FEM: Basic Theory and Implementation

1-D FEM for Elliptic Equation

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Contents

1 Introduction

This notes present Finite Element Method (FEM) for 1d elliptic problem, and aim at the main procedures of such method easily.
• Why Elliptic problem?
• Why 1D case?
• Why FEM?

2 General Finite Element and Finite Element Spaces

2.1 Finite Element

In the next, we introduce a concept of finite element generally:

Definition 2.1. *A triple $(K, \mathcal{P}, \mathcal{N})$ is called a **finite element** if it satisfies the following properties:*

(i) *$K \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ is a closed set with piecewise smooth boundary (called the **element**);*

(ii) *$\mathcal{P}$ is a finite-dimensional space of functions on K (called the **space functions**);*

(iii) *$\mathcal{N}$ is finite set of linear functionals $\mathcal{P} \to \mathbb{R}$ and forms a basis for $\mathcal{P}'$ (called the **degree of freedom**).*

Remark 2.1. *The name finite element is clear due to the finite dimensional space $\mathcal{P}$, which uniquely determined by the DOFs, over an element K.*

Lemma 2.1. *Suppose that $\mathcal{P}$ is finite dimensional and let $\{\psi_1, \cdots, \psi_n\}$ be a basis of $\mathcal{P}$. Then there exists uniquely a set of functionals $\{f_1, \cdots, f_n\} \subset \mathcal{P}'$ such that*

$$f_i(\psi_j) = \delta_{i,j}, \quad i, j = 1, \cdots, n.$$

Moreover, $\{f_i\}$ form a basis for $\mathcal{P}'$, which is called the dual basis of $\{\psi_i\}$. Thus $\dim\mathcal{P} = \dim\mathcal{P}'$.

Proof. The coordinate functionals $g_1, \cdots, g_n$ are given by

$$\left\langle g_i, \sum_{k=0}^{n} \lambda_k \psi_k \right\rangle = \lambda_i.$$

Clearly, $g_i \in \mathcal{P}'$. Thus $\langle g_i, \psi_k \rangle = \delta_{i,k}$. Then for any $v \in \mathcal{P}$ we have $\langle g_i - f_i, v \rangle = 0$, which implies $g_i = f_i$.

We now show that $\{f_i\}$ are linearly independent. Indeed, suppose that there exists $a_i \in \mathbb{R}$ such that $\sum_{i=1}^n a_i f_i = 0$, where 0 denotes the zero functional. Thus

$$a_k = \sum_{i=1}^{n} a_i \langle f_i, \psi_k \rangle = \langle 0, \psi_k \rangle = 0, \quad k = 1, \cdots, n.$$

We now show that $\mathcal{P}' = \mathrm{span}\{f_1, \cdots, f_n\}$. For any $f \in \mathcal{P}'$, let $b_i = f(\psi_i)$ for $i = 1, \cdots, n$. Thus we have $f = \sum_{i=1}^n b_i f_i$. In fact, by definition

$$f(\psi_k) = \sum_{i=1}^{n} b_i f_i(\psi_k), \quad k = 1, \cdots, n,$$

which implies $f(v) = \sum_{i=1}^n b_i f_i(v), \ \forall v \in \mathcal{P}$. □

Remark 2.2. *It is clear that the number of elements in $\mathcal{N}$ is equal to the dimension of $\mathcal{P}$.*

Remark 2.3. *Any function $v \in \mathcal{P}$ is uniquely determined by an arbitrary assignment of values to the DOFs. In fact, suppose that $\mathcal{P} = \operatorname{span}\{\psi_1, \cdots, \psi_n\}$ and $\mathcal{N} = \{\mathcal{N}_1, \cdots, \mathcal{N}_n\}$. Let the dual basis $\mathcal{P}' = \operatorname{span}\{f_1, \cdots, f_n\}$. Then it is clear that $v = \sum_{i=1}^n f_i(v)\psi_i$. We suppose that $\mathcal{N}_i = \sum_{j=1}^n c_{ij} f_j$. Thus*

$$\mathcal{N}_i(v) = \sum_{j=1}^n c_{ij} f_i(v), \ i = 1, \cdots, n.$$

It is known that the matrix (c_{ij}) is nonsingular, then the coefficients $\{f_i(v)\}$ of v is uniquely determined by the assignment of values to $\{\mathcal{N}_i(v)\}$.

Definition 2.2. *Let $(K, \mathcal{P}, \mathcal{N})$ be a finite element. A basis $\{\psi_1, \cdots, \psi_n\}$ for $\mathcal{P}$ is called the **nodal basis** for $\mathcal{P}$ if it is dual to $\mathcal{N}$, i.e., $\mathcal{N}_i(\psi_j) = \delta_{i,j}$.*

Example 2.1. *Let $K = (0, 1)$, $\mathcal{P}$ be the space of linear polynomials, and $\mathcal{N} = \{\ell_0, \ell_1\}$ satisfying*

$$\ell_0(p) = p(0), \ \ell_1(p) = p(1), \quad \forall p \in \mathcal{P}.$$

The triple $(K, \mathcal{P}, \mathcal{N})$ is a finite element. The nodal basis of $\mathcal{P}$ is

$$\varphi_0(x) = 1 - x, \quad \varphi_1(x) = x.$$

Remark 2.4. *Why nodal basis?*

The discussion above states a general framework to define a finite element. However, in implementations and many applications, the finite element will be specified as

(i) Element K is an interval in 1d case, a triangle or parallelogram in 2d case, and a tetrahedron or parallelepiped in 3d case.

(ii) $\mathcal{P}$ is a finite dimensional space containing polynomials.

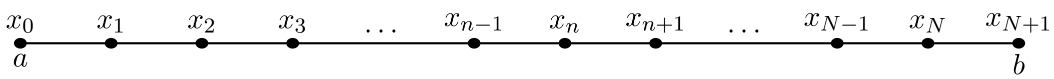
(iii) $\mathcal{N}$ contains functionals that map $v \in \mathcal{P}$ to $v(p)$ for some $p \in K$ (referred to **Lagrange** type), or to $D^\alpha v(p)$ for some $p \in K$ and derivatives D^α (referred to **Hermite** type).

Remark 2.5. *We only consider the Lagrange element, i.e., $\mathcal{P}$ is space of polynomials and $\mathcal{N}$ is Lagrange type.*

Global Degree of Freedom?

2.2 Assembled Finite Element Space

Let $\{x_n\}_{n=0}^{N+1}$ be a grid on the interval $I = (a, b)$. Each **element** $I_n = (x_{n-1}, x_n)$ is the subinterval of I with length of $h_n = |x_n - x_{n-1}|$. We also denote $h = \max_{1 \leqslant n \leqslant N+1} h_n$, which is a parameter to measure how fine the partition is.



Let $\mathcal{T}_h = \{I_n : 1 \leqslant n \leqslant N+1\}$, which is called a partition of the interval I. The idea to construct the approximation space is by defining a finite element triple $(I_n, \mathcal{P}(I_n), \mathcal{N}(I_n))$ on each element I_n, and then assemble them in a way of single-valued DOFs to an **Assembled Finite Element Space** with the properties: for any $v(x)$ defined on I,

(i) $v|_{I_n} \in \mathcal{P}(I_n)$, $n = 1, \cdots, N+1$.

(ii) If $T_1, T_2 \subset \mathcal{T}_h$ and share the same node $p \in \bar{T}_1 \cap \bar{T}_2$ that corresponding DOFs applied on, then the corresponding DOFs applied on $v|_{T_1}$ and $v|_{T_2}$ over p are the same value.

The assembled finite element space is denoted as $(\mathcal{T}_h, \mathcal{P}, \mathcal{N})$

Remark 2.6. *Theoretically, each $\mathcal{P}(I_n)$ or $\mathcal{N}(I_n)$ may be different. However, this configuration is not suitable for analysis and implementations. We will assume that $\mathcal{P}(I_n) \in \mathbb{P}_k$ for some $k \in \mathbb{N}$.*

2.3 X_h^k Space

Remark 2.7. *Compared to 2 or 3d cases, the 1d case is easy to give a partition over I.*

We consider the finite-dimensional subspace of $H^1(I)$.
Let X_h^k be the space of piecewise polynomials in I, i.e.,

$$X_h^k := \{v(x) \in C^0(\bar{I}) : v|_{I_n} \in \mathbb{P}_k,\ n = 1, \cdots, N+1\}.$$

Lemma 2.2. *We have $X_h^k \subset H^1(I)$, and more generally,*

$$\{v \in C^0(\bar{I}) : v|_{I_n} \in H^1(I_n), n = 1, \cdots, N+1\} \subset H^1(I).$$

Proof. For any $v \in \{v \in C^0(\bar{I}) : v|_{I_n} \in H^1(I_n), n = 1, \cdots, N+1\}$, it is clear that $v \in L^2(I)$ because continuity implies square integrability on the bounded domain I. It remains to show that the weak derivative of v also belongs to $L^2(I)$. Since $v|_{I_n} \in H^1(I_n)$, we define a piecewise derivative by

$$g|_{I_n}(x) = (v|_{I_n})'(x), \quad x \in I_n,\ n = 1, \cdots, N+1.$$

Obviously, $g \in L^2(I)$, as each piece $(v|_{I_n})' \in L^2(I_n)$ and the intervals I_n are disjoint and cover I. We claim that g is the derivative of v. Indeed, for any test function $\phi(x) \in C_0^\infty(I)$, we have

$$\begin{aligned}
\int_I g(x)\phi(x)\mathrm{d}x &= \sum_{n=1}^{N+1} \int_{I_n} g|_{I_n}(x)\phi(x)\mathrm{d}x = \sum_{n=1}^{N+1} \int_{I_n} (v|_{I_n})'(x)\phi(x)\mathrm{d}x \\
&= \sum_{n=1}^{N+1} [v(x)\phi(x)]\big|_{x_{n-1}}^{x_n} - \sum_{n=1}^{N+1} \int_{I_n} (v|_{I_n})(x)\phi'(x)\mathrm{d}x \\
&= \sum_{n=1}^{N+1} \left(v(x_n^-)\phi(x_n^-) - v(x_{n-1}^+)\phi(x_{n-1}^+)\right) - \sum_{n=1}^{N+1} \int_{I_n} (v|_{I_n})(x)\phi'(x)\mathrm{d}x.
\end{aligned}$$

Due to the continuity of v across element interfaces, we have $v(x_n^-) = v(x_n^+)$ for $n = 1, \cdots, N$, and since $\phi \in C_0^\infty(I)$ we have $\phi(x_0) = \phi(x_{N+1}) = 0$. Hence, the sum of boundary terms cancels out, yielding

$$\int_I g(x)\phi(x)\mathrm{d}x = -\int_I v(x)\phi'(x)\mathrm{d}x,$$

which confirms that g is the weak derivative of v. Therefore, $v \in H^1(I)$. □

2.3.1 Basis

2.3.2 Error Analysis

3 Programing Considerations

We only consider the Lagrange type elements

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain, and its triangulation $\mathcal{T}_h$ consists of elements $\{K_i\}_{i=1}^{N_h}$, where $N_h \in \mathbb{Z}^+$ denotes the total number of elements. Each element K_i is closed, i.e., $\bar{K}_i = K_i$. Thus

$$\bar{\Omega} \subseteq \bigcup_{K_i \in \mathcal{T}_h} K_i, \quad \mathrm{m}\,(K_i \cap K_j) = 0 \text{ for } i \neq j,$$

where $\mathrm{m}(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}^d$. Furthermore, we assume that for $i \neq j$ there does not exist $C \subset \Omega$ with non zero measure in $\mathbb{R}^{d'}$ for $1 \leqslant d' < d$ such that

$$C \subset K_i,\ K_i \cap K_j \subseteq C, \text{ and } K_i \cap K_j \neq C.$$

We suppose that K_i contains the nodes $\{\mathbf{x}_\ell^{(i)}\}_{\ell=1}^{L_i}$, each of which corresponds to a **local nodal basis function** (Lagrange type), or local DOF, denoted as $\phi_\ell^{(i)}(\mathbf{x}) : K_i \to \mathbb{R}$. We denote it as the set $S^{(i)} = \{\mathbf{x}_\ell^{(i)}\}_{\ell=1}^{L_i}$. We suppose that

$$\phi_\ell^{(i)} \in \mathcal{P}^{(i)},$$

where $\mathcal{P}^{(i)}$ is a linear space consists of functions dimensional of L_i. The nodes set $S^{(i)}$ are chosen to be $\mathcal{P}^{(i)}-$unisolvent, i.e., each $\phi_\ell^{(i)}$ can be uniquely determined in a Lagrange way:

$$\phi_\ell^{(i)}(\mathbf{x}_j^{(i)}) = \delta_{\ell j}, \quad \ell, j = 1, \cdots, L_i.$$

Let the all nodes constitute the set

$$S := \bigcup_{i=1}^{N_h} S^{(i)},$$

where we assemble elements in a way of single-valued DOFs, i.e., the common nodes on adjacent elements are treated as the same. We suppose further that $S = \{\mathbf{x}_\ell\}_{\ell=1}^{L}$. Each $\mathbf{x}_\ell$ corresponds to a **global nodal basis function** (Lagrange type), or global DOF, denoted as $\varphi_\ell(\mathbf{x}) : \Omega \to \mathbb{R}$, satisfying

$$\varphi_\ell|_{K_i} \in \mathcal{P}^{(i)}, \quad i = 1, \cdots, N_h.$$
$$\varphi_\ell(\mathbf{x}_j) = \delta_{\ell j}, \quad \ell, j = 1, \cdots, L.$$

Remark 3.1. *When the nodal basis functions are of Hermite type and global single-valued DOFs are applied, the global nodes are also constructed to account for the (directional) derivatives on which the DOFs act, treating each such derivative as a separate node.*

Generally, we have $L < \sum_{i=1}^{N_h} L_i$. Thus, it is clear that

By the definition of Lagrange type nodal basis functions, it is clear that

Lemma 3.1. *A nodal basis function for a node in the interior of K_i vanishes outside K_i.*

Lemma 3.2. *A nodal basis function for a node on the boundary of K_i is supported in the adjacent elements that have the same type DOFs applied on the node.*

4 Elliptic Problem

We consider the elliptic problem:

4.1 Typical Model: Possion Equation with Homogeneous Dirichlet Boundary

A two-point boundary value problem with homogeneous Dirichlet boundary condition:

$$\begin{cases} -\dfrac{d^2u}{dx^2} = f(x), \quad x \in I := (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{4.1}$$

where $f \in L^2(I)$. The problem (4.1) is also called the *strong problem*. Let $V := H_0^1(I) = \{v \in H^1(I) : v(0) = v(1) = 0\}$. The restriction on boundary values makes sense due to the embedding theorem, which tells that

$$\forall v \in H^1(I),\ \exists \bar{v} \in C(\bar{I}) \text{ s.t. } v = \bar{v} \text{ a.e. in } I.$$

The *variational problem* (or known as *weak problem*):

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ (u', v') = (f, v), \quad \forall v \in V, \end{cases} \tag{4.2}$$

where $(\cdot, \cdot)$ stands for the $L^2(I)-$inner product. Let J be the linear functional:

$$J(v) = \frac{1}{2}(u', v') - (f, v).$$

Then the *minimization problem*:

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ J(u) \leqslant J(v), \quad \forall v \in V. \end{cases} \tag{4.3}$$

The term minimization problem corresponds the "principle of minimum potential energy" in mechanics. It tells us that some of differential equations like (4.1) may originates from minimizing the potential energy in some physical problems.

Theorem 4.1. *Under proper regularity assumptions, the three problems above are equivalent:*
1). the solution of (4.1) *is a solution of* (4.2)*;*
2). the solution of (4.2) *is a solution of* (4.3)*;*
3). the solution of (4.3) *is a solution of* (4.1)*.*

The existence and uniqueness of these three problem can be considered respectively.

For (4.1), its existence can be represented using Green's function (see [Evans (2010), p.35 Chapter 2, Theorem 12]), and the uniqueness is guaranteed by the *strong maximum principle* of harmonic functions (see [Evans (2010), pp. 27-28, Theorem 4&5]).

For (4.3), its existence and uniqueness are guaranteed by that J is strongly convex and is a linear functional over a linear space.

For (4.2), its existence and uniqueness are guaranteed by the well known *Lax-Milgram Lemma*, whose general description reads

Lemma 4.1 (Lax-Milgram)**.** *Let V be a Hilbert space, endowed with the norm $\|\cdot\|_V$. Consider the problem: $\forall f \in V'$,*

$$\begin{cases} \textit{Find } u \in V, \textit{ such that} \\ a(u, v) = < f, v >, \quad \forall v \in V, \end{cases}$$

where $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is a bilinear form. If furthermore, $a(\cdot,\cdot)$ satisfies

$$\begin{aligned} &\text{Continuity}: \quad \exists \gamma > 0 \ s.t. \ |a(u, v)| \leqslant \gamma \|u\|_V \|v\|_V, \quad \forall u, v \in V, \\ &\text{Coercivity}: \quad \exists \alpha > 0 \ s.t. \ a(v, v) \geqslant \alpha \|v\|_V^2, \quad \forall v \in V. \end{aligned}$$

Then the problem admits a unique solution u, which satisfies

$$\|u\|_V \leqslant \frac{1}{\alpha} \sup_{v \in V, v \neq 0} \frac{< f, v >}{\|v\|_V}.$$

Theorem 4.2. *Problem* (4.2) *admits an unique solution.*

4.2 Other Boundary Conditions

5 $P1-$ FEM

The finite element method (FEM) is a numerical technique, arguably the most robust and popular, for solving differential equations. FEM is a numerical method general based on the *Galerkin approximation* (or *Galerkin method* or *Galerkin framework*), to approximate with constructing finite elements (piecewise approximation). Galerkin method is to approximate the weak problem with finite dimensional subspace constructed. For (4.2),

$$\begin{cases} \text{Find } u_h \in V_h \text{ such that} \\ \left(\dfrac{du_h}{dx}, \dfrac{dv_h}{dx}\right) = (f, v_h), \quad \forall v_h \in V_h, \end{cases} \tag{5.1}$$

where V_h is a finite dimensional subspace of V.

We divide the interval $[0,1]$ into $N+2$ grid

$$0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1.$$

We denote the subintervals $I_j = [x_{j-1}, x_j]$ for $1 \leqslant j \leqslant N+1$, with length $h_j = x_j - x_{j-1}$. Let $h = \max_{1\leqslant j\leqslant N+1} h_j$. The mesh size h is used to measure how fine the partition is.

We define the finite element space

$$V_h = \{v \in C[0,1] : v \text{ is linear on each subinterval } I_j, \text{ and } v(0) = v(1) = 0\}.$$

Theorem 5.1. *$V_h \subset V$.*

Proof. It is sufficient to show that for any $v \in V_h$ we have $v \in H^1(I)$, i.e.,

$$\int_0^1 \frac{dv}{dx}\phi \mathrm{d}x = -\int_0^1 v\frac{d\phi}{dx}\mathrm{d}x, \quad \forall \phi \in C_0^\infty(I).$$

In fact,

$$\begin{aligned} \int_0^1 \frac{dv}{dx}\phi \mathrm{d}x = \sum_{j=1}^{N+1}\int_{I_j}\frac{dv}{dx}\phi\mathrm{d}x &= \sum_{j=1}^{N+1}\left(\phi(x_j)v(x_j) - \phi(x_{j-1})v(x_{j-1}) - \int_{I_j} v\frac{d\phi}{dx}\mathrm{d}x\right) \\ &= \phi(1)v(1) - \phi(0)v(0) - \sum_{j=1}^{N+1}\int_{I_j} v\frac{d\phi}{dx}\mathrm{d}x = -\int_0^1 v\frac{d\phi}{dx}\mathrm{d}x. \end{aligned}$$

□

Theorem 5.2. *$\dim(V_h) = N$.*

Proof. For any $v_h \in V_h$, we observe that on each subinterval I_j for $j = 1, \cdots, N+1$, $v|_{I_j}$ is a linear polynomial and thus uniquely determined by 2 parameters, known as the *degree of freedom*. Since there are $N+1$ subintervals, this initially gives a total of $2(N+1)$ degrees of freedom. However, imposing N continuity conditions at the subinterval boundaries and 2 boundary conditions reduces the count by $N+2$, leaving $2(N+1) - N - 2 = N$ degrees of freedom. Consequently, the dimension of the space is N. □

Remark 5.1. *Why nodal basis functions?*

Let us introduce the linear basis function $\phi_j(x)$ for $1 \leqslant j \leqslant N$, which satisfies the properties

$$\phi_j\left(x_i\right)=\left\{\begin{array}{ll}1, & \text{if } i=j, \\ 0, & \text{if } i \neq j.\end{array}\right.$$

Then $\phi_j(x) \in V_h$ and $\{\phi_1(x), \cdots, \phi_N(x)\}$ is linear independent and thus, by dimension argument, constitutes a basis for V_h, i.e., $V_h = \text{span}\{\phi_1, \cdots, \phi_N\}$. Consequently, $\forall v_h \in V_h$, there is an unique representation

$$v_h(x)=\sum_{j=1}^N v_j \phi_j(x), \quad x \in [0,1],$$

where $v_j = v_h(x_j)$. More specifically, ϕ_j is given by

$$\phi_j(x)=\left\{\begin{array}{cc}\frac{x-x_{j-1}}{h_j}, & \text{if } x \in\left[x_{j-1}, x_j\right], \\ \frac{x_{j+1}-x}{h_{j+1}}, & \text{if } x \in\left[x_j, x_{j+1}\right], \\ 0, & \text{elsewhere}.\end{array}\right. \tag{5.2}$$

With the constructed piecewise linear space $V_h = \text{span}\{\phi_1, \cdots, \phi_N\}$, we set the solution u_h of (5.1) as

$$u_h(x)=\sum_{j=1}^N u_j \phi_j(x), \quad u_j=u_h\left(x_j\right).$$

Substituting u_h in (5.1) and choosing $v = \phi_i(x)$ in (5.1) for each $i = 1, \cdots, N$, we obtain

$$\sum_{j=1}^N\left(\frac{d \phi_j}{d x}, \frac{d \phi_i}{d x}\right) u_j=\left(f, \phi_i\right) \quad 1 \leq i \leq N,$$

which is a linear system of N equations with N unknowns u_j:

$$\mathbf{A}\mathbf{u} = \mathbf{F},$$

where $\mathbf{u} = [u_1, \cdots, u_N]^{\mathrm{T}}$, $\mathbf{F} = [F_1, \cdots, F_N]^{\mathrm{T}}$ with elements $F_i = (f, \phi_i)$, and $\mathbf{A} = (a_{i,j})$ is an $N \times N$ matrix with elements $a_{i,j} = (\frac{d\phi_j}{dx}, \frac{d\phi_i}{dx})$.

The matrix $\mathbf{A}$ is called the *stiffness matrix* and $\mathbf{F}$ the *load vector*. We can explicitly calculate the elements in $\mathbf{A}$:

$$\begin{aligned}
a_{j,j} &= \left(\frac{d\phi_j}{dx}, \frac{d\phi_j}{dx}\right) = \int_{x_{j-1}}^{x_j} \frac{1}{h_j^2} dx + \int_{x_j}^{x_{j+1}} \frac{1}{h_{j+1}^2} dx = \frac{1}{h_j} + \frac{1}{h_{j+1}}, \quad 1 \leq j \leq N, \\
a_{j-1,j} &= \left(\frac{d\phi_j}{dx}, \frac{d\phi_{j-1}}{dx}\right) = \int_{x_{j-1}}^{x_j} \frac{-1}{h_j^2} dx = -\frac{1}{h_j}, \quad 2 \leq j \leq N, \\
a_{j,j-1} &= \left(\frac{d\phi_{j-1}}{dx}, \frac{d\phi_j}{dx}\right) = a_{j-1,j} = -\frac{1}{h_j}, \quad 2 \leq j \leq N, \\
a_{i,j} &= \left(\frac{d\phi_j}{dx}, \frac{d\phi_i}{dx}\right) = 0, \quad \text{if} \quad |j-i| > 1.
\end{aligned}$$

Thus the matrix $\mathbf{A}$ is tri-diagonal. Let $\mathbf{v} = [v_1, \cdots, v_N]^{\mathrm{T}}$, and we note that

$$\mathbf{v}^{\mathrm{T}}\mathbf{A}\mathbf{v} = \sum_{i,j=1}^N a_{i,j} v_i v_j = \sum_{i,j=1}^N v_j \left(\frac{d\phi_j}{dx}, \frac{d\phi_i}{dx}\right) v_i = \left(\sum_{j=1}^N v_j \frac{d\phi_j}{dx}, \sum_{i=1}^N v_i \frac{d\phi_i}{dx}\right) = \left(\frac{dv_h}{dx}, \frac{dv_h}{dx}\right) \geqslant 0,$$

where we denote $v_h(x) = \sum_{j=1}^N v_j \phi_j(x)$. Thus the equality holds if and only if $\frac{dv_h}{dx} \equiv 0$, which is equivalent to $v_h(x)$ is constant, and by $v_h(0) = 0$ we have $v_h(x) \equiv 0$, or $\mathbf{v} = \mathbf{0}$. Therefore $\mathbf{A}$ is positive definite, which guarantees the linear system has a unique solution.

- **A** is symmetric: $a_{i,j} = a_{j,i}$,
- **A** is sparse: $a_{i,j} = 0$ for $|i - j| > 1$,
- **A** is positive definite.

In a particular case: $h_j = h = \frac{1}{N+1}$, we have

$$\mathbf{A} = \frac{1}{h} \begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{bmatrix}_{N \times N}$$

Theorem 5.3. *Eigenvalue of* **A** *is*

5.1 Error Estimate For $P1-$FEM

Let $u \in C(\bar{I})$. We denote u_I the interpolation of u into V_h at nodes $\{x_j\}_{j=0}^N$, i.e., $u_I \in V_h$ and

$$u_I(x_j) = u(x_j), \quad j = 0, \cdots, N.$$

It is evident that $u_I(x) = \sum_{j=0}^N u(x_j)\phi_j(x)$.

5.1.1 Interpolation Error bounded by $L^\infty-$norm

Theorem 5.4.

$$\|u - u_I\|_\infty \leqslant \frac{h^2}{8} \max_{x \in \bar{I}} |u''(x)|.$$

6 $P2-$ FEM

7 Implementation in General Framework

7.1 Target Problem

Let $I := (a, b)$ be an interval of $\mathbb{R}$, whose boundary is $\partial I := \{a, b\}$. We consider the elliptic boundary value problem of the form:

$$\begin{cases} Lu = f & \text{in } I, \\ Bu = 0 & \text{on } \partial I, \end{cases}$$

where f is a given function, u is the unknown, B is an affine boundary operator, and L is the second order linear operator defined by

$$Lw := -(a(x)w'(x))' + (b(x)w(x))' + c(x)w'(x) + d(x)w(x).$$

This problem can generally be reformulated in a weak (or variational) form. The weak form can be derived after multiplication of the differential equation by a suitable set of *test functions* and performing an integration upon the domain. Most often, the integration by parts

$$\int_I u'(x)v(x)\mathrm{d}x = -\int_I u(x)v'(x)\mathrm{d}x + u(b)v(b) - u(a)v(a),$$

is used with the aim of reducing the order of differentiation for the solution u.

As a result, we obtain a problem that reads

$$\begin{cases} \text{Find } u \in W \text{ s.t.} \\ \mathcal{A}(u,v) = \mathcal{F}(v), \quad \forall v \in V, \end{cases}$$

where W is the space of admissible solutions and V is the space of test functions. Both W and V can be assumed to be Hilbert spaces. $\mathcal{F} \in V'$ that accounts for the right hand side f as well as for possible non-homogeneous boundary terms. Finally, $\mathcal{A}(\cdot,\cdot)$ is a bilinear form corresponding to the differential operator L.

Remark 7.1. *The boundary conditions of u can be enforced directly in the definition of W (the case of the so-called essential boundary conditions). Otherwise, they can be achieved indirectly through a suitable choice of the bilinear form $\mathcal{A}$ as well as the functional $\mathcal{F}$ (natural boundary conditions).*

Remark 7.2. *We suppose that $W = V$.*

To the operator L we may associate the following bilinear form

$$a(w,v) := \int_I \left[a(x)w'(x)v'(x) - b(x)w(x)v'(x) + c(x)w'(x)v(x) + d(x)w(x)v(x)\right] \mathrm{d}x.$$

Example 7.1. *Homogeneous Dirichlet problem*

$$\begin{cases} Lu(x) = f(x), \ x \in I, \\ u(a) = 0, \ u(b) = 0. \end{cases}$$

Let $V = H_0^1(I)$, $\mathcal{A}(u,v) = a(u,v)$, $\mathcal{F}(v) = \int_I fv\mathrm{d}x$. We have the weak form:

$$\begin{cases} \textit{Find } u \in V \textit{s.t.} \\ a(u,v) = \mathcal{F}(v), \quad \forall v \in V. \end{cases}$$

Example 7.2. *Neumann problem*

$$\begin{cases} Lu(x) = f(x), \ x \in I, \\ u'(a) = g_a, \ u'(b) = g_b. \end{cases}$$

We consider the general problem

$$\begin{cases} Lu(x) = f(x), \ x \in I, \\ \alpha_0 u(a) + \beta_0 u'(a) = \gamma_0, \\ \alpha_1 u(b) + \beta_1 u'(b) = \gamma_1. \end{cases}$$

7.2 Finite Element Spaces

$$X_h^k := \{v_h \in C^0(\bar{I}) : v_h|_K \in \mathbb{P}_k \ \ \forall K \in \mathcal{T}_h\}$$

7.3 Finite Element Discretization

7.4 Boundary Treatment

7.5 Finite Element Method

References

[Evans (2010)] Evans L C. Partial differential equations[M]. American Mathematical Society, Second Edition, 2010.