Exercise 2.14. *Consider the elliptic problem*

$$-u_{xx} + u_x + u = f, \quad \forall x \in (a,b),$$
$$u(a) = u(b) = 0,$$

and its finite difference schema

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \frac{u_{i+1} - u_{i-1}}{2h} + u_i = f_i, \quad \forall i = 1, \cdots, N-1, \qquad (1)$$
$$u_0 = u_N = 0,$$

in an uniform mesh $\{x_i\}_{i=0}^N$*,* $x_i = a + ih$*,* $h = (b-a)/N$*.*
1) Derive an estimate for the truncation error;
2) Establish an a priori estimate for $\|u_h\|_1$*;*
3) Prove the existence and uniqueness of the solution of the finite difference schema;
4) Derive an error estimate for $\|e_h\|_1$*, where* $e_i = u(x_i) - u_i$*.*

Solution. 1). Let the operator $Lu = -u_{xx} + u_x + u$ and the discrete operator L_h on $\{u_i\}_{i=1}^{N-1}$ as

$$L_h u_i = -\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \frac{u_{i+1} - u_{i-1}}{2h} + u_i.$$

Then the truncation error $R_i = L_h[u(x_i)] - [Lu](x_i)$. By the Tylor development

$$u(x_{i+1}) = u(x_i) + hu'(x_i) + \frac{h^2}{2}u''(x_i) + \frac{h^3}{3!}u^{(3)}(x_i) + \frac{h^4}{4!}u^{(4)}(\xi_i), \text{ for some } \xi_i \in (x_i, x_{i+1}),$$

$$u(x_{i-1}) = u(x_i) - hu'(x_i) + \frac{h^2}{2}u''(x_i) - \frac{h^3}{3!}u^{(3)}(x_i) + \frac{h^4}{4!}u^{(4)}(\eta_i), \text{ for some } \eta_i \in (x_{i-1}, x_i),$$

we obtain that $R_i = O(h^2)$ as $h \to 0$ for $i = 1, \cdots, N-1$.
2). Note that $L_h u_i = -\left((u_i)_{\bar{x}}\right)_{\hat{x}} + \frac{1}{2}\left((u_i)_{\bar{x}} + (u_i)_x\right) + u_i$, then multiplying both sides of the finite difference schema $L_h u_i = f_i$ by $u_i h_i$ yields

$$-\left((u_i)_{\bar{x}}\right)_{\hat{x}} u_i h_i + \frac{1}{2}\left((u_i)_{\bar{x}} + (u_i)_x\right) u_i h + u_i^2 h = f_i u_i h_i, \quad \forall i = 1, \cdots, N-1.$$

Summing in i gives

$$-\left(((u_h)_{\bar{x}})_{\hat{x}}, u_h\right)_{I_h} + \frac{1}{2}\left((u_h)_{\bar{x}}, u_h\right)_{I_h} + \frac{1}{2}\left((u_h)_x, u_h\right)_{I_h} + (u_h, u_h)_{I_h} = (f_h, u_h)_{I_h}.$$

In virtue of discrete integral by parts (4), discrete Green formula (5) and the fact that $u_0 = u_N = 0$, we have

$$-\left(((u_h)_{\bar{x}})_{\hat{x}}, u_h\right)_{I_h} = \left((u_h)_{\bar{x}}, (u_h)_{\bar{x}}\right)_{I_h^+}, \quad \left((u_h)_{\bar{x}}, u_h\right)_{I_h} = -\left((u_h)_x, u_h\right)_{I_h}.$$

Thus

$$\left((u_h)_{\bar{x}}, (u_h)_{\bar{x}}\right)_{I_h^+} + (u_h, u_h)_{I_h} = (f_h, u_h)_{I_h}.$$

Using the fact that $u_0 = u_N = 0$, it is equivalent to

$$\left((u_h)_{\bar{x}}, (u_h)_{\bar{x}}\right)_{I_h^+} + (u_h, u_h)_{\bar{I}_h} = (f_h, u_h)_{\bar{I}_h}.$$

By the definition of the discrete inner norm (3), the left-hand side of the above formula is $\|u_h\|_1^2$. By the discrete Cauchy-Schwarz inequality (6), and the discrete Poincaré inequality (7): $\|u_h\|_0 \leqslant C|u_h|_1 \leqslant C\|u_h\|_1$, we have

$$\|u_h\|_1^2 \leqslant \|f_h\|_0 \|u_h\|_0 \leqslant C\|f_h\|_0 \|u_h\|_1 \implies \|u_h\|_1 \leqslant C\|f_h\|_0.$$

3). The finite difference schema is equivalent to solve the linear system:

$$\mathbf{D}\mathbf{u} = \mathbf{f},$$

where $\mathbf{u} = [u_1, \cdots, u_{N-1}]^{\mathrm{T}}$, $\mathbf{f} = [f_1, \cdots, f_{N-1}]^{\mathrm{T}}$ and

$$\mathbf{D} = \begin{bmatrix} 1+\frac{2}{h^2} & -\frac{1}{h^2}+\frac{1}{2h} & & & \\ -\frac{1}{h^2}-\frac{1}{2h} & 1+\frac{2}{h^2} & -\frac{1}{h^2}+\frac{1}{2h} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{1}{h^2}-\frac{1}{2h} & 1+\frac{2}{h^2} & -\frac{1}{h^2}+\frac{1}{2h} \\ & & & -\frac{1}{h^2}-\frac{1}{2h} & 1+\frac{2}{h^2} \end{bmatrix}.$$

Note that $\mathbf{D}$ is strictly diagonally dominant, i.e.,

$$\sum_{j=1, j\neq i}^{N-1} |D_{ij}| < |D_{ii}|, \quad i = 1, \cdots, N-1.$$

Then $\mathbf{D}$ is nonsingular, which leads to the existence and uniqueness of the solution of the finite difference schema.

4). It is obvious that

$$\begin{cases} L_h e_i = R_i, \ i = 1, \cdots, N-1, \\ e_0 = e_N = 0. \end{cases}$$

By 1) and 2) we have $\|e_h\|_1 \leqslant C\|R_h\|_0 = O(h^2)$ as $h \to 0$. $\square$

Appendix: Notations for Discrete Representation

Let $I = [a, b]$. We define the discrete grid points as

$$a = x_0 < x_1 < \cdots < x_N = b.$$

We introduce the following sets:

$$I_h = \{x_1, \cdots, x_{N-1}\},\ \bar{I}_h = \{x_0, x_1, \cdots, x_N\},\ I_h^+ = \{x_1, \cdots, x_N\}.$$

The grid spacing is defined as

$$h_i = x_i - x_{i-1}, \quad i = 1, \cdots, N.$$

Additionally, we define the averaged grid spacing:

$$\begin{aligned} \bar{h}_i &= \frac{1}{2}(h_i + h_{i+1}),\ i = 1, \cdots, N-1, \\ \bar{h}_0 &= \frac{1}{2}h_1, \quad \bar{h}_N = \frac{1}{2}h_N. \end{aligned}$$

A discrete function defined on $\bar{I}_h$ is denoted as

$$v_h = \{v_0, v_1, \cdots, v_N\}.$$

We define the following difference operators:

$$\begin{aligned} (v_i)_{\bar{x}} &:= v_{i,\bar{x}} := \frac{v_i - v_{i-1}}{h_i},\ i = 1, \cdots, N, \\ (v_i)_x &:= v_{i,x} := \frac{v_{i+1} - v_i}{h_{i+1}},\ i = 0, \cdots, N-1, \\ (v_i)_{\hat{x}} &:= v_{i,\hat{x}} := \frac{v_{i+1} - v_i}{\bar{h}_i},\ i = 0, \cdots, N-1. \end{aligned}$$

The discrete inner products are given by

$$(u_h, v_h)_{I_h} = \sum_{i=1}^{N-1} u_i v_i \bar{h}_i,\ (u_h, v_h)_{\bar{I}_h} = \sum_{i=0}^{N} u_i v_i \bar{h}_i,\ (u_h, v_h)_{I_h^+} = \sum_{i=1}^{N} u_i v_i h_i. \tag{2}$$

We define the discrete norms as follows:

$$\begin{aligned} &\|v_h\|_c := \max_{\bar{I}_h} |v_i|,\ \|v_h\|_0 := (v_h, v_h)_{\bar{I}_h}^{1/2}, \\ &|v_h|_1 := ((v_h)_{\bar{x}}, (v_h)_{\bar{x}})_{I_h^+}^{1/2},\ \|v_h\|_1^2 = \|v_h\|_0^2 + |v_h|_1^2. \end{aligned} \tag{3}$$

The discrete integral by parts:

$$\sum_{i=m+1}^{n} v_i (w_i)_{\bar{x}} h_i = -\sum_{i=m}^{n-1} (v_i)_x w_i h_{i+1} + v_n w_n - v_m w_m,\ \text{for some } 0 \leqslant m < n \leqslant N. \tag{4}$$

The discrete Green formula:

$$\sum_{i=m+1}^{n-1} ((u_i)_{\bar{x}})_{\hat{x}}\, v_i \bar{h}_i = -\sum_{i=m+1}^{n} (u_i)_{\bar{x}} (v_i)_{\bar{x}} h_i + (u_n)_{\bar{x}} v_n - (u_m)_x v_m,\ \text{for some } 0 \leqslant m < n \leqslant N. \tag{5}$$

The discrete Cauchy-Schwarz inequality states that

$$|(u_h, v_h)_{\bar{I}_h}| \leqslant (u_h, u_h)_{\bar{I}_h}^{1/2} (v_h, v_h)_{\bar{I}_h}^{1/2}. \tag{6}$$

If $v_0 = 0$ (or $v_N = 0$ or $v_0 = v_N = 0$), the discrete Poincaré inequality holds:

$$\|v_h\|_c \leqslant C|v_h|_1, \quad \|v_h\|_0 \leqslant C|v_h|_1, \tag{7}$$

where C is a constant depending only on a and b.