdots,x_n)$的全体有理复系数多项式P在 $C(\bar{\Omega})$中稠密. (如果 $c=c_1+ic_2$，其中c_1和 c_2 是有理数，$c\in\mathbf{C}$ 是有理复数.)因此，$C(\bar{\Omega})$是可分的.

证明 由 Stone-Weierstrass 定理，x 的多项式的全体构成的集合在 $C(\bar{\Omega})$中稠密. 在紧集$\bar{\Omega}$上任何多项式能够用可数集合 P 中的元素来一致逼近，所以P也在 $C(\bar{\Omega})$中稠密. ▌

1.30 定理 (Ascoli-Arzela 定理) 设Ω是 $\mathbf{R}^n$ 中的有界区域. $C(\bar{\Omega})$的子集合K在 $C(\bar{\Omega})$中是准紧的，假如下面两个条件成立:

(i) 存在常数M使得对一切 $\phi\in K$ 和 $x\in\Omega$, $|\phi(x)|\leqslant M$.

(ii) 对每个 $\varepsilon>0$ 存在 $\delta>0$ 使得如果 $\phi\in K$, x, $y\in\Omega$ 而且 $|x-y|<\delta$，则 $|\phi(x)-\phi(y)|<\varepsilon$.

下列定理是上面所介绍的那些空间的一个直接的嵌入定理.

1.31 定理 设 m 是一个非负整数而令 $0<\nu<\lambda\leqslant 1$. 则存在下列嵌入关系:

$$C^{m+1}(\bar{\Omega})\longrightarrow C^m(\bar{\Omega}), \tag{3}$$

$$C^{m,\lambda}(\bar{\Omega})\longrightarrow C^m(\bar{\Omega}), \tag{4}$$

$$C^{m,\lambda}(\bar{\Omega})\longrightarrow C^{m,\nu}(\bar{\Omega}). \tag{5}$$

如果Ω是有界的，则嵌入(4)和(5)是紧的．如果Ω是凸的，就有更进一步的嵌入关系

$$C^{m+1}(\bar{\Omega})\longrightarrow C^{m,1}(\bar{\Omega}), \tag{6}$$

$$C^{m+1}(\bar{\Omega})\longrightarrow C^{m,\nu}(\bar{\Omega}). \tag{7}$$

如果Ω是有界凸区域，则嵌入(3)和(7)是紧的．

证明　从明显的不等式

$$\|\phi;C^m(\bar{\Omega})\|\leqslant\|\phi;C^{m+1}(\bar{\Omega})\|,$$

$$\|\phi;C^m(\bar{\Omega})\|\leqslant\|\phi;C^{m,\lambda}(\bar{\Omega})\|$$

立即得到嵌入(3)和(4)的存在性．为了证实(5)，我们注意到对于$|\alpha|\leqslant m$,

$$\sup_{\substack{x,y\in\Omega\\0<|x-y|<1}}\frac{|D^\alpha\phi(x)-D^\alpha\phi(y)|}{|x-y|^\nu}$$

$$\leqslant\sup_{x,y\in\Omega}\frac{|D^\alpha\phi(x)-D^\alpha\phi(y)|}{|x-y|^\lambda}$$

和

$$\sup_{\substack{x,y\in\Omega\\|x-y|\geqslant1}}\frac{|D^\alpha\phi(x)-D^\alpha\phi(y)|}{|x-y|^\nu}\leqslant2\sup_{x\in\Omega}|D^\alpha\phi(x)|,$$

由此我们得到

$$\|\phi;C^{m,\nu}(\bar{\Omega})\|\leqslant3\|\phi;C^{m,\lambda}(\bar{\Omega})\|.$$

如果Ω是凸的且$x,y\in\Omega$，则由平均值定理，在联结x和y的线段上存在一点$z\in\Omega$使得$D^\alpha\phi(x)-D^\alpha\phi(y)=(x-y)\cdot\nabla D^\alpha\phi(z)$，其中$\nabla u=(D_1u,D_2u,\cdots,D_nu)$．于是

$$|D^\alpha\phi(x)-D^\alpha\phi(y)|\leqslant n|x-y|\,\|\phi;C^{m+1}(\bar{\Omega})\|, \tag{8}$$

所以

$$\|\phi;C^{m,1}(\overline{\Omega})\|\leqslant n\|\phi;C^{m+1}(\overline{\Omega})\|.$$

因此证明了(6)，从(5)和(6)就得到(7).

现在假定Ω是有界的. 如果A是$C^{0,\lambda}(\overline{\Omega})$中的有界集，则存在$M$使得对一切$\phi\in A$, $\|\phi;C^{0,\lambda}(\overline{\Omega})\|\leqslant M$. 但对一切$\phi\in A$和一切$x$, $y\in\Omega$, $|\phi(x)-\phi(y)|\leqslant M|x-y|^{\lambda}$，根据定理1.30，$A$是$C(\overline{\Omega})$中的准紧集. 这就证明了$m=0$时(4)是紧的. 如果$m\geqslant1$而且$A$在$C^{m,\lambda}(\overline{\Omega})$中是有界的，则$A$在$C^{0,\lambda}(\overline{\Omega})$中有界而且存在序列$\{\phi_j\}\subset A$使得在$C(\overline{\Omega})$中$\phi_j\to\phi$. 但$\{D_1\phi_j\}$在$C^{0,\lambda}(\overline{\Omega})$中也是有界的，所以$\{\phi_j\}$有一个子序列，仍记作$\{\phi_j\}$使得在$C(\overline{\Omega})$中$D_1\phi_j\to\psi_1$. $C(\overline{\Omega})$中的收敛性就是在Ω上的一致收敛，我们有$\psi_1=D_1\phi$. 我们可以继续以这样的方式抽取子序列，直到我们得到一个子序列，它在$C(\overline{\Omega})$中对每个α, $0\leqslant|\alpha|\leqslant m$，都有$D^\alpha\phi_j\to D^\alpha\phi$. 这就证明了(4)的紧性. 关于(5)的紧性我们论证如下，对$C^{m,\lambda}(\overline{\Omega})$的有界子集中的一切$\phi$

$$\begin{aligned}\frac{|D^\alpha\phi(x)-D^\alpha\phi(y)|}{|x-y|^\nu}&=\left(\frac{|D^\alpha\phi(x)-D^\alpha\phi(y)|}{|x-y|^\lambda}\right)^{\nu/\lambda}\\&\quad\times|D^\alpha\phi(x)-D^\alpha\phi(y)|^{1-\nu/\lambda}\\&\leqslant\text{const}\,|D^\alpha\phi(x)-D^\alpha\phi(y)|^{1-\nu/\lambda}.\end{aligned}\tag{9}$$

因为(9)式表明任何在$C^{m,\lambda}(\overline{\Omega})$中有界且在$C^m(\overline{\Omega})$中收敛的序列也在$C^{m,\nu}(\overline{\Omega})$中收敛，由(4)的紧性就得到(5)的紧性.

最后，如果Ω是凸且有界的，把连续嵌入(6)和$\lambda=1$时的紧嵌入(4)和(5)结合起来就得到(3)和(7)的紧性. ▌

在比Ω的凸性更弱的条件下能得到嵌入(6)和(7)的存在性以及(3)和(7)的紧性. 例如，如果任何点对$x,y\in\Omega$能用Ω中长度不超过$|x-y|$的某个固定倍数的可求长弧连起来的话，那么我们能得到一个和(8)类似的不等式，从而完成证明. 请读者证明(6)不是紧的.

$\mathbf{R}^n$ 中的 Lebesgue 测度

1.32 本专著所研究的许多函数空间是由 $\mathbf{R}^n$ 的区域上的 Lebesgue 意义下可积函数组成的. 我们假定多数读者是熟悉 Lebesgue 测度和积分的，即使如此本书中还是包含了 Lebesgue 理论的一个简短的讨论，特别是和以后要研究的 L^p 空间和 Sobolev 空间有关的那些方面. 所有的证明都略去了. 关于 Lebesgue 理论以及更一般的测度和积分的更完全和更系统的讨论，读者可以参考任何一本关于积分论的标准著作，例如 Munroe 的书[48].

1.33 $\mathbf{R}^n$ 中一组子集 Σ 叫做一个 σ-代数，如果下列条件成立：

(i) $\mathbf{R}^n \in \Sigma$.

(ii) 如果 $A \in \Sigma$，则 $A^c = \{x \in \mathbf{R}^n:\ x \notin A\} \in \Sigma$.

(iii) 如果 $A_j \in \Sigma$, $j=1, 2, \cdots$，则 $\bigcup_{j=1}^{\infty} A_j \in \Sigma$.

从(i)到(iii)立得

(iv) $\varnothing \in \Sigma$.

(v) 如果 $A_j \in \Sigma$, $j=1, 2, \cdots$，则 $\bigcap_{j=1}^{\infty} A_j \in \Sigma$.

(vi) 如果 $A, B \in \Sigma$，则 $A - B = A \cap B^c \in \Sigma$.

1.34 Σ 上的一个测度 μ 指的是 Σ 上的一个函数，函数值或取在 $\mathbf{R} \cup \{+\infty\}$ 中（正测度）或取在 $\mathbf{C}$ 中（复测度），这个函数在如下意义下是可数可加的，当 $A_j \in \Sigma$, $j=1, 2, \cdots$，而且对 $j \neq k$, $A_j \cap A_k = \varnothing$ 时

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j).$$

（对复测度而言，对任何这种序列 $\{A_j\}$，若右端的级数收敛，一定是

绝对收敛的.)如果 μ 是一个正测度，又如果 $A, B\in\Sigma$ 而且 $A\subset B$，则 $\mu(A)\leqslant\mu(B)$. 还有，如果 $A_j\in\Sigma$，$j=1, 2, \cdots$ 和 $A_1\subset A_2\subset\cdots$ 则 $\mu(\bigcup_{j=1}^{\infty}A_j)=\lim_{j\to\infty}\mu(A_j)$.

1.35 定理 存在一个由 $\mathbf{R}^n$ 中的子集合组成的 σ-代数 Σ 以及一个具有下列性质的 Σ 上的正测度 μ:

(i) $\mathbf{R}^n$ 中每个开集属于 Σ.

(ii) 如果 $A\subset B$, $B\in\Sigma$，而且 $\mu(B)=0$，则 $A\in\Sigma$ 而且 $\mu(A)=0$.

(iii) 如果 $A=\{x\in\mathbf{R}^n: a_j\leqslant x_j\leqslant b_j, j=1, 2, \cdots, n\}$，则 $A\in\Sigma$ 而且 $\mu(A)=\prod_{j=1}^{n}(b_j-a_j)$.

(iv) μ 是一个平移不变量，即，如果 $x\in\mathbf{R}^n$ 而且 $A\in\Sigma$，则

$$x+A=\{x+y: y\in A\}\in\Sigma \text{ 而且 } \mu(x+A)=\mu(A).$$

Σ 的元素叫做 $\mathbf{R}^n$ 的(Lebesgue)可测子集；μ 叫做 $\mathbf{R}^n$ 中的(Lebesgue)测度.（当我们所需要的只是 $\mathbf{R}^n$ 的测度时，在这些术语中将不用“Lebesgue”这个词)，对于 $A\in\Sigma$，我们把 $\mu(A)$ 叫做 A 的测度或 A 的体积，因为 Lebesgue 测度是 $\mathbf{R}^3$ 中体积概念的自然推广. 我们不去考虑“测度”和“体积”形式上的区别，对于在几何上容易形象化的集合(球，立方体区域)，我们常常采用“体积”这一术语，而且写作 vol$A$ 来代替 $\mu(A)$. 在 $\mathbf{R}^1$ 和 $\mathbf{R}^2$ 中术语长度和面积比体积更合适.

1.36 如果 $B\subset A\subset\mathbf{R}^n$ 而且 $\mu(B)=0$，则任何在集合 $A-B$ 的一切点上成立的条件叫做在 A 中几乎处处(a.e.)成立的条件. 易见 $\mathbf{R}^n$ 中可数集的测度为零，但其逆不真.

在可测集上定义而且在 $\mathbf{R}\cup\{-\infty, \infty\}$ 中取值的函数叫做可测函数，如果对一切实数 a，集合

$$\{x: f(x)>a\}$$

是可测的. 可测函数的一些比较重要的性质列在下面的定理中.

1.37 定理 (a) 如果 f 是可测函数, 则 $|f|$ 也是可测函数.

(b) 如果 f 和 g 是实值可测函数, 则 $f+g$ 和 fg 也是实值可测函数.

(c) 如果 $\{f_n\}$ 是一个可测函数序列, 则 $\sup\limits_n f_n$, $\inf\limits_n f_n$, $\limsup\limits_{n\to\infty} f_n$, $\liminf\limits_{n\to\infty} f_n$ 都是可测函数.

(d) 如果 f 是定义在可测集上的连续函数, 则 f 是可测函数,

(e) 如果 f 是 $\mathbf{R}$ 到 $\mathbf{R}$ 中的连续函数, 而 g 是实值可测函数, 则由 $f\circ g(x)=f(g(x))$ 定义的复合函数 $f\circ g$ 是可测函数.

(f) (Lusin 定理) 如果 f 是可测函数, 而对 $x\in A^c$, $\mu(A)<\infty$, $f(x)=0$, 又如果 $\varepsilon>0$, 则存在一个函数 $g\in C_0(A)$ 使得

$$\sup_{x\in\mathbf{R}^n}|g(x)|\leqslant\sup_{x\in\mathbf{R}^n}|f(x)| \text{ 而且 } \mu\{x\in\mathbf{R}^n: f(x)\neq g(x)\}<\varepsilon.$$

1.38 设 $A\subset\mathbf{R}^n$, 由

$$\chi_A=\begin{cases}1 & \text{当 } x\in A \text{ 时}\\ 0 & \text{当 } x\notin A \text{ 时}\end{cases}$$

定义的函数 χ_A 叫做 A 的特征函数. $\mathbf{R}^n$ 上的一个实值函数 s 叫做简单函数, 如果 s 的值域是有限个实数. 如果对于一切 x, $s(x)\in\{a_1, \cdots, a_m\}$, 则显然有 $s=\sum\limits_{j=1}^{m}a_j\chi_{A_j}$, 其中 $A_j=\{x\in\mathbf{R}^n: s(x)=a_j\}$, 而且 s 是可测的当且仅当 A_1, A_2, $\cdots$, A_m 是可测的. 由于下面的逼近定理, 简单函数在积分论中是一个非常有用的工具.

1.39 定理 给定一个定义域为 $A\subset\mathbf{R}^n$ 的实值函数, 存在一个简单函数的序列 $\{s_n\}$, $\{s_n\}$ 在 A 上点收敛到 f. 如果 f 是有界的, 则可以选到一致收敛到 f 的简单函数序列 $\{s_n\}$. 如果 f 是可测函数, 则每个 s_n 都可以选为可测函数. 如果 f 是非负的, 那么可以选到序列 $\{s_n\}$ 使得在每一点上 $\{s_n\}$ 是单调增加的.

Lebesgue 积分

1.40 现在我们能够给定义在可测集 $A\subset\mathbf{R}^n$ 上的实值可测函数的(Lebesgue)积分下定义了. 对于简单函数 $s=\sum_{j=1}^{m}a_j\chi_{A_j}$, 其中 $A_j\subset A$, A_j 是可测的, 我们定义

$$\int_A s(x)dx=\sum_{j=1}^{m}a_j\mu(A_j) \tag{10}$$

如果 f 是非负可测函数, 我们定义

$$\int_A f(x)dx=\sup\int_A s(x)dx \tag{11}$$

其中上确界是对在 A 外等于零而在 A 中满足 $0\leqslant s(x)\leqslant f(x)$ 的可测简单函数 $s(x)$ 取的. 如果 $f(x)$ 是非负简单函数, 则由(10)和(11)给出的 $\int_A f(x)dx$ 的两个定义是一致的. 注意非负函数的积分可以是 $+\infty$.

如果 f 是实值可测函数, 我们令 $f=f^+-f^-$, 其中
$f^+=\max(f,0)$ 而 $f^-=-\min(f,0)$, 它们都是非负可测函数. 假如右端的两个积分至少有一个是有限的, 我们定义

$$\int_A f(x)dx=\int_A f^+(x)dx-\int_A f^-(x)dx.$$

如果上式中的右端的两个积分都是有限的, 我们说 f 是 A 上的(Lebesgue)可积函数, A 上的可积函数类记作 $L^1(A)$.

1.41 **定理** 假定下面出现的一切函数和集合都是可测的.

(*a*) 如果 f 在 A 上有界而且 $\mu(A)<\infty$, 则 $f\in L^1(A)$.

(*b*) 如果对一切 $x\in A$, $a\leqslant f(x)\leqslant b$, 又如果 $\mu(A)<\infty$, 则

$$a\mu(A)\leqslant\int_A f(x)dx\leqslant b\mu(A).$$

(*c*) 如果对一切 $x\in A$, $f(x)\leqslant g(x)$, 又如果 f, g 的积分都存

在，则

$$\int_A f(x)dx \leqslant \int_A g(x)dx.$$

(d) 如果 $f, g \in L^1(A)$，则 $f+g \in L^1(A)$ 而且

$$\int_A (f+g)(x)dx = \int_A f(x)dx + \int_A g(x)dx.$$

(e) 如果 $f \in L^1(A)$ 而 $c \in \mathbf{R}$，则 $cf \in L^1(A)$，而且

$$\int_A (cf)(x)dx = c\int_A f(x)dx.$$

(f) 如果 $f \in L^1(A)$，则 $|f| \in L^1(A)$，而且

$$\left|\int_A f(x)dx\right| \leqslant \int_A |f(x)|dx.$$

(g) 如果 $f \in L^1(A)$ 而 $B \subset A$，则 $f \in L^1(B)$；此外，如果对一切 $x \in A$，$f(x) \geqslant 0$，则

$$\int_B f(x)dx \leqslant \int_A f(x)dx.$$

(h) 如果 $\mu(A)=0$，则 $\int_A f(x)dx = 0$.

(i) 如果 $f \in L^1(A)$，又如果对一切 $B \subset A$ 有 $\int_B f(x)dx = 0$，则在 A 上几乎处处有 $f(x)=0$.

1.42 定理 如果 f 是 $L^1(\mathbf{R}^n)$ 中的一个元素或是 $\mathbf{R}^n$ 上的非负可测函数，则由

$$\lambda(A) = \int_A f(x)dx$$

定义的集合函数 λ 是可数可加的，因此在由 $\mathbf{R}^n$ 的 Lebesgue 可测子集构成的 σ-代数上 $\lambda(A)$ 是一个测度.

积分的这种可加性的一个推论是对于积分来说测度为零的子集可以忽略，即，如果 f 和 g 是 A 上的可测函数而且在 A 上 $f(x) = g(x)$ a. e.，则 $\int_A f(x)dx = \int_A g(x)dx$. 因此，$L^1(A)$ 中两个几乎处

处相等的函数被看作是一样的.

下面三个定理涉及积分和极限过程的交换问题.

1.43 定理 (单调收敛定理) 设 $A\subset\mathbf{R}^n$ 是可测集, 而$\{f_n\}$是一个可测函数序列, 对一切 $x\in A$ 满足 $0\leqslant f_1(x)\leqslant f_2(x)\leqslant\cdots$. 则

$$\lim_{n\to\infty}\int_A f_n(x)dx=\int_A(\lim_{n\to\infty}f_n(x))dx.$$

1.44 (Fatou 引理) 设 $A\subset\mathbf{R}^n$ 是可测集, 又设$\{f_n\}$是一个非负可测函数序列. 则

$$\int_A(\liminf_{n\to\infty}f_n(x))dx\leqslant\liminf_{n\to\infty}\int_A f_n(x)dx.$$

1.45 定理 (控制收敛定理)设 $A\subset\mathbf{R}^n$ 是可测集, 又设可测函数序列$\{f_n\}$在 A 上逐点收敛到一个极限函数. 如果存在一个函数 $g\in L^1(A)$使得对一切 n 和一切 $x\in A$ 有 $|f_n(x)|\leqslant g(x)$, 则

$$\lim_{n\to\infty}\int_A f_n(x)dx=\int_A(\lim_{n\to\infty}f_n(x))dx.$$

1.46 可测集 $A\subset\mathbf{R}^n$上复值函数的积分定义如下. 令$f=u+iv$, 其中 u 和 v 是实值的, f 叫做可测的当且仅当 u, v 都是可测函数. 假如$|f|=(u^2+v^2)^{1/2}$在 1.40 节所说的意义下属于 $L^1(A)$, 我们就说 f 在 A 上是可积的, 且记作 $f\in L^1(A)$. 对于 $f\in L^1(A)$而且只是对这种 f, f 的积分由

$$\int_A f(x)dx=\int_A u(x)dx+i\int_A v(x)dx$$

定义. 容易验证 $f\in L^1(A)$当且仅当 $u, v\in L^1(A)$. 定理 1.37($a, b, d—f$), 定理 1.41($a, d—i$), 定理 1.42[假定 $f\in L^1(\mathbf{R}^n)$] 以及定理 1.45 都可以推广到 f 是复的情形.

下面的定理使我们能够借助于 Lebesgue 测度 μ 来表示某些复测度, 它是定理 1.42 的逆定理.

1.47 定理 (Radon-Nikodym 定理) 设 λ 是定义在 $\mathbf{R}^n$ 上

的 Lebesgue 可测子集构成的 σ-代数 Σ 上的复测度. 假定对一切使 $\mu(A)=0$ 的 $A\in\Sigma$, $\lambda(A)=0$, 则存在 $f\in L^1(\mathbf{R}^n)$使得对一切 $A\in\Sigma$

$$\lambda(A)=\int_A f(x)dx.$$

函数 f 由 λ 在差一个零测集的意义下完全确定.

1.48 如果 f 是定义在 $\mathbf{R}^{n+m}$ 的子集 A 上的函数,我们可以认为 f 是依赖于一对变量 (x, y) 的函数, $x\in\mathbf{R}^n$ 而 $y\in\mathbf{R}^m$, f 在 A 上的积分通常表为

$$\int_A f(x, y)dxdy,$$

或者,如果还要求在 $\mathbf{R}^{n+m}$ 都有积分的话,还可表为

$$\int_{\mathbf{R}^{n+m}} f(x, y)\chi_A(x, y)dxdy,$$

其中 χ_A 是 A 的特征函数. 特别, 如果 $A\subset\mathbf{R}^n$, 我们可以写为

$$\int_A f(x)dx=\int_A f(x_1, \cdots, x_n)dx_1\cdots dx_n.$$

1.49 定理 (Fubini 定理) 设 f 是 $\mathbf{R}^{n+m}$ 上的可测函数而且假定积分

$$\begin{aligned} I_1 &=\int_{\mathbf{R}^{n+m}} |f(x, y)|dxdy, \\ I_2 &=\int_{\mathbf{R}^m}\left(\int_{\mathbf{R}^n} |f(x, y)|dx\right)dy, \\ I_3 &=\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^m} |f(x, y)|dy\right)dx \end{aligned} \tag{12}$$

中至少有一个存在且有限, 则

(*a*) 对于几乎一切 $y\in\mathbf{R}^m$, $f(\cdot, y)\in L^1(\mathbf{R}^n)$,

(*b*) 对于几乎一切 $x\in\mathbf{R}^n$, $f(x, \cdot)\in L^1(\mathbf{R}^m)$,

(*c*) $\int_{\mathbf{R}^n} f(x, \cdot)dx\in L^1(\mathbf{R}^m)$,

(d) $\int_{\mathbf{R}^m} f(\cdot, y)dy \in L^1(\mathbf{R}^n)$，而且

(e) $I_1 = I_2 = I_3$。

广义函数和弱导数

1.50 在后面几章中我们需要 Schwartz 广义函数论[60]中的某些基本概念和技巧. 这里把与我们有关的广义函数论的那些方面作一个简单的介绍，特别重要的是可积函数的弱导数或广义函数意义下的导数概念. Sobolev 空间的一种标准定义就是用这种导数作为语言表示出来的(3.1节). 对于下面要引进的空间$\mathscr{D}(\Omega)$和$\mathscr{D}'(\Omega)$以及这些空间的推广的更全面的讨论，除文献[60]外介绍读者去看 Rudin 的[59]和 Yosida 的[69].

1.51 设 Ω 是 $\mathbf{R}^n$ 中的一个区域. $C_0^\infty(\Omega)$中的一个函数序列$\{\phi_n\}$叫做在 $\mathscr{D}(\Omega)$空间的意义下收敛到函数 $\phi \in C_0^\infty(\Omega)$，假如满足以下的条件:

(i) 存在 $K \subset\subset \Omega$ 使得对一切 n, $\operatorname{supp}(\phi_n - \phi) \subset K$，而且

(ii) 对于每个多重指标 α, $\lim\limits_{n\to\infty} D^\alpha \phi_n(x) = D^\alpha \phi(x)$在$K$上是一致的.

在向量空间 $C_0^\infty(\Omega)$ 上存在一个局部凸拓扑，关于这个拓扑一个线性泛函是连续的充要条件是当在 $\mathscr{D}(\Omega)$空间意义下 $\phi_n \to \phi$ 时在 $\mathbf{C}$ 中就有 $T(\phi_n) \to T(\phi)$. 这个 TVS 就叫做 $\mathscr{D}(\Omega)$，而它的元素叫做试验函数. $\mathscr{D}(\Omega)$不是一个可赋范的空间. (我们不去过问上面所说的拓扑的唯一性问题. 这个拓扑唯一地决定了$\mathscr{D}(\Omega)$的对偶空间，对于我们说来有这一点就足够了.)

1.52 $\mathscr{D}(\Omega)$的对偶空间 $\mathscr{D}'(\Omega)$叫做(Schwartz)广义函数空间. 作为 $\mathscr{D}(\Omega)$ 的对偶赋予 $\mathscr{D}'(\Omega)$ 以弱星拓扑，从而关于这种拓扑 $\mathscr{D}'(\Omega)$是一个局部凸 TVS. 我们综述 $\mathscr{D}'(\Omega)$中向量空间的运算

和收敛运算如下：如果 $S, T, T_n \in \mathscr{D}'(\Omega)$ 而 $c \in \mathbf{C}$，则

$$(S+T)(\phi)=S(\phi)+T(\phi),\ \phi\in\mathscr{D}(\Omega),$$
$$(cT)(\phi)=cT(\phi),\qquad \phi\in\mathscr{D}(\Omega),$$

在 $\mathscr{D}'(\Omega)$ 中 $T_n \to T$ 当且仅当对一切 $\phi \in \mathscr{D}(\Omega)$ 在 $\mathbf{C}$ 中 $T_n(\phi) \to T(\phi)$.

1.53 Ω 上几乎处处有定义的函数 u 叫做在 Ω 上局部可积的，假如对每个可测集 $A \subset\subset \Omega$，$u \in L^1(A)$. 这时我们记作 $u \in L^1_{\mathrm{loc}}(\Omega)$. 与每个 $u \in L^1_{\mathrm{loc}}(\Omega)$ 对应存在一个由

$$T_u(\phi)=\int_\Omega u(x)\phi(x)dx,\qquad \phi\in\mathscr{D}(\Omega) \tag{13}$$

定义的广义函数 $T_u \in \mathscr{D}'(\Omega)$. 显然这样定义的 T_u 是 $\mathscr{D}(\Omega)$ 上的线性泛函. 为了看出它是连续的，假定在 $\mathscr{D}(\Omega)$ 中 $\phi_n \to \phi$. 则存在 $K \subset\subset \Omega$ 使得对 $n=1, 2, 3, \cdots$, $\mathrm{supp}(\phi_n-\phi) \subset K$. 于是

$$|T_u(\phi_n)-T_u(\phi)|\leqslant\sup_{x\in K}|\phi_n(x)-\phi(x)|\int_K|u(x)|dx.$$

因为在 K 上一致地有 $\phi_n \to \phi$，上述不等式的右端当 $n\to\infty$ 时趋于零.

1.54 并非每个广义函数 $T \in \mathscr{D}'(\Omega)$ 都可对某个 $u \in L^1_{\mathrm{loc}}(\Omega)$ 表为[由(13)定义的]形式 $T=T_u$. 实际上，假定 $0 \in \Omega$，读者可以相信不可能存在 Ω 上的局部可积函数 δ 使得对一切 $\phi \in \mathscr{D}(\Omega)$

$$\int_\Omega \delta(x)\phi(x)dx=\phi(0).$$

但是，易见由

$$\delta(\phi)=\phi(0) \tag{14}$$

给出的定义在 $\mathscr{D}(\Omega)$ 上的线性泛函 δ 是连续的，因此是 Ω 上的一个广义函数.

1.55 设 $u \in C^1(\Omega)$ 而 $\phi \in \mathscr{D}(\Omega)$. 因为在 Ω 的某个紧子集外 ϕ 恒等于零，对变量 x_j 分部积分得到

$$\int_\Omega\left(\frac{\partial}{\partial x_j}u(x)\right)\phi(x)dx=-\int_\Omega u(x)\left(\frac{\partial}{\partial x_j}\phi(x)\right)dx$$

类似地，如果 $u\in C^{|\alpha|}(\Omega)$，分部积分 $|\alpha|$ 次就导致

$$\int_\Omega(D^\alpha u(x))\phi(x)dx=(-1)^{|\alpha|}\int_\Omega u(x)D^\alpha\phi(x)dx.$$

这就启发了以下关于广义函数 $T\in\mathscr{D}'(\Omega)$ 的导数 $D^\alpha T$ 的定义：

$$(D^\alpha T)(\phi)=(-1)^{|\alpha|}T(D^\alpha\phi) \tag{15}$$

因为当 $\phi\in\mathscr{D}(\Omega)$ 时 $D^\alpha\phi\in\mathscr{D}(\Omega)$，$D^\alpha T$ 是 $\mathscr{D}(\Omega)$ 上的一个泛函．显然 $D^\alpha T$ 在 $\mathscr{D}(\Omega)$ 上是线性的．我们将证明 $D^\alpha T$ 是连续的，因而是 Ω 上的广义函数．为此，假定在 $\mathscr{D}(\Omega)$ 中 $\phi_n\to\phi$，则对某个 $K\subset\subset\Omega$

$$\operatorname{supp}(D^\alpha(\phi_n-\phi))\subset\operatorname{supp}(\phi_n-\phi)\subset K.$$

而且

$$D^\beta[D^\alpha(\phi_n-\phi)]=D^{\beta+\alpha}(\phi_n-\phi)$$

当 $n\to\infty$ 时对每个多重指标 β 在 K 上一致收敛到零．因此在 $\mathscr{D}(\Omega)$ 中 $D^\alpha\phi_n\to D^\alpha\phi$．因为 $T\in\mathscr{D}'(\Omega)$，就得到在 $\mathbf{C}$ 中

$D^\alpha T(\phi_n)=(-1)^{|\alpha|}T(D^\alpha\phi_n)\to(-1)^{|\alpha|}T(D^\alpha\phi)=D^\alpha T(\phi)$，于是 $D^\alpha T\in\mathscr{D}'(\Omega)$．

我们已经证明 $\mathscr{D}'(\Omega)$ 中的每个广义函数在定义(15)的意义下在 $\mathscr{D}'(\Omega)$ 中有任意阶的导数，而且从 $\mathscr{D}'(\Omega)$ 到 $\mathscr{D}'(\Omega)$ 中的映射是连续的．如果在 $\mathscr{D}'(\Omega)$ 中 $T_n\to T$，又如果 $\phi\in\mathscr{D}(\Omega)$，则

$$D^\alpha T_n(\phi)=(-1)^{|\alpha|}T_n(D^\alpha\phi)\to(-1)^{|\alpha|}T(D^\alpha\phi)=D^\alpha T(\phi).$$

1.56 例子 (1)如果 $0\in\Omega$ 而 $\delta\in\mathscr{D}'(\Omega)$ 是由(14)定义的，则 $D^\alpha\delta$ 由

$$D^\alpha\delta(\phi)=(-1)^{|\alpha|}D^\alpha\phi(0)$$

给出．

(2) 如果 $\Omega=\mathbf{R}$（即 $n=1$）且 $H\in L^1_{\mathrm{loc}}(\Omega)$ 是由

$$H(x)=\begin{cases}1 & 当\ x\geqslant 0\\ 0 & 当\ x<0\end{cases}$$

定义的，则导数$(T_H)'$是δ，因为如果$\phi\in\mathscr{D}(\mathbf{R})$在区间$[-a,a]$中有支集，则

$$(T_H)'(\phi)=-T_H(\phi')=-\int_0^a\phi'(x)dx=\phi(0)=\delta(\phi).$$

1.57 现在我们来定义局部可积函数弱导数的概念．设$u\in L^1_{\mathrm{loc}}(\Omega)$．使得在$\mathscr{D}'(\Omega)$中$T_{v_\alpha}=D^\alpha(T_u)$成立的函数$v_\alpha\in L^1_{\mathrm{loc}}(\Omega)$不一定存在．如果这样的$v_\alpha$存在，在差一个零测集的意义下$v_\alpha$是唯一的，而且把$v_\alpha$叫做$u$的弱偏导数或广义函数意义下的偏导数，用$D^\alpha u$来记．于是假如$v_\alpha\in L^1_{\mathrm{loc}}(\Omega)$对一切$\phi\in\mathscr{D}(\Omega)$满足

$$\int_\Omega u(x)D^\alpha\phi(x)dx=(-1)^{|\alpha|}\int_\Omega v_\alpha(x)\phi(x)dx$$

则在弱(广义函数)的意义下$D^\alpha u=v_\alpha$．

如果u是具有通常(古典)意义下连续偏导数$D^\alpha u$的充分光滑的函数，则$D^\alpha u$也是u的广义函数意义下的偏导数．当然$D^\alpha u$可以在古典意义下不存在而在广义函数意义下存在，例如$\mathbf{R}$上除有限个点外有有界导数的连续函数u，具有广义函数意义下的导数．在定理3.16中我们将证明具有弱导数的函数能用光滑函数来逼近．

1.58 最后我们注意到能用光滑函数去乘Ω上的广义函数．如果$T\in\mathscr{D}'(\Omega)$而且$\omega\in C^\infty(\Omega)$，积$\omega T\in\mathscr{D}'(\Omega)$由

$$(\omega T)(\phi)=T(\omega\phi),\ \phi\in\mathscr{D}(\Omega)$$

定义．如果对某个$u\in L^1_{\mathrm{loc}}(\Omega)$，$T=T_u$，则$\omega T=T_{\omega u}$．容易验证Leibniz法则(见1.1节)对$D^\alpha(\omega T)$成立．

第二章　空间 $L^p(\Omega)$

定义和基本性质

2.1　设 Ω 是 $\mathbf{R}^n$ 中一个区域，又设 p 是正实数. 我们用 $L^p(\Omega)$ 表示定义在 Ω 上，所有满足

$$\int_\Omega |u(x)|^p dx<\infty \tag{1}$$

的可测函数 u 构成的函数类. 在 $L^p(\Omega)$ 中，我们把在 Ω 上几乎处处相等的函数看成是一样的. 因此 $L^p(\Omega)$ 中的元素实际上是满足(1)的可测函数的等价类. 两个函数是等价的，如果它们几乎处处相等. 但是，为了方便起见，我们忽略了这种差别，当 u 满足(1)时就写作 $u\in L^p(\Omega)$，当在 Ω 中 $u(x)=0$ a.e. 时，就说在 $L^p(\Omega)$ 中 $u=0$. 显然，如果 $u\in L^p(\Omega)$ 且 $c\in\mathbf{C}$，则 $cu\in L^p(\Omega)$. 而且，如果 $u, v\in L^p(\Omega)$，那么由于

$$\begin{aligned}|u(x)+v(x)|^p&\leqslant(|u(x)|+|v(x)|)^p\\&\leqslant 2^p(|u(x)|^p+|v(x)|^p),\end{aligned}$$

因此 $u+v\in L^p(\Omega)$，所以 $L^p(\Omega)$ 是一个向量空间.

2.2　假如 $1\leqslant p<\infty$，我们现在来验证由

$$\|u\|_p=\left\{\int_\Omega |u(x)|^p dx\right\}^{\frac{1}{p}}$$

定义的泛函 $\|\cdot\|_p$ 是 $L^p(\Omega)$ 上的一个范数. （如果 $0<p<1$ 它不是范数.）在论证中凡是可能发生区域混淆的时候，我们将用 $\|\cdot\|_{p,\Omega}$ 来代替 $\|\cdot\|_p$. 显然 $\|u\|_p\geqslant 0$，而且等号当且仅当在 $L^p(\Omega)$ 中 $u=0$ 时成立. 而且

$$\|cu\|_p=|c|\,\|u\|_p,\quad c\in\mathbf{C}.$$

剩下还要证明，当 $1\leqslant p<\infty$ 时

$$\|u+v\|_p\leqslant\|u\|_p+\|v\|_p \tag{2}$$

这就是众所周知的 Minkowski 不等式．对于 $p=1$，条件(2)一定成立，因为

$$\int_\Omega|u(x)+v(x)|dx\leqslant\int_\Omega|u(x)|dx+\int_\Omega|v(x)|dx.$$

当 $1<p<\infty$ 时，我们用 p' 表示数 $p/(p-1)$，所以 $1<p'<\infty$，且

$$(1/p)+(1/p')=1.$$

p' 叫做 p 的共轭指数．

2.3 定理 (Hölder 不等式) 如果 $1<p<\infty$ 而且 $u\in L^p(\Omega)$，$v\in L^{p'}(\Omega)$，则 $uv\in L^1(\Omega)$，并且

$$\int_\Omega|u(x)v(x)|dx\leqslant\|u\|_p\|v\|_{p'} \tag{3}$$

证明 对于 $t\geqslant0$，函数 $f(t)=(t^p/p)+(1/p')-t$ 的最小值为 0，而且这个最小值只在 $t=1$ 达到．对于非负数 a, b，令 $t=ab^{-p'/p}$，得到

$$ab\leqslant(a^p/p)+(b^{p'}/p'), \tag{4}$$

等号当且仅当 $a^p=b^{p'}$ 时成立，如果 $\|u\|_p=0$ 或 $\|v\|_{p'}=0$，那么在 Ω 中 $u(x)v(x)=0$ a. e. 所以(3)成立．不然的话，在(4)式中令 $a=|u(x)|/\|u\|_p$ 和 $b=|v(x)|/\|v\|_{p'}$，在 Ω 上积分就得到(3)，(3)中等号当且仅当在 Ω 中 $|u(x)|^p$ 和 $|v(x)|^{p'}$ 几乎处处成比例时才成立．∎

我们指出用同样的方式能证明有限和或无限和形式的Hölder不等式

$$\Sigma|a_kb_k|\leqslant\{\Sigma|a_k|^p\}^{\frac{1}{p}}\{\Sigma|b_k|^{p'}\}^{\frac{1}{p'}}.$$

2.4 定理 (Minkowski 不等式) 如果 $1\leqslant p<\infty$，则

$$\|u+v\|_p\leqslant\|u\|_p+\|v\|_p. \tag{5}$$

证明 $p=1$ 的情形我们早已证明，所以假定 $1<p<\infty$．我们还可

以假定 $u, v\in L^p(\Omega)$，因为不然的话(5)式的右端是无穷．分别应用 Hölder 不等式得到

$$\begin{aligned}&\int_\Omega |u(x)+v(x)|^p dx\\ \leqslant&\int_\Omega |u(x)+v(x)|^{p-1}(|u(x)|+|v(x)|)dx\\ \leqslant&\left\{\int_\Omega |u(x)+v(x)|^p dx\right\}^{1/p'}(\|u\|_p+\|v\|_p).\end{aligned}$$

由于 $\|u+v\|_p<\infty$，在不等式两边消去 $\left\{\int_\Omega |u(x)+v(x)|^p dx\right\}^{\frac{1}{p'}}$，就得到不等式(5)． ▮

2.5 Ω 上的可测函数 u 叫做在 Ω 上本性有界，假如存在一个常数 K 使得在 Ω 上 $|u(x)|\leqslant K$ a.e.. 常数 K 的下确界叫做 u 在 Ω 上的本性上确界并用 $\operatorname{ess\,sup}_{x\in\Omega}|u(x)|$ 来表示．我们用 $L^\infty(\Omega)$ 表示 Ω 上全体本性有界函数 u 组成的向量空间，和前面一样，把 Ω 上几乎处处相等的函数看成是一样的．容易验证，由

$$\|u\|_\infty=\operatorname*{ess\,sup}_{x\in\Omega}|u(x)|$$

定义的泛函 $\|\cdot\|_\infty$ 是 $L^\infty(\Omega)$ 上的一个范数．而且，显然可以把 Hölder 不等式(3)推广到包括 $p=1$，$p'=\infty$ 和 $p=\infty, p'=1$ 这两种情形．

下面两个定理建立了 $0<p<1$ 情形的 Hölder 和 Minkowski 不等式的逆形式．以后在证明某些 L^p-空间的一致凸性时将要用到后一个不等式．

2.6　定理　设 $0<p<1$，因此 $p'=p/(p-1)<0$．假定 $f\in L^p(\Omega)$，和

$$0<\int_\Omega |g(x)|^{p'}dx<\infty$$

则

$$\int_\Omega |f(x)g(x)|dx \geqslant \left\{\int_\Omega |f(x)|^p dx\right\}^{\frac{1}{p}} \left\{\int_\Omega |g(x)|^{p'} dx\right\}^{\frac{1}{p'}} \tag{6}$$

证明 我们可以假定 $fg \in L^1(\Omega)$，否则(6)的左端成为无穷大．令 $\phi = |g|^{-p}$ 和 $\psi = |fg|^p$，因此 $\phi\psi = |f|^p$．则 $\psi \in L^q(\Omega)$，其中 $q = \dfrac{1}{p} > 1$，而且由于 $p' = -pq'$，其中 $q' = q/(q-1)$，我们有 $\phi \in L^{q'}(\Omega)$．直接利用 Hölder 不等式(3)我们有

$$\begin{aligned}\int_\Omega |f(x)|^p dx &= \int_\Omega \phi(x)\psi(x)dx \leqslant \|\psi\|_q \|\phi\|_{q'} \\ &= \left\{\int_\Omega |f(x)g(x)|dx\right\}^p \left\{\int_\Omega |g(x)|^{p'} dx\right\}^{1-p}\end{aligned}$$

开 p 次方，再除以右端最后那个因子，就得到(6)式．∎

2.7 定理 设 $0 < p < 1$．如果 $u, v \in L^p(\Omega)$，则

$$\| |u| + |v| \|_p \geqslant \|u\|_p + \|v\|_p. \tag{7}$$

证明 如果在 $L^p(\Omega)$ 中 $u = v = 0$，则 (7) 式总是成立的．否则，(7)的左端大于零．应用逆 Hölder 不等式(6)，得到

$$\begin{aligned}\| |u| + |v| \|_p^p &= \int_\Omega (|u(x)| + |v(x)|)^{p-1}(|u(x)| + |v(x)|)dx \\ &\geqslant \left\{\int_\Omega (|u(x)| + |v(x)|)^{(p-1)p'} dx\right\}^{1/p'} (\|u\|_p + \|v\|_p) \\ &= \| |u| + |v| \|_p^{p/p'} (\|u\|_p + \|v\|_p)\end{aligned}$$

消去 $\| |u| + |v| \|_p^{p/p'}$ 就得到(7)．∎

下面的定理给出了体积有界的区域上的 L^p-空间的一个有用的嵌入结果和这个嵌入的一些推论．

2.8 定理 假定 $\operatorname{vol}\Omega = \int_\Omega 1\, dx < \infty$ 而且 $1 \leqslant p \leqslant q \leqslant \infty$．如果 $u \in L^q(\Omega)$，则 $u \in L^p(\Omega)$ 并且

$$\|u\|_p \leqslant (\operatorname{vol}\Omega)^{(1/p)-(1/q)} \|u\|_q. \tag{8}$$

因此

$$L^q(\Omega)\to L^p(\Omega). \tag{9}$$

如果 $u\in L^\infty(\Omega)$，则

$$\lim_{p\to\infty}\|u\|_p=\|u\|_\infty. \tag{10}$$

最后，如果对于 $1\leqslant p<\infty$，$u\in L^p(\Omega)$，又如果存在一个常数 K 使得对于一切 p

$$\|u\|_p\leqslant K, \tag{11}$$

则 $u\in L^\infty(\Omega)$ 且

$$\|u\|_\infty\leqslant K \tag{12}$$

证明 当 $p=q$ 时，(8)和(9)是平凡的.当$1\leqslant p<q\leqslant\infty$且$u\in L^q(\Omega)$时，Hölder 不等式给出

$$\int_\Omega|u(x)|^p dx\leqslant\left\{\int_\Omega|u(x)|^q dx\right\}^{\frac{p}{q}}\left\{\int_\Omega 1dx\right\}^{1-\frac{p}{q}}$$

由此立得(8)和(9). 如果 $u\in L^\infty(\Omega)$，从(8)式我们得到

$$\limsup_{p\to\infty}\|u\|_p\leqslant\|u\|_\infty. \tag{13}$$

另一方面，对于任何 $\varepsilon>0$ 存在一个集合 $A\subset\Omega$，它的测度 $\mu(A)$ 为正，使得

$$|u(x)|\geqslant\|u\|_\infty-\varepsilon,\text{ 如果 } x\in A.$$

因此

$$\int_\Omega|u(x)|^p dx\geqslant\int_A|u(x)|^p dx\geqslant\mu(A)(\|u\|_\infty-\varepsilon)^p.$$

由此得到 $\|u\|_p\geqslant(\mu(A))^{\frac{1}{p}}(\|u\|_\infty-\varepsilon)$，因此

$$\liminf_{p\to\infty}\|u\|_p\geqslant\|u\|_\infty \tag{14}$$

从(13)和(14)就得到等式(10).

现在假定对于 $1\leqslant p<\infty$ (11) 式成立. 如果 $u\notin L^\infty(\Omega)$ 或者 $u\in L^\infty(\Omega)$ 但 (12) 式不成立，那么能找到常数 $K_1>K$ 和集合

$A\subset\Omega,\mu(A)>0$，使得对于 $x\in A$, $|u(x)|\geqslant K_1$. 用得到(14)式的同样的论证方法可以证明

$$\liminf_{p\to\infty}\|u\|_p\geqslant K_1,$$

这和(11)式矛盾. ▌

2.9 推论 对于 $1\leqslant p\leqslant\infty$ 和任何区域 Ω, $L^p(\Omega)\subset L^1_{\mathrm{loc}}(\Omega)$.

$L^p(\Omega)$ 的完备性

2.10 定理 如果 $1\leqslant p\leqslant\infty$, 则 $L^p(\Omega)$ 是一个 Banach 空间.

证明 先假定 $1\leqslant p<\infty$, 又设 $\{u_n\}$ 是 $L^p(\Omega)$ 中的一个 Cauchy 序列. 存在 $\{u_n\}$ 的一个子序列 $\{u_{n_j}\}$ 使得

$$\|u_{n_{j+1}}-u_{n_j}\|_p\leqslant 1/2^j,\quad j=1,2,\cdots.$$

设 $v_m(x)=\sum_{j=1}^{m}|u_{n_{j+1}}(x)-u_{n_j}(x)|$. 则

$$\|v_m\|_p\leqslant\sum_{j=1}^{m}(1/2^j)<1,\quad m=1,2,\cdots.$$

令 $v(x)=\lim_{m\to\infty}v_m(x)$, 在某些 x 处 $v(x)$ 可以是无穷，由 Fatou 引理 1.44 我们得到

$$\int_\Omega|v(x)|^p dx\leqslant\liminf_{m\to\infty}\int_\Omega|v_m(x)|^p dx\leqslant 1.$$

因此在 Ω 中 $v(x)<\infty$ a.e. 而且级数

$$u_{n_1}(x)+\sum_{j=1}^{\infty}(u_{n_{j+1}}(x)-u_{n_j}(x))\tag{15}$$

在 Ω 中几乎处处收敛到极限 $u(x)$. 当 $u(x)$ 作为(15)式的极限没有定义时就令 $u(x)=0$. 缩并(15)式，我们有

$$\lim_{m\to\infty}u_{n_m}(x)=u(x)\quad\text{a.e.}\ .\ 在\Omega中$$

对于任何 $\varepsilon>0$ 存在 N 使得当 $m,n\geqslant N$ 时，$\|u_m-u_n\|_p<\varepsilon$. 因此，再次利用 Fatou 引理，当 $n\geqslant N$ 时，

$$\int_\Omega |u(x)-u_n(x)|^p dx = \int_\Omega \lim_{j\to\infty} |u_{n_j}(x)-u_n(x)|^p dx$$

$$\leqslant \liminf_{j\to\infty} \int_\Omega |u_{n_j}(x)-u_n(x)|^p dx \leqslant \varepsilon^p.$$

于是 $u=(u-u_n)+u_n \in L^p(\Omega)$并且当 $n\to\infty$ 时 $\|u-u_n\|_p \to 0$. 所以 $L^p(\Omega)$是完备的.

最后，如果$\{u_n\}$是 $L^\infty(\Omega)$中的 Cauchy 序列，则存在一个零测集 $A\subset\Omega$ 使得如果 $x\notin A$，则对一切 $n, m=1, 2, \cdots$

$$|u_n(x)| \leqslant \|u_n\|_\infty, \quad |u_n(x)-u_m(x)| \leqslant \|u_n-u_m\|_\infty.$$

由于$\{\|u_n\|_\infty\}$在 $\mathbf{R}$ 中有界，u_n 在 $\Omega \sim A$ 上一致收敛到一个有界函数 u. 对于 $x\in A$，令 $u(x)=0$，我们有 $u\in L^\infty(\Omega)$并且当 $n\to\infty$ 时 $\|u_n-u\|_\infty \to 0$. 因此 $L^\infty(\Omega)$是完备的. ▌

2.11 推论 如果 $1\leqslant p\leqslant\infty$，$L^p(\Omega)$中的 Cauchy 序列有一个在 Ω 上几乎处处点收敛的子序列.

2.12 推论

对于内积

$$(u, v) = \int_\Omega u(x)\overline{v(x)}dx,$$

$L^2(\Omega)$是一个 Hilbert 空间. $L^2(\Omega)$的 Hölder 不等式实际上正好是众所周知的 Schwarz 不等式

$$|(u, v)| \leqslant \|u\|_2 \|v\|_2.$$

用连续函数来逼近，可分性

2.13 定理 如果 $1\leqslant p<\infty$，则 $C_0(\Omega)$在 $L^p(\Omega)$中稠密.

证明 设 $u\in L^p(\Omega)$，又设 $\varepsilon>0$. 我们证明存在一个函数 $\phi\in C_0(\Omega)$使得$\|u-\phi\|_p<\varepsilon$. 令 $u=u_1-u_2+i(u_3-u_4)$，其中 u_j，$1\leqslant j\leqslant 4$，是非负实值函数. 我们找到 $\phi_j\in C_0(\Omega)$使得$\|\phi_j-u_j\|_p<\varepsilon/4$，$1\leqslant j\leqslant 4$. 那么$\|u-\phi_1+\phi_2-i(\phi_3-\phi_4)\|_p<\varepsilon$. 所以不失一般性，我们假定

u 是非负实值函数. 由定理 1.39, 存在一个在 Ω 上点收敛到 u 的单调增加、非负简单函数的序列 $\{s_n\}$. 因为 $0\leqslant s_n(x)\leqslant u(x)$, 我们有 $s_n\in L^p(\Omega)$. 由于 $(u(x)-s_n(x))^p\leqslant(u(x))^p$, 根据控制收敛定理 1.45, 在 $L^p(\Omega)$ 中 $s_n\to u$. 因此, 我们就可以挑选 $s\in\{s_n\}$ 使得 $\|u-s\|_p<\frac{\varepsilon}{2}$. 由于 s 是简单函数而且 $p<\infty$, s 的支集一定有有限体积. 我们还可以假定对所有的 $x\in\Omega^c$, $s(x)=0$. 应用Lusin 定理 1.37(f)我们得到一个函数 $\phi\in C_0(\Omega)$ 使得对所有的 $x\in\Omega$

$$|\phi(x)|\leqslant\|s\|_\infty$$

并且

$$\operatorname{vol}\{x\in\Omega: s(x)\neq\phi(x)\}<(\varepsilon/4\|s\|_\infty)^p$$

因此由定理 2.8 我们有

$$\begin{aligned}\|s-\phi\|_p&\leqslant\|s-\phi\|_\infty(\operatorname{vol}\{x\in\Omega: s(x)\neq\phi(x)\})^{1/p}\\&<2\|s\|_\infty(\varepsilon/4\|s\|_\infty)=\varepsilon/2.\end{aligned}$$

由此得到 $\|u-\phi\|_p<\varepsilon$. ▮

2.14 事实上, 以上证明说明对于 $1\leqslant p<\infty$, $L^p(\Omega)$ 中简单函数组成的集合在 $L^p(\Omega)$ 中是稠密的. 由定理 1.39 的直接推论得到, 这一事实在 $L^\infty(\Omega)$ 中也对.

2.15 定理 如果 $1\leqslant p<\infty$, 则 $L^p(\Omega)$ 是可分的.

证明 对 $m=1,2,\cdots$设

$$\overline{\Omega}_m=\left\{x\in\Omega: \operatorname{dist}(x,\operatorname{bdry}\Omega)\geqslant\frac{1}{m}\text{并且}|x|\leqslant m\right\}$$

因此 $\overline{\Omega}_m$ 是 Ω 的紧子集. 设 P 是 $\mathbf{R}^n$ 上系数为有理复数的全体多项式的集合. 设 $P_m=\{\chi_{\overline{\Omega}_m}f: f\in P\}$, 其中 $\chi_{\overline{\Omega}_m}$ 是 $\overline{\Omega}_m$ 的特征函数, 由推论 1.29 P_m 在 $C(\overline{\Omega}_m)$ 中稠密. 而且, $\bigcup_{m=1}^{\infty}P_m$ 是可数的.

如果 $u\in L^p(\Omega)$, $\varepsilon>0$, 则存在 $\phi\in C_0(\Omega)$ 使得 $\|u-\phi\|_p<\varepsilon/2$.

如果 $1/m<\text{dist}(\text{supp}\,\phi,\ \text{bdry}\,\Omega)$，则存在 $f\in P_m$ 使得 $\|\phi-f\|_\infty<(\varepsilon/2)(\text{vol}\,\overline{\Omega}_m)^{-1/p}$ 由此得到

$$\|\phi-f\|_p\leqslant\|\phi-f\|_\infty(\text{vol}\,\overline{\Omega}_m)^{1/p}<\varepsilon/2$$

所以 $\|u-f\|_p<\varepsilon$. 因此可数集合 $\bigcup_{m=1}^{\infty}P_m$ 在 $L^p(\Omega)$ 中稠密，从而 $L^p(\Omega)$ 是可分的. ▌

2.16 $C(\Omega)$是 $L^\infty(\Omega)$的真闭子空间，在 $L^\infty(\Omega)$中不稠密. 因此 $C_0(\Omega)$和 $C_0^\infty(\Omega)$都不在 $L^\infty(\Omega)$中稠密，从而 $L^\infty(\Omega)$不是可分的.

软化子(Mollifiers)，用光滑函数来逼近

2.17 设 J 是 $C_0^\infty(\mathbf{R}^n)$中的非负、实值函数并且具有性质

(i) 当 $|x|\geqslant1$ 时，$J(x)=0$

(ii) $\int_{\mathbf{R}^n}J(x)dx=1.$

例如，我们可以取

$$J(x)=\begin{cases}k\exp[-1/(1-|x|^2)] & \text{当 } |x|<1 \text{ 时}\\ 0 & \text{当 } |x|\geqslant1 \text{ 时,}\end{cases}$$

其中 $k>0$，选 k 使得满足条件(ii). 如果 $\varepsilon>0$，函数 $J_\varepsilon(x)=\varepsilon^{-n}J(x/\varepsilon)$是属于 $C_0^\infty(\mathbf{R}^n)$的非负函数，而且满足

(i) 当 $|x|\geqslant\varepsilon$ 时，$J_\varepsilon(x)=0$,

(ii) $\int_{\mathbf{R}^n}J_\varepsilon(x)dx=1.$

J_ε 叫做软化子，而卷积

$$J_\varepsilon*u(x)=\int_{\mathbf{R}^n}J_\varepsilon(x-y)u(y)dy, \tag{16}$$

对于使(16)式右端有意义的函数 u 有定义，$J_\varepsilon*u(x)$叫做 u 的一个光滑化(mollification)或者正则化(regularization). 在下面的引理中我们总括了光滑化的一些性质.

2.18 引理 设 u 是定义在 $\mathbf{R}^n$ 上的函数，u 在区域 Ω 外恒等于零.

(a) 如果 $u\in L^1_{\mathrm{loc}}(\overline{\Omega})$，则 $J_\varepsilon * u\in C^\infty(\mathbf{R}^n)$.

(b) 如果还有 supp $u\subset\subset\Omega$ 那么假如

$\varepsilon<\operatorname{dist}(\operatorname{supp} u, \operatorname{bdry}\Omega)$ 就有 $J_\varepsilon * u\in C_0^\infty(\Omega)$.

(c) 如果 $u\in L^p(\Omega)$，其中 $1\leqslant p<\infty$，则 $J_\varepsilon * u\in L^p(\Omega)$. 而且有

$$\|J_\varepsilon * u\|_p\leqslant\|u\|_p \quad \text{和} \quad \lim_{\varepsilon\to 0+}\|J_\varepsilon * u-u\|_p=0.$$

(d) 如果 $u\in C(\Omega)$ 且 $G\subset\subset\Omega$，则在 G 上一致地有

$$\lim_{\varepsilon\to 0+} J_\varepsilon * u(x)=u(x).$$

(e) 如果 $u\in C(\overline{\Omega})$，则在 Ω 上一致地有 $\lim\limits_{\varepsilon\to 0+} J_\varepsilon * u(x)=u(x)$.

证明 由于 $J_\varepsilon(x-y)$ 是 x 的一个无穷次可微函数，当 $|y-x|\geqslant\varepsilon$ 时等于 0，又由于对于每个多重指标 α 和一切在 $\mathbf{R}^n$ 的紧集上可积的函数有

$$D^\alpha(J_\varepsilon * u)(x)=\int_{\mathbf{R}^n} D_x^\alpha J_\varepsilon(x-y)u(y)\,dy,$$

由此得到结论 (a) 和 (b) 是正确的.

假定 $u\in L^p(\Omega)$. 如果 $1<p<\infty$，我们设 $p'=p/(p-1)$，从而由 Hölder 不等式得到

$$\begin{aligned}|J_\varepsilon * u(x)|&=\left|\int_{\mathbf{R}^n} J_\varepsilon(x-y)u(y)\,dy\right|\\&\leqslant\left\{\int_{\mathbf{R}^n} J_\varepsilon(x-y)\,dy\right\}^{1/p'}\left\{\int_{\mathbf{R}^n} J_\varepsilon(x-y)\,|u(y)|^p\,dy\right\}^{1/p}\\&=\left\{\int_{\mathbf{R}^n} J_\varepsilon(x-y)\,|u(y)|^p\,dy\right\}^{1/p}.\end{aligned}\tag{17}$$

因此由 Fubini 定理

$$\int_\Omega |J_\varepsilon * u(x)|^p dx \leqslant \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} J_\varepsilon(x-y)\,|u(y)|^p dy dx$$
$$= \int_{\mathbf{R}^n} |u(y)|^p dy \int_{\mathbf{R}^n} J_\varepsilon(x-y)dx = \|u\|_p^p. \tag{18}$$

设 $\eta>0$. 由定理 2.13 存在 $\phi\in C_0(\Omega)$ 使得 $\|u-\phi\|_p<\eta/3$. 于是由(18)式, $\|J_\varepsilon*u-J_\varepsilon*\phi\|_p<\eta/3$. 现在

$$|J_\varepsilon*\phi(x)-\phi(x)| = \left|\int_{\mathbf{R}^n} J_\varepsilon(x-y)(\phi(y)-\phi(x))dy\right|$$
$$\leqslant \sup_{|y-x|<\varepsilon} |\phi(y)-\phi(x)| \tag{19}$$

由于 ϕ 在 Ω 上一致连续, 当 $\varepsilon\to 0+$ 时(19) 式的右端趋于零. 由于 $\operatorname{supp}\phi$ 是紧的, 我们可以选择 ε 充分小使得 $\|J_\varepsilon*\phi-\phi\|_p<\eta/3$. 所以对于这样的 ε 我们有 $\|J_\varepsilon*u-u\|_p<\eta$, 这就得到($c$). 如果 $p=1$, 不必用 Hölder 不等式直接从(16)式就得到(18)式, 而(c)的证明的其余部分和上面说的一样. 在(19)式中用 u 来代替 ϕ 就可证明(d)和(e). ∎

2.19 定理 如果 $1\leqslant p<\infty$, 则 $C_0^\infty(\Omega)$ 在 $L^p(\Omega)$ 中稠密.

证明是定理 2.13 和引理 2.18(b, e)的直接推论.

$L^p(\Omega)$ 中的准紧集(precompact sets)

2.20 在研究 L^p-空间时下面的定理所起的作用类似于研究连续函数空间时 Ascoli-Arzela 定理 1.30 所起的作用. 如果 u 是在区域 $\Omega\subset\mathbf{R}^n$ 上a.e. 定义的函数. 我们用 $\tilde{u}$ 表示 u 在 Ω 外为零的零延拓, 即

$$\tilde{u}(x)=\begin{cases} u(x) & \text{当 } x\in\Omega \text{ 时} \\ 0 & \text{当 } x\in\mathbf{R}^n\sim\Omega \text{ 时}. \end{cases}$$

2.21 定理 设 $1\leqslant p<\infty$. 有界子集 $K\subset L^p(\Omega)$ 在 $L^p(\Omega)$ 中是准紧的当且仅当对于每一个 $\varepsilon>0$ 存在数 $\delta>0$ 和子集合 $G\subset\subset\Omega$

使得对于一切 $u\in K$ 和一切 $h\in\mathbf{R}^n$, $|h|<\delta$ 有

$$\int_\Omega |\tilde{u}(x+h)-\tilde{u}(x)|^p dx<\varepsilon^p \tag{20}$$

和

$$\int_{\Omega\sim\overline{G}} |u(x)|^p dx<\varepsilon^p. \tag{21}$$

证明 只要对特殊情形 $\Omega=\mathbf{R}^n$ 证明本定理就够了,因为把这种特殊情形下的定理应用到集合 $\widetilde{K}=\{\tilde{u}:u\in K\}$ 上去就得到一般 Ω 上的定理.

我们先假定 K 在 $L^p(\mathbf{R}^n)$ 中是准紧的. 设给定了 $\varepsilon>0$. 由于 K 有一个有限 $(\varepsilon/6)$-网(定理 1.18), 又由于 $C_0(\mathbf{R}^n)$ 在 $L^p(\mathbf{R}^n)$ 中稠密(定理 2.13), 存在一个具有紧支集的连续函数的有限集合 S, 使得对每个 $u\in K$ 存在 $\phi\in S$ 满足 $\|u-\phi\|_p<\varepsilon/3$. 由于 S 是有限的, 存在 $r>0$ 使得对一切 $\phi\in S$ 都有 $\operatorname{supp}\phi\subset\overline{B}_r$, 其中 B_r 是球 $\{x\in\mathbf{R}^n: |x|<r\}$. 令 $G=B_r$ 我们得到(21). 此外, 对所有的 x, $\phi(x+h)-\phi(x)$ 是一致连续的, 而且假如 $|h|<1$ 的话, 在 B_{r+1} 外恒等于零. 因此,

$$\lim_{|h|\to 0}\int_{\mathbf{R}^n} |\phi(x+h)-\phi(x)|^p dx=0. \tag{22}$$

由于 S 是有限的, (22)式对于 $\phi\in S$ 是一致成立的. 对于 $u\in K$, 设 u 平移 h 以后得到 $T_h u$:

$$T_h u(x)=u(x+h). \tag{23}$$

如果 $\phi\in S$ 满足 $\|u-\phi\|_p<\varepsilon/3$, 则也有 $\|T_h u-T_h\phi\|_p<\varepsilon/3$. 因此由(22), 对于(与 $u\in K$ 无关的)充分小的 $|h|$, 我们有

$$\begin{aligned}\|T_h u-u\|_p&\leqslant\|T_h u-T_h\phi\|_p+\|T_h\phi-\phi\|_p+\|\phi-u\|_p\\&<(2\varepsilon/3)+\|T_h\phi-\phi\|_p<\varepsilon.\end{aligned}$$

从而得到(20). [这种论证表明在 $L^p(\Omega)$ 中平移是连续的.]

为了证明充分性, 设给定 $\varepsilon>0$, 又选 $G\subset\subset\mathbf{R}^n$ 使得对所有的

$u \in K$

$$\int_{\mathbf{R}^n \sim \overline{G}} |u(x)|^p dx < \varepsilon/3. \tag{24}$$

对于任何 $\eta > 0$, 由(16)式定义的函数$J_\eta * u$ 属于 $C^\infty(\mathbf{R}^n)$, 特别是属于 $C(\overline{G})$的. 如果 $\phi \in C_0(\mathbf{R}^n)$, 则由 Hölder 不等式

$$|J_\eta * \phi(x) - \phi(x)|^p = \left| \int_{\mathbf{R}^n} J_\eta(y)(\phi(x-y) - \phi(x)) dy \right|^p$$
$$\leqslant \int_{B_\eta} J_\eta(y) |T_{-y}\phi(x) - \phi(x)|^p dy$$

其中 $T_h\phi$ 由(23)式给出. 因此

$$\|J_\eta * \phi - \phi\|_p \leqslant \sup_{h \in B_\eta} \|T_h\phi - \phi\|_p. \tag{25}$$

如果 $u \in L^p(\mathbf{R}^n)$, 设$\{\phi_n\}$是 $C_0(\mathbf{R}^n)$中的一个序列, 在 L^p 的范数下收敛到 u. 由引理 2. 18(c), $\{J_\eta * \phi_n\}$是 $L^p(\mathbf{R}^n)$中的一个收敛到 $J_\eta * u$ 的 Cauchy 序列. 还因为在 $L^p(\mathbf{R}^n)$中 $T_h\phi_n \to T_h u$, (25)式可以扩充到所有的 $u \in L^p(\mathbf{R}^n)$:

$$\|J_\eta * u - u\|_p \leqslant \sup_{h \in B_\eta} \|T_h u - u\|_p.$$

现在(20)式就蕴涵着对于 $u \in K$ 一致地有$\lim_{|h| \to 0} \|T_h u - u\|_p = 0$. 因此对于 $u \in K$ 一致地有 $\lim_{\eta \to 0} \|J_\eta * u - u\|_p = 0$. 现在固定 $\eta > 0$ 使得对所有的 $u \in K$ 有

$$\int_{\overline{G}} |J_\eta * u(x) - u(x)|^p dx \leqslant \frac{\varepsilon}{3 \cdot 2^p} \tag{26}$$

我们证明 $\{J_\eta * u: u \in K\}$在 $\overline{G}$ 上满足 Ascoli-Arzela 定理 1. 30 的条件, 因此是 $C(\overline{G})$中的准紧集. 由(19)式我们有

$$|J_\eta * u(x)| \leqslant (\sup_{x \in \mathbf{R}^n} J_\eta(x))^{1/p} \|u\|_p$$

因为K是 $L^p(\Omega)$中的有界集以及 η 是固定的, 所以对于 $x \in \mathbf{R}^n$ 和 $u \in K$, $J_\eta * u$ 是一致有界的. 类似地

$$|J_\eta * u(x+h) - J_\eta * u(x)| \leqslant (\sup_{x \in \mathbf{R}^n} J_\eta(x))^{1/p} \|T_h u - u\|_p$$

因此对于 $x\in\mathbf{R}^n$ 和 $u\in K$，一致地有$\lim\limits_{|h|\to 0}J_\eta * u(x+h)=J_\eta * u(x)$.

因此 $\{J_\eta * u: u\in K\}$ 是 $C(\overline{G})$ 中的准紧集，从而由定理 1.18，存在 $C(\overline{G})$ 中函数的有限集 $\{\psi_1,\cdots,\psi_m\}$ 使得如果 $u\in K$，则对于某个 j，$1\leqslant j\leqslant m$，和所有的 $x\in\overline{G}$，有

$$|\psi_j(x)-J_\eta * u(x)|^p<\frac{\varepsilon}{3\cdot 2^p\cdot \operatorname{vol}\overline{G}} \tag{27}$$

用 $\widetilde{\psi}_j$ 表示 ψ_j 在 $\overline{G}$ 外为零的零延拓，从(24)，(26)，(27)和不等式 $(|a|+|b|)^p\leqslant 2^p(|a|^p+|b|^p)$，得到

$$\begin{aligned}\int_{\mathbf{R}^n}|u(x)-\widetilde{\psi}_j(x)|^p dx&=\int_{\mathbf{R}^n\sim\overline{G}}|u(x)|^p dx+\int_{\overline{G}}|u(x)-\psi_j(x)|^p dx\\&<\frac{\varepsilon}{3}+2^p\int_{\overline{G}}(|u(x)-J_\eta * u(x)|^p\\&\quad+|J_\eta * u(x)-\psi_j(x)|^p)dx\\&<\frac{\varepsilon}{3}+2^p\left(\frac{\varepsilon}{3\cdot 2^p}+\frac{\varepsilon}{3\cdot 2^p\cdot\operatorname{vol}\overline{G}}\operatorname{vol}\overline{G}\right)=\varepsilon.\end{aligned}$$

因此 K 在 $L^p(\mathbf{R}^n)$ 中有一个有限的 ε-网，就是 $\{\widetilde{\psi}_j: 1\leqslant j\leqslant m\}$，从而根据定理 1.18 是准紧的. ▌

2.22 定理 设 $1\leqslant p<\infty$，又设 $K\subset L^p(\Omega)$. 假设存在一个具有下列性质的 Ω 的子区域序列 $\{\Omega_j\}$：

(a) 对每个 j，$\Omega_j\subset\Omega_{j+1}$；

(b) 对于每个 j，由 K 中的函数在 Ω_j 上的限制构成的集合是 $L^p(\Omega_j)$ 中的准紧集；

(c) 对每个 $\varepsilon>0$，存在 j 使得对一切 $u\in K$ 有

$$\int_{\Omega\sim\Omega_j}|u(x)|^p dx<\varepsilon.$$

则 K 是 $L^p(\Omega)$ 中的准紧集.

证明 设 $\{u_n\}$ 是 K 中的一个序列. 那么根据(b)存在一个子序列 $\{u_n^{(1)}\}$ 使得其限制 $\{u_n^{(1)}|_{\Omega_1}\}$ 在 $L^p(\Omega_1)$ 中收敛. 只要 $\{u_n^{(1)}\},\cdots,\{u_n^{(k)}\}$

一经选定，我们就可以选$\{u_n^{(k)}\}$的一个子序列使得$\{u_n^{(k+1)}|_{\Omega_{k+1}}\}$在$L^p(\Omega_{k+1})$中收敛. 因此由(a)，对于$1\leqslant j\leqslant k+1$，$\{u_n^{(k+1)}|_{\Omega_j}\}$也在$L^p(\Omega_j)$中收敛.

设$v_n=u_n^{(n)}$，$n=1,2,\cdots$，显然$\{v_n\}$是$\{u_n\}$的一个子序列. 给定$\varepsilon>0$，[由于(c)]存在j使得除去开始的$j-1$项外，对一切$m,n=1,2,\cdots$

$$\int_{\Omega\sim\Omega_j}|v_n(x)-v_m(x)|^p dx<\varepsilon/2 \tag{28}$$

$\{v_n\}$是$\{u_n^{(j)}\}$的子序列，所以$\{v_n|_{\Omega_j}\}$是$L^p(\Omega_j)$中的一个Cauchy序列. 于是对充分大的n,m，我们有

$$\int_{\Omega_j}|v_n(x)-v_m(x)|^p dx<\varepsilon/2 \tag{29}$$

把(28)和(29)结合起来，我们看到$\{v_n\}$是$L^p(\Omega)$中的一个Cauchy序列，因此在$L^p(\Omega)$中是收敛的. 因此K是$L^p(\Omega)$中的准紧集. ∎

我们指出，定理2.22正好是一个众所周知的定理对我们适用的背景，该定理说：紧算子序列的算子-范数极限是紧的.

$L^p(\Omega)$的一致凸性

2.23 对于$1<p<\infty$，空间$L^p(\Omega)$是一致凸的，其范数$\|\cdot\|_p$满足在1.19节中所说的条件. 这个结果属于Clarkson [19]，它可以通过$L^p(\Omega)$的一组不等式来得到，这组不等式推广了$L^2(\Omega)$中的平行四边形定律. 在定理2.28中将给出这些不等式，为了证明这些不等式我们准备了下面的引理.

2.24 引理 如果$1\leqslant p<\infty$且$a\geqslant 0$，$b\geqslant 0$，则

$$(a+b)^p\leqslant 2^{p-1}(a^p+b^p). \tag{30}$$

证明 如果$a=0$，(30)显然成立. 如果$a>0$，(30)可以改写为

$$(1+x)^p\leqslant 2^{p-1}(1+x^p), \tag{31}$$

其中 $0\leqslant x=b/a$. 函数 $f(x)=(1+x)^p/(1+x^p)$ 满足 $f(0)=1=\lim\limits_{x\to\infty}f(x)$而且当 $0<x<\infty$时，$f(x)>1$. 因此对于 $x\geqslant0$, f 在其仅有的一个临界点 $x=1$ 处取到 f 的最大值. 由于 $f(1)=2^{p-1}$, 立得(31)式. ▮

2.25 引理 如果 $0<s<1$, 函数 $f(x)=(1-s^x)/x$ 是 $x>0$ 的降函数.

证明 $f'(x)=(1/x^2)(g(s^x)-1)$, 其中 $g(t)=t-t\ln t$. 因为 $0<s^x<1$, 又因为对于 $0<t\leqslant1$, $g'(t)=-\ln t\geqslant0$, 由此得到 $g(s^x)<g(1)=1$, 因此 $f'(x)<0$. ▮

2.26 引理 如果 $1<p\leqslant2$ 且 $0\leqslant t\leqslant1$, 则

$$\left|\frac{1+t}{2}\right|^{p'}+\left|\frac{1-t}{2}\right|^{p'}\leqslant\left(\frac{1}{2}+\frac{1}{2}t^p\right)^{1/(p-1)},\tag{32}$$

其中 $p'=p/(p-1)$ 是 p 的共轭指数.

证明 因为如果 $p=2$ 或 $t=0$ 或 $t=1$, (32)中等号显然成立, 我们可以假定 $1<p<2$ 和 $0<t<1$. 变换 $t=(1-s)/(1+s)$ 把区间 $0<t<1$ 映到区间 $1>s>0$ 上, 它把(32)化为等价的不等式

$$\frac{1}{2}[(1+s)^p+(1-s)^p]-(1+s^{p'})^{p-1}\geqslant0.\tag{33}$$

如果我们记

$$\binom{p}{0}=1\ 和\binom{p}{k}=\frac{p(p-1)(p-2)\cdots(p-k+1)}{k!},\quad k\geqslant1,$$

(33)式左端的幂级数展开为

$$\begin{aligned}&\frac{1}{2}\sum_{k=0}^{\infty}\binom{p}{k}s^k+\frac{1}{2}\sum_{k=0}^{\infty}\binom{p}{k}(-s)^k-\sum_{k=0}^{\infty}\binom{p-1}{k}s^{p'k}\\
&=\sum_{k=0}^{\infty}\binom{p}{2k}s^{2k}-\sum_{k=0}^{\infty}\binom{p-1}{k}s^{p'k}\\
&=\sum_{k=1}^{\infty}\left\{\binom{p}{2k}s^{2k}-\binom{p-1}{2k-1}s^{p'(2k-1)}-\binom{p-1}{2k}s^{2p'k}\right\}\end{aligned}$$

对于 $0\leqslant s\leqslant 1$，后一级数是收敛的．我们通过证明对于 $0<s<1$ 级数中的每一项都是正的来证明(33)．第 k 项可以写成：

$$\frac{p(p-1)(2-p)(3-p)\cdots(2k-1-p)}{(2k)!}s^{2k}$$

$$-\frac{(p-1)(2-p)(3-p)\cdots(2k-1-p)}{(2k-1)!}s^{p'(2k-1)}$$

$$+\frac{(p-1)(2-p)\cdots(2k-p)}{(2k)!}s^{2kp'}$$

$$=\frac{(2-p)(3-p)\cdots(2k-p)}{(2k-1)!}s^{2k}$$

$$\times\left[\frac{p(p-1)}{2k(2k-p)}-\frac{p-1}{2k-p}s^{p'(2k-1)-2k}+\frac{p-1}{2k}s^{2kp'-2k}\right]$$

$$=\frac{(2-p)(3-p)\cdots(2k-p)}{(2k-1)!}s^{2k}$$

$$\times\left[\frac{1-s^{(2k-p)/(p-1)}}{(2k-p)/(p-1)}-\frac{1-s^{2k/(p-1)}}{2k/(p-1)}\right]$$

由于 $p<2$，上式第一个因子是正的；由于 $0<(2k-p)/(p-1)<2k/(p-1)$，由引理 2.25，方括号中的因子是正的．于是就证明了(33)，因此也证明了(32)．▌

2.27 引理 设 $z,w\in\mathbf{C}$．如果 $1<p\leqslant 2$，则

$$\left|\frac{z+w}{2}\right|^{p'}+\left|\frac{z-w}{2}\right|^{p'}\leqslant\left(\frac{1}{2}|z|^p+\frac{1}{2}|w|^p\right)^{1/(p-1)},\tag{34}$$

其中 $p'=p/(p-1)$．如果 $2\leqslant p<\infty$，则

$$\left|\frac{z+w}{2}\right|^{p}+\left|\frac{z-w}{2}\right|^{p}\leqslant\frac{1}{2}|z|^p+\frac{1}{2}|w|^p.\tag{35}$$

证明 因为当 $z=0$ 或 $w=0$ 时(34)显然成立，而且(34)对于 z 和 w 是对称的，我们可以假定 $|z|\geqslant|w|>0$．这时能把(34)重写为

$$\left|\frac{1+re^{i\theta}}{2}\right|^{p'}+\left|\frac{1-re^{i\theta}}{2}\right|^{p'}\leqslant\left(\frac{1}{2}+\frac{1}{2}r^p\right)^{1/(p-1)},\tag{36}$$

其中 $w/z=re^{i\theta}$，$r\geqslant 0$，$0\leqslant\theta<2\pi$．如果 $\theta=0$，则在引理 2.26 中早

已证明了(36). 我们证明，对于固定的 r，函数

$$f(\theta)=|1+re^{i\theta}|^{p'}+|1-re^{i\theta}|^{p'}$$

对于 $0\leqslant\theta<2\pi$ 在 $\theta=0$ 处有一个最大值，利用这一结果来完成(36)式的证明. 因为

$$f(\theta)=(1+r^2+2r\cos\theta)^{p'/2}+(1+r^2-2r\cos\theta)^{p'/2}$$

显然有 $f(2\pi-\theta)=f(\pi-\theta)=f(\theta)$，所以只要在区间 $0\leqslant\theta\leqslant\pi/2$，研究 f 就行了. 由于 $p'\geqslant2$，在这个区间上，我们有

$$f'(\theta)=-p'r\sin\theta[(1+r^2+2r\cos\theta)^{(p'/2)-1}$$
$$-(1+r^2-2r\cos\theta)^{(p'/2)-1}]\leqslant0$$

因此 f 的最大值在 $\theta=0$ 处达到，这就证明了(36).

如果 $2\leqslant p<\infty$，则 $1<p'\leqslant2$，在(34)中交换 p 和 p' 并利用引理 2.24，我们有

$$\begin{aligned}\left|\frac{z+w}{2}\right|^p+\left|\frac{z-w}{2}\right|^p&\leqslant\left(\frac{1}{2}|z|^{p'}+\frac{1}{2}|w|^{p'}\right)^{1/(p'-1)}\\&=\left(\frac{1}{2}|z|^{p'}+\frac{1}{2}|w|^{p'}\right)^{p/p'}\\&\leqslant2^{(p/p')-1}\left(\left(\frac{1}{2}\right)^{p/p'}|z|^p+\left(\frac{1}{2}\right)^{p/p'}|w|^p\right)\\&=\frac{1}{2}|z|^p+\frac{1}{2}|w|^p,\end{aligned}$$

所以(35)也得到了证明. ▌

2.28 定理 (Clarkson 不等式)设 $u,v\in L^p(\Omega)$. 对于 $1<p<\infty$ 设 $p'=p/(p-1)$. 如果 $2\leqslant p<\infty$，则

$$\left\|\frac{u+v}{2}\right\|_p^p+\left\|\frac{u-v}{2}\right\|_p^p\leqslant\frac{1}{2}\|u\|_p^p+\frac{1}{2}\|v\|_p^p,\tag{37}$$

$$\left\|\frac{u+v}{2}\right\|_p^{p'}+\left\|\frac{u-v}{2}\right\|_p^{p'}\geqslant\left(\frac{1}{2}\|u\|_p^p+\frac{1}{2}\|v\|_p^p\right)^{p'-1}.\tag{38}$$

如果 $1<p\leqslant2$，则

$$\left\|\frac{u+v}{2}\right\|_p^{p'}+\left\|\frac{u-v}{2}\right\|_p^{p'}\leqslant\left(\frac{1}{2}\|u\|_p^p+\frac{1}{2}\|v\|_p^p\right)^{p'-1}, \tag{39}$$

$$\left\|\frac{u+v}{2}\right\|_p^{p}+\left\|\frac{u-v}{2}\right\|_p^{p}\geqslant\frac{1}{2}\|u\|_p^p+\frac{1}{2}\|v\|_p^p. \tag{40}$$

证明 对于 $2\leqslant p<\infty$，在(35)式中取 $z=u(x)$和 $w=v(x)$并在 Ω 上积分就得到(37)式. 为了对 $1<p\leqslant 2$ 证明(39)式，我们首先注意到对于任何 $u\in L^p(\Omega)$，$\||u|^{p'}\|_{p-1}=\|u\|_p^{p'}$. 利用相应于指数 $p-1<1$ 的逆 Minkowski 不等式和 $z=u(x)$，$w=v(x)$的(34)式，得到

$$\begin{aligned}
\left\|\frac{u+v}{2}\right\|_p^{p'}+\left\|\frac{u-v}{2}\right\|_p^{p'}&=\left\|\left|\frac{u+v}{2}\right|^{p'}\right\|_{p-1}+\left\|\left|\frac{u-v}{2}\right|^{p'}\right\|_{p-1}\\
&\leqslant\left(\int_\Omega\left(\left|\frac{u(x)+v(x)}{2}\right|^{p'}\right.\right.\\
&\qquad\left.\left.+\left|\frac{u(x)-v(x)}{2}\right|^{p'}\right)^{p-1}dx\right)^{1/(p-1)}\\
&\leqslant\left(\int_\Omega\left(\frac{1}{2}|u(x)|^p+\frac{1}{2}|v(x)|^p\right)dx\right)^{p'-1}\\
&=\left(\frac{1}{2}\|u\|_p^p+\frac{1}{2}\|v\|_p^p\right)^{p'-1},
\end{aligned}$$

这就是(39)式.

对于 $2\leqslant p<\infty$ 能用与证明(39)同样的方法来证明不等式(38)，不过要用相应于 $p-1\geqslant 1$ 的直接 Minkowski 不等式(5)来代替逆不等式，用不等式

$$\left(\left|\frac{\xi+\eta}{2}\right|^{p'}+\left|\frac{\xi-\eta}{2}\right|^{p'}\right)^{p-1}\geqslant\frac{1}{2}|\xi|^p+\frac{1}{2}|\eta|^p,$$

来代替(34). 在(34)式中用 p' 代替 p，$\xi+\eta$ 代替 z，$\xi-\eta$代替 w 就可以得到这个不等式. 最后，能从(35)的一个类似的修正得到(40). 我们要指出，在 $p=2$ 的情形，四个 Clarkson 不等式全都化为平行四边形定律

$$\|u+v\|_2^2+\|u-v\|_2^2=2\|u\|_2^2+2\|v\|_2^2. \blacksquare$$

2.29 推论 如果 $1<p<\infty$, 则 $L^p(\Omega)$ 是一致凸的.

证明 设 $u, v\in L^p(\Omega)$ 满足 $\|u\|_p=\|v\|_p=1$ 和 $\|u-v\|_p\geqslant\varepsilon>0$.

如果 $2\leqslant p<\infty$, 从(37)我们有

$$\left\|\frac{u+v}{2}\right\|_p^p\leqslant 1-\frac{\varepsilon^p}{2^p}.$$

如果 $1<p\leqslant 2$, 从(39)我们有

$$\left\|\frac{u+v}{2}\right\|_p^{p'}\leqslant 1-\frac{\varepsilon^{p'}}{2^{p'}}.$$

这两种情形中都存在 $\delta=\delta(\varepsilon)>0$ 使得 $\|(u+v)/2\|_p\leqslant 1-\delta$. ∎

对于 $1<p<\infty$, 由于 $L^p(\Omega)$ 是一致凸的,根据定理1.20, $L^p(\Omega)$ 是自反的. 在计算了 $L^p(\Omega)$ 的对偶空间之后我们将给出一个关于 $L^p(\Omega)$ 自反性的一个直接证明.

$L^p(\Omega)$ 的赋范对偶

2.30 设 $1\leqslant p\leqslant\infty$, 又设 p' 是 p 的共轭指数. 对于每个元素 $v\in L^{p'}(\Omega)$ 我们能够经由

$$L_v(u)=\int_\Omega u(x)v(x)dx,\qquad u\in L^p(\Omega)$$

来定义一个 $L^p(\Omega)$ 上的线性泛函 L_v. 根据 Hölder 不等式

$|L_v(u)|\leqslant\|u\|_p\|v\|_{p'}$ 因此 $L_v\in[L^p(\Omega)]'$ 且

$$\|L_v;[L^p(\Omega)]'\|\leqslant\|v\|_{p'}.\tag{41}$$

我们证明在(41)中等号一定成立. 如果 $1<p\leqslant\infty$, 设当 $v(x)\neq 0$ 时 $u(x)=|v(x)|^{p'-2}\overline{v(x)}$ 而在其它情形 $u(x)=0$. 则 $u\in L^p(\Omega)$, 而且 $L_v(u)=\|u\|_p\|v\|_{p'}$. 现在假定 $p=1$, 则 $p'=\infty$. 如果 $\|v\|_{p'}=0$, 令 $u(x)=0$. 否则令 $0<\varepsilon<\|v\|_\infty$, 又设 A 是 Ω 的这样的可测子集合使得 $0<\mu(A)<\infty$ 且在 A 上 $|v(x)|\geqslant\|v\|_\infty-\varepsilon$. 设对于 $x\in A$, $u(x)=|v(x)|^{-1}\overline{v(x)}$; 其它地方 $u(x)=0$. 那么 $u\in L^1(\Omega)$ 而且 $L_v(u)\geqslant\|u\|_1(\|v\|_\infty-\varepsilon)$. 因此我们已经证明了

$$\|L_v; [L^p(\Omega)]'\| = \|v\|_{p'} \tag{42}$$

所以把 v 映到 L_v 的算子 L 是 $L^{p'}(\Omega)$ 到 $[L^p(\Omega)]'$ 的子空间上的一个等距同构算子.

2.31 自然要问, 同构 L 的值域是否是整个 $[L^p(\Omega)]'$, 即, 是否 $L^p(\Omega)$ 上的每一个连续线性泛函都是对于某个 $v\in L^{p'}(\Omega)$ 的形为 L_v 的泛函, 我们要证在 $1\leqslant p<\infty$ 时确实是这样的. 对于 $p=2$, 这是 Hilbert 空间的 Riesz 表示定理的一个直接推论. 对于一般的 p, 利用 Radon-Nikodym 定理(见Rudin[58]或定理 8.18) 能给出一个直接的证明. 我们将根据 $L^p(\Omega)$ 的一致凸性和变分论证给出一个更初等的证明. Hewitt 和Stromberg[32]也用过这种证法. 最后, 为了从 $p>1$ 的情形得到 $p=1$ 的情形我们将利用一个极限论证.

2.32 引理 设 $1<p<\infty$. 如果 $L\in[L^p(\Omega)]'$ 且 $\|L; [L^p(\Omega)]'\|=1$, 则存在唯一的 $w\in L^p(\Omega)$ 使得 $\|w\|_p=L(w)=1$. 另一方面, 如果给定了 $w\in L^p(\Omega)$ 和 $\|w\|_p=1$, 则存在唯一的 $L\in[L^p(\Omega)]'$ 使得 $\|L; [L^p(\Omega)]'\|=L(w)=1$.

证明 首先设给定 $L\in[L^p(\Omega)]'$ 且 $\|L\|=1$. 存在一个序列 $\{w_n\}\in L^p(\Omega)$ 使得 $\|w_n\|=1$ 而且 $\lim\limits_{n\to\infty}|L(w_n)|=1$. 我们可以假定对每个 n, $L(w_n)|>\frac{1}{2}$, 而且用一个适当的模为 1 的复数乘上 w_n 来代替 w_n, 使 $L(w_n)>0$. 假如序列$\{w_n\}$不是 $L^p(\Omega)$ 中的 Cauchy 序列, 那么存在 $\varepsilon>0$ 使得对某些任意大的 m 和 n, $\|w_n-w_m\|_p\geqslant\varepsilon$, 根据一致凸性我们有 $\left\|\frac{1}{2}(w_n+w_m)\right\|_p\leqslant1-\delta$, 其中 δ 是一个固定的正数. 因此

$$1\geqslant L\left(\frac{w_n+w_m}{\|w_n+w_m\|_p}\right)=\left\|\frac{w_n+w_m}{2}\right\|_p^{-1}L\left(\frac{w_n+w_m}{2}\right)$$
$$\geqslant\frac{1}{1-\delta}\ \frac{1}{2}[L(w_n)+L(w_m)] \tag{43}$$

由于最后的表达式当 $n, m\to\infty$ 时趋于 $1/(1-\delta)$，我们就得到一个矛盾．因此 $\{w_n\}$ 是 $L^p(\Omega)$ 中的Cauchy 序列，所以收敛到 $L^p(\Omega)$ 中的一个元素 w．显然 $\|w\|_p=1$ 而且 $L(w)=\lim\limits_{n\to\infty}L(w_n)=1$．把(43)用到两个不同的元素 w 上去就得到 w 的唯一性．

现在假定给了 $w\in L^p(\Omega)$ 和 $\|w\|_p=1$．如 2.30 节指出的那样，由

$$L_v(u)=\int_\Omega u(x)v(x)dx, \qquad u\in L^p(\Omega), \tag{44}$$

定义的泛函 L_v 满足 $L_v(w)=\|w\|_p^p=1$ 和 $\|L_v;[L^p(\Omega)]'\|=\|v\|_{p'}=\|w\|_p^{p/p'}=1$，其中

$$v(x)=\begin{cases}|w(x)|^{p-2}\overline{w(x)} & \text{当 } w(x)\neq 0 \text{ 时}\\ 0 & \text{其它 } x \text{ 处}\end{cases} \tag{45}$$

所以，余下要证明，如果 $L_1, L_2\in[L^p(\Omega)]'$ 满足 $\|L_1\|=\|L_2\|=L_1(w)=L_2(w)=1$，则 $L_1=L_2$．假定不是如此．那么存在 $u\in L^p(\Omega)$ 使得 $L_1(u)\neq L_2(u)$．用 u 的一个适当的倍数代替 u，可以假定 $L_1(u)-L_2(u)=2$．那么用 u 和 w 的一个适当的倍数之和来代替 u，我们能够使得 $L_1(u)=1$ 和 $L_2(u)=-1$．如果 $t>0$，则 $L_1(w+tu)=1+t$；由于 $\|L_1\|=1$，所以 $\|w+tu\|_p\geqslant 1+t$．类似地，$L_2(w-tu)=1+t$．所以 $\|w-tu\|_p\geqslant 1+t$．如果 $1<p\leqslant 2$，Clarkson 不等式(40)给出

$$1+t^p\|u\|_p^p=\left\|\frac{(w+tu)+(w-tu)}{2}\right\|_p^p+\left\|\frac{(w+tu)-(w-tu)}{2}\right\|_p^p$$

$$\geqslant\frac{1}{2}\|w+tu\|_p^p+\frac{1}{2}\|w-tu\|_p^p\geqslant(1+t)^p. \tag{46}$$

如果 $2\leqslant p<\infty$，Clarkson 不等式(38)给出

$$1+t^{p'}\|u\|_p^{p'}=\left\|\frac{(w+tu)+(w-tu)}{2}\right\|_p^{p'}$$

$$+\left\|\frac{(w+tu)-(w-tu)}{2}\right\|_p^{p'}\geqslant\left(\frac{1}{2}\left\|w+tu\right\|_p^p\right.$$

$$\left.+\frac{1}{2}\left\|w-tu\right\|_p^p\right)^{p'-1}\geqslant(1+t)^{p'}. \tag{47}$$

方程(46)和(47)不可能对一切 $t>0$ 都成立，除非 $\|u\|_p=0$，而这是不可能的．因此 $L_1=L_2$．∎

2.33 定理 ($L^p(\Omega)$的 Riesz表示定理) 设$1<p<\infty$，又设 $L\in[L^p(\Omega)]'$．那么存在 $v\in L^{p'}(\Omega)$使得对于一切 $u\in L^p(\Omega)$

$$L(u)=\int_\Omega u(x)v(x)dx.$$

而且，$\|v\|_{p'}=\|L;[L^p(\Omega)]'\|$，因此$[L^p(\Omega)]'\cong L^{p'}(\Omega)$．

证明 如果 $L=0$，我们可以取 $v=0$．因此假定 $L\neq0$，而且不失一般性假定$\|L;[L^p(\Omega)]'\|=1$．由引理 2.32，存在 $w\in L^p(\Omega)$，$\|w\|_p=1$ 使得 $L(w)=1$．设 v 是由(45)给出的．则由(44)定义的 L_v满足$\|L_v;[L^p(\Omega)]'\|=1$ 和 $L_v(w)=1$．根据引理2.32，再一次有 $L=L_v$．由于 $\|v\|_{p'}=1$，这就完成了证明．∎

2.34 定理 ($L^1(\Omega)$ 的 Riesz 表示定理) 设 $L\in[L^1(\Omega)]'$．那么存在 $v\in L^\infty(\Omega)$使得对一切 $u\in L^1(\Omega)$

$$L(u)=\int_\Omega u(x)v(x)dx,$$

而且 $\|v\|_\infty=\|L;[L^1(\Omega)]'\|$．因此$[L^1(\Omega)]'\cong L^\infty(\Omega)$．

证明 我们再次假定 $L\neq0$ 和$\|L;[L^1(\Omega)]'\|=1$．暂时假定Ω具有有限体积．则由定理 2.8，如果 $1<p<\infty$，有 $L^p(\Omega)\subset L^1(\Omega)$，而且对于任何 $u\in L^p(\Omega)$有

$$|L(u)|\leqslant\|u\|_1\leqslant(\operatorname{vol}\Omega)^{1-(1/p)}\|u\|_p.$$

因此 $L\in[L^p(\Omega)]'$，根据定理 2.33，存在 $v_p\in L^{p'}(\Omega)$使得

$$\|v_p\|_{p'}\leqslant(\operatorname{vol}\Omega)^{1-(1/p)}, \tag{48}$$

而且对一切 $u\in L^p(\Omega)$

$$L(u)=\int_{\Omega}u(x)v_p(x)dx \tag{49}$$

由于 $1<p<\infty$ 时 $C_0^\infty(\Omega)$ 在 $L^p(\Omega)$ 中稠密，又由于对于任何满足 $1<p,q<\infty$ 的 p, q 和任何 $\phi\in C_0^\infty(\Omega)$ 我们有

$$\int_{\Omega}\phi(x)v_p(x)dx=L(\phi)=\int_{\Omega}\phi(x)v_q(x)dx,$$

由此得到在 Ω 上 $v_p=v_q$ a.e. 因此对于每个 p, $1<p<\infty$，在(49)中可以用一个属于 $L^{p'}(\Omega)$ 的函数 v 去代替 v_p，而且按照 (48)，v 满足

$$\|v\|_{p'}\leqslant(\mathrm{vol}\,\Omega)^{1-(1/p)}.$$

再用定理 2.8，得到 $v\in L^\infty(\Omega)$ 和

$$\|v\|_\infty\leqslant\lim_{p'\to\infty}(\mathrm{vol}\,\Omega)^{1-(1/p)}=1. \tag{50}$$

2.30 节一开始的论证证明了(50)中等号一定成立.

如果 Ω 没有有限体积，我们仍然可写 $\Omega=\bigcup_{j=1}^{\infty}G_j$，其中 $G_j=\{x\in\Omega: j-1\leqslant|x|<j\}$ 有有限体积. 集合 G_j 互不相交. 设 $\chi_j(x)$ 是 G_j 的特征函数. 如果 $u_j\in L^1(G_j)$，令 $\tilde{u}_j$ 表示 u_j 在 G_j 外为零的零延拓，即，当 $x\in G_j$ 时 $\tilde{u}_j(x)=u_j(x)$，否则 $\tilde{u}_j(x)=0$. 设 $L_j(u_j)=L(\tilde{u}_j)$，则 $L_j\in[L^1(G_j)]'$ 且 $\|L_j;\ [L^1(G_j)]'\|\leqslant1$. 根据上面研究过的有限体积的情形，存在 $v_j\in L^\infty(G_j)$ 使得 $\|v_j\|_{\infty,G_j}\leqslant1$ 而

$$L_j(u_j)=\int_{G_j}u_j(x)v_j(x)dx=\int_{\Omega}\tilde{u}_j(x)v(x)dx,$$

其中，对于 $x\in G_j\ (j=1,2,\cdots)\ v(x)=v_j(x)$，因此 $\|v\|_\infty\leqslant1$. 如果 $u\in L^1(\Omega)$，我们令 $u=\sum_{j=1}^{\infty}\chi_j u$；由控制收敛定理，这个级数在 $L^1(\Omega)$ 中依范数收敛. 由于

$$L\left(\sum_{j=1}^{k}\chi_j u\right)=\sum_{j=1}^{k}L_j(\chi_j u)=\int_{\Omega}\sum_{j=1}^{k}\chi_j(x)u(x)v(x)dx,$$

根据控制收敛定理取极限，得到

$$L(u)=\int_{\Omega}u(x)v(x)dx.$$

然后，和有限体积的情形一样，立即得到$\|v\|_{\infty}=1$. ∎

2.35 定理 $L^p(\Omega)$是自反的当且仅当$1<p<\infty$.

证明 设$X=L^p(\Omega)$，其中$1<p<\infty$. 由于$X'\cong L^{p'}(\Omega)$，对于任何$w\in X''$，对应着$\widetilde{w}\in[L^{p'}(\Omega)]'$使得$w(v)=\widetilde{w}(\widetilde{v})$，其中

$$v(u)=\int_{\Omega}\widetilde{v}(x)u(x)dx, \qquad u\in X.$$

类似地，对应于$\widetilde{w}\in[L^{p'}(\Omega)]'$存在$u\in X$使得

$$\widetilde{w}(\widetilde{v})=\int_{\Omega}\widetilde{v}(x)u(x)dx, \qquad \widetilde{v}\in L^{p'}(\Omega).$$

由此得到对于一切$v\in X'$

$$w(v)=\widetilde{w}(\widetilde{v})=\int_{\Omega}\widetilde{v}(x)u(x)dx=v(u)=J_Xu(v),$$

J_X是X到X''中的自然等距同构(见1.13节). 这就证明了J_X把X映射到X''上，因此$X=L^p(\Omega)$是自反的.

因为$L^1(\Omega)$是可分的，而$L^1(\Omega)$的对偶等距同构于$L^\infty(\Omega)$，它是不可分的，所以$L^1(\Omega)$和$L^\infty(\Omega)$都不可能是自反的. ∎

2.36 对于空间$L^\infty(\Omega)$，象定理2.33那样的Riesz表示定理不可能成立，因为如果成立的话，定理2.35的论证就表明了$L^1(\Omega)$是自反的. $L^\infty(\Omega)$的对偶大于$L^1(\Omega)$. $[L^\infty(\Omega)]'$可以和Ω上绝对连续、有界全变差的有限可加集合函数构成的空间看成是一样的. 关于细节读者可以参看Yosida[69, p. 118].

第三章 空间$W^{m,p}(\Omega)$

定义和基本性质

本章介绍整数次 Sobolev 空间并建立它们的某些基本性质. 这些空间是定义在任意区域$\Omega\subset\mathbf{R}^n$上的, 而且都是空间 $L^p(\Omega)$ 的向量子空间.

3.1 我们定义一个泛函$\|\cdot\|_{m,p}$如下:

$$\|u\|_{m,p}=\left\{\sum_{0\leqslant|\alpha|\leqslant m}\|D^\alpha u\|_p^p\right\}^{\frac{1}{p}}, \text{ 如果 } 1\leqslant p<\infty, \tag{1}$$

$$\|u\|_{m,\infty}=\max_{0\leqslant|\alpha|\leqslant m}\|D^\alpha u\|_\infty, \tag{2}$$

其中m是非负整数,而$1\leqslant p\leqslant\infty$. 对于任何使右端有意义的函数$u$,$\|\cdot\|_p$当然是$L^p(\Omega)$-范数. (在可能碰到区域混淆的情形我们记作$\|u\|_{m,p,\Omega}$来代替$\|u\|_{m,p}$.)显然(1)或(2)在任何使右端取有限值的函数构成的向量空间上定义了一个范数, 假如把这个空间中的在Ω中几乎处处相等的函数看作是一样的话. 相应于任何给定的m和p值,我们研究以下三个空间①:

$H^{m,p}(\Omega)\equiv\{u\in C^m(\Omega):\|u\|_{m,p}<\infty\}$ 关于范数 $\|\cdot\|_{m,p}$ 的完备化;

$W^{m,p}(\Omega)\equiv\{u\in L^p(\Omega):D^\alpha u\in L^p(\Omega), 0\leqslant|\alpha|\leqslant m$, 其中 $D^\alpha u$ 是 1.57节中弱的(或广义函数意义的)偏导数$\}$;

$W_0^{m,p}(\Omega)\equiv C_0^\infty(\Omega)$在空间$W^{m,p}(\Omega)$中的闭包.

配上适当的范数(1)或(2),这些空间就叫做Ω上的 Sobolev 空间. 显然, $W^{0,p}(\Omega)=L^p(\Omega)$, 而且如果$1\leqslant p<\infty$, 由定理 2.19,有

① 下面的"≡"表示右端是左端的定义——译者注.

$W_0^{0,p}(\Omega)=L^p(\Omega)$. 对于任何 m, 嵌入串

$$W_0^{m,p}(\Omega)\longrightarrow W^{m,p}(\Omega)\longrightarrow L^p(\Omega)$$

也是显然的. 我们将在定理3.16中证明对于任何区域 Ω, $H^{m,p}(\Omega)=W^{m,p}(\Omega)$. 1964 年 Meyers 和 Serrin [46] 发表的这个结果澄清了在这之前在文献中存在的关于这些空间之间的关系的许多混乱. 竟有这么长的时间没有发现这个初等结果真令人惊讶.

空间 $W^{m,p}$ 是由 Sobolev 在[62, 63]中引进的, 它是和曾为其他作者, 尤其是Morrey[47]和 Deny 与 Lions[21]研究过的许多有关的空间一起引进的. 许多不同的记号($W^{m,p}$, $H^{m,p}$, $P^{m,p}$, L_m^p 等等)曾被(而且正在)用来表示这些空间以及它们的变种, 而且在它们广泛地与 Sobolev 的名字联系在一起之前, 有时还冠以别人的名字来称呼它们, 例如, 象"Beppo Levi 空间"之类的名词.

近来已经作出了基本空间 $W^{m,p}(\Omega)$ 的许多推广, 大部分文献来源于苏联. 我们特别要指出如下推广: 允许 m 为任何实值 (见第七章)而且把它解释为与分数次导数相应的推广, 还有在 L^p-范数中引进权函数的带权空间, 在各种坐标方向上含有不同阶导数和不同 L^p-范数的空间 $W^{m,p}$(各向异性空间). 还有, 按照众所周知的 $L^p(\Omega)$ 空间的推广"Orlicz 空间"为模式而得到的, Orlicz-Sobolev 空间(第八章).

在本专著中不可能研究所有可能的推广.

3.2 定理 $W^{m,p}(\Omega)$ 是 Banach 空间.

证明 设 $\{u_n\}$ 是 $W^{m,p}(\Omega)$ 中的一个 Cauchy 序列. 则对于 $0\leqslant|\alpha|\leqslant m$, $\{D^\alpha u_n\}$ 是 $L^p(\Omega)$ 中的一个 Cauchy 序列. 因为 $L^p(\Omega)$ 是完备的, 在 $L^p(\Omega)$ 中存在函数 u 和 u_α, $0\leqslant|\alpha|\leqslant m$, 使得当 $n\to\infty$ 时, 在 $L^p(\Omega)$ 中 $u_n\to u$ 和 $D^\alpha u_n\to u_\alpha$. 现在 $L^p(\Omega)\subset L^1_{\mathrm{loc}}(\Omega)$, 所以按照 1.53 节的结论, u_n 决定一个广义函数 $T_{u_n}\in\mathscr{D}'(\Omega)$. 对任何 $\phi\in\mathscr{D}(\Omega)$ 由 Hölder 不等式, 有

$$|T_{u_n}(\phi)-T_u(\phi)|\leqslant\int_\Omega|u_n(x)-u(x)|\ \ |\phi(x)|dx$$
$$\leqslant\|\phi\|_{p'}\|u_n-u\|_p,$$
其中 $p'=p/(p-1)$(或当 $p=1$ 时 $p'=\infty$,或当 $p=\infty$时 $p'=1$). 因此对于一切 $\phi\in\mathscr{D}(\Omega)$当 $n\to\infty$时,$T_{u_n}(\phi)\to T_u(\phi)$. 类似地,对于一切 $\phi\in\mathscr{D}(\Omega)$,$T_{D^\alpha u_n}(\phi)\to T_{u_\alpha}(\phi)$. 由此得到,对于一切 $\phi\in\mathscr{D}(\Omega)$
$$T_{u\alpha}(\phi)=\lim_{n\to\infty}T_{D^\alpha u_n}(\phi)=\lim_{n\to\infty}(-1)^{|\alpha|}T_{u_n}(D^\alpha\phi)=(-1)^{|\alpha|}T_u(D^\alpha\phi),$$
因此对于 $0\leqslant|\alpha|\leqslant m$, 在 Ω 上在广义函数意义下 $u_u=D^\alpha u$, 因此 $u\in W^{m,p}(\Omega)$. 因为 $\lim\limits_{n\to\infty}\|u_n-u\|_{m,p}=0$, 所以 $W^{m,p}(\Omega)$ 是完备的. ▌

当古典偏导数存在而且连续时,广义函数意义下的偏导数和古典偏导数是一致的;因此集合
$$S=\{\phi\in C^m(\Omega):\ \|\phi\|_{m,p}<\infty\}$$
显然包含在$W^{m,p}(\Omega)$中. 因为$W^{m,p}(\Omega)$是完备的,S 中的恒等算子延拓成 S 的完备化 $H^{m,p}$和 S 在$W^{m,p}(\Omega)$中的闭包之间的一个等距同构. 因此自然把 $H^{m,p}(\Omega)$ 和这个闭包看成是一样的,从而得到下面的推论.

3.3 推论 $H^{m,p}(\Omega)\subset W^{m,p}(\Omega)$.

3.4 把$W^{m,p}(\Omega)$ 看作空间 $L^p(\Omega)$ 的笛卡尔积空间的一个闭子空间就非常容易得到空间 $W^{m,p}(\Omega)$ 的一些重要性质. 设 $N\equiv N(n,m)=\sum\limits_{0\leqslant|\alpha|\leqslant m}1$ 是满足 $0\leqslant|\alpha|\leqslant m$ 的多重指标的数目. 对于 $1\leqslant p\leqslant\infty$, 设 $L^p_N=\prod\limits_{j=1}^{N}L^p(\Omega)$. 在 L^p_N 中 $u=(u_1,\cdots,u_N)$的范数由
$$\|u;L^p_N\|=\begin{cases}\left(\sum\limits_{j=1}^{N}\|u_j\|_p^p\right)^{\frac{1}{p}} & \text{如果 } 1\leqslant p<\infty\\ \max\limits_{1\leqslant j\leqslant N}\|u_j\|_\infty & \text{如果 } p=\infty\end{cases}$$

给出. 由于定理 1.22, 2.10, 2.15, 2.29 和 2.35, L^p_N是一个 Banach 空间, 当 $1\leqslant p<\infty$ 时它是可分的, 当 $1<p<\infty$ 时它是自反的而且是一致凸的。

我们假定满足 $0\leqslant|\alpha|\leqslant m$ 的 N 个多重指标 α 以某种合适的方式顺次排列起来, 所以对于每一个 $u\in W^{m,p}(\Omega)$可以与一个由

$$Pu=(D^\alpha u)_{0\leqslant|\alpha|\leqslant m} \tag{3}$$

给出的、在 L^p_N中有明确定义(well-defined)的向量联系起来. 因为 $\|Pu;L^p_N\|=\|u\|_{m,p}$, 所以 P 是 $W^{m,p}(\Omega)$到子空间 $W\subset L^p_N$上的一个等距同构. 因为$W^{m,p}(\Omega)$是完备的, 所以W是 L^p_N 的闭子空间. 由定理 1.21, 当 $1\leqslant p<\infty$时W是可分的, 当 $1<p<\infty$ 时W是自反的而且是一致凸的. 所以对于$W^{m,p}(\Omega)=P^{-1}(W)$ 同样的结论一定成立.

3.5 定理 $W^{m,p}(\Omega)$当 $1\leqslant p<\infty$ 时是可分的, 当 $1<p<\infty$时是自反的而且是一致凸的. 因此, 特别说来, $W^{m,2}(\Omega)$是一个可分 Hilbert 空间, 其内积为

$$(u,v)_m=\sum_{0\leqslant|\alpha|\leqslant m}(D^\alpha u,D^\alpha v)$$

其中 $(u,v)=\int_\Omega u(x)\overline{v(x)}dx$是 $L^2(\Omega)$ 中的内积.

对偶性, 空间 $W^{-m,p'}(\Omega)$

3.6 在以下几节中, 对于固定的Ω, m和 p, 数 N, 空间 L^p_N和W以及算子 P 都取作 3.4 节所规定的那样. 我们还定义

$$\langle u,v\rangle=\int_\Omega u(x)v(x)dx,$$

上式对一切使右端有意义的 u, v 有定义. 对于给定的 p, p' 永远表示其共轭指数:

$$p'=\begin{cases}\infty & \text{当} p=1 \text{时}\\ p/(p-1) & \text{当} 1<p<\infty \text{时}\\ 1 & \text{当} p=\infty \text{时}.\end{cases}$$

3.7 引理 设 $1\leqslant p<\infty$. 对于每个 $L\in(L_N^p)'$ 有唯一的 $v\in L_N^{p'}$ 与之对应使得对于一切 $u\in L_N^p$

$$L(u)=\sum_{j=1}^{N}\langle u_j, v_j\rangle$$

而且

$$\|L;(L_N^p)'\|=\|v;L_N^{p'}\| \tag{4}$$

于是 $(L_N^p)'\cong L_N^{p'}$.

证明 若 $1\leqslant j\leqslant N$ 且 $w\in L^p(\Omega)$, 设 $w_{(j)}=(0,\cdots,0,w,0,\cdots,0)$ 是 L_N^p 的元素，它的第 j 个分量是 w，其余分量都是 0. 令 $L_j(w)=L(w_{(j)})$, 我们知道 $L_j\in(L^p(\Omega))'$, 因此由定理 2.33 和 2.34 存在(唯一的) $v_j\in L^{p'}(\Omega)$ 使得对一切 $w\in L^p(\Omega)$

$$L(w_{(j)})=L_j(w)=\langle w, v_j\rangle.$$

如果 $u\in L_N^p$, 则

$$L(u)=L\left(\sum_{j=1}^{N}u_{j(j)}\right)=\sum_{j=1}^{N}L(u_{j(j)})=\sum_{j=1}^{N}\langle u_j, v_j\rangle.$$

由(对于函数及有限和的) Hölder 不等式，我们有

$$|L(u)|\leqslant\sum_{j=1}^{N}\|u_j\|_p\|v_j\|_{p'}\leqslant\|u;L_N^p\|\,\|v;L_N^{p'}\|$$

所以 $\|L;(L_N^p)'\|\leqslant\|v;L_N^{p'}\|$. 事实上这两个范数是相等的，我们证明如下：当 $1<p<\infty$ 且 $1\leqslant j\leqslant N$ 时，设

$$u_j(x)=\begin{cases}|v_j(x)|^{p'-2}\overline{v_j(x)} & \text{当 } v_j(x)\neq 0 \text{ 时}\\ 0 & \text{当 } v_j(x)=0 \text{ 时}.\end{cases}$$

容易验证 $|L(u_1,\cdots,u_N)|=\|v;L_N^{p'}\|^{p'}=\|u;L_N^p\|\,\|v;L_N^{p'}\|$.

当 $p=1$ 时，我们选 k 使得 $\|v_k\|_\infty=\max\limits_{1\leqslant j\leqslant N}\|v_j\|_\infty$, 对任何 $\varepsilon>0$, 存

在一个体积有限且非0的可测集$A\subset\Omega$，使得对于$x\in A$有$|v_k(x)|\geqslant\|v_k\|_\infty-\varepsilon$. 令

$$u(x)=\begin{cases}\overline{v_k(x)}/|v_k(x)| & \text{当}x\in A\text{ 且 }v_k(x)\neq 0\text{ 时}\\ 0 & \text{当 }x\text{ 是}\Omega\text{的其它点时.}\end{cases}$$

则

$$L(u_{(k)})=\langle u,v_k\rangle=\int_A|v_k(x)|dx\geqslant(\|v_k\|_\infty-\varepsilon)\|u\|_1$$
$$=(\|v;L_N^\infty\|-\varepsilon)\|u_{(k)};L_N^1\|$$

因为ε是任意的，因此在这种情形下也一定推得(4). ▮

3.8 定理 设$1\leqslant p<\infty$. 对于每个$L\in(W^{m,p}(\Omega))'$存在一个元素$v\in L_N^{p'}$使得把向量v写成$(v_\alpha)_{0\leqslant|\alpha|\leqslant m}$的形式时，对一切$u\in W^{m,p}(\Omega)$，有

$$L(u)=\sum_{0\leqslant|\alpha|\leqslant m}\langle D^\alpha u,v_\alpha\rangle.\tag{5}$$

而且

$$\|L;(W^{m,p}(\Omega))'\|=\inf\|v;L_N^{p'}\|=\min\|v;L_N^{p'}\|,\tag{6}$$

下确界是在对于一切$u\in W^{m,p}(\Omega)$使(5)式成立的所有的$v\in L_N^{p'}$构成的集合上取的，而且一定在这个集合上取到.

证明 线性泛函L^*定义如下；

$$L^*(Pu)=L(u),\qquad u\in W^{m,p}(\Omega)$$

它是定义在由(3)定义的算子P的值域上的. 因为P是一个等距同构，$L^*\in W'$而且

$$\|L^*;W'\|=\|L;(W^{m,p}(\Omega))'\|.$$

由 Hahn-Banach 定理，存在一个L^*到L_N^p全体的保范延拓$\tilde{L}$，从而由引理3.7，存在$v\in L_N^{p'}$使得如果$u=(u_\alpha)_{0\leqslant|\alpha|\leqslant m}\in L_N^p$，则

$$\tilde{L}(u)=\sum_{0\leqslant|\alpha|\leqslant m}\langle u_\alpha,v_\alpha\rangle.$$

因此对于$u\in W^{m,p}(\Omega)$我们得到

$$L(u)=L^*(Pu)=\widetilde{L}(Pu)=\sum_{0\leq|\alpha|\leq m}\langle D^\alpha u, v_\alpha\rangle$$

而且

$$\|L;(W^{m,p}(\Omega))'\|=\|L^*;W'\|=\|\widetilde{L};(L_N^p)'\|=\|v;L_N^{p'}\|. \qquad (7)$$

现在对一切 $u\in W^{m,p}(\Omega)$ 使(5)式成立的任何元素 $v\in L_N^{p'}$ 与 L^* 的一个延拓 L 相对应，所以范数 $\|v;L_N^{p'}\|$ 不小于 $\|L;(W^{m,p}(\Omega))'\|$. 把这一点和(7)式结合起来，就得到(6)式. ▌

我们指出，至少，当 $1<p<\infty$ 时，满足(5)和(6)的元素 $v\in L_N^{p'}$ 是唯一的. 因为 L_N^p 和 $L_N^{p'}$ 是一致凸的，通过和引理 2.32 类似的论证得到定义在 L_N^p 的闭子空间上的线性泛函有唯一的到 L_N^p 的保范延拓.

3.9 对于 $1\leq p<\infty$，空间 $(W^{m,p}(\Omega))'$ 的每一个元素 L 是一个广义函数 $T\in\mathscr{D}'(\Omega)$ 在 $W^{m,p}(\Omega)$ 上的延拓. 为了证明这一点，假定 L 是对某个 $v\in L_N^{p'}$ 由(5)式给出的，而且由下式定义 $T_{v_\alpha}, T\in\mathscr{D}'(\Omega)$,

$$T_{v_\alpha}(\phi)=\langle\phi, v_\alpha\rangle, \qquad \phi\in\mathscr{D}(\Omega), 0\leq|\alpha|\leq m,$$

$$T=\sum_{0\leq|\alpha|\leq m}(-1)^{|\alpha|}D^\alpha T_{v_\alpha} \qquad (8)$$

对于一切 $\phi\in\mathscr{D}(\Omega)\subset W^{m,p}(\Omega)$ 有

$$T(\phi)=\sum_{0\leq|\alpha|\leq m}T_{v_\alpha}(D^\alpha\phi)=L(\phi)$$

所以 L 显然是 T 的一个延拓. 而且，按照(6)我们有

$$\|L;(W^{m,p}(\Omega))'\|=\min\{\|v;L_N^{p'}\|: L \text{ 延拓了由(8)给出的} T\}.$$

对于 $L\in(W_0^{m,p}(\Omega))'$，上面的注解也适用，因为任何这样的泛函有一个到 $W^{m,p}(\Omega)$ 的保范延拓.

现在假定 T 是 $\mathscr{D}'(\Omega)$ 中对某个 $v\in L_N^{p'}, 1\leq p'<\infty$ 的形为(8)的元素. 那么 T 在 $W^{m,p}(\Omega)$ 上的连续延拓可能不是唯一的. 但是我们证明 T 在 $W_0^{m,p}(\Omega)$ 上的连续延拓是唯一的. 当 $u\in W_0^{m,p}(\Omega)$ 时，设 $\{\phi_n\}$ 是 $C_0^\infty(\Omega)=\mathscr{D}(\Omega)$ 中的一个序列，使得当 $n\to\infty$ 时，

$\|\phi_n-u\|_{m,p}\to 0$，则当 $k,n\to\infty$ 时

$$|T(\phi_k)-T(\phi_n)|\leqslant\sum_{0\leqslant|\alpha|\leqslant m}|T_{v_\alpha}(D^\alpha\phi_k-D^\alpha\phi_n)|$$

$$\leqslant\sum_{0\leqslant|\alpha|\leqslant m}\|D^\alpha(\phi_k-\phi_n)\|_p\|v_\alpha\|_{p'}$$

$$\leqslant\|\phi_k-\phi_n\|_{m,p}\|v;L_N^{p'}\|\to 0.$$

所以 $\{T(\phi_n)\}$ 是 C 中的一个 Cauchy 序列，因而收敛到一个极限，我们把它记作 $L(u)$，因为以下事实是显然的，即如果另有 $\{\psi_n\}\subset\mathscr{D}(\Omega)$ 而且 $\|\psi_n-u\|_{m,p}\to 0$，则当 $n\to\infty$ 时 $T(\phi_n)-T(\psi_n)\to 0$. 这样定义的泛函 L 是线性的而且属于 $(W_0^{m,p}(\Omega))'$，因为如果象前面所说的 $u=\lim\limits_{n\to\infty}\phi_n$，则

$$|L(u)|=\lim_{n\to\infty}|T(\phi_n)|\leqslant\lim_{n\to\infty}\|\phi_n\|_{m,p}\|v;L_N^{p'}\|=\|u\|_{m,p}\|v;L_N^{p'}\|.$$

所以我们已经证明了下面的定理.

3.10 定理 设 $1\leqslant p<\infty$. 对偶空间 $(W_0^{m,p}(\Omega))'$ 等距同构于由对于某些 $v\in L_N^{p'}$ 满足(8)式的广义函数 $T\in\mathscr{D}'(\Omega)$ 组成的 Banach 空间，该空间由

$$\|T\|=\inf\{\|v;L_N^{p'}\|:\ v\text{ 满足(8)}\}$$

来赋范. 一般说来，如果 $W_0^{m,p}(\Omega)$ 是 $W^{m,p}(\Omega)$ 的真子空间，不能期望 $(W^{m,p}(\Omega))'$ 会有如此简单的表征.

3.11 如果 $m=1,2,\cdots$ 以及 $1\leqslant p<\infty$，设 p' 表示 p 的共轭指数，用 $W^{-m,p'}(\Omega)$ 来表示上述定理中所说的 Ω 上的广义函数所构成的 Banach 空间. (空间 $W^{-m,p'}(\Omega)$ 的完备性是上述定理所断言的等距同构的一个推论.) 当 $1<p<\infty$ 时，$W^{-m,p'}(\Omega)$ 显然是可分的，而且是自反的.

3.12 设 $1<p<\infty$，每个元素 $v\in L^{p'}(\Omega)$ 通过 $L_v(u)=\langle u,v\rangle$ 决定 $(W_0^{m,p}(\Omega))'$ 中的一个元素 L_v，因为

$$|L_v(u)|=|\langle u,v\rangle|\leqslant\|v\|_{p'}\|u\|_p\leqslant\|v\|_{p'}\|u\|_{m,p}.$$

我们把 L_v 的范数定义为 $v\in L^{p'}(\Omega)$ 的 $(-m,p')$-范数，即

$$\|v\|_{-m,p'}=\|L_v;(W_0^{m,p}(\Omega))'\|=\sup_{\substack{u\in W\\ \|u\|_{m,p}\leqslant 1}}|\langle u,v\rangle|,$$

在等式的右端我们已经用 W 来表示 $W_0^{m,p}(\Omega)$，为了书写简单起见，我们继续这样做．显然，对任何 $u\in W$ 和 $v\in L^{p'}(\Omega)$，我们有

$$\|v\|_{-m,p'}\leqslant\|v\|_{p'},$$

$$|\langle u,v\rangle|=\|u\|_{m,p}\left|\left\langle\frac{u}{\|u\|_{m,p}},v\right\rangle\right|\leqslant\|u\|_{m,p}\|v\|_{-m,p'}. \tag{9}$$

上式是 Hölder 不等式的一个推广．

设 $V=\{L_v:\ v\in L^{p'}(\Omega)\}$．因此 V 是 $W'=(W_0^{m,p}(\Omega))'$ 的向量子空间．我们证明事实上 V 在 W' 中稠密．易见这等价于证明：如果 $F\in W''$ 对一切 $L_v\in V$ 满足 $F(L_v)=0$，则在 W'' 中 $F=0$．因为 W 是自反的，存在 $f\in W$ 相应于 $F\in W''$ 使得对一切 $v\in L^{p'}(\Omega)$，$\langle f,v\rangle=L_v(f)=F(L_v)=0$．因为 $f\in L^p(\Omega)$．由此得到在 Ω 中 $f(x)=0$ a.e.．因此在 W 中 $f=0$ 从而在 W'' 中 $F=0$.

设 $H^{-m,p'}(\Omega)$ 表示 $L^{p'}(\Omega)$ 关于范数 $\|\cdot\|_{-m,p'}$ 的完备化．那么我们有

$$H^{-m,p'}(\Omega)\cong(W_0^{m,p}(\Omega))'\cong W^{-m,p'}(\Omega).$$

特别，与每个 $v\in H^{-m,p'}(\Omega)$ 相应，存在 $T_v\in W^{-m,p'}(\Omega)$ 使得对一切 $\phi\in\mathscr{D}(\Omega)$ 和一切使 $\lim\limits_{n\to\infty}\|v_n-v\|_{-m,p'}=0$ 的序列 $\{v_n\}\subset L^{p'}(\Omega)$ 有 $T_v(\phi)=\lim\limits_{n\to\infty}\langle\phi,v_n\rangle$．反之，任何 $T\in W^{-m,p'}(\Omega)$ 存在某个 $v\in H^{-m,p'}(\Omega)$ 使得 $T=T_v$. 而且，由(9)式，$|T_v(\phi)|\leqslant\|\phi\|_{m,p}\|v\|_{-m,p'}$.

3.13 对于 $1<p<\infty$ 的情形，通过类似于 3.12 节中的论证，对偶空间 $(W^{m,p}(\Omega))'$ 可以表征为 $L^{p'}(\Omega)$ 关于范数

$$\|v\|_{-m,p'}^{\sim}=\sup_{\substack{u\in W^{m,p}(\Omega)\\ \|u\|_{m,p}\leqslant 1}}|\langle u,v\rangle|$$

的完备化．

用Ω上的光滑函数来逼近

我们希望证明$\{\phi\in C^\infty(\Omega):\ \|\phi\|_{m,p}<\infty\}$在$W^{m,p}(\Omega)$中稠密. 为此我们需要下面关于无穷次可微单位分解的标准存在定理.

3.14 定理 设A是$\mathbf{R}^n$中任意一个子集，又设$\mathscr{O}$是$\mathbf{R}^n$中覆盖住A的一组开集,即,使得$A\subset\bigcup_{U\in\mathscr{O}}U$. 那么存在一个由具有下列性质的函数$\psi\in C_0^\infty(\mathbf{R}^n)$构成的函数族$\Psi$:

(i) 对一切$\psi\in\Psi$和一切$x\in\mathbf{R}^n$, $0\leqslant\psi(x)\leqslant 1$.

(ii) 如果$K\subset\subset A$, 除可能有有限个ψ外所有的$\psi\in\Psi$在K上恒等于0.

(iii) 对每一个$\psi\in\Psi$存在$U\in\mathscr{O}$使得$\operatorname{supp}\psi\subset U$.

(iv) 对一切$x\in A$, $\sum_{\psi\in\Psi}\psi(x)=1$.

这样的一个函数族Ψ叫做A的从属于$\mathscr{O}$的一个C^∞-单位分解.

证明 因为在许多教科书中可以找到证明，我们在这里只给出一个轮廓,把细节留给读者. 首先假定A是紧的,所以$A\subset\bigcup_{j=1}^{N}U_j$,其中$U_1,\cdots,U_N\in\mathscr{O}$. 能够构造紧集$K_1\subset U_1,\cdots,K_N\subset U_N$使$A\subset\bigcup_{j=1}^{N}K_j$. 对$1\leqslant j\leqslant N$存在非负值函数$\phi_j\in C_0^\infty(U_j)$使得对$x\in K_j$, $\phi_j(x)>0$. 那么能够构造一个在$\mathbf{R}^n$上无穷次可微的正函数ϕ，而且对于$x\in A$满足$\phi(x)=\sum_{j=1}^{N}\phi_j(x)$. 于是$\Psi=\{\psi_j:\ \psi_j(x)=\phi_j(x)/\phi(x), 1\leqslant j\leqslant N\}$就具有所要的性质. 现在假定$A$是开集，则$A=\bigcup_{j=1}^{\infty}A_j$,其中

$$A_j=\left\{x\in A:\ |x|\leqslant j\quad\text{而且}\quad \operatorname{dist}(x,\operatorname{bdry}A)\geqslant\frac{1}{j}\right\}$$

是紧的. 对每个j,集族

$$\mathscr{O}_j=\{U\cap(A_{j+1}\text{ 的内点}\cap A_{j-2}^c):\ U\in\mathscr{O}\}$$

盖住 A_j，所以存在一个 A_j 的从属于 $\mathscr{O}_j$ 的有限 C^∞-单位分解 Ψ_j。在每一点，和式 $\sigma(x)=\sum_{j=1}^{\infty}\sum_{\phi\in\Psi_j}\phi(x)$ 只包含有限多个非 0 项，而且在每个 $x\in A$ 上是正的。函数族 $\Psi=\{\psi$: 当 $x\in A$ 时对于在某个 Ψ_j 中的某个 ϕ, $\psi(x)=\phi(x)/\sigma(x)$, 当 $x\notin A$ 时, $\psi(x)=0\}$ 具有所规定的性质. 最后，如果 A 是任意的，则 $A\subset B=\bigcup_{U\in\mathscr{O}}U$，其中 B 是开集. 对 B 做出的任何单位分解同样也是 A 的单位分解. ∎

3.15 引理 设 J_ε 是 2.17 节中定义的软化子，又设 $1\leqslant p<\infty$ 而且 $u\in W^{m,p}(\Omega)$. 如 $\Omega'\subset\subset\Omega$，则在 $W^{m,p}(\Omega')$ 中 $\lim_{\varepsilon\to0+}J_\varepsilon*u=u$.

证明 设 $\varepsilon<\operatorname{dist}(\Omega',\operatorname{bdry}\Omega)$，对任何 $\phi\in\mathscr{D}(\Omega')$ 我们有

$$\begin{aligned}\int_{\Omega'}J_\varepsilon*u(x)D^\alpha\phi(x)dx&=\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\tilde{u}(x-y)J_\varepsilon(y)D^\alpha\phi(x)dxdy\\&=(-1)^{|\alpha|}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}D_x^\alpha\tilde{u}(x-y)J_\varepsilon(y)\phi(x)dxdy\\&=(-1)^{|\alpha|}\int_{\Omega'}J_\varepsilon*D^\alpha u(x)\phi(x)dx,\end{aligned}$$

其中 $\tilde{u}$ 是在 Ω 外为 0 的 u 的零延拓. 因此在 Ω' 中在广义函数意义下 $D^\alpha J_\varepsilon*u=J_\varepsilon*D^\alpha u$. 因为对 $0\leqslant|\alpha|\leqslant m$, $D^\alpha u\in L^p(\Omega)$，由引理2.18(c)我们有

$$\lim_{\varepsilon\to0+}\|D^\alpha J_\varepsilon*u-D^\alpha u\|_{p,\Omega'}=\lim_{\varepsilon\to0+}\|J_\varepsilon*D^\alpha u-D^\alpha u\|_{p,\Omega'}=0.$$

于是

$$\lim_{\varepsilon\to0+}\|J_\varepsilon*u-u\|_{m,p,\Omega'}=0.\ ∎$$

3.16 定理 (Meyers 和 Serrin[46]) 如果 $1\leqslant p<\infty$，则

$$H^{m,p}(\Omega)=W^{m,p}(\Omega).$$

证明 由于推论 3.3 只要证明 $W^{m,p}(\Omega)\subset H^{m,p}(\Omega)$ 就够了，即证明 $\{\phi\in C^m(\Omega):\ \|\phi\|_{m,p}<\infty\}$ 在 $W^{m,p}(\Omega)$ 中是稠密的. 如果

$u\in W^{m,p}(\Omega)$，而 $\varepsilon>0$，实际上我们证明存在 $\phi\in C^\infty(\Omega)$ 使得 $\|u-\phi\|_{m,p}<\varepsilon$. 对 $k=1,2,\cdots$，设

$$\Omega_k=\left\{x\in\Omega:\ |x|<k \text{ 且 } \operatorname{dist}(x,\ \operatorname{bdry}\Omega)>\frac{1}{k}\right\},$$

又设 $\Omega_0=\Omega_{-1}=\varnothing$ 是空集. 则

$$\mathscr{O}=\{U_k:U_k=\Omega_{k+1}\cap(\overline{\Omega_{k-1}})^c,k=1,2,\cdots\}$$

是一个盖住 Ω 的开子集族，设 Ψ 是 Ω 的从属于 $\mathscr{O}$ 的一个 C^∞ 单位分解. 设 ψ_k 表示支集包含在 U_k 中的有限个函数 $\psi\in\Psi$ 的和. 则在 Ω 上 $\psi_k\in C_0^\infty(U_k)$ 而且 $\sum\limits_{k=1}^{\infty}\psi_k(x)=1$.

如 $0<\varepsilon<1/(k+1)(k+2)$，则 $J_\varepsilon*(\psi_k u)$ 在 $\Omega_{k+2}\cap(\Omega_{k-2})^c=\mathscr{V}_k\subset\subset\Omega$ 中有支集，因为 $\psi_k u\in W^{m,p}(\Omega)$ 我们可以选 ε_k，$0<\varepsilon_k<1/(k+1)(k+2)$ 使得

$$\|J_{\varepsilon_k}*(\psi_k u)-\psi_k u\|_{m,p,\Omega}=\|J_{\varepsilon_k}*(\psi_k u)-\psi_k u\|_{m,p,\mathscr{V}_k}<\varepsilon/2^k.$$

设 $\phi=\sum\limits_{k=1}^{\infty}J_{\varepsilon_k}*(\psi_k u)$. 在任何 $\Omega'\subset\subset\Omega$ 上，在和式中只可能有有限项不为 0. 因此 $\phi\in C^\infty(\Omega)$，对于 $x\in\Omega_k$ 我们有

$$u(x)=\sum_{j=1}^{k+2}\psi_j(x)u(x),\qquad \phi(x)=\sum_{j=1}^{k+2}J_{\varepsilon_j}*(\psi_j u)(x).$$

因此

$$\|u-\phi\|_{m,p,\Omega_k}\leqslant\sum_{j=1}^{k+2}\|J_{\varepsilon_j}*(\psi_j u)-\psi_j u\|_{m,p,\Omega}<\varepsilon.$$

由单调收敛定理 1.43，$\|u-\phi\|_{m,p,\Omega}<\varepsilon$. ∎

我们指出定理不能推广到 $p=\infty$ 的情形，例如，若 $\Omega=\{x\in\mathbf{R}: -1<x<1\}$ 而 $u(x)=|x|$，则 $u\in W^{1,\infty}(\Omega)$，但是 $u\notin H^{1,\infty}(\Omega)$. 事实上，如果 $\varepsilon<\frac{1}{2}$，任何函数 $\phi\in C^1(\Omega)$ 都不能使 $\|\phi'-u'\|_\infty<\varepsilon$.

用 $\mathbf{R}^n$ 上的光滑函数来逼近

3.17 刚才证明了 $W^{m,p}(\Omega)$ 中的元素总能用Ω上的光滑函数来逼近，现在我们要问是否在实际上能够用具有一切阶或具有直到m阶有界导数的有界函数来逼近. 也就是，我们问是否对于任何$k\geqslant m$，$C^k(\overline{\Omega})$ 在 $W^{m,p}(\Omega)$ 中稠密. 下面的例子表明这个问题的回答可以是否定的:

设 $\Omega=\{(x,y)\in\mathbf{R}^2: 0<|x|<1,\ 0<y<1\}$，则由

$$u(x,y)=\begin{cases}1 & \text{当 } x>0 \text{ 时}\\ 0 & \text{当 } x<0 \text{ 时}\end{cases}$$

规定的函数显然属于 $W^{1,p}(\Omega)$. 但是读者可以验证，对于充分小的 $\varepsilon>0$，没有函数 $\phi\in C^1(\overline{\Omega})$ 能够满足$\|u-\phi\|_{1,p}<\varepsilon$. 由于这种区域造成的困难就在于区域位于部分边界 (线段 $x=0, 0<y<1$) 的两边.

我们说区域Ω具有**线段性质**，如果对于每个$x\in\operatorname{bdry}\Omega$，存在一个开集$U_x$和一个非 0 向量 y_x，使得 $x\in U_x$，又如果 $z\in\overline{\Omega}\cap U_x$，则对于 $0<t<1$，$z+ty_x\in\Omega$. 具有这种性质的区域一定有$(n-1)$ 维边界，而且不能同时位于其边界的任何给定部分的两边.

下面的定理说明，线段性质足以保证 $C_0^\infty(\mathbf{R}^n)$ 在 $W^{m,p}(\Omega)$ 中稠密，因而特别说来，也保证对于任何 m，$C^k(\overline{\Omega})$ 在 $W^{m,p}$ 中稠密.

3.18 定理 如果Ω具有线段性质，那么对 $1\leqslant p<\infty$，$C_0^\infty(\mathbf{R}^n)$ 中的函数在Ω上的限制构成的集合在 $W^{m,p}(\Omega)$ 中稠密.

证明 设 f 是满足

(i) 当 $|x|\leqslant 1$ 时 $f(x)=1$,

(ii) 当 $|x|\geqslant 2$ 时 $f(x)=0$,

(iii) 对于一切 x 和 $0\leqslant|\alpha|\leqslant m$，$|D^\alpha f(x)|\leqslant M$ (常数) 的

$C_0^\infty(\mathbf{R}^n)$ 中一个固定的函数. 对 $\varepsilon>0$，令 $f_\varepsilon(x)=f(\varepsilon x)$. 那么当

$|x|\leqslant\frac{1}{\varepsilon}$ 时 $f_\varepsilon(x)=1$ 而当 $\varepsilon\leqslant 1$ 时 $|D^\alpha f_\varepsilon(x)|\leqslant M\varepsilon^{|\alpha|}\leqslant M$. 如果 $u\in W^{m,p}(\Omega)$，则 $u_\varepsilon=f_\varepsilon\cdot u$ 属于 $W^{m,p}(\Omega)$ 且具有有界支集. 因为对于 $0<\varepsilon\leqslant 1$ 和 $|\alpha|\leqslant m$

$$|D^\alpha u_\varepsilon(x)|=\left|\sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}D^\beta u(x)D^{\alpha-\beta}f_\varepsilon(x)\right|\leqslant M\sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}|D^\beta u(x)|,$$

令 $\Omega_\varepsilon=\left\{x\in\Omega:|x|>\frac{1}{\varepsilon}\right\}$，我们有

$$\begin{aligned}\|u-u_\varepsilon\|_{m,p,\Omega}&=\|u-u_\varepsilon\|_{m,p,\Omega_\varepsilon}\leqslant\|u\|_{m,p,\Omega_\varepsilon}+\|u_\varepsilon\|_{m,p,\Omega_\varepsilon}\\&\leqslant\mathrm{const}\|u\|_{m,p,\Omega_\varepsilon}.\end{aligned}$$

当 $\varepsilon\to 0$ 时，右端趋于 0. 因此任何 $u\in W^{m,p}(\Omega)$ 能够用 $W^{m,p}(\Omega)$ 中的具有有界支集的函数来逼近.

所以现在我们可以假定 $K=\{x\in\Omega:u(x)\neq 0\}$ 是有界的. 于是集合 $F=\bar{K}\sim(\bigcup_{x\in\mathrm{bdry}\,\Omega}U_x)$ 是紧的而且包含在 Ω 中，$\{U_x\}$ 是在线段性质定义中指出的开集合的集族. 存在一个开集 U_0 使得 $F\subset\subset U_0\subset\subset\Omega$. 因为 $\bar{K}$ 是紧的，存在有限个集合 U_x；我们把它们重新命名为 $U_1,U_2\cdots,U_k$，使得 $\bar{K}\subset U_0\cup U_1\cup\cdots\cup U_k$. 而且，我们可以找到另外一些开集 $\widetilde{U}_0,\widetilde{U}_1,\cdots,\widetilde{U}_k$ 使得 $\widetilde{U}_j\subset\subset U_j$，$0\leqslant j\leqslant k$，仍有 $\bar{K}\subset\widetilde{U}_0\cup\widetilde{U}_1\cup\cdots\cup\widetilde{U}_k$.

设 Ψ 是从属于 $\{\widetilde{U}_j;0\leqslant j\leqslant k\}$ 的一个 C^∞ 单位分解，又设 ψ_j 是支集在 $\widetilde{U}_j$ 中的有限个函数 $\psi\in\Psi$ 的和. 令 $u_j=\psi_j u$. 假设对每个 j 我们能够找到 $\phi_j\in C_0^\infty(\mathbf{R}^n)$ 使得

$$\|u_j-\phi_j\|_{m,p,\Omega}<\frac{\varepsilon}{k+1}. \tag{10}$$

那么令 $\phi=\sum_{j=0}^{k}\phi_j$，我们得到

$$\|\phi-u\|_{m,p,\Omega}\leqslant\sum_{j=0}^{k}\|\phi_j-u_j\|_{m,p,\Omega}<\varepsilon.$$

由引理3.15，对于 $j=0$ 能找到满足(10)的一个函数 $\phi_0\in C_0^\infty(\mathbf{R}^n)$，

因为 supp $u_0\subset\widetilde{U}_0\subset\subset\Omega$. 因此，剩下来要对$1\leqslant j\leqslant k$ 找满足(10)的 ϕ_j. 对于固定的 j，我们把 u_j 延拓成为Ω 外恒等于 0. 于是$u_j\in W^{m,p}(\mathbf{R}^n\sim\Gamma)$，其中$\Gamma=\widetilde{U}_j\cap\mathrm{bdry}\,\Omega$. 设 y 是线段性质定义中与集合 U_j 相应的非 0 向量(图 1)，设 $\Gamma_t=\Gamma-ty$，其中选 t 使得 $0<t<\min(1,\operatorname{dist}(\widetilde{U}_j,R^n\sim U_j)/|y|)$. 则由于线段性质 $\Gamma_t\subset U_j$ 且$\Gamma_t\cap\overline{\Omega}$ 是空集. 设 $u_{j,t}(x)=u_j(x+ty)$，则 $u_{j,t}\in W^{m,p}(\mathbf{R}^n\sim\Gamma_t)$. 在$L^p(\Omega)$中平移是连续的，所以对于$|\alpha|\leqslant m$，当 $t\to0+$ 时，在 $L^p(\Omega)$ 中 $D^\alpha u_{j,t}\to D^\alpha u_j$，因此当 $t\to0+$时，在 $W^{m,p}(\Omega)$中 $u_{j,t}\to u_j$，所以只要找 $\phi_j\in C_0^\infty(\mathbf{R}^n)$使得$\|u_{j,t}-\phi_j\|_{m,p}$充分小就行了. 但是$\Omega\cap U_j\subset\subset\mathbf{R}^n\sim\Gamma_t$，所以由引理 3.15 对适当小的 $\delta>0$ 我们可以取 $\phi_j=J_\delta*u_{j,t}$. 这就完成了证明. ▮

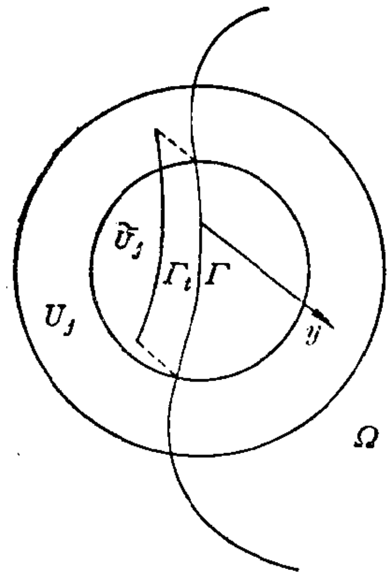

图 1

3.19 推论 $W_0^{m,p}(\mathbf{R}^n)=W^{m,p}(\mathbf{R}^n)$

用 $C_0^\infty(\Omega)$中的函数来逼近；(m,p')-极集(polar sets)

推论 3.19 使人联想到下面的问题：对什么样的区域 Ω，$W^{m,p}(\Omega)=W_0^{m,p}(\Omega)$，即什么时候$C_0^\infty(\Omega)$在$W^{m,p}(\Omega)$中稠密？这个问题的部分回答可以用 $W^{-m,p'}(\Omega)$ 中的广义函数的性质来系统地陈述. 下面给出的方法是属于 Lions [39] 的.

3.20 以下讨论中我们始终假定 $1<p<\infty$而且 p' 是 p 的共轭数. 设F是 $\mathbf{R}^n$ 的闭子集，假如对一切在F上恒等于 0 的 $\phi\in\mathscr{D}(\mathbf{R}^n)$，$T(\phi)=0$，我们就说广义函数 $T\in\mathscr{D}'(\mathbf{R}^n)$的支集在$F$中 (supp$T\subset F$). 如果支集在$F$中的 $W^{-m,p'}(\mathbf{R}^n)$ 的广义函数T只能是零广义函数$T=0$，我们就说闭集F是(m,p')-极集.

如果F具有正测度,那么F不可能是$(m\ p')$-极集，因为具有正测度的F的任何紧子集的特征函数属于 $L^{p'}(\mathbf{R}^n)$，因此属于$W^{-m,p'}(\mathbf{R}^n)$.

以后我们将证明，如果 $mp>n$,则在如下的意义下 $W^{m,p}(\mathbf{R}^n)\to C(\mathbf{R}^n)$ （见定理 5.4）：即，如果 $u\in W^{m,p}(\mathbf{R}^n)$，则存在 $u_0\in C(\mathbf{R}^n)$使得 $u(x)=u_0(x)$ a.e.，而且

$$|u_0(x)|\leqslant \text{const}\|u\|_{m,p},$$

常数与x和u无关,由此得到,由$\delta_x(\phi)=\phi(x)$给出的 Dirac 广义函数δ_x属于 $(W^{m,p}(\mathbf{R}^n))'=(W_0^{m,p}(\mathbf{R}^n))'\cong W^{-m,p'}(\mathbf{R}^n)$. 因此,当 $mp>n$ 时,除非F是空集,集合F不可能是(m,p')-极集.

3.21 因为$W^{m+1,p}(\mathbf{R}^n)\to W^{m,p}(\mathbf{R}^n)$，任何$W^{m,p}(\mathbf{R}^n)$上的有界线性泛函也是 $W^{m+1,p}(\mathbf{R}^n)$ 上的有界线性泛函，即$W^{-m,p'}(\Omega)\subset W^{-m-1,p'}(\Omega)$. 因此任何 $(m+1,p')$-极集也是 (m,p')-极集. 当然反过来说,一般是不对的.

设映射 $u\to\tilde{u}$ 表示u在区域 $\Omega\subset\mathbf{R}^n$外的零延拓:

$$\tilde{u}(x)=\begin{cases}u(x) & \text{当 } x\in\Omega\text{时}\\ 0 & \text{当 } x\in\Omega^c\text{ 时}\end{cases}\tag{11}$$

下面的引理证明了该映射把 $W_0^{m,p}(\Omega)$ (等距地)映射到 $W^{m,p}(\mathbf{R}^n)$ 中去.

3.22 引理 设$u\in W_0^{m,p}(\Omega)$. 如果$|\alpha|\leqslant m$,则在 $\mathbf{R}^n$ 中在广义函数意义下 $D^\alpha\tilde{u}=(D^\alpha u)^\sim$. 因此 $\tilde{u}\in W^{m,p}(\mathbf{R}^n)$.

证明 设 $C_0^\infty(\Omega)$ 中的序列 $\{\phi_n\}$ 在 $W_0^{m,p}(\Omega)$ 中收敛到u. 如果$\psi\in\mathscr{D}(\mathbf{R}^n)$,对于$|\alpha|\leqslant m$,我们有

$$\begin{aligned}(-1)^{|\alpha|}\int_{\mathbf{R}^n}\tilde{u}(x)D^\alpha\psi(x)dx&=(-1)^{|\alpha|}\int_\Omega u(x)D^\alpha\psi(x)dx\\&=\lim_{n\to\infty}(-1)^{|\alpha|}\int_\Omega\phi_n(x)D^\alpha\psi(x)dx\end{aligned}$$

$$=\lim_{n\to\infty}\int_{\Omega} D^{\alpha}\phi_n(x)\psi(x)dx=\int_{\mathbf{R}^n}(D^{\alpha}u)^{\sim}(x)\psi(x)dx.$$

于是在 $\mathbf{R}^n$ 上在广义函数意义下 $D^{\alpha}\tilde{u}=(D^{\alpha}u)^{\sim}$ 因此这些局部可积函数在 $\mathbf{R}^n$ 中是几乎处处相等的，由此得到 $\|\tilde{u}\|_{m,p,\mathbf{R}^n}=\|u\|_{m,p,\Omega}$. ▌

现在我们给出映射(11)把 $W_0^{m,p}(\Omega)$ 等距地映射到 $W^{m,p}(\mathbf{R}^n)$ 上的关于Ω的一个必要充分条件.

3.23 定理 $C_0^{\infty}(\Omega)$ 在 $W^{m,p}(\mathbf{R}^n)$ 中稠密当且仅当Ω 的余集 Ω^c 是(m,p')-极集.

证明 先假设 $C_0^{\infty}(\Omega)$ 在 $W^{m,p}(\mathbf{R}^n)$ 中稠密. 设 $T\in W^{-m,p'}(\mathbf{R}^n)$ 具有包含在Ω^c 中的支集. 如果 $u\in W^{m,p}(\mathbf{R}^n)$, 则存在一个序列$\{\phi_n\}\subset C_0^{\infty}(\Omega)$ 在 $W^{m,p}(\mathbf{R}^n)$ 中收敛到 u. 因此 $T(u)=\lim_{n\to\infty}T(\phi_n)=0$, 所以 $T=0$, 因而 Ω^c 是(m,p')-极集.

反之，若 $C_0^{\infty}(\Omega)$ 在 $W^{m,p}(\mathbf{R}^n)$ 中不稠密，则存在 $u\in W^{m,p}(\mathbf{R}^n)$ 使得对于一切 $\phi\in C_0^{\infty}(\Omega)$, $\|u-\phi\|_{m,p,\mathbf{R}^n}\geq k>0$, k 与ϕ 无关，根据 Hahn-Banach 延拓定理存在$T\in W^{-m,p'}(\mathbf{R}^n)$使得对一切$\phi\in C_0^{\infty}(\Omega)$, $T(\phi)=0$, 但 $T(u)\neq 0$. 因为 $\operatorname{supp}T\subset\Omega^c$ 但 $T\neq 0$, 所以 Ω^c 不可能是(m,p')-极集. ▌

对于可微函数来说以下事实是显然的，即，在矩形上一阶导数恒等于零蕴含着在矩形上函数恒等于常数. 作为我们研究 $W_0^{m,p}(\Omega)$ 和 $W^{m,p}(\Omega)$ 是可能恒同的最后一个准备工作就是首先把这一事实推广到广义函数，而后再推广到局部可积函数.

3.24 引理 设$B=(a_1,b_1)\times(a_2,b_2)\times\cdots\times(a_n,b_n)$是 $\mathbf{R}^n$ 中一开的长方形盒子，设 $\phi\in\mathscr{D}(B)$. 若$\int_B\phi(x)dx=0$, 则 $\phi(x)=\sum_{j=1}^{n}\phi_j(x)$, 其中 $\phi_j\in\mathscr{D}(B)$而且对每个固定的$(x_1,\cdots,x_{j-1},x_{j+1},\cdots,x_n)\in\mathbf{R}^{n-1}$

$$\int_{a_j}^{b_j}\phi_j(x_1,\cdots,x_j,\cdots,x_n)dx_j=0. \tag{12}$$

证明 对于 $1\leqslant j\leqslant n$ 选函数 $u_j\in C_0^\infty(a_j,b_j)$ 使得

$$\int_{a_j}^{b_j}u_j(t)dt=1.$$

设

$$B_j=(a_j,b_j)\times(a_{j+1},b_{j+1})\times\cdots\times(a_n,b_n),$$

$$\psi_j(x_j,\cdots,x_n)=\int_{a_1}^{b_1}dt_1\int_{a_2}^{b_2}dt_2\cdots\int_{a_{j-1}}^{b_{j-1}}\phi(t_1,\cdots,t_{j-1},x_j,\cdots,x_n)dt_{j-1},$$

$$\omega_j(x)=u_1(x_1)\cdots u_{j-1}(x_{j-1})\psi_j(x_j,\cdots,x_n).$$

则 $\psi_j\in\mathscr{D}(B_j)$ 而 $\omega_j\in\mathscr{D}(B)$. 而且

$$\int_{B_j}\psi_j(x_j,\cdots,x_n)dx_j\cdots dx_n=\int_B\phi(x)dx=0.$$

设 $\phi_1=\phi-\omega_2$, $\phi_j=\omega_j-\omega_{j+1}(2\leqslant j\leqslant n-1)$, $\phi_n=\omega_n$.

显然,对于 $1\leqslant j\leqslant n$, $\phi_j\in\mathscr{D}(B)$,而且 $\phi=\sum_{j=1}^{n}\phi_j$. 最后,

$$\begin{aligned}&\int_{a_1}^{b_1}\phi_1(x_1,\cdots,x_n)dx_1\\&=\int_{a_1}^{b_1}\phi(x_1,\cdots,x_n)dx_1-\psi_2(x_2,\cdots,x_n)\int_{a_1}^{b_1}u_1(x_1)dx_1=0,\end{aligned}$$

$$\int_{a_j}^{b_j}\phi_j(x_1,\cdots,x_n)dx_j=u_1(x_1)\cdots u_{j-1}(x_{j-1})$$

$$\times\left(\int_{a_j}^{b_j}\psi_j(x_j,\cdots,x_n)dx_j-\psi_{j+1}(x_{j+1},\cdots,x_n)\int_{a_j}^{b_j}u_j(x_j)dx_j\right)=0$$

$$2\leqslant j\leqslant n-1$$

$$\begin{aligned}\int_{a_n}^{b_n}\phi_n(x_1,\cdots,x_n)dx_n&=u_1(x_1)\cdots u_{n-1}(x_{n-1})\int_{a_n}^{b_n}\psi_n(x_n)dx_n\\&=u_1(x_1)\cdots u_{n-1}(x_{n-1})\int_B\phi(x)dx=0.\ \blacksquare\end{aligned}$$

3.25 推论 如果 $T\in\mathscr{D}'(B)$ 而且对于 $1\leqslant j\leqslant n$, $D_jT=0$, 则存在

常数 k 使得对于一切 $\phi\in\mathscr{D}(B)$

$$T(\phi)=k\int_B\phi(x)dx.$$

证明 首先注意到如果 $\int_B\phi(x)dx=0$，则 $T(\phi)=0$，因为由上面的引理，我们可以把 $\phi(x)$ 写成 $\phi=\sum_{j=1}^{n}\phi_j$，其中 $\phi_j\in\mathscr{D}(B)$ 且满足(12)，因此，$\phi_j=D_j\theta_j$，其中 $\theta_j\in\mathscr{D}(B)$ 由

$$\theta_j(x)=\int_{a_j}^{x_j}\phi_j(x_1,\cdots,x_{j-1},t,x_{j+1},\cdots,x_n)dt$$

定义，因此 $T(\phi)=\sum_{j=1}^{n}T(D_j\theta_j)=-\sum_{j=1}^{n}D_jT(\theta_j)=0.$

现在假设 $T\neq0$. 则存在 $\phi_0\in\mathscr{D}(B)$ 使 $T(\phi_0)=k_1\neq0$. 因此 $\int_B\phi_0(x)dx=k_2\neq0$，而且 $T(\phi_0)=k\int_B\phi_0(x)dx$，其中 $k=k_1/k_2$. 如果 $\phi\in\mathscr{D}(B)$ 是任意的，设 $K(\phi)=\int_B\phi(x)dx$. 则

$$\int_B\left(\phi(x)-\frac{K(\phi)}{k_2}\phi_0(x)\right)dx=0$$

所以 $T(\phi-[K(\phi)/k_2]\phi_0)=0$，由此得到

$$T(\phi)=\frac{K(\phi)}{k_2}T(\phi_0)=kK(\phi)=k\int_B\phi(x)dx.\ \blacksquare$$

应该指出这个推论能推广到 $\mathbf{R}^n$ 中任何开的连通集 B，这种推广通过一个从属于 Ω 的某个开覆盖的单位分解来实现. 这个开覆盖是由包含在 Ω 中开的长方形盒子构成. 但我们将来不需要这种推广.

下面的引理证明了开集 Ω 上不同的局部可积函数决定 Ω 上不同的广义函数.

3.26 引理 设 $u\in L^1_{\mathrm{loc}}(\Omega)$ 对于一切 $\phi\in\mathscr{D}(\Omega)$ 满足 $\int_\Omega u(x)\phi(x)dx=0$. 则在 Ω 中 $u(x)=0$ a.e..

证明 如果 $\psi\in C_0(\Omega)$，则对充分小的正 ε，软化子 $J_\varepsilon*\psi$ 属于 $\mathscr{D}(\Omega)$. 由引理 2.18，当 $\varepsilon\to0+$ 时，在 Ω 上一致地有 $J_\varepsilon*\psi\to\psi$. 因此对一切 $\psi\in C_0(\Omega)$，$\int_\Omega u(x)\psi(x)dx=0$.

设 $K\subset\subset\Omega$，又设 $\varepsilon>0$. 设 χ_K 是 K 的特征函数. 则 $\int_K|u(x)|dx<\infty$，存在 $\delta>0$ 使得对于任何可测集 $A\subset K$，$\mu(A)<\delta$，我们有 $\int_A|u(x)|dx<\frac{\varepsilon}{2}$（例如，参看 Munroe 的书[48, p. 136]）.

由 Lusin 定理 1.37(f) 存在 $\psi\in C_0(\Omega)$，其中 $\operatorname{supp}\psi\subset K$ 且对于一切 x，$|\psi(x)|\leqslant1$，使得

$$\mu(\{x\in\Omega:\ \psi(x)\neq\chi_K(x)\operatorname{sgn}\overline{u(x)}\})<\delta,$$

这里

$$\operatorname{sgn}v(x)=\begin{cases}v(x)/|v(x)| & \text{当 } v(x)\neq0\text{ 时}\\ 0 & \text{当 } v(x)=0\text{ 时}.\end{cases}$$

因此

$$\int_K|u(x)|dx=\int_\Omega u(x)\chi_K(x)\operatorname{sgn}\overline{u(x)}dx$$

$$=\int_\Omega u(x)\psi(x)dx+\int_\Omega u(x)[\chi_K(x)\operatorname{sgn}\overline{u(x)}-\psi(x)]dx$$

$$\leqslant2\int_{\{x\in\Omega:\psi(x)\neq\chi_K(x)\operatorname{sgn}\overline{u(x)}\}}|u(x)|dx<\varepsilon.$$

因为 ε 是任意的，在 K 中 $u(x)=0$ a. e. 因此在 Ω 中 $u(x)=0$ a. e.. ∎

3.27 推论 如果 B 是在引理 3.24 中所说的长方形盒子，而且 $u\in L^1_{\mathrm{loc}}(B)$，对于 $1\leqslant j\leqslant n$ 有弱导数 $D_ju=0$，则对某个常数 k，在 B 中 $u(x)=k$ a. e..

证明 因为 $D_jT_u=0$，$1\leqslant j\leqslant n$，由推论 3.25 我们有

$$\int_B u(x)\phi(x)dx=T_u(\phi)=k\int_B\phi(x)dx.$$

因此在B中 $u(x)-k=0$ a. e.. ∎

3.28 定理 (1) 若 $W^{m,p}(\Omega)=W_0^{m,p}(\Omega)$, 则 Ω^c 是 (m,p')-极集. (2) 若 Ω^c 既是$(1,p)$-极集又是(m,p')-极集, 则 $W^{m,p}(\Omega)=W_0^{m,p}(\Omega)$.

证明 (1) 假定 $W^{m,p}(\Omega)=W_0^{m,p}(\Omega)$. 我们首先推出 Ω^c 的测度一定为零. 如其不然, 应该存在某个开长方形盒子 $B\subset\mathbf{R}^n$, 它和 Ω, Ω^c 的交集的测度均为正. 设 u 是在 $B\cap\Omega$ 上恒等于1的 $C_0^\infty(\mathbf{R}^n)$中的函数在Ω上的限制. 则$u\in W^{m,p}(\Omega)$, 从而$u\in W_0^{m,p}(\Omega)$. 由引理 3.22, $\tilde{u}\in W^{m,p}(\mathbf{R}^n)$ 而且对于 $1\leqslant j\leqslant n$, 在 $\mathbf{R}^n$ 中在广义函数意义下 $D_j\tilde{u}=(D_ju)^\sim$. 现在$(D_ju)^\sim$ 在B上恒等于零, 因此 $D_j\tilde{u}$ 作为一个广义函数也就在B上恒等于零. 由推论 3.27, $\tilde{u}$ 在B中必须几乎处处等于常数. 因为在$B\cap\Omega$上 $\tilde{u}(x)=1$ 而在 $B\cap\Omega^c$上 $\tilde{u}(x)=0$, 我们得到一个矛盾. 因此 Ω^c 的测度为零.

现在如果 $v\in W^{m,p}(\mathbf{R}^n)$ 而且 u 是 v 在Ω上的限制, 则 u 属于 $W^{m,p}(\Omega)$,因此根据假定 u 属于$W_0^{m,p}(\Omega)$. 由引理3.22,$\tilde{u}\in W^{m,p}(\mathbf{R}^n)$ 而且能用 $C_0^\infty(\Omega)$ 中的函数来逼近, 但在Ω上 $v(x)=\tilde{u}(x)$, 即在 $\mathbf{R}^n$ 中几乎处处相等. 因此 v 和 $\tilde{u}$ 有相同的广义导数从而在$W^{m,p}(\mathbf{R}^n)$ 中重合. 所以 $C_0^\infty(\Omega)$ 在 $W^{m,p}(\mathbf{R}^n)$中稠密从而由定理 3.23, Ω^c 是 (m,p')-极集.

(2) 现在假定 Ω^c 是$(1,p)$-极集而且是 (m,p')-极集. 设 $u\in W^{m,p}(\Omega)$. 我们证明 $u\in W_0^{m,p}(\Omega)$. 因为 $\tilde{u}\in L^p(\mathbf{R}^n)$, 与 $D_j\tilde{u}$ 相应的广义函数 $T_{D_j\tilde{u}}$ 属于 $W^{-1,p}(\mathbf{R}^n)$. 因为$(D_ju)^\sim\in L^p(\mathbf{R}^n)\subset H^{-1,p}(\mathbf{R}^n)$, 所以 $T_{(D_ju)^\sim}\in W^{-1,p}(\mathbf{R}^n)$, 因此 $T_{D_j\tilde{u}-(D_ju)^\sim}\in W^{-1,p}(\mathbf{R}^n)$. 但是在$\Omega$上 $D_j\tilde{u}-(D_ju)^\sim$ 等于零, 所以 $\operatorname{supp}T_{D_j\tilde{u}-(D_ju)^\sim}\subset\Omega^c$. 因为 Ω^c 是 $(1,p)$-极集, 在 $\mathbf{R}^n$ 上在广义函数意义下 $D_j\tilde{u}=(D_ju)^\sim$. 通过对$|\alpha|$做归纳法, 类似地我们能够证明对于$|\alpha|\leqslant m$ 在广义函数意义下 $D^\alpha\tilde{u}=(D^\alpha u)^\sim$. 所以 $\tilde{u}\in W^{m,p}(\mathbf{R}^n)$, 因此由定理 3.23, 由

于 Ω^c 是(m,p')-极集，$\tilde{u}$ 在Ω上的限制 u 属于 $W_0^{m,p}(\Omega)$. ∎

如果(m,p')-极性蕴含着$(1,p)$-极性，那么定理 3.28 相当于断言：Ω^c 的(m,p')-极性是 $W^{m,p}(\Omega)$ 和 $W_0^{m,p}(\Omega)$ 相等的充要条件. 现在我们来考察这种可能性，首先建立包含着极性的重要性质的两个引理. 第一个引理证明了(m,p')-极性是一种局部性质.

3.29 引理 $F\subset\mathbf{R}^n$ 是 (m,p')-极集当且仅当对一切紧集 $K\subset\mathbf{R}^n$，$F\cap K$ 是(m,p')-极集.

证明 显然对于一切紧集 K，F 的 (m,p')-极性蕴含着 $F\cap K$ 的 (m,p')-极性. 现在我们证明其逆. 设 $T\in W^{-m,p'}(\mathbf{R}^n)$ 是由(8)式给出的而且 T 的支集在 F 中. 我们必须证明 $T=0$. 设 $f\in C_0^\infty(\mathbf{R}^n)$，它满足：当 $|x|\leqslant 1$ 时 $f(x)=1$，而当 $|x|\geqslant 2$ 时 $f(x)=0$. 对于 $\varepsilon>0$，设 $f_\varepsilon(x)=f(\varepsilon x)$ 所以当 $\varepsilon\to 0+$ 时对 x 一致地有 $D^\alpha f_\varepsilon(x)=\varepsilon^{|\alpha|}D^\alpha f(\varepsilon x)\to 0$. 则 $f_\varepsilon T\in W^{-m,p'}(\mathbf{R}^n)$ 而且对任何 $\phi\in\mathscr{D}(\mathbf{R}^n)$ 我们有

$$
\begin{aligned}
&|T(\phi)-f_\varepsilon T(\phi)|=|T(\phi)-T(f_\varepsilon\phi)|\\
&=\left|\sum_{0\leqslant|\alpha|\leqslant m}\int_{\mathbf{R}^n}v_\alpha(x)D^\alpha[\phi(x)(1-f_\varepsilon(x))]dx\right|\\
&=\left|\sum_{0\leqslant|\alpha|\leqslant m}\sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}\int_{\mathbf{R}^n}v_\alpha(x)D^\beta\phi(x)D^{\alpha-\beta}(1-f_\varepsilon(x))dx\right|\\
&\leqslant\sum_{0\leqslant|\beta|\leqslant m}\int_{R^n}\left|w_\beta(x)D^\beta\phi(x)\right|dx\leqslant\|\phi\|_{m,p}\|w;L_N^{p'}\|,
\end{aligned}
$$

其中

$$
\begin{aligned}
w_\beta(x)&=\sum_{\substack{|\alpha|\leqslant m\\ \alpha\geqslant\beta}}\binom{\alpha}{\beta}v_\alpha(x)D^{\alpha-\beta}(1-f_\varepsilon(x))\\
&=v_\beta(x)(1-f_\varepsilon(x))-\sum_{\substack{|\alpha|\leqslant m\\ \alpha\geqslant\beta,\,\alpha\neq\beta}}\binom{\alpha}{\beta}v_\alpha(x)D^{\alpha-\beta}f_\varepsilon(x).
\end{aligned}
$$

因为对于 $|x|\leqslant\dfrac{1}{\varepsilon}$，$f_\varepsilon(x)=1$，我们有 $\lim\limits_{\varepsilon\to 0+}\|w_\beta\|_p=0$. 因此

当 $\varepsilon\to 0+$ 时在 $W^{-m,p'}(\mathbf{R}^n)$ 中 $f_\varepsilon T\to T$. 由于 $f_\varepsilon T$ 在 K 中有紧支集, 根据假定 $f_\varepsilon T=0$, 因此 $T=0$. ▌

3.30 引理 如果 $p'\leqslant q'$ 且 $F\subset\mathbf{R}^n$ 是 (m,p')-极集, 则 F 也是 (m,q')-极集.

证明 设 $K\subset\mathbf{R}^n$ 是紧的. 由引理 3.29, 只要证明 $F\cap K$ 是 (m,q')-极集就够了. 设 G 是 $\mathbf{R}^n$ 中包含 K 的有界开集. 根据引理 2.8, $W_0^{m,p}(G)\to W_0^{m,q}(G)$, 所以 $W^{-m,q'}(G)\subset W^{-m,p'}(G)$. 任何支集在 $K\cap F$ 中的广义函数 $T\in W^{-m,q'}(\mathbf{R}^n)$ 也属于 $W^{-m,q'}(G)$, 因此属于 $W^{-m,p'}(G)$. 因为 $K\cap F$ 是 (m,p')-极集, $T=0$. 因此 $K\cap F$ 是 (m,q')-极集. ▌

3.31 定理 设 $p\geqslant 2$. 则 $W^{m,p}(\Omega)=W_0^{m,p}(\Omega)$ 当且仅当 Ω^c 是 (m,p')-极集.

证明 若 Ω^c 是 (m,p')-极集, 因为 $p'\leqslant p$, 所以 Ω^c 是 (m,p)-极集, 因此 Ω^c 也是 $(1,p)$-极集. 现在由定理 3.28 就得到所要结果. ▌

3.32 能够利用 Sobolev 嵌入定理(定理 5.4) 把定理 3.31 推广到包括某些 $p<2$ 的情形. 当 $(m-1)p<n$ 时, 嵌入定理给出

$$W^{m,p}(\mathbf{R}^n)\to W^{1,q}(\mathbf{R}^n),\ q=np/[n-(m-1)p],$$

这也蕴含着 $W^{-1,q'}(\mathbf{R}^n)\subset W^{-1,p'}(\mathbf{R}^n)$. 如果还有 $p\geqslant 2n/(n+m-1)$, 则 $q'\leqslant p$, 所以由引理 3.30, 如果 Ω^c 是 (m,p')-极集, 则 Ω^c 是 $(1,p)$-极集. 注意到假如 $m>1$ 则 $2n/(n+m-1)<2$. 另一方面, 如果 $(m-1)p\geqslant n$, 则 $mp>n$, 而且, 如同在 3.20 节中指出的那样, 除非 Ω^c 是空集, 这时 Ω^c 明显地是 $(1,p)$-极集, Ω^c 不能是 (m,p')-极集.

因此仅对于由 $1\leqslant p\leqslant\min\left(\dfrac{n}{m-1},\dfrac{2n}{n+m-1}\right)$ 给出的 p 值才不知道 Ω^c 的 (m,p')-极性能否蕴含着 Ω^c 的 $(1,p)$-极性, 因此也等价于不知道 $W^{m,p}(\Omega)$ 和 $W_0^{m,p}(\Omega)$ 是否恒同.

3.33 当$W_0^{m,p}(\Omega)\neq W^{m,p}(\Omega)$时，$W_0^{m,p}(\Omega)$是$W^{m,p}(\Omega)$的闭子空间. 在Hilbert空间的情形,$p=2$,我们可以考虑由所有的$v\in W^{m,p}(\Omega)$使得对一切$\phi\in C_0^\infty(\Omega)$,$(v,\ \phi)_m=0$的那种函数组成的空间$W_0^\perp$，每个$u\in W^{m,p}(\Omega)$能够唯一地分解成$u=u_0+v$的形式,其中$u_0\in W_0^{m,p}(\Omega)$而$v\in W_0^\perp$. 利用分部积分可以证明，任何$v\in W_0^\perp$在弱的意义下一定满足

$$\sum_{0\leqslant|\alpha|\leqslant m}(-1)^{|\alpha|}D^{2\alpha}v(x)=0,$$

因此在Ω中a. e. 等于零.

坐标变换

3.34 设$\boldsymbol{\Phi}$是区域$\Omega\subset\mathbf{R}^n$到区域$G\subset\mathbf{R}^n$上的一一的变换，$\boldsymbol{\Phi}$有逆$\boldsymbol{\Psi}=\boldsymbol{\Phi}^{-1}$. 记$y=\boldsymbol{\Phi}(x)$以及

$$\begin{aligned}
y_1&=\phi_1(x_1,\cdots,x_n), & x_1&=\psi_1(y_1,\cdots,y_n)\\
y_2&=\phi_2(x_1,\cdots,x_n), & x_2&=\psi_2(y_1,\cdots,y_n)\\
&\ \ \vdots & &\ \ \vdots\\
y_n&=\phi_n(x_1,\cdots,x_n), & x_n&=\psi_n(y_1,\cdots,y_n),
\end{aligned}$$

如果函数$\phi_1,\cdots,\phi_n$属于$C^m(\overline{\Omega})$而函数$\psi_1,\cdots,\psi_n$属于$C^m(\bar{G})$我们就把$\boldsymbol{\Phi}$叫做是m-光滑的.

如果u是定义在Ω上的可测函数,我们能够用

$$Au(y)=u(\boldsymbol{\Psi}(y))\tag{13}$$

定义G上的一个可测函数. 假设$\boldsymbol{\Phi}$是1-光滑的,所以对一切$x\in\Omega$,对于某些常数$c,C,0<c\leqslant C$,

$$c\leqslant|\det\boldsymbol{\Phi}'(x)|\leqslant C\tag{14}$$

[当然，这里$\boldsymbol{\Phi}'(x)$表示Jacobi矩阵$\partial(y_1,\cdots,y_n)/\partial(x_1,\cdots,x_n)$]. 不难看出，由(13)定义的算子$A$把$L^p(\Omega)$有界地变换到$L^p(G)$上,而且有一个有界的逆算子. 事实上(对于$1\leqslant p<\infty$)

$$c^{\frac{1}{p}}\|u\|_{p,\Omega}\leqslant\|Au\|_{p,G}\leqslant C^{\frac{1}{p}}\|u\|_{p,\Omega}.$$

对于 Sobolev 空间我们建立一个类似的结果.

3.35 定理 设Φ是 m-光滑的,其中 $m\geqslant 1$. 则A把$W^{m,p}(\Omega)$有界地变换到 $W^{m,p}(G)$上而且有一个有界的逆算子.

证明 我们现在证明对任何 $u\in W^{m,p}(\Omega)$ 不等式 $\|Au\|_{m,p,G}\leqslant$ const$\|u\|_{m,p,\Omega}$ 成立,常数只依赖于变换Φ. 利用算子 A^{-1} 把定义在G上的函数变换到定义在Ω上的函数,用类似的方法能建立逆不等式$\|Au\|_{m,p,G}\geqslant$const$\|u\|_{m,p,\Omega}$.

由定理 3.16 对于任何 $u\in W^{m,p}(\Omega)$存在一个 $C^\infty(\Omega)$ 的序列$\{u_n\}$ 在 $W^{m,p}(\Omega)$的范数下收敛到 u, 对于这种光滑的 u_n 用归纳法容易验证

$$D^\alpha(Au_n)(y)=\sum_{|\beta|\leqslant|\alpha|}M_{\alpha\beta}(y)[A(D^\beta u_n)](y),\tag{15}$$

其中$M_{\alpha\beta}$是一个次数不超过$|\alpha|$的由Ψ的各个分量构成的多项式,其中有关Ψ分量的导数的次数不超过$|\beta|$. 如果 $\phi\in\mathscr{D}(\Omega)$, 我们从(15)和分部积分得到

$$\begin{aligned}(-1)^{|\alpha|}&\int_G(Au_n)(y)D^\alpha\phi(y)dy\\&=\sum_{|\beta|\leqslant|\alpha|}\int_G[A(D^\beta u_n)](y)M_{\alpha\beta}(y)dy,\end{aligned}\tag{16}$$

或,用$\Phi(x)$来代替 y 而把Ω上的积分表为

$$\begin{aligned}&(-1)^{|\alpha|}\int_\Omega u_n(x)(D^\alpha\phi)(\Phi(x))|\det\Phi'(x)|dx\\&=\sum_{|\beta|\leqslant|\alpha|}\int_\Omega D^\beta u_n(x)M_{\alpha\beta}(\Phi(x))|\det\Phi'(x)|dx.\end{aligned}\tag{17}$$

因为对$|\beta|\leqslant m$ 在 $L^p(\Omega)$中 $D^\beta u_n\to u$. 当 $n\to\infty$时,我们可以通过(17)式取极限,因此我们得到用 u 代替 u_n 的(16)式. 因此对于任何$u\in W^{m,p}(\Omega)$ 在弱的意义下(15)式成立. 从(15)式和(14)式我们得到

$$\int_G |D^\alpha(Au)(y)|^p dy$$

$$\leqslant \left(\sum_{|\beta|\leqslant|\alpha|} 1\right)^p \max_{|\beta|\leqslant|\alpha|}\left(\sup_{y\in G}|M_{\alpha\beta}(y)|^p \int_G |(D^\beta u)(\Psi(y))|^p dy\right)$$

$$\leqslant \text{const} \max_{|\beta|\leqslant|\alpha|}\int_\Omega |D^\beta u(x)|^p dx.$$

由此得到 $\|Au\|_{m,p,G} \leqslant \text{const}\|u\|_{m,p,\Omega}$. ∎

在以后各章中特别重要的是与非奇异的线性变换Φ，或更一般地，仿射变换(由非奇异的线性变换和平移组成)相应的上述定理. 对于这种变换 $\det\Phi'(x)$是非零常数.

第四章　内插和延拓定理

区域的几何性质

4.1　定义在区域Ω上的Sobolev 空间的很多性质，特别是这些空间的嵌入性质，依赖于Ω的正则性．这些正则性通常表示为任一给定区域可能满足也可能不满足的几何条件．我们下面指出五个这样的几何条件，包括在第3.17节已经遇到的线段性质，并且考虑它们之间的关系．首先，我们引进某些有用的几何概念和符号．

给定点$x\in\mathbf{R}^n$，中心为x的开球B_1，不包含x的开球B_2，集合$C_x=B_1\cap\{x+\lambda(y-x):y\in B_2,\lambda>0\}$称为顶点在$x$的$\mathbf{R}^n$内有限锥．我们还把顶点在0的有限锥$C_0$经过平移变换后得到的顶点在$x$的有限锥记作$x+C_0=\{x+y:y\in C_0\}$.

给定线性无关向量$y_1,y_2,\cdots,y_n\in\mathbf{R}^n$，集合$P=\left\{\sum_{j=1}^{n}\lambda_jy_j:0<\lambda_j<1,\ 1\leqslant j\leqslant n\right\}$是一个有一个顶点在原点的平行多面体．类似地，$x+P$为$P$的平移变换，它有一个顶点在$x$。显然，我们说$x+P$的中心，指的是点$c(x+P)=x+\frac{1}{2}(y_1+\cdots\cdots+y_n)$．每一个有一顶点在$x$的平行多面体都包含一个顶点在$x$的有限锥，反之，它也包含在这样的一个有限锥内．

集合$S\subset\mathbf{R}^n$的一个开覆盖$\mathscr{O}$称为是局部有限的，如果$\mathbf{R}^n$内任一紧集合最多只能和$\mathscr{O}$的有限个元素相交．这种局部有限的集合组必定是可数的，因此它的元素可排成序列．如果S是闭的，则S的任一开覆盖有局部有限的子覆盖．

我们现在定义开区域 $\Omega\subset\mathbf{R}^n$ 可能具有的五个正则性.

4.2 Ω具有线段性质，如果存在bdryΩ 的局部有限开覆盖$\{U_j\}$并且存在对应的非零向量序列 $\{y_j\}$ 使得若对某一 j，$x\in\overline{\Omega}\cap U_j$，则对 $0<t<1, x+ty_j\in\Omega$.

4.3 Ω具有锥性质，如果存在有限锥C使每一点 $x\in\Omega$ 是一个包含于Ω 内且全等于C的有限锥 C_x 的顶点.（注意 C_x 不需要由C经平移变换得到，而只要是刚体运动.）

4.4 Ω具有一致锥性质，如果 bdryΩ 有局部有限的开覆盖$\{U_j\}$，以及对应的有限锥序列$\{C_j\}$，每一 C_j 全等于某一固定的有限锥 C，使:

(i) 对某一有限的M，每一 U_j 直径小于M.

(ii) 对某一 $\delta>0$，$\bigcup\limits_{j=1}^{\infty}U_j\supset\Omega_\delta\equiv\{x\in\Omega:\mathrm{dist}(x,\mathrm{bdry}\Omega)<\delta\}$.

(iii) 对每个 j，$\bigcup\limits_{x\in\Omega\cap U_j}(x+C_j)\equiv Q_j\subset\Omega$.

(iv) 对某一有限 R，集合 Q_j 中的任意 $R+1$ 个交于空集.

4.5 Ω具有强局部 Lipschitz 性质，如果存在正数 δ 和 M，bdryΩ 的一个局部有限开覆盖$\{U_j\}$，以及对每一 U_j 有一个 $n-1$ 个实变量的实值函数 f_j，使得以下条件成立:

(i) 对某一有限 R，集合 U_j 中的任意 $R+1$ 个交于空集.

(ii) 对所有满足 $|x-y|<\delta$ 的点对 $x, y\in\Omega_\delta=\{x\in\Omega:\mathrm{dist}(x,\mathrm{bdry}\Omega)<\delta\}$存在 j 使得

$$x,y\in\mathscr{V}_j=\{x\in U_j:\mathrm{dist}(x,\mathrm{bdry}U_j)>\delta\}.$$

(iii) 每一函数 f_j 满足关于常数M的 Lipschitz 条件:

$$|f(\xi_1,\cdots,\xi_{n-1})-f(\eta_1,\cdots,\eta_{n-1})|\leqslant M|(\xi_1-\eta_1,\cdots,\xi_{n-1}-\eta_{n-1})|.$$

(iv) 对某一 U_j 内的笛卡尔坐标系$(\xi_{j,1},\cdots,\xi_{j,n})$，集合$\Omega\cap U_j$ 由不等式

$$\xi_{j,n}<f_j(\xi_{j,1},\cdots\cdots,\xi_{j,n-1})$$

表示.

我们注意到，如果Ω为有界，以上相当复杂的条件简化为Ω有局部 Lipschitz 边界，即，Ω的边界上每一点x有一个邻域U_x使 bdry $\Omega\cap U_x$是一个 Lipschitz 连续函数的图.

4.6 Ω具有一致C^m-正则性，如果存在 bdryΩ的一个局部有限开覆盖$\{U_j\}$，以及对应的m-光滑一一变换的序列$\{\Phi_j\}$（见第3.34节），Φ_j把U_j映到$B=\{y\in\mathbf{R}^n:|y|<1\}$上，使：

(i) 对某一$\delta>0$，$\bigcup_{j=1}^{\infty}\Psi_j\left(\left\{y\in\mathbf{R}^n:|y|<\frac{1}{2}\right\}\right)\supset\Omega_\delta$， 其中$\Psi_j=\Phi_j^{-1}$.

(ii) 对某一有限R，集合U_j中的任意$R+1$个交于空集.

(iii) 对每一j，$\Phi_j(U_j\cap\Omega)=\{y\in B:y_n>0\}$.

(iv) 如果$(\phi_{j,1},\cdots,\phi_{j,n})$和$(\psi_{j,1},\cdots,\psi_{j,n})$分别表示$\Phi_j$和$\Psi_j$的分量，则存在有限$M$，使对所有的$\alpha$，$|\alpha|\leqslant m$，对所有的$i$，$1\leqslant i\leqslant n$，和对所有的$j$，我们有

$$|D^\alpha\phi_{j,i}(x)|\leqslant M,\qquad x\in U_j$$
$$|D^\alpha\psi_{j,i}(y)|\leqslant M,\qquad y\in B.$$

4.7 除了锥性质以外，所有以上其他性质都要求Ω只在它的边界的一边. 在第3.17节中提到的二维区域Ω. 即:

$$\Omega=\{(x,y)\in\mathbf{R}^2:0<|x|<1,0<y<1\}$$

具有锥性质但其余四个都不具备. 读者可以验证对任意的区域Ω

一致C^m-正则性$(m\geqslant1)$

$\Rightarrow$ 强局部 Lipschitz 性质

$\Rightarrow$ 一致锥性质

$\Rightarrow$ 线段性质.

第五章中大部分重要的嵌入结果只要求锥性质，然而有一个

要求强局部 Lipschitz 性质. 尽管锥性质不蕴涵以上其它性质，然而它对有界区域“几乎”蕴涵强局部 Lipschitz 性质，确切意义为下面 Gagliardo 的定理[24].

4.8 定理 (Gagliardo[24]) 设Ω为$\mathbf{R}^n$中具有锥性质的有界区域，对于每一$\rho>0$，存在Ω的开子集的有限集族$\{\Omega_1, \Omega_2, \cdots, \Omega_m\}$，使得$\Omega=\bigcup_{j=1}^{m}\Omega_j$并且每个$\Omega_j$对应一个$\overline{\Omega}_j$的子集$A_j$，直径不超过$\rho$，以及一个有一顶点在$O$的开平行多面体$P_j$，使得$\Omega_j=\bigcup_{x\in A_j}(x+P_j)$. 此外，如果$\rho$充分小，则每一个$\Omega_j$具有强局部 Lipschitz 性质.

证明 设C_0为顶点在O的有限锥，使任意的$x\in\Omega$都是与C_0全等的有限锥$C_x\subset\Omega$的顶点. 显然可以选有限个有限锥C_1, $\cdots$, C_k，每一个顶点在O(每个的锥角小于C_0的锥角) 使得任何顶点在O的与C_0全等的有限锥必须包含锥C_j中的一个，$1\leqslant j\leqslant k$. 对每一C_j设P_j是一个开平行多面体，它的顶点在原点并且$P_j\subset C_j$. 则对每一$x\in\Omega$存在j, $1\leqslant j\leqslant k$，使

$$x+P_j\subset x+C_j\subset C_x\subset\Omega.$$

因为Ω为开的并且$\overline{x+P_j}$是紧的. 因此对充分接近于x的所有的y, $y+P_j\subset\Omega$. 因此对所有的$x\in\Omega$，我们能找到$y\in\Omega$使对某个j, $1\leqslant j\leqslant k$, $x\in y+P_j\subset\Omega$. (因此任意的有锥性质的区域可以表示为有限多个平行多面体的平移的并.)

设$\widetilde{A}_j=\{x\in\overline{\Omega}: x+P_j\subset\Omega\}$. 如果对每一$j$, $\operatorname{diam}\widetilde{A}_j\leqslant\rho$，我们取$m=k$，令$A_j=\widetilde{A}_j$和$\Omega_j=\bigcup_{x\in A_j}(x+P_j)$，注意定理的第一部分已经证明，否则我们把$\widetilde{A}_j$分解为集合$A_{ji}$的有限并使得$\operatorname{diam}A_{ji}\leqslant\rho$. 令对应的$P_{ji}=P_j$，把全部集合$A_{ji}$重新排列成有限序列$A_1, \cdots, A_m$. 把对应的$P_{ji}$重新命名为$P_1, \cdots, P_m$，最后令$\Omega_j=$

$\bigcup\limits_{x\in A_j}(x+P_j)$得到同样结果.（图 2 试图对于情形

$$\Omega=\{(x,y)\in\mathbf{R}^2:\ 0<|x|<1,0<y<1\},$$

$$C_0=\left\{(x,y)\in\mathbf{R}^2:\ x>0,\ y>0,\ x^2+y^2<\frac{1}{4}\right\},$$

$$\rho=13/16$$

表明这些说法. 在此情形Ω可被仅四个开子集 Ω_j 所覆盖，它们仅对应于两个不同的平行多面体.）

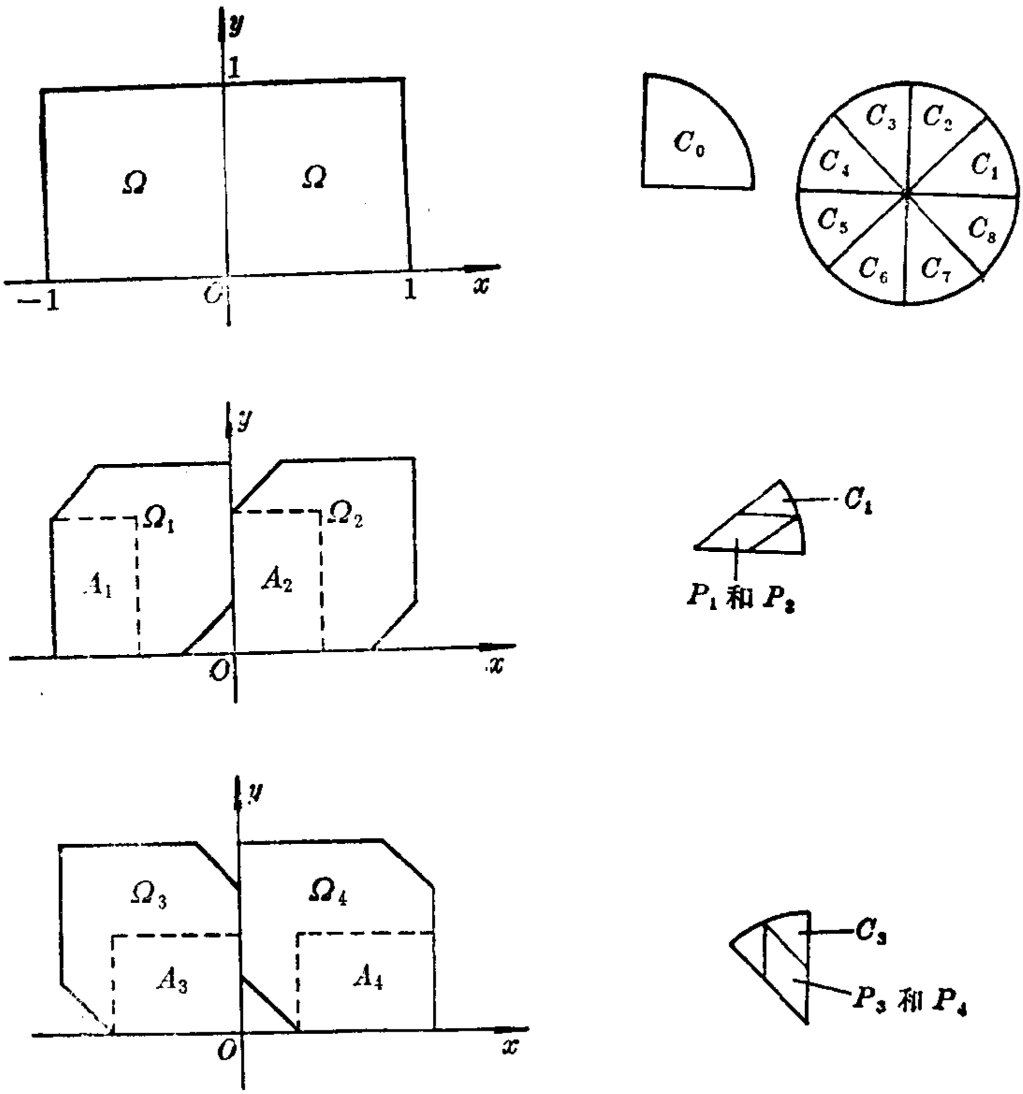

图 2

剩下还要说明如果ρ充分小，则每一 Ω_j 有强局部 Lipschitz

性质．为了符号简单我们假设$\Omega=\bigcup_{x\in A}(x+P)$．其中 $\operatorname{diam}A\leqslant\rho$ 和P是一个固定的平行多面体，我们说明Ω有强局部 Lipschitz 性质．

对P的每个顶点v_j设$Q_j=\{y=v_j+\lambda(x-v_j):x\in P,\ \lambda>0\}$是由$P$生成的顶点为$v_j$的无限锥．则$P=\bigcap Q_j$，这里是对$P$的所有$2^n$个顶点取交集．设$\Omega_{(j)}=\bigcup_{x\in A}(x+Q_j)$，令$\delta=\operatorname{dist}(P$的中心，$\operatorname{bdry}P)$并且令$B$为半径$\sigma=\delta/2$的任意球．对任意的固定$x\in\Omega$，$B$不能与$x+P$的相对两面相交，所以我们可以找到$P$的顶点$v_j$使得$x+v_j$是$x+P$与$B$相交的所有的面的公共点（如果这样的面存在）．则$B\cap(x+P)=B\cap(x+Q_j)$，现设$x,y\in A$并假设$B$能和$x+P$与$y+P$的相应对面相交，即，存在$P$对面上两点$a$和$b$使$x+a\in B$及$y+b\in B$．则

$$
\begin{aligned}
\rho\geqslant\operatorname{dist}(x,y)&=\operatorname{dist}(x+b,y+b)\\
&\geqslant\operatorname{dist}(x+b,x+a)-\operatorname{dist}(x+a,y+b)\\
&\geqslant 2\delta-2\sigma=\delta
\end{aligned}
$$

因此若$\rho<\delta$，则对任意的$x,y\in A$，B不能与$x+P$和$y+P$的相应对面相遇．因此对某个固定的j，$B\cap(x+P)=B\cap(x+Q_j)$与$x\in A$无关，从而$B\cap\Omega=B\cap\Omega_{(j)}$．

在B内取坐标$\xi=(\xi',\ \xi_n)=(\xi_1,\cdots,\ \xi_{n-1},\xi_n)$使$\xi_n$轴为从$P$的中心到点$v_j$的向量的方向．则$(x+Q_j)\cap B$在$B$内由形如$\xi_n<f_x(\xi')$的不等式确定，其中$f_x$满足一个常数与$x$无关的 Lipschitz 条件．因此$\Omega_{(j)}\cap B$，因之$\Omega\cap B$，由$\xi_n<f(\xi')$所确定，其中$f(\xi')=\sup_{x\in A}f_x(\xi')$本身是一个 Lipschitz 连续函数．由于对$\operatorname{bdry}\Omega$上任一点的邻域$B$都可以这样做，所以$\Omega$有强局部 Lipschitz 性质．▌

中间导数的内插不等式

4.9 我们考虑根据函数 $u\in W^{m,p}(\Omega)$ 及其 $|\alpha|=m$ 阶导数 $D^\alpha u$ 的 L^p-范数来确定导数 $D^\beta u(|\beta|\leqslant m)$ 的 L^p-范数上界的问题，这样的内插不等式曾被很多作者得到，诸如Ehrling[23], Nirenberg[53, 54], Browder[11, 12], 和 Gagliardo[24, 25]，并且可以做大量推广．涉及非整数值 m 的 $W^{m,p}(\Omega)$ 的定义的推广通过适当的内插论证就可完成（见第七章）．

从一个简单的一维内插不等式开始是适当的，对下面更一般的定理来说，它仍然是典型的并且提供了证明的基础．

4.10 引理 设 $-\infty\leqslant a<b\leqslant\infty$，设 $1\leqslant p<\infty$，并且设 $0<\varepsilon_0<\infty$．存在有限常数 $K=K(\varepsilon_0,p,b-a)$，它对 $0<b-a\leqslant\infty$ 连续地依赖于 $b-a$，使对所有满足 $0<\varepsilon\leqslant\varepsilon_0$ 的 ε，以及对所有在开区间 (a,b) 上二次连续可微的函数 f

$$\int_a^b |f'(t)|^p dt\leqslant K\varepsilon\int_a^b |f''(t)|^p dt+K\varepsilon^{-1}\int_a^b |f(t)|^p dt. \qquad (1)$$

并且，如果 $b-a=\infty$，则可找到 $K=K(p)$ 使(1)对所有的正数 ε 成立．

证明 只要对实值函数 f 证明(1)就够了，因为，假如已经证明了这一点，把任意 f 写成 $f=u+iv$, u,v 为实值，我们得

$$\int_a^b |f'(t)|^p dt=\int_a^b [u'(t)^2+v'(t)^2]^{p/2}dt$$

$$\leqslant\max(1,2^{(p-2)/2})\int_a^b [|u'(t)|^p+|v'(t)|^p]dt$$

$$\leqslant 2K\max(1,2^{(p-2)/2})\left\{\varepsilon\int_a^b |f''(t)|^p dt+\varepsilon^{-1}\int_a^b |f(t)|^p dt\right\}.$$

不失一般性我们还假定 $\varepsilon_0=1$，因为假定引理已在这种情形证明了，由于 $0<\varepsilon/\varepsilon_0<1$，所以由(1)我们得

$$\int_a^b |f'(t)|^p dt \leqslant K\cdot(\varepsilon/\varepsilon_0)\int_a^b |f''(t)|^p dt$$
$$+K\cdot(\varepsilon_0/\varepsilon)\int_a^b |f(t)|^p dt.$$

随之，它导出(1)，$K=K(\varepsilon_0,p,b-a)=K(1,p,b-a)\max(\varepsilon_0,\varepsilon_0^{-1})$.

因此我们假定 f 为实值并且 $\varepsilon_0=1$. 暂时假定 $a=0$ 和 $b=1$. 如果 $0<\xi<\frac{1}{3}$ 和 $\frac{2}{3}<\eta<1$，则存在 $\lambda\in(\xi,\eta)$ 使

$$|f'(\lambda)|=\left|\frac{f(\eta)-f(\xi)}{\eta-\xi}\right|\leqslant 3|f(\xi)|+3|f(\eta)|.$$

因此对任意的 $x\in(0,1)$

$$|f'(x)|=|f'(\lambda)+\int_\lambda^x f''(t)dt|$$
$$\leqslant 3|f(\xi)|+3|f(\eta)|+\int_0^1 |f''(t)|dt.$$

对 ξ 在$\left(0,\frac{1}{3}\right)$上和对 η 在$\left(\frac{2}{3},1\right)$上积分以上不等式得

$$\frac{1}{9}|f'(x)|\leqslant\int_0^{1/3}|f(\xi)|d\xi+\int_{2/3}^1|f(\eta)|d\eta+\frac{1}{9}\int_0^1|f''(t)|dt$$
$$\leqslant\int_0^1|f(t)|dt+\frac{1}{9}\int_0^1|f''(t)|dt.$$

由 Hölder 不等式

$$|f'(x)|^p\leqslant 2^{p-1}\cdot 9^p\int_0^1|f(t)|^p dt+2^{p-1}\int_0^1|f''(t)|^p dt,$$

因此

$$\int_0^1|f'(t)|^p dt\leqslant K_p\int_0^1|f''(t)|^p dt+K_p\int_0^1|f(t)|^p dt,$$

其中 $K_p=2^{p-1}\cdot 9^p$. 对任意有限区间 (a,b)，作变量替换得

$$\int_a^b|f'(t)|^p dt\leqslant K_p(b-a)^p\int_a^b|f''(t)|^p dt$$
$$+K_p(b-a)^{-p}\int_a^b|f(t)|^p dt. \tag{2}$$

因为 $0<\varepsilon\leqslant 1$, 所以存在正整数 n 使

$$\frac{1}{2}\varepsilon^{1/p}\leqslant 1/n\leqslant\varepsilon^{1/p}.$$

对 $j=0,1,\cdots,n$, 令 $a_j=a+(b-a)j/n$. 注意到 $a_j-a_{j-1}=(b-a)/n$, 由(2)我们得

$$\begin{aligned}\int_a^b|f'(t)|^p dt&=\sum_{j=1}^n\int_{a_{j-1}}^{a_j}|f'(t)|^p dt\\&\leqslant K_p\sum_{j=1}^n\left\{\left(\frac{b-a}{n}\right)^p\int_{a_{j-1}}^{a_j}|f''(t)|^p dt\right.\\&\quad\left.+\left(\frac{n}{b-a}\right)^p\int_{a_{j-1}}^{a_j}|f(t)|^p dt\right\}\\&\leqslant\widetilde{K}(p,b-a)\left\{\varepsilon\int_a^b|f''(t)|^p dt\right.\\&\quad\left.+\varepsilon^{-1}\int_a^b|f(t)|^p dt\right\},\end{aligned}\tag{3}$$

其中 $\widetilde{K}(p,b-a)=K_p\max[(b-a)^p,2^p(b-a)^{-p}]$.

现设

$$K(1,p,b-a)=\begin{cases}\max\limits_{1\leqslant s\leqslant 2}\widetilde{K}(p,s) & \text{如果}\quad b-a\geqslant 1\\ \max\limits_{b-a\leqslant s\leqslant 2}\widetilde{K}(p,s) & \text{如果}\quad 0<b-a<1\end{cases}$$

则对 $0<b-a\leqslant\infty$, $K(1,p,b-a)$ 有限并且连续依赖于 $b-a$. 对 $b-a<1$, (1)直接由(3)得出. 对 $1\leqslant b-a\leqslant\infty$, 区间 (a,b) 可以分成(可能是无穷多个)子区间, 每个子区间的长度在 1 和 2 之间, 从而把(3)用于每一个子区间, 求和得(1).

最后, 设 $b-a=\infty$. 作为典型我们假定 a 为有限和 $b=\infty$ 其余的情形是类似的. 对给定的 $\varepsilon>0$, 设 $a_j=a+j\varepsilon^{1/p}$, $j=0,1,2,\cdots$. 则 $a_j-a_{j-1}=\varepsilon^{1/p}$, 用(2)式得

$$\int_a^\infty|f'(t)|^p dt=\sum_{j=1}^\infty\int_{a_{j-1}}^{a_j}|f'(t)|^p dt$$

$$\leqslant K_p\varepsilon\sum_{j=1}^{\infty}\int_{a_{j-1}}^{a_j}|f''(t)|^p dt$$

$$+K_p\varepsilon^{-1}\sum_{j=1}^{\infty}\int_{a_{j-1}}^{a_j}|f(t)|^p dt,$$

它就是(1)，$K=K_p$ 只依赖于 p. ▌

4.11 对 $1\leqslant p<\infty$ 和对整数 j, $0\leqslant j\leqslant m$，在 $W^{m,p}(\Omega)$ 上我们引进泛函 $|\cdot|_{j,p}$ 如下：

$$|u|_{j,p}=|u|_{j,p,\Omega}=\left\{\sum_{|\alpha|=j}\int_\Omega|D^\alpha u(x)|^p dx\right\}^{1/p}.$$

显然，$|u|_{0,p}=\|u\|_{0,p}=\|u\|_p$ 是 u 在 $L^p(\Omega)$ 内的范数并且

$$\|u\|_{m,p}=\left\{\sum_{0\leqslant j\leqslant m}|u|_{j,p}^p\right\}^{1/p}.$$

如果 $j\geqslant1$，$|\cdot|_{j,p}$ 是一个半范数——它有范数的所有性质除了 $|u|_{j,p}=0$ 并不蕴涵 u 在 $W^{m,p}(\Omega)$ 内等于零；例如，u 可以是一个有限体积的区域 Ω 上的非零常数．在我们下面研究的某些情形，$|\cdot|_{m,p}$ 对空间 $W_0^{m,p}(\Omega)$ 是一个等价范数，特别地，当 Ω 为有界时是如此．

现在我们建立形如

$$|u|_{j,p}\leqslant K\varepsilon|u|_{m,p}+K\varepsilon^{-j/(m-j)}|u|_{0,p} \tag{4}$$

的内插不等式，其中 $0\leqslant j\leqslant m-1$．下面的引理说明一般说来我们只要在特殊情形 $j=1$, $m=2$ 建立(4)就够了，这一简化在下面的三个内插定理中都要用到．

4.12 引理 设 $0<\delta_0<\infty$, 设 $m\geqslant2$, 并设

$$\varepsilon_0=\min(\delta_0,\delta_0^2,\cdots,\delta_0^{m-1}).$$

假设对给定的 p, $1\leqslant p<\infty$, 以及对给定的 $\Omega\subset\mathbf{R}^n$，存在常数 $K=K(\delta_0,\ p,\ \Omega)$ 使对所有的有限的 δ，$0<\delta\leqslant\delta_0$，并且对所有的 $u\in W^{2,p}(\Omega)$，我们有

$$|u|_{1,p}\leqslant K\delta|u|_{2,p}+K\delta^{-1}|u|_{0,p}. \tag{5}$$

则存在常数 $K=K(\varepsilon_0, m, p, \Omega)$ 使对所有的有限的 ε, $0<\varepsilon\leqslant\varepsilon_0$, 所有的整数 j, $0\leqslant j\leqslant m-1$, 所有的 $u\in W^{m,p}(\Omega)$, 我们有

$$|u|_{j,p}\leqslant K\varepsilon|u|_{m,p}+K\varepsilon^{-j/(m-j)}|u|_{0,p}. \tag{6}$$

证明 因为对 $j=0$, (6)式是明显的, 我们只考虑 $1\leqslant j\leqslant m-1$ 的情形. 证明通过对 m 和 j 双重归纳完成. 出现在推导中的常数 $K_1, K_2, \cdots$ 可以依赖于 δ_0(或 ε_0), m, p, 和 Ω. 我们首先对 m 归纳证明(6)式 $j=m-1$ 的情形, 这样(5)就是 $m=2$ 的特殊情形. 假定对某个 k, $2\leqslant k\leqslant m-1$,

$$|u|_{k-1,p}\leqslant K_1\delta|u|_{k,p}+K_1\delta^{-(k-1)}|u|_{0,p} \tag{7}$$

对所有的 δ, $0<\delta\leqslant\delta_0$, 和所有的 $u\in W^{k,p}(\Omega)$ 成立, 如果 $u\in W^{k+1,p}(\Omega)$, 我们证明(7)式用 $k+1$ 代替 k 时仍成立(常数 K_1 不同), 如果 $|\alpha|=k-1$, 我们从(5)得

$$|D^\alpha u|_{1,p}\leqslant K_2\delta|D^\alpha u|_{2,p}+K_2\delta^{-1}|D^\alpha u|_{0,p}.$$

把这个不等式与(7)式联立, 对 $0<\eta\leqslant\delta_0$ 我们得

$$\begin{aligned}|u|_{k,p}&\leqslant K_3\sum_{|\alpha|=k-1}|D^\alpha u|_{1,p}\\&\leqslant K_4\delta|u|_{k+1,p}+K_4\delta^{-1}|u|_{k-1,p}\\&\leqslant K_4\delta|u|_{k+1,p}+K_4K_1\delta^{-1}\eta|u|_{k,p}\\&\quad+K_4K_1\delta^{-1}\eta^{1-k}|u|_{0,p}.\end{aligned}$$

不妨假定 $2K_1K_4\geqslant1$. 因此我们可以取 $\eta=\delta/2K_1K_4$, 因此得

$$\begin{aligned}|u|_{k,p}&\leqslant 2K_4\delta|u|_{k+1,p}+(\delta/2K_1K_4)^{-k}|u|_{0,p}\\&\leqslant K_5\delta|u|_{k+1,p}+K_5\delta^{-k}|u|_{0,p}.\end{aligned}$$

这就完成了归纳, 对 $0<\delta\leqslant\delta_0$ 建立了(7)式, 从而对 $j=m-1$ 和 $0<\varepsilon\leqslant\delta_0$ 建立了(6)式.

我们现在对 j 作反向归纳证明:

$$|u|_{j,p}\leqslant K_6\delta^{m-j}|u|_{m,p}+K_6\delta^{-j}|u|_{0,p} \tag{8}$$

对 $1\leqslant j\leqslant m-1$ 和 $0<\delta\leqslant\delta_0$ 成立. 注意 (7) 式当 $k=m$ 是(8)式 $j=m-1$ 的特殊情形. 因此假定(8)式对某个 $j(2\leqslant j\leqslant m-1)$ 成立. 我们证明它用 $j-1$ 代替 j 时仍成立 (常数 K_6 不同). 从(7)和(8)我们得

$$\begin{aligned}|u|_{j-1,p}&\leqslant K_7\delta|u|_{j,p}+K_7\delta^{1-j}|u|_{0,p}\\&\leqslant K_7\delta\{K_6\delta^{m-j}|u|_{m,p}+K_6\delta^{-j}|u|_{0,p}\}+K_7\delta^{1-j}|u|_{0,p}\\&\leqslant K_8\delta^{m-(j-1)}|u|_{m,p}+K_8\delta^{-(j-1)}|u|_{0,p}.\end{aligned}$$

所以(8)式成立, 由(8)式中令 $\delta=\varepsilon^{1/(m-j)}$ 并注意到如果 $\delta\leqslant\delta_0$ 则 $\varepsilon\leqslant\varepsilon_0$ 便得到(6)式. ∎

4.13 定理 存在常数 $K=K(m,p,n)$ 使对任意的 $\Omega\subset\mathbf{R}^n$, 任意的 $\varepsilon>0$, 任意的整数 j, $0\leqslant j\leqslant m-1$, 和任意的 $u\in W_0^{m,p}(\Omega)$,

$$|u|_{j,p}\leqslant K\varepsilon|u|_{m,p}+K\varepsilon^{-j/(m-j)}|u|_{0,p}. \tag{9}$$

证明 由引理 3.22 在 Ω 外延拓为零的算子是 $W_0^{m,p}(\Omega)$ 到 $W^{m,p}(\mathbf{R})$ 的等距同态. 因此只要对 $\Omega=\mathbf{R}^n$ 建立(9)式就够了. 同样, 由引理 4.12 我们只要考虑 $j=1$, $m=2$ 的情形. ($j=0$, $m=1$ 的情形是平凡的.) 对任意的 $\varepsilon>0$ 和任意的 $\phi\in C_0^\infty(\mathbf{R}^n)$ 我们由引理 4.10 得

$$\int_{-\infty}^{+\infty}\left|\frac{\partial}{\partial x_j}\phi(x)\right|^p dx_j\leqslant K\varepsilon^p\int_{-\infty}^{+\infty}\left|\frac{\partial^2}{\partial x_j^2}\phi(x)\right|^p dx_j+K\varepsilon^{-p}\int_{-\infty}^{+\infty}|\phi(x)|^p dx_j$$

对 x 的其余分量积分, 我们得

$$\|D_j\phi\|_p^p\leqslant K\varepsilon^p\|D_j^2\phi\|_p^p+K\varepsilon^{-p}\|\phi\|_p^p.$$

从而

$$\begin{aligned}|\phi|_{1,p}^p&\leqslant K\varepsilon^p\sum_{j=1}^n\|D_j^2\phi\|_p^p+nK\varepsilon^{-p}|\phi|_{0,p}^p\\&\leqslant K\varepsilon^p|\phi|_{2,p}^p+nK\varepsilon^{-p}|\phi|_{0,p}^p.\end{aligned}$$

只要取 p 次方根并且注意到 $C_0^\infty(\mathbf{R}^n)$ 在 $W^{m,p}(\mathbf{R}^n)$ 内稠密即可得到

(9)式的 $j=1, m=2$ 的情形. ∎

4.14 定理 (Ehrling[23], Nirenberg[53], Gagliardo[24]) 设 $\Omega\subset\mathbf{R}^n$ 具有一致锥性质(4.4 节), 并且设 ε_0 是有限正数, 则存在常数 $K=K(\varepsilon_0, m, p, \Omega)$ 使对任意的 ε, $0<\varepsilon\leqslant\varepsilon_0$, 任意的整数 j, $0\leqslant j\leqslant m-1$, 以及任意的 $u\in W^{m,p}(\Omega)$

$$|u|_{j,p}\leqslant K\varepsilon|u|_{m,p}+K\varepsilon^{-j/(m-j)}|u|_{0,p} \tag{10}$$

证明 $m=1$ 情形是平凡的; 再者由引理 4.12 只要对 $j=1$, $m=2$ 建立(10)式就够了. 此外, 由引理 4.10 证明中第二段所用的讨论我们可以假定 $\varepsilon_0=1$.

在证明中我们始终利用第 4.4 节中描述 Ω 具有一致锥条件的符号. 如果 δ 是该节条件(ii)中的常数并如果 $\lambda=(\lambda_1, \cdots, \lambda_n)$ 是 n 重整数, 我们考虑立方体

$$H_\lambda=\{x\in\mathbf{R}^n: \lambda_k\delta/2\sqrt{n}\leqslant x_k\leqslant(\lambda_k+1)\delta/2\sqrt{n}\},$$

则 $\mathbf{R}^n=\bigcup\limits_{\lambda}H_\lambda$, 并且 $\operatorname{diam}H_\lambda=\delta/2$. 设 $\Omega_0=\bigcup\limits_{H_\lambda\subset\Omega}H_\lambda$. 于是 $\Omega\sim\Omega_\delta\subset\Omega_0\subset\Omega$. 如果集合 $U_1, U_2, \cdots$ 和 $Q_1, Q_2\cdots$ 如第 4.4 节, 则

$$\Omega=\bigcup_{j=1}^{\infty}(U_j\cap\Omega)\cup\Omega_0=\bigcup_{j=1}^{\infty}Q_j\cup\Omega_0$$

我们证明对任意的 $u\in W^{2,p}(\Omega)$

$$|u|^p_{1,p,\Omega_0}\leqslant K_1\varepsilon^p|u|^p_{2,p,\Omega_0}+K_1\varepsilon^{-p}|u|^p_{0,p,\Omega_0} \tag{11}$$

以及对 $j=1, 2, 3, \cdots$

$$|u|^p_{1,p,U_j\cap\Omega}\leqslant K_2\varepsilon^p|u|^p_{2,p,Q_j}+K_2\varepsilon^{-p}|u|^p_{0,p,Q_j}, \tag{12}$$

其中 K_2 与 j 无关. 因为任意 $R+2$ 个集合 Ω_0, Q_1, Q_2, $\cdots$ 交于空集, (11)和(12)式蕴涵

$$|u|^p_{1,p,\Omega}\leqslant|u|^p_{1,p,\Omega_0}+\sum_{j=1}^{\infty}|u|^p_{1,p,U_j\cap\Omega}$$

$$\leqslant\max(K_1, K_2)\Big\{\varepsilon^p|u|^p_{2,p,\Omega_0}+\varepsilon^p\sum_{j=1}^{\infty}|u|^p_{2,p,Q_j}$$

$$+\varepsilon^{-p}|u|_{0,p,\Omega_0}^p+\varepsilon^{-p}\sum_{j=1}^{\infty}|u|_{0,p,Q_j}^p\Big\}$$

$$\leqslant(R+1)\max(K_1,K_2)\{\varepsilon^p|u|_{2,p,\Omega}^p+\varepsilon^{-p}|u|_{0,p,\Omega}^p\},$$

取 p 次方根即得(10)式($j=1, m=2$ 情形). 所以剩下要验证(11)和(12).

如果 $u\in C^\infty(\Omega)\cap W^{2,p}(\Omega)$, 把它看作是从 $\lambda_k\delta/2\sqrt{n}$ 到 $(\lambda_k+1)\delta/2\sqrt{n}$ 区间上的 x_k 的函数, 我们用引理 4.10 于 u, 然后在类似的区间上对其余变量积分, 得到对任意的 $H_\lambda\subset\Omega$

$$\int_{H_\lambda}|D_k u(x)|^p dx\leqslant K_3\varepsilon^p\int_{H_\lambda}|D_k^2 u(x)|^p dx$$

$$+K_3\varepsilon^{-p}\int_{H_\lambda}|u(x)|^p dx, \tag{13}$$

其中 K_3 只依赖于 p 和 H_λ 的边长(也就是通过 δ 和 n 依赖于Ω), 把(13)式对 $1\leqslant k\leqslant n$ 求和, 我们得

$$|u|_{1,p,H_\lambda}^p\leqslant K_3\varepsilon^p|u|_{2,p,H_\lambda}^p+nK_3\varepsilon^{-p}|u|_{0,p,H_\lambda}^p. \tag{14}$$

因为立方体H不重叠, 我们把(14)式对所有的立方体 $H_\lambda\subset\Omega$ 求和得到(11)式, 其中 $K_1=nK_3$. 因为 $C^\infty(\Omega)\cap W^{2,p}(\Omega)$ 在 $W^{2,p}(\Omega)$ 内稠密, (11)式对所有的 $u\in W^{2,p}(\Omega)$ 成立.

(12)式中的常数只依赖于 p, M和锥 C_j 的大小(见第 4.4节). 预料到这一点, 并且注意到所有锥 C_j 与锥 C 全等, 大小完全由 C 确定, 为了简单起见, 我们在考虑(12)式时去掉所有的下标 j. 设 ξ 是 C 内一个方向的单位向量, 并且设 $\Omega_\xi=\{y+t\xi\colon y\in\Omega\cap U, 0\leqslant t\leqslant h\}$, 其中 h 是锥 C 的高(图 3), 因此由一致锥性质条件(iii), $(\Omega\cap U)\subset\Omega_\xi\subset Q$. 任一平行于 ξ 的直线 L 或者与 Ω_ξ 不相交或者与 Ω_ξ 交于长为 ρ 的区间, 其中, 由一致锥性质的条件(i), $h\leqslant\rho\leqslant h+\operatorname{diam}U\leqslant h+M$. 由引理 4.10, 如果 $u\in C^\infty(\Omega)\cap W^{2,p}(\Omega)$,

$$\int_{L\cap\Omega_\xi}|D_\xi u|^p ds\leqslant K_4\varepsilon^p\int_{L\cap\Omega_\xi}|D_\xi^2 u|^p ds$$

$$+K_4\varepsilon^{-p}\int_{L\cap\Omega_\xi}|u|^p ds, \tag{15}$$

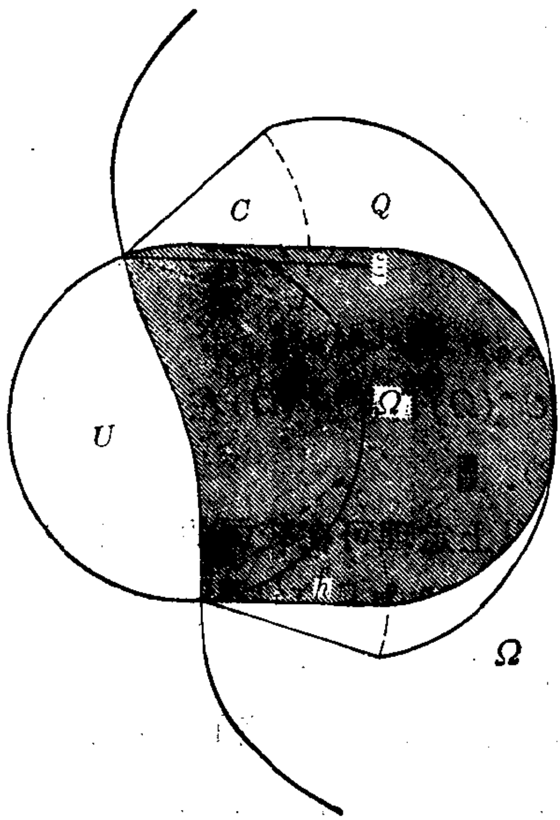

图 3

其中 D_ξ 表示方向 ξ 的微商且 K_4 可以选得只依赖于 p, h 和M, 即只依赖于 p和Ω. 把 Ω_ξ 投影到垂直于 ξ 的超平面上, 在此投影上积分(15)式得

$$\int_{\Omega\cap U}|D_\xi u(x)|^p dx\leqslant\int_{\Omega_\xi}|D_\xi u(x)|^p dx$$

$$\leqslant K_4\,\varepsilon^p\int_{\Omega_\xi}|D_\xi^2 u(x)|^p dx+K_4\varepsilon^{-p}\int_{\Omega_\xi}|u(x)|^p dx$$

$$\leqslant K_4\,\varepsilon^p\int_Q|D_\xi^2 u(x)|^p dx+K_4\varepsilon^{-p}\int_Q|u(x)|^p dx. \tag{16}$$

现在设 $\xi_1, \cdots, \xi_n$ 是 $\mathbf{R}^n$ 内的单位向量基, 它们每一个的方向包含在锥C内. 对于 $1\leqslant k\leqslant n$, $D_k u(x)=\sum_{j=1}^n a_j D_{\xi_j}u(x)$, 其中常数 a_j 满足 $|a_j|\leqslant 1/V$, $1\leqslant j\leqslant n$, V 为 $\xi_1, \cdots\xi_n$ 张成的平行多面体的体积.

(读者可以证实这一断言，它是线代数的简单练习.) V 的下界可以由锥 C 的立体角所确定——也就是说，$\xi_1, \cdots, \xi_n$ 随着覆盖的小块 U 变化，总可以选得使 V 与 U 无关. 由(16)式得

$$\int_{\Omega\cap U}|D_k u(x)|^p dx \leqslant K_5 \sum_{j=1}^{n}\int_{\Omega\cap U}|D_{\xi_j} u(x)|^p dx$$

$$\leqslant K_5 \sum_{j=1}^{n}\left\{K_4\varepsilon^p \int_Q |D_{\xi_j}^2 u(x)|^p dx + K_4\varepsilon^{-p}\int_Q |u(x)|^p dx\right\}$$

$$\leqslant K_6\varepsilon^p |u|_{2,p,Q}^p + K_6\varepsilon^{-p}|u|_{0,p,Q}^p.$$

对 k 求和并利用 $C^\infty(\Omega)\cap W^{2,p}(\Omega)$ 在 $W^{2,p}(\Omega)$ 内的稠密性即得所要的不等式(12). ∎

如果 Ω 有界，以上定理可在较弱的假定下证明

4.15 定理 设 Ω 是 $\mathbf{R}^n$ 内具有锥性质的有界区域，则定理 4.14 的结论对 Ω 成立.

证明 由定理 4.8 存在 Ω 的开子集的有限集族 $\{\Omega_1, \cdots, \Omega_k\}$ 使 $\Omega=\bigcup_{j=1}^{k}\Omega_j$ 并使每一个 Ω_j 是某一个开平行多面体的平移的并集. 显然只要对每个 Ω_j 证明类似于(11)的不等式就够了. 所以不失一般性我们假定 $\Omega=\bigcup_{x\in A}(x+P)$，其中 A 是 $\mathbf{R}^n$ 内的有界集，P 是一个有一个顶点在原点的开平行多面体，设 $\xi_1, \cdots, \xi_n$ 是从原点出发的 P 的 n 条边方向上的单位向量，并且设 l 是这些边的最小长度，则平行于一个向量 ξ_j 的任何直线 L 与 Ω 的交集或者是空集或者是线段的有限集族，每一线段长度介于 l 和 $\operatorname{diam}\Omega$ 之间. 如同(16)式一样由此得到对光滑函数 u

$$\int_\Omega |D_{\xi_j} u(x)|^p dx \leqslant K_1\varepsilon^p\int_\Omega |D_{\xi_j}^2 u(x)|^p dx + K\varepsilon^{-p}\int_\Omega |u(x)|^p dx.$$

因为 $\{\xi_1, \xi_2, \cdots, \xi_n\}$ 是 $\mathbf{R}^n$ 内的一组基，类似于(16)式后面的论证，现在就能证明，对所有 $u\in W^{2,p}(\Omega)$

$$|u|_{1,p,\Omega}^{p} \leqslant K_2 \varepsilon^{p} |u|_{2,p,\Omega}^{p} + K_2 \varepsilon^{-p} |u|_{0,p,\Omega}^{p}$$

成立. ▌

4.16 推论 在下列空间上:

(i) $W_0^{m,p}(\Omega)$ 对任意区域 Ω,

(ii) $W^{m,p}(\Omega)$ 对任意具有一致锥性质的区域 Ω,

(iii) $W^{m,p}(\Omega)$ 对任意具有锥性质的有界区域 Ω,

由

$$((u))_{m,p,\Omega} = \{|u|_{m,p,\Omega}^{p} + |u|_{0,p,\Omega}^{p}\}^{1/p}$$

定义的泛函 $((\cdot))_{m,p,\Omega}$ 是等价于通常范数 $\|\cdot\|_{m,p,\Omega}$ 的一个范数.

4.17 定理 (Ehrling[23], Browder[12]) 如果 $\Omega \subset \mathbf{R}^n$ 具有一致锥性质或者如果它有界且具有锥性质, 并且如果 $1 \leqslant p < \infty$, 则存在常数 $K = K(m, p, \Omega)$ 使对 $0 \leqslant j \leqslant m$ 和任意的 $u \in W^{m,p}(\Omega)$,

$$\|u\|_{j,p} \leqslant K \|u\|_{m,p}^{j/m} \|u\|_{0,p}^{(m-j)/m}. \tag{17}$$

此外, (17)式对所有的 $u \in W_0^{m,p}(\Omega)$ 成立, 常数 $K = K(m, p, n)$ 与 Ω 无关.

证明 不等式(17)对 $j=0$ 或 $j=m$ 是显然的. 对 $0<j<m$, 我们接连地应用(10)式得

$$\|u\|_{j,p} \leqslant K_1 \varepsilon \|u\|_{m,p} + K_1 \varepsilon^{-j/(m-j)} \|u\|_{0,p} \tag{18}$$

对于所有 ε, $0<\varepsilon \leqslant 1$, 以及所有 $u \in W^{m,p}(\Omega)$ 成立, K_1 只依赖于 m, p, 和 Ω. (由定理 4.13, 同样的不等式对所有 $u \in W_0^{m,p}(\Omega)$ 成立, K_1 只依赖于 m, p 和 n) 如果我们在(18)中令

$$\varepsilon = (\|u\|_{0,p} / \|u\|_{m,p})^{(m-j)/m}$$

就对 $u \neq 0$ 得到了不等式(17). ▌

我们注意(17)式同样代数地蕴涵(18)式: 特别地, 在第二章不等式(4)

$$ab \leqslant (a^p/p) + (b^{p'}/p'), \quad (1/p) + (1/p') = 1$$

中, 令 $p = m/j$, $p' = m/(m-j)$, $a = (\varepsilon \|u\|_{m,p})^{j/m}$ 和

$b=\varepsilon^{-j/m}\|u\|_{0,p}^{(m-j)/m}$，就可以看出 (17) 式的右端不超过 (18) 式的右端.

包含紧子区域的内插不等式

4.18 函数 $u\in W^{m,p}(\Omega)$ 的中间导数 $D^\beta u(|\beta|\leqslant m-1)$ 的 $L^p(\Omega)$-范数的上界可以由半范数 $|u|_{m,p,\Omega}$ 和 u 在一个适当的子区域上的 L^p-范数表示出来，这个子区域的闭包是有界区域 Ω 的紧子集. 我们建立某些这一类的混合内插不等式，方法与上面推导内插不等式的思路一致.（例如参看 Agmon 的工作[6].）

4.19 引理 设 (a,b) 是 $\mathbb{R}$ 上的有限开区间并且设 $1\leqslant p<\infty$. 则存在有限常数 $K=K(p,b-a)$，并且对所有的正数 ε，存在数 $\delta=\delta(\varepsilon,b-a)$ 满足 $0<2\delta<b-a$，使所有的 (a,b) 上的连续可微函数 f 满足

$$\int_a^b|f(t)|^p dt\leqslant K\varepsilon\int_a^b|f'(t)|^p dt+K\int_{a+\delta}^{b-\delta}|f(t)|^p dt. \tag{19}$$

此外，可以选与 $b-a$ 无关的 K 和 δ 的固定值使 (19) 式对所有的区间 (a,b) 都成立，只要区间的长度在两个固定的正的界之间：$0<l_1\leqslant b-a\leqslant l_2<\infty$.

证明 证明与引理 4.10 类似，暂时假设 $a=0$ 和 $b=1$，并且令 $\frac{1}{3}<\eta<\frac{2}{3}$. 如果 $0<x<1$，则

$$|f(x)|=\left|f(\eta)+\int_\eta^x f'(t)dt\right|\leqslant|f(\eta)|+\int_0^1|f'(t)|dt$$

在 $\left(\frac{1}{3},\frac{2}{3}\right)$ 上对 η 积分，导致

$$|f(x)|\leqslant 3\int_{1/3}^{2/3}|f(\eta)|d\eta+\int_0^1|f'(t)|dt,$$

由 Hölder 不等式如果 $p>1$，

$$|f(x)|^p\leqslant 3\cdot 2^{p-1}\int_{1/3}^{2/3}|f(t)|^p dt+2^{p-1}\int_0^1|f'(t)|^p dt.$$

在(0, 1)上对 x 积分，我们得

$$\int_0^1|f(x)|^p dt\leqslant K_p\int_0^1|f'(t)|^p dt+K_p\int_{1/3}^{2/3}|f(t)|^p dt,$$

其中 $K_p=3\cdot 2^{p-1}$. 作变量替换 $a+t(b-a)\to t$ 得到在任意的有限区间 (a, b) 上，

$$\int_a^b|f(t)|^p dt\leqslant K_p(b-a)^p\int_a^b|f'(t)|^p dt$$

$$+K_p\int_{a+(b-a)/3}^{b-(b-a)/3}|f(t)|^p dt.$$

对于给定的 $\varepsilon>0$ 取正整数 n 使 $n^{-p}\leqslant\varepsilon$. 对于 $j=0,\ 1,\ \cdots,\ n$ 设 $a_j=a+(b-a)j/n$, 并且取 δ 使 $0<\delta\leqslant(b-a)/3n$. 则

$$\int_a^b|f(t)|^p dt=\sum_{j=1}^n\int_{a_{j-1}}^{a_j}|f(t)|^p dt$$

$$\leqslant K_p\sum_{j=1}^n\left\{\left(\frac{b-a}{n}\right)^p\int_{a_{j-1}}^{a_j}|f'(t)|^p dt+\int_{a_{j-1}+\delta}^{a_j-\delta}|f(t)|^p dt\right\}$$

$$\leqslant K_p\max(1,\ (b-a)^p)\left\{\varepsilon\int_a^b|f'(t)|^p dt+\int_{a+\delta}^{b-\delta}|f(t)|^p dt\right\}$$

这就是所要的不等式. ▮

读者可以确信，与引理 4.10 不同，如果$(a,\ b)$是无限区间，(19)式右端第二个积分在紧子区间上取，那末引理 4.19 是不能推广到这种情形的.

4.20 定理 设Ω是 $\mathbf{R}^n$ 内具有线段性质的有界区域，则存在常数 $K=K(p,\Omega)$使得对任意的正数 ε，对应于一个区域 $\Omega_\varepsilon\subset\subset\Omega$ 使

$$|u|_{0,p,\Omega}\leqslant K\varepsilon|u|_{1,p,\Omega}+K|u|_{0,p,\Omega_\varepsilon}\tag{20}$$

对所有的 $u\in W^{1,p}(\Omega)$ 成立.

证明 证明与定理 4.14 类似. 因为Ω有界，bdryΩ 的局部有限

开覆盖 $\{U_j\}$ 和线段性质叙述中有关的非零向量构成的对应集合 $\{y_j\}$ (第 4.2 节) 都是有限集合. 所以可以找到开集 $\mathscr{V}_j\subset\subset U_j$ 使 $\mathrm{bdry}\Omega\subset\bigcup\limits_j\mathscr{V}_j$. (见定理 3.14 证明的第一部分.)不仅如此, 对某个 $\delta>0$, $\Omega_\delta=\{x\in\Omega:\ \mathrm{dist}(x,\ \mathrm{bdry}\Omega)<\delta\}\subset\bigcup\limits_j\mathscr{V}_j$. 所以我们可以写 $\Omega=\bigcup\limits_j(\mathscr{V}_j\cap\Omega)\cup\widetilde{\Omega}$, 其中 $\widetilde{\Omega}\subset\subset\Omega$. 所以只要证明对每一个 j

$$|u|^p_{0,p,\mathscr{V}_j\cap\Omega}\leqslant K_1\varepsilon^p|u|^p_{1,p,\Omega}+K_1|u|^p_{0,p,\Omega_{\varepsilon,j}}$$

对某个 $\Omega_{\varepsilon,j}\subset\subset\Omega$ 成立就够了, 为了简单起见我们去掉所有的下标 j.

考虑集合 Q, Q_η, $0\leqslant\eta<1$, 其定义为

$$Q=\{x+ty:\ x\in U\cap\Omega,\ 0<t<1\},$$
$$Q_\eta=\{x+ty:\ x\in\mathscr{V}\cap\Omega,\ \eta<t<1\}.$$

如果 $\eta>0$, $Q_\eta\subset\subset Q$. 由线段性质, $Q\subset\Omega$ 并且任意的平行于 y 过 $\mathscr{V}\cap\Omega$ 上一点的直线 L 与 Q_0 交于一个或几个区间, 每个区间的长度在 $|y|$ 与 $\mathrm{diam}\Omega$ 之间. 由引理 4.19 存在 $\eta>0$ 和常数 K 使对任意的 $u\in C^\infty(\Omega)$ 和任意的这样的直线 L

$$\int_{L\cap Q_0}|u(x)|^p ds\leqslant K_1\varepsilon^p\int_{L\cap Q_0}|D_y u(x)|^p ds + K_1\int_{L\cap Q_\eta}|u(x)|^p ds,$$

D_y 表示在 y 的方向求导数. 我们把 Q_0 投影在垂直于 y 的超平面上, 在此投影上积分这个不等式, 从而得

$$\begin{aligned}|u|^p_{0,p,\Omega\cap\mathscr{V}}&\leqslant|u|^p_{0,p,Q_0}\leqslant K_1\varepsilon^p|u|^p_{1,p,Q_0}+K_1|u|^p_{0,p,Q_\eta}\\&\leqslant K_1\varepsilon^p|u|^p_{1,p,\Omega}+K_1|u|^p_{0,p,\Omega_\varepsilon},\end{aligned}$$

其中 $\Omega_\varepsilon=\Omega_\eta\subset\subset\Omega$. 由稠密性, 这个不等式对任意的 $u\in W^{1,p}(\Omega)$ 都成立. ∎

4.21 推论 如果Ω是有界的且具有锥性质，定理 4.20 的结论也成立.

证明 如前所示，具有锥性质的区域Ω不一定具有线段性质. 但是由定理 4.8, Ω是具有强局部 Lipschitz 性质的区域的有限并. 我们把具有强局部 Lipschitz 性质的有界区域具有线段性质的证明留给读者，这样就完成了证明. ▌

4.22 引理 设 Ω_0, Ω 是 $\mathbf{R}^n$ 内的区域，$\Omega_0 \subset\subset \Omega$. 则存在一个具有锥性质的区域 Ω' 使 $\Omega_0 \subset \Omega' \subset\subset \Omega$.

证明 因为$\overline{\Omega}_0$是Ω的紧子集，存在 $\delta>0$ 使 $\operatorname{dist}(\overline{\Omega}_0, \operatorname{bdry}\Omega)>\delta$. 区域 $\Omega'=\{y\in\mathbf{R}^n$: 对某一个 $x\in\Omega_0$, $|y-x|<\delta\}$显然有所要的性质. ▌

4.23 定理 设Ω是 $\mathbf{R}^n$ 内有界区域或者具有线段性质或者具有锥性质，设 $0<\varepsilon_0<\infty$, 设 $1\leqslant p<\infty$, 并且设 j 和 m 为满足 $0\leqslant j\leqslant m-1$ 的整数. 则存在常数 $K=K(\varepsilon_0, m, p, \Omega)$, 对每一个 ε, $0<\varepsilon\leqslant\varepsilon_0$, 存在一个区域 Ω_ε, $\Omega_\varepsilon\subset\subset\Omega$, 使得对所有的 $u\in W^{m,p}(\Omega)$

$$|u|_{j,p,\Omega}\leqslant K\varepsilon|u|_{m,p,\Omega}+K\varepsilon^{-j/(m-j)}|u|_{0,p,\Omega_\varepsilon}. \qquad (21)$$

证明 我们把定理 4.20 或者它的推论用于$D^\beta u$, $|\beta|=m-1$, 得

$$|u|_{m-1,p,\Omega}\leqslant K_1\varepsilon|u|_{m,p,\Omega}+K_1|u|_{m-1,p,\Omega_\varepsilon}, \qquad (22)$$

其中 $\Omega_\varepsilon\subset\subset\Omega$. 由引理 4.22 我们可以假定 Ω_ε 有锥性质. 对于 $0<\varepsilon\leqslant\varepsilon_0$, 由定理 4.15 我们有

$$|u|_{m-1,p,\Omega_\varepsilon}\leqslant K_2\varepsilon|u|_{m,p,\Omega_\varepsilon}+K_2\varepsilon^{-(m-1)}|u|_{0,p,\Omega_\varepsilon}. \qquad (23)$$

把(22)(23)联立起来，我们得(21)式的 $j=m-1$ 情形. 我们用对 j 向下归纳来完成证明. 假定 (21) 对某个 $j\geqslant 1$ 成立，把 ε 换成 ε^{m-j}(随之而来K和 Ω_ε 也作替换)，我们得

$$|u|_{j,p,\Omega}\leqslant K_3\varepsilon^{m-j}|u|_{m,p,\Omega}+K_3\varepsilon^{-j}|u|_{0,p,\Omega'_\varepsilon}. \qquad (24)$$

同样由(21) 式把 j 和m分别换成 $j-1$ 和 j(这个情形已经证明)，我们有

$$|u|_{j-1,p,\Omega}\leqslant K_4\varepsilon|u|_{j,p,\Omega}+K_4\varepsilon^{-(j-1)}|u|_{0,p,\Omega''_\varepsilon}. \tag{25}$$

把(24)和(25)联立起来，我们得

$$|u|_{j-1,p,\Omega}\leqslant K_5\varepsilon^{m-(j-1)}|u|_{m,p,\Omega}+K_5\varepsilon^{-(j-1)}|u|_{0,p,\Omega_\varepsilon},$$

其中 $K_5=K_4(K_3+1)$ 和 $\Omega_\varepsilon=\Omega'_\varepsilon\cup\Omega''_\varepsilon$. 把 ε 换成 $\varepsilon^{1/(m-j+1)}$，我们便完成了归纳. ▌

延 拓 定 理

4.24 设 Ω 是 $\mathbf{R}^n$ 中的区域. 对于给定的 m 和 p, 将 $W^{m,p}(\Omega)$ 映入 $W^{m,p}(\mathbf{R}^n)$ 的线性算子 E 称为 Ω 的简单 (m,p)-延拓算子，如果存在常数 $K=K(m,p)$ 使对所有的 $u\in W^{m,p}(\Omega)$，以下条件成立:

(i) 在 Ω 上 $Eu(x)=u(x)$ a. e.,

(ii) $\|Eu\|_{m,p,\mathbf{R}^n}\leqslant K\|u\|_{m,p,\Omega}$.

E 称为 Ω 的强 m-延拓算子，如果 E 是把 Ω 上 a. e. 定义的函数映入 $\mathbf{R}^n$ 上 a. e. 定义的函数的线性算子并且如果对所有的 p, $1\leqslant p<\infty$, 所有的 k, $0\leqslant k\leqslant m$, E 在 $W^{k,p}(\Omega)$ 上的限制是一个 Ω 的简单 (k,p)-延拓算子. 最后，E 称为 Ω 的全延拓算子，如果对所有的 m, E 都是 Ω 的强 m-延拓算子.

4.25 对于一个区域 Ω 即使一个简单 (m,p)-延拓算子的存在，也保证 $W^{m,p}(\Omega)$ 继承到很多 $W^{m,p}(\mathbf{R}^n)$ 所具有的性质. 例如，如果已知嵌入 $W^{m,p}(\mathbf{R}^n)\to L^q(\mathbf{R}^n)$ 成立，则按照以下的不等式串可以得到嵌入 $W^{m,p}(\Omega)\to L^q(\Omega)$:

$$\|u\|_{0,q,\Omega}\leqslant\|Eu\|_{0,q,\mathbf{R}^n}\leqslant K_1\|Eu\|_{m,p,\mathbf{R}^n}\leqslant K_1K\|u\|_{m,p,\Omega}.$$

但是在第五章我们将不用这个方法证明 Sobolev 嵌入定理，因为我们将在比存在一个 (m,p)-延拓算子所需的假设更为弱的关于 Ω 的假设下来证明 Sobolev 定理.

我们将对以上定义的三种类型的每一个构造延拓算子，这些构造方法中有两个是基于对光滑边界的逐次反射，它属于 Lichenstein[35]和较后的 Hestenes[31]与 Seeley[61]. 第三个构造法属于 Calderón[14], 包括使用关于奇异积分的 Calderón-Zygmund 定理. 它不如反射法明晰, 得到的结果也弱些, 但对区域Ω的正则性要求少一些. 除了特别简单的区域外, 所有的构造都要求以这种方式选择的从属于 bdryΩ 的开覆盖的单位分解: 即使得单位分解中的函数有一致有界的导数. 因为这样, 有界边界的区域(外区域和有界区域)看来更容易满足我们的延拓定理的条件, 例外是半空间, 四分之一空间, 等等, 以及它们的光滑的像.

4.26 定理 设Ω或者是

(i) $\mathbf{R}^n$ 内半空间, 或者是

(ii) $\mathbf{R}^n$ 内具有一致 C^m-正则性的区域, 并且还有有界的边界.

对任意的正数m存在Ω的强 m-延拓算子 E. 此外, 如果α和γ是多重指标, $|\gamma|\leqslant|\alpha|\leqslant m$, 存在线性算子 $E_{\alpha\gamma}$, 对于$1\leqslant j\leqslant m-|\alpha|$, 这个算子从$W^{j,p}(\Omega)$到$W^{j,p}(\mathbf{R}^n)$是连续的, 使得如果$u\in W^{|\alpha|,p}(\Omega)$, 则

$$D^\alpha Eu(x)=\sum_{|\gamma|\leqslant|\alpha|}E_{\alpha\gamma}D^\gamma u(x)\qquad \text{a. e. 在 } \mathbf{R}^n. \tag{26}$$

证明 首先设Ω是半空间 $\mathbf{R}^n_+=\{x\in\mathbf{R}^n: x_n>0\}$. 对于 a. e. 定义在 $\mathbf{R}^n_+$上的函数 u, 我们定义 a. e. 在 $\mathbf{R}^n$ 的延拓算子 Eu 和 $E_\alpha u$, $|\alpha|\leqslant m$, 如下:

$$Eu(x)=\begin{cases}u(x) & x_n>0\\ \sum_{j=1}^{m+1}\lambda_j u(x_1,\cdots,x_{n-1},-jx_n) & \text{如果 } x_n\leqslant 0,\end{cases}$$

$$E_\alpha u(x)=\begin{cases}u(x) & \text{如果 } x_n>0\\ \sum_{j=1}^{m+1}(-j)^{\alpha_n}\lambda_j u(x_1,\cdots,x_{n-1},-jx_n) & \text{如果 } x_n\leqslant 0,\end{cases}\tag{27}$$

其中系数 $\lambda_1, \cdots, \lambda_{m+1}$ 是 $(m+1)\times(m+1)$ 线性方程组

$$\sum_{j=1}^{m+1}(-j)^k\lambda_j=1,\ \ k=0,1,\cdots,m.$$

的唯一解. 如果 $u\in C^m(\overline{\mathbf{R}_+^n})$, 则容易验证 $Eu\in C^m(\mathbf{R}^n)$ 并且

$$D^\alpha Eu(x)=E_\alpha D^\alpha u(x),\ \ |\alpha|\leqslant m.$$

因此

$$\int_{\mathbf{R}^n}|D^\alpha Eu(x)|^p dx=\int_{\mathbf{R}_+^n}|D^\alpha u(x)|^p dx$$

$$+\int_{\mathbf{R}_-^n}\left|\sum_{j=1}^{m+1}(-j)^{\alpha_n}\lambda_j D^\alpha u(x_1,\cdots,x_{n-1},-jx_n)\right|^p dx$$

$$\leqslant K(m,p,\alpha)\int_{\mathbf{R}_+^n}|D^\alpha u(x)|^p dx.$$

通过定理 3.18, 把以上不等式推广到函数 $u\in W^{k,p}(\mathbf{R}_+^n)$, $m\geqslant k\geqslant|\alpha|$. 因此 E 是 $\mathbf{R}_+^n$ 的强 m-延拓算子. 因为 $D^\beta E_\alpha u(x)=E_{\alpha+\beta}D^\beta u(x)$, 类似的计算表明 E_α 是一个强 $(m-|\alpha|)$-延拓. 这样,定理已对半空间证明了(这里 $E_{\alpha\alpha}=E_\alpha$, 当 $\alpha\neq\gamma$ 时 $E_{\alpha\gamma}=0$).

现在假设 Ω 是一致 C^m-正则的并且有有界边界. 则参照第4.6节, bdryΩ 的开覆盖 $\{U_j\}$, 对应的从 U_j 到 B 上的 m-光滑映射 Φ_j 都是有限集族, 譬如 $1\leqslant j\leqslant N$. 设 $Q=\{y\in\mathbf{R}^n: |y'|=\left(\sum_{j=1}^{n-1}y_j^2\right)^{1/2}<\frac{1}{2}, |y_n|<\sqrt{3}/2\}$. 则

$$\left\{y\in\mathbf{R}^n: |y|<\frac{1}{2}\right\}\subset Q\subset B=\{y\in\mathbf{R}^n: |y|<1\}.$$

由第 4.6 节的条件(i), 对于某个 $\delta>0$, 开集 $\mathscr{V}_j=\Psi_j(Q)$, $1\leqslant j\leqslant N$, 形成了 $\Omega_\delta=\{x\in\Omega: \operatorname{dist}(x, \operatorname{bdry}\Omega)<\delta\}$ 的一个开覆盖. 存在与 bdryΩ 分开的 Ω 的开集 $\mathscr{V}_0$, 使 $\Omega\subset\bigcup_{j=0}^N\mathscr{V}_j$. 由定理 3.14 可以找到无限次可微函数 $\omega_0, \omega_1, \cdots, \omega_N$ 使 supp$\omega_j\subset\mathscr{V}_j$ 并且对所有

的 $x\in\Omega$，$\sum_{j=0}^{N}\omega_j(x)=1$.（注意如果 Ω 无界，$\operatorname{supp}\omega_0$ 不必是紧的.）

因为 Ω 是一致 C^m-正则的，它有线段性质并且 $C_0^\infty(\mathbf{R}^n)$ 内的函数在 Ω 上的限制是在 $W^{k,p}(\Omega)$ 内稠密的. 如果 $\phi\in C_0^\infty(\mathbf{R}^n)$ 则 ϕ 在 Ω 上与函数 $\sum_{j=0}^{N}\phi_j$ 一致，其中 $\phi_j=\omega_j\cdot\phi\in C_0^\infty(\mathscr{V}_j)$.

对于 $j\geqslant 1$ 和 $y\in B$ 设 $\psi_j(y)=\phi_j(\Psi_j(y))$ 则 $\psi_j\in C_0^\infty(Q)$. 我们在 Q 外部延拓 ψ_j 恒等于零. 如同(27) 一样定义 E(和 E_α)，我们有 $E\psi_j\in C_0^m(Q)$，在 $Q_+=\{y\in Q: y_n>0\}$ 上有 $E\psi_j=\psi_j$，并且

$$\|E\psi_j\|_{k,p,Q}\leqslant K_1\|\psi_j\|_{k,p,Q_+},\quad 0\leqslant k\leqslant m,$$

其中 K_1 依赖于 k, m, 和 p. 如果 $\theta_j(x)=E\psi_j(\Phi_j(x))$，则 $\theta_j\in C_0^m(\mathscr{V}_j)$ 并且如果 $x\in\Omega$ 则 $\theta_j(x)=\phi_j(x)$. 可以归纳地验证，如果 $|\alpha|\leqslant m$，则

$$D^\alpha\theta_j(x)=\sum_{|\beta|\leqslant|\alpha|}\sum_{|\gamma|\leqslant|\alpha|}a_{j;\alpha\beta}(x)[E_\beta(b_{j;\beta\gamma}\cdot(D^\gamma\phi_j\circ\Psi_j))](\Phi_j(x)).$$

其中 $a_{j;\alpha\beta}\in C^{m-|\alpha|}(\bar{U}_j)$ 和 $b_{j;\beta\gamma}\in C^{m-|\beta|}(\bar{B})$ 依赖于变换 Φ_j 和 $\Psi_j=\Phi_j^{-1}$ 并且满足

$$\sum_{|\beta|\leqslant|\alpha|}a_{j;\alpha\beta}(x)b_{j;\beta\gamma}(\Phi_j(x))=\begin{cases}1 & \text{如果 } \gamma=\alpha\\ 0 & \text{其它.}\end{cases}$$

由定理 3.35 对 $k\leqslant m$ 我们有，

$$\|\theta_j\|_{k,p,\mathbf{R}^n}\leqslant K_2\|E\psi_j\|_{k,p,Q}\leqslant K_1K_2\|\psi_j\|_{k,p,Q_+}\leqslant K_3\|\phi_j\|_{k,p,\Omega},$$

其中 K_3 可以选得与 j 无关. 算子 $\widetilde{E}$ 定义为

$$\widetilde{E}\phi(x)=\phi_0(x)+\sum_{j=1}^{N}\theta_j(x),$$

如果 $x\in\Omega$ 显然满足 $\widetilde{E}\phi(x)=\phi(x)$，并且

$$\|\widetilde{E}\phi\|_{k,p,\mathbf{R}^n}\leqslant\|\phi_0\|_{k,p,\Omega}+K_3\sum_{j=1}^{N}\|\phi_j\|_{k,p,\Omega}$$

$$\leqslant K_4(1+NK_3)\|\phi\|_{k,p,\Omega} \tag{28}$$

其中

$$K_4=\max_{0\leqslant j\leqslant N}\ \max_{|\alpha|\leqslant m}\ \sup_{x\in R^n}\ |D^\alpha\omega_j(x)|<\infty.$$

这样 $\widetilde{E}$ 是 Ω 的强 m-延拓算子，同样

$$D^\alpha\widetilde{E}\phi(x)=\sum_{|\gamma|\leqslant|\alpha|}(E_{\alpha\gamma}D^\gamma\phi)(x),$$

其中如果 $\alpha\neq\gamma$

$$E_{\alpha\gamma}v(x)=\sum_{j=1}^{N}\sum_{|\beta|\leqslant|\alpha|}a_{j;\alpha\beta}(x)[E_\beta(b_{j;\beta\gamma}\cdot(v\cdot\omega_j)\circ\Psi_j)](\Phi_j(x)),$$

并且

$$E_{\alpha\alpha}v(x)=(v\cdot\omega_0)(x)+\sum_{j=1}^{N}\sum_{|\beta|\leqslant|\alpha|}a_{j;\alpha\beta}(x)[E_\beta(b_{j;\beta\alpha}\cdot(v\cdot\omega_j)\circ\Psi_j)](\Phi_j(x)).$$

我们注意如果 $x\in\Omega$，对于 $\alpha\neq\gamma$，$E_{\alpha\gamma}v(x)=0$ 和 $E_{\alpha\alpha}v(x)=v(x)$. 显然 $E_{\alpha\gamma}$ 是线性算子. 由 $a_{j;\alpha\beta}$ 和 $b_{j;\beta\gamma}$ 的可微性，对 $1\leqslant j\leqslant m-|\alpha|$，从 $W^{j,p}(\Omega)$ 到 $W^{j,p}(\mathbf{R}^n)$，$E_{\alpha\gamma}$ 是连续的，这就完成了证明. ▌

以上定理包含了延拓函数的导数的表示式(26)，因为在第七章我们研究分数次空间时要用到它.

以上证明中的反射法可以用来得到对具有强局部 Lipschitz 性质的区域的全延拓算子（参看 Stein [64a]）. 我们给出一个属于 Seeley[61]的对于光滑有界区域的推导.

4.27 引理 存在实序列 $\{a_k\}_{k=0}^\infty$ 使对所有的非负整数 n 我们有

$$\sum_{k=0}^{\infty}2^{nk}a_k=(-1)^n \tag{29}$$

和

$$\sum_{k=0}^{\infty} 2^{nk} |a_k| < \infty. \tag{30}$$

证明 对固定的 N, 设 $a_{k,N}, k=0, 1, 2, \cdots, N$, 是线性方程组

$$\sum_{k=0}^{N} 2^{nk} a_{k,N} = (-1)^n, \quad n=0, 1, 2, \cdots, N \tag{31}$$

的解. 依据范得蒙(Vandermonde)行列式

$$V(x_0, x_1, \cdots x_N) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_N \\ x_0^2 & x_1^2 & \cdots & x_N^2 \\ \vdots & \vdots & & \vdots \\ x_0^N & x_1^N & \cdots & x_N^N \end{vmatrix} = \prod_{\substack{i,j=0 \\ i<j}}^{N} (x_j - x_i)$$

由克莱姆(Cramer)法则给出(31)的解是

$$\begin{aligned} a_{k,N} &= \frac{V(1,2,\cdots,2^{k-1}, -1, 2^{k+1}, \cdots, 2^N)}{V(1, 2, \cdots, 2^N).} \\ &= \left\{ \prod_{\substack{i,j=0 \\ i,j\neq k \\ i<j}}^{N} (2^j - 2^i) \prod_{i=0}^{k-1} (-1-2^i) \prod_{j=k+1}^{N} (2^j+1) \right\} \\ &\quad \times \left\{ \prod_{\substack{i,j=0 \\ i<j}}^{N} (2^j - 2^i) \right\}^{-1} \\ &= A_k B_{k,N}, \end{aligned}$$

其中

$$A_k = \prod_{i=0}^{k-1} \frac{1+2^i}{2^i - 2^k}, \qquad B_{k,N} = \prod_{j=k+1}^{N} \frac{1+2^j}{2^j - 2^k},$$

如果 $l > m$ 应理解为 $\prod_{i=l}^{m} p_i = 1$. 现在

$$|A_k| \leqslant \prod_{i=1}^{k-1} \frac{2^{i+1}}{2^{k-1}} = 2^{(3k-k^2)/2}.$$

同样

$$\log B_{k,N}=\sum_{j=k+1}^{N}\log\left(1+\frac{1+2^k}{2^j-2^k}\right)$$

$$<\sum_{j=k+1}^{N}\frac{1+2^k}{2^j-2^k}<(1+2^k)\sum_{j=k+1}^{N}\frac{1}{2^{j-1}}<4,$$

其中我们用了对于 $x>0$ 成立的不等式 $\log(1+x)<x$. 因此增序列 $\{B_{k,N}\}_{N=0}^{\infty}$ 收敛于极限 $B_k\leqslant e^4$, 设 $a_k=A_kB_k$ 则

$$|a_k|\leqslant e^4 2^{(3k-k^2)/2}.$$

则对任意的 n

$$\sum_{k=0}^{\infty}2^{nk}|a_k|\leqslant e^4\sum_{k=0}^{\infty}2^{(2nk+3k-k^2)/2}<\infty.$$

在(31)式中令 N 趋于无穷,我们便完成了证明.▌

4.28 定理 设 Ω 或者是

(i) $\mathbf{R}^n$ 内的半空间,或者是

(ii) $\mathbf{R}^n$ 内的区域,它对所有的 m 具有一致 C^m-正则性,且有有界边界.

则对 Ω 存在全延拓算子.

证明 只要对半空间 $\mathbf{R}_+^n$ 证明定理就够了;对满足(ii)的 Ω 的证明如同定理 4.26 那样立即可得.

对于任意的 m 和 p, 函数 $\phi\in C_0^\infty(\mathbf{R}^n)$ 在 $\mathbf{R}_+^n$ 的限制在 $W^{m,p}(\mathbf{R}_+^n)$ 内稠密,我们只对这样的函数定义延拓算子. 设 f 是实值函数,在 $[0,\infty)$ 无穷次可微, 当 $0\leqslant t\leqslant\frac{1}{2}$, $f(t)=1$, 当 $t\geqslant1$, $f(t)=0$. 如果 $\phi\in C_0^\infty(\mathbf{R}^n)$, 设

$$E\phi(x)=\begin{cases}\phi(x) & \text{当 } x\in\overline{\mathbf{R}_+^n}\\ \sum_{k=0}^{\infty}a_kf(-2^kx_n)\phi(x',-2^kx_n), & \text{当 } x\in\mathbf{R}_-^n,\end{cases}\tag{32}$$

其中 $\{a_k\}$ 是上面引理所作的序列, $x'=(x_1,\cdots,x_{n-1})$. 则因为对任

意特定的 $x\in\mathbf{R}_-^n=\{x\in\mathbf{R}^n: x_n<0\}$，(32)中的和号只有有限多个非零项. 显然 $E\phi$ 在 $\mathbf{R}^n$ 是有明确定义的. 不仅如此，$E\phi$ 有紧支集并且属于 $C^\infty(\overline{\mathbf{R}}_+^n)\cap C^\infty(\overline{\mathbf{R}}_-^n)$. 如果 $x\in\mathbf{R}_-^n$, 我们有

$$
\begin{aligned}
D^\alpha E\phi(x)=&\sum_{k=0}^{\infty}a_k\sum_{j=0}^{\alpha_n}\left\{\binom{\alpha_n}{j}(-2^k)^{\alpha_n}f^{(\alpha_n-j)}(-2^kx_n)\right.\\
&\left.\times D_n^jD^{\alpha'}\phi(x',-2^kx_n)\right\}\\
=&\sum_{k=0}^{\infty}\psi_k(x).
\end{aligned}
$$

因为当 $-x_n>1/2^{k-1}$ 时 $\psi_k(x)=0$，由(30)式得以上级数当 x_n 从下面趋于零时绝对且一致收敛. 因此由(29)式

$$
\begin{aligned}
\lim_{x_n\to0-}D^\alpha E\phi(x)&=\sum_{k=0}^{\infty}(-2^k)^{\alpha_n}a_kD^\alpha\phi(x',0+)\\
&=D^\alpha\phi(x',0+)=\lim_{x_n\to0+}D^\alpha E\phi(x)\\
&=D^\alpha E\phi(0).
\end{aligned}
$$

因此 $E\phi\in C_0^\infty(\mathbf{R}^n)$. 此外, 如果 $|\alpha|\leqslant m$,

$$
|\psi_k(x)|^p\leqslant K_1^p|a_k|^p2^{kmp}\sum_{|\beta|\leqslant m}|D^\beta\phi(x',-2^kx_n)|^p,
$$

其中 K_1 依赖于 $m,p,n,$ 和 f. 因此

$$
\begin{aligned}
\|\psi_k\|_{0,p,\mathbf{R}_-^n}&\leqslant K_1|a_k|2^{km}\left\{\sum_{|\beta|\leqslant m}\int_{\mathbf{R}_-^n}|D^\beta\phi(x',-2^kx_n)|^pdx\right\}^{1/p}\\
&=K_1|a_k|2^{km}\left\{(1/2^k)\sum_{|\beta|\leqslant m}\int_{\mathbf{R}_+^n}|D^\beta\phi(y)|^pdy\right\}^{1/p}\\
&\leqslant K_1|a_k|2^{km}\|\phi\|_{m,p,\mathbf{R}_+^n}.
\end{aligned}
$$

由(30)得

$$
\begin{aligned}
\|D^\alpha E\phi\|_{0,p,\mathbf{R}_-^n}&\leqslant K_1\|\phi\|_{m,p,\mathbf{R}_+^n}\sum_{k=0}^{\infty}2^{km}|a_k|\\
&\leqslant K_2\|\phi\|_{m,p,\mathbf{R}_+^n}.
\end{aligned}
$$

把它和对于 $\|D^\alpha E\phi\|_{0,p,\mathbf{R}_+^n}$ 的类似的明显不等式联立，我们得

$$\|E\phi\|_{m,p,\mathbf{R}^n}\leqslant K_3\|\phi\|_{m,p,\mathbf{R}_+^n},$$

其中 $K_3=K_3(m,p,n)$，证毕.∎

4.29 在定理 4.26(类似地，在定理 4.28)中bdryΩ服从有界这一限制，因此覆盖$\{\mathscr{V}_j\}$有限. 这个有限性在证明中有两处用到，首先用在断言常数 K_4 的存在性，其次用在得到(28) 的最后的不等式. 这后一个用处在证明中不是基本的，因为，即使覆盖$\{\mathscr{V}_j\}$不是有限的，(28)还是可以从有限相交性质得到[4.6 节，条件(ii)]. 定理 4.26 和 4.28 可推广到任意的适当正则的区域，这些区域存在一个从属于覆盖$\{\mathscr{V}_j\}$的单位分解$\{\omega_j\}$，对任意给定的 α, $D^\alpha\omega_j$对 j 一致地在 $\mathbf{R}^n$ 上有界，读者可以发现，用以上技巧作出一些没有被以上定理包括的区域的延拓算子是很有兴味的，例如，四分之一空间，带形，方盒子，和它们的光滑的像.

这里可以提一下，虽然 Calderón 延拓定理(定理 4.32)的证明方法和以上的反射的方法很不一样，但是证明用到了与定理 4.26 相类似的单位分解. 因此，以上考虑对它也适用. 这个定理在一个加强形式的一致锥条件下证明，如果Ω有有界边界，它归结为第 4.4 节的一致锥条件.

在给出 Calderón 定理之前，我们给出两个在证明中用到的关于卷积算子的熟知的结果. 第一个是 W. H. Young 的定理的特殊情形.

4.30 定理 (Young) 设$1\leqslant p<\infty$ 并且假设 $u\in L^1(\mathbf{R}^n)$ 和 $v\in L^p(\mathbf{R}^n)$. 则卷积

$$u*v(x)=\int_{\mathbf{R}^n}u(x-y)v(y)\,dy,$$

$$v*u(x)=\int_{\mathbf{R}^n}v(x-y)u(y)\,dy$$

有明确定义而且对几乎所有的 $x\in\mathbf{R}^n$相等. 不仅如此，$u*v\in L^p(\mathbf{R}^n)$ 并且

$$\|u*v\|_p \leqslant \|u\|_1 \|v\|_p. \tag{33}$$

证明 如果 $p=1$，证明是 Fubini 定理的简单推论，因此我们假定 $1<p<\infty$. 设 $w\in L^{p'}(\Omega)$，则

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} w(x)\int_{\mathbb{R}^n} u(x-y)v(y)\,dy\,dx\right| \\
&= \left|\int_{\mathbb{R}^n} w(x)\int_{\mathbb{R}^n} u(y)v(x-y)\,dy\,dx\right| \\
&\leqslant \int_{\mathbb{R}^n} |u(y)|\,dy\int_{\mathbb{R}^n} |v(x-y)|\,|w(x)|\,dx \\
&\leqslant \int_{\mathbb{R}^n} |u(y)|\,dy\left\{\int_{\mathbb{R}^n} |v(x-y)|^p\,dx\right\}^{1/p} \\
&\quad \times \left\{\int_{\mathbb{R}^n} |w(x)|^{p'}\,dx\right\}^{1/p'} \\
&= \|u\|_1 \|v\|_p \|w\|_{p'}
\end{aligned}$$

因为 w 可以选得无处为零，因此 $u*v(x)$ 和 $v*u(x)$ 必须 a. e. 有限. 此外，泛函

$$F_{u*v}(w)=\int_{\mathbb{R}^n} u*v(x)w(x)\,dx$$

属于 $[L^{p'}(\mathbb{R}^n)]'$，所以由定理2. 33，存在 $\lambda\in L^p(\mathbb{R}^n)$，$\|\lambda\|_p\leqslant\|u\|_1\|v\|_p$，使得对所有的 $w\in L^{p'}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \lambda(x)w(x)\,dx=\int_{\mathbb{R}^n} u*v(x)w(x)\,dx.$$

因此 $\lambda=u*v\in L^p(\mathbb{R}^n)$ 并且证明了(33). $u*v$ 与 $v*u$ 相等的证明是初等的. ▌

下面的定理是 Caldlrón 和Zygmund 对包含不可积奇性核的卷积的著名不等式的特殊情形[16]，对我们的目的来说它是适用的. 证明很长，可以在很多地方找到(例如，Stein 和 Weiss [65]). 这里省略了. 这个不等式和基于这个不等式的延拓定理在本专著里今后没有用到.

设 $B_R=\{x\in\mathbb{R}^n\colon |x|\leqslant R\}$，$S_R=\{x\in\mathbb{R}^n\colon |x|=R\}$，并设 $d\sigma_R$是

S_R 上的面积元[Lebesgue$(n-1)$-测度]. 函数 g 称为在 $B_R\sim\{0\}$ 上 μ 阶齐次，如果对所有的 $x\in B_R\sim\{0\}$ 和 $0<t\leqslant 1$, $g(tx)=t^\mu g(x)$.

4.31 定理 (Calderón-Zygmund 不等式) 设

$$g(x)=G(x)|x|^{-n},$$

其中

(i) G 在 $\mathbf{R}^n\sim\{0\}$ 上有界并且有紧支集,

(ii) 对某个 $R>0$, G 在 $B_R\sim\{0\}$ 上 0 阶齐次, 并且

(iii) $\int_{S_R} G(x)d\sigma_R=0$.

如果 $1<p<\infty$ 并且 $u\in L^p(\mathbf{R}^n)$ 则卷积积分的主值

$$u*g(x)=\lim_{\varepsilon\to 0+}\int_{\mathbf{R}^n\sim B_\varepsilon} u(x-y)g(y)dy$$

对几乎所有的 $x\in\mathbf{R}^n$ 存在, 并且存在常数 $K=K(G,p)$ 使对所有这样的 u

$$\|u*g\|_p\leqslant K\|u\|_p.$$

反之，如果 G 满足(i)和(ii)并且如果对所有的 $u\in C_0^\infty(\mathbf{R}^n)$, $u*g$ 存在，则 G 满足(iii).

4.32 定理 (Calderón 延拓定理) 设 Ω 是 $\mathbf{R}^n$ 内一个区域，它具有作了如下改变的一致锥性质(第 4.4 节):

(i) bdryΩ 的开覆盖 $\{U_j\}$ 要求有限, 并且

(ii) 集合 U_j 不要求有界.

则对任意的 $m\in\{1,2,\cdots\}$ 和任意的 p, $1<p<\infty$，存在一个 Ω 的简单 (m,p)-延拓算子 $E=E(m,p)$.

证明 设 $\{U_1, U_2,\cdots, U_N\}$ 是一致锥性质给出的 bdryΩ 的开覆盖, 并且设 U_0 是与 bdryΩ 分开的开子集, 使得 $\Omega\subset\bigcup_{j=0}^N U_j$ [由第 4.4 节条件(ii)，这样的 U_0 是存在的.]设 $\omega_0,\omega_1,\cdots,\omega_N$ 是对 Ω 的

C^∞-单位分解，$\mathrm{supp}\,\omega_j\subset U_j$. 对于 $1\leqslant j\leqslant N$ 我们将定义算子 E_j，使得如果 $u\in W^{m,p}(\Omega)$，则 $E_ju\in W^{m,p}(\mathbf{R}^n)$ 并且满足

$$
\begin{aligned}
&E_ju=u, \qquad \text{在 } U_j\cap\Omega,\\
&\|E_ju\|_{m,p,\mathbf{R}^n}\leqslant K_{m,p,j}\|u\|_{m,p,\Omega}.
\end{aligned}
\tag{34}
$$

所要的延拓算子显然由

$$
Eu=\omega_0u+\sum_{j=1}^N\omega_jE_ju
$$

给出. 我们把 $x\in\mathbf{R}^n$ 写成极坐标形式 $x=\rho\sigma$，其中 $\rho\geqslant0$，σ 是单位向量. 设 C_j 是在一致锥性质叙述中与 U_j 对应的锥，它有顶点为0. 设 ϕ_j 是定义在 $\mathbf{R}^n\sim\{0\}$ 上的函数，并且满足

(i) 对所有的 $x\neq0$，$\phi_j(x)\geqslant0$，

(ii) $\mathrm{supp}\,\phi_j\subset -C_j\cup\{0\}$，

(iii) $\phi_j\in C^\infty(\mathbf{R}^n\sim\{0\})$，

(iv) 对某个 $\varepsilon>0$，ϕ_j 在 $B_\varepsilon\sim\{0\}$ 是 $m-n$ 阶齐次的. 既然 $\rho^{n-1}\phi_j$ 在 $B_\varepsilon\sim\{0\}$ 上是 $m-1\geqslant0$ 阶齐次的，所以函数 $\psi_j(x)=(\partial/\partial\rho)^m[\rho^{n-1}\phi_j(x)]$ 在 $B_\varepsilon\sim\{0\}$ 上等于零. 因此 ψ_j 在 $x=0$ 处延拓为零以后，属于 $C_0^\infty(-C_j)$. 定义

$$
\begin{aligned}
E_ju(y)=K_j\Big\{(-1)^m&\int_S\int_0^\infty\phi_j(\rho\sigma)\rho^{n-1}\left(\frac{\partial}{\partial\rho}\right)^m u(y-\rho\sigma)d\rho d\sigma\\
&-\int_S\int_0^\infty\psi_j(\rho\sigma)u(y-\rho\sigma)d\rho d\sigma\Big\},
\end{aligned}
\tag{35}
$$

其中 $\int_S\cdot d\sigma$ 表示在单位球上的积分. 常数 K_j 马上就要确定. 如果 $y\in U_j\cap\Omega$，则暂时假定 $u\in C^\infty(\Omega)$，由第 4.4 节条件(iii) $u(y-\rho\sigma)$ 在 $\rho\sigma\in\mathrm{supp}\,\phi_j$ 上无穷次可微，分部积分 m 次得

$$
(-1)^m\int_0^\infty\rho^{n-1}\phi_j(\rho\sigma)\left(\frac{\partial}{\partial\rho}\right)^m u(y-\rho\sigma)d\rho
$$

$$
\begin{aligned}
&=\sum_{k=0}^{m-1}(-1)^{m-k}\left(\frac{\partial}{\partial\rho}\right)^{k}[\rho^{n-1}\phi_j(\rho\sigma)]\\
&\quad\times\left(\frac{\partial}{\partial\rho}\right)^{m-k-1}u(y-\rho\sigma)\Big|_{\rho=0}^{\rho=\infty}\\
&\quad+\int_0^\infty\left(\frac{\partial}{\partial\rho}\right)^{m}[\rho^{n-1}\phi_j(\rho\sigma)]u(y-\rho\sigma)d\rho\\
&=\left(\frac{\partial}{\partial\rho}\right)^{m-1}[\rho^{n-1}\phi_j(\rho\sigma)]\Big|_{\rho=0}u(y)+\int_0^\infty\psi_j(\rho\sigma)\\
&\quad\times u(y-\rho\sigma)d\rho.
\end{aligned}
$$

因此

$$
E_ju(y)=K_ju(y)\int_S\left(\frac{\partial}{\partial\rho}\right)^{m-1}[\rho^{n-1}\phi_j(\rho\sigma)]\Big|_{\rho=0}d\sigma.
$$

因为$(\partial/\partial\rho)^{m-1}[\rho^{n-1}\phi_j(\rho\sigma)]$在 0 附近是 0 阶齐次的，如果 ϕ_j 不恒等于零,以上积分就不等于零.因此可以选K_j使对于 $y\in U_j\cap\Omega$, 和所有的 $u\in C^\infty(\Omega)$, $E_ju(y)=u(y)$. 因为 $C^\infty(\Omega)$在 $W^{m,p}(\Omega)$内稠密, 对于所有的 $u\in W^{m,p}(\Omega)$在$U_j\cap\Omega$ 上 $E_ju(y)=u(y)$a. e.. 剩下还要说明(34)式成立, 即对任意的 α, $|\alpha|\leqslant m$,

$$
\|D^\alpha E_ju\|_{0,p,\mathbf{R}^n}\leqslant K_\alpha\|u\|_{m,p,\Omega}.
$$

(35)中最后一个积分形如 $\theta_j*u(y)$, 其中 $\theta_j(x)=\psi_j(x)|x|^{1-n}$ 因为 $\theta_j\in L^1(\mathbf{R}^n)$并且有紧支集, 通过Young 定理 4.30 和用光滑函数对 u 作适当的逼近得:

$$
\begin{aligned}
\|D^\alpha(\theta_j*u)\|_{0,p,\mathbf{R}^n}&=\|\theta_j*(D^\alpha u)\|_{0,p,\mathbf{R}^n}\\
&\leqslant\|\theta_j\|_{0,1,\mathbf{R}^n}\|D^\alpha u\|_{0,p,\Omega}.
\end{aligned}
$$

剩下还要说明(35) 中第一个积分定义了从 $W^{m,p}(\Omega)$到 $W^{m,p}(\mathbf{R}^n)$的有界映射. 因为 $(\partial/\partial\rho)^m=\sum\limits_{|\alpha|=m}(m!/\alpha!)\sigma^\alpha D^\alpha$ 我们得

$$
\begin{aligned}
&\int_S\int_0^\infty\phi_j(\rho\sigma)\rho^{n-1}\left(\frac{\partial}{\partial\rho}\right)^{m}u(y-\rho\sigma)d\rho d\sigma\\
&\quad=\sum_{|\alpha|=m}\frac{m!}{\alpha!}\int_{\mathbf{R}^n}\phi_j(x)D_x^\alpha u(y-x)\sigma^\alpha dx
\end{aligned}
$$

$$= \sum_{|\alpha|=m} \xi_\alpha * D^\alpha u,$$

其中 $\xi_\alpha=(-1)^{|\alpha|}(m!/\alpha!)\sigma^\alpha\phi_j$ 在 $B_\varepsilon\sim\{0\}$ 上是 $m-n$ 阶齐次的并且属于 $C^\infty(\mathbf{R}^n-\{0\})$. 现在显然只要说明对任意的 β, $|\beta|\leqslant m$,

$$\|D^\beta(\xi_\alpha * v)\|_{0,p,\mathbf{R}^n} \leqslant K_{\alpha,\beta}\|v\|_{0,p,\Omega}. \tag{36}$$

如果 $|\beta|\leqslant m-1$, 则 $D^\beta\xi_\alpha$ 在 $B_\varepsilon\sim\{0\}$ 上是不超过 $1-n$ 阶齐次的. 因此属于 $L^1(\mathbf{R}^n)$. 不等式(36)由 Young 定理的论证可得. 因此我们只要考虑 $|\beta|=m$ 的情形, 在此情形我们对某个 γ, $|\gamma|=m-1$ 和某个 i, $1\leqslant i\leqslant n$, 记 $D^\beta=(\partial/\partial x_i)D^\gamma$. 暂时假设 $v\in C_0^\infty(\Omega)$. 则我们可以记

$$D^\beta(\xi_\alpha * v)(x)=[D^\gamma\xi_\alpha] * \left[\left(\frac{\partial}{\partial x_i}\right)v\right](x)$$

$$=\int_{\mathbf{R}^n} D_i v(x-y)D^\gamma\xi_\alpha(y)dy$$

$$=\lim_{\delta\to 0+}\int_{\mathbf{R}^n\sim B_\delta} D_i v(x-y)D^\gamma\xi_\alpha(y)dy.$$

我们现在在最后的积分中分部积分解脱作用在 v 上的 D_i 并在积分中得到 $D^\beta\xi_\alpha$. 积分项是 $v(x-\cdot)$ 与一个在零点附近 $1-n$ 阶齐次的函数的乘积在球 S_δ 上的面积分. 当$\delta\to 0+$这个面积分必须对某个常数K趋于 $Kv(x)$. 注意 $D_i v(x-y)=-(\partial/\partial y_i)v(x-y)$, 我们现在有

$$D^\beta(\xi_\alpha * v)(x)$$

$$=\lim_{\delta\to 0+}\int_{\mathbf{R}^n} v(x-y)D^\beta\xi_\alpha(y)dy+Kv(x).$$

现在 $D^\beta\xi_\alpha$ 在原点附近是$(-n)$阶齐次的, 因此由定理 4.31 的最后的结论, $D^\beta\cdot\xi_\alpha$ 满足该定理关于奇异核 g 的所有条件. 由于 $p>1$, 对于任意的 $v\in L^p(\Omega)$我们有(在Ω之外看成恒等于零)

$$\|D^\beta\xi_\alpha * v\|_{0,p,\mathbf{R}^n} \leqslant K_{\alpha,\beta}\|v\|_{0,p,\Omega}.$$

证毕. ▌

第五章　$W^{m,p}(\Omega)$的嵌入

Sobolev 嵌入定理

5.1　主要是由于Sobolev空间的嵌入性质才使得Sobolev空间在分析中，特别是在微分算子和积分算子的研究中如此的有用。空间$W^{m,p}(\Omega)$的最重要的嵌入性质通常总括在一个叫做Sobolev嵌入定理的定理之中。核心的结果应归功于Sobolev[63]，但是我们的讲法(定理5.4)包括了其他一些作者的改进，尤其是Morrey[47]和Gagliardo[24]。

对于$\mathbf{R}^n$中具有锥性质的区域Ω，大多数嵌入结果是成立的，但是另一些嵌入结果不需要这种限制；然而某些嵌入定理要求强的局部Lipschitz性质。特别是如果Ω只有锥性质，那么就可能不存在$W^{m,p}(\Omega)$到Ω上的一致连续函数空间的嵌入，通过对3.17节第二段中给出的例子的研究就能够看到这一点。

5.2　Sobolev嵌入定理断言$W^{m,p}(\Omega)$到下列类型的空间的嵌入是存在的，这些空间是：

(i) $W^{j,q}(\Omega)$，$j\leqslant m$，特别是$L^q(\Omega)$。

(ii) $C_B^j(\Omega)=\{u\in C^j(\Omega): D^\alpha u$ 在Ω上有界，$|\alpha|\leqslant j\}$，这个空间比$C^j(\overline{\Omega})$大，其元素不必在Ω上一致连续。然而，在范数

$$\|u;C_B^j(\Omega)\|=\max_{0\leqslant|\alpha|\leqslant j}\sup_{x\in\Omega}|D^\alpha u(x)|$$

下，$C_B^j(\Omega)$是一个Banach空间。

(iii) $C^{j,\lambda}(\overline{\Omega})$(见1.27节)特别是$C^j(\overline{\Omega})$。

(iv) $W^{j,q}(\Omega^k)$，特别是$L^q(\Omega^k)$。这里Ω^k表示Ω和$\mathbf{R}^n$中的一个k维平面的交集，看作$\mathbf{R}^k$中的一个区域。

因为，严格地说，$W^{m,p}(\Omega)$ 中的元素不是在 Ω 上处处有定义的函数，而是在 Ω 去掉一个零测集上有定义且互相相等的那种函数构成的等价类，故我们必须说清楚 $W^{m,p}(\Omega)$ 嵌入到(ii)——(iv)类型的空间中去是什么意思. 在把 $W^{m,p}(\Omega)$ 嵌入到(ii)或(iii)的情形，嵌入的意思就是指"等价类"$u\in W^{m,p}(\Omega)$ 中应该包含有一个函数，它属于要嵌入的目标即连续函数空间，而且在连续函数空间中为一个常数乘以 $\|u\|_{m,p,\Omega}$ 所界定. 因此，举例说，$W^{m,p}(\Omega)\to C^j(\overline{\Omega})$ 的意思就是说，每个 $u\in W^{m,p}(\Omega)$，当我们把它看作一个函数时，能够在 Ω 中的一个零测集上重新定义它的值使得修改以后的函数 $\tilde{u}$[它在 $W^{m,p}(\Omega)$ 中等于 u]属于 $C^j(\overline{\Omega})$，而且满足 $\|\tilde{u}; C^j(\overline{\Omega})\|\leqslant K\|u\|_{m,p,\Omega}$，$K$ 与 u 无关.

在解释嵌入 $W^{m,p}(\Omega)\to W^{j,q}(\Omega^k)$ 时，其中 $k<n$，就更要小心了. 根据定理 3.16，每个元素 $u\in W^{m,p}(\Omega)$ 是 $C^\infty(\Omega)$ 中一个函数序列$\{u_n\}$在 $W^{m,p}(\Omega)$ 中的一个极限，函数 u_n 在 Ω^k 上具有属于 $C^\infty(\Omega^k)$ 的迹. 上述嵌入意味着这些迹在 $W^{j,q}(\Omega^k)$ 收敛到一个函数 $\tilde{u}$，它满足 $\|\tilde{u}\|_{j,q,\Omega^k}\leqslant K\|u\|_{m,p,\Omega}$，$K$ 与 u 无关.

我们注意到一个有趣的事实(虽然以后我们用不着它)，即嵌入 $W^{m,p}(\Omega)\to W^{j,q}(\Omega)$ 等价于简单的包含关系 $W^{m,p}(\Omega)\subset W^{j,q}(\Omega)$，无疑嵌入关系推出包含关系. 为了证明包含关系也能推出嵌入关系，假定 $W^{m,p}(\Omega)\subset W^{j,q}(\Omega)$ 而且设 I 是由 $Iu=u$ 定义的 $W^{m,p}(\Omega)$ 到 $W^{j,q}(\Omega)$ 中的线性算子. 如果在 $W^{m,p}(\Omega)$ 中(因此在 $L^p(\Omega)$ 中)$u_n\to u$ 而且在 $W^{j,q}(\Omega)$ 中(因此在 $L^q(\Omega)$ 中)$Iu_n\to v$，那么如果必要时通过取子序列的方法，由 2.11 节的推论，我们在 Ω 中有 $u_n(x)\to u(x)$ a.e.，而且在 Ω 中 $u_n(x)=Iu_n(x)\to v(x)$ a.e.. 于是在 Ω 中 $u(x)=v(x)$ a.e.，即 $Iu=v$，而且，由泛函分析中的闭图定理，I 是连续的.

5.3 设 $\mathbf{R}^n$ 中的区域 Ω 具有由一个有限锥 C 所规定的锥性质(见

4.3 节)，C 可以看作是和 C 有同样顶点的无限锥 C^* 和中心也在这个顶点的球 B 的交集. 把 B 的半径称为 C 的高. 把 C^* 和球心在 C 的顶点的单位球所交的曲面的面积[$(n-1)$ 维测度]叫做 C 的开度(opening). 这些几何参数显然是 C 的刚体变换下的不变量.

在断言对于具有锥性质的区域 Ω 的形如

$$W^{m,p}(\Omega) \to X \tag{1}$$

的嵌入成立时,(其中 X 是由定义在 Ω 上的函数组成的一个Banach空间), 意思是说, (1)的嵌入常数, 亦即使不等式

$$\|u;X\| \leqslant K\|u\|_{m,p,\Omega}$$

对所有的 $u \in W^{m,p}(\Omega)$ 均成立的常数，可以选得只通过维数 n 和一些不变量(例如锥 C 的在刚体运动下不变的那些参数)来和 Ω 发生依赖关系.

5.4 定理 (Sobolev 嵌入定理) 设 Ω 是 $\mathbf{R}^n$ 中一个区域，又设 Ω^k 是 Ω 和 $\mathbf{R}^n$ 中的一个 k 维平面相交得到的 k 维区域，$1 \leqslant k \leqslant n$. (因此有 $\Omega^n \equiv \Omega$.) 设 j 和 m 是非负整数, 还设 p 满足 $1 \leqslant p < \infty$.

第 I 部分 如果 Ω 具有锥性质, 那么存在下列嵌入:

情形 A 假定 $mp < n$ 而且 $n - mp < k \leqslant n$. 则

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega^k), \qquad p \leqslant q \leqslant \frac{kp}{n-mp}, \tag{2}$$

特别有,

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega), \qquad p \leqslant q \leqslant \frac{np}{n-mp}, \tag{3}$$

或者

$$W^{m,p}(\Omega) \to L^q(\Omega), \qquad p \leqslant q \leqslant \frac{np}{n-mp}. \tag{4}$$

而且, 如果 $p=1$, 因而 $m<n$, 对于 $k=n-m$ 嵌入(2)也存在.

情形 B 假定 $mp=n$. 那么对于每个 k, $1 \leqslant k \leqslant n$,

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega^k), \qquad p \leqslant q < \infty, \tag{5}$$

所以特别有

$$W^{m,p}(\Omega)\to L^q(\Omega),\qquad p\leqslant q<\infty. \tag{6}$$

而且，若 $p=1$，因而 $m=n$，则对于 $q=\infty$ 嵌入(5)和(6)同样存在；事实上，

$$W^{j+n,1}(\Omega)\to C_B^j(\Omega). \tag{7}$$

情形 C 假定 $mp>n$，那么

$$W^{j+m,p}(\Omega)\to C_B^j(\Omega). \tag{8}$$

第 II 部分 如果 Ω 具有强局部 Lipschitz 性质，那么能够把第 I 部分情形 C 改善如下：

情形 C′ 假定 $mp>n>(m-1)p$. 则

$$W^{j+m,p}(\Omega)\to C^{j,\lambda}(\overline{\Omega}),\qquad 0<\lambda\leqslant m-\frac{n}{p}. \tag{9}$$

情形 C″ 假定 $n=(m-1)p$. 则

$$W^{j+m,p}(\Omega)\to C^{j,\lambda}(\overline{\Omega}),\quad 0<\lambda<1. \tag{10}$$

还有，如果 $n=m-1$ 和 $p=1$，则对 $\lambda=1$ (10)同样成立.

第 III 部分 假如把接受嵌入的 W-空间都替换成相应的 W_0-空间，那么第 I 和 II 部分的一切结论对任意区域都成立.

5.5 附注 (1) 嵌入(2)—(8)本质上应归功于 Sobolev[62, 63]，但是他的证明中并未包括(2)中 $q=kp/(n-mp)$，或(3)与(4)中 $q=np/(n-mp)$ 的情形. 嵌入 (9) 和(10) 可以在 Morrey 的工作[47]中找到其原型.

(2) 包含着低维平面上函数迹的类型(2)和(5)的嵌入能够用一种合理的方式推广到使它能应用到更一般的光滑流形的迹上去. 例如，见定理 5.22.

(3) 定理的第 III 部分是第 I 和 II 部分应用到 $\mathbf{R}^n$ 上去的直接的推论，因为按照引理 3.22，把函数在 Ω 外延拓为零的算子将 $W_0^{m,p}(\Omega)$ 等距地映射到 $W^{m,p}(\mathbf{R}^n)$ 中.

(4) 假定对于 $\Omega=\mathbf{R}^n$ 嵌入定理的所有结论都已得到证明.

那么就可推出，对于任何满足 4.32 节 Calderon 延拓定理要求的 Ω，这些嵌入定理也都成立. 例如，如果 $W^{m,p}(\mathbf{R}^n)\to L^q(\mathbf{R}^n)$，又如果 E 是 Ω 的一个 (m,p)-延拓算子，则对于任何 $u\in W^{m,p}(\Omega)$ 我们有

$$\|u\|_{0,q,\Omega}\leqslant\|Eu\|_{0,q,\mathbf{R}^n}\leqslant K_1\|Eu\|_{m,p,\mathbf{R}^n}\leqslant K_1K_2\|u\|_{m,p,\Omega},$$

K_1 和 K_2 与 u 无关. 然而我们将不用这种延拓的论证来证明嵌入定理.

(5) 只要对特殊情形 $j=0$ 证明(2),(3),(5),(7)—(10)中的每一个嵌入关系就够了. 例如，如果已经证明 $W^{m,p}(\Omega)\to L^q(\Omega)$，那么对任何 $u\in W^{j+m,p}(\Omega)$，对于 $|\alpha|\leqslant j$ 我们就有 $D^\alpha u\in W^{m,p}(\Omega)$，由此 $D^\alpha u\in L^q(\Omega)$，因此 $u\in W^{j,q}(\Omega)$；而且

$$\begin{aligned}\|u\|_{j,q}&=\left(\sum_{|\alpha|\leqslant j}\|D^\alpha u\|_{0,q}^q\right)^{\frac{1}{q}}\\&\leqslant K_1\left(\sum_{|\alpha|\leqslant j}\|D^\alpha u\|_{m,p}^p\right)^{\frac{1}{p}}\leqslant K_2\|u\|_{j+m,p}.\end{aligned}$$

因此，在证明中我们总是限定 $j=0$.

(6) 当 Ω^k (或 Ω)具有有限体积时，由定理 2.8 得到除去定理中所断言的 q 值外对于 $1\leqslant q<p$ 嵌入(2)—(6)成立. 以后将证明(6.38节)，除非 Ω 具有有限体积，不可能有形如 $W^{m,p}(\Omega)\to L^q(\Omega)$ 的嵌入，其中 $q<p$.

嵌入定理的证明

5.6 这里给出的证明是属于 Gagliardo[24]的. 虽然证明稍微长了一点，但是所包含的技巧是十分初等的，把比简单微积分稍多一点的知识和灵活地应用 Hölder 不等式结合起来就是这种技巧的基础. 而且，Gagliardo 的证明以最大可能的一般性建立了嵌入定理，对于某些不具有锥性质的区域也能够推广这个证明去得到

嵌入结果(见定理 5.35—5.37).

证明是通过一系列辅助引理来实现的. 在每个引理的证明中出现的常数 K_1, K_2, …都可以依赖于在引理的叙述中所提到的如 K 那样的参数.

5.7 引理 设

$$R=\{x\in\mathbf{R}^n: a_i<x_i<b_i;\ 1\leqslant i\leqslant n\}$$

与

$$R'=\{x'=(x_1,\ \cdots,\ x_{n-1})\in\mathbf{R}^{n-1}:\ a_i<x_i<b_i;\ 1\leqslant i\leqslant n-1\}$$

分别是 $\mathbf{R}^n$ 和 $\mathbf{R}^{n-1}$ 中有界的开长方体. 如果 $a_n<\zeta<b_n$ 而且 $p\geqslant 1$, 则对于一切 $u\in C^\infty(R)\cap W^{1,p}(R)$, 有

$$\|u(\cdot,\zeta)\|_{0,p,R'}\leqslant K\|u\|_{1,p,R}, \tag{11}$$

其中 $K=K(p,b_n-a_n)$. 因此迹映射 $u\to u(\cdot,\zeta)$ 就扩张成 $W^{1,p}(R)$ 到 $L^p(R_\zeta^{n-1})$ 中的一个嵌入, 其中 $R_\zeta^{n-1}=R\cap\{x\in R^n:\ x_n=\zeta\}$.

证明 由定理3.18, $C^\infty(\bar{R})$ 在 $W^{1,p}(R)$ 中稠密, 所以可假定 $u\in C^\infty(\bar{R})$. 因此 $\int_{R'}|u(x',\cdot)|^p dx'$ 属于 $C^\infty([a_n,b_n])$, 由积分中值定理, 对某个 $\sigma\in[a_n,b_n]$ 我们有

$$\|u\|_{0,p,R}^p=\int_{a_n}^{b_n}\left(\int_{R'}|u(x',x_n)|^p dx'\right)dx_n$$

$$=(b_n-a_n)\int_{R'}|u(x',\sigma)|^p dx'$$

现在由 Hölder 不等式

$$|u(x',\zeta)|^p=\left|u(x',\sigma)+\int_\sigma^\zeta D_n u(x',t)\,dt\right|^p$$

$$\leqslant 2^{p-1}\left[|u(x',\sigma)|^p+|\zeta-\sigma|^{p-1}\int_\sigma^\zeta|D_n u(x',t)|^p dt\right]$$

在 R' 上积分就导至

$$\|u(\cdot,\zeta)\|_{0,p,R'}^p\leqslant 2^{p-1}[\|u(\cdot,\sigma)\|_{0,p,R'}^p+(b_n-a_n)^{p-1}\|D_n u\|_{0,p,R}^p]$$

$$\leqslant 2^{p-1}[(b_n-a_n)^{-1}\|u\|_{0,p,R}^p+(b_n-a_n)^{p-1}\|D_n u\|_{0,p,R}^p],$$

由此就得到(11)，其中常数 $K=[2^{p-1}\max((b_n-a_n)^{-1}, (b_n-a_n)^{p-1})]^{1/p}$. 我们注意到$K$连续依赖于 b_n-a_n，但当 b_n-a_n 趋于零或无穷时 K 可以趋于无穷大. ▌

5.8 引理 设R如上面引理中所述，则

$$W^{n,1}(R)\longrightarrow C(\bar{R}).$$

嵌入常数只依赖于 n 和R的大小.

证明 设 x 是R中的任意一点，又设 R' 如上述引理所述. 如果 $u\in C^{\infty}(\bar{R})$且$|\alpha|\leqslant n-1$，则由引理 5.7，有

$$\|D^{\alpha}u(\cdot,x_n)\|_{0,1,R'}\leqslant K_1\|D^{\alpha}u\|_{1,1,R}.$$

因此

$$\|u(\cdot,x_n)\|_{n-1,1,R'}\leqslant K_2\|u\|_{n,1,R},$$

K_2依赖于 b_n-a_n，在更低维的长方体上逐次地重复这种论证就导至

$$\|u(\cdot,x_2,\cdots,x_n)\|_{1,1,(a_1,b_1)}\leqslant K_3\|u\|_{n,1,R},$$

K_3 依赖于 b_j-a_j, $2\leqslant j\leqslant n$. 由积分中值定理，存在 $\sigma\in[a_1,b_1]$使得.

$$\|u(\cdot,x_2,\cdots,x_n)\|_{0,1,(a_1,b_1)}=(b_1-a_1)\,|u(\sigma,x_2,\cdots,x_n)|.$$

因此

$$\begin{aligned}|u(x)|&\leqslant|u(\sigma,x_2,\cdots,x_n)|+\int_{\sigma}^{x_1}|D_1u(t,x_2,\cdots,x_n)|\,dt\\&\leqslant[1/(b_1-a_1)]\|u(\cdot,x_2,\cdots,x_n)\|_{0,1,(a_1,b_1)}\\&\quad+\|D_1u(\cdot,x_2,\cdots,x_n)\|_{0,1,(a_1,b_1)}\leqslant K\|u\|_{n,1,R}.\end{aligned}\tag{12}$$

现在假定 $u\in W^{n,1}(R)$. 由定理3.18，u 是 $C^{\infty}(\bar{R})$里的函数序列在 $W^{n,1}(R)$中的极限. 从(12)得出这个序列在$\bar{R}$上一致收敛到一个函数 $\tilde{u}\in C(\bar{R})$. 由于在 R中 $\tilde{u}(x)=u(x)$a.e., 引理得证. ▌

现在把我们的注意力转向更一般的区域. 下面Gagliardo的引理是他证明嵌入定理的基础，这个引理本质上是属于组合性质的.

5.9 引理 设Ω是$\mathbf{R}^n$中一个区域，$n\geqslant 2$. 设k是满足$1\leqslant k\leqslant n$的整数，又设$\kappa=(\kappa_1,\kappa_2,\cdots,\kappa_k)$表示满足$1\leqslant\kappa_1<\kappa_2<\cdots<\kappa_k\leqslant n$的$k$重整数. 设$S$是由全体$\binom{n}{k}$个这样的$k$重数构成的集合. 还给定$x\in\mathbf{R}^n$, 设$x_\kappa$表示点$(x_{\kappa_1},\cdots,x_{\kappa_k})\in\mathbf{R}^k$; $dx_\kappa=dx_{\kappa_1}\cdots dx_{\kappa_k}$.

对于给定的$\kappa\in S$, 设E_κ是由相应于x_κ的分量的坐标轴张起来的$\mathbf{R}^n$中的k维平面:

$$E_\kappa=\{x\in\mathbf{R}^n: x_i=0 \text{ 如果 } i\notin\kappa\};$$

对于任何集合$G\subset\mathbf{R}^n$, 设G_κ是G在E_κ上的投影; 特别

$$\Omega_\kappa=\{x\in E_\kappa: \exists y\in\Omega \text{ 使 } y_\kappa=x_\kappa\}.$$

设F_κ是依赖于x_κ的k个分量的函数而且属于$L^\lambda(\Omega_\kappa)$, 其中$\lambda=\binom{n-1}{k-1}$. 那么由

$$F(x)=\prod_{\kappa\in S}F_\kappa(x_\kappa)$$

给出的定义在Ω上的函数F属于$L^1(\Omega)$, 而且$\|F\|_{1,\Omega}\leqslant\prod_{\kappa\in S}\|F_\kappa\|_{\lambda,\Omega_\kappa}$, 即

$$\left[\int_\Omega|F(x)|dx\right]^\lambda\leqslant\prod_{\kappa\in S}\int_{\Omega_\kappa}|F_\kappa(x_\kappa)|^\lambda dx_\kappa. \tag{13}$$

证明 对于$\kappa\in S$和$\xi_\kappa\in\mathbf{R}^k$, 设$\Omega(\xi_\kappa)$表示$\Omega$和平面$x_\kappa=\xi_\kappa$相交的$k$维平截面:

$$\Omega(\xi_\kappa)=\{x\in\Omega: x_\kappa=\xi_\kappa\}.$$

我们对n做归纳法证明(13)式，因此首先考虑$n=2$的情形. 因为对于任何n, $k=n$时(13)式总是对的，所以可以假定$k=1$. 对于$n=2, k=1$有$\lambda=1$, S只有两个元素$\kappa=1$和$\kappa=2$. 因此

$$\int_\Omega|F_1(x_1)F_2(x_2)|dx_1dx_2=\int_{\Omega_1}dx_1\int_{\Omega(x_1)}|F_1(x_1)F_2(x_2)|dx_2$$

$$=\int_{\Omega_1}|F_1(x_1)|dx_1\int_{(\Omega(x_1))_2}|F_2(x_2)|dx_2$$

$$\leqslant\int_{\Omega_1}|F_1(x_1)|dx_1\int_{\Omega_2}|F_2(x_2)|dx_2,$$

因为对任何x_1显然有$(\Omega(x_1))_2\subset\Omega_2$. 这就是 $n=2,k=1$ 时的(13)式. (进行类似的计算就可得到任意 n 和 $k=1$ 时的(13)式.)

现在假定对 $n=N-1$(13)式已经证明. 我们来研究 $n=N$ 的情形, 如前所述, 可以假定 $2\leqslant k\leqslant N-1$. 于是$\lambda=\binom{N-1}{k-1}$. 设 $\mu=\binom{N-2}{k-1}$ 而 $\nu=\binom{N-2}{k-2}$. (13)式左端的被积函数是$\binom{N}{k}$个因子$|F_\kappa|$的积, 而每个 F_κ 属于相应的空间 $L^\lambda(\Omega_\kappa)$. 实际上这些因子中的$\binom{N-1}{k}$个因子是和 x_N 无关的. 把这些因子叫做是与 $\kappa\in A\subset S$ 相应的因子. 对$(N-1)$维区域 $\Omega(x_N)$ 应用归纳法假设, 而且注意到$(\Omega(x_N))_\kappa\subset\Omega_\kappa$ 就得到

$$\begin{aligned}&\int_{\Omega(x_N)}\prod_{\kappa\in A}|F_\kappa(x_\kappa)|^{\lambda/\mu}dx_1\cdots dx_{N-1}\\&\leqslant\prod_{\kappa\in A}\left[\int_{(\Omega(x_N))_\kappa}|F_\kappa(x_\kappa)|^\lambda dx_\kappa\right]^{1/\mu}\\&\leqslant\prod_{\kappa\in A}\left[\int_{\Omega_\kappa}|F_\kappa(x_\kappa)|^\lambda dx_\kappa\right]^{1/\mu}.\end{aligned}\tag{14}$$

剩下$\binom{N}{k}-\binom{N-1}{k}=\lambda$ 个因子$|F_\kappa|$依赖于 x_N, 所以当限于 $\Omega(x_N)$ 上时, 只依赖于 $k-1$ 个变量. 在 $\Omega(x_N)$上再次应用归纳法假设, 但这一次用 $k-1$ 代替 k, 我们得到

$$\begin{aligned}&\int_{\Omega(x_N)}\prod_{\kappa\in S\sim A}|F_\kappa(x_\kappa)|^{\lambda/\nu}dx_1\cdots dx_{N-1}\\&\leqslant\prod_{\kappa\in S\sim A}\left[\int_{(\Omega(x_N))_\kappa}|F_\kappa(x_\kappa)|^\lambda dx_{\kappa_1}\cdots dx_{\kappa_{k-1}}\right]^{1/\nu}.\end{aligned}\tag{15}$$

现在 $\mu+\nu=\lambda$ 所以由 Hölder 不等式和(14), (15),

$$\int_{\Omega(x_N)}\prod_{\kappa\in S}|F_\kappa(x_\kappa)|dx_1\cdots dx_{N-1}$$

$$\leqslant\prod_{\kappa\in A}\left[\int_{\Omega\kappa}|F_\kappa(x_\kappa)|^\lambda dx_\kappa\right]^{1/\lambda}$$

$$\times\prod_{\kappa\in S\sim A}\left[\int_{(\Omega(x_N))_\kappa}|F_\kappa(x_\kappa)|^\lambda dx_{\kappa_1}\cdots dx_{\kappa_{k-1}}\right]^{1/\lambda}. \tag{16}$$

由于 $S\sim A$ 包含 λ 个元素，利用(多个函数形式的)Hölder 不等式得到

$$\int_{\Omega_N}\prod_{\kappa\in S\sim A}\left[\int_{(\Omega(x_N))_\kappa}|F_\kappa(x_\kappa)|^\lambda dx_{\kappa_1}\cdots dx_{\kappa_{k-1}}\right]^{1/\lambda}dx_N$$

$$\leqslant\prod_{\kappa\in S\sim A}\left[\int_{\Omega_N}\int_{(\Omega(x_N))_\kappa}|F_\kappa(x_\kappa)|^\lambda dx_\kappa\right]^{1/\lambda}$$

$$\leqslant\prod_{\kappa\in S\sim A}\left[\int_{\Omega\kappa}|F_\kappa(x_\kappa)|^\lambda dx_\kappa\right]^{1/\lambda}. \tag{17}$$

把(17)式插入到(16)式中去就得到

$$\int_\Omega\prod_{\kappa\in S}|F_\kappa(x_\kappa)|dx=\int_{\Omega_N}dx_N\int_{\Omega(x_N)}\prod_{\kappa\in S}|F_\kappa(x_\kappa)|dx_1\cdots dx_{N-1}$$

$$\leqslant\prod_{\kappa\in S}\left[\int_{\Omega_\kappa}|F_\kappa(x_\kappa)|^\lambda dx_\kappa\right]^{1/\lambda}.$$

这就完成了归纳法并证明了(13). ▌

5.10 引理 设Ω是$\mathbf{R}^n$中具有锥性质的有界区域. 如果 $1\leqslant p<n$, 则 $W^{1,p}(\Omega)\to L^q(\Omega)$，其中 $q=np/(n-p)$. 嵌入常数可以选得只和 m, p, n 和决定Ω的锥性质的锥C有关.

证明 我们必须证明, 对于任何 $u\in W^{1,p}(\Omega)$

$$\|u\|_{0,q,\Omega}\leqslant K\|u\|_{1,p,\Omega}, \tag{18}$$

$K=K(m,p,n,C)$. 由定理 4.8 知，Ω 可以表为有限个子区域的并集，每个子区域都有强局部 Lipschitz 性质（因此具有线段性质)，而且每个子区域都是一个相应的平行多面体的平移集的并集. 回顾一下定理 4.8 的证明，就知道子区域的数目和相应的平行多面体的大小依赖于 n 和 C，所以只对这些子区域中的一个证明(18)就行了.

通过定理 3.35 和一个合适的非奇异线性变换，实际上我们可以假定子区域所包含的平行多面体是边长为 2 个单位的立方体 Q，而且 Q 的边平行于坐标轴. 因此今后我们假定 $\Omega=\bigcup\limits_{x\in A}(x+Q)$，$A\subset\Omega$，而且 Ω 具有线段性质. 由定理 3.18，只要对 $u\in C^\infty(\overline{\Omega})$ 证实(18)就行了.

对 $x\in\Omega$，令 $w_i(x)$ 表示 Ω 和通过 x 平行于 x_i 坐标轴的直线的交. 显然 $w_i(x)$ 包含一个端点在 x 的单位长线段，比如线段 $x+te_i$，$0\leqslant t<1$，其中 e_i 是沿着 x_i 轴的单位向量.

设 $\gamma=(np-p)/(n-p)$，因而 $\gamma\geqslant 1$. 对于 $u\in C^\infty(\overline{\Omega})$，分部积分给出

$$
\begin{aligned}
&\int_0^1|u(x+(1-t)e_i)|^\gamma dt\\
&=|u(x)|^\gamma-\gamma\int_0^1 t|u(x+(1-t)e_i)|^{\gamma-1}\\
&\quad\times\frac{d}{dt}|u(x+(1-t)e_i)|dt. \qquad (19)
\end{aligned}
$$

设 $\hat{x}_i=(x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_n)$ 并令

$$F_i(\hat{x}_i)=\sup_{y\in w_i(x)}|u(y)|^{p/(n-p)}.$$

则(19)给出

$$
\begin{aligned}
|F_i(\hat{x}_i)|^{n-1}\leqslant&\int_{w_i(x)}|u(x)|^\gamma dx_i\\
&+\gamma\int_{w_i(x)}|u(x)|^{\gamma-1}|D_iu(x)|dx_i. \qquad (20)
\end{aligned}
$$

在 Ω_i（Ω 在平面 $x_i=0$ 上的投影）上积分，就导至

$$
\begin{aligned}
&\int_{\Omega_i}|F_i(\hat{x}_i)|^{n-1}d\hat{x}_i\\
&\leqslant\int_\Omega|u(x)|^\gamma dx+\gamma\int_\Omega|u(x)|^{\gamma-1}|D_iu(x)|dx.
\end{aligned}
$$

如果 $p>1$ 则 $\gamma>1$, 因为 $(\gamma-1)p'=q$, 应用 Hölder 不等式就得到

$$\|F_i\|_{0,n-1,\Omega_i}^{n-1}\leqslant\gamma\left[\int_\Omega(|u(x)|+|D_iu(x)|)^p dx\right]^{1/p}$$

$$\times\left[\int_\Omega|u(x)|^{(\gamma-1)p'}dx\right]^{1/p'}$$

$$\leqslant 2^{(p-1)/p}\gamma\|u\|_{1,p,\Omega}\|u\|_{0,q,\Omega}^{q/p'}.$$

现在把引理 5.9 用到函数 F_i 上去, $1\leqslant i\leqslant n$, 注意到 $k=n-1$ 所以这个引理的指数 λ 就是 $n-1$:

$$\|u\|_{0,q,\Omega}^q=\int_\Omega|u(x)|^{np/(n-p)}dx\leqslant\int_\Omega\prod_{i=1}^n F_i(\hat{x}_i)\,dx$$

$$\leqslant\prod_{i=1}^n\|F_i\|_{0,n-1,\Omega_i}$$

$$\leqslant(2^{(p-1)/p}\gamma\|u\|_{1,p,\Omega}\|u\|_{0,q,\Omega}^{q/p'})^{n/(n-1)}.$$

由于 $(n-1)q/n-q/p'=1$, 消去因子就得到(18)式. 因为 $u\in C^\infty(\overline{\Omega})$ 而且 Ω 是有界的, 所以 $\|u\|_{0,q,\Omega}$ 是有限的, 因此消去因子是合法的, 由于 $C^\infty(\overline{\Omega})$ 在 $W^{1,p}(\Omega)$ 中稠密, 根据连续性(18)就可以扩充到对 $W^{1,p}(\Omega)$ 的所有元素都成立. ▌

5.11 附注 设 $u\in C_0^\infty(\mathbf{R}^n)$, 又设 q, γ 如上述引理的证明中所述. 从恒等式

$$\int_0^\infty\frac{d}{dt}|u(x+te_i)|^\gamma dt=-|u(x)|^\gamma$$

得到

$$\sup_{y\in w_i(x)}|u(y)|^\gamma\leqslant\gamma\int_{-\infty}^\infty|u(x)|^{\gamma-1}|D_iu(x)|dx_i,$$

其中 $w_i(x)$ 是过 x 点平行于 x_i 轴的直线. 把这个不等式和(20)比较一下, 重做一下上述引理证明中的计算, 这时就可以得到

$$\|u\|_{0,q,\mathbf{R}^n}\leqslant K|u|_{1,p,\mathbf{R}^n}, \tag{21}$$

其中 $|\cdot|_{1,p}$ 是 4.11节中所定义的半范数, 不等式(21)叫做 Sobolev 不等式。

5.12 引理 设Ω是$\mathbf{R}^n$中具有锥性质的有界区域. 如果$mp<n$, 则对于$p\leqslant q\leqslant np/(n-mp)$, $W^{m,p}(\Omega)\to L^q(\Omega)$. 嵌入常数可以选得只依赖于$m,p,n,q$和决定$\Omega$的锥性质的锥$C$.

证明 设$q_0=np/(n-mp)$. 首先对m做归纳法来证明$W^{m,p}(\Omega)\to L^{q_0}(\Omega)$. 注意引理5.10证实了$m=1$时是对的.

所以, 假定当$n>(m-1)p$时, 对于$r=np/(n-mp+p)$有$W^{m-1,p}(\Omega)\to L^r(\Omega)$. 如果$u\in W^{m,p}(\Omega)$, $n>mp$, 则u和$D_ju(1\leqslant j\leqslant n)$属于$W^{m-1,p}(\Omega)$. 由此得到$u\in W^{1,r}(\Omega)$而且

$$\|u\|_{1,r,\Omega}\leqslant K_1\|u\|_{m,p,\Omega}.$$

因为$mp<n$, 我们有$r<n$, 所以由引理5.10我们有$W^{1,r}(\Omega)\to L^{q_0}(\Omega)$, 其中$q_0=nr/(n-r)=np/(n-mp)$而且

$$\|u\|_{0,q_0,\Omega}\leqslant K_2\|u\|_{1,r,\Omega}\leqslant K_3\|u\|_{m,p,\Omega}. \tag{22}$$

这就完成了归纳法的证明.

现在假定$p\leqslant q\leqslant q_0$, 令

$$s=(q_0-q)p/(q_0-p)\text{和 }t=p/s=(q_0-p)/(q_0-q)$$

利用Hölder不等式和(22)式就得到

$$\begin{aligned}\|u\|^q_{0,q,\Omega}&=\int_\Omega|u(x)|^s|u(x)|^{q-s}dx\\&\leqslant\left[\int_\Omega|u(x)|^{st}dx\right]^{1/t}\left[\int_\Omega|u(x)|^{(q-s)t'}dx\right]^{1/t'}\\&=\|u\|^{p/t}_{0,p,\Omega}\|u\|^{q_0/t'}_{0,q_0,\Omega}\leqslant K_3^{q_0/t'}\|u\|^q_{m,p,\Omega}.\ \blacksquare\end{aligned} \tag{23}$$

5.13 推论 如果$mp=n$, 则对于$p\leqslant q<\infty$, $W^{m,p}(\Omega)\to L^q(\Omega)$. 这里的嵌入常数还可以依赖于$\mathrm{vol}\,\Omega$.

证明 如果$q\geqslant p'=p/(p-1)$, 则$q=ns/(n-ms)$, 其中$s=pq/(p+q)$满足$1\leqslant s<p$. 由定理2.8, $W^{m,p}(\Omega)\to W^{m,s}(\Omega)$, 嵌入常数依赖于$\mathrm{vol}\,\Omega$. 因$ms<n$, 由引理5.12, $W^{m,s}(\Omega)\to L^q(\Omega)$. 如果$p\leqslant q\leqslant p'$, 则通过如(23)式所示的那种在$W^{m,p}(\Omega)\to L^p(\Omega)$和

$W^{m,p}(\Omega)\to L^{p'}(\Omega)$中间作内插就可得到所要的嵌入. ∎

对于$mp=n$和$q\geqslant p$的情形，下面的引理从引理5.12和推论5.13中去掉了Ω是有界的这一限制，从而说明嵌入常数可以不依赖于$\mathrm{vol}\Omega$.

5.14 引理 设Ω是$\mathbf{R}^n$中具有锥性质的任意区域. 如果$mp<n$，则对于$p\leqslant q\leqslant \dfrac{np}{n-mp}$, $W^{m,p}(\Omega)\to L^q(\Omega)$. 如果$mp=n$，则对于$p\leqslant q<\infty$, $W^{m,p}(\Omega)\to L^q(\Omega)$. 如果$p=1$而且$m=n$，则$W^{m,p}(\Omega)\to C_B^0(\Omega)$. 这些嵌入的嵌入常数可以依赖于$m$, p, n, q和决定Ω的锥性质的锥C.

证明 我们用边长为单位的立方体对$\mathbf{R}^n$作田字形划分. 如果$\lambda=(\lambda_1,\cdots,\lambda_n)$是一个$n$重整数，设$H_\lambda=\{x\in\mathbf{R}^n:\ \lambda_i\leqslant x_i\leqslant\lambda_{i+1},\ 1\leqslant i\leqslant n\}$. 则$\mathbf{R}^n=\bigcup\limits_{\lambda}H_\lambda$.

如定理4.8证明的第一段中所指出的，即使是具有锥性质的无界区域Ω也能表为有限个子区域的并集，比如说$\Omega=\bigcup\limits_{j=1}^{N}\Omega_j$，使得$\Omega_j=\bigcup\limits_{x\in A_j}(x+P_j)$，其中$A_j\subset\Omega$，且$P_j$是有一个顶点在原点的平行多面体. 数目$N$和平行多面体的大小依赖于$n$和决定$\Omega$的锥性质的锥$C$. 对每个$\lambda$和$1\leqslant j\leqslant N$，设

$$\Omega_{\lambda,j}=\bigcup_{x\in A_j\cap H_\lambda}(x+P_j).$$

区域$\Omega_{\lambda,j}$显然有以下性质:

(i) $\Omega=\bigcup\limits_{\lambda,j}\Omega_{\lambda,j}$;

(ii) $\Omega_{\lambda,j}$是有界的;

(iii) 存在一个只依赖于$P_1,\cdots,P_N$(因此只依赖于n和C)的有限锥C'使得每个$\Omega_{\lambda,j}$具有由C'所决定的锥性质;

(iv) 存在一个依赖于 n 和 C 的正整数 R 使得任何 $R+1$ 个区域 $\Omega_{\lambda,j}$ 的交为空集;

(v) 存在依赖于 n 和 C 的常数 K' 和 K'' 使得对每个 $\Omega_{\lambda,j}$,

$$K' \leqslant \operatorname{vol} \Omega_{\lambda,j} \leqslant K''.$$

假定 $mp<n$, 又设 $u \in W^{m,p}(\Omega)$. 如果 $p \leqslant q \leqslant np/(n-mp)$, 则由(ii), (iii)和引理 5.12, 我们有

$$\|u\|_{0,q,\Omega_{\lambda,j}} \leqslant K \|u\|_{m,p,\Omega_{\lambda,j}}, \tag{24}$$

其中 $K=K(m,p,n,q,C)$ 和 λ, j 无关. 因此由(i)和(iv), 又因为 $q \geqslant p$,

$$\|u\|_{0,q,\Omega}^q \leqslant \sum_{\lambda,j} \|u\|_{0,q,\Omega_{\lambda,j}}^q \leqslant K^q \sum_{\lambda,j} [\|u\|_{m,p,\Omega_{\lambda,j}}^p]^{q/p}$$

$$\leqslant K^q \left[\sum_{\lambda,j} \|u\|_{m,p,\Omega_{\lambda,j}}^p \right]^{q/p} \leqslant K^q R^{q/p} \|u\|_{m,p,\Omega}^q.$$

因此 $W^{m,p}(\Omega) \to L^q(\Omega)$, 其嵌入常数为 $KR^{1/p}$.

如果 $mp=n$, 由于推论 5.13 对于任何使 $p \leqslant q < \infty$ 的 q, (24)式成立, 而且由于(V), 常数 K 可以选得和 λ, j 无关, 继续做(24)式后面的证明, 就得到 $n=mp$ 时的嵌入结果.

最后, 如果 $p=1$ 而且 $m=n$, 根据引理 5.8 和通过一个非奇异的线性变换, 对任何平行多面体 $P \subset \Omega$, 我们有 $W^{n,1}(P) \to C^0(\bar{P})$, 嵌入常数只依赖于 n 和 P 的大小. 因此, 由于分解 $\Omega = \bigcup \Omega_{\lambda,j}$, 有 $W^{n,1}(\Omega) \to C_B^0(\Omega)$. ∎

现在我们已经对 $k=n$ 的情形证明了定理 5.4 第一部分的情形 A 和 B, 在通过研究迹嵌入的方法来完成其它情形($k<n$)的证明之前, 我们先来证明连续函数空间的嵌入, 即第 I 部分情形 C 和第 II 部分.

5.15 引理 设 Ω 是 $\mathbf{R}^n$ 中具有锥性质的区域. 如果 $mp>n$, 则 $W^{m,p}(\Omega) \to C_B^0(\Omega)$, 嵌入常数只依赖于 m, p, n 和决定 Ω 的锥

性质的锥 C.

证明 假定对任何 $\phi\in C^{\infty}(\Omega)$ 我们能证明

$$\sup_{x\in\Omega}|\phi(x)|\leqslant K\|\phi\|_{m,p,\Omega}, \tag{25}$$

其中 $K=K\ (m,\ p,\ n,C)$. 那么如果 $u\in W^{m,p}(\Omega)$, 则根据定理 3.16, 在 $C^{\infty}(\Omega)$ 中有一个序列 $\{\phi_n\}$ 在 $W^{m,p}(\Omega)$ 的范数下收敛到 u. 因为 $\{\phi_n\}$ 是 $W^{m,p}(\Omega)$ 中的 Cauchy 序列, (25) 式蕴涵着 $\{\phi_n\}$ 收敛到 Ω 上的一个连续函数. 因此 u 一定和 $C_B^0(\Omega)$ 中的一个函数 a. e. 相等. 所以只要证明 (25) 就够了.

首先假定 $m=1$, 因而 $p>n$. 设 $x\in\Omega$, 又设 $C_x\subset\Omega$ 是一个顶点在 x 的和 C 全等的有限锥. 设 h 是 C 的高. 设 $(r,\ \theta)$ 表示 $\mathbf{R}^n$ 中原点在 x 的球极坐标, 因而 C_x 是由 $0<r<h$, $\theta\in A$ 所规定的锥. 在这种坐标系里的体积元素表为 $r^{n-1}\omega(\theta)drd\theta$. 我们有

$$\phi(x)=\phi(0,\theta)=\phi(r,\theta)-\int_0^r\frac{d}{dt}\phi(t,\theta)\,dt,$$

由此得出, 对 $0<r<h$,

$$|\phi(x)|\leqslant|\phi(r,\theta)|+\int_0^h|\operatorname{grad}\phi(t,\theta)|\,dt.$$

用 $r^{n-1}\omega(\theta)$ 乘这个不等式的两边, 并对 r 在 $(0,h)$ 上积分以及对 θ 在 A 上积分, 得到

$$\begin{aligned}(\mathrm{vol}C_x)|\phi(x)|&\leqslant\int_{C_x}|\phi(y)|\,dy+\frac{h^n}{n}\int_{C_x}\frac{|\operatorname{grad}\phi(y)|}{|x-y|^{n-1}}dy\\&\leqslant(\mathrm{vol}C_x)^{1/p'}\|\phi\|_{0,p,C_x}\\&\quad+\frac{h^n}{n}\|\operatorname{grad}\phi\|_{0,p,C_x}\left|\int_{C_x}|x-y|^{-(n-1)p'}dy\right|^{1/p'},\end{aligned}$$

最后的不等式是由两次应用 Hölder 不等式得到的. 因为 $p>n$, 有 $(n-1)(1-p')>-1$, 所以

$$\int_{C_x}|x-y|^{-(n-1)p'}dy=\int_A\omega(\theta)d\theta\int_0^h r^{(n-1)(1-p')}dr<\infty.$$

因此

$$|\phi(x)|\leqslant K\|\phi\|_{1,p,C_x}\leqslant K\|\phi\|_{1,p,\Omega},$$

$K=K(m,p,n,C_x)=K(m,p,n,C)$. 因此对 $m=1$ 的情形证明了(25)式.

如果 $m>1$ 但 $p>n$, 我们仍有

$$|\phi(x)|\leqslant K\|\phi\|_{1,p,C_x}\leqslant K\|\phi\|_{m,p,C_x}\leqslant K\|\phi\|_{m,p,\Omega}.$$

如果 $p\leqslant n<mp$, 则存在一个满足 $1\leqslant j\leqslant m-1$ 的整数 j 使得 $jp\leqslant n<(j+1)p$. 如果 $jp<n$, 令 $r=np/(n-jp)$; 如果 $jp=n$, 选 $r>\max(n,p)$. 无论那种情形, 根据上面已经证明的结果和引理 5.14 我们有

$$\begin{aligned}|\phi(x)|&\leqslant K_1\|\phi\|_{1,r,C_x}\leqslant K_1\|\phi\|_{m-j,r,C_x}\\&\leqslant K\|\phi\|_{m,p,C_x}\leqslant K\|\phi\|_{m,p,\Omega}.\end{aligned}$$

常数只依赖于 m,p,n 和 C. 这就完成了证明. ▌

5.16 推论 如果 $mp>n$, 则对于 $p\leqslant q\leqslant\infty$, $W^{m,p}(\Omega)\to L^q(\Omega)$. 嵌入常数只依赖于 m,p,n,q 和锥 C.

证明 我们已经证明对所有的 $u\in W^{m,p}(\Omega)$

$$\|u\|_{0,\infty,\Omega}=\operatorname*{ess\,sup}_{x\in\Omega}|u(x)|\leqslant K\|u\|_{m,p,\Omega}.$$

如果 $p\leqslant q<\infty$, 我们有

$$\begin{aligned}\|u\|_{0,q,\Omega}^q&=\int_\Omega|u(x)|^p|u(x)|^{q-p}dx\\&\leqslant K^{q-p}\|u\|_{m,p,\Omega}^{q-p}\|u\|_{0,p,\Omega}^p\leqslant K^{q-p}\|u\|_{m,p,\Omega}^q.\end{aligned}$$ ▌

5.17 引理 设 Ω 是 $\mathbf{R}^n$ 中具有强局部 Lipschitz 性质的区域, 又假定 $mp>n\geqslant(m-1)p$. 那么对于

(i) $0<\lambda\leqslant m-n/p$(若 $n>(m-1)p$), 或

(ii) $0<\lambda<1$(若 $n=(m-1)p$), 或

(iii) $0<\lambda\leqslant1$(若 $p=1, n=m-1$),

有 $W^{m,p}(\Omega)\to C^{0,\lambda}(\overline{\Omega})$. 特别有 $W^{m,p}(\Omega)\to C^0(\overline{\Omega})$. 嵌入常数依赖于 m,p,n 和描述 Ω 的强局部 Lipschitz 性质中规定的参数 δ, M

(见 4.5 节).

证明 设 $u\in W^{m,p}(\Omega)$. 强局部 Lipschitz 性质蕴涵着锥性质，所以根据引理 5.15, 我们可以假定 u 在 Ω 上连续而且满足

$$\sup_{x\in\Omega}|u(x)|\leqslant K_1\|u\|_{m,p,\Omega}. \tag{26}$$

所以只要进一步证明对适当的 λ

$$\sup_{\substack{x,y\in\Omega\\ x\neq y}}\frac{|u(x)-u(y)|}{|x-y|^{\lambda}}\leqslant K_2\|u\|_{m,p,\Omega} \tag{27}$$

就行了.

因为 $mp>n\geqslant(m-1)p$, 根据引理 5.14, 有 $W^{m,p}(\Omega)\to W^{1,r}(\Omega)$, 其中

(i) 如果 $n>(m-1)p$, 则 $r=np/(n-mp+p)$ 且 $1-(n/r)=m-(n/p)$, 或者

(ii) 如果 $n=(m-1)p$, 则取 r 使得 $p<r<\infty$ 而且 $0<1-(n/r)<1$, 或者

(iii) 如果 $p=1$ 且 $n=m-1$, 则取 $r=\infty$, $1-(n/r)=m-(n/p)=1$,

所以只要对 $m=1$ 证明 (27) 就够了; 即, 我们要证明如果 $n<p\leqslant\infty$ 而且 $0<\lambda\leqslant1-(n/p)$, 则

$$\sup_{\substack{x,y\in\Omega\\ x\neq y}}\frac{|u(x)-u(y)|}{|x-y|^{\lambda}}\leqslant K_3\|u\|_{1,p,\Omega}. \tag{28}$$

暂时假定 Ω 是一个立方体，不失一般性我们还可以假定其边长为 1. 对 $0<t<1$, Ω_t 表示其表面和 Ω 的表面平行，边长为 t 的立方体，且使 $\overline{\Omega}_t\subset\Omega$. 设 $u\in C^\infty(\Omega)$.

设 $x,y\in\Omega$, $|x-y|=\sigma<1$. 则存在一个固定的立方体 Ω_σ, $x,y\in\overline{\Omega}_\sigma\subset\Omega$. 如果 $z\in\Omega_\sigma$, 则

$$u(x)=u(z)-\int_0^1\frac{d}{dt}u(x+t(z-x))\,dt,$$

所以

$$|u(x)-u(z)|\leqslant\sqrt{n}\,\sigma\int_0^1|\operatorname{grad}u(x+t(z-x))|\,dt.$$

因此

$$\left|u(x)-\frac{1}{\sigma^n}\int_{\Omega_\sigma}u(z)dz\right|\leqslant\left|\frac{1}{\sigma^n}\int_{\Omega_\sigma}(u(x)-u(z))dz\right|$$

$$\leqslant\frac{\sqrt{n}}{\sigma^{n-1}}\int_{\Omega_\sigma}dz\int_0^1|\operatorname{grad}u(x+t(z-x))|\,dt$$

$$=\frac{\sqrt{n}}{\sigma^{n-1}}\int_0^1 t^{-n}\,dt\int_{\Omega_{t\sigma}}|\operatorname{grad}u(z)|\,dz$$

$$\leqslant\frac{\sqrt{n}}{\sigma^{n-1}}\|\operatorname{grad}u\|_{0,p,\Omega}\int_0^1(\operatorname{vol}\Omega_{t\sigma})^{1/p'}t^{-n}dt \tag{29}$$

$$\leqslant K_4\sigma^{1-(n/p)}\|\operatorname{grad}u\|_{0,p,\Omega},$$

其中 $K_4=K_4(n,\ p)=\sqrt{n}\int_0^1 t^{-n/p}dt<\infty$. 用 y 代替 x 就有一个类似的不等式，所以

$$|u(x)-u(y)|\leqslant 2K_4|x-y|^{1-(n/p)}\|\operatorname{grad}u\|_{0,p,\Omega}.$$

对于 Ω 是一个立方体的情形，对于 $0<\lambda\leqslant 1-(n/p)$ (28) 式成立，所以，通过一个非奇异的线性变换知道当 Ω 是一个平行多面体的情形，(28)式也成立.

现在假定 Ω 具有强局部 Lipschitz 性质. 设 $\delta, M,\ \Omega_\delta, U_j$ 和 $\mathscr{V}_j$ 如同在 4.5 节中规定的一样. 存在一个直径为 δ 的平行多面体 P, P 的大小只依赖于 δ 和 M，使得对于每个 j 对应一个与 P 全等的且有一个顶点在原点的平行多面体 P_j，使得对一切 $x\in\mathscr{V}_j\cap\Omega$ 我们有 $x+P_j\subset\Omega$. 而且存在只依赖于 δ 和 P 的常数 δ_0 和 δ_1, $\delta_0\leqslant\delta$，使得若 $x, y\in\mathscr{V}_j\cap\Omega$ 和 $|x-y|<\delta_0$，则存在 $z\in(x+P_j)\cap(y+P_j)$，其中 $|x-z|+|y-z|\leqslant\delta_1|x-y|$. 把(28)式应用到 $x+P_j$ 和 $y+P_j$ 上去就得到：如果 $u\in C^\infty(\Omega)$，则

$$|u(x)-u(y)|\leqslant|u(x)-u(z)|+|u(y)-u(z)|$$

$$\leqslant K_5|x-z|^\lambda\|u\|_{1,p,\Omega}$$

$$+K_5|y-z|^\lambda\|u\|_{1,p,\Omega}$$

$$\leqslant 2^{1-\lambda}K_5\delta_1^\lambda\ |x-y|^\lambda\|u\|_{1,p,\Omega}. \tag{30}$$

现在设 $x,y\in\Omega$ 是任意的. 如果 $|x-y|<\delta_0\leqslant\delta$, $x,y\in\Omega_\delta$, 则对于某个 j 有 $x,y\in\mathscr{V}_j$, 从而估计式(30)成立. 如果 $|x-y|<\delta_0$, $x\in\Omega_\delta$, $y\in\Omega\sim\Omega_\delta$, 则对某个 j, $x\in\mathscr{V}_j$ 而且把(28)式再次应用到 $x+P_j$ 和 $y+P_j$ 上去就得到(30). 如果 $|x-y|<\delta_0$ 而且 $x,y\in\Omega\sim\Omega_\delta$, 则把(28)式应用到 $x+P'$, $y+P'$ 上去就得到(30), 其中 P' 是任何和 P 全等的, 有一个顶点在原点的平行多面体. 最后, 如果 $|x-y|\geqslant\delta_0$, 则有

$$|u(x)-u(y)|\leqslant|u(x)|+|u(y)|$$

$$\leqslant K_6\|u\|_{1,p,\Omega}\leqslant K_6\delta_0^{-\lambda}|x-y|^\lambda\|u\|_{1,p,\Omega}$$

这就对 $u\in C^\infty(\Omega)$ 完成了(28)的证明. 所以根据定理 3.16, 对于所有的连续函数, (28)式也对. ▌

现在, 除定理 5.4 情形 A 和 B (相应于 $k<n$) 的迹嵌入外, 我们已经证明了嵌入定理 5.4. 为了证明迹嵌入, 我们需要下面的内插结果.

5.18 引理 设 Q 是 $\mathbf{R}^n$ 中边长为 l, 其边平行于坐标轴的立方体. 如果 $p>1$, $q\geqslant1$ 以及 $mp-p<n<mp$, 则存在常数 $K=K(p,q,m,n,l)$ 使得对一切 $u\in W^{m,p}(Q)$ (在 Q 中 a. e.) 有

$$|u(x)|\leqslant K\|u\|_{0,q,Q}^s\|u\|_{m,p,Q}^{1-s}, \tag{31}$$

其中 $s=(mp-n)q/[np+(mp-n)q]$.

证明 只要对 $u\in C^\infty(\bar{Q})$ 证明(31)就行了. 由于 $\bar{Q}$ 的每一点是一个包含在 $\bar{Q}$ 中, 其边和 Q 的边平行而且边长为 $\dfrac{l}{2}$ 的立方体的角点, 不失一般性我们可以假定 x 本身是 Q 的一个角点, 比如说 $Q=\{y\in\mathbf{R}^n:\ x_i<y_i<x_i+l;\ 1\leqslant i\leqslant n\}$.

根据引理 5.17, 对于 $y\in Q$ 我们有

$$|u(x)|-|u(y)|\leqslant|u(x)-u(y)|\leqslant K_1|x-y|^{m-(n/p)}\|u\|_{m,p,Q}. \tag{32}$$

设$U=\|u\|_{m,p,Q}$，我们可以假定U是正的；设 $\rho=|x-y|$ 而 $\zeta=[|u(x)|/K_1U]^{p/(mp-n)}$. 暂时假定 $\zeta\leqslant l$. 对于 $\rho\leqslant\zeta$ 我们有

$$|u(y)|\geqslant|u(x)|-K_1U\rho^{m-(n/p)}\geqslant 0.$$

把上面的不等式取 p 幂，在 Q 上对 y 积分得到

$$\begin{aligned}\int_Q|u(y)|^q dy&\geqslant K_2\int_0^\zeta(|u(x)|-K_1U\rho^{m-(n/p)})^q\rho^{n-1}d\rho\\&=K_2\zeta^n|u(x)|^q\int_0^1(1-\sigma^{m-(n/p)})^q\sigma^{n-1}d\sigma\\&=K_3|u(x)|^{q+(np/(mp-n))}U^{-np/(mp-n)},\end{aligned}$$

由此立即得到(31)。

另一方面，如果 $\zeta>l$，则从(32)我们得到

$$\begin{aligned}|u(y)|&\geqslant|u(x)|-K_1U\rho^{m-(n/p)}\\&\geqslant|u(x)|-|u(x)|(\rho/l)^{m-(n/p)}\\&\geqslant 0\qquad 当\ \rho\leqslant l.\end{aligned}$$

如果 $t>0$，则

$$\begin{aligned}\int_Q|u(y)|^t dy&\geqslant K_2\int_0^l|u(x)|^t(1-(\rho/l)^{m-(n/p)})^t\rho^{n-1}d\rho\\&=K_4|u(x)|^t\end{aligned}$$

令 $t=[(mp-n)q+np]/mp$. 则用一下 Hölder 不等式

$$\begin{aligned}&|u(x)|^{[(mp-n)q+np]/mp}\\&\leqslant(1/K_4)\int_\Omega[|u(y)|^q]^{(mp-n)/mp}[|u(y)|^p]^{n/mp}dy\\&\leqslant(1/K_4)\|u\|_{0,q,\Omega}^{q(mp-n)/mp}\|u\|_{0,p,\Omega}^{n/m}.\end{aligned}$$

因为$\|u\|_{0,p,\Omega}\leqslant\|u\|_{m,p,\Omega}$，所以立即得到(31)式. ∎

我们指出：对 $p=1, m=n$ 的情形，上面的引理还成立. 这时从引理 5.14 我们可得 $W^{n,1}(\Omega)\to L^\infty(\Omega)$，所以在$Q$中$|u(x)|\leqslant K\|u\|_{n,1,Q}$ a.e.，这就是 $p=1, m=n$ 时的(31)式.

5.19 引理 设Ω是$\mathbf{R}^n$中具有锥性质的区域，又设Ω^k表示Ω和某个k维平面的交，其中$1\leqslant k\leqslant n(\Omega^n\equiv\Omega)$. 如果$n\geqslant mp$且$n-mp<k\leqslant n$，那么

$$W^{m,p}(\Omega)\to L^q(\Omega^k), \tag{33}$$

其中当$n>mp$时$p\leqslant q\leqslant kp/(n-mp)$，或者当$n=mp$时$p\leqslant q<\infty$. 如果$p=1, n>m$，而且$n-m\leqslant k\leqslant n$，则对于$1\leqslant q\leqslant k/(n-m)$，(33)式成立.

嵌入常数只依赖于m, p, k, n, q和决定Ω的锥性质的锥C.

证明 只要对有界区域Ω, $n>mp$和$q=kp/(n-mp)$的情形证明上述结论就行了，因为用推论5.13和引理5.14中对$k=n$的情形所用的同样的方法就可把结论推广到其它情形. 和引理5.10的证明一样，我们还可以假定Ω是边长为2单位的坐标立方体的并集.

设$\mathbf{R}_0^k$是$\mathbf{R}^n$的一个k维坐标子空间，Ω^k在$\mathbf{R}_0^k$上有一个一对一的投影Ω_0^k. 暂时假定$p>1$. 设ν是小于mp的最大整数. 那么$mp-p<\nu<mp$而且由于$n-mp<k$我们有$n-\nu\leqslant k$. (注意如果$p=1$，对$k=n-m, \nu=m$，同样的结论成立.) 设$\mu=\binom{k}{n-\nu}$又设$E_i(1\leqslant i\leqslant\mu)$表示维数为$n-\nu$的$\mathbf{R}_0^k$的各种坐标子空间. 设$\Omega_i$表示$\Omega_0^k$(因此$\Omega^k$)在$E_i$上的投影. 还有，对每个$x\in\Omega_i$设$\Omega_{i,x}$表示$\Omega$和通过$x$点垂直于$E_i$的$\nu$维平面的交. 于是$\Omega_{i,x}$包含一个边长为单位，有一个顶点在$x$的$\nu$维坐标立方体. 根据引理5.18，对于$q=q_0=np/(n-mp)$，对于$u\in C^\infty(\Omega)$我们有

$$\sup_{y\in\Omega_{i,x}}|u(y)|^{(n-\nu)p/(n-mp)}\leqslant K_1\|u\|_{0,q_0,\Omega_{i,x}}^{(mp-\nu)q_0/mp}\|u\|_{m,p,\Omega_{i,x}}^{\nu/m}. \tag{34}$$

设dx^i和dx_*^i分别表示E_i和E_i的正交余集中的体积元素. 将(34)式在Ω_i上积分，利用Hölder不等式就导致

$$\int_{\Omega_i}\sup_{y\in\Omega_{i,x}}|u(y)|^{(n-\nu)p/(n-mp)}dx^i$$

$$\leqslant K_1\int_{\Omega_i}\left[\int_{\Omega_{i,x}}|u(x)|^{q_0}dx_*^i\right]^{(mp-\nu)/mp}$$

$$\times\left[\int_{\Omega_{i,x}}\sum_{|\alpha|\leqslant m}|D^\alpha u(x)|^p dx_*^i\right]^{\nu/mp}dx^i$$

$$\leqslant K_1\left[\int_\Omega|u(x)|^{q_0}dx\right]^{(mp-\nu)/mp}$$

$$\times\left[\int_\Omega\sum_{|\alpha|\leqslant m}|D^\alpha u(x)|^p dx\right]^{\nu/mp}$$

$$=K_1\|u\|_{0,q_0,\Omega}^{q_0(mp-\nu)/mp}\|u\|_{m,p,\Omega}^{\nu/m}. \tag{35}$$

最后，把引理 5.9 应用到 $\mathbf{R}_0^k$ 的子空间 E_i 上去. 注意到引理 5.9 中的常数 λ 在这里等于$\binom{k-1}{n-\nu-1}$. 令 $dx^{(k)}$ 表示 $\mathbf{R}_0^k$ 中的体积元素，而令 $q=kp/(n-mp)$，我们得到

$$\|u\|_{0,q,\Omega^k}^q\leqslant K_2\int_{\Omega_0^k}\prod_{i=1}^{\mu}\sup_{y\in\Omega_{i,x}}|u(y)|^{q/\mu}dx^{(k)}$$

$$\leqslant K_2\prod_{i=1}^{\mu}\left[\int_{\Omega_i}\sup_{y\in\Omega_{i,x}}|u(y)|^{q\lambda/\mu}dx^i\right]^{1/\lambda}. \tag{36}$$

因为 $q\lambda/\mu=(n-\nu)p/(n-mp)$，从(35)和(36)以及引理 5.14 得到

$$\|u\|_{0,q,\Omega^k}\leqslant K_3\prod_{i=1}^{\mu}\|u\|_{0,q_0,\Omega}^{q_0(mp-\nu)/mp\lambda q}\|u\|_{m,p,\Omega}^{\nu/m\lambda q}$$

$$\leqslant K_4\left[\|u\|_{m,p,\Omega}^{q_0(mp-\nu)/mp}\|u\|_{m,p,\Omega}^{\nu/m}\right]^{\mu/\lambda q}=K_4\|u\|_{m,p,\Omega}.$$

这就证实了所要的嵌入定理. ▮

现在我们已完成定理 5.4 的证明.

$W^{m,p}(\Omega)$ 中的函数在 Ω 边界上的迹

5.20 研究定义在区域 Ω 上的微分算子的边值问题时，重要的问题是决定定义在 Ω 的边界上的函数空间，包括决定 $W^{m,p}(\Omega)$ 中函数 u 的迹 $u|_{\mathrm{bdry}\,\Omega}$. 例如，如果 $W^{m,p}(\Omega)\to C(\overline{\Omega})$，那么 $u|_{\mathrm{bdry}\,\Omega}$ 显然属于 $C(\mathrm{bdry}\,\Omega)$. 下面我们概述一个对于这种迹的 L^q 嵌入定

理，它可以作为定理 5.4 的推论而得到.

在算子 $u\to u|_{\mathrm{bdry}\,\Omega}$ 作用下 $W^{m,p}(\Omega)$ 的象的表征问题为许多作者广泛地研究过. 包含分数 m 次 Sobolev 空间在内的解决方法将在第七章中给出(特别见定理 7.53). 第七章所用的方法是属于 Lions 的[37, 38].

5.21 设 Ω 是 $\mathbf{R}^n$ 中具有一致 C^m-正则性的区域 Ω. 因此存在 bdry Ω 的一个局部有限开覆盖 $\{U_j\}$ 和相应的 m-光滑变换 $\boldsymbol{\Psi}_j$ 把 $B=\{y\in\mathbf{R}^n\colon |y|<1\}$ 映射到 U_j 上，使得 $U_j\cap \mathrm{bdry}\,\Omega=\boldsymbol{\Psi}_j(B_0)$; $B_0=\{y\in B\colon y_n=0\}$. 如果 f 是一个支集在 U_j 中的函数，我们可以通过

$$\int_{\mathrm{bdry}\,\Omega} f(x)d\sigma=\int_{U_j\cap\mathrm{bdry}\,\Omega} f(x)d\sigma$$
$$=\int_{B_0} f\circ\boldsymbol{\Psi}_j(y',0)\mathbf{J}_j(y')dy'$$

来定义 f 在 bdryΩ 上的积分，其中 $y'=(y_1,\cdots,y_{n-1})$, 而且如果 $x=\boldsymbol{\Psi}_j(y)$, 则

$$\mathbf{J}_j(y')=\left\{\sum_{k=1}^{n}\left(\frac{\partial(x_1,\cdots,\hat{x}_k,\cdots,x_n)}{\partial(y_1,\cdots,y_{n-1})}\right)^2\right\}^{1/2}\Bigg|_{y_n=0}.$$

如果 f 是定义在 bdryΩ 上的任意函数，我们可以令

$$\int_{\mathrm{bdry}\,\Omega} f(x)d\sigma=\sum_j\int_{\mathrm{bdry}\,\Omega} f(x)v_j(x)d\sigma,$$

其中 $\{v_j\}$ 是 bdryΩ 的从属于 $\{U_j\}$ 的一个单位分解.

5.22 定理 设 Ω 是 $\mathbf{R}^n$ 中具有一致 C^m-正则性的区域，而且假定存在一个 Ω 的简单 (m,p)-延拓算子 E. 如果 $mp<n$ 而

$$p\leqslant q\leqslant(n-1)p/(n-mp),\ \text{则}$$
$$W^{m,p}(\Omega)\to L^q(\mathrm{bdry}\,\Omega). \tag{37}$$

如果 $mp=n$, 则对 $p\leqslant q<\infty$ (37) 式成立.

证明 嵌入(37)应该在下列意义下解释：如果 $u\in W^{m,p}(\Omega)$, 则 Eu

在 5.2 节最后一段所说的意义下有一个在 bdryΩ 上的迹，而且 $\|Eu\|_{0,q,\mathrm{bdry}\,\Omega}\leqslant K\|u\|_{m,p,\Omega}$，$K$ 与 u 无关. [注意因为 $C_0(\mathbf{R}^n)$ 在 $W^{m,p}(\Omega)$ 中稠密，$\|Eu\|_{0,q,\mathrm{bdry}\,\Omega}$ 和所用的特殊的延拓算子 E 无关.]

只要对 $mp<n$ 和 $q=(n-1)p/(n-mp)$ 证明定理就够了. 存在常数 K_1 使得对于一切 $u\in W^{m,p}(\Omega)$

$$\|Eu\|_{m,p,\mathbf{R}^n}\leqslant K_1\|u\|_{m,p,\Omega}.$$

根据一致 C^m-正则性条件(4.6 节)，存在常数 K_2，使得对每个 j 和一切 $y\in B, x=\boldsymbol{\Psi}_j(y)\in U_j$，有

$$|J_j(y')|\leqslant K_2 \quad \text{以及} \quad \left|\frac{\partial(y_1,\cdots,y_n)}{\partial(x_1,\cdots,x_n)}\right|\leqslant K_2.$$

注意到在 $\mathbf{R}^n$ 上 $0\leqslant v_j(x)\leqslant 1$，而且把定理 5.4 的嵌入(2)用到 B 上去，对于 $u\in W^{m,p}(\Omega)$，我们有

$$\begin{aligned}\int_{\mathrm{bdry}\,\Omega}|Eu(x)|^q d\sigma &\leqslant\sum_j\int_{U_j\cap\mathrm{bdry}\,\Omega}|Eu(x)|^q d\sigma\\ &\leqslant K_2\sum_j\|Eu\circ\boldsymbol{\Psi}_j\|^q_{0,q,B_0}\\ &\leqslant K_3\sum_j(\|Eu\circ\boldsymbol{\Psi}_j\|^p_{m,p,B})^{q/p}\\ &\leqslant K_4\left(\sum_j\|Eu\|^p_{m,p,U_j}\right)^{q/p}\\ &\leqslant K_4R\|Eu\|^q_{m,p,\mathbf{R}^n}\\ &\leqslant K_5\|u\|^q_{m,p,\Omega}.\end{aligned}$$

上述倒数第二个不等式利用了覆盖 $\{U_j\}$ 具有的有限交性质. 因为对所有的 i, j, $|D^\alpha\Psi_{j,i}(y)|\leqslant\mathrm{const}$, 其中 $\boldsymbol{\Psi}_j=(\Psi_{j,1},\cdots,\Psi_{j,n})$，所以常数 K_4 和 j 无关. 这就完成了证明. ∎

作为 Bananh 代数的 $W^{m,p}(\Omega)$

给了 $W^{m,p}(\Omega)$ 中的函数 u 和 v，其中 Ω 是 $\mathbf{R}^n$ 中的区域，一般

说不能指望它们的逐点乘积 uv 是属于 $W^{m,p}(\Omega)$ 的. [在 Ω 中 $(uv)(x)=u(x)v(x)$ a.e..] 然而直接应用 Sobolev 嵌入定理就可证明: 假如 $mp>n$ 而且 Ω 具有锥性质, 则 uv 属于 $W^{m,p}(\Omega)$.

5.23 定理 设 Ω 是 $\mathbf{R}^n$ 中具有锥性质的区域. 如果 $mp>n$, 则存在一个依赖于 m, p, n 和决定 Ω 的锥性质的有限锥 C 的常数 K^*, 使得对于所有的 $u, v\in W^{m,p}(\Omega)$, 在 Ω 中 a.e. 逐点定义的乘积 uv 属于 $W^{m,p}(\Omega)$ 而且满足

$$\|uv\|_{m,p,\Omega}\leqslant K^*\|u\|_{m,p,\Omega}\|v\|_{m,p,\Omega}. \tag{38}$$

特别地, $W^{m,p}(\Omega)$ 关于逐点相乘和等价范数

$$\|u\|^*_{m,p,\Omega}=K^*\|u\|_{m,p,\Omega}$$

是一个可交换的 Banach 代数.

证明 为了证实(38), 只要证明, 如果 $|\alpha|\leqslant m$, 则

$$\int_\Omega|D^\alpha[u(x)v(x)]|^p dx\leqslant K_\alpha\|u\|^p_{m,p,\Omega}\|v\|^p_{m,p,\Omega},$$

其中 $K_\alpha=K_\alpha(m, p, n, C)$. 我们暂时假定 $u\in C^\infty(\Omega)$. 根据广义导数的 Leibniz 法则, 即

$$D^\alpha[uv]=\sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}D^\beta uD^{\alpha-\beta}v,$$

只要证明, 对任何的 $\beta\leqslant\alpha$, $|\alpha|\leqslant m$, 有

$$\int_\Omega|D^\beta u(x)D^{\alpha-\beta}v(x)|^p dx\leqslant K_{\alpha,\beta}\|u\|^p_{m,p,\Omega}\|v\|^p_{m,p,\Omega},$$

其中 $K_{\alpha,\beta}=K_{\alpha,\beta}(m,p,n,C)$. 根据嵌入定理, 对每个 β, $|\beta|\leqslant m$, 存在一个常数 $K(\beta)=K(\beta, m, p, n, C)$ 使得对任何 $w\in W^{m,p}(\Omega)$, 假如 $(m-|\beta|)p\leqslant n$ 和 $p\leqslant r\leqslant np/(n-(m-|\beta|)p)$ [或如果 $(m-|\beta|)p=n$ 则 $p\leqslant r<\infty$], 就有

$$\int_\Omega|D^\beta w(x)|^r dx\leqslant K(\beta)\|w\|^r_{m,p,\Omega}, \tag{39}$$

或者假如 $(m-|\beta|)p>n$, 则有

$$|D^\beta w(x)|\leqslant K(\beta)\|w\|_{m,p,\Omega}\quad \text{在}\,\Omega\,\text{中 a.e.}.$$

设 k 是使 $(m-k)p>n$ 的最大整数. 因为 $mp>n$, 所以 $k\geqslant 0$, 如果 $|\beta|\leqslant k$, 则 $(m-|\beta|)p>n$, 所以

$$\int_\Omega |D^\beta u(x)D^{\alpha-\beta}v(x)|^p dx\leqslant K(\beta)^p\|u\|^p_{m,p,\Omega}\|D^{\alpha-\beta}v\|^p_{0,p,\Omega}$$
$$\leqslant K(\beta)^p\|u\|^p_{m,p,\Omega}\|v\|^p_{m,p,\Omega}.$$

类似地, 如果 $|\alpha-\beta|\leqslant k$, 则

$$\int_\Omega |D^\beta u(x)D^{\alpha-\beta}v(x)|^p dx\leqslant K(\alpha-\beta)^p\|u\|^p_{m,p,\Omega}\|v\|^p_{m,p,\Omega}.$$

如果 $|\beta|>k$ 和 $|\alpha-\beta|>k$, 则实际上 $|\beta|\geqslant k+1$ 且 $|\alpha-\beta|\geqslant k+1$, 所以 $n\geqslant(m-|\beta|)p$ 且 $n\geqslant(m-|\alpha-\beta|)p$. 而且

$$\frac{n-(m-|\beta|)p}{n}+\frac{n-(m-|\alpha-\beta|)p}{n}$$
$$=2-\frac{(2m-|\alpha|)p}{n}<2-\frac{mp}{n}<1.$$

因此存在正数 r, r', 满足 $(1/r)+(1/r')=1$, 使得

$$p\leqslant rp<\frac{np}{n-(m-|\beta|)p},\qquad p\leqslant r'p<\frac{np}{n-(m-|\alpha-\beta|)p}.$$

因此根据 Hölder 不等式和(39)我们有

$$\int_\Omega |D^\beta u(x)D^{\alpha-\beta}v(x)|^p dx$$
$$\leqslant\left[\int_\Omega |D^\beta u(x)|^{rp}dx\right]^{1/r}\left[\int_\Omega |D^{\alpha-\beta}v(x)|^{r'p}dx\right]^{1/r'}$$
$$\leqslant[K(\beta)]^{1/r}[K(\alpha-\beta)]^{1/r'}\|u\|^p_{m,p,\Omega}\|v\|^p_{m,p,\Omega}.$$

这就对 $u\in C^\infty(\Omega)$, $v\in W^{m,p}(\Omega)$ 证明了(38)式.

如果 $u\in W^{m,p}(\Omega)$, 则根据定理3.16存在 $C^\infty(\Omega)$ 中的序列 $\{u_n\}$ 在 $W^{m,p}(\Omega)$ 中收敛到 u. 于是由上述论证 $\{u_nv\}$ 是 $W^{m,p}(\Omega)$ 中的一个 Cauchy序列, 所以收敛到 $W^{m,p}(\Omega)$ 中的一个元素 w. 因为 $mp>n$,

可以假定 u, v 在 Ω 上连续且有界. 因此, 当 $n\to\infty$ 时

$$\|w-uv\|_{0,p,\Omega}\leqslant\|w-u_nv\|_{0,p,\Omega}+\|(u_n-u)v\|_{0,p,\Omega}$$
$$\leqslant\|w-u_nv\|_{0,p,\Omega}+\|v\|_{0,\infty,\Omega}\|u_n-u\|_{0,p,\Omega}\to 0.$$

因此在 $L^p(\Omega)$ 中 $w=uv$, 所以在广义函数的意义下 $w=uv$. 所以在 $W^{m,p}(\Omega)$ 中 $w=uv$ 而且

$$\|uv\|_{m,p,\Omega}=\|w\|_{m,p,\Omega}\leqslant\limsup_{n\to\infty}\|u_nv\|_{m,p,\Omega}=\|u\|_{m,p,\Omega}\|v\|_{m,p,\Omega}$$

这就完成了定理的证明. ▌

我们指出 Banach 代数 $W^{m,p}(\Omega)$ 有一个么元 (identity) 当且仅当 Ω 是有界的, 即, 函数 $e(x)\equiv 1$ 属于 $W^{m,p}(\Omega)$ 当且仅当 $\mathrm{vol}\,\Omega<\infty$, 但是体积有限又具有锥性质的无界区域是不存在的.

反例和非嵌入定理

5.24 研究一下 Sobolev 嵌入定理 5.4 的陈述, 读者可能会推测几个可能推广的方向. 在探讨当不满足定理 5.4 中所述区域的条件时证明嵌入定理的可能性之前, 我们先来构造一些例子, 这些例子表明对于定理 5.4 中所考虑的区域, 甚至是任何区域, 定理 5.4 在某些方面给出了“最好可能”的嵌入结果.

设 Ω 是 $\mathbf{R}^n$ 中的任意区域, 不失一般性假定原点属于 Ω. 设 $R>0$ 使闭球 $\overline{B_{2R}}$ 包含在 Ω 中. (这里 $B_R=\{x\in\mathbf{R}^n:|x|<R\}$). 在下面的每一个例子中我们作一个只依赖于 $\rho=|x|$ 的函数 $u\in C^\infty(B_{3R}\sim\{0\})$. 如果 $f\in C^\infty(0,\infty)$ 满足: 当 $t\leqslant R$ 时 $f(t)=1$ 而当 $t\geqslant 2R$ 时 $f(t)=0$, 则由

$$w(x)=\begin{cases}0 & \text{当 } \rho=|x|\geqslant 3R\\ f(\rho)u(\rho) & \text{当 } 0<\rho<3R\end{cases}$$

定义的函数 w 在 Ω 中有紧支集并属于 $W^{m,p}(\Omega)$ 当且仅当 $u\in W^{m,p}(B_R)$.

5.25 例 设 k 是一整数, $1\leqslant k\leqslant n$, 假定 $mp<n$ 和 $q>kp/(n-$

mp). 我们造一个 $u\in W^{m,p}(B_R)$, 但 $u\notin L^q(B_R^k)$, 其中 $B_R^k=\{x\in B_R: x_{k+1}=\cdots=x_n=0\}$. 因此当 $q>kp/(n-mp)$时, 就不可能有形如 $W^{m,p}(\Omega)\to L^q(\Omega^k)$的嵌入.

设 $u(x)=\rho^\lambda$, 其中 $\rho=|x|$而指数 λ 将在下面规定. 通过对 $|\alpha|$做归纳法容易验证

$$D^\alpha u(x)=P_\alpha(x)\rho^{\lambda-2|\alpha|}, \tag{40}$$

其中P_α是 x 的分量的$|\alpha|$次齐次多项式. 因此$|D^\alpha u(x)|\leqslant K_\alpha\rho^{\lambda-|\alpha|}$而且

$$\int_{B_R}|D^\alpha u(x)|^p dx\leqslant \text{const}\int_0^R \rho^{(\lambda-|\alpha|)p+n-1}d\rho.$$

所以假如

$$(\lambda-m)p+n>0, \tag{41}$$

则 u 属于$W^{m,p}(B_R)$. 另一方面, 如果 $\sigma=(x_1^2+\cdots+x_k^2)^{1/2}$, 则

$$\int_{B_R^k}|u(x)|^q dx_1\cdots dx_k=\text{const}\int_0^R \sigma^{\lambda q+k-1}d\sigma,$$

所以如果

$$\lambda q+k<0, \tag{42}$$

则 $u\notin L^q(B_R^k)$. 由于 $q>kp/(n-mp)$, 就能选到所要求的 λ 使得 λ 既满足(41)又满足(42). ▌

由于上面构造的函数 u 在 0 附近是无界的, 所以当 $mp<n$ 时不可能有形如$W^{m,p}(\Omega)\to C_B^0(\Omega)$的嵌入.

5.26 例 假定 $p>1$ 和 $mp=n$. 我们构造 $u\in W^{m,p}(B_R)$使得 $u\notin L^\infty(B_R)$. 因此当 $mp=n$ 而且Ω具有锥性质时, 对于$p\leqslant q<\infty$成立的嵌入 $W^{m,p}(\Omega)\to L^q(\Omega)$ 不能推广成 $W^{m,p}(\Omega)\to L^\infty(\Omega)$ 或 $W^{m,p}(\Omega)\to C_B^0(\Omega)$, 除非 $p=1$ 和 $n=m$(参看定理 8.25).

设 $u(x)=\log(\log 4R/\rho)$, 其中 $\rho=|x|$. 显然 $u\notin L^\infty(B_R)$. 由归纳法容易验证

$$D^\alpha u(x)=\sum_{j=1}^{|\alpha|}P_{\alpha,j}(x)\rho^{-2|\alpha|}[\log(4R/\rho)]^{-j}, \tag{43}$$

其中 $P_{\alpha,j}$ 是 x 的分量的 $|\alpha|$ 次齐次多项式. 因为 $p=n/m$ 有

$$|D^\alpha u(x)|^p\leqslant\sum_{j=1}^{|\alpha|}K_{\alpha,j}\rho^{-|\alpha|n/m}[\log(4R/\rho)]^{-jp},$$

所以

$$\int_{B_R}|D^\alpha u(x)|^p dx\leqslant\text{const}\sum_{j=1}^{|\alpha|}\int_0^R[\log(4R/\rho)]^{-jp}\rho^{(-|\alpha|n/m)+n-1}d\rho.$$

如果 $|\alpha|<m$, 上述不等式的右端一定是有限的. 如果 $|\alpha|=m$. 令 $\sigma=\log(4R/\rho)$, 我们有

$$\int_{B_R}|D^\alpha u(x)|^p dx\leqslant\text{const}\sum_{j=1}^{|\alpha|}\int_{\log_4}^{\infty}\sigma^{-jp}d\sigma.$$

由于 $p>1$, 它是有限的. 因此 $u\in W^{m,p}(B_R)$. ▌

有趣的是, 对于 $mp=n$ 的情形, 上面的函数 u 与 m 和 p 的选取无关.

5.27 例 假定 $mp>n>(m-1)p$, 又设 $\lambda>m-(n/p)$. 我们构造 $u\in W^{m,p}(B_R)$ 使得 $u\notin C^{0,\lambda}(\bar{B}_R)$. 因此当 $mp>n>(m-1)p$ 和 $\lambda>m-(n/p)$ 时不可能有形如 $W^{m,p}(\Omega)\to C^{0,\lambda}(\overline{\Omega})$ 的嵌入.

和例 5.25 一样, 取 $u(x)=\rho^\mu$, $\rho=|x|$. 假如 $\mu>m-(n/p)$ 由(41)式有 $u\in W^{m,p}(B_R)$. 现在 $|u(x)-u(0)|/|x-0|^\lambda=\rho^{\mu-\lambda}$, 所以当 $\mu<\lambda$ 时 $u\notin C^{0,\lambda}(\bar{B}_R)$. 因此如果我们选满足 $m-(n/p)<\mu<\lambda$ 的 μ, 则 u 具有所要的性质. ▌

5.28 例 假定 $(m-1)p=n$ 和 $p>1$. 我们构造 $u\in W^{m,p}(B_R)$ 使得 $u\notin C^{0,1}(\bar{B}_R)$. 因此除非 $p=1$, $m-1=n$, 当 Ω 具有强局部 Lipshitz 性质时对 $0<\lambda<1$ 成立的嵌入 $W^{m,p}(\Omega)\to C^{0,\lambda}(\overline{\Omega})$ 不能推广到 $\lambda=1$ 的情形.

设 $u(x)=\rho\log(\log 4R/\rho)$, 其中 $\rho=|x|$. 由于当 $x\to 0$ 时,

$$|u(x)-u(0)|/|x-0|=\log(\log 4R/\rho)\to\infty,$$
显然 $u\notin C^{0,1}(\bar{B}_R)$. 由(40)和(43)我们有

$$D^\alpha u(x)=\sum_{j=1}^{|\alpha|}P_{\alpha,j}(x)\rho^{1-2|\alpha|}[\log(4R/\rho)]^{-j},$$

其中 $P_{\alpha,j}$ 是$|\alpha|$次齐次多项式. 因此

$$|D^\alpha u(x)|^p\leqslant\sum_{j=1}^{|\alpha|}k_{\alpha,j}\rho^{p(1-|\alpha|)}[\log(4R/\rho)]^{-jp}.$$

和例 5.26 一样, 由此推出 $u\in W^{m,p}(B_R)$. ▌

5.29 上面的例子表明, 除了以前明确说过的那些例子外, 甚至对于很规则的区域也可能没有定理 5.4 中所考虑的形式的嵌入. 余下的问题是要研究一下对于没有锥性质的不规则区域能否有这种形式的嵌入关系, 我们将要证明定理 5.4 能推广到某些不规则区域, 但是结论要弱一点. 不过我们首先要说明如果区域"太不规则"的话推广是不可能的.

如果Ω在无穷远变窄的话, 即如果

$$\lim_{\substack{|x|\to\infty\\ x\in\Omega}}\operatorname{dist}(x,\operatorname{bdry}\Omega)=0$$

的话, $\mathbf{R}^n$ 中的无界区域Ω可以有光滑边界而没有锥性质. 下面的定理表明对于任何这样的体积有限的无界区域 Ω定理 5.4 的第 I 和 II 部分完全失效.

5.30 定理 设Ω是 $\mathbf{R}^n$ 中体积有限的无界区域, 又设 $q>p$. 则 $W^{m,p}(\Omega)$不嵌入到 $L^q(\Omega)$中.

证明 我们构造一个函数 $u(x)$, 它只依赖于 x 到原点的距离 $\rho=|x|$, 当 ρ 增加时 $u(x)$的增长快得足以使 $u(x)\notin L^q(\Omega)$ 但仍有 $u\in W^{m,p}(\Omega)$.

不失一般性我们假定 $\operatorname{vol}\Omega=1$. 设 $A(\rho)$表示 Ω 和中心在原点半径为 r 的球面相交的曲面的面积[$(n-1)$-维测度]. 那么

$$\int_0^\infty A(\rho)d\rho=1.$$

设 $r_0=0$, 对于 $n=1, 2, \cdots$由

$$\int_{r_n}^\infty A(\rho)d\rho=1/2^n=\int_{r_{n-1}}^{r_n} A(\rho)d\rho$$

来定义 r_n. 显然 r_n 随着 n 趋于无穷而趋于无穷. 设 $\Delta r_n=r_{n+1}-r_n$ 而且固定 ε 使得 $0<\varepsilon<[1/(mp)]-[1/(mq)]$. 一定存在一个增长的序列$\{n_j\}_{j=1}^\infty$ 使得 $\Delta r_{n_j}\geqslant 2^{-\varepsilon n_j}$. 否则除可能有有限个 n 值外的一切 n 都有 $\Delta r_n<2^{-\varepsilon n}$, 由此应该有$\sum_{n=0}^\infty \Delta r_n<\infty$, 这就导致矛盾. 为方便起见我们假定 $n_1\geqslant 1$, 所以对一切 $j, n_j\geqslant j$. 设 $a_0=0$, $a_j=r_{n_j+1}$, 和 $b_j=r_{n_j}$. 注意到 $a_{j-1}\leqslant b_j<a_j$ 而且 $a_j-b_j=\Delta r_{n_j}\geqslant 2^{-\varepsilon n_j}$.

设 f 是一个具有以下性质的 $\mathbf{R}$ 上的非负无穷可微函数:

(i) 对一切 $t, 0\leqslant f(t)\leqslant 1$,

(ii) 如果 $t\leqslant 0, f(t)=0$, 如果 $t>1, f(t)=1$,

(iii) 如果$1\leqslant k\leqslant m$, 则对一切 t, $|(d/dt)^k f(t)|\leqslant M$.

如果 $x\in\Omega$ 而 $\rho=|x|$. 令

$$u(x)=\begin{cases}2^{n_{j-1}/q} & \text{对于 } a_{j-1}\leqslant\rho\leqslant b_j\\ 2^{n_{j-1}/q}+(2^{n_j/q}-2^{n_{j-1}/q})f\left(\dfrac{\rho-b_j}{a_j-b_j}\right) & \text{对于 } b_j\leqslant\rho\leqslant a_j\end{cases}$$

显然 $u(x)\in C^\infty(\Omega)$. 记 $\Omega_j=\{x\in\Omega: a_{j-1}\leqslant\rho\leqslant a_j\}$, 我们有

$$\begin{aligned}\int_{\Omega_j}|u(x)|^p dx&=\left\{\int_{a_{j-1}}^{b_j}+\int_{b_j}^{a_j}\right\}[u(x)]^p A(\rho)d\rho\\&\leqslant 2^{n_{j-1}p/q}\int_{a_{j-1}}^\infty A(\rho)d\rho+2^{n_j p/q}\int_{b_j}^{a_j}A(\rho)d\rho\\&=\frac{1}{2}[2^{-n_{j-1}(1-p/q)}+2^{-n_j(1-p/q)}]\leqslant 2^{-(j-1)(1-p/q)}.\end{aligned}$$

因为 $p<q$, 由上述不等式必然得出

$$\int_{\Omega}|u(x)|^p dx=\sum_{j=1}^{\infty}\int_{\Omega_j}|u(x)|^p dt<\infty.$$

如果 $1\leqslant k\leqslant m$, 我们还有

$$\int_{\Omega_j}\left|\frac{d^k u}{dx^k}\right|^p dx=\int_{b_j}^{a_j}\left|\frac{d^k u}{d\rho^k}\right|^p A(\rho)d\rho$$

$$\leqslant M^p 2^{n_j p/q}[a_j-b_j]^{-kp}\int_{b_j}^{a_j}A(\rho)d\rho$$

$$=\frac{1}{2}M^p 2^{-n_j(1-p/q-\varepsilon kp)}\leqslant\frac{1}{2}M^p 2^{-Cn_j},$$

其中 $C=1-p/q-\varepsilon kp$, 因为 $\varepsilon<[1/(mp)]-[1/(mq)]$, 所以 $C>0$. 因此对于 $|\alpha|\leqslant m$, $D^\alpha u\in L^p(\Omega)$, 这就是说 $u\in W^{m,p}(\Omega)$. 可是, $u\notin L^q(\Omega)$, 因为对于每个 j 我们有

$$\int_{\Omega_j}|u(x)|^q dx\geqslant 2^{n_{j-1}}\int_{a_{j-1}}^{a_j}A(\rho)d\rho$$

$$=2^{n_{j-1}}[2^{-n_{j-1}-1}-2^{-n_j-1}]\geqslant\frac{1}{4},$$

所以 $W^{m,p}(\Omega)$不可能嵌入到 $L^q(\Omega)$中去. ▌

上述定理的结论能推广到体积无限但满足

$$\limsup_{N\to\infty}\operatorname{vol}\{x\in\Omega: N\leqslant|x|\leqslant N+1\}=0$$

的无界区域Ω上去(见 6.35 节).

5.31 对于有很尖的边界尖点的区域定理 5.4 的第 I 和第 II 部分也完全失效. 如果Ω是 $\mathbf{R}^n$ 中的区域而 x_0 是Ω边界上的一点, 设 $B_r=B_r(x_0)$ 表示中心在 x_0 半径为 r 的一个开球, 设 $\Omega_r=B_r\cap\Omega$, 设 $S_r=(\operatorname{bdry}B_r)\cap\Omega$, 又设 $A(r,\Omega)$是 S_r 的曲面面积[$(n-1)$维测度]. 如果对于任何实数 k, 有

$$\lim_{r\to 0+}\frac{A(r,\Omega)}{r^k}=0,\tag{44}$$

我们就说Ω在 $x_0\in\operatorname{bdry}\Omega$有一个指数尖点(exponential cusp).

5.32 定理 如果Ω是 $\mathbf{R}^n$ 中的区域, Ω在 $\operatorname{bdry}\Omega$上有一个指数

尖点 x_0, 又如果 $q>p$, 则 $W^{m,p}(\Omega)$ 不嵌入到 $L^q(\Omega)$ 中.

证明 我们构造 $u\in W^{m,p}(\Omega)$ 但不属于 $L^q(\Omega)$, 因为 $|u(x)|^q$ 在 x_0 附近很快变为无界. 不失一般性我们可以假定 $x_0=0$, 所以 $\rho=|x|$, 设 $\Omega^*=\{y=x/|x|^2: x\in\Omega, |x|<1\}$. 易见 Ω^* 是无界的且有有限体积, 而且

$$A(r, \Omega^*)=r^{2(n-1)}A(1/r, \Omega).$$

设 t 满足 $p<t<q$. 根据定理 5.30, 存在一个函数 $\tilde{v}\in C^m(0, \infty)$ 使得

(i) 如果 $0<r\leqslant 1$, 则 $\tilde{v}(r)=0$,

(ii) 如果 $0\leqslant j\leqslant m$, 则 $\int_1^\infty |v^{(j)}(r)|^t A(r, \Omega^*)dr<\infty$,

(iii) $\int_1^\infty |\tilde{v}(r)|^q A(r, \Omega^*)dr=\infty$.

[如果 $r=|y|$, 则 $v(y)=\tilde{v}(r)$ 定义 $v\in W^{m,p}(\Omega^*)$ 但是 $v\notin L^q(\Omega^*)$.]

设 $x=y/|y|^2$, 所以 $\rho=|x|=1/|y|=1/r$. 令 $\lambda=2n/q$, 定义 $u(x)=\tilde{u}(\rho)=r^\lambda\tilde{v}(r)=|y|^\lambda v(y)$. 由此可见对于 $|\alpha|=j\leqslant m$

$$|D^\alpha u(x)|\leqslant|\tilde{u}^{(j)}(\rho)|\leqslant\sum_{i=0}^{j}c_{ij}r^{\lambda+j+i}\tilde{v}^{(i)}(r),$$

其中系数 c_{ij} 只依赖于 λ. 现在对 $|x|\geqslant 1$, $u(x)$ 等于 0, 所以

$$\int_\Omega |u(x)|^q dx=\int_0^1 |\tilde{u}(\rho)|^q A(\rho, \Omega)d\rho$$

$$=\int_1^\infty |\tilde{v}(r)|^q A(r, \Omega^*)dr=\infty.$$

另一方面, 如果 $0\leqslant|\alpha|=j\leqslant m$, 我们有

$$\int_\Omega |D^\alpha u(x)|^p dx\leqslant\int_0^1 |\tilde{u}^{(j)}(\rho)|^p A(\rho, \Omega)d\rho$$

$$\leqslant \text{const}\sum_{i=0}^{j}\int_1^\infty |\tilde{v}^{(i)}(r)|^p r^{(\lambda+j+i)p-2n}A(r, \Omega^*)dr. \tag{45}$$

如果$(\lambda+2m)p\leqslant 2n$，则由于$p<t$和$\mathrm{vol}\Omega^*<\infty$，通过应用 Hölder 不等式就知道(45)中所有的积分都是有限的，所以$u\in W^{m,p}(\Omega)$。否则，设

$$k=[(\lambda+2m)p-2n][t/(t-p)]+2n.$$

由(44)存在$a\leqslant 1$使得如果$\rho\leqslant a$，则$A(\rho,\Omega)\leqslant\rho^k$。由此得到，如果$r\geqslant 1/a$，则

$$r^{k-2n}A(r,\Omega^*)\leqslant r^{k-2}\rho^k=r^{-2}.$$

因此

$$\begin{aligned}&\int_1^\infty|\tilde{v}^{(i)}(r)|^p r^{(\lambda+j+i)p-2n}A(r,\Omega^*)dr\\&=\int_1^\infty|\tilde{v}^{(i)}(r)|^p r^{(k-2n)(t-p)/t}A(r,\Omega^*)dr\\&\leqslant\left\{\int_1^\infty|\tilde{v}^{(i)}(r)|^t A(r,\Omega^*)dr\right\}^{p/t}\left\{\int_1^\infty r^{k-2n}A(r,\Omega^*)dr\right\}^{(t-p)/t},\end{aligned}$$

它是有限的．因此$u\in W^{m,p}(\Omega)$，从而完成了证明．▌

有尖点区域的嵌入定理

5.33 已经证明对于很不规则的区域定理 5.4 完全失效，现在我们要证明对于虽然没有锥性质但不是太不规则的区域定理 5.4 中所考虑的某些类型的嵌入是成立的．这类问题曾为一些作者研究过，例如可以看 Globenko[26, 27]和 Maz'ja[44, 45]的工作，下面给出的处理方法是本书作者在一篇论文[1]中所给出的．

我们考虑区域$\Omega\subset\mathbf{R}^n$，$\Omega$的边界仅仅是由$(n-1)$维曲面构成的，又假定$\Omega$位于边界的一侧．$x_0\in\mathrm{bdry}\,\Omega$，如果不可能存在顶点在$x_0$，体积为正又包含在$\Omega$中的有限开锥的话，就说$\Omega$在$x_0\in\mathrm{bdry}\,\Omega$处有一个尖点(cusp)．当然区域$\Omega$没有任何尖点也并不蕴涵着$\Omega$具有锥性质．我们暂时考虑叫做标准尖点 (standard cusps) 区域的一族特殊区域，这些区域具有幂尖性的尖点 (cusps of power sharpness) (不如指数尖点 (cusps of exponential sharpness)那么尖的尖点).

5.34 如果 $1\leqslant k\leqslant n-1$ 而 $\lambda>1$, 设 $Q_{k,\lambda}$ 表示由不等式

$$
\begin{aligned}
&x_1^2+\cdots+x_k^2<x_{k+1}^{2\lambda}, \quad x_{k+1}>0, \cdots, x_n>0,\\
&(x_1^2+\cdots+x_k^2)^{1/\lambda}+x_{k+1}^2+\cdots+x_n^2<a^2,
\end{aligned}
\tag{46}
$$

规定的 $\mathbf{R}^n$ 中的标准尖点区域, 其中 a 是 $\mathbf{R}^n$ 中具有单位体积的球的半径. 我们注意到 $a<1$, $Q_{k,\lambda}$ 具有由 $x_{k+1}, \cdots, x_n$ 轴张起来的轴向平面和由 $x_{k+2}, \cdots, x_n$ 轴张起来的垂直平面(尖点平面). 如果$k=n-1$, 只有原点是 $Q_{k,\lambda}$ 的顶点. 为了计算简单起见把 $Q_{k,\lambda}$ 的外边界曲面取成(46)的形式. 球面或者不包括原点的适当的有界曲面都可以作为外表面.

与标准尖点区域 $Q_{k,\lambda}$ 相对应, 我们考虑一个借助于笛卡尔坐标, 由

$$
\begin{aligned}
&y_1^2+\cdots+y_k^2<y_{k+1}^2, \; y_{k+1}>0, \cdots, y_n>0,\\
&y_1^2+\cdots+y_n^2<a^2
\end{aligned}
$$

所规定的与 $Q_{k,\lambda}$ 相应的标准锥 $Q_k=Q_{k,1}$. 图 4 说明在 $\mathbf{R}^2$ 和 $\mathbf{R}^3$ 中的标准尖点区域和与其相应的标准锥. 在 $\mathbf{R}^3$ 中尖点区域 $Q_{2,2}$ 只有一个尖点在原点, 而 $Q_{1,2}$ 有一条沿着 x_3 轴的尖点线.

在 $\mathbf{R}^n$ 中采用广义"柱"坐标$(r_k, \phi_1, \cdots, \phi_{k-1}, y_{k+1}, \cdots, y_n)$是方便的, $r_k\geqslant 0$, $-\pi\leqslant\phi_1\leqslant\pi$, $0\leqslant\phi_2, \cdots, \phi_{k-1}\leqslant\pi$, 而

$$
\begin{aligned}
y_1&=r_k\sin\phi_1\sin\phi_2\cdots\sin\phi_{k-1},\\
y_2&=r_k\cos\phi_1\sin\phi_2\cdots\sin\phi_{k-1},\\
y_3&=\qquad\quad r_k\cos\phi_2\cdots\sin\phi_{k-1},\\
&\vdots\\
y_k&=\qquad\qquad\qquad\; r_k\cos\phi_{k-1},
\end{aligned}
\tag{47}
$$

利用这些坐标, Q_k 表为

$$
\begin{aligned}
&0\leqslant r_k<y_{k+1}, \; y_{k+1}>0, \cdots, y_n>0,\\
&r_k^2+y_{k+1}^2+\cdots+y_n^2<a^2.
\end{aligned}
$$

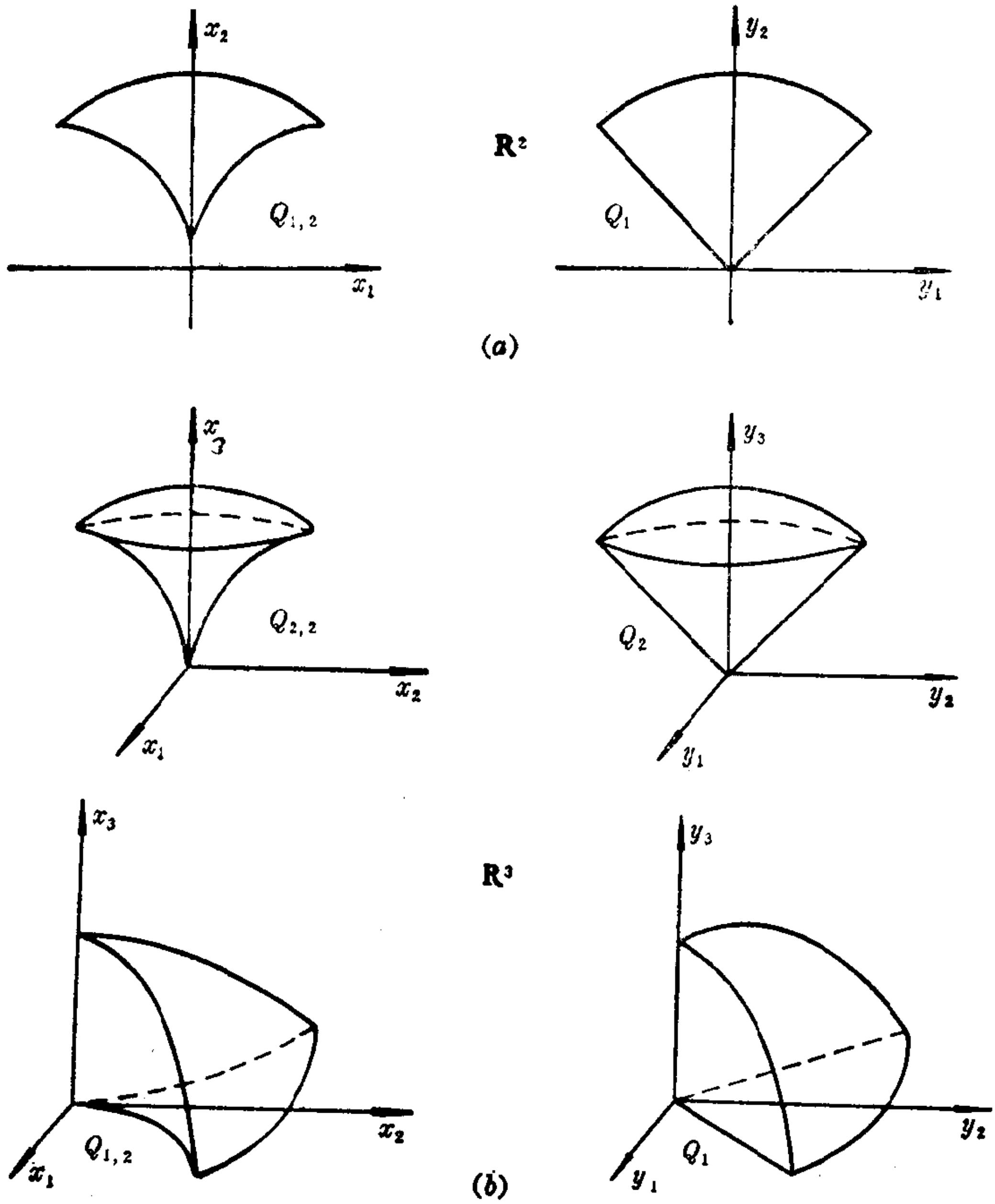

图 4. $\mathbf{R}^2$ 中(a)和 $\mathbf{R}^3$ 中(b)的标准尖点区域和标准锥

利用一对一的变换

$$
\begin{aligned}
x_1 &= r_k^{\lambda}\sin\phi_1\sin\phi_2\cdots\sin\phi_{k-1},\\
x_2 &= r_k^{\lambda}\cos\phi_1\sin\phi_2\cdots\sin\phi_{k-1},\\
x_3 &= \qquad\quad r_k^{\lambda}\cos\phi_2\cdots\sin\phi_{k-1},\\
&\ \ \vdots\\
x_k &= \qquad\qquad\qquad\quad r_k^{\lambda}\cos\phi_{k-1},
\end{aligned}
\tag{48}
$$

$$x_{k+1}=y_{k+1}$$
$$\vdots$$
$$x_n=y_n$$

可以把标准尖点区域 $Q_{k,\lambda}$ 变换到与之相应的锥 Q_k 中去，(48) 的 Jacobi 行列式为

$$\left|\frac{\partial(x_1,\cdots,x_n)}{\partial(y_1,\cdots,y_n)}\right|=\lambda r_k^{(\lambda-1)k}. \tag{49}$$

现在我们叙述三个定理，它们把定理 5.4 中考虑的嵌入类型(除迹嵌入外)推广到可与标准尖点区域相比较的边界不规则的区域. 这些定理的证明将在本章后面给出.

5.35 定理 设 Ω 是 $\mathbf{R}^n$ 中具有下列性质的区域：存在 Ω 的一族开子集合 Γ 使得

(i) $\Omega=\bigcup_{G\in\Gamma}G$;

(ii) Γ 具有有限交性质，即存在正整数 N 使得 Γ 中任意 $N+1$ 个不同的集合的交为空集;

(iii) 至多有一个集合 $G\in\Gamma$ 具有锥性质;

(iv) 存在正常数 $\nu>mp-n$ 和 A，使得对于任何没有锥性质的 $G\in\Gamma$ 存在一个一对一的函数 ψ 把 G 映到一个标准的尖点区域 $Q_{k,\lambda}$ 上去，其中 $(\lambda-1)k\leqslant\nu$，且使得对于所有的 $i,j(1\leqslant i,j\leqslant n)$，所有的 $x\in G$ 和所有的 $y\in Q_{k,\lambda}$，有

$$\left|\frac{\partial\psi_j}{\partial x_i}\right|\leqslant A \quad 和 \quad \left|\frac{\partial(\psi^{-1})_j}{\partial y_i}\right|\leqslant A.$$

于是

$$W^{m,p}(\Omega)\to L^q(\Omega),$$
$$p\leqslant q\leqslant\frac{(\nu+n)p}{\nu+n-mp}. \tag{50}$$

[如果 $\nu=mp-n$，则对于 $p\leqslant q<\infty$，(50) 成立(如果 $p=1$ 则 $q=$

∞). 如果 $\nu<mp-n$, 则对于 $p\leqslant q\leqslant\infty$, (50) 成立.]

5.36 定理 设 Ω 是 $\mathbf{R}^n$ 中具有下列性质的区域: 存在正常数 $\nu<mp-n$ 和 A 使得对于每个 $x\in\Omega$ 存在一个开集 G, $x\in G\subset\Omega$ 和一个一对一的映射 ψ, 把 G 映到一个标准尖点区域 $Q_{k,\lambda}$ 上, 其中 $(\lambda-1)k\leqslant\nu$, 而且使得对所有的 $i, j(1\leqslant i, j\leqslant n)$, 所有的 $x\in G$, 和所有的 $y\in Q_{k,\lambda}$, 有

$$\left|\frac{\partial\psi_j}{\partial x_i}\right|\leqslant A \qquad \text{和} \qquad \left|\frac{\partial(\psi^{-1})_j}{\partial y_i}\right|\leqslant A.$$

则

$$W^{m,p}(\Omega)\to C_B^0(\Omega). \tag{51}$$

更一般地, 如果 $\nu<(m-j)p-n$, 其中 $0\leqslant j\leqslant m-1$, 则

$$W^{m,p}(\Omega)\to C_B^j(\Omega).$$

5.37 定理 设 Ω 是 $\mathbf{R}^n$ 中具有下列性质的区域: 存在正常数 ν, δ 和 A 使得对每一对满足 $|x-y|\leqslant\delta$ 的点 x, $y\in\Omega$, 存在一个开集 G, 使 $x, y\in G\subset\Omega$, 还存在一个一对一的映射 ψ, 把 G 映到某个标准尖点区域 $Q_{k,\lambda}$ 上去, 其中 $(\lambda-1)k\leqslant\nu$, 还使得对所有的 i, j $(1\leqslant i, j\leqslant n)$, 所有的 $x\in G$, 和所有的 $y\in Q_{k,\lambda}$, 有

$$\left|\frac{\partial\psi_j}{\partial x_i}\right|\leqslant A \qquad \text{和} \qquad \left|\frac{\partial(\psi^{-1})_j}{\partial y_i}\right|\leqslant A.$$

假定对某个使得 $0\leqslant j\leqslant m-1$ 的 j, 我们有 $(m-j-1)p<\nu+n<(m-j)p$. 则

$$W^{m,p}(\Omega)\to C^{j,\mu}(\overline{\Omega}),$$
$$0<\mu\leqslant m-j-[(n+\nu)/p]. \tag{52}$$

如果 $(m-j-1)p=\nu+n$, 则对 $0<\mu<1$, (52) 成立. 两种情形都有 $W^{m,p}(\Omega)\to C^j(\overline{\Omega})$.

5.38 附注 (1) 读者可以构造与 5.25——5.28 节中类似的例子来说明上述三个定理对于所考虑的区域说来给出了最好可能的

嵌入.

(2) 下面的例子有助于说明定理 5.35: 设 $\Omega=\{(x_1, x_2, x_3)\in\mathbf{R}^3: x_2>0, x_2^2<x_1<3x_2^2\}$. 令 $a=(3/4\pi)^{1/3}$ 是 $\mathbf{R}^3$ 中具有单位体积的球的半径,容易验证变换

$$y_1=x_1-2x_2^2,\ y_2=x_2,\ y_3=x_3-(k/a),\quad k=0,\pm1,\pm2,\cdots,$$

像定理 5.35 的叙述中所要求的函数 ψ 那样地把 Ω 的子区域 G_k 变换到标准尖点区域 $Q_{1,2}$ 上去. 而且 $\{G_k\}_{k=-\infty}^{\infty}$ 具有有限交性质,除 Ω 上的一个具有锥性质的集合外盖住 Ω. 因此如果 $mp<4$ 则对于 $p\leqslant q\leqslant 4p(4-mp)$,如果 $mp=4$ 则对于 $p\leqslant q<\infty$, 如果 $mp>4$,则对于 $p\leqslant q\leqslant\infty$,有 $W^{m,p}(\Omega)\to L^q(\Omega)$.

包含带权范数的嵌入不等式

5.39 经过 (47) 和 (48) 把一个标准尖点区域映到与它相应的标准锥上去的技巧是证明定理 5.35 的主要技巧. 这种变换把形如 Jacobi 行列式 (49) 的权因子引入到那些有关的积分中去. 因此对于这种标准锥形区域我们一定会得到相应于由到锥的轴向平面的距离之幂给出的带权 L^p-范数的嵌入不等式. 在把嵌入定理 5.4 推广到更一般的包含带权范数的 Sobolev 空间去的时候, 这种不等式也是有用的.

我们从 $\mathbf{R}$ 中的固定开区间 $(0,T)$ 上的连续可微函数的某些一维不等式开始.

5.40 引理 如果 $\nu>0$ 而 $u\in C^1(0,T)$, 又如果 $\int_0^T|u'(t)|\,t^\nu dt<\infty$, 则 $\lim\limits_{t\to0+}|u(t)|\,t^\nu=0$.

证明 设给定 $\varepsilon>0$ 而且固定 s, $0<s<T/2$, 使它小到足以使任何满足 $0<t<s$ 的 t 有

$$\int_t^s |u'(\tau)|\tau^\nu d\tau < \varepsilon/3.$$

存在 $\delta, 0<\delta<s$，使得

$$\delta^\nu |u'(T/2)| < \varepsilon/3$$

且

$$(\delta/s)^\nu \int_s^{T/2} |u'(\tau)|\tau^\nu d\tau < \varepsilon/3.$$

如果 $0<t\leqslant\delta$，我们有

$$|u(t)| \leqslant |u(T/2)| + \int_t^{T/2} |u'(\tau)| d\tau$$

所以

$$t^\nu |u(t)| \leqslant \delta^\nu |u(T/2)| + \int_t^s |u'(\tau)|\tau^\nu d\tau$$

$$+ (\delta/s)^\nu \int_s^{T/2} |u'(\tau)|\tau^\nu d\tau < \varepsilon.$$

因此

$$\lim_{t\to 0+} t^\nu |u(t)| = 0. \quad \blacksquare$$

5.41 引理 如果 $\nu>0, p\geqslant 1$，而且 $u\in C^1(0,T)$，则

$$\int_0^T |u(t)|^p t^{\nu-1} dt \leqslant \frac{\nu+1}{\nu T}\int_0^T |u(t)|^p t^\nu dt$$

$$+\frac{p}{\nu}\int_0^T |u(t)|^{p-1}|u'(t)|\, t^\nu dt. \tag{53}$$

证明 不失一般性可以假定(53)的右端是有限的，而且假定$p=1$. 分部积分给出

$$\int_0^T |u(t)| \left[\nu t^{\nu-1} - \frac{\nu+1}{T} t^\nu\right] dt$$

$$= -\int_0^T \left[t^\nu - \frac{1}{T} t^{\nu+1}\right] \frac{d}{dt}|u(t)|\, dt;$$

引理 5.40 保证要积分的项在 $t=0$ 处等于零. 移项并估计一下右端的项就给出

$$\nu\int_0^T |u(t)|\, t^{\nu-1}dt$$

$$\leqslant\frac{\nu+1}{T}\int_0^T |u(t)|\, t^\nu dt+\int_0^T |u'(t)|\, t^\nu dt,$$

这就是 $p=1$ 时的(53)式. ▮

5.42 引理 如果 $\nu>0$, $p\geqslant1$, 而 $u\in C^1(0,T)$, 我们有下面一对不等式

$$\sup_{0<t<T}|u(t)|^p\leqslant\frac{2}{T}\int_0^T |u(t)|^p dt$$

$$+p\int_0^T |u(t)|^{p-1}|u'(t)|\,dt, \tag{54}$$

$$\sup_{0<t<T}|u(t)|^p t^\nu\leqslant\frac{\nu+3}{T}\int_0^T |u(t)|^p t^\nu dt$$

$$+2p\int_0^T |u(t)|^{p-1}|u'(t)|\, t^\nu dt. \tag{55}$$

证明 只要证 $p=1$ 的情形就可以了. 如果 $0<t\leqslant T/2$, 分部积分得到

$$\int_0^{T/2} \left|u\left(t+\frac{T}{2}-\tau\right)\right|d\tau=\frac{T}{2}|u(t)|$$

$$-\int_0^{T/2} \tau\frac{d}{d\tau}\left|u\left(t+\frac{T}{2}-\tau\right)\right|d\tau,$$

由此得到

$$|u(t)|\leqslant\frac{2}{T}\int_0^T |u(\sigma)|\,d\sigma$$

$$+\int_0^T |u'(\sigma)|\,d\sigma.$$

对于 $T/2\leqslant t<T$, 从对 $\int_0^{T/2} |u(t+\tau-T/2)|\,d\tau$ 的分部积分得到同样的不等式. 这就证明了 $p=1$ 时的 (54) 式. 在这个不等式中用 $u(t)t^\nu$ 代替 $u(t)$, 得到

$$\begin{aligned}\sup_{0<t<T}|u(t)|t^\nu\leqslant&\frac{2}{T}\int_0^T|u(t)|t^\nu dt\\&+\int_0^T[|u'(t)|t^\nu+\nu|u(t)|t^{\nu-1}]dt\\\leqslant&\frac{2}{T}\int_0^T|u(t)|t^\nu dt\\&+\int_0^T|u'(t)|t^\nu dt\\&+\nu\left\{\frac{\nu+1}{\nu T}\int_0^T|u(t)|t^\nu dt\right.\\&\left.+\frac{1}{\nu}\int_0^T|u'(t)|t^\nu dt\right\}.\end{aligned}$$

这里为了得到最后的那个不等式已经用到了(53)式. 这就是 $p=1$ 时的(55)式. ∎

5.43 现在我们转向 $\mathbf{R}^n$, $n\geqslant2$. 如果 $x\in\mathbf{R}^n$, 我们将利用球极坐标表示

$$x=(\rho,\phi)=(\rho,\phi_1,\cdots,\phi_{n-1}),$$

其中 $\rho\geqslant0$, $-\pi\leqslant\phi_1\leqslant\pi$, $0\leqslant\phi_2,\cdots,\phi_{n-1}\leqslant\pi$, 而且

$$\begin{aligned}x_1&=\rho\sin\phi_1\sin\phi_2\cdots\sin\phi_{n-1},\\x_2&=\rho\cos\phi_1\sin\phi_2\cdots\sin\phi_{n-1},\\x_3&=\rho\cos\phi_2\cdots\sin\phi_{n-1},\\&\vdots\\x_n&=\rho\cos\phi_{n-1}.\end{aligned}$$

体积元素是

$$dx=dx_1dx_2\cdots dx_n=\rho^{n-1}\prod_{j=1}^{n-1}\sin^{j-1}\phi_jd\rho d\phi,$$

其中 $d\phi=d\phi_1\cdots d\phi_{n-1}$.

对于 $1\leqslant k\leqslant n$ 定义函数 $r_k=r_k(x)$如下:

$$r_1(x)=\rho|\sin\phi_1|\prod_{j=2}^{n-1}\sin\phi_j,$$

$$r_k(x)=\rho\prod_{j=k}^{n-1}\sin\phi_j,\quad k=2,3,\cdots,n-1,$$

$$r_n(x)=\rho.$$

对于 $1\leqslant k\leqslant n-1$, $r_k(x)$是由 $x_{k+1},\cdots,x_n$ 轴张成的坐标平面到 x 的距离; $r_n(x)$正好是原点到 x 点的距离. 关于使用形如

$$P=\prod_{j=k}^{m}P_j$$

的连乘积符号，今后约定，如果 $m<k$，则 $P=1$.

设 Q 是 $\mathbf{R}^n$ 中由不等式

$$0<\rho<a,\quad -\beta_1<\phi_1<\beta_1,\quad 0\leqslant\phi_j<\beta_j,$$
$$j=2,3,\cdots,n-1 \tag{56}$$

规定的锥形开区域，其中 $0<\beta_i\leqslant\pi$. [与任何 $\beta_i=\pi$ 相对应，(56) 中的不等号"<"应代之以"⩽". 如果所有的 $\beta_i=\pi$，则第一个不等式应该用 $0\leqslant\rho<a$ 来代替.] 应该注意，任何(5.34 节引入的)标准锥 Q_k 都有(56)式的形式，其中参数 β_i, $1\leqslant i\leqslant n-1$，应作某种选择.

下面的引理用一种对我们说来是合适的方式推广了引理 5.41.

5.44 引理 设 Q 由(56)式规定，又设 $p\geqslant1$，假定或者 $m=k=1$，或者 $2\leqslant m\leqslant n$ 且 $1\leqslant k\leqslant n$. 还假定 $1-k<\nu_1\leqslant\nu\leqslant\nu_2<\infty$. 那么存在一个和 ν, a 无关的常数 $K=K(m,k,n,p,\nu_1,\nu_2,\beta_1,\cdots,\beta_{n-1})$ 使得对于一切函数 $u\in C^1(Q)$ 有

$$\int_Q|u(x)|^p[r_k(x)]^\nu[r_m(x)]^{-1}dx$$
$$\leqslant K\int_Q|u(x)|^{p-1}[(1/a)|u(x)|$$
$$+|\operatorname{grad}u(x)|][r_k(x)]^\nu dx. \tag{57}$$

证明 仍然只对 $p=1$ 证明 (57) 就够了. 设 $Q_+=\{x=(\rho,\phi)\in$

$Q:\phi_1\geqslant 0\}$, $Q_-=\{x\in Q:\phi_1\leqslant 0\}$. 则 $Q=Q_+\cup Q_-$. 我们只对Q_+(不过我们把 Q_+ 仍叫做 Q)证明(57); 类似地可以证明对 Q_- 的(57)式,所以对给定的 Q (57)式成立,因此,假定 $Q=Q_+$.

对于 $k\leqslant m$, 我们可以把(57)写成形式(取 $p=1$)

$$\begin{aligned}&\int_Q|u|\prod_{j=2}^{k-1}\sin^{j-1}\phi_j\prod_{j=k}^{m-1}\sin^{\nu+j-1}\phi_j\\&\quad\times\prod_{j=m}^{n-1}\sin^{\nu+j-2}\phi_j\rho^{\nu+n-2}d\rho d\phi\\&\leqslant K\int_Q\left[\frac{1}{a}|u|+|\operatorname{grad}u|\right]\prod_{j=2}^{k-1}\sin^{j-1}\phi_j\\&\quad\times\prod_{j=k}^{n-1}\sin^{\nu+j-1}\phi_j\rho^{\nu+n-1}d\rho d\phi.\end{aligned}\tag{58}$$

对于 $k>m\geqslant 2$ 我们可以把(57)写成形式

$$\begin{aligned}&\int_Q|u|\prod_{j=2}^{m-1}\sin^{j-1}\phi_j\prod_{j=m}^{k-1}\sin^{j-2}\phi_j\\&\quad\times\prod_{j=k}^{n-1}\sin^{\nu+j-2}\phi_j\rho^{\nu+n-2}d\rho d\phi\\&\leqslant K\int_Q\left[\frac{1}{a}|u|+|\operatorname{grad}u|\right]\prod_{j=2}^{k-1}\sin^{j-1}\phi_j\\&\quad\times\prod_{j=k}^{n-1}\sin^{\nu+j-1}\phi_j\rho^{\nu+n-1}d\rho d\phi.\end{aligned}\tag{59}$$

由于在引理的陈述中对 ν, m 和 k 的限制, (58)和(59)都是

$$\begin{aligned}&\int_Q|u|\prod_{j=1}^{i-1}\sin^{\mu_j}\phi_j\prod_{j=i}^{n-1}\sin^{\mu_j-1}\phi_j\rho^{\nu+n-2}d\rho d\phi\\&\leqslant K\int_Q\left[\frac{1}{a}|u|+|\operatorname{grad}u|\right]\\&\quad\times\prod_{j=1}^{n-1}\sin^{\mu_j}\phi_j\rho^{\nu+n-1}d\rho d\phi\end{aligned}\tag{60}$$

的特殊情形，其中 $1\leqslant i\leqslant n$，$\mu_j\geqslant 0$，而当 $j\geqslant i$ 时 $0<\mu_j^*\leqslant\mu_j$. 我们对 i 用向后归纳法来证明(60). 把 u 看作是定义在 $(0, a)$ 上 ρ 的函数，把引理 5.41 用上去，再对带有适当权的其余变量积分，就得到 $i=n$ 时的(60). 所以假定对 $i=k+1$ 已经证明，其中 $1\leqslant k\leqslant n-1$. 我们现在证明对 $i=k$ 时(60)也成立.

如果 $\beta_k<\pi$，我们有

$$\sin\phi_k\leqslant\phi_k\leqslant K_1\sin\phi_k, 0\leqslant\phi_k\leqslant\beta_k, \tag{61}$$

其中 $K_1=K_1(\beta_k)$. 由引理 5.41，又由于

$$|\partial u/\partial\phi_k|\leqslant\rho|\operatorname{grad} u|\prod_{j=k+1}^{n-1}\sin\phi_j,$$

我们有

$$\begin{aligned}
&\int_0^{\beta_k}|u(\rho,\phi)|\sin^{\mu_k-1}\phi_k d\phi_k\\
&\leqslant\int_0^{\beta_k}|u|\phi_k^{\mu_k-1}d\phi_k\\
&\leqslant K_2\int_0^{\beta_k}\left[|u|+|\operatorname{grad} u|\rho\prod_{j=k+1}^{n-1}\sin\phi_j\right]\phi_k^{\mu_k}d\phi_k\\
&\leqslant K_3\int_0^{\beta_k}\left[|u|+|\operatorname{grad} u|\rho\right.\\
&\qquad\left.\times\prod_{j=k+1}^{n-1}\sin\phi_j\right]\sin^{\mu_k}\phi_k d\phi_k.
\end{aligned}\tag{62}$$

注意到 K_2 从而 K_3 依赖于 β_k，但是在引理的条件下可以选得与 μ_k 无关，从而与 ν 无关. 如果 $\beta_k=\pi$，利用 $\int_0^{\pi}=\int_0^{\pi/2}+\int_{\pi/2}^{\pi}$，以及代替(61)利用不等式

$$\begin{aligned}
&\sin\phi_k\leqslant\phi_k\leqslant(\pi/2)\sin\phi_k \quad 当\ 0\leqslant\phi_k\leqslant\pi/2\ 时,\\
&\sin\phi_k\leqslant\pi-\phi_k\leqslant(\pi/2)\sin\phi_k \quad 当\ \pi/2\leqslant\phi_k\leqslant\pi\ 时
\end{aligned}\tag{63}$$

就得到(62). 利用(62)和归纳法假设，现在有

$$\int_Q |u| \prod_{j=1}^{k-1} \sin^{\mu_j}\phi_j \prod_{j=k}^{n-1} \sin^{\mu_j-1}\phi_j \rho^{\nu+n-2} d\rho d\phi$$

$$\leqslant \int_0^a \rho^{\nu+n-2} d\rho \prod_{j=1}^{k-1} \int_0^{\beta_j} \sin^{\mu_j}\phi_j d\phi_j$$

$$\times \prod_{j=k+1}^{n-1} \int_0^{\beta_j} \sin^{\mu_j-1}\phi_j d\phi_j \int_0^{\beta_k} |u| \sin^{\mu_k-1}\phi_k d\phi_k$$

$$\leqslant K_3 \int_Q |\mathrm{grad} u| \prod_{j=1}^{n-1} \sin^{\mu_j}\phi_j \rho^{\nu+n-1} d\rho d\phi$$

$$+ K_3 \int_Q |u| \prod_{j=1}^{k} \sin^{\mu_j}\phi_j$$

$$\times \prod_{j=k+1}^{n-1} \sin^{\mu_j-1}\phi_j \rho^{\nu+n-2} d\rho d\phi$$

$$\leqslant K \int_Q \left[\frac{1}{a}|u| + |\mathrm{grad}\, u|\right] \prod_{j=1}^{n-1} \sin^{\mu_j}\phi_j \rho^{\nu+n-1} d\rho d\phi.$$

这就用归纳法证明了(60)，因而证明了引理. ▮

在下面的引理中，对于区域Q和适当的带权L^p-范数我们得到一个类似于引理 5.10 的嵌入不等式.

5.45 引理 设Q由 (56) 规定，又设$p \geqslant 1$和$1 \leqslant k \leqslant n$. 假定$\max(1-k, p-n) < \nu_1 < \nu_2 < \infty$. 则存在一个与$a$无关的常数$K = K(k, n, p, \nu_1, \nu_2, \beta_1, \cdots, \beta_{n-1})$使得对于一切满足$\nu_1 \leqslant \nu \leqslant \nu_2$的$\nu$，和一切函数$u \in C^1(Q) \cap C(\bar{Q})$我们有

$$\left\{\int_Q |u(x)|^q [r_k(x)]^\nu dx\right\}^{1/q}$$

$$\leqslant K \left\{\int_Q [(1/a^p)|u(x)|^p + |\mathrm{grad}\, u(x)|^p] \times [r_k(x)]^\nu dx\right\}^{1/p}, \qquad (64)$$

其中$q = (\nu+n)p/(\nu+n-p)$.

证明 设 $\delta = (\nu+n-1)p/(\nu+n-p)$, $s = (\nu+n-1)/\nu$, $s' =$

$(\nu+n-1)/(n-1)$. 由 Hölder 不等式和引理 5.44($m=k$ 的情形)我们有

$$\begin{aligned}
&\int_Q |u(x)|^q[r_k(x)]^\nu dx \\
\leqslant &\left\{\int_Q |u|^\delta r_k{}^{\nu-1}dx\right\}^{1/s} \\
&\times\left\{\int_Q |u|^{n\delta/(n-1)}r_k{}^{n\nu/(n-1)}dx\right\}^{1/s'} \\
\leqslant &K_1\left\{\int_Q |u|^{\delta-1}[(1/a)|u|+|\operatorname{grad} u|]r_k^\nu dx\right\}^{1/s} \\
&\times\left\{\int_Q |u|^{n\delta/(n-1)}r_k{}^{n\nu/(n-1)}dx\right\}^{1/s'}. \qquad (65)
\end{aligned}$$

为了估计上式中最后一个积分,我们采用记号

$$\rho^*=(\phi_1,\cdots,\phi_{n-1}),$$
$$\phi_j^*=(\rho,\phi_1,\cdots,\hat{\phi}_j,\cdots,\phi_{n-1}),$$
$$j=1,2,\cdots,n-1.$$

其中“$\wedge$”表示省略该分量①, 设

$$Q_0^*=\{\rho^*:(\rho,\rho^*)\in Q,\ 对于\ 0<\rho<a\},$$
$$Q_j^*=\{\phi_j^*:(\rho,\phi)\in Q,\ 对于\ 0<\phi_j<\beta_j\}.$$

Q_0^* 和 Q_j^* $(1\leqslant j\leqslant n-1)$ 都是 R^{n-1} 中的区域. 我们在 Q_0^* 和 Q_j^* 上分别定义函数如下:

$$\begin{aligned}
[F_0(\rho^*)]^{n-1}&=[F_0(\phi_1,\cdots,\phi_{n-1})]^{n-1} \\
&=\sup_{0<\rho<a}[|u|^\delta\rho^{\nu+n-1}]\prod_{i=k}^{n-1}\sin^\nu\phi_i \\
&\quad\times\prod_{i=2}^{n-1}\sin^{i-1}\phi_i,
\end{aligned}$$

$$[F_j(\phi_j^*)]^{n-1}=[F_j(\rho,\phi_1,\cdots,\hat{\phi}_j,\cdots,\phi_{n-1})]^{n-1}$$

① 本书后面单独使用符号 $\hat{\phi}_j$时,表示$(\phi_1,\cdots,\phi_{j-1},\phi_{j+1},\cdots,\phi_{n-1})$——译者注.

$$= \sup_{0<\phi_j<\beta_j} [|u|^{\delta} \sin^{\nu+j-1}\phi_j] \rho^{\nu+n-2}$$

$$\times \prod_{i=k}^{n-1} \sin^{\nu}\phi_i \prod_{i=2}^{j-1} \sin^{i-1}\phi_i$$

$$\times \prod_{i=j+1}^{n-1} \sin^{i-2}\phi_i.$$

那么有

$$|u|^{n\delta/(n-1)} r_k^{n\nu/(n-1)} \rho^{n-1} \prod_{i=2}^{n-1} \sin^{i-1}\phi_i$$

$$\leqslant F_0(\rho^*) \prod_{j=1}^{n-1} F_j(\phi_j^*).$$

应用组合引理 5.9，得到

$$\int_Q |u|^{n\delta/(n-1)} r_k^{n\nu/(n-1)} dx$$

$$\leqslant \int_Q F_0(\rho^*) \prod_{j=1}^{n-1} F_j(\phi_j^*) d\rho d\phi$$

$$\leqslant \left\{ \int_{Q_0^*} [F_0(\rho^*)]^{n-1} d\phi \right.$$

$$\left. \times \prod_{j=1}^{n-1} \int_{Q_j^*} [F_j(\phi_j^*)]^{n-1} d\rho d\hat{\phi}_j \right\}^{1/(n-1)}. \tag{66}$$

根据引理 5.42，又由于$|\partial u/\partial \rho| \leqslant |\mathrm{grad}\, u|$，

$$\sup_{0<\rho<a} |u|^{\delta} \rho^{\nu+n-1}$$

$$\leqslant K_2 \int_0^a |u|^{\delta-1} [(1/a)|u|$$

$$+ |\mathrm{grad}\, u|] \rho^{\nu+n-1} d\rho,$$

其中对于 $1-n<\nu_1\leqslant\nu\leqslant\nu_2<\infty$ 的 ν 来说K与 ν 无关. 由此得到

$$\int_{Q_0^*} [F_0(\rho^*)]^{n-1} d\phi$$

$$\leqslant K_2\int_Q |u|^{\delta-1}[(1/a)|u|+|\operatorname{grad} u|]r_k^\nu dx. \tag{67}$$

类似地，利用引理 5.44 中的不等式(61)或(63)，从引理 5.42 得到

$$\sup_{0<\phi_j<\beta_j}|u|^\delta\sin^{\nu+j-1}\phi_j$$

$$\leqslant K_{2,j}\int_0^{\beta_j}|u|^{\delta-1}\left[\ |u|+\left|\frac{\partial u}{\partial\phi_j}\right|\ \right]\sin^{\nu+j-1}\phi_j d\phi_j$$

$$\leqslant K_{2,j}\int_0^{\beta_j}|u|^{\delta-1}[|u|+|\operatorname{grad} u|$$

$$\times\rho\prod_{i=j+1}^{n-1}\sin\phi_i]\sin^{\nu+j-1}\phi_j d\phi_j,$$

由于

$$|\partial u/\partial\phi_j|\leqslant\rho\prod_{i=j+1}^{n-1}\sin\phi_i.$$

因此

$$\int_{Q_j^*}[F_j(\phi_j^*)]^{n-1}d\rho d\hat{\phi}_j$$

$$\leqslant K_{2,j}\int_Q|\operatorname{grad} u||u|^{\delta-1}r_k^\nu dx$$

$$+K_{2,j}\int_Q|u|^\delta r_k^\nu r_{j+1}^{-1}dx$$

$$\leqslant K_{3,j}\int_Q|u|^{\delta-1}[(1/a)|u|+|\operatorname{grad} u|]r_k^\nu dx \tag{68}$$

这里，为了得到最后一个不等式，我们已经再次用了引理 5.44. 注意到对于允许的 ν 值可以选常数 $K_{2,j}$ 和 $K_{3,j}$ 和 ν 无关. 把 (67) 和(68)代入(66)然后又代入(65)就导至

$$\int_Q|u|^q r_k^\nu dx$$

$$\leqslant K_4\left\{\int_Q|u|^{\delta-1}[(1/a)|u|\right.$$

$$+|\operatorname{grad} u|r_k^\nu dx\Big\}^{1/s+n/(n-1)s'}$$

$$\leqslant K_4\left\{\left\{\int_Q |u|^q r_k^\nu dx\right\}^{(p-1)/p}\right.$$

$$\times\left\{2^{p-1}\int_Q[(1/a^p)|u|^p+|\operatorname{grad} u|^p]\right.$$

$$\left.\left.\times r_k^\nu dx\right\}^{1/p}\right\}^{(\nu+n)/(\nu+n-1)}.$$

因为$(\nu+n-1)/(\nu+n)-(p-1)/p=1/q$，又由于$u$在$Q$上有界，而且$\nu>1-n$，$\int_Q|u|^q r_k^\nu dx$是有限的，消去后就得到(64)．▌

5.46 附注 (1) $u\in C(\bar{Q})$这一假定只是用来保证上面所说的消去的合法性．事实上对于任何$u\in C^1(Q)$引理都成立．

(2) 如果$1-k<\nu_1<\nu_2<\infty$而且$\nu_1\leqslant\nu\leqslant\nu_2$，其中$\nu\leqslant p-n$，则对于任何满足$1\leqslant q<\infty$的$q$，(64)都成立．只要对大的$q$证明这点就行了．如果$q\geqslant(\nu+n)/(\nu+n-1)$，则对于满足$1\leqslant s<p$的某个$s$，$q=(\nu+n)s/(\nu+n-s)$．因此

$$\left\{\int_Q|u|^q r_k^\nu dx\right\}^{s/q}$$

$$\leqslant K\int_Q[(1/a^s)|u|^s+|\operatorname{grad} u|^s]r_k^\nu dx$$

$$\leqslant K\left\{2^{(p-s)/s}\int_Q[(1/a^p)|u|^p+|\operatorname{grad} u|^p]r_k^\nu dx\right\}^{s/p}$$

$$\times\left\{\int_Q r_k^\nu dx\right\}^{(p-s)/p},$$

由于右端的最后一个因子是有限的，故得(64)．

(3) 如果$\nu=m$是一正整数，那么能用如下方法非常简单地得到(64)．设$y=(x,z)=(x_1,\cdots,x_n,z_1,\cdots,z_m)$表示$\mathbf{R}^{n+m}$中的一个点，而且对于$x\in Q$定义$u^*(y)=u(x)$．如果

$$Q^*=\{y\in\mathbf{R}^{n+m}:y=(x,z),x\in Q,0<z_j<r_k(x),1\leqslant j\leqslant m\},$$

那么 Q^* 在 $\mathbf{R}^{n+m}$ 中具有锥性质，由此根据定理5.4，令 $q=(n+m)p/(n+m-p)$，我们有

$$\begin{aligned}&\left\{\int_Q |u|^q r_k^m dx\right\}^{1/q}\\&=\left\{\int_{Q^*}|u^*(y)|^q dy\right\}^{1/q}\\&\leqslant K\left\{\int_{Q^*}[(1/a^p)|u^*(y)|^p+|\operatorname{grad} u^*(y)|^p]dy\right\}^{1/p}\\&=K\left\{\int_Q[(1/a^p)|u|^p+|\operatorname{grad} u|^p]r_k^m dx\right\}^{1/p},\end{aligned}$$

这里用了 $|\operatorname{grad} u^*(y)|=|\operatorname{grad} u(x)|$，$u^*$ 与 z 无关.

（4）假定 $u\in C_0^\infty(\mathbf{R}^n)$，或更一般地，对于上述引理中的 ν，

$$\int_{\mathbf{R}^n}|u(x)|^p[r_k(x)]^\nu dx<\infty.$$

如果取 $\beta_i=\pi$，$1\leqslant i\leqslant n-1$，在(64)中令 $a\to\infty$，我们得到

$$\begin{aligned}&\left\{\int_{\mathbf{R}^n}|u(x)|^q[r_k(x)]^\nu dx\right\}^{1/q}\\&\leqslant K\left\{\int_{\mathbf{R}^n}|\operatorname{grad} u(x)|^p[r_k(x)]^\nu dx\right\}^{1/p},\end{aligned}$$

这个不等式推广了5.11节中给出的Sobolev不等式.

现在我们把引理5.15推广到允许带权范数的情形. 这里与其处理上述引理中考虑过的特殊情形 Q 还不如处理具有锥性质的任何区域来得方便. 下面的初等结果是需要的.

5.47 引理 设 $z\in\mathbf{R}^k$，又设 Ω 是 $\mathbf{R}^k$ 中体积有限的区域. 如果 $0\leqslant\nu<k$，则

$$\int_\Omega|x-z|^{-\nu}dx\leqslant K(\operatorname{vol}\Omega)^{1-\nu/k}, \tag{69}$$

其中常数 K 依赖于 ν 和 k 但不依赖于 z 或 Ω.

证明 设 B 是 $\mathbf{R}^k$ 中中心在 z 而体积等于 $\operatorname{vol}\Omega$ 的球. 容易验证

(69)的左边不超过$\int_B |x-z|^{-\nu}dx$,但是对于$\Omega=B$,(69)显然成立.∎

5.48 引理 设Ω是$\mathbf{R}^n$中具有锥性质的区域. 设$1\leqslant k\leqslant n$,又设P是$\mathbf{R}^n$中一个$(n-k)$维平面. 用$r(x)$表示x到P的距离,如果$0\leqslant\nu<p-n$,则对于所有的$u\in C^1(\Omega)$有

$$\sup_{x\in\Omega}|u(x)|\leqslant K\left\{\int_\Omega[|u(x)|^p + |\operatorname{grad} u(x)|^p][r(x)]^\nu dx\right\}^{1/p}, \tag{70}$$

其中常数K可以依赖于ν, n, p, k和决定Ω的锥性质的锥C但是不依赖于u.

证明 贯穿证明的始终A_i和K_i表示依赖于(70)中的K所允许依赖的参数ν, n, p, k和C中某几个的常数. 只要证明如果C是顶点在原点又包含在Ω中的有限锥,则

$$|u(0)|\leqslant K\left\{\int_C[|u(x)|^p+|\operatorname{grad}u(x)|^p]\times[r(x)]^\nu dx\right\}^{1/p}. \tag{71}$$

对于$0\leqslant j\leqslant n$,设A_j表示C在$\mathbf{R}^j$上投影的j维Lebesgue测度当取遍$\mathbf{R}^n$的所有j维子空间$\mathbf{R}^j$时的上确界. 记$x=(x', x'')$,其中$x'=(x_1,\cdots,x_{n-k})$而$x''=(x_{n-k+1},\cdots,x_n)$,不失一般性我们可以假定$P$与相应于$x''$的分量的坐标轴正交. 定义

$$S=\{x'\in R^{n-k}:(x', x'')\in C \quad 对于某些\ x''\in\mathbf{R}^k\}$$

$$R(x')=\{x''\in\mathbf{R}^k:(x', x'')\in C\} \quad 对于每个\ x'\in S.$$

对于$0\leqslant t\leqslant 1$我们用C_t来表示锥$\{tx: x\in C\}$,所以$C_t\subset C$而且当$t=1$时$C_t=C$. 对于C_t我们定义一些和前面对C所定义的量相似的量$A_{t,j}$, S_t和$R_t(x')$. 显然$A_{t,j}=t^jA_j$. 如果$x\in C$,我们有

$$u(x)=u(0)+\int_0^1\frac{d}{dt}u(tx)dt,$$

所以

$$|u(0)|\leqslant|u(x)|+|x|\int_0^1|\operatorname{grad} u(tx)|dt.$$

令 $V=\operatorname{vol} C$ 而 $a=\sup\limits_{x\in C}|x|$，把上述不等式在 C 上积分，得到

$$\begin{aligned}V|u(0)|&\leqslant\int_C|u(x)|dx+a\int_C\int_0^1|\operatorname{grad} u(tx)|dt\,dx\\&=\int_C|u(x)|dx+a\int_0^1 t^{-n}dt\\&\qquad\times\int_{C_t}|\operatorname{grad} u(x)|dx.\end{aligned}\tag{72}$$

设 z 表示 x 在 P 上的正交投影，则 $r(x)=|x''-z''|$. 由于 $0\leqslant\nu<p-n$ 故有 $p>1$，所以根据引理 5.47

$$\begin{aligned}&\int_{C_t}[r(x)]^{-\nu/(p-1)}dx\\&=\int_{S_t}dx'\int_{R_t(x')}|x''-z''|^{-\nu/(p-1)}dx''\\&\leqslant K_1\int_{S_t}[A_{t,k}]^{1-\nu/k(p-1)}dx'\\&\leqslant K_1[A_{t,k}]^{1-\nu/k(p-1)}A_{t,n-k}\\&=K_2t^{n-\nu/(p-1)}.\end{aligned}$$

由此得到

$$\begin{aligned}&\int_{C_t}|\operatorname{grad} u(x)|dx\\&\leqslant\left\{\int_{C_t}|\operatorname{grad} u(x)|^p[r(x)]^\nu dx\right\}^{1/p}\\&\qquad\times\left\{\int_{C_t}[r(x)]^{-\nu/(p-1)}dx\right\}^{1/p'}\\&\leqslant K_3t^{n-(\nu+n)/p}\left\{\int_C|\operatorname{grad} u(x)|^p[r(x)]^\nu dx\right\}^{1/p}.\end{aligned}\tag{73}$$

因此，由于 $\nu<p-n$

$$\int_0^1 t^{-n}dt\int_{C_t}|\operatorname{grad}u(x)|dx$$

$$\leqslant K_4\left\{\int_C|\operatorname{grad}u(x)|^p[r(x)]^\nu dx\right\}^{1/p}. \tag{74}$$

类似地，

$$\int_C|u(x)|dx\leqslant\left\{\int_C|u(x)|^p[r(x)]^\nu dx\right\}^{1/p}$$

$$\times\left\{\int_C[r(x)]^{-\nu/(p-1)}dx\right\}^{1/p'}$$

$$\leqslant K_5\left\{\int_C|u(x)|^p[r(x)]^\nu dx\right\}^{1/p}. \tag{75}$$

从(72)，(74)和(75)就得到(71).▊

5.49 引理 假定引理 5.48 的一切条件都满足，此外还假定 Ω 有强局部 Lipschitz 性质. 则对于一切 $u\in C^1(\Omega)$，有

$$\sup_{\substack{x,y\in\Omega\\x\neq y}}\frac{|u(x)-u(y)|}{|x-y|^\mu}$$

$$\leqslant K\left\{\int_\Omega[|u(x)|^p+|\operatorname{grad}u(x)|^p][r(x)]^\nu dx\right\}^{1/p}, \tag{76}$$

其中 $\mu=1-(\nu+n)/p$ 满足 $0<\mu<1$ 而 K 与 u 无关.

证明 证明和引理 5.17 中证明(28)式一样，只是在(29)式中用到的是不等式

$$\int_{\Omega_{t\sigma}}|\operatorname{grad}u(z)|dz$$

$$\leqslant K_1t^{n-(\nu+n)/p}\left\{\int_\Omega|\operatorname{grad}u(z)|^p[r(z)]^\nu dz\right\}^{1/p} \tag{77}$$

当 $\nu=0$ 时的特殊情形. 现在的证明中要用(77)式的一般情形，应用和前面得到(73)同样的方法就得到(77). ▊

定理 5.35—5.37 的证明

5.50 引理 设$\bar{\nu}\geqslant 0$. 若$\bar{\nu}>p-n$, 设$1\leqslant q\leqslant(\bar{\nu}+n)p/(\bar{\nu}+n-p)$; 否则设 $1\leqslant q<\infty$, 那么存在常数 $K=K(n,p,\bar{\nu})$ 使得对于一切标准尖点区域 $Q_{k,\lambda}$(见 5.34 节)有$(\lambda-1)k\equiv\nu\leqslant\bar{\nu}$, 而且对于一切 $u\in C^1(Q_{k,\lambda})$ 有

$$\|u\|_{0,q,Q_{k,\lambda}}\leqslant K\|u\|_{1,p,Q_{k,\lambda}}. \tag{78}$$

证明 由于每个 $Q_{k,\lambda}$ 具有线段性质, 只要对 $u\in C^1(\bar{Q}_{k,\lambda})$证明(78)就行了. 我们首先对给定的 k 和 λ 证明(78), 然后再证明可以选 K 使之与 k,λ 无关.

先假定 $\bar{\nu}>p-n$. 只要对

$$q=(\bar{\nu}+n)/(\bar{\nu}+n-p)$$

证明(78)就够了. 对于 $u\in C^1(\bar{Q}_{k,\lambda})$定义 $\tilde{u}(y)=u(x)$, 其中 y 和 x 是通过(47) 和 (48)联系起来的. 因此 $\tilde{u}\in C^1(Q_k)\cap C(\bar{Q}_k)$, 其中 Q_k 是与 $Q_{k,\lambda}$ 相应的标准锥. 根据引理 5.45, 又由于 $q\leqslant(\nu+n)p/(\nu+n-p)$, 我们有

$$\begin{aligned}\|u\|_{0,q,Q_{k,\lambda}}&=\left\{\lambda\int_{Q_k}|\tilde{u}(y)|^q[r_k(y)]^\nu dy\right\}^{1/q}\\&\leqslant K_1\left\{\int_{Q_k}[|\tilde{u}(y)|^p+|\mathrm{grad}\tilde{u}(y)|^p]\right.\\&\qquad\left.\times[r_k(y)]^\nu dy\right\}^{1/q}\end{aligned} \tag{79}$$

现在如果 $1\leqslant j\leqslant k$, 则 $x_j=r_k^{\lambda-1}y_j$; 如果 $k+1\leqslant j\leqslant n$, 则 $x_j=y_j$. 因为 $r_k^2=y_1^2+y_2^2+\cdots+y_k^2$, 我们有

$$\frac{\partial x_j}{\partial y_i}=\begin{cases}\delta_{ij}r_k^{\lambda-1}+(\lambda-1)r_k^{\lambda-3}y_iy_j & \text{如果 } 1\leqslant i,j\leqslant k\\ \delta_{ij} & \text{其它情形,}\end{cases}$$

由于在 Q_k 上 $r_k(y)\leqslant 1$, 由此得到

$$|\mathrm{grad}\,\tilde{u}(y)|\leqslant K_2|\mathrm{grad}\,u(x)|.$$

因此这时从(79)就得到(78). 对于 $\bar{\nu}\leqslant p-n$ 及任意 q 的情形，根据附注 5.46(2),证明是类似的.

为了证明: 假如 $\nu=(\lambda-1)k\leqslant\bar{\nu}$, 那么(78)中的常数 K 能选得和 k,λ 无关,我们指出只要证明存在一个常数 $\widetilde{K}$ 使得对任何这种 k,λ 和一切 $v\in C^1(Q_k)\cap C(\bar{Q}_k)$ 有

$$\left\{\iint_{Q_k}|v(y)|^q[r_k(y)]^\nu dy\right\}^{1/q}$$
$$\leqslant\widetilde{K}\left\{\iint_{Q_k}[|v(y)|^p+|\operatorname{grad} v(y)|^p]\right.$$
$$\left.\times[r_k(y)]^\nu dy\right\}^{1/p}. \tag{80}$$

事实上，只要证明带有 $\widetilde{K}$ 的(80)，其中 $\widetilde{K}$ 依赖于 k,因为我们可以在 k 的有限多个允许值上取最大值 $\widetilde{K}(k)$. 我们分三种情形来证明.

情形 I　$\bar{\nu}<p-n$, $1\leqslant q<\infty$,根据引理 5.48，对于 $0\leqslant\nu\leqslant\bar{\nu}$有

$$\sup_{x\in Q_k}|v(x)|\leqslant K(\nu)\left\{\iint_{Q_k}[|v(y)|^p\right.$$
$$\left.+|\operatorname{grad} v(y)|^p][r_k(y)]^\nu dy\right\}^{1/p}. \tag{81}$$

因为当 ν 增加时右边的积分减少,我们有 $K(\nu)\leqslant K(\bar{\nu})$, 由(81)和 Q_k 的有界性就得到(80).

情形 II　$\bar{\nu}>p-n$, 只要讨论 $q=(\bar{\nu}+n)p/(\bar{\nu}+n-p)$ 就够了, 从引理 5.45 我们得到

$$\left\{\iint_{Q_k}|v|^s r_k^\nu dy\right\}^{1/s}$$
$$\leqslant K_1\left\{\iint_{Q_k}[|v|^p+|\operatorname{grad} v|^p]r_k^\nu dy\right\}^{1/p}, \tag{82}$$

其中 $s=(\nu+n)p/(\nu+n-p)\geqslant q$, 对于 $p-n<\nu_0\leqslant\nu\leqslant\bar{\nu}$, K_1 与 ν

无关，根据 Hölder 不等式，又因为在 Q_k 上 $r_k(y)\leqslant 1$，有

$$\left\{\int_{Q_k}|v|^q r_k^\nu dy\right\}^{1/q}$$
$$\leqslant\left\{\int_{Q_k}|v|^s r_k^\nu dy\right\}^{1/s}[\mathrm{vol}\,Q_k]^{(s-q)/sq},$$

所以如果 $\nu_0\leqslant\nu\leqslant\bar{\nu}$，则从(82)得到(80)。

如果 $p-n<0$，我们可以取 $\nu_0=0$，因而证得(80)。否则的话，$p\geqslant n\geqslant 2$。固定 $\nu_0=(\bar{\nu}-n+p)/2$，能找到 ν_1 使得 $0\leqslant\nu_1\leqslant p-n$（或如果 $p=n$ 则 $\nu_1=0$），使得对于 $\nu_1\leqslant\nu\leqslant\nu_0$，有

$$1\leqslant t=\frac{(\nu+n)(\bar{\nu}+n)p}{(\nu+n)(\bar{\nu}+n)+(\bar{\nu}-\nu)p}\leqslant\frac{p}{1+\varepsilon_0},$$

其中 $\varepsilon_0>0$ 只依赖于 $\bar{\nu}$，n 和 p。因为后一不等式我们还可以假定 $t-n<\nu_1$。因为 $(\nu+n)t/(\nu+n-t)=q$，又由于引理 5.45 和 Hölder 不等式，有

$$\left\{\int_{Q_k}|v|^q r_k^\nu dy\right\}^{1/q}$$
$$\leqslant K_2\left\{\int_{Q_k}[|v|^t+|\mathrm{grad}v|^t]r_k^\nu dy\right\}^{1/t}$$
$$\leqslant 2^{(p-1)/pt}K_2\left\{\int_{Q_k}[|v|^p+|\mathrm{grad}v|^p]r_k^\nu dy\right\}^{1/p}[\mathrm{vol}Q_k]^{(p-t)/pt}, \tag{83}$$

其中对于 $\nu_1\leqslant\nu\leqslant\nu_0$，$K_2$ 与 ν 无关。

在 $\nu_1>0$ 的情形用情形 I 的方法能对 $0\leqslant\nu\leqslant\nu_1$ 得到一个类似的（一致）估计。把它与(82)，(83)结合起来，就证明了 $\nu_1>0$ 时的(80)。

情形 III　$\bar{\nu}=p-n$，$1\leqslant q<\infty$。固定 $s\geqslant\max(q,n/(n-1))$ 又设 $t=(\nu+n)s/(\nu+n+s)$，所以 $s=(\nu+n)t/(\nu+n-t)$。于是，对于 $0\leqslant\nu\leqslant\bar{\nu}$ 有 $1\leqslant t\leqslant ps/(p+s)<p$。因此我们能选 $\nu_1\geqslant 0$ 使得 $t-n<\nu_1<p-n$，余下的证明与情形 II 类似，这就完成了证

明. ▌

5.51 定理 5.35 的证明 通过和引理 5.12 的证明中使用过的一样的论证，只要考虑特殊情形 $m=1$就行了. 设 q 满足：当 $\nu+n>p$ 时，$p\leqslant q\leqslant(\nu+n)p/(\nu+n-p)$，当 $\nu+n\leqslant p$ 时，$p\leqslant q<\infty$. 显然，如果 $n>p$，则$q<np/(n-p)$，所以，不论在哪种情形根据定理5.4，对于一切 $u\in C^1(\Omega)$和具有锥性质的Γ的元素G（如果这种G存在的话），都有

$$\|u\|_{0,q,G}\leqslant K_1\|u\|_{1,p,G}.$$

如果$G\in\Gamma$ 没有锥性质，又如果 ψ: $G\to Q_{k,\lambda}$，其中$(\lambda-1)k\leqslant\nu$，是定理的陈述中所规定的 1-光滑映射，那么由定理 3.35 和引理 5.50，有

$$\begin{aligned}\|u\|_{0,q,G}&\leqslant K_2\|u\circ\psi^{-1}\|_{0,q,Q_{k,\lambda}}\leqslant K_3\|u\circ\psi^{-1}\|_{1,p,Q_{k,\lambda}}\\&\leqslant K_4\|u\|_{1,p,G}\end{aligned}$$

其中 K_4 与G 无关. 所以注意到 $q/p\geqslant1$，我们有

$$\begin{aligned}\|u\|^q_{0,q,\Omega}&\leqslant\sum_{G\in\Gamma}\|u\|^q_{0,q,G}\leqslant K_5\sum_{G\in\Gamma}(\|u\|^p_{1,p,G})^{q/p}\\&\leqslant K_5\left(\sum_{G\in\Gamma}\|u\|^p_{1,p,G}\right)^{q/p}\leqslant K_5N^{q/p}\|u\|^q_{1,p,\Omega},\end{aligned}$$

为了得到最后的不等式，我们已经用了Γ的有限交性质. 通过完备化就得到嵌入(50). [如果 $\nu<mp-n$. 则要求对 $q=\infty$的情形，(50)成立，这是下面要证明的定理 5.36 的一个推论.]▌

5.52 引理 设 $0\leqslant\bar{\nu}<mp-n$. 则存在常数 $K=K(m,p,n,\bar{\nu})$使得如果 $Q_{k,\lambda}$ 是任何满足$(\lambda-1)k=\nu\leqslant\bar{\nu}$ 的标准尖点区域，又如果 $u\in C^m(Q_{k,\lambda})$，则

$$\sup_{x\in Q_{k,\lambda}}|u(x)|\leqslant K\|u\|_{m,p,Q_{k,\lambda}}. \tag{84}$$

证明 只要对 $m=1$ 的情形证明引理就行了；通过和引理 5.15 证明的最后一段中所用的同样的论证就得到对于一般 m 的证明.

如果$u\in C^1(Q_{k,\lambda})$, $(\lambda-1)k=\nu\leqslant\bar{\nu}$, 根据引理 5.48 和利用引理 5.50 的证明第二段中的方法, 就有

$$\begin{aligned}\sup_{x\in Q_{k,\lambda}}|u(x)|&=\sup_{y\in Q_k}|\tilde{u}(y)|\\&\leqslant K_1\left\{\int_{Q_k}[|\tilde{u}(y)|^p+|\operatorname{grad}\tilde{u}(y)|^p][r_k(y)]^\nu dy\right\}^{1/p}\\&\leqslant K_2\left\{\int_{Q_{k,\lambda}}[|u(x)|^p+|\operatorname{grad}u(x)|^p]dx\right\}^{1/p}.\end{aligned}\tag{85}$$

因为对于 $y\in Q_k$, $r_k(y)\leqslant 1$, 假如 $0\leqslant\nu=(\lambda-1)k\leqslant\bar{\nu}$ 的话, 显然能选到与 k,λ 无关的 K_1, 从而选到与 k, λ 无关的 K_2. ▌

5.53 定理 5.36 的证明 只要证明(51) 就行了. 设 $u\in C^m(\Omega)$. 如果 $x\in\Omega$, 则对于某个区域 G, $x\in G\subset\Omega$, 而对 G, 如定理的陈述中所规定的那样, 存在 1-光滑变换 ψ: $G\to Q_{k,\lambda}$, $(\lambda-1)k\leqslant\nu$. 因此

$$\begin{aligned}|u(x)|&\leqslant\sup_{x\in G}|u(x)|=\sup_{y\in Q_{k,\lambda}}|u\circ\psi^{-1}(y)|\\&\leqslant K_1\|u\circ\psi^{-1}\|_{m,p,Q_{k,\lambda}}\leqslant K_2\|u\|_{m,p,G}\\&\leqslant K_2\|u\|_{m,p,\Omega},\end{aligned}\tag{86}$$

其中 K_1 和 K_2 与 G 无关. 证明的其余部分类似于引理 5.15 证明中的第一段. ▌

5.54 定理 5.37 的证明 和引理 5.17 的证明一样, 只要证明当 $j=0$ 和 $m=1$ 时 (52) 成立就行了, 这就是说证明当 $\nu+n<p$ 和 $0<\mu\leqslant 1-(\nu+n)/p$ 时

$$\sup_{\substack{x,y\in\Omega\\x\neq y}}\frac{|u(x)-u(y)|}{|x-y|^\mu}\leqslant K\|u\|_{1,p,\Omega}\tag{87}$$

成立. 由于(86)对于 $x,y\in\Omega$, $|x-y|>\delta$, (87)成立. 如果$|x-y|<\delta$, 则存在使 $x,y\in G\subset\Omega$ 的 G 以及一个 1-光滑变换 ψ, ψ 把 G 映到标准尖点区域 $Q_{k,\lambda}$ 上, k, λ 满足定理的条件 $(\lambda-1)k\leqslant\nu$. 用引理 5.52 的证明中使用过的同样的方法, 从引理 5.49 就能导出不等式(87). 详细证明留给读者. ▌

第六章　$W^{m,p}(\Omega)$的紧嵌入

Rellich-Kondrachov 定理

6.1　设Ω是 $\mathbf{R}^n$ 中一个区域，Ω_0 是Ω的一个子区域．又设 $X(\Omega)$ 为 $W^{m,p}(\Omega)$ 可能嵌入的目标空间，即 $X(\Omega)$ 是空间 $C_B^j(\Omega)$，$C^{j,\lambda}(\overline{\Omega})$，$L^q(\Omega^k)$或者 $W^{j,q}(\Omega^k)$中的一个，这里 Ω^k，$1\leqslant k\leqslant n$，是 Ω与 $\mathbf{R}^n$ 中一个 k 维超平面的交．由于线性限制算子 $i_{\Omega_0}: u\to u|_{\Omega_0}$ 是从 $X(\Omega)$ 到 $X(\Omega_0)$ 的有界算子［事实上 $\|i_{\Omega_0}u;\ X(\Omega_0)\|\leqslant\|u; X(\Omega)\|$］，因此对于任何一个嵌入

$$W^{m,p}(\Omega)\to X(\Omega), \tag{1}$$

利用这个限制能够产生一个嵌入

$$W^{m,p}(\Omega)\to X(\Omega_0), \tag{2}$$

并且(2)的嵌入常数不超过(1)的嵌入常数．

如果Ω满足 Sobolev 嵌入定理 5.4 的假设并且 Ω_0是有界的，那么除去某些特殊的情况之外所有的嵌入(2)(对应于定理 5.4 断言的嵌入）都是紧的．这些紧嵌入结果中最重要的部分来源于 Rellich[57] 的一个引理，对于 Sobolev 空间的情况 Kondrachov [33]给出了证明．紧嵌入在分析中有很多重要的应用，尤其是对于证明在有界区域上线性椭圆型偏微分算子谱的离散性．

在下面的定理中我们扼要的叙述 $W^{m,p}(\Omega)$的各种紧嵌入．

6.2　定理　(Rellich-Kondrachov 定理)　设Ω是 $\mathbf{R}^n$ 中的一个区域，Ω_0 是Ω的有界子区域，Ω^k 是Ω与 $\mathbf{R}^n$ 中一个k 维超平面的交．又设 j, m 是整数，$j\geqslant 0$，$m\geqslant 1$；p 是实数，$1\leqslant p<\infty$．

I　如果Ω具有锥形性质并且 $mp\leqslant n$，则下面的嵌入是紧的：

$$W^{j+m,p}(\Omega)\to W^{j,q}(\Omega_0^k),\ 0<n-mp<k\leqslant n,$$
$$1\leqslant q<kp/(n-mp), \tag{3}$$
$$W^{j+m,p}(\Omega)\to W^{j,q}(\Omega_0^k),\ \ n=mp,\ \ 1\leqslant k\leqslant n,$$
$$1\leqslant q<\infty. \tag{4}$$

II 如果Ω具有锥形性质并且 $mp>n$ 则下面的嵌入是紧的:

$$W^{j+m,p}(\Omega)\to C_B^j(\Omega_0), \tag{5}$$
$$W^{j+m,p}(\Omega)\to W^{j,q}(\Omega_0^k),\ \ 1\leqslant q\leqslant\infty. \tag{6}$$

III 如果Ω具有强局部 Lipschitz 性质则下面的嵌入是紧的:

$$W^{j+m,p}(\Omega)\to C^j(\overline{\Omega}_0),\ mp>n, \tag{7}$$
$$W^{j+m,p}(\Omega)\to C^{j,\lambda}(\overline{\Omega}_0),\ mp>n\geqslant(m-1)p,$$
$$0<\lambda<m-(n/p). \tag{8}$$

IV 若Ω是$\mathbf{R}^n$中一个任意的区域,在嵌入(3)—(8)中用$W_0^{j+m,p}(\Omega)$代替 $W^{j+m,p}(\Omega)$所得到的嵌入也是紧的.

6.3 附注 (1) 若 X,Y,Z 是三个空间, 对它们我们有嵌入 $X\to Y$, $Y\to Z$, 如果这两个嵌入中有一个是紧的,则合成的嵌入$X\to Z$是紧的.例如若 $Y\to Z$ 是紧的,那么X中任意一个有界序列$\{u_i\}$ 在 Y 中是有界的, 因而它就有一个子序列在 Z 中收敛.

由引理 3.22 我们知道扩张算子 $u\to\tilde{u}$ (这里 $\tilde{u}(x)=u(x)$, $x\in\Omega$; $\tilde{u}(x)=0, x\notin\Omega$.) 定义了一个嵌入 $W_0^{j+m,p}(\Omega)\to W^{j+m,p}(\mathbf{R}^n)$, 因而定理 6.2 的第 IV 部分结论可以由前面的 I—III 部分得到(取 Ω 为 $\mathbf{R}^n$).

(2) 只要对 $j=0$ 的情况证明了嵌入(3)—(8)的紧性, 那么对任意正整数 j 嵌入(3)—(8) 都是紧的, 例如假设对于嵌入(3)在 $j=0$ 的情况已经证明了它的紧性, 对于$j\geqslant1$, 考虑 $W^{j+m,p}(\Omega)$中一个有界序列$\{u_i\}$那么对每一个满足$|\alpha|\leqslant j$ 的 α, $\{D^\alpha u_i\}$ 是$W^{m,p}(\Omega)$中的有界集. 由假设我们知道 $\{D^\alpha u_i\}$ 在 $L^q(\Omega_0^k)$ (q 满足(3)中的条件)中是准紧的. 所以能够从 $\{u_i\}$ (用有限归纳法)抽出一个子

序列 $\{u'_i\}$ 使得对每一个满足 $|\alpha|\leqslant j$ 的 α, $\{D^\alpha u'_i\}$ 在 $L^q(\Omega_0^k)$ 中收敛, 因而 $\{u'_i\}$ 在 $W^{j,p}(\Omega_0^k)$ 中收敛, 因此嵌入(3)是紧的.

(3) 由于 Ω_0 是有界的, 对 $1\leqslant q<\infty$, 有 $C_B^0(\Omega_0^k)\to L^q(\Omega_0^k)$; 事实上 $\|u\|_{0,q,\Omega_0^k}\leqslant\|u;\ C_B^0(\Omega_0^k)\|[\mathrm{vol}_k\Omega_0^k]^{1/q}$. 因而嵌入(6)的紧性可以从嵌入(5)的紧性推出来($j=0$).

(4) 为了证明定理 6.2, 若 Ω 具有锥性质总可以假设有界子集 Ω_0 也具有锥性质. 如 C 是决定 Ω 的锥性质的一个有限锥; 设 $\widetilde{\Omega}$ 是所有包含在 Ω 中与 Ω_0 有非空交的与 C 全等的有限锥的并, 则 $\Omega_0\subset\widetilde{\Omega}\subset\Omega$, 并且 $\widetilde{\Omega}$ 是有界的, $\widetilde{\Omega}$ 具有锥性质. 若嵌入 $W^{m,p}(\Omega)\to X(\widetilde{\Omega})$ 是紧的, 因此, 限制在 Ω_0 上即得 $W^{m,p}(\Omega)\to X(\Omega_0)$ 也是紧嵌入.

如果 Ω 是有界的, 那么在定理的叙述中 Ω_0 可以等于 Ω.

6.4 定理 6.2III 的证明 如果 $mp>n\geqslant(m-1)p$ 并且 $0<\lambda<(m-n)/p$, 那么就存在一个 μ, $\lambda<\mu<m-(n/p)$. 因为 Ω_0 是有界的, 由定理 1.31 我们知道嵌入 $C^{0,\mu}(\overline{\Omega}_0)\to C^{0,\lambda}(\overline{\Omega}_0)$ 是紧的. 应用定理 5.4 和限制算子 有 $W^{m,p}(\Omega)\to C^{0,\mu}(\overline{\Omega})\to C^{0,\mu}(\overline{\Omega}_0)$. 利用附注 6.3(1)可知对于 $j=0$ 嵌入(8)是紧的.

如果 $mp>n$, 设 j^* 是满足条件 $(m-j^*)p>n\geqslant(m-j^*-1)p$ 的非负整数, 那么我们就可以得到一个嵌入串:

$$W^{m,p}(\Omega)\to W^{m-j^*,p}(\Omega)\to C^{0,\mu}(\overline{\Omega}_0)\to C(\overline{\Omega}_0),\tag{9}$$

其中 $0<\mu<m-j^*-(n/p)$. 由定理 1.31 知道(9)式中最后一个嵌入是紧的. 这样, 对 $j=0$ 嵌入(7)是紧的.

6.5 定理 6.2II 的证明 如附注 6.3(4)所指明的, 可以假设 Ω_0 具有锥性质. 又由于 Ω_0 是有界的, 由定理 4.8 Ω_0 可以表示为下述有限和: $\Omega_0=\bigcup\limits_{k=1}^{M}\Omega_k$, 其中每个 Ω_k 有强局部 Lipschitz 性质.

如果 $mp>n$ 那么 $W^{m,p}(\Omega)\to W^{m,p}(\Omega_k)\to C(\overline{\Omega}_k)$, 上面已经证明

了最后一个嵌入是紧的. 若$\{u_i\}$是$W^{m,p}(\Omega)$中的一个有界序列,我们可以(对k进行有限归纳)从中抽出一个子序列$\{u'_i\}$,对每个$k(1\leqslant k\leqslant M)$,$\{u'_i\}$在$\Omega_k$上的限制在$C(\overline{\Omega}_k)$上收敛. 那么$\{u'_i\}$就在$C_B^0(\Omega_0)$上收敛,这就对$j=0$证明了嵌入(5)是紧的. 由附注6.3(3)可得嵌入(6)也是紧的.

6.6 引理 设Ω是$\mathbf{R}^n$中的一个区域,Ω_0是Ω的一个子区域,Ω_0^k是Ω_0与$\mathbf{R}^n$中k维超平面的交$(1\leqslant k\leqslant n)$. 设对于$1\leqslant q_1<q_0$并有嵌入

$$W^{m,p}(\Omega)\to L^{q_0}(\Omega_0^k), \tag{10}$$

$$W^{m,p}(\Omega)\to L^{q_1}(\Omega_0^k). \tag{11}$$

又设嵌入(11)是紧的. 若$q_1\leqslant q<q_0$那么嵌入

$$W^{m,p}(\Omega)\to L^{q}(\Omega_0^k) \tag{12}$$

(存在并且)是紧的.

证明 令$\lambda=q_1(q_0-q)/q(q_0-q_1)$并且$\mu=q_0(q-q_1)/q(q_0-q_1)$. 显然$\lambda>0$, $\mu\geqslant 0$. 利用Hölder不等式和嵌入(10),对任意$u\in W^{m,p}(\Omega)$存在一个常数K使:

$$\begin{aligned}\|u\|_{0,q,\Omega_0^k}&\leqslant\|u\|_{0,q_1,\Omega_0^k}^{\lambda}\|u\|_{0,q_0,\Omega_0^k}^{\mu}\\&\leqslant K\|u\|_{0,q_1,\Omega_0^k}^{\lambda}\|u\|_{m,p,\Omega}^{\mu}.\end{aligned} \tag{13}$$

若$\{u_i\}$是$W^{m,p}(\Omega)$中一有界序列,由于嵌入(11)是紧的,所以存在一个收敛的子序列$\{u'_i\}$,它也就是$L^{q_1}(\Omega_0^k)$中的一个Cauchy序列. 由(13)可知$\{u'_i\}$也是$L^q(\Omega_0^k)$中的Cauchy序列,因而嵌入(12)是紧的.

6.7 定理6.2I的证明 首先我们对$j=0$的情况讨论嵌入(3),暂时设$k=n$, $q_0=np/(n-mp)$,由引理6.6我们知道为了证明嵌入

$$W^{m,p}(\Omega)\to L^q(\Omega_0),\quad 1\leqslant q<q_0 \tag{14}$$

是紧的,只需要对$q=1$证明嵌入(14)是紧的. 对$j=1,2,3,\cdots$令

$$\Omega_j=\{x\in\Omega_0:\operatorname{dist}(x,\operatorname{bdry}\Omega)>2/j\}.$$

设S是在$W^{m,p}(\Omega)$中有界的函数集合，我们指出S(当限制在Ω_0上)满足定理2.21的条件以说明S在$L^1(\Omega_0)$中是准紧的. 设给定$\varepsilon>0$，对每个$u\in W^{m,p}(\Omega)$令

$$\tilde{u}(x)=\begin{cases}u(x) & x\in\Omega_0\\ 0 & \text{其他}\end{cases}$$

由 Hölder 不等式和嵌入$W^{m,p}(\Omega)\to L^{q_0}(\Omega_0)$，我们有

$$\begin{aligned}\int_{\Omega_0\sim\Omega_j}|u(x)|dx&\leqslant\left\{\int_{\Omega_0\sim\Omega_j}|u(x)|^{q_0}dx\right\}^{\frac{1}{q_0}}\left\{\int_{\Omega_0\sim\Omega_j}1dx\right\}^{1-\frac{1}{q_0}}\\&\leqslant K_1\|u\|_{m,p,\Omega}[\mathrm{vol}(\Omega_0\sim\Omega_j)]^{1-\frac{1}{q_0}},\end{aligned}$$

其中K_1不依赖于u. 由于Ω_0有有限体积，可以选取j足够大使得对每个$u\in S$,

$$\int_{\Omega_0\sim\Omega_j}|u(x)|dx<\varepsilon,$$

并且对每个$h\in\mathbf{R}^n$,

$$\int_{\Omega_0\sim\Omega_j}|\tilde{u}(x+h)-\tilde{u}(x)|dx<\varepsilon/2. \tag{15}$$

若$|h|<1/j$, $x\in\Omega_j$, $0\leqslant t\leqslant1$则$x+th\in\Omega_{2j}$，如果$u\in C^\infty(\Omega)$，可以推出

$$\begin{aligned}\int_{\Omega_j}|u(x+h)-u(x)|dx&\leqslant\int_{\Omega_j}dx\int_0^1\left|\frac{du(x+th)}{dt}\right|dt\\&\leqslant|h|\int_0^1dt\int_{\Omega_{2j}}|\mathrm{grad}\,u(y)|dy\\&\leqslant|h|\|u\|_{1,1,\Omega_0}\leqslant K_2|h|\|u\|_{m,p,\Omega},\end{aligned}\tag{16}$$

这里K_2是不依赖于u的. 由于$C^\infty(\Omega)$在$W^{m,p}(\Omega)$中稠，(16)式对于$u\in W^{m,p}(\Omega)$也是成立的. 因此只要$|h|$足够小从(15)，(16)可得

$$\int_{\Omega_0}|\tilde{u}(x+h)-\tilde{u}(x)|dx<\varepsilon.$$

由定理2.21，S在$L^1(\Omega_0)$是准紧的，那么嵌入(14)就是紧的.

下面假设 $k<n, p>1$，可以选取一个 r 满足 $1<r<p, n-mr<k$，令 ν 是比 mr 小的最大整数，$s=kr/(n-mr)$，$q=nr/(n-mr)$. 由于我们可以假设 Ω_0 具有锥性质，我们从引理 5.19 证明中的不等式(35)(36)可得

$$\begin{aligned}\|u\|_{0,1,\Omega_0^k} &\leqslant K_3\|u\|_{0,s,\Omega_0^k}\\ &\leqslant K_4\|u\|_{0,q,\Omega_0^k}^{\lambda}\|u\|_{m,r,\Omega_0}^{1-\lambda}\\ &\leqslant K_5\|u\|_{0,q,\Omega_0}^{\lambda}\|u\|_{m,p,\Omega}^{1-\lambda},\end{aligned}\tag{17}$$

这里 $\lambda=n(mr-\nu)/mr(n-\nu)$ 满足 $0<\lambda<1$，并且常数 K_3, K_4, K_5 不依赖于 u，注意 $1<q<q_0$. 我们已经证明了如果 $\{u_i\}$ 是 $W^{m,p}(\Omega)$ 中的一个有界序列，一定可以抽出一个在 $L^q(\Omega_0)$ 中收敛的子序列 $\{u'_i\}$. 由(17)式 $\{u'_i\}$ 在 $L^1(\Omega_0^k)$ 中是一个 Cauchy 序列，因此嵌入 $W^{m,p}(\Omega)\to L^1(\Omega_0^k)$ 是紧的，由引理 6.6 可知：对 $1\leqslant q<kp/(n-mp)$ 嵌入 $W^{m,p}(\Omega)\to L^q(\Omega_0^k)$ 是紧的.

最后，假设 $p=1$，$0\leqslant n-m<k<n$. 显然 $n-m+1\leqslant k<n$，因此 $2\leqslant m\leqslant n$. 由定理 5.4 有嵌入 $W^{m,1}(\Omega)\to W^{m-1,r}(\Omega)$，其中 $r=n/(n-1)>1$. 由上面的证明对于 $k\geqslant n-(m-1)>n-(m-1)r$ 嵌入 $W^{m-1,r}(\Omega)\to L^1(\Omega_0^k)$ 是紧的. 这就足以完成嵌入 (3) 紧性的证明.

下面我们接着证明嵌入(4)的紧性，如果 $n=mp$，$p>1$，$1\leqslant q<\infty$，我们可以选取 r 使得 $1\leqslant r<p, k>n-mr>0, kr/(n-mr)>q$. 仍然假设 Ω_0 具有锥性质，我们有

$$W^{m,p}(\Omega)\to W^{m,r}(\Omega_0)\to L^q(\Omega_0^k).\tag{18}$$

在(3)中已经证明 (18) 中后面一个嵌入是紧的. 如果 $p=1$，$n=m\geqslant 2$，那么取 $r=n/(n-1)>1$，于是 $n=(n-1)r$，对 $1\leqslant q<\infty$ 我们有

$$W^{n,1}(\Omega)\to W^{n-1,r}(\Omega)\to L^q(\Omega_0^k),$$

像(18)的证明一样可证上式后面的嵌入是紧的. 最后，对于 $n=m$

$=p=1$，那么必然 $k=1$．任意取 $q_0>1$，完全按照上面考虑的(3)式中 $k=n$ 的情形，我们可以证明 $W^{1,1}(\Omega)\to L^1(\Omega_0)$ 的紧性．因为当 $1\leqslant q<\infty$，$W^{1,1}(\Omega)\to L^q(\Omega_0)$，再由引理 6.6，所有这些嵌入是紧的．▌

6.8 读者可以发现，将定理 6.2 推广到定理 5.35—5.37 中给出的嵌入是有益的．

两个反例

6.9 从 Rellich-Kondrachov 定理 6.2 的叙述中产生两个非常明显的问题．第一个问题是定理能不能推广到区域 Ω_0 是无界的情况？

第二个问题是"极端情况"

$$W^{j+m,p}(\Omega)\to W^{j,q}(\Omega_0^k),\quad 0<n-mp<k\leqslant n,\quad q=kp/(n-mp) \tag{19}$$

和

$$W^{j+m,p}(\Omega)\to C^{j,\lambda}(\overline{\Omega}_0)\qquad mp>n>(m-1)p,\quad \lambda=m-(n/p) \tag{20}$$

是不是紧的？

第一个问题将在这一章的后面研究．目前我们至少对 $k=n$ 的情况可以确切地给以否定的回答，除掉 Ω_0 是拟有界的情况，即除掉 Ω_0 满足

$$\lim_{\substack{x\in\Omega_0\\|x|\to\infty}}\operatorname{dist}(x,\operatorname{bdry}\Omega_0)=0$$

的情况．

6.10 例 设 Ω 是 $\mathbf{R}^n$ 中无界区域，而且 Ω 不是拟有界的．那么就存在由 Ω 中相互分离的开球组成的序列 $\{B_i\}$，这些开球有相同的正的半径．设 $\phi_1\in C_0^\infty(B_1)$，并且设 $\|\phi_1\|_{k,p,B_1}=A_{k,p}>0$ $k=0,1,$

2, …; $p\geqslant 1$ 令 φ_i 是φ_1 的一个使 φ_i 的支集在 B_i 中的平移. 那么显然对于任何固定的 j, m, p, $\{\phi_i\}$ 是空间 $W_0^{j+m,p}(\Omega)$ 中的有界集 .但是对于任意一个 q

$$\|\phi_i-\phi_k\|_{j,q,\Omega}=[\|\phi_i\|_{j,q,B_i}^q+\|\phi_k\|_{j,q,B_k}^q]^{1/q}=2^{1/q}A_{j,q}>0,$$

因此$\{\phi_i\}$在$W^{j,q}(\Omega)$中不能有收敛的子序列. 因而嵌入$W_0^{j+m,p}(\Omega)\to W^{j,q}(\Omega)$不可能是紧的. 定理 6.2 其他嵌入的非紧性可以类似地证明. ▌

下面我们说明, 6.9 节中提出的第二个问题, 其答案总是否定的.

6.11 例 设Ω是 $\mathbf{R}^n$中任意一个区域, Ω_0是Ω的任一有界子区域, Ω_0^k是 $\mathbf{R}^n$ 中 k 维超平面与 Ω_0 的交, 比如(不失一般性)可以认为此 k 维超平面是坐标轴 $x_1, x_2, \cdots x_k$ 所张成的. 设$\{a_1, a_2, \cdots\}$是Ω_0^k 中的相异点列, $\{r_1, r_2\cdots\}$ $(0<r_i\leqslant 1)$是一个数列, 并且具有下面性质, $B_{r_i}(a_i)=\{x\in\mathbf{R}^n: |x-a_i|<r_i\}\subset\Omega_0$, $B_{r_i}(a_i)$ 互不相交.

设 $\phi\in C_0^\infty(B_1(0))$; 并且满足下面条件:

(i) 对每个非负正数 h, 每个实数 $q\geqslant 1$ 及每个 $k(1\leqslant k\leqslant n)$ 我们有

$$|\phi|_{h,q,\mathbf{R}^k}=|\phi|_{h,q,\mathbf{R}^k\cap B_1(0)}$$
$$=\left\{\sum_{\substack{|\alpha|=h\\ \alpha_{k+1}=\cdots=\alpha_n=0}}\|D^\alpha\phi\|_{0,q,\mathbf{R}^k\cap B_1(0)}^q\right\}^{1/q}=A_{h,q,k}>0.$$

(ii) 存在一个 $a\in B_1(0)$, $a\neq 0$, 对每个非负的整数 h

$$|D_1^h\phi(a)|=B_h>0. \tag{21}$$

固定 $p\geqslant 1$ 及整数 $j\geqslant 0$ 和整数 $m\geqslant 1$. 对每一个整数 i 令

$$\phi_i(x)=r_i^{j+m-n/p}\phi((x-a_i)/r_i).$$

显然 $\phi_i\in C_0^\infty(B_{r_i}(a_i))$, 通过简单的计算可得

$$|\phi_i|_{h,q,\mathbf{R}^k}=r_i^{j+m-n/p-h+k/q}A_{h,q,k}. \tag{22}$$

如果 $h\leqslant j+m$ 从(22)式及 $r_i\leqslant 1$ 可以推出

$$|\phi_i|_{h,p,\mathbf{R}^n}\leqslant A_{h,p,n},$$

因而$\{\phi_i\}$是 $W^{j+m,p}(\Omega)$ 中的一个有界序列.

假设 $mp<n$, $n-mp<k\leqslant n$, 取 $q=kp/(n-mp)$, 我们从(22)式可得到

$$\|\phi_i\|_{j,q,\Omega_0^k}\geqslant|\phi_i|_{j,q,\mathbf{R}^k}=A_{j,q,k}.$$

由于函数$\{\phi_i\}$的支集是两两不相交的, 我们有

$$\|\phi_i-\phi_h\|_{j,q,\Omega_0^k}\geqslant 2^{1/q}A_{j,q,k}>0,$$

因而$\{\phi_i\}$在 $W^{j,q}(\Omega_0^k)$ 中没有收敛的子序列. 那么嵌入(19)不可能是紧的.

另一方面, 设$mp>n>(m-1)p$, $\lambda=m-(n/p)$, 令$b_i=a_i+r_ia$, 由(21)式得

$$|D_1^j\phi_i(b_i)|=r_i^{m-(n/p)}|D_1^j\phi(a)|=r_i^\lambda B_j>0.$$

令 $c_i=a_i+ar_i/|a|$, 这样 $c_i\in\operatorname{bdry}B_{r_i}(a_i)$ 并且 $|b_i-c_i|=(1-|a|)r_i$. 又由于$\{\phi_i\}$的支集是两两不相交的,

$$\begin{aligned}
&\|\phi_i-\phi_h;C^{j,\lambda}(\overline{\Omega}_0)\|\\
&\geqslant\max_{|\alpha|=j}\sup_{\substack{x,y\in\Omega_0\\x\neq y}}\frac{|D^\alpha(\phi_i(x)-\phi_h(x))-D^\alpha(\phi_i(y)-\phi_h(y))|}{|x-y|^\lambda}\\
&\geqslant\frac{|D_1^j\phi_i(b_i)-D_1^j\phi_h(b_i)-D_1^j\phi_i(c_i)+D_1^j\phi_h(c_i)|}{|b_i-c_i|^\lambda}\\
&=\frac{B_j}{(1-|a|)^\lambda}>0.
\end{aligned}$$

因此$\{\phi_i\}$在空间 $C^{j,\lambda}(\overline{\Omega}_0)$ 中不可能有收敛的子序列, 即嵌入(20)不可能是紧的. ▌

$W_0^{m,p}(\Omega)$ 在无界区域上的紧嵌入

6.12 设Ω是 $\mathbf{R}^n$ 中的一个无界区域, 我们将考查下面给定的嵌入

$$W_0^{m,p}(\Omega)\to L^p(\Omega) \tag{23}$$

是不是紧的，如果(23)是紧的，那由附注 6.3(2)和 6.7 节第二段可以推出嵌入

$$W_0^{j+m,p}(\Omega)\to W^{j,q}(\Omega^k),\ 0<n-mp<k\leqslant n,\ p\leqslant q<kp/(n-mp),$$

和

$$W_0^{j+m,p}(\Omega)\to W^{j,q}(\Omega^k),\ n=mp,\ 1\leqslant k\leqslant n,\ p\leqslant q<\infty$$

也是紧的.

在例 6.10 中我们已经指出除去 Ω 是拟有界的之外嵌入(23)不可能是紧的. 我们在定理 6.13 中对Ω给出一个几何条件，它充分保证嵌入(23)是紧的. 在定理 6.16 中我们对 (23) 的紧性给出一个充分必要的分析条件. 这两个定理来源于 Adams的工作[2].

令 Ω_r 表示集合 $\{x\in\Omega: |x|\geqslant r\}$. 在下面讨论中所涉及的任意的立方体$H$的表面都平行于坐标平面.

6.13 定理 令 ν 是一整数，满足 $1\leqslant\nu\leqslant n$, $mp>\nu$ (或者 $p=m=\nu=1$). 设对任给的 $\varepsilon>0$, 存在两个数 h 和 r 且 $0<h\leqslant 1, r\geqslant 0$ 使对每一个边长为 h 且与 Ω_r 的交非空的立方体 $H\in\mathbf{R}^n$ 有

$$\mu_{n-\nu}(H,\Omega)/h^{n-\nu}\geqslant h^p/\varepsilon,$$

其中 $\mu_{n-\nu}(H,\Omega)$ 是 $H\sim\Omega$ 在H的所有 $(n-\nu)$-维表面上投影 $P(H\sim\Omega)$的最大面积(即$(n-\nu)$-维测度), 则嵌入(23)是紧的.

6.14 上面的定理指出对给定的拟有界区域 Ω 嵌入 (23) 的紧性从本质上看依赖区域边界 bdryΩ 的维数. 我们考虑 $\nu=1$, $\nu=n$ 两种极端情况. 对于 $\nu=n$ 定理对Ω所加的条件就是Ω拟有界性这个最小限制. 因此如果 $mp>n$, 则嵌入(23) 对任意拟有界的Ω是紧的. 如果 $p>1$, Ω是拟有界区域而且其边界由没有有限聚点的孤立点构成，还可以证明除去 $mp>n$ 之外嵌入 (23) 不可能是紧的.

如果$\nu=1$，定理 6.13 的条件对 m, p 没有要求，但是要求边界 bdryΩ“实质上是 $n-1$ 维的”. 任意边界是由适当规则的 $n-1$ 维曲面组成的拟有界区域，是满足此条件的. 例如在 $\mathbf{R}^2$ 中一个“刺猬状”的区域（见图 5)就是这样的区域，它是由$\mathbf{R}^2$ 除去集合S_k $(k=1,2\cdots)$得到的,

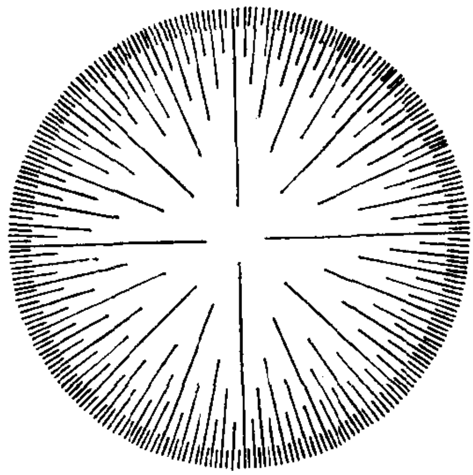

图 5 “刺猬状”的区域

$$S_k=\{(r,\theta):r\geqslant k,\ \theta=n\pi/2^k,\ n=1,2,\cdots,2^{k+1}\}.$$

注意这个区域虽然是拟有界的，但是它是单连通的，此区域的外部是空的. 一般地，如果ν是比 mp 小的最大整数，定理 6.13 中的条件要求在某种意义上，Ω的边界的至少 $n-\nu$ 维的部分界住一个拟有界区域.

6.15 设H是 $\mathbf{R}^n$ 中边长为 h 的正方体，E是H的一个闭子集. 对给定的 m,p 我们定义在 $C^\infty(H)$上的泛函 $I_H^{m,p}$:

$$I_H^{m,p}(u)=\sum_{1\leqslant j\leqslant m}h^{jp}|u|_{j,p,H}^p=\sum_{1\leqslant|\alpha|\leqslant m}h^{|\alpha|p}\int_H|D^\alpha u(x)|^p dx.$$

我们用$C^{m,p}(H,E)$表示E在H中的“$(m-p)$-容量”，$C^{m,p}(H,E)$是这样定义的:

$$C^{m,p}(H,E)=\inf_{u\in C^\infty(H,E)}\frac{I_H^{m,p}(u)}{\|u\|_{0,p,H}^p}$$

其中 $C^\infty(H,E)$是所有 $C^\infty(H)$ 中在E的一个邻域等于零的函数 u 组成的集合. 显然 $C^{m,p}(H,E)\leqslant C^{m+1,p}(H,E)$，并且对于 $E\subset F\subset H$ 有 $C^{m,p}(H,E)\leqslant C^{m,p}(H,F)$.

定理 6.13 可以简化为下面定理的形式，此定理利用区域的“$(m-p)$-体积”做为嵌入(23)紧性的条件.

6.16 定理 嵌入 (23) 是紧的充分必要条件是 Ω 满足下面条

件：对任给 $\varepsilon>0$ 存在 $h\leqslant 1, r\geqslant 0$，使对任意以 h 为边长，与 Ω_r 的交是非空的 n 维正方体 H 满足不等式

$$C^{m,p}(H, H\sim\Omega)\geqslant h^p/\varepsilon.$$

（显然这个条件包括了 Ω 是一个拟有界区域．）

6.17 引理 对任意以 h 为边长的 n 维-正方体 H 及 H 的任意一个体积为正的可测子集 A，和任意的 $u\in C^1(H)$ 存在一个常数 $K=K(n,p)$ 使下面不等式成立：

$$\|u\|_{0,p,H}^p\leqslant\frac{2^{p-1}h^n}{\mathrm{vol}A}\|u\|_{0,p,A}^p+K\frac{h^{n+p}}{\mathrm{vol}A}\|\mathrm{grad}u\|_{0,p,H}^p. \tag{24}$$

证明 令 $y\in A, x=(\rho,\phi)\in H$，其中 (ρ,ϕ) 是以 y 为原点的球极坐标，其中体积元由 $dx=\omega(\phi)\rho^{n-1}d\rho d\phi$ 给出，H 的边界用 $\rho=f(\phi)$，$\phi\in\Sigma$ 表示，显然 $f(\phi)\leqslant\sqrt{n}h$．因为

$$u(x)=u(y)+\int_0^\rho\frac{d}{dr}u(r,\phi)dr,$$

利用 Hölder 不等式

$$\begin{aligned}\int_H|u(x)|^pdx&\leqslant 2^{p-1}|u(y)|^ph^n+2^{p-1}\int_H\left|\int_0^\rho\frac{d}{dr}u(r,\phi)dr\right|^pdx\\&\leqslant 2^{p-1}h^n|u(y)|^p+2^{p-1}\int_\Sigma\omega(\phi)d\phi\int_0^{f(\phi)}\rho^{n+p-2}d\rho\\&\qquad\times\int_0^\rho|\mathrm{grad}u(r,\phi)|^pdr\\&\leqslant 2^{p-1}h^n|u(y)|^p+\frac{2^{p-1}}{n+p-1}(\sqrt{n}h)^{n+p-1}\int_H\frac{|\mathrm{grad}u(z)|^p}{|z-y|^{n-1}}dz.\end{aligned}$$

由上式在 A 上对 y 积分，用引理 5.47 可得

$$(\mathrm{vol}A)\|u\|_{0,p,H}^p\leqslant 2^{p-1}h^n\|u\|_{0,p,A}^p+Kh^{n+p}\int_H|\mathrm{grad}u(x)|^pdx,$$

由此可立刻得到(24)式．∎

6.18 定理 6.16 的证明 （必 要 性） 设 Ω 不满足定理中所叙述的条件．那么就存在一个有限的常数 $K_1=1/\varepsilon$，使得对每一个 h

$(0<h\leqslant 1)$都可以找到一个边长为 h 的正方体序列$\{H_j\}$，$\{H_j\}$中的正方体与Ω都是相交的，而且它们之间是两两不相交的；满足

$$C^{m,p}(H_j, H_j\sim\Omega)<K_1h^p.$$

由“$m-p$-容量”的定义可知对于每一个 H_j 存在 $u_j\in C^\infty(H_j, H_j\sim\Omega)$使得$\|u_j\|^p_{0,p,H_j}=h^n$，$\|\mathrm{grad}u_j\|^p_{0,p,H_j}\leqslant K_1h^n$ 并且 $\|u_j\|^p_{m,p,H_j}\leqslant K_2(h)$. 令 $A_j=\{x\in H_j;\ |u_j(x)|<\frac{1}{2}\}$. 由引理 6.17 可得

$$h^n\leqslant\frac{2^{p-1}h^n}{\mathrm{vol}A_j}\cdot\frac{\mathrm{vol}A_j}{2^p}+\frac{KK_1}{\mathrm{vol}A_j}h^{2n+p},$$

由上式可得 $\mathrm{vol}A_j\leqslant K_3h^{n+p}$. 我们选取 h 充分小使 $K_3h^p<\frac{1}{3}$，这样 $\mathrm{vol}A_j\leqslant\frac{1}{3}\mathrm{vol}H_j$. 在 $C_0^\infty(H_j)$中选取函数 w_j，使在 H_j 的子集 S_j (S_j 的体积不小于 H_j 的$\frac{2}{3}$)上 $w_j(x)=1$，并满足

$$\sup_j\max_{|\alpha|\leqslant m}\sup_{x\in H_j}|D^\alpha w_j(x)|=K_4<\infty.$$

那么 $v_j=u_jw_j\in C_0^\infty(H_j\cap\Omega)\subset C_0^\infty(\Omega)$ 并且在 $S_j\cap(H_j\sim A_j)$ 上 $|v_j(x)|\geqslant\frac{1}{2}$，$S_j\cap(H_j\sim A_j)$的体积不小于$h^n/3$，因此$\|v_j\|^p_{0,p,H_j}\geqslant h^n/3\cdot 2^p$. 另一方面，如果$|\alpha|$，$|\beta|\leqslant m$，

$$\int_{H_j}|D^\alpha u_j(x)|^p\cdot|D^\beta w_j(x)|^p dx\leqslant K_4^pK_2(h).$$

因此$\{v_j\}$是 $W_0^{m,p}(\Omega)$ 中的有界序列. 由于函数 v_j 的支集是相互不交的，则有 $\|v_j-v_k\|^p_{0,p,\Omega}\geqslant 2h^n/3\cdot 2^p$ 因而嵌入 (23) 不可能是紧的.

（充分性） 现在我们假设Ω满足定理中叙述的条件. 对任给的 $\varepsilon>0$ 都可以找到 $r\geqslant 0$，$h\leqslant 1$ 使每一个边长为 h 与 Ω_r 相交的正方体有 $C^{m,p}(H, H\sim\Omega)\geqslant h^p/\varepsilon^p$. 那么对每个函数 $u\in C_0^\infty(\Omega)$我们可得

$$\|u\|_{0,p,H}^{p}\leqslant(\varepsilon^{p}/h^{p})I_{H}^{m,p}(u)\leqslant\varepsilon^{p}\|u\|_{m,p,H}^{p}.$$

由于 Ω_r 的某一邻域可以分割成这样一些正方体，由上式的和可得

$$\|u\|_{0,p,\Omega_r}\leqslant\varepsilon\|u\|_{m,p,\Omega}.$$

那么由定理2.22和6.2立刻可以推出 $W_0^{m,p}(\Omega)$ 中的任意有界集 S 在 $L^p(\Omega)$ 中是准紧的. ▌

6.19 引理 对 $\mathbf{R}^n$ 中边长为 h 的任意正方体 H，存在一个不依赖于 h 的常数 K 对每一个满足 $p\leqslant q\leqslant np/(n-mp)$ 的 q（如果 $n=mp$, $p\leqslant q<\infty$; 如果 $n<mp$, $p\leqslant q\leqslant\infty$）及任意 $u\in C^\infty(H)$ 有下面不等式成立

$$\|u\|_{0,q,H}\leqslant K\left\{\sum_{|\alpha|\leqslant m}h^{|\alpha|p-n+np/q}\|D^\alpha u\|_{0,p,H}^{p}\right\}^{1/p}. \tag{25}$$

证明 我们可以假设 H 的中心在坐标原点，以 $\widetilde{H}$ 表示与 H 有相同中心、边长为单位的正方体，对于 $\widetilde{H}$ 由 Sobolev 嵌入定理，不等式(25)是成立的. 对于每一个 $u\in C^\infty(H)$ 我们对应地定义一个 $\tilde{u}\in C^\infty(\widetilde{H})$, $\tilde{u}(y)=u(x)$, 其中 $x=hy$. 因而有

$$\left\{\int_{\widetilde{H}}|\tilde{u}(y)|^q dy\right\}^{1/q}\leqslant K\left\{\sum_{|\alpha|\leqslant m}\int_{\widetilde{H}}|D_y^\alpha\tilde{u}(y)|^p dy\right\}^{1/p},$$

由自变量的代换可得

$$h^{-n/q}\left\{\int_H|u(x)|^q dx\right\}^{1/q}\leqslant K\left\{\sum_{|\alpha|\leqslant m}h^{|\alpha|p-n}\int_H|D_x^\alpha u(x)|^p dx\right\}^{1/p},$$

由此即得(25)式. ▌

6.20 引理 假设 $mp>n$(或者 $m=p=n=1$)，那么对 $\mathbf{R}^n$ 中任意边长为 h 的正方体 H，及 $C^\infty(H)$ 中在 H 内某一点 y 的邻域内等于零的函数 u，存在一个常数 $K=K(m,p,n)$ 满足

$$\|u\|_{0,p,H}^{p}\leqslant KI_H^{m,p}(u).$$

证明 这个证明与引理5.15的证明有点相似. 首先设 $p\leqslant n<mp$, (ρ,ϕ) 表示中心在点 y 的极坐标. 那么

$$u(\rho,\phi)=\int_0^\rho \frac{d}{dt}u(t,\phi)dt.$$

如果 $n>(m-1)p$，令 $q=np/(n-mp+p)$，这样 $q>n$. 其他情况我们令 $q>n$ 是任意的. 若$(\rho,\phi)\in H$，由 Hölder 不等式可得

$$|u(\rho,\phi)|^q\rho^{n-1}\leqslant(\sqrt{n}h)^{n-1}\int_0^\rho\left|\frac{d}{dt}u(t,\phi)\right|^q t^{n-1}dt$$

$$\times\left\{\int_0^{\sqrt{n}h}t^{-(n-1)/(q-1)}dt\right\}^{q-1}$$

$$\leqslant K_1h^{q-1}\int_0^\rho\left|\frac{d}{dt}u(t,\phi)\right|^q t^{n-1}dt.$$

以 $m-1$ 代替 m 利用引理 6.19 可以推出

$$\|u\|_{0,q,H}^q\leqslant K_2h^q\int_H|\mathrm{grad}u(x)|^q dx$$

$$\leqslant K_2h^q\sum_{|\alpha|=1}\|D^\alpha u\|_{0,q,H}^q$$

$$\leqslant K_3h^q\sum_{|\alpha|=1}\left\{\sum_{|\beta|\leqslant m-1}h^{|\beta|p-n+np/q}\|D^{\alpha+\beta}u\|_{0,p,H}^p\right\}^{q/p}. \quad (26)$$

进一步应用 Hölder 不等式得

$$\|u\|_{0,p,H}^p\leqslant\|u\|_{0,q,H}^p(\mathrm{vol}H)^{(q-p)/q}$$

$$\leqslant K_3^{p/q}\sum_{1\leqslant|\gamma|\leqslant m}h^{|\gamma|p}\|D^\gamma u\|_{0,p,H}^p=KI_H^{m,p}(u).$$

对于 $p>n$(或者 $p=n=1$)可以由(26) 式取 $q=p$ 直接得到引理的结果

$$\|u\|_{0,p,H}^p\leqslant KI_H^{1,p}(u)\leqslant KI_H^{m,p}(u).\ \blacksquare$$

6.21 定理 6.13 的证明 设H是具有下面性质的正方体，对 $mp>\nu$(或 $m=p=\nu=1$) $\mu_{n-\nu}(H,\ \Omega)/h^{n-\nu}\geqslant h^p/\varepsilon$. 设$P$是定理中所叙述的最大投影，$E=P(H\sim\Omega)$. 不失一般性，我们可以设包含 E 的H的$(n-\nu)$维表面F是平行于坐标平面 $x_{\nu+1},\cdots,x_n$ 的. 对于 E 中的每个点 $x=(x',x'')$，其中 $x'=(x_1,\cdots,x_\nu)$，$x''=(x_{\nu+1},\cdots,x_n)$，

以 $H_{x''}$ 表示具有下面性质的边长为 h 的 ν-维立方体，它是过点 x 垂直于 F 的 ν 维平面与 H 的交. 由 P 的定义我们知道存在一个 $y\in H_{x''}\sim\Omega$. 如果 $u\in C^\infty(H, H\sim\Omega)$，那么 $u(\cdot, x'')\in C^\infty(H_{x''}, y)$. 对 $u(\cdot, x'')$ 应用引理 6.20, 我们可得

$$\int_{H_{x''}}|u(x', x'')|^p dx'$$

$$\leqslant K_1\sum_{1\leqslant|\alpha|\leqslant m}|h|^{|\alpha|p}\int_{H_{x''}}|D^\alpha u(x', x'')|^p dx',$$

其中 K_1 不依赖于 h, x'' 及 u. 对上面不等式在 E 上积分，记

$$H'=\{x' : x=(x', x'')\in H \text{ 对某些 } x''\},$$

我们可得到

$$\|u\|^p_{0,p,H'\times E}\leqslant K_1 I^{m,p}_{H'\times E}(u)\leqslant K_1 I^{m,p}_H(u).$$

再应用引理 6.17, 取 $A=H'\times E$, 这样 $\operatorname{vol}A=h^\nu\mu_{n-\nu}(H, \Omega)$. 这就得

$$\|u\|^p_{0,p,H}\leqslant K_2\frac{h^{n-\nu}}{\mu_{n-\nu}(H, \Omega)}I^{m,p}_H(u).$$

其中 K_2 不依赖于 h. 由此可以推出

$$C^{m,p}(H, H\sim\Omega)\geqslant\frac{\mu_{n-\nu}(H, \Omega)}{K_2 h^{n-\nu}}\geqslant\frac{h^p}{\varepsilon K_2}.$$

因此如果 Ω 满足定理 6.13 中的假设，则它满足定理 6.16 的假设.▌

下面两个关于插值的引理使我们能够将定理 6.13 推广为空间 $W^{m,p}_0$ 到连续函数空间的嵌入.

6.22 引理 设 $1\leqslant p<\infty$ 并且 $0<\mu\leqslant 1$，则存在一个常数 $K=K(n, p, \mu)$ 对任意 $u\in C^\infty_0(\mathbf{R}^n)$ 有

$$\sup_{x\in\mathbf{R}^n}|u(x)|\leqslant K\|u\|_{0,p,\mathbf{R}^n}\left\{\sup_{\substack{x,y\in\mathbf{R}^n\\ x\neq y}}\frac{|u(x)-u(y)|}{|x-y|^\mu}\right\}^{1-\lambda}, \qquad (27)$$

其中 $\lambda=p\mu/(n+p\mu)$.

证明 我们可以设

$$\sup_{x\in\mathbf{R}^n}|u(x)|=N>0,\ \sup_{x,y\in\mathbf{R}^n}\frac{|u(x)-u(y)|}{|x-y|^\mu}=M<\infty.$$

令 ε 满足 $0<\varepsilon\leqslant N/2$. 则存在一个点 $x_0\in\mathbf{R}^n$ 使 $|u(x_0)|\geqslant N-\varepsilon\geqslant N/2$. 由于对所有的 $x\in\mathbf{R}^n$, $|u(x_0)-u(x)|/|x_0-x|^\mu\leqslant M$, 因此对满足 $|x-x_0|\leqslant(N/4M)^{1/\mu}=r$ 的 x 有下式成立:

$$|u(x)|\geqslant|u(x_0)|-M|x_0-x|^\mu\geqslant\frac{1}{2}|u(x_0)|.$$

因此

$$\int_{\mathbf{R}^n}|u(x)|^p dx\geqslant\int_{B_r(x_0)}\left(\frac{|u(x_0)|}{2}\right)^p dx$$

$$\geqslant K_1\left(\frac{N-\varepsilon}{2}\right)^p\left(\frac{N}{4M}\right)^{n/\mu}.$$

由于上式对任意小的 $\varepsilon>0$ 成立, 所以我们有

$$\|u\|_{0,p,\mathbf{R}^n}\geqslant(K_1^{1/p}/2\cdot 4^{n/\mu p})N^{1+n/\mu p}M^{-n/\mu p},$$

由此可立即得到(27)式.▌

6.23 引理 设 Ω 是 $\mathbf{R}^n$ 中一个任意区域, $0<\lambda<\mu\leqslant 1$, 对于任意函数 $u\in C^{0,\mu}(\overline{\Omega})$ 我们有

$$\|u;C^{0,\lambda}(\overline{\Omega})\|\leqslant 3^{1-\lambda/\mu}\|u;C(\overline{\Omega})\|^{1-\lambda/\mu}\|u;C^{0,\mu}(\overline{\Omega})\|^{\lambda/\mu}. \tag{28}$$

证明 令 $p=\mu/\lambda$, $p'=p/(p-1)$,

$$A_1=\|u;C(\overline{\Omega})\|^{1/p},\quad B_1=\sup_{\substack{x,y\in\Omega\\x\neq y}}\left\{\frac{|u(x)-u(y)|}{|x-y|^\mu}\right\}^{1/p},$$

$$A_2=\|u;C(\overline{\Omega})\|^{1/p'},\quad B_2=\sup_{\substack{x,y\in\Omega\\x\neq y}}|u(x)-u(y)|^{1/p'}.$$

显然 $A_1^p+B_1^p=\|u;C^{0,\mu}(\overline{\Omega})\|$, $B_2^{p'}\leqslant 2\|u;C(\overline{\Omega})\|$, 我们有

$$\|u;C^{0,\lambda}(\overline{\Omega})\|=\|u;C(\overline{\Omega})\|+\sup_{\substack{x,y\in\Omega\\x\neq y}}\frac{|u(x)-u(y)|}{|x-y|^\lambda}$$

$$\leqslant A_1A_2+B_1B_2$$

$$\leqslant\{A_1^p+B_1^p\}^{\frac{1}{p}}\{A_2^{p'}+B_2^{p'}\}^{1/p'}$$
$$\leqslant\|u;C^{0,\mu}(\overline{\Omega})\|^{\lambda/\mu}(3\|u;C(\overline{\Omega})\|)^{1-\lambda/\mu}.$$

这就是所要证明的. ▌

6.24 定理 若Ω满足定理 6.13 的条件,则下面的嵌入是紧的:

$$W_0^{j+m,p}(\Omega)\to C^j(\overline{\Omega}),\ \text{当}\ mp>n \tag{29}$$
$$W_0^{j+m,p}(\Omega)\to C^{j,\lambda}(\overline{\Omega}),\ \text{当}\ mp>n\geqslant(m-1)p\ \text{及}\ 0<\lambda<m-(n/p). \tag{30}$$

证明 只要对 $j=0$ 的情况进行证明即可得定理的结论. 若 $mp>n$, 设 j^*是满足$(m-j^*)p>n\geqslant(m-j^*-1)p$ 的非负整数. 那么我们有下面的嵌入串

$$W_0^{m,p}(\Omega)\to W_0^{m-j^*,p}(\Omega)\to C^{0,\mu}(\overline{\Omega})\to C(\overline{\Omega}),$$

其中 $0<\mu<m-j^*-(n/p)$. 如果$\{u_i\}$是 $W_0^{m,p}(\Omega)$中的有界序列,那么$\{u_i\}$也是 $C^{0,\mu}(\overline{\Omega})$中的有界序列. 由定理 6.13$\{u_i\}$有一个子序列$\{u_i'\}$在 $L^p(\Omega)$中是收敛的, 利用(27)式(用完备化的方法(27)式也可以应用于函数 u_i)可知$\{u_i'\}$是$C(\overline{\Omega})$中的Cauchy序列,因而它是收敛的. 因而对于 $j=0$ 嵌入(29)是紧的. 此外, 对于 $mp>n\geqslant(m-1)p$(即 $j^*=0$), $0<\lambda<\mu$ 由 (28) 式 $\{u_i'\}$ 是空间 $C^{0,\lambda}(\overline{\Omega})$中的一个 Cauchy 序列,因而嵌入(30)也是紧的. ▌

$W_0^{m,p}(\Omega)$的一个等价范数

6.25 确定对于怎样的无界区域Ω嵌入 $W_0^{m,p}(\Omega)\to L^p(\Omega)$ 是紧的这个问题与确定Ω是一个怎样的区域,半范数$|\cdot|_{m,p,\Omega}$, 即

$$|u|_{m,p,\Omega}=\left\{\sum_{|\alpha|=m}\|D^\alpha u\|_{0,p,\Omega}^p\right\}^{1/p}$$

实际是 $W_0^{m,p}(\Omega)$的范数(也就是等价于范数$\|\cdot\|_{m,p,\Omega}$)这个问题有密切的联系. 对于任何有界区域的情况现在我们证明上面给出的半范数的确是 $W_0^{m,p}(\Omega)$的一个等价范数.

6.26 一个区域 $\Omega\in\mathbf{R}^n$，如果它是在两个平行的超平面之间我们说它有有限宽. 设Ω是有有限宽的区域，不失一般性可认为它是在超平面 $x_n=0$ 和 $x_n=d$ 之间. 令 $x=(x',x_n)$，其中 $x'=(x_1,\cdots,x_{n-1})$，对任意的 $\phi\in C_0^\infty(\Omega)$

$$\phi(x)=\int_0^{x_n}\frac{d}{dt}\phi(x',t)\,dt,$$

因此

$$\begin{aligned}\|\phi\|_{0,p,\Omega}^p&=\int_{\mathbf{R}^{n-1}}dx'\int_0^d|\phi(x)|^p dx_n\\&\leqslant\int_{\mathbf{R}^{n-1}}dx'\int_0^d x_n^{p-1}dx_n\int_0^d|D_n\phi(x',t)|^p dt\\&\leqslant(d^p/p)\,|\phi|_{1,p,\Omega}^p,\end{aligned}\tag{31}$$

并且

$$|\phi|_{1,p,\Omega}^p\leqslant\|\phi\|_{1,p,\Omega}^p=\|\phi\|_{0,p,\Omega}^p+|\phi|_{1,p,\Omega}^p\leqslant(1+(d^p/p))\,|\phi|_{1,p,\Omega}^p.$$

对导数 $D^\alpha\phi$，$|\alpha|\leqslant m-1$，连续地应用上面的不等式，推出

$$|\phi|_{m,p,\Omega}\leqslant\|\phi\|_{m,p,\Omega}\leqslant K|\phi|_{m,p,\Omega},\tag{32}$$

利用完备性，(32)式对所有 $u\in W_0^{m,p}(\Omega)$ 成立. 不等式(31)通常称为 Poincare 不等式.

6.27 $\mathbf{R}^n$ 中的无界区域 Ω，若

$$\lim_{x\in\Omega}\ \sup_{|x|\to\infty}\ \mathrm{dist}(x,\mathrm{bdry}\,\Omega)<\infty,$$

则称Ω是拟柱形的，显然任意一个拟有界区域正如同每一个(无界的)有限宽的区域一样也是拟柱形的，读者可以自己去构造一个反例证明若Ω不是一个拟柱形的区域那么 $|\cdot|_{m,p,\Omega}$ 与 $W_0^{m,p}(\Omega)$ 的范数是不等价的. 下面的定理类似于定理 6.13.

6.28 定理 设存在满足下面条件的常数K,R,h 和 ν：$0<K\leqslant1$，$0\leqslant R<\infty$，$0<h<\infty$，$1\leqslant\nu\leqslant n$，ν 是一个整数，$\nu<p$ 或者$\nu=p=1$，对于每一个 $\mathbf{R}^n$ 中边长为h 且与 $\Omega_R=\{x\in\Omega,\ |x|\geqslant R\}$ 有非空交的正方体H有

$$\mu_{n-\nu}(H,\Omega)/h^{n-\nu}\geqslant K,$$

其中$\mu_{n-\nu}(H,\Omega)$如定理 6.13 的叙述中所定义的那样. 则$|\cdot|_{m,p,\Omega}$与$\|\cdot\|_{m,p,\Omega}$是空间$W_0^{m,p}(\Omega)$的等价范数.

证明 从第 6.26 节讨论中我们知道，为了证明定理的结论只需对于$u\in C_0^\infty(\Omega)$证明$\|u\|_{0,p,\Omega}\leqslant K_1|u|_{1,p,\Omega}$. 设$H$是边长为$h$与$\Omega_R$有非空交的立方体，由于$\nu<p$(或者$\nu=p=1$)定理 6.13 的证明(6.21 节)中已指出对所有$u\in C_0^\infty(H)$

$$C^{1,p}(H,H\sim\Omega)\geqslant\mu_{n-\nu}(H,\Omega)/K_2h^{n-\nu}\geqslant K/K_2$$

K_2不依赖于函数u. 因而

$$\|u\|_{0,p,H}^p\leqslant(K_2/K)I_H^{1,p}(u)=K_3|u|_{1,p,H}^p. \tag{33}$$

将(33)式对包含Ω_R的某个邻域的田字形划分中的所有正方体H求和，得

$$\|u\|_{0,p,\Omega_R}^p\leqslant K_3|u|_{1,p,\Omega}^p. \tag{34}$$

余下来的是要证明

$$\|u\|_{0,p,B_R}^p\leqslant K_4|u|_{1,p,\Omega}^p,$$

其中$B_R=\{x\in\mathbf{R}^n:|x|<R\}$. 设$(\rho,\phi)$表示$\mathbf{R}^n$中一点$x$的球坐标$(\rho\geqslant0,\phi\in\Sigma)$，$dx=\rho^{n-1}\omega(\phi)d\rho d\phi$表示体积元. 对空间$C^\infty(\mathbf{R}^n)$中的任意一个函数，我们有

$$u(\rho,\phi)=u(\rho+R,\phi)-\int_\rho^{R+\rho}\frac{d}{dt}u(t,\phi)dt,$$

所以

$$|u(\rho,\phi)|^p\leqslant2^{p-1}|u(\rho+R,\phi)|^p$$
$$+2^{p-1}R^{p-1}\rho^{1-n}\int_\rho^{R+\rho}|\operatorname{grad}u(t,\phi)|^pt^{n-1}dt,$$

因此得

$$\|u\|_{0,p,B_R}^p=\int_\Sigma\omega(\phi)d\phi\int_0^R|u(\rho,\phi)|^p\rho^{n-1}d\rho$$

$$\leqslant 2^{p-1}\int_{\Sigma}\omega(\phi)d\phi\int_0^R|u(\rho+R,\phi)|^p(\rho+R)^{n-1}d\rho$$

$$+2^{p-1}R^p\int_{\Sigma}\omega(\phi)d\phi\int_0^{2R}|\mathrm{grad}u(t,\phi)|^p t^{n-1}dt.$$

所以由(34)对于 $u\in C_0^\infty(\Omega)$ 我们可得

$$\|u\|_{0,p,B_R}^p\leqslant 2^{p-1}\|u\|_{0,p,B_{2R}\sim B_R}^p+2^{p-1}R^p|u|_{1,p,B_{2R}}^p$$
$$\leqslant 2^{p-1}\|u\|_{0,p,\Omega_R}^p+2^{p-1}R^p|u|_{1,p,\Omega}^p\leqslant K_4|u|_{1,p,\Omega}^p. \blacksquare$$

无界区域——在无穷远处的衰减

6.29 在我们对某些无界区域建立嵌入

$$W_0^{m,p}(\Omega)\longrightarrow L^p(\Omega) \tag{35}$$

的紧性时，空间 $W_0^{m,p}(\Omega)$ 的元素在 Ω 的边界等于零(在某种推广的意义下)起着关键作用. 对于 $W^{m,p}(\Omega)$ 的元素不再有这个性质，还留下这样一个问题：什么时候嵌入(如果存在的话)

$$W^{m,p}(\Omega)\to L^p(\Omega) \tag{36}$$

对无界的 Ω 是紧的？或者甚至对有界的、相当不规则的以至不存在嵌入

$$W^{m,p}(\Omega)\to L^q(\Omega) \tag{37}$$

(对任意的 $q>p$)的区域 Ω 嵌入(36)是紧的？

注意如果 Ω 有有限体积，对某个 $q>p$ 嵌入(37)存在，用6.7节第一部分中的方法可以证明嵌入(36)的紧性. 但由定理5.30可知对 $q>p$，Ω 是有有限体积的无界界区域嵌入(37)是不能存在的.

6.30 例 对 $j=1,2,\cdots$ 令 B_j 表示 $\mathbf{R}^n$ 中半径为 r_j 的开球，并且假设当 $j\neq K$ 时 $\bar{B}_j\cap\bar{B}_K$ 是空集. 记 $\Omega=\bigcup_{j=1}^{\infty}B_j$; Ω 可以是有界的也可以是无界的. 定义一个函数序列 $\{u_j\}$

$$u_j(x)=\begin{cases}(\mathrm{vol}B_j)^{-1/p}, & x\in\bar{B}_j,\\ 0 & x\notin\bar{B}_j,\end{cases}$$

显然$\{u_j\}$在$W^{m,p}(\Omega)$中是有界的，但是当j趋于无穷大，无论$r_j\to 0$怎样快，$\{u_j\}$在$L^p(\Omega)$都不是准紧的．因此嵌入(36)不可能是紧的．（注意由定理6.13我们知道当$\lim\limits_{j\to\infty}r_j=0$时嵌入(35)是紧的．）如果$\Omega$是有界的对任意$q>p$嵌入(37)是不可能存在的．

6.31 使得嵌入(36)是紧的无界区域是存在的(参看6.48节)．Adams和Fournier[3]给出了这样区域的一个例子，并且Adams和Fournier在工作[4]为一般问题的研究打下了基础．后一文章的方法在下一节将要用到．首先我们考虑嵌入(37)紧性的必要条件．这些条件包括对任意无界区域在无穷远处的快速衰减（见定理6.40）．在证明中运用这些技巧我们可以得到一个比定理5.30的说法更强的定理(即定理6.36)和这个断言的逆命题(参看附注5.5(6))，

如果Ω有有限体积则对$1\leqslant q<p$有嵌入

$$W^{m,p}(\Omega)\to L^q(\Omega).$$

嵌入(36)紧性的充分条件将在定理6.47中给出．这个条件可应用到很多有界和无界的区域，可用到Rellich-Kondachov定理及其推广所不能用的区域(即指数尖点——参看例6.49)．

6.32 设T是由边长为h的闭n-正方体组成的$\mathbf{R}^n$的一个田字形划分，若H是T中的一个正方体，用$N(H)$表示以$3h$为边长与H有相同中心的正方体，其表面与H的表面平行．称$N(H)$为H的邻域．显然$N(H)$是T中3^n个与H相交的正方体的并，将

$$F(H)=N(H)\sim H$$

称为H的边框．

设Ω是$\mathbf{R}^n$中的一个区域，T是上面给定的田字形划分，设$\lambda>0$．如果T中的正方体H满足

$$\mu(H\cap\Omega)>\lambda\mu(F(H)\cap\Omega),$$

则称H是λ-丰满的(关于区域Ω),这里μ表示$\mathbf{R}^n$中的n维Lebesgue测度(为了记法的简单我们用μ代替"vol", 在下面的讨论中这个记号多次用到.)如果H不是λ-丰满的则称H为λ-窄小的.

6.33 定理 假设存在一个紧嵌入

$$W^{m,p}(\Omega)\to L^q(\Omega), \tag{38}$$

其中$q\geqslant p$. 则对任意的$\lambda>0$和$\mathbf{R}^n$的任意一个由固定边长的正方体组成的田字形划分T,只有有限多个λ-丰满的正方体.

证明 若定理不成立,则对每个$\lambda>0$存在$\mathbf{R}^n$的边长为h的正方体组成的田字形划分,它包含有一个λ-丰满的正方体序列$\{H_j\}_{j=1}^{\infty}$. 如有必要, 抽取子序列, 我们总可以假设$N(H_j)\cap N(H_k)=\phi$, $j\neq k$. 对每一个j存在一个具有下面性质的函数$\phi_j\in C_0^\infty(N(H_j))$

(i) $|\phi_j(x)|\leqslant 1$, 对所有$x\in\mathbf{R}^n$,

(ii) $\phi_j(x)=1$, 如果$x\in H_j$,

(iii) $|D^\alpha\phi_j(x)|\leqslant M$, 对所有$x\in\mathbf{R}^n$及$0\leqslant|\alpha|\leqslant m$,

其中$M=M(n,m,h)$是与j无关的常数. 令$\psi=c_j\phi_j$, 其中正常数c_j根据下面条件选取:

$$\|\psi_j\|_{0,q,\Omega}^q\geqslant c_j^q\int_{H_j\cap\Omega}|\phi_j(x)|^q dx=c_j^q\mu(H_j\cap\Omega)=1.$$

然而因为H_j是λ-丰满的

$$\begin{aligned}\|\psi_j\|_{m,p,\Omega}^p&=c_j^p\sum_{0\leqslant|\alpha|\leqslant m}\int_{N(H_j)\cap\Omega}|D^\alpha\phi_j(x)|^p dx\\&\leqslant M^p c_j^p\mu(N(H_j)\cap\Omega)\\&<M^p c_j^p\mu(H_j\cap\Omega)[1+(1/\lambda)]\\&=M^p[1+(1/\lambda)]c_j^{p-q},\end{aligned}$$

而 $\mu(H_j\cap\Omega)\leqslant\mu(H_j)=h^n$, 这样$c_j\geqslant h^{-n/q}$, 又由于$p-q\leqslant 0$, 由上式可得$\{\psi_j\}$是$W^{m,p}(\Omega)$中的有界集. 又因为函数$\psi_j$的

支集是两两不相交的，$\{\psi_j\}$ 在 $L^q(\Omega)$ 中不可能是准紧的. 这就与嵌入 (38) 的紧性产生矛盾. 因此 T 只能有有限个 λ-丰满的正方体. ∎

6.34 推论 设对某个 $q>p$ 嵌入(38)是存在的. 若 T 是边长为 h 的正方体组成的 $\mathbf{R}^n$ 的一个田字形划分，如果给定的 $\lambda>0$ 则存在一个 $\varepsilon>0$ 使得对所有 T 中 λ-丰满的 H 满足 $\mu(H\cap\Omega)\geqslant\varepsilon$

证明 若不然则存在 λ-丰满的正方体序列 $\{H_j\}$，而 $\lim\limits_{j\to\infty}\mu(H_j\cap\Omega)=0$. 如果 c_j 如上面证明中所定义的那样，我们有 $\lim\limits_{j\to\infty}c_j=\infty$，由于 $p<q$，则 $\lim\limits_{j\to\infty}\|\psi_j\|_{m,p,\Omega}=0$. 又因在空间 $L^q(\Omega)$ 中 $\{\psi_j\}$ 与 0 的距离是大于 1 的，这就与嵌入(38)的连续性矛盾. ∎

6.35 让我们进一步考查上面推论所包含的意思. 如果对某个 $q>p$ 嵌入(38)存在则下面两个结论中必有一个成立.

(*a*) 存在 $\mathbf{R}^n$ 中的由固定边长的正方体组成的 $\mathbf{R}^n$ 的田字形划分 T 和 $\varepsilon>0$ 使对 T 中无穷多个 H 满足

$$\mu(H\cap\Omega)\geqslant\varepsilon.$$

(*b*) 对任意 $\lambda>0$ 及对任意由固定边长的正方体组成的 $\mathbf{R}^n$ 的田字形划分 T，T 中仅包含有限个 λ-丰满的正方体.

我们将在定理 6.37 中证明结论(*b*)包含了 Ω 有有限的体积. 由定理 5.30 因而(*b*)与对 $q>p$ 嵌入(38)存在是矛盾的. 另一方面条件(*a*)意味着当 N 趋向无穷时 $\mu(\{x\in\Omega:N\leqslant|x|\leqslant N+1\}$ 不趋向于零. 所以我们证明了下面比定理 5.30 更强的定理.

6.36 定理 设 Ω 是 $\mathbf{R}^n$ 中的一个无界区域，满足条件

$$\lim_{N\to\infty}\sup\ \mathrm{vol}\{x\in\Omega:N\leqslant|x|\leqslant N+1\}=0.$$

则对任意的 $q>p$，嵌入(38)是不存在的.

6.37 定理 假设嵌入(38)对于某个 $q\geqslant p$ 是紧的. 则 Ω 有有限的体积.

证明 设T是由边长为1的立方体组成的 $\mathbf{R}^n$ 的田字形划分，取$\lambda=1/[2(3^n-1)]$．令P是T中有限个λ-丰满的正方体的和，显然 $\mu(P\cap\Omega)\leqslant\mu(P)<\infty$。设$H$是一个$\lambda$-窄小的正方体．又设$H_1$是包含在边框$F(H)$中$3^n-1$个正方体中的一个且使 $\mu(H_1\cap\Omega)$为最大．因此

$$\mu(H\cap\Omega)\leqslant\lambda\mu(F(H)\cap\Omega)\leqslant\lambda(3^n-1)\mu(H_1\cap\Omega)$$
$$=\frac{1}{2}\mu(H_1\cap\Omega).$$

如果H_1也是λ-窄小的，我们可以选取 $H_2\in T$, $H_2\subset F(H_1)$使得

$$\mu(H_1\cap\Omega)\leqslant\frac{1}{2}\mu(H_2\cap\Omega).$$

假设可以用上面方法得到一个无限λ-窄小的正方体序列 $\{H, H_1 H_2\cdots\}$那么

$$\mu(H\cap\Omega)\leqslant\frac{1}{2}\mu(H_1\cap\Omega)\leqslant\cdots\leqslant(1/2^j)\mu(H_j\cap\Omega)$$
$$\leqslant 1/2^j$$

对任意的j成立(因为$\mu(H_j\cap\Omega)\leqslant\mu(H_j)\leqslant1$)．因此$\mu(H\cap\Omega)=0$．用$P_\infty$表示可以构造成无穷序列的$\lambda$-窄小的正方体$H\in T$的并，于是有 $\mu(P_\infty\cap\Omega)=0$。

令P_j表示T中具有下面性质的λ-窄小的正方体H的并；由H出发用上面的方法得到的某个序列终止于第j步(即H_j是λ-丰满的)．任何特定的λ-丰满的正方体H'只有当H包含于边长为$2j+1$，中心在H'中的正方体时，才能作为由H开始的序列的末尾H_j．因此，在P_j中正方体以H'为其序列的第j个元素H_j的个数最多是$(2j+1)^n$个，那么

$$\mu(P_j\cap\Omega)=\sum_{H\subset P_j}\mu(H\cap\Omega)$$
$$\leqslant(1/2^j)\sum_{H\subset P_j}\mu(H_j\cap\Omega)$$

$$\leqslant [(2j+1)^n/2^j] \sum_{H' \in P} \mu(H' \cap \Omega)$$
$$= [(2j+1)^n/2^j] \mu(P \cap \Omega),$$

因此$\sum_{j=1}^{\infty} \mu(P_j \cap \Omega) < \infty$. 由于 $\mathbf{R}^n = P \cup P_\infty \cup P_1 \cup P_2 \cdots$，我们有 $\mu(\Omega) < \infty$. ▌

假设 $1 \leqslant q < p$. 如果 $\mathrm{vol}\,\Omega < \infty$, 那么由定理 2.8 可知嵌入

$$W^{m,p}(\Omega) \to L^q(\Omega) \tag{39}$$

是存在的. 下面我们证明其逆命题.

6.38 定理 假设对某一对 p, q, $1 \leqslant q < p$, 嵌入(39)是存在的, 则 Ω 有有限的体积.

证明 设 T, λ 与前面定理证明引入的 T, λ 意义一样. 这里仍然用 P 表 T 中 λ-丰满的正方体的并, 如果我们能够证明 $\mu(P \cap \Omega)$ 是有限的, 那么由前面定理证明中同样的推导可知 $\mathrm{vol}\,\Omega$ 是有限的.

于是我们假定$\mu(P \cap \Omega)$是无限的. 那么就存在序列$\{H_j\}_{j=1}^{\infty}$, H_j 是 T 中λ-丰满的正方体, 而且$\sum_{j=1}^{\infty} \mu(H_j \cap \Omega) = \infty$. 以 L 表示 T 中所有正方体的中心组成的点阵, 我们可以将 L 分为 3^n 个子点阵$\{L_i\}_{i=1}^{3^n}$使每个子点阵中点的坐标在每个坐标方向上都以 3 为周期. 对于每个 i 用 T_i 表示 T 中中心在 L_i 中的正方体组成的集合. 因而必然存在某一个 i 使得 $\sum_{\lambda-\text{丰满的}, H \in T_i} \mu(H \cap \Omega) = \infty$. 因此我们可以假设序列$\{H_j\}$的每个元素 H_j 都在某个 T_i 中, 因此 $N(H_j) \cap N(H_k)$是不重叠的.

取整数 j_1 使得

$$2 \leqslant \sum_{j=1}^{j_1} \mu(H_j \cap \Omega) < 4.$$

设 ϕ_j 就是在证明定理 6.33 中引进的 ϕ_j, 令

$$\psi_1(x)=2^{-1/p}\sum_{j=1}^{j_1}\phi_j(x).$$

由于 ϕ_j 的支集是分离的，正方体 H_j 是 λ-丰满的，对于 $|\alpha|\leqslant m$ 我们有

$$\begin{aligned}\|D^\alpha\psi_1\|_{0,p,\Omega}^p&=\frac{1}{2}\sum_{j=1}^{j_1}\int_\Omega|D^\alpha\phi_j(x)|^p dx\\&\leqslant\frac{1}{2}M^p\sum_{j=1}^{j_1}\mu(N(H_j)\cap\Omega)\\&\leqslant\frac{1}{2}M^p(1+(1/\lambda))\sum_{j=1}^{j_1}\mu(H_j\cap\Omega)\\&<2M^p(1+(1/\lambda)).\end{aligned}$$

另一方面

$$\|\psi_1\|_{0,q,\Omega}^q\geqslant 2^{-q/p}\sum_{j=1}^{j_1}\mu(H_j\cap\Omega)\geqslant 2^{1-q/p}.$$

这样定义了 j_1, ψ_1 之后，我们可以进一步用归纳法引进 j_2, $j_3\cdots$ 和 ψ_2, ψ_3, $\cdots$ 使得

$$2^k\leqslant\sum_{j=j_{k-1}}^{j_k}\mu(H_j\cap\Omega)<2^{k+1}$$

$$\psi_k=2^{-k/p}k^{-2/p}\sum_{j=j_{k-1}}^{j_k}\phi_j(x).$$

象上面一样对 $|\alpha|\leqslant m$ 我们有

$$\|D^\alpha\psi_k\|_{0,p,\Omega}^p<(2/k^2)M^p(1+(1/\lambda)),$$
$$\|\psi_k\|_{0,q,\Omega}^q\geqslant 2^{k(1-q/p)}(1/k)^{2q/p}.$$

因此 $\psi=\sum_{k=1}^\infty\psi_k$ 是属于空间 $W^{m,p}(\Omega)$ 的但是不属于空间 $L^q(\Omega)$，这与(39)矛盾．因此 $\mu(p\cap\Omega)<\infty$，这就是所要证的．∎

6.39 如果对某个 $q\geqslant p$ 存在紧嵌入(38)，我们已经证明了 Ω 的体积是有限的．实际上正如我们下面要证明的，更重要的是：当

$R\to\infty$时 $\mu(\{x\in\Omega:|x|\geqslant R\})$必须以很快的速度趋于零.

如果Q是$\mathbf{R}^n$某田字形划分中一些正方体H的并, 我们以明显的方式将邻域, 边框的记号推广到Q:

$$N(Q)=\bigcup_{H\subset Q}N(H),\quad F(Q)=N(Q)\sim Q.$$

给定$\delta>0$, 令$\lambda=\delta/3^n(1+\delta)$. 如果$Q$中所有的正方体$H$是$\lambda$-窄小的则$Q$本身在下面意义下是$\delta$-窄小的

$$\mu(Q\cap\Omega)\leqslant\delta\mu(F(Q)\cap\Omega). \tag{40}$$

为了看清这一点, 注意当H跑过包含Q的诸正方体时, $F(H)$最多把$N(Q)$覆盖3^n次. 因此

$$\mu(Q\cap\Omega)=\sum_{H\subset Q}\mu(H\cap\Omega)\leqslant\lambda\sum_{H\subset Q}\mu(F(H)\cap\Omega)$$

$$\leqslant 3^n\lambda\mu(N(Q)\cap\Omega)=3^n\lambda[\mu(Q\cap\Omega)+\mu(F(Q)\cap\Omega)]$$

由上式移项(因为$\mu(\Omega)<\infty$, 上式右端第一项可以移至左端), 并注意$3^n\lambda/(1-3^n\lambda)=\delta$可以得到(40)式.

若S是$\mathbf{R}^n$中任意一个可测集, 以Q表示我们的田字形划分中所有与S内相交①的正方体H的并. 且我们定义$F(S)=F(Q)$. 如果S与田字形划分中有限个λ-丰满的正方体有一个正的距离, 那么Q是由λ-窄小的正方体组成的, 由(40)我们可得

$$\mu(S\cap\Omega)\leqslant\mu(Q\cap\Omega)\leqslant\delta\mu(F(S)\cap\Omega). \tag{41}$$

6.40 定理 假设对于某个$q\geqslant p$存在一个紧嵌入(38). 对任意$r\geqslant 0$, 令$\Omega_r=\{x\in\Omega:|x|>r\}$, $S_r=\{x\in\Omega:|x|=r\}$, 并且以A_r表示S_r($n-1$维测度)的面积则有

(a) 对于任给的$\varepsilon,\delta>0$存在一个R使得如果$r>R$则有

$$\mu(\Omega_r)\leqslant\delta\mu(\{x\in\Omega:r-\varepsilon\leqslant|x|\leqslant r\}).$$

(b) 若A_r是正的而且当r趋于无穷时总是非增的, 则对任意

① 这里"内相交"是指H的内点与S相交——译者注.

的 $\varepsilon>0$

$$\lim_{r\to\infty}\frac{A_{r+\varepsilon}}{A_r}=0.$$

证明 对于给定的 $\varepsilon>0$, 以 T 表示由边长为 $\varepsilon/2\sqrt{n}$ 的正方体组成的 $\mathbf{R}^n$ 的一个田字形划分. 则 T 中任意一个与 Ω_r 内相交的正方体一定包含在 $\Omega_{r-\varepsilon/2}$ 内并且

$$F(\Omega_r)\subset\{x\in\Omega: r-\varepsilon\leqslant|x|\leqslant r\}.$$

对于给定 $\delta>0$, 令 $\lambda=\delta/3^n(1+\delta)$, R 表示一个充分大的数使 T 中的所有有限个 λ-丰满的正方体都包含在中心在原点半径为 $R-\varepsilon/2$ 的球中. 这样对任意的 $r\geqslant R$, T 中所有与 Ω_r 内相交的正方体都是 λ-窄小的, 由(41)式就可以推出结论(a).

为了证明结论(b), 取 R_0 使 A_r 在 (R_0,∞) 上为非增的. 对于固定的 $\varepsilon',\delta>0$, 令 $\varepsilon=\varepsilon'/2$. 设 R 为(a)中的 R. 如果 $r>\max(R_0+\varepsilon', R)$, 则

$$\begin{aligned}A_{r+\varepsilon'}&\leqslant(1/\varepsilon)\int_{r+\varepsilon}^{r+2\varepsilon}A_s ds\leqslant(1/\varepsilon)\,\mu(\Omega_{r+\varepsilon})\\&\leqslant(\delta/\varepsilon)\,\mu\{x\in\Omega: r\leqslant|x|\leqslant r+\varepsilon\}\\&=(\delta/\varepsilon)\int_r^{r+\varepsilon}A_s ds\leqslant\delta A_r\end{aligned}$$

由于 ε' 和 δ 是任意的, 由此可得结论(b). ∎

6.41 推论 若对某个 $q\geqslant p$ 存在一个紧嵌入(38), 则对任意的 k 有

$$\lim_{r\to\infty}e^{kr}\mu(\Omega_r)=0.$$

证明 固定 k, 令 $\delta=e^{-(k+1)}$. 那么由定理 6.40(a)的结论可知存在一个 R, 使当 $r\geqslant R$ 时则有 $\mu(\Omega_{r+1})\leqslant\delta\mu(\Omega_r)$. 因此如果 j 是一个正整数, $0\leqslant t<1$, 则有

$$\begin{aligned}e^{k(R+j+t)}\mu(\Omega_{R+j+t})&\leqslant e^{k(R+j+1)}\mu(\Omega_{R+j})\\&\leqslant e^{k(R+1)}e^{kj}\delta^j\mu(\Omega_R)\end{aligned}$$

$$=e^{k(R+1)}\mu(\Omega_R)e^{-j}.$$

当 j 趋向无穷时最后一项是趋于零的. ∎

6.42 附注 (1) 用 $\mathbf{R}^n$ 中的任意一个范数 ρ 代替范数 $\rho(x)=|x|$, 定理 6.40(a)证明中的论证仍然是成立的, 如果用新的范数 ρ 定义 A_r, 并且

$$\mu(\{x\in\Omega:r\leqslant\rho(x)\leqslant r+\varepsilon\})=\int_r^{r+\varepsilon}A_s ds,$$

同样结论(b)也成立. 例如如果 $\rho(x)=\max\limits_{1\leqslant i\leqslant n}|x_i|$ 结论 (a), (b)都是正确的.

(2)如果 A_r 有一个等价的, 正的非增的控制函数, 即存在一个正的非增函数 $f(r)$和一个常数 $M>0$, 使得对充分大的 r 满足

$$A_r\leqslant f(r)\leqslant MA_r,$$

则结论(b)仍然成立.

(3)定理 6.33 比定理 6.40 更明确, 因为后面定理的结论是整体性的, 然而显然嵌入(38)的紧性依赖于Ω的局部性质. 下面通过两个例子来解释这个事实.

6.43 例 设 $f\in C^1([0,\infty))$是一个正的非增的函数, 而且有有界的导数 f'. 我们考虑平面区域(图 6(a))

$$\Omega=\{(x,y)\in\mathbf{R}^2:x>0,0<y<f(x)\}. \tag{42}$$

关于 $\mathbf{R}^2$ 上的上确界范数, 即 $\rho(x,y)=\max(|x|,|y|)$, 对充分大的 s 我们有 $A_s=f(s)$. 因而当而且仅当对任意的 $\varepsilon>0$ 有

$$\lim_{s\to\infty}\frac{f(s+\varepsilon)}{f(s)}=0 \tag{43}$$

时Ω满足定理 6.40 结论(b)[因为 f 是单调的, 结论(a)同样满足]. 例如 $f(x)=\exp(-x^2)$是满足(43)的但 $f(x)=e^{-x}$ 是不满足(43)的. 我们将要(在 6.48 节)证明如果条件(43)满足则嵌入

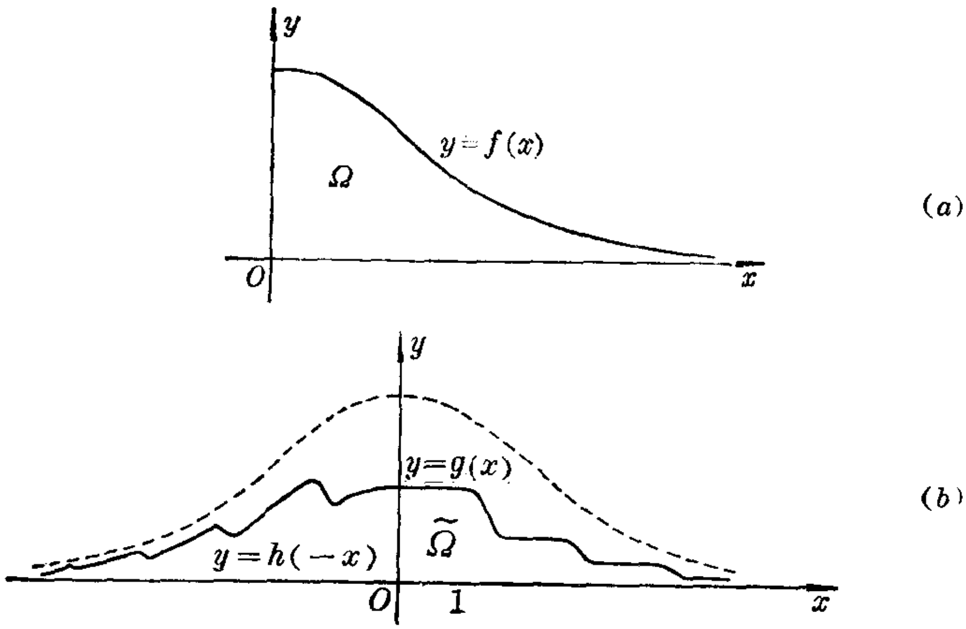

图 6 (a) 例 6.43 中的区域 Ω, (b) 例 6.44 中的区域 $\widetilde{\Omega}$

$$W^{m,p}(\Omega)\to L^p(\Omega) \tag{44}$$

是紧的，因此对于(42)型的区域，(43)是嵌入(38)紧的充分必要条件.

6.44 例 设 f 满足例 6.43 中的条件，还满足 $f'(0)=0$ 又设 $g\in C^1([0,\infty))$是一个正的非增的函数并且满足

(i) $g(0)=\dfrac{1}{2}f(0)$, $g'(0)=0$,

(ii) 对所有的 $x\geqslant 0$, $g(x)<f(x)$,

(iii) $g(x)$在无穷多个相互不相交的单位长的区间上等于常数. 令 $h(x)=f(x)-g(x)$, 我们考虑区域(图 6(b))

$$\widetilde{\Omega}=\{(x,y)\in\mathbf{R}^2:\quad \text{当 } x\geqslant 0,\ 0<y<g(x),$$
$$\text{当 } x<0,\ 0<y<h(-x)\}.$$

对充分大的 s 我们仍有 $A_s=f(s)$, 所以如果(43)式成立，$\widetilde{\Omega}$就满足定理 6.40 的结论.

但是如果 T 是 $\mathbf{R}^2$ 的一个田字形划分，T 中的正方形边长为

$\frac{1}{4}$,其边平行于坐标轴,而且T中有一个正方形的中心在原点,则T有无穷多个$\frac{1}{3}$-丰满的正方形,其中心在正的x轴上.由定理6.33,嵌入(44)不可能是紧的.

无界区域——$W^{m,p}(\Omega)$的紧嵌入

6.45 从上面例子可以看出对于无界区域Ω上嵌入

$$W^{m,p}(\Omega)\to L^p(\Omega) \tag{45}$$

的紧性的任何充分条件必须包含:当r趋于无穷时Ω_r的每一部分的局部体积迅速的衰减,一个方便的办法是根据Ω上的流动来表示这样的局部衰减.

我们讲Ω上的一个*流动*意思就是有一个连续可微的映射$\boldsymbol{\Phi}$:$U\to\Omega$,其中U是$\Omega\times\mathbf{R}$中包含$\Omega\times\{0\}$的一个开集,并且对每个$x\in\Omega$有$\boldsymbol{\Phi}(x,0)=x$.

对固定的$x\in\Omega$,曲线$t\to\boldsymbol{\Phi}(x,t)$称为这个流动的一条*流线*,对于固定的$t$映射$\boldsymbol{\Phi}_t$:$x\to\boldsymbol{\Phi}(x,t)$将$\Omega$的子集映射到$\Omega$,这个映射的函数行列式是

$$\det\boldsymbol{\Phi}'_t(x)=\left.\frac{\partial(\boldsymbol{\Phi}_1,\cdots,\boldsymbol{\Phi}_n)}{\partial(x_1,\cdots,x_n)}\right|_{(x,t)}.$$

有时要求流动$\boldsymbol{\Phi}$具有性质$\boldsymbol{\Phi}_{s+t}=\boldsymbol{\Phi}_s\circ\boldsymbol{\Phi}_t$,但是我们不需要这个性质,所以没有作这个假定.

6.46 例 设Ω是由(42)式给出的区域,定义流动

$$\boldsymbol{\Phi}(x,y,t)=(x-t,[f(x-t)/f(x)]y),\ 0<t<x.$$

这个流动是朝向线$x=0$的,且当区域变宽时流线散开(见图7).对$t>0$,$\boldsymbol{\Phi}_t$是局部放大的:

$$\det\boldsymbol{\Phi}'_t(x,y)=f(x-t)/f(x).$$

注意如果f满足(43)式则$\lim\limits_{x\to\infty}\det\boldsymbol{\Phi}'_t(x,y)=\infty$.

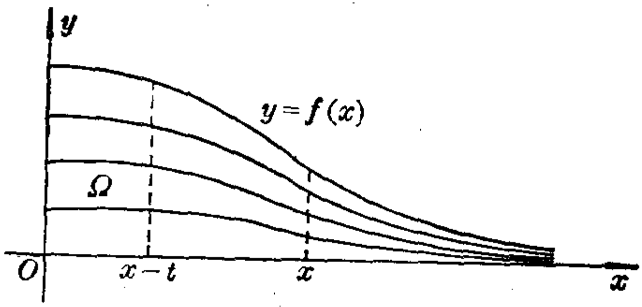

图 7　例 6.46 中给的流动 Φ 的流线

对 $N=1,2,\cdots$ 令 $\Omega_N^*=\{(x,y)\in\Omega:0<x<N\}$. Ω_N^*是有界的并且具有锥形性质, 因此嵌入

$$W^{1,p}(\Omega_N^*)\to L^p(\Omega_N^*)$$

是紧的, 根据下面的定理由这个紧性和流动 Φ 的性质就能够得到嵌入(45)的紧性.

6.47　定理　假设Ω是 $\mathbf{R}^n$ 中的有界开集并且具有下面性质:

(a) 存在Ω的一个开的子集序列 $\{\Omega_N^*\}_{N=1}^{\infty}$, 使 $\Omega_N^*\subset\Omega_{N+1}^*$, 对每一个$N$嵌入

$$W^{1,p}(\Omega_N^*)\to L^p(\Omega_N^*)$$

是紧的.

(b)存在一个流动 $\Phi: U\to\Omega$, 如果 $\Omega_N=\Omega\sim\Omega_N^*$ 则

(i) 对每个 N, $\Omega_N\times[0,1]\subset U$,

(ii) 对所有的 t, Φ_t 是一对一的,

(iii) 对所有$(x,t)\in U$, $|(\partial/\partial t)\Phi(x,t)|\leqslant M$(常数),

(c)函数 $d_N(t)=\sup\limits_{x\in\Omega_N}|\det\Phi_t'(x)|^{-1}$满足

(i) $\lim\limits_{N\to\infty}d_N(1)=0$,

(ii) $\lim\limits_{N\to\infty}\int_0^1 d_N(t)dt=0$.

则嵌入(45)是紧的.

证明　由于我们有嵌入 $W^{m,p}(\Omega)\to W^{1,p}(\Omega)\to L^p(\Omega)$, 因而只要

证明后面的嵌入是紧的. 设 $u\in C^1(\Omega)$对每个 $x\in\Omega_N$有

$$u(x)=u(\Phi_1(x))-\int_0^1\frac{\partial}{\partial t}u(\Phi_t(x))dt$$

而且

$$\begin{aligned}\int_{\Omega_N}|u(\Phi_1(x))|dx&\leqslant d_N(1)\int_{\Omega_N}|u(\Phi_1(x))||\det\Phi_1'(x)|dx\\&=d_N(1)\int_{\Phi_1(\Omega_N)}|u(y)|dy\\&\leqslant d_N(1)\int_\Omega|u(y)|dy.\end{aligned}$$

还有
$$\begin{aligned}&\int_{\Omega_N}\left|\int_0^1\frac{\partial}{\partial t}u(\Phi_t(x))dt\right|dx\\&\leqslant\int_{\Omega_N}dx\int_0^1\left|\mathrm{grad}u(\Phi_t(x))\right|\left|\frac{\partial}{\partial t}\Phi_t(x)\right|dt\\&\leqslant M\int_0^1 d_N(t)dt\int_{\Omega_N}\left|\mathrm{grad}u(\Phi_t(x))\right|\left|\det\Phi_t'(x)\right|dx\\&\leqslant M\left\{\int_0^1 d_N(t)dt\right\}\left\{\int_\Omega\left|\mathrm{grad}u(y)\right|dy\right\}.\end{aligned}$$

取 $\delta_N=\max(d_N(1),\ M\int_0^1 d_N(t)dt$, 我们有

$$\begin{aligned}\int_{\Omega_N}|u(x)|dx&\leqslant\delta_N\int_\Omega(|u(y)|+|\mathrm{grad}u(y)|)dy\\&\leqslant\delta_N\|u\|_{1,1,\Omega},\end{aligned}\tag{46}$$

并且有 $\lim\limits_{N\to\infty}\delta_N=0$.

假设 u 是实的, 而且 $u\in C^1(\Omega)\cap W^{1,p}(\Omega)$. 对 $|u|^p$ 的广义导数.

$$D_j(|u|^p)=p|u|^{p-1}\cdot\mathrm{sgn}u\cdot D_ju$$

的积分应用 Hölder 不等式得

$$\int_\Omega|D_j(|u(x)|^p)|dx\leqslant p\|D_ju\|_{0,p,\Omega}\|u\|_{0,p,\Omega}^{p-1}\leqslant p\|u\|_{1,p,\Omega}^p.$$

由于 $|u|^p \in W^{1,1}(\Omega)$，由定理 3.16 可知存在函数序列 $\phi_j \in C^1(\Omega) \cap W^{1,1}(\Omega)$，使得 $\lim\limits_{j\to\infty} \|\phi_j - |u|^p\|_{1,1,\Omega} = 0$. 那么由 (46) 式得

$$\int_{\Omega_N} |u(x)|^p dx = \lim_{j\to\infty} \int_{\Omega_N} \phi_j(x) dx \leqslant \lim_{j\to\infty} \sup \delta_N \|\phi_j\|_{1,1,\Omega}$$

$$= \delta_N \| |u|^p \|_{1,1,\Omega} \leqslant K\delta_N \|u\|^p_{1,p,\Omega}, \tag{47}$$

其中 $K = K(n, p)$. 对于任意的(复值的)函数 $u \in C^1(\Omega) \cap W^{1,p}(\Omega)$ 分别对 u 的实部和虚部应用不等式(47)，可得不等式(47)对函数 u 也是成立的(常数 K 可能不同).

如果 S 是 $W^{1,p}(\Omega)$ 中的有界集，那么对任给 $\varepsilon > 0$ 由式(47)可以找到一个 N 使对所有 $u \in S$

$$\int_{\Omega_N} |u(x)|^p dx < \varepsilon.$$

由于嵌入 $W^{1,p}(\Omega \sim \Omega_N) \to L^p(\Omega \sim \Omega_N)$ 是紧的，由定理 2.22 可得 S 在 $L^p(\Omega)$ 中是准紧的，因此嵌入 $W^{1,p}(\Omega) \to L^p(\Omega)$ 是紧的.

6.48 例 我们仍然考虑例 6.43, 6.46 讨论过的区域 Ω 和在例 6.46 中给出的流动 Φ. 当 $0 \leqslant t \leqslant 1$ 时，我们有

$$d_N(t) = \sup_{x \geqslant N} \frac{f(x)}{f(x-t)} \leqslant 1,$$

由式(43)，当 $t > 0$

$$\lim_{N\to\infty} d_N(t) = 0.$$

由控制收敛定理我们有

$$\lim_{N\to\infty} \int_0^1 d_N(t) dt = 0.$$

f' 是有界的这个假设保证了速度 $|(\partial/\partial t)\Phi(x, y, t)|$ 在 U 上是有界的. 因而区域 Ω 满足定理 6.47 的条件，对于这样的区域 Ω 嵌入 (45)是紧的.

6.49 例 对于某些有界区域，定理 6.47 也能够用于证明嵌入

(45)的紧性,对这些有界区域既不能应用 Rellich-Kondrachov定理,也不能用这个定理证明的技巧. 例如我们考虑

$$\Omega=\{(x,y)\in\mathbf{R}^2: 0<x<2,\ 0<y<f(x)\}.$$

其中 $f\in C^1(0,2)$ 是正的,非减的,有有界的导数 f',并且满足

$$\lim_{x\to 0+} f(x)=0.$$

令 $U=\{(x,y,t)\in\mathbf{R}^3: (x,y)\in\Omega,\ -x<t<2-x\}$,

定义下面的流动

$$\Phi: U\to\Omega$$

$$\Phi(x,y,t)=\left(x+t,\ \frac{f(x+t)}{f(x)}y\right).$$

因此 $\det\Phi_t'(x,y)=f(x+t)/f(x)$. 如果 $\Omega_N^*=\{(x,y)\in\Omega,\ x>1/N\}$,则

$$d_N(t)=\sup_{0<x\leqslant 1/N}\left|\frac{f(x)}{f(x+t)}\right|$$

对 $0\leqslant t\leqslant 1$ 满足 $d_N(t)\leqslant 1$,如果 $t>0$, $\lim\limits_{N\to\infty}d_N(t)=0$. 由控制收敛定理也有 $\lim\limits_{N\to\infty}\int_0^1 d_N(t)=0$. 由于 Ω_N^* 是有界的而且具有锥性质,又由于 f' 的有界性保证了 $\dfrac{\partial\Phi}{\partial t}$ 的有界性,因此由定理 6.47 我们可得嵌入

$$W^{m,p}(\Omega)\to L^p(\Omega) \tag{48}$$

的紧性.

假设对任意的 k, $\lim\limits_{x\to 0+}f(x)/x^k=0$. [例如,令 $f(x)=e^{-\frac{1}{x}}$.]那么区域 Ω 在原点就有一个指数的尖点,由定理 5.32 我们知道对任意的 $q>p$ 嵌入

$$W^{m,p}(\Omega)\to L^q(\Omega)$$

是不存在的. 因此不能用 6.7 节的方法证明嵌入(48) 的紧性.

6.50 附注 (1) 容易设想比上面的例子更一般的区域，定理 6.47 也是能够应用的，虽然选取一个适当的流动可能是困难的. 一个有很多(可能是无穷多)个无界的分支的连通区域也能够用一个适当的流动来描述，只要每一分支是正则的，且其体积相当快的衰减. 从例 6.44 中给出的区域 $\widetilde{\Omega}$ 可以看出一个条件是不够的. 对于每部分体积都是单调衰减的无界区域，定理 6.40 实质上是定理 6.47 的逆定理，定理 6.40 的证明可以分别地应用于证明每个分支的体积以所要求的方式衰减.

(2) 由于对于无界区域 Ω 为了使 $W^{m,p}(\Omega)$ 紧嵌入 $L^p(\Omega)$，必须有有限的体积，定理 6.47 不能推广到使 $W^{m,p}(\Omega)$ 紧嵌入 $L^q(\Omega)$ $(q>p)$，$C_B(\Omega)$，等等. ——这些嵌入是不存在的.

Hilbert-Schmidt 嵌入

6.51 在可分的 Hilbert 空间 X 中一个完全的标准正交系是一个满足下面条件的元素序列 $\{e_i\}_{i=1}^{\infty}$

$$(e_i,\ e_j)_X=\begin{cases}1, & i=j,\\ 0, & i\neq j\end{cases}$$

[这里 $(\cdot,\cdot)_X$ 是空间 X 的内积]，而且对每个 $x\in X$ 有

$$\lim_{k\to\infty}\|x-\sum_{i=1}^{k}(x, e_i)_X e_i; X\|=0. \tag{49}$$

因此 $x=\sum_{i=1}^{\infty}(x,e_i)_X e_i$，这个级数对于 X 中的范数是收敛的. 我们知道每一个可分的 Hilbert 空间都具有这样的完全正交系. 从(49)式可以推出 Parseval 等式

$$\|x;\ X\|^2=\sum_{i=1}^{\infty}|(x,\ e_i)_X|^2.$$

假设X和Y是两个可分的 Hilbert 空间，$\{e_i\}_{i=1}^{\infty}$ $\{f_i\}_{i=1}^{\infty}$分别是它们的完全标准正交系，又假设A是一个由X到Y的线性算子. 令A^*是A的伴随算子A^*把Y映射到X，其定义如下：

$$(x, A^*y)_X = (Ax, y)_Y, \ x \in X, \ y \in Y$$

定义

$$|||A|||^2 = \sum_{i=1}^{\infty} \|Ae_i; Y\|^2, \ |||A^*|||^2 = \sum_{i=1}^{\infty} \|A^*f_i; X\|^2.$$

如果$|||A|||$是有限的，则称A为 Hilbert-Schmidt 算子，称$|||A|||$为算子的 Hilbert-Schmidt 范数(回忆一下，算子A的范数由

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax; Y\|}{\|x; X\|}$$

给出.)我们必须验证上面定义的合理性.

6.52 引理 范数$|||A|||$和$|||A^*|||$不依赖于正交系$\{e_i\}$，$\{f_i\}$的选取，而且

$$|||A||| = |||A^*||| \geqslant \|A\|.$$

证明 由 Parseval 等式

$$|||A|||^2 = \sum_{i=1}^{\infty} \|Ae_i; Y\|^2 = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |(Ae_i, f_j)_Y|^2$$

$$= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |(e_i, A^*f_j)_X|^2 = \sum_{j=1}^{\infty} \|A^*f_j; X\|^2 = |||A^*|||.$$

因此$|||A|||$, $|||A^*|||$是不依赖于$\{e_i\}$; $\{f_i\}$的. 对于任意的$x \in X$，我们有

$$\|Ax; Y\|^2 = \left\|\sum_{i=1}^{\infty} (x, e_i)_X Ae_i; Y\right\|^2$$

$$\leqslant \left(\sum_{i=1}^{\infty} |(x, e_i)_X| \, \|Ae_i; Y\|\right)^2$$

$$\leqslant \left(\sum_{i=1}^{\infty} |(x, e_i)_X|^2\right)\left(\sum_{j=1}^{\infty} \|Ae_j, Y\|^2\right) = \|x; X\|^2 |||A|||^2.$$

所以$\|A\| \leqslant |||A|||$，这就是所要求的. ▮

我们留给读者自己去证明下面的结论:

(a) 设 X, Y, Z 是可分的 Hilbert 空间,A,B 分别是从 X 到 Y, Y 到 Z 的有界线性算子,如果 A,B中有一个是 Hilbert-Schmidt 算子则 $B\circ A$ 是 X 到 Z 的 Hilbert-Schmidt 算子. (如 A 是 Hilbert-Schmidt 算子, 则 $\|\!|B\circ A|\!\|\leqslant\|B\|\,|\!\|A|\!\|$.)

(b) 每一个 Hilbert-Schmidt 算子是紧的.

下面的定理是属于 Maurin[43] 的, 这个定理对于微分算子的特征函数展开是寓义深远的.

6.53 定理 设 Ω 是 $\mathbf{R}^n$ 中具有锥性质的有界集, m, k 是非负的整数 $k>n/2$. 则嵌入映射

$$W^{m+k,2}(\Omega)\to W^{m,2}(\Omega) \tag{50}$$

是 Hilbert-Schmidt 算子. 类似地对任意有界区域 Ω 嵌入

$$W_0{}^{m+k,2}\to W_0{}^{m,2}(\Omega) \tag{51}$$

也是 Hilbert-Schmidt 算子.

证明 对于给定的 $y\in\Omega, \alpha(|\alpha|\leqslant m)$, 在 $W^{m+k,2}(\Omega)$ 上定义如下的线性泛函 T_y^α

$$T_y^\alpha(u)=D^\alpha u(y).$$

由于 $k>n/2$, 由 Sobolev 嵌入定理 5.4 可知 T_y^α 在 $W^{m+k,2}(\Omega)$ 上是有界的, 存在一个不依赖于 α, y 的常数 K 使:

$$|T_y^\alpha(u)|\leqslant\max_{0\leqslant|\alpha|\leqslant m}\sup_{x\in\Omega}|D^\alpha u(x)|\leqslant K\|u\|_{m+k,2,\Omega}. \tag{52}$$

由 Hilbert 空中间的 Riesz 表示定理, 存在一个 $v_y^\alpha\in W^{m+k,2}(\Omega)$ 使

$$D^\alpha u(y)=T_y^\alpha(u)=(u, v_y^\alpha)_{m+k}, \tag{53}$$

其中 $(\cdot, \cdot)_{m+k}$ 是空间 $W^{m+k,2}(\Omega)$ 中的内积, 而且

$$\|v_y^\alpha\|_{m+k,2,\Omega}=\|T_y^\alpha;[W^{m+k,2}(\Omega)]'\|\leqslant K. \tag{54}$$

如果 $\{e_i\}_{i=1}^\infty$ 是空间 $W^{m+k,2}(\Omega)$ 中一完全标准正交系, 则

$$\|v_y^\alpha\|^2{}_{m+k,2,\Omega}=\sum_{i=1}^{\infty}|(e_i,v_y^\alpha)_{m+k}|^2=\sum_{i=1}^{\infty}|D^\alpha e_i(y)|^2.$$

因此,

$$\sum_{i=1}^{\infty}\|e_i\|^2{}_{m,2,\Omega}\leqslant\sum_{|\alpha|\leqslant m}\int_\Omega\|v_y^\alpha\|^2{}_{m+k,2,\Omega}dy.$$

$$\leqslant\sum_{|\alpha|\leqslant m}K\mathrm{vol}\Omega<\infty \tag{55}$$

因此嵌入(50)是 Hilbert-Schmidt 算子. 对嵌入(51)的证明是完全相同的,只是不用定理 5.4 对区域Ω正则性的要求.

下面的 Maurin 定理的推广是属于 Clark 的[17].

6.54 定理 设μ是定义于区域 $\Omega\in\mathbf{R}$ 上一个非负可测函数. 又设 $W_0{}^{m,2;\mu}(\Omega)$ 是由空间 $C_0{}^\infty(\Omega)$ 按照下面带权的范数完备化得到的 Hilbert 空间

$$\|u\|_{m,2;\mu}=\left\{\sum_{|\alpha|\leqslant m}\int_\Omega|D^\alpha u(x)|^2\mu(x)dx\right\}^{1/2}.$$

对于 $y\in\Omega$, 令 $\tau(y)=\mathrm{dist}\,(y,\mathrm{bdry}\Omega)$. 设 ν 是一个非负的整数使

$$\int_\Omega[\tau(y)]^{2\nu}\mu(y)dy<\infty. \tag{56}$$

如果 $k>\nu+n/2$, 那么嵌入

$$W_0{}^{m+k,2}(\Omega)\to W_0{}^{m,2,\mu}(\Omega) \tag{57}$$

(存在并且)是 Hilbert-Schmidt 算子.

证明 设$\{e_i\}$, T_y^α, v_y^α 如前一定理中定义的那样. 如果 $y\in\Omega$, 在 bdryΩ 上选取一点 y_0 使 $\tau(y)=|y-y_0|$. 如果 ν 是一个正整数,而且$u\in C_0^\infty(\Omega)$,利用带有余项的Taylor公式我们对于满足$|y-y_\beta|\leqslant\tau(y)$的某点 y_β 有

$$D^\alpha u(y)=\sum_{|\beta|=\nu}(1/\beta!)D^{\alpha+\beta}u(y_\beta)(y-y_\beta)^\beta.$$

如果$|\alpha|\leqslant m$ 并且 $k>\nu+n/2$ 我们由 Sobolev 嵌入定理[像在(52)

中一样]可以得到

$$|D^\alpha u(y)| \leqslant K\|u\|_{m+k,2}[\tau(y)]^\nu. \tag{58}$$

由于完备性对任何 $u \in W_0{}^{m+k,2}(\Omega)$ 不等式(58)也是成立的，如果 $\nu=0$，直接从(52)可知不等式(58)也成立. 因此由(53)和(54)可得

$$\|v_y^\alpha\|_{m+k,2} = \sup_{\|u\|_{m+k,2}=1} |D^\alpha u(y)| \leqslant K[\tau(y)]^\nu.$$

最后由(56)式得到

$$\sum_{i=1}^{\infty} \|e_i\|^2_{m,2;\mu} \leqslant \sum_{|\alpha|\leqslant m} \int_\Omega \|v_y^\alpha\|^2_{m+k,2} \mu(y) dy$$

$$\leqslant K^2 \sum_{|\alpha|\leqslant m} \int_\Omega [\tau(y)]^{2\nu} \mu(y) dy < \infty.$$

因此嵌入(57)是 Hilbert-Schmidt 算子. ∎

6.55附注 选取各种不同的 μ, ν，导至形为(51)式的 Maurin定理的各种推广. 如果 $\mu(x) \equiv 1$ 且 $\nu=0$，我们得到对于体积有限的无界区域上显然的推广. 如果 $\mu(x) \equiv 1$，$\nu > 0$，Ω可以是无界区域甚至可以有无限体积，但是由(56)式它必是拟有界的[当然拟有界性不足以保证(56)式成立]. 如果 μ 是 Ω 的一个有界子区域 Ω_0 的特征函数，并且 $\nu=0$，我们得到 Hilbert-Schmidt 嵌入

$$W_0{}^{m+k,2}(\Omega) \to W^{m,2}(\Omega_0), \quad k > n/2.$$

第七章　分数次空间

概　要

7.1　本章我们推广 Sobolev 空间的概念到允许非整数 m. 为作到这一点，存在不止一种方法. 不同的途径可引导到相同或不同的空间族. 主要的空间族如下:

(i)　空间 $W^{s,p}(\Omega)$——它可由"实内插"法定义，亦可通过包含高阶导数的一阶差分的固有范数表征.

(ii)　空间 $L^{s,p}(\Omega)$——它可由"复内插"法定义，但是在 $\Omega=\mathbf{R}^n$ 时亦可通过 Fourier 变换表征.

(iii)　Besov 空间 $B^{s,p}(\Omega)$——通过类似 $W^{s,p}(\Omega)$ 的固有范数定义，但包含二阶差分而非一阶差分.

(iv)　Nikols'kii 空间 $H^{s,p}(\Omega)$——赋予包含 L^p 度量的 Hölder 条件的范数.

仅仅空间 $W^{s,p}(\Omega)$ 和 $L^{s,p}(\Omega)$ 当 $s=m$(整数)时和 $W^{s,m}(\Omega)$ 重合. $B^{s,p}(\Omega)$ 和 $W^{s,p}(\Omega)$ 当 $p=2$ 时对所有的 s 重合，但当 $p\neq 2$ 时仅对非整数 s 重合. 空间 $H^{s,p}(\Omega)$ 总是大于(但接近于) $W^{s,p}(\Omega)$, 从嵌入的观点看最简单最完全的结果是对空间 $B^{s,p}(\Omega)$ 和 $H^{s,p}(\Omega)$ 的情形获得的 (见定理 7.70 和 7.73 节). 但是，正是借助于空间 $W^{s,p}(\Omega)$，在 5.20 节提出的表征 $W^{m,p}(\Omega)$ 中的函数在一光滑流形上迹的问题有解答(定理 7.53). 基于这一理由，本章我们集中力量阐明空间 $W^{s,p}(\Omega)$ 的性质，而对其它空间族的类似性质仅给以简短的描述.

本章大约一半篇幅按 Lions 的"迹内插法"展开，在此基础上

我们研究空间 $W^{s,p}(\Omega)$. 这些空间在7.36引进. 迹内插法是本质上等价的几个 Banach 空间实内插法中的一个, 对此现已有可观的文献. 这些方法的描述可在 Butzer 和 Berens [13] Stein 和 Weiss[65] 的著作中找到, 有兴趣的读者可以参考 Peetre[56] 和 Grisvard[28]的关于内插在分数次 Sobolev 空间这一方面的应用的著作. 在 Stein[64*a*]中亦给出这些空间的论述. 本章大部份材料循沿 Lions[37], [38]和 Lions 与 Magenes[40].

Bochner 积分

7.2 本章我们将经常使用定义在 $\mathbf{R}$ 中一个区间上取值在Banach 空间的函数 f 的积分的概念. 因此我们从 Bochner 积分的一个简短的讨论开始, 有关我们的断言的细节及其证明, 读者可查阅例如 Yosida 的书[69].

设 B 是一个带范数 $\|\cdot\|_B$ 的 Banach 空间. 设 $\{A_1, A_2, \cdots, A_m\}$ 是 $\mathbf{R}$ 中一族有限个互不相交的可测子集, 每一子集有有穷测度, 设 $\{b_1, b_2, \cdots, b_n\}$ 是 B 中一族对应点, 由

$$f(t)=\sum_{j=1}^{m}\chi_{A_j}(t)b_j$$

定义的 $\mathbf{R}$ 到 B 的函数称为简单函数, 其中 χ_A 是 A 的特征函数. 对简单函数, 我们显然定义

$$\int_{\mathbf{R}}f(t)dt=\sum_{j=1}^{m}\mu(A_j)b_j,$$

其中 $\mu(A)$ 表示 A 的(Lebesgue)测度.

令 A 是 $\mathbf{R}$ 中可测集, f 是 A 到 B 几乎处处定义的任意函数. 函数 f 称为在 A(强)可测的, 若存在一个支集在 A 中的简单函数序列 $\{f_n\}$ 满足

$$\lim_{n\to\infty}\|f_n(t)-f(t)\|_B=0 \qquad \text{a. e. 于 } A. \tag{1}$$

令$\langle\cdot,\cdot\rangle$表示$B$和它的对偶空间$B'$之间的元素偶(即$b'(b)=\langle b, b\rangle'$, $b\in B$, $b'\in B'$). 可以证明任何值域可分的函数是可测的，只要对每一$b'\in B'$数值函数$\langle f(\cdot), b'\rangle$在$A$上可测.

假如满足(1)的简单函数列f_n可选得满足

$$\lim_{n\to\infty}\int_A\|f_n(t)-f(t)\|_B dt=0.$$

则称f在A上(Bochner)可积，并且定义

$$\int_A f(t)dt=\lim_{n\to\infty}\int_R f_n(t)dt. \tag{2}$$

[(2)右端的积分在B中按范数拓扑收敛到一个极限，它不依赖于逼近序列f_n的选择.]

当且仅当$\|f(\cdot)\|_B$在A上(Lebesgue)可积时，可测函数f在A上是可积的. 事实上

$$\left\|\int_A f(t)dt\right\|_B\leqslant\int_A\|f(t)\|_B dt.$$

7.3 设$-\infty\leqslant a<b\leqslant\infty$. 我们用$L^p(a, b; B)$表示$(a, b)$到$B$的满足$\|f(\cdot)\|_B\in L^p(a, b)$的可测函数$f$的（等价类的）空间. 空间$L^p(a, b; B)$赋以下述范数是 Banach 空间:

$$\|f; L^p(a, b; B)\|=\begin{cases}\left\{\int_a^b\|f(t)\|_B^p dt\right\}^{1/p}, & \text{当 } 1\leqslant p<\infty\\ \operatorname*{ess\,sup}_{t\in(a,b)}\|f(t)\|_B & \text{当 } p=\infty.\end{cases}$$

类似的，若$f\in L^p(c, d; B)$对任何满足$a<c<d<b$的c, d均成立，就记$f\in L^p_{\text{loc}}(a, b; B)$，当$p=1$时称$f$局部可积.

一个局部可积函数g称为局部可积函数f的j次广义导数，只要对任意数值检验函数$\phi\in\mathscr{D}(a, b)=C_0^\infty(a, b)$有

$$\int_a^b\phi^{(j)}(t)f(t)dt=(-1)^j\int_a^b\phi(t)g(t)dt.$$

算子半群和抽象 Cauchy 问题

7.4 以下几节讨论 Banach 空间中算子半群理论的若干方面，这些在后面"迹空间"和分数次空间 $W^{s,p}(\Omega)$ 的研究中需要. 我们根据Zaidman 的著作[70]进行论述.

7.5 设 B 是一个 Banach 空间，$L(B)$ 是定义域为 B，值域在 B 中的有界线性算子的 Banach 空间，并分别用 $\|\cdot\|_B$ 和 $\|\cdot\|_{L(B)}$ 表示 B 和 $L(B)$ 中的范数.

定义域为 $[0,\infty)$，值域在 $L(B)$ 中的函数 G 称为 B 上(强)连续半群，只要

(i) $G(0)=I$, I 是 B 上等同算子,

(ii) 对任意 $s, t\geqslant 0$, $G(s)G(t)=G(s+t)$,

(iii) 对每一 $b\in B$, 函数 $G(\cdot)b$ 从 $[0,\ \infty)$ 到 B(按范数拓扑)连续.

我们注意(ii)意味算子 $G(s)$ 和 $G(t)$ 可交换. 又(iii) 意味对每一 $t_0\geqslant 0$，集 $\{t:\|G(t)\|_{L(B)}>t_0\}$ 在 $\mathbf{R}$ 中是开集，从而可测. 若 $0\leqslant t_0<t_1<\infty$，则对每一 $b\in B$, $G(\cdot)b$ 在 $[t_0, t_1]$ 一致连续，因此存在常数 K_b 满足 $\|G(t)b\|_B\leqslant K_b$, $t_0\leqslant t\leqslant t_1$. 由泛函分析的一致有界性原理，存在常数 K 使对所有 $b\in B$ 及所有 $t\in[t_0,\ t_1]$，$\|G(t)\|_{L(B)}\leqslant K$. 于是 $\|G(\cdot)\|_{L(B)}\in L^\infty_{\text{loc}}(0,\infty)$，我们引伸这一结果成下列引理.

7.6 引理 (*a*) 极限

$$\lim_{t\to\infty}(1/t)\log\|G(t)\|_{L(B)}=\delta_0$$

存在且有穷.

(*b*) 对每一 $\delta>\delta_0$，存在常数 M_δ，使当 $t\geqslant 0$ 有

$$\|G(t)\|_{L(B)}\leqslant M_\delta e^{\delta t}.$$

证明 令 $N(t)=\log\|G(t)\|_{L(B)}$, 由于

$$\|G(s+t)\|_{L(B)}\leqslant\|G(s)\|_{L(B)}\|G(t)\|_{L(B)},$$

我们得N的次可加性

$$N(s+t)\leqslant N(s)+N(t).$$

令 $\delta_0=\inf\limits_{t>0}N(t)/t$. 显然 $0\leqslant\delta_0<\infty$. 给定 $\varepsilon>0$, 选 $r>0$ 使 $N(r)/r<\delta_0+\varepsilon$, 若 $t\geqslant 2r$, 取 k 是满足$(k+1)r\leqslant t<(k+2)r$ 的整数, 则

$$\delta_0\leqslant\frac{N(t)}{t}\leqslant\frac{N(kr)+N(t-kr)}{t}\leqslant\frac{k}{t}N(r)+\frac{1}{t}N(t-kr).$$

现有 $t-kr\in[r,2r)$, 前面业已指出 $N(t-kr)$有界, 设界为 K. 这样就有

$$\delta_0\leqslant\frac{N(t)}{t}\leqslant\frac{kr}{t}(\delta_0+\varepsilon)+\frac{K}{t}\leqslant\left(1-\frac{r}{t}\right)(\delta_0+\varepsilon)+\frac{K}{t}.$$

右端当 $t\to\infty$ 时趋于 $\delta_0+\varepsilon$, 而 $\varepsilon>0$ 任意, 这就推出(a).

若 $\delta>\delta_0$, 存在 t_δ, 当 $t\geqslant t_\delta$ 时有 $N(t)\leqslant\delta t$, 或等价地有 $\|G(t)\|_{L(B)}\leqslant e^{\delta t}$. 现在为推出结论$(b)$, 只需令

$$M_\delta=\max(1,\sup_{0\leqslant t\leqslant t_\delta}\|G(t)\|_{L(B)}). \blacksquare$$

7.7 对给定的 $b\in B$, 商$(G(t)b-b)/t$当$t\to0+$时在B中可能(强)收敛亦可能不(强)收敛. 令 $D(A)$是所有这样的极限存在的元素的集, 对 $b\in D(A)$令

$$Ab=\lim_{t\to0+}\frac{G(t)b-b}{t}=\lim_{t\to0+}\frac{G(t)b-G(0)b}{t}.$$

显然 $D(A)$是B的一个线性子空间, 而 A 是一个从 $D(A)$到B内的线性算子. 称 A 为半群G 的无穷小生成子. 注意 A 与$G(t)$在 $D(A)$上可交换.

7.8 引理 (a)对每一$b\in B$

$$\lim_{t\to0+}(1/t)\int_0^t G(\tau)b\,d\tau=b.$$

(*b*) 对每一$b\in B$和$t>0$我们有

$$\int_0^t G(\tau)bd\tau\in D(\varLambda)\text{且}\ \varLambda\int_0^t G(\tau)bd\tau=G(t)b-b.$$

(*c*) 对每一$b\in D(\varLambda)$和$t>0$我们有

$$\int_0^t G(\tau)\varLambda bd\tau=G(t)b-b.$$

(*d*) $D(\varLambda)$在B中稠密.

(*e*) $\varLambda$是B中闭算子，即B的图象$\{(b,\varLambda b):b\in D(\varLambda)\}$是$B\times B$中一个闭子空间.

证明 令$b\in B$. 由$G(\cdot)b$的连续性我们有

$$\lim_{\tau\to0+}\|G(\tau)b-b\|_B=\lim_{\tau\to0+}\|G(\tau)b-G(0)b\|_B=0.$$

只要注意到 $b=\frac{1}{t}\int_0^t bd\tau$ 就可得结论(a).

对固定的t我们有

$$\begin{aligned}&\lim_{s\to0+}\frac{G(s)-G(0)}{s}\int_0^t G(\tau)\,bd\tau\\&=\lim_{s\to0+}\frac{1}{s}\int_0^t[G(s+\tau)-G(\tau)]bd\tau\\&=\lim_{s\to0+}\left(\frac{1}{s}\int_s^{s+t}G(\tau)bd\tau-\frac{1}{s}\int_0^t G(\tau)bd\tau\right)\\&=\lim_{s\to0+}\left(\frac{1}{s}\int_t^{s+t}G(\tau)bd\tau-\frac{1}{s}\int_0^s G(\tau)bd\tau\right)\\&=\lim_{s\to0+}\frac{1}{s}\int_0^s G(\tau)G(t)bd\tau-b=G(t)b-b.\end{aligned}$$

这就证明了(b). 若$b\in D(\varLambda)$, 则当$s\to0+$

$$\begin{aligned}&\left\|\int_0^t G(\tau)\left(\frac{G(s)b-b}{s}-\varLambda b\right)d\tau\right\|_B\\&\leqslant t\sup_{0<\tau<t}\|G(\tau)\|_{L(B)}\left\|\frac{G(s)b-b}{s}-\varLambda b\right\|_B\to0,\end{aligned}$$

于是

$$A\int_0^t G(\tau)bd\tau=\lim_{s\to 0+}\int_0^t G(\tau)\frac{G(s)b-b}{s}d\tau=\int_0^t G(\tau)Abd\tau,$$

这就证明了(c)．(d)是(a)和(b)的直接推论．

设$b_n\in D(A)$，$b_n\to b$且在B中，$Ab_n\to b_0$，则由(c)

$$G(t)b_n-b_n=\int_0^t G(\tau)Ab_n d\tau,$$

我们令$n\to\infty$，并可象在(c)中那样验证交换极限和积分的次序的合理性，从而得

$$G(t)b-b=\int_0^t G(\tau)b_0 d\tau,$$

于是由(a)

$$\lim_{t\to 0+}\frac{G(t)b-b}{t}=\lim_{t\to 0+}\frac{1}{t}\int_0^t G(\tau)b_0 d\tau=b_0,$$

因此$b\in D(A)$并且$Ab=b_0$，这就证明了(e)．▮

7.9 注 由A的闭性可知$D(A)$赋予范数$\|b;D(A)\|=\|b\|_B+\|Ab\|_B$构成Banach空间．这样就有显然的嵌入$D(A)\to B$．

下面的定理论及对一阶微分算子的典型Cauchy初值问题的抽象提法．

7.10 定理 设A是Banach空间B上连续半群G的无穷小生成子．令$a\in D(A)$，f是$[0,\infty)$到B内的连续可微函数．则存在唯一的$[0,\infty)$到$D(A)$的连续函数u，它有$(0,\infty)$到B内的连续导数u'，满足

$$u'(t)-Au(t)=f(t),\ t\geq 0 \tag{3}$$
$$u(0)=a$$

并且u由

$$u(t)=G(t)a+\int_0^t G(t-\tau)f(\tau)d\tau \tag{4}$$

给出．

证明 （唯一性） 我们必须指出当$f(t)\equiv 0$，$a=0$时(3)仅有的解

是 $u(t)\equiv 0$. 若 u 是任一解而 $t>\tau$, 我们有

$$\frac{\partial}{\partial\tau}G(t-\tau)\,u(\tau)=\lim_{s\to 0}\frac{G(t-\tau-s)u(\tau+s)-G(t-\tau)u(\tau)}{s}$$

$$=\lim_{s\to 0}\frac{G(t-\tau-s)-G(t-\tau)}{s}u(\tau)$$

$$+\lim_{s\to 0}G(t-\tau-s)\frac{u(\tau+s)-u(\tau)}{s}$$

$$=-G(t-\tau)Au(\tau)+G(t-\tau)u'(\tau)=0.$$

于是 $G(t-\tau)u(\tau)=G(t)u(0)=0,\ t>\tau$. 令 $t\to\tau+$, 我们得 $u(\tau)=G(0)u(\tau)=0$ 对所有 $\tau\geqslant 0$ 成立.

（存在性） 我们验证由(4)给定的 u 满足(3). 首先注意 $u(0)=a$ 且 $(d/dt)G(t)a=AG(t)a$. 因此只需证明函数

$$g(t)=\int_0^t G(t-\tau)f(\tau)d\tau$$

从$[0,\ \infty)$到 B 内连续可微, 在 $D(A)$ 取值且满足 $g'(t)=Ag(t)+f(t)$.

今有

$$\frac{g(t+s)-g(t)}{s}$$

$$=\frac{1}{s}\int_0^{t+s}G(t+s-\tau)f(\tau)d\tau-\frac{1}{s}\int_0^t G(t-\tau)f(\tau)d\tau$$

$$=\frac{1}{s}\int_{-s}^{t}G(t-\tau)f(\tau+s)d\tau-\frac{1}{s}\int_0^t G(t-\tau)f(\tau)d\tau$$

$$=\int_0^t G(t-\tau)\frac{f(\tau+s)-f(\tau)}{s}d\tau+\frac{1}{s}\int_t^{t+s}G(\tau)f(t+s-\tau)d\tau.$$

因 f 从$[0,\infty)$到 B 内连续可微, 故

$$g'(t)=\int_0^t G(t-\tau)f'(\tau)d\tau+G(t)f(0)$$

存在且从$[0,\infty)$到 B 内连续, 另一方面,

$$\frac{g(t+s)-g(t)}{s}=\int_0^t \frac{G(s)-G(0)}{s}G(t-\tau)f(\tau)d\tau$$

$$+\frac{1}{s}\int_t^{t+s} G(t+s-\tau)f(\tau)d\tau$$

$$=\frac{G(s)-G(0)}{s}g(t)+\frac{1}{s}\int_0^s G(s-\tau)f(t+\tau)d\tau.$$

由引理 7.8(a)和f 的连续性,后一积分当 $s\to 0+$ 时收敛到$f(t)$,再加上 $g'(t)$的存在性,这就保证

$\lim\limits_{s\to 0+}[G(s)-G(0)]g(t)/s$ 存在,即$g(t)\in D(A)$,

并且 $g'(t)=Ag(t)+f(t)$,这正是所要证明的. ▮

Lions 的迹空间

7.11 设B_1 和B_2是分别带有范数$\|\cdot\|_{B_1}$ 和$\|\cdot\|_{B_2}$的两个 Banach 空间,X 是一个拓扑向量空间, B_1 和 B_2 连续嵌入其中(即对X中任一开集U, $B_i\cap U$ 在 B_i 中是开集, $i=1, 2$). B_1 和 B_2 的向量和

$$B_1+B_2=\{b_1+b_2\in X: b_1\in B_1,\ b_2\in B_2\}$$

赋以范数

$$\|u; B_1+B_2\|=\inf_{\substack{b_i\in B_i\\ b_1+b_2=u}}(\|b_1\|_{B_1}+\|b_2\|_{B_2})$$

是 Banach 空间.

令 $1\leqslant p\leqslant\infty$, 对每一实数 ν, 令 t^ν 表示定义在$[0, \infty)$的实值函数 $t^\nu(t)=t^\nu$, $0\leqslant t<\infty$.

用 $W(p, \nu; B_1, B_2)$(当不致引起混淆时用 W)表示从$[0, \infty)$到 B_1+B_2 内的满足条件.

$$t^\nu f\in L^p(0, \infty; B_1)\text{且 } t^\nu f'\in L^p(0, \infty; B_2)$$

的可测函数 f(的等价类) 的空间. f' 表示f 的广义导数. 空间 W 赋以范数

$$\|f\|_W=\|f; W(p, \nu; B_1, B_2)\|$$

$$=\max(\|t^\nu f;\ L^p(0,\ \infty;B_1)\|, \|t^\nu f';L^p(0,\infty;B_2)\|)$$

是 Banach 空间.

作为这种结构的一个例子，读者可以验证 $W(p,0;W^{1,p}(\mathbf{R}^n), L^p(\mathbf{R}^n))$同构于 Sobolev 空间 $W^{1,p}(\Omega)$，其中 $\Omega=\{(x,t)=(x_1,\cdots,x_n,t)\in\mathbf{R}^{n+1}:t>0\}$.

我们将证明对某些 p 和 ν 的值，W 中的函数 f 在 B_1+B_2 中具有迹 $f(0)$.

7.12 引理 设 $f\in W$，则存在 $b\in B_1+B_2$ 在$(0,\infty)$a.e. 满足

$$f(t)=b+\int_1^t f'(\tau)d\tau. \tag{5}$$

因此 f 几乎处处等于一个从$(0,\infty)$到 B_1+B_2 内的连续函数.

证明 由于 $t^\nu f\in L^p(0,\infty;B_1)$，因此 $f\in L^p{}_{\mathrm{loc}}(0,\ \infty;B_1)$，同理 $f'\in L^p{}_{\mathrm{loc}}(0,\infty;B_2)$，由

$$v(t)=f(t)-\int_1^t f'(\tau)d\tau$$

几乎处处定义在$(0,\ \infty)$到 B_1+B_2 内的函数 v 属于 $L^p{}_{\mathrm{loc}}(0,\infty;B_1+B_2)$，于是对任意 $b'\in(B_1+B_2)'$，数值函数 $\langle v(\cdot),b'\rangle$属于 $L^p{}_{\mathrm{loc}}(0,\infty)$. 对每一 $\phi\in C_0^\infty(0,\infty)$，我们有

$$\int_0^\infty \frac{d}{dt}\langle v(t),b'\rangle\phi(t)\,dt=-\int_0^\infty\langle v(t),b'\rangle\phi'(t)dt$$

$$=-\langle\int_0^\infty v(t)\phi'(t)dt,b'\rangle$$

$$=-\langle\int_0^\infty f(t)\,\phi'(t)dt-\int_0^\infty\phi'(t)dt\int_1^t f'(\tau)d\tau,b'\rangle$$

$$=\langle\int_0^\infty f'(t)\phi(t)dt-\int_0^\infty f'(t)\phi(t)dt,b'\rangle=0.$$

(积分和元素偶$\langle\cdot,\cdot\rangle$次序的交换是合理的，因为 v 在 ϕ 的支集上

可积，可用简单函数逼近，而对简单函数，上述次序交换显然可以.）由推论 3.27，对每一$b'\in(B_1+B_2)'$，$\langle v(t), b'\rangle$ 在$(0,\ \infty)$ a.e. 是一常数，于是 $v(t)=b$, a.e. 于$(0, \infty)$，b 是B_1+B_2 中一固定向量，(5)立刻由此推出. 显然(5)中的积分从$(0, \infty)$到 B_2 内连续，从而到 B_1+B_2 内连续. ▌

7.13 引理 假设$(1/p)+\nu<1$，则(5)式右端当 $t\to0+$ 时在 B_1+B_2 中收敛. 其极限就定义为f 在 $t=0$ 的迹 $f(0)$.

证明 设 $0<s<t$，对 $1<p<\infty$ 我们有

$$\left\|\int_s^t f'(\tau)d\tau\right\|_{B_2}\leqslant\int_s^t\left\|\tau^\nu f'(\tau)\right\|_{B_2}\tau^{-\nu}d\tau$$
$$\leqslant\left\|t^\nu f'; L^p(0, \infty; B_2)\right\|\left(\int_0^t\tau^{-\nu p/(p-1)}\,d\tau\right)^{(p-1)/p}.$$

后一因子由于$(1/p)+\nu<1$ 当$t\to0+$时趋于 0.（当 $p=1$ 或 $p=\infty$ 时推理作相应修改.）于是 $\int_1^t f'(\tau)d\tau$ 当 $t\to0+$ 时在 B_2 中收敛，这就证明了引理. ▌

7.14 给定实数 p 和 ν，$1\leqslant p\leqslant\infty$，$\theta=(1/p)+\nu<1$，我们用 $T(p, \nu; B_1, B_2)$（或简单地用 T）表示 $W=W(p, \nu; B_1, B_2)$中的函数的迹$f(0)$所组成的空间，它称为W 的迹空间，T 赋以范数

$$\|u\|_T=\inf_{\substack{u=f(0)\\ f\in W}}\|f\|_W$$

是 Banach 空间.

T 是B_1+B_2 的子空间且拓扑地“处于 B_1 和 B_2 之间”，其意义后面将会变得明朗.

在展开迹空间T的一些性质的讨论之前，我们准备一个后面需要的关于W的引理.

7.15 引理 若 $1\leqslant p<\infty$，W 中从$(0, \infty)$到 B_1 无穷次可微函数f 组成的子空间在W中稠密.

证明 在变换

$$t=e^{\tau}, f(e^{\tau})=\tilde{f}(\tau)$$

之下，我们有 $f\in W$ 当且仅当

$$\int_{-\infty}^{\infty}(e^{\theta p\tau}\|\tilde{f}(\tau)\|_{B_1}^p+e^{(\theta-1)p\tau}\|\tilde{f}'(\tau)\|_{B_2}^p)d\tau<\infty,$$

其中 $\theta=(1/p)+\nu$. 设 J_ε 是 2.17 节中的软化子，则正象引理 2.18 中一样，$J_\varepsilon*\tilde{f}$ 从 R 到 B_1 无穷次可微，且

$$\lim_{\varepsilon\to 0+}\int_{-\infty}^{\infty}(e^{\theta p\tau}\|J_\varepsilon*\tilde{f}(\tau)-\tilde{f}(\tau)\|_{B_1}^p$$
$$+e^{(\theta-1)p\tau}\|(J_\varepsilon*\tilde{f})'(\tau)-\tilde{f}'(\tau)\|_{B_2}^p)d\tau=0.$$

于是 $f_n(t)=J_{1/n}*\tilde{f}(\log t)$ 在 $0<t<\infty$ 无穷次可微，取值在 B_1 中. 因此在 W 中 $f_n\to f$，引理证完. ▌

现着手研究迹空间 T 的内插性质.

7.16 引理 设 $\theta=(1/p)+\nu$ 满足 $0<\theta<1$，则

(a) 每一 $u\in T$ 满足

$$\|u\|_T=\inf_{\substack{f\in W\\ f(0)=u}}\|t^\nu f;L^p(0,\infty;B_1)\|^{1-\theta}\|t^\nu f';L^p(0,\infty;B_2)\|^\theta; \quad (6)$$

(b) 若 $u\in B_1\cap B_2$，则 $u\in T$ 且

$$\|u\|_T\leqslant K\|u\|_{B_1}^{1-\theta}\|u\|_{B_2}^\theta, \quad (7)$$

其中 K 是一个不依赖于 u 的常数.

证明 (a) 固定 $u\in T$ 和 $\varepsilon>0$，令 $f\in W$ 满足 $f(0)=u$ 和 $\|f\|_W\leqslant\|u\|_T+\varepsilon$，令

$$R=\|t^\nu f;L^p(0,\infty;B_1)\|, S=\|t^\nu f';L^p(0,\infty;B_2)\|.$$

对任意 $\lambda>0$，函数 $f_\lambda(t)=f(\lambda t)$ 也属于 W 且满足 $f_\lambda(0)=u$. 而且有

$$\|t^\nu f_\lambda;L^p(0,\infty;B_1)\|=\lambda^{-\theta}R,$$
$$\|t^\nu(f_\lambda)';\ L^p(0,\infty;\ B_2)\|=\lambda^{1-\theta}S.$$

当选择 $\lambda=R/S$，两表达式都等于 $R^{1-\theta}S^\theta$，因此

$$\begin{aligned}\max(R, S) = \|f\|_W &\leqslant \|u\|_T + \varepsilon \\ &\leqslant \inf_{\lambda>0} \|f_\lambda\|_W + \varepsilon \\ &\leqslant \inf_{\lambda>0} \max(\lambda^{-\theta}R, \lambda^{1-\theta}S) + \varepsilon \\ &\leqslant R^{1-\theta}S^\theta + \varepsilon \leqslant \max(R, S) + \varepsilon.\end{aligned}$$

因 ε 是任意的，(6)立刻得证.

(b) 令 $\phi \in C^\infty((0, \infty))$ 满足 $\phi(0)=1$，当 $t \geqslant 1$，$\phi(t)=0$；对所有 $t \geqslant 0$，$|\phi(t)| \leqslant 1$ 且 $|\phi'(t)| \leqslant K_1$. 若 $u \in B_1 \cap B_2$，令 $f(t) = \phi(t)u$，则有 $u = f(0)$. 今有

$$\|t^\nu f;\ L^p(0, \infty; B_1)\| \leqslant K_2 \|u\|_{B_1},$$

其中 $K_2 = \left\{\int_0^1 t^{\nu p} dt\right\}^{1/p}$. 类似地

$$\|t^\nu f';\ L^p(0, \infty; B_2)\| \leqslant K_1 K_2 \|u\|_{B_2}.$$

因此 $f \in W$ 且从(6)推出(7). ∎

上面的引理提出了当 $0<\theta<1$ 时 T“处于 B_1 和 B_2 之间”的意义. 有时就说对应于所有这样的 θ 的空间 T 组成内插于 B_1 和 B_2 之间的 Banach 空间的一个外壳. T 的许多性质可根据下述内插定理从 B_1 和 B_2 的相应性质推出.

7.17 定理 假设 $B_1 \cap B_2$ 在 B_1 和 B_2 中稠密. 设 $\widetilde{B}_1$, $\widetilde{B}_2$ 和 $\widetilde{X}$ 是跟 7.11 节中的 B_1, B_2 和 X 有同样性质的三个空间. 设 L 是定义在 $B_1 \cap B_2$，取值在 $\widetilde{B}_1 \cap \widetilde{B}_2$ 的线性算子，又设对任意 $v \in B_1 \cap B_2$ 有

$$\|Lv\|_{\widetilde{B}_1} \leqslant K_1 \|v\|_{B_1}, \tag{8}$$

$$\|Lv\|_{\widetilde{B}_2} \leqslant K_2 \|v\|_{B_2}. \tag{9}$$

则 L 具有唯一的到 B_1 和 B_2（从而到 B_1+B_2）分别满足 (8) 和 (9) 的连续延拓（仍记作 L），满足

$$\|Lu\|_{\widetilde{B}_1+\widetilde{B}_2} \leqslant \max(K_1, K_2)\|u\|_{B_1+B_2}.$$

若 $0<\theta=(1/p)+\nu<1$，则对任意 $u \in T = T(p, \nu;\ B_1,\ B_2)$ 我们有 $Lu \in \widetilde{T} = T(p, \nu; \widetilde{B}_1, \widetilde{B}_2)$ 且

$$\|Lu\|_{\widetilde{T}}\leqslant K_1^{1-\theta}K_2^{\theta}\|u\|_T. \tag{10}$$

证明 L定义在 B_1+B_2 上，从而定义在T上．由(6)对 $u\in T$ 有

$$\|Lu\|_{\widetilde{T}}=\inf_{\substack{\widetilde{f}\in W\\ \widetilde{f}(0)=Lu}}\|t^{\nu}\widetilde{f};L^p(0,\infty;\widetilde{B}_1)\|^{1-\theta}\|t^{\nu}\widetilde{f}';L^p(0,\infty;\widetilde{B}_2)\|^{\theta}.$$

又对任给 $\varepsilon>0$，存在一个元素 $f\in W$，满足 $f(0)=u$ 且

$$\|t^{\nu}f;L^p(0,\infty;B_1)\|^{1-\theta}\|t^{\nu}f';L^p(0,\infty;B_2)\|^{\theta}<\|u\|_T+\varepsilon.$$

对 $t\geqslant 0$ 令 $\widetilde{f}(t)=Lf(t)$，于是 $\widetilde{f}(0)=Lu$ 且

$$\|t^{\nu}\widetilde{f};L^p(0,\infty;\widetilde{B}_1)\|^{1-\theta}\|t^{\nu}f';L^p(0,\infty;\widetilde{B}_2)\|^{\theta}$$

$$\leqslant K_1^{1-\theta}\|t^{\nu}f;L^p(0,\infty;B_1)\|^{1-\theta}K_2^{\theta}\|t^{\nu}f';L^p(0,\infty;B_2)\|^{\theta}.$$

因此

$$\|Lu\|_{\widetilde{T}}\leqslant K_1^{1-\theta}K_2^{\theta}(\|u\|_T+\varepsilon).$$

因为ε 任意，(10)得证．▮

若用$\|L\|_{L(S,\widetilde{S})}$ 表示S到 $\widetilde{S}$ 内连续线性算子的 Banach 空间 $L(S,\widetilde{S})$中元素的范数，则我们可把(10)写成形式

$$\|L\|_{L(T,\widetilde{T})}\leqslant\|L\|_{L(B_1,\widetilde{B}_1)}^{1-\theta}\|L\|_{L(B_2,\widetilde{B}_2)}^{\theta}.$$

7.18 引理 设 $B_1\cap B_2$ 在 B_1 和 B_2 中稠密，又设存在同时属于 $L(B_1)$和 $L(B_2)$，值域在 $B_1\cap B_2$ 的线性算子序列$\{P_j\}_{j=1}^{\infty}$，再设对每一 $b_i\in B_i$，$i=1,2$

$$\lim_{j\to\infty}\|P_jb_i-b_i\|_{B_i}=0.$$

则对任一 $u\in T$ 我们有

$$\lim_{j\to\infty}\|P_ju-u\|_T=0.$$

特别，$B_1\cap B_2$ 在T中稠密．

证明 固定 $u\in T$，选择 $f\in W$ 满足 $f(0)=u$．令 $f_j(t)=P_jf(t)$．若 $b_i\in B_i$，$i=1,2$，存在一个整数 $j_0=j_0(b_i)$，若$j\geqslant j_0$有

$$\|P_jb_i-b_i\|_{B_i}\leqslant 1.$$

因此 $\{P_jb_i\}$在 B_i 有与 j 无关的界，$i=1,2$，由一致有界原理，存在常数 K_1 和 K_2 使对每一 j 有

$$\|P_j\|_{L(B_i)}\leqslant K_i.$$

由此

$$\|f_j(t)\|_B\leqslant K_1\|f(t)\|_B,\ \|f_j'(t)\|_{B_2}\leqslant K_2\|f'(t)\|_{B_2}.$$

因对几乎所有的 $t>0$，当 $j\to\infty$，$f_j(t)\to f(t)$(在B_1)且 $f_j'(t)\to f'(t)$(在B_2)，由控制收敛定理 $t^\nu f_j\to t^\nu f$(在$L^p(0,\infty;B_1)$)且 $t^\nu f_j'\to t^\nu f'$ (在 $L^p(0,\infty;B_2)$)．因此 $f_j\to f$(在 W)，$P_ju=P_jf(0)\to f(0)=u$ (在 T)．由于 $t^\nu f_j$ 和 $t^\nu f_j'$ 在 $B_1\cap B_2$ 取值，P_ju 属于 $B_1\cap B_2$． ▌

我们引述一个把迹空间的对偶空间表示成另一迹空间的定理，由于其证明相当长而省略，有兴趣的读者可在 Lions 的著作〔37〕中找到，那里研究比之上面稍许广泛的迹空间．(下一定理是〔37〕中第 II 章定理 1.1 的特殊情形．)

7.19 定理 设 B_1 和 B_2 是自反 Banach 空间且满足引理 7.18 的条件．若 $1<p<\infty$，$(1/p)+\nu=\theta$ 满足 $0<\theta<1$，则 $(1/p')-\nu=1-(1/p)-\nu=1-\theta$ 且

$$[T(p,\nu;B_1,B_2)]'\cong T(p',-\nu;B_2',B_1').$$

特别，$T(p,\nu;B_1,B_2)$ 自反．

我们现在证明两个 L^p 空间之间的迹空间的嵌入定理，它在把 Sobolev 嵌入定理的某些方面推广到分数次空间时将起关键作用(见定理 7.57)．设Ω是R^n中一个区域，则 $B_1=L^q(\Omega)$ 和 $B_2=L^p(\Omega)$ 连续嵌入拓扑向量空间 $X=L^1_{\mathrm{loc}}(\Omega)$．(子集 $U\subset L^1_{\mathrm{loc}}(\Omega)$ 称为开集，若对每一 $u\in U$ 存在 $\varepsilon>0$ 和 $K\subset\subset\Omega$ 使对每一 v，只要 $\|v-u\|_{0,1,K}<\varepsilon$，及 $v\in L^1_{\mathrm{loc}}(\Omega)$，就有 $v\in U$．)

7.20 定理 设 p，q，θ 满足 $1\leqslant p\leqslant q<\infty$，$0<\theta<1$，$\theta=(1/p)+\nu$．则

$$T(p,\nu,L^q(\Omega),L^p(\Omega))\to L^r(\Omega), \tag{11}$$

其中

$$1/r=[(1-\theta)/q]+(\theta/p).$$

证明 设 $f\in C^\infty([0,\infty])$，从等式

$$f(0)=f(t)-\int_0^t f'(\tau)d\tau$$

易得

$$\begin{aligned}|f(0)|&\leqslant\int_0^1|f(t)|dt+\int_0^1|f'(t)|dt\\&\leqslant\left\{\left(\int_0^\infty t^{\nu p}|f(t)|^p dt\right)^{1/p}+\left(\int_0^\infty t^{\nu p}|f'(t)|^p dt\right)^{1/p}\right\}\\&\quad\times\left(\int_0^1 t^{-\nu p'}dt\right)^{1/p'}\\&=K_1(\|t^\nu f\|_{0,p,(0,\infty)}+\|t^\nu f'\|_{0,p,(0,\infty)}),\end{aligned}$$

其中 由 $\theta=(1/p)+\nu<1$ 知 $K_1<\infty$. 由类似于引理 7.16 (a) 的证明中用的齐次性考虑可得

$$|f(0)|\leqslant 2K_1\|t^\nu f\|_{0,p,(0,\infty)}^{1-\theta}\ \|t^\nu f'\|_{0,p,(0,\infty)}^{\theta}. \tag{12}$$

现设 $f\in W(p,\ \nu;\ L^q(\Omega),\ L^p(\Omega))$，并暂设 f 从 $(0,\infty)$ 到 $L^q(\Omega)$ 无穷次可微. 令 $\tilde f(x,t)=f(t)(x)$，$0\leqslant t<\infty$，$x\in\Omega$. 由(12)我们有 $\tilde f(x,0)=\lim\limits_{t\to0+}\tilde f(x,t)$ 对几乎所有 $x\in\Omega$ 满足

$$\begin{aligned}&|\tilde f(x,0)|^r\\&\leqslant K_2\left(\int_0^\infty t^{\nu p}|\tilde f(x,t)|^p dt\right)^{(1-\theta)r/p}\left(\int_0^\infty t^{\nu p}\left|\frac{\partial}{\partial t}\tilde f(x,t)\right|^p dt\right)^{\theta r/p},\end{aligned}$$

由 Hölder 不等式,

$$\begin{aligned}\int_\Omega|\tilde f(x,0)|^r dx&\leqslant K_2\left(\int_\Omega\left(\int_0^\infty t^{\nu p}|\tilde f(x,t)|^p dt\right)^{(1-\theta)rs/p}dx\right)^{1/s}\\&\quad\times\left(\int_\Omega\left(\int_0^\infty t^{\nu p}\left|\frac{\partial}{\partial t}\tilde f(x,t)\right|^p dt\right)^{\theta rs'/p}dx\right)^{1/s'},\end{aligned}$$

其中 $(1/s)+(1/s')=1$，若取 s 满足 $(1-\theta)rs=q$，从而 $\theta rs'=p$，则有

$$\|\tilde{f}(\cdot,0)\|_{0,r,\Omega}\leqslant K_3\left(\int_\Omega\left(\int_0^\infty t^{\nu p}|\tilde{f}(x,t)|^p dt\right)^{q/p} dx\right)^{(1-\theta)/q}$$

$$\times\left(\int_\Omega\int_0^\infty t^{\nu p}\left|\frac{\partial}{\partial t}\tilde{f}(x,t)\right|^p dtdx\right)^{\theta/p}$$

$$= K_3\left\|\int_0^\infty t^{\nu p}|f(t)|^p dt\right\|_{0,q/p,\Omega}^{(1-\theta)/p}\|t^\nu f'; L^p(0,\infty,L^p(\Omega))\|^\theta$$

$$\leqslant K_3\left(\int_0^\infty t^{\nu p}\left\||f(t)|^p\right\|_{0,q/p,\Omega}dt\right)^{(1-\theta)/p}\|t^\nu f'; L^p(0,\infty;L^p(\Omega))\|^\theta$$

$$= K_3\|t^\nu f; L^p(0,\infty;L^q(\Omega))\|^{1-\theta}\|t^\nu f'; L^p(0,\infty;L^p(\Omega))\|^\theta.$$

由W中无穷次可微函数的稠密性(引理 7.15)，上述不等式对任意$f\in W$成立．由引理 7.16(a)

$$\|u\|_{0,r,\Omega}\leqslant \inf_{\substack{f\in W\\ f(0)=u}} K_3\|t^\nu f; L^p(0,\infty,L^q(\Omega))\|^{1-\theta}$$

$$\times\|t^\nu f'; L^p(0,\infty;L^p(\Omega))\|^\theta$$

$$=K_3\|u\|_T.$$

这就建立了嵌入(11)． ▮

7.21 注 证明稍许修改，只要用$L^p(\Omega)\cap L^\infty(\Omega)$在$L^\infty(\Omega)$中的闭包代替$L^q(\Omega)$，上述定理可推广到$q=\infty$的情形．

迹空间的半群表征

7.22 设B是一个 Banach 空间，G是B上的连续半群；它一致有界，即存在常数M使

$$\|G(t)\|_{L(B)}\leqslant M,\ 0\leqslant t<\infty.$$

设Λ是G的无穷小生成子，Λ在B中的定义域 $D(\Lambda)$ 赋以范数

$$\|u;D(\Lambda)\|=\|u\|_B+\|\Lambda u\|_B$$

是一个Banach 空间，并且是B的稠密向量子空间．空间$B_1=D(\Lambda)$和$B_2=X=B$满足 7.11 节的条件，只要$\theta=(1/p)+\nu<1$，我们可

相应地构造迹空间 $T=T(p,\nu;\ D(\Lambda),B)$. 定理 7.24 用包含半群 G 的一个明显的范数表征 T. 但首先我们推导一个后面需要的 Hardy, Littlewood 和 Póllya [28] 的不等式.

7.23 引理 设 f 是一个$(0,\infty)$上 a.e. 定义的数值函数,令

$$g(t)=(1/t)\int_0^t f(\xi)d\xi.$$

若 $1\leqslant p<\infty$, $(1/p)+\nu=\theta<1$, 则

$$\int_0^\infty t^{\nu p}|g(t)|^p dt\leqslant[1/(1-\theta)^p]\int_0^\infty t^{\nu p}|f(t)|^p dt. \qquad (13)$$

证明 我们总可假定(13)右端有穷. 在变换 $t=e^\tau, f(e^\tau)=\tilde{f}(\tau)$, $\xi=e^\sigma, g(e^\tau)=\tilde{g}(\tau)$之下, (13)变为

$$\int_{-\infty}^\infty e^{\theta p\tau}|\tilde{g}(\tau)|^p d\tau\leqslant[1/(1-\theta)^p]\int_{-\infty}^\infty e^{\theta p\tau}|\tilde{f}(\tau)|^p d\tau. \qquad (14)$$

注意

$$\tilde{g}(\tau)=e^{-\tau}\int_{-\infty}^\tau \tilde{f}(\sigma)e^\sigma d\sigma.$$

令 $E(\tau)=e^{\theta\tau}$,

$$F(\tau)=\begin{cases}e^{(\theta-1)\tau} & \text{当 } \tau>0\\ 0 & \text{当 } \tau\leqslant 0.\end{cases}$$

则 $E\cdot\tilde{g}=F*(E\cdot\tilde{f})$, 由 Young 定理 4.30

$$\|E\cdot\tilde{g}\|_{0,p,\mathbf{R}}\leqslant\|F\|_{0,1,\mathbf{R}}\|E\tilde{f}\|_{0,p,\mathbf{R}}.$$

由于$\int_{-\infty}^\infty|F(\tau)|d\tau=1/(1-\theta)$, 上式正是(14). ∎

7.24 定理 设 Λ 是 Banach 空间 B 上一致有界连续半群 G 的无穷小生成子. 若 $1\leqslant p<\infty$ 且 $0<(1/p)+\nu<1$, 则 $T=T(p,\ v;\ D(\Lambda),B)$和范数

$$\|u\|_{T^\bullet}=(\|u\|_B^p+\int_0^\infty t^{(\nu-1)p}\|G(t)u-u\|_B^p dt)^{1/p} \qquad (15)$$

有穷的所有 $u\in B$ 的空间 T^0 重合. 范数 $\|\cdot\|_T$ 和 $\|\cdot\|_{T^\bullet}$ 等价.

证明 首先设 $u\in T$, 选择 $f\in W$ 使 $f(0)=u$. 暂且假设 f 从$(0,\ \infty)$

到 $D(\Lambda)$ 内无穷次可微. 令 $f'(t)-\Lambda f(t)=h(t)$. 若 $t\geqslant \varepsilon>0$.

由定理 7.10 得

$$f(t)=G(t-\varepsilon)f(\varepsilon)+\int_{\varepsilon}^{t}G(t-\tau)h(\tau)d\tau.$$

因此

$$G(t-\varepsilon)f(\tau)-f(\varepsilon)=\int_{\varepsilon}^{t}f'(\tau)d\tau-\int_{\varepsilon}^{t}G(t-\tau)h(\tau)d\tau.$$

在上式中令 $\varepsilon\to 0+$, 由定义 $f(\varepsilon)\to f(0)$, 我们得

$$G(t)f(0)-f(0)=\int_{0}^{t}f'(\tau)d\tau-\int_{0}^{t}G(t-\tau)h(\tau)d\tau. \tag{16}$$

现在(16)对所有 $f\in W$ 成立, 因为由引理7.15, f 是一个从$(0,\infty)$到$D(\Lambda)$内无穷次可微函数序列$\{f_n\}$的极限. 因此对 $u\in T$ 和任意 $f\in W, f(0)=u$ 有

$$G(t)u-u=\int_{0}^{t}f'(\tau)d\tau-\int_{0}^{t}G(t-\tau)h(\tau)d\tau,$$
$$h(\tau)=f'(\tau)-\Lambda f(\tau).$$

这样一来

$$\left\|\frac{G(t)u-u}{t}\right\|_{B}\leqslant\frac{1}{t}\int_{0}^{t}\left\|f'(\tau)\right\|_{B}d\tau+\frac{M}{t}\int_{0}^{t}\left\|h(\tau)\right\|_{B}d\tau,$$

其中我们利用了G 的一致有界性. 应用引理7.23, 令$\theta=(1/p)+\nu$, 我们得

$$\int_{0}^{\infty}t^{(\nu-1)p}\|G(t)u-u\|_{B}^{p}dt$$
$$\leqslant\frac{1}{(1-\theta)^{p}}\int_{0}^{\infty}t^{\nu p}(\|f'(t)\|_{B}+M\|h(t)\|_{B})^{p}dt$$
$$\leqslant\frac{2^{p-1}(M+1)^{p}}{(1-\theta)^{p}}(\|t^{\nu}f';L^{p}(0,\infty;B)\|^{p}+\|t^{\nu}\Lambda f;L^{p}(0,\infty;B)\|^{p})$$
$$\leqslant\frac{2^{p}(M+1)^{p}}{(1-\theta)^{p}}\|f\|_{W}^{p}.$$

因为这对任意满足 $f(0)=u$ 的 $f\in W$ 成立，我们有

$$\int_0^\infty t^{(\nu-1)p}\|G(t)u-u\|_B^p dt\leqslant\left(\frac{2M+2}{1-\theta}\right)^p\|u\|_T^p.$$

另外，B 上的等同算子提供 B 到 B 内和 $D(\Lambda)$ 到 B 内的嵌入，每一嵌入有嵌入常数 1．由定理 7.17，我们同样有 $T\to B$，且 $\|u\|_B\leqslant\|u\|_T$．因此 $u\in T$ 导出 $u\in T^0$，并且

$$\|u\|_{T^0}\leqslant\left(1+\frac{2M+2}{1-\theta}\right)\|u\|_T.$$

反之，设 $u\in T^0$．令 $\Phi\in C^\infty([0,\infty])$，它满足 $\Phi(0)=1$，当 $t\geqslant 1$ 时 $\Phi(t)=0$，当 $t\geqslant 0$ 时 $|\Phi(t)|\leqslant 1$，$|\Phi'(t)|\leqslant K_1$．令

$$f(t)=\Phi(t)g(t),$$

其中

$$g(t)=(1/t)\int_0^t G(\tau)ud\tau,\ t>0. \tag{17}$$

为了证明 $u\in T$ 且 $\|u\|_T\leqslant K_2\|u\|_{T^0}$，只要证明 $f\in W$ 且

$$\|f\|_W\leqslant K_2\|u\|_{T^0}. \tag{18}$$

[因由引理 7.8（a），$f(0)=\lim\limits_{t\to 0+}\Phi(t)g(t)=u$.] 而这可由证明 $t^\nu g\in L^p(0,1;D(\Lambda))$ 和 $t^\nu g'\in L^p(0,1;B)$ 且带适当的被 $K_3\|u\|_{T^0}$ 界住的范数而办到.

由引理 7.8（b），$\int_0^t G(\tau)ud\tau\in D(\Lambda)$ 且

$$\Lambda\int_0^t G(\tau)ud\tau=G(t)u-u.$$

于是

$$\int_0^1 t^{\nu p}\|g(t);D(\Lambda)\|^p dt$$

$$=\int_0^1 t^{(\nu-1)p}\left(\left\|\int_0^t G(\tau)ud\tau\right\|_B+\left\|\Lambda\int_0^t G(\tau)ud\tau\right\|_B\right)^p dt$$

$$\leqslant 2^{p-1}M^p\|u\|_B^p\int_0^1 t^{\nu p}dt+2^{p-1}\int_0^\infty t^{(\nu-1)p}\|G(t)u-u\|_B^p dt\quad .$$

$$\leqslant 2^{p-1}\max(M^p/\theta p, 1)\|u\|_T^p{}_0.$$

因为

$$g'(t)=(1/t)G(t)u-(1/t^2)\int_0^t G(\tau)ud\tau$$

$$=(1/t)(G(t)u-u)-(1/t^2)\int_0^t (G(\tau)u-u)d\tau,$$

又因为

$$\int_0^1 t^{\nu p}\left\|\frac{G(t)u-u}{t}\right\|_B^p dt\leqslant\|u\|_T^p,$$

并由引理 7.23，将 ν 代之以 $\nu-1$，

$$\int_0^1 t^{\nu p}\left\|(1/t^2)\int_0^t (G(\tau)u-u)d\tau\right\|_B^p dt$$

$$\leqslant[1/(2-\theta)^p]\int_0^\infty t^{(\nu-1)p}\|G(t)u-u\|_B^p dt$$

$$\leqslant[1/(2-\theta)^p]\|u\|_T^p.$$

因此我们有

$$\int_0^1 t^{\nu p}\|g'(t)\|_B^p d\tau\leqslant K_4\|u\|_T^p.$$

于是 $g\in W$，(18)成立，证明完毕. ▌

7.25 为我们的目的，尚需对定理 7:24 的作稍微的推广. 设 A_1, $A_2, \cdots, A_n$ 是交换的，一致有界的 B 上的连续半群 $G_1, G_2, \cdots, G_n$ 的无穷小生成子的有限族.

$$\|G_j(t)\|_{L(B)}\leqslant M_j;\ 1\leqslant j\leqslant n,\ t\geqslant 0$$

$$G_j(s)G_k(t)=G_k(t)G_j(s);\ 1\leqslant j,k\leqslant n,\ s,t\geqslant 0.$$

令 B^n 表示乘积空间 $B\times B\times\cdots\times B$(n 个因子)，B^n 赋以范数

$$\|(b_1,b_2,\cdots,b_n)\|_{B^n}=\sum_{j=1}^n\|b_j\|_B$$

是一个 Banach 空间. A 是从 $D(A)=\bigcap_{j=1}^m D(A_j)$ 到 B^n 内由

$$Au=(A_1u, A_2u, \cdots, A_nu)$$

定义的算子.

我们留给读者推广引理 7.8，证明$D(\Lambda)$在B中稠密，Λ是闭算子，从而赋以了范数

$$\|u;D(\Lambda)\|=\|u\|_B+\|\Lambda u\|_{B^n}=\|u\|_B+\sum_{j=1}^{n}\|\Lambda_j u\|_B,$$

$D(\Lambda)$是一个 Banach 空间.

7.26 定理 设$0<(1/p)+\nu<1, 1\leqslant p<\infty$，则$T=T(p,\nu;D(\Lambda),B)$和范数

$$\|u\|_{T^*}=(\|u\|_B^p+\sum_{j=1}^{n}\int_0^\infty t^{(\nu-1)p}\|G_j(t)u-u\|_B^p dt)^{1/p}$$

有穷的所有$u\in B$组成的空间T^0重合. 范数$\|\cdot\|_T$和$\|\cdot\|_{T^*}$等价.

证明 证明跟定理 7.24 的证明差不多，只是代替(17)给的$g(t)$我们置

$$g(t)=(1/t^n)\int_0^t\int_0^t\cdots\int_0^t G_1(\tau_1)G_2(\tau_2)\cdots G_n(\tau_n)u d\tau_1 d\tau_2\cdots d\tau_n.$$

细节留给读者.

7.27 例 令 $B=L^p(\mathbf{R}^n), 1\leqslant p<\infty$，对$u\in B$令

$$(G_j(t)u)(x)=u(x_1,\cdots,x_j+t,\cdots,x_n);\quad j=1,2,\cdots,n.$$

显然G_j是$L^p(\mathbf{R}^n)$上交换、一致有界($M_j=1$)且连续的半群.（事实上若允许$t<0$则它们是群.）相应的无穷小生成子满足

$$D(\Lambda_j)=\{u\in L^p(\mathbf{R}^n): D_j u\in L^p(\mathbf{R}^n)\},$$

$$\Lambda_j u=D_j u, u\in D(\Lambda_j).$$

相应地，$D(\Lambda)=\bigcap_{j=1}^{n}D(\Lambda_j)=W^{1,p}(\mathbf{R}^n)$. 由定理 7.26，范数

$$\Big(\|u\|_{0,p,\mathbf{R}^n}^p+\sum_{j=1}^{n}\int_0^\infty t^{(\nu-1)p}\int_{\mathbf{R}^n}|u(x_1,\cdots,x_j+t,\cdots x_n)$$

$$-u(x_1,x_2,\cdots,x_n)|^p dx dt\Big)^{1/p}$$

和空间 $T=T(p,\nu;W^{1,p}(\mathbf{R}^n),L^p(\mathbf{R}^n))$上的范数$\|u\|_T$ 等价，只要 $0<(1/p)+\nu<1$.

高 次 迹

7.28 至此，我们仅考虑了本身及其一阶导数 f' 满足从$[0,\infty]$到不同的 Banach 空间的可积条件的函数的迹 $f(0)$. 我们现在推广迹的概念以获得值 $f^{(j)}(0)$, $0\leqslant j\leqslant m-1$, 只要 $f, f', \cdots, f^{(m)}$满足一定的可积性条件. 作为这种推广的一个结果，后面我们将可表征 $W^{m,p}(\Omega)$中的函数在正则区域Ω边界上的迹.

7.29 设B是一个 Banach 空间，$\Lambda_1, \cdots, \Lambda_n$ 是B上交换、一致有界、连续半群$G_1\cdots$, G_n 的无穷小生成子. 对每一重指标α 我们用归纳法定义B上的子空间 $D(\Lambda^\alpha)$ 和 $D(\Lambda^\alpha)$ 上相应线性算子 Λ^α 如下:

若 $\alpha=(0,0,\cdots,0)$，则 $D(\Lambda^\alpha)=B$, $\Lambda^\alpha=I$, B上等同算子.

若 $\alpha=(0,\cdots,1,\cdots,0)$(1 在第 j 个位置)，则 $D(\Lambda^\alpha)=D(\Lambda_j)$且 $\Lambda^\alpha=\Lambda_j$.

若对所有满足$|\beta|\leqslant r$的β已经定义了$D(\Lambda^\beta)$和Λ^β，设$|\alpha|=r+1$，则

$$D(\Lambda^\alpha)=\{u: u\in D(\Lambda^\beta)\text{且对所有 }\beta<\alpha,\quad \Lambda^{\alpha-\beta}u\in D(\Lambda^\beta)\},$$

$$\Lambda^\alpha=\Lambda_1^{\alpha_1}\cdots\Lambda_n^{\alpha_n}.$$

设 k 是一个正整数，令 $D(\Lambda^k)=\bigcap_{|\alpha|\leqslant k} D(\Lambda^\alpha)$. 把验证下述事实的任务留给读者（例如用关于 k 的归纳法.）.：$D(\Lambda^k)$在B中稠密，$\Lambda^k=(\Lambda^\alpha)_{|\alpha|\leqslant k}$ 是 $D(\Lambda^k)$到 $\prod_{|\alpha|\leqslant k} B$ 内的闭算子，从而 $D(\Lambda^k)$赋以范数

$$\|u:D(\Lambda^k)\|=\sum_{|\alpha|\leqslant k}\|\Lambda^\alpha u\|_B$$

是 Banach 空间.

7.30 对正整数 m 和实数 p, $1\leqslant p\leqslant\infty$, 用 $W^m=W^m(p,\nu;A;B)$ 表示满足

$$t^\nu f^{(k)}\in L^p(0,\infty;D(A^{m-k})),\ 0\leqslant k\leqslant m$$

的 $(0,\ \infty)$ 到 B 内可测函数(的等价类)的空间, $f^{(k)}$ 是 f 的广义导数 d^kf/dt^k. 空间 W^m 赋以范数

$$\|f\|_{W^m}=\max_{0\leqslant k\leqslant m}\|t^\nu f^{(k)};L^p(0,\infty;D(A^{m-k}))\|$$

是 Banach 空间. 注意 $W^1=W(p,\nu;D(A),B)$, $D(A)$ 和 7.25 节的一样.

7.31 引理 设 $f\in W^m$, $m\geqslant 1$. 若 $0\leqslant k\leqslant m-1$, α 是重指标, $|\alpha|+k\leqslant m-1$, 则函数 $f_{\alpha k}=A^\alpha f^{(k)}\in W^1$ 且

$$\|f_{\alpha k}\|_{W^1}\leqslant\|f\|_{W^m}.$$

证明 对 $1\leqslant p<\infty$ 我们有

$$\begin{aligned}
&\|t^\nu f_{\alpha k};L^p(0,\infty;D(A)\|^p\\
&=\int_0^\infty t^{\nu p}\Big(\|A^\alpha f^{(k)}(t)\|_B+\sum_{j=1}^n\|A_jA^\alpha f^{(k)}(t)\|_B\Big)^p dt\\
&\leqslant\int_0^\infty t^{\nu p}\Big(\sum_{|\beta|\leqslant m-k}\|A^\beta f^{(k)}(t)\|_B\Big)^p dt\\
&=\|t^\nu f^{(k)};L^p(0,\infty;D(A^{m-k}))\|^p\leqslant\|f\|^p{}_{W^m}.
\end{aligned}$$

还有,

$$\begin{aligned}
\|t^\nu f'_{\alpha k};L^p(0,\ \infty;B)\|^p&=\int_0^\infty t^{\nu p}\|A^\alpha f^{(k+1)}(t)\|_B^p dt\\
&\leqslant\int_0^\infty t^{\nu p}\Big(\sum_{|\beta|\leqslant m-k-1}\|A^\beta f^{(k+1)}(t)\|_B\Big)^p dt\\
&=\|t^\nu f^{(k+1)};L^p(0,\infty;D(A^{m-k-1}))\|^p\leqslant\|f\|_{W^m}^p,
\end{aligned}$$

于是引理得证. ▌

7.32 让我们此后假定: $0<(1/p)+\nu<1$, 用 T^0 表示对应 W^1 的迹空间 $T(p,\nu;D(A),B)$. 由定理 7.26, 可在 T^0 上取范数

$$\|u\|_{T^0}=(\|u\|_B^p+\|u\|_G^p)^{1/p},$$

其中

$$\|u\|_G=\left(\sum_{j=1}^n\int_0^\infty t^{(\nu-1)p}\,\|G_j(t)u-u\|_B^p dt\right)^{1/p}.$$

高次迹空间可定义如下：对 $k=0, 1, 2, \cdots$ 我们定义 $T^k=T^k(p,\nu;\Lambda;B)$ 是所有满足 $\Lambda^\alpha u\in T^0(|\alpha|\leqslant k)$ 的 $u\in D(\Lambda^k)$ 的元素组成的空间. 空间 T^k 赋以范数

$$\|u\|_{T^k}=(\|u;D(\Lambda^k)\|^p+\sum_{|\alpha|=k}\|\Lambda^\alpha u\|_G^p)^{1/p}$$

是 Banach 空间.

由引理 7.31 和 7.13 推出若 $f\in W^m$, $|\alpha|\leqslant m-k-1$, 则 $\Lambda^\alpha f^{(k)}(0)$ 在 T^0 中存在, 且

$$\|\Lambda^\alpha f^{(k)}(0)\|_{T^0}\leqslant K_{\alpha k}\|f\|_{W^m},$$

其中 $K_{\alpha k}$ 是依赖于 α 和 k 的常数. 因此 $f^{(k)}(0)\in T^{m-k-1}$ 且

$$\|f^{(k)}(0)\|_{T^{m-k-1}}=(\|f^{(k)}(0),D(\Lambda^{m-k-1})\|^p$$

$$+\sum_{|\beta|=m-k-1}\|\Lambda^\beta f^k(0)\|_G^p)^{1/p}\leqslant K_k\|f\|_{W^m}.$$

由此推出线性映射

$$f\to(f(0),f'(0)\cdots,f^{(m-1)}(0)) \tag{19}$$

从 W^m 到 $T^{m-1}\times T^{m-2}\times\cdots\times T^0=\prod_{k=0}^{m-1}T^{m-k-1}$ 连续, 即

$$\sum_{k=0}^{m-1}\|f^{(k)}(0)\|_{T^{m-k-1}}\leqslant K\|f\|_{W^m}.$$

我们下面证明这一映射是映上的(参看 Lions[38]).

7.33 定理 映射(19)的象是 $\prod_{k=0}^{m-1}T^{m-k-1}$. 若 $(u_0,u_1,\cdots u_{m-1})\in\prod_{k=0}^{m-1}T^{m-k-1}$ 则存在 $f\in W^m$ 使 $f^{(k)}(0)=u_k$, $0\leqslant k\leqslant m-1$, 且

$$\|f\|_{W^m}\leqslant K_0\sum_{k=0}^{m-1}\|u_k\|_{T^{m-k-1}}$$

证明 证明类似于定理 7.24 的第 2 部分，自然更复杂．为简单计我们将仔细讨论 $n=1$ 的情形，这样 Λ_j 成为 Λ 而 Λ^α 成为 Λ^k（$|\alpha|=k$）．

假使我们对每一k，$0\leqslant k\leqslant m-1$，构造了函数 $f_k\in W^m$，满足 $f_k^{(k)}(0)=u_k$，

且

$$\|f_k\|_{W^m}\leqslant K_k\|u_k\|_{T^{m-k-1}},\ 0\leqslant k\leqslant m-1.$$

令 $\lambda_{r,k}$，$0\leqslant r\leqslant m-1$，满足非奇异方程组

$$\sum_{r=0}^{m-1} r^j\lambda_{r,k}=\begin{cases}1 & 若\ j=k\\ 0 & 0\leqslant j\leqslant m-1,\ j\neq k.\end{cases}$$

则函数

$$g_k(t)=\sum_{r=0}^{m-1}\lambda_{r,k}f_k(rt)$$

满足

$$g_k^{(k)}(0)=u_k,\ g_k^{(j)}(0)=0,\ 0\leqslant j\leqslant m-1,\ j\neq k.$$

进而容易验证

$$\|g_k\|_{W^m}\leqslant\widetilde{K}_k\|f_k\|_{W^m}.$$

因此函数 $f(t)=\sum_{k=0}^{m-1}g_k(t)$ 具有定理陈述中所要求的性质．这样我们只需构造 f_k．

以下证明中，我们将广泛使用$[0,\infty)$上算子值函数的卷积．如果对 $t\geqslant0$，$F_1(t)$和 $F_2(t)$属于 $L(B)$，我们用

$$F_1*F_2(t)b=\int_0^t F_1(t-\tau)F_2(\tau)b\,d\tau,\ b\in B$$

定义$[0,\infty)$到 $L(B)$内的 F_1*F_2．（假定所有我们使用的算子是可交换的．）如果 F_1 从$[0,\infty)$到 $L(B)$内连续可微，显然有

$$(F_1 * F_2)'(t) = F'_1 * F_2(t).$$

我们用 $F^{((m))}$ 表示有 m 个因子的卷积 $F*F*\cdots*F$；由于对相互交换的因子 $*$ 是结合的，$F^{((m))}$ 被一意确定. 若 $I(t)=I$ 表示 B 上的等同映射，显然有

$$I^{((m))}(t) = [t^{m-1}/(m-1)!\ ]I.$$

若 G 是无穷小生成子为 Λ 的连续半群，由引理 7.8 我们有

$$\Lambda(I*G) = G - I.$$

又当下式两端限制在 $D(\Lambda)$ 的元素上，

$$I*\Lambda G = G - I.$$

给定　$u = u_k \in T^{m-k-1}$，我们定义

$$f_k(t) = [(m+k)!/k!\ ]\phi(t)g(t),$$

其中 $\phi \in C([0,\infty))$ 满足：当 $t \leqslant \frac{1}{2}$，$\phi(t)=1$，当 $t \geqslant 1$，$\phi(t)=0$，并且 $|\phi^{(j)}(t)| \leqslant K_1$，$0 \leqslant j \leqslant m$　又其中

$$g(t) = t^{-m} I^{((k+1))} * G^{((m))}(t)u. \tag{20}$$

[注意当 $m=1$ 且 $k=0$ 时这里给出的 g 和定理 7.24 证明中(17)给出的相同.] 我们必须验证

$$f_k^{(k)}(0) = u, \tag{21}$$

且

$$\|f_k\|_{W^m} \leqslant K_2 \|u\|_{T^{m-k-1}}. \tag{22}$$

由于当 $t \leqslant \frac{1}{2}$ 时 ϕ 是常数，为证(21)只需指出

$$g^{(k)}(0) = [k!/(m+k)!]u.$$

首先有

$$\begin{aligned} g(t) &= t^{-m} I^{((k+1))} * (G - I + I)^{((m))}(t)u \\ &= t^{-m} \sum_{j=0}^{m} \binom{m}{j} I^{((k+1+m-j))} * (G-I)^{((j))}(t)u. \end{aligned}$$

因为 $t^{-m} I^{((k+1+m-j))}(t) = (t^{k-j}/(k+m-j)!)I$，当 $j>0$ 时其 k

次导数为0，我们有

$$g^{(k)}(0)=\left(\frac{d}{dt}\right)^k\frac{t^k}{(k+m)!}\bigg|_{t=0}\quad I*(G-I)^{((0))}u=\frac{k!}{(k+m)!}u.$$

为建立(22)，显然只需证明

$$\int_0^1 t^{\nu p}\|\Lambda^i g^{(j)}(t)\|_B^p dt\leqslant K_3\|u\|_{T^{m-k-1}}^p \tag{23}$$

对任意满足 $0\leqslant j\leqslant m$ 及 $0\leqslant i\leqslant m-j$ 的 i，j 成立，我们区别三种情形.

情形1　设 $0\leqslant j\leqslant k$　且 $m-k\leqslant i\leqslant m-j$. 令 $w=\Lambda^{m-k-1}u$，于是 $w\in B$，令 $l=k+1+i-m$，于是 $l\geqslant 1$ 且 $k+1-l\geqslant j$. 现在

$$\Lambda^i g(t)=t^{-m}I^{((k+1-l))}*\Lambda^l I^{((l))}*G^{((l))}*G^{((m-l))}(t)w.$$

因为 $\Lambda(I*G)=G-I$，我们有

$$\Lambda^l I^{((l))}*G^{((l))}=(G-I)^{((l))},$$

于是

$$\Lambda^i g(t)=t^{-m}I^{((k+1-l))}*G^{((m-l))}*(G-I)^{((l))}(t)w.$$

因为 $k+1-l\geqslant j$，对 $t>0$ 我们有

$$\Lambda^i g^{(j)}(t)=\sum_{r=0}^{j}w_r(t),$$

其中

$$w_r(t)=\widetilde{K}_r t^{-m-r}I^{((k+1-l-j+r))}*G^{((m-l))}*(G-I)^{((l))}(t)w.$$

现在

$$\begin{aligned}&\|I^{((k+1-l-j+r))}*G^{((m-l))}*(G-I)^{((l-1))}\|_{L(B)}\\&\leqslant K_4 t^{(k+1-l-j+r)+(m-l)+(l-1)-1}=K_4 t^{2m-i-j+r-2},\end{aligned}$$

因此

$$\begin{aligned}\|w_r(t)\|_B&\leqslant K_5 t^{-m-r}\int_0^t(t-\tau)^{2m-i-j+r-2}\|G(\tau)w-w\|_B d\tau\\&\leqslant K_5 t^{m-i-j-2}\int_0^t\|G(\tau)w-w\|_B d\tau.\end{aligned}$$

因为 $i\leqslant m-j$，因此我们对 $0<t\leqslant 1$ 得

$$\|\Lambda^i g^{(j)}(t)\|_B\leqslant K_6 t^{-2}\int_0^t \|G(\tau)w-w\|_B d\tau.$$

由引理 7.23 (ν 代之以 $\nu-1$)

$$\begin{aligned}
&\int_0^1 t^{\nu p}\|\Lambda^i g^{(j)}(t)\|_B^p dt\\
&\leqslant K_6\int_0^1 t^{(\nu-1)p}\left(1/t\int_0^t \|G(\tau)w-w\|_B d\tau\right)^p dt\\
&\leqslant K_7\int_0^\infty t^{(\nu-1)p}\ \|G(t)w-w\|_B^p dt\\
&\leqslant K_7\|w\|_{T^0}^p\leqslant K_7\|u\|_{T^{m-k-1}}
\end{aligned}$$

情形 2　设 $0\leqslant j\leqslant k$ 且 $0\leqslant i\leqslant m-k-1$. 则 $w=\Lambda^i u\in B$ 且

$$\Lambda^i g(t)=t^{-m}I^{((k+1))}*G^{((m))}(t)w.$$

因此

$$\Lambda^i g^{(j)}(t)=\sum_{r=0}^{j}w_r(t)=\sum_{r=0}^{j}\widetilde{K}_r t^{-m-r}I^{((k+1-j+r))}*G^{((m))}(t)w.$$

今有

$$\|w_r(t)\|_B\leqslant K_8 t^{-m-r+(k+1-j+r)-1+m}\|w\|_B=K_8 t^{k-j}\|w\|_B.$$

于是

$$\|\Lambda^i g^{(j)}(t)\|_B\leqslant K_9 t^{k-j}\|w\|_B,$$

且

$$\int_0^1 t^{\nu p}\|\Lambda^i g^{(j)}(t)\|_B^p dt\leqslant K_9^p\|w\|_B^p=K_9^p\|\Lambda^i u\|_B\leqslant K_9^p\|u\|_{T^{m-k-1}}.$$

情形 3　设 $k+1\leqslant i\leqslant m$ 且 $0\leqslant i\leqslant m-j$，则 $i\leqslant m-k-1$ 且

$\tilde{u}=\Lambda^i u\in T^{m-k-1-i}$,

$$\|\tilde{u}\|_{T^{m-k-1-i}}\leqslant\|u\|_{T^{m-k-1}}.$$

令 $h(t)=\Lambda^i g(t)$，则

$$h(t)=t^{-m}I^{((k+1))}*G^{((m))}(t)\tilde{u}. \tag{24}$$

在这种情形下为证(23)只需指出

$$\int_0^1 t^{\nu p}\|h^{(j)}(t)\|_B^p dt \leqslant K_{10}\|\tilde{u}\|_{T^{m-k-1-i}}^p. \tag{25}$$

由 $G=\Lambda(I*G)+I$ 得

$$h(t)=t^{-m}I^{((k+2))}*G^{((m))}(t)\Lambda\tilde{u}+t^{-m}I^{((k+2))}*G^{((m-1))}(t)\tilde{u}.$$

再重复 $m-1$ 次这一推理得

$$h(t)=\sum_{l=0}^{m-1}t^{-m}I^{((k+2+l))}*G^{((m-l))}(t)\Lambda\tilde{u}+t^{-m}I^{((k+1+m))}(t)\tilde{u}.$$

为了证明(25)，我们可以忽略

$$t^{-m}I^{((k+1+m))}(t)\tilde{u}=[t^k/(k+m)!]\tilde{u}$$

这一项，因为这一项的 j 阶导数当 $t>0$ 时为 0．相应地我们考虑①

$$h(t)\sim\sum_{l=0}^{m-1}t^{-m}I^{((k+2+l))}*G^{((m-l))}\Lambda\tilde{u}. \tag{26}$$

再重复从(24)导出(26)的推理 $m-k-i-2$ 次，每一次丢掉对 $h^{(j)}$ 没有贡献的次数 $\leqslant j-1$ 的多项式的项，这就导出

$$h(t)\sim\sum_{l=0}^{m-1}t^{-m}I^{((k+2+(m-k-i-2)+l))}*G^{((m-l))}(t)\Lambda^{m-k-1-i}\tilde{u}.$$

令 $w=\Lambda^{m-k-1-i}\tilde{u}=\Lambda^{m-k-1}u$．上述和中的项形如

$$w_l(t)=t^{-m}I^{((m+l-i))}*G^{((m-l))}(t)w,$$

其中 $0\leqslant l\leqslant m-1$．注意 $m+l-i\geqslant j$．为证(25)只需指出

$$\int_0^1 t^{\nu p}\|w_l^{(j)}(t)\|_B^p dt\leqslant K_{11}\|w\|_{T^0}^p. \tag{27}$$

这里我们必须再区分两种情况 $i\leqslant m-j-1$ 和 $i=m-j$.

若 $i\leqslant m-j-1$ 则 $w_l^{(j)}$ 是形如

$$t^{-m-r}I^{((m+l-i-j+r))}*G^{((m-l))}(t)w$$

项的线性组合，它在 B 中的范数不超过

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① 这里"～"表示 $h(t)$ 是右端和式中诸项的线性组合与次数 $\leqslant j-1$ 的多项式之和——译者注.

$$K_{12}t^{-m-r+(m+l-i-j+r-1)+m-l}\|w\|_B \leqslant K_{12}t^{m-j-i-1}\|w\|_T,$$

而(27)立刻导出，

若 $i=m-j$, 我们有

$$w_l(t)=t^{-m}I^{((j+l))}*G^{((m-l-1))}*(G-I+I)(t)w$$
$$=t^{-m}I^{((j+l))}*G^{((m-l-1))}*(G-I)(t)w$$
$$+t^{-m}I^{((j+l+1))}*G^{((m-l-1))}(t)w,$$

对后一项再重复这一手续 $m-l-1$ 次就得

$$w_l(t)=\sum_{s=0}^{m-l-1} t^{-m}I^{((j+l+s))}*G^{((m-l-s-1))}*(G-I)(t)w$$
$$+t^{-m}I^{((j+m))}(t)w.$$

可以再次丢掉对(27)左端没有贡献的项．因此，为建立(27)，只需用

$$w_{ls}(t)=t^{-m}I^{((j+l+s))}*G^{((m-l-s-1))}*(G-I)(t)w$$

代替 w_l 来建立(27)．而 $w_{ls}^{(j)}(t)$ 是对 $0\leqslant r<j$ 形如

$$t^{-m-r}I^{((l+s+r))}*G^{((m-l-s-1))}*(G-I)(t)w$$

的项的线性组合．象在情形 1 一样，对 $0<t\leqslant 1$ 有

$$\|w_{ls}^{(j)}(t)\|_B\leqslant K_{13}t^{-2}\int_0^t\|G(\tau)w-w\|_B d\tau,$$

再用引理 7.23 就推得(27)成立．这就完成了证明．∎

我们注意对一般的 n 的证明本质上类似于上面对 $n=1$ 给出的．代替(20)我们用(加一适当倍数的)

$$g(t)=t^{-mn}I^{((k+1))}*G_1^{((m))}*\cdots*G_n^{((m))}(t)u.$$

7.34 例 令 $B=L^p(\mathbf{R}^n)$ 而 $G_j\ (1\leqslant j\leqslant n)$ 如例 7.27 给出的那样，于是 $\Lambda_j=D_j$．显然

$$D(\Lambda^k)=\{u\in L^p(\mathbf{R}^n): D^\alpha u\in L^p(\mathbf{R}^n),\ |\alpha|\leqslant k\}=W^{k,p}(\mathbf{R}^n).$$

对每一 $u\in L^p(\mathbf{R}_+^{n+1})$，由

$$\tilde{u}(t)(x_1,\cdots,x_n)=u(x_1,\cdots,x_n,t)$$

a.e. 定义由 $[0,\infty)$ 到 $L^p(\mathbf{R}^n)$内的 $\tilde{u}$，则只要 $\tilde{u}^{(k)}\in L^p(0,\ \infty;\ W^{m-k,p}(\mathbf{R}^n))$, $0\leqslant k\leqslant m$, 就有 $u\in W^{m,p}(\mathbf{R}_+{}^{n+1})$.

相应地

$$W^{m,p}(\mathbf{R}_+^{n+1})\simeq W^m(p,0;A;L^p(R^n)),$$

$$A=(D_1,\cdots,D_n), \text{若 } 1<p<\infty, \text{映射 } \gamma$$

$$\gamma: u\to(u(\cdot,\cdots,\cdot,0), D_{n+1}u(\cdot,\cdots,\cdot,0),\cdots,$$
$$D_{n+1}^{m-1}u(\cdot,\cdots,\cdot,0))$$

是 $W^{m,p}(\mathbf{R}_+{}^{n+1})/\ker\gamma$ 到乘积 $\prod\limits_{k=0}^{m-1}T^{m-k-1}$ 上的一个同构和同胚，

其中

$$T^k=T^k(p,0;A,L^p(\mathbf{R}^n))$$
$$=\{v\in W^{k,p}(\mathbf{R}^n):D^\alpha v\in T^0,\ |\alpha|\leqslant k\},$$

且

$$\|v\|_{T^k}=\left\{\sum_{|\alpha|\leqslant k}\|D^\alpha v\|_{0,p,\mathbf{R}^n}^p+\sum_{|\alpha|=k}\sum_{j=1}^n\int_0^\infty t^{-p}\right.$$
$$\left.\times\int_{\mathbf{R}^n}|D^\alpha v(x_1,\cdots,x_j+t,\cdots,x_n)-D^\alpha v(x_1,\cdots,x_n)|^p dx\,dt\right\}^{1/p}.$$

空间 $W^{s,p}(\Omega)$

7.35 我们现在对 $\mathbf{R}^n$ 中任意区域，任意值 s 和 $1<p<\infty$定义空间 $W^{s,p}(\Omega)$. 这些空间对整数值 s 同第三章定义的 $W^{m,p}(\Omega)$和 $W^{-m,p}(\Omega)$一致. 对 $s\geqslant0$ 定义可推广到 $p=1$ 和 $p=\infty$，但我们暂不理会这些极限值.

空间 $B_1=W^{1,p}(\Omega)$和 $B_2=X=L^p(\Omega)$显然满足 7.11 节中设置的条件. 对 $0<\theta<1$ 令

$$T^{\theta,p}(\Omega)=T(p,\nu;W^{1,p}(\Omega),L^p(\Omega)),$$

其中 $\nu+(1/p)=\theta$, 记 $W=W(p,\nu;W^{1,p}(\Omega),L^p(\Omega))$, 我们写出 u

在 $T^{\theta,p}(\Omega)$ 中的范数

$$\|u;T^{\theta,p}(\Omega)\| = \inf_{\substack{f\in W\\ u=f(0)}} \max\left\{\left(\int_0^\infty t^{\nu p}\|f(t)\|_{1,p,\Omega}^p dt\right)^{1/p}, \left(\int_0^\infty t^{\nu p}\|f'(t)\|_{0,p,\Omega}^p dt\right)^{1/p}\right\}. \tag{28}$$

7.36 设 $s\geqslant 0$ 任意. 若 $s=m$ 是一个整数, 我们定义 $W^{s,p}(\Omega)=W^{m,p}(\Omega)$. 若 s 不是一个整数, 记 $s=m+\sigma$, m 是一个整数而 $0<\sigma<1$. 在这种情形, 定义空间 $W^{s,p}(\Omega)$ 由所有这样的函数 (的等价类) $u\in W^{m,p}(\Omega)$ 所组成, 其广义导数 $D^\alpha u$(任意 $|\alpha|=m$) 属于 $T^{1-\sigma,p}(\Omega)$. 则 $W^{s,p}(\Omega)$ 赋以范数

$$\|u\|_{s,p,\Omega}=\{\|u\|_{m,p,\Omega}^p+\sum_{|\alpha|=m}\|D^\alpha u;T^{1-\sigma,p}(\Omega)\|^p\}^{1/p} \tag{29}$$

是Banach 空间.

7.37 由

$$Pu=(u,(D^\alpha u)_{|\alpha|=m})$$

给定的算子(满足 $|\alpha|=m$ 的重指标 α 以某种适当的次序排列)是从 $W^{s,p}(\Omega)$ 到(乘积)Banach 空间

$$S=W^{m,p}(\Omega)\times\prod_{|\alpha|=m}T^{1-\sigma,p}(\Omega)$$

的一个闭子空间上的一个等距同构, 这里 S 赋以范数

$$\|(u,(v_\alpha)_{|\alpha|=m});S\|=\left\{\|u\|_{m,p,\Omega}^p+\sum_{|\alpha|=m}\|v_\alpha;T^{1-\sigma,p}(\Omega)\|^p\right\}^{1/p}.$$

因为 $W^{m,p}(\Omega)$ 和 $T^{1-\sigma,p}(\Omega)$ 是自反的(定理 3.5 和 7.19), 由定理 1.21 和 1.22 推出 $W^{s,p}(\Omega)$ 也是自反的.

7.38 定理 对任何 $s\geqslant 0$, $C_0^\infty(\mathbf{R}^n)$ 在 $W^{s,p}(\mathbf{R}^n)$ 中稠密.

证明 这个结果对 $s=0$, $1,2,\cdots$ 已经证明(定理 2.19 和 3.18)特别说来 $W^{1,p}(\mathbf{R}^n)$ 在 $L^p(\mathbf{R}^n)$ 中稠密[即, 对于 $L^p(\mathbf{R}^n)$ 中的拓扑稠密]. 若 $s=m+\sigma>0$, m 是整数, 而 $0<\sigma<1$, 定理可证明如下.

设 $\psi\in C^\infty(\mathbf{R})$, 满足: 当 $t\leqslant 0$, $\psi(t)=1$, 当 $t\geqslant 1$, $\psi(t)=0$, 对

$j=1, 2, 3, \cdots$令 $\psi_j \in C_0^\infty(\mathbf{R}^n)$定义如下:

$$\psi_j(x)=\psi(|x|-j).$$

设 J_ε 是 2.17 节中引入的软化子. 若 u 是 $\mathbf{R}^n$ 上 a.e. 定义的函数, 令

$$P_j u=J_{1/j}*(\psi_j \cdot u), j=1, 2, \cdots$$

显然, 当$m=0, 1, 2, \cdots, P_j$ 是从$W^{m,p}(\mathbf{R}^n)$到$W^{m,p}(\mathbf{R}^n)$内的有界线性算子. 其值域在 $C_0^\infty(\mathbf{R}^n)$中, 由引理 2.18 和 3.15 推出若 $u\in W^{m,p}(\mathbf{R}^n)$, 则

$$\lim_{j\to\infty}\|P_j u-u\|_{m,p,\mathbf{R}^n}=0.$$

由引理 7.18 推出若 $0<\theta<1$ 且 $u\in T^{\theta,p}(\mathbf{R}^n)$ 则

$$\lim_{j\to\infty}\|p_j u-u; T^{\theta,p}(\mathbf{R}^n)\|=0.$$

因为

$$D^\alpha P_j u=J_{1/j}*(D^\alpha(\psi_j\cdot u))=P_j D^\alpha u+J_{1/j}*\omega_j,$$

其中

$$\omega_j=\sum_{\beta<\alpha}\binom{\alpha}{\beta}D^{\alpha-\beta}\psi_j D^\beta u,$$

由于当 $u\in W^{|\alpha|,p}(\mathbf{R}^n)$时

$$\lim_{j\to\infty}\|\omega_j\|_{1,p,\mathbf{R}^n}=0,$$

于是对任何 $u\in W^{s,p}(\mathbf{R}^n)$, 当$|\alpha|=m$ 时我们有

$$\lim_{j\to\infty}\|D^\alpha P_j u-D^\alpha u;\ T^{1-\sigma,p}(\mathbf{R}^n)\|=0.$$

因此

$$\lim_{j\to\infty}\|P_j u-u\|_{s,p,\Omega}=0,$$

于是证明完成. ∎

7.39 用 $W_0^{s,p}(\Omega)$ 表示 $C_0^\infty(\Omega)$ 在空间 $W^{s,p}(\Omega)$ $(s\geqslant 0)$ 中的闭包. 由上述定理, $W_0^{s,p}(\mathbf{R}^n)=W^{s,p}(\mathbf{R}^n)$. 对 $s<0$, 我们定义

$$W^{s,p}(\Omega)=[W_0^{-s,p'}(\Omega)]', (1/p)+(1/p')=1.$$

由自反性知对任意实的 s

$$[W^{s,p}(\mathbf{R}^n)]' \cong W^{-s,p'}(\mathbf{R}^n).$$

注意到当 $s<0$ 时，$\mathscr{D}(\Omega)=C_0^\infty(\Omega)$ 是 $W_0^{-s,p'}(\Omega)$ 的稠密子空间，$W^{s,p}(\Omega)$ 是 Ω 上的广义函数的空间，除此之外，关于 $W^{s,p}(\Omega)$ 的结构我们不再有什么进一步的解释.

空间 $W^{s,p}(\Omega)$ 的许多性质仅对 $\Omega=\mathbf{R}^n$ 可以方便地证明，然后必须借助保持微分性质的延拓算子把定义在 Ω 上的函数延拓到 $\mathbf{R}^n$ 上而对一般区域推导出来(参看 4.24 节). 对分数 $s=m+\sigma$，空间 $W^{s,p}(\Omega)$ 适当的延拓由内插而获得. 对 Ω 的强 $(m+1)$-延拓算子自然是需要的. 例如假定 Ω 满足定理 4.26 的假设(还可参看 4.29节).

7.40 定理 若 $s=m+\sigma$，m 是整数，$0<\sigma<1$，如果存在对 $\mathbf{R}^n$ 中区域 Ω 的强 $(m+1)$-延拓算子 E，则 $C_0^\infty(\mathbf{R}^n)$ 中的函数在 Ω 上的限制的集合在 $W^{s,p}(\Omega)$ 稠密.

证明 (回忆结论对 $W^{m,p}(\Omega)$ 成立，只需假设 Ω 具有线段性质.) 证明依照定理 7.38 证明的同一线索，只是代替算子 P_j 我们使用算子

$$\widetilde{P}_j u = R_\Omega P_j E u, \qquad u \text{ 定义在 } \Omega,$$

其中 R_Ω 是把 $\mathbf{R}^n$ 上的函数限制在 Ω 上的算子. 证明的细节留给读者. ∎

下面的局部化定理，除要求存在对 Ω 的强 $(m+1)$-延拓算子 E 外，还需要一个定理 4.26 提供的导数 $D^\alpha Eu(x)$ 的通过 u 的导数的表达式. 下述定理的假设必然对满足定理 4.26 条件的任何区域成立.

7.41 定理 设 Ω 是 $\mathbf{R}^n$ 中的区域，对 Ω 存在一个 $(m+1)$-强延拓算子 E，且对 $|\gamma|\leqslant|\alpha|=m$ 存在从 $W^{1,p}(\Omega)$ 到 $W^{1,p}(\mathbf{R}^n)$ 内和从 $L^p(\Omega)$ 到 $L^p(\mathbf{R}^n)$ 内的连续线性算子 $E_{\alpha\gamma}$，若 $u\in W^{m,p}(\Omega)$ 则有

$$D^{\alpha}Eu(x)=\sum_{|\gamma|\leqslant m}E_{\alpha\gamma}D^{\gamma}u(x) \qquad \text{a. e. 于 } \mathbf{R}^n. \tag{30}$$

若 $s=m+\sigma>0$, $0\leqslant\sigma<1$, 则 $W^{s,p}(\Omega)$ 同 $W^{s,p}(\mathbf{R}^n)$ 中的函数在 Ω 上的限制的集合重合.

证明 若 $\sigma=0$, 结果是对 Ω 的强 m-延拓存在性的直接推论. 今设 $0<\sigma<1$. 若 $u\in W^{s,p}(\Omega)$, 则 $u\in W^{m,p}(\Omega)$ 且 $Eu\in W^{m,p}(\mathbf{R}^n)$ 满足

$$\|Eu\|_{m,p,\mathbf{R}^n}\leqslant K_1\|u\|_{m,p,\Omega}\leqslant K_1\|u\|_{s,p,\Omega}. \tag{31}$$

若 $|\gamma|\leqslant m$ 则 $D^{\gamma}u\in T^{1-\sigma,p}(\Omega)$ 并且

$$\|D^{\gamma}u;T^{1-\sigma,p}(\Omega)\|\leqslant K_2\|u\|_{s,p,\Omega}. \tag{32}$$

(这由定义知 $|\gamma|=m$ 时成立, 由引理 7.16 知当 $|\gamma|<m$ 时亦成立.) 因 $E_{\alpha\gamma}$ 从 $W^{1,p}(\Omega)$ 到 $W^{1,p}(\mathbf{R}^n)$ 和从 $L^p(\Omega)$ 到 $L^p(\mathbf{R}^n)$ 都是线性连续的, 由定理 7.17, 它从 $T^{1-\sigma,p}(\Omega)$ 到 $T^{1-\sigma,p}(\mathbf{R}^n)$ 也是连续的. 由(30)和(32)对 $|\alpha|=m$ 我们有

$$\|D^{\alpha}Eu;\ T^{1-\sigma,p}(\mathbf{R}^n)\|\leqslant K_3\|u\|_{s,p,\Omega}.$$

结合(31), 我们得

$$\|Eu\|_{s,p,\mathbf{R}^n}\leqslant K_4\|u\|_{s,p,\Omega},$$

因此 u 是 $Eu\in W^{s,p}(\mathbf{R}^n)$ 在 Ω 上的限制.

反之, 对任何 m, 从 $\mathbf{R}^n$ 到 Ω 上的限制算子 R_Ω 从 $W^{m,p}(\mathbf{R}^n)$ 到 $W^{m,p}(\Omega)$ 连续, 从而由定理 7.17 从 $W^{s,p}(\mathbf{R}^n)$ 到 $W^{s,p}(\Omega)$ 亦连续, 于是 $u\in W^{s,p}(\mathbf{R}^n)$ 在 Ω 上的限制 $R_\Omega u$ 属于 $W^{s,p}(\Omega)$. ∎

我们注意在定理的条件下, 延拓算子 E 对任何 s, $0\leqslant s\leqslant m+1$, 从 $W^{s,p}(\Omega)$ 到 $W^{s,p}(\mathbf{R}^n)$ 内是连续的.

$W^{s,p}(\Omega)$ 的一个内在范数

7.42 我们现在研究对 $W^{s,p}(\Omega)$ $(s\geqslant 0)$ 构造一个新范数的可能性, 这个新范数等价于"迹范数"(29)(s 不是一个整数), 但它借助

于元素的内在性质来表达．考虑到例 7.27，最方便莫如由情形 $\Omega=R^n$ 开始. 按照 Lions 和 Magenes[34]，我们定义带内在范数的新空间 $\widetilde{W}^{s,p}(\Omega)$，然后指出至少对适当正则的区域 Ω，$\widetilde{W}^{s,p}(\Omega)$ 跟 $W^{s,p}(\Omega)$ 重合.

7.43 对 $0<\theta<1$，$1\leqslant p<\infty$ 令 $\widetilde{T}^{\theta,p}(\Omega)$ 表示这样的函数(的等价类) u 的空间，对于它，范数

$$\|u;\widetilde{T}^{\theta,p}(\Omega)\|=\left\{\|u\|_{0,p,\Omega}^p+\int_\Omega\int_\Omega\frac{|u(x)-u(y)|^p}{|x-y|^{n-1+(1-\nu)p}}dxdy\right\}^{1/p} \tag{33}$$

有限，$\nu+(1/p)=\theta$.

7.44 引理 空间 $\widetilde{T}^{\theta,p}(\mathbf{R}^n)$ 同 Banach 空间 $T^{\theta,p}(\mathbf{R}^n)$ 重合，两空间的范数等价.

证明 $T^{\theta,p}(\mathbf{R}^n)$ 中元素的范数[在(28)]曾被定义为它在迹空间 $T(p,\nu;W^{1,p}(\mathbf{R}^n),\ L^p(\mathbf{R}^n))$ 中的范数．由例 7.27，我们可以取范数为

$$\|u\|_T=\left\{\|u\|_{0,p,\mathbf{R}^n}^p+\sum_{j=1}^n\int_0^\infty t^{(\nu-1)p}\int_{\mathbf{R}^n}|u(x_1,\cdots,x_j+t,\cdots,x_n)\right.$$
$$\left.-u(x_1,\cdots,x_n)|^pdxdt\right\}^{1/p}.$$

让我们把(33)给出的范数记作 $\|u\|_{\widetilde{T}}$.

设 $u\in T^{\theta,p}(\mathbf{R}^n)$．令 $\lambda=\dfrac{1}{2}[(n-1)+(1-\nu)p]$ 且写 $u(x)-u(y)$ 成如下形式:

$$\sum_{j=1}^n[u(y_1,\cdots,y_{j-1},x_j,x_{j+1},\cdots,x_n)$$
$$-u(y_1,\cdots y_{j-1},y_j,x_{j+1},\cdots,x_n)],$$

我们有

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\frac{|u(x)-u(y)|^p}{|x-y|^{n-1+(1-\nu)p}}dxdy\leqslant K_1\sum_{j=1}^nQ_j,$$

其中

$$Q_j=\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}|u(y_1,\cdots,y_{j-1},x_j,\cdots,x_n)-u(y_1,\cdots,y_j,x_{j+1},\cdots,x_n)|^p$$

$$\times\frac{dxdy}{\left(\sum_{k=1}^{n}(x_k-y_k)^2\right)^\lambda}$$

于是 $$Q_j=\int_{\mathbf{R}^j}dy_1\cdots dy_j$$

$$\times\int_{\mathbf{R}^{n+1-j}}dx_j\cdots dx_n|u(y_1,\cdots,y_{j-1},x_j,\cdots,x_n)$$

$$-u(y_1,\cdots,y_j,x_{j+1},\cdots,x_n)|^pR_j,\qquad(34)$$

其中 $$R_j=\int_{\mathbf{R}^{n-j}}\int_{\mathbf{R}^{j-1}}\frac{dx_1\cdots dx_{j-1}dy_{j+1}\cdots dy_n}{\left(\sum_{k=1}^{n}(x_k-y_k)^2\right)^\lambda}.$$

令 $\rho^2=(x_1-y_1)^2+\cdots+(x_{j-1}-y_{j-1})^2+(x_{j+1}-y_{j+1})^2+\cdots+(x_n-y_n)^2$，则因 $\lambda>0$ 且 $n-1-2\lambda<0$，

$$R_j=K_2\left(\int_0^{|x_j-y_j|}+\int_{|x_j-y_j|}^{\infty}\right)\frac{\rho^{n-2}}{[\rho^2+(x_j-y_j)^2]^\lambda}d\rho$$

$$\leqslant\frac{K_2}{|x_j-y_j|^{2\lambda}}\int_0^{|x_j-y_j|}\rho^{n-2}d\rho+K_2\int_{|x_j-y_j|}^{\infty}\rho^{n-2-2\lambda}d\rho$$

$$=K_3|x_j-y_j|^{(\nu-1)p}.\qquad(35)$$

在(34)中令

$$y_i=z_i,\ 1\leqslant i\leqslant j,\ x_j=t+y_j,\ x_i=z_i,\ j+1\leqslant i\leqslant n,$$

由(35)得

$$Q_j\leqslant 2K_3\int_0^\infty t^{(\nu-1)p}dt$$

$$\times\int_{\mathbf{R}^n}|u(z_1,\cdots,z_j+t,\cdots,z_n)-u(z_1,\cdots,z_n)|^pdz.$$

于是 $u\in\widetilde{T}^{\theta,p}(\mathbf{R}^n)$，且 $\|u\|_{\widetilde{T}}\leqslant K_4\|u\|_T$.

反之，设 $u\in\widetilde{T}^{\theta,p}(\mathbf{R}^n)$. 令 $x'=(x_2,\cdots,x_n)$和 $z'=(z_2,\cdots,z_n)$，在中心为$x'\in\mathbf{R}^{n-1}$，半径为 $\frac{1}{2}t$ 的球 $D(t,x')$上对 z' 积分不等式

$$
\begin{aligned}
&|u(x_1+t,x')-u(x_1,x')|^p\\
&\leqslant K_5\left(\left|u(x_1+t,x')-u\left(x_1+\frac{1}{2}t,z'\right)\right|^p\right.\\
&\qquad\left.+\left|u\left(x_1+\frac{1}{2}t,z'\right)-u(x_1,x')\right|^p\right),
\end{aligned}
$$

得 $|u(x_1+t,x')-u(x_1,x')|^p\leqslant(K_6/t^{n-1})[I_t(t,x)+I_t(0,x)]$，其中对 $s=t$，或 $s=0$

$$
I_t(s,x)=\int_{D(t;x')}\left|u(x_1+s,x')-u\left(x_1+\frac{1}{2}t,z'\right)\right|^p dz'.
$$

今有

$$
\begin{aligned}
&\int_0^\infty t^{(\nu-1)p}dt\int_{\mathbf{R}^n}\frac{1}{t^{n-1}}I_t(t,x)dx\\
&=\int_{\mathbf{R}^{n-1}}dx'\int_0^\infty\frac{1}{t^{2\lambda}}dt\\
&\quad\times\int_{D(t;x')}dz'\int_{-\infty}^{\infty}\left|u(x_1+t,\ x')-u\left(x_1+\frac{1}{2}t,z'\right)\right|^p dx_1\\
&=\int_{\mathbf{R}^{n-1}}dx'\int_0^\infty\frac{1}{t^{2\lambda}}dt\\
&\quad\times\int_{D(t;x')}dz'\int_{-\infty}^{\infty}\left|u(x_1,\ x')-u\left(x_1-\frac{1}{2}t,z'\right)\right|^p dx_1\\
&=\int_{\mathbf{R}^n}dx\int_{\mathbf{R}^{n-1}}dz'\int_{2|z'-x'|}^{\infty}\frac{\left|u(x_1,\ x')-u\left(x_1-\frac{1}{2}t,z'\right)\right|^p}{t^{2\lambda}}dt\\
&=2\int_{\mathbf{R}^n}dx\int_{\mathbf{R}^{n-1}}dz'\int_{-\infty}^{x_1-|z'-x'|}\frac{|u(x)-u(z)|^p}{[2(x_1-z_1)]^{2\lambda}}dz_1\\
&\leqslant\frac{2}{2^\lambda}\int_{\mathbf{R}^n}dx\int_{\mathbf{R}^n}\frac{|u(x)-u(z)|^p}{|x-z|^\lambda}dz,
\end{aligned}
$$

其中.我们作了替换$z_1=x_1-\frac{1}{2}t, dz_1=-\frac{1}{2}dt$, 并在倒第二行的内层积分中利用了$|x_1-z_1|\geqslant|x'-z'|$从而$|x_1-z_1|\geqslant(x-z)/\sqrt{2}$这一事实. 类似的不等式对$I_t(0,x)$亦成立. 于是

$$\begin{aligned}&\int_0^\infty t^{(\nu-1)p}\int_{\mathbf{R}^n}|u(x_1+t,x_2,\cdots,x_n)\\&-u(x_1,\cdots,x_n)|^p dxdt\\&\leqslant K_7\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\frac{|u(x)-u(z)|^p}{|x-z|^{n-1+(1-\nu)p}}dxdz.\end{aligned}$$

对于其他变量$x_2,\cdots,x_n$作的差, 类似的不等式也成立, 组合这些不等式我们即得$\|u\|_T\leqslant K_8\|u\|_{\tilde{T}}$. ▌

为推广上述引理到一般区域Ω, 我们需要下列延拓引理.

7.45 引理 设Ω是$\mathbf{R}^n$中的一个半空间或是$\mathbf{R}^n$中一个一致C^1正则且带有界边界的区域. 则存在一个把$L^p(\Omega)$映射到$L^p(\mathbf{R}^n)$的线性算子E使

$$Eu(x)=u(x)\quad \text{a.e. 于}\Omega,$$

并且若$0<\theta<1$, $u\in\tilde{T}^{\theta,p}(\Omega)$, 则$Eu\in\tilde{T}^{\theta,p}(\mathbf{R}^n)$且

$$\|Eu;\tilde{T}^{\theta,p}(\mathbf{R}^n)\|\leqslant K\|u;\tilde{T}^{\theta,p}(\Omega)\|,$$

K不依赖于u.

证明 这里的证明十分类似于定理4.26的证明. 我们从情形$\Omega=\mathbf{R}_+^n=\{x\in\mathbf{R}^n: x_n>0\}$开始. 记$x'=(x_1,\cdots,x_{n-1})$, 对$u\in L^p(\Omega)$置

$$Eu(x)=\begin{cases}u(x) & \text{a.e. 于 }\mathbf{R}_+^n.\\ u(x',-x_n) & \text{a.e 于 }\mathbf{R}^n\sim\mathbf{R}_+^n.\end{cases}$$

则

$$\begin{aligned}\|Eu\|_{0,p,\mathbf{R}^n}^p=&\int_{\mathbf{R}^{n-1}}dx'\left\{\int_0^\infty|u(x)|^p dx_n+\right.\\&\left.+\int_{-\infty}^0\left|u(x',-x_n)\right|^p dx_n\right\}\end{aligned}$$

$$=2\|u\|^p_{0,p\mathbf{R}^n_+}.$$

又令 $2\lambda=n-1+(1-\nu)p=n+(1-\theta)p>0$, 我们有

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\frac{|Eu(x)-Eu(y)|^p}{(x-y)^{2\lambda}}dxdy=I_{+-}+I_{+-}+I_{-+}+I_{--},$$

其中

$$I_{++}=\int_{\mathbf{R}^n_+}\int_{\mathbf{R}^n_+}\frac{|u(x)-u(y)|^p}{(x-y)^{2\lambda}}dxdy$$

$$I_{+-}=\int_{\mathbf{R}^{n-1}}dx'\int_{\mathbf{R}^{n-1}}dy'\int_0^\infty dx_n\int_{-\infty}^0\frac{|u(x)-u(x',-y_n)|^p}{[|x'-y'|^2+(x_n-y_n)^2]^\lambda}dy_n$$

$$=\int_{\mathbf{R}^{n-1}}dx'\int_{\mathbf{R}^{n-1}}dy'\int_0^\infty dx_n\int_0^\infty\frac{|u(x)-u(y)|^p}{[|x'-y'|^2+(x_n+y_n)^2]^\lambda}dy_n$$

$$\leqslant\int_{\mathbf{R}^n_+}\int_{\mathbf{R}^n_+}\frac{|u(x)-u(y)|^p}{|x-y|^{2\lambda}}dxdy$$

[因为当 $x_n\geqslant0$ 且 $y_n\geqslant0$, $(x_n+y_n)^2\geqslant(x_n-y_n)^2$], 又类似的不等式对 I_{-+} 和 I_{--} 成立. 于是

$$\|Eu;\widetilde{T}^{\theta,p}(\mathbf{R}^n)\|\leqslant4^{1/p}\|u;\widetilde{T}^{\theta,p}(\mathbf{R}^n_+)\|.$$

现设 Ω 是一致 C^1- 正则且带有界边界的区域. 则 4.6 节中的 bdry Ω 的开覆盖 $\{U_j\}$ 和相应的 U_j 到 $B=\{y\in\mathbf{R}^n:|y|<1\}$ 上的1-光滑映射族 $\{\Phi_j\}$ 是有限集, 比如, $1\leqslant j\leqslant N$. 还可假定集 U_j 有界. 设 U_0 是 Ω 的离开 bdry Ω 的有界开子集, 使 $\Omega\subset\bigcup\limits_{j=0}^N U_j$. 令 $\{\omega_j\}_{j=0}^N$ 是对 Ω 的从属于 $\{U_j\}$ 的 C^∞-单位分解. 给定 $u\in L^p(\Omega)$, 令 u_j 由 $u_j(x)=\omega_j(x)u(x)$ 在 Ω 上 a.e. 定义. 显然 $u_j\in L^p(\Omega)$ 且 $\|u_j\|_{0,p,\Omega}\leqslant\|u\|_{0,p,\Omega}$. 若 $u\in\widetilde{T}^{\theta,p}(\Omega)$, 则对 $1\leqslant j\leqslant N$

$$\int_\Omega\int_\Omega\frac{|u_j(x)-u_j(y)|^p}{|x-y|^{2\lambda}}dxdy\leqslant K_1\int_\Omega\int_\Omega\frac{|u(x)-u(y)|^p}{|x-y|^{2\lambda}}dxdy$$

$$+K_1\int_{\Omega\cap U_j}|u(y)|^pdy\int_{U_j}\frac{|\omega_j(x)-\omega_j(y)|^p}{|x-y|^{2\lambda}}dx.$$

因 U_j 有界, 由引理 5.47, 对 $y\in\Omega\cap U_j$ 我们有

$$\int_{U_j} \frac{|\omega_j(x)-\omega_j(y)|^p}{|x-y|^{2\lambda}} dx \leqslant K_2 \int_{U_j} |x-y|^{\nu p+1-n} dx \leqslant K_3,$$

而且 K_3 可选得不依赖于所包含的 j 的有限个值. 于是 $u_j \in \widetilde{T}^{\theta,p}(\Omega)$ 且

$$\|u_j; \widetilde{T}^{\theta,p}(\Omega)\| \leqslant K_4 \|u; \widetilde{T}^{\theta,p}(\Omega)\|.$$

因为对所有不属于集 $\bigcup_{j=1}^{N} U_j$ 的点 $x \in \Omega$ 有 $\omega_0(x)=1$, 上述不等式对 u_0 亦成立.

对 $1 \leqslant j \leqslant N$, 令 v_j 在 $\mathbf{R}_+^n$ 上由

$$v_j(y) = \begin{cases} u_j \circ \boldsymbol{\Psi}_j(y) & \text{若 } y \in B \cap \mathbf{R}_+^n \\ 0 & \text{若 } y \in \mathbf{R}_+^n \sim B. \end{cases}$$

定义, 其中 $\boldsymbol{\Psi}_j = \boldsymbol{\Phi}_j^{-1}$, 则 $v_j \in \widetilde{T}^{\theta,p}(\mathbf{R}_+^n)$. 事实上, 令 $y = \boldsymbol{\Phi}_j(x)$, $\eta = \boldsymbol{\Phi}_j(\xi)$, 我们有

$$\begin{aligned}
\|v_j; \widetilde{T}^{\theta,p}(\mathbf{R}_+^n)\|^p &= \int_{\mathbf{R}_+^n \cap B} |u_j(\boldsymbol{\Psi}_j(y))|^p dy \\
&\quad + \int_{\mathbf{R}_+^n \cap B} \int_{\mathbf{R}_+^n \cap B} \frac{|u_j(\boldsymbol{\Psi}_j(y)) - u_j(\boldsymbol{\Psi}_j(\eta))|^p}{|y-\eta|^{2\lambda}} dy d\eta \\
&= \int_\Omega |u_j(x)|^p |\det \boldsymbol{\Phi}'_j(x)| dx \\
&\quad + \int_\Omega \int_\Omega \frac{|u_j(x) - u_j(\xi)|^p}{|\boldsymbol{\Phi}_j(x) - \boldsymbol{\Phi}_j(\xi)|^{2\lambda}} |\det \boldsymbol{\Phi}'_j(x)| \\
&\quad \times |\det \boldsymbol{\Phi}'_j(\xi)| dx d\xi \\
&\leqslant K_5 \|u_j; \widetilde{T}^{\theta,p}(\Omega)\|^p, \qquad (36)
\end{aligned}$$

这是因为 $|\det \boldsymbol{\Phi}'_j|$ 有界, 并且 $\boldsymbol{\Psi}_j$ 在 B 上 1-光滑.

$$\begin{aligned}
|x-\xi| = |\boldsymbol{\Psi}_j(y) - \boldsymbol{\Psi}_j(\eta)| &\leqslant K_6 |y-\eta| \\
&= K_6 |\boldsymbol{\Phi}_j(x) - \boldsymbol{\Phi}_j(\xi)|.
\end{aligned}$$

今有 $Ev_j \in \widetilde{T}^{\theta,p}(\mathbf{R}^n)$, 并且

$$\|Ev_j; \widetilde{T}^{\theta,p}(\mathbf{R}^n)\| \leqslant K_7 \|v_j; \widetilde{T}^{\theta,p}(\mathbf{R}_+^n)\|.$$

还有 $\operatorname{supp} Ev_j \subset\subset B$. 我们在 $\mathbf{R}^n$ 上由

$$w_j(x)=\begin{cases}Ev_j(\Phi_j(x)) & \text{若 } x\in U_j\\ 0 & \text{若 } x\in\mathbf{R}^n\sim U_j.\end{cases}$$

a. e. 定义 w_j. 则显然 $w_j(x)=u_j(x)$ a. e. 于 Ω, $\operatorname{supp} w_j\subset\subset U_j$, 且由导出(36)的类似的计算得

$$\|w_j(x);\widetilde{T}^{\theta,p}(\mathbf{R}^n)\|\leqslant K_8\|Ev_j;\widetilde{T}^{\theta,p}(\mathbf{R}^n)\|.$$

最后令

$$E^*u(x)=u_0(x)+\sum_{j=1}^N w_j(x).$$

显然 E^*具有引理叙述中对E所要求的性质. ∎

应当注意，在 4.29 节中对延拓定理 4.26 和 4.28 的减弱的假设所作的注释同样适用于上述引理.

7.46 推论 在引理 7.45 的条件下，空间 $\widetilde{T}^{\theta,p}(\Omega)$ 和 $T^{\theta,p}(\Omega)$ 重合，并且它们的范数等价.

证明 两个空间的重合性从下述事实推出，它们分别同相互重合的空间 $\widetilde{T}^{\theta,p}(\mathbf{R}^n)$ 和 $T^{\theta,p}(\mathbf{R}^n)$ 中的函数在Ω上的限制重合. 若 $u\in\widetilde{T}^{\theta,p}(\Omega)$ 且E是上述引理中的延拓算子，我们有

$$\begin{aligned}\|u;T^{\theta,p}(\Omega)\|&\leqslant\|Eu;T^{\theta,p}(\mathbf{R}^n)\|\leqslant K_1\|Eu;\widetilde{T}^{\theta,p}(\mathbf{R}^n)\|\\&\leqslant K_2\|u;\widetilde{T}^{\theta,p}(\Omega)\|.\end{aligned}$$

反向不等式由同样的方法推出，只是代替E利用定理 4.6 (情形 $m=1$) 中构造的 1-延拓算子，从定理 7.41 的证明可以看出，它是一个对 $T^{\theta,p}(\Omega)$ 的延拓算子. ∎

7.47 对 $s\geqslant 0$，令 $\widetilde{W}^{s,p}(\Omega)$ 是按 7.36 中构造 $W^{s,p}(\Omega)$ 的同样方式构造的空间，只是代替 $T^{1-\theta,p}(\Omega)$ 而使用 $\widetilde{T}^{1-\theta,p}(\Omega)$. 由于推论 7.46，我们证明了下述定理.

7.48 定理 设Ω是 $\mathbf{R}^n$，或 $\mathbf{R}^n$ 中的一个半空间，或 $\mathbf{R}^n$ 中一个一致 C^1-正则且有有界边界的区域. 则空间 $\widetilde{W}^{s,p}(\Omega)$ 和 $W^{s,p}(\Omega)$ 对每一 $s\geqslant 0$ 代数上和拓扑上重合. 特别若 $s=m+\sigma$，m 是整数，

$0<\sigma<1$，则由

$$\|u\|_{\tilde{s},p,\Omega}=\left\{\|u\|_{m,p,\Omega}^p+\sum_{|\alpha|=m}\int_\Omega\int_\Omega\frac{|D^\alpha u(x)-D^\alpha u(y)|^p}{|x-y|^{n+\sigma p}}dxdy\right\}^{1/p}$$

给的范数和 $W^{s,p}(\Omega)$ 上原来的范数 $\|\cdot\|_{s,p,\Omega}$ 等价.

7.49 附注 正是上面记为 $\widetilde{W}^{s,p}(\Omega)$ 的空间在文献中经常碰到，并常记为 $W^{s,p}(\Omega)$. 空间 $\widetilde{W}^{s,\infty}$ 显然可用类似的方式定义，它由范数

$$\|u\|_{s,\infty,\Omega}=\max\left(\|u\|_{m,\infty,\Omega},\max_{|\alpha|=m}\ \operatorname*{ess\ sup}_{\substack{x,y\in\Omega\\x\neq y}}\frac{|D^\alpha u(x)-D^\alpha u(y)|}{|x-y|^\sigma}\right)$$

有限的 $u\in W^{m,\infty}(\Omega)$ 组成.

嵌 入 定 理

7.50 我们已经看到(例 7.34)，若 $1<p<\infty$，则线性映射

$$u\to\gamma u=(\gamma_0 u,\cdots,\gamma_{m-1}u);\ \gamma_j u=D_n^j u(\cdot,\cdots,\cdot,0)$$

建立了 $W^{m,p}(\mathbf{R}_+^n)/\ker\gamma$ 到 $\prod_{k=0}^{m-1}T^{m-k-1}(p,0;\Lambda;L^p(\mathbf{R}^{n-1}))$ 上的一个同构和同胚，其中 $\Lambda=(D_1,\cdots,D_{n-1})$. 因为 $D(\Lambda^k)=W^{k,p}(\mathbf{R}^{n-1})$ 且

$$T^0(p,0;\Lambda;L^p(\mathbf{R}^{n-1}))=T^{1/p,p}(\mathbf{R}^{n-1}),$$

我们有 $\quad T^k(p,0;\Lambda;L^p(\mathbf{R}^{n-1}))=W^{k+1-1/p,p}(\mathbf{R}^{n-1}).$

这样 γ 实际上是一个从 $W^{m,p}(\mathbf{R}_+^n)/\ker\gamma$ 到 $\prod_{k=0}^{m-1}W^{m-k-1/p,p}(\mathbf{R}^{n-1})$ 上的一个同构和同胚. 特别，$W^{m,p}(\mathbf{R}_+^n)$ 中的函数在 $\mathbf{R}^{n-1}=\mathrm{bdry}\ \mathbf{R}_+^n$ 上的迹属于并构成整个空间 $W^{m-1/p,p}(\mathrm{bdry}\ \mathbf{R}_+^n)$. [这一现象有时描写为在边界上失去 $1/p$ 次导数.] 这一结果可以推广到边界光滑的区域.

7.51 若 Ω 是 $\mathbf{R}^n$ 中的一个区域，它有一致 C^m-正则性和有界边界，则 bdryΩ 的开覆盖 $\{U_j\}$ 和相应的从 $B=\{y\in\mathbf{R}^n: |y|<1\}$ 到集 U_j 上的 m-光滑映射族 $\{\overline{\Psi}_j\}$（参看 4.6 节）是有限族，比如 $1\leqslant j\leqslant r$. 若 $\{\omega_j\}$ 是从属 $\{U_j\}$ 的对 badyΩ 的单位分解，又若 u 是定义在 baryΩ 的一个函数，我们在 $\mathbf{R}^{n-1}$ 上定义 $\theta_j u$ 如下：

$$\theta_j u(y')=\begin{cases}(\omega_j u)(\overline{\Psi}_j(y',0)) & \text{若 } |y'|<1\\ 0 & \text{其余,}\end{cases}$$

其中 $y'=(y_1,\cdots,y_{n-1})$.

对 $s\geqslant 0$ 和 $1<p<\infty$，我们定义 $W^{s,p}(\mathrm{bdry}\,\Omega)$ 是满足

$$\theta_j u\in W^{s,p}(\mathbf{R}^{n-1})\qquad(1\leqslant j\leqslant r)$$

的函数 $u\in L^p(\mathrm{bdry}\,\Omega)$（见 5.21 节）的类. 空间 $W^{s,p}(\mathrm{bdry}\,\Omega)$ 赋以范数

$$\|u\|_{s,p,\mathrm{bdry}\,\Omega}=\left\{\sum_{j=1}^{r}\|\theta_j u\|_{s,p,\mathbf{R}^{n-1}}^{p}\right\}^{1/p}$$

是 Banach 空间.

乍看起来，如上定义的空间 $W^{s,p}(\mathrm{bdry}\,\Omega)$ 依赖于定义中使用的特殊的覆盖 $\{U_j\}$，映射 $\{\overline{\Psi}_j\}$ 和单位分解 $\{\omega_j\}$. 可以验证，对不同的族 $\{\widetilde{U}_j\}$，$\{\widetilde{\Psi}_j\}$ 和 $\{\widetilde{\omega}_j\}$，我们构造出的是带等价范数的同一空间.（我们略去细节，参看 Lions 和 Magenes[40].）同样可以推出 $C^\infty(\mathrm{bdry}\,\Omega)$ 在 $W^{s,p}(\mathrm{bdry}\,\Omega)$ 中稠密.

7.52 令 $u\in C_0^\infty(\mathbf{R}^n)$［这种函数在 $W^{m,p}(\Omega)$ 上的限制在 $W^{m,p}(\Omega)$ 中稠密.］令 γ 表示线性映射

$$u\to\gamma u=(\gamma_0 u,\cdots,\gamma_{m-1}u),\ \gamma_j u=\left.\frac{\partial^j u}{\partial n^j}\right|_{\mathrm{bdry}\,\Omega},\tag{37}$$

这里 $\partial^j/\partial n^j$ 表示沿 bdryΩ 外法向的 j 次方向导数. 利用对 bdryΩ 一个邻域从属于开覆盖 $\{U_j\}$ 的一个单位分解，我们可以证明 7.50 节结果的下列（到 Ω）的推广.

7.53 定理 设 $1<p<\infty$，又设 Ω 满足上面描述的条件. 则由(37)给出的映射 γ 可连续延拓成一个从 $W^{m,p}(\Omega)/\ker\gamma$ 到

$$\prod_{k=0}^{m-1} W^{m-k-1/p,p}(\mathrm{bdry}\,\Omega)$$

上的一个同构和同胚.

7.54 下述定理的一个直接推论是映射 γ 的核 $\ker\gamma$，即满足 $\gamma u=0$ 的 $u\in W^{m,p}(\Omega)$ 全体，正好是空间 $W_0^{m,p}(\Omega)$. 我们再一次采用 7.30 节的记号. 令 W_0^m 表示 W^m 中在一个区间 $[0,\varepsilon)$ $(\varepsilon>0$ 可依赖 $f)$ 为 0 的函数 f 的集合在 $W^m=W^m(p,\gamma,A;B)$ 中的闭包.

7.55 定理 若 $f\in W^m$ 且对 $0\leqslant k\leqslant m-1$，$f^{(k)}(0)=0$，则 $f\in W_0^m$. 这样 W_0^m 是从 W^m 到 $\prod_{k=0}^{m-1}T^{m-k-1}$ 上的映射

$$f\to(f(0),f'(0),\cdots,f^{(m-1)}(0))$$

的核.

证明 设 $f\in W^m$ 满足 $f(0)=\cdots=f^{(m-1)}(0)=0$. 令 $\psi\in C^\infty(\mathbf{R})$ 满足：当 $t\leqslant1$ $\psi(t)=0$，当 $t\geqslant2$ $\psi(t)=1$，$0\leqslant\psi(t)\leqslant1$ 且对所有 t，$0\leqslant k\leqslant m$，$|\psi^{(k)}(t)|\leqslant K_1$. 令 $f_n(t)=\psi(nt)f(t)$. 显然 $f_n\in W_0^m$. 我们必须证明，当 $n\to\infty$，$f(t)-f_n(t)=(1-\psi(nt))f(t)\to0$（在 W^m），即必须证明对每一 k，$0\leqslant k\leqslant m$，每一重指标 α，$|\alpha|\leqslant m-k$，我们有当 $n\to\infty$

$$\int_0^\infty t^{\nu p}\left\|A^\alpha(f-f_n)^{(k)}(t)\right\|_B^p dt\to0.$$

因 $f\in W^m$，当 $n\to\infty$ 有

$$\int_0^\infty t^{\nu p}\left\|(1-\psi(nt))A^\alpha f^{(k)}(t)\right\|_B^p dt\leqslant\int_0^{2/n} t^{\nu p}\left\|A^\alpha f^{(k)}(t)\right\|_B^p dt\to0.$$

因此我们仅需指出若 $1\leqslant j\leqslant k$，则当 $n\to\infty$

$$\int_0^\infty t^{\nu p}\left[\left(\frac{d}{dt}\right)^j(1-\psi(nt))\right]^p\left\|A^\alpha f^{(k-j)}(t)\right\|_B^p dt\to0. \qquad (38)$$

但(38)左端不超过

$$n^{jp}\int_{1/n}^{2/n} t^{\nu p}\left\|A^{\alpha} f^{(k-j)}(t)\right\|_{B}^{p} dt \tag{39}$$

的常数倍. 因为 $f(0)=f'(0)=\cdots=f^{(m-1)}(0)=0$ 及 $k-j\leqslant m-1$, 我们有(其中 $p^{-1}+(p')^{-1}=1$)

$$\begin{aligned}\left\|A^{\alpha} f^{(k-j)}(t)\right\|_{B}^{p} &\leqslant\left\{\frac{1}{(j-1)!}\int_{0}^{t}(t-\tau)^{j-1}\left\|A^{\alpha} f^{(k)}(\tau)\right\|_{B} d\tau\right\}^{p} \\ &\leqslant \frac{t^{(j-1)p}}{[(j-1)!]^{p}}\int_{0}^{t} \tau^{\nu p}\left\|A^{\alpha} f^{(k)}(\tau)\right\|_{B}^{p} d\tau \\ &\quad\times\left\{\int_{0}^{t} \tau^{-\nu p'} d\tau\right\}^{p/p'} \\ &\leqslant K_{2} t^{jp-\nu p-1}\int_{0}^{t} \tau^{\nu p}\left\|A^{\alpha} f^{(k)}(\tau)\right\|_{B}^{p} d\tau .\end{aligned}$$

由此推出(39)不超过下式的常数倍:

$$\begin{aligned}n^{jp}\int_{1/n}^{2/n} t^{jp-1} dt\int_{0}^{t} \tau^{\nu p}\left\|A^{\alpha} f^{(k)}(\tau)\right\|_{B}^{p} d\tau &\leqslant\left(2^{jp}/jp\right) \\ \times\int_{0}^{2/n} \tau^{\nu p}\left\|A^{\alpha} f^{(k)}(\tau)\right\|_{B}^{p} d\tau &\rightarrow 0 .\end{aligned}$$

这是因为 $f\in W^{m}$. ∎

7.56 $W^{m,p}(\Omega)$中的函数在 $\operatorname{bdry}\Omega$ 上的迹的表征在对定义在Ω上的微分算子的非齐次边值问题的研究中有重要的应用. 定理7.53在下述意义下包含了对$W^{m,p}(\Omega)$的“正”和“反”两方面的嵌入定理: 若 $u\in W^{m,p}(\Omega)$, 则迹 $v=u|_{\operatorname{bdry}\Omega}$ 属于 $W^{m-1/p,p}(\operatorname{bdry}\Omega)$ 且

$$\|v\|_{m-1/p,p,\operatorname{bdry}\Omega}\leqslant K_{1}\|u\|_{m,p,\Omega};$$

反之, 若 $v\in W^{m-1/p,p}(\operatorname{bdry}\Omega)$, 则存在 $u\in W^{m,p}(\Omega)$满足

$$v=u|_{\operatorname{bdry}\Omega}$$

且

$$\|u\|_{m,p,\Omega}\leqslant K_{2}\|U\|_{m-1/p,p,\operatorname{bdry}\Omega}.$$

在开始介绍对空间 $W^{s,p}(\Omega)$的非常一般的嵌入定理之前, 我

们指出如何从对整数 s 的情形的嵌入定理和内插定理 7.17 获得一些(但非全部)对这些空间的嵌入.

7.57 定理 设Ω是 $\mathbf{R}^n$ 中一个具有锥性质的区域，设 $s>0$ 且 $1<p<n$.

(a) 若 $n>sp$, 则当 $p\leqslant r\leqslant np/(n-sp)$, $W^{s,p}(\Omega)\to L^r(\Omega)$.

(b) 若 $n=sp$, 则当 $p\leqslant r<\infty$, $W^{s,p}(\Omega)\to L^r(\Omega)$.

(c) 若$n<(s-j)p$, j为某一非负整数, 则$W^{s,p}(\Omega)\to C_B^j(\Omega)$.

证明 结论对整数 s 业已证明，我们可设 s 不是一个整数而 $s=m+\sigma$, m是一个整数而 $0<\sigma<1$. 首先设 $m=0$, 则

$$
\begin{aligned}
W^{\sigma,p}(\Omega) &= L^p(\Omega)\cap T^{1-\sigma,p}(\Omega)\\
&= L^p(\Omega)\cap T(p,1-\sigma-(1/p);W^{1,p}(\Omega),\ L^p(\Omega)).
\end{aligned}
$$

今等同算子从$W^{1,p}(\Omega)$到$L^{np/(n-p)}$和从$L^p(\Omega)$到$L^p(\Omega)$(明显地)连续. 由定理 7.17, 它从 $T(p,1-\sigma-(1/p);\ W^{1,p}(\Omega);L^p(\Omega))$ 到 $T(p,\ 1-\sigma-(1/p);\ L^{np/(n-p)}(\Omega),\ L^p(\Omega))$亦连续. 由定理 7.20 只要 $n>\sigma p$, 后者嵌入 $L^{np/(n-\sigma p)}$. 因此

$$W^{\sigma,p}(\Omega)\to L^{np/(n-\sigma p)}(\Omega).$$

对一般的 m, 我们推证如下. 设 $u\in W^{m+\sigma,p}(\Omega)$. 若$|\alpha|=m$, 则 $D^\alpha u\in W^{\sigma,p}(\Omega)\to L^{np/(n-\sigma p)}(\Omega)$. 若$|\alpha|\leqslant m-1$, 则 $D^\alpha u\in W^{1,p}(\Omega)\to L^{np/(n-\sigma p)}$. 因此 $W^{m+\sigma,p}\to W^{m,np/(n-\sigma p)}(\Omega)$. 若 $n>sp$, 由定理 5.4, 我们有 $W^{m,np/(n-\sigma p)}(\Omega)\to L^{np/(n-sp)}(\Omega)$, 因此(a)得证. 若 $n=sp$, 则对任何满足 $p\leqslant r<\infty$的r, $W^{m,np/(n-\sigma p)}(\Omega)\to L^r(\Omega)$, 于是 (b) 得证. 若$(s-j)p>n$, 则$(m-j)np/(n-\sigma p)>n$, 从而 $W^{m,np/(n-\sigma p)}\to C_B^j(\Omega)$, 而(c)得证. ▌

上述定理中限制 $p<n$ 是不自然的, 加上它仅仅是为了得到一个非常简单的证明.

下述定理含有上面引入的所有嵌入结果作为特殊情形. 它包括了几位作者所得到的结果，特别是 Besov[9, 10], Uspenskii

[67,68]和 Lizorkin[41]的结果．定理对 $\mathbf{R}^n$ 陈述，但显然可以推广到充分正则的区域，例如满足定理 7.41 的条件的区域．我们不打算给任何证明．

7.58 定理 设 $s>0, 1<p\leqslant q<\infty$，$1\leqslant k\leqslant n$．令 $\chi=s-(n/p)+(k/q)$．

若

（i） $\chi\geqslant 0$ 且 $p<q$，或

（ii） $\chi>0$ 且 χ 不是整数，或

（iii） $\chi\geqslant 0$ 且 $1<p\leqslant 2$，

则（正嵌入定理）

$$W^{s,p}(\mathbf{R}^n)\to W^{\chi,q}(\mathbf{R}^k). \tag{40}$$

嵌入(40)当 $p=q>2$ 且 χ 为非负整数未必成立．（特别，一般不可能加强定理 5.4 的部分 I 的情形 A 到允许 $k=n-mp$．）

反之，若 $p=q$ 且若

（iv） $\chi=s-(n-k)/p>0$ χ 不是整数，或

（v） $\chi\geqslant 0$ 且 $p\geqslant 2$，

则我们有逆嵌入

$$W^{\chi,p}(\mathbf{R}^k)\to W^{s,p}(\mathbf{R}^n)$$

其意义是每一 $u\in W^{\chi,p}(\mathbf{R}^k)$ 是一个函数 $w\in W^{s,p}(\mathbf{R}^n)$ 在 $\mathbf{R}^k$ 上的迹（即 $u=w|_{\mathbf{R}^k}$），且满足

$$\|w\|_{s,p,\mathbf{R}^n}\leqslant K\|u\|_{\chi,p,\mathbf{R}^k},$$

k 不依赖于 u．（迹应理解为 5.2 节的意义．）

Bessel 位势——空间 $L^{s,p}(\Omega)$

7.59 我们将不加证明而概述构造分数次空间的另一方法，它起源于由 Aronszajn 和 Smith[7]（及他们的合作者——Adams 等人[5]和 Aronszajn 等人[81]）作的 Bessel 位势的研究．这一方法

为 Calderón[13]和 Lions 与 Magenes[40]所介绍. 所得空间记为 $L^{s,p}(\Omega)$(Lions 与 Magenes 记为 $H^{s,p}(\Omega)$, 但不要同 7.73 节将定义的 Nikol'skii 的 H-空间混淆.) 对整数 s(若 $1<p<\infty$)和所有 s(当 $p=2$)它同空间 $W^{s,p}(\Omega)$重合.

空间 $L^{s,p}(\mathbf{R}^n)$ 直接通过适度广义函数的 Fourier 变换定义, 然后指出对整数 s 和 $1<p<\infty$, $L^{s,p}(\mathbf{R}^n)$和 $W^{s,p}(\mathbf{R}^n)$同构且同胚. 对满足 $s_1\leqslant s\leqslant s_2$ 的 s, 空间 $L^{s,p}(\mathbf{R}^n)$可以等同于由复内插法内插于 $L^{s_1,p}(\mathbf{R}^n)$和 $L^{s_2,p}(\mathbf{R}^n)$之间的一个中间空间 (见 Calderón [15]或 Lions[36]), 复内插法不同于前面描述的 Lions 的迹方法. 然而复内插法提供一个方法对 $\Omega\subset\mathbf{R}^n$ 把 $L^{s,p}(\Omega)$ 定义成形如 $W^{m,p}(\Omega)$(m 为整数)的空间的中间空间.

空间 $L^{s,p}(\Omega)$讨论中断言的证明, 以及它们同空间 $W^{s,p}(\Omega)$的关系可在上面提到的 Calderón, Lions, 及 Lions 和 Magenes 的文章中找到.

7.60 首先我们引进适度广义函数的概念. 我们用 $\mathscr{S}(\mathbf{R}^n)$表示 $\mathbf{R}^n$ 上速减函数空间, 所谓函数 ϕ 速减, 是指它使

$$\sup_{x\in\mathbf{R}^n}|x^\alpha D^\beta\phi(x)|<\infty$$

对所有重指标 α 和 β 成立. 空间 $\mathscr{S}(\mathbf{R}^n)$赋以由下述收敛概念表征的局部凸拓扑: 若对所有 α 和 β

$$\lim_{j\to\infty}x^\alpha D^\beta\phi_j(x)=0(\text{在 }\mathbf{R}^n\text{ 上一致}),$$

则说序列$\{\phi_j\}$在 $\mathscr{S}(\mathbf{R}^n)$中收敛到 0. 很易验证 Fourier 变换

$$\mathscr{F}\phi(y)=(2\pi)^{-n/2}\int_{\mathbf{R}^n}e^{-ix\cdot y}\phi(x)dx$$

和逆 Fourier 变换

$$\mathscr{F}^{-1}\phi(x)=(2\pi)^{-n/2}\int_{\mathbf{R}}e^{ix\cdot y}\phi(y)dy$$

都从 $\mathscr{S}(\mathbf{R}^n)$到 $\mathscr{S}(\mathbf{R}^n)$连续, 并且因为 $\mathscr{F}^{-1}\mathscr{F}\phi=\mathscr{F}\mathscr{F}^{-1}\phi=\phi$,

它们事实上是 $\mathscr{S}(\mathbf{R}^n)$ 到 $\mathscr{S}(\mathbf{R}^n)$ 上的同构和同胚.

显然，由 $\mathscr{S}(\mathbf{R}^n)$ 的定义，$\mathscr{D}(\mathbf{R}^n)\to\mathscr{S}(\mathbf{R}^n)$. 因此对偶空间 $\mathscr{S}'(\mathbf{R}^n)$ 由这样的广义函数 $T\in\mathscr{D}'(\mathbf{R}^n)$ 组成，它具有到 $\mathscr{S}(\mathbf{R}^n)$ 上的连续延拓. 例如，若 $1\leqslant p\leqslant\infty$ 且 $f\in L^p(\mathbf{R}^n)$，则

$$T_f(\phi)=\int_{\mathbf{R}^n} f(x)\phi(x)dx$$

定义 $T_f\in\mathscr{S}'(\mathbf{R}^n)$，这一事实对任何在无穷远"缓增"的函数 f 都成立，即这样的函数 f，对某一有限的 k，$|f(x)|\leqslant \mathrm{const}|x|^k$ a. e. 在无穷远点的某个邻域内成立. $\mathscr{S}'(\mathbf{R}^n)$ 中的元素因此称为适度广义函数. $\mathscr{S}'(\mathbf{R}^n)$ 赋以作为 $\mathscr{S}(\mathbf{R}^n)$ 的对偶的弱-星拓扑，它连同这一拓扑是一个局部凸拓扑向量空间.

正的和逆的 Fourier 变换由

$$\mathscr{F}T(\phi)=T(\mathscr{F}\phi),\mathscr{F}^{-1}(T(\phi))=T(\mathscr{F}^{-1}\phi)$$

推广到 $\mathscr{S}'(\mathbf{R}^n)$. 两者都是从 $\mathscr{S}'(\mathbf{R}^n)$ 到 $\mathscr{S}'(\mathbf{R}^n)$ 上的同构和同胚，并且 $\mathscr{F}^{-1}\mathscr{F}T=\mathscr{F}\mathscr{F}^{-1}T=T$.

7.61 给定一个 $\mathbf{R}^n$ 上的适度广义函数和一个复数 z，u 的 z 次 Bessel 位势用 $J^z u$ 表示并定义为

$$J^z u=\mathscr{F}^{-1}((1+|\cdot|^2)^{-z/2}\mathscr{F}u).$$

显然 J^z 是从 $\mathscr{S}'(\mathbf{R}^n)$ 到 $\mathscr{S}'(\mathbf{R}^n)$ 内的一个一一对应. 若 $\mathrm{Re}z>0$ 且 $1\leqslant p<\infty$，或若 $\mathrm{Re}z\geqslant0$ 且 $1<p<\infty$，则 J^z 连续变换 $L^p(\mathbf{R}^n)$ 到 $L^p(\mathbf{R}^n)$ 内，而 $D^\alpha J^{z+|\alpha|}$ 亦然.

7.62 对实数 s 和 $1\leqslant p\leqslant\infty$，令 $L^{s,p}(\mathbf{R}^n)$ 表示 $L^p(\mathbf{R}^n)$ 在线性映射 J^s 之下的值域. 这样对每一 s，$L^{s,p}(\mathbf{R}^n)\to\mathscr{S}'(\mathbf{R}^n)$ 且对 $s\geqslant0$，$L^{s,p}(\mathbf{R}^n)\to L^p(\mathbf{R}^n)$ 成立. 若 $u\in L^{s,p}(\mathbf{R}^n)$，则存在唯一的 $\tilde{u}\in L^p(\mathbf{R}^n)$，$u=J^s\tilde{u}$，我们定义

$$\|u;L^{s,p}(\mathbf{R}^n)\|=\|\tilde{u}\|_{0,p,\mathbf{R}^n}.$$

赋以这个范数，$L^{s,p}(\mathbf{R}^n)$ 是一个 Banach 空间. 我们综述它的一

些性质.

7.63 定理 (a) 若 $s\geqslant 0$ 且 $1\leqslant p<\infty$，则 $\mathscr{D}(\mathbf{R}^n)$在 $L^{s,p}(\mathbf{R}^n)$ 中稠密.

(b) 若 $1<p<\infty$ 且 $p'=p/(p-1)$，则 $[L^{s,p}(\mathbf{R}^n)]'\cong L^{-s,p'}(\mathbf{R}^n)$.

(c) 若 $t<s$，则 $L^{s,p}(\mathbf{R}^n)\to L^{t,p}(\mathbf{R}^n)$.

(d) 若 $t\leqslant s$ 且或 $1<p<q\leqslant np/[n-(s-t)p]<\infty$ 或 $p=1$ 且 $1\leqslant q<n/(n-s+t)$，则 $L^{s,p}(\mathbf{R}^n)\to L^{t,q}(\mathbf{R}^n)$.

(e) 若 $0\leqslant\mu\leqslant s-(n/p)<1$，则 $L^{s,p}(\mathbf{R}^n)\to C^{0,\mu}(\mathbf{R}^n)$.

(f) 若 s 是非负整数，$1<p<\infty$，则 $L^{s,p}(\mathbf{R}^n)$ 同 $W^{s,p}(\mathbf{R}^n)$ 重合，两空间的范数等价，若 $p=2$ 这一结论对任何 s 成立.

(g) 若 $1<p<\infty$ 且 $\varepsilon>0$，则对任何 s 我们有

$$L^{s+\varepsilon,p}(\mathbf{R}^n)\to W^{s,p}(\mathbf{R}^n)\to L^{s-\varepsilon,p}(\mathbf{R}^n).$$

7.64 现在我们叙述 Calderón[15]和 Lions[36]的复内插法，空间 $L^{s,p}(\mathbf{R}^n)$可由此产生.

设 B_0 和 B_1 是 Banach 空间，二者都象 7.11 节中一样嵌入一个拓扑向量空间 X，空间 B_0+B_1 象在该节那样定义. 用$F(B_0,B_1)$表示复变量 $z=\sigma+i\tau$ 的取值在 B_0+B_1 内的函数 f 的空间，f 满足下述条件

(i) f 在带形 $0<\sigma<1$ 内全纯,

(ii) f 在带形 $0\leqslant\sigma\leqslant 1$ 内连续且有界,

(iii) 对 $\tau\in\mathbf{R}$，$f(i\tau)\in B_0$，映射 $\tau\to f(i\tau)$从 $\mathbf{R}$ 到 B_0 内连续，且 $\lim\limits_{|\tau|\to\infty}f(i\tau)=0$.

(iv) 对 $\tau\in\mathbf{R}$，$f(1+i\tau)\in B_1$，映射 $\tau\to f(1+i\tau)$从 $\mathbf{R}$ 到 B_1内连续，且 $\lim\limits_{|\tau|\to\infty}f(1+i\tau)=0$.

$F(B_0,B_1)$赋以范数

$$\|f:F(B_0,B_1)\|=\max\left\{\sup_{\tau\in\mathbf{R}}\|f(i\tau)\|_{B_0},\ \sup_{\tau\in\mathbf{R}}\|f(1+i\tau)\|_{B_1}\right\}$$

是一个 Banach 空间.

对 $0\leqslant\sigma\leqslant1$ 令

$$\begin{aligned}B_\sigma&=[B_0;B_1]_\sigma\\&=\{u\in B_0+B_1:\text{对某一 }f\in F(B_0,B_1),u=f(\sigma)\}.\end{aligned}$$

B_σ 赋以范数

$$\|u\|_{B\sigma}=\|u;[B_0;B_1]_\sigma\|=\inf_{\substack{f\in F(B_0,B_1)\\f(\sigma)=u}}\|f;F(B_0,B_1)\|$$

是一个嵌入 B_0+B_1 内的 Banach 空间.

中间空间 B_σ 具有类似于 Lions 的迹空间的内插特征. 若 C_0, C_1 和 Y 有类似于 B_0, B_1 和 X 的性质, L 是从 B_0+B_1 到 C_0+C_1 内的线性映射, 满足

$$\|Lu\|_{C_0}\leqslant k_0\|u\|_{B_0},\quad \|Lu\|_{C_1}\leqslant k_1\|u\|_{B_1},$$

则对任一 $u\in B_\sigma$ 有 $Lu\in C_\sigma$ 且

$$\|Lu\|\leqslant K_0^{1-\sigma}K_1^\sigma\|u\|_{B_0}.$$

下列定理可以在 Calderón 或 Lions 的文章[15][36]中找到.

7.65 定理 对任何实的 s_0 和 s_1, $0\leqslant\sigma\leqslant1$, 我们有

$$[L^{s_0,p}(\mathbf{R}^n);L^{s_1,p}(\mathbf{R}^n)]_\sigma=L^{(1-\sigma)s_0+\sigma s_1,p}(\mathbf{R}^n)$$

我们注意对由迹内插得到的$W^{s_0,p}(\mathbf{R}^n)$和$W^{s_1,p}(\mathbf{R}^n)$间的中间空间的相应陈述并非对所有的 s_0 和 s_1 都有效, 尽管对某些值, 特别是若 s_0, s_1 是相邻的整数, 它是有效的.

7.66 上述定理暗示如何对任意区域 $\Omega\subset\mathbf{R}^n$ 定义空间 $L^{s,p}(\Omega)$.

若 $s\geqslant0$, m 是满足 $s\leqslant m<s+1$ 的整数, 定义

$$L^{s,p}(\Omega)=[W^{m,p}(\Omega);L^p(\Omega)]_{(m-s)/m}.$$

若 Ω 充分正则, 以致具有强 m-延拓算子, 则由内插推理表明:

$L^{s,p}(\Omega)$同 $L^{s,p}(\mathbf{R}^n)$中的函数在Ω上的限制的空间重合. 只要 $0\leqslant s_0$, $s_1\leqslant m$, 定理 7.65 对空间 $L^{s,p}(\Omega)$. 亦有效.

对 s 的负值, 空间 $L^{s,p}(\Omega)$的定义可按与 $W^{s,p}(\Omega)$ 同样的方式引进. 用 $L_0^{s,p}(\Omega)(s>0)$表示$\mathscr{D}(\Omega)$在 $L^{s,p}(\Omega)$ 中的闭包, 对 $1<p<\infty$, $s<0$, 定义空间$L^{s,p}(\Omega)$为$[L_0^{-s,p'}]'$, 其中$(1/p)+(1/p')=1$.

只要Ω适当正则, $L^{s,p}(\Omega)$具有定理 7.63 对 $L^{s,p}(\mathbf{R}^n)$ 陈述的所有性质.

其它分数次空间

7.67 对空间 $W^{s,p}(\mathbf{R}^n)$ 的一般嵌入定理的某些缺陷 (参看定理 7.58) 导致 Besov[9. 10]构造空间族 $B^{s,p}(\mathbf{R}^n)$, 当 s 是正整数时它异于 $W^{s,p}(\mathbf{R}^n)$, 当 s 是正整数, 它自然地补充了 $W^{s,p}(\mathbf{R}^n)$, 其意义后面将明确.

对 $s>0$ 和 $1\leqslant p\leqslant\infty$, $B^{s,p}(\mathbf{R}^n)$定义如下. 令 $s=m+\sigma$, m 是非负整数, $0<\sigma\leqslant 1$, 空间 $B^{s,p}(\mathbf{R}^n)$由 $W^{m,p}(\mathbf{R}^n)$ 中这样的 u 组成: 对它, 下面定义的范数$\|u; B^{s,p}(\mathbf{R}^n)\|$有限. 若 $1\leqslant p<\infty$,

$$\|u; B^{s,p}(\mathbf{R}^n)\|$$
$$=\Big\{\|u\|_{m,p,\mathbf{R}^n}^p$$
$$+\sum_{|\alpha|=m}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\frac{|D^\alpha u(x)-2D^\alpha u((x+y)/2)+D^\alpha u(y)|^p}{|x-y|^{n+\sigma p}}dxdy\Big\}^{1/p}.$$

若 $p=\infty$,

$$\|u; B^{s,\infty}(\mathbf{R}^n)\|=\max\Big\{\|u\|_{m,\infty\mathbf{R}^n},$$
$$\max_{|\alpha|=m}\ \operatorname*{ess\,sup}_{\substack{x,y\in\mathbf{R}^n\\ x\neq y}}\frac{|D^\alpha u(x)-2D^\alpha u((x+y)/2)+D^\alpha u(y)|}{|x-y|^\alpha}\Big\},$$

$B^{s,p}(\mathbf{R}^n)$赋以上面的范数是 Banach 空间. 若 $1\leqslant p<\infty$, $C_0^\infty(\mathbf{R}^n)$ 在 $B^{s,p}(\mathbf{R}^n)$中稠密.

7.68 引理 若 $1\leqslant p<\infty$, $s>0$ 是非整数, 则空间 $W^{s,p}(\mathbf{R}^n)$和 $B^{s,p}(\mathbf{R}^n)$重合,且有等价范数.

证明 对定义在 $\mathbf{R}^n$ 上的函数我们由

$$\Delta_z u(x)=u(x+z)-u(x)$$

定义差分算子. 二阶差分算子 Δ_z^2 定义为

$$\Delta_z^2 u(x)=\Delta_z\Delta_z u(x)=u(x+2z)-2u(x+z)+u(x).$$

等式

$$\Delta_z u=(1/2^k)\Delta_{2^k z}u-\frac{1}{2}\sum_{j=0}^{k-1}(1/2^j)\Delta_{2^j z}^2 u \tag{41}$$

容易由展开右端的和来验证.

显然$B^{s,p}(\mathbf{R}^n)$中函数 u 的范数等价于

$$\left\{\|u\|_{m,p,\mathbf{R}^n}^p+\sum_{|\alpha|=m}\int_{\mathbf{R}^n}|z|^{-n-\sigma p}dz\int_{\mathbf{R}^n}|\Delta_z^2 D^\alpha u(x)|^p dx\right\}^{1/p}, \tag{42}$$

而由定理 7.48, $W^{s,p}(\mathbf{R}^n)$中 u 的范数可以表示成如下形式:

$$\left\{\|u\|_{m,p,\mathbf{R}^n}^p+\sum_{|\alpha|=m}\int_{\mathbf{R}^n}|z|^{-n-\sigma p}dz\int_{\mathbf{R}^n}|\Delta_z D^\alpha u(x)|^p dx\right\}^{1/p}, \tag{43}$$

显然,(42)被(43)的常数倍界住,我们必须再证明反面的断言.

因此设 $u\in C_0^\infty(\mathbf{R}^n)$, 利用(41), 我们有

$$\left\{\int_{\mathbf{R}^n}|z|^{-n-\sigma p}dz\int_{\mathbf{R}^n}|\Delta_z D^\alpha u(x)|^p dx\right\}^{1/p}$$

$$\leqslant(1/2^k)\left\{\int_{\mathbf{R}^n}|z|^{-n-\sigma p}dz\int_{\mathbf{R}^n}|\Delta_{2^k z}D^\alpha u(x)|^p dx\right\}^{1/p}$$

$$+\frac{1}{2}\sum_{j=0}^{k-1}(1/2^j)\left\{\int_{\mathbf{R}^n}|z|^{-n-\sigma p}dz\int_{\mathbf{R}^n}|\Delta_{2^j z}^2 D^\alpha u(x)|^p dx\right\}^{1/p}$$

$$=(1/2^{k(1-\sigma)})\left\{\int_{\mathbf{R}^n}|\rho|^{-n-\sigma p}d\rho\int_{\mathbf{R}^n}|\Delta_\rho D^\alpha u(x)|^p dx\right\}^{1/p}$$

$$+\frac{1}{2}\sum_{k=0}^{k-1}(1/2^{j(1-\sigma)})\left\{\int_{\mathbf{R}^n}|\rho|^{-n-\sigma p}d\rho\int_{\mathbf{R}^n}|\Delta_\rho^z D^\alpha u(x)|^p dx\right\}^{1/p}.$$

(在第一个积分中作了替换 $\rho=2^k z$, 在第二个令 $\rho=2^j z$.) 因为 s 是非整数, 我们有 $\sigma<1$, 因此 k 可取得充分大使 $k(1-\sigma)>1$. 尔后推出(43)被(42)的常数倍界住. 因 $C_0^\infty(\mathbf{R}^n)$ 在 $B^{s,p}(\mathbf{R}^n)$ 中稠密, 引理得证. ▌

7.69 若 s 是整数而 $p=2$, 空间 $W^{s,p}(\mathbf{R}^n)$ 和 $B^{s,2}(\mathbf{R}^n)$ 重合. 对 $p\neq 2$, s 是整数, 它们相异, 但对任 $\varepsilon>0$, 我们有

$$W^{s+\varepsilon,p}(\mathbf{R}^n)\to B^{s,p}(\mathbf{R}^n)\to W^{s,p}(\mathbf{R}^n)\quad 若\ 1<p\leqslant 2$$

$$B^{s+\varepsilon,p}(\mathbf{R}^n)\to W^{s,p}(\mathbf{R}^n)\to B^{s,p}(\mathbf{R}^n)\quad 若\ p\geqslant 2.$$

人们对空间 $B^{s,p}(\mathbf{R}^n)$ 有兴趣在于它的嵌入特征. 它对嵌入是"封闭系统", 同时填补了空间 $W^{s,p}(\mathbf{R}^n)$ 嵌入系统的空隙.

7.70 定理 设 $s>0, 1\leqslant p\leqslant q\leqslant\infty, 1\leqslant k\leqslant n, k$ 是整数. 设

$$r=s-(n/p)+(k/q)>0$$

则

$$B^{s,p}(\mathbf{R}^n)\to B^{r,q}(\mathbf{R}^k).$$

反之, 若 $p=q, r=[sp-(n-k)]/p>0$, 则逆嵌入

$$B^{r,p}(\mathbf{R}^k)\to B^{s,p}(\mathbf{R}^n)$$

成立, 即每一元素 $u\in B^{r,p}(\mathbf{R}^k)$ 是某一元素 $v\in B^{s,p}(\mathbf{R}^n)$ 的迹 $u=v|_{\mathbf{R}^k}$, 且满足

$$\|v;B^{s,p}(\mathbf{R}^n)\|\leqslant K\|u;B^{r,p}(\mathbf{R}^k)\|,$$

K 不依赖于 u.

7.71 定理 若 $s>0, 1\leqslant p\leqslant\infty, 1\leqslant k\leqslant n,\ r=[sp-(n-k)]/p$, 则

$$W^{s,p}(\mathbf{R}^n)\to B^{r,p}(\mathbf{R}^k)$$

且反之

$$B^{r,p}(\mathbf{R}^k)\to W^{s,p}(\mathbf{R}^n).$$

7.72 上面的定义和定理可以推广到适当的区域 $\Omega\subset\mathbf{R}^n$ 和包含在 $\overline{\Omega}$ 中的 k 维光滑流形 Ω^k. 对 $1\leqslant p<\infty$, $B^{s,p}(\Omega)$ 中的范数是

$$\|u;B^{s,p}(\Omega)\|=\Big\{\|u\|_{m,p,\Omega}^p$$

$$+\sum_{|\alpha|=m}\int_{\Omega}\int_{\Omega_x}\frac{|D^\alpha u(x)-2D^\alpha u((x+y)/2)+D^\alpha u(y)|^p}{|x-y|^{n+\sigma p}}dydx\Big\}^{1/p}$$

其中 $\Omega_x=\{y\in\Omega;(x+y)/2\in\Omega\}$.

7.73 具有类似于 Besov 空间嵌入性质的另一空间是 Nikol'skii [49—51]引进的空间 $H^{s,p}(\Omega)$. 具有按 $L^p(\Omega)$ 度量的 Hölder 条件的范数的这些空间，其研究早于(分数次) W-或 B-空间并且推动了后两者的发展.

我们再一次令 $s=m+\sigma$, $m\geqslant 0$ 是整数, $0<\sigma\leqslant 1$, 对 $1\leqslant p<\infty$ 和 $\Omega\subset\mathbf{R}^n$, 函数 u 只要范数

$$\|u;H^{s,p}(\Omega)\|=\Big\{\|u\|_{0,p,\Omega}^p$$

$$+\sum_{|\alpha|=m}\sup_{\substack{h\in\mathbf{R}^n\\ \eta>0\\ 0<|h|<\eta}}\int_{\Omega_\eta}\frac{|\Delta_h^2D^\alpha u(x)|^p}{|h|^{\sigma p}}dx\Big\}^{1/p}$$

是有限的，则 u 属于 $H^{s,p}(\Omega)$, 其中 $\Omega_\eta=\{x\in\Omega;\operatorname{dist}(x,\ \operatorname{bdry}\Omega)\geqslant 2\eta\}$. 对 $p=\infty$ 的情况要作明显的修改，且事实上 $H^{s,\infty}(\Omega)=B^{s,\infty}(\Omega)$. 类似于引理 7.68 的推理表明若 s 是非整数，则 $H^{s,p}(\Omega)$ 的范数中的二阶差分 Δ^2 可以用一阶差分 Δ 代替，而不改变空间.

空间 $H^{s,p}(\Omega)$ 大于对应空间 $W^{s,p}(\Omega)$, 但若 $\varepsilon>0$, 我们有

$$H^{s+\varepsilon,p}(\Omega)\to W^{s,p}(\Omega)\to H^{s,p}(\Omega).$$

空间 $H^{s,p}(\mathbf{R}^n)$ 跟 Besov 空间一样具有嵌入的封闭系统，即定理 7.70 中的 B 处处可代之以 H 而成立. 强延拓定理可以对光滑有界区域上的 H-空间证明，于是嵌入定理可以推广到这类区域和在其内的光滑流形上的迹.

对空间 $H^{s,p}(\mathbf{R}^n)$和 $B^{s,p}(\mathbf{R}^n)$ 的嵌入定理可借助这类空间中的函数用多复变量指数型整函数逼近的技术来证明(例如,可参看 Nikol'skii[49]).

7.74 或者为空间本身或者为推进分析中其它问题的解决，人们对上述空间作了为数众多的推广. 我们指出两个方面的推广. 其一在于用加权范数代替通常的 L^p 范数.其二在于在用沿不同坐标方向的积分定义范数时使用不同的 s,p 值(各向异性空间). 有兴趣的读者可参考两篇优秀的综述文章(Nikol'skii,[52], Sobolev 和 Nikol'skii[64]) 及其中的关于多实变量可微函数空间的全貌的进一步知识的文献.

第八章 Orlicz空间和Orlicz-Sobolev 空间

引 言

8.1 在这最后一章我们介绍一些涉及空间 $L^p(\Omega)$代以更一般的空间 $L_A(\Omega)$的新近结果，这里通常由凸函数 t^p 扮演的角色由更一般的凸函数 $A(t)$ 承担. 空间 $L_A(\Omega)$ 称为 Orlicz 空间，它在 Krasnosel'skii 和 Rutickii 的文献 [34] 和 Luxemburg 的博士论文[42]中得到深入的研究，读者可以参考它们当中的任何一个，以便了解下面概述的材料的更完全的发展. 上述文献还包含 Orlicz 空间对非线分析中一些问题应用的例子.

依照 Krasnosel'skii 和 Rutickii[34]，我们使用“N-函数”类中的函数 A 来定义 Orlicz 空间，这个类不如Luxemburg[42] 使用的 Yourg 函数类广阔(还可参看 O'Neill[55])；例如，它从 Orlicz 空间中排除了 $L^1(\Omega)$和 $L^\infty(\Omega)$，但 N-函数便于处理并对我们的目的是足够的，仅在定理8. 35 的证明中，必须援引更一般的Young 函数;

若在 Sobolev 空间 $W^{m,p}(\Omega)$的定义中 $L^p(\Omega)$扮演的角色用 Orlicz 空间 $L_A(\Omega)$ 代替，所得的空间记为 $W^mL_A(\Omega)$ 并称为 Orlicz-Sobolev空间，Sobolev 空间的许多性质主要地由 Donaldson 和 Trudinger [22] 推广到 Orlicz-Sobolev 空间. 本章我们介绍一些这类结果.

指出 Sobolev 嵌入定理 5. 4 中的一个空隙可以由考虑 Orlicz 空间而被填补也多少有些兴趣，明确说来，该定理情形 B 对 $\mathbf{R}^n$ 内

“正则”区域Ω上的$W^{m,p}(\Omega)$当$mp=n$的嵌入没有提供“最好的”目标空间．我们对$p\leqslant q<\infty$有$W^{m,p}(\Omega)\to L^q(\Omega)$，但$W^{m,p}(\Omega)\not\to L^\infty(\Omega)$．在定理8.25中构造了一个$W^{m,p}(\Omega)$到一个Orlicz空间的最佳嵌入．这个结果属于Trudinger[66]．

N-函数

8.2 设a是定义在$[0,\infty)$上具有下列性质的实值函数；

(a) $a(0)=0$，当$t>0, a(t)>0$．$\lim\limits_{t\to\infty} a(t)=\infty$；

(b) a非减，即$s>t\geqslant 0$蕴涵$a(s)\geqslant a(t)$；

(c) a右连续，即若$t\geqslant 0$，则$\lim\limits_{s\to t+} a(s)=a(t)$

则由下式定义的$[0,\infty)$上的实值函数

$$A(t)=\int_0^t a(\tau)d\tau \tag{1}$$

称为N-函数．

不难验证任何这样的N-函数A具有下列性质：

(i) A在$[0,\infty)$连续；

(ii) A严格增加，即$s>t\geqslant 0$蕴涵$A(s)>A(t)$；

(iii) A是凸的，即若$s,t\geqslant 0$且$0<\lambda<1$则

$$A(\lambda s+(1-\lambda)t)\leqslant \lambda A(s)+(1-\lambda)A(t);$$

(iv) $\lim\limits_{t\to 0+} A(t)/t=0$，$\lim\limits_{t\to\infty} A(t)/t=\infty$；

(v) 若$s>t>0$，则$A(s)/s>A(t)/t$．

性质(i)，(iii)和(iv)可被用来定义N-函数，因为由它们可推出A通过具有性质(a)—(c)的a的形如(1)的表达式的存在性．

下面是N-函数的例子：

$A(t)=t^p$，$1<p<\infty$，

$A(t)=e^t-t-1$，

$$A(t)=e^{(t^p)}-1,\ \ 1<p<\infty,$$
$$A(t)=(1+t)\log(1+t)-t.$$

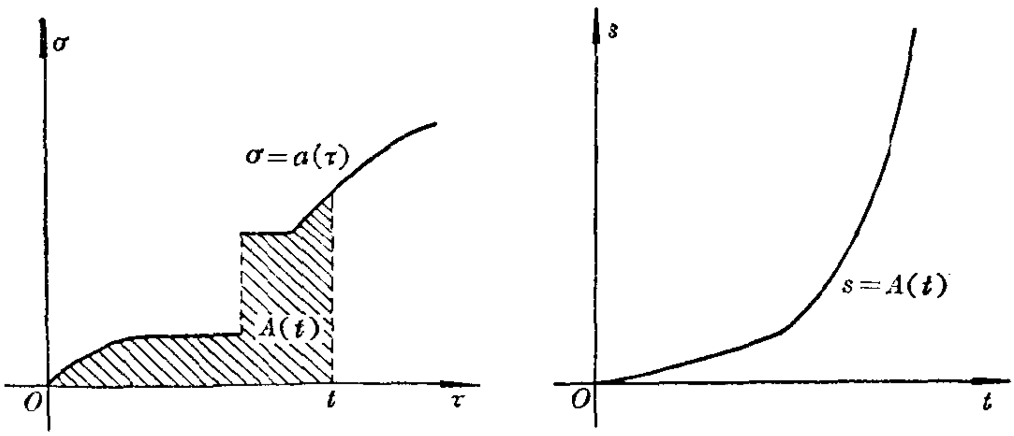

图 8

显然 $A(t)$用图形 $\sigma=a(\tau)$下由 $\tau=0$ 到 $\tau=t$ 的面积表示(图 8). A 的图形中的直线段对应 a 的常数区间，而 A的图形的角点对应 a 的图形的不连续(即垂直跳跃)点.

8.3 给定满足(a)—(c)的 a, 我们定义

$$\tilde{a}(s)=\sup_{a(t)\leqslant s} t. \tag{2}$$

不难验证这样定义的 $\tilde{a}$ 也满足(a)—(c)并且 a 可由 $\tilde{a}$ 依照

$$a(t)=\sup_{\tilde{a}(s)\leqslant t} s \tag{3}$$

来复原(若 a 严格增加,则 $\tilde{a}=a^{-1}$). 由

$$A(t)=\int_0^t a(\tau)d\tau,\ \ \tilde{A}(s)=\int_0^s \tilde{a}(\sigma)d\sigma \tag{4}$$

给定的 N-函数 A 和 $\tilde{A}$ 称为互补的；每一个称为另一个的补(N-函数).

下面是互补对的例子:

$$A(t)=t^p/p,\ \ \tilde{A}(s)=s^{p'}/p'\quad 1<p<\infty,$$
$$(1/p)+(1/p')=1;$$
$$A(t)=e^t-t-1,\ \ \tilde{A}(s)=(1+s)\log(1+s)-s.$$

$\tilde{A}(s)$用图形 $\sigma=a(\tau)$ [或更正确地说 $\tau=\tilde{a}(\sigma)$] 左边从 $\sigma=0$

到 $\sigma=s$ 的面积表示，见图 9，显然我们有

$$st\leqslant A(t)+\widetilde{A}(s),\tag{5}$$

(5)称为 Young 不等式，当且仅当 $t=\tilde{a}(s)$ 或 $s=a(t)$ 时(5)中等号成立，把(5)写成形式

$$\widetilde{A}(s)\geqslant st-A(t),$$

并且注意当 $t=\tilde{a}(s)$ 时出现等号，我们有

$$\widetilde{A}(s)=\max_{t\geqslant 0}(st-A(t)).$$

这个关系可用作 A 的补 N-函数 $\widetilde{A}$ 的定义.

因为 A 和 $\widetilde{A}$ 严格增加，它们有反函数且(5)蕴涵对任意 $t\geqslant 0$

$$A^{-1}(t)\widetilde{A}^{-1}(t)\leqslant A(A^{-1}(t))+\widetilde{A}(\widetilde{A}^{-1}(t))=2t.$$

另外，$A(t)\leqslant ta(t)$，再考虑图 9，对 $t>0$ 我们有

$$\widetilde{A}(A(t)/t)<(A(t)/t)t=A(t).\tag{6}$$

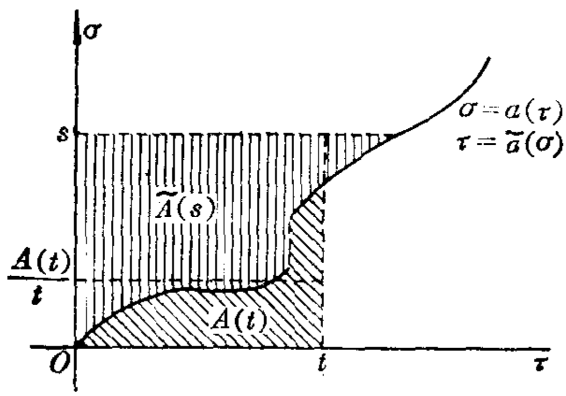

图 9

(6)中 $A(t)$ 代以 t，我们得

$$\widetilde{A}(t/A^{-1}(t))<t.$$

因此，对任意 $t>0$ 我们有

$$t<A^{-1}(t)\widetilde{A}^{-1}(t)\leqslant 2t.\tag{7}$$

8.4 我们将需要 N-函数之间的一些偏序关系，若 A 和 B 是两个 N-函数，我们说 B 全局控制 A，只要存在一个正常数 k 使

$$A(t)\leqslant B(kt)\tag{8}$$

对所有 $t\geqslant 0$ 成立. 类似地说 B 在无穷远控制 A，若存在正常数 k 和 t_0 使(8) 对所有 $t\geqslant t_0$ 成立. 两个 N-函数 A 和 B 称为是全局(或在无穷远附近)等价的，若其中每一个全局(或在无穷远附近)控制另一个，即 A 和 B 在无穷远附近等价，当且仅当存在两个正常数 k_1,k_2 和 t_0 使若 $t\geqslant t_0$，则 $B(k_1t)\leqslant A(t)\leqslant B(k_2t)$. 若

$$0<\lim_{t\to\infty}\frac{B(t)}{A(t)}<\infty,$$

则必是这种情形.

若 A 和 B 分别有补 N-函数 $\widetilde{A}$ 和 $\widetilde{B}$，则 B 全局(或在无穷远附近)控制 A，当且仅当 $\widetilde{A}$ 全局(或在无穷远附近)控制 $\widetilde{B}$. 类似地，A 和 B 等价当且仅当 $\widetilde{A}$ 和 $\widetilde{B}$ 等价.

8.5 若 B 在无穷远附近控制 A，而 A 和 B 不是在无穷远附近等价的，则说 A 本质上在无穷远附近增加慢于 B. 当且仅当对每一 $k>0$

$$\lim_{t\to\infty}\frac{A(kt)}{B(t)}=0 \tag{9}$$

会出现这种情形. 读者可以验证(9)等价于条件

$$\lim_{t\to\infty}\frac{B^{-1}(t)}{A^{-1}(t)}=0.$$

令 $1<p<\infty$. 此后我们用 A_p 表示 N-函数

$$A_p(t)=t^p/p,\quad 0\leqslant t<\infty. \tag{10}$$

若 $1<p<q<\infty$，则 A_p 本质上在无穷远处附近慢于 A_q. 但 A_q 不全局控制 A_p.

8.6 一个 N-函数被说成满足全局 Δ_2-条件，假若存在一个正常数 k 使对每一 $t>0$，

$$A(2t)\leqslant kA(t). \tag{11}$$

易见当且仅当对每一 $r>0$ 存在一个正常数 $k=k(r)$ 使对所有 $t\geqslant 0$

$$A(rt)\leqslant kA(t) \tag{12}$$

时(11)成立. 类似地,说 A 在无穷远附近满足 Δ_2-条件,假若存在 $t_0>0$ 使(11)[或等价地(12),对 $r>1$]当 $t\geqslant t_0$ 时成立. 显然 t_0 可用任何更小的正数 t_1 代替,因若 $t_1\leqslant t\leqslant t_0$,则

$$A(rt)\leqslant[A(rt_0)/A(t_1)]A(t).$$

若 A 满足全局(或在无穷远附近)Δ_2-条件且若 B 全局(或在无穷远附近)等价于 A,则 B 亦满足 Δ_2-条件,显然 N-函数 $A_p(t)=t^p/p, 1<p<\infty$ 满足全局 Δ_2-条件. 可以验证, A 满足全局(或在无穷远附近)Δ_2-条件当且仅当存在一个有限常数 c 使

$$(1/c)ta(t)\leqslant A(t)\leqslant ta(t)$$

对所有 $t\geqslant 0$(或对所有 $t\geqslant t_0>0$)成立,其中 A 由(1)给定.

Orlicz 空间

8.7 设 Ω 是 $\mathbf{R}^n$ 中的一个区域而 A 是一个 N-函数. Orlicz 类 $K_A(\Omega)$ 是所有定义在 Ω 上且满足

$$\int_\Omega A(|u(x)|)dx<\infty$$

的可测函数(的对 Ω 上 a. e. 相等关系等价类)u 的集合. 因 A 是凸的,$K_A(\Omega)$ 总是函数的凸集,但它未必是一个向量空间;例如可以存在 $u\in K_A(\Omega)$ 和 $\lambda>0$ 使 $\lambda u\notin K_A(\Omega)$.

我们称二元对 (A,Ω) Δ-正则,假若

(a) A 满足全局 Δ_2-条件,或

(b) A 在无穷远附近满足 Δ_2-条件且 Ω 有有限体积.

8.8 引理 $K_A(\Omega)$ 是一个向量空间(对于逐点相加和数乘)当且仅当 (A,Ω) 是 Δ-正则的.

证明 因为 A 是凸的,我们有

(i) 只要 $u\in K_A(\Omega)$ 且 $|\lambda|\leqslant 1$, $\lambda u\in K_A(\Omega)$,

(ii) 若$u\in K_A(\Omega)$蕴涵对每一复数λ, $\lambda u\in K_A(\Omega)$, 则 $u, v\in K_A(\Omega)$蕴涵$u+v\in K_A(\Omega)$.

这就推出: 当且仅当若$u\in K_A(\Omega)$且$|\lambda|>1$蕴涵$\lambda u\in K_A(\Omega)$, $K_A(\Omega)$是一个向量空间

若A满足全局Δ_2-条件且$|\lambda|>1$, 则由 (12) 对$u\in K_A(\Omega)$我们有

$$\int_\Omega A(|\lambda u(x)|)dx\leqslant k(|\lambda|)\int_\Omega A(|u(x)|)dx<\infty.$$

类似的, 若A满足在无穷远附近的Δ_2-条件且$\mathrm{vol}\Omega<\infty$, 对$|\lambda|>1$, $u\in K_A(\Omega)$和某一$t_0>0$, 我们有

$$\begin{aligned}\int_\Omega A(|\lambda u(x)|)dx&=\left(\int_{\{x:|u(x)|\geqslant t_0\}}+\int_{\{x:|u(x)|<t_0\}}\right)A(|\lambda u(x)|)dx\\&\leqslant k(|\lambda|)\int_\Omega A(|u(x)|)dx+A(|\lambda|t_0)\mathrm{vol}\Omega<\infty.\end{aligned}$$

由此看出在两种情形$K_A(\Omega)$都是向量空间.

现设(A, Ω)不是Δ-正则的, 若$\mathrm{vol}\Omega<\infty$, 还设$t_0>0$给定. 存在正数序列$\{t_j\}$使

(i) $A(2t_j)\geqslant 2^jA(t_j)$, 且

(ii) 若$\mathrm{vol}\Omega<\infty$, $t_j\geqslant t_0>0$.

设$\{\Omega_j\}$是Ω的互不相交的可测子集序列使

$$\mathrm{vol}\Omega_j=\begin{cases}1/2^jA(t_j) & \text{若 } \mathrm{vol}\Omega=\infty\\ A(t_0)\mathrm{vol}\Omega/2^jA(t_j) & \text{若 } \mathrm{vol}\Omega<\infty.\end{cases}$$

令

$$u(x)=\begin{cases}t_j & \text{若 } x\in\Omega_j\\ 0 & \text{若 } x\in\Omega\sim\left(\bigcup_{j=1}^\infty\Omega_j\right).\end{cases}$$

则

$$\int_\Omega A(|u(x)|)dx=\sum_{j=1}^\infty A(t_j)\mathrm{vol}\Omega_j$$

$$
=\begin{cases}1 & \text{若 } \mathrm{vol}\Omega=\infty \\ A(t_0)\mathrm{vol}\Omega & \text{若 } \mathrm{vol}\Omega<\infty.\end{cases}
$$

但

$$
\int_\Omega A(|2u(x)|)dx \geqslant \sum_{j=1}^{\infty} 2^j A(t_j)\mathrm{vol}\Omega_j=\infty;
$$

于是 $K_A(\Omega)$ 不是一个向量空间. ∎

8.9 Orlicz 空间 $L_A(\Omega)$ 被定义为 Orlicz 类 $K_A(\Omega)$ 的线性包, 即包含 $K_A(\Omega)$ 的最小向量空间(关于逐点加法和数乘). 显然 $L_A(\Omega)$ 由元素 $u\in K_A(\Omega)$ 的所有数积 λu 组成. 于是 $K_A(\Omega)\subset L_A(\Omega)$; 当且仅当 (A,Ω) Δ-正则, 这两个集合相等.

读者可以验证泛函

$$
\|u\|_A=\|u\|_{A,\Omega}=\inf\left\{k>0: \int_\Omega A\left(\frac{|u(x)|}{k}\right)dx\leqslant 1\right\} \tag{13}
$$

是 $L_A(\Omega)$ 上的一个范数. (这个范数由 Luxemburg[42] 引进.) 对 $\|u\|_A>0$, (13) 中的下确界可以达到; 事实上, 在不等式

$$
\int_\Omega A\left(\frac{|u(x)|}{k}\right)dx\leqslant 1 \tag{14}
$$

中令 k 减少趋于 $\|u\|_A$, 由单调收敛定理我们得

$$
\int_\Omega A\left(\frac{|u(x)|}{\|u\|_A}\right)dx\leqslant 1. \tag{15}
$$

(15) 中的等号可能不成立, 但若 (14) 的等号成立, 则 $k=\|u\|_A$.

8.10 定理 $L_A(\Omega)$ 赋以范数 (13) 是 Banach 空间.

完备性的证明十分类似于在定理 2.10 中给的空间 $L^p(\Omega)$ 完备性的证明, 细节留给读者. [我们注意若 $1<p<\infty$ 且 A_p 由 (10) 给出, 则

$$
L^p(\Omega)=L_{A_p}(\Omega)=K_{A_p}(\Omega).
$$

并且 $\|u\|_{A_p}\Omega=p^{-1/p}\|u\|_{p,\Omega}$.]

8.11 若 A 和 $\widetilde{A}$ 是互补 N-函数, Hölder 不等式的一个推广

$$\left|\int_\Omega u(x)v(x)dx\right|\leqslant 2\|u\|_{A,\Omega}\|v\|_{\tilde{A},\Omega} \tag{16}$$

可应用 Young 不等式(5)于$|u(x)|/\|u\|_A$和$|v(x)|/\|v\|_{\tilde{A}}$并在 Ω上积分而得到.

下述对 Orlicz 空间的初等嵌入定理与引理 2.8 相应的结果类似.

8.12 定理 当且仅当

(*a*) B 全局控制 A, 或

(*b*) B 在无穷远附近控制 A 且 $\mathrm{vol}\,\Omega<\infty$, 嵌入 $L_B(\Omega)\to L_A(\Omega)$成立.

证明 若 $A(t)\leqslant B(kt)$对所有 $t\geqslant 0$ 成立且若 $u\in L_B(\Omega)$; 则

$$\int_\Omega A\left(\frac{|u(x)|}{k\|u\|_B}\right)dx\leqslant\int_\Omega B\left(\frac{|u(x)|}{\|u\|_B}\right)dx\leqslant 1.$$

于是 $u\in L_A(\Omega)$且$\|u\|_A\leqslant k\|u\|_B$.

若 $\mathrm{vol}\,\Omega<\infty$, 令 $t_1=A^{-1}((2\mathrm{vol}\Omega)^{-1})$. 若$B$在无穷远附近控制 A, 则存在正数 t_0 和 k 使对 $t\geqslant t_0$ 有 $A(t)\leqslant B(kt)$. 显然当 $t\geqslant t_1$ 有

$$A(t)\leqslant K_1B(kt),$$

其中 $K_1=\max(1, A(t_0)/B(kt_1))$. 若给定 $u\in L_B(\Omega)$, 令$\Omega'(u)=\{x\in\Omega:|u(x)/2K_1k\|u\|_B<t_1\}$, $\Omega''(u)=\Omega\sim\Omega'(u)$, 则

$$\int_\Omega A\left(\frac{|u(x)|}{2K_1k\|u\|_B}\right)dx=\left(\int_{\Omega'(u)}+\int_{\Omega''(u)}\right)A\left(\frac{|u(x)|}{2K_1k\|u\|_B}\right)dx$$

$$\leqslant\frac{1}{2\mathrm{vol}\Omega}\int_{\Omega'(u)}dx+K_1\int_{\Omega''(u)}B\left(\frac{|u(x)|}{2K_1\|u\|_B}\right)dx$$

$$\leqslant\frac{1}{2}+\frac{1}{2}\int_\Omega B\left(\frac{|u(x)|}{\|u\|_B}\right)dx\leqslant 1.$$

于是 $u\in L_A(\Omega)$, 且$\|u\|_A\leqslant 2kK_1\|u\|_B$.

反之, 设无论是(*a*)还是(*b*)都不成立, 则存在点 $t_j>0$ 使

$$A(t_j)\geqslant B(jt_j),\ j=1,2,\cdots.$$

当 $\mathrm{vol}\Omega<\infty$, 我们可设

$$t_j \geqslant (1/j)B^{-1}(1/\mathrm{vol}\Omega).$$

令 Ω_j 是Ω的体积为 $1/B(jt_j)$ 的子区域, 令

$$u_j(x)=\begin{cases} jt_j & 若\ x\in\Omega_j \\ 0 & 若\ x\in\Omega\sim\Omega_j. \end{cases}$$

则

$$\int_\Omega A(|u_j(x)|/j)dx \geqslant \int_\Omega B(|u_j(x)|)dx=1.$$

于是$\|u_j\|_B=1$ 但$\|u_j\|_A\geqslant j$, 这说明 $L_B(\Omega)$不嵌入 $L_A(\Omega)$. ∎

8.13 我们说 $L_A(\Omega)$中的函数序列 u_j 平均收敛到 $u\in L_A(\Omega)$, 若

$$\lim_{j\to\infty}\int_\Omega A(|u_j(x)-u(x)|)dx=0.$$

A 的凸性蕴涵对 $0<\varepsilon\leqslant 1$ 有

$$\int_\Omega A(|u_j(x)-u(x)|)dx \leqslant \varepsilon\int_\Omega A(|u_j(x)-u(x)|/\varepsilon)dx,$$

由此显然 $L_A(\Omega)$中的依范数收敛蕴涵平均收敛. 当且仅当二元对(A,Ω)是 Δ-正则时, 其逆成立, 即平均收敛蕴涵依范数收敛. 证明与引理 8.8 的类似, 将其留给读者.

8.14 用 $E_A(\Omega)$ 表示Ω上有界且支集在$\overline{\Omega}$有界的函数 u 的空间在 $L_A(\Omega)$内的闭包, 若 $u\in K_A(\Omega)$, 则由

$$u_j(x)=\begin{cases} u(x) & 当\ |u(x)|\leqslant j\quad 且\ |x|\leqslant j,\ x\in\Omega \\ 0 & 其余 \end{cases} \tag{17}$$

定义的序列 $\{u_j\}$ 在 Ω a.e. 收敛. 因为 $A(|u(x)-u_j(x)|)\leqslant A(|u(x)|)$, 由控制收敛定理 u_j 在 $L_A(\Omega)$中平均收敛到 u 因此若(A,Ω) 是 Δ-正则的, 则 $E_A(\Omega)=K_A(\Omega)=L_A(\Omega)$若$(A,\Omega)$不是$\Delta$-正则的, 我们有

$$E_A(\Omega)\subset K_A(\Omega)\subsetneqq L_A(\Omega), \tag{18}$$

于是在这种情形 $E_A(\Omega)$是 $L_A(\Omega)$的一个真闭子空间, 为验证(18)

中第一个包含关系，令 $u\in E_A(\Omega)$ 给定，取有界且支集有界的函数 v 满足 $\|u-v\|_A<\frac{1}{2}$，利用 A 的凸性和(15)式，我们得

$$\frac{1}{\|2u-2v\|_A}\int_\Omega A(|2u(x)-2v(x)|)dx$$

$$\leqslant\int_\Omega A\left(\frac{|2u(x)-2v(x)|}{\|2u-2v\|_A}\right)dx\leqslant 1$$

因此 $2u-2v\in K_A(\Omega)$．因为 $2v$ 显然属于 $K_A(\Omega)$ 而 $K_A(\Omega)$ 是凸集，我们有 $u=\frac{1}{2}(2u-2v)+\frac{1}{2}(2v)$ 属于 $K_A(\Omega)$．

8.15 引理 $E_A(\Omega)$ 是 $K_A(\Omega)$ 的最大线性子空间．

证明 设 S 是 $K_A(\Omega)$ 的线性子空间而令 $u\in S$．则对任何数 λ，$\lambda u\in K_A(\Omega)$．若 $\varepsilon>0$ 而 u_j 由(17)式给定，则前节已指出 u_j/ε 在 $L_A(\Omega)$ 中平均收敛到 u/ε，因此对充分大的 j，

$$\int_\Omega A(|u_j(x)-u(x)|/\varepsilon)dx\leqslant 1,$$

从而 u_j 在 $L_A(\Omega)$ 中收敛到 u，于是 $S\subset E_A(\Omega)$．▌

8.16 定理 设 Ω 有有限体积，又设 N-函数 A 在无穷远附近增加本质上慢于 B，则

$$L_B(\Omega)\to E_A(\Omega).$$

证明 因为 $L_B(\Omega)\to L_A(\Omega)$ 业已建立，我们只需指出 $L_B(\Omega)\subset E_A(\Omega)$．因为 $L_B(\Omega)$ 是 $K_B(\Omega)$ 的线性包而 $E_A(\Omega)$ 是 $K_A(\Omega)$ 的最大线性子空间，只需指出每当 $u\in K_B(\Omega)$ 和 λ 是一个数，就有 $\lambda u\in K_A(\Omega)$．但存在一个正数 t_0 使 $A(|\lambda|t)\leqslant B(t)$ 对所有 $t\geqslant t_0$ 成立，于是

$$\int_\Omega A(|\lambda u(x)|)dx=\left\{\int_{\{x:|u(x)|\leqslant t_0\}}+\int_{\{x:|u(x)|>t_0\}}\right\}$$

$$\times A(|\lambda u(x)|)dx\leqslant A(|\lambda|t_0)\operatorname{vol}\Omega+\int_\Omega B(|u(x)|)dx<\infty,$$

因此定理得证. ▌

Orlicz 空间中的对偶

8.17 引理 对固定的 $v\in L_{\widetilde{A}}(\Omega)$, 由

$$L_v(u)=\int_\Omega u(x)v(x)dx \tag{19}$$

定义的线性泛函属于$[L_A(\Omega)]'$. 用 $\|L_v\|$ 表示它在该空间中的范数, 我们有

$$\|v\|_{\widetilde{A}}\leqslant\|L_v\|\leqslant 2\|v\|_{\widetilde{A}}. \tag{20}$$

证明 由 Hölder 不等式(16)推出若 $u\in L_A(\Omega)$, 则

$$|L_v(u)|\leqslant 2\|u\|_A\|v\|_{\widetilde{A}}.$$

从而 L_v 在 $L_A(\Omega)$ 上有界且(20)中第二个不等式成立.

为建立第一个不等式可设在 $L_{\widetilde{A}}(\Omega)$ 中 $v\neq 0$, 即$\|L_v\|=K>0$. 令

$$u(x)=\begin{cases}\widetilde{A}\left(\dfrac{|v(x)|}{K}\right)\Big/\dfrac{v(x)}{K} & \text{当}\quad v(x)\neq 0\\ 0 & \text{当}\quad v(x)=0.\end{cases}$$

若$\|u\|_A>1$, 则对充分小的 $\varepsilon>0$ 我们有

$$\frac{1}{\|u\|_A-\varepsilon}\int_\Omega A(|u(x)|)dx\geqslant\int_\Omega A\left(\frac{|u(x)|}{\|u\|_A-\varepsilon}\right)>1.$$

令 $\varepsilon\to 0^+$, 利用不等式(6)我们得

$$\|u\|_A\leqslant\int_\Omega A(|u(x)|)dx=\int_\Omega A\left(\widetilde{A}\left(\frac{|v(x)|}{K}\right)\Big/\frac{|v(x)|}{K}\right)dx$$

$$<\int_\Omega\widetilde{A}\left(\frac{v(x)}{K}\right)dx=\frac{1}{\|L_v\|}\int_\Omega u(x)v(x)dx\leqslant\|u\|_A.$$

这一矛盾表明$\|u\|_A\leqslant 1$. 今有

$$\|L_v\|=\sup_{\|u\|_A\leqslant 1}|L_v(u)|\geqslant\|L_v\|\left|\int_\Omega\widetilde{A}\left(\frac{|v(x)|}{\|L_v\|}\right)dx\right|,$$

于是

$$\int_{\Omega}\widetilde{A}\left(\frac{|v(x)|}{\|L_v\|}\right)dx\leqslant 1. \qquad (21)$$

即得$\|v\|_{\widetilde{A}}\leqslant\|L_v\|$. ∎

我们注意当 L_v 限制作用在 $E_A(\Omega)$时引理仍成立. 此时为得到(20)中的第一个不等式, 取$\|L_v\|$是 L_v 在 $[E_A(\Omega)]'$ 中的范数, 上述证明中的 u 代以 $\chi_n u$, χ_n 是 $\Omega_n=\{x\in\Omega:|x|\leqslant n$ 且 $|u(x)|\leqslant n\}$的特征函数. 显然 $\chi_n u$ 属于 $E_A(\Omega)$, $\|\chi_n u\|_A\leqslant 1$ 而(21)变为

$$\int_{\Omega}\chi_n(x)\widetilde{A}\left(\frac{|v(x)|}{\|L_v\|}\right)dx\leqslant 1.$$

因为 χ_n 在 Ω a. e. 增加收敛到 1, 由单调收敛定理得

$$\int_{\Omega}\widetilde{A}\left(\frac{|v(x)|}{\|L_v\|}\right)dx\leqslant 1,$$

于是又得$\|v\|_{\widetilde{A}}\leqslant\|L_v\|$.

8.18 定理 $E_A(\Omega)$的对偶空间$[E_A(\Omega)]'$同构且同胚于 $L_{\widetilde{A}}(\Omega)$.

证明 我们已经指出每一元素 $v\in L_{\widetilde{A}}(\Omega)$ 由 (19) 式确定既在 $L_A(\Omega)$上又在 $E_A(\Omega)$上的一个有界线性泛函, 两种情形下其范数和$\|v\|_{\widetilde{A}}$至多相差因子2. 余下要指出的是 $E_A(\Omega)$ 上每一有界线性泛函对某一 $v\in L_{\widetilde{A}}(\Omega)$有形式 L_v.

设 $L\in[E_A(\Omega)]'$ 给定. 我们由

$$\lambda(S)=L(\chi_S)$$

在有有限体积的 Ω 的可测子集上定义一个复测度 λ, 其中 χ_S 是 S 的特征函数; 因为

$$\int_{\Omega}A(|\chi_S(x)|A^{-1}(1/\mathrm{vol}S))dx=\int_S(1/\mathrm{vol}S)dx=1, \qquad (22)$$

我们有

$$|\lambda(S)|\leqslant\|L\|\|\chi_S\|_A=\|L\|[1/A^{-1}(1/\mathrm{vol}S)].$$

因为右端随 vol$S$ 趋于零, 测度 λ 对于 Lebesgue 测度绝对连续,

由 Radon-Nikodym 定理 1.47，λ 可表成形式

$$\lambda(S)=\int_S v(x)dx,$$

其中 v 在 Ω 上可积，于是

$$L(u)=\int_\Omega u(x)v(x)dx$$

对可测简单函数 u 成立.

设 $u\in E_A(\Omega)$，可找到 a.e. 收敛到 u 的可测简单函数列 u_j 在 Ω 满足 $|u_j(x)|\leqslant|u(x)|$. 因为 $|u_j(x)v(x)|$ a.e. 收敛到 $|u(x)v(x)|$，由 Fatou 引理 1.44 得

$$\left|\int_\Omega u(x)v(x)dx\right|\leqslant\sup_j\int_\Omega|u_j(x)v(x)|dx$$
$$=\sup_j|L(|u_j|\operatorname{sgn}v)|\leqslant\|L\|\sup_j\|u_j\|_A\leqslant\|L\|\|u\|_A.$$

故线性泛函

$$L_v(u)=\int_\Omega u(x)v(x)dx$$

在 $E_A(\Omega)$ 上有界，由引理 8.17 后的注释知 $v\in L_{\tilde{A}}(\Omega)$. 因为 L_v 和 L 在可测简单函数上取相同的值，它们组成一个在 $E_A(\Omega)$ 中稠密的集[见定理 8.20(a)的证明]，故两泛函在 $E_A(\Omega)$ 上重合而定理得证.∎

Hahn-Banach 延拓定理的一个简单应用指出若 $E_A(\Omega)$ 是 $L_A(\Omega)$ 的一个真子空间[即，若 (A,Ω) 不是 Δ-正则]，则存在一个 $L_A(\Omega)$ 上的有界线性泛函对任何 $v\in L_{\tilde{A}}(\Omega)$ 不能由(19)给出，作为一个直接推论我们有

8.19 定理 当且仅当 $(A,\ \Omega)$ 和 $(\tilde{A},\ \Omega)$ 都 Δ-正则时 $L_A(\Omega)$ 自反.

我们略去 orlicz 空间一致凸性的任何讨论，这个题目在 Luxemburg 的论文[42]中讨论过.

可分性和紧性定理

下面我们推广逼近定理 2.13, 2.15 和 2.19.

8.20 定理 (*a*) $C_0(\Omega)$在 $E_A(\Omega)$中稠密

(*b*) $E_A(\Omega)$可分.

(*c*) 若 J_ε 是 2.17 节中引入的软化子, 则对任何 $u\in E_A(\Omega)$, 我们有 $\lim\limits_{\varepsilon\to 0+} J_\varepsilon * u = u$(在 $E_A(\Omega)$),

(*d*) C_0^∞ 在 $E_A(\Omega)$中稠密.

证明 (*a*)可用定理 2.13 中使用的同样方法证明. 首先用简单函数对 $u\in E_A(\Omega)$进行逼近时, 我们可以假设 u 在 Ω 有界且有有界支集. 这时, 为指出简单函数依 $E_A(\Omega)$中的范数收敛到 u, 而利用控制收敛推理是需要的(细节留给读者).

由在定理 2.15 中给的同样的证明可由 (*a*) 推出(*b*). 现考虑(*c*). 若 $u\in E_A(\Omega)$, 令 u 在 Ω 外为零而把 u 延拓到 $\mathbf{R}^n$. 令

$$v\in L_{\widetilde{A}}(\Omega),$$

则

$$\begin{aligned}&\left|\int_\Omega (J_\varepsilon * u(x) - u(x))v(x)dx\right|\\ &\leqslant \int_{\mathbf{R}^n} J(y)dy \int_\Omega |u(x-\varepsilon y)-u(x)|\,|v(x)|dx\\ &\leqslant 2\|v\|_{\widetilde{A},\Omega}\int_{|y|\leqslant 1} J(y)\|u_{\varepsilon y}-u\|_{A,\Omega}dy,\end{aligned}$$

这里利用了 Hölder 不等式(16), 并令 $u_{\varepsilon y}(x)=u(x-\varepsilon y)$. 由(20)和定理 8.18

$$\begin{aligned}\|J_\varepsilon * u - u\|_{A,\Omega} &= \sup_{\|v\|_{\widetilde{A},\Omega}\leqslant 1}\left|\int_\Omega (J_\varepsilon * u(x)-u(x))v(x)dx\right|\\ &\leqslant 2\int_{|y|\leqslant 1} J(y)\|u_{\varepsilon y}-u\|_{A,\Omega}dy.\end{aligned}$$

给定 $\delta>0$，我们可找到 $\tilde{u}\in C_0(\Omega)$ 使 $\|u-\tilde{u}\|_{A,\Omega}<\delta/6$. 显然 $\|u_{\varepsilon y}-\tilde{u}_{\varepsilon y}\|_{A,\Omega}<\delta/6$，而对充分小的 ε，$\|\tilde{u}_{\varepsilon y}-\tilde{u}\|_{A,\Omega}<\delta/6$ 对任意 $y:|y|\leqslant 1$ 成立，于是 $\|J_\varepsilon * u-u\|_{A,\Omega}<\delta$ 而(c)被建立.

(d)是(a)和(c)的直接推论. ▌

我们注意 $L_A(\Omega)$ 是不可分的，除非 $L_A(\Omega)=E_A(\Omega)$，即除非 $(A,\ \Omega)$ 是 Δ-正则的. 这个事实的一个证明可在 Krasnosel'skii 和 Rutickii 的著作中找到，[34，第 II 章，定理 10.2.]

8.21 一个可测函数序列 u_j 被说成在 Ω 上依测度收敛到函数 u，只要对任意 $\varepsilon>0$ 和 $\delta>0$ 存在一个整数 M 使当 $j>M$ 有

$$\operatorname{vol}\{x\in\Omega:|u_j(x)-u(x)|>\varepsilon\}\leqslant\delta,$$

显然这时亦存在一个整数 N 使当 $j,k\geqslant N$ 则

$$\operatorname{vol}\{x\in\Omega:|u_j(x)-u_k(x)|\geqslant\varepsilon\}\leqslant\delta.$$

8.22 定理 设 Ω 的体积有限，又设 N-函数 B 在无穷远附近本质上增加比 A 更慢. 若序列 $\{u_j\}$ 在 $L_A(\Omega)$ 中有界并且在 Ω 上依测度收敛，则它在 $L_B(\Omega)$ 中收敛.

证明 固定 $\varepsilon>0$ 而令 $v_{j,k}(x)=[u_j(x)-u_k(x)]/\varepsilon$. 显然 $\{v_{j,k}\}$ 在 $L_A(\Omega)$ 中有界；比如设 $\|v_{j,k}\|_{A,\Omega}\leqslant K$. 令存在正数 t_0，使当 $t>t_0$ 有

$$B(t)\leqslant\frac{1}{4}A(t/K).$$

令 $\delta=1/4B(t_0)$，并令

$$\Omega_{j,k}=\{x\in\Omega:|v_{j,k}(x)|\geqslant B^{-1}(1/2\operatorname{vol}\Omega)\}.$$

因 $\{u_j\}$ 依测度收敛，存在一个整数 N 使当 $j,\ k\geqslant N$ 有 $\operatorname{vol}\Omega_{jk}\leqslant\delta$ 令

$$\Omega'_{j,k}=\{x\in\Omega_{j,k}:|v_{j,k}(x)|\geqslant t_0\},$$
$$\Omega''_{j,k}=\Omega_{j,k}\sim\Omega'_{j,k}.$$

对 $j,k\geqslant N$ 我们有

$$\int_{\Omega} B(|v_{j,k}(x)|)dx = \Big(\int_{\Omega\sim\Omega_{j,k}} + \int_{\Omega'_{j,k}} + \int_{\Omega''_{j,k}}\Big) B(|v_{j,k}(x)|)dx$$
$$\leqslant \frac{\operatorname{vol}\Omega}{2\operatorname{vol}\Omega} + \frac{1}{4}\int_{\Omega'_{j,k}} A\Big(\frac{|v_{j,k}(x)|}{K}\Big)dx + \delta B(t_0) \leqslant 1,$$

因此$\|u_j-u_k\|_{B,\Omega}\leqslant\varepsilon$，从而$\{u_j\}$在$L_B(\Omega)$中收敛. ∎

当我们希望推广 Rellich-Kondra chov 定理 6.2 到 Orlicz-Sobolev 空间的嵌入时将会用到下述定理:

8.23 定理 设Ω体积有限，又设N-函数B在无穷远附近本质上增加比A更慢，则$L_A(\Omega)$的任何有界子集S若在$L^1(\Omega)$中准紧，则它在$L_B(\Omega)$中亦准紧.

证明 因Ω体积有限，显然$L_A(\Omega)\to L^1(\Omega)$. 若$\{u_j^*\}$是$S$中的一个序列，它有子序列$\{u_j\}$在$L^1(\Omega)$中收敛；比如就是$u_j\to u$，在$L^1(\Omega)$中成立. 令$\varepsilon$，$\delta>0$，则存在一个整数$N$使当$j\geqslant N$有$\|u_j-u\|_{1,\Omega}\leqslant\varepsilon\delta$. 于是 $\operatorname{vol}\{x\in\Omega:|u_j(x)-u(x)|\geqslant\varepsilon\}\leqslant\delta$，于是$\{u_j\}$在$\Omega$上依测度收敛，因此也在$L_B(\Omega)$中收敛. ∎

Sobolev 嵌入定理的一个极限情形

8.24 若$mp=n$且$p>1$，Sobolev 嵌入定理 5.4 提供的不是$W^{m,p}(\Omega)$可以嵌入的最好(即最小的)目标空间. 事实上这时对适当正则的Ω有

$$W^{m,p}(\Omega)\to L^q(\Omega) \qquad p\leqslant q<\infty,$$

但(见例 5.26)

$$W^{m,p}(\Omega)\not\subset L^\infty(\Omega).$$

若嵌入的目标空间扩大到包括 Orlicz 空间，则可找到最好的目标空间. 我们首先考虑有界区域Ω. 下列定理当$m=1$时的情形已被 Trudinger[66]建立.

8.25 定理 设Ω是$\mathbf{R}^n$中一个有锥性质的区域. 设$mp=n$且$p>1$. 令

$$A(t)=\exp[t^{n/(n-m)}]-1=\exp[t^{p/(p-1)}]-1. \tag{23}$$

则存在嵌入

$$W^{m,p}(\Omega)\to L_A(\Omega).$$

证明 设$x\in\Omega$, C是包含在Ω中顶点在x的一个有限锥. 设$u\in C^m(\bar{C})$. 对函数$f(t)=u(y+t(x-y))$应用Taylor公式

$$f(1)=\sum_{j=0}^{m-1}\frac{f^{(j)}(0)}{j!}+\frac{1}{(m-1)!}\int_0^1(1-t)^{m-1}f^{(m)}(t)dt,$$

并注意

$$f^{(j)}(t)=\sum_{|\alpha|=j}\frac{j!}{\alpha!}D^\alpha u(y+t(x-y))(x-y)^\alpha,$$

我们得

$$|u(x)|\leqslant\sum_{|\alpha|\leqslant m-1}\frac{1}{\alpha!}|D^\alpha u(y)|\,|x-y|^{|\alpha|}$$

$$+\sum_{|\alpha|=m}\frac{m}{\alpha!}|x-y|^m\int_0^1(1-t)^{m-1}|D^\alpha u(y+t(x-y))|dt.$$

令V和h分别是C的体积和高. 用(ρ,θ)表示$y\in C$的原点在x的球面极坐标, 这样C用$0<\rho<h,\theta\in\Sigma$表示, 体积元dy写成形式$\rho^{n-1}\omega(\theta)d\rho d\theta$. 而

$$|u(x)|=\frac{1}{V}\int_C|u(x)|dy\leqslant\frac{1}{V}\sum_{|\alpha|\leqslant m-1}\frac{h^{|\alpha|}}{\alpha!}\int_C|D^\alpha u(y)|dy$$

$$+\frac{1}{V}\sum_{|\alpha|=m}\frac{m}{\alpha!}\int_\Sigma\omega(\theta)d\theta\int_0^h\rho^{n+m-1}d\rho$$

$$\times\int_0^1(1-t)^{m-1}|D^\alpha u((1-t)\rho,\theta)|dt$$

$$\leqslant K_1\Big\{\|u\|_{m-1,1,C}+\sum_{|\alpha|=m}\int_\Sigma\omega(\theta)d\theta\int_0^h\rho^{n-1}d\rho$$

$$\times \int_0^\rho \sigma^{m-1} |D^\alpha u(\sigma,\theta)| d\sigma \Big\}$$

$$= K_1 \Big\{ \|u\|_{m-1,1,C} + \sum_{|\alpha|=m} \int_\Sigma \omega(\theta) d\theta$$

$$\times \int_0^h \sigma^{m-1} |D^\alpha u(\sigma, \theta)| d\theta \int_0^h \rho^{n-1} d\rho \Big\}$$

$$\leqslant K_2 \Big\{ \|u\|_{m-1,1,C} + \sum_{|\alpha|=m} \int_C \frac{|D^\alpha u(z)|}{|z-x|^{n-m}} dz \Big\}.$$

由稠密性，上面的不等式对所有 $u \in W^{m,1}(C)$ 成立. 特别，对任何 $u \in W^{m,p}(\Omega)$ 和几乎所有的 $x \in \Omega$，我们有

$$|u(x)| \leqslant K_2 \Big\{ \|u\|_{m-1,1,\Omega} + \sum_{|\alpha|=m} \int_\Omega \frac{|D^\alpha u(y)|}{|x-y|^{n-m}} dy \Big\},$$

其中 K_2 依赖 m, n 和确定 Ω 的锥性质的锥高 h 和体积 V.

我们希望对任意 $s>1$ 估计 $\|u\|_{0,s}$. 相应地，若 $v \in L^{s'}(\Omega)$，$s' = s/(s-1)$，则

$$\int_\Omega |u(x)v(x)| dx \leqslant K_2 \|u\|_{m-1,1} \int_\Omega |v(x)| dx$$

$$+ K_2 \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{|D^\alpha u(y)| \, |v(x)|}{|x-y|^{n-m}} dy dx$$

$$\leqslant K_2 \|u\|_{m-1,1} \|v\|_{0,s'} (\mathrm{vol}\,\Omega)^{1/s}$$

$$+ K_2 \sum_{|\alpha|=m} \Big\{ \int_\Omega \int_\Omega \frac{|v(x)|}{|x-y|^{n-(m/s)}} dy dx \Big\}^{1-(1/p)}$$

$$\times \Big\{ \int_\Omega \int_\Omega \frac{|D^\alpha u(y)|^p \, |v(x)|}{|x-y|^{(n-m)/s}} dy dx \Big\}^{1/p}.$$

今由引理 5.47 我们有当 $0 \leqslant \nu < n$ 则

$$\int_\Omega \frac{1}{|x-y|^\nu} dx \leqslant K_3(\nu, n) (\mathrm{vol}\,\Omega)^{1-(\nu/n)}.$$

事实上细察引理的证明知 $K_3(\nu,n) = K_4/(n-\nu)$，K_4 仅依赖于 n. 因此

$$\int_\Omega \int_\Omega \frac{|v(x)|}{|x-y|^{n-(m/s)}} dy dx \leqslant K_4 \frac{s}{m} (\mathrm{vol}\,\Omega)^{m/sn} \int_\Omega |v(x)| dx$$

$$\leqslant K_5 s(\mathrm{vol}\,\Omega)^{(1/sp)+(1/s)}\|v\|_{0,s'}.$$

又有

$$\int_\Omega\int_\Omega \frac{|D^\alpha u(y)|^p|v(x)|}{|x-y|^{(n-m)/s}}dydx$$

$$\leqslant\int_\Omega |D^\alpha u(y)|^p dy\cdot\|v\|_{0,s'}\cdot\left\{\int_\Omega \frac{1}{|x-y|^{n-m}}dx\right\}^{1/s}$$

$$\leqslant\|D^\alpha u\|_{0,p}^p\|v\|_{0,s'}(K_5(\mathrm{vol}\,\Omega)^{1/p})^{1/s}.$$

因此

$$\int_\Omega |u(x)v(x)|dx\leqslant K_2\|u\|_{m-1,1}\|v\|_{0,s'}(\mathrm{vol}\,\Omega)^{1/s}$$

$$+K_6\sum_{|\alpha|=m}s^{(p-1)/p}\|D^\alpha u\|_{0,p}\|v\|_{0,s'}(\mathrm{vol}\,\Omega)^{1/s}.$$

因为 $s^{(p-1)/p}>1$ 且 $W^{m-1,1}(\Omega)\to W^{m,p}(\Omega)$，于是

$$\|u\|_{0,s}=\sup_{v\in L^{s'}(\Omega)}\frac{\int_\Omega |u(x)v(x)|dx}{\|v\|_{0,s'}}$$

$$\leqslant K_7 s^{(p-1)/p}(\mathrm{vol}\,\Omega)^{1/s}\|u\|_{m,p},$$

常数 K_7 仅依赖于 m,n 和确定 Ω 的锥性质的锥. 令 $s=nk/(n-m)=pk/(p-1)$，我们得

$$\int_\Omega |u(x)|^{pk/(p-1)}dx\leqslant\mathrm{vol}\,\Omega\left\{\frac{pk}{p-1}\right\}^k\{K_7\|u\|_{m,p}\}^{pk/(p-1)}$$

$$=\mathrm{vol}\,\Omega\left\{\frac{k}{e^{p/(p-1)}}\right\}^k\left\{eK_7\left(\frac{p}{p-1}\right)^{(p-1)/p}\|u\|_{m,p}\right\}^{pk/(p-1)}.$$

因为 $e^{p/(p-1)}>e$，级数 $\sum_{k=1}^{\infty}(1/k!)(k/e^{p(p-1)})^k$ 收敛到一个有限和 K_8. 令 $K_9=\max(1,K_8\,\mathrm{vol}\,\Omega)$ 且令

$$K_{10}=eK_9K_7[p/(p-1)]^{(p-1)/p}\|u\|_{m,p}=K_{11}\|u\|_{m,p}.$$

则注意到 $K_9\geqslant1$ 和 $pk/(p-1)>1$ 得

$$\int_\Omega\left(\frac{|u(x)|}{K_{10}}\right)^{pk/(p-1)}dx\leqslant\frac{\mathrm{vol}\,\Omega}{K_9^{pk/(p-1)}}\left(\frac{k}{e^{p/(p-1)}}\right)^k$$

$$<\frac{\operatorname{vol}\Omega}{K_9}\left(\frac{k}{e^{p/(p-1)}}\right)^k.$$

展开 $A(t)$成幂级数，我们得

$$\int_\Omega A\left(\frac{|u(x)|}{K_{10}}\right)dx=\sum_{k=1}^{\infty}\frac{1}{k!}\int_\Omega\left(\frac{|u(x)|}{K_{10}}\right)^{pk/(p-1)}dx$$

$$<\frac{\operatorname{vol}\Omega}{K_9}\sum_{k=1}^{\infty}\frac{1}{k!}\left(\frac{k}{e^{p/(p-1)}}\right)^n\leqslant 1.$$

因此 $u\in L_A(\Omega)$且

$$\|u\|_A\leqslant K_{10}=K_{11}\|u\|_{m,p},$$

其中 K_{11} 依赖于 $n,m,\operatorname{vol}\Omega$和确定$\Omega$的锥性质的锥. ▌

上面的定理建立的嵌入在下述意义下是“最佳可能”的，即若存在一个形如

$$W_0^{m,p}(\Omega)\to L_B(\Omega)$$

的嵌入，则A在无穷远附近控制 B. 这一事实对情形$m=1$, $n=p>1$ 的证明可在 Hempel 等的注记[30] 中找到，一般情形留给读者作为练习.

定理 8.25 可推广到分数次空间. 对这方面的结果，读者可参考 Grisvard[28]和 Peetre[56].

8.26 若Ω无界(有锥性质)从而体积无穷，则由(23)给定的N-函数在零点可能减小得不够快，以致不能保证每一 $u\in W^{m,p}(\Omega)$ $(mp=n)$是 $L_A(\Omega)$的成员. 设 k_0 是满足 $k_0\geqslant p-1$ 的最小整数并定义修改了的N-函数 A_0 如下:

$$A_0(t)=\exp(t^{p/(p-1)})-\sum_{j=0}^{k_0-1}(1/j!)t^{jp/(p-1)}.$$

显然 A_0 在无穷远附近等价于 A, 于是对体积有限的任何区域Ω, $L_A(\Omega)$和 $L_{A_0}(\Omega)$ 重合且有等价的范数. 然而 A_0 尚具有进一步的性质，当 $0<r\leqslant 1$

$$A_0(rt)\leqslant r^{k_0p/(p-1)}A_0(t)\leqslant r^pA_0(t). \tag{24}$$

我们证明若 $mp=n$ 且 Ω 有锥性质(但可能无界),则

$$W^{m,p}(\Omega)\to L_{A_0}(\Omega).$$

象在引理 5.14 的证明中所做的,我们可以写 Ω 成可数个子区域 Ω_j 之和,每一 Ω_j 有被某一不依赖于 j 的固定的锥确定的锥性质,存在某两个常数 K_1 和 K_2 使

$$0<K_1\leqslant \operatorname{vol}\Omega_j\leqslant K_2,$$

并且对某一正整数 R,任 $R+1$ 个子区域 Ω_j 的交集是空集. 由定理 8.25 推出若 $u\in W^{m,p}(\Omega)$,则

$$\|u\|_{A_0,\Omega_j}\leqslant K_3\|u\|_{m,p,\Omega_j},$$

其中 K_3 不依赖于 j. 取 $r=R^{1/p}\|u\|_{m,p,\Omega_j}^{-1}\|u\|_{m,p,\Omega}$ 利用(24)并利用区域 Ω_j 的有限交性质,我们有

$$\int_\Omega A_0\left(\frac{|u(x)|}{R^{1/p}K_3\|u\|_{m,p,\Omega}}\right)dx\leqslant\sum_{j=1}^\infty\int_{\Omega_j}A_0\left(\frac{|u(x)|}{R^{1/p}K_3\|u\|_{m,p,\Omega}}\right)dx$$

$$\leqslant\sum_{j=1}^\infty\frac{\|u\|_{m,p,\Omega_j}^p}{R\|u\|_{m,p,\Omega}^p}\leqslant 1.$$

因此 $\|u\|_{A_0,\Omega}\leqslant R^{1/p}K_3\|u\|_{m,p,\Omega}$ 这就是所要证明的.

我们注意若 $k_0>p-1$,上述结果可稍许改进. 即 A_0 代以 N-函数 $\max(t^p,A_0(t))$.

Orlicz-Sobolev 空间

8.27 对 $\mathbf{R}^n$ 中一个给定的区域 Ω 和一个给定的 N-函数 A, Orlicz-Sobolev 空间 $W^mL_A(\Omega)$, 由 $L_A(\Omega)$中的这些函数(的等价类)u 组成: 对满足 $|\alpha|\leqslant m$ 的所有 α,它们的广义函数导数 $D^\alpha u$ 也属于 $L_A(\Omega)$. 类似地可定义空间 $W^mE_A(\Omega)$. 用与定理 3.2 的证明中所使用的同样的方法可以验证 $W^mL_A(\Omega)$赋以范数

$$\|u\|_{m,A}=\|u\|_{m,A,\Omega}=\max_{0\leqslant|\alpha|\leqslant m}\|D^\alpha u\|_{A,\Omega}\tag{25}$$

是一个 Banach 空间,并且 $W^mE_A(\Omega)$是 $W^mL_A(\Omega)$的一个闭子空

间从而对(25)也是一个 Banach 空间. 应当记住，当且仅当(A, Ω) Δ-正则时 $W^mE_A(\Omega)$和 $W^mL_A(\Omega)$重合. 若 $1<p<\infty$, $A_p(t)=t^p$ 则 $W^mL_{A_p}(\Omega)=W^mE_{A_p}(\Omega)=W^{m,p}(\Omega)$，后一空间的范数和前两个的等价.

象通常 Sobolev 空间的情形一样，$W_0^mL_A(\Omega)$ 是 $C_0^\infty(\Omega)$ 在 $W^mL_A(\Omega)$中的闭包. [对 $W_0^mE_A(\Omega)$，类似的定义显然在所有情形导致同一空间.]

Orlicz-Sobolev 空间的许多性质可由对普通 Sobolev 空间同样性质的证明的十分直接的推广得到. 在下述定理中我们总结其中的一些，至于证明方法读者可参阅第三章中对应的结果. 细节可以在 Donaldson 和 Trudinger 的文章[2]中找到.

8.28 定理 (*a*) $W^mE_A(\Omega)$是可分的(定理 3.5).

(*b*) 若(A, Ω)和$(\widetilde{A}, \Omega)$是 Δ-正则的，则

$$W^mE_A(\Omega)=W^mL_A(\Omega)$$

是自反的(定理 3.5).

(*c*) 空间$[W^mE_A(\Omega)]'$ 中每一元素 L 由

$$L(u)=\sum_{0\leqslant|\alpha|\leqslant m}\int_\Omega D^\alpha u(x)v_\alpha(x)dx$$

对某些函数 $v_\alpha\in L_{\widetilde{A}}(\Omega)$, $0\leqslant|\alpha|\leqslant m$，给出(定理 3.8).

(*d*) $C^\infty(\Omega)\cap W^mE_A(\Omega)$在 $W^mE_A(\Omega)$ 中稠密(定理 3.16).

(*e*) 若Ω有线段性质，则 $C^\infty(\overline{\Omega})$在 $W^mE_A(\Omega)$ 中稠密(定理 3.18).

(*f*) $C_0^\infty(\mathbf{R}^n)$在 $W^mE_A(\mathbf{R}^n)$中稠密. 于是

$$W_0^mL_A(\mathbf{R}^n)=W^mE_A(\mathbf{R}^n)$$

(定理 3.18).

Orlicz-Sobolev 空间的嵌入定理

8.29 与五、六两章中对空间 $W^{m,p}(\Omega)$ 得到的类似的嵌入结果

对 Orlicz-Sobolev 空间 $W^m L_A(\Omega)$和 $W^m E_A(\Omega)$亦成立. 这方面的首批结果由 Dankert[20]和 Donaldson 得到. 跟定理 5.4 和 6.2 平行的一个相当一般的嵌入定理由 Donaldson 和 Trudinger [22]表述,下面我们就来叙述它.

象通常 Sobolev 空间的情形一样, 大多数嵌入结果是对具有锥性质的区域得到的. 给出(广义) Hölder 连续性估计的嵌入是例外, 这时需要强局部 Lipschitz 性质. 下面一些结果仅对有界区域得到. 当涉及到一般的 Orlicz 空间时, 推广通常 Sobolev 空间的类似结果到无界区域所使用的方法(见引理 5.14) 看来不能采用直接的方式. 在这个意义下嵌入的结果仍欠完备.

8.30 我们暂只涉及 $W^1 L_A(\Omega)$的嵌入, $W^m L_A(\Omega)$的嵌入总结在定理 8.40 中. 下面Ω总理解为是 $\mathbf{R}^n$ 中的一个区域.

设A是一个给定的N-函数. 我们总假设

$$\int_0^1 \frac{A^{-1}(t)}{t^{(n+1)/n}} dt < \infty, \tag{26}$$

若有必要, A 代以另一在无穷远附近等价的N-函数以适合(26) [若Ω体积有限,从嵌入理论的观点看来,(26)对A不算什么限制,因为在无穷远附近等价的N-函数确定相同的 Orlicz 空间.]

还假设

$$\int_1^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}} dt = \infty. \tag{27}$$

例如若 $A = A_p$ 由(10)给定, (27)恰好当 $p \leqslant n$ 时成立. 当(27)满足,我们由

$$A_*^{-1}(t) = \int_0^t \frac{A^{-1}(\tau)}{\tau^{(n+1)/n}} d\tau, \ t \geqslant 0 \tag{28}$$

定义A的 Sobolev 共轭 A_*. 易于验证 A_* 是一个N-函数. 若 $1 < p < n$, 令 $q = np/(n-p)$, 我们有

$$A_{p*}(t)=q^{1-q}p^{-q/p}A_q(t).$$

同样易见对情形 $p=n$，$A_{n*}(t)$ 在无穷远附近等价于 N-函数

$$e^t-t-1.$$

在叙述第一个嵌入定理之前，我们准备今后证明中需要的一个技术性引理.

8.31 引理 设 $u\in W^{1,1}_{\mathrm{loc}}(\Omega)$ 且 f 在 $\mathbf{R}$ 上满足 Lipschitz 条件. 则 $g\in W^{1,1}_{\mathrm{loc}}(\Omega)$，其中 $g(x)=f(|u(x)|)$，且

$$D_jg(x)=f'(|u(x)|)\mathrm{sgn}u(x)\cdot D_ju(x).$$

证明 因为 $|u|\in W^{1,1}_{\mathrm{loc}}(\Omega)$ 且 $D_j|u(x)|=\mathrm{sgn}u(x)D_ju(x)$，只需对正实值函数 u 建立引理，于是 $g(x)=f(u(x))$. 令 $\phi\in\mathscr{D}(\Omega)$. 又令 $\{e_j\}_{j=1}^n$ 是 $\mathbf{R}^n$ 中的通常的基，我们得

$$\begin{aligned}
&-\int_\Omega f(u(x))D_j\phi(x)dx\\
&=-\lim_{h\to 0}\int_\Omega f(u(x))\frac{\phi(x)-\phi(x-he_j)}{h}dx\\
&=\lim_{h\to 0}\int_\Omega \frac{f(u(x+he_j))-f(u(x))}{h}\phi(x)dx\\
&=\lim_{h\to 0}\int_\Omega Q(x,h)\frac{u(x+he_j)-u(x)}{h}\phi(x)dx,
\end{aligned}$$

其中，因为 f 是 Lipschitz 连续的，对每一 h 函数 $Q(\cdot,h)$ 在 Ω 上由

$$Q(x,h)=\begin{cases}\dfrac{f(u(x+he_j))-f(u(x))}{u(x+he_j)-u(x)} & \text{当 } u(x+he_j)\neq u(x)\\ f'(u(x)) & \text{其余}\end{cases}$$

a.e. 定义，此外，$\|Q(\cdot,h)\|_{\infty,\Omega}\leqslant K$，$K$ 为一不依赖于 h 的常数. 泛函分析中一个熟知的定理保证对 h 的某一趋于 0 的序列，$Q(\cdot,h)$ 依 L^∞ 的弱-星拓扑收敛到 $f'(u(x))$，另外因为 $u\in W^{1,1}(\mathrm{supp}\phi)$. 我们有在 $L^1(\mathrm{supp}\phi)$ 中

$$\lim_{h\to 0}\frac{u(x+he_j)-u(x)}{h}\phi(x)=D_ju(x)\phi(x),$$

由此

$$-\int_\Omega f(u(x))D_j\phi(x)dx=\int_\Omega f'(u(x))D_ju(x)\phi(x)dx.$$

这显然蕴涵引理.

8.32 定理 设Ω在$\mathbf{R}^n$中有界且有锥性质，若(26)和(27)成立，则

$$W^1L_A(\Omega)\to L_{A_*}(\Omega).$$

并且，若B是任何在无穷远附近本质上增加比A_*更慢的N-函数,则嵌入

$$W^1L_A(\Omega)\to L_B(\Omega)$$

是紧的.

证明 由(28)定义的函数 $s=A_*(t)$满足微分方程

$$A^{-1}(s)\frac{ds}{dt}=s^{(n+1)/n}, \tag{29}$$

于是由(7)式右边一个不等式,

$$\frac{ds}{dt}\leqslant s^{1/n}\widetilde{A}^{-1}(s).$$

因此 $\sigma(t)=[A_*(t)]^{(n-1)/n}$ 满足微分不等式

$$\frac{d\sigma}{dt}\leqslant\frac{n-1}{n}\widetilde{A}^{-1}((\sigma(t))^{n/(n-1)}). \tag{30}$$

设 $u\in W^1L_A(\Omega)$ 并暂且假定u在Ω上有界且在$L_A(\Omega)$中异于零. 则$\int_\Omega A_*(|u(x)|/\lambda)dx$ 当λ从零增至无穷时连续地从无穷减小至零,从而对某一正数λ取值为1. 这样

$$\int_\Omega A_*\left(\frac{|u(x)|}{K}\right)dx=1,\quad K=\|u\|_{A_*}. \tag{31}$$

令 $f(x)=\sigma(|u(x)|/K)$. 显然 $u\in W^{1,1}(\Omega)$而σ在$|u|/K$ 的值域

上满足 Lipschitz 条件，于是由引理 8.31，f 属于 $W^{1,1}(\Omega)$. 由定理 5.4 我们有 $W^{1,1}(\Omega)\to L^{n/(n-1)}(\Omega)$ 且

$$\|f\|_{0,n/(n-1)}\leqslant K_1\left\{\sum_{j=1}^{n}\|D_jf\|_{0,1}+\|f\|_{0,1}\right\}$$

$$=K_1\left\{\sum_{j=1}^{n}\frac{1}{K}\int_\Omega\sigma'\left(\frac{|u(x)|}{K}\right)|D_ju(x)|dx+\int_\Omega\sigma\left(\frac{|u(x)|}{K}\right)dx\right\}. \tag{32}$$

由(31)和 Hölder 不等式(16)，我们得

$$1=\left\{\int_\Omega A_*\left(\frac{|u(x)|}{K}\right)dx\right\}^{(n-1)/n}=\|f\|_{0,n/(n-1)}$$

$$\leqslant\frac{2K_1}{K}\sum_{j=1}^{n}\left\|\sigma'\left(\frac{|u|}{K}\right)\right\|_{\widetilde{A}}\|D_ju\|_A+K_1\int_\Omega\sigma\left(\frac{|u(x)|}{K}\right)dx. \tag{33}$$

利用(30)，我们有

$$\left\|\sigma'\left(\frac{|u|}{K}\right)\right\|_{\widetilde{A}}\leqslant\frac{n-1}{n}\left\|\widetilde{A}^{-1}\left(\left(\sigma\left(\frac{|u|}{K}\right)\right)^{n/(n-1)}\right)\right\|_{\widetilde{A}}$$

$$=\frac{n-1}{n}\inf\left\{\lambda>0:\ \int_\Omega\widetilde{A}\left(\frac{\widetilde{A}^{-1}(A_*(|u(x)|/K))}{\lambda}\right)dx\leqslant1\right\}.$$

设 $\lambda>1$ 则

$$\int_\Omega\widetilde{A}\left(\frac{\widetilde{A}^{-1}(A_*(|u(x)|/K))}{\lambda}\right)dx\leqslant\frac{1}{\lambda}\int_\Omega A_*\left(\frac{|u(x)|}{K}\right)dx$$

$$=\frac{1}{\lambda}<1.$$

于是

$$\left\|\sigma'\left(\frac{|u|}{K}\right)\right\|_{\widetilde{A}}\leqslant\frac{n-1}{n}. \tag{34}$$

令 $g(t)=A_*(t)/t$, $h(t)=\sigma(t)/t$. 容易验证 h 在有限区间有界且 $\lim\limits_{t\to\infty}g(t)/h(t)=\infty$. 于是存在常数 t_0 使当 $t\geqslant t_0$ 有 $h(t)\leqslant g(t)/2K_1$. 令 $K_2=K_1\sup\limits_{0\leqslant t\leqslant t_0}h(t)$，对所有 $t\geqslant0$ 我们有

$$\sigma(t)\leqslant(1/2K_1)A_*(t)+(K_2/K_1)t.$$

因此

$$K_1\int_\Omega \sigma\left(\frac{|u(x)|}{K}\right)dx\leqslant\frac{1}{2}\int_\Omega A_*\left(\frac{|u(x)|}{K}\right)dx+\frac{K_2}{K_1}\int_\Omega |u(x)|\,dx$$

$$\leqslant\frac{1}{2}+\frac{K_3}{K}\|u\|_A, \tag{35}$$

其中因为Ω体积有限 $K_3=2K_2\|1\|_{\tilde{A}}<\infty$.

联合(33)—(35), 我们得

$$1\leqslant(2K_1/K)(n-1)\|u\|_{1,A}+\frac{1}{2}+(K_3/K)\|u\|_A,$$

从而

$$\|u\|_{A_*}=K\leqslant K_4\|u\|_{1,A}. \tag{36}$$

我们注意 K_4 可以依赖 n, A, vol Ω 和确定Ω的锥性质的锥.

为推广(36)到任意 $u\in W^1L_A(\Omega)$令

$$u_k(x)=\begin{cases}u(x) & \text{当} |u(x)|\leqslant k\\ k\,\mathrm{sgn}\,u(x) & \text{当} |u(x)|>k,\end{cases} \tag{37}$$

显然 u_k 有界并由引理 8.31 亦属于 $W^1L_A(\Omega)$, 并且 $\|u_k\|_{A_*}$随 k 增加但被 $K_4\|u\|_{1,A}$ 界住, 于是 $\lim\limits_{k\to\infty}\|u_k\|_{A_*}=K$ 存在且

$$K\leqslant K_4\|u\|_{1,A}.$$

由 Fatau 引理

$$\int_\Omega A_*\left(\frac{|u(x)|}{K}\right)dx\leqslant\lim_{k\to\infty}\int_\Omega A_*\left(\frac{|u_k(x)|}{K}\right)dx\leqslant1,$$

因此 $u\in L_{A_*}(\Omega)$且(36)成立.

因Ω体积有限我们有

$$W^1L_A(\Omega)\to W^{1,1}(\Omega)\to L^1(\Omega),$$

由定理 6.2, 后一嵌入是紧的. $W^1L_A(\Omega)$的一个有界子集S在$L_{A*}(\Omega)$中有界并且在$L^1(\Omega)$中准紧, 由定理 8.23, 它在 $L_B(\Omega)$准紧, 只要B在无穷远附近本质上增加比A更慢. ▌

定理 8.32 可推广到任意(甚至无界)区域 Ω, 只要W代以W_0.

8.33 定理 设Ω是$\mathbf{R}^n$中任一区域，若N-函数A满足(26)和(27)，则

$$W_0^1L_A(\Omega)\to L_{A_*}(\Omega).$$

并且，若Ω_0是Ω的一个有界子区域，则对任何在无穷远附近本质上增加比A更慢的N-函数B嵌入

$$W_0^1L_A(\Omega)\to L_B(\Omega_0)$$

存在且是紧的.

证明 若$u\in W_0^1L_A(\Omega)$，则前面的证明中的函数f可用$C_0^\infty(\Omega)$中的元素在$W^{1,1}(\Omega)$中来逼近．由Sobolev不等式(5.11节)(32)中右端的项$\|f\|_{0,1}$取消后成立．因此(35)不再需要，而证明不要求Ω体积有限，锥性质也不需要，因为Sobolev不等式对所有$u\in C_0^\infty(\mathbf{R}^n)$成立．紧性的推理与前面类似. ▌

8.34 附注 定理8.32在下述意义下不是最优的：对某些A，L_{A_*}未必是$W^1L_A(\Omega)$可以嵌入其中的最小Orlicz空间．例如若$A(t)=A_n(t)=t^n/n$，如前面已指出的，$A_*(t)$在无穷远附近等价于e^t-t-1但该N-函数在无穷远附近本质上增加比$\exp(t^{n/(n-1)})-1$更慢，于是定理8.25给出一个比定理8.32更深刻的结果．Donaldson和Trudinger[22]断言定理8.32可用定理8.25的方法改进，只要对$p<n$，A在无穷远附近控制A_p，但若对某一$p<n$，A_p在无穷远附近控制A，定理8.32给出最佳结果.

8.35 定理 设Ω在$\mathbf{R}^n$中有锥性质，设A是满足

$$\int_1^\infty \frac{A^{-1}(\tau)}{\tau^{(n+1)/n}}d\tau<\infty \tag{38}$$

的N-函数，则

$$W^1L_A(\Omega)\to C_B(\Omega)=C(\Omega)\cap L^\infty(\Omega).$$

证明 设C是包含在Ω内的一个有限锥．我们将指出存在一个依赖于n，A和C的大小的常数K_1使

$$\|u\|_{\infty,C}\leqslant K_1\|u\|_{1,A,C}. \tag{39}$$

这样，我们可设A满足(26)而不失去一般性，因为否则，设B是一个满足(26)且在无穷远附近和A等价的N-函数，则$W^1L_A(C)\to W^1L_B(C)$，由定理 8.12，嵌入常数依赖于A，B和 vol$C$。因B满足(38)，我们将有

$$\|u\|_{\infty,C}\leqslant K_2\|u\|_{1,B,C}\leqslant K_3\|u\|_{1,A,C}.$$

今Ω可表示成一些这样的全等的有限锥C的并，

(39)显然蕴涵

$$\|u\|_{\infty,\Omega}\leqslant K_1\|u\|_{1,A,\Omega}. \tag{40}$$

因A满足(26)和(38)我们有

$$\int_0^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}dt=K_4<\infty.$$

令
$$\Lambda^{-1}(t)=\int_0^t \frac{A^{-1}(\tau)}{\tau^{(n+1)/n}}d\tau,$$

则Λ^{-1}一对一映射$[0,\infty)$到$[0,K_4)$上而且有凸反函数Λ. 对$t\geqslant K_4$，我们令$\Lambda(t)=\infty$，即把Λ的定义域延拓到$(0,\infty)$. 函数Λ是所谓 Young 函数 (参看 Luxemburg[42]或 O'Neill[55])，尽管不是在 8.2 节定义的N-函数，易见 Luxemburg 范数

$$\|u\|_{\Lambda,C}=\inf\{k>0: \int_\Omega \Lambda(|u(x)|/k)dx\leqslant 1\}$$

仍是一个在$L^\infty(C)$上跟通常范数等价的一个范数；事实上，

$$(1/K_4)\|u\|_{\infty,C}\leqslant\|u\|_{\Lambda,C}\leqslant[1/\Lambda^{-1}(1/\mathrm{vol}C)]\|u\|_{\infty,C}. \tag{41}$$

并且，$s=\Lambda(t)$满足微分方程(29)，于是定理 8.32 的证明可以用于这里从而对$u\in W^1L_A(C)$得到

$$\|u\|_{\Lambda,C}\leqslant K_5\|u\|_{1,A,C} \tag{42}$$

不等式(39)现在即由(41)和(42)推出.

由定理 8.28(d)一个元素$u\in W^1E_A(\Omega)$可用Ω上的连续函数依范数逼近，由(40)推出u必定在Ω内 a.e.同一个连续函数重合

(参看引理 5.15 证明的第一节).

假定可以构造一个 N-函数使在 0 附近 $B(t)=A(t)$, B 在无穷远附近本质上增加比 A 更慢, 并且

$$\int_1^\infty \frac{B^{-1}(t)}{t^{(n+1)/n}}dt\leqslant 2\int_1^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}dt<\infty.$$

则由定理 8.16, $u\in W^1L_A(C)$蕴涵 $u\in W^1E_B(C)$, 于是 $W^1L_A(\Omega)\subset C(\Omega)$, 这正是所要证明的.

因此只剩下构造一个这样的 N-函数 B. 令 $1<t_1<t_2<\cdots$满足

$$\int_{t_k}^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}dt=\frac{1}{2^{2k}}\int_1^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}dt.$$

我们归纳地定义序列$\{s_k\}(s_k\geqslant t_k)$和函数 $B^{-1}(t)$如下:

令 $s_1=t_1$ 且当 $0\leqslant t\leqslant s_1$, $B^{-1}(t)=A^{-1}(t)$. 设已选择 s_1, s_2, $\cdots s_{k-1}$ 并当 $0\leqslant t\leqslant s_{k-1}$ 已定义 $B^{-1}(t)$, 我们沿着斜率为 $(A^{-1})'(s_{k-1}-)$(因为 A^{-1} 凹, 它总存在)的直线向 s_{k-1} 右侧延续 $B^{-1}(t)$直到点t'_k, 这里 $B^{-1}(t'_k)=2^{k-1}A^{-1}(t'_k)$. 因为 $\lim\limits_{t\to\infty}A^{-1}(t)/t=0$, 这样的 t'_k 必定存在. 若 $t'_k\geqslant t_k$ 令$s_k=t'_k$. 不然令 $s_k=t_k$ 而令 $B^{-1}(t)=2^{k-1}A^{-1}(t)$以从 t'_k 到 s_k 延拓B^{-1}. 显然B^{-1}是凹的而B是一个 N-函数. 而且在 0 附近 $B(t)=A(t)$, 又因

$$\lim_{t\to\infty}\frac{B^{-1}(t)}{A^{-1}(t)}=\infty,$$

B 在无穷远附近本质上增加比 A 更慢. 最后

$$\begin{aligned}\int_1^\infty \frac{B^{-1}(t)}{t^{(n+1)/n}}dt&\leqslant\int_1^{s_1} \frac{A^{-1}(t)}{t^{(n+1)/n}}dt\\&\quad+\sum_{k=2}^\infty\int_{s_{k-1}}^{s_k}\frac{2^{k-1}A^{-1}(t)}{t^{(n+1)/n}}dt\\&\leqslant\int_1^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}dt+\sum_{k=2}^\infty 2^{k-1}\int_{t_{k-1}}^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}dt\end{aligned}$$

$$=2\int_1^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}dt.$$

这正是所要证明的. ▌

8.36 定理 设Ω是$\mathbf{R}^n$内具有强局部Lipschitz性质的一个区域，若N-函数A满足

$$\int_1^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}dt<\infty, \tag{43}$$

则存在一个常数K使对任何$u\in W^1L_A(\Omega)$(由前一定理可设它是连续的)和所有$x,y\in\Omega$我们有

$$|u(x)-u(y)|\leqslant K\|u\|_{1,A,\Omega}\int_{|x-y|^{-n}}^\infty \frac{A^{-1}(t)}{t^{(n+1)/n}}dt. \tag{44}$$

证明 我们对单位棱长的立方体Ω建立(44)；用引理5.17证明中的方法可以推广到更一般的强Lipschitz区域，象在该引理中那样，用Ω_σ表示Ω的一个平行子立方体，对$x\in\overline{\Omega}_\sigma$得到

$$\left|u(x)-\frac{1}{\sigma^n}\int_{\Omega_\sigma}u(z)dz\right|\leqslant\frac{\sqrt{n}}{\sigma^{n-1}}\int_0^1 t^{-n}dt\int_{\Omega_{t\sigma}}|\mathrm{grad}u(z)|dz.$$

由(22), $\|1\|_{\widetilde{A},\Omega_{t\sigma}}=1/\widetilde{A}^{-1}(t^{-n}\sigma^{-n})$, 由Hölder不等式和(7)推得

$$\begin{aligned}\int_{\Omega_{t\sigma}}|\mathrm{grad}u(z)|dz&\leqslant2\|\mathrm{grad}u\|_{A,\Omega_{t\sigma}}\|1\|_{\widetilde{A},\Omega_{t\sigma}}\\&\leqslant2\|u\|_{1,A,\Omega}/\widetilde{A}^{-1}(t^{-n}\sigma^{-n})\\&\leqslant2\sigma^n t^n\|u\|_{1,A,\Omega}A^{-1}(t^{-n}\sigma^{-n}).\end{aligned}$$

因此

$$\begin{aligned}\left|u(x)-\frac{1}{\sigma^n}\int_{\Omega_\sigma}u(z)dz\right|&\leqslant2\sqrt{n}\,\sigma\|u\|_{1,A,\Omega}\int_0^1A^{-1}\left(\frac{1}{t^n\sigma^n}\right)dt\\&=\frac{2}{\sqrt{n}}\|u\|_{1,A,\Omega}\int_{\sigma^{-n}}^\infty\frac{A^{-1}(\tau)}{\tau^{(n+1)/n}}d\tau.\end{aligned} \tag{45}$$

若$x,y\in\Omega$且$\sigma=|x-y|<1$，存在一个子立方体Ω_σ, $x,y\in\overline{\Omega}_\sigma\subset\Omega$. 对$x$和$y$应用(45)，我们得

$$|u(x)-u(y)|\leqslant\frac{4}{\sqrt{n}}\|u\|_{1,A,\Omega}\int_{|x-y|^{-n}}^\infty\frac{A^{-1}(t)}{t^{(n+1)/n}}dt.$$

对 $|x-y|\geqslant 1$, (44) 由 (40) 和 (43) 导出. ∎

8.37 用 M 表示 $t>0$ 的这样正的连续增加的函数的类: 当 t 减小至 0 时它趋于 0. 若 $\mu\in M$, 空间 $C_\mu(\overline{\Omega})$ 由这样的函数 $u\in C(\overline{\Omega})$ 组成: u 的范数

$$\|u;C_\mu(\overline{\Omega})\|=\|u;C(\overline{\Omega})\|+\sup_{\substack{x,y\in\Omega\\ x\neq y}}\frac{|u(x)-u(y)|}{\mu(|x-y|)}$$

有限. $C_\mu(\overline{\Omega})$ 赋以上述范数是一个 Banach 空间. 上面的定理断言, 若 (43) 成立, 则

$$W^1L_A(\Omega)\to C_\mu(\overline{\Omega}),$$

$$\mu(t)=\int_{t^{-n}}^{\infty}\frac{A^{-1}(\tau)}{\tau^{(n+1)/n}}d\tau. \tag{46}$$

若 $\mu,\nu\in M$ 且 $\mu/\nu\in M$, 则对有界集 Ω, 象在定理 1.31 中一样, 我们有嵌入

$$C_\mu(\overline{\Omega})\to C_\nu(\overline{\Omega})$$

是紧的. 因此若 μ 由 (46) 给定, 嵌入

$$W^1L_A(\Omega)\to C_\nu(\overline{\Omega})$$

亦如是.

下面是一个迹嵌入定理, 它推广了引理 5.19(的 $m=1$ 的情形).

8.38 定理 设 Ω 是 $\mathbf{R}^n$ 内一个有锥性质的有界区域, 令 Ω^k 表示 Ω 同 $\mathbf{R}^n$ 内一个 k 维平面的交, 设 A 是满足 (26) 和 (27) 的一个 N-函数, 而 A_* 由 (28) 给定. 令 $1\leqslant p<n$, 由 $B(t)=A(t^{1/p})$ 给定的函数 B 是一个 N-函数. 若或者 $n-p<k\leqslant n$ 或者 $p=1$ 而 $n-1\leqslant k\leqslant n$, 则

$$W^1L_A(\Omega)\to L_{A_*^{k/n}}(\Omega^k),$$

其中 $A_*^{k/n}(t)=[A_*(t)]^{k/n}$.

并且, 若 $p>1$ 而 C 是一个在无穷远附近增加本质上比 $A_*^{k/n}$

更慢的 N-函数，则嵌入

$$W^1L_A(\Omega)\to L_C(\Omega^k) \tag{47}$$

是紧的.

证明 验证$A_*^{k/n}$是一个N-函数的问题留给读者. 设$u\in W^1L_A(\Omega)$是一个有界函数，则

$$\int_{\Omega^k} A_*^{k/n}\left(\frac{|u(y)|}{K}\right)dy=1$$

$$K=\|u\|_{A_*^{k/n},\Omega^k}. \tag{48}$$

我们希望证明

$$K\leqslant K_1\|u\|_{1,A,\Omega}, \tag{49}$$

K_1 不依赖于 u. 因为(49)对特殊情形 $k=n$ 是已知的(定理 8.32)，不失一般性我们可以假设

$$K\geqslant\|u\|_{A_*,\Omega}=\|u\|_{A_*^{n/n},\Omega^n}. \tag{50}$$

令 $\omega(t)=[A_*(t)]^{1/q}$, $q=np(n-p)$. 由引理 5.19(情形 $m=1$)我们有

$$\begin{aligned}
&\left\|\omega\left(\frac{|u|}{K}\right)\right\|^p_{kp/(n-p),\Omega^k}\\
&\leqslant K_2\left\{\sum_{j=1}^n\left\|D_j\omega\left(\frac{|u|}{K}\right)\right\|^p_{p,\Omega}+\left\|\omega\left(\frac{|u|}{K}\right)\right\|^p_{p,\Omega}\right\}\\
&=K_2\left\{\frac{1}{K^p}\sum_{j=1}^n\int_\Omega\left|\omega'\left(\frac{|u(x)|}{K}\right)\right|^p|D_ju(x)|^pdx+\right.\\
&\qquad\qquad\left.\int_\Omega\left|\omega\left(\frac{|u(x)|}{K}\right)\right|^pdx\right\}.
\end{aligned}$$

利用(48)并注意$\||v|^p\|_{B,\Omega}\leqslant\|v\|^p_{A,\Omega}$ 我们得

$$\begin{aligned}
1&=\left(\int_{\Omega^k}\left(A_*\left(\frac{|u(y)|}{K}\right)\right)^{k/n}dy\right)^{(n-p)/k}=\left\|\omega\left(\frac{|u|}{K}\right)\right\|^p_{kp/(n-p),\Omega^k}\\
&\leqslant\frac{2K_2}{K^p}\sum_{j=1}^n\left\|\left(\omega'\left(\frac{|u|}{K}\right)\right)^p\right\|_{\widetilde{B},\Omega}\||D_ju|^p\|_{B,\Omega}+K_2\left\|\omega\left(\frac{|u|}{K}\right)\right\|^p_{p,\Omega}
\end{aligned}$$

$$\leqslant \frac{2nK_2}{K^p}\left\|\left(\omega'\left(\frac{|u|}{K}\right)\right)^p\right\|_{\widetilde{B},\Omega}\|u\|_{1,A,\Omega}^p+K_2\left\|\omega\left(\frac{|u|}{K}\right)\right\|_{p,\Omega}^p. \tag{51}$$

令 $B^{-1}(t)=[A^{-1}(t)]^p$ 并且利用(29)和(7), 我们有

$$\begin{aligned}[\omega'(t)]^p &= (1/q^p)[A_*(t)]^{p(1-q)/q}[A_*'(t)]^p \\ &= (1/q^p)A_*(t)[1/B^{-1}(A_*(t))] \\ &\leqslant (1/q^p)\widetilde{B}^{-1}(A_*(t)).\end{aligned}$$

由(50)推出

$$\int_\Omega \widetilde{B}\left(\left(\frac{\omega'(|u(x)|/K)}{1/q}\right)^p\right)dx \leqslant \int_\Omega A_*\left(\frac{|u(x)|}{K}\right)dx \leqslant 1.$$

于是

$$\|(\omega'(|u|/K))^p\|_{\widetilde{B},\Omega} \leqslant 1/q^p. \tag{52}$$

今令 $g(t)=A_*(t)/t^p$, $h(t)=(\omega(t)/t)^p$, 易于验证 $\lim\limits_{t\to\infty} g(t)/h(t)=\infty$. 为看出 $h(t)$在 0 附近是有界的, 令$s=A_*(t)$, 而考虑

$$(h(t))^{1/p}=\frac{(A_*(t))^{1/q}}{t}=\frac{s^{(1/p)-(1/n)}}{\int_0^s \frac{A^{-1}(\tau)}{\tau^{(n+1)/n}}d\tau} \leqslant \frac{s^{1/p}}{\int_0^s \frac{[B^{-1}(\tau)]^{1/p}}{\tau}d\tau}.$$

因为B是一个 N-函数, $\lim\limits_{\tau\to 0+} B^{-1}(\tau)/\tau=\infty$. 因此对充分小的 t 值我们有

$$(h(t))^{1/p} \leqslant \frac{s^{1/p}}{\int_0^s \tau^{-1+1/p}d\tau}=\frac{1}{p}.$$

因此存在一个常数 K_3 使当 $t\geqslant 0$

$$(\omega(t))^p \leqslant (1/2K_2)A_*(t)+K_3t^p.$$

利用(50), 我们现在得到

$$\begin{aligned}\left\|\omega\left(\frac{|u|}{K}\right)\right\|_{p,\Omega}^p &\leqslant \frac{1}{2K_2}\int_\Omega A_*\left(\frac{|u(x)|}{K}\right)dx+\frac{K_3}{K^p}\int_\Omega |u(x)|^p dx \\ &\leqslant \frac{1}{2K_2}+\frac{2K_3}{K^p}\||u|^p\|_{B,\Omega}\|1\|_{\widetilde{B},\Omega}\end{aligned}$$

$$\leqslant \frac{1}{2K_2} + \frac{K_4}{K^p} \|u\|_{A,\Omega}^p. \tag{53}$$

从(51)—(53)推出不等式

$$1 \leqslant \frac{2nK_2}{K^p} \frac{1}{q^p} \|u\|_{1,A,\Omega}^p + \frac{1}{2} + \frac{K_4 K_2}{K^p} \|u\|_{A,\Omega}^p,$$

因此(49)成立．推广(49)到任意 $u \in W^1 L_A(\Omega)$ 可象定理 8.32 的证明那样．

因 $B(t) = A(t^{1/p})$ 是一个 N-函数并且 Ω 是有界的，我们有 $W^1 L_A(\Omega) \to W^{1,p}(\Omega) \to L^1(\Omega^k)$，由定理 6.2, 后一嵌入是紧的(只要 $p>1$)．(47)的紧性现可由定理 8.23 推出．∎

8.39 我们以 Donaldson 和 Trudinger [22] 的一般 Orlicz-Sobolev 空间嵌入定理结束本章．对一个给定的 N-函数 A 我们定义 N-函数序列 $B_0, B_1, B_2, \cdots$ 如下:

$$B_0(t) = A_0(t)$$

$$(B_k)^{-1}(t) = \int_0^t \frac{(B_{k-1})^{-1}(\tau)}{\tau^{(n+1)/n}} d\tau, k = 1, 2, \cdots$$

在每一步我们假定

$$\int_0^1 \frac{(B_k)^{-1}(\tau)}{\tau^{(n+1)/n}} d\tau < \infty, \tag{54}$$

若有必要，B_k 代之以在无穷远附近与 B_k 等价的满足(54)的 N-函数．令 $J = J(A)$ 是满足

$$\int_1^\infty \frac{(B_J)^{-1}(\tau)}{\tau^{(n+1)/n}} d\tau < \infty$$

的最小非负整数，显然 $J(A) \leqslant n$.

若 μ 属于 8.37 节中定义的类 M，我们定义空间 $C_\mu^m(\overline{\Omega})$ 由所有这样的函数 $u \in C^m(\overline{\Omega})$ 组成：对于它，$D^\alpha u \in C_\mu(\overline{\Omega})$，$|\alpha| \leqslant m$．空间 $C_\mu^m(\overline{\Omega})$ 赋以范数

$$\|u; C_\mu^m(\overline{\Omega})\| = \max_{|\alpha| \leqslant m} \|D^\alpha u; C_\mu(\overline{\Omega})\|$$

是一个 Banach 空间.

8.40 定理 设Ω是 $\mathbf{R}^n$ 内一个具有锥性质的有界区域. 设A是一个 N-函数.

(*a*) 若 $m \leqslant J(A)$, 则 $W^m L_A(\Omega) \to L_{B_m}(\Omega)$ 且嵌入 $W^m L_A(\Omega) \to L_C(\Omega)$ 对任何在无穷远附近本质上增加比 B_m 更慢的 N-函数 C 是紧的.

(*b*) 若 $m > J(A)$, 则 $W^m L_A(\Omega) \to C_B(\Omega) = C(\Omega) \cap L^\infty(\Omega)$.

(*c*) 若Ω还有强局部 Lipschitz 性质, 且若 $m > J = J(A)$, 则 $W^m L_A(\Omega) \to C_\mu^{m-J-1}(\overline{\Omega})$, 其中

$$\mu(t) = \int_{t^{-n}}^{\infty} \frac{(B_J)^{-1}}{\tau^{(n+1)/n}} d\tau.$$

并且, 只要 $\nu \in M$ 又 $\mu/\nu \in M$, 嵌入 $W^m L_A(\Omega) \to C^{m-J-1}(\overline{\Omega})$ 和 $W^m L_A(\Omega) \to C_\nu^{m-J-1}(\overline{\Omega})$ 是紧的.

8.41 附注 上述定理可直接从定理 8.32, 8.35 和 8.36 推出. 并且, 若在(*a*)中用 E_A 代替 L_A, 我们得 $W^m E_A(\Omega) \to E_{B_m}(\Omega)$, 因为若 $u \in W^1 E_A(\Omega)$, 在定理 8.32 证明中由(37)定义的序列$\{u_k\}$收敛到 u. 若 $W^m L_A(\Omega)$ 处处代之以 $W_0^m L_A(\Omega)$. 定理 8.40 对Ω不加限制亦成立.

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