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A Series of Comprehensive Studies in Mathematics

George G. Lorentz
Manfred v. Golitschek
Yuly Makovoz

Constructive Approximation

Advanced Problems

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A Series of Comprehensive Studies in Mathematics

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George G. Lorentz
Manfred v. Golitschek
Yuly Makovoz

Constructive Approximation

Advanced Problems

With 10 Figures

Springer

George G. Lorentz
Department of Mathematics
The University of Texas
at Austin
Austin, TX 78712-1082, USA
combs@math.utexas.edu

Yuly Makovoz
Department of Mathematics
University of Massachusetts
at Lowell
Lowell, MA 01854, USA
makovozy@woods.uml.edu

Manfred v. Golitschek
Universität Würzburg
Institut für Angewandte
Mathematik und Statistik
D-97074 Würzburg, Germany
goli@mathematik.uni-wuerzburg.de

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Preface

In the last 30 years, Approximation Theory has undergone wonderful development, with many new theories appearing in this short interval. This book has its origin in the wish to adequately describe this development, in particular, to rewrite the short 1966 book of G. G. Lorentz, "Approximation of Functions." Soon after 1980, R. A. DeVore and Lorentz joined forces for this purpose. The outcome has been their "Constructive Approximation" (1993), volume 303 of this series. References to this book are given as, for example [**CA**, p. 201].

Later, M. v. Golitschek and Y. Makovoz joined Lorentz to produce the present book, as a continuation of the first.

Completeness has not been our goal. In some of the theories, our exposition offers a selection of important, representative theorems, some other cases are treated more systematically. As in the first book, we treat only approximation of functions of one real variable. Thus, functions of several variables, complex approximation or interpolation are not treated, although complex variable methods appear often.

Most of the chapters of the present book can be read independently of each other. They fall into groups: Chapters 1–6 deal with polynomial and spline approximation – in some sense they continue the themes of [**CA**]. Chapters 7–10 contain a fairly complete theory of rational approximation. Chapters 12–14 treat widths and entropy of classes of functions. But even within the groups, chapters are more or less independent, except that it is advisable to read Chapter 3 before Chapter 4, while Chapter 7 is indispensable for Chapters 8 and 10, and Chapter 13 for Chapter 14. Most of the information about Banach function spaces needed in the two volumes of **CA** can be found in [**CA**, Chapter 2], in §7 of Chapter 1 of the present volume, and in the book of Bennett and Sharpley [B-1988]. We also provide a quick new look at some of the important approximation theorems: for polynomials in §7 of Chapter 1, for splines in §1 of Chapter 6.

Related branches of Analysis: Fourier Series, Orthogonal Polynomials, Potential Theory, Functional Analysis, even Number Theory are our allies. We use their methods; some of the needed results are collected for the reader in the four Appendices.

For the development of the Approximation Theory, one cannot be sufficiently thankful to the Russian (Soviet) mathematicians: to Chebyshev, A. A. Markov, Bernstein, Kolmogorov and others, who built its foundations.

At present Approximation Theory is popular worldwide, with the new theories of splines, of rational approximation, of wavelets.

We are very grateful to A. A. Pekarskii (Grodno, Belarus), who has prepared for us Chapter 10, which deals with complex methods in rational approximation. Our colleagues, Berens, R. A. Lorentz, Stahl, Erdélyi, Lubinsky, Totik have helped us with concrete problems. We are also indebted to Blatt, Buslaev, Chui, Jetter, Maiorov, Shechtman, Varga and others for useful advice. Margaret Combs at the Department of Mathematics, The University of Texas, has very ably typed many chapters of the book.

The book has an extensive bibliography, which can also serve as Author's Index. Each quoted journal article is followed by the number of page, where it is referred to in the text. There is also a Subject Index.

The authors would be grateful for any comments or proposals of corrections from the readers.

The Authors

Contents

Chapter 1. Problems of Polynomial Approximation 1

§ 1. Examples of Polynomials of Best Approximation 1
§ 2. Distribution of Alternation Points of Polynomials of Best Approximation 4
§ 3. Distribution of Zeros of Polynomials of Best Approximation 11
§ 4. Error of Approximation 20
§ 5. Approximation on $(-\infty, \infty)$ by Linear Combinations of Functions $(x-c)^{-1}$ 23
§ 6. Weighted Approximation by Polynomials on $(-\infty, \infty)$ 28
§ 7. Spaces of Approximation Theory 33
§ 8. Problems and Notes 37

Chapter 2. Approximation Problems with Constraints 39

§ 1. Introduction 39
§ 2. Growth Restrictions for the Coefficients 39
§ 3. Monotone Approximation 43
§ 4. Polynomials with Integral Coefficients 49
§ 5. Determination of the Characteristic Sets 59
§ 6. Markov-Type Inequalities 64
§ 7. The Inequality of Remez 73
§ 8. One-sided Approximation by Polynomials 76
§ 9. Problems 81
§ 10. Notes 82

Chapter 3. Incomplete Polynomials 85

§ 1. Incomplete Polynomials 85
§ 2. Incomplete Chebyshev Polynomials 89
§ 3. Incomplete Trigonometric Polynomials 92
§ 4. Sequences of Polynomials with Many Real Zeros 98
§ 5. Problems 104
§ 6. Notes 104

Chapter 4. Weighted Polynomials 105

§ 1. Essential Sets of Weighted Polynomials 105
§ 2. Weighted Chebyshev Polynomials 109
§ 3. The Equilibrium Measure 117
§ 4. Determination of Minimal Essential Sets 125
§ 5. Weierstrass Theorems and Oscillations 131
§ 6. Weierstrass Theorem for Freud Weights 134
§ 7. Problems 141
§ 8. Notes 141

Chapter 5. Wavelets and Orthogonal Expansions 145

§ 1. Multiresolutions and Wavelets 145
§ 2. Scaling Functions with a Monotone Majorant 151
§ 3. Periodization 156
§ 4. Polynomial Schauder Bases 160
§ 5. Orthonormal Polynomial Bases 164
§ 6. Problems and Notes 172

Chapter 6. Splines 175

§ 1. General Facts 175
§ 2. Splines of Best Approximation 181
§ 3. Periodic Splines 189
§ 4. Convergence of Some Spline Operators 196
§ 5. Notes 202

Chapter 7. Rational Approximation 205

§ 1. Introduction 205
§ 2. Best Rational Approximation 210
§ 3. Rational Approximation of $|x|$ 217
§ 4. Approximation of e^x on $[-1, 1]$ 221
§ 5. Rational Approximation of e^{-x} on $[0, \infty)$ 227
§ 6. Approximation of Classes of Functions 231
§ 7. Theorems of Popov 235
§ 8. Properties of the Operator of Best Rational Approximation in C and L_p 242
§ 9. Approximation by Rational Functions with Arbitrary Powers ... 248
§ 10. Problems 251
§ 11. Notes 252

Chapter 8. Stahl's Theorem 255

§ 1. Introduction and Main Result 255
§ 2. A Dirichlet Problem on $[1/2, 1/\rho_n]$ 256

§ 3. The Second Approach to the Dirichlet Problem ... 263
§ 4. Proof of Theorem 1.1 ... 271
§ 5. Notes ... 276

Chapter 9. Padé Approximation ... 277

§ 1. The Padé Table ... 277
§ 2. Convergence of the Rows of the Padé Table ... 282
§ 3. The Nuttall-Pommerenke Theorem ... 290
§ 4. Problems ... 296
§ 5. Notes ... 296

Chapter 10. Hardy Space Methods in Rational Approximation ... 299

§ 1. Bernstein-Type Inequalities for Rational Functions ... 300
§ 2. Uniform Rational Approximation in Hardy Spaces ... 308
§ 3. Approximation by Simple Functions ... 314
§ 4. The Jackson-Rusak Operator; Rational Approximation of Sums of Simple Functions ... 320
§ 5. Rational Approximation on $\mathbb{T}$ and on $[-1,1]$ ... 322
§ 6. Relations Between Spline and Rational Approximation in the Spaces L_p, $0 < p < \infty$ ... 332
§ 7. Problems ... 341
§ 8. Notes ... 341

Chapter 11. Müntz Polynomials ... 345

§ 1. Definitions and Simple Properties ... 345
§ 2. Müntz-Jackson Theorems ... 347
§ 3. An Inverse Müntz-Jackson Theorem ... 353
§ 4. The Index of Approximation ... 360
§ 5. Markov-Type Inequality for Müntz Polynomials ... 362
§ 6. Problems ... 365
§ 7. Notes ... 366

Chapter 12. Nonlinear Approximation ... 369

§ 1. Definitions and Simple Properties ... 369
§ 2. Varisolvent Families ... 371
§ 3. Exponential Sums ... 376
§ 4. Lower Bounds for Errors of Nonlinear Approximation ... 383
§ 5. Continuous Selections from Metric Projections ... 386
§ 6. Approximation in Banach Spaces: Suns and Chebyshev Sets ... 390
§ 7. Problems ... 395
§ 8. Notes ... 396

Chapter 13. Widths I 399

§ 1. Definitions and Basic Properties 399
§ 2. Relations Between Different Widths 407
§ 3. Widths of Cubes and Octahedra 410
§ 4. Widths in Hilbert Spaces 412
§ 5. Applications of Borsuk's Theorem 418
§ 6. Variational Problems and Spectral Functions 423
§ 7. Results of Buslaev and Tikhomirov 432
§ 8. Classes of Differentiable Functions on an Interval 441
§ 9. Classes of Analytic Functions 443
§ 10. Problems 445
§ 11. Notes 447

Chapter 14. Widths II: Weak Asymptotics for Widths of Lipschitz Balls, Random Approximants 449

§ 1. Introduction 449
§ 2. Discretization 451
§ 3. Weak Equivalences for Widths. Elementary Methods 453
§ 4. Distribution of Scalar Products of Unit Vectors 461
§ 5. Kashin's Theorems 465
§ 6. Gaussian Measures 469
§ 7. Linear Widths of Finite Dimensional Balls 472
§ 8. Linear Widths of the Lipschitz Classes 478
§ 9. Problems 481
§ 10. Notes 481

Chapter 15. Entropy 485

§ 1. Entropy and Capacity 485
§ 2. Elementary Estimates 489
§ 3. Linear Approximation and Entropy 492
§ 4. Relations Between Entropy and Widths 497
§ 5. Entropy of Classes of Analytic Functions 502
§ 6. The Birman-Solomyak Theorem 506
§ 7. Entropy Numbers of Operators 509
§ 8. Notes 514

Chapter 16. Convergence of Sequences of Operators 517

§ 1. Introduction 517
§ 2. Simple Necessary and Sufficient Conditions 518
§ 3. Geometric Properties of Dominating Sets 523
§ 4. Strict Dominating Systems; Minimal Systems; Examples 528
§ 5. Shadows of Sets of Continuous Functions 536
§ 6. Shadows in Banach Function Spaces 541

§ 7. Positive Contractions 545
§ 8. Contractions 547
§ 9. Notes 551

Chapter 17. Representation of Functions by Superpositions 553

§ 1. The Theorems of Kolmogorov 553
§ 2. Proof of the Theorems 555
§ 3. Functions Not Representable by Superpositions 559
§ 4. Linear Superpositions 562
§ 5. Notes 564

Appendix 1. Theorems of Borsuk and of Brunn-Minkowski 567

§ 1. Borsuk's Theorem 567
§ 2. The Brunn-Minkowski Inequality 572

Appendix 2. Estimates of Some Elliptic Integrals 575

Appendix 3. Hardy Spaces and Blaschke Products 581

§ 1. Hardy Spaces 581
§ 2. Conjugate Functions and Cauchy Integrals 584
§ 3. Atomic Decompositions in Hardy Spaces 587
§ 4. Blaschke Products 591

Appendix 4. Potential Theory and Logarithmic Capacity 595

§ 1. Logarithmic Potentials 595
§ 2. Equilibrium Distribution and Logarithmic Capacity 603
§ 3. The Dirichlet Problem and Green's Function 614
§ 4. Balayage Methods 615

Bibliography 621

Author Index 641

Subject Index 647

Chapter 1. Problems of Polynomial Approximation

This chapter contains a discussion of some important problems of approximation, mainly by algebraic polynomials. We begin with properties of polynomials of best approximation: some examples in §1, distribution of their alternation points on the interval in §2, distribution of their zeros in the complex plane in §3. In §4, as an exception, we discuss approximation by entire functions, and the error of approximation in Banach spaces. In §§5–6, we give a solution of a problem of Bernstein, about the weighted polynomial approximation on $(-\infty, \infty)$. Spaces for approximation problems are found in §7.

§ 1. Examples of Polynomials of Best Approximation

Polynomials of best uniform approximation on the circle $\mathbb{T}$ or an interval $[a, b]$ are described by the theorems of Chebyshev (see, for example, [**CA**, Theorem A, p.58, or Theorem 5.1, p.74]). Only in exceptional cases can they be given explicitly. Here are some examples.

Let $n_1, n_2, \ldots$ be odd integers ≥ 3, we write $N_k = n_1 n_2 \cdots n_k$.

Theorem 1.1. *Let f be a continuous function on $\mathbb{T}$, with the Fourier series of one of the forms*

$$a_0 + \sum_{k=1}^{\infty} a_k \cos N_k t \quad , \quad \sum_{k=1}^{\infty} b_k \sin N_k t \ . \tag{1.1}$$

Then the trigonometric polynomials of best approximation to f are precisely the partial sums of (1.1). *(In particular, the series converge uniformly.)*

Proof. Consider for example, the first series (1.1). The statement of the theorem asserts that the partial sum $S_{k-1}(t) = \sum_{j=0}^{k-1} a_j \cos N_j t$ is the best approximation to f, among all polynomials of degree $\leq n$, for each $n = N_{k-1}, \ldots, N_k - 1$. The difference $R_k = f - S_{k-1}$ has the following properties. Its Fourier series is $\sum_{j=k}^{\infty} a_j \cos N_j t$, and by Fejér's theorem, R_k lies in the closed span of these cosines. Hence R_k has period $2\pi/N_k$. In addition, since all N_j are odd, R_k is odd about the center $c := c_\ell$ of each of the intervals

$$I_\ell = \left[\frac{\pi}{2N_k} + \frac{2\pi\ell}{N_k} \quad , \quad \frac{\pi}{2N_k} + \frac{2\pi(\ell+1)}{N_k}\right] \quad , \quad \ell = 0, \ldots, N_k - 1 \ ,$$

that is, it satisfies $R_k(c-t) = -R_k(c+t)$. It follows that the absolute maximum M of R_k on $\mathbb{T}$ and its absolute minimum $-M$ are taken on each I_ℓ at points symmetric about c_ℓ. We get enough alternation points to apply Chebyshev's theorem. Similarly for the second series (1.1). □

If the coefficients a_k in the first series (1.1) are of the same sign, we can obtain an explicit formula for the error of approximation $E_n(f)$. We explain this for the algebraic case. Let $a_k \geq 0$, $\sum a_k < +\infty$ and let f on $[-1,1]$ be given by

$$f = \sum_{j=1}^{\infty} a_j C_{N_j} , \tag{1.2}$$

where C_n are the Chebyshev polynomials [**CA**, §6, Chapter 3]. By the standard substitution $x = \cos t$ and Theorem 1.1, $S_n(x) = \sum_{j=1}^{k} a_j C_{N_j}(x)$, $N_k \leq n < N_{k+1}$ is the polynomial of best approximation to f from $\mathcal{P}_n$ and

$$E_n(f) = f(1) - S_n(1) = \sum_{j=k+1}^{\infty} a_j , \qquad N_k \leq n < N_{k+1} , \ k = 1, 2, \ldots . \tag{1.3}$$

Here is another concrete example, known already to Chebyshev.

Theorem 1.2. *Let $f(x) = (x-a)^{-1}$, $x \in [-1,1]$, where $a > 1$. Then for* $c := a - \sqrt{a^2 - 1} < 1$,

$$E_n\left((x-a)^{-1}\right) = \frac{4c^{n+2}}{(1-c^2)^2} . \tag{1.4}$$

Proof. The formula $x = \frac{1}{2}(w + w^{-1})$ defines a one to one map of the complex x-plane split by $[-1,1]$ onto the disk $|w| < 1$. To each $x \in [-1,1]$ correspond two values of w on $|w| = 1$, related by $w_1 = w_2^{-1}$. (See [**CA**, §2, Chapter 4].) Let $0 < c < 1$ be given by $a = \frac{1}{2}(c + c^{-1})$, that is, by $c = a - \sqrt{a^2 - 1}$. Then

$$\Phi(x) = \frac{M}{2}\left(w^n \frac{c-w}{1-cw} + w^{-n}\frac{1-cw}{c-w}\right) \tag{1.5}$$

defines a function on $\mathbb{C}$. We note that $w^k + w^{-k}, k = 0, 1, \ldots$ is a polynomial in x of degree k and that

$$(1+c^2)(1-\frac{x}{a}) = (\frac{1}{w} - c)(w - c) .$$

Therefore

$$\begin{aligned} \Phi(x) &= -\frac{M}{2}\left\{w^{n-1}\frac{w-c}{w^{-1}-c} + w^{1-n}\frac{w^{-1}-c}{w-c}\right\} \\ &= -\frac{M}{2}\left\{w^{n-1}(w-c)^2 + w^{1-n}(w^{-1}-c)^2\right\}\left(1-\frac{x}{a}\right)^{-1}(1+c^2)^{-1} . \end{aligned} \tag{1.6}$$

We see that $\Phi(x)$ has the form

$$\Phi(x) = \frac{A}{x-a} - P_n(x) ,$$

where P_n is a polynomial of degree n with real coefficients. Since

$$A = \lim_{x\to a}(x-a)\Phi(x) = \frac{M}{2}\lim_{w\to c}\frac{(w-c)(cw-1)}{2cw}w^{-n}\frac{1-cw}{c-w} = \frac{M(1-c^2)^2}{4c^{n+2}} ,$$

we select $M = 4c^{n+2}(1-c^2)^{-2}$, and have then

$$\Phi(x) = \frac{1}{x-a} - P_n(x) .$$

As w moves on the upper semicircle $|w| = 1$ counterclockwise, x moves on $[-1,1]$ from 1 to -1. By (1.5), $\Phi(x) = \frac{M}{2}(\Psi(w)+\Psi(w)^{-1})$, where $\Psi(w) = w^n(c-w)(1-cw)^{-1}$. Since $|\Psi(w)| = 1$ on $|w| = 1$, we have $|\Phi(x)| \le M$, $|w| = 1$. The function Ψ has $n+1$ zeros inside $|w| = 1$, and because of symmetry, $\arg\Psi(w)$ changes from π to $(n+2)\pi$ on the upper semi-circle. If $\arg\Psi(w) = k\pi$, $k = 1,\dots,n+2$, then $\Phi(x) = M$ or $\Phi(x) = -M$ for even or for odd k, respectively. By Chebyshev's theorem on $[-1,1]$, P_n is the polynomial of best approximation for $(x-a)^{-1}$, and the error of approximation is M. □

Let P_n, P_{n+1} be two polynomials of best approximation to $f \in C[-1,1]$, and let $P_n \ne P_{n+1}$. Then:

(1.7) $Q := P_n - P_{n+1}$ has $n+1$ distinct zeros in the open interval $(-1,1)$.

Indeed, let $-1 \le x_1 < \cdots < x_{n+2} \le 1$ be $n+2$ alternation points for P_n (from [**CA**, Theorem 5.1, p.74]). If for instance $f(x_j) - P_n(x_j) > 0$, then

$$f(x_j) - P_n(x_j) = \|f - P_n\| > \|f - P_{n+1}\| \ge f(x_j) - P_{n+1}(x_j) , \tag{1.8}$$

so that $Q(x_j) < 0$. Similarly $Q(x_{j+1}) > 0$. Thus, Q changes sign on each of the intervals $[x_j, x_{j+1}]$.

Can it happen that all polynomials of best approximation to $f \in C[-1,1] \setminus \mathcal{P}$ have a common zero of high multiplicity p? This is impossible even for $p = 2$ – it would contradict (1.7). However, this phenomenon can occur infinitely often.

There is a sequence $p_n \to \infty$, a function $f \in C(\mathbb{T})$ and a point c with the property that for infinitely many n, the best approximation T_n to f has a zero of multiplicity p_n at c. According to Zeller, this may be established as follows. Using the notation of Theorem 1.1, we put

$$f(t) = \sum_{k=1}^{\infty} a_k \cos^{q_k} N_k t \quad , \qquad \sum |a_k| < \infty \tag{1.9}$$

where q_k are odd positive integers which tend to infinity. The partial sum $S_k(t) := \sum_{i=1}^{k} a_i \cos^{q_i} N_i t$ is a polynomial of degree $N_k q_k$. If

$$N_k q_k + 2 \le N_{k+1} \quad , \qquad k = 1, 2, \dots \; , \tag{1.10}$$

the proof of Theorem 1.1 still applies and we get that S_k is the best approximation to f among all polynomials of degree $\le N_k q_k$. Thus one can take $p_n = q_k$ when $n = N_k q_k$ and c to be the center of any interval I_ℓ.

It is more difficult to find examples for a specific sequence p_n. It is known that one can achieve, in the above statement, for infinitely many n multiplicities $p_n \ge C \log n$ of the zero c (Lorentz [1978]).

§ 2. Distribution of Alternation Points of Polynomials of Best Approximation

Let $f \in C(\mathbb{T})$ be a function that is not a trigonometric polynomial, let for some n, $T_n^* \in \mathcal{T}_n$ be the trigonometric polynomial of best uniform approximation of f. By Chebyshev's theorem [**CA**, Theorem 5.1, p.74] there exists a group of $N \ge 2n+2$ *alternation points*:

$$-\pi \le t_1 < \cdots < t_N < \pi \quad , \quad t_{N+1} := t_1 + 2\pi \; . \tag{2.1}$$

They are characterized by the equations $|f(t_k) - T_n^*(t_k)| = \|f - T_n^*\|$, $k = 1, \dots, N$, with alternating signs of the differences $f(t_k) - T_n^*(t_k)$. Obviously, there is a group with the largest possible $N = N(n)$. This $N(n)$ is necessarily even. If $E_{n+1}(f) < E_n(f)$, then $N(n) = 2n+2$, otherwise $N(n) \ge 2n+4$. How are the points t_k distributed on $\mathbb{T}$? Perhaps, uniformly? Are there some irregularities? Similar questions apply to algebraic polynomials of best approximation P_n^* to $f \in C[-1,1]$ and their groups of $n+2$ alternation points. It proves that for a given n, regularity of the distribution of the points (2.1) can be guaranteed only if *the difference $E_{n-1}(f) - E_n(f)$ is not too small.* And this happens for many n. In fact, *if $f \in C(\mathbb{T}) \setminus \mathcal{T}$, and if $\gamma > 1$, then for infinitely many n,*

$$E_n(f) - E_{n+1}(f) \ge (n+1)^{-\gamma} E_n(f) \; . \tag{2.2}$$

Indeed, since $E_n(f) \to 0$, the infinite product $\prod\{E_{n+1}(f)/E_n(f)\}$ diverges to zero, consequently the series $\sum\{1 - E_{n+1}(f)/E_n(f)\}$ diverges. Infinitely often, the n-th term of the series is $\ge (n+1)^{-\gamma}$, and we obtain (2.2).

In what follows, without attempting completeness, we give results dealing with the regularity of distribution (Theorems 2.2, 2.4) as well as of its irregularity (Theorem 2.7).

One way to measure the distribution of points (2.1), with $N = 2n+2$, is by examining the quantity

$$\Delta_n(f) = \max_{1 \le k \le 2n+2} (t_{k+1} - t_k) \; .$$

Lemma 2.1. *For a polynomial $T_n \in \mathcal{T}_n$, let there exist $2n$ points*

$$-\pi \le t_1 < \cdots < t_{2n} < \pi \ , \qquad t_{2n+1} := t_1 + 2\pi$$

at which T_n alternates in sign: $T_n(t_k)T_n(t_{k+1}) < 0$, $k = 1, \ldots, 2n$. Then

$$\Delta < 4\sqrt{2}\left\{\left(\frac{6\|T_n\|}{m}\right)^{1/n} - 1\right\} , \tag{2.3}$$

where $\Delta = \max(t_{k+1} - t_k)$ and $m = \min|T_n(t_k)|$.

Proof. Let $\beta \in (0, \pi/2)$ be defined by $\tan(\beta/2) = \Delta/8$; this implies $0 < \beta < \Delta/4$. We first show the existence of a polynomial $Q \in \mathcal{T}_n$ with the properties

$$|Q(t)| \le 1 \quad , \quad \beta \le |t| \le \pi \ , \tag{2.4}$$

$$Q(0) = \|Q\| \ge \frac{1}{2}\left(1 + \sqrt{2}\tan\frac{\beta}{2}\right)^n \ . \tag{2.5}$$

We take $Q(t) = C_n\left(\cos^2\frac{t}{2}/\cos^2\frac{\beta}{2}\right)$, where C_n is the Chebyshev polynomial. We have (2.4), and, since for $x > 1$,

$$C_n(x) \ge \frac{1}{2}\left(x + \sqrt{x^2 - 1}\right)^n \ge \frac{1}{2}\left(1 + \sqrt{2(x-1)}\right)^n \ ,$$

we have

$$Q(0) = C_n\left(\cos^{-2}\frac{\beta}{2}\right) \ge \frac{1}{2}\left(1 + \sqrt{2}\tan\frac{\beta}{2}\right)^n \ .$$

Each of the intervals (t_k, t_{k+1}), $k = 1, \ldots, 2n$ contains exactly one zero of T_n. We select a j with $\Delta = t_{j+1} - t_j$, then $[t_j, t_{j+1}]$ contains a subinterval I of length 2β, on which T_n does not vanish. By replacing $T_n(t)$ by $T_n(-t)$ if necessary, we can assume that $T_n(t_j) < 0$ and (by translation) that $x_j = -\beta$ is the only zero of T_n in $[t_j, t_{j+1}]$. We have $t_j < -\beta$ and $t_{j+1} > \beta$, and we choose I so that $0 \in I$.

In this situation, we prove that

$$\frac{m}{2}Q(0) \le \frac{m}{2} + T_n(0) \ .$$

Indeed, if this is not true, then since $Q(0) > 0$, there is a λ, $0 < \lambda < m/2$ with $\lambda Q(0) = \frac{m}{2} + T_n(0)$. Let $S_n \in \mathcal{T}_n$ be defined by $S_n := \frac{m}{2} + T_n - \lambda Q$. We have $S_n(0) = 0$ and $S_n(x_j) > 0$, moreover sign $T_n(t_k) =$ sign $S_n(t_k)$ for all k. The polynomials S_n have also exactly one zero in each interval (t_k, t_{k+1}). As $S_n(t_j) < 0$, $S_n(x_j) > 0$, we will have $S_n(t) > 0$ on I, in contradiction to $S_n(0) = 0$. This proves our inequality. From it we deduce

$$Q(0) \le 1 + \frac{2\|T_n\|}{m} \le \frac{3\|T_n\|}{m} \ ,$$

and (2.3) follows immediately. □

Theorem 2.2 (i) (Tashev [1981]). *For each function $f \in C(\mathbb{T})$ that is not a polynomial, and for each selection of $2n+2$ alternation points for the polynomial of best approximation for f in $\mathcal{T}_n$,*

$$\liminf_{n\to\infty} \frac{n\Delta_n(f)}{\log n} < 6 . \tag{2.6}$$

(ii) *If in addition, f is analytic, then*

$$\liminf_{n\to\infty} (n\Delta_n(f)) < +\infty . \tag{2.7}$$

Proof. (i) Let T_n^*, $n = 1, 2, \ldots$ be the polynomials of best approximation to f. For n satisfying (2.2), with n replaced by $n-1$, we apply Lemma 2.1 to the polynomial $T := T_{n-1}^* - T_n^*$ and the alternation points t_k of T_{n-1}^*. Then $\|T\| \le 2E_{n-1}(f)$, while $m = \min|T(t_k)| \ge E_{n-1}(f) - E_n(f) \ge n^{-\gamma}E_{n-1}(f)$. Substituting this into (2.3), we get, with strong equivalences,

$$\Delta \le 4\sqrt{2}\{(12n^\gamma)^{1/n} - 1\} \approx 4\sqrt{2}\,\frac{1}{n}\,\log(12n^\gamma) \approx 4\sqrt{2}\,\gamma\,\frac{\log n}{n} .$$

For γ close to 1, the right-hand side is $< 6\log n/n$.

(ii) Here we have $E_n(f) \le \rho^n$ for large n and some $0 < \rho < 1$. Let $\rho < \rho_1 < 1$. Then we must have $E_n(f) \le \rho_1 E_{n-1}(f)$, that is,

$$E_{n-1}(f) - E_n(f) \ge (1-\rho_1)E_{n-1}(f)$$

for infinitely many n. For these n we apply Lemma 2.1 to $T := T_{n-1}^* - T_n^*$ in the same way as in (i) and obtain

$$\Delta \le 4\sqrt{2}\left\{\left(\frac{12}{1-\rho_1}\right)^{1/n} - 1\right\} \le C\frac{1}{n} . \qquad \square$$

Another main result of this section will be Theorem 2.4 of Kadec [1960]. This theorem asserts that at least for the subsequence of n's which satisfy (2.2), the alternation points x_k are approximately equally distributed.

Let S be a trigonometric polynomial of degree $n+1$ which has $2n+2$ local extrema at points $-\pi \le t_1 < \cdots < t_{2n+2} < \pi$, whose values $S(t_i)$, $i = 1, 2, \ldots, 2n+2$ alternate in sign and satisfy $|S(t_i)| \ge 1$, $i = 1, \ldots, 2n+2$. We want to estimate $M(S, I)$, the number of the points t_i which lie in an arbitrary interval $I \subset \mathbb{T}$. Let $a := \pi - |I|/2$. We take any $a_1 \in (0, a)$ and any integer $m, 1 \le m < n$ for which $(\cos a_1 - \cos a + 1)^m > \|S\|$ (assuming that such a_1, m exist).

Lemma 2.3. *Under the conditions described above*

$$M(S, I) \le \frac{|I|}{\pi}n + \frac{2}{\pi}a_1 m + \frac{2}{\pi}(a - a_1)n + 5 . \tag{2.8}$$

Proof. One may assume that $I = [-\pi, \pi) \setminus [-a, a]$, $a > 0$. We compare S with the trigonometric polynomial of degree n, $U(t) := \sin(n-m)t(1-\cos a + \cos t)^m$. We count the number of zeros of $S-U$. There are altogether $\leq 2n+2$ zeros. Since $|U(t)| < 1$ for $t \in I$ and $|S(t_i)| \geq 1$, $t_i \in I$, there are at least $M(S,I) - 1$ zeros of $S - U$ on I.

On $I_1 := (-a_1, a_1)$, we have $(1 - \cos a + \cos t)^n > L := (\cos a_1 - \cos a + 1)^m$ and hence the maxima of $|U(t)|$ on this interval are $> L$. There are $\geq |I_1|(n-m)/\pi - 1$ of these and the corresponding $U(t)$ alternate in sign. Since $|S(t)| < L$, $-\pi \leq t < \pi$, between any two maxima of $|U(t)|$ on I_1 we have a zero of $S - U$. It follows that

$$M(S,I) + |I_1|(n-m)/\pi \leq 2n + 5 \ .$$

Since $2\pi - |I_1| = |I| + 2(a - a_1)$, this gives (2.8). □

Now suppose that $f \in C(\mathbb{T})$ and that T_n^* is the trigonometric polynomial of best approximation to f of degree $\leq n$. We consider only n for which $E_{n+1}(f) < E_n(f)$. Then each group of alternation points (2.1) has $N = 2n+2$ members. Given an arbitrary interval I, let $A_n(f,I)$ stand for the number of alternation points which are in I.

Theorem 2.4. *For each $f \in C(\mathbb{T}) \setminus \mathcal{T}$ there are infinitely many integers n such that for each interval I,*

$$A_n(f,I) = \frac{|I|}{\pi} n + O(\sqrt{n \log n}) \ . \tag{2.9}$$

Proof. If (2.9) is true for some intervals I, it is true also for their translates, their disjoint unions and their complements. Therefore, it is enough to prove (2.9) for all I of the form $I = \mathbb{T} \setminus [-a, a]$, with $\frac{\pi}{4} \leq a \leq \frac{3\pi}{4}$. The complements $I' = [-a, a]$ of these I have length $\frac{\pi}{2} \leq |I'| \leq \frac{3\pi}{2}$. The inequality with $\leq$ instead of $=$ in (2.9) for all such I, when applied to the I', yields the opposite inequality $\geq$ for I. It suffices therefore to prove (2.9) with $\leq$ for the restricted class of I's.

From (2.2) we know that for infinitely many values of n we have $\lambda_n^{-1} := E_n(f) - E_{n+1}(f) \geq E_n(f) n^{-2}$. For any of these n we define the polynomial $S := \lambda_n(T_{n+1}^* - T_n^*)$. We have sign $S = \operatorname{sign}(f - T_n^*)$ at each alternation point of $f - T_n^*$. Hence, S has $2n+2$ local extrema; they alternate in sign and have absolute value ≥ 1. We take $a_1 := a - 6(\log n/n)^{1/2}$, $m := [\sqrt{n \log n}]$. Since $a \in [\pi/4, 3\pi/4]$ and $a_1 \to a$ as $n \to \infty$, we have for n sufficiently large, by Lagrange's formula

$$\cos a_1 - \cos a \geq \left(\sin \frac{\pi}{4}\right)(a - a_1) = 3\sqrt{2}(\log n/n)^{1/2}$$

and

$$\begin{aligned}\log(\cos a_1 - \cos a + 1)^m &\geq (n \log n)^{1/2} \log\left(1 + 3\sqrt{2}(\log n/n)^{1/2}\right) \\ &\geq (n \log n)^{1/2} 3(\log n/n)^{1/2} > \log(2n^2) \ ,\end{aligned}$$

whereas $\|S\| \le 2\lambda_n E_n(f) \le 2n^2$. Therefore the conditions of Lemma 2.3 are satisfied and from (2.8) we obtain $M(S,I) \le |I|n/\pi + O(\sqrt{n\log n})$. This completes the proof since $A_n(f,I) \le M(S,I)+2$. □

Remark. The remainder in (2.9) is not the best possible. In fact, Blatt [1992] proves (2.9) with the error $O(\log^2 n)$. The proof is much more difficult. It is not known whether this estimate can be improved.

Corollary 2.5. *We compare the alternation points* (2.1) *of T_n^* with the equidistant points $s_k^{(n)} := 2\pi k/n$, $k = 0,\dots,n$. For infinitely many n, $x_k^{(n)}$ is approximately $s_k^{(n)}$:*

$$\left|x_k^{(n)} - s_k^{(n)}\right| \le \text{const}\sqrt{\frac{\log n}{n}} \quad , \qquad k = 1,\dots,n\,. \tag{2.10}$$

Indeed, since (2.9) is valid uniformly for all subintervals I of $\mathbb{T}$, we can take $I = [0, x_k^{(n)}]$. On one hand, I contains $k = s_k^{(n)} n/(2\pi)$ points $x_j^{(n)}$. On the other hand, this number is $(x_k^{(n)}/2\pi)n + O(\sqrt{n\log n})$. □

Corollary 2.6. *A formula similar to* (2.9) *holds for algebraic polynomials P_n^* of best approximation to $f \in C[-1,1]$ if we use the measure $d\mu = (1-x^2)^{-1/2}\,dx$:*

$$A_n(f,I) = \frac{\mu I}{\pi} n + O(\sqrt{n\log n}) \quad , \qquad I \subset [-1,+1]\,. \tag{2.11}$$

The reason is that the map $x = \cos t$ transforms f into an even function $g(t) = f(\cos t)$, P_n^* into an even trigonometric polynomial $T_n^*(t) = P_n^*(\cos t)$, a set $e \subset [-1,+1]$ of measure μe into a set $e_1 \subset [0,\pi]$ of Lebesgue measure $m e_1 = \mu e$, and an alternation point of f and P_n^* into two alternation points of g, T_n^*, symmetric with respect to 0. □

In the statements (2.6) and (2.9), one cannot replace a subsequence of the $n = 1,2,\dots$ by all $n \to \infty$. For certain large n, the alternating points may be distributed in a very bizarre manner. In fact, (restricting ourselves to the algebraic case) we shall show that for some functions $f \in C[-1,1]$, and for properly chosen large n, the alternating sets $\Delta_n : -1 \le x_1^{(n)} < \cdots < x_{n+2}^{(n)} \le 1$ of f consist of almost arbitrarily chosen points $x_k^{(n)}$.

Theorem 2.7 (Lorentz; Saff and Totik). *Let $\Delta_n' : -1 \le y_1^{(n)} < \cdots < y_{n+2}^{(n)} \le 1$ be, for each $n = 1,2,\dots$, arbitrary distributions of points in $[-1,1]$, let $\varepsilon_k \to 0$ be arbitrary real numbers decreasing to zero. There exists then an entire function f with real values on $[-1,1]$, with the following property. For some increasing sequence of integers $n_k \to \infty$, all alternation sets $\Delta_{n_k} : -1 \le x_1^{(n_k)} < \cdots < x_{n_k+2}^{(n_k+2)} \le 1$ of f consist of n_k+2 points which satisfy*

$$|x_j^{(n_k)} - y_j^{(n_k)}| < \varepsilon_k \ , \qquad j = 1, \ldots, n_k + 2 \ . \tag{2.12}$$

This was proved by Lorentz [1984₂] for points $y_j^{(n)}$ clustering towards -1 and $+1$, by Saff and Totik [1989] in the general case. Our proof is modelled after the first paper, we replace the incomplete polynomials used there by the S of the following lemma.

Lemma 2.8. *Let* $\Delta_n' : -1 \le y_1 < \cdots < y_{n+2} \le 1$ *be given. Then:* (i) *the supremum*

$$K(\Delta_n') := \sup\{\|P\| : P \in \mathcal{P}_n \ , \ (-1)^j P(y_j) \le 1 \ , \ j = 1, \ldots, n+2\}$$

is finite, (ii) *if*

$$0 < \delta < \tfrac{1}{4} \min_j (y_{j+1} - y_j) \ , \tag{2.13}$$

then for all sufficiently large integers m there is a polynomial $S \in \mathcal{P}_m$ with the properties

(a) $\|S\| = 1$

(b) $(-1)^j S(y_j) \ge 1 - \delta, \quad j = 1, \ldots, n+2$

(c) $(-1)^j S(x) > 0, \quad x \in I^{(j)} := [y_j - \delta, y_j + \delta] \cap [-1, 1],$

(d) $|S(x)| < \delta, \quad x \in [-1, 1] \setminus \cup I^{(j)}.$

Proof. (i) Since the space $\mathcal{P}_n$ is $(n+1)$-dimensional, there exist real numbers $a_1, \ldots, a_{n+2}$, not all zero, so that

$$a_1 P(y_1) + \cdots + a_{n+2} P(y_{n+2}) = 0 \tag{2.14}$$

for all $P \in \mathcal{P}_n$. In particular, for $P(x) = P_j(x) := \prod_{k \ne j, j+1} (x - y_k)$, $j = 1, \ldots, n+1$, we get

$$a_j P_j(y_j) + a_{j+1} P_j(y_{j+1}) = 0 \ .$$

Since $P_j(y_j) P_j(y_{j+1}) > 0$, this implies that none of the a_j is zero, for otherwise they would all vanish. Consequently, $a_j a_{j+1} < 0$ for all j. We may assume that $a_j = (-1)^j b_j$, $b_j > 0$. From (2.14), for each polynomial of (i) and each $j = 1, \ldots, n+2$,

$$(-1)^j P(y_j) = -b_j^{-1} \sum_{k \ne j} b_k (-1)^k P(y_k) \ge -b_j^{-1} \sum_k b_k \ .$$

Together with $(-1)^j P(y_j) \le 1$, this gives $|P(y_j)| \le M$, with M independent of P and j. Thus, the polynomials P of (i) form a bounded set in the norm $\|P\|^* = \max_{1 \le j \le n+1} |P(y_j)|$, hence in any other norm.

(ii) Since the $I^{(j)}$ are disjoint, the polynomials $S := Q / \|Q\|$ with $m = 2p$ and

$$Q(x) = \sum_{j=1}^{n+2} (-1)^j \left(1 - \tfrac{1}{4}(x - y_j)^2\right)^p$$

for large p have the properties (a)–(d). □

Let $m =: m(\Delta'_n, \delta)$ be the smallest $m > n$ for which this is true.

Proof of Theorem 2.7. If the numbers n_i, δ_i, $i = 1, \dots, k-1$ are known, we put $K_k := K(\Delta'_{n_{k-1}})$ and select $\delta_k > 0$ so that

$$\delta_k < \varepsilon_k \ , \quad \delta_k < \delta_{k-1} \ , \quad \delta_k(2 + 3K_k) < \tfrac{1}{2} \ . \tag{2.15}$$

Then we take $n_k := m(\Delta'_{n_{k-1}}, \delta_k)$. This produces the polynomial $S_k \in \mathcal{P}_{n_k}$ and the intervals $I_k^{(j)} = [y_j^{(n_{k-1})} - \delta_k, y_j^{(n_{k-1})} + \delta_k]$, $j = 1, \dots, n_{k-1} + 2$ of Lemma 2.8(ii). The function f will be

$$f(x) := \sum_{i=1}^{\infty} a_i S_i(x) \ , \tag{2.16}$$

where $a_1 = 1$ and a_i are defined by induction,

$$0 < a_{i+1} < \tfrac{1}{2}\delta_i a_i \ , \qquad i = 1, 2, \dots \ .$$

Then $\sum_{i=k+1}^{\infty} a_i < \delta_k a_k$, $k = 1, 2, \dots$. The a_k we take decreasing so fast that f will be an entire function.

For fixed k and $n := n_{k-1}$ let $Q_n := \sum_{i=1}^{k-1} a_i S_i$; we write $y_j := y_j^{(n)}$ and $I^{(j)}$ for the intervals of the set Δ'_n.

Since $Q_n \in \mathcal{P}_n$, we have

$$E_n(f) \le \|f - Q_n\| \le \sum_{i=k}^{\infty} a_i \le (1 + \delta_k) a_k \ .$$

Comparing Q_n with the polynomial P_n^* of best approximation to f, we have

$$\begin{aligned}(-1)^j \big(f(y_j) - Q_n(y_j)\big) &\ge (-1)^j a_k S_k(y_j) - \sum_{i=k+1}^{\infty} a_i \\ &\ge (1 - 2\delta_k) a_k \ .\end{aligned}$$

The same computation yields

$$(-1)^j (f(x) - P_n^*(x)) \ge -\tfrac{1}{2} a_k \ , \qquad x \in I^{(j)} \ .$$

It follows that

$$(-1)^j \big(Q_n(y_j) - P_n^*(y_j)\big) \le E_n(f) - (1 - 2\delta_k) a_k \le 3\delta_k a_k \ , \qquad j = 1, \dots, n+2 \ .$$

From Lemma 2.8(i),

$$\|Q_n - P_n^*\| \le 3K_k \delta_k a_k \ .$$

We try to determine the alternation sets of f, P_n^*. If $x \in [-1,1] \setminus \cup I^{(j)}$,

$$\begin{aligned}|f(x) - P_n^*(x)| &\le |f(x) - Q_n(x)| + \|Q_n - P_n^*\| \\ &\le a_k |S_k(x)| + \sum_{i=k+1}^{\infty} a_i + 3K_k\delta_k a_k \\ &\le (2\delta_k + 3K_k\delta_k)a_k < \tfrac{1}{2}a_k \ .\end{aligned}$$

Similarly, for $x = y_j$, $j = 1, \ldots, n+2$,

$$\begin{aligned}(-1)^j\big(f(y_j) - P_n^*(y_j)\big) &\ge (-1)^j\big(f(y_j) - Q_n(y_j)\big) - \|Q_n - P_n^*\| \\ &\ge (-1)^j a_k S_k(y_j) - \sum_{i=k+1}^{\infty} a_i - 3K_k\delta_k a_k \\ &\ge a_k - (2\delta_k + 3K_k\delta_k)a_k > \tfrac{1}{2}a_k \ .\end{aligned}$$

Therefore, each alternation set Δ_{n_k} of f, P_n^* has exactly $n_k + 2$ points $-1 \le x_1^{(n_k)} < \cdots < x_{n_k+2}^{(n_k)} \le 1$ and

$$|x_j^{(n_k)} - y_j^{(n_k)}| < \delta_k < \varepsilon_k \ , \qquad j = 1, \ldots, n_k + 2 \ . \qquad \square$$

§ 3. Distribution of Zeros of Polynomials of Best Approximation

In this section, $P_n^* \in \mathcal{P}_n$ will stand for the polynomial of best uniform approximation for a function $f \in C(I)$, $I = [-1,1]$. Let E_ρ, $\rho > 1$ be the ellipse with foci $-1, 1$ and the sum of half-axes ρ (see [**CA**, p.100]). Its boundary ∂E_ρ is given by $z = \frac{1}{2}(w + w^{-1}), |w| = \rho$. The inversion of this equation is the function $w = z + \sqrt{z^2 - 1}$, which has two single valued branches in $\mathbb{C} \setminus I$. The branch with $w(\infty) = \infty$ has $\lim_{z\to\infty}(w/z) = 2$; it has $\sqrt{z^2-1} > 0$ for $z > 1$ and maps $\mathbb{C} \setminus I$ onto the outside of the unit circle. The second, with $w(\infty) = 0$, maps $\mathbb{C} \setminus I$ onto the inside of this circle. If nothing is said to the contrary, we shall always take the first branch.

By $\|f\|$, $\|f\|_A$, $\|f\|_\rho$ we denote the uniform norm of f on I, on $A \subset \mathbb{C}$, and on ∂E_ρ. Thus, Bernstein's inequality [**CA**, (2.8), p.101] reads

$$\|P_n\|_\rho \le \|P_n\|\rho^n \ . \tag{3.1}$$

We would like to study the distribution of (real and complex) zeros of the P_n^* in $\mathbb{C}$. The problem can be stated in much greater generality. Let $A \subset \mathbb{C}$ be compact and let A° be the interior of A. The fundamental theorem of Mergelyan (see Gaier, [A-1980]) asserts that if A has a simply connected complement, then each function $f \in C(A)$ that is analytic on A° is uniformly approximable by polynomials. In this situation, the polynomials P_n^* are defined for each $f \in C(A)$. What is the distribution of the zeros of the P_n^* in $\mathbb{C}$?

This has been answered by Blatt and Saff [1986], see also Blatt, Saff and Simkani [1988], in a development initiated by Borwein [1984]. They proved that if f has a singularity at a point of the boundary ∂A of A, then each point of this boundary is a limit point of the zeros of P_n^* (compare Theorem 3.3); moreover, there is some regularity in the distribution of zeros (Theorem 3.7). First results of this type go back to Jentzsch (see Landau-Gaier [B-1986]) who proved already in 1914 that the limit points of the zeros of the partial sums of a power series with radius of convergence ρ, $0 < \rho < \infty$, cover the circle $|z| = \rho$.

We shall restrict ourselves to the case $A = I := [-1, 1]$, which is a good illustration of the methods of Blatt and Saff; except for Lemma 3.2, we use the ideas of their proofs. In the more general case, one replaces the function $\log(z + \sqrt{z^2 - 1})$ of the present section by the Green function G of the set $A \subset \mathbb{C}$, the ellipses E_ρ by the level curves of $|G|$. Also the logarithmic capacity $\gamma(A)$ of A appears: in Theorem 3.1 one replaces the constant 2 by $\gamma(A)^{-1}$. (See also Notes.)

Returning to the case $A = I$, on I itself, there are usually not many zeros of P_n^*: for instance none at all, for large n, if f does not vanish on I. More can be said if $f \in C(I)$ is analytic inside an ellipse $E_\rho, \rho > 1$. *Then the P_n^* converge to f locally uniformly inside E_ρ.* Indeed, by [**CA**, Theorem 8.1, p.229] there are polynomials $P_n \in \mathcal{P}_n$ with the properties: $|P_n(x) - f(x)| \le C\rho_1^{-n}$, $x \in I$ for each ρ_1, $1 < \rho_1 < \rho$, and $P_n(z) \to f(z)$ inside E_ρ. The first inequality holds for the P_n^* instead of the P_n, and together with (3.1) applied to $P_n - P_n^*$, we obtain $|P_n(z) - P_n^*(z)| \le 2C(\rho_2/\rho_1)^n$, $z \in E_{\rho_2}$, $1 < \rho_2 < \rho_1$, so that $P_n^*(z) \to f(z)$ inside E_ρ. If f has N zeros on I, then by Rouché's theorem, this will also be the number of zeros of P_n^* in E_ρ provided n is large and ρ is close to 1. Thus, we shall discuss the zeros of P_n^* outside of $[-1, 1]$.

A simple generalization of (3.1) is the inequality for $P_n \in \mathcal{P}_n$,

$$\left| \frac{P_n(z)}{(z + \sqrt{z^2 - 1}\,)^n} \right| \le \|P_n\|_\rho \rho^{-n} \quad , \qquad z \notin E_\rho \ , \ \rho \ge 1 \ , \tag{3.2}$$

where $z + \sqrt{z^2 - 1}$ means the branch of this function that is infinite at $z = \infty$. For this branch, $|z + \sqrt{z^2 - 1}| = \rho$ on ∂E_ρ, $\rho > 1$. Indeed, the function under the absolute value in (3.2) is analytic outside of E_ρ, also at $z = \infty$; a reference to the maximum modulus principle establishes the inequality.

Making $z \to \infty$ in (3.2) we obtain for the polynomial $P_n(z) = a_n z^n + \cdots \in \mathcal{P}_n$ the inequality

$$|a_n| \le \left(\frac{2}{\rho}\right)^n \|P_n\|_\rho \quad , \qquad \rho \ge 1 \ . \tag{3.3}$$

For $\rho = 1$, if the P_n are bounded on $I := [-1, 1]$ we obtain $\limsup |a_n|^{1/n} \le 2$. This can be used to separate analytic from non-analytic functions in $C(I)$:

Theorem 3.1. *A function $f \in C(I)$ is analytic on I if and only if the leading coefficients of its polynomials of best approximation $P_n^*(z) := a_n z^n + \cdots$ satisfy*

(3.4) $$\limsup_{n\to\infty} |a_n|^{1/n} < 2 \; ;$$

it is non-analytic if and only if

(3.5) $$\limsup_{n\to\infty} |a_n|^{1/n} = 2 \; .$$

Proof. If f is analytic, then for some $\rho > 1$, $P_n^*(z) \to f(z)$ uniformly on E_ρ, and (3.3) leads to (3.4).

Conversely, let (3.4) be satisfied. The norm of the Chebyshev polynomial C_n is $\|C_n\| = 1$, and its leading coefficient is 2^{n-1}. Hence $R_{n-1} = P_n^* - 2^{-n+1}a_n C_n$ are polynomials of degree $n-1$. They satisfy

$$\|P_n^* - R_{n-1}\| = 2^{-n+1}|a_n| < q^n \quad , \qquad n \geq n_0$$

for some $0 < q < 1$ and n_0. Therefore

$$E_{n-1}(f) = \|f - P_{n-1}^*\| \leq \|f - R_{n-1}\| \leq \|f - P_n^*\| + \|P_n^* - R_{n-1}\| \leq E_n(f) + q^n \; .$$

It follows that $E_{n-1}(f) \leq q^n + q^{n+1} + \cdots = (1-q)^{-1}q^n$. By [**CA**, Theorem 8.1, p.229] f is analytic inside $E_{1/q}$. □

What we would like to have in the following theorems, is the "normal case", when instead of (3.5) we have

(3.6) $$\lim |a_n|^{1/n} = 2 \; .$$

If this is not so, our results hold for any subsequence $P_{n_k}^*$ with the property $\lim_{k\to\infty} |a_{n_k}|^{1/n_k} = 2$. However, we shall usually suppress the subscript k and say instead that $n \to \infty$ "through a normal sequence".

The functions

$$F_n(z) := P_n^*(z) w^{-n} \; , \qquad w = z + \sqrt{z^2 - 1}$$

are analytic in $\mathbb{C} \setminus I$, including the point ∞, where their values are

(3.7) $$F_n(\infty) = \lim_{z\to\infty} \left(P_n^*(z) w^{-n}\right) = \frac{a_n}{2^n} \; , \qquad n = 1, 2, \ldots \; .$$

We let $n \to \infty$ through a fixed normal sequence, and derive an estimate for $\log |F_n(z)|$.

Lemma 3.2. *Let numbers $\rho > 1$, $\gamma > 0$, and an arc A of ∂E_ρ be given. For a normal sequence of n, there is an n_0 so that for each $n \geq n_0$ there are points $z_n \in A$ with the property that*

(3.8) $$\log |F_n(z_n)| \geq -n\gamma \; .$$

Proof. The inequality of Jensen (see Hille [B-1962])

$$\log |f(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{it})| \, dt$$

is valid for each function $f(u)$ that is analytic on the disk $|u| \le r$. It has a companion, obtainable by the substitution $w = u^{-1}$:

$$\log |H(\infty)| \le \frac{1}{2\pi} \int_0^{2\pi} \log |H(\rho e^{it})| \, dt , \tag{3.9}$$

which is valid when $H(w)$ is analytic for $|w| \ge \rho$, including the point ∞.

Let $\varepsilon > 0$ be fixed, to be selected later. The following is true for n_0 sufficiently large. From (3.6) and (3.7) we derive

$$\log |F_n(\infty)| = \log \frac{|a_n|}{2^n} \ge \log e^{-n\varepsilon} = -n\varepsilon , \qquad n \ge n_0 . \tag{3.10}$$

From (3.1), if $C = \sup_n \|P_n^*\|$,

$$\log |F_n(z)| \le \log \frac{C\rho^n}{|w|^n} = \log C , \qquad z \in \partial E_\rho . \tag{3.11}$$

Finally, if (3.8) is not true, then for some arbitrary large n,

$$\log |F_n(z)| < -n\gamma , \qquad z \in A . \tag{3.12}$$

The transformation $z = \frac{1}{2}(w+w^{-1})$, sends ∂E_ρ into the circle $|w| = \rho > 1$; the arc A into an arc B of this circle; and the $F_n(z)$ into functions $H_n(w)$ analytic for $|w| \ge \rho$, and with properties similar to (3.10)–(3.12). But this leads to a contradiction, for ε sufficiently small, which proves the lemma:

$$\begin{aligned} -n\varepsilon \le \log |H_n(\infty)| &\le \frac{1}{2\pi} \int_0^{2\pi} \log |H_n(\rho e^{it})| \, dt \\ &\le (1 - |B|/2\pi) \log C + \frac{1}{2\pi} \int_B (-n\gamma) \, dt = (1 - |B|/2\pi) \log C - n\gamma \frac{|B|}{2\pi} . \square \end{aligned}$$

As a simple corollary of the lemma, the sequence P_n^* (in case that f is not analytic on $[-1,1]$) cannot converge uniformly on any open subset of $\mathbb{C}$.

Theorem 3.3. *If $f \in C(I)$ is not analytic on $I := [-1,1]$, then each point $x_0 \in I$ is a limit point of the set of z with $P_n^*(z) = a$, for any fixed complex number a.*

Proof. It is sufficient to prove this when $a = 0$, the general case will then follow by considering the function $f - a$.

Let U be a disk with center x_0. Then $U \subset E_\rho$ for some $\rho > 1$. We assume that the $P_n^*(z)$ have no zeros in U and derive a contradiction. We select single-valued branches of the functions $G_n(z) := P_n^*(z)^{1/n}$ on U. Since they are bounded on U, they form a normal family. If necessary, we replace the G_n by a subsequence, U by a smaller disk with center x_0, and are then left with the case that the sequence $G_n(z)$ converges uniformly on U to an analytic function $G(z)$ on U.

We note that if A is an arc of an ellipse ∂E_{ρ_1}, $\rho_1 > 1$, then $|G(z_0)| > 1$ at some point $z_0 \in A$. Indeed, from (3.8), for properly chosen $\gamma > 0$, and infinitely many n,

$$\frac{1}{n} \log |P_n^*(z_n)| = \frac{1}{n} \log |F_n(z_n)| + \log \rho_1 \geq \log \rho_1 - \gamma =: c > 0 .$$

We can assume $z_n \to z_0 \in A$, then $|G(z_0)| \geq e^c > 1$.

Moreover, at x_0, the values $P_n^*(x_0)$ are bounded, hence

$$|G(x_0)| = \lim_{n \to \infty} |P_n^*(x_0)|^{1/n} \leq 1 .$$

In particular, G is not identically zero. The G_n have no zeros in U. By Rouché's theorem, this property is inherited by G. This enables us to apply the minimum modulus principle: $|G(z)|$ does not attain its minimum inside U. Thus, the value $|G(x_0)|$ is not minimal. There must exist a disk $U_1 \subset U$ where $|G(z)| < 1$. The presence of an arc $A \subset U_1$ yields a contradiction. □

We write $D_\rho := \mathbb{C} \setminus E_\rho$ for the complements of the closed ellipses E_ρ, in particular $D_1 := \mathbb{C} \setminus I$. Let $z_k := z_{k,n}$, $k = 1, \ldots, n$ be all zeros of P_n^*, and z_k', z_k'' those of them that lie in E_ρ or in D_ρ, respectively. For a set $A \subset \mathbb{C}$, $N_n(A)$ is the number of the $z_k \in A$; in particular, let $N_n' := N_n(E_\rho)$, $N_n'' := N_n(D_\rho)$.

Theorem 3.4. *If $f \in C[-1,1]$ is not analytic on $[-1,1]$, then for each normal sequence of n,*

$$\lim_{n \to \infty} \frac{1}{n} N_n(D_\rho) = 0 \quad , \qquad \rho > 1 . \tag{3.13}$$

Proof. Let z_k, $k = 1, \ldots, N_n''$ be all zeros of P_n^* in D_ρ, let $1 < \rho_1 < \rho$. We let w_k correspond to the z_k under the map $w = z + \sqrt{z^2 - 1}$, $|w| > 1$. We define

$$G(w) = w^{-n} P_n^* \left(\tfrac{1}{2}(w + w^{-1})\right) \prod_{k=1}^{N_n''} \frac{\rho_1^2 - \bar{w}_k w}{w - w_k} \, , \qquad |w| \geq \rho_1 \, .$$

This function is analytic in $|w| \geq \rho_1$, also at ∞, and for $|w| = \rho_1$, $|\rho_1^2 - \bar{w}_k w| = \rho_1 |w - w_k|$. Applying the maximum modulus principle, we get

$$|G(\infty)| \leq \max_{|w|=\rho_1} |G(w)| \, .$$

This yields

$$|a_n| \frac{1}{2^n} \prod_{k=1}^{N_n''} |w_k| \leq \rho_1^{-n} \|P_n^*\|_{\rho_1} \rho_1^{N_n''} \, . \tag{3.14}$$

Here $\|P_n^*\|_{\rho_1} \leq C\rho_1^n$, where $C = \sup_n \|P_n^*\|$, and $|w_k| \geq \rho$ for $k = 1, \ldots, N_n''$. We obtain $(\rho/\rho_1)^{N_n''} \leq C 2^n / |a_n|$, in other words,

$$\frac{N_n''}{n}\log(\rho/\rho_1) \le \frac{1}{n}\log C + \log\frac{2}{|a_n|^{1/n}} \ . \tag{3.15}$$

Making $n \to \infty$ and using (3.6), we arrive at (3.13). □

Using facts from §2, we can show that normal subsequences are essential for the validity of Theorem 3.4.

We shall give some explicit formulas for the zeros of the P_n^*.

Theorem 3.5. *Let g be a continuous bounded function on $\mathbb{C}$. If $z_k := z_{k,n}$, $k = 1,\dots,n$ are all zeros of P_n^*, then for a normal subsequence of n,*

$$\lim_{n\to n}\frac{1}{n}\sum_{k=1}^{n} g(z_{k,n}) = \frac{1}{\pi}\int_{-1}^{1} g(x)\frac{dx}{\sqrt{1-x^2}} \ . \tag{3.16}$$

Proof. Because of (3.6), we can assume that $a_n \neq 0$ for all n. For some $\rho > 1$ we take the factorization of P_n^*,

$$P_n^*(z) = e^{i\alpha_n}P_n(z)Q_n(z) \ , \tag{3.17}$$

where $e^{i\alpha_n} = a_n/|a_n|$, and $Q_n(z) = \prod_1^{N_n''}(z - z_k'')$ is the monic polynomial that contains all zeros of P_n^* in D_ρ. Thus, P_n has the form $P_n(z) = |a_n|z^{N_n'} + \cdots$. We first prove that

$$\lim_{n\to\infty}\frac{P_n(z)^{1/N_n'}}{z+\sqrt{z^2-1}} = 1 \tag{3.18}$$

uniformly on compact subsets of D_ρ if the branch of the function under the limit sign is properly selected.

We take the branch of $z+\sqrt{z^2-1}$ that is infinite at ∞. Since P_n has no zeros in D_ρ, we select the branch of

$$h_n(z) = P_n(z)^{1/N_n'}(z+\sqrt{z^2-1}\,)^{-1}$$

for which $h_n(\infty) > 0$. This is possible and gives

$$h_n(\infty) = \tfrac{1}{2}|a_n|^{1/N_n'} = \tfrac{1}{2}(|a_n|^{1/n})^{n/(n-N_n'')} \to 1 \ . \tag{3.19}$$

We show that the h_n are uniformly bounded in D_ρ. Let δ be the distance from D_ρ to $I = [-1,1]$. Then

$$|P_n(x)| = \frac{|P_n^*(x)|}{|Q_n(x)|} \le \delta^{-N_n''}\|P_n^*\| \le C\delta^{-N_n''} \ , \qquad x \in I \ ,$$

where C does not depend on n; by Bernstein's inequality,

$$|P_n(z)| \le C\rho^n\delta^{-N_n''} \ , \qquad z \in \partial E_\rho \ ,$$

and therefore on this set, by Theorem 3.4,

$$|h_n(z)| \le C^{1/N'_n} \rho^{n/N'_n - 1} \delta^{-N''_n/N'_n} \to 1 \ . \tag{3.20}$$

Together with (3.19) this shows that the h_n are uniformly bounded on D_ρ. There is a subsequence, convergent on compact subsets of D_ρ to an analytic function h. From (3.19) and (3.20), $h(\infty) = 1$ and $h(z) = 1$, $z \in \partial E_\rho$, hence $h(z) = 1$ on D_ρ. We have proved (3.18) for a subsequence.

If the original normal sequence was h_{n_k}, we have proved (3.18) for some subsequence $h_{n_{k_i}}$. However, the same argument applies to any subsequence of the normal sequence h_{n_k}. This is possible only if the original sequence is convergent, and we obtain the full statement of (3.18).

Taking logarithmic derivatives in (3.18),

$$\lim_{n\to\infty} \frac{1}{N'_n} \sum_{z_{k,n} \in E_\rho} \frac{1}{z - z_{k,n}} = \frac{1}{\sqrt{z^2-1}}$$

locally uniformly in D_ρ. (The branch of $\sqrt{z^2-1}$ on the right satisfies $\sqrt{z^2-1} > 0$ for $z > 1$.) We multiply both sides with some polynomial $Q(z)$ and integrate over ∂E_{ρ_1}, $\rho_1 > \rho$:

$$\lim_{n\to\infty} \frac{1}{N'_n} \sum_{z_{k,n} \in E_\rho} Q(z_{k,n}) = \frac{1}{2\pi i} \int_{\partial E_{\rho_1}} \frac{Q(z)dz}{\sqrt{z^2-1}} \ . \tag{3.21}$$

The integral on the right does not depend upon $\rho_1 > 1$. Its limit for $\rho_1 \to 1$ is the integral over $[-1,1]$ covered twice, once taken in the negative direction on $[-1,1]$, with the value $i\sqrt{1-x^2}$ for the square root, once in the positive direction with value $-i\sqrt{1-x^2}$. This yields

$$\frac{1}{\pi} \int_{-1}^{+1} \frac{Q(x)dx}{\sqrt{1-x^2}} \tag{3.22}$$

as the limit of the integral (3.21) for $\rho_1 \to 1$.

For a bounded continuous function g on $\mathbb{C}$ we put

$$S_n(g) = \frac{1}{N'_n} \sum_{z_{k,n} \in E_\rho} g(z_{k,n}) \quad , \quad J(g) = \int_I g \, d\mu \ ,$$

where $d\mu = \frac{1}{\pi}(1 - x^2)^{-1/2} dx$ (and $\mu(I) = 1$). If g is given, we select a polynomial Q to have $|g(x) - Q(x)| < \varepsilon$, $x \in I$, then ρ so close to 1 that $|g(z) - Q(z)| < \varepsilon$, $z \in E_\rho$, and finally an n_0 so that $|S_n(Q) - J(Q)| < \varepsilon$, $n \ge n_0$. Then we shall have $|J(Q) - J(g)| < \varepsilon$, $|S_n(Q) - S_n(g)| < \varepsilon$, hence

$$|S_n(g) - J(g)| < 3\varepsilon \quad , \quad n \ge n_0 \ .$$

We can now use Theorem 3.4 and replace $S_n(g)$ by the average $\frac{1}{n}\sum_1^n g(z_{k,n})$, obtaining

$$\left| \frac{1}{n} \sum_1^n g(z_{k,n}) - J(g) \right| < 4\varepsilon \quad , \quad n \ge n_0 \ . \qquad \square$$

Relation (3.16) has the following interpretation. Let $d\mu_n$ for $n = 1, 2, \ldots$ be the discrete measure on $\mathbb{C}$ which assigns to each $z_{k,n}$ the measure $1/n$. The average in (3.16) becomes $\int_{\mathbb{C}} g\, d\mu_n$. Likewise, we extend $d\mu$ to $\mathbb{C}$ by assigning measure zero to the set $\mathbb{C} \setminus I$. Then (3.16) means that

$$\lim_{n\to\infty} \int_{\mathbb{C}} g\, d\mu_n = \int_{\mathbb{C}} g\, d\mu \ , \tag{3.23}$$

in other words, that $\mu_n \to \mu$ weakly on the set of all bounded continuous functions on $\mathbb{C}$.

For further results, it is convenient to use

Lemma 3.6. *Let λ_n, $n = 1, 2, \ldots$, and λ be positive measures on $\mathbb{R}$, with total mass 1, let λ be absolutely continuous. If for all bounded continuous functions h on $\mathbb{R}$,*

$$\lim_{n\to\infty} \int h\, d\lambda_n = \int h\, d\lambda \ , \tag{3.24}$$

then for each set $G \subset \mathbb{R}$ which is the union of finitely many (compact or non-compact) intervals,

$$\lim_{n\to\infty} \lambda_n(G) = \lambda(G) \ . \tag{3.25}$$

Proof. Let $G(\varepsilon)$ be the ε-neighborhood of the set G. We put $h_\varepsilon(x) = 1$, if $x \in G$; $= 0$, if $x \notin G(\varepsilon)$, and let h_ε be linear elsewhere. Then $\limsup \lambda_n(G) \le \lim \int h_\varepsilon\, d\lambda_n = \int h_\varepsilon d\lambda \le \lambda(G(\varepsilon))$, and making $\varepsilon \to 0$, we get $\limsup \lambda_n(G) \le \lambda(G)$. For $h = 1$, (3.24) yields $\lambda(\mathbb{R}) = \lim \lambda_n(\mathbb{R})$. If G' is the complement of G, then

$$\begin{aligned}\lambda(\mathbb{R}) &= \lim_{n\to\infty} \{\lambda_n(G) + \lambda_n(G')\} \le \limsup \lambda_n(G) + \limsup \lambda_n(G') \\ &\le \lambda(G) + \lambda(G') = \lambda(\mathbb{R}) \ .\end{aligned}$$

This is possible only if $\limsup \lambda_n(G) = \lambda(G)$. Also for any subsequence of λ_n this is valid, and (3.24) follows. □

We need some single-valued continuous functions of $z \in \mathbb{C}$. One of them is $x = \operatorname{Re} z$. If $z = \frac{1}{2}(w + w^{-1})$ with $w = \rho e^{i\phi}$, $\rho \ge 1$, $-\pi \le \phi \le \pi$, then ρ is uniquely determined by z. Since $x = \frac{1}{2}(\rho + \rho^{-1}) \cos\phi$, the function $\cos\phi$, and consequently $|\phi|$, are continuous and single-valued functions of $z \in \mathbb{C}$.

For the zeros $z_{k,n}$, $k = 1, \ldots, n$ of P_n^* we denote the corresponding x, ϕ by $x_{k,n}$, $\phi_{k,n}$. Let $N(\cdots)$ be the number of $k = 1, \ldots, n$ which satisfy the condition in the brackets.

Theorem 3.7. *If $f \in C[-1, 1]$ is not analytic on $[-1, 1]$, then for the zeros of a normal subsequence of its polynomials of best approximation P_n^*, for $-1 \le a < b \le 1$, $0 \le \alpha < \beta \le \pi$,*

$$(3.26) \qquad \lim_{n\to\infty} \frac{1}{n} N(a \le x_{k,n} \le b) = \frac{1}{\pi}\ (\arcsin b - \arcsin a)\ ,$$

$$(3.27) \qquad \lim_{n\to\infty} \frac{1}{n} N(\alpha \le |\phi_{k,n}| \le \beta) = \frac{\beta-\alpha}{\pi}\ .$$

If, in addition, f is real-valued and if the P_n^* have only $o(n)$ zeros on I, then

$$(3.28) \qquad \lim_{n\to\infty} \frac{1}{n} N(\alpha \le \phi_{k,n} \le \beta) = \frac{\beta-\alpha}{2\pi}\quad , \qquad -\pi \le \alpha < \beta \le \pi\ .$$

Proof. We take $g(z) = h(\operatorname{Re} z)$ in (3.16), where h is a bounded continuous function on $\mathbb{R}$. Let $\lambda = \mu$, and let λ_n be measures of $\mathbb{R}$ induced by the μ_n on $\mathbb{C}$; that is, for $A \subset \mathbb{R}$, let $\lambda_n(A) = \mu_n(z : \operatorname{Re} z \in A)$. Then $\lambda_n[a,b] = \frac{1}{n}N(a \le x_{k,n} \le b)$, $\lambda[a,b] = \frac{1}{\pi}(\arcsin b - \arcsin a)$ so that (3.26) follows from (3.16) and Lemma 3.6.

Relation (3.27) is proved similarly, by taking $g(z) = h(\cos\phi)$.

Finally, if f satisfies the additional conditions, (3.13) implies that P_n^* have only $o(n)$ zeros on $\mathbb{R}$. The remaining $z_{k,n}$ split into pairs of conjugate zeros. If $z_{k,n} = \bar{z}_{\ell,n}$ are not real, their arguments are uniquely defined and satisfy $\phi_{k,n} = -\phi_{\ell,n}$. In (3.27) we can assume that $0 \le \alpha < \beta \le \pi$. Then $N(\alpha \le |\phi_{k,n}| \le \beta) = 2N(\alpha \le \phi_{k,n} \le \beta)$ and we get (3.28). □

The restriction that the P_n^* have only $o(n)$ zeros on $[-1,1]$ is a weak one; it is satisfied if f vanishes only on a subset of λ-measure zero of $[-1,1]$, see Theorem 4.3 of Chapter 3.

The following two diagrams by R.S. Varga show the complex zeros of P_{20}^* and of P_{40}^* for the function $f(x) = \sqrt{x}$, $x \in [0,1]$.

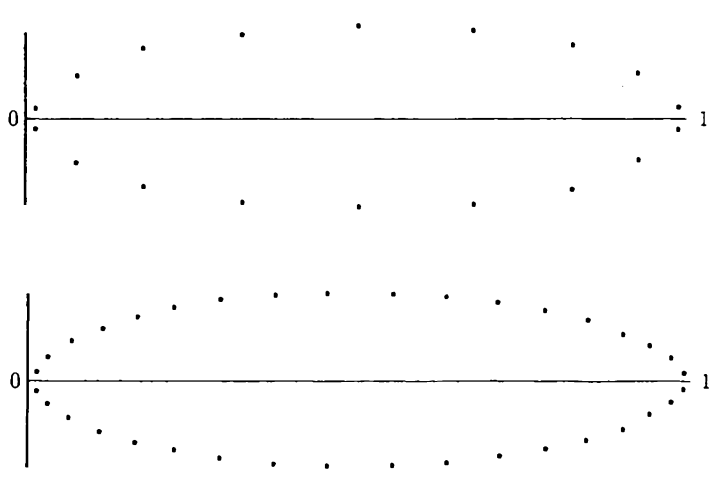


Fig. 3.1. Zeros of $P_{20}^*(x)$, and of $P_{40}^*(x)$, best approximations to $f(x) = \sqrt{x}$ on $[0,1]$.

§ 4. Error of Approximation

If a function $f \in C[-1,+1]$ is given, and $E_n(f)$, $n = 0,1,\dots$ are its errors of polynomial approximation, what are then the properties of the sequence $E_n(f)$? Clearly, the sequence is decreasing, and from Weierstrass' theorem it follows that $E_n(f) \to 0$. Are there some additional properties of the sequence? Bernstein [A-1952-54] shows that there are none. His arguments apply to linear approximation in an arbitrary separable Banach space X.

Let X_n be a strictly increasing sequence of subspaces $X_1 \subset \cdots \subset X_n \subset \cdots$ of X with $\dim X_n = n$. We suppose that $\cup X_n$ is dense in X. Let $E_n(f)$, $n = 0,1,\dots$ stand for the error of approximation of $f \in X$ by the elements of X_n, with $E_0(f) = \|f\|$. We have $E_n(f) \to 0$ for each $f \in X$.

By c_0 we denote the Banach space of all real sequences $y = (y_k)_0^\infty$, $y_k \to 0$, with the norm $\|y\| = \sup_k |y_k|$. We can study the space X by means of the map $\Lambda(f)$ of X into c_0 given by

$$\Lambda(f) := (E_n f)_0^\infty \ .$$

This map is by no means one to one, but it is continuous:

$$\|\Lambda(f) - \Lambda(g)\| = \sup_k |E_k(f) - E_k(g)| \le \sup_k E_k(f-g) = \|f-g\| \ .$$

Lemma 4.1. *A bounded sequence $f_m \in X$ with the property $\Lambda(f_m) \to y \in c_0$ with convergence in the norm of the space c_0, must contain a convergent subsequence $f_{m_i} \to f \in X$, with $\Lambda(f) = y$.*

Proof. Let $y = (y_n)_0^\infty$. For each $\varepsilon > 0$, we have $|E_n(f_m) - y_n| < \frac{1}{2}\varepsilon$ for all large m and all n. Since $|y_n| < \frac{1}{2}\varepsilon$ for large n, there is an N with the property

$$(4.1) \qquad E_n(f_m) < \varepsilon \quad \text{for} \quad n,m > N \ .$$

However, $E_n(f_m) \to 0$ for $n \to \infty$ and $m = 1,\dots,N$. We can assume that (4.1) holds for $n > N$ and all m.

Since the set $F = (f_m)$ is bounded, it follows that it can be approximated, with an error $< \varepsilon$, by a bounded subset F_n of X_n, which consists of best approximations of the f_m, $m = 1,2,\dots$ from X_n. The sets F_n are precompact, therefore F is precompact. Hence (f_m) contains a convergent subsequence $f_{m_i} \to f \in X$, and by continuity of Λ we have $\Lambda(f) = y$. □

Theorem 4.2. *In the situation described, for each sequence $(y_n)_0^\infty$ of real numbers satisfying*

$$(4.2) \qquad y_0 \ge \cdots \ge y_n \ge \cdots \ , \qquad \lim y_n = 0$$

there exists an element $f \in X$ with the property

$$(4.3) \qquad E_n(f) = y_n \ , \qquad n = 0,1,\dots \ .$$

Proof. Each $f \in X$ has an element of best approximation $p \in X_m$ with $E_m(f) = \|f - p\|$ (see also [CA, §1, Chapter 3]). This p is not necessarily unique. However, the following is true: p is a best approximation to f from X_m if and only if 0 is a best approximation to $f - p$. In this case, 0 is also a best approximation to f from X_{m-1} and we have $E_{m-1}(f - p) = E_m(f)$.

For the proof of (4.3) it is sufficient to construct, for each $m = 1, 2, \ldots$, an element $p_m \in X_m$ with the properties

$$E_j(p_m) = y_j \ , \qquad j = 1, \ldots, m-1 \ . \tag{4.4}$$

Indeed, then Lemma 4.1, applied to the sequence $(p_m)_1^\infty$, yields a required $f \in X$.

We prove the following fact. For each j, $0 \le j \le m$, there is a $q_j \in X_m$ for which

$$E_k(q_j) = y_k \ , \qquad j \le k < m \ , \ E_{j-1}(q_j) = y_j \ . \tag{4.5}$$

For $j = m$, there is no k with $j \le k < m$, and (4.5) is trivially true. For $j = 0$, (4.5) gives the desired $p_m = q_0$. We assume that (4.5) holds for some j, $1 \le j \le m$ and derive (4.5) with j replaced by $j - 1$. This will establish the theorem.

If q_j satisfies (4.5), we select a $q \in X_j \setminus X_{j-1}$ and consider the continuous function

$$\phi(\lambda) := E_{j-1}(q_j + \lambda q) \ , \qquad \lambda \ge 0 \ . \tag{4.6}$$

Then $\phi(0) = y_j$, moreover $\phi(\lambda) \ge \lambda E_{j-1}(q) - y_j$. Hence $\phi(\lambda) > y_{j-1}$ for large λ. We select λ so that $\phi(\lambda) = y_{j-1}$. Let r be a best approximation to $q_j + \lambda q$ from X_{j-1}. For $q_{j-1} := q_j + \lambda_q - r$ we have $E_{j-1}(q_{j-1}) = y_{j-1}$, moreover, by (4.5), $E_k(q_{j-1}) = E_k(q_j) = y_j$, $j \le k < m$. Finally, by the remarks above, $E_{j-2}(q_{j-1}) = E_{j-1}(q_{j-1}) = y_{j-1}$. □

On $\mathbb{R}$, we can measure the error of approximation by a function. A very old theorem of Carleman [1927] deals with approximation of continuous (in general, unbounded) functions on $\mathbb{R} = (-\infty, \infty)$ by entire functions.

Theorem 4.3. *For each $f \in C(\mathbb{R})$ and each function $\varepsilon \in C(\mathbb{R})$, $\varepsilon(x) > 0$, $x \in \mathbb{R}$, there exists an entire function e with the property*

$$|f(x) - e(x)| < \varepsilon(x) \ , \qquad x \in \mathbb{R} \ . \tag{4.7}$$

Proof (by Scheinberg [1976]). We use the following fact:

(4.8) *For each strictly positive function $\varepsilon \in C(\mathbb{R})$ there exists an entire function e_0 such that $0 < e_0(x) \le \varepsilon(x)$, $x \in \mathbb{R}$.*

To prove this, it suffices to find an entire function e_1 without zeros, for which $e_1(x) \ge 1/\varepsilon(x)$, $x \in \mathbb{R}$. Let $M_k := \max_{|x| \le k+1}(1/\varepsilon(x))$. We define

$$e_2(x) := M_1 + \sum_{k=2}^{\infty} \left(\frac{x^2}{k+1}\right)^{n_k} ,$$

where n_k are increasing positive integers which satisfy $(k^2(k+1)^{-1})^{n_k} > M_k$, $k = 2, \ldots$. Then $e_2(x) > M_1 \geq 1/\varepsilon(x)$, if $|x| \leq 2$. If $2 \leq k \leq |x| \leq k+1$, we see that

$$e_2(x) \geq \left(\frac{k^2}{k+1}\right)^{n_k} \geq M_k \geq \frac{1}{\varepsilon(x)} .$$

We can then select $e_1 = \exp e_2$, proving (4.8).

As a consequence, Theorem 4.3 follows from its special case when $\varepsilon(x)$ is constant or even $\varepsilon(x) = 1$. For $f \in C(\mathbb{R})$ implies $f/e_0 \in C(\mathbb{R})$, and the relation $|f(x)/e_0(x) - e(x)| < 1$ implies $|f(x) - e_0(x)e(x)| < \varepsilon(x)$, $x \in \mathbb{R}$.

For a constant $\varepsilon > 0$ in (4.7) we shall use the *Weierstrass integrals*

$$g_n(z) := (W_n f)(z) := \frac{n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-\frac{n^2}{2}(z-t)^2}\, dt \tag{4.9}$$

(Weierstrass used them in 1885 to establish his polynomial approximation theorem.)

If $f \in L_1(\mathbb{R})$, then $g_n(z)$ is an entire function. The symmetric kernel

$$W_n(x,t) = \frac{n}{\sqrt{2\pi}} \exp\left(-\frac{n^2}{2}(x-t)^2\right)$$

satisfies all conditions of [**CA**, Theorem 2.1, p.5]. Therefore

$$(W_n f)(x) \to f(x) \text{ uniformly on } \mathbb{R} , \tag{4.10}$$

whenever $f \in C(\mathbb{R})$ has compact support.

Let $1 = \sum_{-\infty}^{\infty} \phi_j$ be a partition of unity on $\mathbb{R}$ with continuous functions ϕ_j of support $I_j = [j-1, j+1]$, $j \in \mathbb{Z}$.

Let $f \in C(\mathbb{R})$. Then $f = \sum f\phi_j$. We examine the functions $g_{n,j} := W_n(f\phi_j)$. If the disk $D_k = \{z : |z| \leq k\}$, $k = 1, 2, \ldots$ and the interval I_j, $j \in \mathbb{Z}$ are at a distance $\geq k+1$ from each other, and if $z = x+iy \in D_k$, $t \in I_j$, then

$$\mathrm{Re}(z-t)^2 = (t-x)^2 - y^2 \geq (k+1)^2 - k^2 \geq 1 .$$

Therefore for each k and all j with $|j| > 2k+1$,

$$|g_{n,j}(z)| \leq \|f\phi_j\|_{I_j} \frac{2n}{\sqrt{2\pi}}\, e^{-n^2/2} , \qquad z \in D_k . \tag{4.11}$$

For each $j \in \mathbb{Z}$ we select a positive integer n_j so that the right-hand side of (4.11) is $< \varepsilon 2^{-|j|-2}$ and that in addition, for $h_j := g_{n_j,j}$,

$$|(f\phi_j)(x) - h_j(x)| < \varepsilon 2^{-|j|-2} , \qquad x \in \mathbb{R} .$$

(This is possible because of (4.10).) Then the series $H = \sum_{j=-\infty}^{\infty} h_j$ converges uniformly on each disk D_k, $k = 1, 2, \ldots$, represents an entire function, and we have

$$|f(x) - H(x)| \le \sum_{-\infty}^{\infty} |(f\phi_j)(x) - h_j(x)| \le \varepsilon \sum 2^{-|j|-2} < \varepsilon \,, \qquad x \in \mathbb{R} \,. \quad \square$$

§ 5. Approximation on $(-\infty, \infty)$ by Linear Combinations of Functions $(x-c)^{-1}$

Properties of functions $(z-c)^{-1}$ on the unit circle, due to Akhiezer (see his book [A-1965, problems 1 and 2]), expressed in Theorems 5.1 and 5.2, are of independent interest. They lead to Theorem 5.3, which provides an excellent approach to the solution of Bernstein's problem of §6.

First we discuss the degree of approximation of z^s, $s = 0, 1, \ldots$, on the circle $|z| = 1$ by linear combinations of functions

$$z^{s-1}, \ldots, z, 1, \frac{1}{z - z_1}, \ldots, \frac{1}{z - z_n} \,, \tag{5.1}$$

where the z_j are distinct and satisfy $|z_j| \ne 1, j = 1, \ldots, n$. (If $s = 0$, only the terms $\frac{1}{z - z_j}$ appear in (5.1). We can compute *exactly* the degree of approximation $E_n(z^s)_p$ both in the uniform norm and in the L_p metric

$$\|f\|_p = \left\{ \frac{1}{2\pi} \int_{\mathbb{T}} |f(z)|^p \, dt \right\}^{1/p} \,, \qquad p > 0 \,.$$

A trivial remark is that for $|z_0| \ne 1$,

$$\left| \frac{z\bar{z}_0 - 1}{z - z_0} \right| = \left| \frac{1 - z\bar{z}_0}{1 - \bar{z}z_0} \right| = 1 \,, \qquad |z| = 1 \,. \tag{5.2}$$

We order the sequence z_j so that $|z_j| > 1$, $j = 1, \ldots, k$, and $|z_j| < 1$, $j = k+1, \ldots, n$, and put

$$M := \frac{1}{|z_1 \cdots z_k|} \,.$$

Theorem 5.1. *We have, for the approximation by linear combinations of the functions* (5.1)

$$E(z^s)_\infty = E(z^s)_p = M \,, \qquad p > 0 \,.$$

In other words, for $0 < p \le +\infty$,

$$\min_{A_j} \left\| \frac{z^N + A_1 z^{N-1} + \cdots + A_N}{(z - z_1) \cdots (z - z_n)} \right\|_p = M \,, \tag{5.3}$$

where $N = n + s \geq n$. *An extremal polynomial* $P(z) = z^N + \cdots + A_N$ *is given by*

$$P(z) = \frac{z^{N-n}(z\bar{z}_1 - 1)\cdots(z\bar{z}_k - 1)(z - z_{k+1})\cdots(z - z_n)}{\bar{z}_1 \cdots \bar{z}_k} .$$

Proof. Let Q be any polynomial of degree N with leading coefficient one. Then with $\omega(z) = (z - z_1)\cdots(z - z_n)$, for $0 < p < \infty$,

$$\frac{1}{2\pi}\int_{\mathbb{T}} \left|\frac{Q}{\omega}\right|^p dt \leq \max_{|z|=1} \left|\frac{Q(z)}{\omega(z)}\right|^p .$$

On the other hand, on $|z| = 1$,

$$\left|\frac{P(z)}{\omega(z)}\right| = M .$$

Therefore

$$\min_Q \left\|\frac{Q}{\omega}\right\|_p \leq \min_Q \left\|\frac{Q}{\omega}\right\|_\infty \leq M ,$$

and all that we have to prove is that

$$M \leq \left\|\frac{Q}{\omega}\right\|_p , \qquad 0 < p < +\infty .$$

Now $Q = \prod_1^N (z - \alpha_j)$, where we assume $|\alpha_j| < 1$ for $j = 1, \ldots, q$, $|\alpha_j| \geq 1$, $j > q$. Then

$$Q(z) = (z - \alpha_1)\cdots(z - \alpha_q)S(z) , \qquad |S(0)| \geq 1 .$$

In view of (5.2), the function Q/ω on $|z| = 1$ has modulus equal to that of

$$R(z) = \frac{(1 - \bar{\alpha}_1 z)\cdots(1 - \bar{\alpha}_q z)S(z)}{(z - z_1)\cdots(z - z_k)(1 - \bar{z}_{k+1}z)\cdots(1 - \bar{z}_n z)} .$$

Since R has no zeros or poles in $|z| < 1$, there exists a single valued branch of $R(z)^p$ in $|z| \leq 1$. We have $R(0) = S(0)/|z_1 \cdots z_k|$, and obtain for this branch

$$\begin{aligned} M^p &= \frac{1}{|z_1 \cdots z_k|^p} \leq \left|\frac{S(0)}{z_1 \cdots z_k}\right|^p = |R(0)|^p \\ &= \left|\frac{1}{2\pi i}\int_{|z|=1} R(z)^p \frac{dz}{z}\right| \leq \frac{1}{2\pi}\int_{\mathbb{T}} |R(z)|^p \, dt \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} \left|\frac{Q}{\omega}\right|^p dt . \end{aligned}$$

□

We shall now discuss when the system of functions

$$\psi_k(t) = \frac{1}{e^{it} - z_k}\,, \qquad k = \pm 1, \pm 2, \ldots, t \in \mathbb{T} \tag{5.4}$$

is complete in the space $C(\mathbb{T})$ or $L_p(\mathbb{T})$. We shall assume that $|z_k| \neq 1$, $z_k \neq 0$ and order the points so that $|z_{-k}| > 1$, and $|z_k| < 1$, $k = 1, 2, \ldots$. We do not exclude the case that there is a finite number of z_k or of z_{-k}.

Theorem 5.2 (Akhiezer). *Linear combinations of the functions ψ_k are dense in $C(\mathbb{T})$ or $L_p(\mathbb{T})$, $p > 0$, if and only if the following two series are divergent:*

$$\sum_{k=1}^{\infty} \{1 - |z_k|\} = +\infty \quad , \quad \sum_{k=1}^{\infty} \left\{1 - \left|\frac{1}{z_{-k}}\right|\right\} = +\infty\,. \tag{5.5}$$

These conditions imply in particular, that there are infinitely many of the z_k and of the z_{-k}.

Proof. It is necessary to see whether the functions e^{imt}, $m = 0, \pm 1, \ldots$ lie in the closed span of the ψ_k. First we consider the space $C(\mathbb{T})$. From (5.3) we have the identity, for the supremum norm on $\mathbb{T}$.

$$\begin{aligned} &\min_{A_k, B_k} \left\| z^m - A_1 z^{m-1} - \cdots - A_m - {\sum_{k=-n_2}^{n_1}}' \frac{B_k}{z - z_k} \right\| \\ &\quad = \prod_{k=1}^{n_2} \left|\frac{1}{z_{-k}}\right|\,, \qquad m = 0, 1, \ldots\,, \end{aligned} \tag{5.6}$$

where the $'$ in $\sum$ means that the term $k = 0$ is not present. Similarly,

$$\begin{aligned} &\min_{A_k, B_k} \left\| z^{-m} - A_1 z^{-(m-1)} - \cdots - A_m - {\sum_{k=-n_2}^{n_1}}' \frac{B_k}{z - z_k} \right\| \\ &\quad = \prod_{k=1}^{n_2} |z_k|\,, \qquad m = 0, 1, \ldots\,. \end{aligned} \tag{5.7}$$

To prove (5.7), we replace the function under the sign of the norm, for $|z| = 1$, by its conjugate. We then obtain an expression

$$z^m - \cdots - \bar{A}_{m-1} z - C_0 - {\sum_{k=-n_2}^{n_1}}' \frac{C_k}{z - (1/\bar{z}_k)}\,, \tag{5.8}$$

where C_k are connected with A_m and B_k by means of the formulas

$$C_k = -\bar{B}_k\, \bar{z}_k^{-2}\,, \quad k \neq 0\,, \quad C_0 = \bar{A}_m - {\sum}' \bar{B}_k\, \bar{z}_k^{-1}\,.$$

One sees that the C_k are arbitrary if the B_k and A_m are arbitrary, and conversely. Applying Theorem 5.1 to the function (5.8), we obtain (5.7).

Now, equation (5.6) can be written

$$(5.9)\qquad \min_{A_k, B_k} \left\| e^{imt} - A_1 e^{i(m-1)t} - \cdots - A_m - \sum_{-n_2}^{n_1}{}' B_k \psi_k(t) \right\| = \prod_{k=1}^{n_2} \left| \frac{1}{z_{-k}} \right| ;$$

and (5.7) allows a similar interpretation.

The conditions (5.5) are sufficient: they imply that the products in (5.6), (5.7) converge to zero for $n_1 \to \infty$, $n_2 \to \infty$. Putting $m = 0, 1, \ldots$ in (5.9), we obtain in turn that all functions $1, e^{it}, \ldots$ lie in the closed span of the ψ_n, and from (5.7) we draw a similar conclusion about $e^{-it}, e^{-2it}, \ldots$.

Conditions (5.5) are also necessary. From (5.9) with $m = 0$ we see that if 1 is approximable by linear combinations of the ψ_k, the second of the series (5.5) must diverge. This assumed, (5.7) shows that if e^{-it} is approximable in this way, then also the first series (5.5) must diverge.

For the spaces L_p, $p > 0$ the proof is the same. It remains to note that linear combinations of the exponentials e^{imt} are dense in the spaces $C(\mathbb{T})$ and $L_p(\mathbb{T})$. □

Remark 1. In the series (5.5), one can replace $|z_k|$ and $|1/z_{-k}|$ by their squares, for the convergence to zero of products (5.6) or (5.7) and of their squares are equivalent.

Remark 2. Conditions (5.5) show that omission of any finite set of the z_k will not destroy the possibility of approximation. It is easy to see that the addition of the function $\psi_0 \equiv 1$ does not change the conclusions of Theorem 5.2.

We shall use Theorem 5.2 for approximation on $\mathbb{R}$. We need a simple remark.

(5.10) *Let c_k, $k = 1, 2, \ldots$ be complex numbers with* $\operatorname{Im} c_k > 0$, $c_k \neq i$. *Then the series*

$$\sum_{k=1}^{\infty} \left\{ 1 - \left| \frac{i - c_k}{i + c_k} \right|^2 \right\} \quad \text{and} \quad \sum \frac{\operatorname{Im} c_k}{1 + |c_k|^2}$$

converge or diverge at the same time.

This follows from

$$\left| \frac{i - c_k}{i + c_k} \right|^2 = 1 - \frac{4 \operatorname{Im} c_k}{|c_k + i|^2}$$

and the inequalities

$$|c_k|^2 + 1 \le |c_k + i|^2 \le 2(|c_k|^2 + 1) .$$

To formulate our main theorem, (see Bernstein [A-1926, p.65]), let c_k, $k = \pm 1, \pm 2, \ldots$ be a sequence of numbers that are not real. We assume that $\operatorname{Im} c_k > 0$, and $\operatorname{Im} c_{-k} < 0$ for $k = 1, 2, \ldots$. Let X, equipped with the supremum norm, be the space of all continuous functions f on $(-\infty, +\infty)$, for which $\lim_{|x|\to\infty} f(x)$ exists.

Theorem 5.3 (Akhiezer). *The span of the functions*

$$\phi_k(x) = \frac{1}{x - c_k}\ , \qquad k = \pm 1, \pm 2, \ldots, \qquad \phi_0(x) = 1 \tag{5.11}$$

is dense in X *if and only if*

$$\sum_{k=1}^{\infty} \frac{\operatorname{Im} c_k}{1 + |c_k|^2} = +\infty \quad , \quad \sum_{k=1}^{\infty} \frac{\operatorname{Im} c_{-k}}{1 + |c_{-k}|^2} = -\infty\ . \tag{5.12}$$

This remains true if one omits from (5.11) *any finite subset of the* ϕ_k, *not containing* ϕ_0.

Proof. The map $x \to t$ given by $x = \tan(t/2)$ or $ix = (e^{it} - 1)/(e^{it} + 1)$ is a 1–1 map of $\mathbb{T}$ onto $(-\infty, +\infty)$; $f \in X$ is equivalent to $g(t) = f(\tan(t/2)) \in C(\mathbb{T})$. Approximating f on $(-\infty, +\infty)$ by the linear combinations of the ϕ_k amounts to approximating it by linear combinations of the functions 1 and $(1 - ix)/(x - c_k)$. But

$$\frac{1 - ix}{x - c_k} = -\frac{2}{i + c_k} \frac{1}{e^{it} - z_k}\ , \qquad z_k = \frac{i - c_k}{i + c_k}\ , \qquad k = \pm 1, \pm 2, \ldots\ . \tag{5.13}$$

Here $|z_k| \neq 1$, for the c_k are not real; for $k = 1, 2, \ldots$, $\operatorname{Im} c_k > 0$, hence $|z_k| < 1$, and $\operatorname{Im} c_{-k} < 0$, so that $|z_{-k}| > 1$. Our theorem follows immediately from the remarks to Theorem 5.2 and from (5.10). □

In particular, if $c_{-k} = \bar{c}_k$, $c_k = a_k + ib_k$ and $b_k > 0$, $k = 1, 2, \ldots$, then (5.12) becomes

$$\sum \frac{b_k}{1 + a_k^2 + b_k^2} = +\infty\ . \tag{5.14}$$

If the function f is real and g approximates f, then $\operatorname{Re} g$ approximates f with at most the same error. Thus, for the approximation of a real function we can use the real sums

$$A + \sum_{k=1}^{n} \left(\frac{A_k}{x - c_k} + \frac{\bar{A}_k}{x - \bar{c}_k} \right) = A + \sum_{k=1}^{n} \frac{B_k x + C_k}{(x - a_k)^2 + b_k^2}\ .$$

Examples. The following sequences c_k satisfy (5.14):

(a) $c_k = ik$, $k = 1, 2, \ldots$;
(b) $c_k = ib_k$, where b_k decreases to 0 and $\sum b_k = \infty$;
(c) $c_k = ib_k$, where b_k increases to ∞ and $\sum b_k^{-1} = \infty$;
(d) $c_k = k^\alpha + ik^\beta$, where $0 < \beta \leq \alpha \leq \frac{\beta+1}{2}$, or $0 < \alpha \leq \beta \leq 1$.

§ 6. Weighted Approximation by Polynomials on $(-\infty, \infty)$

Let Φ be a continuous function on $\mathbb{R}$, which satisfies

$$(6.1) \qquad c_0 \le \Phi(x) < \infty \quad , \quad \lim_{x \to \pm\infty} \frac{x^n}{\Phi(x)} = 0 \quad , \qquad n = 0, 1, \ldots \, ,$$

where $c_0 > 0$ is a constant. In particular, this implies that $\Phi(x) \to \infty$ for $x \to \pm\infty$. We denote by C_Φ the space of all continuous complex valued functions on $\mathbb{R}$ for which

$$(6.2) \qquad \lim_{x \to \pm\infty} \frac{f(x)}{\Phi(x)} = 0 \, .$$

With the norm

$$(6.3) \qquad \|f\|_\Phi = \sup_{x \in \mathbb{R}} \frac{|f(x)|}{\Phi(x)} \, ,$$

C_Φ is a Banach space, which contains all polynomials. All spaces C_Φ contain the subspace C_0 of functions f with $f(x) \to 0$ for $x \to \pm\infty$.

Examples of subsets of C_0 that approximate any function $f \in C_0$ are: (a) continuous functions with compact support; (b) linear combinations of $(x \pm ki)^{-1}$, $k = 1, 2, \ldots$ and the function 1 (see Theorem 5.3).

Bernstein [A-1952-54] calls a function Φ satisfying (6.1) a *weight function*, if each $f \in C_\Phi$ is approximable by polynomials (with complex coefficients) in the norm (6.3). Since C_0 is dense in C_Φ, this is equivalent to the requirement of the density of the polynomial set $\mathcal{P}$ in C_0. Bernstein proposed to find necessary and sufficient conditions characterizing weight functions. If Φ is a weight function and $\Phi(x) \le \Phi_1(x)$, $x \in \mathbb{R}$, then also Φ_1 is a weight function. If $0 < C_1 \le \Phi_1(x)/\Phi_2(x) \le C_2$, then Φ_1 and Φ_2 simultaneously are or are not weight functions. The same is true for $\Phi(x)$ and $A\Phi(Bx)$, where $A, B > 0$ are constants.

A source of weight functions is the theory of orthogonal polynomials. Thus, theorems about Laguerre and Hermite polynomials (see Szegö [B-1975]) show that e^x and e^{x^2} are weight functions on $\mathbb{R}_+$ and $\mathbb{R}$, correspondingly. However, this is not the main path to the solution of Bernstein's problem, which proved to be both difficult and interesting. All known solutions of the problem depend in some way on the complex function theory. Very approximately, Φ is a weight function if

$$(6.4) \qquad \mu(\Phi) := \int_{-\infty}^{\infty} \frac{\log \Phi(x)}{1 + x^2} \, dx = \infty \, .$$

We shall see that this is a necessary condition; it is not quite sufficient, if Φ does not behave regularly. We have to use a more complicated functional,

$$(6.5) \qquad \lambda(\Phi) := \sup_{P \in \mathcal{M}_\Phi} \int_{-\infty}^{\infty} \frac{\log |P(x)|}{1 + x^2} \, dx \, ,$$

where $\mathcal{M}_\Phi$ stands for the set of all (complex valued) polynomials P of arbitrary degree, which are majorized by Φ:

$$|P(x)| \le \Phi(x) , \qquad x \in \mathbb{R} .$$

The integral (6.5) exists even if P has real zeros; and obviously $\lambda(\Phi) \le \mu(\Phi)$.

Bernstein himself gave first significant examples of weight functions; different complete solutions of the problem have been given by Pollard [1953], Akhiezer and Bernstein [1953], and somewhat later by Mergelyan [1956_1]. The solution of the second two authors is usually believed to be the most satisfying.

We shall need the value of certain integrals of the type appearing in (6.5). Let $P(z)$ be a polynomial with no zeros in the half plane $\operatorname{Im} z > 0$, and let $\operatorname{Im} c > 0$. Then

$$\int_{-\infty}^{\infty} \frac{\log |P(x)|\, dx}{(x-c)(x-\bar{c})} = \frac{\pi}{\operatorname{Im} c} \log |P(c)| . \tag{6.6}$$

Indeed, we can restrict $\log P(z)$ in $\operatorname{Im} z > 0$ to one of its single valued branches. The residue at $z = c$ of a quotient $f(z)/g(z)$, if g has a simple zero at c, is $f(c)/g'(c)$. We can approximate the integral (6.6) by one over a closed curve consisting of the interval $[-r, r]$, a semicircle of radius r in the upper half plane with large r, and of small semicircles with centers at the real zeros of P. Therefore, by the calculus of residues,

$$\int_{-\infty}^{\infty} \frac{\log P(x)\, dx}{(x-c)(x-\bar{c})} = 2\pi i \frac{\log P(c)}{2i \operatorname{Im} c} = \frac{\pi}{\operatorname{Im} c} \log P(c) ,$$

and taking real parts, we get (6.6).

For an arbitrary polynomial P we have

$$\int_{-\infty}^{\infty} \frac{\log |P(x)|}{1+x^2}\, dx \ge \pi \log |P(i)| . \tag{6.7}$$

To derive this, we factor $P(z) = P_1(z) \prod (z - a_j)$, where P_1 has no zeros in the upper half plane and for each a_j, $\operatorname{Im} a_j > 0$. For P_1 we can use (6.6), with $c = i$, for each of the factors $z - a_j$ we have,

$$\int_{-\infty}^{\infty} \frac{\log |x - a_j|}{1+x^2}\, dx = \pi \log |-i - a_j| \ge \pi \log |i - a_j| .$$

Adding these relations, we obtain (6.7).

By $\mathcal{N}_\Phi$ we denote the set of polynomials P^* that have no zeros in $\operatorname{Im} z \ge 0$ and satisfy, with c_0 from (6.1),

$$1 \le |P^*(x)| \le \sqrt{1 + c_0^{-2}}\; \Phi(x) . \tag{6.8}$$

Lemma 6.1. *For each polynomial $P \in \mathcal{M}_\Phi$ there exists a polynomial $P^* \in \mathcal{N}_\Phi$ of the same degree for which*

$$|P(x)| \le |P^*(x)| \,, \qquad x \in \mathbb{R} \,. \tag{6.9}$$

Proof. For a polynomial $P(z) = a_0 + \cdots + a_n z^n$ we put $\widehat{P}(z) := \bar{a}_0 + \cdots + \bar{a}_n z^n$. Plainly, $\widehat{P}(x) = \overline{P(x)}$, $x \in \mathbb{R}$. The polynomial $R(z) := 1 + P(z)\widehat{P}(z)$ has real coefficients; hence its zeros occur in conjugate pairs. Also, R has no real zeros, since $R(x) \ge 1$ for $x \in \mathbb{R}$. If we combine all factors $z - c$ of R with $\operatorname{Im} c < 0$ into P^*, and multiply by the proper constant, we will have $R = P^* \widehat{P^*}$, hence

$$|P(x)|^2 < 1 + P(x)\overline{P}(x) = R(x) = P^*(x)\overline{P^*(x)} = |P^*(x)|^2 \,, \qquad x \in \mathbb{R}$$

and

$$|P^*(x)| \le \sqrt{1 + \Phi(x)^2} \le \sqrt{1 + c_0^{-2}}\, \Phi(x) \,. \qquad \square$$

We can derive a necessary condition for a weight function:

Theorem 6.2. *If Φ is a weight function, then*

$$\lambda\big(\sqrt{1+x^2}\, \Phi\big) = \infty \,. \tag{6.10}$$

This implies that $\mu(\sqrt{1+x^2}\, \Phi) = \infty$, hence that $\mu(\Phi) = \infty$.

Proof. Let $\Psi(x) = \sqrt{1+x^2}\, \Phi(x)$. Since $\frac{1}{x-i} \in C_0$, we can find, for each $\varepsilon > 0$, a polynomial P satisfying

$$\frac{1}{\Phi(x)} \left| \frac{1}{x-i} - P(x) \right| < \varepsilon \,, \qquad x \in \mathbb{R} \,.$$

Then $|1 - (x-i)P(x)| < \varepsilon \Psi(x)$, that is, $Q(x) = \varepsilon^{-1}[1 - (x-i)P(x)] \in \mathcal{M}_\Psi$. By (6.7),

$$\int_{-\infty}^{\infty} \frac{\log |Q(x)|}{1+x^2}\, dx \ge \pi \log |Q(i)| = \pi \log \frac{1}{\varepsilon} \,.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\lambda(\Psi) = \infty$. $\square$

A sufficient condition for a weight function is given by

Theorem 6.3. *If $\lambda(\Phi) = \infty$, then $\Phi(x)/\sqrt{1+x^2}$, (and therefore also Φ) is a weight function.*

Proof. By Theorem 5.3, each function $f \in C_0$ can be uniformly approximated on the whole real axis, in the norm (6.3), by linear combinations of fractions $\frac{1}{x-c}$, $\operatorname{Im} c \ne 0$, and 1. Thus, it will be sufficient to discuss the approximation by polynomials of $\frac{1}{x-c}$, $\operatorname{Im} c > 0$. (Here and later, P_n, Q_n will be polynomial in $\mathcal{P}$, not necessarily of degree $\le n$.)

By the assumption, there exists a sequence of polynomials $P_n \in \mathcal{P}$ for which $|P_n(x)| \le \Phi(x)$, $x \in \mathbb{R}$ and

$$\int_{-\infty}^{\infty} \frac{\log |P_n(x)|\, dx}{1+x^2} \to \infty \,. \tag{6.11}$$

Invoking Lemma 6.1 we can assume that the P_n have no zeros in $\operatorname{Im} z \geq 0$. Our purpose is to find polynomials $Q_n \in \mathcal{P}$ for which

$$\sup_x \left\{ \frac{\sqrt{1+x^2}}{\Phi(x)} \left| \frac{1}{x-c} - Q_{n-1}(x) \right| \right\} \to 0 \,, \qquad n \to \infty \,.$$

Instead of this, we prove a stronger relation

$$\sup_x \left\{ \frac{\sqrt{1+x^2}}{|P_n(x)|} \left| \frac{1}{x-c} - Q_{n-1}(x) \right| \right\} \to 0 \,, \qquad n \to \infty \,. \tag{6.12}$$

Since $P_n(c) \neq 0$,

$$Q_n(x) := \frac{1}{x-c} - \frac{P_n(x)}{(x-c)P_n(c)}$$

defines a polynomial. For this Q_n, the quantity (6.12) does not exceed

$$\sup_x \left\{ \frac{\sqrt{1+x^2}}{|x-c|} \frac{1}{|P_n(c)|} \right\} \leq \frac{C}{|P_n(c)|} \,,$$

where C depends only on c. It remains to show that $|P_n(c)| \to \infty$. This follows from (6.6). On $\mathbb{R}$, $(x-c)(x-\bar{c})$ is positive and weakly equivalent to $1+x^2$. Therefore, the integral in (6.6) exceeds a constant multiple of the integral (6.11). □

Finally we need

Theorem 6.4 (Pollard's Criterion). *A function Φ is a weight function if and only if $\mu(\Phi) = \infty$ and if there is a sequence P_n of polynomials and a constant $M > 0$ for which*

$$\lim_{n\to\infty} P_n(x) = \Phi(x) \quad , \quad |P_n(x)| \leq M\Phi(x) \,, \qquad x \in \mathbb{R} \,. \tag{6.13}$$

Proof. (a) *Necessity.* We know already that $\mu(\Phi) = \infty$ is necessary. To construct the polynomials P_n, let f_n be continuous functions defined as follows: $f_n(x) = \Phi(x)$ on $[-n, n]$; $f_n(x) = 0$ outside of $[-n-1, n+1]$; on the remaining two intervals, $0 \leq f_n(x) \leq \Phi(x)$. Then $f_n \in C_0$, and $\|f_n\|_\Phi = 1$.

For each n, we take a P_n to satisfy $\|f_n - P_n\|_\Phi \leq \frac{1}{2^n}$, $n = 1, 2, \ldots$. Then $\|P_n\|_\Phi \leq \frac{1}{2} + \|f_n\|_\Phi = \frac{3}{2}$, so that (6.13) is satisfied with $M = \frac{3}{2}$. Also, on $[-n, n]$, $|\Phi(x) - P_n(x)| \leq \frac{1}{2^n}\Phi(x)$, this implies $P_n(x) \to \Phi(x)$ for each fixed x.

(b) *Sufficiency.* For the polynomials P_n^*, which correspond to the P_n by means of Lemma 6.1, we have, because of $1 + |P_n(x)|^2 = |P_n^*(x)|^2$ and of (6.8),

$$\lim |P_n^*(x)| = \sqrt{1+\Phi(x)^2} \quad , \quad 1 \leq |P_n^*(x)| \leq M_1\Phi(x) \,. \tag{6.14}$$

The functions $\log|P_n^*(x)|$ are positive. By Fatou's theorem,

$$\int_{-\infty}^{\infty} \frac{\log\sqrt{1+\Phi(x)^2}}{1+x^2}\,dx \le \liminf \int_{-\infty}^{\infty} \frac{\log|P_n^*(x)|}{1+x^2}\,dx \,.$$

Since $\mu(\Phi)=\infty$, the integral on the left is divergent, and we obtain $\lambda(\Phi)=\infty$. An application of Theorem 6.3 shows that Φ is a weight function. □

The proof just given, and Theorem 6.2 imply that $\lambda(\Phi)=\infty$ holds for each weight function. Therefore:

Theorem 6.5 (Akhiezer and Bernstein). *A function Φ which satisfies* (6.1) *is a weight function if and only if $\Phi(x)/\sqrt{1+x^2}$ is a weight function and if and only if $\lambda(\Phi)=\infty$.*

As an example, we take the functions $e^{|x|}$ and $\operatorname{ch} x$. They are comparable, because their quotient lies between two non-zero constants. Both satisfy $\mu(\Phi)=\infty$. The polynomials P_n of (6.13) for $\operatorname{ch} x$ are provided by $P_n(x)=1+\frac{x^2}{2!}+\cdots+\frac{x^{2n}}{(2n)!}$. Hence both are weight functions.

A little more difficult is the treatment of the functions $\Phi(x)=\exp(|x|/\phi(x))$, where

$$\text{(6.15)} \qquad \phi(x) \text{ is } \log^\alpha(2+|x|) \text{ or } \log(2+|x|)\log^\alpha\log(4+|x|), \text{ or } \cdots \,.$$

If $\alpha>1$, Φ is not a weight function, because $\mu(\Phi)<\infty$. The functions (6.15) are even and satisfy $0<\phi(x)\to\infty$ for $x\to\infty$, and

$$\text{(6.16)} \qquad B\phi(x)\le\phi(\sqrt{x})\,,\quad x>0\,,\qquad \phi(x)x^{-\varepsilon}\to 0\,,\quad x\to\infty$$

for some $B>0$ and each $\varepsilon>0$. The following theorem shows that Φ is a weight function if $\alpha=1$ in (6.15).

Theorem 6.6. *If $\phi(x)>0$, $x\in\mathbb{R}$ is an even function increasing for $x\ge 0$ which satisfies* (6.16), *then $\Phi(x):=\exp(x/\phi(x))$ is a weight function if*

$$\text{(6.17)} \qquad \sum_{k=0}^{\infty}\phi(2^k)^{-1}=\infty\,.$$

Proof. We shall construct a sequence of polynomials P_n with the properties

$$\text{(6.18)} \qquad 1\le P_n(x)\le A\Phi(x/B)\,,\qquad x\in\mathbb{R}\,,\ n=1,2,\ldots$$

for some $A>0$ and with $B\ge 1$ from (6.16), and

$$\text{(6.19)} \qquad \int_{-\infty}^{\infty}\frac{\log P_n(x)\,dx}{1+x^2}\to\infty\,,\qquad \text{for } n\to\infty\,.$$

We put

$$P_n(x) := \sum_{k=0}^{n} u_{2^k}(x) , \qquad u_m(x) := \frac{1}{m!}\left(\frac{x}{\phi(m)}\right)^m , \quad m = 1, 2, \dots .$$

Plainly, $P_n(x)$ is even and at least 1 for $x \in \mathbb{R}$. We define the disjoint intervals $I_k := [2^k\phi(2^k), 2^{k+1}\phi(2^k)]$, $k = 0, 1, \dots$. On I_k, for $m = 2^k$, $x/\phi(m) \ge m$, and by Stirling's formula, if $k \le n$, $P_n(x) \ge u_m(x) \ge m^m/m! \ge e^m/\sqrt{m}$, and $\log P_n(x) \ge Cm$. Therefore,

$$\int_{I_k} (1 + x^2)^{-1} \log P_n(x)\, dx \ge Cm|I_k|\big(2^k\phi(2^k)\big)^{-2} \ge C\phi(2^k)^{-1} .$$

The integral (6.19) exceeds $C\sum_0^n \phi(2^k)^{-1} \to \infty$ as $n \to \infty$.

To prove (6.18), we show that $Q_N(x) := \sum_1^N u_m(x) \le A\Phi(x/B)$. Let $x > 0$ be fixed. In the range $x^{1/2} < m \le N$, by (6.16), $\phi(m) \ge \phi(x^{1/2}) \ge B\phi(x)$ and

$$\Sigma_1 := \sum_{x^{1/2}<m\le N} u_m(x) \le \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{x/B}{\phi(x)}\right)^m \le \Phi(x/B) .$$

In the range $1 \le m \le x^{1/2}$, $u_m(x) \le \phi(0)^{-m}\frac{x^m}{m!} \le \phi(0)^{-m}(ex/m)^m$, and

$$\Sigma_2 := \sum_{1\le m\le x^{1/2}} u_m(x) \le x^{1/2}C^{x^{1/2}}x^{\frac{1}{2}x^{1/2}} .$$

Comparing $\log\Phi(x)$ with $\log\Sigma_2$, we see that, for large x, $\Sigma_2 \le \Phi(x)$, hence $\Sigma_2 \le C\Phi(x)$ for all x. This proves (6.18), and establishes that $A\Phi(x/B)$ and $\Phi(x)$ as well are weight functions. □

§ 7. Spaces of Approximation Theory

Die Funktionenräume C, L_2, L_1
hat Gott geschaffen. Alle anderen
Funktionenräume sind Menschenwerk.
(After Kronecker)

The main theorems of the approximation theory are usually studied for the traditional spaces L_p, $1 \le p \le \infty$; in addition, one replaces L_∞ by C. This is a completely justified point of view. Another would be to seek the most general classes of spaces, for which the theorems apply, and if possible, with the same or reasonably simple proofs. In this section we shall try to answer this problem. (There exist already *conditional* answers: they postulate the validity of certain approximation theorems for the space Y, derive other theorems for Y.)

The spaces under investigation will be *Banach function* (B.f.) *spaces* on $\mathbb{T}$ with the Lebesgue measure μ, and their special case, the *rearrangement-invariant* (r.i.) *spaces* on $\mathbb{T}$, and the means of approximation, the trigonometric polynomials $T_n \in \mathcal{T}_n$ of degrees $n = 0, 1, \ldots$. Spaces L_p, $1 \le p \le \infty$, the Orlicz and the Lorentz spaces are all r.i. For the definitions and properties of the B.f. and r.i. spaces see the book of Bennett and Sharpley [B-1988] and [**CA**, Chapter 1].

In what follows, X, Y will stand for spaces of this type. We shall first indicate the desirable properties of a space Y and then give conditions under which they are valid and how they can be proved.

1. The space Y should be *separable*. Otherwise approximation on Y by trigonometric polynomials will hardly be possible.

2. The *Weierstrass theorem* for Y: for each $f \in Y$ and each $\varepsilon > 0$ there is a T_n with the property

$$\|f - T_n\|_Y < \varepsilon \,. \tag{7.1}$$

3. The *Hardy-Littlewood-Pólya theorem* for Y: If $K(x,t)$ is a kernel on $\mathbb{T}^2$, and if

$$\int_{\mathbb{T}} |K(x,t)|\, dx \le M \ \text{ a.e.}, \quad \int_{\mathbb{T}} |K(x,t)|\, dt \le M \ \text{ a.e.} \tag{7.2}$$

then the operator $U(f)$, given by

$$U(f)(x) = \int_{\mathbb{T}} K(x,t) f(t)\, dt \tag{7.3}$$

is defined on Y and maps this space into itself, with norm $\|U\| \le M$.

4. The *Jackson theorem* for Y: If $E_n(f)$ is the error of the trigonometric approximation of $f \in Y$, given by

$$E_n(f) = \inf_{T_n \in \mathcal{T}_n} \|f - T_n\|_Y \,, \tag{7.4}$$

then

$$E_n(f) \le C_r \omega_r(f, 1/n)_Y \,, \tag{7.5}$$

where $\omega_r(f,h)_Y$ is the r-th modulus of smoothness for Y:

(7.6)

$$\omega_r(f,h)_Y := \|\Delta^r f(x,h)\|_Y \quad , \quad \Delta^r f(x,h) := \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} f(x+kh)$$

and C_r is a constant.

5. One has for $r = 1, 2, \ldots$, $f \in Y$

$$\omega_r(f,h)_Y \to 0 \ \text{ for } \ h \to 0 \,, \ h > 0 \,. \tag{7.7}$$

6. The *Bernstein inequality* for Y: for $T_n \in \mathcal{T}_n$, $r = 1, 2, \ldots$,

$$T_n^{(r)}(x) \prec n^r T_n(x) \ , \tag{7.8}$$

where "$\prec$" is the Hardy-Littlewood-Pólya quasi-inequality relation. Also desirable is

$$\|T_n^{(r)}\|_Y \le n^r \|T_n\|_Y \ . \tag{7.9}$$

7. The *converse approximation theorem* for Y; this estimates the smoothness of f by means of its errors of approximation:

$$\omega_r(f, 1/n)_Y \le C_r n^{-r} \sum_{k=1}^{n} k^{r-1} E_k(f)_Y \ . \tag{7.10}$$

We now give the answers:

1a. A B.f. space X on $\mathbb{T}$ with the Lebesgue measure is the closure of its subset consisting of all bounded measurable functions. Therefore, by a theorem (Bennett and Sharpley [B-1988, Corollary 5.6, p.29]) due to several authors, X is separable if and only if X has *absolutely continuous norm*. This means the following. For each $f \in X$, one has $\|f\chi_e\|_X \to 0$ for $\mu e \to 0$.

If X does not have this property, we replace X by X_c – the space of all functions $f \in X$, approximable in X by continuous functions. From the Weierstrass theorem for C and the inequality $\|\cdot\|_Y \le C\|\cdot\|_\infty$ it follows that X_c is separable. One cannot claim that X_c is a B.f. space, since $f \in X_c$ and $|g| \le f$ a.e. do not necessarily imply that $g \in X_c$.

Examples. 1) One has $(L_\infty)_c = C$.

2) The Lorentz spaces $\Lambda(\phi, p)$, $1 \le p < \infty$ are separable.

3) The Lorentz spaces $M(\phi, p)$, $1 \le p < \infty$ are not separable.

The space $M(\phi, p)_c$ consists of all f with

$$\int_0^c f^* \, dt = o\left(\int_0^c \phi \, dt\right) \tag{7.11}$$

(for spaces Λ, M see [**CA**, p.23]).

For a separable space X one has $X_c = X$. Indeed, by Lusin's theorem, for each $f \in X$ and each $\varepsilon > 0$, there is a continuous function g with the properties $g(x) = f(x)$ except for a set e, $\mu e < \varepsilon$, and $|g| \le \|f\|_\infty$. Then

$$\|f - g\|_X \le \|(f-g)\chi_e\|_X \le 2\|f\chi_e\|_X \ .$$

Theorem 7.1. *A B.f. space X is separable if and only if X has absolutely continuous norm. If X is not separable, then X_c is.*

2a. From **1a** and the classical Weierstrass theorem we get:

Theorem 7.2. *The Weierstrass theorem holds for each separable B.f. space X, and if X is not separable, then for X_c.*

3a. Theorem 7.3. (i) *The Hardy-Littlewood-Pólya theorem holds for each r.i. space X;* (ii) *it holds for each space X_c if in addition $K(x,t)$ is continuous on $\mathbb{T}^2$.*

An equivalent form of (i) has been proved for r.i. *sequence* spaces in the first (1934) edition of the book Hardy, Littlewood, Pólya [B-1968] and for the space of functions, as formulated above, in Lorentz [A-1953, p.78]. It holds also for operators given abstractly (O'Neil-Weiss [1963], [**CA**, Theorem 4.4, p.33]). The proofs, however, are essentially the same for all three cases. For (ii), we note that by [**CA**, Theorem 4.2, p.30] the operator U maps the space $C(\mathbb{T})$ into itself. If $f \in X_c$, then for each $\varepsilon > 0$ there is a $g \in C$ with $\|f - g\|_X < \varepsilon$. By (i), $\|Uf - Ug\|_X \le M\varepsilon$, and since $Ug \in C$ we get $Uf \in X_c$. □

4a. A proof of Jackson's theorem for a r.i. space X is given in [**CA**, (2.13), p.205]. For this to be valuable, we must have (7.6).

5a. Theorem 7.4. *Relation* (7.7) *holds for each separable r.i. space X, and if X is not separable, then for X_c.*

Proof. It is sufficient to prove (7.7) for all $f \in X_c$. The translation operator $f(t) \to f(x+t)$ for a fixed $x \in \mathbb{T}$ maps X into itself with norm one, hence the difference operator $f(t) \to \Delta^r f(t,h)$ maps X_c into itself with norm $\le 2^r$. Since $\Delta^r f(t,h) \to 0$ in X for each continuous f, by the Banach-Steinhaus theorem, the same happens for all $f \in X_c$. □

6a. According to Lorentz [1984] and [**CA**, p.102] one has

Theorem 7.5. *Relations* (7.8) *and therefore* (7.9) *hold for each r.i. space X.*

7a. Theorem 7.6. *Relation* (7.10) *holds for each r.i. space X.*

Proof. Let $W^r X$ be the space of all functions f on $\mathbb{T}$ for which $f, f', \dots, f^{(r-1)}$ are absolutely continuous and $f^{(r)} \in X$. An argument of Johnen [1972], [**CA**, p.177] and Theorem 7.4 yield the equivalence of the K-functional $K(f, t^r, X, W^r X) =: K(f, t^r)$ and of $\omega_r(f,t)_X$:

$$C_r \omega_r(f,t)_X \le K(f, t^r, X, W^r X) \le C_r' \omega_r(f,t)_X \; . \tag{7.12}$$

We then use a theorem of Butzer and Scherer [1972] given in [**CA**, Theorem 5.1, p.216]; in the theorem we select $\Phi_n := \mathcal{T}_n$, $Y = W^r X$. Then the inequality [**CA**, (5.5), p.216] is the Bernstein inequality (7.9), and therefore we have the relation [**CA**, (5.8), p.217], namely

$$K(f, n^{-r}) \le C_r n^{-r} \sum_{k=1}^{n} k^{r-1} E_k(f)_X \ .$$

Combining this with (7.12), we get (7.10). □

Conclusion. The main approximation theorems on $\mathbb{T}$ hold for each separable r.i. space X, and for each X_c, if X is r.i.

The case of spaces on a compact interval $[a, b]$ is similar. It would be interesting to decide whether our ideas apply to the known estimates of widths or entropy in Sobolev spaces.

§ 8. Problems and Notes

Problems

8.1. There is an equidistribution of the alternation points in Theorem 2.4 *for all n:*

$$A_n(f, I) = \frac{|I|}{\pi}(1 + o(1))n \ ,$$

if $E_{n+1} \le (1 - \varepsilon_n)E_n$, $n = 1, 2, \ldots$, where $\varepsilon_n \to 0$ and $\log(1/\varepsilon_n) = o(n)$.

8.2. Let $f(t) = \sum_{k=0}^{\infty} a_k \cos kt$, where a_k are positive, decreasing, and satisfy $a_k^2 \le a_{k+1}a_{k-1}$, $k \ge 1$. Then the degree of approximation of f can be estimated by

$$E_n(f) \le \sum_{k=n+1}^{\infty} a_k \le 4eE_n(f)$$

(Newman and Rivlin [1976]).

8.3. Let c_k, $k = 1, \ldots, n$ be distinct, not real numbers. Show that the functions $1/(x - c_k)$ form a Haar system on $\mathbb{R}$.

Notes

8.4. If $f \in C[-1, 1]$ is odd, so are its best approximation polynomials P_n^* (therefore they are of odd degree), consequently they vanish at 0. It has been conjectured that this can be inverted: if $P_{2k+1}^*(0) = 0$ for $k = 0, 1, \ldots$, then f must be odd. The conjecture is still open, although Saff and Varga [1980] proved it for entire funtions of a certain type.

8.5. Newman and Rivlin [1976] gave a very exact estimate for the degree of the uniform approximation of x^N on $[0, 1]$ by polynomials of degree $\le n$ for $n < N$.

They show that

$$\frac{1}{4e} P_{N,n} \le E_n(x^N)_\infty \le P_{N,n} , \tag{8.1}$$

where

$$P_{N,n} = \frac{1}{2^{N-1}} \sum_{j>(N+n)/2} \binom{N}{j} .$$

One can interpret $P_{N,n}$ as a probability, and then one gets that $P_{N,n}$ can be well approximated by the probability integral

$$\frac{1}{\sqrt{2\pi}} \int_{n/\sqrt{N}}^{\infty} e^{-t^2/2}\, dt .$$

In particular, if $N = N(n)$, then $E_n(x^N) \to 0$ if and only if $N = o(n^2)$. One can obtain this latter result also from the theorems of Chapter 11.

8.6. Theorem 3.4 is not true for all $n \to \infty$ instead of the normal subsequences. There exists a non-analytic function $f \in C[-1,1]$ and a sequence of its polynomials $P_{n_k}^*$ so that, for each $\rho > 1$, all zeros of the $P_{n_k}^*$, $k \ge K(\rho)$ lie outside of E_ρ (Ivanov, Saff and Totik [1991]).

8.7. The history of Bernstein's problem is interesting. Bernstein himself [1924] studied functions

$$\Phi(x) = c_0 + c_1 x^2 + \cdots + c_n x^{2n} + \cdots , \qquad c_n \ge 0 , \ c_0 = 1 \tag{8.2}$$

and established that Φ is a weight function if and only if it is an entire function of genus > 0. For the class (8.2), this proved to be equivalent to $\mu(\Phi) = +\infty$. P. Hall [1939] proved that $\mu(\Phi) = +\infty$ is necessary for any Φ. The solution of Mergelyan [1956_1] is similar to Theorem 6.6; his necessary and sufficient condition is

$$\int_{-\infty}^{\infty} (1+x^2)^{-1} \sup_{P \in \mathcal{M}_\Phi} |\log P(x)|\, dx = \infty . \tag{8.3}$$

For details and references see the expository articles of Akhiezer [1956] and Mergelyan [1956_2]. For an abstract approach to Bernstein's problem see Nachbin [A-1967].

Chapter 2. Polynomial Approximation with Constraints

§ 1. Introduction

In this chapter we discuss the approximation of functions f by polynomials $P_n(x) = \sum_0^n a_i x^i$ which satisfy some additional restrictions: We restrict the size of the coefficients a_i in §2; we assume in §3 that the P_n and the approximated functions f are monotone. In §§4 – 5, the coefficients a_i are integers. To be approximable, a function f must possess integral values $f(x)$ at integral points x of its domain of definition. There may be other, less obvious restrictions for f. This makes the two sections essentially algebraic in character.

In §6, improvements of Markov inequalities are given, which are possible for some P_n. The main assumption is that P_n should have none or few zeros in certain critical regions. These regions are the unit disk D_1, or even two disks D_ρ^+, D_ρ^- of radius ρ and centers $1-\rho$, $-1+\rho$ with arbitrary small $\rho > 0$. Then one has $\|P_n^{(r)}\|_\infty \le Cn^r \|P_n\|_\infty$ with C independent of n. Given $|P_n(x)| \le M$ on a subset $e \subset [-1,1]$, the useful Remez inequality of §7 estimates the norm $\|P_n\|_\infty$ in terms of M and the measure of e. In §8, we treat two-sided and one-sided approximation, with the (trigonometric) polynomials S_n, T_n restricted by $S_n(x) \le f(x) \le T_n(x)$ or by $S_n(x) \le f(x)$.

§ 2. Growth Restrictions for the Coefficients

We treat here the following question: For which sequences $A = (A_k)_1^\infty$ of non-negative real numbers are the algebraic polynomials

$$\mathcal{P}_A = \{P(x) = \sum_{k=0}^{n} a_k x^k : \ |a_k| \le A_k, \ k = 1, \dots, n, \ n = 1, 2, 3, \dots\}$$

dense in $C[a,b]$? By Λ we shall denote any increasing sequence $\Lambda : 0 < k_1 < \cdots < k_j < \cdots$ of positive integers. We shall see that the problems of this section are related to the density or non-density properties of *Müntz polynomials* $Q(x) = \alpha_0 + \sum_{j=1}^m \alpha_j x^{k_j}$ in $C[0,1]$. They are treated in [**CA**, Chapter 11, §§5,6]; see also Chapter 11.

Stafney [1967] and Roulier [1972] proved the necessity conditions in the next theorem. Their sufficiency and other results of this section were given by v. Golitschek [1973], and v. Golitschek and Leviatan [1977].

Theorem 2.1. *The polynomial space $\mathcal{P}_A$ is dense in $C[0,1]$ if and only if there exists a sequence $\Lambda = (k_j)_{j=1}^{\infty}$ with the properties*

$$\lim_{j\to\infty}(A_{k_j})^{1/k_j} = \infty \text{ and } \sum_{j=1}^{\infty} 1/k_j = \infty. \tag{2.1}$$

Proof. (a) *Necessity.* Assume that Λ with the property (2.1) does not exist. Then

(2.2) *there exists a sequence $\Lambda = (k_j)_1^{\infty}$ and a number $M \ge 1$ for which*

$$\sum_{j=1}^{\infty} 1/k_j < \infty \text{ and } (A_k)^{1/k} \le M \text{ for all } k \notin \Lambda.$$

Indeed, if (2.2) were not true, that is, if the sums $\sum_{k\in\mathcal{N}_M} 1/k$ diverge for all sets $\mathcal{N}_M := \{k : A_k^{1/k} > M\}$, $M \ge 1$, then we take from $\mathcal{N}_1$ numbers $k_1 < \ldots < k_{n_1}$ so that $1 \le \sum_{i=1}^{n_1} 1/k_i \le 2$; we take from $\mathcal{N}_2$ numbers ($k_{n_1} <$) $k_{n_1+1} < \cdots < k_{n_2}$ so that $1 \le \sum_{j=n_1+1}^{n_2} 1/k_j \le 2$, and so on. The sequence $(k_j)_1^{\infty}$ constructed in this way satisfies (2.1), a contradiction to the above assumption.

We denote by Λ_n and Λ_n', respectively, the subsets of $\{1,\ldots,n\}$ that belong or do not belong to Λ. According to [**CA**, Theorem 5.1, p. 345], the condition $\sum 1/k_j < \infty$ implies that the polynomials $Q(x) = \alpha_0 + \sum_1^m \alpha_j x^{k_j}$ are not dense in $C[0,1]$; hence they are not dense in $C(J)$, where $J := [0, 1/(2M)]$. There exists a function $f^* \in C(J)$ with $\|f^* - Q\|_{C(J)} \ge 2$ for all Q. Now let $P(x) := \sum_{k=0}^n a_k x^k$, $P \in \mathcal{P}_A$. Since $|a_k x^k| \le A_k/(2M)^k \le 2^{-k}$, $x \in J$, for all k, we have

$$\begin{aligned}\|f^* - P\|_{C(J)} &\ge \|f^*(x) - a_0 - \sum_{k\in\Lambda_n} a_k x^k\|_{C(J)} - \sum_{k\in\Lambda_n'} \|a_k x^k\|_{C(J)}\\ &\ge 2 - 1 = 1.\end{aligned}$$

This shows that the set $\mathcal{P}_A$ is not dense in C(J) therefore not dense in $C[0,1]$.

(b) *Sufficiency.* Let Λ be a sequence which satisfies (2.1). We prove more than needed:

(2.3) *For each $f \in C[0,1]$, $f(0) = 0$, and each $0 < \varepsilon < 1$ there is a polynomial $P_n(x) = \sum_1^n a_k x^k$ for which $|a_k| \le \varepsilon A_k$ for $k \in \Lambda$, $a_k = 0$ for $k \notin \Lambda$, and $\|f - P_n\| < \varepsilon$.*

The set of all f with this property is a linear subset of $C[0,1]$. Plainly, it is sufficient to prove (2.3) for all monomials x^q, $q \ge 1$.

For large r we have $k_r \ge 2q+1$. For r of this type, we select $s = s(r)$ so that

$$\log(1/\varepsilon) \le \sum_{j=r}^{s} \frac{1}{k_j} \le 1 + \log(1/\varepsilon).$$

Let $\mu_\infty = \mu(x^q)_\infty$ be the error of uniform approximation of x^q on $[0,1]$ by linear combinations of powers x^{k_j}, $r \le j \le s$. We have for some c_j,

$$\mu_\infty = \|x^q - \sum_{j=r}^{s} c_j x^{k_j}\|.$$

By [**CA**, Theorem 5.5, p.347] and the inequality

$$(k_j - q)/(k_j + q) \le \exp(-2q/k_j), \quad r \le j \le s,$$

we obtain

$$\mu_\infty \le \prod_{j=r}^{s} \frac{k_j - q}{k_j + q} \le \varepsilon^{2q}. \tag{2.4}$$

To estimate the coefficients c_i, $i = r, r+1, \ldots, s$, we compare μ_∞ with the L_2-error of approximation on [0,1] (see [**CA**, Theorem 5.4, p.346]):

$$\begin{aligned}\mu(x^{k_i})_2 &:= \min_{\alpha_j} \|x^{k_i} - \alpha_0 x^q - \sum_{j=r, j\ne i}^{s} \alpha_j x^{k_j}\|_2 \\ &= \frac{1}{\sqrt{1+2k_i}} \frac{k_i - q}{k_i + q + 1} \prod_{j=r, j\ne i}^{s} \frac{|k_j - k_i|}{k_j + k_i + 1}.\end{aligned}$$

From the definition of the μ_∞ it follows that

$$|c_i| \; \mu(x^{k_i})_2 \le \mu_\infty$$

and thus

$$|c_i| \le 3 \; \varepsilon^{2q} \; \sqrt{1+2k_i} \prod_{j=r, j\ne i}^{s} \frac{k_j + k_i + 1}{|k_j - k_i|}. \tag{2.5}$$

(We have used that $k_i \ge 2q+1$ and $(k_i + q + 1)/(k_i - q) \le 3$.) For convenience we set $m := k_i$. We estimate the product in (2.5) by splitting it into two factors, corresponding to $k_j < 2m$ and $k_j \ge 2m$. For the first of them, by Stirling's formula,

$$\prod_{k_j < 2m, j \ne i} \frac{k_j + m + 1}{|k_j - m|} \le \prod_{\nu=1, \nu\ne m}^{2m-1} \frac{\nu + m + 1}{|\nu - m|} \le \frac{m^2 (3m)^{2m-2}}{(m!)^2} < \frac{(3e)^{2m}}{9m},$$

and for the second factor,

$$\begin{aligned}\prod_{k_j \ge 2m} \frac{k_j + m + 1}{k_j - m} &= \prod_{k_j \ge 2m} \left(1 + \frac{2m+1}{k_j - m}\right) \le \prod_{k_j \ge 2m} \left(1 + \frac{5m}{k_j}\right) \\ &\le \exp\left(\sum_{k_j \ge 2m} \frac{5m}{k_j}\right) \le (e/\varepsilon)^{5m}.\end{aligned}$$

It follows that

$$|c_i| \le \frac{3\ \sqrt{1+2k_i}}{9k_i} 3^{2k_i} e^{7k_i} \varepsilon^{2q-5k_i} \le \left(9e^7\varepsilon^{-5}\right)^{k_i}.$$

We can take r so large that

$$9e^7\varepsilon^{-6} \le (A_{k_i})^{1/k_i}, \quad \text{for all } i \ge r.$$

Then we have $|c_i| \le \varepsilon^{k_i} A_{k_i} \le \varepsilon A_{k_i}$, $i = r, \dots, s = s(r)$. This and (2.4) yield (2.3) for $f(x) = x^q$. □

On an interval $[a, b]$, $a > 0$, or if f vanishes in a neighborhood of zero, the rate of the increase of the coefficients can be much slower.

Theorem 2.2. *Let $0 < a < 1$. For each $f \in C[0,1]$ with $f(x) = 0$ for $0 \le x \le a$, there exists a sequence of polynomials $P_n(x) = \sum_{k=1}^n a_{k,n}x^k$ such that uniformly on $[0,1]$*

$$\lim_{n\to\infty} \|f - P_n\| = 0 \tag{2.6}$$

and

$$|a_{k,n}| \le a^{-k}\|f\|, \quad k = 1, \dots, n. \tag{2.7}$$

Proof. Let $\varepsilon > 0$ be given. There exists a $0 < \delta < 1 - a$ so that $|f(x)| \le \varepsilon/2$ for $0 \le x \le a + \delta$. Now there is a function $g \in C[0,1]$, $g(x) = 0$ for $0 \le x \le a + \delta$, so that $\|g\| \le \|f\|$ and $\|f - g\| \le \varepsilon/2$. We choose an integer p so large that $(2e)^{1/p} \le (a + \delta)/a$, set $R := (a + \delta)^p$.

For $N \in \mathbb{N}$ we define $m = m(N) \in \mathbb{N}$ by $RN < m \le RN + 1$. The function $G(x) := g(x^{1/p})$ is in $C[0,1]$ and vanishes in $0 \le x \le R$. Its Bernstein polynomial of degree $\le N$ is of the form

$$B_N(G, x) = \sum_{j=m}^{N} G(j/N)\binom{N}{j} x^j(1-x)^{N-j} =: \sum_{j=m}^{N} b_j x^j$$

where

$$b_j = b_{j,N} = \sum_{i=m}^{j} (-1)^{j-i} G(i/N)\binom{N}{i}\binom{N-i}{j-i}.$$

This implies that

$$|b_j| \le \|f\| \sum_{i=m}^{j} \binom{N}{i}\binom{N-i}{j-i} \le \|f\| \sum_{i=m}^{j} \frac{N^j}{i!(j-i)!} \le \|f\| \frac{(2N)^j}{j!},$$

and the inequality $j! \ge j^j e^{-j}$ yields

$$|b_j| \le \|f\|(2e/R)^j \le \|f\| a^{-jp}, \quad RN < j \le N.$$

The polynomial $P_n(x) := B_N(G, x^p) = \sum_{j=m}^{N} b_j x^{jp}$ is of degree $\le n := Np$; its coefficients $a_{k,n}$ vanish for $k \ne jp$, $j = m, \dots, N$. For $k = jp$ we obtain (2.7) since $|a_{k,n}| = |b_j| \le \|f\| a^{-k}$. By [**CA**, Theorem 3.2, p.308], $B_N(G,x)$ converge to $G(x)$ uniformly on $[0,1]$, as $N \to \infty$, hence $P_n(x)$ converges to $g(x)$ uniformly on $[0,1]$ and $\|f - P_n\| < \varepsilon$ for large $N = n/p$. □

Corollary 2.3. *Let* $0 < a < b$. *Each* $f \in C[a,b]$, $f(a) = 0$, *is the uniform limit of some sequence of polynomials* $P_n(x) = \sum_{k=1}^{n} a_{k,n} x^k$, *whose coefficients are bounded by*

$$|a_{k,n}| \le a^{-k} \|f\|_{C[a,b]}, \quad k = 1, \dots, n. \tag{2.8}$$

It is obvious that the number a on the right-hand sides of (2.7) and (2.8) cannot be replaced by a larger number a', since this would imply that f is analytic in the interval $[a, a')$.

§ 3. Monotone Approximation

In this section we approximate functions by monotone, convex, etc. polynomials. We define $\mathcal{M}_n \subset \mathcal{P}_n$ for $[-1,1]$ to consist of all increasing polynomials of degree $\le n$, that is, of all $P_n \in \mathcal{P}_n$ with $P_n'(x) \ge 0$, $x \in [-1,1]$. More general is the set $\mathcal{M}_n^* := \mathcal{M}_n^*(k_1, \dots, k_p;\ \varepsilon_1, \dots, \varepsilon_p)$ with given integers $0 < k_1 < \cdots < k_p \le n$, $\varepsilon_i = \pm 1$, $i = 1, \dots, p$. It consists of all $P_n \in \mathcal{P}_n$ that satisfy

$$\varepsilon_i P_n^{(k_i)}(x) \ge 0, \quad x \in [-1,1], \quad i = 1, \dots, p. \tag{3.1}$$

For example, $\mathcal{M}_n = \mathcal{M}_n^*(1; +1)$.

First papers on this subject are by Lorentz and Zeller [1968],[1969],[1970], R.A.Lorentz [1971], Lorentz [1972]. This was followed by several beautiful results of later authors, but some questions still remain without answer.

The existence of monotone polynomials of best approximation is standard. Their uniqueness and characterization were given by Lorentz and Zeller for $\mathcal{M}_n$, by R.A.Lorentz for $\mathcal{M}_n^*$. Thus, we have :

Theorem 3.1. *For each* $f \in C[-1,1]$ *there is a unique polynomial of best uniform approximation to* f *from* $\mathcal{M}_n$ *and from* $\mathcal{M}_n^*$.

It is immaterial whether the function f itself is monotone or not. The proof (which we do not reproduce, see Lorentz, Jetter and Riemenschneider [A-1983, p.127]) is based on properties of Birkhoff interpolation.

Estimates of the error of monotone approximation of f in the uniform norm on $[-1,1]$,

$$E_n^*(f) := \min\{\|f - P_n\| : \ P_n \in \mathcal{M}_n\} \tag{3.2}$$

will depend, of course, on the assumption that f is itself monotone.

We shall discuss Jackson type estimates for the errors. We begin with some simple lemmas which use spline approximation.

Lemma 3.2. *Let $f \in C[-1,1]$ and $n \geq 1$. Let L be the piecewise linear function on $[-1,1]$, which interpolates f at the points $x_j := -1 + jh$, $j = 0, \dots, n$, $h := 2/n$. Then,*

$$\|f - L\| \leq \omega_2(f, h/2) = \omega_2(f, 1/n). \tag{3.3}$$

Proof. We denote $F := f - L$. Let the maximum $\|F\| = \|f - L\|$ of $|F|$ be attained at x^*, $x^* \in (x_{k-1}, x_k)$ for some k. Let $h_0 := \min(x^* - x_{k-1}, x_k - x^*)$. At least one of the points $x^* \pm h_0$ coincides with x_{k-1} or x_k, hence at least one of the values $F(x^* \pm h_0)$ vanishes. This implies that

$$|F(x^* - h_0) - 2F(x^*) + F(x^* + h_0)| \geq |F(x^*)| = \|f - L\|. \tag{3.4}$$

The three points $x^*, x^* \pm h_0$ lie in $I_k := [x_{k-1}, x_k]$ and L is linear in I_k. It follows that

$$|f(x^* - h_0) - 2f(x^*) + f(x^* + h_0)| = |F(x^* - h_0) - 2F(x^*) + F(x^* + h_0)|.$$

This and (3.4) yield (3.3) since $h_0 \leq h/2$. □

The term $\omega(L, h^2)$ can be much larger than $\omega_2(L, h)$. But we have

Lemma 3.3. *If in the situation of Lemma 3.2 the function $f \in C[-1,1]$ satisfies $f(x_{k+1}) = f(x_k)$ for some k, then*

$$\omega(L, h^2) \leq 2\omega_2(L, h). \tag{3.5}$$

Proof. Let

$$\alpha_j := L(x_{j+1}) - L(x_j) = f(x_{j+1}) - f(x_j), \ j = 0, \dots, n-1. \tag{3.6}$$

Then we have $\alpha_k = 0$ and Then we have $\alpha_k = 0$ and

$$\omega(L, h^2) = h \max_{0 \leq j < n} |\alpha_j| \quad , \quad \omega_2(L, h) = \max_{0 \leq j < n-1} |\alpha_{j+1} - \alpha_j|. \tag{3.7}$$

We choose α_j so that $h|\alpha_j| = \omega(L, h^2)$. Then,

$$\omega(L, h^2) = h|\alpha_j| = h|\alpha_j - \alpha_k| \leq hn\omega_2(L, h) = 2\omega_2(L, h).$$ □

For the needs of approximation on $\mathbb{T}$, we prove:
If $g(t) = f(\cos t)$, for $f \in C[-1,1]$, then

$$\omega_2(g, h) \leq \omega(f, h^2) + \omega_2(f, h). \tag{3.8}$$

Indeed, if $x = \cos t$, and $x, x + h \in [-1,1]$, then we put $h_1 := \cos(t + h) - \cos t$ with $|h_1| \leq |h|$ and have

$$\Delta_h^2 g(t) = \{f(\cos t) - 2f(\cos(t+h)) + f(2\cos(t+h) - \cos t)\}$$
$$+ \{f(\cos(t+2h)) - f(2\cos(t+h) - \cos t)\}.$$

The first term does not exceed $\omega_2(f, h_1)$, the second is $\leq \omega(f, h^2)$, since the difference in the arguments in it is $\leq h^2$.

To monotone functions $f(x)$ on the interval $[-1, 1]$, $f \in C[-1, 1]$, there correspond, by means of the substitution $x = \cos t$, *bell-shaped functions* $g(t)$ on $[-\pi, \pi]$, that means even functions that increase on $[-\pi, 0]$.

Theorem 3.4. *If the function $g \in C(\mathbb{T})$ is bell-shaped, then for each even n there is a bell-shaped trigonometric polynomial $T_n \in \mathcal{T}_n$ so that in the uniform norm on $\mathbb{T}$,*

$$\|g - T_n\| \leq C\omega_2(g, 1/n), \tag{3.9}$$

where C is an absolute constant.

Proof. We can assume that $n/2$ is odd so that $m := 1 + n/2$ is an even integer. Let $L(t)$ be the piecewise linear function on $\mathbb{T}$ that interpolates g at the knots $t_k := 2\pi k/m$, $k = 0, \ldots, \pm m/2$. Let $J_n(L) = K_n * L$ be the Jackson integral of L with the kernel

$$K_n(t) = \lambda_n \left(\frac{\sin(mt/2)}{\sin(t/2)}\right)^4, \quad \int_{\mathbb{T}} K_n(t)dt = 1,$$

with the normalizing constant λ_n. By Jackson's theorem [**CA**,p.204],

$$\|L - J_n(L)\| \leq C\omega_2(L, 1/n).$$

Using Lemma 3.2 for the function $g(\pi x)$ we get

$$\|g - L\| \leq \omega_2(g, \pi/m) \leq C\omega_2(g, 1/n).$$

This implies that

$$\omega_2(L, 1/n) \leq 4\|L - g\| + \omega_2(g, 1/n) \leq C'\omega_2(g, 1/n)$$

for some constant C'. It follows from the above inequalities that (3.9) is satisfied for $T_n := J_n(L)$ and a new constant C.

It remains to show that $T_n = J_n(L)$ is bell-shaped. Let $\ell_k(t)$ be the function which is 1 for $|t| \leq t_k$, is 0 for $t_{k+1} \leq |t| \leq \pi$, and is obtained by linear interpolation on the two remaining intervals of $[-\pi, \pi]$. Since $L - L(\pi)$ is a linear combination with non-negative coefficients of the ℓ_k, $k = 0, \ldots, \frac{1}{2}m - 1$, it is sufficient to show that the $J_n(\ell_k)$ are bell-shaped. We have

$$J_n(\ell_k, x) = \int_{-\pi}^{\pi} \ell_k(x+t)K_n(t)dt, \tag{3.10}$$

and we will show that $J_n'(\ell_k, x)$ is negative for $0 < x < \pi$, $k = 0, 1, \ldots, \frac{1}{2}m - 1$. Let

$$F(y) = \int_{y-\pi/m}^{y+\pi/m} K_n(t)dt.$$

We observe that $F(-y) = F(y)$ and $F(2\pi - y) = F(y)$ for all y. Since $\sin^4 \frac{m}{2}(y + \frac{\pi}{m}) = \sin^4 \frac{m}{2}(y - \frac{\pi}{m}) = \cos^4 \frac{my}{2}$, we have for $0 < y < \pi$

$$\begin{aligned}
F'(y) &= K_n(y + \pi/m) - K_n(y - \pi/m) \\
&= \lambda_n \cos^4 \frac{my}{2} \left(\sin^{-4} \frac{1}{2}(y + \frac{\pi}{m}) - \sin^{-4} \frac{1}{2}(y - \frac{\pi}{m}) \right) < 0,
\end{aligned}$$

that is, F decreases for $0 < y < \pi$.

Let $\mu_k > 0$ be the slope of ℓ_k on the interval $[-t_{k+1}, -t_k]$, that is, $\mu_k = m/(2\pi)$. Differentiating (3.10) we obtain

$$\begin{aligned}
J_n'(\ell_k, x) &= \int_{-\pi}^{\pi} \ell_k'(x + t) K_n(t)dt \\
&= \mu_k \int_{-x-t_{k+1}}^{-x-t_k} K_n(t)dt - \mu_k \int_{-x+t_k}^{-x+t_{k+1}} K_n(t)dt \\
&= \mu_k F(-x - t_k - \pi/m) - \mu_k F(-x + t_k + \pi/m) \\
&= \mu_k F(x + t_k + \pi/m) - \mu_k F(x - t_k - \pi/m).
\end{aligned}$$

This leads to the inequality $J_n'(\ell_k, x) \le 0$, $0 \le x \le \pi$, since $F(x + t) \le F(x - t) = F(t - x)$ for all $0 \le x, t \le \pi$. Indeed, if $x + t < \pi$, we use the fact that F decreases in $[0, \pi]$. And if $\pi \le t + x \le 2\pi$, we replace t, x by $t' := \pi - t$, $x' := \pi - x$. □

Theorem 3.5. *If $f \in C[-1, 1]$ is increasing, then*

$$E_n^*(f) \le C\omega_2(f, 1/n), \tag{3.11}$$

and therefore if in addition $f \in C^1[-1, 1]$, then

$$E_n^*(f) \le Cn^{-1}\omega(f', 1/n), \tag{3.12}$$

for some absolute constant C.

Proof. Let L be the piecewise linear function on $[-1, 1]$ that interpolates f at the knots $x_j := -1 + jh$, $j = 0, \dots, n$, $h := 2/n$. Let α_k be the minimum of the numbers α_j, $j = 0, 1, \dots, n - 1$ defined in (3.6). Then $\alpha_k \ge 0$ and $L_1(x) := L(x) - \alpha_k x/h$ is increasing on $[-1, 1]$. Hence the function $g(t) := L_1(\cos t)$ is bell-shaped. Let $T_n \in \mathcal{T}_n$ be the bell-shaped trigonometric polynomial of Theorem 3.4 for g. The polynomial

$$P_n(x) := \alpha_k x/h + T_n(\arccos x)$$

is increasing on $[-1, 1]$ and satisfies

$$\|f - P_n\| \le \|f - L\| + \|g - T_n\| \le \omega_2(f, 1/n) + C\omega_2(g, 1/n).$$

Since $L_1(x_{k+1}) = L_1(x_k)$, we have (3.5) for L_1 and thus, using the inequality (3.8),

$$\omega_2(g, 1/n) \le \omega(L_1, h^2) + \omega_2(L_1, h) \le 3\omega_2(L_1, h) = 3\omega_2(L, h).$$

From this we deduce (3.11), since

$$\omega_2(L, h) \le \omega_2(f, h) + 4\|f - L\| \le C'\omega_2(f, 1/n). \qquad \square$$

For the errors of best approximation of a monotone function f by monotone and by non-restricted algebraic polynomials one has the same estimates (3.11), (3.12) (and (3.19) for $f \in C^r[-1,1]$, all $r \ge 0$). The difference could be only in the value of the constants. Of course, for each fixed monotone function $f \in C[-1,1]$, one has $E_n(f) \le E_n^*(f)$, and one could think that $E_n(f) \sim E_n^*(f)$. But this is not correct; this follows from the following theorem of Lorentz and Zeller [1969]:

Theorem 3.6. *For each $k = 1, 2, \ldots$ there exists a function $f \in C^k[-1,1]$, $f^{(k)} \ge 0$ with the property that*

$$\limsup_{n\to\infty} \frac{E_n^*(f)}{E_n(f)} = \infty \ , \tag{3.13}$$

where $E_n^*(f)$ is the error of approximation of f by $P_n \in \mathcal{P}_n$ with $P_n^{(k)} \ge 0$.

Proof. We begin by constructing an elementary singular situation which we shall use inductively.

(*) *For each $b > 0$ there exist a polynomial p of degree $\le 3k + 4$ and a polynomial P of degree $N > 3k + 4$ so that, in the uniform norm on $I := [-1,1]$,*

$$\begin{aligned} &\|p\| \le 1, \quad \|P\| \le 1, \quad P^{(k)} \ge 0, \quad P^{(k)}(0) = 0, \\ &p^{(k)}(0) < -b\ \|p - P\|. \end{aligned} \tag{3.14}$$

Let $\nu := k+2$ if k is odd, $\nu := k+1$ if k is even. We take, for some $0 < a < 1$,

$$p(x) := (x^2 - a^2)^\nu x^k.$$

Let $g(x) := 0$ on $[-a, a]$, $:= p(x)$ elsewhere on I. Clearly $g \in C^k(I)$, $g^{(k)} \ge 0$. We can approximate $g^{(k)}$ arbitrarily closely by a non-negative polynomial Q of degree $> 2k + 4$ with $Q(0) = 0$. Then we approximate $g^{(k-1)}$ by $\int_0^x Q(t)dt$ on I. After k steps we get a polynomial P satisfying $\|g - P\| < \varepsilon$, $P^{(k)} \ge 0$, $P^{(k)}(0) = 0$. We take $\varepsilon := a^{2\nu+k}$. Since $\|p - g\| = \max_{|x|\le a} |p(x)| \le a^{2\nu+k}$, we have then

$$\|p - P\| \le 2a^{2\nu+k}, \quad p^{(k)}(0) = -k!a^{2\nu}.$$

For a given b we can achieve the last inequality in (3.14) by taking $0 < a < 1$ small enough.

We now define sequences $b_j \to \infty$ of positive numbers, of polynomials p_j of degree $\le 4k+3$, and of polynomials P_j of degree N_j. We can assume that $4k+3 < N_1 < N_2 < \dots$. If b_{j-1}, p_{j-1}, P_{j-1} are known, we take $b_j := (2j+2)N_{j-1}^{2k}$ and define p_j and P_j by means of (*).

The function f will be given by the series

$$f := \sum_{j=1}^{\infty} c_j P_j$$

where the c_j satisfy $0 < c_j < N_j^{-2k}/j!$ and

$$\sum_{j=n+1}^{\infty} c_j \le c_n \|p_n - P_n\|, \quad n = 1, 2, \dots. \tag{3.15}$$

The function f is in $C^k[-1,1]$ since by Markov's inequality, $\|c_j P_j^{(k)}\| \le c_j N_j^{2k} \|P_j\| < 1/j!$. In addition, $f^{(k)} \ge 0$ on I. For each n let

$$f_n := \sum_{j=1}^{n} c_j P_j, \quad \Pi_n := \sum_{j=1}^{n-1} c_j P_j + c_n p_n.$$

Using (3.15) we have

$$\|f - f_n\| = \| \sum_{j=n+1}^{\infty} c_j P_j \| \le c_n \|p_n - P_n\|.$$

Since Π_n is a polynomial of degree N_{n-1} and $f_n - \Pi_n = c_n(P_n - p_n)$, we obtain

$$E_{N_{n-1}}(f) \le \|f - \Pi_n\| \le \|f - f_n\| + \|f_n - \Pi_n\| \le 2c_n \|p_n - P_n\|. \tag{3.16}$$

On the other hand, let R be the polynomial of degree $\le N_{n-1}$ with $R^{(k)} \ge 0$ on I which satisfies $E^*_{N_{n-1}}(f_n) = \|f_n - R\|$. Since $P_j^{(k)}(0) = 0$, it follows by (3.14) that $\Pi_n^{(k)}(0) = c_n p_n^{(k)}(0)$ is negative and

$$|\Pi_n^{(k)}(0)| > c_n b_n \|p_n - P_n\|. \tag{3.17}$$

Then, Markov's inequality yields

$$\begin{aligned} |\Pi_n^{(k)}(0)| &\le |\Pi_n^{(k)}(0) - R^{(k)}(0)| \le N_{n-1}^{2k} \|\Pi_n - R\| \\ &\le N_{n-1}^{2k} \left(\|f_n - R\| + \|f_n - \Pi_n\| \right) \\ &= (2n+2)^{-1} b_n \left(E^*_{N_{n-1}}(f_n) + c_n \|P_n - p_n\| \right). \end{aligned}$$

This and (3.17) imply

$$E^*_{N_{n-1}}(f_n) \ge (2n+2) b_n^{-1} |\Pi_n^{(k)}(0)| - c_n \|P_n - p_n\| \ge (2n+1) c_n \|P_n - p_n\|,$$

and we obtain the inequalities

$$E^*_{N_n-1}(f) \geq E^*_{N_n-1}(f_n) - \|f - f_n\| \geq 2nc_n\|p_n - P_n\|.$$

By using (3.16),

$$E^*_{N_n-1}(f) \geq nE_{N_n-1}(f)$$

and the theorem follows. □

Among the newer difficult results we mention

Theorem 3.7 (Hu, Leviatan and Yu [1994]). *A convex continuous function f on $[-1,1]$ possesses approximating convex algebraic polynomials for which*

$$|f(x) - P_n(x)| \leq C\omega_3(f, 1/n), \tag{3.18}$$

where C is an absolute constant independent of n and f.

It is remarkable that this is exact: one cannot replace here $\omega_3(f,1/n)$ by $\omega_4(f,1)$ (and in (3.11), $\omega_2(f,1/n)$ by $\omega_3(f,1)$) without making the constant C dependent of f. See Shvedov [1979] and Note **10.1**.

Theorem 3.8 (DeVore [1977_2]). *An increasing function $f \in C^r[-1,1]$, $r = 1,2,\ldots$, can be approximated by increasing polynomials P_n with the error*

$$|f(x) - P_n(x)| \leq C_r n^{-r}\omega(f^{(r)}, 1/n). \tag{3.19}$$

§ 4. Polynomials with Integral Coefficients

When is a function $f \in C[a,b]$ approximable (with arbitrarily small error) by polynomials $Q(x) = \alpha_n x^n + \cdots + \alpha_0$ with coefficients $\alpha_0, \ldots, \alpha_n$ that are arbitrary integers? Not always. If $[a,b]$ contains an integer k, an obvious necessary condition is that $f(k)$ be itself an integer. But there are less obvious necessary conditions. If both 1 and -1 are in $[a,b]$ then $f(1) \pm f(-1)$ must be even integers. In the less trivial situation of Example 4.5, when both $\sqrt{2}$ and $-\sqrt{2}$ are in $[a,b]$, then $(f(\sqrt{2})+f(-\sqrt{2})-2f(0))/4$ and $\sqrt{2}(f(\sqrt{2})-f(-\sqrt{2}))/4$ must be integers.

Now $\sqrt{2}$ is an algebraic integer, that is, a zero of a polynomial with integral coefficients and leading coefficient one. This example is an indication that the theory must depend on results from Algebra and Number Theory. Several papers appeared on our subject, the most important of them are by Fekete [1923, 1954], and some of his publications in Hebrew. They deal with approximation of functions $f \in C(A)$ on a compact set $A \subset \mathbb{C}$. However, we are most interested in the real case, and the paper of Hewitt and Zuckerman [1959], dealing with it, suits us perfectly. We use this paper in the following two sections. The true relation between the papers mentioned remains unresolved.

By $\mathcal{Q}_n$ we denote the set of all polynomials $Q(x) = \alpha_n x^n + \cdots + \alpha_0$ with $\alpha_k \in \mathbb{Z}$, $k = 0, \ldots, n$, $\alpha_n \neq 0$. We write $\mathcal{Q} := \bigcup_0^\infty \mathcal{Q}_n$ and let R stand for monic polynomials in $\mathcal{Q}_n$, that is, those with $\alpha_n = 1$.

We begin with a simple theorem.

Theorem 4.1. *A function $f \in C[0,1]$ is approximable from $\mathcal{Q}$ if and only if $f(0)$ and $f(1)$ are both integers.*

Proof (by Kantorovich). We approximate f by means of a modified Bernstein polynomial $B_n^*(f)$,

$$B_n^*(f,x) = \sum_{k=0}^{n} \left[f\left(\frac{k}{n}\right) \binom{n}{k} \right] x^k (1-x)^{n-k},$$

where $[c]$ stands for the integral part of a real number c. Then, if $f(0)$ and $f(1)$ are both integers, by [**CA**, Theorem 3.2, p.308]

$$\begin{aligned} |f(x) - B_n^*(f,x)| &\leq |f(x) - B_n(f,x)| + \sum_{k=1}^{n-1} x^k (1-x)^{n-k} \\ &\leq C\omega(f, n^{-1/2}) + \frac{1}{n} \sum_{k=1}^{n-1} \binom{n}{k} x^k (1-x)^{n-k} \\ &\leq C\omega(f, n^{-1/2}) + n^{-1}. \end{aligned}$$

□

As we shall see, an essential property of an interval $[a,b]$ is the existence of non-trivial polynomials $Q \in \mathcal{Q}$ with $0 \leq Q(x) < 1$ on $[a,b]$. We denote this set of polynomials by $\mathcal{B}(a,b)$. The next theorem is essentially due to S.Kakeya (see Okada [1923, p.30]).

Theorem 4.2. *The set $\mathcal{B}(a,b)$ is non-empty if and only if $b - a < 4$. More exactly:* (i) *If $b - a \geq 4$, then each non-constant polynomial $Q \in \mathcal{Q}$ has norm ≥ 2 on $[a,b]$.* (ii) *If $b - a < 4$, there exists a monic polynomial $R \in \mathcal{B}(a,b)$.*

Proof. (i) The leading coefficient of the Chebyshev polynomial C_n is 2^{n-1}. Hence,

$$P_n(x) := \frac{(b-a)^n}{2^{2n-1}} C_n((2x - b - a)/(b-a)) \tag{4.1}$$

is the monic polynomial which among all monic polynomials of degree n has the least uniform norm on $[a,b]$. For any $Q_n \in \mathcal{Q}_n$ we therefore have

$$\|Q_n\| \geq \|P_n\| = \frac{(b-a)^n}{2^{2n-1}} \geq 2.$$

(ii) Since $q := (b-a)/4 < 1$, we can take n_0 so large that $q^{n_0} < (1-q)/6$. Let P_n be defined by (4.1). Its leading coefficient is 1 and $\|P_n\| = 2q^n$. We put

$$S_n := P_n + a_1 P_{n-1} + \cdots + a_{n-n_0} P_{n_0}, \quad n \geq n_0. \tag{4.2}$$

The coefficients a_j we define inductively as follows. If b_1 is the coefficient of x^{n-1} in P_n, we put $a_1 = [b_1] - b_1$. The two highest coefficients of $P_n + a_1 P_{n-1}$ are integers. If b_2 is the coefficient of x^{n-2} in this polynomial, we select $a_2 = [b_2] - b_2$, and so on. In this way, $|a_j| \leq 1$, $j = 1, \ldots, n - n_0$, and the monic polynomial S_n in (4.2) is of the form $S_n(x) = \sum_{j=0}^{n} c_{n,j} x^j$, where all $c_{n,j}$, $j \geq n_0$ are integers. Then

$$\|S_n\| \leq \sum_{j=0}^{n-n_0} \|P_{n-j}\| \leq 2 \sum_{j=0}^{n-n_0} q^{n-j} \leq 2q^{n_0}/(1-q) < 1/3.$$

Also, $S_n = R_n + P_n^*$, where $R_n \in \mathcal{Q}$ and the $P_n^*(x) = \sum_{j<n_0} (c_{n,j} - [c_{n,j}]) x^j$ are polynomials of bounded degrees $< n_0$ and with bounded coefficients. A subsequence of the P_n^* will be uniformly convergent on $[a, b]$. For any two of its terms $P_{n_1}^*, P_{n_2}^*$ with large n_1, n_2, $\|P_{n_2}^* - P_{n_1}^*\| < 1/3$. If $n_2 > n_1$, then $R_{n_2} - R_{n_1}$ is a monic polynomial in $\mathcal{Q}$, and

$$\|R_{n_2} - R_{n_1}\| \leq \|S_{n_2} - S_{n_1}\| + 1/3 \leq \|S_{n_2}\| + \|S_{n_1}\| + 1/3 < 1.$$

Thus $(R_{n_2} - R_{n_1})^2$ has the desired properties. □

Corollary 4.3. *A function $f \in C[a, b]$, $b - a \geq 4$, is approximable from $\mathcal{Q}$ if and only if it is a polynomial from $\mathcal{Q}$.*

Proof. Indeed, let $Q_n \to f$ uniformly. We can assume that the Q_n are not constants, then $\|Q_n - Q_{n_0}\| < 2$, $n \geq n_0$ for some n_0. Then by Theorem 4.2 (i), $Q_{n_0} = Q_n = f$. □

Because of the last corollary, we shall, in what follows, always assume that $b - a < 4$.

The logarithmic capacity of $[a, b]$ is $\gamma([a, b]) = (b - a)/4$, hence the condition $b - a < 4$ means that $\gamma([a, b]) < 1$. In Fekete's famous paper [1923] the interval $[a, b]$ is replaced by an arbitrary compact set A of the complex plane. Fekete and Szegö proved that the logarithmic capacity, the transfinite diameter and the Chebyshev constant of a compact set $A \subset \mathbb{C}$ are equal (see Theorem 2.4 of Appendix 4). From this it follows similarly to the proof of the last corollary that, if $\gamma(A) \geq 1$, then no function $f \in C(A)$ can be uniformly approximated by polynomials with integral coefficients unless it is identical with such a polynomial.

There may be many functions in $C[a, b]$, approximable by polynomials $Q_n \in \mathcal{Q}_n$ and we would like to describe them. Some of the zeros $Z(Q)$ of the polynomials $Q \in \mathcal{B}(a, b)$ may lie in $[a, b]$. Let $J := J[a, b]$ be the set of points which *are zeros of each of the* $Q \in \mathcal{B}(a, b)$:

$$J[a, b] := \bigcap_{Q \in \mathcal{B}(a,b)} Z(Q) \cap [a, b]. \tag{4.3}$$

Clearly, the set J is finite. As an example, all integers k, $a \le k \le b$ belong to J, for if $0 \le Q(k) < 1$, then necessarily $Q(k) = 0$. There exists a polynomial $Q_0 \in \mathcal{B}(a,b)$, whose zeros in $[a,b]$ are exactly the set J. For if some Q has in addition to J extra zeros $x_1, \dots, x_q$ in $[a,b]$, we take for each j a $Q_j \in \mathcal{B}(a,b)$ that does not vanish at x_j, and for a large p put $Q_0 := Q^p + Q_1^p + \cdots + Q_q^p$.

We can formulate:

Theorem 4.4 (First Main Theorem). *A function $f \in C[a,b]$ is approximable from $\mathcal{Q}$ if and only if on $J[a,b]$, the function f is identical with some polynomial Q with integral coefficients.*

Proof. The condition is necessary, for if $Q_n \to f$ uniformly on $[a,b]$, then $\|Q_n - Q_{n_0}\| < 1$ for all $n > n_0$ and some n_0, so that $(Q_n - Q_{n_0})^2 \in \mathcal{B}(a,b)$ unless $Q_n = Q_{n_0}$. If follows that $Q_n(x) = Q_{n_0}(x), x \in J$, for all $n \ge n_0$, hence $f(x) = Q_{n_0}(x)$ on J.

To prove the sufficiency, we first note that for each real λ, for $0 < c < 1$, and each $\varepsilon > 0$, there is a $Q \in \mathcal{Q}$ for which

$$|\lambda y - Q(y)| < \varepsilon, \quad 0 \le y \le c. \tag{4.4}$$

This follows from Theorem 4.1, for we can extend λy to a function for $0 \le y \le 1$ that vanishes at 0 and 1. Without loss of generality, we can assume that $f(x) = 0$ on J, for otherwise we would consider $f - Q$ with a proper Q instead of f. We take a polynomial $Q_0 \in \mathcal{B}(a,b)$ that within $[a,b]$ vanishes exactly on J. We can approximate f by a function $f^* \in C[a,b]$ which coincides with Q_0 in some neighborhood of each point $x \in J$, so that $\|f - f^*\| < \varepsilon$. On the other hand, $f^* = gQ_0$ with $g \in C[a,b]$. It follows that $g(x) = 1$ on J. If $0 \in [a,b]$ then $0 \in J$ and $g(0) = 1$. In this case, but also if $0 \notin [a,b]$, there exists a polynomial $P(x) = \sum_0^n a_j x^j$ with $a_0 = 1$ and real coefficients a_j, $1 \le j \le n$, so that $\|g - P\| < \varepsilon$.

Let $M := \max\{|a|, |b|, 1\}$ and $\varepsilon_n := \varepsilon M^{-n}/n$. For each a_j, $j = 1, \dots, n$, we select, according to (4.4), a polynomial $Q_j \in \mathcal{Q}$ to that $|a_j y - Q_j(y)| < \varepsilon_n$ for $0 \le y \le c$, where c $:= \max_{a \le x \le b} Q_0(x)$. Then also

$$|a_j Q_0(x) - Q_j(Q_0(x))| < \varepsilon_n, \quad a \le x \le b.$$

With the polynomial $Q(x) = Q_0(x) + \sum_{j=1}^n x^j Q_j(Q_0(x))$ we have

$$\|PQ_0 - Q\| = \max_{a \le x \le b} \Big|\sum_{j=1}^n x^j (a_j Q_0(x) - Q_j(Q_0(x))\Big| \le \varepsilon,$$

and thus

$$\|f - Q\| \le \|f - f^*\| + \|f^* - PQ_0\| + \|PQ_0 - Q\| < 3\varepsilon. \qquad \square$$

Example 4.5. *A function $f \in C[-\sqrt{2}, \sqrt{2}]$ is approximable from $\mathcal{Q}$ if and only if the numbers*

$$(4.5) \qquad \begin{aligned} &f(0), \quad \frac{1}{2}(f(1) \pm f(-1)), \\ &\frac{1}{4}(f(\sqrt{2}) + f(-\sqrt{2}) - 2f(0)), \quad \frac{\sqrt{2}}{4}(f(\sqrt{2}) - f(-\sqrt{2})) \end{aligned}$$

are integers.

Proof. The polynomial $Q_0(x) = (x(x^2-1)(x^2-2))^2$ is in $\mathcal{B}(I)$, $I := [-\sqrt{2}, \sqrt{2}]$, hence $J \subset \{0; \pm 1; \pm\sqrt{2}\}$.

For an arbitrary $Q \in \mathcal{Q}$, $Q(x) = \sum_0^n a_k x^k$, we have $Q(0) = a_0 \in \mathbb{Z}$ and

$$\sum_{k=2,k \text{ even}}^{n} a_k = \frac{1}{2}(Q(1) + Q(-1) - 2Q(0)) \in \mathbb{Z},$$

$$\sum_{k=2,k \text{ even}}^{n} a_k 2^{(k-2)/2} = \frac{1}{4}(Q(\sqrt{2}) + Q(-\sqrt{2}) - 2Q(0)) \in \mathbb{Z},$$

$$\sum_{k=1,k \text{ odd}}^{n} a_k = \frac{1}{2}(Q(1) - Q(-1)) \in \mathbb{Z},$$

$$\sum_{k=1,k \text{ odd}}^{n} a_k 2^{(k-1)/2} = \frac{\sqrt{2}}{4}(Q(\sqrt{2}) - Q(-\sqrt{2})) \in \mathbb{Z}.$$

This implies that the numbers

$$(4.6) \qquad \begin{aligned} &Q(0), \quad \frac{1}{2}(Q(1) \pm Q(-1), \\ &\frac{1}{4}(Q(\sqrt{2}) + Q(-\sqrt{2}) - 2Q(0)), \quad \frac{\sqrt{2}}{4}(Q(\sqrt{2}) - Q(-\sqrt{2})) \end{aligned}$$

are integers. In particular, if $Q \in \mathcal{B}(I)$, that is, $0 \le Q(x) < 1$ in $[-\sqrt{2}, \sqrt{2}]$, it follows from (4.6) that Q vanishes at $0, \pm 1, \pm\sqrt{2}$. Hence, $J = \{0; \pm 1; \pm\sqrt{2}\}$.

If $f \in C(I)$ is approximable from $\mathcal{Q}$, then f must satisfy the conditions (4.5) because of (4.6). Conversely, if f satisfies (4.5) then the polynomial $Q \in \mathcal{P}_4$, which interpolates f at $0, \pm 1, \pm\sqrt{2}$, is in $\mathcal{Q}$, hence f is approximable from $\mathcal{Q}$ by Theorem 4.4. □

In some of the proofs of this and the next section we shall need the following three simple lemmas from Algebra. The first of them is Gauss's Lemma. A polynomial with integral coefficients is *primitive* if its coefficients have no common divisor greater than one. A polynomial with integral coefficients is called *irreducible* if it is not the product of two polynomials with integral coefficients, both of degree at least one.

Lemma 4.6. *The product of two primitive polynomials is again a primitive polynomial.*

Proof. Let $P(x) = a_m x^m + \cdots + a_0$ and $Q(x) = b_r x^r + \cdots + b_0$ be two primitive polynomials. Their product $S := PQ$ has the coefficients

$$c_k := \sum_{i=0}^{m} a_i b_{k-i}, \quad k = 0, 1, \ldots, m+r$$

where we have set $b_j := 0$ if $j \le -1$ and if $j \ge r+1$. Let p be any integer greater than 1. Since P and Q are primitive, there exist non-negative integers $l \le m$ and $q \le r$ such that p divides all a_i, $l+1 \le i \le m$, but not a_l, and p divides all b_j, $q+1 \le j \le r$, but not b_q. This implies, for $k = l+q$, that p divides each of the terms $a_i b_{k-i}$, $i = 0, 1, \ldots, m$, except of $a_l b_{k-l}$. Hence, p does not divide c_k. Since p is arbitrary, we have shown that PQ is primitive. □

Lemma 4.7. *If a polynomial S with integral coefficients is the product $S = P_0 Q_0$ of two polynomials with rational coefficients, then S is also the product of two polynomials which have integral coefficients and are rational multiples of P_0 and Q_0, respectively.*

Proof. Since the coefficients of P_0 and Q_0 are rational numbers, there exist positive integer k_0, k_1, ℓ_0, ℓ_1 for which $P := \frac{k_1}{k_0} P_0$ and $Q := \frac{\ell_1}{\ell_0} Q_0$ are primitive polynomials, with integral coefficients. By Lemma 4.6 the polynomial $S_0 := PQ$ is also primitive. By construction we have $k_1 \ell_1 S = k_0 \ell_0 S_0$. Since S has integral coefficients and since S_0 is primitive it follows that $k_0 \ell_0 = q k_1 \ell_1$ for some integer q. Hence $S = qS_0 = qPQ$. □

Lemma 4.8. (i) *If two polynomials P_0 and P_1 with integral coefficients have a common zero, then they have a common factor of degree ≥ 1, also with integral coefficients.*

(ii) *For each algebraic integer $c \in \mathbb{C}$ there exists a unique irreducible monic polynomial R_c with integral coefficients for which $R_c(c) = 0$. All zeros of R_c are of multiplicity* 1.

Proof. (i) We may suppose that P_0 and P_1 are primitive and that the degree ∂P_0 of P_0 is larger than the degree of P_1. If $\partial P_0 \le \partial P_1$ we would replace $P_0(x)$ by $(x-p)^p P_0(x)$ with a large integer p. We apply the euclidean algorithm and obtain polynomials $Q_0, Q_1, \ldots$ and $P_2, P_3, \ldots$ with rational coefficients such that

$$\begin{aligned} P_0 &= P_1 Q_1 + P_2, \quad \partial P_2 < \partial P_1 \\ P_1 &= P_2 Q_2 + P_3, \quad \partial P_3 < \partial P_2 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ P_{l-1} &= P_l Q_l + P_{l+1}, \quad \partial P_{l+1} < \partial P_l \\ P_l &= P_{l+1} Q_{l+1} \end{aligned}$$

for some $l \ge 0$. Since P_0 and P_1 have a common zero, all $P_2, \ldots, P_{l+1}$ have the same zero. In particular, $\partial P_{l+1} \ge 1$. All these polynomials have rational coefficients. The polynomial P_{l+1} is a common divisor of $P_l, \ldots, P_1, P_0$, in particular $P_0 = P_{l+1}Q_0^*$ and $P_1 = P_{l+1}Q_1^*$ for some polynomials Q_0^* and Q_1^* with rational coefficients. By Lemma 4.7 there are rational numbers α_0 and α_1 so that the polynomials

$$\alpha_0 P_{l+1}, \ \frac{1}{\alpha_0}Q_0^*, \ \alpha_1 P_{l+1}, \ \frac{1}{\alpha_1}Q_1^*$$

have integral coefficients. Since P_0 and P_1 are primitive, $\alpha_0 P_{l+1}$ and $\alpha_1 P_{l+1}$ are also primitive. This is only possible if $\alpha_0 = \alpha_1$. Hence $\alpha_0 P_{l+1}$ is a common divisor of P_0 and P_1.

(ii) Clearly, there exists an irreducible monic polynomial $R = R_c$ with integral coefficients which vanishes at c. It follows from (i) that there is only one such polynomial. Suppose that R_c has a zero of multiplicity ≥ 2. Then R_c and R_c' have a common zero, hence a common factor of degree ≥ 1 with integral coefficients. This is impossible since R_c is irreducible. □

An *algebraic integer* α is a zero of a monic polynomial R with integral coefficients. Here and later we denote by $Z(R)$ the complete set of zeros of R. Sometimes we have a *real complete set of zeros*, when all zeros of R are real. For example, $\{k\}$, $k \in \mathbb{Z}$, and $\{\pm\sqrt{2}\}$ are real complete sets, with the polynomials $R = x - k$, $R = x^2 - 2$, respectively.

We have: all elements of $J[a, b]$ are algebraic integers. For if $\alpha \in J$, then all $Q \in \mathcal{B}(a, b)$ vanish at α, in particular the monic polynomial R of Theorem 4.2 (ii). Our purpose will be to describe J more precisely.

For each interval $[a, b]$, let $J' := J'[a, b]$ be the union of all real complete sets of zeros contained in $[a, b]$. In other words: let $\mathcal{M}(a, b)$ be the set of all monic $R \in \mathcal{Q}$ with all their zeros contained in $[a, b]$, then

$$J'[a, b] = \bigcup_{R \in \mathcal{M}(a,b)} Z(R). \tag{4.7}$$

We call $J'[a, b]$ the *characteristic set of* $[a, b]$. Our purpose is to prove that $J = J'$. For the general case of approximation on a compact set $A \subset \mathbb{C}$, the relation $J = J'$ is due to Fekete [1954] , but we follow the proofs of Hewitt and Zuckerman [1959], suitable for the real interval $[a, b]$. At present it is not even immediate that J' is finite; this follows from

Lemma 4.9. *For $b - a < 4$,*

$$J'[a, b] \subset J[a, b].$$

Proof. An arbitrary polynomial in $\mathcal{M}(a, b)$ is the finite product of of irreducible polynomials in $\mathcal{M}(a, b)$. Hence it suffices to show that the zeros $Z(R)$ of an irreducible polynomial $R(x) = x^n + b_1 x^{n-1} + \cdots + b_n$ in $\mathcal{M}(a, b)$ are contained in $J[a, b]$.

Let Q be any polynomial in $\mathcal{B}(a, b)$. If r_i, $i = 1, \ldots, n$, are all zeros of R, we define

$$R^*(x) = \prod_{j=1}^{n}(x - Q(r_j)) = x^n + c_1 x^{n-1} + \cdots + c_n.$$

The coefficients c_j depend on $r_1, \ldots, r_n$,

$$c_j(r_1, \ldots, r_n) = \sum a^{(j)}_{i_1 \ldots i_n} r_1^{i_1} \cdots r_n^{i_n},$$

with integral coefficients $a^{(j)}_{i_1 \ldots i_n}$. As functions of free variables $r_1, \ldots, r_n$, the c_j are symmetric polynomials: they do not change under permutations of the $r_1, \ldots, r_n$. Important examples of symmetric polynomials are

$$B_k = \sum_{i_1 < \cdots < i_k} r_{i_1} \cdots r_{i_k}, \quad k = 1, \ldots, n.$$

For example, $B_1 = r_1 + \cdots + r_n$, $B_2 = r_1 r_2 + r_1 r_3 + \cdots + r_{n-1} r_n$ and $B_n = r_1 \cdots r_n$. By the main theorem on symmetric polynomials there is for each c_j, $j = 1, \ldots, n$, a polynomial with integral coefficients

$$U_j(t_1, \ldots, t_n) = \sum b^{(j)}_{i_1 \ldots i_n} t_1^{i_1} \cdots t_n^{i_n}$$

such that

$$c_j(r_1, \ldots, r_n) = U_j(B_1, \ldots, B_n).$$

Since

$$R(x) = \prod_{j=1}^{n}(x - r_j) = x^n + \sum_{i=1}^{n}(-1)^i B_i x^{n-i},$$

all $B_i(r_1, \ldots, r_n)$ are integers, and also $c_1, \ldots, c_n$ are integers. In particular, $c_n = \pm \prod Q(r_j)$ is an integer. Since $0 \le Q(r_j) < 1$, $j = 1, \ldots, n$, $c_n = 0$ and one of the $Q(r_j)$ is zero. By Lemma 4.8 (i), R and Q have a common factor of degree ≥ 1 with integral coefficients. This factor is R itself since R is irreducible: we have $Z(R) \subset Z(Q)$. This implies that $Z(R) \subset J[a,b]$ since $Q \in \mathcal{B}(a,b)$ is arbitrary. □

Let $m := |J'[a,b]|$ be the number of points of $J'[a,b]$. It is clear now that there exists a unique maximal polynomial $R_0 = R_0[a,b]$ of degree m in $\mathcal{M}(a,b)$, for which $J'[a,b] = Z(R_0)$. Indeed, R_0 is the product of all irreducible polynomials of $\mathcal{Q}$, which produce the complete sets of zeros in $J'[a,b]$.

It is convenient to use

Proposition 4.10 (Khinchin [1923]). (Principle of induction in the continuum). *Let Π be a property of points $d \in \mathbb{R}$, which d may have: $d \in \Pi$, or not have, $d \notin \Pi$. Then all points of an interval $[a,b]$ have Π if:*

(a) *The point a has property Π;*
(b) *If $c \in [a,b)$ has Π, then for some $\delta > 0$, all $d \in [c, c+\delta]$ have Π;*
(c) *If all d, $a \le d < c$ have Π, then also $c \in \Pi$.*

Proof. If $d \notin \Pi$ for some $d \in [a,b]$, we let c_0 be the infimum of these d. By (c), $c_0 \in \Pi$, and then (b) gives a contradiction. □

In what follows, a will be fixed, and the property Π of d will be $J[a,d] = J'[a,d]$. It is easy to see that this property satisfies (a) and (b); the latter because for each c, $a \le c < b$, there exists a $\delta > 0$ for which $J[a,c+\delta] = J[a,c]$ and $J'[a,c+\delta] = J'[a,c]$. That also (c) holds, will follow from Theorem 4.4 and the following two lemmas.

Lemma 4.11. *Let $a < c \le b$, $b - a < 4$, and let $J[a,d] = J'[a,d]$ for all $a \le d < c$. (i) If $c \notin J[a,c]$ then $J[a,c] = J'[a,c]$.*
(ii) If $c \in J[a,c]$ then $J[a,c] \subset J'[a,c] \cup Z(R_c)$, where R_c is the unique irreducible monic polynomial in $\mathcal{Q}$ satisfying $R_c(c) = 0$.

Proof. Let $Q_0 \in \mathcal{B}(a,c)$ be a polynomial whose zeros in $[a,c]$ are exactly the set $J[a,c]$. For $p = 1,2,\ldots$, the polynomials $Q_p := (1 - Q_0)^p$ have integral coefficients. They satisfy $0 < Q_p(x) \le 1$ for $a \le x \le c$, and $Q_p(x) = 1$ for $x \in [a,c]$ if and only if $x \in J[a,c]$. Let $c_0 := \max\{x \in J[a,c] : x < c\}$ and let $\tilde{Q}_0 \in \mathcal{B}(a,c_0)$ be a polynomial whose zeros in $[a,c_0]$ are exactly the set $J[a,c_0]$.

(i) If $c \notin J[a,c]$ then $J[a,c] = J[a,c] \cap [a,c_0]$ and, for sufficiently large p, the polynomials $Q_p\tilde{Q}_0$ are in $\mathcal{B}(a,c)$, their zeros on $[a,c_0]$ coincide with those of $\tilde{Q}_0$. Hence, $J[a,c] \cap [a,c_0] = J[a,c_0]$. Since $J'[a,c_0] \subset J'[a,c] \subset J[a,c]$, it follows that $J[a,c] = J[a,c_0] = J'[a,c_0] = J'[a,c]$.

(ii) Let $c \in J[a,c]$. For each $\delta > 0$, $\|Q_p\tilde{Q}_0\|_{C[a,c-\delta]} \to 0$ as $p \to \infty$. For sufficiently large p, the polynomials $Q_p\tilde{Q}_0R_c$ are in $\mathcal{B}(a,c)$. Since Q_p is positive on $[a,c]$, this implies that

$$J[a,c] \cap [a,c_0] \subset \Big(Z(\tilde{Q}_0) \cap [a,c_0]\Big) \cup Z(R_c) = J[a,c_0] \cup Z(R_c).$$

This yields (ii) since $J[a,c] = (J[a,c] \cap [a,c_0]) \cup \{c\}$ and $J[a,c_0] = J'[a,c_0] \subset J'[a,c]$. □

The next lemma is obtained by counting polynomials of a certain kind.

Lemma 4.12. *Let $R \in \mathcal{Q}$ be an irreducible monic polynomial of degree n. If $\alpha_1,\ldots,\alpha_s$ are some real zeros of R, with $s < n$, then there exists a $Q \in \mathcal{Q}$ with*

$$0 < Q(\alpha_j) < 1, \quad j = 1,\ldots,s.$$

Proof. We look for Q among the polynomials of the set $\mathcal{Q}_N^*$ (where N is a positive integer), consisting of

$$Q(x) = c_0x^{n-1} + c_1x^{n-2} + \cdots + c_{n-1}, \quad |c_k| \le N, \quad k = 0,\ldots,n-1.$$

There are $(2N+1)^n$ distinct polynomials in $\mathcal{Q}_N^*$. If $\beta := \max\{1, |\alpha_1|,\ldots,|\alpha_s|\}$ then $|Q(\alpha_j)| \le n\beta^{n-1}N$, $j = 1,\ldots,s$. For two distinct $Q_1, Q_2 \in \mathcal{Q}_N^*$ it is impossible that $Q_1(\alpha_j) = Q_2(\alpha_j)$, for then, by Lemma 4.8 (i), R would have a common factor of degree ≥ 1 with $Q_1 - Q_2$. The $(2N+1)^n$ points

$y = (Q(\alpha_1), \ldots, Q(\alpha_s))$, $Q \in \mathcal{Q}_N^*$ of the space $\mathbb{R}^s$ are contained in a cube of side length $2n\beta^{n-1}N$. For a positive integer M, we break up this cube into $(2M)^s$ smaller cubes of side lengths $n\beta^{n-1}N/M$. We assume that

$$(2M)^s < (2N+1)^n, \tag{4.8}$$

then there will be two distinct points y which lie in one of the smaller cubes. Let $Q_1, Q_2 \in \mathcal{Q}_N^*$ correspond to two such points. Then $Q := Q_1 - Q_2 \in Q$ is a polynomial of degree $\leq n-1$ for which

$$0 < |Q(\alpha_j)| \leq M^{-1}n\beta^{n-1}N, \ \ j = 1, \ldots, s.$$

We take $M = [N^{n/s}]$. Both (4.8) and $M^{-1}n\beta^{n-1}N < 1$ are satisfied if N is sufficiently large, and then Q is the required polynomial. □

Theorem 4.13 (Second Main Theorem). *For $b - a < 4$,*

$$J[a,b] = J'[a,b]. \tag{4.9}$$

Proof. If Π is the property of $c \in [a,b]$ expressed by

$$J[a,c] = J'[a,c], \tag{4.10}$$

it remains only to prove (c) of the principle of induction in the continuum. Therefore we assume that for some c, $J[a,d] = J'[a,d]$, $a < d < c$, and try to prove (4.10).

If $c \notin J[a,c]$, then Lemma 4.11 (i) reduces to (4.10).

Let $c \in J[a,c]$ and let R_c be the unique monic irreducible polynomial of Lemma 4.8, with $R_c(c) = 0$, and let n be its degree. The other extreme case is when all n zeros $\alpha_1, \ldots, \alpha_n$ of R_c belong to $[a,c]$. Then they are a subset of the characteristic set $J'[a,c]$, and again (4.10) follows from Lemma 4.11.

It remains the case when not all n of the α_j are in $[a,c]$. Let for instance $\alpha_1, \ldots, \alpha_s$ be those of them that belong to $J[a,c]$. By Lemma 4.8, R_c is a divisor of each polynomial $Q \in \mathcal{B}(a,c)$, hence $J[a,c] = J'[a,c] \cup \{\alpha_1, \ldots, \alpha_s\}$. By the definition of $J'[a,c]$ it follows that $\alpha_1, \ldots, \alpha_s$ do not belong to $J'[a,c]$. We want to prove that this is impossible.

Let $R_0 = R_0[a,c]$ be the polynomial which produces the complete sets of zeros in $J'[a,c]$.

The polynomial Q of Lemma 4.12, constructed for R_c and the $\alpha_1, \ldots, \alpha_s$, we replace by $Q_1 := Q^p$, taking p so large that for a prescribed ε, $0 < \varepsilon < 1$, one has $0 < \delta < |Q_1(\alpha_j)R_0(\alpha_j)| < \varepsilon$, $j = 1, \ldots, s$. By U_j we denote closed neighborhoods of the α_j, mutually disjoint and disjoint with $J'[a,c]$, so that

$$\delta < |Q_1(x)R_0(x)| < \varepsilon, \quad x \in U_j, \ \ j = 1, \ldots, s.$$

We put

$$f(x) = \begin{cases} 0, & \text{if } x = \alpha_j \\ Q_1(x)R_0(x), & \text{if } x \notin \cup U_j \end{cases}$$

and extend f by linearity onto $[a,c]$. Plainly, $|f(x)| < \varepsilon$ for $x \in \cup U_j$. This function vanishes on $J'[a,c] \cup \{\alpha_1, \dots, \alpha_s\} = J[a,c]$. Hence by Theorem 4.4, for some $Q_2 \in \mathcal{Q}$,

$$|f(x) - Q_2(x)| < \delta, \quad x \in [a,c].$$

We examine the polynomial $Q := Q_1 R_0 - Q_2$. Outside of $\cup U_j$, $|Q(x)| < \delta$; on $\cup U_j$,

$$|Q(x)| \le |f(x) - Q_2(x)| + |f(x)| + |Q_1(x) R_0(x)| < \delta + 2\varepsilon < 1,$$

if ε is small. Thus, $Q \in \mathcal{B}(a,c)$. As a consequence, Q vanishes on $J[a,c]$, in particular $Q(\alpha_j) = 0$, $j = 1, \dots, s$. Therefore

$$|Q_1(\alpha_j) R_0(\alpha_j)| = |Q(\alpha_j) + Q_2(\alpha_j)| = |Q_2(\alpha_j) - f(\alpha_j)| < \delta,$$

which is a contradiction. We have proved (4.9). □

How to decide whether a given function $f \in C[a,b]$, $b - a < 4$, is approximable from $\mathcal{Q}$ or not? This is answered by Theorem 4.4. A more computational criterion is given by

Theorem 4.14. *A function $f \in C[a,b]$, $b-a < 4$, is approximable by polynomials with integral coefficients if and only if the Lagrange interpolation polynomial for f on the characteristic set $J' := J'[a,b] = J[a,b]$ belongs to $\mathcal{Q}$.*

Proof. Only the necessity of the condition is to be proved. Let $R_0 = R_0[a,b]$ be the polynomial of degree $m := |J'|$ in $\mathcal{M}(a,b)$, for which $J' = Z(R_0)$. Let $Q \in \mathcal{Q}$ be the polynomial of Theorem 4.4, with $Q(x) = f(x)$ on J'. Division $Q = R_0 Q_0 + L$ yields a polynomial $L \in \mathcal{Q}$ of degree $\le m - 1$, which agrees with f on J' and is therefore the Lagrange interpolation of f on J'. □

How to find the characteristic set $J'[a,b]$? In the next section we shall prove some general theorems which allow to determine it in many cases.

§ 5. Determination of the Characteristic Sets

Here, following Hewitt and Zuckerman [1959], we determine the characteristic set $J[a,b] = J'[a,b]$ in the case when $-2 \le a < b \le 2$, $b - a < 4$. We begin with a theorem by Kronecker [1857].

Theorem 5.1. *Let $R(z) = z^m + a_{m-1} z^{m-1} + \cdots + a_0$, $m \ge 2$, be a polynomial with integral coefficients.* (i) *If all zeros of R lie on the unit circle $|z| = 1$, then they are zeros of some polynomial $z^n - 1$.*
(ii) *If all zeros of R lie in the open interval $(-2,2)$, then for some $n > 2$, they are among the numbers*

$$2\cos(2\pi k/n), \quad 1 \le k < n/2. \tag{5.1}$$

Proof. (i) The coefficients $a_0, \dots, a_{m-1}$ of R are integers and satisfy $|a_k| \le \binom{m}{k}$. Hence there exist only a finite number of polynomials R of a given degree m with the properties of the theorem.

Let $\eta_1, \dots, \eta_m$ be the zeros of such a polynomial R, $R(z) = \prod_1^m (z - \eta_j)$. Then $(-1)^m R(z) R(-z) =: R_1(z^2)$ is an even monic polynomial of degree $2m$ with integral coefficients, and R_1 has the properties of the theorem. The zeros of R_1 are η_j^2, $j = 1, \dots, m$. By the same argument we see that the polynomials

$$R_k(z) := \prod_{j=1}^{m} (z - \eta_j^{2^k}), \quad k = 1, 2, \dots,$$

have integral coefficients, and they have the properties of the theorem. We conclude that there exist positive integers k and l, $k < l$, for which $R_k = R_l$. Setting $\kappa := 2^k$ and $\lambda := 2^l$ we see that the set of zeros $\{\eta_1^\kappa, \eta_2^\kappa, \dots, \eta_m^\kappa\}$ of R_k and the set of zeros $\{\eta_1^\lambda, \eta_2^\lambda, \dots, \eta_m^\lambda\}$ of R_l are identical. This implies that for some indices $i_1, i_2, \dots, i_s$

$$\eta_1^\lambda = \eta_{i_1}^\kappa, \quad \eta_{i_1}^\lambda = \eta_{i_2}^\kappa, \quad \dots, \quad \eta_{i_s}^\lambda = \eta_1^\kappa.$$

We eliminate from these $s+1$ equations the s numbers $\eta_{i_1}, \dots, \eta_{i_s}$ and obtain the equation

$$\eta_1^{n_1} = 1, \quad n_1 := \lambda^{s+1} - \kappa^{s+1}.$$

But κ and λ are distinct positive integers. Therefore, η_1 is a zero of the polynomial $z^{n_1} - 1$. The same argument applies for the next zero η_2, producing an equation $z^{n_2} = 1$ which it satisfies, and so on. Then we define $n = n_1 n_2 \cdots$.

(ii) The zeros of R are of the form $\xi_j = 2\cos(2\pi\alpha_j)$, for some $0 < \alpha_j < 1/2$, $j = 1, \dots, m$. We define the polynomial R^* by $R^*(z) := z^m R(z + z^{-1})$. Then R^* is a monic polynomial of degree $2m$, with integral coefficients. Its $2m$ zeros $e^{\pm 2\pi i \alpha_j}$, $j = 1, \dots, m$, lie on the unit circle. By (i) they are zeros of some polynomial $z^n - 1$. It follows that the α_j are among the numbers k/n, $1 \le k < n/2$, and the zeros ξ_i of R are among those in (5.1). □

By $(r, q) \ge 1$ we denote the greatest common divisor of two integers r, q. Thus $(r, q) = 1$ means that r, q are relatively prime. We shall use the following known fact. Two integers m and n are relatively prime if and only if there exist integers u and v with

$$mu + nv = 1, \quad u, v \in \mathbb{Z}. \tag{5.2}$$

Lemma 5.2. *Let k and n be integers for which $(k, n) = 1$. Then for each positive integer N there exist prime numbers $p_1, p_2, \dots, p_s$ with the properties*

$$k \equiv p_1 \cdots p_s \ (\text{mod. } n), \quad (p_i, n) = 1, \quad p_i > N, \quad i = 1, \dots, s. \tag{5.3}$$

Proof. Let m be the product of all prime numbers $p \le N$ that are relatively prime with n, hence m is also relatively prime with n. By (5.2) there exist integers u and v for which $mu+nv = 1$. We set $\ell := kmu+nv$. Let $\ell = p_1 \cdots p_s$ be the product of the prime numbers $p_1, \dots, p_s$. Since $\ell = k + (1-k)nv$ and $(k,n) = 1$, we have $(p_i, n) = 1$ and $k \equiv p_1 \cdots p_s \pmod{n}$. Since $\ell = 1 + mu(k-1)$, ℓ is not divisible by any prime number $\leq N$, so that all p_i are $> N$. This yields (5.3). □

Lemma 5.3. *Let c be an algebraic integer and*

$$R_c(z) = z^m + a_{m-1}z^{m-1} + \cdots + a_0 \tag{5.4}$$

the unique irreducible monic polynomial with integral coefficients satisfying $R_c(c) = 0$. *For each prime number p there exists a polynomial S with integral coefficients, which satisfies*

$$R_c(c^p) + pS(c) = 0 \ \textit{ and } \ \partial S \leq m-1. \tag{5.5}$$

Proof. Using Fermat's theorem we have $a_j^p \equiv a_j$ (mod. p), $j = 0, 1, \dots, m-1$, and thus

$$\begin{aligned}(R_c(z))^p &= z^{mp} + a_{m-1}^p z^{(m-1)p} + \cdots + a_0^p + pS_0(z)\\ &= z^{mp} + a_{m-1}z^{(m-1)p} + \cdots + a_0 + pS_1(z)\\ &= R_c(z^p) + pS_1(z)\end{aligned}$$

where S_0 and S_1 are polynomials with integral coefficients. This yields $R_c(c^p) + pS_1(c) = 0$. We define the polynomial S as follows: if $\partial S_1 \leq m-1$ then $S := S_1$, else S is the remainder in $S_1(z) = R_c(z)Q(z) + S(z)$, for some appropriate polynomial Q. Both cases yield (5.5). □

The number theoretic Euler function ϕ is defined by $\phi(1) = 1$, while for $n > 1$, $\phi(n)$ is the number of k, $1 \leq k < n$ that are relatively prime with n. Since $(k,n) = 1$ implies $(n-k, n) = 1$, the k with the above properties appear in pairs, k, $n-k$, except when $k = n/2$ is an integer prime with n which happens if and only if $n = 2$. Thus, *$\phi(n)$ is even for all $n > 2$.*

The polynomials $Y_1(z) := z - 1$ and

$$Y_n(z) := \prod_{k=1,(k,n)=1}^{n} \left(z - e^{2\pi ik/n}\right), \quad n \geq 2, \tag{5.6}$$

are called the *cyclotomic polynomials.* For example, $Y_2(z) = z + 1$, $Y_3(z) = z^2 + z + 1$, $Y_4(z) = z^2 + 1$. The degree of Y_n is $\phi(n)$. With $e^{2\pi ik/n}$ also $e^{-2\pi ik/n} = e^{2\pi i(n-k)/n}$ is a zero of Y_n. It follows that the product of all zeros of Y_n is one, that Y_n is a real polynomial and that $Y_n(z) = z^{\phi(n)}Y_n(1/z)$. Therefore the function $z^{-\phi(n)/2}Y_n(z)$ is invariant under change of variables $z \to 1/z$. This fact implies that $Y_n(z)$ has the form

$$Y_n(z) = z^{\phi(n)} + 1 + \sum_{k=1}^{\phi(n)/2} b_k \left(z^k + z^{\phi(n)-k}\right), \quad b_k \in \mathbb{Z}. \tag{5.7}$$

Theorem 5.4. *The cyclotomic polynomials Y_n have integral coefficients and are irreducible.*

Proof. Let c be one of the zeros of Y_n. Since c is a zero of the monic polynomial $z^n - 1$, c is an algebraic integer. Let R_c be the unique irreducible monic polynomial with $R_c(c) = 0$. We want to show that $c^n = 1$ implies $Y_n = R_c$. For this purpose we first show that $R_c(c^p) = 0$ for all large primes p.

Let k, $1 \le k \le n-1$, be relatively prime with n. The polynomial $R_c(z^k)$ is also monic and has integral coefficients. Hence there exist polynomials Q_k and P_k, both with integral coefficients, with the properties

$$R_c(z^k) = R_c(z)Q_k(z) + P_k(z), \quad \partial P_k \le m-1.$$

Let $P_k(z) = a_{m-1}^{(k)} z^{m-1} + \cdots + a_0^{(k)}$. Since R_c vanishes at c,

$$R_c(c^k) = P_k(c) = a_{m-1}^{(k)} c^{m-1} + \cdots + a_0^{(k)}. \tag{5.8}$$

We set $N := \max\{|a_j^{(k)}| : j = 0, \ldots, m-1,\ k = 1, \ldots, n-1 \text{ with } (k,n) = 1\}$.

For any prime number p with $(p,n) = 1$, we define the integer $k = k(p)$, $1 \le k \le n-1$, by the relation $p \equiv k$ (mod. n). Then we have $(k,n) = 1$ and since $c^n = 1$, $c^p = c^k$.

Let S be the polynomial in (5.5). Using the equality $c^p = c^k$ and the relations (5.5) and (5.8) we deduce that

$$0 = R_c(c^k) - R_c(c^p) = R_c(c^k) + pS(c) = a_{m-1}^{(k)} c^{m-1} + \cdots + a_0^{(k)} + pS(c).$$

It follows that the polynomial $a_{m-1}^{(k)} z^{m-1} + \cdots + a_0^{(k)} + pS(z)$ has integral coefficients, is of degree $\le m-1$ and has the zero c. It is the zero polynomial, since otherwise, by Lemma 4.8 (i), R_c could not be irreducible. We conclude that the coefficients $a_j^{(k)}$, $j = 0, \ldots, m-1$, are divisible by p. If the prime number p is larger than N, all $a_j^{(k)}$, $j = 0, \ldots, m-1$, are zero and thus

$$R_c(c^p) = 0 \text{ for all primes } p > N,\ (p,n) = 1. \tag{5.9}$$

Let $p_1 > N$, $p_2 > N$ be two primes relatively prime with n. Since c^{p_1} is a zero of R_c and of Y_n, R_c is also the irreducible polynomial of Lemma 4.8 (ii) for c^{p_1}. Hence, by (5.9), $R_c(c^{p_1 p_2}) = 0$. By induction it follows that $R_c(c^\ell) = 0$ is valid for all products $\ell := p_1 \cdots p_s$ of prime numbers $p_i > N$, $(p_i, n) = 1$.

We want to show that $R_c = Y_n$. It suffices to prove that R_c and Y_n have the same zeros. For $n = 1, 2, 3, \ldots$ one of the zeros of Y_n is $c := e^{2\pi i/n}$, and all other zeros are c^k, where $1 \le k < n$ and $(k,n) = 1$. By Lemma 5.2 there exist prime numbers $p_1, \ldots, p_s$ with the properties (5.3). Since $c^n = 1$ we have $R_c(c^k) = R_c(c^{p_1 \cdots p_s}) = 0$: each zero of Y_n is a zero of R_c. But R_c is irreducible, hence $R_c = Y_n$. □

For $n \ge 3$, formula (5.7) for Y_n implies the existence of a unique monic polynomial Y_n^* of degree $\phi(n)/2$, with integral coefficients, for which

(5.10) $$Y_n(z) = z^{\phi(n)/2} Y_n^*(z + z^{-1}), \quad n \geq 3.$$

For example, $Y_3^*(x) = x + 1$, $Y_4^*(x) = x$. We claim that *the Y_n^* are irreducible.* To prove this, we assume to the contrary that $Y_n^*(x) = S(x)T(x)$, where S and T are monic polynomials of positive degrees s and t with integral coefficients. Then we would have $s + t = \phi(n)/2$ and

$$Y_n(z) = z^s S(z + z^{-1}) \ z^t T(z + z^{-1}).$$

Hence, Y_n would be reducible, a contradiction to Theorem 5.4.

The zeros of the polynomials Y_n^* are given exactly by

(5.11) $$\mathcal{Z}_n := \{2\cos(2\pi j/n), \quad 1 \leq j < n/2, \quad (j, n) = 1\}.$$

For example, we have

$$\mathcal{Z}_3 = \{-1\}, \ \mathcal{Z}_4 = \{0\}, \ \mathcal{Z}_5 = \{2\cos(2\pi/5); 2\cos(4\pi/5)\}, \ \mathcal{Z}_6 = \{1\}.$$

In addition, we have $\mathcal{Z}_1 = \{2\}$ and $\mathcal{Z}_2 = \{-2\}$. Each set of zeros $\mathcal{Z}_n$ is a complete family of zeros and belongs to $J'[a, b]$ if $\mathcal{Z}_n \subset [a, b]$. Now we can prove that there are no other numbers in $J'[a, b]$ if $[a, b]$ is a subinterval of $[-2, 2]$.

Theorem 5.5. *Let $-2 \leq a < b \leq 2$, $b - a < 4$. The characteristic set $J'[a, b]$ is the union of the sets $\mathcal{Z}_n$, $n \geq 1$, contained entirely in $[a, b]$.*

Proof. If $\xi \in J'[a, b]$ is an integer, it belongs to one of the sets $\mathcal{Z}_n$, $n = 1, 2, 3, 4, 6$. Let $\xi \in J'[a, b]$ be not an integer. Using the definition (4.7) of $J'[a, b]$ there exists a monic polynomial R with integral coefficients which vanishes at ξ and has all its zeros in $[a, b]$. By Lemma 4.8 (ii) there exists a unique irreducible monic polynomial $R_\xi \in \mathcal{Q}$ which vanishes at ξ. By Lemma 4.8 (i) R_ξ is a factor of R. All zeros of R_ξ lie therefore in $[a, b]$. By Theorem 5.1 (ii), ξ is of the form $\xi = 2\cos(2\pi j/n)$. We can take j and n so that $(j, n) = 1$ and $1 \leq j < n/2$, then ξ will be a zero of the polynomial Y_n^*. Since Y_n^* is also irreducible, $Y_n^* = R_\xi$ and all zeros $\mathcal{Z}_n$ of Y_n^* lie in $[a, b]$. □

Only finitely many $\mathcal{Z}_n$ can be contained in $[a, b]$ of Theorem 5.5. For example, if $b < 2$, then the $\mathcal{Z}_n$ with $4\sin^2 \frac{\pi}{n} < 2 - b$, hence with $n > 2\pi/\sqrt{2 - b}$ do not satisfy $\mathcal{Z}_n \subset [a, b]$. Therefore the set J' of Theorem 4.14 can be obtained by finitely many simple steps.

We can treat also some intervals not contained in $[-2, 2]$. If $[a, b]$ contains three or less consecutive integers, for example, $r - 1, r, r + 1$ in its interior, then the approximation problem for $f \in C[a, b]$ reduces to the case of Theorem 5.5 by translation by $-r$: the interval $[a - r, b - r]$ will be contained in $[-2, 2]$. The problem remains open if there are four such integers.

§ 6. Markov-Type Inequalities

The inequalities of Bernstein and Markov can be improved for special classes of polynomials. For example, Erdös [1940] proved that in the uniform norm on $[-1,1]$, which we denote in this section by $\|\cdot\|$,

$$\|P_n'\| \le \frac{en}{2}\|P_n\|,$$

for all real polynomials P_n of degree $\le n$ which have only real zeros, all of them outside of $(-1,1)$. Later, Lorentz [1963] defined the class

$$P_n(x) = \sum_{j=0}^{n} a_j(1-x)^j(1+x)^{n-j}, \quad a_j \ge 0, \quad j=1,\dots,n, \tag{6.1}$$

of polynomials with positive coefficients in $1-x$ and $1+x$ and showed that they satisfy

$$\|P_n^{(r)}\| \le C_r n^r \|P_n\|, \tag{6.2}$$

$r = 1,2,\dots$, with a constant $C_r > 0$ depending only on r. He also proved the matching Bernstein's inequality

$$|P_n^{(r)}(x)| \le C_r \left(\frac{n}{1-x^2}\right)^{r/2} \|P_n\|, \quad -1 < x < 1. \tag{6.3}$$

See [**CA**, Chapter 4, §4].

In this section we shall first consider the classes $\mathcal{P}_n^k$ of real algebraic polynomials of degree $\le n$ which have at most k zeros (counting multiplicities) in the open unit disk $|z| < 1$. If $k = 0$, that is, if all zeros of P_n lie outside of $|z| < 1$, Lorentz observed ([**CA**, **4**, p.109]) that either P_n or $-P_n$ is of the type (6.1). Hence the improved Markov inequality (6.2) holds for the class $\mathcal{P}_n^0$.

P. Borwein [1985] proved the next theorem for all polynomials $P_n \in \mathcal{P}_n^k$ which have at least $n-k$ real zeros outside of $(-1,1)$. That this is valid for all $P_n \in \mathcal{P}_n^k$ has been shown by Erdélyi [1989_1].

From Bernstein's inequality one sees (compare (6.9)) that the critical areas for the estimates of the type $\|P_n^{(r)}\| = \mathcal{O}(n^r)\|P_n\|$ on $[-1,1]$ are close to the points ± 1. In Theorem 6.5 we shall give estimates of Erdélyi [1989_1] when the number of zeros of P_n is restricted in the domains $D_\rho^\pm$,

$$D_\rho^+ := \{z \in \mathbb{C} : |z-1+\rho| < \rho\}, \quad D_\rho^- := \{z \in \mathbb{C} : |z+1-\rho| < \rho\}.$$

Theorem 6.1. *For $r = 1,2,\dots$ there exist positive numbers C_r with the following property: For each $n, k \in \mathbb{N}$, and each $P_n \in \mathcal{P}_n^k$,*

$$\|P_n^{(r)}\| \le C_r (k+1)^r n^r \|P_n\|. \tag{6.4}$$

Proof. The next two lemmas will be used in the Proof of Theorem 6.1. The first of them is a simple application of a Markov-type inequality for Müntz polynomials of Chapter 11:

Lemma 6.2. *If* $Q_n(x) = x^{n-k-r}S(x)$, $S \in \mathcal{P}_{k+r}$, $n > k+r$ *and* $r \in \mathbb{N}$, *then*

$$|Q_n^{(r)}(1)| \le C_r(k+1)^r n^r \|Q_n\|_{C[0,1]}, \tag{6.5}$$

where C_r *depends only on* r.

Proof. An application of Theorem 5.1 of Chapter 11 for the exponents $\lambda_j := n-k-r+j$, $j = 0, 1, \dots, k+r$, yields

$$\|xQ_n'(x)\|_{C[0,1]} \le \sigma \|Q_n\|_{C[0,1]},$$

where $\sigma := 11\sum_0^{k+r}\lambda_j \le 11(k+r+1)n$. This proves (6.5) for $r = 1$. The polynomials $x^r Q_n^{(r)}$, $r = 0, 1, \dots$, have the same exponents λ_j. Therefore

$$\|x(x^r Q_n^{(r)})'\|_{C[0,1]} \le \sigma \|x^r Q_n^{(r)}\|_{C[0,1]},$$

and thus, by induction

$$\begin{aligned} \|x^{r+1}Q_n^{(r+1)}\|_{C[0,1]} &\le \sigma\|x^r Q_n^{(r)}\|_{C[0,1]} + \|rx^r Q_n^{(r)}\|_{C[0,1]} \\ &\le (\sigma + r)\|x^r Q_n^{(r)}\|_{C[0,1]} \le \|Q_n\|_{C[0,1]}\prod_{j=0}^{r}(\sigma + j). \end{aligned} \tag{6.6}$$

Since $\prod_0^{r-1}(\sigma + j) \le (\sigma + r)^r \le C_r(k+1)^r n^r$, for some C_r, (6.6) implies (6.5) for all $r \in \mathbb{N}$. □

Lemma 6.3. *If* $U_n \in \mathcal{P}_n$ *has at most* k *zeros in the disk* $|z - 1/2| < 1/2$ *and if* $n > k + r$, *then*

$$|U_n^{(r)}(1)| \le C_r(k+1)^r n^r \|U_n\|_{C[0,1]}, \tag{6.7}$$

where C_r *is the number of the last lemma.*

Proof. Using again [**CA**, **4**, p.109] and a linear transformation we see that the polynomial U_n has a representation

$$U_n(x) = q_k(x)\sum_{j=0}^{n-k} a_j(1-x)^j x^{n-k-j}, \quad a_j \ge 0, \quad j = 0, \dots, n-k,$$

with some polynomial $q_k \in \mathcal{P}_k$. Let $S(x) := q_k(x)\sum\limits_{j=0}^{r} a_j(1-x)^j x^{r-j}$. Then $S \in \mathcal{P}_{k+r}$ and

$$\begin{aligned}U_n^{(r)}(1) &= (x^{n-k-r}q_k(x)\sum_{j=0}^{n-k} a_j(1-x)^j x^{r-j})^{(r)}(1)\\ &= (x^{n-k-r}q_k(x)\sum_{j=0}^{r} a_j(1-x)^j x^{r-j})^{(r)}(1)\\ &= \left(x^{n-k-r}S(x)\right)^{(r)}(1).\end{aligned}$$

Hence, by Lemma 6.2,

$$|U_n^{(r)}(1)| \le C_r(k+1)^r n^r \|x^{n-k-r}S\|_{C[0,1]}.$$

It remains to show that

$$\|x^{n-k-r}S\|_{C[0,1]} \le \|U_n\|_{C[0,1]}.$$

But this is obvious since all $a_j \ge 0$ and thus, for $0 \le x \le 1$,

$$\begin{aligned}|x^{n-k-r}S(x)| &= |q_k(x)| \ \sum_{j=0}^{r} a_j(1-x)^j x^{n-k-j}\\ &\le |q_k(x)| \ \sum_{j=0}^{n-k} a_j(1-x)^j x^{n-k-j} = |U_n(x)|. \qquad \square\end{aligned}$$

Let P_n be the polynomial of Theorem 6.1. We may assume that $|P_n^{(r)}(x_0)| = \|P_n^{(r)}\|$ $(:= \|P_n^{(r)}\|_{C[-1,1]})$ for some $0 \le x_0 \le 1$. Otherwise we would consider $P_n(-x)$. The polynomial $U_n(x) := P_n((1+x_0)x-1)$ has at most k zeros in the disk $|z-1/2| < 1/2$. Hence it satisfies (6.7) which together with

$$|U_n^{(r)}(1)| = (1+x_0)^r |P_n^{(r)}(x_0)| \ge \|P_n^{(r)}\|$$

implies that

$$\|P_n^{(r)}\| \le |U_n^{(r)}(1)| \le C_r(k+1)^r n^r \|U_n\|_{C[0,1]}.$$

This conclude the proof of Theorem 6.1 since $\|U_n\|_{C[0,1]} \le \|P_n\|$. $\square$

For $r=1$, Theorem 6.1 cannot be improved. In fact, Szabados [1981], for some constant $C>0$ and each $k = 1,2,\ldots$, constructed polynomials $P_n \in \mathcal{P}_n^k$, $n = 1,2,\ldots$, which satisfy

$$P_n'(1) \ge C(k+1)n\|P_n\|.$$

For $0 < \rho \le 1$ let $\mathcal{P}_n^k(\rho)$ be the collection of all real polynomials of degree $\le n$ which have at most k zeros in the open disk D_ρ^+. In particular, $\mathcal{P}_n^k(1) = \mathcal{P}_n^k$ are the polynomials of Theorem 6.1.

The linear transformation $x = \rho y + 1 - \rho$ maps the interval $[-1,1]$ onto $[1-2\rho, 1]$ and $|y| < 1$ onto D_ρ^+. Hence, $Q_n(y) := P_n(\rho y + 1 - \rho)$ for $P_n \in \mathcal{P}_n^k(\rho)$

also belongs to $\mathcal{P}_n^k$ and satisfies $\|Q_n\| \le \|P_n\|$. An application of Theorem 6.1 therefore yields

$$(6.8) \qquad |P_n^{(r)}(x)| \le \rho^{-r}\|Q_n^{(r)}\| \le C_r(k+1)^r\rho^{-r}n^r\|P_n\|$$

for $1-2\rho \le x \le 1$. From (6.8) and the Bernstein inequality

$$(6.9) \qquad |P_n^{(r)}(x)| \le C_r(1-x^2)^{-r/2}n^r\|P_n\|, \quad -1 < x < 1,$$

(see [**CA**, (1.6), p.98]) we deduce

Theorem 6.4. *Let $0 < \rho < 1$ and $k \ge 0$. For $P_n \in \mathcal{P}_n^k(\rho)$ and $r \ge 1$,*

$$(6.10) \qquad \|P_n^{(r)}\|_{C[0,1]} \le C_r(k+1)^r\rho^{-r}n^r\|P_n\|_{C[-1,1]}$$

with $C_r > 0$ depending only on r.

In the following, more difficult theorem of Erdélyi the exponent of ρ of (6.10) is reduced from $-r$ to $-r/2$, at the cost of increasing the exponent of $(k+1)$.

Theorem 6.5. *Let $0 < \rho < 1$ and $k \ge 0$. For $P_n \in \mathcal{P}_n^k(\rho)$ and $r \ge 1$,*

$$(6.11) \qquad \|P_n^{(r)}\|_{C[0,1]} \le C_r(k+1)^{2r}\rho^{-r/2}n^r\|P_n\|_{C[-1,1]}$$

with $C_r > 0$ depending only on r.

For the proof of Theorem 6.5 we need two lemmas. The first of them, in a simpler version, appears already in Borwein [1985]. For the set of polynomials

$$\Pi_n^k(\rho) := \{P \in \mathcal{P}_n^k(\rho) : \ \|P\| = 1\},$$

we describe some special properties of a polynomial $Q_n \in \Pi_n^k(\rho)$ with largest possible value $Q_n^{(r)}(1)$:

Lemma 6.6. *For $r \ge 1$, $0 < \rho \le 1$ and $0 \le k \le n$ there exists a polynomial $Q_n \in \Pi_n^k(\rho)$ with the properties*

(i) $\qquad Q_n^{(r)}(1) = \max\{|P^{(r)}(1)| : \quad P \in \Pi_n^k(\rho)\}$;

(ii) $\qquad$ *at most r zeros of Q_n are outside of $\Omega(\rho)$,*

where $\Omega(\rho) := (-\infty, 1] \cup \{z \in \mathbb{C} : |z-(1-\rho)| \le \rho\} = (-\infty, 1] \cup \overline{D_\rho^+}$.

(iii) $\quad$ *If $r = 1$, then all zeros of Q_n lie in the interval $(-\infty, 1]$.*

Proof. The set $\Pi_n^k(\rho)$ is compact in the uniform norm on $[-1,1]$, in particular all coefficients in $\Pi_n^k(\rho)$ are uniformly bounded. The operator $P \to P^{(r)}(1)$ is continuous on $\Pi_n^k(\rho)$. This guarantees the existence of a maximal Q_n in (i). Let Q_n be one of these maximal polynomials, with minimal L_1-norm on $[-1,1]$.

Suppose that Q_n has some $r+1$ zeros outside of $\Omega(\rho)$. We may assume that these are the non-real numbers $z_1, z_2, \dots, z_{2s}$ where $z_j = \overline{z_{j+s}}$ and $|z_j - 1 + \rho| > \rho$, $j = 1, \dots, s$ and the real numbers $1 < z_{2s+1} \le \dots \le z_{r+1}$. The polynomials

$$S_1(x) := \prod_1^s (x - z_j)(x - \overline{z_j}) \ , \ S_2(x) := \prod_{2s+1}^{r+1} (z_j - x)$$

are positive on $[-1, 1]$ and, for sufficiently small $\varepsilon > 0$, the polynomial

$$R_n(x) := Q_n(x) \left(1 - \varepsilon \frac{(1-x)^{r+1}}{S_1(x) S_2(x)}\right)$$

belongs to $\mathcal{P}_n^k(\rho)$, $R_n^{(r)}(1) = Q_n^{(r)}(1)$ and $\|R_n\| \le \|Q_n\| = 1$, $\|R_n\|_1 < \|Q_n\|_1$. This contradicts the definition of Q_n. Therefore, Q_n has the property (ii).

Let $r = 1$. We want to prove that all zeros of Q_n lie in $(-\infty, 1]$. Suppose first that Q_n has a zero α, which is not real. Then its conjugate $\beta := \overline{\alpha}$ is also a zero of Q_n. For sufficiently small $\varepsilon > 0$ the function

$$\Phi(x) := 1 - \frac{\varepsilon(x-1)^2}{(x-\alpha)(x-\beta)} = \frac{|x-\alpha|^2 - \varepsilon(x-1)^2}{|x-\alpha|^2}$$

satisfies $0 < \Phi(x) \le \Phi(1) = 1$ for $-1 \le x \le 1$, and $R_n := \Phi Q_n$ is a polynomial of degree $\le n$ with the properties $R_n'(1) = Q_n'(1)$, $\|R_n\| \le \|Q_n\|$ and $\|R_n\|_1 < \|Q_n\|_1$. This is impossible. The inequality $\|R_n\| < \|Q_n\|$ contradicts the maximality of Q_n. And $\|R_n\| = \|Q_n\|$ implies $R_n \in \Pi_n^k(\rho)$ and then $\|R_n\|_1 < \|Q_n\|_1$ contradicts the definition of Q_n. This argument applies also if there are two zeros α, β of Q_n with $1 < \alpha \le \beta$. It follows that all zeros of Q_n are real, at most one of them is > 1. Let α be the only zero > 1. We consider the polynomial

$$R_n(x) := Q_n(x) \left(1 - \frac{\varepsilon(1-x)}{\alpha - x}\right)$$

for sufficiently small $\varepsilon > 0$. Then R_n belongs to $\mathcal{P}_n^k(\rho)$, $R_n(1) = Q_n(1)$ and $|R_n(x)| < |Q_n(x)|$ for all $-1 \le x < 1$. In addition, since $Q_n'(1) > 0$ and $R_n'(1) = Q_n'(1) - \varepsilon Q_n(1)/(1-\alpha)$, the extremality of Q_n implies that $Q_n(1) < 0$. From this and Rolle's theorem we deduce that Q_n is positive on (α, ∞) and Q_n' is positive on $[\alpha, \infty)$. It follows that Q_n' is positive also on $[1, \alpha)$. Otherwise, $Q_n'(1) > 0$ and $Q_n'(\alpha) > 0$ would imply that Q_n' has a double zero or two distinct zeros in $(1, \alpha)$, which is impossible. Since $Q_n'(x) > 0$ for all $x \ge 1$, $|Q_n|$ is decreasing on $[1, \alpha]$ and $Q_n(x + \varepsilon)$ violates the maximality of Q_n for small $\varepsilon > 0$. □

Lemma 6.7. *Let* $0 < \rho \le 1$. *If* $P_n \in \mathcal{P}_n$ *has all its zeros in* $(-\infty, 1 - 2\rho]$ *and* $r = 1, 2, \dots,$ *then*

$$|P_n^{(r)}(1)| \le \left(2n\rho^{-1/2}\right)^r \|P_n\|. \tag{6.12}$$

Proof. First let $\|P_n\| = |P_n(1)|$. We begin by showing that *in each interval* $[1-a, 1]$, $0 < a < 1$, P_n *has* $\le n\sqrt{2a}$ *zeros*. Indeed, $x = \cos t$ maps $\mathbb{T}$ onto $[-1, 1]$, also $[-\alpha, \alpha] \subset \mathbb{T}$, $0 < \alpha < \pi/2$ onto $[1-a, 1] =: J$, $a = 1 - \cos\alpha$. To each zero of P_n on J there correspond two zeros of the even trigonometric polynomials $T_n(t) := P_n(\cos t)$. Since $|T_n(0)| = \|T_n\|$, Corollary 3.5 of Chapter 3 implies that the number of zeros of P_n is at most $2n\sin(\alpha/2) = n\sqrt{2a}$.

Let $h > 0$ and $a_1 := 2\rho$, $a_k := a_1 + 2\rho(k-1)h$, $k = 2, 3, \ldots$. We denote by N_k the number of zeros x_j of P_n in the interval $(1 - a_{k+1}, 1]$ and put $N_0 := 0$. By the last lemma we have $N_k \le n\sqrt{2a_{k+1}}$ for any $k \ge 1$. We set $I_k := (1 - a_{k+1}, 1 - a_k]$ and get

$$\begin{aligned}
\frac{|P_n'(1)|}{|P_n(1)|} &= \sum_{j=1}^{n} \frac{1}{1-x_j} = \sum_{k=1}^{\infty} \sum_{x_j \in I_k} \frac{1}{1-x_j} \le \sum_{k=1}^{\infty} \frac{N_k - N_{k-1}}{a_k} \\
&= \sum_{k=1}^{\infty} N_k \left(\frac{1}{a_k} - \frac{1}{a_{k+1}} \right) \le \sqrt{2}n \sum_{k=1}^{\infty} \frac{a_{k+1} - a_k}{a_k\sqrt{a_{k+1}}} \\
&\le 2\sqrt{2}\rho h n \sum_{k=1}^{\infty} (a_k)^{-3/2} \le h n \rho^{-1/2} \sum_{k=0}^{\infty} (1 + kh)^{-3/2} \\
&\le 2n\rho^{-1/2}/(1-h).
\end{aligned}$$

We have proved that

$$|P_n'(1)| \le 2n\rho^{-1/2}|P_n(1)|. \tag{6.13}$$

By Rolle's theorem, each of the polynomials P_n', P_n'', $\ldots$, $P_n^{(r-1)}$ has $\le n\sqrt{2a}$ zeros in $[1-a, 1]$, $0 < a < 1$, or is identically zero. By the arguments just used, this implies that $|P_n^{(j)}(1)| \le 2n\rho^{-1/2}|P_n^{(j-1)}(1)|$, $j = 1, 2, \ldots, r$. Therefore we get

$$|P_n^{(r)}(1)| \le \left(2n\rho^{-1/2}\right)^r |P_n(1)|. \tag{6.14}$$

Replacing here $|P_n(1)|$ by $\|P_n\|$, we obtain (6.12).

Now let $|P_n(1)| < \|P_n\|$. Since $|P_n(x)|$ is strictly monotone increasing to ∞ on $[1, \infty)$, there exists a smallest number $x_0 > 1$ for which $|P_n(x_0)| = \|P_n\|$. The linear transformation $x = -1 + (x_0 + 1)(y + 1)/2$ maps the interval $[-1, 1]$ onto $[-1, x_0]$, and P_n is transformed into the polynomial $p(y) = P_n(x)$ which has the property

$$|p(1)| = |P_n(x_0)| = \|P_n\| = \|p\|.$$

Therefore (6.12) is valid for p. Since $|P_n^{(r)}|$ is monotone increasing in $[1 - 2\rho, \infty)$ and thus

$$|P_n^{(r)}(1)| \le |P_n^{(r)}(x_0)| = 2^r |p^{(r)}(1)|/(x_0 + 1)^r \le |p^{(r)}(1)|,$$

relation (6.12) is also valid for P_n. □

Proof of Theorem 6.5. We have to establish that

$$|P_n^{(r)}(x)| \le C_r(k+1)^{2r}\rho^{-r/2}n^r\|P_n\|, \quad 0 \le x \le 1, \ P_n \in \mathcal{P}_n^k(\rho). \tag{6.15}$$

First of all we note that (6.9) guarantees that this is true, even without the factor $(k+1)^{2r}$ and for all $P_n \in \mathcal{P}_n$ if x is not too close to ± 1. Indeed, from (6.9) it follows for $-1+\rho/4 \le x \le 1-\rho/4$

$$|P_n^{(r)}(x)| \le C_r(1-x^2)^{-r/2}n^r\|P_n\| \le C_r\left(2n\rho^{-1/2}\right)^r\|P_n\|, \tag{6.16}$$

Next we show that *for each $P_n \in \mathcal{P}_n^k(\rho)$ and $r = 1,2,\ldots$ there exists a polynomial $P_n^* \in \mathcal{P}_n^k(\rho/2)$ so that*

$$\begin{aligned} |P_n^{(r)}(x)| &\le (4/3)^r|P_n^{*(r)}(1)|, \quad 1-\rho/4 \le x \le 1, \\ \|P_n^*\| &\le \|P_n\|. \end{aligned} \tag{6.17}$$

Indeed, let $\xi \in J := [1-\rho/4, 1]$ be a point where $|P_n^{(r)}(x)|$ attains its maximum on J. Since $D_{\rho/2}^+ \subset D_\rho^+$ and $\xi \ge 1-\rho/4 \ge 3/4$, $z \in D_{\rho/2}^+$ implies $\xi z \in D_\rho^+$. Therefore the polynomial $P_n^*(x) := P_n(\xi x)$ belongs to $\mathcal{P}_n^k(\rho/2)$ and satisfies $|P_n^{*(r)}(1)| = \xi^r\|P_n^{(r)}\|_{C(J)}$. For P_n^* we have (6.17).

We prove (6.15) for $1-\rho/4 \le x \le 1$ by showing that for $P_n^* \in \mathcal{P}_n^k(\rho/2)$ of (6.17) we have

$$|P_n^{*(r)}(1)| \le C_r(k+1)^{2r}\rho^{-r/2}n^r\|P_n^*\|.$$

It is sufficient to establish this for $Q_n \in \Pi_n^k(\rho/2)$ of Lemma 6.6, or, equivalently, to show that

$$Q_n^{(r)}(1) \le C_r(k+1)^{2r}\rho^{-r/2}n^r, \quad Q_n \in \Pi_n^k(\rho). \tag{6.18}$$

We prove (6.18) by induction on r.

Case $r = 1$, $k \ge 0$:
(a) We assume first that the extremal polynomial $Q_n \in \Pi_n^k(\rho)$ of Lemma 6.6 satisfies $|Q_n(1)| = \|Q_n\|$ $(= 1)$. Then Q_n has all its zeros in $(-\infty, 1)$, at most k of them in $(1-2\rho, 1)$. This implies that $Q_n = q_kP_n$, $q_k \in \mathcal{P}_k$, $P_n \in \mathcal{P}_n^0(\rho)$, where P_n and q_k have all their zeros in $(-\infty, 1-2\rho]$ and in $(1-2\rho, 1)$, respectively. We deduce from this that P_n satisfies (6.12) of Lemma 6.7 and

$$|P_n(1)| = \|P_n\|. \tag{6.19}$$

Indeed, let $|P_n(1)| < \|P_n\|$, and let $t_1, t_2, \ldots$ be the zeros of P_n. Then $|P_n(x)| \le |P_n(1)|$ for $1-4\rho \le x \le 1$, because the maximum of each factor $|x-t_i|$ of $|P_n(x)|$ on $[1-4\rho, 1]$ is at $x = 1$. Therefore $\|P_n\| = |P_n(t_1)|$ for some $t_1 \le 1-4\rho$. The same argument for the factors of $|q_k(x)|$ yields $|q_k(1)| < |q_k(t_1)|$ and thus $|Q_n(1)| < |Q_n(t_1)| \le \|Q_n\|$, in contradiction to the assumption $|Q_n(1)| = \|Q_n\|$.

Let $a := 1 - \sqrt{\rho}/(4n)$ and $J = [a, 1]$. The interval J has the length $|J| = \sqrt{\rho}/(4n)$. If Markov's inequality is applied to the polynomial q_k and the interval J one obtains

$$|q_k'(1)| \le \frac{2k^2}{|J|}\|q_k\|_{C(J)} = 8nk^2\rho^{-1/2}\|q_k\|_{C(J)}.$$

Since $|P_n|$ and $|P_n'|$ are monotone increasing on $[1-2\rho, 1]$, it follows from (6.12) and (6.19) that for $x \in J$,

$$2|P_n(x)| \ge 2(|P_n(1)| - (1-x)|P_n'(1)|) \ge 2(|P_n(1)| - \frac{1}{2}|P_n(1)|) = |P_n(1)|.$$

We take $t_2 \in J$ such that $|q_k(t_2)| = \|q_k\|_{C(J)}$. Since $2|P_n(t_2)| \ge |P_n(1)|$, we have

$$|P_n(1)q_k'(1)| \le 16nk^2\rho^{-1/2}|P_n(t_2)q_k(t_2)|.$$

On the other hand, from (6.13) we get

$$|P_n'(1)q_k(1)| \le 2n\rho^{-1/2}|P_n(1)q_k(1)|$$

and thus

$$Q_n'(1) = |P_n'(1)q_k(1) + P_n(1)q_k'(1)| \le 2n(1+8k^2)\rho^{-1/2}\|Q_n\|. \tag{6.20}$$

(b) Let $|Q_n(1)| < \|Q_n\|$. This case can be reduced to case (a) with k replaced by $k+1$. Indeed, similar to the end of the proof of Lemma 6.7 let $x_0 > 1$ be the smallest number for which $|Q_n(x_0)| = \|Q_n\|$. The polynomial

$$Q_n^*(x) := Q_n(-1 + (x_0+1)(x+1)/2)$$

belongs to $\Pi_n^{k+1}(\rho)$ (to $\Pi_n^k(\rho)$ if Q_n does not vanish at $x = 1$) and satisfies

$$(Q_n^*)'(1) = \frac{x_0+1}{2}Q_n'(x_0) \ge Q_n'(1) \tag{6.21}$$

since $Q_n'(x)$ is positive and increasing in $x \ge 1$. Since $\|Q_n^*\| = \|Q_n\| = |Q_n^*(1)| = 1$, since all zeros of Q_n^* lie in $(-\infty, 1)$, at most $k+1$ of them in $(1-2\rho, 1)$, case (a) applies also to Q_n^*: $(Q_n^*)'(1)$ satisfies (6.20) with k replaced by $k+1$. This and (6.21) conclude the proof of (6.18) and thus of Theorem 6.5 for $r = 1$.

Case $r \ge 2$, $k \ge 0$: We assume by induction that Theorem 6.5 is true for derivatives of order $r-1$. The extremal polynomial $Q_n \in \Pi_n^k(\rho)$ of Lemma 6.6 (of uniform norm $\|Q_n\| = 1$) is of the form $Q_n = R_n q_n \phi_n$, where $R_n \in \mathcal{P}_n$ has all its zeros in $(-\infty, 1-2\rho]$, $q_n \in \mathcal{P}_n$ is a monic polynomial whose zeros are not real and lie on the boundary of D_ρ^+, that is on

$$\Gamma(\rho) := \{z \in \mathbb{C} : |z - (1-\rho)| = \rho,\ \ z \ne 1-2\rho,\ \ z \ne 1\},$$

and ϕ_n is a monic polynomial which has all its zeros in D_ρ^+, at $x = 1$, and outside of $\Omega(\rho)$. The degree of ϕ_n is at most $k + 2r$. Indeed, Q_n and thus ϕ_n

have at most k zeros in D_ρ^+, at most r zeros outside of $\Omega(\rho)$ (by Lemma 6.6), and at $x = 1$ a zero of order at most r since $Q_n^{(r)}(1) > 0$.

If z_j is a zero of q_n, that is, $z_j = 1 - \rho + \rho e^{it_j}$, $t_j \in (0, \pi) \cup (\pi, 2\pi)$, then $\overline{z_j}$ is also a zero. Hence

$$(z-z_j)(z-\overline{z_j}) = \alpha_j(1-z)^2+(1-\alpha_j)(z-1+2\rho)^2, \quad 0 < \alpha_j := \cos^2(t_j/2) < 1,$$

is a factor of q_n, and q_n is of even degree $2s$, $0 \le s \le n/2$:

$$\begin{aligned} q_n(z) &= \prod_{j=1}^{s}\{\alpha_j(1-z)^2 + (1-\alpha_j)(z-1+2\rho)^2\} \\ &= \sum_{\ell=0}^{s} a_\ell (1-z)^{2\ell}(z-1+2\rho)^{2s-2\ell}, \quad a_\ell \ge 0. \end{aligned} \tag{6.22}$$

We also need the polynomials

$$\tilde{q}_n(z) := \sum_{\ell=0}^{\min\{s,r\}} a_\ell(1-z)^{2\ell}(z-1+2\rho))^{2s-2\ell},$$

$\tilde{Q}_n := R_n\tilde{q}_n\phi_n$ and $S_n(z) := \tilde{Q}_n((1-\rho/8)z+\rho/8)$ which are all of degree $\le n$. Since $\tilde{q}_n^{(j)}(1) = q_n^{(j)}(1)$, $j = 0, \dots, r$, one has

$$Q_n^{(r)}(1) = \tilde{Q}_n^{(r)}(1) = (1-\rho/8)^{-r}S_n^{(r)}(1). \tag{6.23}$$

The polynomial $\tilde{q}_n$ has degree $2s$ and has a zero of order $2s-2r$ at $z = 1-2\rho$ if $s > r$. Therefore and since $\phi_n \in \mathcal{P}_{2r+k}$, the polynomials $\tilde{Q}_n$ and S_n have at most $4r+k$ zeros outside of $(-\infty, 1-2\rho]$. By Rolle's theorem, this is also true for $S_n^{(r-1)}$, that is, the polynomials $\tilde{Q}_n$ and $S_n^{(r-1)}$ belong to the class $\mathcal{P}_n^{4r+k}(\rho)$. We apply the case $r = 1$ of Theorem 6.5 to $S_n^{(r-1)}$:

$$|S_n^{(r)}(1)| \le C_1(4r+k+1)^2 n\rho^{-1/2}\|S_n^{(r-1)}\|. \tag{6.24}$$

Moreover,

$$\|S_n^{(r-1)}\| = (1-\rho/8)^{r-1}\|\tilde{Q}_n^{(r-1)}\|_{C[-1+\rho/4,1]}. \tag{6.25}$$

Applying the induction hypothesis to $\tilde{Q}_n$ yields

$$\|\tilde{Q}_n^{(r-1)}\|_{C[0,1]} \le C_{r-1}(4r+k+1)^{2r-2}\left(n\rho^{-1/2}\right)^{r-1}\|\tilde{Q}_n\|. \tag{6.26}$$

From (6.16), if applied to the $(r-1)$-st derivative of $\tilde{Q}_n$,

$$|\tilde{Q}_n^{(r-1)}(x)| \le C_{r-1}(2n\rho^{-1/2})^{r-1}\|\tilde{Q}_n\|, \quad -1+\rho/4 \le x \le 1-\rho/4. \tag{6.27}$$

Since the coefficients a_ℓ in (6.22) are non-negative, $|\tilde{q}_n(x)| \le |q_n(x)|$, $-1 \le x \le 1$, and thus

$$\|\tilde{Q}_n\| \le \|Q_n\|. \tag{6.28}$$

By using (6.23)-(6.28) we deduce the inequality

$$|Q_n^{(r)}(1)| \le C_r'(4r+k+1)^{2r}\rho^{-r/2}n^r\|Q_n\|.$$

This concludes the proof of (6.18) and thus also of Theorem 6.5. □

There is a dual to Theorem 6.5, with the disk D_ρ^+ replaced by D_ρ^-. Combining them, we obtain

Theorem 6.8 (Erdélyi [1989_1]). *Let $0<\rho<1$, $k\ge 0$ and $r=1,2,\ldots$. If the real polynomial P_n of degree $\le n$ has at most k zeros in each of the two disks D_ρ^+ and D_ρ^-, then uniformly on $[-1,1]$,*

$$\|P_n^{(r)}\| \le C_r(k+1)^{2r}\left(\frac{n}{\sqrt{\rho}}\right)^r\|P_n\|.$$

We have discussed at considerable length Markov-type inequalities for $\mathcal{P}_n^k(\rho)$. Similar results exist for the Bernstein-type and for mixed inequalities. As a final result (for *the first derivative* of P_n) in this direction we give

Theorem 6.9. *For an absolute constant $C>0$ and for the polynomials $P_n\in\mathcal{P}_n^k$ of Theorem 6.1 one has*

$$|P_n'(x)| \le C\min\left\{n(k+1), \left(\frac{n(k+1)}{1-x^2}\right)^{1/2}\right\}\|P_n\|, \quad -1<x<1. \tag{6.29}$$

See Borwein and Erdélyi [1994], where also the history of the subject is described.

§ 7. The Inequality of Remez

In this section we reproduce Erdélyi's [1989_2] simple proof of the Remez inequality.

Theorem 7.1 (Remez [1936]). *Let $0<\delta<2$ and let P_n be a real algebraic polynomial of degree $\le n$ which satisfies $|P_n(x)|\le 1$ for $-1\le x\le 1$, except on a subset of Lebesgue measure $\le\delta$. Then, in the uniform norm on $[-1,1]$,*

$$\|P_n\| \le C_n\left(\frac{2+\delta}{2-\delta}\right) \tag{7.1}$$

where C_n is the Chebyshev polynomial of degree n. The inequality (7.1) is sharp only for

$$P_n(x) = \pm C_n((\pm 2x+\delta)/(2-\delta)). \tag{7.2}$$

Proof. For fixed δ and n, the set of polynomials of the theorem is compact. Hence there exists one, Q_n, with maximal norm $\|Q_n\|$, and with $|\Omega| \le \delta$, where $\Omega := \{x \in [-1,1] : |Q_n(x)| > 1\}$.

(a) Let us first assume that $Q_n(1) = \|Q_n\|$. Then there exists an interval $[b,1]$, $1-\delta < b < 1$, which belongs to Ω.

We claim that all zeros of Q_n are real and lie in $[-1,1)$. Otherwise, if Q_n vanishes at a non-real point α, then

$$P_n(x) := (1+\varepsilon^2)Q_n(x)\left(1 - \frac{\varepsilon(x-1)^2}{(x-\alpha)(x-\overline{\alpha})}\right)$$

contradicts the maximality of Q_n for sufficiently small $\varepsilon > 0$. Indeed, $\|P_n\| \ge P_n(1) = (1+\varepsilon^2)\|Q_n\|$, and $|P_n(x)| \le |Q_n(x)|$ for all $x \in [-1,b]$ since

$$(1+\varepsilon^2)\left(1 - \frac{\varepsilon(x-1)^2}{(x-\alpha)(x-\overline{\alpha})}\right) < 1 \quad \text{for all} \quad -1 \le x \le b.$$

If Q_n has a real zero α outside of $[-1,1]$, then

$$P_n(x) := (1+\varepsilon^2)Q_n(x)\left(1 - \varepsilon\ \mathrm{sign}(\alpha)\ \frac{1-x}{\alpha - x}\right)$$

again contradicts the maximality of Q_n if $\varepsilon > 0$ is small.

(b) Suppose now that Q_n attains its uniform norm at an interior point $\xi \in (-1,1)$. We may assume that $Q_n(\xi) = \|Q_n\|$. The polynomials

$$R_1(x) := Q_n\left(\frac{\xi-1}{2} + \frac{\xi+1}{2}\ x\right)\ ,\ R_2(x) := Q_n\left(\frac{\xi+1}{2} + \frac{\xi-1}{2}\ x\right)$$

satisfy $R_j(1) = \|R_j\| = \|Q_n\|$, $j = 1,2$. Since Q_n is extremal, the measure $|\Omega_j|$ of $\Omega_j := \{x \in [-1,1] : |R_j(x)| > 1\}$ is $\ge \delta$. On the other hand,

$$|\Omega_1|(\xi+1)/2 + |\Omega_2|(1-\xi)/2 \le \delta,$$

so that $|\Omega_1| = |\Omega_2| = \delta$ which means that R_1 and R_2 are extremal polynomials which attain their uniform norm at 1: it follows from (a) that R_1 and R_2 have all their zeros in $[-1,1]$, which is absurd since the number of zeros of R_1 and R_2 in $[-1,1]$ is equal to the number of zeros of Q_n in $[-1,\xi)$ and in $(\xi,1]$, respectively.

(c) This contradiction proves that either $Q_n(1) = \|Q_n\|$ or $Q_n(-1) = \|Q_n\|$ and that all zeros of Q_n lie in $[-1,1]$. We may assume that

$$Q_n(1) = \|Q_n\| > |Q_n(x)|, \quad -1 < x < 1, \tag{7.3}$$

otherwise we study the polynomial $Q_n(-x)$. We shall prove now that

$$|Q_n(x)| \le 1\ ,\ -1 \le x \le 1-\delta. \tag{7.4}$$

Assume to the contrary that (7.4) does not hold. Then there exist points $-1 \le a_0 < a_1 < a_2 < 1$ so that $Q_n(x) \ge 1$ for $x \in [a_2,1]$, $|Q_n(x)| \le 1$ for $x \in [a_1,a_2]$, $|Q_n(x)| \ge 1$ for $x \in [a_0,a_1]$.

Let $h := \min\{a_1 - a_0, 1 - a_2\}$ and let $x_1, \dots, x_m$ be the zeros of Q_n in $[a_1, a_2]$. Since all zeros of Q_n are in $[-1, 1)$, m is ≥ 1 and the other zeros of Q_n, if there are any, lie in $[-1, a_0)$. We define the polynomials $R(x) := \prod_{j=1}^{m}(x - x_j)$ and $S := Q_n/R$. Since $|R(1+h)| > |R(1)|$, $|R(x+h)| < |R(x)|$ for $x \leq a_0$, and $|S(x-h)| \leq |S(x)|$ for $x \geq a_0 + h$, the polynomial $Q_n^*(x) := R(x+h)S(x)$ has the following properties.

(i) $|Q_n^*(1)| > Q_n(1) = \|Q_n\|$,

(ii) $|Q_n^*(x)| \leq |Q_n(x)|$ for all $-1 \leq x \leq a_0$,

(iii) $|Q_n^*(x-h)| = |R(x)S(x-h)| \leq |Q_n(x)|$ for all $a_1 \leq x \leq 1$.

It follows from (ii) and (iii) that the Lebesgue measure of the set $\{x \in [-1,1] : |Q_n^*(x)| \leq 1\}$ is not smaller than $|[-1,1] \setminus \Omega| \geq 2 - \delta$. This and (i) contradict the maximality of Q_n: (7.4) is valid.

Any real algebraic polynomial P_n of degree $\leq n$, with $\|P_n\| \leq 1$, satisfies

$$|P_n(x)| \leq |C_n(x)| \quad \text{for all } x < -1, \ x > 1, \tag{7.5}$$

with strict equality if and only if $P_n = \pm C_n$. (See Theorem 2.1 of Chapter 4 for a similar result in weighted approximation.) By a linear transformation we deduce from (7.4) and (7.5) that the four polynomials (7.2) are the only extremal polynomials. □

Since $C_n(x) \leq (x + \sqrt{x^2-1})^n$, $x \geq 1$, the maximum on the right-hand side of (7.1) satisfies

$$C_n((2+\delta)/(2-\delta)) \leq \left(\frac{1+\sqrt{2\delta}+\delta/2}{1-\delta/2}\right)^n$$

and, if $0 < \delta \leq 1$,

$$C_n((2+\delta)/(2-\delta)) \leq (1+\sqrt{2\delta}+3.5\delta)^n \leq e^{5n\sqrt{\delta}}. \tag{7.6}$$

Because of the periodicity we may expect a significant improvement in the trigonometric case. This is confirmed by

Theorem 7.2 (Erdélyi [1992]). *Let $0 < \delta \leq \pi/2$ and let T_n be a real trigonometric polynomial of degree $\leq n$ which satisfies $|T_n(t)| \leq 1$ on $\mathbb{T}$ except on a subset of measure $\leq \delta$. Then*

$$\|T_n\|_{C(\mathbb{T})} \leq e^{Cn\delta}, \tag{7.8}$$

for some absolute constant $C > 0$.

The proof of this trigonometric Remez-type inequality needs new ideas, and it is far from being a copy of the methods working in the algebraic case.

§ 8. One-sided Approximation by Polynomials

Let $f(x)$ be a function defined *for each* $x \in A$, $A = [a,b]$ or $A = \mathbb{T}$. Let X stand for a Banach function space with norm $\|\cdot\|_X$, and let f belong to X (or f belong to one of the classes of functions which are elements of X), and let Y_n be an n-dimensional subspace of X. We define by

$$\tilde{E}_n^-(f)_X := \inf\{\|f - S\| : \; S(x) \le f(x), x \in A, S \in Y_n\} \tag{8.1}$$

$$\tilde{E}_n^+(f)_X := \inf\{\|T - f\| : \; f(x) \le T(x), x \in A, T \in Y_n\} \tag{8.2}$$

the errors of one-sided approximation of f, from below and from above, respectively. Let also

$$\tilde{E}_n(f)_X := \inf\{\|T - S\| : \; S(x) \le f(x) \le T(x), x \in A, \;\; S, T \in Y_n\} \tag{8.3}$$

be the error of two-sided approximation. Plainly, if $E_n(f)_X$ is the unrestricted error of approximation, then

$$E_n(f)_X \le \tilde{E}_n^{\pm}(f)_X \le \tilde{E}_n(f)_X \le \tilde{E}_n^-(f)_X + \tilde{E}_n^+(f)_X. \tag{8.4}$$

In this section we take $A := \mathbb{T}$, $Y_n := \mathcal{T}_n$, $X := L_p(\mathbb{T})$, $1 \le p < \infty$ or $X := C(\mathbb{T})$, and will try to determine $\tilde{E}_n(f)$, at least up to weak equivalence, for some f. For $X = C(\mathbb{T})$ this is easy: if $Q_n \in \mathcal{T}_n$ approximates f best, then $S_n := Q_n - E_n(f)$, $T_n := Q_n + E_n(f)$ satisfy

$$S_n(x) \le f(x) \le T_n(x), \quad x \in \mathbb{T}, \tag{8.5}$$

and we deduce

$$E_n(f) \le \tilde{E}_n(f) \le 2E_n(f). \tag{8.6}$$

Using the fact that $f - Q_n$ attains the values $\pm E_n(f)$ at $2n+2$ consecutive points of $\mathbb{T}$ with alternating signs, one can prove that S_n and T_n are the unique best one-sided uniform approximants to f from below and above, respectively.

The following books treat one-sided approximation: Korneichuk, Ligun, and Doronin [A-1982], Sendov and Popov [A-1983], Pinkus [A-1989].

The proofs of the existence of best approximation polynomials in $\mathcal{T}_n$ for the spaces L_p, $1 \le p < \infty$ and of the existence of a best pair in (8.3) are standard.

We say that a Banach function space X is *strictly convex*, if for any two functions $f_1, f_2 \in X$, the relation

$$\|f_1 + f_2\| = \|f_1\| + \|f_2\|$$

holds if and only if one of them is a non-negative multiple of the other (for example, $f_1 = \alpha f_2$, $\alpha \ge 0$). From the conditions of equality for the Minkowski inequality it is known that spaces $L_p(\mathbb{T})$, $1 < p < \infty$ are strictly convex.

Proposition 8.1. *In a strictly convex space X, each function $f \in X$ has a unique best approximation from below (hence also from above).*

Proof. Let $S, S_1 \in Y_n$ be two best approximations to $f \in X \setminus Y_n$ from below. Then also $\frac{1}{2}(S + S_1)$ has this property. Therefore

$$\tilde{E}_n^-(f) = \|f - \frac{1}{2}(S + S_1)\| = \|f - S\| = \|f - S_1\|.$$

This implies $f - S = \alpha(f - S_1)$, and since $f \notin Y_n$, $\alpha = 1$, $S = S_1$. □

For the space $L_1(\mathbb{T})$, continuous functions may have several approximations; for example, to $f(x) := |\cos(nx)|$ all polynomials $S_n(x) := \alpha \cos(nx)$, $-1 \le \alpha \le 1$ are best one-sided approximations from $\mathcal{T}_n$. This cannot happen for differentiable functions.

Lemma 8.2. *If $S_n \in \mathcal{T}_n$ is a best $L_1(\mathbb{T})$ approximation to a differentiable function f on $\mathbb{T}$ from below, then $f - S_n$ has at least $n + 1$ distinct zeros on $\mathbb{T}$.*

Proof. Suppose to the contrary that $x_1 < \cdots < x_r$, $r \le n$ are the only zeros of $f - S_n$. By a theorem of Krein [**CA**, Theorem 9.1, p.80], valid for arbitrary Haar systems, there exists a $T_n \in \mathcal{T}_n$ which has double zeros at some n points of $\mathbb{T}$, including the x_i. We may assume that it satisfies $T_n(x) \ge 0$, $x \in \mathbb{T}$ and $\int_{\mathbb{T}} T_n dx = 3$. We take $a := 1/\pi$, then $\int_{\mathbb{T}}(T_n - a)dx = 1$. The set $U := \{x \in \mathbb{T} : \ T(x) - a < 0\}$ is open and contains all zeros of $f - S_n$. Hence, $K := \mathbb{T} \setminus U$ is compact so that $\eta := \min\{f(x) - S_n(x) : \ x \in K\} > 0$. We choose $\varepsilon > 0$ so small that $\varepsilon(T(x) - \delta) \le \eta$ on $\mathbb{T}$. Then the polynomial $S_n^* := S_n + \varepsilon(T - a)$ satisfies $S_n^* \le f$ and $\int_{\mathbb{T}}(f - S_n^*)dx < \int_{\mathbb{T}}(f - S_n)dx$ which contradicts the minimality of S_n. □

Theorem 8.3 (Bojanic and DeVore [1966]). *Let $f \in C(\mathbb{T})$ be differentiable on $\mathbb{T}$ and let $n \ge 1$. There exist unique polynomials S_n and $T_n \in \mathcal{T}_n$ which satisfy* (8.5) *and*

$$\|f - S_n\|_1 = \tilde{E}_n^-(f)_1, \quad \|T_n - f\|_1 = \tilde{E}_n^+(f)_1. \tag{8.7}$$

Proof. It suffices to prove only the uniqueness of S_n. (The uniqueness of T_n follows then from the approximation of $-f$ from below.)

Suppose that S_n, S are two extremal polynomials. Then $S_n^* := \frac{1}{2}(S_n + S)$ is also extremal and, by Lemma 8.2, $f - S_n^*$ has $n+1$ distinct zeros $(x_i)_1^{n+1}$ in $\mathbb{T}$. Since $S_n(x) \le f(x)$ and $S(x) \le f(x)$, $x \in \mathbb{T}$, the x_i are also zeros of $f - S_n$ and of $f - S$ and therefore $f'(x_i) = S_n'(x_i) = S'(x_i)$. This implies that $S_n - S$ has $n + 1$ zeros of multiplicity ≥ 2, hence $S_n = S$. □

For smooth functions f, one often has the same, up to a constant factor, estimates for the one-sided and the unrestricted errors of approximation.

First results of this kind, for algebraic polynomials, are due to Freud [1955] and for trigonometric polynomials to Ganelius [1956]. The next result compares the error of one-sided approximation of f with the error of unrestricted approximation of f':

Theorem 8.4 (Ganelius). *Let $1 \le p < \infty$. If f is absolutely continuous on $\mathbb{T}$ and $f' \in L_p(\mathbb{T})$, that is, if $f \in W_p^1(\mathbb{T})$, then*

$$\tilde{E}_n(f)_p \le \frac{2\pi}{n+1} E_n(f')_p, \quad n \ge 1. \tag{8.8}$$

Proof. We follow the exposition of Sendov and Popov [A-1983]. We may suppose that $\int_0^{2\pi} f(t)dt = 0$. (Otherwise we consider $f - c$ for some appropriate constant c.) Then f has the representation

$$f(x) = \frac{1}{\pi} \int_0^{2\pi} f'(t)\mathcal{B}_1(x-t)dt \tag{8.9}$$

(see [**CA**,(5.15),p.151]), where $\mathcal{B}_1$ is the Bernoulli spline

$$\mathcal{B}_1(t) := (\pi - t)/2, \quad 0 \le t < 2\pi.$$

We proceed to approximate $-\mathcal{B}_1$ from below and above. Let $S_n, T_n \in \mathcal{T}_n$ be defined by the Hermite interpolation conditions

$$\begin{cases} S_n(t_k) = T_n(t_k) = -\mathcal{B}_1(t_k), & k = 1, 2, \dots, n \\ S_n'(t_k) = T_n'(t_k) = -\mathcal{B}_1'(t_k) = 1/2, & k = 1, 2, \dots, n \\ S_n(0) = -\pi/2, \quad T_n(0) = \pi/2, & \end{cases} \tag{8.10}$$

with $t_k := 2\pi k/(n+1)$. We claim that

$$S_n(x) \le -\mathcal{B}_1(x) \le T_n(x), \quad 0 \le x < 2\pi. \tag{8.11}$$

Indeed, since $(S_n + \mathcal{B}_1)(0+) = 0$ and since $S_n + \mathcal{B}_1$ has n zeros of multiplicity ≥ 2 at the t_k, $S_n' + \mathcal{B}_1'$ has $\ge 2n$ zeros in $(0, 2\pi)$. Since $S_n' + \mathcal{B}_1' \in \mathcal{T}_n$ cannot have more than $2n$ zeros in $(0, 2\pi)$, it follows that all zeros t_k of $S_n + \mathcal{B}_1$ have the exact multiplicity 2 and that $S_n + \mathcal{B}_1$ therefore does not change sign: since $(S_n + \mathcal{B}_1)(2\pi-) = -\pi$, we have $(S_n + \mathcal{B}_1)(x) \le 0$ on $\mathbb{T}$. Similarly, we conclude that $T_n + \mathcal{B}_1$ is non-negative on $\mathbb{T}$.

For $\nu = \pm 1, \pm 2, \dots, \pm n$ one has $\int_0^{2\pi} e^{i\nu t}dt = 0$ and $\sum_{k=0}^n e^{2\pi i k\nu/(n+1)} = 0$. This implies, for each $T \in \mathcal{T}_n$, $T(t) := \sum_{-n}^n c_\nu e^{i\nu t}$, that

$$2\pi \sum_{k=0}^{n} T(2\pi k/(n+1)) = 2\pi(n+1)c_0 = (n+1)\int_0^{2\pi} T(t)dt. \tag{8.12}$$

Sometimes this identity is called the quadrature formula of Hermite. We apply this formula to the polynomial $T_n - S_n$ and use (8.10) and (8.11). Then we get

$$(8.13) \qquad \frac{1}{\pi}\int_0^{2\pi} |T_n(t) - S_n(t)|dt = \frac{2}{n+1}(T_n(0) - S_n(0)) = \frac{2\pi}{n+1}.$$

Let $Q_n \in \mathcal{T}_n$ be the best unrestricted L_p approximation to f', that is, let $E_n(f')_p = \|f' - Q_n\|_p$. We set $g := f' - Q_n$, $g_+(t) := \max\{g(t), 0\}$, and $g_-(t) := \min\{g(t), 0\}$. Let $S_n^*, T_n^* \in \mathcal{T}_n$ be defined by

$$S_n^*(x) := \frac{1}{\pi}\int_0^{2\pi} (\mathcal{B}_1(x-t)Q_n(t) - T_n(x-t)g_+(t) - S_n(x-t)g_-(t))\,dt,$$

$$T_n^*(x) := \frac{1}{\pi}\int_0^{2\pi} (\mathcal{B}_1(x-t)Q_n(t) - S_n(x-t)g_+(t) - T_n(x-t)g_-(t))\,dt.$$

Using (8.9) and (8.11) we see that S_n^* and T_n^* satisfy (8.5), hence $\tilde{E}_n(f)_p \le \|T_n^* - S_n^*\|_p$. Finally, applying the generalized inequality of Minkowski (see [**CA**,(1.6),p.18] and (8.13), we have

$$\begin{aligned} \|T_n^* - S_n^*\|_p &= \pi^{-1}\left\|\int_0^{2\pi} (T_n(x-t) - S_n(x-t))|g(t)|dt\right\|_p \\ &= \pi^{-1}\left\|\int_0^{2\pi} (T_n(t) - S_n(t))|g(x-t)|dt\right\|_p \\ &\le \pi^{-1}\int_0^{2\pi} (T_n(t) - S_n(t))\|g\|_p dt = \frac{2\pi}{n+1}E_n(f')_p. \end{aligned} \qquad \square$$

The error of unrestricted approximation of functions of Sobolev classes $W_p^r(\mathbb{T})$ is estimated in [**CA**, Chapter 7, §§2 and 4]. Thus, Favard's theorem [**CA**, Theorem 4.3, p.214] applied to $f' \in W_p^{r-1}(\mathbb{T})$ yields $E_n(f')_p \le K_{r-1}(n+1)^{-(r-1)}\|f^{(r)}\|_p$ and we obtain

Corollary 8.5. *Let $1 \le p < \infty$. For $r = 1, 2, \dots$ and $f \in W_p^r(\mathbb{T})$,*

$$\tilde{E}_n(f)_p \le 2\pi K_{r-1}(n+1)^{-r}\|f^{(r)}\|_p, \quad n \ge 1,$$

where $K_0 := 1$ and K_r are the Favard numbers.

Similarly, from [**CA**,Corollary 2.4, p.205] we get

Corollary 8.6. *For $r = 1, 2, \dots$ and $f \in W_p^r(\mathbb{T})$,*

$$\tilde{E}_n(f)_p \le C_r n^{-r}\omega(f^{(r)}, n^{-1})_p, \quad n \ge 1,$$

where C_r is a positive number depending only on r.

It has been suggested by Sendov that the so-called *averaged modulus of smoothness* $\tau(f,\delta)_p$ is more appropriate to describe the error of one-sided approximation. This new modulus is defined as follows: for a bounded measurable function f and $\delta > 0$ let

$$\omega(f,x,\delta) := \sup\{|f(t+h) - f(t)| : t, t+h \in [x-\delta/2, x+\delta/2]\}$$

be the local modulus of smoothness of first order, then

$$\tau(f,\delta)_p := \|\omega(f,x,\delta)\|_p. \tag{8.14}$$

The new modulus shares many useful properties with the more popular ones, for example,

(i) $\tau(f,\delta')_p \le \tau(f,\delta)_p, \quad 0 < \delta' < \delta;$

(ii) $\tau(f,k\delta)_p \le k\tau(f,\delta)_p, \quad k = 1,2,\ldots.$

(iii) $\omega(f,\delta)_p \le \tau(f,\delta)_p \le \omega(f,\delta)_\infty.$

Theorem 8.7. *Let $1 \le p < \infty$. For a bounded measurable f on $\mathbb{T}$,*

$$\tilde{E}_n(f)_p \le C\tau(f,n^{-1})_p, \quad n \ge 1, \tag{8.15}$$

where $C \le 8 + 10\pi$.

Proof (by Sendov and Popov [A-1983]). For $h := 1/n$ we define the function

$$U(y) := \sup\{f(t) : y-h \le t \le y+h\}, \quad y \in \mathbb{T}$$

and its Steklov function

$$F(x) := \frac{1}{2h}\int_{x-h}^{x+h} U(y)dy.$$

We have $f(x) \le U(y)$ for $y \in [x-h, x+h]$, hence $f(x) \le F(x)$ for all $x \in \mathbb{T}$ and

$$\begin{aligned} F(x) - f(x) &= \frac{1}{2h}\int_{x-h}^{x+h} \sup\{f(t) - f(x) : y-h \le t \le y+h\}dy \\ &\le \sup\{|f(t) - f(x)| : |x-t| \le 2h\} \le \omega(f,x,4h). \end{aligned}$$

This implies that

$$\|F - f\|_p \le \tau(f,4h)_p.$$

In addition, for $0 < \delta \le h/2$,

$$\begin{aligned} &2h|F(x+\delta) - F(x-\delta)| = \left|\int_{x+h-\delta}^{x+h+\delta} U(y)dy - \int_{x-h-\delta}^{x-h+\delta} U(y)dy\right| \\ &\le 2\delta \ \sup\{|f(t) - f(s)| : \ x - 5h/2 \le s < t \le x + 5h/2\} \\ &= 2\delta \ \omega(f,x,5h). \end{aligned}$$

Hence, for all $x \in \mathbb{T}$ where $F'(x)$ exists,

$$|F'(x)| = \lim_{\delta\to 0+} \frac{|F(x+\delta) - F(x-\delta)|}{2\delta} \le \frac{1}{2h}\ \omega(f,x,5h).$$

Since $F'(x)$ exists almost everywhere on $\mathbb{T}$, we have

$$\|F'\|_p \le \frac{1}{2h}\ \tau(f,5h)_p.$$

Now we apply Theorem 8.4 to F: there exists $T_n \in \mathcal{T}_n$, $T_n(x) \ge F(x) \ge f(x)$ on $\mathbb{T}$, so that

$$\|T_n - F\|_p \le \frac{2\pi}{n+1}\|F'\|_p \le \pi\ \tau(f,5h)_p,$$

and thus

$$\|T_n - f\|_p \le \tau(f,4h)_p + \pi\tau(f,5h)_p \le (4+5\pi)\tau(f,h)_p. \tag{8.16}$$

Replacing f by $-f$, we find a polynomial $S_n \in \mathcal{T}_n$, $S_n(x) \le f(x)$ on $\mathbb{T}$, which also satisfies (8.16). □

This theorem extends to averaged moduli of higher orders $\tau_k(f,\delta)_p$, with a definition similar to (8.14), with $\Delta_h f$ replaced by $\Delta_h^k f$. As an example we quote

Theorem 8.8 (Sendov and Popov [A-1983]). *For $1 \le p < \infty$, $0 < \alpha < k$, $k = 1, 2, \ldots$, and any bounded measurable f on $\mathbb{T}$ one has $\tilde{E}_n(f)_p \le Cn^{-\alpha}$ if and only if $\tau_k(f,\delta)_p \le C_1\delta^\alpha$.*

§ 9. Problems

9.1. Let $P \in \mathcal{P}_n$ be a polynomial with positive coefficients in $1-x$ and $1+x$, that is,

$$P(x) = \sum_{j=0}^{n} a_j(1-x)^j(1+x)^{n-j}, \quad a_j \ge 0,\ j = 0, \ldots, n.$$

In addition, let $0 < \delta < 2$ and let $P(x) \le 1$ on $[-1,1]$ except on a set of measure $\le \delta$. Prove the Remez inequality

$$\|P\|_\infty \le (1-\delta/2)^{-n}$$

and show that equality holds if and only if $P(x) = (1 \pm x)^n/(2-\delta)^n$ (Erdélyi [1990_2]).

9.2. Let f be a bounded measurable function on $\mathbb{T}$. Let $\tau(f,\delta)_p$ be the averaged modulus of smoothness defined by (8.14). Prove that for $1 \le p < \infty$ and $k = 1, 2, \ldots$

$$\tau(f,k\delta)_p \le k\tau(f,\delta)_p, \quad \delta > 0.$$

9.3. Prove (by the standard substitution $x = \cos t$) the analogue of Theorem 8.4 for the error of one-sided approximation by algebraic polynomials in the L_p norms on $[-1,1]$: *If f is absolutely continuous on $[-1,1]$, $f' \in L_p[-1,1]$, then there exist polynomials $P_n, Q_n \in \mathcal{P}_n$ so that $P_n(x) \le f(x) \le Q_n(x)$ for all $x \in [-1,1]$ and*

$$\|Q_n - P_n\|_p \le \frac{2\pi}{n+1} E_{n-1}(f')_p, \quad n \ge 1,$$

where $E_{n-1}(f')_p$ is the L_p error of unrestricted approximation of f' on $[-1,1]$ from $\mathcal{P}_{n-1}$.

9.4. Prove that for a convex function $f \in C[-1,1]$, there exists a sequence of convex algebraic polynomials so that with an absolute constant $c > 0$,

$$|f(x) - P_n(x)| \le c\omega_2(f, 1/n). \qquad \text{(X. M.Yu)}$$

§10. Notes

10.1. Several authors studied monotone (and convex) approximation with an error depending on x. For example, DeVore and Yu [1985] have

$$(10.1) \qquad |f(x) - P_n(x)| \le C\omega_2(f, \sqrt{1-x^2}/n), \quad x \in [-1,1].$$

A general estimate of Shevchuk [1992] extends Theorem 3.8:

$$(10.2) \qquad |f(x) - P_n(x)| \le C(\Delta_n(x))^r \omega_k(f^{(r)}, \Delta_n(x)), \quad x \in [-1,1].$$

Here $\Delta_n(x) := n^{-1}\sqrt{1-x^2} + n^{-2}$, $C = C_{k,r}$, $f \in C^r[-1,1]$, $f' \ge 0$, $k \ge 1$, $n \ge r + k - 1$, and the polynomials P_n are increasing.

Leviatan [1988] replaces ω_2 in (10.1) by ω_2^ϕ, where ω_2^ϕ is the Ditzian-Totik modulus of smoothness (defined in [**CA**,Chapter 6, §6]). Shvedov [1979] extends the estimate (3.11) to $L_p[-1,1]$, $1 \le p < \infty$:

$$(10.3) \qquad \|f - P_n\|_p \le C\omega_2(f, 1/n)_p;$$

see also Yu [1987_1] and Leviatan [1989].

10.2. A generalization of monotone approximation is the theory of comonotone approximation. One says that an algebraic polynomial P is comonotone with $f \in C[-1,1]$ if the interval $[-1,1]$ can be divided into subintervals, on each of which f and P are monotone in the same sense. Jackson type estimates on comonotone approximation are known for $r = 0$ (Iliev [1978] and Newman [1979_1]), and for $r = 1$ (Beatson and Leviatan [1983]): Let $r = 0$ or $r = 1$. Let $f \in C^r[-1,1]$) change monotonicity in the points $-1 < x_1 < \cdots < x_m < 1$.

Then, for each n there exists a polynomial $P_n \in \mathcal{P}_n$ which is comonotone with f and satisfies

$$\|f - P_n\|_\infty \le Cn^{-r}\omega(f^{(r)}, 1/n)$$

where C depends only on m.

10.3. For $0 < \rho \le 1$ let $\mathcal{S}_n^k(\rho)$ be the set of real polynomials of degree $\le n$ which have at most k zeros in the disk $|z| < \rho$. If $\rho = 1$, $\mathcal{S}_n^k(1) = \mathcal{P}_n^k$ are the polynomials of Theorem 6.1. Erdélyi [1990_1] proved that for $k = 0, 1, 2, \dots$ there are positive numbers $c_1(k)$ and $c_2(k)$ depending only on k so that in the uniform norm on $[-1, 1]$,

$$c_1(k)\{n + (1-\rho)n^2\} \le \sup_{P \in \mathcal{S}_n^k(\rho)} \frac{\|P'\|_\infty}{\|P\|_\infty} \le c_2(k)\{n + (1-\rho)n^2\}.$$

10.4. In [1992], Borwein and Erdélyi give a Remez type inequality for polynomials $P_n \in \mathcal{P}_n^k$ of the space $\mathcal{P}_n^k$ of Theorem 6.1.

10.5. For the functions of the form

$$f(x) = \prod_{j=1}^{m} |x - z_j|^{r_j}, \tag{10.4}$$

with real $r_j \ge 1$ and $z_j \in \mathbb{C}$, $j = 1, \dots, m$, Borwein and Erdélyi [1992] (see also Erdélyi, Máté and Nevai [1991]) have established a Markov inequality similar to that of Theorem 6.1: Let $N := \sum_1^m r_j$ and $K := \sum r_j : |z_j| < 1$, then uniformly on $[-1, 1]$,

$$\|f'\|_\infty \le C_2 N(K+1)\|f\|_\infty,$$

where $C_2 > 0$ is an absolute constant. If $z_j \in [-1, 1]$ and $r_j = 1$, then $f'(z_j)$ means the one-sided derivatives of f at z_j from the right or left. Remez-type inequalities for the functions f in (10.4) and for the generalized trigonometric polynomials

$$f(x) = \prod_{j=1}^{m} |\sin((x - z_j)/2)|^{r_j} \tag{10.5}$$

similar to those in §7 are given in Erdélyi [1992].

10.6. In [1994], Kroó and Szabados studied the approximation properties of self-reciprocal polynomials $P_n(x) = \sum_{k=0}^{n} a_k x^k$, $a_k = a_{n-k}$, $k = 0, 1, \dots, n$, so called because $P_n(1/x) = x^{-n}P_n(x)$. The error of uniform approximation, $E_n^s(f)_\infty$ of $f \in C[-1, 1]$ for even n does not exceed

$$\mathcal{O}\left(\omega(f, 1/n) + |f(1)|/\sqrt{\log\log n}\right),$$

and if $f(1) \neq 0$, it is not better than $\mathcal{O}(1/\log n)$. A Markov inequality is $\|P_n'\|_\infty \leq Cn \log n \|P_n\|_\infty$, with the factor not improvable beyond $Cn \log n$. Points ± 1 act as singular points; the behavior of the P_n in $C[-a, a]$, $0 < a < 1$ is more regular.

10.7. For the class of polynomials $P_n \in \mathcal{P}_n^0$, which do not have zeros in the disk $|z| < 1$, Aziz and Dawood [1988] prove the following for $r > 1$:

$$\max_{|z|=r} |P_n(z)| \leq \frac{r^n + 1}{2} \max_{|z|=1} |P_n(z)| - \frac{r^n - 1}{2} \min_{|z|=1} |P_n(z)|.$$

And for P_n with all zeros in $|z| \leq 1$:

$$\min_{|z|=1} |P_n'(z)| \geq n \min_{|z|=1} |P_n(z)|,$$

$$\min_{|z|=r>1} |P_n(z)| \geq r^n \min_{|z|=1} |P_n(z)|.$$

They also give cases of equality.

Chapter 3. Incomplete Polynomials

§ 1. Incomplete Polynomials

We discuss here some properties of algebraic and trigonometric polynomials which have many real zeros. Important examples are polynomials of the form

$$I_n(x) = x^s P_m(x) = \sum_{k=s}^{n} a_k x^k \quad , \quad n = s+m,$$

with a zero of order $s > 0$ at the origin. Polynomials $I_n(x)$ are called *incomplete polynomials.* If $a_s \neq 0$, $a_n \neq 0$, the ratio $\tau = s/n$ is called the *type* of $I_n(z)$. In 1976, Lorentz (see Lorentz [1977]) established some basic properties of these polynomials and raised some questions; several of them were answered by Saff and Varga. This will be the subject of §§1, 2. Also incomplete trigonometric polynomials T_n are of interest; they will help us to derive in §4 properties of the limit function $f = \lim T_n$ from the properties of the zeros of the T_n.

The norm used, unless stated otherwise, is the uniform norm on $[0,1]$. A natural question to start with is: for what functions $f \in C[0,1]$ is the Weierstrass approximation theorem valid if one requires that the approximating polynomials are incomplete? Of course, one must assume that $f(0) = 0$, and one can easily see that this condition is also sufficient. Furthermore, it remains sufficient if one takes some arbitrary sequence $\delta_n \downarrow 0$ and admits only those polynomials for which $s \geq n\delta_n$, $n \geq 1$. Indeed, because of the Weierstrass theorem for ordinary polynomials, it suffices to show that the monomials x^r, $r = 1, 2, \ldots$, are approximable. According to [CA, Theorem 5.5, p.347] one can uniformly approximate x^r, for $r < s \leq n$, by incomplete polynomials I_n so that

$$\text{(1.1)} \qquad \|x^r - I_n(x)\| \leq \prod_{k=s}^{n} \frac{k-r}{k+r} \leq \left(\frac{s}{n}\right)^{2r}.$$

If we take $s = 1 + [n\delta_n]$ in (1.1), the right-hand side converges to zero as $n \to \infty$, hence $\|x^r - I_n(x)\| \to 0$.

However, the answer is different if we fix some θ, $0 < \theta < 1$, and restrict the set of approximating polynomials to those of types $\tau \geq \theta$. One can easily prove that functions $f(x)$ approximable by such polynomials must vanish on some interval $[0, \Delta]$, $\Delta > 0$, depending on θ. It is more difficult to obtain the best value of Δ, which is $\Delta = \theta^2$ (Lorentz [1977], Saff and Varga [1979]).

We begin with a Weierstrass theorem for incomplete polynomials of type $\geq \theta$. The following theorem has been obtained simultaneously by v. Golitschek [1980] and Saff and Varga [1978_1].

Theorem 1.1. *Let $0 < \theta < 1$. For any function $f \in C[0,1]$ with $f(x) = 0$, $0 \leq x \leq \theta^2$, there exists a sequence I_n of incomplete polynomials of type $\geq \theta$ which converge to f, uniformly on $[0,1]$.*

Proof. We follow the first of the above papers. Since the continuously differentiable functions are dense in C[0,1], we may assume that f is such a function with $\|f'\| \leq 1$. It even suffices to consider only functions f which vanish in some larger interval $[0, \delta^2] \supset [0, \theta^2]$, $0 < \theta < \delta < 1$.

Let N be a large integer and let $d > 0$ be defined by $N\delta^{2d} = 1$. The function $F(x) := f(x^{1/d})$ is continuously differentiable in [0,1] and vanishes in $0 \leq x \leq \delta^{2d} = N^{-1}$. Hence, for $N^{-1} \leq x \leq 1$,

$$|F'(x)| = d^{-1}x^{-1+1/d}|f'(x^{1/d})| \leq d^{-1}N^{1-1/d} \leq d^{-1}N. \tag{1.2}$$

Let M be the least integer which satisfies $M \leq N^2/(4e)$. Since F vanishes on $[0, N^{-1}]$, (1.2) is valid for all $x \in [0,1]$ and the Bernstein polynomial $B_M(x) := B_M(F, x)$ of degree M is of the form

$$B_M(x) := \sum_{j=N+1}^{M} F\left(\frac{j}{M}\right)\binom{M}{j}x^j(1-x)^{M-j}.$$

We write $B_M(x) = \sum_{k=N+1}^{M} b_k x^k$, with

$$b_k = \sum_{j=N+1}^{k} (-1)^{k-j} F\left(\frac{j}{M}\right)\binom{M}{j}\binom{M-j}{k-j}.$$

Since $\|f\| = \|F\| \leq 1$, we get using Stirling's inequality

$$\begin{aligned}|b_k| &\leq \sum_{j=0}^{k}\binom{M}{j}\binom{M-j}{k-j} = \sum_{j=0}^{k}\binom{k}{j}\binom{M}{k} = 2^k\binom{M}{k} \\ &\leq \frac{2^k M^k}{k!} \leq \left(\frac{2eM}{k}\right)^k \leq \left(\frac{N}{2}\right)^k = 2^{-k}\delta^{-2dk}.\end{aligned}$$

By [**CA**, Theorem 3.2, p.308],

$$\|F - B_M\| \leq CM^{-1/2}\|F'\|$$

for some constant $C > 0$ and, by (1.2),

$$\|f(x) - \sum_{k=N+1}^{M} b_k x^{dk}\| = \|F - B_M\| \leq Cd^{-1}, \tag{1.3}$$

for another constant C. As a next step the monomials x^{dk} in (1.3) are approximated by incomplete polynomials $I_{n,k}$ of type $\geq \theta$ where the degrees n are taken so large that $\theta n > dM$. We use (1.1) with $\theta n \leq s < \theta n + 1$ and $r = dk$: there exist $I_{n,k}$ for which

$$\|x^{dk} - I_{n,k}(x)\| \leq \left(\frac{s}{n}\right)^{2dk} \leq (\theta + n^{-1})^{2dk}, \quad k = N+1, \dots, M.$$

The polynomial $I_n := \sum_{k=N+1}^{M} b_k I_{n,k}$ is incomplete of type $\geq \theta$. It remains to show that I_n converges to f as $N \to \infty$. We have

$$\begin{aligned}\|f - I_n\| &= \|F(x^d) - \sum_{k=N+1}^{M} b_k I_{n,k}(x)\| \\ &\leq \|F(x^d) - B_M(x^d)\| + \sum_{k=N+1}^{M} |b_k| \|x^{dk} - I_{n,k}(x)\| \\ &\leq Cd^{-1} + \sum_{k=N+1}^{M} 2^{-k} \left(\frac{\theta + n^{-1}}{\delta}\right)^{2dk}.\end{aligned}$$

The last expression tends to zero if $N \to \infty$ since $\delta > \theta$, $n > dM/\theta \to \infty$ and $d = \log N/(2 \log(1/\delta)) \to \infty$. □

We want to show that the number θ^2 in Theorem 1.1 is the best possible. Here, the function

$$G_\theta(u) := |u|^{-1} \left| \frac{1 - \theta + (1+\theta)u}{1 + \theta + (1-\theta)u} \right|^\theta \tag{1.4}$$

will play an important role. It is continuous for $0 < |u| \leq 1$ and tends to $+\infty$ as $u \to 0$. The next lemma is a special case of a theorem of Kemperman and Lorentz [1979].

Lemma 1.2. *Let $0 < \theta < 1$ and let $I_n(z) = z^s P_m(z)$ be an incomplete polynomial of type $\geq \theta$ satisfying*

$$|I_n(x)| \leq 1, \quad \theta^2 \leq x \leq 1. \tag{1.5}$$

Then for all complex numbers $z \notin [\theta^2, 1]$

$$|I_n(z)|^{1/n} \leq G_\theta(u) \tag{1.6}$$

where $|u| < 1$ is defined by

$$z = \frac{1+\theta^2}{2} + \frac{1-\theta^2}{4}(u + u^{-1}). \tag{1.7}$$

Proof. Let $c_1 := -(1+\theta^2)/(1-\theta^2)$, $\xi = (u+u^{-1})/2$. The linear function

$$z = \frac{1}{2}\left(1+\theta^2+(1-\theta^2)\xi\right) = \frac{1-\theta^2}{2}\ (\xi - c_1)$$

maps the interval [-1,1] onto $[\theta^2, 1]$ and the point c_1 onto the origin $z = 0$.

We define the polynomial $S_m \in \mathcal{P}_m$ by the equality

$$I_n(z) = z^s P_m(z) = (\xi - c_1)^s S_m(\xi).$$

Since S_m satisfies $|S_m(\xi)| \le |\xi - c_1|^{-s}$ for $-1 \le \xi \le 1$ it follows from [**CA**, Theorem 2.1, p.100] that for all complex values ξ outside of [-1,1],

$$|S_m(\xi)| \le |\xi - c_1|^{-s}|u|^{-m}\left|\frac{u^{-1}(u-u_1)}{1-u_1u}\right|^s, \tag{1.8}$$

where

$$u_1 := c_1 + \sqrt{c_1^2 - 1} = -(1-\theta)/(1+\theta).$$

From (1.8) we get for all z outside of $[\theta^2, 1]$

$$\begin{aligned} |I_n(z)| &= |(\xi - c_1)^s S_m(\xi)| \le |u|^{-n}\left|\frac{u-u_1}{1-u_1u}\right|^s \\ &= |u|^{-n}\left|\frac{1-\theta+(1+\theta)u}{1+\theta+(1-\theta)u}\right|^s. \end{aligned} \tag{1.9}$$

The function $\{1-\theta+(1+\theta)u\}/\{1+\theta+(1-\theta)u\}$ on the right-hand side of (1.9) is analytic in the unit disk $|u| \le 1$ and its modulus is identically one on the unit circle. By the maximum principle it is ≤ 1 in the unit disk. The inequality (1.6) follows, therefore, from (1.9) since $s \ge \theta n$. □

Let $\Omega = \Omega(\theta)$ be the set of all $z = z(u) \in \mathbb{C}$ given by (1.7) for which

$$|u| < 1\ ,\ G_\theta(u) < 1.$$

This is a bounded open subset of $\mathbb{C}$. Each sequence of incomplete polynomials I_n of types $\ge \theta$ and increasing degrees n converges to zero in $\Omega(\theta)$ if I_n are uniformly bounded on $[\theta^2, 1]$. The convergence is uniform on each compact subset of $\Omega(\theta)$. We shall show in the next theorem that $[0, \theta^2)$ belongs to $\Omega(\theta)$.

Theorem 1.3. *If an incomplete polynomial I_n of degree n and type $\ge \theta$ satisfies* (1.5), *then for each $\delta > 0$, $0 < \delta < 1$, one has*

$$|I_n(x)| \le \rho_0^n, \quad 0 \le x \le \theta^2(1-\delta), \tag{1.10}$$

where $0 < \rho_0 = \rho_0(\theta, \delta) < 1$ is a continuous function of its arguments.

Proof. By means of (1.7) we map the interval $J := (-1, u_1)$ onto $(0, \theta^2)$. For $u \in J$, that is, for $z \in (0, \theta^2)$, we have

$$G_\theta(u) = -u^{-1}\left(\frac{\theta - 1 - (1+\theta)u}{1+\theta+(1-\theta)u}\right)^\theta .$$

The logarithmic derivative of G_θ,

$$\begin{aligned} g_\theta(u) := \frac{G'_\theta(u)}{G_\theta(u)} &= -\frac{1}{u} + \frac{\theta(1+\theta)}{1-\theta+(1+\theta)u} - \frac{\theta(1-\theta)}{1+\theta+(1-\theta)u} \\ &= -\frac{(1-\theta^2)(1+u)^2}{u(1-\theta+(1+\theta)u)(1+\theta+(1-\theta)u)} \end{aligned}$$

is negative in J, and G_θ is strictly decreasing from $G_\theta(-1) = 1$ to $G_\theta(u_1) = 0$. This implies that $0 \le G_\theta(u) < 1$ for $0 \le z < \theta^2$. The function $G_\theta(u)$ depends continuously on z and θ. We can take

$$\rho_0(\theta, \delta) = \max_{0 \le z \le \theta^2(1-\delta)} G_\theta(u).$$

Then $0 < \rho_0 < 1$, and ρ_0 is a continuous function of θ and δ. □

Corollary 1.4. *If I_n is an incomplete polynomial of the type $\ge \theta$, then all points x with $|I_n(x)| = \|I_n\|$ lie in the interval $[\theta^2, 1]$.*

Proof. We may assume that the maximum of $|I_n|$ on $[\theta^2, 1]$ is one. Then I_n satisfies the condition of the last theorem, hence $|I_n(x)| < 1$ on $[0, \theta^2)$. □

If for some $f \in C[0,1]$ and a sequence I_n of incomplete polynomials of types $\ge \theta$ we have $\|f - I_n\| \to 0$, then such I_n are bounded on $[0,1]$. If f is not itself an incomplete polynomial of type $\ge \theta$, then by Theorem 1.3, f must vanish on $[0, \theta^2]$. Conversely, if $f \in C[0,1]$ is a function with the properties $f(x) = 0$, $0 \le x \le \theta^2$, $f(x) > 0$, $\theta^2 < x \le 1$, Theorem 1.1 yields a sequence of incomplete polynomials of type $\ge \theta$ that does not converge to zero at any point $x \in (\theta^2, 1]$.

Corollary 1.5. *The number θ^2 is the best possible in Theorems* 1.1 *and* 1.3.

§ 2. Incomplete Chebyshev Polynomials

For positive integers s, m, let $Q_{s,m}(x)$ be the monic incomplete polynomial of degree $n = s + m$ and type $\tau \ge s/n$ which achieves the minimum

$$E_{s,m} := \|Q_{s,m}\| = \min_{a_k} \|x^{s+m} - \sum_{k=s}^{s+m-1} a_k x^k\| \tag{2.1}$$

in the uniform norm on $[0,1]$. Then

$$C_{s,m}(x) := Q_{s,m}(x)/E_{s,m} = x^s \sum_{k=0}^{m} b_k x^k ,$$

with $b_m = 1/E_{s,m}$, is called the *incomplete Chebyshev polynomial of class* (s,m). It appears already in the book Bernstein [A-1926]; Saff and Varga [1978_2, 1979] gave some of its important properties.

Theorem 2.1. *The extremal polynomial $Q_{s,m}$ is unique for each pair (s,m). It attains its norm at exactly $m+1$ distinct points*

$$\left(\frac{s}{s+m}\right)^2 \le \xi_0^{s,m} < \cdots < \xi_m^{s,m} = 1, \tag{2.2}$$

with alternating signs, $Q_{s,m}(\xi_k^{s,m}) = (-1)^{m-k} E_{s,m}$.

Proof. By definition, $Q_{s,m}$ is incomplete of type τ and by Corollary 1.4, all its extreme points on $[0,1]$ actually lie in the smaller interval $J := [\tau^2, 1]$, that is, $Q_{s,m}$ is also the solution of the minimum problem (2.1) with respect to the uniform norm on J. The m functions $x^s, \cdots, x^{s+m-1}$ are a Haar system on J, hence the extreme polynomial $Q_{s,m}$ is unique and attains its extreme values $\pm E_{s,m}$ at $m+1$ points $\tau^2 \le \xi_0^{s,m} < \cdots < \xi_m^{s,m} \le 1$ with alternating sign. This implies that $Q_{s,m}$ and $Q'_{s,m}$ have m zeros in $[\tau^2, \xi_m^{s,m})$ and cannot have more positive zeros. We conclude that $Q_{s,m}$ increases in $(\xi_m^{s,m}, \infty)$ to $+\infty$, that $\xi_m^{s,m} = 1$ is a maximal point of $Q_{s,m}$ and that there are no other extreme points of $Q_{s,m}$ in $(0,1)$. □

How small is $E_{s,m}$? In the classical case, when $s = 0$, we have $E_{0,m} = 2^{1-2m}$. A simple formula for $E_{s,m}$, $s > 0$, is not known, but we can compute its asymptotic value if $s/(s+m)$ converges.

Theorem 2.2. *If a sequence of pairs (s,m) satisfies $\lim_{m\to\infty} s/(s+m) = \theta$, $0 < \theta < 1$, then*

$$\lim_{m\to\infty} (E_{s,m})^{1/(s+m)} = \frac{1}{4}(1+\theta)^{1+\theta}(1-\theta)^{1-\theta}. \tag{2.3}$$

Proof. We shall need the number

$$\Pi_{s,m} := \prod_{k=s}^{n-1} \frac{n-k}{n+k} = \frac{(n-s)!(n+s-1)!}{(2n-1)!}, \quad n = s+m.$$

By [**CA**, Theorem 5.5, p.347], the error of approximation of x^n by linear combinations of the powers $x^s, \cdots, x^{n-1}$ satisfies $E_{s,m} \le \Pi_{s,m}$. Conversely, $E_{s,m}$ is larger than the corresponding minimal error in the L_2 norm on $[0,1]$: using [**CA**, Theorem 5.4, p.346] we therefore get

$$E_{s,m} \ge \frac{1}{\sqrt{2n+1}} \prod_{k=s}^{n-1} \frac{n-k}{n+k+1} = \frac{n+s}{2n\sqrt{2n+1}}\, \Pi_{s,m}. \tag{2.4}$$

We have to show that $\Pi_{s,m}^{1/n}$ converges to the right-hand side of (2.3). By Stirling's formula we get

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\log\Pi_{s,m} &= \lim_{n\to\infty}\frac{1}{n}\log\frac{(n+s)!(n-s)!}{(2n)!}\\ &= \lim_{n\to\infty}\frac{1}{n}\log\frac{(n+s)^{n+s+1/2}(n-s)^{n-s+1/2}}{(2n)^{2n+1/2}}\\ &= \lim_{n\to\infty}\left(\frac{n+s}{n}\log\frac{n+s}{n}+\frac{n-s}{n}\log\frac{n-s}{n}\right)-\log 4\\ &= (1+\theta)\log(1+\theta)+(1-\theta)\log(1-\theta)-\log 4. \qquad \square\end{aligned}$$

The functions $z = z(u)$ of (1.7) and $G_\theta(u)$ of (1.4) have played an important role in §1. We have shown that all sequences of incomplete polynomials of type $\geq \theta$ that are uniformly bounded on $[\theta^2, 1]$, converge to zero in $\Omega(\theta)$. Our next theorem shows that $\Omega(\theta)$ is the largest open set with this property.

Theorem 2.3 (Saff and Varga [1978_2]). *Let the sequence of pairs (s,m) satisfy $\lim_{m\to\infty} s/(s+m) = \theta$, $0 < \theta < 1$. Then, uniformly on each compact set in $\mathbb{C}$ that does not intersect $[\theta^2, 1]$,*

$$\lim_{m\to\infty}|C_{s,m}(z)|^{1/(s+m)} = G_\theta(u). \tag{2.5}$$

Proof. It is sufficient to show that for any $0 < r < 1$, (2.5) holds uniformly for all z with the corresponding value of u satisfying $|u| \leq r$. We denote this set of z by A_r, and take m so large that A_r is disjoint with the interval $J := [s^2/(s+m)^2, 1]$. Let

$$V_m(u) := \frac{s}{m}\log\frac{|1+\theta+(1-\theta)u|^2}{4}+\frac{1}{m}\log|u^m z^{-s}C_{s,m}(z)|. \tag{2.6}$$

Since $C_{s,m}$ has a zero of order s at $z = 0$, and all its other zeros are on J, the functions $V_m(u)$ are continuous and harmonic for $0 < |u| \leq r$. We shall examine their behavior for $u \to 0$. In this case, $uz \to (1-\theta^2)/4$ and therefore

$$u^m z^{-s}C_{s,m}(z) = \sum_{k=0}^{m} u^{m-k}b_k(uz)^k \to b_m\left(\frac{1-\theta^2}{4}\right)^m = \frac{1}{E_{s,m}}\left(\frac{1-\theta^2}{4}\right)^m.$$

We put

$$V_m(0) := \frac{s}{m}\log\frac{(1+\theta)^2}{4}+\log\frac{1-\theta^2}{4}-\frac{1}{m}\log E_{s,m} \tag{2.7}$$

and obtain a harmonic function V_m on the compact disk $D_r : |u| \leq r$. Next we examine what happens for $|u| \to 1$. The first term in (2.6) is continuous in the disk $|u| \leq 1$, and for $|u| = 1$, $u = e^{it}$, $t \in \mathbb{R}$,

$$|1+\theta+(1-\theta)u|^2 = 2+2\theta^2+2(1-\theta^2)\cos t = 4z.$$

The second term in (2.6) may have values $-\infty$ because of the zeros of $C_{s,m}$, however $|C_{s,m}(z)| \le 1$ holds on $[\theta^2, 1]$, and therefore

$$\limsup_{R\to 1-} \max_{|u|=R} V_m(u) \le \max_{|u|=1}\left(\frac{s}{m}\log z - \frac{s}{m}\log z\right) = 0.$$

Since V_m is harmonic for $|u|<1$, except at the zeros of $C_{s,m}$ outside of $[\theta^2,1]$ where $V_m(u) = -\infty$, it follows by the maximum principle for the harmonic functions that the $V_m(u)$ are all bounded from above by 0 in the disk $|u|<1$. They form a normal family of harmonic functions on D_r, and each sequence of the V_m contains a subsequence that is uniformly convergent on D_r. Any limit function V of a subsequence is bounded from above by 0 and is harmonic on D_r. We can compute $V(0)$:

$$\begin{aligned}\lim_{m\to\infty} V_m(0) &= \frac{\theta}{1-\theta}\log\frac{(1+\theta)^2}{4} + \log\frac{1-\theta^2}{4}\\ &\quad - \frac{1}{1-\theta}\lim_{m\to\infty}(s+m)^{-1}\log E_{s,m} = 0,\end{aligned}$$

by (2.7) and Theorem 2.2. By the maximum principle, V is identically zero. It follows also that $V_m(u) \to 0$, $m\to\infty$, uniformly on D_r. From this we conclude that uniformly on A_r

$$\lim_{m\to\infty}\frac{1}{m}\log|u^m z^{-s} C_{s,m}(z)| = -\frac{\theta}{1-\theta}\log\frac{|1+\theta+(1-\theta)u|^2}{4}.$$

This implies

$$\begin{aligned}\lim_{m\to\infty}\frac{1}{s+m}\log|C_{s,m}(z)| &= (1-\theta)\lim_{m\to\infty}\frac{1}{m}\log|C_{s,m}(z)|\\ &= -\theta\log\frac{|1+\theta+(1-\theta)u|^2}{4} - (1-\theta)\log|u| + \theta\log|z|\\ &= -\log|u| + \theta\log\frac{|4uz|}{|1+\theta+(1-\theta)u|^2}.\end{aligned}$$

The last line is equal to $\log G_\theta(u)$ since

$$\frac{4uz}{1+\theta+(1-\theta)u} = \frac{2(1+\theta^2)u+(1-\theta^2)(u^2+1)}{1+\theta+(1-\theta)u} = 1-\theta+(1+\theta)u.$$

We have established the asymptotic formula (2.5). □

§ 3. Incomplete Trigonometric Polynomials

Theorem 1.3 asserts that a polynomial that has a multiple zero at zero is small on a certain adjacent interval. There are analogues of this for trigonometric polynomials on $\mathbb{T}$ given below in Theorem 3.1. By translation, this theorem is valid not only at $t = 0$ but, more generally, at any point of $\mathbb{T}$. However, much more is true. If T_n has many, say N, zeros in some interval I of length $|I|$, then it is small in that interval. By analogy, we call such trigonometric polynomials with concentrated real zeros *incomplete.* Following v.Golitschek and Lorentz [1985], we find estimates for $|T_n(t)|$ in terms of N and $|I|$. These results will be an essential tool in §4.

We begin with a simple consequence of the results of §1.

Theorem 3.1. *For $0 < \theta < 2$, $0 < \delta < 1$, there exists a function $\rho = \rho(\theta, \delta)$, $0 < \rho < 1$, with the following property. If a trigonometric polynomial T_n of degree n with complex coefficients has a zero of order $\geq n\theta$ at 0, and if*

$$|T_n(t)| \leq 1 \quad \text{for} \quad |\sin(t/2)| \geq \theta/2 \ , \tag{3.1}$$

then

$$|T_n(t)| \leq \rho^n \quad \text{for} \quad |\sin(t/2)| \leq \theta(1-\delta)/2 \ . \tag{3.2}$$

In particular, as $n \to \infty$,

$$T_n(t) \to 0 \quad \text{for} \quad |\sin(t/2)| < \theta/2 \ . \tag{3.3}$$

The interval for t in (3.3) *is the best possible.*

Proof. We set $\rho := \rho_0(\theta/2, \delta)$ where ρ_0 is the continuous function of Theorem 1.3. Let T_n be even, that is,

$$T_n(t) = \sum_{k=0}^{n} a_k \cos^k t = \sum_{k=0}^{n} b_k \sin^{2k}(t/2). \tag{3.4}$$

The substitution $x = \sin^2(t/2)$ transforms $T_n(t)$ into an algebraic polynomial $R_n(x) = \sum_0^n b_k x^k$, which has at $x = 0$ a zero of order $\geq n\theta/2$ and satisfies $|R_n(x)| \leq 1$ for $\theta^2/4 \leq x \leq 1$. Hence we have (3.2) by Theorem 1.3.

If T_n is odd, we apply this result to T_n^2. For an arbitrary polynomial T_n we consider T_n^m for positive integers m and split T_n^m into its even and odd parts. Then (3.2) leads to

$$|T_n(t)|^m \leq 2\rho^{nm} \ \text{for} \ |\sin(t/2)| \leq \theta(1-\delta)/2 \ ,$$

and thus, letting $m \to \infty$, we see that (3.2) holds.

To prove that the interval in (3.3) is the best possible, we use Corollary 1.5 and the substitution $x = \sin^2(t/2)$. □

We need also another, more elementary estimate that is free of restrictions on N/n but is less sharp than (3.2) when this ratio is large.

Lemma 3.2. *If T_n has N zeros in the interval $I \subset \mathbb{T}$ of length $|I|$, then*

$$|T_n(t)| \le \left(\frac{en|I|}{N}\right)^N \|T_n\|_\infty \ , \quad t \in I. \tag{3.5}$$

Proof. Suppose that $\|T_n\|_\infty = 1$. We interpolate T_n at the N zeros by an algebraic polynomial of degree $\le N-1$, which is, of course, the identically zero polynomial. The remainder formula of the interpolation and Bernstein's inequality yield

$$|T_n(t)| \le \|T_n^{(N)}\|_\infty |I|^N/N! \le n^N |I|^N/N!$$

for each $t \in I$. □

A similar result is

Theorem 3.3. *If the function $f \in C^N(I)$ has N zeros in the interval I, then*

$$|f(t)| \le \frac{|I|^{N-1}}{(N-1)!} \int_I |f^{(N)}(s)|ds \ , \quad t \in I. \tag{3.6}$$

Proof. Equation (3.6) is valid for $N = 1$. Let $N \ge 2$. We interpolate f at $N-1$ of the N zeros $x_1, \ldots, x_N$ by an algebraic polynomial of degree $\le N-2$, which is, again, the zero polynomial. The remainder formula of the interpolation yields

$$f(t) = \frac{f^{(N-1)}(u_t)}{(N-1)!} \prod_{k=1}^{N-1} (t - x_k), \quad t \in I, \quad \text{for some } u_t \in I.$$

Since $f^{(N-1)}$ has a zero, u , in I ,

$$|f^{(N-1)}(u_t)| = \left| \int_u^{u_t} f^{(N)}(s)ds \right| \le \int_I |f^{(N)}(s)|ds,$$

which leads to (3.6). □

We now pass to an arbitrary distribution of zeros of T_n in some interval $I = [a, b] \subset \mathbb{T}$. If the number $N(I)$ of zeros of T_n in I is large compared to the length $|I|$ of I, we show that $T_n(t) \to 0$ on I for $n \to \infty$. There are two types of theorems here: (i) when I is taken arbitrarily, and (ii) when I is chosen in an optimal way for a given set of zeros. In the second case, one can take $|I|$ approximately twice larger than in the first.

We begin by noting that a trigonometric polynomial T_n has a zero of order $2r$ $(r = 1, 2, \ldots)$ at 0 if and only if it can be written

$$T_n(t) = \sin^{2r}(t/2) \ Q(t), \tag{3.7}$$

where Q is another trigonometric polynomial. We can see this by writing $T_n = T_n^* + \sin t \; T_{n-1}^{**}$, where T_n^* and T_{n-1}^{**} are even polynomials, and applying (3.4). Similarly, if T_n has zeros $t_1, \dots, t_r$ (not necessarily distinct), each of double multiplicity, then

$$T_n(t) = \prod_{j=1}^{r} \sin^2\left(\frac{t-t_j}{2}\right) Q(t). \tag{3.8}$$

Theorem 3.4. *For all* $0 < \theta < 2$, $0 < \delta < 1$, *there is a continuous function* $\rho_1 = \rho_1(\theta, \delta)$, $0 < \rho_1 < 1$, *with the following properties.*

(i) *If* T_n *has* $\geq n\theta$ *zeros in an interval* $I \subset \mathbb{T}$ *for which*

$$\sin(|I|/4) \leq \theta(1-\delta)/4 \tag{3.9}$$

and if

$$\|T_n\|_\infty \leq 1, \tag{3.10}$$

then

$$|T_n(t)| \leq \rho_1^n, \quad t \in I. \tag{3.11}$$

(ii) *Moreover, there exists an interval* $J \supset I$ *(which depends on* T_n*) with*

$$\sin(|J|/4) \geq \theta(1-\delta)/2, \tag{3.12}$$

so that the inequality (3.11) *is valid for* $t \in J$.

Note that $\sin(|J|/4) \geq 2\sin(|I|/4)$; hence $|J| \geq 2|I|$.

Here is a useful simpler version of Theorem 3.4 (i):

Corollary 3.5. *If* $T_n \in \mathcal{T}_n$, $T_n \neq 0$ *and if a closed interval* $I \subset \mathbb{T}$ *of length* ≤ 2 *has the property that the maximum of* $|T_n(x)|$ *on* I *is equal to* $\|T_n\|$, *then* T_n *has at most* $M := 4n\sin(|I|/4)$ *zeros on* I.

Indeed, if this is not true, then for some $0 < \delta < 1$, T_n has $Z \geq M/(1-\delta)$ zeros on I. We can assume that $\|T_n\| = 1$. We define θ by $n\theta(1-\delta) = M$. Then $0 < \theta < 2$, $Z \geq n\theta$, and $\sin(|I|/4) = \theta(1-\delta)/4$. By Theorem 3.4(i), $|T_n(x)| \leq \rho_1(\theta,\delta)^n < 1$ on I, a contradiction. □

The proofs in this section will be achieved by means of *variation of zeros.*

Lemma 3.6. *If, in the situation of Theorem* 3.4, *we have instead of* (3.9)

$$\sin(|I|/2) \leq \theta(1-\delta)/2 \tag{3.13}$$

and instead of (3.10)

$$|T_n(t)| \leq 1, \quad t \notin I, \tag{3.14}$$

then at each of the endpoints of I we have (with $\rho = \rho(\theta, \delta)$ of Theorem 3.1*)*

$$|T_n(t)| \le \rho^n. \tag{3.15}$$

Proof. We can assume that $I = [0, a]$ and prove (3.15) for $t = a$. Let $0 \le t_1 \le \cdots \le t_r < a$ be the zeros of T_n in $[0, a]$. We put $S_{2n} = T_n^2$. For this polynomial we have (3.8). We compare S_{2n} with

$$S_{2n}^*(t) := \sin^{2r}(t/2)Q(t) = \Phi(t)S_{2n}(t), \tag{3.16}$$

where

$$\Phi(t) := \prod_{j=1}^{r} \frac{\sin^2(t/2)}{\sin^2 \frac{t-t_j}{2}}.$$

It is easy to see that if $0 < t_j < a$, the function

$$\phi(t) := \sin^2(t/2)/\sin^2 \frac{t - t_j}{2}$$

satisfies $\phi(t) \le \phi(a)$ for $-\pi \le t \le 0$ and for $a \le t \le \pi$. Therefore, $\Phi(t) \le \Phi(a)$ for all $t \notin I$. The polynomial $S_{2n}^*(t)/\Phi(a)$ satisfies the assumptions of Theorem 3.1; hence $|S_{2n}^*(t)|/\Phi(a) \le \rho^{2n}$ for $t \in I$. Taking $t = a$, we then obtain $|S_{2n}(a)| \le \rho^{2n}$. □

Proof of Theorem 3.4 (i). We set $\rho_1(\theta, \delta) := \max\{\rho(\theta, \delta); \rho(\delta\theta/2, \delta/2)\}$. Since inequality (3.9) implies (3.13), we have proved (3.11) at the endpoints of the interval $I = [a, b]$.

Now let $a < t < b$, with $T_n(t) \ne 0$. We consider the two intervals $I_1 = [a, t]$, $I_2 = [t, b]$. Condition (3.9) can be written as

$$\sin(|I|/4) \le \frac{N(I)}{4n}(1 - \delta). \tag{3.17}$$

We claim that

$$\sin(|I_i|/2) \le \frac{N(I_i)}{2n}(1 - \delta/2) \ , \ N(I_i) \ge n\delta\theta/2 \tag{3.18}$$

holds for at least one value $i = 1$ or $i = 2$. Indeed, because of the concavity of $\sin(u/2)$ and the equalities $N(I) = N(I_1) + N(I_2)$, $|I_1| + |I_2| = |I|$ it follows that the first inequality of (3.18) is valid for at least one of the i, say for $i = 2$. If (3.18) is incorrect for $i = 2$, then $N(I_2) < n\delta\theta/2$ and thus $N(I_1) \ge N(I) - n\delta\theta/2 \ge n\theta(1 - \delta/2)$. Hence the second inequality of (3.18) is valid for $i = 1$. The first inequality is also true since

$$\begin{aligned} \sin(|I_1|/2) &\le 2\sin(|I|/4) \\ &\le \frac{N(I_1) + n\delta\theta/2}{2n}(1 - \delta) \le \frac{N(I_1)}{2n}(1 - \delta/2). \end{aligned}$$

Applying Lemma 3.6 for the point t as the endpoint of the proper interval I_i, we obtain (3.11). □

Theorem 3.4 (ii) follows from the next result which will be used in §4.

Theorem 3.7. *Let T_n have $N(I) \geq n\theta$ zeros on $I = [a, b]$. If*

$$\sin(|I|/4) < \frac{N(I)}{2n}(1 - \delta), \tag{3.19}$$

and

$$|T_n(t)| \leq 1, \quad t \notin I, \tag{3.20}$$

then there exists an interval $J \supset I$,

$$\sin(|J|/4) \geq \frac{N(J)}{2n}(1 - \delta), \tag{3.21}$$

so that with $\rho = \rho(\theta, \delta)$ from Theorem 3.1,

$$|T_n(t)| \leq \rho^n, \quad t \in J \setminus I. \tag{3.22}$$

Proof. We first show that

$$|T_n(a)T_n(b)| \leq \rho^{2n}. \tag{3.23}$$

Without loss of generality we can assume that $I = [-a, a]$ and that all the zeros t_j of T_n in I are of even multiplicity (otherwise we would consider T_n^2) and different from $\pm a$. We can write T_n in the form (3.8); then with $R(t) := Q(t)Q(-t)$,

$$S_{2n}(t) := T_n(t)T_n(-t) = R(t) \prod_{j=1}^{r} \sin^2 \frac{t - t_j}{2} \sin^2 \frac{t + t_j}{2},$$

we have $|S_{2n}(t)| \leq 1, t \notin I$. The polynomial

$$S_{2n}^*(t) := \sin^{4r}(t/2)R(t) = \Psi(t)S_{2n}(t),$$

where

$$\Psi(t) := \prod_{j=1}^{r} \left(\frac{1 - \cos t}{\cos t_j - \cos t} \right)^2,$$

has a zero at $t = 0$ of order $\geq 2n\theta$. The function $\Psi(t)$ is even and decreases on $[a, \pi]$. Hence $\Psi(t) \leq \Psi(a)$ for $t \notin I$. By Theorem 3.1, $|S_{2n}^*(t)|/\Psi(a) \leq \rho^{2n}, \ t \in I$; putting $t = a$, we obtain (3.23).

Now let J be the largest closed interval $J \supset I$ with the property (3.22). If it is not $\mathbb{T}$, then $J = [\hat{a}, \hat{b}]$. We claim that (3.21) holds, because otherwise we could apply the first part of the proof to the interval J and get a contradiction to the maximality of J. □

We can also derive an L_p-version of this.

Theorem 3.8. *The statements of Theorem 3.7 remain valid for all sufficiently large integers n if one replaces (3.20) by the condition $\|T_n\|_p \le 1$, $1 \le p \le \infty$, and ρ by any ρ_2, $\rho < \rho_2 < 1$.*

Proof. Since $\|T_n\|_p \le 1$ implies $\|T_n'\|_p \le n$, and since T_n has zeros on $\mathbb{T}$ we have $|T_n(t)| \le 2\pi n$. Then the polynomial $T_n^* = T_n/(2\pi n)$ satisfies $\|T_n^*\|_\infty \le 1$. From Theorem 3.7, we derive the existence of a $J \supset I$ with the property (3.21) and with $|T_n^*(t)| \le \rho^n$, for $t \in J \setminus I$, hence with $|T_n(t)| \le 2\pi n\rho^n \le \rho_2^n$, $0 < \rho < \rho_2$, $n \ge n_0$, on the same interval. Thus, (3.22) is valid if one replaces $\rho = \rho(\theta, \delta)$ by any $\rho_2 > \rho$. □

§ 4. Sequences of Polynomials with Many Real Zeros

What we prove in this section, are theorems of the following type: if in some Banach function space we have $S_k \to f$, where the S_k are trigonometric polynomials of degrees $n_k \to \infty$, and if the S_k have many zeros on $\mathbb{T}$ then also f has many zeros on $\mathbb{T}$. One can regard this as a nullity preserving statement. However, the number N_k of zeros of S_k on $\mathbb{T}$, in many cases, will tend to infinity. The proper measure of nullity of S_k is here N_k/n_k; and the measure of nullity of f is the Lebesgue measure mE of the set E where f vanishes. A typical theorem is

Theorem 4.1 (v. Golitschek and Lorentz [1985]). *Let S_k be a sequence of trigonometric polynomials of degrees n_k for which $S_k \to f$ uniformly on $\mathbb{T}$. If N_k denotes the number of zeros of S_k on $\mathbb{T}$ and $n_k \to \infty$, then*

$$m[t \in \mathbb{T} : f(t) = 0] \ \ge\ 2 \limsup_{k\to\infty} (N_k/n_k), \tag{4.1}$$

and 2 is the best possible constant in (4.1).

For the proof of Theorem 4.1 it is convenient to have a generalization of Theorem 3.7. In what follows, for an interval $I \subset \mathbb{T}$, $N(I)$ will denote the number of zeros of T_n in I.

Lemma 4.2. *Let $0 < \delta < 1$, $0 < \sigma < 2\pi$ be given. Then there exists a number $0 < \rho < 1$ with the following property. If $\|T_n\|_\infty \le 1$ and if $\{I_j\}_{j=1}^r$ are disjoint closed intervals with $|I_j| \ge \sigma$, $j = 1, \dots, r$, then there are disjoint closed intervals $\{J_j\}_{j=1}^s$ which cover $\cup_{j=1}^r I_j$ so that*

$$|T_n(t)| \le \rho^n, \quad t \in \cup_{j=1}^s J_j \setminus \cup_{j=1}^r I_j, \tag{4.2}$$

$$\sin(|J_j|/4) \ge \frac{N(J_j)}{2n}(1 - \delta), \quad j = 1, \dots, s. \tag{4.3}$$

Proof. Let θ be any number for which

$$(4.4) \qquad 0 < \theta \le \min\left\{\frac{2}{1-\delta}\sin(\sigma/4), 2\right\},$$

and let $\rho = \rho(\theta, \delta)$ be the number from Theorem 3.7. It is convenient to call $I \subset \mathbb{T}$ a small or a large interval, if, respectively,

$$(4.5) \qquad \sin(|I|/4) < \frac{N(I)}{2n}(1-\delta) \quad \text{or} \quad \sin(|I|/4) \ge \frac{N(I)}{2n}(1-\delta).$$

If all the I_j were large, we could take $J_j := I_j$, $j = 1, \ldots, r$, and would have nothing to prove. We shall reduce the general situation to this by finitely many steps.

For a small interval $I \subset \mathbb{T}$, $|I| \ge \sigma$, we have $N(I) > n\theta$. We can then use Theorem 3.7 and obtain a large interval $J \supset I$ for which $|T_n(t)| \le \rho^n$, $t \in J \setminus I$.

We divide the original intervals $\{I_j\}$ into the groups of small intervals $\{I_j^0\}$ and large intervals $\{J_j^0\}$. After the k-th step we will have the groups $\{I_j^k\}_{j=1}^l$, $\{J_j^k\}_{j=1}^q$. All these intervals will have lengths $\ge \sigma$ and together cover $A = \cup I_j$ and satisfy

$$(4.6) \qquad |T_n(t)| \le \rho^n \quad \text{for} \quad t \in A_k \setminus A,$$

where $A_k = (\cup_j I_j^k) \cup (\cup_j J_j^k)$. Also, the A_k will be increasing.

If, after the $(k-1)$-st step, I_1^{k-1} exists, we cover it with a large interval $J \supset I_1^{k-1}$ as above. First assume that J does not intersect neither the $I_j^{k-1}, j > 1$, nor the J_j^{k-1}. We then let the J_j^k be J together with all J_j^{k-1}, and the I_j^k be the I_j^{k-1}, $j > 1$. This is case (a) : l decreases by 1 , q increases by 1.

If J intersects some of the intervals mentioned, we let J' be the union of J with those I_j^{k-1}, and J_j^{k-1} which intersect J. We then let J' be a new interval I_j^k if J' is small, a new J_j^k if J' is large, and we omit all old I_j^{k-1}, J_j^{k-1} which intersect J'. This is case (b): l+q decreases.

Since 2l+q decreases in each case, this process will stop, and there will be no intervals I_j^k. □

Proof of Theorem 4.1. Let $\varepsilon > 0$ be arbitrary. The S_k are equicontinuous: for all sufficiently small $\sigma > 0$, $|t - t'| \le \sigma$ implies $|S_k(t) - S_k(t')| < \varepsilon$, $k = 1, 2, \ldots$. We fix σ so that $2\pi/\sigma$ is an integer. We also take δ, $0 < \delta < 1$, arbitrarily and by Lemma 4.2 have a $0 < \rho < 1$. Let k be so large that $\rho^{n_k} < \varepsilon$.

We subdivide $\mathbb{T}$ into intervals of length σ. We discard those of them which do not contain zeros of S_k. What remains is a union of closed disjoint intervals I_j of length $\ge \sigma$, with the property

$$(4.7) \qquad |S_k(t)| < \varepsilon, \quad t \in I_j, \quad j = 1, \ldots, r.$$

Now Lemma 4.2 yields intervals J_j, $j = 1, \ldots, s$, which cover $\cup I_j$ and satisfy (4.3) and $|S_k(t)| < \varepsilon$, $t \in J_j$.

Let $A_k = \{t : \ |S_k(t)| < \varepsilon\}$. Since all zeros of S_k are covered by the J_j,

$$(4.8)\quad \sum_{j=1}^{s} |J_j| \geq 4 \sum_{j=1}^{s} \sin(|J_j|/4) \geq \frac{2}{n_k}(1-\delta) \sum_{j=1}^{s} N(J_j) = 2(1-\delta)N_k/n_k.$$

We have proved that

$$(4.9)\qquad mA_k \geq 2(1-\delta)N_k/n_k.$$

This and the uniform convergence $S_k \to f$ on $\mathbb{T}$ yield

$$m[t : |f(t)| < 2\varepsilon] \geq 2(1-\delta)\ \limsup (N_k/n_k).$$

Since ε and δ are arbitrary, we obtain (4.1).

The constant 2 in (4.1) cannot be improved. Indeed, let $f \in C[0,1]$ vanish on $[0, \theta^2/4]$ and be positive on $(\theta^2/4, 1]$. The incomplete polynomials $I_n(x)$ of Theorem 1.1, with $\theta/2$ instead of θ, converge uniformly to f. By substituting $x = \sin^2(t/2)$ we obtain trigonometric polynomials $T_n(t) := I_n(x)$ of degree $\leq n$ which converge to $g(t) := f(x)$, uniformly on $\mathbb{T}$. The T_n have zeros at $t = 0$ of orders $\geq n\theta$, and $g(t)$ is positive outside of the interval $|\sin(t/2)| \leq \theta/2$. This interval has measure $2\arcsin(\theta/2) \geq C\theta$ for all $\theta > 0$ only if $C \leq 2$. □

There exists an unproved conjecture that the lower bound in (4.1) can be replaced by a larger number $4\arcsin(\theta/2)$, where $\theta = \limsup (N_k/n_k)$.

An extension of Theorem 4.1 to the L_p metric is also possible.

Theorem 4.3 (Lorentz [1984₁]). *Let $f \in L_p(\mathbb{T})$, $1 \leq p < \infty$, and let S_k be a sequence of trigonometric polynomials of degrees $n_k \to \infty$ for which either*

$$(4.10)\qquad S_k(t) \to f(t) \ \text{a.e.}, \ \sup_k \|S_k\|_p < \infty \ \text{and} \ p > 1$$

or

$$(4.11)\qquad p = 1 \ \text{and} \ \|S_k - f\|_1 \to 0.$$

Then the function f vanishes on a subset of $\mathbb{T}$ of measure

$$m[t \in \mathbb{T} : f(t) = 0] \geq 2 \limsup_{k\to\infty} (N_k/n_k);$$

the constant 2 is the best possible.

First of all, the polynomials S_k of Theorem 4.3 have equi-absolutely continuous integrals: there exists an increasing function $\omega(h) > 0$ for $h > 0$ with $\omega(h) \to 0$ for $h \to 0$ for which

$$(4.12)\qquad \int_e |S_k(t)|dt \leq \omega(me) \ , \ \ e \subset \mathbb{T}.$$

This follows immediately from (4.11), and if (4.10) is satisfied, then

$$\int_e |S_k(t)|dt \le \|S_k\|_p (me)^{1/p'} \le \text{const}(me)^{1/p'}.$$

We invoke [**CA**, Theorem 2.4, p.101] and derive from the inequalities $S_k^{(l)} \prec n_k^l S_k$, $l = 1, 2, \ldots$, that

$$n_k^{-l} \int_e |S_k^{(l)}(t)|dt \le \omega(me) \ , \ e \subset \mathbb{T}. \tag{4.13}$$

We shall need the notion of proportional covering. Let $\Lambda = \{t_j\}_{j=1}^N$ be points on $\mathbb{T}$, perhaps with repetitions. By $N(A)$ we denote the number of t_j in a set $A \subset \mathbb{T}$. For $\lambda > 0$, $0 < \lambda \le 2\pi/N$, the closed disjoint intervals V_i, $i = 1, \ldots, r$, provide a *proportional λ-covering of Λ* if

$$|V_i| = \lambda N(V_i) \ , \ i = 1, \ldots, r. \tag{4.14}$$

Lemma 4.4. *For each finite set $\Lambda = \{t_j\}_1^N$ and each $0 < \lambda \le 2\pi/N$ there exist proportional λ-coverings of Λ.*

Proof. This is obvious if $\lambda > 0$ is small or if $\lambda = 2\pi/N$, when $V_1 = \mathbb{T}$ is the covering . Starting with a λ-covering for a small $\lambda > 0$, we stretch its intervals proportionally from their centers in a continuous way until some of the V_i get common endpoints. We combine such V_i into single intervals and continue the stretching. This will produce a λ-covering for each $0 < \lambda \le 2\pi/N$. □

We now assume that $\Lambda = \{t_j\}_1^N$, $N = N_k$, is the set of zeros of S_k on $\mathbb{T}$, so that $N_k \le 2n_k$.

Lemma 4.5. *Let V_i be a proportional λ-covering of the set $\{t_j\}_1^{N_k}$ of the zeros of S_k. For sufficiently small $\varepsilon > 0$, let $\lambda = \varepsilon/n_k$, and let $\delta = C\omega(2\varepsilon)$, where C is an absolute constant to be defined later. Then the sets $W = \cup V_i$ and $W^* = \{t \in W : |S_k(t)| \ge \sqrt{\delta}\}$ satisfy*

$$\int_W |S_k(t)|dt \le \varepsilon\delta \tag{4.15}$$

and thus

$$mW^* \le \varepsilon\sqrt{\delta}. \tag{4.16}$$

Proof. For the set W we have, since $N_k \le 2n_k$,

$$mW = \sum |V_i| = \sum \varepsilon N(V_i)/n_k = \varepsilon N_k/n_k \le 2\varepsilon. \tag{4.17}$$

From Theorem 3.3, for $t \in V_i$, we have

$$|V_i||S_k(t)| \le \frac{|V_i|^{r_i}}{(r_i - 1)!} \int_{V_i} |S_k^{(r_i)}(s)|ds = \frac{(\varepsilon r_i n_k^{-1})^{r_i}}{(r_i - 1)!} \int_{V_i} |S_k^{(r_i)}(s)|ds,$$

where $r_i := N(V_i)$. We select $t \in V_i$ so that $|V_i||S_k(t)| = \int_{V_i} |S_k(s)|ds$ and apply Stirling's inequality $r^r/(r-1)! \le \sqrt{r}e^r$. This yields

$$\int_{V_i} |S_k(s)|ds \le \sqrt{r_i}(e\varepsilon n_k^{-1})^{r_i} \int_{V_i} |S_k^{(r_i)}(s)|ds.$$

After summation over all i for which r_i take the same value r, and by an application of (4.13) we obtain

$$\int_{\cup V_i} |S_k(t)|dt \le \sqrt{r}(e\varepsilon)^r \omega(m[\cup V_i]) \le \sqrt{r}(e\varepsilon)^r \omega(mW).$$

Here the union $\cup V_i$ extends over all i with $r_i = r$. Summing over all r,

$$\int_W |S_k(t)|dt \le C\varepsilon\omega(mW) \le C\varepsilon\omega(2\varepsilon) = \varepsilon\delta, \tag{4.18}$$

where we have assumed that $\varepsilon \le 1/(2e)$. □

We shall prove now Theorem 4.3. Let $\varepsilon > 0$ be arbitrary. Since $f \in L_1(\mathbb{T})$, $S_k \in C(\mathbb{T})$ and $S_k \to f$ a.e., it follows from Egorov's theorem that there exists a compact set $F \subset \mathbb{T}$ with complement G, $mG < \varepsilon^2$, so that $S_k \to f$ uniformly on F, and f is continuous on F.

As a consequence, the polynomials S_k are equi-uniformly continuous on F. For the selected $\varepsilon > 0$ there exists a $\sigma > 0$ with the property that $2\pi/\sigma$ is an integer and that

$$|S_k(t) - S_k(s)| < \varepsilon \quad \text{for } t, s \in F \ , \ |t - s| \le \sigma. \tag{4.19}$$

We subdivide $\mathbb{T}$ into $2\pi/\sigma$ closed intervals U of length σ.

Let $k \in \mathbb{N}$ be fixed and let the intervals V_i and the sets W and W^* be as in Lemma 4.5. For our decomposition $\mathbb{T} = F \cup G$, $mG < \varepsilon^2$, we divide the intervals V_i into two disjoint classes: intervals V_i' which contain a point of $F \setminus W^*$, and intervals V_i'' which do not, and are therefore contained in $G \cup W^*$. Each V_i' contains a point $t_0 \in F \setminus W^*$ for which

$$|S_k(t_0)| \le \sqrt{\delta}, \quad t_0 \in V_i'. \tag{4.20}$$

On the other hand, the intervals V_i'' contain at most $\varepsilon_1 n_k$ zeros of S_k where $\varepsilon_1 := \varepsilon + \sqrt{\delta}$, since

$$\sum N(V_i'') = n_k\varepsilon^{-1} \sum |V_i''| \le n_k\varepsilon^{-1} m[G \cup W^*] \le (\varepsilon + \sqrt{\delta})n_k.$$

In an analogy to the proof of Theorem 4.1, we construct disjoint intervals I_j which contain most of the N_k zeros of S_k on $\mathbb{T}$. For each V_i', we take all intervals U that contain a point of type (4.20). On these U, we have $|S_k(t)| < \sqrt{\delta} + \varepsilon = \varepsilon_1$, $t \in U \cap F$, hence $|S_k(t)| < \varepsilon_1$ on the union of V_i' and these U, unless $t \in W^*$. If some of these unions are not disjoint, we combine them into disjoint intervals, obtaining a set $\{I_j\}_j$. Then $|I_j| \ge \sigma$ and $|S_k(t)| \le \varepsilon_1$, $t \in I_j \setminus W^*$ for all j. The I_j contain all but at most $\varepsilon_1 n_k$ zeros of S_k.

We apply Lemma 4.2. This lemma yields disjoint closed intervals $\{J_j\}_{j=1}^s$ which cover $\cup I_j$, and satisfy (4.3) and

$$|S_k(t)| \le \rho^{n_k}\|S_k\|_\infty \le \text{Const } \rho^{n_k} n_k \|S_k\|_1 < 2\varepsilon_1,$$

$t \in J_j \setminus W^*$, for all large k. Let A_k be the subset of $\mathbb{T}$ where $|S_k(t)| < 2\varepsilon_1$. Since all but $\varepsilon_1 n_k$ zeros of S_k are covered by the J_j,

$$\begin{aligned}\sum |J_j| \ge 4\sum \sin(|J_j|/4) &\ge 2n_k^{-1}(1-\delta)\sum N_k(J_j)\\ &\ge 2n_k^{-1}(1-\delta)(N_k - \varepsilon_1 n_k).\end{aligned}$$

This establishes that

$$mA_k \ge 2(1-\delta)\left(\frac{N_k}{n_k} - \varepsilon_1\right) - mW^*. \tag{4.21}$$

For large k and $t \in A_k \cap F$, we have $|f(t) - S_k(t)| < \varepsilon$, hence by (4.21) and (4.16),

$$m[t : |f(t)| < 2\varepsilon_1 + \varepsilon] \;\ge\; 2(1-\delta)\{\limsup (N_k/n_k) - \varepsilon_1\} - \varepsilon\sqrt{\delta} - \varepsilon^2.$$

Since ε and δ are arbitrarily small, this completes the proof of Theorem 4.3. □

There are some interesting applications of Theorems 4.1 and 4.3 (see v. Golitschek and Lorentz [1985]). Using the standard substitution $x = \cos t$ and the measure $d\mu(x) = dx/\sqrt{1-x^2}$, one obtains similar theorems for algebraic polynomials. For example:

Proposition 4.6. *Let $R_k := P_{n_k}$, $k = 1, 2, \ldots$, be a sequence of algebraic polynomials of degrees $\le n_k$, with N_k zeros on [-1,1]. If for some function $f \in L_1(d\mu)$*

$$\int_{-1}^{1} |f(x) - R_k(x)|\,d\mu(x) \to 0,$$

then

$$\mu[x : f(x) = 0] \;\ge\; 2\limsup_{k\to\infty} (N_k/n_k).$$

Proof. The bijection $x = \cos t$ from $[0,\pi]$ onto $[-1,1]$ produces even trigonometric polynomials $S_k(t) := P_{n_k}(\cos t)$ and an even function $g(t) = f(\cos t)$, for which $\|g - S_k\|_1 \to 0$. This map cannot decrease the multiplicity of zeros, hence S_k has $\ge 2N_k$ zeros on $\mathbb{T}$. The set $e_0 = \{t : g(t) = 0\}$ will have measure $me_0 \ge 2\limsup (2N_k/n_k) =: 4\theta$. For its part e_1 on $[0,\pi]$, $me_1 \ge 2\theta$. By substitution, $\mu[x : f(x) = 0] = me_1$. □

Our theorems can be applied to partial sums or means of Fourier series. From Theorem 4.3 we derive: For a function $f \in L_1(\mathbb{T})$, the Fejér means $\sigma_n(f)$ have $o(n)$ zeros if f vanishes only on a set of measure zero. It is not known whether one can replace here $\sigma_n(f)$ by $s_n(f)$.

The Hardy space $H_1(\mathbb{T})$ (see Hoffman [B-1962], also Appendix 3) is a subspace of $L_1(\mathbb{T})$ which, among other things, enjoys the property that $f \in H_1(\mathbb{T})$ is identically zero if it vanishes on a set of positive measure. Therefore: For a nontrivial function $f \in H_1$, if $\|f - P_n\|_1 \to 0$, where P_n are polynomials with N_n zeros on $\mathbb{T}$, one must have $N_n = o(n)$.

§ 5. Problems

5.1. Let $\theta = 1/2$. Prove that the boundary curve of the set $\Omega(\theta)$ of p. 85 intersects the x-axis vertically at $x = -1/8$ and under the angle $\pm\pi/3$ at $x = 1/4$.

5.2. Let $P_n(x) = \sum_{k=s}^{s+m} a_k x^k$ and $\|P\| \le 1$, then

$$|P^{(r)}(x)| \le |C_{s,m}^{(r)}(x)| \text{ for } 0 < x \le \xi_0^{s,m} \text{ and } x \ge 1.$$

§ 6. Notes

6.1. Let $\theta^2 < a < 1$ and let $f \in C[a,1]$. There exists a sequence I_n of incomplete polynomials of type $\ge \theta$ and a number C independent of n with the property that

$$\|f - I_n\|_{C[a,1]} \le C\omega(f; 1/n) \ .$$

(v. Golitschek [1980]). It is unknown if this Jackson-type theorem is valid in the situation of Theorem 1.1.

6.2. Let $0 < \theta < 1$ and $\theta^2 \le c < d \le 1$ be given. Let $N_{s,m}(c,d)$ be the number of extreme points of $C_{s,m}$ which are contained in the interval (c,d). If $s/(s+m) \to \theta$, then

$$\lim_{m\to\infty} \frac{N_{s,m}(c,d)}{s+m} = \frac{1}{(1-\theta)\pi} \int_c^d \frac{1}{x}\sqrt{\frac{x-\theta^2}{1-x}}\, dx.$$

(Saff, Ullman, Varga [1980], see also Chapter 4.)

Chapter 4. Weighted Polynomials

§ 1. Essential Sets of Weighted Polynomials

In this chapter, $A \subset \mathbb{R}$ is a closed real set. We always assume that A is the union of finitely many closed (possibly infinite) intervals. A function w on A is called a *weight on A* if

$$
(1.1) \qquad \begin{cases} \text{(i)} & w \text{ is continuous and non-negative;} \\ \text{(ii)} & w \text{ is positive on some subinterval of } A; \\ \text{(iii)} & xw(x) \to 0, \;\; x \to \pm\infty, \;\; x \in A \;\; \text{if } A \text{ is unbounded.} \end{cases}
$$

Actually, the much weaker assumptions that A is a closed real set and that w is positive on a subset of A of positive logarithmic capacity and satisfies (1.1)(i), (1.1)(iii) suffice in the larger part of this chapter. The full set of assumptions will be used only in the proofs of Theorems 3.3 and 3.5.

Instead of ordinary real algebraic polynomials $P_n \in \mathcal{P}_n$ of degree $\leq n$, we consider the *weighted polynomials* $w(x)^n P_n(x)$. Examples of families of weighted polynomials can be given at once. First, we have the incomplete polynomials of Chapter 3, which correspond to a weight $w(x) = x^\sigma$, $\sigma > 0$, on $[0, 1]$. Another example is provided by the theory of orthogonal polynomials on $(-\infty, \infty)$ with a *Freud weight*

$$w_\sigma(x) = \exp(-|x|^\sigma), \quad \sigma > 0.$$

One considers in this theory functions of the type $w_\sigma(x)P_n(x)$, $P_n \in \mathcal{P}_n$. By the substitution $y = n^{-1/\sigma}x$ they are reduced to *weighted* polynomials $w_\sigma^n Q_n$, $Q_n \in \mathcal{P}_n$. Properties of weighted polynomials, which the reader will find in this chapter, are useful for the theory of orthogonal polynomials. In particular, they have permitted Saff, Mhaskar, Lubinsky and others to prove the long outstanding *Freud conjecture* (see Note 8.4).

For weighted polynomials $w^n P_n$, the exponent n of w^n changes together with the degree of P_n. They provide therefore a different (and more difficult) type of approximation from what is understood under "weighted approximation". Indeed, now the polynomial P_n must balance the exponential oscilliations of w^n.

The three main questions of this chapter parallel those of Chapter 3. The first is: "where on A do the functions $w^n P_n$ live", that is, on what subsets of

A do they attain their uniform norms $\|w^n P_n\|_A$? The second question is concerned with "root asymptotics", for example, with the behavior of $|P_n(z)|^{1/n}$, $z \in \mathbb{C}$, for large n if $\|w^n P_n\|_A = 1$. The third question is: Which functions $f \in C(A)$ are approximable on A by the $w^n P_n$?

For each closed subset B of A we define $\|w^n P_n\|_B := \max_{x \in B} |w(x)^n P_n(x)|$; because of (i), (iii), this maximum is attained even if B is unbounded.

A closed subset B of A is an *essential set for* w if

$$\|w^n P_n\|_B = \|w^n P_n\|_A \tag{1.2}$$

for all $n \geq 1$ and all $P_n \in \mathcal{P}_n$. One obtains an equivalent definition if one requires that (1.2) is valid only for sufficiently large n, since a positive integer power of any function $w^n P_n$ is again a function of the same form and both functions attain their norms at the same points. The space Φ_n of all $\phi := w^n P_n$ of given degree n is linear, but the space Φ of all ϕ of arbitrary degree is not. However, if $\phi = w^n P_n$, $\psi = w^m Q_m$, then $a\phi^m + b\psi^n \in \Phi$ for all real a, b.

For $f \in C(A)$, let $\mathcal{E}(f)$ denote the set of all maximum points of $|f|$ on A. Then:

1. For $q = 2, 3, \ldots$, we have $\mathcal{E}(w^{nq} P_n^q) = \mathcal{E}(w^n P_n)$.

2. If A is unbounded, there exists a compact subset $A_0 \subset A$ so that for each $n = 1, 2, \ldots$ and each $P_n \in \mathcal{P}_n$

$$w(x)^n |P_n(x)| \leq 2^{-n} \|w^n P_n\|_A \ , \quad x \in A \setminus A_0.$$

Indeed, let $I \subset A$ be a compact interval where w is positive, $\lambda := \min\{w(x) : x \in I\} > 0$. Let a be the center of I. We can assume that $\|w^n P_n\|_A = 1$ and have $\|P_n\|_I \leq \lambda^{-n}$. Let C_n be the ordinary Chebyshev polynomial of degree n, $C_n(x) = \frac{1}{2}(x + \sqrt{x^2 - 1})^n + \frac{1}{2}(x - \sqrt{x^2 - 1})^n$. At all points $|x| > 1$, $|C_n|$ is larger than any polynomial of degree n and uniform norm 1 on $[-1, 1]$, and $|C_n(x)| \leq (2|x|)^n$. Hence we have

$$|P_n(x)| \leq \lambda^{-n} |C_n(2(x-a)/|I|)| \leq \{4|x-a|/(\lambda|I|)\}^n \ , \quad x \notin I.$$

The last inequality implies

$$w(x)^n |P_n(x)| \leq \left(\frac{4w(x)|x-a|}{\lambda|I|} \right)^n \|w^n P_n\|_A \ , \quad x \in A \setminus I. \tag{1.3}$$

Now take, for example, $A_0 := I \cup \{x \in A : w(x)|x-a| \geq \lambda|I|/8\}$.

3. For each $P_n \in \mathcal{P}_n, n \geq 1$, and each $x_0 \in \mathcal{E}(w^n P_n)$, there exists $P_{2n} \in \mathcal{P}_{2n}$ with $\mathcal{E}(w^{2n} P_{2n}) = \{x_0\}$. For the proof, one takes $P_{2n}(x) := P_n^2(x) - \varepsilon(x - x_0)^2$, with sufficiently small $\varepsilon > 0$.

There exists the (unique) smallest essential set B. This set, denoted by A_w, will be called the *minimal essential set for the weight* w *on* A. (For the first time used in v. Golitschek, Lorentz, Makovoz [1992].)

Proposition 1.1. *The minimal essential set A_w is compact and is equal to*

$$A_w = \overline{\bigcup_{n\geq 1, P_n \in \mathcal{P}_n} \mathcal{E}(w^n P_n)}. \tag{1.4}$$

Proof. We denote the set on the right-hand side of (1.4) by B. Clearly, B is a closed subset of A which satisfies (1.2) for all n and all $P_n \in \mathcal{P}_n$. Hence, B is an essential set. Because of **3**, each essential set for w must contain all the sets $\mathcal{E}(w^n P_n)$ and therefore also B. This shows that B is the unique minimal essential set for w. By **2**, B is bounded. □

A trivial corollary of Proposition 1.1, **1** and **3** is

4. For $x_0 \in A_w$, $\varepsilon > 0$, $\delta > 0$ there exist $y_0 \in A_w$, $m \in \mathbb{N}$, $P_m \in \mathcal{P}_m$ so that $|y_0 - x_0| < \delta$, $w(y_0)^m P_m(y_0) = \|w^m P_m\|_A = 1$ and

$$w(x)^m |P_m(x)| \leq \varepsilon, \quad x \in A, \quad |x - x_0| \geq \delta. \tag{1.5}$$

Proposition 1.2. (i) *The minimal essential set A_w has no isolated points;* (ii) *the weight w is positive on A_w: $w(x) > 0$.*

Proof. (i) Let $x_0 \in A_w$ be isolated in A_w. We take $\delta > 0$ so small that x_0 is the only point of A_w in the interval $I := [x_0 - \delta, x_0 + \delta]$. A polynomial P_m of **4**, with $\varepsilon := 1/4$ and $y_0 = x_0$, satisfies $w(x)^m |P_m(x)| \leq 1/4$ for all $x \in A_w \setminus \{x_0\}$. Since A has no isolated points and w is continuous, there exists $x^* \in I \cap A$, $x^* \neq x_0$ so that $w(x^*)^m P_m(x^*) \geq 1/2$. Clearly, $x^* \notin A_w$. For $q = 2, 3, \ldots$ we define $\phi_q(x) := w(x)^{mq+1}(x - x_0) P_m(x)^q$, $\phi_q \in \Phi_{mq+1}$. For $x \in A_w$,

$$|\phi_q(x)| = w(x)|x - x_0||w(x)^m P_m(x)|^q \leq (1/4)^q \|w(x)(x - x_0)\|_A,$$

on the other hand, $\phi_q(x^*) \geq (1/2)^q w(x^*)|x^* - x_0|$. Hence, $\phi_q(x^*) > \|\phi_q\|_{A_w}$ for large q, a contradiction.

(ii) The inequality (1.3) of **2** is valid for bounded and unbounded sets A. If $x_0 \in A_w$ and $w(x_0) = 0$, let $\mathcal{U}$ be a neighborhood of x_0, so small that $4w(x)|x - a|/(\lambda|I|) \leq 1/2$ for all $x \in \mathcal{U} \cap A$. By (1.3), $w(x)^n |P_n(x)| \leq 2^{-n} \|w^n P_n\|_A$ for $x \in \mathcal{U} \cap A$ and all $n \geq 1$, $P_n \in \mathcal{P}_n$, which is impossible. □

If w is a weight on A, then $w_1(x) = w(cx)$, $c > 0$, is a weight on $c^{-1}A$, and $A_{w_1} = c^{-1} A_w$. For the symmetric case we have:

Proposition 1.3. *Let $A \subset \mathbb{R}$ be symmetric with respect to zero, let w be an even weight on A. Then* (i) *the minimal essential set A_w is symmetric;*
(ii) *Formula* (1.4) *for A_w is valid with the union restricted to even n and even polynomials P_n;*
(iii) *The function $w^*(x) := w(\sqrt{x})^2$ is a weight on $A^* := \{x \geq 0 : \sqrt{x} \in A\}$; its minimal essential set A_{w^*} is given by*

$$A_w^* = \{x^2 : x \in A_w\}. \tag{1.6}$$

Proof. (i) follows immediately from (1.4) since for $Q_n(x) := P_n(-x)$,

$$\mathcal{E}(w^n Q_n) = \{x : -x \in \mathcal{E}(w^n P_n)\}.$$

(ii) Let $x_0 \in A_w$. For $0 < \varepsilon < 1$ and an arbitrary $\delta > 0$, let $y_0 \in A_w$ and P_m have the properties **4**. If q is a large even integer, all extreme points of the even weighted polynomial $w(x)^{qm}\,(P_m(x)^q + P_m(-x)^q)$ lie in the δ-neighborhoods of x_0 and $-x_0$. This yields (ii).

(iii) The function w^* satisfies (i)-(iii) of (1.1), for example, $xw^*(x) = (\sqrt{x}w(\sqrt{x}))^2 \to 0$ for $x \to \infty$. □

In §2 we shall prove

Proposition 1.4. *The set A_w has a positive capacity,*

$$\gamma(A_w) > 0. \tag{1.7}$$

Example 1. For $A = [0,1]$ and $w(x) = x^\sigma$, $\sigma = \theta/(1-\theta)$, $0 < \theta < 1$, the $w^n P_n$ are essentially the θ-incomplete polynomials of Lorentz originally defined to be $x^k P_{n-k}(x)$, $k/n \to \theta$. It is known that $A_w = [\theta^2, 1]$.

The minimal essential set A_w is not always an interval, but can be very arbitrary, even if A is an interval and w is positive on A:

Example 2. For $A = [-1,1]$ there exists a weight $w > 0$ and a sequence of disjoint closed intervals I_j, $j = 1,2,\ldots$, $I_j \subset A$ numbered from left to right, so that $I_{2j+1} \subset A_w$, $I_{2j} \cap A_w = \emptyset$.

If I_1, I_2 are two disjoint compact intervals, and if I_2 is contained in the δ-neighborhood, $\delta > 0$, of I_1, then, using the properties of the Chebyshev polynomial C_n, $|P_n(x)| \le \rho^n \|P_n\|_{I_1}$, $x \in I_2$, where $\rho = \rho(I_1, I_2) > 1$ depends only on δ and the length of I_1; moreover, for given I_1, $\rho \to 1$ for $\delta \to 0$.

We take a sequence of compact intervals $I_j \subset A := [-1,1]$, $j = 1,2,\ldots$, beginning with $I_1 = [-1,0]$, so that $\rho_j := \rho(I_j, I_{j+1})$ decreases strictly to 1, and take $\rho_j' > \rho_j$, $\rho_j' \to 1$. Let $w(x) = 1$, $x \in I_{2j-1}$, $j = 1,2,\ldots$, $w(x) := 1/\rho_j'$, $x \in I_{2j}$, $j = 1,2,\ldots$, $w(\pm 1) = 1$, we interpolate w linearly outside of the I_j. If C_n is the ordinary Chebyshev polynomial, then $\|w^n C_n\|_A = 1$. The extreme points of the $w^n C_n$ on each I_{2j-1} are dense there, and by Proposition 1.1, $A_w \supset I_{2j-1}$. On each I_{2j}, however, $w(x)^n |P_n(x)| \le (\rho_j/\rho_j')^n < 1$, if $\|w^n P_n\|_A = 1$; and $A_w \cap I_{2j} = \emptyset$. □

In §2, we study the properties of the *Chebyshev polynomials* $C_{w,n}$ of w on A. Their zeros are used to generate the *Chebyshev measure* ν_w of w on A and the negative logarithmic potential

$$L_w(z) := \int_A \log|z-t|\,d\nu_w(t), \quad z \in \mathbb{C}.$$

Theorems 2.5 and 2.7 exhibit the important relations between the functions $C_{w,n}$, L_w and $q(x) := -\log w(x)$.

In §3 we establish relations of A_w, ν_w to the potential-theoretic notions studied by Mhaskar and Saff in connection with the behavior of weighted polynomials. These include the *equilibrium measure* μ_w of w on A, and the corresponding negative logarithmic potential $\Lambda_w(z)$. This allows us to prove that A_w is identical to the support S_w^ν of the measure ν_w. (The *support* S^ν of a measure ν on $\mathbb{R}$ is the smallest closed set so that $\nu = 0$ on its complement.) Moreover, under certain conditions (for example, if A is an interval and $w > 0$ on its interior) one has $\nu_w = \mu_w$.

The needed elements of the potential theory the reader can learn from Appendix 4.

In §4 we determine the minimal essential sets for some special weights: the Jacobi weights on $[-1, 1]$, the Freud weights on $\mathbb{R}$ and $\mathbb{R}_+$. The last two sections, §§5, 6 are devoted to Weierstrass theorems. In §5 we give an elementary (that is, independent of the potential theory) proof based on some oscillation properties of the weight w. In §6 we deal with Weierstrass theorems for the Freud weights w_σ, by using Hermite interpolation.

For further information about weighted polynomials, see the books by Lubinsky and Saff [A-1988], by Totik [A-1994] and reviews by Totik [1992] and Lorentz [1980].

§ 2. Weighted Chebyshev Polynomials

For each weight w on A there exists a monic polynomial $Q_{w,n}(x) = x^n - p(x)$ in $\mathcal{P}_n, n \ge 1$, which solves the minimal problem

$$\|w^n Q_{w,n}\|_A = \min_{p \in \mathcal{P}_{n-1}} \|w(x)^n (x^n - p(x))\|_A =: e_{w,n} =: e_{w,n}^A; \tag{2.1}$$

this follows from the fact that the norm in (2.1) is attained on the compact set A_w. The polynomial $C_{w,n} := C_{w,n}^A := Q_{w,n}/e_{w,n}$ is called the *Chebyshev polynomial for weight w on A*; the leading coefficient of $C_{w,n}$ is $1/e_{w,n}$.

Theorem 2.1. *The Chebyshev polynomials $C_{w,n}$ are unique and for $n \ge 1$ have the following properties:* (i) *$w^n C_{w,n}$ has a sequence of $n+1$ extrema $\xi_0^{(n)} < \xi_1^{(n)} < \cdots < \xi_n^{(n)}$ in A with alternating signs,*

$$(-1)^{n-j} w(\xi_j^{(n)})^n C_{w,n}(\xi_j^{(n)}) = 1, \quad j = 0, 1, \ldots, n, \tag{2.2}$$

and no longer sequence. The $\xi_j^{(n)}$ are contained in A_w.

(ii) *For each $P_n \in \mathcal{P}_n$ with $\|w^n P_n\|_A \le 1$ and $r = 0, 1, \ldots, n-1$,*

$$|P_n^{(r)}(x)| \le |C_{w,n}^{(r)}(x)| \quad \text{if } x \le \xi_0^{(n)} \text{ or } x \ge \xi_n^{(n)}. \tag{2.3}$$

The leading coefficient of P_n satisfies $|a_n| \le 1/e_{w,n}$.

(iii) *For each essential set B for w, $C_{w,n}^B = C_{w,n}^A$ and $e_{w,n}^B = e_{w,n}^A$.*

Proof. (i) We construct the sequence $\xi_j^{(n)}$ in the following way. As $\xi_0^{(n)}$ we take the smallest extreme point of $w^n C_{w,n}$ in A. For $j = 1, 2, \ldots$, $\xi_j^{(n)}$ is the smallest extreme point of $w^n C_{w,n}$ with the properties $\xi_j^{(n)} > \xi_{j-1}^{(n)}$ and $\operatorname{sign}(C_{w,n}(\xi_j^{(n)})) = -\operatorname{sign}(C_{w,n}(\xi_{j-1}^{(n)}))$. This we continue, until the process stops at $j = r$.

Let x_j, for $j = 1, \ldots, r$, be the largest zero of $Q_{w,n}$ in the interval $(\xi_{j-1}^{(n)}, \xi_j^{(n)})$. Let $P \in \mathcal{P}_{n-1}$ be defined by

$$P(x) := Q_{w,n}(\xi_r^{(n)}) \prod_{j=1}^{r} (x - x_j).$$

Now $\tilde{Q}_{w,n} := Q_{w,n} - \varepsilon P$ is a monic polynomial of degree n. A standard argument shows that one has $\|w^n \tilde{Q}_{w,n}\|_A < \|w^n Q_{w,n}\|_A$ for small $\varepsilon > 0$, in contradiction to the minimality of $Q_{w,n}$. The length of an alternation set of $w^n C_{w,n}$ cannot be larger than $n+1$ since $C_{w,n}$ has at most n zeros on $\mathbb{R}$.

We have $C_{w,n}(\xi_n^{(n)}) > 0$ since $C_{w,n}$ is positive for large x and all its zeros lie in $(\xi_0^{(n)}, \xi_n^{(n)})$.

(ii) We prove (2.3) for $x \geq \xi_n^{(n)}$. The proof for $x \leq \xi_0^{(n)}$ is similar. For arbitrary $-1 < \rho < 1$, all n zeros of the polynomial $R_\rho := C_{w,n} - \rho P_n$ lie in the interval $I := (\xi_0^{(n)}, \xi_n^{(n)})$, since $(-1)^{n-j} R_\rho(\xi_j^{(n)}) > 0$, $j = 0, \ldots, n$. Hence $R_\rho(x)$ is positive for $x \geq \xi_n^{(n)}$ and we have (2.3) for $r = 0$ as $\rho \to \pm 1$. But it shows also that $|a_n|$ is not larger than $1/e_{w,n}$, the size of the leading coefficient of $C_{w,n}$.

Let $r = 1, \ldots, n-1$. The leading coefficient of R_ρ is $1/e_{w,n} - \rho a_n$; it is positive. Therefore, $R_\rho^{(r)}(x)$ is positive for large x. By Rolle's theorem, all $n-r$ zeros of $R_\rho^{(r)}$ lie in I. It follows that $R_\rho^{(r)}$ is positive for all $x \geq \xi_n^{(n)}$, and we have (2.3) as $\rho \to \pm 1$.

(iii) is a simple consequence of the uniqueness of $C_{w,n}$. □

If there are several sequences $(\xi_j^{(n)})_{j=0}^n$ with the property (2.2), then we select one of them and call it *the alternation sequence of the Chebyshev polynomial $C_{w,n}$.*

Theorem 2.2. *For the alternation sequence $(\xi_j^{(n)})_{j=0}^n$ of the Chebyshev polynomials $C_{w,n}$ one has*

$$\lim_{n\to\infty} \max_{x \in A_w} \min_{0 \leq j \leq n} |x - \xi_j^{(n)}| = 0. \tag{2.4}$$

As a consequence, we have an improvement of (1.4):

$$A_w = \overline{\bigcup_{n \geq 1} \mathcal{E}(w^n C_{w,n})}; \tag{2.5}$$

the smallest and the largest alternation points converge,

$$\lim_{n\to\infty} \xi_0^{(n)} = \alpha, \quad \lim_{n\to\infty} \xi_n^{(n)} = \beta, \tag{2.6}$$

where $[\alpha, \beta] := co(A_w)$ *is the convex hull of* A_w.

Proof. The set A_w is compact and w is positive and continuous on it. Without loss of generality we can assume that $1 \le w(x) \le M$ on A_w.

Suppose that (2.4) does not hold. Then there is a bounded open interval U for which $\mathrm{U} \cap A_w \ne \emptyset$, but

$$\mathrm{U} \cap \{\xi_j^{(n)} : j = 0, \dots, n\} = \emptyset \tag{2.7}$$

for infinitely many n. By Proposition 1.2(i) A_w has no isolated points, hence there exist two points $x_1 \ne x_2$ in $\mathrm{U} \cap A_w$ and two polynomials P_1 and P_2 of degrees m_1, m_2 for which $(w^{m_\nu} P_\nu)(x_\nu) = \|w^{m_\nu} P_\nu\|_A = 1$, $\nu = 1, 2$. By **3** of §1 we may assume that $\mathcal{E}(w^{m_\nu} P_\nu) = \{x_\nu\}$, $\nu = 1, 2$. We may further assume that $m_1 = m_2 =: m$ for otherwise we can replace P_1 by $P_1^{m_2}$ and P_2 by $P_2^{m_1}$ and set $m := m_1 m_2$.

For a given $n > m$ with the property (2.7) we define the integers q and r by

$$n = qm + r, \quad 0 \le r < m. \tag{2.8}$$

We set

$$h(x) := 3w(x)^n (P_1(x)^q - P_2(x)^q) = 3w(x)^r \left([w(x)^m P_1(x)]^q - [w(x)^m P_2(x)]^q\right).$$

Clearly, $h \in \Phi_n$. We choose n so large that

$$h(x_1) \ge 2, \quad h(x_2) \le -2, \quad |h(x)| < 1 \text{ for } x \in A \setminus \mathrm{U}.$$

Then both functions $w^n(C_{w,n} \pm h)$ are in Φ_n, but one of them has more than n sign changes on A, which is impossible, since $C_{w,n} \pm h \in \mathcal{P}_n$. □

The next lemma will be used below.

Lemma 2.3. *If* $(c_n)_{n=1}^\infty$ *is a real sequence satisfying*

$$c_{m+n} \le c_m + c_n, \quad m, n = 1, 2, \dots, \tag{2.9}$$

then the sequence $(c_n/n)_{n=1}^\infty$ *either converges or tends to* $-\infty$.

Proof. The sequence c_n/n is bounded from above by c_1 since, by using (2.9) repeatedly, one gets $c_n \le n \cdot c_1$. For a fixed m, any $n \ge m$ can be written in the form (2.8), that is, $n = qm + r$, $0 \le r < m$. By (2.9), $c_n \le c_{qm} + c_r \le q \cdot c_m + r \cdot c_1$ and thus,

$$\limsup_{n\to\infty} \frac{c_n}{n} \le \limsup_{n\to\infty} \frac{q \cdot c_m}{n} = \frac{c_m}{m}.$$

It follows that $\liminf (c_n/n) = \limsup (c_n/n) < +\infty$. □

In the next theorem we refer to the logarithmic capacity defined in Appendix 4.

Theorem 2.4. (i) *For each weight w on A there exists the finite limit*

$$a_w := \lim_{n\to\infty} n^{-1} \log e^A_{w,n}; \tag{2.10}$$

(ii) *the logarithmic capacity of the minimal essential set A_w is different from zero, $\gamma(A_w) > 0$.*

(This establishes Proposition 1.4.)

Proof. (i) We apply Lemma 2.3 to $c_n := \log e^A_{w,n}$. For the interval I of **2** of §1 and the positive minimum λ of w on I,

$$e^A_{w,n} \geq e^I_{w,n} \geq \lambda^n \min_{p\in\mathcal{P}_{n-1}} \|x^n - p(x))\|_I = 2(\lambda|I|/4)^n.$$

Hence, $c_n \geq n\log(\lambda|I|/4) + \log 2$ and c_n/n is bounded from below. Since $Q_{w,m}Q_{w,n} \in \mathcal{P}_{m+n}$,

$$e^A_{w,m+n} \leq \|w^{m+n} Q_{w,m}Q_{w,n}\|_A \leq e^A_{w,m} e^A_{w,n}.$$

Hence $c_{m+n} \leq c_m + c_n$, and c_n/n converges.

(ii) We can connect a_w with the logarithmic capacity $\gamma(A_w)$ of A_w and its Chebyshev constant $\mathrm{c}(A_w)$. According to Theorem 2.4 of Appendix 4,

$$\gamma(A_w) = \mathrm{c}(A_w) = \lim_{n\to\infty} (e^{A_w}_n)^{1/n},$$

where $e^{A_w}_n := \min_{p\in\mathcal{P}_{n-1}} \|x^n - p(x)\|_{A_w}$. Since $e^{A_w}_{w,n} \leq \|w\|^n_A e^{A_w}_n$, we get

$$\gamma(A_w) \geq \|w\|^{-1}_A e^{a_w} > 0. \tag{2.11}$$

□

By $\mathcal{M}(K)$ we denote the set of all probability measures on a compact set K, that is, of all non-negative Borel measures of total mass 1 whose support is contained in K. Some of their properties are discussed in §1 of Appendix 4. Our next problem will be to introduce a measure related to the weight w, by means of the Chebyshev polynomials $C_{w,n}$ and their zeros $x^{(n)}_j$, $j = 1, \dots, n$. The *Chebyshev measure* ν_w of w on A will be defined on the convex hull $\mathrm{co}(A_w) = [\alpha, \beta]$. Later we shall compare ν_w with the equilibrium measure μ_w of §3.

To any Borel set $S \subset [\alpha, \beta]$ we assign the measures

$$\nu_n(S) = \sum_{x^{(n)}_j \in S} \frac{1}{n}, \quad n = 1, 2, \dots. \tag{2.12}$$

They belong to $\mathcal{M}[\alpha, \beta]$ Since this set of measures is weakly$*$-compact, each subsequence $\mathcal{N}$ of it has a weakly$*$-convergent subsequence whose weak$*$-limit is again in $\mathcal{M}[\alpha,\beta]$. We shall show in the next theorem that the sequence ν_n is even weakly$*$-convergent.

Theorem 2.5. *Let w be a weight on A. Then,* (i) *the sequence $(\nu_n)_{n=1}^{\infty}$ has a unique weak$*$ limit ν_w, $\nu_w \in \mathcal{M}[\alpha,\beta]$, the Chebyshev measure of w on A.*
(ii) *For all $z \in \mathbb{C} \setminus \operatorname{co}(A_w)$,*

$$\lim_{n\to\infty} n^{-1}\log|C_{w,n}(z)| = L_w(z) - a_w, \tag{2.13}$$

where a_w is the limit (2.10) *and L_w is the negative logarithmic potential of ν_w,*

$$L_w(z) := \int_A \log|z-t|d\nu_w(t) = \int_\alpha^\beta \log|z-t|d\nu_w(t). \tag{2.14}$$

Proof. We prove first that the limit (2.13) exists for each $z>\beta$. We set $c_n := \log|C_{w,n}(z)|$. The polynomial $P_{m,n} := C_{w,m}C_{w,n}$ satisfies $\|w^{m+n}P_{m,n}\|_A \le 1$. By Theorem 2.1 (ii), $|P_{m,n}(z)| \le |C_{w,m+n}(z)|$ and, by taking the logarithm, $c_m + c_n \le c_{m+n}$. We apply Lemma 2.3 to the sequence $(-c_n)_1^\infty$ and prove that the sequence $n^{-1}\log|C_{w,n}(z)|$ converges or tends to $+\infty$. The latter cannot happen. Indeed, let I be the interval in **2** of §1 with center a and let $\lambda>0$ be the minimum of w on I, then

$$n^{-1}\log|C_{w,n}(x)| \le \log\{4|x-a|/(\lambda|I|)\}, \quad x \in \mathbb{R}\setminus I.$$

We now compute the limit (2.13) in a different way, assuming that a subsequence $\nu_n, n \in \mathcal{N} \subset \mathbb{N}$ converges weakly$*$ to some limit $\nu \in \mathcal{M}[\alpha,\beta]$. We have $C_{w,n}(z) = e_{w,n}^{-1}\prod_{i=1}^{n}(z-x_i^{(n)})$ and therefore for all $n \in \mathcal{N}$

$$\begin{aligned} n^{-1}\log|C_{w,n}(z)| &= -\log e_{w,n}^{1/n} + \int_\alpha^\beta \log|z-t|d\nu_n(t) \\ &\to -a_w + \int_\alpha^\beta \log|z-t|d\nu(t), \quad z \in \mathbb{C}\setminus[\alpha,\beta]. \end{aligned}$$

It follows from this and the first part of the proof that all convergent subsequences ν_n produce the same values $\int_\alpha^\beta \log|z-t|d\nu(t)$ for $z>\beta$. Theorem 1.9 of Appendix 4 now yields that all measures ν are identical. This proves (i) and (2.13). □

Example 3. For $A=[-1,1]$ and $w=1$, we have $a_w = -\log 2$, and

$$C_n(x) = C_{w,n}(x) = \cos(n\arccos x), \quad -1 \le x \le 1,$$

are the ordinary Chebyshev polynomials, with the n zeros

$$x_j^{(n)} = \cos\frac{\pi(n+1/2-j)}{n}, \quad j=1,\ldots,n.$$

The corresponding density of the measure is

$$\frac{d\nu(x)}{dx} = \frac{1}{\pi\sqrt{1-x^2}}, \quad -1 < x < 1.$$

Another representation of C_n is

$$C_n(z) = \frac{1}{2}(z + \sqrt{z^2-1})^n + \frac{1}{2}(z - \sqrt{z^2-1})^n, \quad z \in \mathbb{C},$$

where we select the branch of the square root so that $\sqrt{z^2-1} > 0$ for $z > 1$. It follows that

$$\lim_{n\to\infty} n^{-1} \log |C_n(z)| = \log |z + \sqrt{z^2-1}|, \quad z \in \mathbb{C} \setminus [-1,1],$$

and by (2.13)-(2.14),

$$\int_{-1}^{1} \log |z-t| d\nu(t) = \log |z + \sqrt{z^2-1}| - \log 2, \quad z \in \mathbb{C}.$$

A more concrete formula for the measure ν_w in Theorem 2.5 is:

$$\nu_w([c,d]) = \lim_{n\to\infty} \frac{N_n[c,d]}{n}, \tag{2.15}$$

where $N_n[c,d]$ is the number of zeros of $C_{w,n}$ in the interval $[c,d]$. Indeed, for $\delta > 0$ let $f_\delta \in C(\mathbb{R})$ have the values 1 on $[c,d]$, 0 on $(-\infty, c-\delta] \cup [d+\delta, \infty)$, and let f_δ be linear in the two remaining intervals. From $\int f_\delta(t) d\nu_n(t) \to \int f_\delta(t) d\nu_w(t)$, $n \to \infty$, we obtain $\limsup \nu_n([c,d]) \le \nu_w((c-\delta, d+\delta))$. Since $\nu_w((c-\delta,c) \cup (d,d+\delta))$ converges to 0 for $\delta \to 0$, we derive that $\limsup \nu_n([c,d]) \le \nu_w([c,d])$. This is true for all compact intervals $[c,d]$, and $\nu_n([\alpha,\beta]) = \nu_w([\alpha,\beta]) = 1$ for all n. Therefore, (2.15) follows from Lemma 2.6(ii), that is, from the fact that ν_w has no atoms.

Lemma 2.6. *Let S_w^ν be the support of the Chebyshev measure ν_w. Then* (i) *$S_w^\nu \subset A_w$;* (ii) *the measure ν_w has no atoms and S_w^ν has no isolated points.*

Proof. (i) Let $x_0 \in S_w^\nu$. Since S_w^ν is the support of ν_w, $\nu_w(I_\delta) > 0$ for any interval $I_\delta = [x_0 - \delta, x_0 + \delta]$, $\delta > 0$. By (2.15), I_δ contains for large n at least two zeros $x_i^{(n)}$ of $C_{w,n}$, and hence an extreme point of $w^n C_{w,n}$. By Proposition 1.1, $x_0 \in A_w$.

(ii) At an extreme point ξ of $w^n C_{w,n}$, $w(\xi)^n e_{w,n}^{-1} \prod_{j=1}^{n} |\xi - x_j^{(n)}| = 1$, therefore

$$n^{-1} \sum_{j=1}^{n} \log |\xi - x_j^{(n)}| = -\log w(\xi) + a_w + o(1), \quad n \to \infty. \tag{2.16}$$

Now if $\nu_w(\{x_0\}) =: \rho > 0$, then for all large n, I_δ contains an extreme point ξ and $\ge \rho n/2$ zeros of $C_{w,n}$, so that

$$n^{-1}\sum_{j=1}^{n}\log|\xi - x_j^{(n)}| \le \frac{\rho}{2}\log 2\delta + \log(\beta-\alpha).$$

Since δ can be arbitrarily small, (2.16) implies that $\log w(x) = +\infty$, in contradiction to the continuity of w. □

The next theorem parallels Frostman's theorem 3.2 of §3. It relates the behavior of $L_w(x)$ and $q(x) := -\log w(x)$, this latter function is defined on A and continuous on A_w.

Theorem 2.7. *One has*

$$L_w(x) \ge q(x) + a_w, \quad x \in A_w, \tag{2.17}$$

(in particular, L_w is bounded on A_w) and

$$L_w(x) \le q(x) + a_w, \quad x \in A \setminus A_w. \tag{2.18}$$

In addition, L_w is bounded from below on $\mathbb{C}$.

Proof. Let $x \in A_w$, $y > 0$. By the definition of the measure ν_w,

$$L_w(x+iy) = \lim_{n\to\infty} n^{-1}\sum_{j=1}^{n}\log|x+iy-x_j^{(n)}|. \tag{2.19}$$

If $\xi_{k(n)}^{(n)}$ denotes an extreme point of Theorem 2.2 closest to x, then $\xi_{k(n)}^{(n)} \to x$, $n \to \infty$. Hence, with $M := \max\{|t| : t \in A_w\}$, for each fixed $y > 0$ and all large n,

$$\begin{aligned}
&|x+iy-x_j^{(n)}|^2 - (\xi_{k(n)}^{(n)} - x_j^{(n)})^2 \\
&= y^2 + (x-x_j^{(n)})^2 - (\xi_{k(n)}^{(n)} - x)^2 - 2(\xi_{k(n)}^{(n)} - x)(x - x_j^{(n)}) - (x - x_j^{(n)})^2 \\
&\ge y^2 - (\xi_{k(n)}^{(n)} - x)^2 - 4M|\xi_{k(n)}^{(n)} - x| > 0,
\end{aligned}$$

so that (2.19) implies

$$L_w(x+iy) \ge \limsup_{n\to\infty} n^{-1}\sum_{j=1}^{n}\log|\xi_{k(n)}^{(n)} - x_j^{(n)}|.$$

As in (2.16), the sum is $q(\xi_{k(n)}^{(n)}) + a_w + o(1)$. The continuity of q and the upper semi-continuity of L_w yield (2.17).

We prove (2.18) for $x \in A$ in any of the complementary intervals I to A_w. They are of the form (ξ_1, ξ_2), $\xi_1, \xi_2 \in A_w$, (β, ∞), $(-\infty, \alpha)$. They do not contain extreme points $\xi_j^{(n)}$ of $w^n C_{w,n}$, hence contain at most one zero $x_{k(n)}^{(n)}$ of this function (there are no zeros in infinite intervals I). Passing, if necessary, to a subsequence, we can assume that $x_{k(n)}^{(n)} \to x^*$. Let $x \in A \setminus A_w$, $x \ne x^*$, and let $y > 0$. For all sufficiently large n, $|x - x_j^{(n)}| \ge c > 0$, and then

$$n^{-1}\log|Q_{w,n}(x+iy)| = n^{-1}\log|Q_{w,n}(x)| + n^{-1}\sum_{j=1}^{n}\log\left|1+\frac{iy}{x-x_j^{(n)}}\right|$$
$$\le n^{-1}\log|Q_{w,n}(x)| + n^{-1}\sum_{j=1}^{n}\log(1+|y|/c).$$

Then

$$L_w(x+iy) \le \limsup n^{-1}\log|C_{w,n}(x)| + a_w + \log(1+|y|/c).$$

Here $n^{-1}\log|C_{w,n}(x)| \le q(x)$ for all $x \in A$ by the definition of $C_{w,n}$. By Lemma 1.7(i) of Appendix 4, $L_w(x) \le L_w(x+iy)$. Since $y > 0$ is arbitrary, this yields (2.18).

By (2.17), L_w is bounded from below on A_w ($\supset S_w^\nu$). The same applies to $\mathbb{C}\setminus S_w^\nu$, since L_w is harmonic there and satisfies $L_w(\infty) = \infty$. □

Proposition 2.8. *All polynomials $P_n \in \mathcal{P}_n$, $\|w^n P_n\|_A \le 1$, satisfy*

$$n^{-1}\log|P_n(z)| \le L_w(z) - a_w, \quad z \in \mathbb{C}. \tag{2.20}$$

Proof. Since $S_w^\nu \subset A_w$ we can apply Theorem 1.8 of Appendix 4. □

A desirable property of the minimal essential set A_w is to be an interval. This is the case under the following assumption on w and A:

Theorem 2.9 (Mhaskar and Saff [1985] for S_w^μ of §3 instead of S_w^ν). *Let A be an interval and w be a weight on A. Then A_w is an interval if $q := -\log w$ is continuous and convex in the interior A^o of A.*

Proof. If A_w is not an interval, there exists an open interval $I := (\xi_1, \xi_2)$, $\xi_1, \xi_2 \in A_w$, disjoint with A_w, hence disjoint with S_w^ν. The function L_w is harmonic in I and satisfies

$$L_w'(x) = \int_A \frac{1}{x-t}\,d\nu_w(t), \quad L_w''(x) = -\int_A \frac{1}{(x-t)^2}\,d\nu_w(t), \quad x \in I.$$

Since $L_w''(x) < 0$ in I, L_w is strictly concave in I. Hence $v := L_w - q - a_w$ is also strictly concave in I. An application of Lemma 1.7(ii) of Appendix 4 yields that $v(x)$ is continuous for $\xi_1 \le x \le \xi_2$. Theorem 2.7 now implies that

$$v(\xi_1) = v(\xi_2) = 0, \quad v(x) \le 0 \text{ for } \xi_1 < x < \xi_2. \tag{2.21}$$

This is impossible for the strictly concave function v. □

Corollary 2.10. *The minimal essential sets A_w for the following weights w are compact intervals:* (i) *the Jacobi weights*

$$w(x) = (1-x)^{s_1}(1+x)^{s_2}, \quad s_1 > 0, \; s_2 > 0 \quad \text{on } A = [-1,1], \tag{2.22}$$

(ii) *the Freud weights*

$$w_\sigma(x) = e^{-|x|^\sigma}, \quad \sigma \geq 1, \quad \text{on } A = (-\infty, \infty), \tag{2.23}$$

(iii) *the Freud weights*

$$w_\sigma(x) = e^{-x^\sigma}, \quad \sigma \geq 1, \quad \text{on } A = [0, \infty). \tag{2.24}$$

Indeed, in all three cases the functions $q := -\log w$ are convex in A^o. □

The minimal essential sets A_w for the Freud weights in (ii) and (iii) for $0 < \sigma < 1$ are also compact intervals. This follows from the next theorem which has been communicated to us by Mhaskar (for S_w^μ instead of S_w^ν).

Proposition 2.11. *Let w be a weight on $A := [0, \infty)$. The set A_w is a compact interval, $A_w = [0, \beta]$, if $w(x)$ attains its maximum at $x = 0$, if $q(x) := -\log w(x)$ is continuously differentiable on $(0, \infty)$ and $xq'(x)$ is non-decreasing on $(0, \infty)$.*

Proof. Since w attains its maximum at $x = 0$, $0 \in A_w$. If A_w is not an interval, there exists an open interval $I := (\xi_1, \xi_2)$ disjoint with A_w, hence disjoint with S_w^ν, and with $\xi_1, \xi_2 \in A_w$. We may assume that $0 < \xi_1 < \xi_2$ since, by Proposition 1.2, 0 is not an isolated point of A_w. The function $v := L_w - q - a_w$ of the proof of Theorem 2.9 is again a continuous function on $[\xi_1, \xi_2]$ which satisfies (2.21). In addition, v is continuously differentiable in I and

$$(xL_w'(x))' = L_w'(x) + xL_w''(x) = -\int_0^\beta \frac{t}{(x-t)^2} d\nu_w(t) < 0, \quad x \in I.$$

Thus, $xL_w'(x)$ and hence $xv'(x)$ are strictly decreasing on I. Therefore, from $v(\xi_1) = 0$ and $v(x) \leq 0$, $\xi_1 < x < \xi_2$ it follows that $v'(x) < 0$ on I and $v(\xi_2) < 0$, which contradicts (2.21). □

The Freud weight $w := w_\sigma$, $0 < \sigma < \infty$, on $A = [0, \infty)$ has all the properties of Proposition 2.11. Thus, A_w is an interval of the form $[0, \beta]$.

The Freud weight $w := w_\sigma$, $0 < \sigma < \infty$, on $A = (-\infty, \infty)$ is an even function. By Proposition 1.3, $A_w = \{x \in \mathbb{R} : x = \pm\sqrt{y},\ y \in A_{w^*}\}$, where w^* is the weight on $[0, \infty)$ given by $w^*(x) := w(\sqrt{x})^2 = e^{-2x^{\sigma/2}}$. Clearly, A_{w^*} is an interval $[0, \beta^*]$ by Proposition 2.11. Hence A_w is a symmetric interval.

§ 3. The Equilibrium Measure

Let w be a weight on A, then $A \subset \mathbb{R}$ is the union of finitely many closed intervals and $w \in C(A)$ satisfies (1.1). An extension of the classical potential theory is the potential theory with weights. For a measure $\mu \in \mathcal{M}(A)$, the *energy integral $I_w(\mu)$ with weight w* is defined by

(3.1) $$I_w(\mu) = \int_{A\times A} G(x,t)d\mu(x)d\mu(t),$$

where

(3.2) $$G(x,t) := -\log\{|x-t|w(x)w(t)\} = -\log|x-t| + q(x) + q(t).$$

Because of (1.1)(iii) and the continuity of w on A, $G(x,t)$ is bounded from below on $A \times A$, even if A is unbounded. Hence, the infimum of the energy integrals

(3.3) $$V_w := V_w(A) := \inf_{\mu\in\mathcal{M}(A)} I_w(\mu),$$

is $> -\infty$. Since $w(x) > 0$ on some compact subinterval I of A, the measure $\mu := m/|I|$, where m denotes the ordinary Lebesgue measure on I, is in $\mathcal{M}(A)$ and satisfies $I_w(\mu) < +\infty$ and thus $V_w < +\infty$.

Our first three theorems are due to Mhaskar and Saff [1985].

Theorem 3.1. (i) *There exists a measure $\mu_w \in \mathcal{M}(A)$ called an equilibrium measure, for which*

(3.4) $$I_w(\mu_w) = V_w.$$

(ii) *The support S_w^μ of μ_w has positive capacity; μ_w has no atoms.*
(iii) *The weight w is positive on S_w^μ; S_w^μ is compact.*
(iv) *The measure μ_w with the property* (3.4) *is unique.*

Proof. We select a sequence of $\mu_n \in \mathcal{M}(A)$ for which $I_w(\mu_n) \to V_w$. By the weak$*$-compactness of $\mathcal{M}(A)$, we may assume that the μ_n are weakly$*$-convergent to some $\mu_w \in \mathcal{M}(A)$. We shall prove first the properties (ii), (iii) of μ_w and S_w^μ, and then (3.4).

If $\gamma(S_w^\mu) = 0$, then $\mu_w(S_w^\mu) = 0$ (see **3** of §2 in Appendix 4), which is impossible since $\mu_w(S_w^\mu) = \mu_w(A) = 1$. If μ_w has an atom, the product measure $\mu_w \times \mu_w$ of the diagonal $\Delta := \{(x,t) : x = t \in S_w^\mu\}$ is positive and the integral of $G(x,t)$ over Δ equals $+\infty$ since $G(x,t) = +\infty$ on Δ. On the other hand, the integral of $G(x,t)$ over $S_w^\mu \times S_w^\mu \setminus \Delta$ is bounded from below since $G(x,t)$ is bounded from below. This implies $V_w = +\infty$, a contradiction, and proves part (ii) of the theorem.

Suppose that $w(x_0) = 0$ for some $x_0 \in S_w^\mu$. By continuity, there is a compact neighborhood U_1 of x_0 in A for which $\delta := \mu_w(U_1) < 1$ and $G(x,t) \geq I_w(\mu_w)+2$, $(x,t) \in A\times U_1$. Hence, for all large n, $G(x,t) \geq I_w(\mu_n)+1$, $(x,t) \in A \times U_1$. We have $\delta > 0$ since $x_0 \in S_w^\mu$. Moreover, $\mu_n(U_1) \to \mu_w(U_1)$ since μ_w has no atoms (see the proof of (2.15)). This implies that $\delta_n := \mu_n(U_1) \to \delta$. We now set $U := A \setminus U_1$ and define the new measures $\mu'_n \in \mathcal{M}(A)$ by setting, for each Borel set $C \subset A$, $\mu'_n(C) := (1-\delta_n)^{-1}\mu_n(C \cap U)$. Then, for large n,

$$\begin{aligned}
I_w(\mu_n') &= \int_{A\times A} G(x,t)d\mu_n'(x)d\mu_n'(t) = \int_{U\times U} G(x,t)d\mu_n'(x)d\mu_n'(t)\\
&= (1-\delta_n)^{-2}\int_{U\times U} G(x,t)d\mu_n(x)d\mu_n(t)\\
&\le (1-\delta_n)^{-2}\left(I_w(\mu_n) - [I_w(\mu_n)+1][\mu_n(U_1)+\mu_n(U_1)\mu_n(U)]\right)\\
&= I_w(\mu_n) - (1-\delta_n)^{-2}(2\delta_n - \delta_n^2).
\end{aligned}$$

For $n\to\infty$ this implies that $\limsup I_w(\mu_n') \le V_w - (1-\delta)^{-2}(2\delta-\delta^2) < V_w$ contradicting the definition of the sequence μ_n. This contradiction proves that $w > 0$ on S_w^μ.

Let A be unbounded. By (1.1)(iii), $\rho := \max\{\|w\|_A, \|tw(t)\|_A\} < \infty$,

$$|x-t|w(x)w(t) \le \rho w(x)(1+|x|) =: h(x) \quad \text{for all } t\in A,$$

and

$$\inf_{t\in A; t\ne x} G(x,t) \ge -\log h(x) \to \infty \quad \text{for } |x|\to\infty,\ x\in A.$$

Therefore, there exist $x_0\in S_w^\mu$ and a compact neighborhood U_1 of x_0 in A for which $G(x,t)\ge I_w(\mu_w)+2$, $(x,t)\in A\times U_1$. We have shown above that this is impossible. This completes the proof of (iii).

Let $\log_R|x| := \max\{\log|x|, \log R\}$, $R>0$. From the inequality $\log_R|x| \ge \log|x|$, $x\in\mathbb{R}$, we get for any fixed $R>0$

$$I_w(\mu_n) \ge -\int_{A\times A}\log_R|x-t|d\mu_n(x)d\mu_n(t) + 2\int_A q(x)d\mu_n(x). \tag{3.5}$$

The first integrand belongs to $C(A\times A)$ for every R, and q is continuous on S_w^μ by (iii), so that we can pass to the limit as $n\to\infty$ on both sides of (3.5). After this we let $R\to 0$. and obtain $V_w \ge I_w(\mu_w)$ since μ_w has no atoms. The opposite inequality follows from the definition of V_w: we obtain (3.4).

We do not prove the uniqueness of μ_w. (Proofs of our theorems of Chapter 4 will not use this fact.) □

We introduce the number

$$\alpha_w := \int_A q(t)d\mu_w(t) - V_w. \tag{3.6}$$

It is finite since q is continuous on S_w^μ. An important function is the negative logarithmic potential of μ_w,

$$\Lambda_w(z) := \int_A \log|z-t|d\mu_w(t) = \int_{S_w^\mu}\log|z-t|d\mu_w(t). \tag{3.7}$$

It has values $-\infty\le\Lambda_w(z)<\infty$ on $\mathbb{C}$, is finite on $\mathbb{C}\setminus S_w^\mu$. By Theorem 1.1 of Appendix 4, Λ_w is harmonic in $\mathbb{C}\setminus S_w^\mu$, and upper semi-continuous on $\mathbb{C}$, that is, it satisfies $\Lambda_w(z_0)\ge\limsup_{z\to z_0}\Lambda_w(z)$.

An important tool of this section is a generalization of Frostman's theorem in potential theory:

Theorem 3.2. *For the function Λ_w one has*

$$\Lambda_w(x) \le q(x) + \alpha_w \quad \textit{quasi everywhere on} \quad A, \tag{3.8}$$

$$\Lambda_w(x) \ge q(x) + \alpha_w \quad \textit{everywhere on} \quad S_w^\mu. \tag{3.9}$$

As a corollary, $\Lambda_w(z)$ is bounded from below on $\mathbb{C}$.

The original Frostman theorem (Theorem 2.1 of Appendix 4) corresponds to the weight $w(x) = 1$, $q(x) = 0$ on A. The deep proof of this theorem applies also in the present situation, with small changes.

Proof. The function $v(x) := q(x) + \alpha_w - \Lambda_w(x)$ is lower semi-continuous which implies that the sets $S_k := \{x \in A : v(x) \le -1/k\}$, $k = 1, 2, \dots$, are closed. They are even compact. To show this, we assume that A is unbounded and $x \to \pm\infty$, $x \in A$. Then $v(x) \to \infty$ since, by (1.1)(iii), $q(x) - \log|x| \to +\infty$, and $\Lambda_w(x) = \log|x| + o(1)$ for large $|x|$ since S_w^μ is compact.

From the definitions (3.1)–(3.3) and (3.7), we have

$$\begin{aligned}
\int_A \Lambda_w(x) d\mu_w(x) &= \int_{A\times A} \log|x-t| d\mu_w(t) d\mu_w(x) \\
&= -V_w - \int_{A\times A} \log\{w(x)w(t)\} d\mu_w(x) d\mu_w(t) \\
&= -V_w + 2\int_A q(x) d\mu_w(x),
\end{aligned}$$

hence by (3.6), $\int_A \Lambda_w(x) d\mu_w(x) = \alpha_w + \int_A q(x) d\mu_w(x)$ and

$$\int_A v(x) d\mu_w(x) = 0. \tag{3.10}$$

In contradiction to (3.8) we assume that $\gamma(B') > 0$. Then, by **1** (iv) of Appendix 4, at least one of the sets S_k has positive capacity since $B' = \cup_{k=1}^\infty S_k$. Let $S := S_n$ be such a set with $\gamma(S) > 0$. We take an ε with $0 < 2\varepsilon < 1/n$. Since μ_w has no atoms and since v satisfies (3.10), there exist some $x_0 \in S_w^\mu$ and some open neighborhood U of x_0 such that $v(x) > -\varepsilon$ for all $x \in E := U \cap S_w^\mu$. One has $\delta := \mu_w(E) > 0$ as $x_0 \in S_w^\mu$. The sets E and S are disjoint. We construct a signed Borel measure σ on A as follows: Since $\gamma(S) > 0$, there exists a non-negative measure ν on S for which $\nu(S) = \delta$ and

$$-\int_{S\times S} \log|x-t| d\nu(x) d\nu(t) < +\infty; \tag{3.11}$$

see the definition (2.3) of $\gamma(S)$ in Appendix 4. We set

$$\sigma := \begin{cases} \nu & \text{on } S \\ -\mu_w & \text{on } E \\ 0 & \text{elsewhere on } A, \end{cases}$$

that is, for a Borel set C we set $\sigma(C) = \nu(C \cap S) - \mu_w(C \cap E)$. Then $\sigma(S) = \nu(S) = \delta$, $\sigma(E) = -\mu_w(E) = -\delta$ and $\sigma(A) = 0$. For each $0 < \eta < 1$, $\mu_w + \eta\sigma$ is a non-negative Borel measure; it is a probability measure on A because $(\mu_w + \eta\sigma)(A) = 1$. From the symmetry of $G(x,t)$ one obtains

$$\begin{aligned} \delta I_w &:= I_w(\mu_w + \eta\sigma) - I_w(\mu_w) \\ &= 2\eta \int_{A \times A} G(x,t) d\mu_w(t) d\sigma(x) + \eta^2 \int_{A \times A} G(x,t) d\sigma(x) d\sigma(t). \end{aligned}$$

The above integrals are finite because of (3.11) and because $w > 0$ on S_w^μ. One has

$$\int_A G(x,t) d\mu_w(t) = -\Lambda_w(x) + q(x) + \int_A q(t) d\mu_w(t) = v(x) + V_w ,$$

so that

$$\begin{aligned} \delta I_w &= 2\eta \int_A (v(x) + V_w) d\sigma(x) + \mathcal{O}(\eta^2) \\ &= 2\eta \int_A v(x) d\sigma(x) + \mathcal{O}(\eta^2) \\ &= 2\eta \left(\int_S v(x) d\nu(x) - \int_E v(x) d\mu_w(x) \right) + \mathcal{O}(\eta^2) \\ &\le -\frac{\eta\delta}{n} + \mathcal{O}(\eta^2) \end{aligned}$$

since $\int_S v(x) d\nu(x) \le -\delta/n$ and $-\int_E v(x) d\mu_w(x) < \varepsilon\delta \le \delta/(2n)$. Hence, $\delta I_w < 0$, if $\eta > 0$ is small. But this is a contradiction to the minimal property of $I_w(\mu_w)$, hence the assumption that $\gamma(S_n) > 0$ for some $n \ge 1$ is wrong, which proves (3.8).

We have shown above that $\gamma(S_k) = 0$, $k = 1, 2, \cdots$, and that the sets S_k are compact. Hence, by **3** of Appendix 4, §2, one has $m(S_k) = 0$ for the ordinary Lebesgue measure and $\mu_w(S_k) = 0$ for all k. Since $B' = \cup_{k=1}^\infty S_k$, it follows that $m(B') = \mu_w(B') = 0$; the latter implies that $\Lambda_w(x) - q(x) \le \alpha_w$ holds μ_w-almost everywhere. If at some $x_1 \in S_w^\mu$, $\Lambda_w(x_1) - q(x_1) < \alpha_w$, then, as $\Lambda_w - q$ is upper semi-continuous on A, this inequality holds in some neighborhood $E := U(x_1) \cap S_w^\mu$. The set E has positive μ_w-measure since $x_1 \in S_w^\mu$. This yields $\int_{S_w^\mu} (\Lambda_w(x) - q(x)) d\mu_w(x) < \alpha_w$, which contradicts (3.10): we have (3.9), with equality in (3.9) quasi everywhere on S_w^μ.

The function Λ_w is upper semi-continuous on $\mathbb{C}$ and harmonic in $\mathbb{C} \setminus S_w^\mu$. By (3.9), Λ_w is bounded from below on the support of μ. By Theorem 1.4 (i) of Appendix 4, this property extends to the whole of $\mathbb{C}$. □

Theorem 3.3. *Let w be a weight on A. Then the support S_w^μ of the equilibrium measure μ_w is an essential set for w so that $S_w^\mu \supset A_w$.*

Proof. We have to prove that the inequality $w(x)^n|P_n(x)| \le 1$ for all $x \in S_w^\mu$ implies the same inequality for all $x \in A$. The assumption is equivalent to

$$n^{-1}\log|P_n(x)| \le q(x), \quad x \in S_w^\mu. \tag{3.12}$$

By (3.9), this implies

$$n^{-1}\log|P_n(x)| \le \Lambda_w(x) - \alpha_w, \quad x \in S_w^\mu. \tag{3.13}$$

Let Z be the set of the zeros of P_n including the point ∞ if the degree of P_n is less than n. Since $\Lambda_w(z)$ is bounded from below on $\mathbb{C}$, the function $v(z) := n^{-1}\log|P_n(z)| - \Lambda_w(z)$ is bounded from above on $\mathbb{C}^* := \mathbb{C} \cup \{\infty\}$; and it is harmonic in $G := \mathbb{C}^* \setminus (S_w^\mu \cup \mathrm{Z})$. Using (3.12) and Lemma 1.7(iii) of Appendix 4, and then (3.9), we get

$$\begin{aligned}\limsup_{z\to x_0, z\in G} v(z) &\le q(x_0) + \limsup_{x\to x_0, x\in S_w^\mu}(-\Lambda_w(x))\\ &\le q(x_0) + \limsup_{x\to x_0, x\in S_w^\mu}(-q(x) - \alpha_w) = -\alpha_w.\end{aligned}$$

By Theorem 2.11 of Appendix 4, $v(z) \le -\alpha_w$ for all $z \in G$, in particular, we have (3.13) for all $x \in A$.

From (3.8) we get that (3.12) is valid for all $x \in A$ except for a set of capacity zero. However, A is a union of intervals, and each of its points is a limit of points of A not belonging to this exceptional set. By continuity, (3.12) is valid for all $x \in A$. □

Lemma 3.4. *The negative logarithmic potentials L_w, Λ_w of the Chebyshev measure ν_w of §2 and of the equilibrium measure μ_w, respectively, and the numbers a_w, α_w satisfy*

$$L_w(z) - \Lambda_w(z) \le a_w - \alpha_w \quad \textit{for all } z \in \mathbb{C}. \tag{3.14}$$

Moreover, $\alpha_w \le a_w$.

Proof. Both functions L_w, Λ_w are defined and finite on $\mathbb{C}$, by Theorems 2.7 and 3.2. For the polynomials $C_{w,n} := C_{w,n}^A$, we have

$$n^{-1}\log|C_{w,n}(x)| \le \Lambda_w(x) - \alpha_w \text{ everywhere on } S_w^\mu, \tag{3.15}$$

since $n^{-1}\log|C_{w,n}(x)| \le q(x)$, $x \in A$, and by (3.9), $q(x) \le \Lambda_w(x) - \alpha_w$ for all $x \in S_w^\mu$. Moreover, by Theorem 3.2, $\Lambda_w(z)$ is bounded from below on $\mathbb{C}$. Then Theorem 1.8 of Appendix 4 implies that (3.15) is valid in the whole complex plane.

By Theorem 2.5(ii), for $z \notin \mathrm{co}(A_w) = [\alpha, \beta]$, we have

$$n^{-1}\log|C_{w,n}(z)| \to L_w(z) - a_w, \quad n \to \infty.$$

Since (3.15) is valid for all $z \in \mathbb{C}$, we derive the inequality (3.14) for all $z \in \mathbb{C}\setminus \mathrm{co}(A_w)$.

Using (1.9) of Appendix 4 for L_w and Λ_w, we get (3.14) also for $z \in S_w^\mu$, hence for all $z \in \mathbb{C}$. Making $z \to \infty$ in (3.14), we obtain $\alpha_w \leq a_w$. □

Another of our main theorems is:

Theorem 3.5. *Let w be a weight on A. Then the minimal essential set A_w for w on A and the support S_w^ν of the Chebyshev measure ν_w are identical.*

Proof. By Proposition 2.6, $S_w^\nu \subset A_w$. Let $S := A_w \setminus S_w^\nu$ be not empty. By Theorem 2.2 and since S_w^ν and A_w are compact, there exist $n \geq 1$ and an extreme point $\xi \in S$ of $w^n C_{w,n}$. We shall prove that

$$L_w(x) > q(x) + a_w, \quad x \in \mathrm{U} \cap A \tag{3.16}$$

for some neighborhood U of ξ. Indeed, let Z_n be the set of the zeros of $C_{w,n}$. The function

$$v_n(z) := n^{-1} \log |C_{w,n}(z)| - L_w(z)$$

is harmonic in $G_n := \mathbb{C}^* \setminus (S_w^\nu \cup Z_n)$. By Proposition 2.8, $v_n(z) \leq -a_w$ for all $z \in \mathbb{C}$. It follows from the maximum principle that $v_n(\xi) < -a_w$. This implies $L_w(\xi) > q(\xi) + a_w$ since $\xi \in G_n$ and $n^{-1} \log |C_{w,n}(\xi)| = q(\xi)$. Since L_w and q are continuous on S, we have (3.16). From this and (3.14) we have

$$\Lambda_w(x) > q(x) + \alpha_w, \quad x \in \mathrm{U} \cap A.$$

Since A is a union of intervals, U contains a subinterval of A. This contradicts (3.8). □

It is an interesting problem to compare the Chebyshev measure ν_w for w with the equilibrium measure μ_w. We need

Lemma 3.6. *There exists a point $\xi \in A_w$, at which the negative logarithmic potential Λ_w of the equilibrium measure μ_w satisfies*

$$\Lambda_w(\xi) \geq q(\xi) + a_w. \tag{3.17}$$

Proof. We write $[c, d] := \mathrm{co}(S_w^\mu)$ and construct the monic polynomials Q_n of degree n, $n \geq 1$, as follows: Let $t_k := t_k^{(n)} \in S_w^\mu$, $k = 1, \ldots, n$, be defined by $\mu_w((-\infty, t_k]) = k/n$. Since μ_w has no atoms, this is possible and we have $c < t_1 < \cdots < t_n = d$. We set

$$Q_n(z) := \prod_{k=1}^{n} (z - t_k^{(n)}).$$

To any Borel set $S \subset [c, d]$ we assign the measures

$$\mu_n(S) = \sum_{t_k^{(n)} \in S} \frac{1}{n}, \quad n = 1, 2, \ldots.$$

They belong to $\mathcal{M}[c,d]$ and converge weakly* to μ_w. Since

$$n^{-1}\log|Q_n(z)| = \int_c^d \log|z-t|d\mu_n(t), \quad z \in G := \mathbb{C}\setminus[c,d],$$

we have

$$\lim_{n\to\infty} n^{-1}\log|Q_n(z)| = \Lambda_w(z), \quad z \in G. \tag{3.18}$$

Since Q_n is a monic polynomial, $\|w^n Q_n\|_A \geq e^A_{w,n}$. Let $x_n \in A_w$ be some point at which

$$|w(x_n)^n Q_n(x_n)| = \|w^n Q_n\|_A. \tag{3.19}$$

Taking a subsequence n, if necessary, we may assume that $x_n \to \xi$, $n \to \infty$, for some $\xi \in A_w$.

Let $y > 0$ be fixed. Then, $|Q_n(\xi+iy)| \geq |Q_n(x_n)|$ for all large n and by (3.18), (3.19) and the continuity of q on A_w,

$$\begin{aligned}\Lambda_w(\xi+iy) &= \lim_{n\to\infty} n^{-1}\log|Q_n(\xi+iy)| \\ &\geq q(\xi) + \lim_{n\to\infty} n^{-1}\log|w(x_n)^n Q_n(x_n)| \\ &\geq q(\xi) + \lim_{n\to\infty} n^{-1}\log e^A_{w,n} = q(\xi) + a_w.\end{aligned}$$

Since y is arbitrary, by Lemma 1.7(i) of Appendix 4, we have (3.17). □

Sometimes, with the help of the solution of a Dirichlet problem, one can improve the statement of Frostman's Theorem 3.2 to equality instead of q.e. equality. If $K \subset \mathbb{R}$ is compact and of positive capacity, $\gamma(K) > 0$, and $q(x)$ is a continuous function on K, this solution is a function $F(z)$, $z \in \mathbb{C}^*$ which is continuous on $\mathbb{C}^*$, harmonic on $\mathbb{C}^*\setminus K$ and on K satisfies $F(x) = q(x)$. This solution exists and is unique if K is regular (see Appendix 4, §3, **1**). In Appendix 4, §3, **2** we give a general definition of Green's function $g(z) = g(z,\infty)$ of $\mathbb{C}^*\setminus K$. For regular K, g is continuous on $\mathbb{C}$. Then $\Lambda := F + g$ has the following properties. It is continuous on $\mathbb{C}$, harmonic on $\mathbb{C}\setminus K$, coincides with q on K and satisfies

$$\Lambda(z) = \log|z| + \text{const} + o(1) \quad \text{for } z \to \infty.$$

We shall have $q := -\log w$, and assume therefore that $w \neq 0$ on K.

Proposition 3.7. *If $w(x) > 0$ on some regular set K, $S^\mu_w \subset K \subset A$, then*

$$\Lambda_w(x) = q(x) + \alpha_w, \quad x \in S^\mu_w, \tag{3.20}$$

and Λ_w is continuous on $\mathbb{C}$.

Proof. The function $v := \Lambda_w - \Lambda$, with Λ from **2** is harmonic in $G := \mathbb{C}^*\setminus K$, bounded from above on $\mathbb{C}^*$ and continuous on $\mathbb{C}^*\setminus S^\mu_w$. By upper semicontinuity of Λ_w, we have, using (3.8):

$$\limsup_{z\to x, z\in G} v(z) \le v(x) = \Lambda_w(x) - q(x) \le \alpha_w \quad \text{for quasi all } x \in K.$$

We apply Theorem 2.11 of Appendix 4 to the function v and the region G and get $v(z) \le \alpha_w$ for all $z \in G$. Returning to Λ_w, we use (1.9) of Appendix 4 and get

$$\begin{aligned}\Lambda_w(x) &= \limsup_{y\to 0+} \Lambda(x+iy) = \limsup_{y\to 0+}(\Lambda(x+iy) + v(x+iy)) \\ &\le q(x) + \alpha_w, \quad x \in K.\end{aligned} \tag{3.21}$$

Together with (3.9) this yields (3.20).

It remains to prove that Λ_w is continuous at any point $x_0 \in S_w^\mu$. Let $(z_m)_1^m \subset \mathbb{C}$ converge to x_0. By the upper semi-continuity of Λ_w and (3.20),

$$\limsup_{m\to\infty} \Lambda_w(z_m) \le \Lambda_w(x_0) = q(x_0) + \alpha_w.$$

We obtain the opposite inequality from Lemma 1.3 of Appendix 4 and the continuity of q on S_w^μ: Since μ_w has no atoms,

$$\liminf_{m\to\infty} \Lambda_w(z_m) \ge \liminf_{x\to x_0, x\in S_w^\mu} \Lambda_w(x) = q(x_0) + \alpha_w.$$

This shows that Λ_w is continuous at x_0. □

Theorem 3.8. *If $w(x) > 0$ on some regular set K, $S_w^\mu \subset K \subset A$, then*

$$a_w = \alpha_w, \quad L_w = \Lambda_w, \quad A_w = S_w^\nu = S_w^\mu, \tag{3.22}$$

and the Chebyshev measure and the equilibrium measure for w coincide: $\nu_w = \mu_w$.

Proof. By Lemma 3.6, for some $\xi \in A_w$, $\Lambda_w(\xi) \ge q(\xi) + a_w$. Combining this with (3.20), we get $a_w \le \alpha_w$, and by Lemma 3.4, $a_w = \alpha_w$. Now Lemma 3.4 yields also $L_w(z) \le \Lambda_w(z)$, $z \in \mathbb{C}$. Applying the maximum modulus principle to $u := L_w - \Lambda_w$, with $v(\infty) = 0$, $v(z) \le 0$, $z \in \mathbb{C}$, we get $L_w = \Lambda_w$ and from Theorems 1.5, 1.6 of Appendix 4 that $\mu_w = \nu_w$. □

Corollary 3.9. *The statement of Theorem* 3.8 *holds in particular if A is a finite union of intervals, and if w has only finitely many zeros on A.*

§ 4. Determination of Minimal Essential Sets

In this section we shall assume that A is a closed interval with finite or infinite endpoints a, b and that w is a weight on A. Moreover, we shall assume that it is known that $\nu_w = \mu_w$ and that $A_w = S_w^\nu = S_w^\mu =: [\alpha, \beta]$ is a compact interval, hence $q(x) := -\log w(x)$ is continuous on $[\alpha, \beta]$. We shall try to find the endpoints α, β.

From Proposition 3.1 of Appendix 4 (with $[c, d] := [\alpha, \beta]$) we know that the interval $[\alpha, \beta]$ is regular and the solution of the Dirichlet Problem **2** of §3 for $G := \mathbb{C}^* \setminus [\alpha, \beta]$ and the boundary values $q(x)$ on $[\alpha, \beta]$ is of the form

$$\Lambda(z) = H(\Psi^{-1}(z)) + g(z, \infty), \tag{4.1}$$

with the *Poisson integral*

$$H(re^{it}) = \frac{1 - r^2}{2\pi} \int_0^{2\pi} \frac{q(\Psi(e^{i\phi}))}{1 - 2r\cos(\phi - t) + r^2} d\phi, \quad 0 \le r < 1, \ t \in \mathbb{R}, \tag{4.2}$$

the mapping

$$\Psi(u) = \frac{1}{2}(\alpha + \beta) + \frac{1}{4}(\beta - \alpha)(u + u^{-1}), \quad 0 \le |u| \le 1, \tag{4.3}$$

and its inverse

$$\Psi^{-1}(z) = \frac{2}{\beta - \alpha}\left(z - \frac{\alpha + \beta}{2} + \sqrt{\left(z - \frac{\alpha + \beta}{2}\right)^2 - \frac{(\beta - \alpha)^2}{4}}\right). \tag{4.4}$$

In (4.4) we have to select the branch of the square root which is positive for $z > \beta$. The Green's function for $G = \mathbb{C}^* \setminus [\alpha, \beta]$ is given by

$$g(z, \infty) = \log\left|z - \frac{\alpha + \beta}{2} + \sqrt{\left(z - \frac{\alpha + \beta}{2}\right)^2 - \frac{(\beta - \alpha)^2}{4}}\right| - \log\frac{\beta - \alpha}{2}. \tag{4.5}$$

The function (4.1) attains the boundary values $\Lambda(x) = q(x)$, $x \in [\alpha, \beta]$. Therefore by Theorem 3.2, the negative logarithmic potential of $\nu_w = \mu_w$ is given by

$$L_w(z) = \Lambda_w(z) = H(\Psi^{-1}(z)) + g(z, \infty) + a_w. \tag{4.6}$$

In particular L_w is continuous on $\mathbb{C}$ and satisfies

$$L_w(x) = q(x) + a_w, \quad x \in A_w = [\alpha, \beta]. \tag{4.7}$$

There exists only one function of this type: this follows from the maximum modulus theorem.

A step towards finding α, β is

Theorem 4.1. *Let $A = [a, b]$, let w be a weight on A, with $q(x) = -\log w(x)$ continuously differentiable on (a, b). If $A_w = [\alpha, \beta]$ is a compact interval, then:*
(i) *either $\beta = b$ or*

$$\frac{1}{\pi}\int_\alpha^\beta q'(x)\sqrt{\frac{x - \alpha}{\beta - x}}\,dx = 1; \tag{4.8}$$

(ii) *either* $\alpha = a$ *or*

$$\frac{1}{\pi}\int_\alpha^\beta q'(x)\sqrt{\frac{\beta - x}{x-\alpha}}dx = -1. \tag{4.9}$$

(The first possibility in (i) *or* (ii) *is not present if* b *or* a *is a zero of* w.*)*

Proof. We shall discuss only the case (i), because (ii) follows then by replacing $[a,b]$ by $[-b,-a]$, and $w(x)$ by $w(-x)$.

Let $\beta < b$, then $I := [\beta, \beta+\delta] \subset A$ for small $\delta > 0$. From (4.3), for $u = e^{i\phi}$,

$$\Psi(e^{i\phi}) = \frac{1}{2}(\alpha+\beta) + \frac{1}{2}(\beta-\alpha)\cos\phi.$$

Integrating by parts and substituting $x = \Psi(e^{i\phi})$ yields

$$\frac{1}{\pi}\int_\alpha^\beta q'(x)\sqrt{\frac{x-\alpha}{\beta - x}}dx = -\frac{1}{\pi}\int_0^\pi \frac{q(\Psi(e^{i\phi})) - q(\beta)}{1-\cos\phi}d\phi. \tag{4.10}$$

Applying the Poisson integral formula to the constant function $H(1) = q(\beta)$ leads to

$$H(1) = \frac{1-r^2}{2\pi}\int_0^{2\pi}\frac{q(\beta)}{1-2r\cos\phi + r^2}d\phi \quad \text{for all } 0 \le r < 1. \tag{4.11}$$

By l'Hospital's rule,

$$\lim_{\phi\to 0}\frac{q(\Psi(e^{i\phi})) - q(\beta)}{1-\cos\phi} = -\frac{\beta-\alpha}{2}q'(\beta).$$

It follows that $(|q(\Psi(e^{i\phi})) - q(\beta)|)/(1-\cos\phi)$, $0 \le \phi \le 2\pi$, is bounded by some positive number M. The integral

$$\int_0^{2\pi}\left(\frac{q(\Psi(e^{i\phi})) - q(\beta)}{2r(1-\cos\phi)} - \frac{q(\Psi(e^{i\phi})) - q(\beta)}{1-2r\cos\phi+r^2}\right)d\phi$$

converges to zero for $r \to 1-$, since its absolute value is bounded from above by

$$\frac{M(1-r)^2}{2r}\int_0^{2\pi}\frac{d\phi}{1-2r\cos\phi+r^2} = \frac{M\pi(1-r)}{r(1+r)}.$$

This, (4.2) and (4.11) yield

$$\begin{aligned} H'(1) &:= \lim_{r\to 1-}\frac{H(r) - H(1)}{r-1} \\ &= \lim_{r\to 1-} -\frac{1+r}{2\pi}\int_0^{2\pi}\frac{q(\Psi(e^{i\phi})) - q(\beta)}{1-2r\cos\phi + r^2}d\phi \\ &= -\frac{1}{2\pi}\int_0^{2\pi}\frac{q(\Psi(e^{i\phi})) - q(\beta)}{1-\cos\phi}d\phi = -\frac{1}{\pi}\int_0^{\pi}\frac{q(\Psi(e^{i\phi})) - q(\beta)}{1-\cos\phi}d\phi. \end{aligned}$$

Comparing this result with (4.10) we obtain

$$H'(1) = \frac{1}{\pi}\int_\alpha^\beta q'(x)\sqrt{\frac{x-\alpha}{\beta-x}}dx,$$

that is, $H'(1)$ is equal to the left-hand side of (4.8).

It remains to show that $H'(1) = 1$. We introduce the function

$$G(r) := H(r) - \log|r| - q(\Psi(r)) = L_w(x) - q(x) - a_w, \quad x = \Psi(r), \quad 0 < r \le 1.$$

If $r = 1$, then $x = \beta$, hence $G(1) = 0$. Since $\Psi'(1) = 0$, we have $G'(1) = H'(1) - 1$, so that it remains to prove that $G'(1) = 0$.

From Theorem 2.7, for $x > \beta$ (or, equivalently, $r < 1$) we have $L_w(x) \le q(x) + a_w$, hence $G(r) \le 0$, so that $G'(1) \ge 0$.

Next, $L_w(\beta) = q(\beta) + a_w$, and $L_w(x) = \int_\alpha^\beta \log|x - t| d\nu_w(t)$ increases for $x \ge \beta$. Therefore,

$$G(r) = L_w(x) - q(x) - a_w \ge L_w(\beta) - q(x) - a_w = q(\beta) - q(x)$$

and for $x = \Psi(r)$, $r < 1$, since $G(1) = 0$,

$$\begin{aligned} G'(1) &:= \lim_{r\to 1-} \frac{G(r) - G(1)}{r - 1} \le \liminf_{r\to 1} \frac{q(\beta) - q(x)}{r - 1} \\ &= \liminf_{r\to 1} \frac{(q(\beta) - q(x))}{(\beta - x)} \frac{(\beta - x)}{(r - 1)} = -q'(\beta)\Psi'(1) = 0, \end{aligned}$$

and we get $G'(1) \le 0$. □

In the symmetric case of Theorem 4.1, $w(-x) = w(x)$ on $A = (-\infty, \infty)$, (4.8) and (4.9) can be replaced by the single condition

$$\frac{1}{\pi} \int_{-\beta}^{\beta} q'(x) \sqrt{\frac{x + \beta}{\beta - x}} dx = \frac{2}{\pi} \int_0^\beta \frac{xq'(x)}{\sqrt{\beta^2 - x^2}} dx = 1. \tag{4.12}$$

In this case, additional assumptions on q' allow us to determine β unconditionally, by means of the Mhaskar-Saff-Rakhmanov equation (4.13):

Theorem 4.2. *Let w be an even positive weight on $(-\infty, \infty)$ which attains its maximum at $x = 0$. If $q := -\log w$ is continuously differentiable on $(0, \infty)$ and if $xq'(x)$ is increasing on $(0, \infty)$, then the minimal essential set A_w is an interval $[-\beta, \beta]$, where β is the unique root of the equation*

$$\frac{2}{\pi} \int_0^1 \frac{\beta x q'(\beta x)}{\sqrt{1 - x^2}} dx = 1. \tag{4.13}$$

Proof. By Proposition 1.3(iii), the function $w^*(x) := w(\sqrt{x})^2 =: e^{-q^*(x)}$ is a weight on $[0, \infty)$. The function $xq^{*\prime}(x) = \sqrt{x}q'(\sqrt{x})$ is monotone increasing on $(0, \infty)$, and w^* attains its maximum at $x = 0$. By Proposition 2.11 and Proposition 1.3, the minimal essential set A_{w^*} is an interval $[0, \beta^*]$ and A_w is a symmetric interval, $[-\beta, \beta]$, where $\beta = \sqrt{\beta^*}$. By Theorem 4.1, β is a root of (4.12), which is equivalent to (4.13). Since $xq'(x)$ is monotone increasing, (4.13) has a unique solution. □

Corollary 4.3. *From the proof it follows that the minimal essential set* A_{w^*} *for the weight* $w^*(x) = w(\sqrt{x})^2$ *on* $[0, \infty)$ *is* $[0, \beta^2]$, *with* β *defined by* (4.13).

Example 4. Jacobi Weights. This is the case when $A = [-1, 1]$ and, with $s_1 > 0$, $s_2 > 0$,

$$w(x) = (1-x)^{s_1}(1+x)^{s_2} \ , \quad -1 \le x \le 1. \tag{4.14}$$

By Corollary 2.10, the minimal essential set A_w is an interval, $A_w =: [\alpha, \beta]$, with $-1 < \alpha < \beta < 1$.

Theorem 4.4 (Saff, Ullman, Varga [1980]). *For the Jacobi weights* (4.14),

$$\alpha = \theta_2^2 - \theta_1^2 - \sqrt{\Delta} \ , \quad \beta = \theta_2^2 - \theta_1^2 + \sqrt{\Delta} \tag{4.15}$$

where $\theta_1 := s_1/(1+s_1+s_2)$, $\theta_2 := s_2/(1+s_1+s_2)$ *and* $\Delta := (\theta_2^2 - \theta_1^2)^2 - 2\theta_1^2 - 2\theta_2^2 + 1$.

Proof. By Theorem 4.1, α, β satisfy the two equations

$$\begin{aligned} &\frac{1}{\pi}\int_\alpha^\beta \left(\frac{s_1}{x-1} + \frac{s_2}{x+1}\right)\sqrt{\frac{x-\alpha}{\beta-x}}\,dx = -1 \\ &\frac{1}{\pi}\int_\alpha^\beta \left(\frac{s_1}{x-1} + \frac{s_2}{x+1}\right)\sqrt{\frac{\beta-x}{x-\alpha}}\,dx = 1. \end{aligned} \tag{4.16}$$

We shall show that the numbers α, β in (4.15) are the unique solution of this system.

For $\eta = \pm 1$, elementary calculations yield

$$\int_\alpha^\beta \frac{1}{x+\eta}\sqrt{\frac{x-\alpha}{\beta-x}}\,dx = \pi - \pi\sqrt{\frac{\alpha+\eta}{\beta+\eta}}$$

$$\int_\alpha^\beta \frac{1}{x+\eta}\sqrt{\frac{\beta-x}{x-\alpha}}\,dx = \pi\sqrt{\frac{\beta+\eta}{\alpha+\eta}} - \pi.$$

Hence, the system (4.16) is equivalent to

$$s_1\left(1 - \sqrt{\frac{\alpha-1}{\beta-1}}\right) + s_2\left(1 - \sqrt{\frac{\alpha+1}{\beta+1}}\right) = -1$$

$$s_1\left(\sqrt{\frac{\beta-1}{\alpha-1}} - 1\right) + s_2\left(\sqrt{\frac{\beta+1}{\alpha+1}} - 1\right) = 1.$$

With the numbers θ_1, θ_2, the system for α and β becomes

$$\theta_1\sqrt{\frac{1-\alpha}{1-\beta}}+\theta_2\sqrt{\frac{1+\alpha}{1+\beta}}=1 \tag{4.17}$$

$$\theta_1\sqrt{\frac{1-\beta}{1-\alpha}}+\theta_2\sqrt{\frac{1+\beta}{1+\alpha}}=1. \tag{4.18}$$

The solution of (4.17)-(4.18) is not difficult: we multiply (4.17) with the factor $(1-\beta)\sqrt{(1+\alpha)(1+\beta)}$, and subtract from it (4.18) multiplied with $(1-\alpha)\sqrt{(1+\alpha)(1+\beta)}$. We obtain

$$2\theta_2=\sqrt{(1+\alpha)(1+\beta)}.$$

Similarly, we multiply (4.17) and (4.18) with the factors $(1+\beta)\sqrt{(1-\alpha)(1-\beta)}$ and $(1+\alpha)\sqrt{(1-\alpha)(1-\beta)}$, respectively, and subtract the results. This yields

$$2\theta_1=\sqrt{(1-\alpha)(1-\beta)}.$$

It follows that

$$\alpha\beta=2\theta_1^2+2\theta_2^2-1\ ,\ \ \alpha+\beta=2\theta_2^2-2\theta_1^2,$$

and further that α and β are the two solutions of the quadratic equation

$$(z-\alpha)(z-\beta)=z^2-2z(\theta_2^2-\theta_1^2)+2\theta_1^2+2\theta_2^2-1=0.$$

This establishes (4.15). □

Example 5. Freud Weights. The Freud weights

$$w_\sigma(x):=\exp(-|x|^\sigma),\ \ x\in\mathbb{R},\quad \sigma>0, \tag{4.19}$$

have the properties of Theorem 4.2, with $q(x)=|x|^\sigma$. The endpoint β_σ of the minimal essential set $A_{w_\sigma}=[-\beta_\sigma,\beta_\sigma]$ is the unique solution of

$$\frac{2}{\pi}\int_0^1\frac{\sigma\beta^\sigma x^\sigma}{\sqrt{1-x^2}}\ dx=1,$$

that is,

$$\beta_\sigma^{-\sigma}=\frac{2\sigma}{\pi}\int_0^1\frac{x^\sigma}{\sqrt{1-x^2}}\ dx=\frac{2\sigma}{\pi}\int_0^{\pi/2}\sin^\sigma x\ dx. \tag{4.20}$$

We need the formulas $\Gamma(\sigma/2)\Gamma((\sigma+1)/2)=\sqrt{\pi}2^{1-\sigma}\Gamma(\sigma)$, and

$$\int_0^{\pi/2}\sin^\sigma x\ dx=\frac{\sqrt{\pi}\Gamma((\sigma+1)/2)}{\sigma\Gamma(\sigma/2)},\quad \int_0^{\pi/2}\sin^{\sigma-1}x\ dx=\frac{\sqrt{\pi}\Gamma(\sigma/2)}{2\Gamma((\sigma+1)/2)}.$$

They imply that

$$\int_0^{\pi/2}\sin^{\sigma-1}x\ dx=\frac{\pi}{2\sigma}\left(\int_0^{\pi/2}\sin^\sigma x\ dx\right)^{-1}.$$

From (4.20) we therefore get (Mhaskar and Saff [1984])

$$\beta_\sigma = \left(\frac{2^{\sigma-2}\,\Gamma(\sigma/2)^2}{\Gamma(\sigma)}\right)^{1/\sigma} = \left(\int_0^{\pi/2} \sin^{\sigma-1} x \; dx\right)^{1/\sigma}. \tag{4.21}$$

Example 6. The minimal essential set for the Freud weight w_σ on $[0,\infty)$ is $[0,\beta_\sigma^*]$ where

$$\beta_\sigma^* := 2^{1/\sigma}\beta_{2\sigma}^2. \tag{4.22}$$

Indeed, the weight $w(x) := w_{2\sigma}(2^{-1/(2\sigma)}x)$ on $\mathbb{R}$ satisfies $w_\sigma(x) = w(\sqrt{x})^2$, $x \geq 0$, and (4.22) follows from Corollary 4.3 and the remark preceding Proposition 1.3.

§ 5. Weierstrass Theorems and Oscillations

This section is devoted to Weierstrass theorems that are elementary in the sense that they do not depend on Potential Theory. Also, we do not need the assumption that A_w is an interval.

We denote by $C_0(B)$, $B \subset A$ the subspace of all functions $f \in C(A)$ that vanish outside of B. A set $G \subset A$, open in A, is called a *Weierstrass set* for w if each function $f \in C_0(G)$ is approximable by weighted polynomials: for $n = 1, 2, \ldots$ there exist $\phi_n = w^n P_n$, $P_n \in \mathcal{P}_n$, so that

$$\lim_{n\to\infty} \|f - \phi_n\|_A = 0. \tag{5.1}$$

A Weierstrass set G is called *maximal* if each Weierstrass set for w is a subset of G.

Proposition 5.1. *There exists a unique maximal Weierstrass set for w, denoted by G_w. It is contained in the interior A_w^o of the minimal essential set A_w.*

Proof. Each Weierstrass set G is contained in A_w^o. Indeed, if x_0 is a point of G we can take $f \in C_0(G)$, $f \geq 0$, with a unique maximum at x_0, $f(x_0) = \|f\|_A = 1$; then in any neighborhood of x_0, if $w^n P_n = \phi_n$ approximates f well, we shall have a point x_1 with $\phi_n(x_1) = \|\phi_n\|_A$, hence $x_0 \in A_w$. Since $x_0 \in G$ is arbitrary, $G \subset A_w$, hence $\overline{G} \subset A_w$ and $G \subset A_w^o$.

Let $I_1 = (c_1, d_1)$, $I_2 = (c_2, d_2)$ be *Weierstrass intervals.* If $I_1 \subset I_2$, $I_2 \subset I_1$ or $I_1 \cap I_2 = \emptyset$ then $I := I_1 \cup I_2$ is again a Weierstrass set. Suppose now that $c_1 < c_2 < d_1 < d_2$. We define the continuous function h on $\mathbb{R}$ by

$$h(x) := \begin{cases} 1, & \text{for } x \in (-\infty, c_2], \\ & \text{linear on } [c_2, d_1], \\ 0, & \text{for } x \in [d_1, \infty). \end{cases}$$

If $f \in C_0(I)$, we set $f_1 := f \cdot h$, $f_2 := f - f_1$. Then f_1 and f_2 are approximable by the ϕ_n, hence $f = f_1 + f_2$ is approximable. This proves that $I = I_1 \cup I_2$ is also a Weierstrass interval. By induction, the union of finitely many Weierstrass intervals is a Weierstrass set.

Let G_0 be the union of all Weierstrass sets for w. Then G_0 is also the union of all subintervals of A_w^o which are Weierstrass sets. Let $\Omega := A \setminus G_0$ and, for $\delta > 0$, $K := K(\delta) := \{x \in G_0 : \min_{t\in\Omega} |t - x| \geq \delta\}$. If $\Omega = \emptyset$, that is, if $G_0 = A$, we take $K := A$. The set $K \subset G_0$ is compact. Hence there exist finitely many Weierstrass intervals $I_1, \ldots, I_m$ which cover K. Their union G is also a Weierstrass set, and $G \supset K$: each function $f \in C_0(K)$ is approximable. Since $\delta > 0$ is arbitrary, each $f \in C(A)$ which vanishes on Ω is approximable: G_0 is the maximal Weierstrass set for w. □

The ideal situation would be when $G_w = A_w^o$. The first example of this are the incomplete polynomials, where this statement has been conjectured by Lorentz, and proved by Saff and Varga [1978] and v.Golitschek [1980]. Further extensions to Jacobi weights were obtained by He and Li [1991].

The next lemma is about oscillating weighted polynomials.

Lemma 5.2. *For each interval $I \subset A_w$ there exist weighted polynomials $\psi_n := w^n Q_n$, $Q_n \in \mathcal{P}_n$, $n = 1, 2, \ldots$, that satisfy $\|\psi_n\|_A = 1$ and*

$$\lim_{n\to\infty} \|\psi_n\|_{A\setminus I} = 0, \quad \lim_{n\to\infty} M_n(I) = \infty, \tag{5.2}$$

where $M_n(I)$ denotes the maximal length of an increasing sequence (x_i) of points of I with $\psi_n(x_i)$ of alternating sign and with $|\psi_n(x_i)| = 1$.

Proof. Let $I := [c, d]$, let $I_0 := [c + \delta, d - \delta]$, $\delta > 0$ be a nondegenerate subinterval of I.

Since $I_0 \subset A_w$, there exist $N \geq 1$ and $P \in \mathcal{P}_N$, for which $\mathcal{E}(w^N P) \cap I_0^o$ is nonempty. By **3** of §1 we may assume that $\mathcal{E}(w^N P) = \{x_0\}$. Hence there exists a weight w_1 on A for which $w_1(x) = w(x)$, $x \in I_0$, $w_1(x) > w(x)$, $x \in A \setminus I_0$, and $x_0 \in A_{w_1}$. Since A_{w_1} has no isolated points, $A_{w_1} \cap I_0$ consists of infinitely many points. By Theorem 2.2, (2.4), the Chebyshev polynomials $C_{w_1,n}$ and $\psi_n := w^n C_{w_1,n}$ satisfy all requirements of the lemma. □

We assume in the rest of this section that $A = [a, b]$ is a closed interval and $w(x) > 0$, $x \in (a, b)$. The endpoints of A may be finite or not. The infinite a, b are always zeros of w. Fixing n, we shall show that the ordinary Chebyshev theorem is valid on A for functions $f \in C_0(A_w)$, and $\phi \in \Phi_n$ from the set Φ_n of all $\phi := w^n P_n$. We shall prove: for each $f \in C_0(A_w) \setminus \Phi_n$, there exists a best approximation $\phi_n \in \Phi_n$ and $n + 2$ points $a \leq x_1 < \cdots < x_{n+2} \leq b$ so that for $\sigma = 1$ or $\sigma = -1$

$$f(x_j) - \phi_n(x_j) = \sigma(-1)^j \|f - \phi_n\|_{C[a,b]}, \quad j = 1, \ldots, n + 2. \tag{5.3}$$

If a, b are not zeros of w, then $w(x) \geq C > 0$ on $[a, b]$, and therefore Φ_n is a Haar system.

Let now b be a zero of w (and a not a zero). Let

$$d = \operatorname{dist}(f, \Phi_n) = \inf_{\phi \in \Phi_n} \|f - \phi\|_{C[a,b]}.$$

If Φ_n^0 is a bounded subset of Φ_n, then $|\phi(x)| \le M$ for $\phi \in \Phi_n^0$. This is true also on any interval $[a, b-\delta]$. Since $w(x) \ge C > 0$ there, we have that the polynomials $P_n \in \mathcal{P}_n$ corresponding to the $\phi \in \Phi_n^0$ are uniformly bounded on $[a, b-\delta]$, and therefore their coefficients are uniformly bounded. As $w(x)^n x^k \to 0$ for $x \to b$, $0 \le k \le n$ (by (1.1),(iii) if $b = \infty$) we conclude that $\phi(x) \to 0$ for $x \to b$ uniformly for all $\phi \in \Phi_n^0$. The same is true for $f - \phi$.

We see that the distance d remains unchanged if in its definition we replace Φ_n by a proper Φ_n^0, and $[a, b]$ by an interval $[a, b-\delta]$. On the latter interval, Φ_n is a Haar system, and we have again (5.3). The same argument applies if a, or if a and b are zeros.

We shall extend the sequence $x_1 < \cdots < x_{n+2}$, adding $x_0 = a$ if a is a zero of w (then $x_1 > a$) and $x_{n+3} = b$ if b is one. The extended sequence has $n + 2 + z$ elements, where z is the number of zeros of w among the a, b.

Definition. *The weight w on $A = [a, b]$ has property (E) if, for some $n_0 \ge 1$, no $\phi \in \Phi_n$, $\phi \ne 0$, $n \ge n_0$, can have $n + z$ local extrema in $A^o = (a, b)$.*

A point $x_0 \in (a, b)$ is a *local extremum* of a function $g \in C(A)$ if $g(x) \le g(x_0)$ (or $g(x) \ge g(x_0)$) in some neighborhood of x_0.

Let $G = A_w^o$, $f \in C_0(G) \setminus \Phi_n$, let $\varepsilon > 0$ be fixed, let $\delta = \delta(\varepsilon) > 0$ be selected so that $\omega(f, \delta) < \varepsilon/3$. For a best approximant $\phi_n \in \Phi_n$ to f on A, with $\|f - \phi_n\|_A \ge \varepsilon$, we select an extended alternation sequence (x_j) of $n + 2 + z$ points. We call an interval (x_j, x_{j+1}) *singular*, if it contains a subinterval $I \subset A_w$ of length δ.

Lemma 5.3. *For the best approximant $\phi_n \in \Phi_n$ to $f \in C_0(G) \setminus \Phi_n$, with $\|f - \phi_n\|_A \ge \varepsilon$, let (x_j) be an extended alternation sequence of $n + 2 + z$ points which has no singular intervals. Then the differences $\phi_n(x_{j+1}) - \phi_n(x_j)$ alternate in sign and have absolute values $\ge \varepsilon/3$.*

Proof. For a function $g \in C(A)$ we write $\Delta_j g := g(x_{j+1}) - g(x_j)$.

For some j let $x_{j+1} - x_j \le \delta$. Then $|\Delta_j f| < \varepsilon/3$, and from (5.3), $|\Delta_j \phi_n| \ge \varepsilon/3$. On the other hand, $x_{j+1} - x_j > \delta$, then both x_j and x_{j+1} are at a distance $\le \delta$ from $A \setminus G$. As a consequence, $|f(x_j)| < \varepsilon/3$, $|f(x_{j+1})| < \varepsilon/3$. Thus $|\Delta_j f| < 2\varepsilon/3$, and again $|\Delta_j \phi_n| \ge \varepsilon/3$. In both cases, sign $\Delta_j \phi_n =$ $-$sign $\Delta_{j+1}(f - \phi_n)$. □

In particular, for Φ_n with the property (E), there must be singular intervals in any extended alternation sequence for functions $f \in C_0(A_w^o)$. The following theorem (v. Golitschek, Lorentz, Makovoz [1992]) is a slight improvement (by the notion of minimal essential sets) of one due to v. Golitschek [1990]. For a similar theorem, see Borwein and Saff [1992].

Theorem 5.4. *Let w be a weight on $A = [a, b]$ which satisfies $w(x) > 0$, $a < x < b$, and has the property (E). Then A_w^o is the maximal Weierstrass set.*

Proof. Otherwise, for an infinite sequence $\mathcal{N}$ of the n, $\|f - \phi_n\|_A \geq \varepsilon > 0$, for the best approximants $\phi_n \in \Phi_n$ to some $f \in C_0(G)$, $G := A_w^o$. For each $n \in \mathcal{N}$, there exist singular intervals for ϕ_n, f; since each of them has length $\geq \delta$, there are at most $m := (b-a)/\delta$ of them. Since G is bounded, by replacing $\mathcal{N}$ by a subsequence, we may assume that there exists an interval $I \subset G$, $|I| \geq \delta/2$, which for each $n \in \mathcal{N}$ is contained in a singular interval.

There are $\ell \leq m+1$ closed intervals J of A, complementary to singular intervals. They contain all the points x_j, $j = 1, \ldots, n+2$. Let p_i be the number of these x_j in J_i, then $\sum_1^\ell p_i = n+2$. In each J_i, the x_j produce at least $p_i - 2$ alternating local maxima and minima of ϕ_n.

Let $\psi_n \in \Phi_n$ be the function of Lemma 5.2 for the interval I (with δ replaced by $\delta/2$), which has $M_n(I)$ alternating extrema on I. The function $\phi_n^* := \phi_n + \lambda_n \psi_n$, $\lambda_n := 1 + \|\phi_n\|_A$, will have on A, for all large $n \in \mathcal{N}$, a sequence of at least

$$\sum_{i=1}^{\ell}(p_i - 2) + M_n(I) - 2 \geq (n+2) - 2(m+1) + M_n(I) - 2 \geq n + z$$

alternating local extrema in (a, b), and this contradicts (E). □

Corollary 5.5. *The last theorem applies to the incomplete polynomials of Example 1 in §1 (for $z = 1$), to the Jacobi weights (4.14) on $A = [-1, 1]$ (for $z = 2$), to the Freud weights $w(x) = e^{-x^2}$ on $\mathbb{R}$ (for $z = 2$), to the Freud weights $w(x) = e^{-x}$ on $[0, \infty)$ (for $z = 1$).*

Indeed, all these weights have property (E) since the first derivative ϕ' of any nonzero weighted polynomial $\phi = w^n P_n \in \Phi_n$ has at most $n + z - 1$ distinct zeros in A^o. □

§ 6. Weierstrass Theorem for Freud Weights

We have seen in §4 that the minimal essential set for the Freud weight $w_\sigma(x) = e^{-|x|^\sigma}$ on $\mathbb{R}$ is the interval $[-\beta_\sigma, \beta_\sigma]$, $\sigma > 0$. The value of β_σ is determined by (4.21). We shall prove in Theorem 6.3 that $(-\beta_\sigma, 0) \cup (0, \beta_\sigma)$ is a Weierstrass set for w_σ, for all $0 < \sigma < \infty$, and that $(-\beta_\sigma, \beta_\sigma)$ is the maximal Weierstrass set for w_σ if $1 < \sigma < \infty$. The remaining cases have been settled by Lubinsky and Totik [1994]: the maximal Weierstrass set for w_1 is $(-\beta_1, \beta_1) = (-\frac{\pi}{2}, \frac{\pi}{2})$, and is $(-\beta_\sigma, 0) \cup (0, \beta_\sigma)$ if $0 < \sigma < 1$. The latter are examples of weights for which the maximal Weierstrass set and the interior of the minimal essential set are not equal.

We shall state and prove our results first for $A = [0, \infty)$ and the weight $w_\sigma(x)$. According to Example 6 of §4, the minimal essential set is then $[0, \beta_\sigma^*]$, $\beta_\sigma^* = 2^{1/\sigma}\beta_{2\sigma}^2$.

Lemma 6.1. *For each $\sigma > 1/2$ there exists $x_0 > 0$, $x_0 < \beta_\sigma^*$, with the following property: If $f \in C([0, \infty))$, vanishes outside of $[0, x_0)$, then*

$$\lim_{n\to\infty} \min_{P_n \in \mathcal{P}_n} \|f - w_\sigma^n P_n\|_{[0,\infty)} = 0. \tag{6.1}$$

Proof. By **2** of §1 there exists some c $> \beta_\sigma^*$ so that for each $P_n \in \mathcal{P}_n$, $n \geq 1$,

$$w_\sigma(x)^n |P_n(x)| \leq 2^{-n}\|w_\sigma^n P_n\|_{[0,\mathrm{c}]}, \quad x \in (c, \infty).$$

Therefore, since $f(x) = 0$, $x \geq \beta_\sigma^*$, it suffices to prove the convergence (6.1) for the uniform norm on $[0, c]$.

According to [**CA**, Theorem 5.5, p. 347] one can approximate the monomials $x^{j\sigma}$, $j = 1, 2, \ldots$, on $[0, 1]$ by polynomials $P_{n,j} \in \mathcal{P}_n$ such that

$$\|x^{j\sigma} - P_{n,j}(x)\|_{[0,1]} \leq \prod_{k=1}^{n} \frac{|k - j\sigma|}{k + j\sigma}. \tag{6.2}$$

All factors in (6.2) are less than 1, even less than $e^{-j\sigma/k}$ for $k > j\sigma$. Hence

$$\|x^{j\sigma} - P_{n,j}(x)\|_{[0,1]} \leq \prod_{j\sigma < k \leq n} e^{-j\sigma/k} \leq e\left(\frac{j\sigma}{n}\right)^{j\sigma}, \quad \text{for } n > j\sigma. \tag{6.3}$$

To prove the second inequality of (6.3) we take the logarithm on both sides and use

$$\log\frac{n}{j\sigma} = \int_{j\sigma}^{n} \frac{dx}{x} \leq 1/(j\sigma) + \sum_{j\sigma<k\leq n} 1/k.$$

For the polynomials $Q_{n,j}(x) := \mathrm{c}^{j\sigma}P_{n,j}(x/\mathrm{c})$ of degree n, (6.3) yields

$$\left|x^{j\sigma} - Q_{n,j}(x)\right| \leq e\left(\frac{\mathrm{c}j\sigma}{n}\right)^{j\sigma}, \quad 0 \leq x \leq c. \tag{6.4}$$

Let $K_\sigma := 3e\sigma^\sigma \mathrm{c}^\sigma$. We choose $\rho > 0$ so small that

$$\delta_n := e \sum_{1\leq j\leq \rho n} K_\sigma^j \left(\frac{j}{n}\right)^{(\sigma-1)j}$$

converges to zero as $n \to \infty$. We shall prove the lemma for $x_0 := (\rho/(6e))^{1/\sigma}$. For fixed n let $N = N(n)$ be the largest integer $\leq \rho n$. Let S_N be the N-th partial sum of $w_\sigma(x)^{-3n} = e^{3nx^\sigma}$, that is,

$$S_N(x) := \sum_{j=0}^{N} \frac{(3n)^j}{j!}\, x^{j\sigma}.$$

The function $F_n := w_\sigma^{3n} S_N$ satisfies $0 < F_n(x) \le 1$ for all $x \in [0, \infty)$. Using Stirling's formula $j! \ge j^j e^{-j}$ we obtain for $0 \le x \le x_0$ and for $j > N$ that

$$\frac{(3n)^j}{j!} x^{j\sigma} \le \left(\frac{3enx_0^\sigma}{N+1}\right)^j \le 2^{-j}.$$

Hence, $F_n(x) \to 1$ uniformly on $[0, x_0]$. Let

$$R_{2n} := 1 + \sum_{j=1}^{N} \frac{(3n)^j}{j!} Q_{n,j}.$$

Clearly, $R_{2n} \in \mathcal{P}_{2n}$ and by (6.4), on $0 \le x \le c$,

$$|S_N(x) - R_{2n}(x)| \le e \sum_{j=1}^{N} \frac{(3n)^j}{j!} \left(\frac{cj\sigma}{n}\right)^{j\sigma} \le e \sum_{j=1}^{N} K_\sigma^j \left(\frac{j}{n}\right)^{(\sigma-1)j} = \delta_n.$$

This yields $\|F_n - w_\sigma^{3n} R_{2n}\|_{[0,c]} \le \delta_n$. As a final step we take a sequence $L_n \in \mathcal{P}_n$ which converges to f uniformly on $[0, c]$. The properties of F_n and f imply that $\|f - L_n F_n\|_{[0,c]} \to 0$. Then $P_{3n} := L_n R_{2n}$ is a polynomial of degree $\le 3n$ and

$$\begin{aligned}\|f - w_\sigma^{3n} P_{3n}\|_{[0,c]} &\le \|f - L_n F_n\|_{[0,c]} + \|L_n\|_{[0,c]} \|F_n - w_\sigma^{3n} R_{2n}\|_{[0,c]} \\ &\le \|f - L_n F_n\|_{[0,c]} + \|L_n\|_{[0,c]} \delta_n \to 0\end{aligned}$$

as $n \to \infty$. This proves the lemma for the degrees $3n$. For the degrees $3n + k$, $k = 1, 2$, we first approximate f/w_σ^k by weighted polynomials $\phi_{3n} = w_\sigma^{3n} P_{3n}$. Then the weighted polynomials $\phi_{3n+k} := w_\sigma^k \phi_{3n}$ converge to f. □

Theorem 6.2 (Lubinsky and Saff [1988]). *Let $f \in C[0, \infty)$ vanish on $[\beta_\sigma^*, \infty)$ and let $f(0) = 0$ if $0 < \sigma \le 1/2$. Then f has the property* (6.1).

Proof. We shall give a complete proof for $\sigma \ge 1/3$ and sketch the proof for $0 < \sigma < 1/3$. We use here a paper of v. Golitschek [1991_2] who simplified the original proof of Lubinsky and Saff.

It is enough to consider functions f which satisfy $f(0) = 0$. If $f(0) \ne 0$ and $\sigma > 1/2$, then we subtract from f a function f_0 of Lemma 6.1, with $f_0(0) = f(0)$, and approximate f_0 and $f - f_0$ separately by the weighted polynomials.

By $\mathbb{C}_\sigma$, we mean the Riemann surface of the analytic function $q(z) := z^\sigma$ which satisfies $q(x) = x^\sigma > 0$ for $x > 0$. The function $w(z) := \exp(-q(z))$ is the analytic continuation of w_σ onto $\mathbb{C}_\sigma$, that is, for $z = re^{it}$, $r \ge 0$, $t \in \mathbb{R}$, we have

$$w(z) = w(re^{it}) = \exp(-r^\sigma e^{i\sigma t}). \tag{6.5}$$

By Proposition 1.3, the minimal essential set for $w := w_\sigma$ on $[0, \infty)$ is $[0, b]$, where $b := \beta_\sigma^*$. We denote the Chebyshev polynomials for w on $[0, \infty)$

by $C_{w,n}$. Let $(\xi_j^{(n)})_{j=0}^n$ be the alternation sequence of $C_{w,n}$ (see Theorem 2.1). Then the zeros $(x_j^{(n)})_{j=1}^n$ of $C_{w,n}$ satisfy

$$0 \le \xi_0^{(n)} < x_1^{(n)} < \xi_1^{(n)} < \cdots < x_n^{(n)} < \xi_n^{(n)} \le b. \tag{6.6}$$

We have even

$$\xi_0^{(n)} = 0 \ , \quad \lim_{n\to\infty} \xi_n^{(n)} = b. \tag{6.7}$$

Indeed, the second relation of (6.7) follows from (2.6). To prove the first part of (6.7), assume that $\xi_0^{(n)} > 0$. Then $|C_{w,n}|$ and w are strictly decreasing on $[0, \xi_0^{(n)}]$, hence

$$w(0)^n |C_{w,n}(0)| > w(\xi_0^{(n)})^n |C_{w,n}(\xi_0^{(n)})| = 1,$$

a contradiction.

Let $\varepsilon > 0$ be fixed. Since f vanishes for $x = 0$ and for $x \ge b$, there exists an entire function g for which

$$\|f(x) - x(x-b)g(x)\|_{[0,b]} < \varepsilon/2.$$

Since $\xi_n^{(n)} \to b$, the entire functions $f_n(z) := z(z - \xi_n^{(n)})g(z)$ converge uniformly on $[0, b]$ to $z(z-b)g(z)$, hence, for large n,

$$\|f - f_n\|_{[0,b]} < \varepsilon. \tag{6.8}$$

From now on we assume that $\sigma \ge 1/3$. We define

$$\delta := \delta(\sigma) := \begin{cases} \pi/(2\sigma), & \text{if } 1/3 \le \sigma \le 1 \\ \pi/2, & \text{if } 1 < \sigma \le 3 \\ (2k+1)\pi/(2\sigma), & \text{if } 2k-1 < \sigma \le 2k+1, \quad k \ge 2, \end{cases}$$

and construct the closed paths $\Gamma_n := \Gamma_n(\sigma) := \Gamma_n^1 \cup \Gamma_n^2 \cup \Gamma_n^3$ in $\mathbb{C}_\sigma$ as follows.

$$\begin{aligned} \Gamma_n^1 &:= \{\eta = \xi_n^{(n)} e^{it} : \ -\delta \le t \le \delta\}, \\ \Gamma_n^2 &:= \{\eta = re^{i\delta} : \ \xi_n^{(n)} \ge r \ge 0\}, \\ \Gamma_n^3 &:= \{\eta = re^{-i\delta} : \ 0 \le r \le \xi_n^{(n)}\}. \end{aligned}$$

It will be crucial that

$$\cos(\delta\sigma) \le 0, \quad \frac{\pi}{2} \le \delta \le \frac{3\pi}{2}, \quad \cos\delta \le 0 \tag{6.9}$$

and that the interior of Γ_n intersects the positive x-axis in the interval $(0, \xi_n^{(n)})$.

We show in our next step that the Chebyshev polynomials $C_{w,n}$ satisfy

$$M_n := \int_{\Gamma_n} \frac{|d\eta|}{|w(\eta)^n C_{w,n}(\eta)|} \to 0 \ , \quad n \to \infty. \tag{6.10}$$

To prove this, we consider first the points $\eta \in \Gamma_n^2 \cup \Gamma_n^3$, that is,

$$\eta = re^{\pm i\delta}, \ \ 0 \le r \le \xi_n^{(n)}.$$

From $w(0) = |C_{w,n}(0)| = 1$ and $\cos\delta \le 0$ it follows that

$$\frac{1}{|C_{w,n}(\eta)|} = \frac{|C_{w,n}(0)|}{|C_{w,n}(re^{\pm i\delta})|} \le \prod_{k=1}^{n} \frac{x_k^{(n)}}{|ir + x_k^{(n)}|} \le \left(1 + (r/b)^2\right)^{-n/2}.$$

The first inequality (6.9) implies

$$|w(\eta)|^{-1} = |w(re^{\pm i\delta})|^{-1} = \exp\left(r^\sigma \cos(\delta\sigma)\right) \le 1.$$

This yields

$$\int_{\Gamma_n^2 \cup \Gamma_n^3} \frac{|d\eta|}{|w(\eta)^n C_{w,n}(\eta)|} \le 2\int_0^b \left(1 + (r/b)^2\right)^{-n/2} dr \to 0$$

as $n \to \infty$. For $\eta \in \Gamma_n^1$, that is, $\eta = \xi_n^{(n)} e^{it}, \ \ -\delta \le t \le \delta$, we have

$$\frac{|C_{w,n}(\xi_n^{(n)})|}{|C_{w,n}(\eta)|} = \prod_{k=1}^{n} \frac{|\xi_n^{(n)} - x_k^{(n)}|}{|\xi_n^{(n)} e^{it} - x_k^{(n)}|} \le 1.$$

Hence we get

$$\begin{aligned}
\int_{\Gamma_n^1} \frac{|d\eta|}{|w(\eta)^n C_{w,n}(\eta)|} &= \int_{\Gamma_n^1} \frac{|w(\xi_n^{(n)})^n C_{w,n}(\xi_n^{(n)})||d\eta|}{|w(\eta)^n C_{w,n}(\eta)|} \\
&\le \int_{\Gamma_n^1} \frac{|w(\xi_n^{(n)})|^n |d\eta|}{|w(\eta)|^n} = \xi_n^{(n)} \int_{-\delta}^{\delta} |\exp\{n(\xi_n^{(n)})^\sigma (e^{i\sigma t} - 1)\}| dt \\
&= \xi_n^{(n)} \int_{-\delta}^{\delta} \exp\{-2n(\xi_n^{(n)})^\sigma \sin^2(\sigma t/2)\} dt.
\end{aligned}$$

The last expression tends to zero as $n \to \infty$ since $\xi_n^{(n)} \to b$. This completes the proof of (6.10).

Next we define the contour integrals

$$\Delta_n(x) := \frac{1}{2\pi i} \int_{\Gamma_n} \frac{f_n(\eta) C_{w,n}(x) d\eta}{(x - \eta) w(\eta)^n C_{w,n}(\eta)}, \quad 0 \le x < \infty. \tag{6.11}$$

Since f_n has zeros at the intersection points $\eta = 0$ and $\eta = \xi_n^{(n)}$ of Γ_n with the x-axis, Δ_n is a continuous function on $0 \le x < \infty$. In addition,

$$\lim_{n\to\infty} \|w^n \Delta_n\|_{[0,\infty)} = 0. \tag{6.12}$$

To prove (6.12), let $D_0 := \{\eta = re^{it} : 0 < r \le b, \ \pi/2 \le t \le 3\pi/2\}$ be the half disk with center 0 and radius b, let $K_n : |\eta| = \xi_n^{(n)}$ be the circle with center 0 and radius $\xi_n^{(n)}$. Since $|\eta| \leq |\eta - x|$ for all $\eta \in D_0$, $x \geq 0$, and since $|\eta - \xi_n^{(n)}| \leq 2|\eta - x|$ for all $\eta \in K_n$, $x \geq 0$, the supremum

$$c_0 := \sup_{\eta,x,n} \left\{ \frac{|f_n(\eta)|}{|\eta - x|} : \quad \eta \in D_0 \cup K_n, \ x > 0, \ x \neq \xi_n^{(n)}, \ n \geq 1 \right\}$$

is finite: $c_0 \leq 2b \max\{|g(\eta)| : |\eta| \leq b\}$. Therefore, $|f_n(\eta)|/|\eta - x| \leq c_0$ for all $x > 0$, $x \neq \xi_n^{(n)}$, all $\eta \in \Gamma_n$. From this we get for $x > 0$, $x \neq \xi_n^{(n)}$,

$$|\Delta_n(x)| \leq \frac{1}{2\pi} \int_{\Gamma_n} \frac{|f_n(\eta) C_{w,n}(x)||d\eta|}{|(x-\eta) w(\eta)^n C_{w,n}(\eta)|} \leq \frac{c_0}{2\pi} \int_{\Gamma_n} \frac{|C_{w,n}(x)||d\eta|}{|w(\eta)^n C_{w,n}(\eta)|}$$

and since $w(x)^n |C_{w,n}(x)| \leq 1$,

$$|w(x)^n \Delta_n(x)| \leq \frac{c_0}{2\pi} \int_{\Gamma_n} \frac{|d\eta|}{|w(\eta)^n C_{w,n}(\eta)|}.$$

It is simple to see that this inequality is also true for $x = \xi_n^{(n)}$. The last integral is equal to M_n and converges to zero by (6.10). This yields (6.12).

Next we show that the functions Δ_n are of the form

$$\Delta_n(x) = \begin{cases} R_n(x) - w(x)^{-n} f_n(x), & \text{if } 0 \leq x \leq \xi_n^{(n)} \\ R_n(x), & \text{if } x > \xi_n^{(n)}, \end{cases} \tag{6.13}$$

where R_n is the algebraic polynomial of degree $\leq n-1$ which interpolates $w^{-n} f_n$ at the zeros $x_k^{(n)}$, $k = 1, \ldots, n$, of $C_{w,n}$.

Indeed, for $x > 0$, $x \neq x_k^{(n)}$, $k = 1, \ldots, n$, the integrand in (6.11), as a function of $\eta \in \mathbb{C}_\sigma$, has a pole at x with the residue $-w(x)^{-n} f_n(x)$ and has poles at the points $x_k^{(n)}$, $k = 1, \ldots, n$, with the residues

$$w(x_k^{(n)})^{-n} f_n(x_k^{(n)}) l_k(x), \quad l_k(x) := \prod_{\ell=1, \ell \neq k}^{n} \frac{x - x_\ell^{(n)}}{x_k^{(n)} - x_\ell^{(n)}}. \tag{6.14}$$

The sum of the residues (6.14),

$$R_n(x) := \sum_{k=1}^{n} w(x_k^{(n)})^{-n} f_n(x_k^{(n)}) l_k(x),$$

is the Lagrange interpolation polynomial of $w^{-n} f_n$ at the interpolation points $x_\ell^{(n)}$, $\ell = 1, \ldots, n$. If $0 < x < x_n^{(n)}$, all of the above poles lie inside of Γ_n. If $x > x_n^{(n)}$, the poles $x_k^{(n)}$, $k = 1, \ldots, n$, are inside Γ_n and x is outside Γ_n. Hence, by the residue theorem, we obtain (6.13). By continuity, (6.13) is also valid at $x = 0$ and $x = \xi_n^{(n)}$.

Multiplying (6.13) by $w(x)^n$ and using the fact that $f(x) = 0$ for $x \geq b$, (6.13) and the convergence in (6.12) yield

$$\limsup_{n \to \infty} \|f - w^n R_n\|_{[0,\infty)} \leq \varepsilon. \tag{6.15}$$

This completes the proof of Theorem 6.2 for $\sigma \geq 1/3$ since $\varepsilon > 0$ is arbitrary.

Let $0 < \sigma < 1/3$. There exist a positive integer m, $m < 1/\sigma$, and $\delta := (2m+1)\pi$, for which $\cos(\delta\sigma) \le 0$, and of course, $\cos\delta = -1$. The sequence M_n defined by (6.10) converges to 0; the contour integrals (6.11) satisfy (6.12) and, instead of (6.13), are of the form

$$\Delta_n(x) = \begin{cases} R_n(x) - G_n(x), & \text{if } 0 \le x \le \xi_n^{(n)} \\ R_n(x), & \text{if } x > \xi_n^{(n)}, \end{cases} \tag{6.16}$$

where

$$G_n(x) := f_n(x) \sum_{j=-m}^{m} \exp\left(nx^\sigma e^{2\pi i j\sigma}\right), \quad x \ge 0,$$

and where R_n is the algebraic polynomial of degree $\le n-1$ which interpolates G_n at the zeros $x_k^{(n)}$, $k = 1, \dots, n$, of $C_{w,n}$. It remains to prove that

$$\lim_{n\to\infty} \|f_n - w^n G_n\|_{[0,b]} = 0. \tag{6.17}$$

This follows from the properties of $f_n(x)$ on $[0, b]$ and from

$$|(f_n - w^n G_n)(x)| \le 2|f_n(x)| \sum_{j=1}^{m} |\exp\left(nx^\sigma \cos 2\pi j\sigma\right)|, \quad 0 \le x \le b.$$

Now we get (6.1) similarly as for $\sigma \ge 1/3$. □

Theorem 6.3 (Lubinsky and Saff [1988]). *Let $f \in C(\mathbb{R})$ be zero outside of $(-\beta_\sigma, \beta_\sigma)$ and let $f(0) = 0$ if $0 < \sigma \le 1$. Then,*

$$\lim_{n\to\infty} \min_{P_n \in \mathcal{P}_n} \|f - w_\sigma^n P_n\|_{\mathbb{R}} = 0. \tag{6.18}$$

Proof. If f is assumed to be even, f is approximable on $\mathbb{R}$ by the $w_\sigma^n P_n$ since, by Theorem 6.2, $F(x) := f(2^{-1/\sigma}\sqrt{x})$, $0 \le x < \infty$, is approximable on $[0, \infty)$ by the $w_{\sigma/2}^n P_n$.

If $f \in C(\mathbb{R})$ is odd and vanishes outside of $(-\beta_\sigma, \beta_\sigma)$, and since we may assume that $f \in C^1(\mathbb{R})$, the function $\tilde{f}(x) := f(x)/(x w_\sigma(x))$ is even and in $C(\mathbb{R})$. Therefore, $\tilde{f}$ is approximable by the $w_\sigma^n P_n$, then f is also approximable. An arbitrary f with the properties of Theorem 6.3 is approximable since its even and odd parts are approximable. □

Remark 1. Lubinsky and Totik [1994] prove that for $0 < \sigma < 1$, the origin 0 does not belong to any Weierstrass set of w_σ.

Remark 2. Weierstrass theorems for arbitrary weights have been given by Totik [A-1994]. For instance, if A is the finite union of compact disjoint intervals, if $w \in C^{1+\varepsilon}$, $\varepsilon > 0$ and if $q(x)$ is convex in each of them, then the interior of S_w^μ (or of A_w, by Corollary 3.9) is the maximal Weierstrass set for w on A. See also Note **8.2**.

§ 7. Problems

7.1. Prove that the weights w and Cw (with a constant $C > 0$) produce in Theorem 2.5 and Theorem 3.1 the same measures ν_w, μ_w and the same functions L_w, Λ_w.

7.2. If $w \in C(A)$ is positive on some subinterval $I \subset A$, then

$$e^{a_w} = \lim_{n\to\infty} e_{w,n}^{1/n} \geq \frac{|I|}{4} \min_{x\in I} w(x).$$

7.3. Let $A = [a, b]$ and $[\alpha, \beta] := \mathrm{co}(A_w)$. Using the formula

$$\frac{d^2}{dx^2} L_w(x) = -\int_\alpha^\beta \frac{1}{(x-t)^2} d\nu_w(t)$$

prove that the conclusion of Theorem 2.9 is still valid if $q = -\log w \in C^2(\alpha, \beta)$ and $q''(x) > -1/(\beta - \alpha)^2$ on (α, β).

7.4. Let $w = 1$ be the unit weight on $A = [-1, 1]$. Show that the limit in (2.10) is $a_w = -\log 2$ and that the Chebyshev measure ν_w has the density

$$\frac{d\nu_w}{dx}(x) = \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1.$$

7.5. Prove that w_σ, $\sigma = 2, 4, 6, \ldots$, have property (E).

§ 8. Notes

8.1. Formulas for Some Equilibrium Measures. The equilibrium measures μ_w for the Jacobi weights (4.14) have been found by Saff, Ullman, Varga [1980]: If α and β are the numbers given in (4.15), then

$$\frac{d\mu_w(x)}{dx} = \frac{(s_1 + s_2 + 1)}{\pi} \frac{\sqrt{(x-\alpha)(\beta - x)}}{1 - x^2}, \quad \alpha \leq x \leq \beta. \tag{8.1}$$

As a corollary, for the incomplete polynomials of Example 1 in §1,

$$\frac{d\mu_w(x)}{dx} = \frac{1}{(1-\theta)\pi x} \sqrt{\frac{x - \theta^2}{1 - x}}, \quad \theta^2 \leq x \leq 1. \tag{8.2}$$

The density of the equilibrium measure for the Freud weight $w = w_\sigma$ on $\mathbb{R}$, $\sigma > 0$, is

$$\frac{d\mu_w(x)}{dx} = \frac{1}{\beta_\sigma} v(\sigma, x/\beta_\sigma) \tag{8.3}$$

(Mhaskar and Saff [1984,1985]), where β_σ is given by (4.21) and where

$$v(\sigma,t) := \frac{\sigma}{\pi} \int_{|t|}^{1} \frac{u^{\sigma-1}}{\sqrt{u^2-t^2}} du, \quad -1 \le t \le 1, \tag{8.4}$$

$v(\sigma, t) := 0$ elsewhere, is the *Ullman distribution.*

8.2. V. Totik [A-1994] proved the following general Weierstrass theorems for the weighted polynomials $w^n P_n$ on A:

(i) Let S^w denote the set of all points x_0 which possess a neighborhood $\mathcal{U}(x_0)$ where the density of the equilibrium measure, $v_w(x) := d\mu_w(x)/dx$ exists and is positive and continuous. Then S^w is a Weierstrass set.

(ii) Let $w \in C^{1+\varepsilon}(A)$ for some $\varepsilon > 0$. Then the union of the interiors of the supports of the weights w^λ, $\lambda > 1$, is a Weierstrass set for w.

8.3. Orthogonal Polynomials on $\mathbb{R}$. In the late 1960's, G.Freud began to study the properties of the polynomials $P_n(w^2, x)$ of degree n, $n = 0, 1, \ldots$, which satisfy

$$\int_{-\infty}^{\infty} w(x)^2 P_n(w^2, x) P_m(w^2, x) dx = \delta_{n,m}, \quad n, m = 0, 1, \ldots . \tag{8.5}$$

The results of this chapter have important applications in this theory. Typical examples of weights are the Freud weights, $w_\sigma(x)$, and

$$w_{\sigma,\beta}(x) := |x|^\beta \exp(-|x|^\sigma), \quad \sigma > 0, \beta > -1/2.$$

One of Freud's problems was the description of the distribution of zeros of the polynomials $P_n(w_\sigma^2)$. This has been answered independently by Mhaskar and Saff [1984] and Rakhmanov [1984]. They obtained (2.15) with the measure $\nu_w = \mu_w$ given by (8.3). In the second paper the asymptotic value of the largest zero $x_n(w_\sigma)$ of $P_n(w_\sigma^2)$ has been found, namely

$$\lim_{n\to\infty} n^{-1/\sigma} x_n(w_\sigma) = \beta_\sigma. \tag{8.6}$$

Several authors have generalized the conditions on the weight w for which this asymptotic formula is valid. This includes all weights $w_{\sigma,\beta}$. See Lubinsky and Saff [A-1988].

8.4. Freud's Conjecture. It concerns the coefficients $(A_k)_0^\infty$ of the recurrence formula

$$xP_n(w_\sigma^2, x) = A_{n+1} P_{n+1}(w_\sigma^2, x) + A_n P_{n-1}(w_\sigma^2, x), \quad n = 1, 2, \cdots . \tag{8.7}$$

Freud conjectured that

$$\lim_{n\to\infty} n^{-1/\sigma} A_n = \beta_\sigma/2 \tag{8.8}$$

is valid for all σ and proved it for $\sigma = 2, 4, 6$. A. Magnus [1986] verified it for all even positive integers. The conjecture has been completely proved in the three papers by Knopfmacher, Lubinsky and Nevai [1988], Lubinsky and Saff [1988] and Lubinsky, Mhaskar and Saff [1988]. Particularly important is the second paper. They established (8.8) for all weights $w_{\sigma,\beta}$, and also for some more general weights.

Relation (8.8) is "quotient asymptotics" for the the leading coefficients $c_n := c_n(\sigma)$ of the $P_n(w_\sigma^2, x)$, since (8.7) implies that $A_{n+1} = c_n/c_{n+1}$. Later, Lubinsky and Saff [A-1988] obtained a stronger result

$$\lim_{n\to\infty} c_n(\sigma)\beta_\sigma^{n+1/2}\sqrt{\pi}2^{-n}e^{-n/\sigma}n^{(n+1/2)/\sigma} = 1 \tag{8.9}$$

("strong asymptotics of the c_n"). For a relatively simple proof see Totik [A-1994].

Chapter 5. Wavelets and Orthogonal Expansions

§ 1. Multiresolutions and Wavelets

From a certain point of view, the wavelets are a chapter of the theory of the orthogonal expansions (or orthogonal series, see the book of this name by Kashin and Sahakian [B-1984]). For other points of view, see Note 6.3. A sequence of functions $(\phi_n)_0^\infty$ in the real Hilbert space $L_2(\mathbb{R})$ is *orthonormal* if $\langle \phi_n, \phi_m \rangle = \int_{\mathbb{R}} \phi_n \phi_m \, dx = \delta_{n,m}$, $n, m = 0, 1, \ldots$. An orthonormal sequence is an *orthonormal basis* (o.n.b.) for a subspace $X \subset L_2(\mathbb{R})$ if X is the closure of the span of the ϕ_n, or, equivalently, if the orthonormal expansion $\sum_{n=0}^{\infty} c_n \phi_n$, $c_n = \langle f, \phi_n \rangle$ of each $f \in L_2$ converges to some $g \in X$ in the L_2-norm $\| \cdot \|$. Then g is the orthogonal projection of f onto X. If $X = L_2(\mathbb{R})$, then (ϕ_n) is a *complete orthonormal basis* (c.o.n.b.).

We shall need some basic concepts and facts concerning Hilbert spaces. We shall need, for instance, the orthogonal sum of spaces (which is the closure of the sum if there are infinitely many of them), and the Fourier transform, which for $f \in L_1(\mathbb{R})$ is given by

$$\hat{f}(y) = \int_{\mathbb{R}} f(x) e^{-iyx} \, dx \, , \qquad y \in \mathbb{R} \, , \tag{1.1}$$

and is extended by continuity from $L_2 \cap L_1$ to L_2, also its properties such as the Plancherel formula; see Helson [B-1983], Zygmund [B-1959].

With a function f, all of its dilates-translates

$$f_{k,\ell}(x) := 2^{k/2} f(2^k x - \ell) \, , \qquad k, \ell \in \mathbb{Z} \tag{1.2}$$

belong to L_2. One has $\|f_{k,\ell}\| = \|f\|$, moreover

$$\hat{f}_{k,\ell}(y) := \widehat{(f_{k,\ell})}(y) = 2^{-k/2} \hat{f}(2^{-k} y) e^{-i\ell 2^{-k} y} \, . \tag{1.3}$$

An o.n.b. for $L_2(\mathbb{R})$ is called a *wavelet basis* if it consists of dilation-translations of one (equivalently, of each) of its elements ψ, and ψ is called a *wavelet*. Wavelet theory was initiated by Y. Meyer around 1986 (see [B-1990]); see the books of Chui [B-1992], Daubechies [B-1992].

One way to construct wavelets is to start with a *multiresolution*. Let $\phi \in L_2(\mathbb{R})$ be a real function with $\|\phi\| = 1$. We assume that for $k = 0$ and therefore for each $k \in \mathbb{Z}$ the functions $(\phi_{k,\ell})$, $\ell \in \mathbb{Z}$ are orthonormal and denote by V_k

the closed span of this sequence in $L_2(\mathbb{R})$. Thus, the translates of ϕ produce, in this sense, V_0. We further assume that

$$(1.4) \qquad \begin{cases} \text{(a)} & \cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \\ \text{(b)} & \overline{\cup_{k\in\mathbb{Z}} V_k} = L_2(\mathbb{R}) , \\ \text{(c)} & \cap_{k\in\mathbb{Z}} V_k = \{0\} \end{cases}$$

Then the sequence of spaces $(V_k)_{k\in\mathbb{Z}}$ is called a *multiresolution* and ϕ is called its *scaling function.*

Multiresolutions exist. The simplest choice of a scaling function is the characteristic function $\phi(x) := \chi_{[0,1]}(x)$, $x \in \mathbb{R}$; properties (1.4) can be easily checked.

For a multiresolution, we introduce spaces W_k, $k \in \mathbb{Z}$ to be the orthogonal complements, defined by $W_k := V_{k+1} \ominus V_k$, or, what is the same, by $V_{k+1} = V_k \oplus W_k$. Then

$$(1.5) \qquad V_{k+1} = V_\ell \oplus W_\ell \oplus \cdots \oplus W_k , \qquad \ell \le k , \ \ell, k \in \mathbb{Z} .$$

It follows that

$$V_{k+1} = \bigoplus_{\ell\le k} W_\ell , \qquad k \in \mathbb{Z} .$$

Indeed, since V_{k+1} is closed, from (1.5) we have $U := \overline{\bigoplus_{\ell\le k} W_\ell} \subset V_{k+1}$. Conversely, let $g \in V_{k+1}$. We can write $g = g_1 + g_2$, $g_1, g_2 \in V_{k+1}$ with $g_1 \in U$, $g_2 \perp U$. Then $g_2 \perp W_\ell$ for all $\ell \le k$. From (1.5) we derive $g_2 \in V_\ell$, $\ell \le k$, and then from (1.4c) that $g_2 = 0$. Hence $g \in U$ and $V_{k+1} \subset U$.

Using a similar argument, from (1.4b) we now obtain for any $k \in \mathbb{Z}$

$$(1.6) \qquad L_2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j = V_k \bigoplus_{j\ge k} W_j .$$

This is the standard definition of a multiresolution. In some constructions with wavelets only "upper" multiresolutions $(V_k)_{k=0}^{\infty}$ shall be used (see, for example, §§3,5). Note that under the assumptions (1.4) the linear subspace $Y := \overline{\bigcup_{k\ge p} V_k}$ of $L_2(\mathbb{R})$ does not depend on p. We can therefore work with V_p and the W_j, $j \ge p$. Important is also another modification that will be adopted in this section: we shall omit the assumption (1.4b). Then instead of (1.6) we shall have

$$(1.7) \qquad Y = V_0 \bigoplus_{k\ge 0} W_k = \bigoplus_{k\in Z} W_k .$$

We call a multiresolution *complete* if $Y = L_2(\mathbb{R})$. We shall not assume the completeness in §1; later, in §§2–3 it will be seen that it can be derived from some properties of ϕ. Accordingly, we shall call ψ a wavelet if the $(\psi_{k,\ell})$ form an o.n.b. for Y.

We have:

1. The space V_0 consists of all functions $f \in L_2(\mathbb{R})$ for which there exists a sequence of (real) numbers $(c_\ell) \subset \ell_2(\mathbb{Z})$ so that

$$f(x) = \sum_{\ell \in \mathbb{Z}} c_\ell \phi(x - \ell) = \sum c_\ell \phi_{0,\ell}(x) \ , \qquad \sum c_\ell^2 = \|f\|^2 \ .$$

2. Relations for functions in $L_2(\mathbb{R})$ are often equivalent to relations for their Fourier transforms. For example, if $f, g \in L_2(\mathbb{R})$, $(c_\ell) \in \ell_2(\mathbb{Z})$, $k \in \mathbb{Z}$, then

$$f(x) = 2^{k/2} \sum_{\ell \in \mathbb{Z}} c_\ell g(2^k x - \ell) = \sum c_\ell g_{k,\ell}(x) \tag{1.8}$$

is equivalent to

$$\hat{f}(y) = C(y/2^k)\hat{g}(y/2^k) \ , \qquad y \in \mathbb{R} \ ,$$

where $C(y) := 2^{-k/2} \sum c_\ell e^{-i\ell y}$ is a 2π-periodic function on $\mathbb{R}$.

3. The change of variables $x \to 2^p x$ produces from a function $f(x)$ in V_k (or W_k) the function $f(2^p x)$ in V_{k+p} (or W_{k+p}), $k, p \in \mathbb{Z}$.

4. If for a finite set of functions $\{g_j\}$, their translates $\{(g_j)_{0,\ell}\}$ are an o.n.b. for V_k (or W_k), then $\{(g_j)_{s,\ell}\}$ are an o.n.b. for V_{k+s} (or W_{k+s}). This follows from the identity

$$\|f_{k,0} - \sum c_{j,\ell}(g_j)_{k,\ell}\| = \|f_{k+s,0} - \sum c_{j,\ell}(g_j)_{k+s,\ell}\| \ ,$$

valid for any set of coefficients $(c_{j,\ell})_{\ell \in \mathbb{Z}}$ in $\ell_2(\mathbb{Z})$.

Lemma 1.1. *For a given $f \in L_2$, the functions $f_{0,\ell}$, and therefore for each k the functions $(f_{k,\ell})_{\ell \in \mathbb{Z}}$, are orthonormal if and only if*

$$\sum |\hat{f}(y - 2\pi\ell)|^2 = 1 \quad \textit{a.e. on} \ \mathbb{T} \ . \tag{1.9}$$

Proof. By Plancherel's theorem, the function F, equal to the sum (1.9), satisfies

$$\frac{1}{2\pi} \int_{\mathbb{T}} F(y)\, dy = \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(y)|^2\, dy = \frac{1}{2\pi} \|\hat{f}\|^2 = \|f\|^2 \ ,$$

so that $F \in L_1(\mathbb{T})$. From (1.3), the scalar product $(f_{0,\ell}, f_{0,\ell'})$ equals

$$\frac{1}{2\pi} \langle \hat{f}_{0,\ell}, \hat{f}_{0,\ell'} \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(y)|^2 e^{i(\ell' - \ell)y}\, dy = \frac{1}{2\pi} \int_{\mathbb{R}} F(y) e^{i(\ell' - \ell)y}\, dy \ .$$

By the properties of Fourier series, the last expression is equal to $\delta_{\ell,\ell'}$, $\ell, \ell' \in \mathbb{Z}$ if and only if $F(y) = 1$ a.e. □

For a scaling function of a multiresolution we have $\phi \in V_0 \subset V_1$. There exists therefore a sequence of coefficients $(a_\ell)_\ell$ in the real space $l_2(\mathbb{Z})$ for which

$$\phi(x) = \sqrt{2} \sum_{\ell \in \mathbb{Z}} a_\ell \phi(2x - \ell) \ , \qquad \sum a_\ell^2 = 1 \ ; \tag{1.10}$$

this is the *refinement equation* for ϕ.

We shall look for a wavelet ψ, $\|\psi\| = 1$ in the space $W_0 \subset V_1$ of the multiresolution (V_k). This ψ must be of the form

$$\psi(x) = \sqrt{2}\sum_{\ell\in\mathbb{Z}} b_\ell \phi(2x-\ell)\ , \qquad \sum b_\ell^2 = 1\ . \tag{1.11}$$

We select

$$b_\ell := (-1)^\ell a_{1-\ell}\ . \tag{1.12}$$

The main result of this section is Theorem 1.3 which asserts that ψ with coefficients (1.12) is indeed a wavelet, so that

$$W_0 = \overline{\operatorname{lin}}_\ell\, \psi_{0,\ell}\quad ,\quad W_k = \overline{\operatorname{lin}}_\ell\, \psi_{k,\ell}\ , \qquad k\in\mathbb{Z}\ .$$

By **2** and (1.3), relations (1.10) and (1.11) are equivalent to

$$\begin{cases} \hat\phi(y) = A(y/2)\hat\phi(y/2)\ , & \hat\psi(y) = B(y/2)\hat\phi(y/2) \\ A(y) := \dfrac{1}{\sqrt2}\sum a_\ell e^{-i\ell y}\ , & B(y) := \dfrac{1}{\sqrt2}\sum b_\ell e^{-i\ell y}\ . \end{cases} \tag{1.13}$$

We derive some identities for A, B. From (1.12),

$$B(y) = -e^{-iy}\overline{A(y+\pi)}\ . \tag{1.14}$$

Let $\Phi(y) := \sum|\hat\phi(y+2\ell\pi)|^2$. From the orthogonality of the $\phi_{0,\ell}$, $\ell\in\mathbb{Z}$ and Lemma 1.1, $\Phi(y) = 1$ a.e. This yields

$$|A(y)|^2 + |A(y+\pi)|^2 = 1 \ \text{ a.e. } \ y\in\mathbb{R}\ . \tag{1.15}$$

Indeed, from (1.13),

$$1 = \Phi(y) = \sum_\ell \left|A\left(\frac{y+2\ell\pi}{2}\right)\right|^2 \left|\hat\phi\left(\frac{y+2\ell\pi}{2}\right)\right|^2\ .$$

Since A has period 2π, the sum of terms with even ℓ is $|A(y/2)|^2\Phi(y/2) = |A(y/2)|^2$, while the sum with odd ℓ is $|A(y/2+\pi)|^2$. This gives (1.15). From this and (1.14) we derive new identities

$$|B(y)|^2 + |B(y+\pi)|^2 = 1 \ \text{ a.e.} \tag{1.16}$$

$$A(y)\overline{B(y)} + A(y+\pi)\overline{B(y+\pi)} = 0 \ \text{ a.e.} \tag{1.17}$$

Lemma 1.2. *If the integer translates $f_{0,\ell}$ of a function $f\in L_2(\mathbb{R})$, $\|f\| = 1$ are orthonormal, then the space $X_0 := \overline{\operatorname{lin}}_{\ell\in\mathbb{Z}}(f_{0,\ell})$ has also the o.n.b. $(f^1_{0,2\ell}, f^2_{0,2\ell})_{\ell\in\mathbb{Z}}$, where*

$$f^1(x) := \sum_\ell a_\ell f(x-\ell) \quad ;\quad f^2(x) := \sum_\ell b_\ell f(x-\ell)\ . \tag{1.18}$$

Proof. (a) We prove that the $f^1_{0,2\ell}$ (and the $f^2_{0,2\ell}$) are orthonormal. This means that $\langle f^1, f^1_{0,2\ell}\rangle = \delta_{0,\ell}$, $\ell \in \mathbb{Z}$. By (1.3), this scalar product is equal to

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}^1(y)\overline{\hat{f}^1_{0,2\ell}(y)}\,dy &= \frac{1}{\pi}\int_{\mathbb{R}} |\hat{f}(y)|^2|A(y)|^2 e^{2i\ell y}\,dy \\ &= \frac{1}{\pi}\int_0^\pi \sum_{j\in\mathbb{Z}} |\hat{f}(y+j\pi)|^2|A(y+j\pi)|^2 e^{2i\ell y}\,dy\ .\end{aligned}$$

Separating even and odd j, and using Lemma 1.1, we obtain for this

$$\frac{1}{\pi}\int_0^\pi (|A(y)|^2 + |A(y+\pi)|^2)e^{2i\ell y}\,dy = \delta_{0,\ell}\ .$$

(b) Similarly we prove that $f^1_{0,2\ell}$ and $f^2_{0,2\ell'}$ are orthogonal for all $\ell, \ell' \in \mathbb{Z}$. This will follow from

$$\langle f^1, f^2_{0,2\ell}\rangle = \frac{1}{2\pi}\langle \hat{f}^1, \hat{f}^2_{0,2\ell}\rangle = 0\ , \qquad \ell \in \mathbb{Z}\ .$$

For the last scalar product we get from (1.3) and (1.13)

$$\begin{aligned}\int_{\mathbb{R}} \hat{f}^1(y)\overline{\hat{f}^2_{0,2\ell}(y)}\,dy &= 2\int_{\mathbb{R}} |\hat{f}(y)|^2 A(y)\overline{B(y)}\,e^{2i\ell y}\,dy \\ &= 2\int_0^\pi \sum_{j\in\mathbb{Z}} |\hat{f}(y+j\pi)|^2 A(y+j\pi)\overline{B(y+j\pi)}\,e^{2i\ell y}\,dy\ .\end{aligned}$$

Separating odd and even j, using Lemma 1.1 and (1.17) we get that this is equal to

$$2\int_0^\pi \left(A(y)\overline{B(y)} + A(y+\pi)\overline{B(y+\pi)}\right) e^{2i\ell y}\,dy = 0\ .$$

(c) Since X_0 is closed and invariant under integer translations, all functions $f^1_{0,2\ell}$, $f^2_{0,2\ell}$ belong to it. It remains to show that all translations $f_{0,\ell}$ belong to the closed span of these functions. It is sufficient to prove this only for $\ell = 0$, $\ell = 1$. Indeed, we have

$$\begin{cases} f(x) = \dfrac{1}{\sqrt{2}}\displaystyle\sum_{\ell\in\mathbb{Z}}(a_{2\ell}f^1(x+2\ell) + b_{2\ell}f^2(x+2\ell))\ , \\ f(x-1) = \dfrac{1}{\sqrt{2}}\displaystyle\sum_{\ell\in\mathbb{Z}}(a_{2\ell+1}f^1(x+2\ell) + b_{2\ell+1}f^2(x+2\ell))\ . \end{cases}$$

We prove this by comparing the Fourier transforms of both sides. For example, the Fourier transform of the second sum is equal to

$$\frac{1}{\sqrt{2}}\sum a_{2\ell+1}\hat{f}^1(y)e^{2i\ell y}+\frac{1}{\sqrt{2}}\sum b_{2\ell+1}\hat{f}^2(y)e^{2i\ell y}$$

$$=\frac{1}{\sqrt{2}}e^{-iy}\hat{f}(y)\left\{A(y)\sum a_{2\ell+1}e^{i(2\ell+1)y}+B(y)\sum b_{2\ell+1}e^{i(2\ell+1)y}\right\}$$

$$=e^{-iy}\hat{f}(y)=\hat{f}_{0,1}(y)\ .$$

We have used here **2**, the identity

$$\sum a_{2\ell+1}e^{i(2\ell+1)y}=\frac{1}{2}\left(\sum a_\ell e^{i\ell y}-\sum a_\ell e^{i\ell(y+\pi)}\right)$$

and the relations (1.15) and (1.16). □

As a corollary we have

Theorem 1.3 (Mallat [1989]). *If ϕ is a scaling function of the multiresolution (V_k), and if (a_ℓ) are the coefficients of its refinement formula* (1.10), *then the function ψ given by* (1.11) *and* (1.12) *is a wavelet with*

$$(1.19)\qquad \overline{\operatorname*{lin}_{\ell\in\mathbb{Z}}}\{\psi_{0,\ell}\}=W_0\quad,\quad\overline{\operatorname*{lin}_{\ell\in\mathbb{Z}}}\{\psi_{k,\ell}\}=W_k\ ,\ k\in\mathbb{Z}\ ,\quad\overline{\operatorname*{lin}_{k,\ell\in\mathbb{Z}}}\{\psi_{k,\ell}\}=Y\ .$$

Proof. Taking $f=\phi$ in Lemma 1.2, we get $f^1(x)=\frac{1}{\sqrt{2}}\phi(x/2)$, $f^2(x)=\frac{1}{\sqrt{2}}\psi(x/2)$, so that $\phi_{-1,\ell}$, $\psi_{-1,\ell}$ is an o.n.b. for V_0. By **4**, $\phi_{0,\ell}$, $\psi_{0,\ell}$ is an o.n.b. for V_1. But $\phi_{0,\ell}$ is an o.n.b. for V_0, and we obtain that $\psi_{0,\ell}$ is an o.n.b. for $W_0=V_1\ominus V_0$. □

Example. If $\phi=\chi_{[0,1]}$, then $\psi=\chi_{[0,1/2]}-\chi_{[1/2,1]}$. The wavelet basis with this ψ is the well known *Haar orthogonal system*, which possesses many interesting properties, not shared by the trigonometric system. See Kashin and Sahakian [B-1984, Chapter 3].

We shall discuss two useful orthonormal bases in the space $Y=\overline{\bigcup_{k=0}^\infty V_k}$ of (1.7). The simpler of them consists of the functions (with the corresponding spaces for which they are o.n.b.'s):

$$(\mathcal{V})\qquad\left\{\begin{array}{cccccccc}\varphi_{0,\ell}; & \psi_{0,\ell}; & \psi_{1,\ell}; & \ldots; & \psi_{k,\ell}; & \ldots\\ V_0, & W_0, & W_1, & \ldots, & W_k, & \ldots\end{array}\right.\qquad \ell\in\mathbb{Z}\ .$$

In the *packet bases* of Coifman, Meyer, Quake and Wickerhauser [1989], the amount of dilations is decreased at the beginning of the basis. For this purpose we employ the operators, defined for $f\in L_2(\mathbb{R})$,

$$(1.20)\qquad L(f,x):=\sqrt{2}\sum a_\ell f(2x-\ell)\ ;\ M(f,x)=\sqrt{2}\sum b_\ell f(2x-\ell)\ .$$

By means of the change of variables $x/2\to x$ and the relation $f_{-1,0}(x)=\frac{1}{\sqrt{2}}f(x/2)$, the statement of Lemma 1.2 can be reformulated: the integer translates of

$$L(f_{-1,0},x) = \sqrt{2}\sum a_\ell f_{-1,0}(2x-\ell) \quad , \quad M(f_{-1,0},x) = \sqrt{2}\sum b_\ell f_{-1,0}(2x-\ell)$$

form an o.n.b. for $X_0 := \overline{\text{lin}}\{f_{0,\ell}\}$. Then **4** yields:

5. If $f_{0,\ell}$ is an o.n.b. in X_0, then the integer translates of Lf, Mf are an o.n.b. in $X_1 := \overline{\text{lin}}\{f_{1,\ell}\}$. In this way we obtain a *packet basis* $(\mathcal{W}_s)$, which will be important in §§2,3,5.

Theorem 1.4. *For each multiresolution $(V_k)_0^\infty$ with the scaling function ϕ, formulas $w^0 = \phi$ and*

$$w^{2n} = L(w^n) \quad , \quad w^{2n+1} = M(w^n) \ , \qquad n = 0, 1, \ldots \tag{1.21}$$

uniquely define all functions $w^n \in L_2(\mathbb{R})$, $n = 0, 1, \ldots$.

For each $k = 0, 1, \ldots$, the integer translates of the first 2^k functions w^n, $0 \le n < 2^k$ are an o.n.b. for V_k; and the last 2^{k-1} of these w^n, $2^{k-1} \le n < 2^k$ form an o.n.b. for W_{k-1}.

Proof. The soundness of this definition (that is, the fact that each w^m appears on the left in (1.21) exactly once) and all other statements follow by induction. Thus, $V_0 = \overline{\text{lin}}\ \{w^0_{0,\ell}\}$, and $W_0 = \overline{\text{lin}}\ \{w^1_{0,\ell}\}$. If at the k − 1-st step the w^n, $n = 0, \ldots, 2^k - 1$ are defined, orthonormal and if $W_{k-1} = \overline{\text{lin}}_{2^{k-1} \le n < 2^k}\{w^n_{0,\ell}\}_\ell$, then at the k-th step formulas (1.21), applied to the w^n, $2^{k-1} \le n < 2^k$ give exactly the w^n, $2^k \le n < 2^{k+1}$. They are orthonormal and by the definition of W_k and **5** satisfy

$$W_k = \overline{\text{lin}}_{2^{k-1} \le n < 2^k}\{w^n_{1,\ell}\}_\ell = \overline{\text{lin}}_{2^k \le n < 2^{k+1}}\{w^n_{0,\ell}\}_\ell \ . \qquad \square$$

To obtain a packet basis of order $s = 1, 2, \ldots$ we begin with the translates $w^n_{0,\ell}$ of the first 2^s functions w^n, $0 \le n < 2^s$. The last 2^{s-1} of these define W_{s-1}. Hence their dilates $w^n_{1,\ell}$ give W_s. At the k-th step the functions $w^n_{k,\ell}$, $2^{s-1} \le n < 2^s$ are an o.n.b for W_{s+k-1}.

Corollary 1.5. *For $s = 0, 1, \ldots$, the following functions*

$$(\mathcal{W}_s) \quad \underbrace{w^0_{0,\ell}, \ldots, w^{2^s-1}_{0,\ell}}_{V_s}; \underbrace{w^{2^{s-1}}_{1,\ell}, \ldots, w^{2^s-1}_{1,\ell}}_{W_s}; \ldots; \underbrace{w^{2^{s-1}}_{k,\ell}, \ldots, w^{2^s-1}_{k,\ell}}_{W_{s+k-1}}; \ldots, \quad \ell \in \mathbb{Z}$$

form an orthonormal basis for $Y = V_s \oplus W_s \oplus \cdots \oplus W_{s+k} \oplus \cdots$.

Clearly, $(\mathcal{V})$ is a special case: $(\mathcal{V}) = (\mathcal{W}_1)$.

§ 2. Scaling Functions with a Monotone Majorant

In this and the next section (with the main Theorems 2.5 and 3.4) we shall examine the convergence problems of orthogonal expansions for wavelet bases. We have had the privilege of using a manuscript of R.A. Lorentz.

A function $\rho(x) > 0$, $0 \le x < \infty$ is a *monotone decreasing majorant* if it is continuous, decreasing and satisfies $\int_0^\infty \rho(x)\,dx < \infty$. Then we also have $\sum_0^\infty \rho(\ell) < \infty$, and for all $\alpha > 0$,

$$\alpha \sum_{\ell=1}^{\infty} \rho(\alpha\ell) \le \int_0^\infty \rho(x)\,dx. \tag{2.1}$$

We say that a function $f \in L_1(\mathbb{R})$ belongs to $L_1^M(\mathbb{R})$, if for some majorant ρ, $|f(x)| \le \rho(|x|)$, a.e. on $\mathbb{R}$. Similarly, $\ell_1^M(\mathbb{Z})$ consists of all sequences $(c_\ell)_{\ell\in\mathbb{Z}}$ for which $|c_\ell| \le \rho(|\ell|)$, $\ell \in \mathbb{Z}$. We have:

1. If $f, g \in L_1^M(\mathbb{R})$, with orthonormal $g_{0,\ell}$, and if $\sum c_\ell g(x-\ell)$ is the orthonormal expansion of $f(x)$, then the coefficients (c_ℓ) belong to $\ell_1^M(\mathbb{Z})$.

Indeed, assuming that $\ell \ge 0$, for the majorants ρ, ρ_1 of f, g we have

$$\begin{aligned}
|c_\ell| &\le \int_{\mathbb{R}} |f(x)|\,|g(x-\ell)|\,dx \le \Big(\int_{-\infty}^{\ell/2} + \int_{\ell/2}^{\infty}\Big)\rho(|x|)\rho_1(|x-\ell|)\,dx \\
&\le \rho_1(\ell/2)\int_{-\infty}^{\ell/2} \rho(|x|)\,dx + \rho(\ell/2)\int_{\ell/2}^{\infty} \rho_1(|x-\ell|)\,dx \\
&\le 2\|\rho\|_1\rho_1(\ell/2) + 2\|\rho_1\|_1\rho(\ell/2) =: \rho_2(\ell)\ ,
\end{aligned}$$

so that (c_ℓ) belongs to $\ell_1^M(\mathbb{Z})$.

2. If $g \in L_1^M(\mathbb{R})$, and $\sum c_\ell g(x-\ell)$ is an orthonormal expansion of a function f with $(c_\ell) \in \ell_1^M(\mathbb{Z})$, then also f belongs to $L_1^M(\mathbb{R})$. The proof is similar to that of **1**.

We shall now *assume that the scaling function of (not necessarily complete) multiresolution* (V_k) *satisfies*

$$\phi \in L_1^M(\mathbb{R}) \text{ and } \int_{\mathbb{R}} \phi\,dt = 1\ . \tag{2.2}$$

Then:

3. The coefficients $(a_\ell), (b_\ell)$ of the formulas (1.10), (1.11), (1.20) belong to $\ell_1^M(\mathbb{Z})$.

4. The functions φ, ψ, and for any fixed N, the w^n, $n = 0, 1, \dots, N$ belong to $L_1^M(\mathbb{R})$, with a common majorant.

5. The series $\sum a_\ell \phi(x-\ell)$, $\sum b_\ell \psi(2x-\ell)$, $(a_\ell), (b_\ell) \in \ell_1^M(\mathbb{Z})$ or (1.20) with $\phi \in L_1^M(\mathbb{R})$ all converge absolutely and uniformly on compact subsets of $\mathbb{R}$.

6. One has

$$\widetilde{\phi}(x) := \sum_{\ell} \phi(x-\ell) = 1 \tag{2.3}$$

with uniform convergence on compact subsets of $\mathbb{R}$.

Proof. From the equations $\hat{\phi}(y) = A(y/2)\hat{\phi}(y/2)$ and $A(y) = \frac{1}{2}\sum a_\ell e^{i\ell y}$ for $y = 0$ we get $A(0) = 1$, that is $\sum a_\ell = 2$. Now relation (1.15) implies $A(\pi) = 0$, that is, $\sum(-1)^\ell a_\ell = 0$. We find that $\sum a_{2\ell} = \sum a_{2\ell+1} = 1$. Then, using (2.1) and 5 we get

$$\widetilde{\phi}(x) = \sum_\ell \sum_j a_j \phi(2x - 2\ell - j) = \sum_\ell \sum_m a_{m-2\ell}\phi(2x - m)$$

$$= \sum_m \phi(2x - m) \sum_\ell a_{m-2\ell} = \sum_m \phi(2x - m) = \widetilde{\phi}(2x) \ .$$

In other words, $\widetilde{\phi}(x) = \widetilde{\phi}(x/2) = \widetilde{\phi}(2^{-n}x)$ for $n = 1, 2, \ldots$. We get $\widetilde{\phi}(x) = \widetilde{\phi}(0) = \text{const}$. From

$$\int_0^1 \tilde{\phi}(x)^2 \, dx = \int_0^1 \sum_{\ell,k} \phi(x - \ell)\phi(x - k) \, dx$$

$$= \int_{\mathbb{R}} \sum \phi(x - \ell)\phi(x) \, dx = \|\phi\|_2^2 = 1$$

we get $\tilde{\phi}(x) = \pm 1$, and from (2.2), $\tilde{\phi}(x) = 1$.

7. Each function $g \in L_1(\mathbb{R})$ orthogonal to the integer translates of ϕ has mean value zero. In fact, the sum $\sum |\varphi(x - \ell)|$ is uniformly bounded, and by the dominated convergence theorem,

$$\int_{\mathbb{R}} g(x) \, dx = \int_{\mathbb{R}} \lim_{M \to \infty} g(x) \sum_{\ell=-M}^{M} \phi(x - \ell) \, dx$$

$$= \lim_{M \to \infty} \sum_{\ell=-M}^{M} \int_{\mathbb{R}} g(x)\phi(x - \ell) \, dx = 0 \, .$$

Multiresolutions with ϕ satisfying (2.2) *are necessarily complete.* This will follow from the next theorems about the convergence of the partial sums of orthogonal expansions for the basis $(\mathcal{V})$ of §1. The bases $(\mathcal{W}_s)$ will also be treated, they are needed in §5.

Let $U_k(f)$, $k = 0, 1, \ldots$ stand for the sum of the orthogonal expansion of $f \in L_2(\mathbb{R})$ for the group $\phi_{0,\ell}, \psi_{0,\ell}, \ldots, \psi_{k,\ell}$, $\ell \in \mathbb{Z}$ of $(\mathcal{V})$. Since $V_0 \oplus W_0 \oplus \cdots \oplus W_{k-1} = V_k$, this is equal to the sum of the expansion of f with respect to the $\varphi_{k,\ell}$, $\ell \in \mathbb{Z}$. Therefore from $U_k(f) = \sum_\ell \langle f, \phi_{k,\ell}\rangle \phi_{k,\ell}$ we get

$$\text{(2.4)} \qquad \begin{cases} U_k(f)(x) = \displaystyle\int_{\mathbb{R}} \Lambda_k(x,t) f(t) \, dt \\ \Lambda_k(x,t) := \displaystyle\sum_\ell \phi_{k,\ell}(x)\phi_{k,\ell}(t) = 2^k \sum_\ell \phi(2^k x - \ell)\phi(2^k t - \ell) \, . \end{cases}$$

Lemma 2.1. *For each $g \in L_1^M(\mathbb{R})$,*

$$\Omega_g(x,t) := \sum |g(x-\ell)g(t-\ell)| \le \tau(|x-t|) \tag{2.5}$$

where τ is a monotone decreasing majorant which depends on g.

Proof. Let ρ be the majorant of g. Assuming that $t \le x$, we have

$$\Omega_g(x,t) \le \Big(\sum_{t\le\ell\le x} + \sum_{\ell<t} + \sum_{\ell>x}\Big)|g(x-\ell)g(t-\ell)| =: \Sigma_1 + \Sigma_2 + \Sigma_3 \ .$$

In Σ_1, for each ℓ, either $\ell - t \ge \frac{1}{2}(x-t)$, or $x - \ell \ge \frac{1}{2}(x-t)$. Therefore by (2.1)

$$\Sigma_1 \le \rho\left(\tfrac{1}{2}|x-t|\right) 4\|\rho\|_1 \ .$$

For Σ_2, $x - t \le x - \ell$, so that $\Sigma_2 \le \rho(|x-t|)\cdot 2\|\rho\|_1$ and similarly for Σ_3. We define $\tau(x) := 8\|\rho\|_1\rho(x/2)$, and (2.5) follows. □

Now let P_k be any of the operators defined as follows. It is the sum $P_k := U_k + U_k^*$, where $U_k^*(f)$ is the sum of the expansion of f with respect to *some of the $\psi_{k+1,\ell}$, $\ell \in S$*, and S is any subset of $\mathbb{Z}$. The kernel Λ_k^* of U_k^* is similar to Λ_k, with the $\phi_{k,\ell}$, $\ell \in \mathbb{Z}$ replaced by the $\psi_{k+1,\ell}$, $\ell \in S$. Then

$$P_k(f,x) = \int_{\mathbb{R}} Q_k(x,t)f(t)\,dt \ , \qquad Q_k := \Lambda_k + \Lambda_k^* \ . \tag{2.6}$$

Theorem 2.2. *The kernels Q_k satisfy*

$$\int_{\mathbb{R}} |Q_k(x,t)|\,dt \le C \ , \tag{2.7}$$

where the constant C depends only on ϕ.

Proof. For Λ_k^*, by (2.5), and similarly for Λ_k,

$$\begin{aligned}\int_{\mathbb{R}} |\Lambda_k^*(x,t)|\,dt &\le 2^k \int_{\mathbb{R}} \Omega_\psi(2^kx, 2^kt)\,dt = \int_{\mathbb{R}} \Omega_\psi(2^kx,u)\,du \\ &\le \int_{\mathbb{R}} \tau(|2^kx - t|)\,dt = 2\|\tau\|_1 \ .\end{aligned} \tag{2.8}$$

Theorem 2.3. *If the scaling function of a multiresolution (V_k) satisfies* (2.2) *then:* (i) *for each bounded function $f \in C(\mathbb{R})$,*

$$P_k(f,x) \to f(x) \quad \textit{uniformly on compact sets.} \tag{2.9}$$

(ii) *For each of the spaces $C_0(\mathbb{R})$ (space of continuous functions vanishing at ∞), $L_p(\mathbb{R})$, $1 \le p < \infty$, and each rearrangement-invariant space X in which continuous functions with compact support are dense,*

$$\|P_kf - f\|_X \to 0 \ , \qquad k \to \infty \ , \ f \in X \ . \tag{2.10}$$

Proof. We require a standard theorem for integral operators

$$P_k(f,x) = \int_{\mathbb{R}} A_k(x,t) f(t)\, dt \;.$$

For a continuous kernel $A_k(x,t)$ on $\mathbb{R}^2$, the following conditions imply (i):
(a) $\int_{\mathbb{R}} |A_k(x,t)|\, dt \le M$ for $k = 1,2,\ldots$, $x \in \mathbb{R}$;
(b) $\int_{\mathbb{R}} A_k(x,t)\, dt \to 1$, $k \to \infty$
(c) for any $\delta > 0$, uniformly for $x \in \mathbb{R}$,

$$\left\{ \int_{-\infty}^{x-\delta} + \int_{x+\delta}^{\infty} \right\} |A_k(x,t)|\, dt \to 0 \;.$$

We leave the proof to the reader. See also [**CA**, Theorem 2.1, p.5].

These conditions hold for $A_k = Q_k$. Condition (a) follows from Theorem 2.2. Next we show that

$$\int_{\mathbb{R}} Q_k\, dt = \int_{\mathbb{R}} \Lambda_k\, dt + \int_{\mathbb{R}} \Lambda_k^*\, dt = 1 \;.$$

Indeed, due to **6**,

$$\begin{aligned} \int_{\mathbb{R}} \Lambda_k(x,t)\, dt &= 2^k \int_{\mathbb{R}} \sum \phi(2^k x - \ell)\phi(2^k t - \ell)\, dt \\ &= \sum \phi(2^k x - \ell) \int_{\mathbb{R}} \phi(2^k t - \ell) 2^k\, dt = \sum \phi(2^k x - \ell) = 1 \end{aligned}$$

Using **7** with $g(x) = \psi_{k,\ell}(x)$ we conclude, as above, that $\int_{\mathbb{R}} \Lambda^*\, dt = 0$.

Finally, we estimate

$$\begin{aligned} \int_{-\infty}^{x-\delta} |\Lambda_k(x,t)|\, dt &\le \int_{-\infty}^{x-\delta} 2^k \tau(2^k |x-t|)\, dt \\ &= \int_{-\infty}^{-2^k\delta} \tau(|u|)\, du \to 0 \;. \end{aligned}$$

Together with a similar relation for $\int_{x+\delta}^{\infty} |\Lambda_k|\, dt$ this establishes (c) for Λ_k, with a similar estimate for Λ_k^*.

(ii) Condition (2.7) for the symmetric kernel Q_k and the Hardy, Littlewood, Pólya theorem [**CA**, Theorem 4.5, p.34] imply that the operators P_k have uniformly bounded norms on each of the spaces of (ii). Continuous functions with compact support are dense in these spaces, so that (2.10) is a consequence of (2.9). □

Corollary 2.4. *A multiresolution $(V_k)_0^\infty$ is complete in $L_2(\mathbb{R})$ if its scaling function is continuous, has a monotone majorant and has mean value one.*

Theorem 2.3 is valid for any basis $(\mathcal{W}_s)$, $s = 1,2,\ldots$ instead of $(\mathcal{V})$.

Here we define an operator $P_k^*(f)$ to be the sum of the orthogonal expansion of f with respect to some of the elements $w_{j,\ell}^q$, $j \le k$, $\ell \in \mathbb{Z}$ of the basis $(\mathcal{W}_s)$ of Corollary 1.5. Namely, we take *all* of the $w^q_{j,\ell}$ with $j \le k$ (they produce the spaces $V_s, W_s, \dots, W_{s+k}$) and *an arbitrary subset* of the $w^q_{k+1,\ell}$ (that is, some of those that produce W_{s+k+1}).

Theorem 2.5. *For the operators P^*_k in each of the spaces of Theorem 2.3,*

$$\|f - P^*_k(f)\|_X \to 0 \quad \textit{for} \quad k \to \infty .$$

Proof. The proof is based on the facts that for a fixed s, all $w^0 = \varphi$, $w^1, \dots, w^{2^s-1}$ have a common decreasing majorant and that all but the first of them have mean value zero. □

§ 3. Periodization

The *periodization* of a function $f \in L_1(\mathbb{R})$ is the function on $\mathbb{R}$ of period 1 given by

$$\widetilde{f}(x) := \sum f(x - \ell) , \tag{3.1}$$

The series converges a.e. since

$$\int_0^1 \sum |f(x-\ell)|\, dx = \sum \int_0^1 |f(x-\ell)|\, dx = \int_{\mathbb{R}} |f(x)|\, dx < \infty .$$

Another natural domain of definition of $\widetilde{f}$ is $[0,1]$ (or, equivalently, $\mathbb{T}_1$, the circle of length 1). We also have $\widetilde{f+g} = \widetilde{f} + \widetilde{g}$, $\widetilde{af} = a\widetilde{f}$. Thus $f \to \widetilde{f}$ is a linear, L_1-norm decreasing map: $\|\widetilde{f}\|_{L_1[0,1]} \le \|f\|_{L_1(\mathbb{R})}$. This map is onto, since each $f \in L_1[0,1]$ is the periodization of its extension f^+ by zero onto $\mathbb{R}$.

Of particular importance will be the periodization of functions in $L_1^M(\mathbb{R})$. Here apply the remarks **1**–**7** of §2. Moreover,

1. If $f \in L_1^M(\mathbb{R})$, then the series (3.1) converges uniformly. If, in addition, f is continuous, then $\widetilde{f} \in \widetilde{C}[0,1]$, that is, $\widetilde{f} \in C[0,1]$ and $\widetilde{f}(0) = \widetilde{f}(1)$.

2. If $f, g \in L_1^M(\mathbb{R})$ and $f \perp g_{o,\ell}$, $\ell \in \mathbb{Z}$, then also $\widetilde{f}, \widetilde{g}$ are orthogonal.

3. If $f, g \in L_1^M(\mathbb{R})$ and $f(x) = \sum a_\ell g(x-\ell)$ is an orthogonal expansion of f, then $\widetilde{f}(x) = \sum a_\ell \widetilde{g}(x-\ell)$.

For $f \in L_1(\mathbb{R})$ we use the notation $\widetilde{f}_{k,\ell} := \widetilde{f_{k,\ell}}$; if X is the set $X = \{f\}$, we define $\widetilde{X} := \{\widetilde{f}\}$.

4. For a fixed $k = 1, 2, \dots$, and $f \in L_1(\mathbb{R})$, the functions $\widetilde{f}_{k,\ell}$, $\ell \in \mathbb{Z}$ span a space of dimension $\le 2^k$. In fact,

$$\widetilde{f}_{k,\ell}(x) = 2^{k/2} \sum_j f(2^k(x-j) - \ell) = 2^{k/2} \sum_j f(2^k x - 2^k j - \ell) . \tag{3.2}$$

This is equal to $\widetilde{f}_{k,\ell_1}$, if $\ell - \ell_1 \equiv 0 \bmod 2^k$.

5. We give another useful formula, valid for a function $f \in L_1^M(\mathbb{R})$ and a bounded 1-periodic function h on $\mathbb{R}$ (for example, for $h = \widetilde{g}_{k_1,\ell_1}$, $g \in L_1^M(\mathbb{R})$, where $k_1 = 0, 1, \dots,$ and $\ell_1 \in \mathbb{Z}$). For all $k = 0, 1, \dots$, $\ell \in \mathbb{Z}$ we have

$$\begin{aligned} \int_0^1 \widetilde{f}_{k,\ell}(x)h(x)\,dx &= \sum_j \int_0^1 f_{k,\ell}(x-j)h(x)\,dx \\ &= \sum_j \int_{-j}^{-j+1} f_{k,\ell}(x)h(x)\,dx = \int_{\mathbb{R}} f_{k,\ell}(x)h(x)\,dx\,. \end{aligned} \tag{3.3}$$

An example of periodization is $\tilde{\phi}(x) = 1$, if ϕ satisfies (2.2).

6. Useful for our purposes is the *Poisson formula*

$$\widetilde{f}(x) = \sum_{\ell \in \mathbb{Z}} \hat{f}(2\pi\ell)e^{2\pi i\ell x}\,, \tag{3.4}$$

which is valid if $f \in L_1(\mathbb{R}) \cap C(\mathbb{R})$, see Helson [B-1983, §2.4] or Zygmund [B-1959, vol.2, p.68].

Assuming that ϕ satisfies (2.2), we shall discuss the effect of periodization on the bases $(\mathcal{V}) = (\mathcal{W}_1)$ and $(\mathcal{W}_s)$. They become $(\widetilde{\mathcal{V}})$ and $(\widetilde{\mathcal{W}_s})$. Functions of the packet of $(\mathcal{W}_s)$ will become $\widetilde{w}^0 = 1, \dots, \widetilde{w}^{2^s-1}$. To the rest, we apply **4.** Thus, the infinite set of translates $w_{k,\ell}^{2^{s-1}}, \dots, w_{k,\ell}^{2^s-1}$, $\ell \in \mathbb{Z}$ will produce the finite set of elements $\widetilde{w}_{k,\ell}^q$, $q = 2^{s-1}, \dots, 2^s - 1$, $\ell = 0, 1, \dots, 2^k - 1$.

Theorem 3.1. *If the scaling function ϕ of the multiresolution $(\mathcal{V}_k)$ satisfies (2.2), then the basis $(\widetilde{\mathcal{W}_s})$ begins with the packet $\widetilde{w}^0 = 1, \dots, \widetilde{w}^{2^s-1}$, whose translates give $\widetilde{V}_s$. It is followed by the blocks $j = 1, \dots, k, \dots$ which produce the spaces $\widetilde{W}_s, \dots, \widetilde{W}_{s+k-1}, \dots$:*

$$\begin{array}{cccc} \widetilde{w}_{1,0}^{2^{s-1}}, \widetilde{w}_{1,1}^{2^{s-1}}; & \dots; & \widetilde{w}_{k,0}^{2^{s-1}}; \dots; \widetilde{w}_{k,2^k-1}^{2^{s-1}}; & \dots \\ \vdots \quad \vdots & & \vdots \quad \vdots & \\ \underbrace{\widetilde{w}_{1,0}^{2^s-1}, \widetilde{w}_{1,1}^{2^s-1};}_{\widetilde{W}_s} & \dots; & \underbrace{\widetilde{w}_{k,0}^{2^s-1}, \dots, \widetilde{w}_{k,2^k-1}^{2^s-1};}_{\widetilde{W}_{s+k-1}} & \dots\,. \end{array} \tag{3.5}$$

All these functions together form a complete orthonormal basis in $[0,1]$. *(In particular, the spaces* **4** *for $f = w^q$, $q = 2^{s-1}, \dots, 2^s - 1$ have full dimension 2^k). This implies that*

$$L_2[0,1] = \widetilde{V}_s \oplus \widetilde{W}_s \oplus \cdots \oplus \widetilde{W}_{s+k} \oplus \cdots$$

Proof. Let $f := w^q$, $g := w^{q_1}$, and for some k, $k_1 \geq 0$, let $0 \leq \ell < 2^k$, $0 \leq \ell_1 < 2^{k_1}$. In the formula (3.3) we replace $f_{k,\ell}$, $h = \widetilde{g}_{k_1,\ell_1}$ by their representations by means of f, g and make the substitution $2^k x = y$; we obtain, with $k_0 = k_1 - k$,

$$\int_0^1 \widetilde{f}_{k,\ell}\widetilde{g}_{k_1,\ell_1}\,dx = 2^{k_0/2}\sum_j \int_{\mathbb{R}} f(y-\ell)g(2^{k_0}y - 2^{k_0}j - \ell_1)\,dy\ .$$

If $f \neq g$, all integrals under the sum are zero, due to the orthogonality $w^q_{0,\ell} \perp w^{q_1}_{k_0,\ell'}$. If $f = g$, they are still zero if $k_0 \neq 0$. If $k_0 = 0$, then $k = k_1$ and an integral is zero unless $\ell = 2^{k_1}j + \ell_1$ for some $j \in \mathbb{Z}$. This can happen only if $\ell = \ell_1$, $j = 0$, in which case exactly one of the integrals is different from zero. It is equal to $\|w^q\|_2 = 1$. We have proved that $(\widetilde{\mathcal{W}_s})$ is orthonormal.

Finally, $(\mathcal{W}_s)$ is $L_1(\mathbb{R})$-complete, so that for a bounded function h^* on $\mathbb{R}$, $w^q_{k,\ell} \perp h^*$ for all elements $w^q_{k,\ell}$ of $(\mathcal{W}_s)$ implies $h^* = 0$. Let $h \in C[0,1]$, $h(0) = h(1)$ be arbitrary and let $h \perp \widetilde{w}^q_{k,\ell}$ for all elements of $(\widetilde{\mathcal{W}_s})$, let h^* be the 1-periodic extension of h onto $\mathbb{R}$. Then by (3.3),

$$0 = \int_0^1 \widetilde{w}^q_{k,\ell} h\,dx = \int_{\mathbb{R}} w^q_{k,\ell} h^*\,dx\ . \tag{3.6}$$

Consequently, $h^* = 0$ and $h = 0$. Thus, $(\widetilde{\mathcal{W}_s})$ is complete in $L_2[0,1]$. □

As a special case, Theorem 3.1 applies to $(\mathcal{V}) = (\mathcal{W}_1)$:

$$\underbrace{\tilde{\phi} = 1, \tilde{\psi}}_{\widetilde{V}_0};\underbrace{\tilde{\psi}_{1,0}, \tilde{\psi}_{1,1}}_{\widetilde{W}_1};\dots;\underbrace{\tilde{\psi}_{k,0},\dots,\tilde{\psi}_{k,2^k-1}}_{\widetilde{W}_k};\dots \tag{3.7}$$

with the corresponding representation

$$L_2[0,1] = \widetilde{V}_0 \oplus \widetilde{W}_0 \oplus \cdots \oplus \widetilde{W}_k \oplus \cdots\ . \tag{3.8}$$

Stronger than the completeness of $(\widetilde{\mathcal{V}})$ or $(\widetilde{\mathcal{W}_s})$ in $L_2[0,1]$ is the statement

Theorem 3.2. *Linear combinations of the functions of $(\mathcal{V}) = (\mathcal{W}_1)$ or $(\mathcal{W}_s)$ are dense in $C(\mathbb{T}_1)$.*

Proof. We shall approximate a function $f \in \widetilde{C}[0,1]$, $f(0) = f(1) = 0$ by means of $\widetilde{U}_k(f) := \widetilde{U_k(f^+)}$, where U_k is the operator (2.3) and f^+ is the extension of f to $\mathbb{R}$ by zero. Thus $\widetilde{U_k(f^+)}$ is a finite sum of functions whose closed span in $(\widetilde{\mathcal{V}})$ (or in $(\widetilde{\mathcal{W}_s})$) is $\widetilde{V}_k$. Then, with the kernel Λ_k of §2,

$$\begin{cases} \widetilde{U}_k(f,x) = \displaystyle\int_0^1 \tilde{\Lambda}_k(x,t)f(t)\,dt \\ \tilde{\Lambda}_k(x,t) := \displaystyle\sum_\ell \Lambda_k(x-\ell,t) \end{cases} \tag{3.9}$$

We estimate, for some natural $\ell_0 \geq 2$,

(3.10)

$$|f(x) - \widetilde{U}_k(f^+,x)| \leq |f(x) - \sum_{|\ell|\leq\ell_0} U_k(f^+,x-\ell)| + \sum_{|\ell|>\ell_0}\int_0^1 \Lambda_k(x-\ell,t)f(t)\,dt\ .$$

The first term on the right converges to $|f(x) - \sum_{|\ell|\le \ell_0} f^+(x-\ell)| = 0$. For the second term,

$$|\Lambda_k(x-\ell,t)| \le 2^k \sum_j |\phi(2^k(x-\ell)-j)\phi(2^k t - j)|$$
$$\le 2^k \tau(|2^k(x-t-\ell)|) \; .$$

Therefore, if $x,t \in [0,1]$ and $\ell \ge 2$,

$$\int_0^1 |\Lambda_k(x-\ell,t)|\,dt \le 2^k \int_0^1 \tau(|2^k(x-t-\ell)|\,dt \le \int_{\ell-1}^{\ell+1} \tau(u)\,du \; .$$

We see that the second term in (3.10) does not exceed $4\|f\|_\infty \int_{\ell_0}^\infty \tau(u)\,du < \varepsilon$, if ℓ_0 is properly selected. Altogether, this yields $\|f - \widetilde{U}_k(f)\|_\infty < 2\varepsilon$ for large k. □

The main subject in this section are the partial sums $S_m(f) = \sum_{\nu=0}^m \langle f, g_\nu \rangle g_\nu$ of orthogonal expansions with respect to the o.n.b. $(\mathcal{V})$ and $(\mathcal{W}_s)$. We begin by ordering the terms of the orthogonal series. The groups with $k = 0, 1, \dots$ in (3.5) consist only of a finite number of the functions $\widetilde{w}_{k,\ell}^q$; we allow an *arbitrary order* within each group. Thus, for $(\widetilde{\mathcal{W}}_s)$, the first g_ν's in the sum are the elements of the packet in their *natural order*, then follow the elements of groups $k = 0, 1, \dots$, with an arbitrary arrangement of elements within each group. With some symmetric kernel K_m we get

$$S_m(f,x) = \int_0^1 K_m(x,t) f(t)\,dt \; . \tag{3.11}$$

Lemma 3.3. *If $K(x,t) = \sum_{\ell=0}^n \widetilde{g}_{k,\ell}(x)\widetilde{g}_{k,\ell}(t)$, and the $\widetilde{g}_{k,\ell}$ are the periodizations of the translates-dilates of a fixed function $g \in L_1^M(\mathbb{R})$, then with a constant C that depends only on g,*

$$\int_0^1 |K(x,t)|\,dt \le C \; . \tag{3.12}$$

Proof. Using the inequalities

$$\int_0^1 |\widetilde{g}_{k,\ell}(t)|\,dt \le \int_{\mathbb{R}} |g_{k,\ell}(t)|\,dt = 2^{-k/2}\|g\|_1 \; ,$$

we find

$$\int_0^1 |K(x,t)|\,dt \le \sum_{\ell=0}^n 2^{-k/2}\|g\|_1 |\widetilde{g}_{k,\ell}(x)|$$
$$\le \|g\|_1 \sum_{\ell=0}^n \sum_j |g(2^k(x-j)) - \ell|$$
$$\le \|g\|_1 \sum_i |g(2^k x - i)| \le 2\|g\|_1 \|\tau\|_1 =: C \; .$$
□

Theorem 3.4. *For each rearrangement invariant space X on $[0,1]$ in which $C[0,1]$ is dense, for $L_p[0,1]$, $1 \le p < \infty$ and for $C[0,1]$, the operators $S_m(f)$, defined by $(\mathcal{W}_s)$ with a scaling function with the property* (2.2), *are uniformly bounded*

$$\|S_m\| \le C \ . \tag{3.13}$$

The constant C depends only on ϕ and s. For $f \in X$,

$$S_m(f) \to f \ , \qquad m \to \infty \tag{3.14}$$

in the norm of the space X.

Proof (similar to that of Theorem 2.5). Let k be defined by $2^{k+s} \le m < 2^{k+s+1}$. We have $\widetilde{V}_{k+s} = \widetilde{V}_k \oplus \widetilde{W}_k \oplus \cdots \oplus \widetilde{W}_{k+s}$. We apply Lemma 3.3 with $g = \phi_{k+s,0}$ to the sum containing all functions of the packet of $(\mathcal{W}_s)$ and of the groups $j = 0, \dots, 2^k - 1$. For the terms of the group $k+1$, we apply Lemma 3.3 separately for fixed values of q, $2^{s-1} \le q < 2^s$ with $g = \widetilde{w}^q_{k,0}$, for the translations of these functions. Then S_m becomes the sum of $1 + 2^{s-1}$ operators. This yields (3.12), and (3.13) follows from this with the help of [**CA**, Theorem 4.4, p.33]. □

§ 4. Polynomial Schauder Bases

A sequence $\Phi := (\phi_k)_0^\infty \subset X$ whose linear combinations are dense in a Banach space X, is called a *basis* (or a *Schauder basis*) if each element $f \in X$ can be uniquely represented in the form

$$f = \sum_{k=0}^{\infty} c_k \phi_k \ . \tag{4.1}$$

There exist separable spaces X without a basis (this is a consequence of a theorem of Enflo of [**CA**, p.267] and of Proposition 4.1), but these are rather exotic. Here we shall discuss only some special bases in the spaces C and sometimes L_p.

The coefficients c_k in (4.1) are clearly linear functionals $c_k(f)$, $k = 0, 1, \dots$.

Proposition 4.1. *For a basis Φ, the functionals $c_k(f)$ are continuous, $A_n(f) := \sum_{k=0}^n c_k \phi_k$ are linear bounded operators on X, and for some $M > 0$, $\|A_n\| \le M$, $n = 0, 1, \dots$.*

Proof. We define $\|f\|_0 := \sup_n \|A_n(f)\|$. Plainly, this is a new norm on X, and $\|f\| \le \|f\|_0$ for every f. Moreover, the space X_0 obtained by equipping X with the norm $\|\cdot\|_0$ is complete. Indeed, for every k, $|c_k(f)|\,\|\phi_k\| = \|A_k(f) - A_{k-1}(f)\| \le 2\|f\|_0$ so that $|c_k(f)| \le 2\|f\|_0/\|\phi_k\|$. It follows that if (f_j) is a Cauchy sequence in X_0, then $(c_k(f_j))$ is a Cauchy number sequence for every fixed k, hence $\bar{c}_k := \lim_{j\to\infty} c_k(f_j)$ exist. Furthermore, the series $\bar{f} := \sum_{k=0}^{\infty} \bar{c}_k\phi_k$ converges in X_0 and $\|f_j - \bar{f}\|_0 \to 0$, so that X_0 is a complete space. If $I : X \to X_0$ is the identity embedding, then $\|I\| \le 1$. By the Banach theorem, $\|I^{-1}\|$ is also bounded, hence $\|f\|_0 \le M\|f\|$. □

We now prove that the property of being a basis is stable, that is if $(\psi_k)_0^\infty$ is sufficiently close to a basis (ϕ_k), then (ψ_k) is itself a basis.

Theorem 4.2. *Let (ϕ_k) be a basis in a Banach space X and let (c_k) be the associated sequence of coordinate functionals. If (ψ_k) is a sequence of vectors in X for which*

$$\sum_{k=0}^{\infty} \|\phi_k - \psi_k\| \cdot \|c_k\| < 1 , \tag{4.2}$$

then (ψ_k) is also a basis in X.

Proof. Due to (4.2), the series $B(f) := \sum_{k=0}^{\infty} c_k(f)\psi_k$ is convergent for every $f \in X$, and $\|I - B\| < 1$ (I is the identity operator). Therefore B^{-1} exists, so that B is an automorphism. The statement now follows from the fact that $B(\phi_k) = \psi_k$ for each k. □

The simplest basis in $C[0,1]$, the basis of Schauder (Schauder [1927]), is the sequence of functions

$$t,\ 1-t,\ u_{0,0}(t),\ u_{1,0}(t),\ u_{1,1}(t),\ u_{2,0}(t),\ u_{2,1}(t),\ u_{2,2}(t), \dots, \tag{4.3}$$

where $u_{i,\ell}(t)$, $i = 0,1,\dots$, $\ell = 0,1,\dots,2^i - 1$, is defined as a piecewise continuous function with three breakpoints: at $\alpha_{i,\ell} := \ell 2^{-i}$, at $\beta_{i,\ell} = (\ell+1)2^{-i}$, and at $\gamma_{i,j} := (\alpha_{i,\ell} + \beta_{i,\ell})/2$, so that $u_{i,\ell}(t) = 0$ if $t \le \alpha_{i,\ell}$ or $t \ge \beta_{i,\ell}$ and $u_{i,\ell}(\gamma_{i,\ell}) = 1$. Every $f \in C[0,1]$ can be represented by the uniformly convergent series

$$f(t) = c_0 t + c_1(1-t) + \sum_{i=0}^{\infty} \sum_{\ell=0}^{2^i-1} c_{i,\ell} u_{i,\ell}(t) , \tag{4.4}$$

with $c_0, c_1, c_{0,0}, c_{1,0}, c_{1,1}, \dots$, being uniquely determined by f, namely

$$c_0 = f(1) , \quad c_1 = f(0), \quad c_{i,\ell} = f(\gamma_{i,\ell}) - \tfrac{1}{2}\left[f(\alpha_{i,\ell}) + f(\beta_{i,\ell})\right] . \tag{4.5}$$

(The partial sum of the series (4.4) corresponding to $0 \le i \le s-1$ is a piecewise linear continuous function interpolating $f(t)$ at $t = 0$, $t = 1$ and at $2^s - 1$ equidistant breakpoints.)

The partial sums of the series (4.4) are splines of order two. It is now easy to construct bases that consist of algebraic or trigonometric polynomials. More exactly, using e for even, o for odd, we constrict (a) an algebraic basis $(P_k)_{k=0}^\infty$ for $C[-1,1]$; (b) a basis of even trigonometric polynomials $(T_k^e)_{k=0}^\infty$ for $C^e(\mathbb{T})$, that is, for $C[0,\pi]$; (c) a basis $(T_k^o)_{k=0}^\infty$ for $C^o(\mathbb{T})$, that is for functions of $C[0,\pi]$ vanishing at $0,\pi$; (d) a trigonometric basis $(T_k)_0^\infty$ for $C(\mathbb{T})$. (In this and the next section, for polynomials P_k, T_k the subscript k as a rule, does not denote their degree.) In case (a), one repeats the construction (4.3) on $[-1,1]$ and approximates the functions u by polynomials, using Theorem 4.2. For (b), one does the same for $[0,\pi]$, replacing the first two functions of (4.3) by the function 1, and approximating the u by even trigonometric polynomials. For (c), one omits the two functions altogether and approximates by odd polynomials. If (T_k^e), (S_k^o) are two bases of types (b) and (c), one obtains a basis of type (d) by combining them into one sequence: $T_1^o, S_1^e, T_2^o, \dots$. Therefore:

Proposition 4.3. *There exist bases of all types* (a), (b), (c), (d), *that consist of algebraic or of trigonometric polynomials.*

We shall concentrate our attention on the following problem. How slowly can the degrees $\nu_k := \deg P_k$ or $\nu_k := \deg T_k^e$ in cases (a), (b) or $\mu_k := \deg T_k$ in case (d) increase to infinity? In [1914], Faber showed that if $\nu_1 \le \nu_2 \le \cdots$, then $\nu_k = k$ can hold only for finitely many k. In the positive direction, Bochkarev [1985] constructed a trigonometric basis with $\mu_k = 2k$, $k = 0,1,\dots$.

Here, assuming that

$$\deg P_k \le \deg P_{k+1}\ , \quad \deg T_k \le \deg T_{k+1}\ , \qquad k=0,1,\dots \tag{4.6}$$

we shall give estimates of the degrees from below.

Theorem 4.4. *If* $(T_k)_0^\infty$ *is a Schauder basis for* $C(\mathbb{T})$ *consisting of trigonometric polynomials that satisfy* (4.6), *then*

$$\deg T_k \ge \left[\frac{k+1}{2}\right]\ , \qquad k=0,1,\dots\ . \tag{4.7}$$

Proof. We have only to prove that if T_k are linearly independent, then $d := \deg T_k \ge [(k+1)/2]$. Indeed, the polynomials T_k, $k=0,1,\dots,m$ belong to the space $\mathcal{T}_d$ of dimension $1+2d$ and there are $k+1$ of them. Hence $k+1 \le 1+2d$, $d \ge k/2$, hence $d \ge [(k+1)/2]$. □

Our main result is

Theorem 4.5 (Privalov [1987]). (i) *For each trigonometric basis* $(T_k)_0^\infty$ *on* $C(\mathbb{T})$ *one has, for some* $\varepsilon > 0$,

$$\mu_k \ge \tfrac{1}{2}(1+\varepsilon)k\ , \qquad k \ge k_0\ . \tag{4.8}$$

(ii) *For each basis* $(P_k)_0^\infty$ *on* $C[-1,1]$ *or* $(T_k^e)_0^\infty$ *on* $C[0,\pi]$ *one has* $\nu_k \ge (1+\varepsilon)k$, $k \ge k_0$, *for some* $\varepsilon > 0$.

We start with a proof of (i). Our proof, a corrected and simplified version of Privalov's, uses properties of the de la Vallée-Poussin operators V_n, $n = 1,2,\dots$ (see [**CA**, p.273]), which map $C(\mathbb{T})$ into $\mathcal{T}_{2n-1}$, and are given by

$$V_n(f,x) := 2\sigma_{2n-1}(f,x) - \sigma_{n-1}(f,x) ,$$

where $\sigma_m(f,x)$ are the Fejér sums of the Fourier series of f. They are "semi-projectors" with the properties $V_n(T) = T$ for $T \in \mathcal{T}_n$, $V_n(\cos\omega t) = V_n(\sin\omega t) = 0$ for each integer frequency $\omega \geq 2n$, further they satisfy $\|V_n\| \leq 3$.

Lemma 4.6. *Let X_{2m+1}, $m < n$ be a $(2m+1)$-dimensional subspace of $\mathcal{T}_n$ and let U be a linear bounded operator from $C(\mathbb{T})$ into $\mathcal{T}_n$ with the property that $Uf = f$ for $f \in X_{2m+1}$. Then for some absolute constant $C > 0$,*

$$\|U\| \geq C \log \frac{n}{n-m+1} . \tag{4.9}$$

Theorem 4.5(i) follows easily from the lemma. Indeed, if $(T_k)_0^\infty$ is a basis for $C(\mathbb{T})$, we consider

$$A_{2m+1}(f) := \sum_{k=0}^{2m} c_k(f)T_k .$$

The operator A_{2m+1} is a semi-projector: it maps $C(\mathbb{T})$ into $\mathcal{T}_{\mu_{2m}}$ and it preserves the elements of the subspace spanned by $T_0, \dots, T_{2m}$. By (4.9) and Proposition 4.1 we obtain

$$\log \frac{\mu_{2m}}{\mu_{2m} - m} \leq C$$

or $\mu_{2m} \leq C(\mu_{2m} - m)$. In the last inequality, necessarily, $C > 1$, and with $\varepsilon := C/(C-1)$ we derive $\mu_{2m} \geq (1+\varepsilon)m$, which after a change of ε yields (4.8).

Proof of Lemma 4.6. Let $\ell := n - m$; we consider only the cases when $n > 16\ell$, since the constant C in (4.9) can be decreased to include all other n, ℓ. Let $k := [\sqrt{n/(4\ell)}\,]$, $\mu := 4\ell k$ and let f be a trigonometric polynomial of degree $\leq 2k+5$ that does not contain $\cos\omega t$, $\sin\omega t$ with $\omega = k, k+1, k+2, k+3$, has $\|f\| \leq 1$, and for $k-1$-st Fourier sum, and an absolute constant $C_1 > 0$, satisfies

$$s_{k-1}(f,0) \geq C_1 \log k . \tag{4.10}$$

To obtain such an f, we start with a function $g \in C(\mathbb{T})$, $\|g\| \leq 1$ so that $s_{k-1}(g,0) \geq C \log k$. From g we subtract its Fourier terms with $\omega = k, k+1, k+2, k+3$ to get a $\bar{g}$ with $\bar{g} \in C(\mathbb{T})$, $\|\bar{g}\| \leq 17$, $s_{k-1}(g,0) \geq C\log k - 16$. We can take $f := \frac{1}{17}V_{k+3}(\bar{g})$, and adjust the constant C.

Let $f_1 := s_{k-1}(f)$, $f_2 := f - f_1$ and let for $c \in \mathbb{T}$,

$$\varphi_j^c(t) := f_j\big(\mu(t-c)\big)S(t) , \qquad j = 1,2, \quad \varphi^c := \varphi_1^c + \varphi_2^c , \tag{4.11}$$

where $S \in \mathcal{T}_{2\ell k}$, $S \neq 0$ is a trigonometric polynomial for which $\varphi_1^c \in X_{2m+1}$ for all $c \in \mathbb{T}$. To prove that such S exists, we note that the highest possible frequency ω in φ_1^c is $\mu(k-1)+2k\ell = 4\ell k^2 - 2\ell k < n$, so that $\varphi_1^c \in \mathcal{T}_n$. There exist linear functionals $L_1, \ldots, L_{2\ell}$ with the property

$$X_{2m+1} = \{g \in \mathcal{T}_n : L_i(g) = 0\ ,\ i = 1, \ldots, 2\ell\}\ .$$

The condition that $\varphi_1^c \in X_{2m+1}$ for every c is equivalent to

$$L_i(S) = 0,\ L_i\bigl(\cos\omega t\, S(t)\bigr) = 0,\ L_i\bigl(\sin\omega t\, S(t)\bigr) = 0,$$
$$i = 1, \ldots, 2\ell,\ \omega = \mu, 2\mu, \ldots, (k-1)\mu\ .$$

We have $2\ell(2k-1)$ linear homogeneous equations for the $4\ell k+1$ unknown coefficients of S; so there exists a non-zero solution which we normalize by $\|S\| = 1$.

Then $\|\varphi^c\| \le 1$ and we now estimate $\|U\varphi^c\|$ from below. Since $\varphi_1^c \in X_{2m+1}$, we have $U\varphi_1^c = \varphi_1^c$; in particular $U(\varphi_1^c, \mathrm{c}) = f_1(0)S(\mathrm{c})$. We see that, as a function of c, $U(\varphi_1^c, \mathrm{c})$ is in $\mathcal{T}_{2\ell k}$. On the other hand, $U(\varphi_2^c, t)$ can be written in the form

$$\sum_{j=k+4}^{2k+5} \bigl(a_j(t)\cos j\mu\mathrm{c} + b_j(t)\sin j\mu\mathrm{c}\bigr)\ ,$$

where $a_j, b_j \in \mathcal{T}_n$. Hence, the lowest possible frequency, in terms of c, in $U(\varphi_2^c, \mathrm{c})$ is $(k+4)\mu - n = 4\ell k(k+4) - n > 4\ell(k+1)^2 - n + 4\ell k \ge 4\ell k$. It follows that for the trigonometric polynomial $h(\mathrm{c}) := (U\varphi^c, \mathrm{c})$ we have $V_{2\ell k}(h, \mathrm{c}) = f_1(0)S(\mathrm{c})$. Therefore we get, using the fact that $\|V_{2\ell k}\| \le 3$,

$$\max_t \|U\varphi^t\| \ge \max_t |(U\varphi^t, t)| = \|h\| \ge \tfrac{1}{3} f_1(0)\|S\| \ge C\log k\ .$$

For some $\mathrm{c}_0 \in \mathbb{T}$ we have therefore $\|U\varphi^{\mathrm{c}_0}\| \ge C_1 \log(m/(n-m+1))$ while $\|\varphi^{\mathrm{c}_0}\| \le 1$.

The proof of part (ii) of Theorem 4.5 is quite similar. The substitution $x = \cos t$ shows that the two variants of (ii) are equivalent. There is a variant of Lemma 4.6, asserting (4.9) for an operator U that maps $\mathcal{T}_n^e$ into itself and is an identity on a subspace X_{m+1}, $m < n$. Since the operators $V_k(f)$ for even f are even polynomials, the main proof remains the same with some simplifications. □

§ 5. Orthonormal Polynomial Bases

We shall prove here the positive counterpart of Theorem 4.5. What is more, we shall construct not merely a Schauder basis for $C(\mathbb{T})$ that consists of polynomials of low degree, but an *orthonormal* basis for all spaces L_p $(1 \le p < \infty)$ and C on $\mathbb{T}$. The following theorem is due to R.A. Lorentz and Sahakian [1994]. The proofs will depend heavily on the wavelet theory of §§1,2,3. We shall use its notations and results.

As in §4, we assume that the polynomial sequence $(T_m)_0^\infty$ satisfies:

$$\deg T_m \le \deg T_{m+1}\ , \qquad m = 0, 1, \dots . \tag{5.1}$$

Theorem 5.1. *For each $\varepsilon > 0$ there is an orthonormal sequence of trigonometric polynomials $(T_m)_0^\infty$ satisfying* (5.1) *with the properties* (i)

$$\deg T_m \le \max\left(\left[\frac{m+1}{2}\right], (1+\varepsilon)\frac{m}{2}\right)\ , \qquad m = 0, 1, \dots \tag{5.2}$$

(ii) *Let $S_m(f)$ be the partial sum of the orthogonal expansion of f with respect to the T_j. Then for $f \in X$, where X is any of the spaces $X = C(\mathbb{T})$, $L_p(\mathbb{T})$ (and even for any rearrangement invariant space in which $C(\mathbb{T})$ is dense),*

$$\|f - S_m(f)\|_X \to 0\ , \qquad f \in X\ .$$

Many earlier authors have obtained somewhat weaker results. The existence of a Schauder basis on $C(\mathbb{T})$ with $\deg S_m \le (1+\varepsilon)m/2$ for all large m have been established by Privalov [1990]. Wojtaszchyk and Woźnjakowski [1991] have an o.n.b. with $\deg S_m \le 4m/3$. Their proofs depend on some ideas of Bourgain. The first to use wavelets were Offin and Oskolkov [1992], they achieved o.n.b. with $\deg S_m \le (1+\varepsilon)m$.

Instead of the polynomials $\sum c_\ell e^{i\ell y}$ on $\mathbb{T}$, we shall work with those on the circle $\mathbb{T}_1$ of length one (or, equivalently, on $[0,1]$), namely $S(y) := \sum c_\ell e^{i2\pi\ell y}$. The polynomial S is real if $\bar c_\ell = c_{-\ell}$.

For the proof of Theorem 5.1 we need a multiresolution with special properties.

Theorem 5.2. *For each $0 < \varepsilon < 1/3$ there exists a function ϕ with the properties:*

(i) *$\phi \in L_1^M(\mathbb{R})$: ϕ is continuous and has a monotone majorant;*
(ii) *the Fourier transform $\hat\phi \in C^\infty(\mathbb{R})$;*
(iii) *$\frac{1}{\pi}\operatorname{supp}\hat\phi = [-1-\varepsilon, 1+\varepsilon]$, and $\hat\phi(x) = 1$ for $x \in [-1+\varepsilon, 1-\varepsilon]$;*
(iv) *ϕ is a scaling function of a complete multiresolution $(V_k)_{k\in\mathbb{Z}}$.*
(v) *for the functions A, B of* (1.13) *one has*

$$\Gamma' := \frac{1}{\pi}\operatorname{supp} A(y/2) = \bigcup_{r\in\mathbb{Z}} [4r-1-\varepsilon, 4r+1+\varepsilon]\ ; \tag{5.3}$$

$$\Gamma'' := \frac{1}{\pi}\operatorname{supp} B(y/2) = \bigcup_{r\in\mathbb{Z}} [4r+1-\varepsilon, 4r+3+\varepsilon]\ . \tag{5.4}$$

Proof. We begin with $\hat\phi$ rather than ϕ itself. Let G be a function in $C^\infty(\mathbb{R})$ which increases strictly from 0 to $\pi/2$ on $[-\varepsilon\pi, \varepsilon\pi]$, and is constant outside of this interval. Then the function $\sin G(y+\pi)$ is $= 1$ for $y > -(1-\varepsilon)\pi$, is $= 0$ for $y \le -(1+\varepsilon)\pi$, and is strictly increasing between these points. Similar properties has $\cos G(y-\pi)$. We define

(5.5) $$\hat{\phi}(y) := \sin G(y+\pi)\cos G(y-\pi) \in C^\infty(\mathbb{R}) .$$

This function has support $[-1-\varepsilon, 1+\varepsilon]\pi$, is $=1$ on $[-1+\varepsilon, 1-\varepsilon]\pi$ and is monotone on the two intervals $I_1 := [-1-\varepsilon, -1+\varepsilon]\pi$ and $I_2 := [1-\varepsilon, 1+\varepsilon]\pi$. Moreover, it satisfies

(5.6) $$\sum_{\ell\in\mathbb{Z}} \hat{\phi}(y-2\pi\ell)^2 = 1 , \qquad y \in \mathbb{R} .$$

Indeed, if $y \notin I_1 \cup I_2$, then just one of the terms of (5.6) is 1, the rest zero. It is sufficient to examine the case when $y \in I_1 := [-1-\varepsilon, -1+\varepsilon]\pi$. Then the sum reduces to two terms $\ell = 0$, $\ell = -1$ and is equal to

$$\begin{aligned}&\sin^2 G(y+\pi)\cos^2 G(y-\pi) + \sin^2 G(y+3\pi)\cos^2 G(y+\pi)\\ &\quad = \sin^2 G(y+\pi) + \cos^2 G(y+\pi) = 1 .\end{aligned}$$

Since $\hat{\phi} \in L_1(\mathbb{R})$, the function $\phi(x)$, obtained from $\hat{\phi}$ by the inverse Fourier transform, is continuous.

Now $\hat{\phi}$ and all its derivatives vanish at the points $\pm(1+\varepsilon)\pi$. By partial integration, for $x \neq 0$,

(5.7) $$\phi(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\phi}(y)e^{ixy}\,dy = -\frac{e^{-i(1+\varepsilon)\pi x} - e^{i(1+\varepsilon)\pi x}}{2\pi i x}\int_{\mathbb{R}} \hat{\phi}'(y)e^{ixy}\,dy .$$

Repeating this process, we obtain, for $r > 1$, $|\phi(x)| \le C_r(1+|x|^r)^{-1}$, which shows that $\phi \in L_1^M(\mathbb{R})$. We have proved (i), (ii), (iii).

By (5.6) and Lemma 1.1, ϕ satisfies $\|\phi\|_2 = 1$ and defines the space $V_0 = \overline{\mathrm{lin}}\{\phi_{0,\ell}\}$. Let $A(y) := \frac{1}{\sqrt{2}}\sum a_\ell e^{i\ell y}$, $\sum a_\ell^2 = 1$ be the 2π-periodic extension of $\hat{\phi}(2y)$ from $[-\pi,\pi]$ to $\mathbb{R}$. Then we have $\hat{\phi}(y) = A(y/2)\hat{\phi}(y/2)$. By **2** of §1, this implies $\phi(x) = \sqrt{2}\sum_\ell a_\ell\phi(2x-\ell)$. We have

(5.8) $$\hat{\phi}(2y) = A(y)\hat{\phi}(y) , \qquad y \in \mathbb{R} .$$

Indeed, outside of $\pi[-1-\varepsilon, 1+\varepsilon]$, $\hat{\phi}(2y) = \hat{\phi}(y) = 0$. Inside this interval, $A(y) = \hat{\phi}(2y)$. Because $\varepsilon < 1/3$, either $y \in \pi[-1+\varepsilon, 1-\varepsilon]$, when $\hat{\phi}(y) = 1$, or $y \notin (\pi/2)[-1-\varepsilon, 1+\varepsilon]$, in which case $A(y) = \hat{\phi}(2y) = 0$.

From (5.8), $\hat{\phi}(y) = A(y/2)\hat{\phi}(y/2)$. By **2** of §1, this implies $\phi(x) = \sqrt{2}\sum a_\ell\phi(2x-\ell)$. Therefore, $V_0 \subset V_1$, and we deduce condition (1.4a) of the definition of the multiresolution (V_ℓ).

We have $f \in V_k$, $k \in \mathbb{Z}$ if and only if $f(x) = \sum_\ell c_\ell\phi_{k,\ell}(x)$, $(c_\ell) \in l_2$. By **2**, §1 this is equivalent to

(5.9) $$\hat{f}(y) = C(y/2^k)\hat{\phi}(y/2^k) ,$$

where $C(y) = \sum c_\ell e^{-i\ell y}$ is 2π periodic, $(c_\ell) \in l_2$. Moreover, equivalent to this is also $\operatorname{supp}\hat{f} \subset 2^k\pi[-1-\varepsilon, 1+\varepsilon]$. It is clear that (5.9) implies this, and the converse follows as in the proof of (5.8).

The density of $\cup_k V_k$ in L_2 follows from Theorem 2.3. Moreover, if $f \in \cap V_k$, then $f \in V_k$ for each k, $\operatorname{supp} \hat{f} \subset 2^k\pi[-1-\varepsilon, 1+\varepsilon]$; letting $k \to -\infty$ yields $\operatorname{supp} \hat{f} = \{0\}$. Consequently, f is a constant, and the only constant in $L_2(\mathbb{R})$ is zero. This establishes (iv).

Finally, (v) follows from the definition of the function A and the relation $B(y) = -e^{-iy}A(y+\pi)$. □

The construction of the orthogonal system (S_m) of Theorem 5.1 will be via the "*packet-bases*" $(\mathcal{W}_s)$, $s = 1, 2, \ldots$ in the space $L_2(\mathbb{R})$, discussed in §2. If ψ is the wavelet (1.11), with coefficients b_ℓ, the elements w^n of this basis are given by $w^0 = \varphi$, $w^1 = \psi$, and the formulas $w^{2n} = L(w^n)$, $w^{2n+1} = M(w^n)$, where the operators L, M are, for $f \in L_2(\mathbb{R})$,

$$(5.10)\quad (Lf)(x) := \sqrt{2}\sum_{\ell\in\mathbb{Z}} a_\ell f(2x-\ell) \quad , \quad (Mf)(x) := \sqrt{2}\sum_{\ell\in\mathbb{Z}} b_\ell f(2x-\ell)\ .$$

The "packet" of the basis $(\mathcal{W}_s)$ is the set of the functions w^n, $0 \le n \le 2^s - 1$.

To obtain from $(\mathcal{W}_s)$ a basis of orthonormal polynomials on $\mathbb{T}_1$, we use the *process of periodization* $f \to \tilde{f}$, given by $\tilde{f}(x) := \sum_{\ell\in\mathbb{Z}} f(x-\ell)$; from $(\mathcal{W}_s)$ it produces the basis $(\widetilde{\mathcal{W}_s})$ (see Theorem 3.1). That the $\tilde{w}^n := \widetilde{w^n}$ will be trigonometric polynomials, follows from the Poisson formula. In fact, if $f \in L_1^M(\mathbb{R})$ and if the Fourier transform $\hat{f}$ has a compact support contained in $[a, b]$, then according to (3.4), $\tilde{f}$ is a trigonometric polynomial on $\mathbb{T}_1$ of degree $< \max(|a|, |b|)/(2\pi)$. For real f, the support of $\hat{f}$ is symmetric. If in addition f is even, then $\hat{f}$ is real. We write $\operatorname{supp}_+ \hat{f} = \operatorname{supp} \hat{f} \cap [0, \infty)$.

For the calculation of the degrees of the $\tilde{w}^n$, we need the function $\eta(n)$, $n = 0, 1, 2, \ldots$ defined as follows. (See Note 6.5.)

We let $\eta(0) := 0$, $\eta(1) := 1$ and define

If n is even,

$$(5.11)\qquad \eta(n) = \begin{cases} 2\eta(n/2) & \text{if } \eta(n/2) \text{ is even,} \\ 2\eta(n/2)+1 & \text{if } \eta(n/2) \text{ is odd;} \end{cases}$$

and if n is odd,

$$(5.12)\qquad \eta(n) = \begin{cases} 2\eta\left((n-1)/2\right)+1 & \text{if } \eta\left((n-1)/2\right) \text{ is even,} \\ 2\eta\left((n-1)/2\right) & \text{if } \eta\left((n-1)/2\right) \text{ is odd.} \end{cases}$$

The definition is by induction in j. If $\eta(n)$ is known for all n in the range $1 \le n < 2^{j-1}$, then formulas (5.11), (5.12) define $\eta(n)$ uniquely for all n with $2^{j-1} \le n < 2^j$.

For example,

$$\eta(1) = 1\ ;\ \eta(2) = 3\ ,\ \eta(3) = 2\ ;\ \eta(4) = 7\ ,\ \eta(5) = 6\ ,\ \eta(6) = 4\ ,\ \eta(7) = 5\ ;\ldots$$

The function η has interesting properties, it is related to the Grey code of the coding theory. See Note 6.3.

Proposition 5.3. *For each j, $j = 1, 2, \dots$, the function η maps in a 1–1 manner the range $1, 2, \dots, 2^j - 1$ onto itself. As a consequence, it also maps the range $2^{j-1}, \dots, 2^j - 1$ onto itself.*

Proof. It is sufficient to prove that the equation

$$\eta(n) = m \tag{5.13}$$

for m with $1 \le m < 2^j$ has a solution n in the same range; it follows then that the solution is unique.

By induction on j, we establish this in the case when m is odd. The case of even m is similar.

Let $m = 2\mu + 1$, then $1 \le \mu < 2^{j-1}$. By the induction hypothesis, there is a solution v of $\eta(\nu) = \mu$. First case: μ is even. We define $n := 2\nu + 1$ and have $\eta((n-1)/2) = \mu$. We are in the situation: n is odd, $\eta((n-1)/2)$ is even. This corresponds to the first line of (5.12). Consequently, $\eta(n) = 2\eta((n-1)/2) + 1 = 2\mu + 1 = m$; we have found a solution for (5.13). Second case: μ is odd. We define $n := 2\nu$ and have: n is even, $\eta(n/2) = \eta(v) = \mu$ is odd. This is the second row of (5.11), so that $\eta(n) = 2\eta(n/2) + 1 = 2\mu + 1 = m$. □

In particular, we get $\eta(n) < 2n$, $n = 1, 2, \dots$.

We now estimate the support of $\hat{w}^n$ for all n.

Lemma 5.4 *Let $0 < \varepsilon < 1/3$ and $2^s\varepsilon < 1/2$. Then for all $n < 2^s$,*

$$\frac{1}{\pi} \operatorname{supp}_+ \hat{w}^n \subset I_{\eta(n)} := [\eta(n) - \varepsilon_n, \eta(n) + 1 + \varepsilon_n] \tag{5.14}$$

where

$$\varepsilon_n = 2^j\varepsilon \quad \textit{if} \quad 2^{j-1} \le n < 2^j \ , \ j = 0, 1, \dots \ . \tag{5.15}$$

Proof. We proceed by induction on dyadic groups of n's: $2^{j-1} \le n \le 2^j - 1$. The value of η appearing on the right in (5.11) and (5.12) is $\eta(n_1)$, where $n_1 = [n/2]$. With n and n_1 so chosen, $\varepsilon_n = 2\varepsilon_{n_1}$. Using the notations of §1, we let N stand for L (or M) from (1.20), C for A (or B) from (1.13), Γ for Γ' (or Γ'') from (5.3) and (5.4) if n is even or odd, respectively.

We have $w^n = N(w^{n_1})$, hence (see (1.13)) $\hat{w}^n(y) = C(y/2)\hat{w}^{n_1}(y/2)$. Consequently,

$$\frac{1}{\pi} \operatorname{supp}_+ \hat{w}^n(y) = \left(\frac{1}{\pi} \operatorname{supp} C(y/2)\right) \cap \frac{2}{\pi} \operatorname{supp}_+ \hat{w}^{n_1}(y) \ .$$

To start the induction, we use $w^0 = \varphi$ and $\eta(0) = 0$ to ascertain that $\pi^{-1} \operatorname{supp}_+ \hat{w}^0 = [0, 1 + 2^0\varepsilon]$. Similarly, $w^1 = \psi$, $\eta(1) = 1$ and

$$\begin{aligned} \frac{1}{\pi} \operatorname{supp}_+ \hat{w}_1 &= \frac{1}{\pi} \operatorname{supp}_+ \widehat{N(\varphi)} \\ &= [1 - \varepsilon, 2 + 2^1\varepsilon] \subset [\eta(1) - 2^1\varepsilon, \eta(1) + 1 + 2^1\varepsilon] \ . \end{aligned} \tag{5.16}$$

Let $2^{j-1} \le n \le 2^j - 1$, for some j with $2 \le j < s$. Then, by the induction hypothesis,

$$\frac{1}{\pi} \operatorname{supp}_+ \hat{w}^n \subset \Gamma \cap J_{n_1}$$

where $J_{n_1} := [2\eta(n_1) - 2\varepsilon_{n_1}, 2\eta(n_1) + 2 + 2\varepsilon_{n_1}] = [2\eta(n_1) - \varepsilon_n, 2\eta(n_1) + 2 + \varepsilon_n]$.

We consider the four cases:

Case 1: n is even, $\eta(n_1) = \eta(n/2)$ is odd. This is the second line of (5.11), hence $\Gamma = \Gamma'$, $\eta(n) = 2\eta(n_1) + 1$, $\eta(n) \equiv -1 \bmod 4$. Thus for some $r_0 \in \mathbb{Z}$, $\eta(n) = 4r_0 - 1$ and $2\eta(n_1) = 4r_0 - 2$. We have $J_{n_1} = [\eta(n) - 1 - \varepsilon_n, \eta(n) + 1 + \varepsilon_n]$. Since $\varepsilon_n = 2^j \varepsilon < 1/2$, J_{n_1} intersects only one interval $[4r - 1 - \varepsilon, 4r + 1 + \varepsilon]$ of Γ', namely one with $r = r_0$, that is, $[\eta(n) - \varepsilon, \eta(n) + 2 + \varepsilon]$. This is guaranteed by the inequalities $\eta(n) - 1 - \varepsilon_n > \eta(n) - 2 - \varepsilon$ and $\eta(n) + 1 + \varepsilon_n < \eta(n) + 4 - \varepsilon$. The intersection is

$$[\eta(n) - \varepsilon, \eta(n) + 1 + \varepsilon_n] \; . \tag{5.17}$$

Case 2: n is even, $\eta(n_1)$ is even. This is the first line of (5.11), when $\Gamma = \Gamma'$, $\eta(n) = 2\eta(n_1)$, $\eta(n) \equiv 0 \bmod 4$, and the intersection of Γ' with $J_{n_1} = [\eta(n) - 2\varepsilon_n, \eta(n) + 1 + 2\varepsilon_n]$ is

$$[\eta(n) - 2\varepsilon_n, \eta(n) + 1 + \varepsilon] \; . \tag{5.18}$$

Case 3: n is odd, $\eta(n_1) = \eta((n-1)/2)$ is even — is the first line of (5.12). Here $\Gamma = \Gamma''$, $\eta(n) = 2\eta(n_1) + 1$, $\eta(n) \equiv 1 \bmod 4$. The intersection of Γ'' with J_{n_1} produces a single interval, namely (5.17).

Case 4: n is odd, $\eta(n_1)$ is odd — this is the second line of (5.12). Now $\Gamma = \Gamma''$, $\eta(n) = 2\eta(n_1)$, $\eta(n) \equiv 2 \bmod 4$. The intersection of J_{n_1} with Γ'' is here an interval of the type (5.18).

Since $\varepsilon \le \varepsilon_n$, the intervals of all of these cases are contained in $I_{\eta(n)}$ so that the induction is complete. □

To the w^n correspond $\tilde{w}^n$, which are real trigonometric polynomials on $\mathbb{T}_1$. The set of all frequencies j of a real trigonometric polynomial $T(x) = \sum_{j \in \mathbb{Z}} c_j e^{-2\pi i j x}$, that is, the set of $j \ge 0$ with $c_j \ne 0$, we call the *spectrum* $\operatorname{Sp}(T)$ of T. By Poisson's formula, $j \in \operatorname{Sp}(\tilde{w}^n)$ if and only if j is an interior point of $(1/2\pi) \operatorname{supp}_+ \hat{w}^n$.

We also need the spectra of the elements $\tilde{w}^n_{k,\ell} := \widetilde{w^n_{k,\ell}}$ of the packet base $(\widetilde{\mathcal{W}_s})$ of §3. By (1.1), the support of $\hat{w}^n_{k,\ell}(y)$ is independent of ℓ and is equal to the support of $\hat{w}^n(y/2^k)$. This yields

$$\frac{1}{2\pi} \operatorname{supp}_+ \hat{w}^n_{k,\ell}(y) = 2^{k-1} \frac{1}{\pi} \operatorname{supp}_+ \hat{w}^n(y) \subset 2^{k-1} I_{\eta(n)} \; , \qquad n = 0, 1, 2, \dots \; .$$

Selecting an ε with $2^{-s-3} \le \varepsilon < 2^{-s-2}$, we get:

Theorem 5.5. *The spectrum of* $\tilde{w}^n$, $n = 0, 1, \dots, 2^s - 1$, *consists of one single point, namely,* $\eta^*(n) := [(\eta(n) + 1)/2]$. *Thus* $\tilde{w}^0 = 1$ *and*

$$\tilde{w}^n(x) = \sqrt{2}\cos 2\pi(\eta^*(n)x + \alpha_n)) , \qquad n = 1, 2, \dots, 2^s - 1 . \tag{5.19}$$

For the spectrum of the element $\tilde{w}^n_{k,\ell}$ *of* $(\widetilde{\mathcal{W}_s})$, $n < 2^s$, $k = 1, 2, \dots$, $0 \le \ell < 2^k$ *we have (with the open interval* $I^0_{\eta(n)}$*)*

$$\mathrm{Sp}(\tilde{w}^n_{k,\ell}) \subset 2^{k-1} I^0_{\eta(n)} ; \tag{5.20}$$

in particular,

$$\deg \tilde{w}^n_{k,\ell} \le 2^{k-1}(\eta(n) + 2) . \tag{5.21}$$

Proof of Theorem 5.1. We can now prove the existence of a c.o.n.b. $(T_m)_0^\infty$ with the properties (i), (ii) of the theorem, in particular with

$$\deg T_m \le \max\left(\left[\frac{m+1}{2}\right], (1+\varepsilon)\frac{m}{2}\right) , \qquad m = 0, 1, \dots \tag{5.22}$$

(In the paper of R.A. Lorentz and Sahakian [1994], this inequality is given not quite correctly, with the first term under max missing.)

Let $0 < \varepsilon < 1/3$, we select an s with $2^{-s-1} \le \varepsilon < 2^{-s}$, and use the packet basis $(\widetilde{\mathcal{W}_s})$ of §3; it consists of trigonometric polynomials on $\mathbb{T}_1$. The basis begins with the packet

$$\tilde{w}^0 = 1, \tilde{w}^1, \tilde{w}^2, \tilde{w}^3, \dots, \tilde{w}^{2^s-1} . \tag{5.23}$$

Then follow blocks $j = 1, \dots$; the j-th block consists of 2^{s+j-1} elements $\tilde{w}^n_{j,\ell}$, $2^{s-1} \le n < 2^s$, $\ell = 0, 1, \dots, 2^j - 1$. We arrange them in rows:

$$\begin{cases} \tilde{w}^{2^{s-1}}_{k,0}, & \tilde{w}^{2^{s-1}}_{k,1}, & \dots, & \tilde{w}^{2^{s-1}}_{k,2^k-1} \\ \vdots & \vdots & & \vdots \\ \tilde{w}^n_{k,0}, & \tilde{w}^n_{k,1}, & \dots, & \tilde{w}^n_{k,2^k-1} \\ \vdots & \vdots & & \vdots \\ \tilde{w}^{2^s-1}_{k,0}, & \tilde{w}^{2^s-1}_{k,1}, & \dots, & \tilde{w}^{2^s-1}_{k,2^k-1} \end{cases} \tag{5.24}$$

We rearrange the packet (5.23) according to the increasing values of $\eta(n)$, $0 \le n < 2^s$. Thus, the new function in position ν will be $\tilde{w}^n$ with $n = \eta^{-1}(\nu)$. We rearrange the rows of each block (5.24), so that the new row ν, $2^{s-1} \le \nu < 2^s$ will be the old row n with $n = \eta^{-1}(\nu)$.

We then relabel the elements of $(\widetilde{\mathcal{W}_s})$ by $T_0, T_1, \dots$. Then $T_0, T_1, \dots, T_{2^s-1}$ will be the elements $\tilde{w}^n$ in their new order. The elements $\tilde{w}^n_{k,\ell}$ of the blocks (5.24) we label by the remaining T_m lexicographically: first with respect to k, then with respect to the new row number ν, then with respect to ℓ.

By (5.19), the elements of (5.23) will become T_m with

$$T_0(x) = 1 , \quad T_{2m}(x) = \sqrt{2}\cos(mx + \alpha_m) , \quad T_{2m+1}(x) = \sqrt{2}\cos(mx + \beta_m) ,$$

which conforms with (5.22).

An element $\tilde{w}^n_{k,\ell}$ will be in the k-th block (5.24) and the new row $\nu = \eta(n)$. In the rearranged $(\widetilde{\mathcal{W}}_s)$, it will be preceded by

$$2^s + \sum_{j=1}^{k-1} 2^{s+j-1} + 2^k(\nu - 2^{s-1}) + \ell = 2^k\nu + \ell$$

elements. This yields $\tilde{w}^n_{k,\ell} = T_m$, $m = 2^k\eta(n) + \ell + 1$. On the other hand, since $\eta(n) \geq 2^{s-1}$, from (5.21) we get

$$\deg T_m \leq 2^{k-1}(\eta(n) + 2) \leq \frac{m}{2}\left(1 + \frac{2}{\eta(n)}\right) \leq \frac{m}{2}(1 + 4\varepsilon) \ , \qquad m \geq 2^s \ .$$

Adjusting ε, we get (5.22).

The sums $S_m(f)$ satisfy (ii) by Theorem 3.4. □

Corollary 5.6. *It follows from* (5.20) *(for proper $\varepsilon > 0$ and large n) that the partial sums S_n of the orthogonal expansion for (T_j) reproduce all trigonometric polynomials of degree $\leq n/3$.*

This has interesting applications. Since $\|U_n\| \leq C$ with a constant $C > 0$, it follows that the $S_n(f)$ provide excellent polynomial approximation:

$$\|f - S_n(f)\|_\infty \leq C_1 E_{n/3}(f) \ , \qquad f \in C(\mathbb{T}) \ . \tag{5.25}$$

In fact, if T^* is the polynomial of best approximation to f of degree $m = [n/3]$, then $\|f - T^*\| = E_{n/3}(f)$, $T^* = S_n(T^*)$, and (5.25) follows from

$$\|f - S_n(f)\| \leq \|f - T^*\| + \|S_n(T^* - f)\| \ .$$

We can also derive a theorem about c.o.n.b. consisting of algebraic polynomials on $[-1, 1]$. (We have not seen a proof of a theorem of this type in the literature.) For this purpose we need the basis $(\widetilde{\mathcal{V}})$ of §3 on $\mathbb{T}_1$, constructed for the function ϕ of Theorem 5.2:

$$(\widetilde{\mathcal{V}}) \ : \qquad 1; \tilde{\psi}_{0,0}; \tilde{\psi}_{1,0}, \tilde{\psi}_{1,1}; \ldots; \tilde{\psi}_{k,0}, \ldots, \tilde{\psi}_{k,2^k-1}; \ldots$$

If we relabel the elements of $(\widetilde{\mathcal{V}})$ by $T_0, T_1, \ldots$, we obtain

$$\deg T_m \leq (1 + \varepsilon)m \ , \qquad m = 0, 1, \ldots . \tag{5.26}$$

It will be sufficient to construct an o.n.b. of even trigonometric polynomials with the property (5.26) which spans $L_2^e(\mathbb{T})$.

Theorem 5.7. *There exists a c.o.n.b. on* $[-1,1]$, *which consists of polynomials* $P_m \in \mathcal{P}$, *with closed span* $L_2^e(T_1)$, *and with the properties given in Theorem 5.1, but with* (5.2) *replaced by* (5.26).

Proof. Using formulas (1.1), (1.2), (1.13), (1.14) we obtain, for ϕ of Theorem 5.2, that for $k = 0, 1, \ldots$, $\ell = 0, \ldots, 2^k - 1$,

$$\hat{\psi}_{k,\ell}(y) = 2^{-k/2}\hat{\phi}(2^{-k-1}y)A(2^{-k-1}y+\pi)e^{i(\ell-1/2)2^{-k}y} .$$

Here the functions A and $\hat{\phi}$ are even and positive, while for $0 \le \ell, \ell' \le 2^k - 1$, $\ell + \ell' = 2^k - 1$,

$$i\left(\ell - \tfrac{1}{2}\right)2^{-k}y = -i\left(\ell' - \tfrac{1}{2}\right)2^{k}y ,$$

whenever $y = 2\pi r$, $r \in \mathbb{Z}$. We let

$$\psi_{k,\ell}^e := \psi_{k,\ell} + \psi_{k,\ell'} \quad ; \quad \psi_{k,\ell}^o := \psi_{k,\ell} - \psi_{k,\ell'} .$$

It follows from the Poisson formula (3.4) that

$$\bar{\psi}_{k,\ell}^e := \overline{\psi_{k,\ell}^e} := \bar{\psi}_{k,\ell} + \bar{\psi}_{k,\ell'} , \qquad \bar{\psi}_{k,\ell}^o := \bar{\psi}_{k,\ell} - \bar{\psi}_{k,\ell'}$$

are, respectively, even and odd polynomials T_m^e, T_m^o; we normalize them by $\|T_m^e\| = \|T_m^o\| = 1$. As in the proof of Theorem 5.1, the orthogonal expansions with respect to T_m^e, T_m^o will converge in each space X, and the polynomials T_m^e alone will suffice for all even functions of X. □

We conjecture that Theorem 5.7 holds with the upper bound (5.2) instead of (5.26).

§ 6. Problems and Notes

Problems

6.1. Formula (1.12) is far from the unique way to construct wavelets in a given multiresolution.
A function (1.11) is a wavelet of the multiresolution (V_k) if and only if the corresponding function B has the property $B(y) = \lambda(y)\overline{A(y+\pi)}$, where λ is 2π periodic with $\lambda(y) + \lambda(y+\pi) = 0$ and $|\lambda(y)| = 1$ a.e.

6.2. Prove that all spaces $\widetilde{W}_s$, $s < 0$ in Theorem 3.1 contain only the zero function.

Notes

6.3. Another, more general approach to wavelet theory is via the approximation from shift-invariant spaces, on $\mathbb{R}$ (compare [**CA**, §7 of Chapter 13]), or in several dimensions. We can quote only a few examples, de Boor [1993], de Boor, DeVore and Ron [1995].

6.4. We have proved that assumption (2.2) is sufficient for the density of the linear combinations of the functions of $(\mathcal{V})$ in $C_0(\mathbb{R})$. Related to this is the necessary and sufficient condition for these functions to be dense in $L_2(\mathbb{R})$, namely

$$\lim_{k\to\infty} \frac{1}{2^{-k}|I|} \int_{2^{-k}I} |\hat{\phi}(y)|^2 \, dy = 1$$

for any interval $I \subset \mathbb{R}$ (see Madych [1992, p.266], compare also Plonka and Tasche [1995]).

6.5. The inverse $g(n) = \eta^{-1}(n)$ of the function η is the Grey code, used in the coding theory (see Hamming [B-1986]). For x, y equal to 0 or 1, let $x \vee y$ be defined by $x \vee y = 1$ if $x \neq y$, $x \vee y = 0$ otherwise. For (finite) dyadic expansions of integers $m = \sum_{j=0}^{\infty} \mu_j 2^j$, $n = \sum_{j=0}^{\infty} \nu_j 2^j$ we define $m \dot{+} n := \sum_{j=0}^{\infty} (\mu_j \vee \nu_j) 2^j$. Then $g = \eta^{-1}$ is given by the formula $g(n) = n \dot{+} [n/2]$.

6.6. In the paper Kilgore, Prestin, Selig [1996], the authors construct, for each $\varepsilon > 0$, an orthonormal Schauder basis $\{P_n\}$ for $C[0,1]$, consisting of algebraic polynomials with $\deg P_n \leq (1+\varepsilon)n$. The orthogonality is with respect to the Chebyshev weight.

Chapter 6. Splines

§ 1. General Facts

Splines were the subject of [CA, Chapters 5, 12, 13 and of §7 of Chapter 7]. In the present chapter we discuss some further facts, mainly of qualitative nature, concerning splines as well as spline spaces needed in Chapters 13, 14, and 15, which deal with widths and entropy.

Let us recall some basic definitions. Given a strictly increasing finite or biinfinite sequence $T^* := (t_i^*)_1^m$ or $T^* := (t_i^*)_{-\infty}^{\infty}$, with $|t_i^*| \to \infty$ for $|i| \to \infty$, we call a function S on $\mathbb{R}$ a *spline of order* r (equivalently of degree $r-1$), with *breakpoints* t_i^* if S is a polynomial of degree $\le r-1$ on each of the intervals (t_i^*, t_{i+1}^*), and on $(-\infty, t_1^*)$, (t_m^*, ∞) if T^* is finite, and on one of them of degree exactly $r-1$. At the points t_i^* the spline can be discontinuous or continuous, with various degrees of smoothness; it is not defined at its discontinuity points. The smoothness k_i of a spline S of order r at its breakpoint t_i^* is $k_i = 0$ if S is discontinuous at t_i^*, otherwise it is the largest integer $0 < k_i \le r$ for which $S \in C^{k_i-1}(\mathcal{U})$ for some neighborhood $\mathcal{U}$ of t_i^*. Thus, $k_i = r$ means that S is a polynomial of degree $\le r-1$ on $\mathcal{U}$. The number $r - k_i$ is sometimes called the *defect* of the spline at t_i^*.

We shall often consider splines defined on a finite interval $[a, b]$. In this case we assume that $a < t_1^* < \ldots < t_m^* < b$. We shall also consider periodic splines on $\mathbb{R}$, assuming that the sequence of the breakpoints is infinite and periodic, with finitely many points on each period. Equivalently, one may define periodic splines as splines on $\mathbb{T}$, with the usual identifications. For $A = \mathbb{R}$, $[a, b]$, or $\mathbb{T}$, we denote by $\mathcal{S}_r(A)$ the set of all splines of order $\le r$ on A. Given A, T^*, and the sequence of integers $\mu := (\mu_i)_1^m$, $1 \le \mu_i \le r$, we define the *Schoenberg space* $\mathcal{S}_r(T^*, \mu, A)$ as the set of all splines of order $\le r$ with smoothness $k_i \ge r - \mu_i$ at all breakpoints $t_i^* \in T^*$. The space $\mathcal{S}_r(T^*, 1, A)$ with *simple breakpoints*, that is, with $\mu_i = 1$ for all i, is the smallest; it is contained in any other space $\mathcal{S}_r(T^*, \mu, A)$. The space with all $\mu_i = r$ (called also the *space of piecewise polynomials*) is the largest; the splines in this space may be discontinuous at every breakpoint.

The space $\mathcal{S}_r(T^*, \mu, A)$ is a linear space of dimension $N := r + \sum_{i=1}^m \mu_i$. For $A = [a, b]$ or $\mathbb{R}$, it can be equivalently defined (see [CA, p.136]) as the set of all functions S of the form

$$S(x) = \sum_{i=1}^{m} \sum_{k=1}^{\mu_i} a_{i,k}(x - t_i^*)_+^{r-k} + \sum_{j=0}^{r-1} b_j x^j, \tag{1.1}$$

where $a_{i,k}$, b_j are real numbers.

Often a slightly different point of view is preferable. Instead of m breakpoints $T^* = (t_i^*)$ with multiplicities μ_i, one takes $n := \sum_{i=1}^m \mu_i$ *(basic) knots* $T : a < t_1 \le \ldots \le t_n < b$, with the same multiplicities:

$$t_1 = \ldots = t_{\mu_1} = t_1^* < t_{\mu_1+1} = \ldots = t_{\mu_1+\mu_2} = t_2^* < \ldots .$$

The terms *breakpoints, knots* serve to distinguish these points of view. However, they produce the same Schoenberg spaces $\mathcal{S}_r := \mathcal{S}_r(T^*, \mu, [a,b]) = \mathcal{S}_r(T, [a,b])$. In order to obtain a spline basis suitable for knots, in addition to the basic knots, we have to select arbitrary *auxiliary knots* $t_{-r+1} \le \ldots \le t_0 \le a$ and $b \le t_{n+1} \le \ldots \le t_{n+r}$. Then we define *B-splines* $N_j(x)$, $j = -r+1, \ldots, n$ by the formula:

$$N_j(x) := N(x; t_j, \ldots, t_{j+r}) := (t_{j+r} - t_j)[t_j, \ldots, t_{r+j}](t-x)_+^{r-1}, \tag{1.2}$$

where $[t_1, \ldots, t_r]f(x)$ is the *divided difference* (see [**CA**, p.120]) of the function f. The new *B-spline basis*, called also the *Schoenberg basis*, consists of all N_j of (1.2). Each $S \in \mathcal{S}_r$ has the unique representation

$$S(x) = \sum_{j=-r+1}^{n} c_j N_j(x), \quad a \le x \le b. \tag{1.3}$$

Thus, the spline space $\mathcal{S}_r$ on $[a,b]$ does not depend on the choice of the auxiliary knots, but the representation (1.3) does. It follows from (1.3) that the N_j form a partition of unity: $\sum N_j(x) = 1$ on A.

One of the properties of the B-splines N_j is that they are locally supported: $\operatorname{supp} N_j = [t_j, t_{j+r}]$, with $N_j(x) > 0$ on (t_j, t_{j+r}).

The N_j that do not vanish on an interval $I = (c, d]$ are linearly independent on it. If $S(x) = 0$ on I, it follows that for all j for which $\operatorname{supp} N_j$ contains interior points of I, the coefficient c_j is zero. In this case, the sum (1.3) breaks up into two parts, one with $\operatorname{supp} N_j$ to the left, the other to the right of I. Similarly, if $S(x) = 0$ on $(-\infty, c)$, then the sum (1.3) reduces to terms with N_j that depend only on the knots $t_k \ge c$. In particular, $S \in \mathcal{S}_r(T, [a,b])$ vanishes for $x \le t_{-r+1}$ or $x > t_{r+n}$. If $t_{-r+1} = \ldots = t_0 = a$, $t_{n+1} = \ldots = t_{n+r} = b$, then all splines $S \in \mathcal{S}_r$ vanish outside of $[a,b]$. For further properties of B-splines, see [**CA**, Chapter 5].

These formal definitions of spline spaces are usually combined with an informal understanding that r *is fixed or at least bounded, n is large.* Thus, in approximating a function f, the assumption $n \to \infty$ may cause the error of approximation to tend to zero, while the choice of r may take care of some local singularities of f.

The following is a list of some of the important properties of splines from [**CA**, Chapter 5]. For the particularly useful spline spaces $\mathcal{S}_{m,r}^*$ on $[0,1]$ we have added some simple proofs.

1. Let $\mathcal{S}_r$ be the Schoenberg space on [0,1] with breakpoints that satisfy, for some $C_1, C_2, \delta > 0$, the inequalities $C_1\delta \le t_{j+1}^* - t_j^* \le C_2\delta$. Then for $S \in \mathcal{S}_r$

and $c = (c_j)_{-r+1}^{m}$ of (1.3), one has (de Boor, see [**CA**, p.145]), similar to the Parseval property of the Fourier coefficients,

$$\|S\|_{L_p} \sim \delta^{1/p}\|c\|_{l_p} = \delta^{1/p}\left(\sum |c_j|^p\right)^{1/p}. \tag{1.4}$$

In particular, the space $S_{m,r}^*$ of piecewise polynomials on $[0,1]$, with breakpoints $t_k^* = k/m$, $k = 1, \dots, m-1$, equipped with the L_p norm, $1 \le p \le \infty$, is isomorphic, up to a factor $\sim m^{-1/p}$, to l_p^{mr}.

This fact we can establish, in an elementary way, using another isomorphism, $I : S \leftrightarrow c$, where $S \in S_{m,r}^*$ with the L_p norm, and $c = (c_j)_1^{mr} \in l_p^{mr}$, with $c_j := S((2j-1)/(2mr))$, $j = 1, \dots, mr$. Indeed, if Q be a polynomial of degree $\le r-1$ (order $\le r$), then for any interval $[a,b]$,

$$C_1(b-a)^r \left(\sum_{k=1}^{r} |Q_k|^p\right)^{1/p} \le \|Q\|_{L_p[a,b]} \le C_2(b-a)^r \left(\sum_{k=1}^{r} |Q_k|^p\right)^{1/p},$$

where $Q_k := Q(a + (2k-1)(b-a)/(2r))$ and C_1, C_2 depend only on p, r. For $[a,b] = [0,1]$, this follows from the fact that any two norms on a finite-dimensional Banach space are equivalent; for arbitrary $[a,b]$, it then follows by a change of variable. Using the above inequalities on each subinterval $[(k-1)/m, k/m]$, we obtain for the operator I,

$$C_1 m^{-1/p}\|c\|_p \le \|S\|_p \le C_2 m^{-1/p}\|c\|_p, \quad S = I^{-1}(c). \tag{1.5}$$

2. From (1.4) and the inequality $\|c\|_{l_q} \le \|c\|_{l_p}$, $q \ge p$, follows a spline analogue of the polynomial Nikolskii inequality [**CA**, p.136]:

$$\|S\|_q \le C\delta^{1/q-1/p}\|S\|_p, \quad 1 \le p \le q \le \infty. \tag{1.6}$$

There is also a spline analogue of Markov's inequality:

$$\|S^{(k)}\|_p \le C\delta^{-k}\|S\|_p, \quad k = 1, \dots, r-1, \quad 1 \le p \le \infty. \tag{1.7}$$

3. Unlike polynomials, non-trivial splines S on $\mathbb{R}$ may vanish identically on some intervals, called *zero intervals.* An interval $[c,d]$ is called a *support interval* if it does not contain zero subintervals and is not contained in any larger interval with this property. (The latter means that for some $\delta > 0$, $[c-\delta, c]$ and $[d, d+\delta]$ are zero intervals.) The support of S is the union of its support intervals. A point of discontinuity c of S is called a *discontinuous zero* if S changes sign at c, that is, if $S(c+h_1)S(c-h_2) < 0$ for all small $h_1, h_2 > 0$. A discontinuous zero is counted as zero of multiplicity one. The multiplicity of a conventional (continuous) zero is the largest integer ℓ for which c is a continuous zero of $S, \dots, S^{(\ell-2)}$ and a continuous or discontinuous zero of $S^{(\ell-1)}$. (It should be noted that some authors, for example, Schumaker [A-1981, p.159], have different definitions of spline zeros and their multiplicities.)

One of the useful theorems about spline zeros is as follows [**CA**, p.160]. For each support interval $I := [c, d] \subset [a, b]$ of $S \in \mathcal{S}_r(T^*, [a, b])$, the number of zeros $Z(S, (c, d))$ of S and the number $k(I)$ of knots (counting their multiplicities) of S in I satisfy (Birkhoff-Lorentz)

$$Z(S, (c, d)) \le k(I) - r - 1. \tag{1.8}$$

(One can prove that for a support interval the right-hand side of (1.8) is ≥ 0.)

4. Another relation between the coefficients $c = (c_j)$ in (1.3) and the spline $S \in \mathcal{S}_r(T, [a, b])$ is given in [**CA**, p.164]:

(1.9) *The number of sign changes of S does not exceed the number of sign changes of the sequence (c_j).*

A linear function space $\mathcal{S}$ on $[a, b]$ of dimension N is a *weak Haar space* if its elements S cannot have N sign changes, that is, if there is no $S \in \mathcal{S}$ and $N + 1$ points $a \le x_1 < \ldots < x_{N+1} \le b$ with $S(x_j)S(x_{j+1}) < 0$ for $j = 1, \ldots, N$. From (1.9) one easily derives:

(1.10) *Every Schoenberg space $\mathcal{S}_r = \mathcal{S}_r(T, \mu, [a, b])$ is a weak Haar space.*

5. Since Schoenberg spaces are not Haar spaces, interpolation problems (Lagrange or Hermite) are not always solvable for them. For a linear function space $\mathcal{S}$ on $[a, b]$ of dimension N, the Hermite interpolation problem can be formulated as follows. Let $a \le x_1 \le x_2 \le \ldots \le x_N \le b$ be the interpolation points in $[a, b]$ and let $y_1, \ldots, y_N$ be arbitrary real data. For each i, let l_i be the number of $x_k = x_i$, $k < i$. The Hermite interpolation problem is solvable for the points x_i if the equations

$$S^{(l_i)}(x_i) = y_i, \quad i = 1, \ldots, N, \quad S \in \mathcal{S} \tag{1.11}$$

are (uniquely) solvable for all data (y_i).

Let t_j, x_i be the points of $[a, b]$,

(1.12)
$$a < t_1 = \ldots = t_{\mu_1} < t_{\mu_1+1} = \ldots \le t_n < b, \quad a \le x_1 \le \ldots \le x_{n+r} \le b,$$

with multiplicities $\mu_1, \ldots, \mu_m$ of the t_j. We shall say that they have the *interlacing property* if

$$x_i < t_i < x_{i+r}, \quad i = 1, \ldots, n. \tag{1.13}$$

If the t_i are the knots of a Schoenberg space $\mathcal{S}_r(T, [a, b])$, we need an additional requirement. We want to prevent the possibility that for $S \in \mathcal{S}_r$ the derivative $S^{(l_i)}$ is not defined at x_i. In other words, if x_i is also a knot, $x_i = t_j^*$, then it is assumed that l_i and the multiplicity k_j of t_j^* satisfy

$$l_i + k_j \le r. \tag{1.14}$$

We have (Karlin and Ziegler, see [**CA**, p.162]):

Proposition 1.1. *An Hermite problem* (1.11) *with the interpolation points* $a \le x_1 \le \ldots \le x_{n+r} \le b$ *and the knots* $a < t_1 \le \ldots \le t_n < b$ *satisfying* (1.14) *is solvable if and only if the interlacing condition holds.*

It is worthwhile to formulate (1.13), and therefore Proposition 1.1, in a different form. For a set $A \subset \mathbb{R}$ and points (1.12), we denote by $X(A)$, $T(A)$ the number of $x_i \in A$ or of $t_i \in A$. For simplicity of notation, let $t_0 := a$, $t_{n+1} := b$. Then the interlacing condition (1.13) is equivalent to the assumption that for each $0 \le p \le n+1$,

$$T[t_0, t_p] \le X[t_0, t_p); \quad T[t_p, t_{n+1}] \le X(t_p, t_{n+1}]. \tag{1.15}$$

In terms of the numbers

$$N(p,q) := r + T(t_p, t_q), \quad 0 \le p < q \le n+1 \tag{1.16}$$

we have:

Proposition 1.2. *For the points* x_i, t_j *of* (1.12), *the interlacing condition* (1.13) *is equivalent to*

$$X[t_0, t_p] \le N(0,p); \quad X[t_p, t_{n+1}] \le N(p, n+1), \quad 1 \le p < n+1 \tag{1.17}$$

or to

$$X[t_p, t_q] \le N(p,q), \quad 0 \le p < q \le n+1. \tag{1.18}$$

Proof. Inequalities (1.15) imply that

$$\begin{aligned} X[t_p, t_q] &= n + r - X[t_0, t_p) - X(t_q, t_{n+1}] \\ &\le n + r - T[t_0, t_p] - T[t_q, t_{n+1}] \quad , \\ &= r + T(t_p, t_q) = N(p,q) \end{aligned}$$

which is (1.18). The latter inequality implies (1.17). Finally, relations (1.17) imply (1.15). For example, from the second inequality (1.17), one derives

$$\begin{aligned} X[t_0, t_p) &= n + r - X[t_p, t_{n+1}] \ge n + r - N(p, n+1) \\ &= n - T(t_p, t_{n+1}) = T[t_0, t_p]. \end{aligned}$$

□

The number $N(p,q)$ is the dimension of the restriction $\mathcal{S}_r[t_p, t_q]$ of $\mathcal{S}_r$ to the interval $[t_p, t_q]$, that is, of a Schoenberg space on $[t_p, t_q]$ with the basic knots $t_i \in T$ satisfying $t_p < t_i < t_q$.

6. To underline the importance of the Schoenberg spaces $\mathcal{S}_r(T^*, \mu, A)$, we shall give their definition which does not mention the breakpoints T^*. We call a spline S_1 *at least as smooth* as a spline S_2, if at each point of A where S_2 has derivatives of all orders $l \le l_0$, also S_1 has derivatives of these orders. Then (Lorentz [1989]):

A finite dimensional spline space on $[a,b]$ is a Schoenberg space if and only if it is closed under uniform convergence, and with each spline S it contains any other spline that is at least as smooth as S.

7. We shall now discuss spline approximation of functions $f \in \mathrm{Lip}(\alpha, p)$ on [0,1]. This is done in [**CA**, §7, Chapter 7] by means of the quasi-interpolants, but the following approach, using the spaces $\mathcal{S}^*_{m,r}$, is simpler. By definition [**CA**, p.51], $f \in \mathrm{Lip}(\alpha,p)$, $\alpha = r+\beta$, $r = 0,1,\dots$, $0 < \beta \le 1$, if $f,\dots,f^{(r-1)}$ are absolutely continuous, $f^{(r)} \in L_p[0,1]$, and $|f|_{\mathrm{Lip}(\alpha,p)} = \sup(t^{-\beta}\omega(f^{(r)},t)_p) < \infty$, where

$$\omega(f^{(r)},t)_p := \sup_{0\le\delta\le t}\left(\int_a^{b-\delta} |f^{(r)}(x+\delta) - f^{(r)}(x)|^p\,dx\right)^{1/p} \le t^\beta.$$

In particular, $f \in B_p^\alpha$ if $\omega(f^{(r)},t)_p \le t^\beta$, $t > 0$.

Proposition 1.3. *For $1 \le p,q \le \infty$, $\alpha > (1/p - 1/q)_+$, and any $f \in B_p^\alpha$, there exist splines $S_m \in \mathcal{S}^*_{m,r+1}$ for which, with $C = C(\alpha,p,q)$,*

$$\|f - S_m\|_q \le Cm^{-\alpha+(1/p-1/q)_+}, \quad m = 1,2,\dots. \tag{1.19}$$

The splines S_m are given by linear operators $U_{m,r}$ mapping $\mathrm{Lip}(\alpha,p)$ *into $\mathcal{S}^*_{m,r+1}$.*

Proof. We need to consider only the case $p \le q$. For $k = 1,\dots,m$, let $A_k := ((k-1)h, kh)$, $h := 1/m$, and let $S = S_m$ be the piecewise polynomial in $\mathcal{S}^*_{m,r+1}$ which for $x \in A_k$ is given by the relations

$$S^{(l)}((k-1)h+) = f^{(l)}((k-1)h), \quad l = 0,\dots,r-1, \quad S^{(r)}(x) = h^{-1}\int_{A_k} f^{(r)}dt$$

(if $r = 0$, the conditions at $(k-1)h$ should be dropped). Let first $p = q$, $0 < \alpha < 1$. Then $r = 0$ and S is a piecewise constant function. By Hölder's inequality, for $x \in A_k$ we have

$$|f(x) - S(x)| \le h^{-1}\int_{A_k} |f(x) - f(t)|\,dt \le h^{-1/p}\left(\int_{A_k} |f(x) - f(t)|^p\,dt\right)^{1/p}.$$

Hence, with $\mathcal{B} = \cup_k(A_k \times A_k)$,

$$\int_0^1 |f - S|^p\,dx \le h^{-1}\int_{\mathcal{B}} |f(x) - f(t)|^p\,dxdt \le h^{-1}\int_{D_h} |f(x) - f(t)|^p\,dxdt,$$

where D_h is the strip $\{(x,t) : 0 \le x,t \le 1,\ |x-t| \le h\}$. Now

$$\begin{aligned}\int_{D_h} |f(x) - f(t)|^p\,dxdt &= 2\int_0^h ds\int_0^{1-s} |f(t+s) - f(t)|^p\,dt \\ &\le 2h^{\alpha p}\int_0^h ds = 2h^{\alpha p+1},\end{aligned}$$

so that $\|f - S\|_p^p \le 2h^{\alpha p}$. This proves (1.19) for $p = q$, $0 < \alpha \le 1$.

Now let $p = q$, $\alpha = r + \beta > 1$. Then $S^{(r)}$ is a piecewise constant function, and by the above argument $\|f^{(r)} - S^{(r)}\|_p \le Ch^\beta$. For $x \in A_k$ we have

$$|f^{(r-1)}(x) - S^{(r-1)}(x)| \le \int_{A_k} |f^{(r)}(t) - S^{(r)}(t)|\,dt$$
$$\le h^{1-1/p}\left(\int_{A_k} |f^{(r)} - S^{(r)}|^p\,dt\right)^{1/p},$$

hence

$$\int_{A_k} |f^{(r-1)} - S^{(r-1)}|^p\,dx \le Ch^p \int_{A_k} |f^{(r)} - S^{(r)}|^p\,dx.$$

Adding these inequalities yields the estimate $\|f^{(r-1)} - S^{(r-1)}\|_p \le Ch^{1+\beta}$. We similarly estimate $\|f^{(l)} - S^{(l)}\|_p$ for $l = r-2, \dots, 0$ and obtain (1.19) for $p = q$.

Let now $p < q$. From $\|f - S_m\|_p \le Cm^{-\alpha}|f|$ we derive, if $2^{k-1} \le l \le 2^k$, $\|S_l - S_{2^k}\|_p \le |f|\,2^{-\alpha k}$, $k = 1, 2, \dots$, and therefore, by (1.6),

$$\|S_l - S_{2^k}\|_q \le |f|\,2^{(-\alpha+1/p-1/q)k}.$$

Since this converges to zero, S_{2^k} is a Cauchy sequence in L_q, which must converge to f. Thus, $\|S_l - f\|_q \le C|f|\,l^{-\alpha+1/p-1/q}$, $l = 1, 2, \dots$. □

There are several books on the theory of splines. Close to the spirit of our exposition is Schumaker [A-1981]. The book Nürnberger [A-1989] treats only splines with simple knots, and mainly their qualitative properties. Two short books of de Boor [A-1978], [A-1990] provide an introduction to B-splines. The books of Korneichuk [A-1984], [A-1991] deal with many different extremal problems for splines, mainly for periodic splines with simple knots. In the books of Bojanov, Hakopian, Sahakian [A-1993] and Lorentz, Jetter, Riemenschneider [A-1983] also Birkhoff splines are discussed, needed in polynomial and spline Birkhoff interpolation. In some applications (for example, in Petrushev, Popov [A-1987] or in Chapters 14, 15 of the present volume) the simplest kind of spline approximation – that by piecewise polynomials – is sufficient.

In the next sections of this chapter we treat: splines of best approximation, the Schoenberg spline operator, periodic splines, and relations between spline and polynomial interpolation.

§ 2. Splines of Best Approximation

In this section we discuss best approximation to a given function f of a Banach space X by splines from a given family. This family is usually a Schoenberg space $\mathcal{S}_r := \mathcal{S}_r(T^*, \mu, [a, b])$. We shall assume in this section that all $\mu_i \le r - 1$, so that the splines $S \in \mathcal{S}_r$ are continuous. The existence of a spline of best approximation to f follows from [**CA**, Theorem 1.1, p.59], since $\mathcal{S}_r$ is a linear finite dimensional space. Moreover, if X is strictly convex (as are all the $L_p[a,b]$ for $1 < p < \infty$), a spline of best approximation is unique. The spaces $C[a,b]$ and $L_1[a,b]$ are not strictly convex. In $C[a,b]$, the uniqueness of best approximation from a given n-dimensional subspace X_n for every f is equivalent to the Haar property of X_n. However, $\mathcal{S}_r$ is not a Haar space because it contains non-trivial functions vanishing on intervals; it is only a weak Haar space. As a consequence, not every $f \in C[a,b]$ has a unique best uniform approximation in $\mathcal{S}_r$. On the other hand, by [**CA**, Theorem 8.6, p.330], for every $f \in C[a,b]$ there is at least one spline S^* of best approximation and $N+1 = r + \sum_1^n \mu_i + 1 = \dim \mathcal{S}_r + 1$ points $a \le x_1^* < \ldots < x_{N+1}^* \le b$ with the alternation property:

$$(2.1) \qquad f(x_i^*) - S^*(x_i^*) = \sigma(-1)^i \|f - S^*\|_\infty, \quad \sigma = \pm 1, \quad i = 1, \ldots, N+1.$$

We call a sequence $(x_i^*)_1^{N+1}$ satisfying (2.1) an *alternation sequence* of length $N+1$. It follows from Theorem 2.3 below that all best approximations S_0 to f from $\mathcal{S}_r$ can be characterized by some alternation property, although not all S_0 have an alternation sequence of length $N+1$.

We shall need two lemmas about weak Haar spaces. By definition, an N-dimensional subspace $\mathcal{F}$ of $C[a,b]$ is a weak Haar space if there is no function in $\mathcal{F}$ with N sign changes. The next lemma shows that under certain conditions a non-trivial function from $\mathcal{F}$ cannot have N sign changes even if one allows equalities. We say that an N-dimensional function space $\mathcal{F}$ *interpolates at the points* $(x_i)_1^N$ if for any sequence $(y_i)_1^N$ of real numbers there is a (unique) $f \in \mathcal{F}$ for which $f(x_i) = y_i$, $i = 1, \ldots, N$.

Lemma 2.1. *Let $\mathcal{F}$ be an N-dimensional weak Haar subspace of $C[a,b]$ and let $a \le x_1 < \ldots < x_{N+1} \le b$ be some $N+1$ points with the property that $\mathcal{F}$ interpolates at every N-point subset of (x_i). Then for each $g \in \mathcal{F}$ with the property*

$$(2.2) \qquad (-1)^i g(x_i) \ge 0, \quad i = 1, \ldots, N+1$$

one has $g = 0$.

Proof. Let $\{g_j\}_1^N$ be a basis of $\mathcal{F}$. Due to the interpolation property of the (x_i), all the determinants

$$D_k := \det\,[g_j(x_i)], \quad j = 1, \ldots, n; \quad i = 1, \ldots, N+1,\ i \ne k$$

are non-zero, and by [**CA**, Theorem 12.1, p.92] they all are of the same sign. Since $\mathcal{F}$ is N-dimensional, the determinant $D := \det\,[g_j(x_i)]_{i,j=1}^{N+1}$, with $g_{N+1} := g$, is equal to zero, hence $\sum_1^{N+1} (-1)^i g(x_i) D_i = D = 0$. Therefore, if (2.2) holds, we have $g(x_i) = 0$ for all i. Since $\mathcal{F}$ interpolates at (x_i), this implies $g = 0$. □

Lemma 2.2. *If $\mathcal{F}$ is an N-dimensional weak Haar subspace of $C[a,b]$, then for every natural $k \le N$ and every sequence $a = x_0 < x_1 < \ldots < x_{k-1} < x_k = b$, there is a non-trivial $g \in \mathcal{F}$ for which*

$$(2.3) \qquad (-1)^i g(x) \ge 0, \quad x \in [x_{i-1}, x_i], \quad i = 1, \ldots, k.$$

Proof. For Haar spaces, this follows from [**CA**, Theorem 9.1, p.80]. Let now $\Phi := (\phi_1, \ldots, \phi_N)$ be a weak Haar system. By [**CA**, Theorem 8.5, p.329], there is a sequence of Haar systems $\Phi_l := (\phi_{1,l}, \ldots, \phi_{N,l})$ on $[a,b]$ with $\|\phi_{i,l} - \phi_i\| \to 0$ for $l \to \infty$, $i = 1, \ldots, N$. We now use the following general fact: If $\Phi = (\phi_i)_1^N$ is a linearly independent system in a Banach space, then for any $\phi = \sum_1^N c_j \phi_j$ one has $|c_j| \le \|\phi\|/d(\Phi)$, where $d(\Phi) := \min_j d_j(\Phi)$, and $d_j(\Phi)$ is the distance from ϕ_j to $\mathrm{lin}\{\phi_i,\, i \ne j\}$. Indeed, if $c_j \ne 0$, then

$$\|\phi\| = \|c_j \phi_j + \sum_{i \ne j} c_i \phi_i\| = |c_j| \|\phi_j + \sum_{i \ne j} (c_i/c_j)\phi_i\| \ge |c_j|\, d(\Phi).$$

For each l, we can find a $g_l := \sum_{i=1}^N c_{i,l}\phi_{i,l}$ that changes signs according to (2.3). Since obviously $d(\Phi_l) \to d(\Phi)$, we have $d(\Phi_l) > (1/2)d(\Phi)$ for sufficiently large l. Therefore the $|c_{i,l}|$ are uniformly bounded by a constant independent of i, l, so that there exists a subsequence g_{l_j} uniformly convergent on $[a,b]$ to some $g \in \mathcal{F}$, and this g has the desired properties. □

The following theorem by Rice [1967] and Schumaker [1968] characterizes, by means of alternation sequences, all splines in $\mathcal{S}_r$ of best approximation to f. With the knots $t_0 := a$, $t_{n+1} := b$, let $N(p,q)$ denote, as in §1, the dimension of the space $\mathcal{S}_r[t_p, t_q]$, $0 \le p < q \le n+1$.

Theorem 2.3. *Let $\mathcal{S}_r = \mathcal{S}_r(T, \mu, [a,b])$ be a Schoenberg space on $[a,b]$ with knots t_j, $j = 1, \ldots, n$. Then $S_0 \in \mathcal{S}_r$ is a spline of best uniform approximation to a function $f \in C[a,b] \setminus \mathcal{S}_r$ if and only if for some $0 \le p < q \le n+1$, the interval $[t_p, t_q]$ contains a sequence of $N(p,q) + 1$ alternating extreme points of $f - S_0$.*

Proof. (a) Let $f - S_0$ have an alternation sequence of length $N(p,q) + 1$ in $[t_p, t_q]$. If there existed a spline $S \in \mathcal{S}_r$ of better approximation, $\|f - S\| < \|f - S_0\|$, then $S_0 - S$ would have $N(p,q)$ sign changes on $[t_p, t_q]$, which is impossible since the restriction $\mathcal{S}_r[t_p, t_q]$ of $\mathcal{S}_r$ to $[t_p, t_q]$ is a weak Haar space of dimension $N(p,q)$.

(b) Let $S_0 \in \mathcal{S}_r$ be a best approximation to $f \notin \mathcal{S}_r$, and let S^* be the special best approximation for which (2.1) holds at some $N+1$ points $(x_i^*)_1^{N+1}$. There exist intervals $[t_p, t_q]$ that contain $N(p,q) + 1$ points x_i^* of (2.1) – for example, the interval $[t_0, t_{n+1}]$. We select a $[t_p, t_q]$ of smallest length $q - p$. Let $x_1 < \ldots < x_{N(p,q)+1}$ be the subsequence of the x_i^* contained in $[t_p, t_q]$. Each subinterval $[t_k, t_l]$ of $[t_p, t_q]$ of length $l - k < q - p$ contains $\le N(k,l)$ points x_i. If we omit any one of the x_i, also $[t_p, t_q]$ will acquire this property. We can then apply Propositions 1.2 and 1.1 and deduce that $\mathcal{S}_r[t_p, t_q]$ interpolates on any subset of $N(p,q)$ points x_i.

Because S_0 and S^* are splines of best approximation and because for each x_i^*, $S_0(x_i^*)$ is at least as close to $f(x_i^*)$ as $S^*(x_i^*)$, we conclude that $f(x_i) - S^*(x_i)$ and $S(x_i) - S^*(x_i)$, $i = 1, \dots, N(p,q)+1$, cannot be of opposite sign. It follows from (2.1) that for some $\sigma_1 = \pm 1$

$$\sigma_1(-1)^i[S_0(x_i) - S^*(x_i)] \geq 0, \quad i = 1, \dots, N(p,q)+1.$$

We apply Lemma 2.1 and deduce that $S_0 - S^* = 0$ on $[t_p, t_q]$. Now (2.1) yields the required property of S_0 on $[t_p, t_q]$. □

The natural problem to characterize those $S_0 \in \mathcal{S}_r$ of best approximation to f that are unique, has been solved by Nürnberger and Singer [1982]. The answer is given in terms of not only alternation sequences but also of the so-called flatness, not discussed here. Instead, we shall give a characterization of those S_0 that are strongly unique, also by Nürnberger (Theorem 2.6). Both theorems appear in the book of Nürnberger [A-1989, p.132-143] which, however, is restricted to splines with simple knots.

By definition [**CA**, p.77], given a subspace G of a Banach space X, an element $g_0 \in G$ is called a *strongly unique* best approximation to $f \in X$ if there exists a constant γ (depending on f and g_0) so that for any $g \in G$

$$\|f - g\| \geq \|f - g_0\| + \gamma\|g - g_0\|. \tag{2.4}$$

Thus, strong uniqueness implies that g_0 is the unique best approximation to f.

For approximations from $\mathcal{S}_r$, we shall give here a definitive test for strong unicity. As in Theorem 2.3, all that matters here is the distribution of the alternation points of the difference $f - S_0$ with respect to the knots. We first prove a general criterion of strong unicity which is closely related to Kolmogorov's theorem [**CA**, Theorem 2.2, p.64].

Theorem 2.4 (Wulbert [1971]). *Let X_n be a finite dimensional subspace of the space $C(A)$ of continuous real-valued functions on the compact Hausdorff topological space A. For a given $f \in C(A) \setminus X_n$, an element $g_0 \in X_n$ is a strongly unique best approximation from X_n if and only if for every non-trivial $g \in X_n$ one has*

$$\min_{x \in A_0} (f(x) - g_0(x))g(x) < 0, \tag{2.5}$$

where

$$A_0 := A(g_0) := \{x \in A : |f(x) - g(x)| = \|f - g_0\|\}.$$

Proof. Let B be the unit ball of X_n. Suppose that (2.5) holds and consider the mapping $\Psi : B \to \mathbb{R}$ given by the formula

$$\Psi(g) = \min_{x \in A_0} \frac{f(x) - g_0(x)}{\|f - g_0\|} g(x).$$

The mapping Ψ is continuous on the compact set B, and since $\Psi(g) < 0$ for all $g \in X_n$, there is some constant $\gamma > 0$ so that $\Psi(g) \le -\gamma$ for all $g \in B$. Equivalently,

$$\min_{x \in A_0} (f(x) - g_0(x))g(x) \le -\gamma \|f - g_0\| \|g\| \tag{2.6}$$

for all $g \in X_n$. It follows from (2.6) that for any given $g \in X_n$ there is some $x^* \in A_0$ for which

$$(f(x^*) - g_0(x^*))(g(x^*) - g_0(x^*)) \le -\gamma \|f - g_0\| \|g - g_0\|.$$

Therefore

$$\|f - g\| \ge |f(x^*) - g(x^*)| = |(f(x^*) - g_0(x^*)) - (g(x^*) - g_0(x^*))|$$

$$= |f(x^*) - g_0(x^*)| + |g(x^*) - g_0(x^*)| \ge \|f - g_0\| + \gamma \frac{\|f - g_0\|}{|f(x^*) - g_0(x^*)|} \|g - g_0\|$$

$$= \|f - g_0\| + \gamma \|g - g_0\|,$$

so that g_0 is strongly unique.

Suppose now that (2.5) fails. Then for some $g \in X_n$, $g \neq 0$, we have $(f(x) - g_0(x))g(x) \ge 0$ on $A(g_0)$. We take any $\gamma > 0$ and show that (2.4) cannot be always true. We can assume that

$$\|g\| < \|f - g_0\|/2, \tag{2.7}$$

and select $0 < \delta < \gamma \|g\|$. It is easy to construct an open set $G \supset A(g_0)$, so that for $x \in G$ either $|g(x)| < \delta$ or

$$(f(x) - g_0(x))g(x) \ge 0 \tag{2.8}$$

On the compact set $A \setminus G$ we have

$$|f(x) - g_0(x)| < \|f - g_0\|. \tag{2.9}$$

We perform a small perturbation and define $g_\varepsilon := g_0 + \varepsilon g$, $0 < \varepsilon < 1$. We have, for $\varepsilon \to 0$, $g_\varepsilon \to g_0$. Also, $A(g_\varepsilon) \to A(g_0)$; more exactly, $A(g_\varepsilon) \subset U(A(g_0))$ for all small $\varepsilon > 0$ and each neighborhood U of $A(g_0)$.

From (2.9), for all small ε,

$$|f(x) - g_\varepsilon(x)| < \|f - g_0\|, \quad x \in A \setminus G. \tag{2.10}$$

For $x \in G$, if $|g(x)| < \delta$, then

$$|f(x) - g_\varepsilon(x)| \ge |f(x) - g_0(x)| + |\varepsilon g(x)| \le \|f - g_0\| + \varepsilon\delta.$$

If $x \in G$ and (2.8) is valid, then

$$|f(x) - g_\varepsilon(x)| \le \max\{|f(x) - g_0(x)|, g(x)\} \le \|f - g_0\|.$$

We have $\|f - g_\varepsilon\| \le \|f - g_0\| + \varepsilon\delta$ for all small ε, and therefore

$$\|f - g_\varepsilon\| \le \|f - g_0\| + \varepsilon\gamma\|g\| = \|f - g_0\| + \gamma\|g_\varepsilon - g_0\|.$$

Thus, (2.4) is wrong for the chosen γ. □

For given $0 \le p < q \le n+1$, let $\mathcal{S}_r^0[t_p, t_q]$ be the subspace of all splines $S \in \mathcal{S}_r = \mathcal{S}_r(T, [a,b])$ vanishing in $[a,b] \setminus [t_p, t_q]$. We denote $N_0(p,q) := \dim \mathcal{S}_r^0[t_p, t_q]$. As we agreed earlier, $t_0 := a$, $t_{n+1} := b$, so that $N_0(0, n+1) = \dim \mathcal{S}_r = n + r := N$. As in §1 we denote by $T[t_p, t_q]$, $T(t_p, t_q)$ the number of knots t_j with $t_p \le t_j \le t_q$ or with $t_p < t_j < t_q$. If $T[t_p, t_q] < r$, then $\mathcal{S}_r^0[t_p, t_q] = \{0\}$.

Lemma 2.5. *The spline space $\mathcal{S}_r^0[t_p, t_q]$ is a weak Haar space of dimension*

$$N_0(p,q) = (T[t_p, t_q] - r)_+ \ . \tag{2.11}$$

Proof. The restriction of $\mathcal{S}_r^0[t_p, t_q]$ to $[t_p, t_q]$ is the linear space of those $S \in \mathcal{S}_r[t_p, t_q]$ that satisfy the conditions $S^{(k)}(t_p) = 0$, $k = 0, \dots, r - \mu_p$ and $S^{(l)}(t_q) = 0$, $l = 0, \dots, r - \mu_q$, where μ_p, μ_q are the multiplicities of the knots t_p, t_q. The derivatives $S^{(k)}(t_p)$, $S^{(l)}(t_q)$ are linearly independent linear functionals on the space $\mathcal{S}_r[t_p, t_q]$ since by [**CA**, Theorem 1.1, p.135] there is a spline $S \in \mathcal{S}_r[t_p, t_q]$ for any combination of their values. Since by (1.16) $\dim \mathcal{S}_r[t_p, t_q] = N(p,q) = r + T(t_p, t_q)$, we have

$$N_0(p,q) = N(p,q) - (r - \mu_p) - (r - \mu_q) = T[t_p, t_q] - r.$$

The support of every B-spline N_j of the space $\mathcal{S}_r[a,b]$ is an interval of the form $I_j := (t_j, t_{j+r})$, and $N_j \in \mathcal{S}_r^0[t_p, t_q]$ if and only if $I_j \subset [t_p, t_q]$. There are exactly $T[t_p, t_q] - r$ of such B-splines, hence they form a basis for $\mathcal{S}_r^0[t_p, t_q]$, and by (1.10) $\mathcal{S}_r^0[t_p, t_q]$ is a weak Haar space. □

We can now prove Nürnberger's theorem [1982]. For all knots t_k we define $\Delta_{k,l} := (t_k, t_l)$, $1 \le k < l \le n$ (if $t_k = t_l$, $\Delta_{k,l}$ will be empty), $\Delta_{0,l} = [t_0, t_l)$, $1 \le l \le n$, $\Delta_{k,n+1} = (t_k, t_{n+1}]$, $1 \le k \le n$, and $\Delta_{0,n+1} = [t_0, t_{n+1}] = [a,b]$.

Theorem 2.6. *For a given $f \in C[a,b] \setminus \mathcal{S}_r$, a spline S_0 is a strongly unique best uniform approximation to f from $\mathcal{S}_r$ if and only if every non-empty interval $\Delta_{k,l}$ contains an alternating sequence for $f - S_0$ of length $\ge N_0(k,l) + 1$.*

Proof. (a) *Necessity.* Suppose that there is an interval $\Delta_{k,l}$ in which one can find an alternation sequence for $f - S_0$ of length at most $m \le N_0(k,l)$. Let A_0 be the set of all points $x \in [a,b]$ where $|f(x) - S_0(x)| = \|f - S_0\|$. We suppose that $1 \le k < l \le n$ (for $k = 0$ or $l = n+1$ our argument will need some obvious minor adjustments). Then there exists a partition $t_k = z_0 < z_1 < \dots < z_m = t_l$ for which the sets $A_i := (z_{i-1}, z_i) \cap A_0$, $i = 1, \dots, m$, are non-empty, with the sign of $f - S_0$ being constant on each A_i and alternating when i advances from 1 to m. By Lemma 2.2, since $\mathcal{S}_r^0[t_k, t_l]$ is a weak Haar space of dimension $N_0(k,l)$, there is an $S \neq 0$ in $\mathcal{S}_r^0[t_k, t_l]$ for which

$$(-1)^i S(x) \geq 0, \qquad x \in [z_{i-1}, z_i], \quad i = 1, \ldots, m.$$

Since $S = 0$ outside $[t_k, t_l]$, replacing S by $-S$ if necessary, we obtain

$$(f(x) - S_0(x))S(x) \geq 0 \qquad \text{for } x \in A_0, \tag{2.12}$$

which implies, by Theorem 2.4, that S_0 is not a strongly unique best approximation to f.

(b) *Sufficiency.* Suppose now that the extreme points of $f - S_0$ have the asserted property but S_0 fails to be a strongly unique best approximation to f. Then (2.12) holds for some non-trivial $S \in \mathcal{S}_r$. We consider three possibilities.

(1) S has no zero intervals. By assumption, there are $N+1$ alternating extreme points of $f - S_0$ in $[a, b]$. One can find, as in the proof of Theorem 2.3, an interval $[t_p, t_q]$ containing $\nu := N(p,q) + 1$ alternating extreme points $(x_i)_1^\nu$ of $f - S_0$, so that $\mathcal{S}_r$ interpolates at every $(\nu - 1)$-point subset of $(x_i)_1^\nu$. Because of (2.12) one has $\sigma(-1)^i S(x_i) \geq 0$ for $i = 1, \ldots, \nu$ and some $\sigma = \pm 1$. Then, by Lemma 2.1, $S = 0$ on $[t_p, t_q]$, contrary to the assumption that S has no zero intervals.

(2) There is a single zero interval, $[t_v, t_{v'}]$, of S. We suppose that $v > 0$ (the case $v = 0$, $v' < n + 1$ is analogous). By assumption, $f - S_0$ has $\geq N_0(0, v) + 1$ alternating extreme points on $[0, t_v)$. Let t_w be the smallest knot with $t_w > t_v$. Choosing arbitrarily at most r additional points on $[t_v, t_w]$, we obtain $N_0(0, w) + 1$ points (x_i) for which, with some $\sigma = \pm 1$,

$$\sigma(-1)^i S(x_i) \geq 0, \qquad i = 1, \ldots, N_0(0, w) + 1.$$

With the restriction of S to $[t_0, t_w]$ and the space $\mathcal{S}_r[t_0, t_w]$, we now find ourselves in the same situation as with S itself and the space $\mathcal{S}_r$ in case (1). By the same argument we conclude that in $[t_0, t_w]$ there is an interval $[t_p, t_q]$ containing $\geq N(p, q) + 1$ points x_i on which S vanishes. This interval $[t_p, t_q]$ is certainly not $[t_v, t_w]$ because the latter contains $\leq r$ points x_i while $N(v, w) = r$. Therefore $[t_p, t_q]$ contains a zero interval, a contradiction.

(3) A similar argument applies when there exist two disjoint zero intervals of S separated by some interval (t_u, t_v). Let t_α, t_β be the largest knot $< t_u$ and the smallest knot $> t_v$. By assumption, the interval (t_u, t_v) contains $\geq N_0(u, v) + 1$ alternating extreme points of $f - S_0$, so we choose some additional points in the intervals (t_α, t_u) and (t_v, t_β), no more than r points in each, to the total of $N(\alpha, \beta) + 1$, and obtain a contradiction as in the case (2). □

We shall conclude by examining the uniqueness problem also for the $L_1[a, b]$ approximation. A Schoenberg space $\mathcal{S}_r = \mathcal{S}_r(T, [a, b])$ is not a unicity subspace of L_1 because by [**CA**, Theorem 10.7, p. 85], no finite dimensional subspace of L_1 has this property. However, we shall show that unicity holds for each *continuous* function f of L_1. This parallels a similar old result by

Jackson [**CA**, Theorem 10.9, p. 86]), where $\mathcal{S}_r$ is replaced by any Haar subspace of $C[a,b]$. As before, we assume that $\mu_i \le r-1$ for $i = 1, \dots, n$, so that $\mathcal{S}_r$ consists of continuous splines.

Theorem 2.7 (Galkin [1974], Strauss [1975]). *The best $L_1[a,b]$-approximation from the space $\mathcal{S}_r(T,[a,b])$ to any continuous function f is unique.*

Proof. We use the following simple observation: If g_1 and g_2 are best L_1 approximations from the same subspace G to some f, and if f, g_1, and g_2 are continuous, then $(f(x)-g_1(x))(f(x)-g_2(x)) \ge 0$ for every $x \in [a,b]$. This follows from the fact that also $(1/2)(g_1+g_2)$ is a best approximation to f and hence

$$\int_a^b \{|f-g_1| + |f-g_2| - |2f-(g_1+g_2)|\}\, dx = 0.$$

Suppose that for some $f \in C[a,b] \setminus \mathcal{S}_r$ we have two different best L_1 approximations from $\mathcal{S}_r$. Without loss of generality we assume that these are $S=0$ and $S_1 \ne 0$. Then $S_0 := (S_1+0)/2 = S_1/2$ is also a best approximation to f. We claim that

(2.13) every zero of $f - S_0$ is a zero of S_0.

Indeed, since S_1 and 0 are best approximations to f, we have $(f(x) - S_1(x))f(x) \ge 0$, that is, $(f(x)-S_0(x))^2 \ge S_0(x)^2$ for all x. Consequently, if $f(x_0) - S_0(x_0) = 0$ for some x_0, we must have $S_0(x_0) = 0$. We derive a contradiction.

The spline S_0 has a *support interval* $A := [t_p, t_q]$ which may be $[a,b]$. (This means that S_0 has no interval zeros in A but is zero in $(t_p-\varepsilon, t_p)$ and in $(t_q, t_q+\varepsilon)$ for small $\varepsilon > 0$.). By (1.8), the number of zeros of S_0 (and hence of $f-S_0$) on (t_p,t_q) does not exceed $T[t_p,t_q] - r - 1 = N_0 - 1$, $N_0 := N_0(p,q)$. Then $f - S_0$ has $\le N_0 - 1$ sign changes inside A. Let x_i be the points where this takes place. We add points $x_0 = t_p$, $x_k = t_q$ and get $k+1 \le N_0+1$ points so that $f - S_0$ is of alternating signs on the intervals (x_{i-1}, x_i). The dimension of the weak Haar space $\mathcal{S}_r^0$ is N_0. Using Lemma 2.2, we obtain a spline $S \in \mathcal{S}_r^0$, $S \ne 0$, which satisfies $S(x)(f(x)-S_0(x)) \ge 0$, $x \in A$. Outside of A, $S(x) = 0$. Let A_0 be the (finite) subset of A where $f(x) - S_0(x) = 0$. Then the inequality

$$\int_A S(f-S)\, dx \le \int_{A_0} |S|\, dx$$

does not hold since the right-hand side is zero while the left-hand side is > 0. This contradicts [**CA**, Theorem 10.4, p.84] which gives a necessary and sufficient condition for $S_0 \in G$ to be a best L_1-approximation to f from a linear subspace $G \subset L_1(A)$. □

An important problem of free knots spline approximation deals with the non-linear space $\Sigma_{n,r}$ which consists of all splines of order $\le r$ and arbitrary knots $t_1 \le \ldots \le t_p$, with the sum of multiplicities $\le n$. See [**CA**, Chapter

12, §§4,8] and Chapter 10, §6 of the present book. In Schumaker [A-1981, Ch. 7] and Braess [A-1986, §2 of Ch. 8] one can find further discussion of this problem.

§ 3. Periodic Splines

Polynomials (other than constants) cannot be periodic on $\mathbb{R}$ but *polynomial splines* can. Here and in §4 we shall deal with 2π-periodic splines. Let $T^* := (t_i^*)_{-\infty}^{\infty}$ be a 2π-periodic biinfinite sequence of breakpoints, that is, $t_{i+m}^* = t_i^* + 2\pi$ for some positive integer m and all integer i. Let $\mu = (\mu_i)$ be the corresponding 2π-periodic sequence of multiplicities. We denote by $\widetilde{\mathcal{S}}_r(T^*, \mu, \mathbb{T})$ the space formed by all 2π-periodic polynomial splines of order r, $r \geq 1$, that belong to $\mathcal{S}_r(T^*, \mu, \mathbb{R})$. As in §1, we can use knots $T := (t_j)$ instead of breakpoints, so that $t_{j+n} = t_j + 2\pi$ for all j, where $n := \sum_1^m \mu_i$.

Theorem 3.1. *The dimension of the Schoenberg space $\widetilde{\mathcal{S}}_r(T^*, \mu, \mathbb{T})$ of 2π-periodic polynomial splines is*

$$\dim \widetilde{\mathcal{S}}_r(T^*, \mu, \mathbb{T}) = n = \sum_{i=1}^{m} \mu_i. \tag{3.1}$$

Proof. (a) Suppose first that T^* contains a breakpoint with $\mu_i = r$. Without loss of generality we may assume that $t_1^* = 0$, $\mu_1 = r$. Then the restriction of the space $\widetilde{\mathcal{S}}_r := \widetilde{\mathcal{S}}_r(T^*, \mu, \mathbb{T})$ to $(0, 2\pi)$ coincides with the space $\mathcal{S}_r(T', \mu', (0, 2\pi))$, where $T' := (t_2^*, \ldots, t_m^*)$, $\mu' := (\mu_2, \ldots, \mu_m)$, so that (see §1) $\dim \widetilde{\mathcal{S}}_r = r + \sum_{i=2}^m \mu_i = \sum_{i=1}^m \mu_i$.

(b) Let now, for the given r and μ, $\mu_i < r$ for all i. For an arbitrary $r \geq 2$, let X_r, Y_r be the subspaces of the space $\widetilde{\mathcal{S}}_r$, of codimension 1, defined by the conditions $S(0) = 0$ or $\int_{\mathbb{T}} S\,dx = 0$, respectively. The operator of differentiation establishes a linear isomorphism between X_r and Y_{r-1}: if $S \in X_r$, then $S' \in Y_{r-1}$, and conversely, if $g \in Y_{r-1}$, then $g_1(x) := \int_0^x g(t)dt \in X_r$. It follows that $\dim X_r = \dim Y_{r-1}$, consequently, $\dim \widetilde{\mathcal{S}}_r = \dim \widetilde{\mathcal{S}}_{r-1}$. If for the given r and μ we have $\nu := \max \mu_i < r$, then applying the above argument $r - \nu$ times we obtain $\dim \widetilde{\mathcal{S}}_r = \dim \widetilde{\mathcal{S}}_\nu$ which reduces the question to the case (a). $\square$

In the rest of this section we shall deal with the Schoenberg spaces $\widetilde{\mathcal{S}}_{n,r}$ of polynomial splines of order r on $\mathbb{T}$, with simple knots $\{k\pi/n\}$, $k = 1, \ldots, 2n$. Thus, $\widetilde{\mathcal{S}}_{n,1}$ is the space of all step functions that are constant on the intervals

$$\Delta_k := \left(\frac{(k-1)\pi}{n}, \frac{k\pi}{n} \right), \quad k = 1, 2, \ldots, 2n,$$

and $\widetilde{\mathcal{S}}_{n,r}$, $r \geq 2$, is the space of C^{r-2} splines S on $\mathbb{T}$ with $S^{(r-1)} \in \widetilde{\mathcal{S}}_{n,1}$. By Theorem 3.1, the space $\widetilde{\mathcal{S}}_{n,r}$ is for every r a linear space of dimension $2n$.

A function $f \perp 1$ in $L_1(\mathbb{T})$ (for instance, the derivative of a function on $\mathbb{T}$) has a unique periodic integral with mean value zero. The same applies to the integral g of f of order m defined by $g^{(m)} = f$. Thus, there is a unique $g \perp 1$ for each m, or equivalently, a unique g up to a constant if the last requirement is dropped.

An important role in the following discussion will be played by the splines $\varphi_{n,r} \in \widetilde{S}_{n,r+1}$, $r \geq 0$, which are related to the Euler splines $\mathcal{E}_r$ of [**CA**, pp.148–150]:

$$\varphi_{n,r}(t) = \begin{cases} K_r n^{-r}\mathcal{E}_r(nt/\pi) & \text{for } r \text{ even} \\ K_r n^{-r}\mathcal{E}_r(nt/\pi - 1/2) & \text{for } r \text{ odd} \end{cases}, \tag{3.2}$$

$r = 0, 1, \ldots,\ n = 1, 2, \ldots,$ where K_r are the Favard constants

$$K_r = \frac{4}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^{l(r+1)}}{(2l+1)^{r+1}}. \tag{3.3}$$

An alternative definition is $\varphi_{n,0}(t) := \operatorname{sign}\sin nt$ and

$$\varphi_{n,r}(t) := \int_{t_{0,r}}^{t} \varphi_{n,r-1}(s)ds, \quad r = 1, 2, \ldots, \tag{3.4}$$

where

$$t_{k,r} := t_k := \begin{cases} k\pi/n & \text{for r even} \\ (k-1/2)\pi/n & \text{for r odd} \end{cases}. \tag{3.5}$$

From the last definition or from [**CA**, pp.148-150] it is not difficult to derive the main properties of the $\varphi_{n,r}$:

(i) $\varphi_{n,r}$ is even if r is odd and odd when r is even, and $\varphi_{n,r} \perp 1$;

(ii) $\varphi_{n,r}$ has period $2\pi/n$, satisfies $\varphi_{n,r}(t + \pi/n) = -\varphi_{n,r}(t)$;

(iii) The only zeros of $\varphi_{n,r}$ are the simple zeros $t_{k,r}$ of (3.5);

(iv) $\|\varphi_{n,r}\|_\infty = K_r n^{-r}$, and $\varphi_{n,r}(t_{n,r+1}) = \pm K_r n^{-r}$ with alternating signs.

The $\varphi_{n,r}$ appear in the theorem of Favard [**CA**, p.213] which concerns with the error of approximation of functions in the unit ball B_∞^r of the Sobolev space $W_\infty^r(\mathbb{T})$:

$$\max_{f \in B_\infty^r} E_{n-1}(f) = K_r n^{-r}, \tag{3.6}$$

and this maximum is attained for $f = \varphi_{n,r}$.

In this section we discuss mainly the $\widetilde{S}_{n,r}$ interpolation of functions on $\mathbb{T}$.

Theorem 3.2. *Space $\widetilde{S}_{n,r}$, $r \geq 1$, interpolates at the points $t_{k,r}$, that is, for any set of real numbers $(y_1, y_2, \ldots, y_{2n})$ there exists a unique spline $S \in \widetilde{S}_{n,r}$ for which*

$$S(t_{k,r}) = y_k, \quad k = 1, \ldots, 2n.$$

Proof. We have to prove that the linear operator U defined by the formula

$$U(S) = (S(t_{1,r}), \dots, S(t_{2n,r}))$$

is a one-to-one map from $\widetilde{S}_{n,r}$ onto $\mathbb{R}^{2n}$. This is obvious for $r = 1$ for the $t_{k,r}$ are the middle points of the Δ_k, so we take $r \geq 2$. Since $\widetilde{S}_{n,r}$ and $\mathbb{R}^{2n}$ are of the same dimension, it suffices to show that $U(S) = 0$ implies $S = 0$. Assuming the contrary, suppose that for some $S_0 \neq 0$ we have $U(S_0) = 0$. Then $S_0 \neq$ const., so that $S_0^{(r-1)} \neq 0$ on, say, Δ_ν. Let $R := \varphi_{n,r-1} - \lambda S_0$, where λ is determined from the condition that $R^{(r-1)}(t) \equiv 0$ on Δ_ν. We have $R(t_{k,r}) = \pm\|\varphi_{n,r}\|_\infty$, with alternating signs, and by Rolle's theorem, $R^{(r-2)}(t)$ should also change sign at least $2n$ times on $\mathbb{T}$, which is impossible since it is a broken line of $2n$ pieces of which one is horizontal. □

The inequalities (3.7) and (3.8) of the next theorem, due to Tikhomirov [1960] and Subbotin [1970], respectively, are spline analogues of the Bernstein inequalities for trigonometric polynomials. Let $\operatorname{Var} S$ denote the variation of $S(t)$ on $\mathbb{T}$.

Theorem 3.3. *For $n, r = 1, 2, \dots$, and every $S \in \widetilde{S}_{n,r+1}$,*

$$\|S^{(r)}\|_\infty \leq K_r^{-1} n^r \max_k |S(t_{k,r+1})| \leq K_r^{-1} n^r \|S\|_\infty. \tag{3.7}$$

For every $S \in \widetilde{S}_{n,r}$,

$$\operatorname{Var} S^{(r-1)} \leq K_r^{-1} n^r \|S\|_1. \tag{3.8}$$

Both inequalities are exact: the equality in (3.7) holds for $S = \varphi_{n,r}$ and in (3.8) for $S = \varphi_{n,r-1}$.

Proof. To prove (3.7), we may assume that $\|S^{(r)}\|_\infty = |S^{(r)}(t)| = 1$ on some Δ_ν. We set $R := \varphi_{n,r} - \lambda S$, where $\lambda = \pm 1$ is chosen to make $R^{(r)}(t) = 0$ on Δ_ν. If we suppose that $\max_k |S(t_{k,r+1})| < \|\varphi_{n,r}\| = K_r n^{-r}$, then $R(t)$ will have $2n$ sign changes at the points $t_{k,r+1}$. By Rolle's theorem, $R^{(r-1)}(t)$ must then have $2n$ sign changes on $\mathbb{T}$, but this is impossible as in the proof of Theorem 3.2.

The inequality (3.8) can be derived from (3.7) as follows. First we make some remarks about $\widetilde{S}_{n,r+1}$. For a given $S \in \widetilde{S}_{n,r+1}$, let $S_k, S_k^{(r)}$ denote $S(t_{k,r+1})$ and the value of $S^{(r)}(t)$ on Δ_k, respectively, $k = 1, \dots, 2n$. Then $S_1^{(r)}$ is a linear functional on $\widetilde{S}_{n,r+1}$ and by Theorem 3.2 it is also a linear functional on the space of vectors $(S_1, \dots, S_{2n})$, hence

$$S_1^{(r)} = a_1 S_1 + \dots + a_{2n} S_{2n},$$

where $a_1, \dots, a_{2n}$ do not depend on S. Taking $S =$const. we get

$$a_1 + \dots + a_{2n} = 0. \tag{3.9}$$

Moreover, by (3.7), $|a_1S_1 + \ldots + a_{2n}S_{2n}| \le K_r^{-1}n^r \max|S_k|$ for all $(S_k)_1^{2n}$. This implies

$$|a_1| + \ldots + |a_{2n}| \le K_r^{-1}n^r.$$

Replacing $S(t)$ in (3.9) by $S(t + (k-1)\pi/n)$, we also get

$$S_k^{(r)} = a_1S_k + \ldots + a_{2n}S_{k+2n-1}, \qquad k = 1, \ldots, 2n. \tag{3.10}$$

If now $S \in \widetilde{S}_{n,r}$, we define $Q(t)$ to be the periodic integral of $S(t) - C$, $C := (2\pi)^{-1}\int_{\mathbb{T}} S\,dt$. Then $Q \in \widetilde{S}_{n,r+1}$, and by (3.10)

$$Q_{k+1}^{(r)} - Q_k^{(r)} = a_1(Q_{k+1} - Q_k) + \ldots + a_{2n}(Q_{k+2n} - Q_{k+2n-1}),$$

or

$$S_{k+1}^{(r+1)} - S_k^{(r+1)} = a_1\int_{t_{k,r+1}}^{t_{k+1,r+1}} (S(t) - C)\,dt + \ldots + a_{2n}\int_{t_{k+2n-1,r+1}}^{t_{k+2n,r+1}} (S(t) - C)\,dt.$$

Due to (3.9), C may be dropped here, and we have

$$\operatorname{Var} S^{(r-1)} = \sum_{k=1}^{2n} \left|S_{k+1}^{(r-1)} - S_k^{(r-1)}\right| \le (|a_1| + \ldots + |a_{2n}|)\int_{\mathbb{T}} |S|dt = K_r^{-1}n^r\|S\|_1.$$

□

Theorem 3.2 allows the use of splines $S \in \widetilde{S}_{n,r}$ for interpolation at the points $t_{k,r}$. For an $f \in C(\mathbb{T})$ we get the interpolating spline

$$U_{n,r}(f,t) = \sum_{k=1}^{2n} f(t_{k,r})\sigma_k(t), \quad r \ge 1,\ n = 1, 2, \ldots,$$

where $\sigma_k \in \widetilde{S}_{n,r}$ are the fundamental interpolating splines, defined by the conditions $\sigma_k(t_{j,r}) = \delta_{k,j}$, $k, j = 1, \ldots, 2n$. Taking $f(t) = 1$, we obtain

$$\sigma_1(t) + \ldots + \sigma_{2n}(t) = 1, \quad t \in \mathbb{T}. \tag{3.11}$$

Moreover, since all the σ_k are translates of σ_1, we get $\int_{\mathbb{T}} \sigma_k\,dt = \pi/n$, $k = 1, \ldots, 2n$. For the remainder we have

$$f(t) - U_{n,r}(f,t) = \sum_{k=1}^{2n}(f(t) - f(t_{k,r}))\sigma_k(t). \tag{3.12}$$

In particular, we apply this to the functions $f \in W_1^r(\mathbb{T})$ which have absolutely continuous derivatives $f^{(k)}$, $k = 0, \ldots, r-1$, with $f^{(r)} \in L_1(\mathbb{T})$. They possess the representation [CA, (5.14), p.151]:

$$f(t) = C + \frac{1}{\pi}\int_{\mathbb{T}} \mathcal{B}_r(t-s)f^{(r)}(s)\,ds, \quad r = 1, 2, \ldots \tag{3.13}$$

where C is the mean value of f and $\mathcal{B}_r$ is the Bernoulli spline. The $\mathcal{B}_r$ are splines of order $r+1$ on $\mathbb{T}$ with the single simple knot $t = 0$. They are defined by $\mathcal{B}_1(t) = (\pi - t)/2$ for $0 \le t < 2\pi$, with $\mathcal{B}_r, r > 1$, being the r-th periodic integral, with mean value zero, of $\mathcal{B}_1(t)$. Since $f^{(r)} \perp 1$, we can replace $\pi^{-1}\mathcal{B}_r(t-s)$ in (3.13) by the kernel

$$H_r(t,s) := \pi^{-1}[\mathcal{B}_r(t-s) - \mathcal{B}_r(t)], \quad r \ge 1. \tag{3.14}$$

Substituting into (3.12) yields the formula

$$f(t) - U_{n,r}(f,t) = \int_{\mathbb{T}} G_{n,r}(t,s) f^{(r)}(s)\, ds, \tag{3.15}$$

with

$$G_{n,r}(t,s) := G(t,s) := H_r(t,s) - \sum_{k=1}^{2n} \sigma_k(t) H_r(t_{k,r}, s)\,. \tag{3.16}$$

For example, on $[0, 2\pi)^2$,

$$H_1(t,s) = \begin{cases} s/(2\pi) - 1 & \text{for } 0 \le t < s \\ s/(2\pi) & \text{for } s \le t < 2\pi \end{cases},$$

and from (3.16), $G_{n,1}(t,s) = +1$ or $G_{n,1}(t,s) = -1$ if, respectively, $(k + 1/2)\pi/n \le s \le t < (k+1)\pi/n$ or $k\pi/n < t \le s < (k+1/2)\pi/n$ for some $k = 1, \ldots, 2n$; $G_{n,1}(t,s) = 0$ in all other cases (except for (t,s) with $t = k\pi/n$ when the $\sigma_k(t)$ are undefined).

We shall need

Lemma 3.4. *Let $\widetilde{\mathcal{S}}_r = \widetilde{\mathcal{S}}_r(T, \mathbb{T})$, $r \ge 3$, be a Schoenberg space with T consisting of $2n+1$ simple knots. If $S \in \widetilde{\mathcal{S}}_r$ $S \ne 0$, has $2n$ distinct zeros on $\mathbb{T}$ that are either (a) all located at the knots or (b) all located in the open intervals Δ between them, at most one zero in each interval, then all zeros are simple, and S has no other zeros.*

Proof. First, S cannot vanish identically on an interval Δ. For otherwise there would be an adjacent interval Δ' which contains a known zero of S other than the common endpoint of Δ and Δ'. Then on Δ', S would have a zero of order $r-2$ at this endpoint and one additional zero, and would vanish on Δ'. Continuing this argument, we arrive, in case (a), at $2n$ intervals Δ containing all given zeros. On the remaining interval S would be also zero, for at its endpoints it has altogether $2(r-1) \ge r$ zeros. In case (b), after $2n-1$ steps, we conclude that S would vanish on all intervals Δ except for at most two intervals. For one interval we argue as before. If two intervals would remain on which S is not identically zero, the closure of their union would be a support interval of **3**, §1. From (1.8) we would get that it contains $\ge r+1 \ge 4$ knots, a contradiction.

So S has a finite number of zeros. If this number were $\geq 2n$ (counting multiplicities), then S' would change sign at least $2n+1$ times on $\mathbb{T}$, and therefore, as a function on $\mathbb{T}$, at least $2n+2$ times. The same would be true for $S^{(r-1)}$ which is again impossible since $S^{(r-1)}$ is a step function with at most $2n+1$ breaks. $\square$

The kernel $G_{n,r}(t,s)$ has been studied by Zhensykbaev [1974] and Korneichuk [1977], see also [A-1987, §5.2.4]. It has remarkable properties described by the following theorem. We shall write $\beta_r = 0$ if r is even, $\beta_r = 1/2$ if r is odd. Thus, $t_{k,r} = (k-\beta_r)\pi/n$. Let $\Delta_k := ((k-1)\pi/n, k\pi/n)$, $\Delta'_l := (t_{l,r}, t_{l+1,r})$, and let $Q_{k,l} := \Delta_k \times \Delta'_l$, $k,l \in \mathbb{Z}$, be the squares in the (t,s)-plane. For $r \geq 3$, the signs of $G_{n,r}(t,s)$ exhibit a checkerboard distribution on the squares $Q_{k,l}$. For $r=2$, the situation is different for the kernel vanishes on all $Q_{k,l}$ with $k \neq l$.

Theorem 3.5. *For $r \geq 2$ there is some $\theta_r = \pm 1$ so that*

(i) $G_{n,r}(t,s) = 0$ *on the lines $t = t_{k,r}$ and $s = k\pi/n$.*

(ii) $\theta_r \sin(nt + \beta_r\pi) G_{n,r}(t,s) \sin ns > 0$ *for $r \geq 3$ if (t,s) is not on the lines (i). For $r = 2$, this holds only on the squares $Q_{k,k}$, with $G_{n,2} = 0$ elsewhere.*

(iii) $G_{n,r}(t,s) = (-1)^r G_{n,r}(s - \beta_r\pi/n,\ t - \beta_r\pi/n)$, $\quad t,s \in \mathbb{R}$ *(for even r this means that $G_{n,r}$ is symmetric).*

Proof. For every fixed $s \in \mathbb{T}$, $H_r(t,s)$, as a function of t, is a C^{r-2} spline with knots at $t=0$ and at $t=s$. It is a spline of order r since the terms containing t^r, namely $-(1/2)(t^r/r!)$, in $\mathcal{B}_r(t-s)$ and $\mathcal{B}_r(t)$ cancel out on each interval Δ_k. As a function of s, $H_r(t,s)$ is a spline of order $r+1$, with a single knot $s=t$. It follows that $G(t,s)$, as a function of t, is a C^{r-2} spline of order r, with knots $(k\pi/n)$ and $t=s$. As a function of s, it is a C^{r-2} spline, with knots (t_k) and $s=t$. Moreover, it is, too, a spline of order r since the polynomial terms containing s^r in (3.16) cancel out due to the identity $\sum_{k=1}^{2n} \sigma_k(t) \equiv 1$.

We have $G(t_j, s) = 0$, $j = 0, \ldots, 2n-1$, for every s, by the definition of G. In particular, $G(t_j, k\pi/n) = 0$ for all j,k. Furthermore, $G(t, k\pi/n) = 0$ for all $t \in \mathbb{T}$ and all k. Indeed, for every k, $G(t,k\pi/n)$ is a C^{r-2} spline with knots only at $(j\pi/n)$, therefore it belongs to $\widetilde{S}_{n,r}$, and by Theorem 3.2 must be identically zero.

It follows from the formula for $H_1(t,s)$, since $\partial H_2(t,s)/\partial t = H_1(t,s)$, that $H_2(t,s)$ as a function of t is continuous, linear and decreasing for $t<s$, linear and increasing for $t>s$, $0 \leq t,s < 2\pi$. By (3.16), $G_{n,2}(t,s) = H_2(t,s) - S$, where S is the broken line interpolating $H_2(t,s)$ at the knots $t = k\pi/n$. Consequently, $G_{n,2}(t,s) < 0$ if t,s belong to the same interval Δ_k (in which case $\sin nt \sin ns > 0$); otherwise $G_{n,2}(t,s) = 0$. This proves (ii) for $r=2$, with $\theta_2 = -1$.

Let $u_s(t) := G(t,s)$. If $r \geq 3$, the function $u_s(t)$ is a C^{r-2} spline of order r, with $2n+1$ knots, vanishing at the $2n$ points (t_k). By Lemma 3.4, for every fixed s, the function $u_s(t)$ changes sign exactly at the points t_k. We see that for $r \geq 2$, $u_s(t)$ has the sign alternation of (ii). If, in addition to this fact, we prove (iii), then (ii) will follow.

We now compare the splines $u_s(t)$ and $v_s(t) := G(s - \beta_r\pi/n,\, t - \beta_r\pi/n)$. Both are C^{r-2} splines with knots at $(k\pi/n)$ and at $t = s$, and with zeros at (t_k). The same is true for $w(t) := u_s(t) - \lambda v_s(t)$, with any $\lambda \in \mathbb{R}$. Moreover, $w(t) = 0$ identically for some $\lambda = \lambda(s)$. This is obvious for $r = 2$ since in this case both u_s and v_s are broken lines that do not vanish only on the interval Δ_k containing s where they have exactly one knot $t = s$. For $r \geq 3$, we take t^* with $v_s(t^*) \neq 0$ and find $\lambda = \lambda(s)$ for which $w(t^*) = 0$. Then, again by Lemma 3.4, $w = 0$ identically. Thus, $u_s = \lambda(s)v_s$, or equivalently

$$G(s,t) = \lambda(s)G(t - \beta_r\pi/n,\, s - \beta_r\pi/n), \quad t, s \in \mathbb{T} \tag{3.17}$$

Using the definition of H and G, the fact that $\mathcal{B}_r$ has mean value zero and the relation $\int_{\mathbb{T}} \sigma_k dt = \pi/n$, we obtain by straightforward computation:

$$A(s) := \int_{\mathbb{T}} G(s,t)\,dt = -2\mathcal{B}_r(s) + 2\sum_{k=1}^{2n} \mathcal{B}_r(t_k)\sigma_k(s),$$

$$B(s) := \int_{\mathbb{T}} G(t - \beta_r\pi/n,\, s - \beta_r\pi/n)\,dt = -n^{-1}\sum_{k=1}^{2n} \left[\mathcal{B}_r(k\pi/n - s) - \mathcal{B}_r(t_k)\right].$$

Both $A(s)$ and $B(s)$ are splines with the knots at $(k\pi/n)$. On each subinterval Δ_k they are represented by polynomials of degree r with the leading coefficients that differ by the factor $(-1)^r$, hence $A(s) - (-1)^r B(s) \in \widetilde{\mathcal{S}}_{n,r}$. We further observe that $A(t_j) = B(t_j) = 0$ for $j = 1, \dots, 2n$. For A, this follows from the properties of of σ_k; for B from the fact that the sets $\{t_k\}_k$ and $\{(k\pi/n - t_j)_j\}$ are identical on $\mathbb{T}$. Since splines of $\widetilde{\mathcal{S}}_{n,r}$ are uniquely determined by their values at (t_j), we have identically $A(s) - (-1)^r B(s) = 0$. Integrating (3.17), we derive from this fact that $\lambda(s) = (-1)^r$ for all s (note that $A(s)$ vanishes only on a finite set of points). This completes the proof of the theorem. □

With Korneichuk, we use Theorem 3.5 to prove

Theorem 3.6. *If $f \in W_\infty^r(\mathbb{T})$, $r \geq 1$, and $\|f^{(r)}\|_\infty \leq 1$, then for every t,*

$$|f(t) - U_{n,r}(f,t)| \leq |\varphi_{n,r}(t)|. \tag{3.18}$$

Proof. For f as defined and for every t, we have by (3.15)

$$|f(t) - U_{n,r}(f,t)| \leq \int_{\mathbb{T}} |G(t,s)|\,ds.$$

Due to (ii) of Theorem 3.5, the integral on the right equals

$$\left|\int_{\mathbb{T}} G(t,s)\text{sign } \sin ns\,ds\right| = |\varphi_{n,r}(t) - U_{n,r}(\varphi_{n,r}, t)| = |\varphi_{n,r}(t)|$$

since $\varphi_{n,r}(t_{k,r}) = 0$ and therefore $U_{n,r}(\varphi_{n,r}) = 0$. □

As an immediate corollary we get for $1 \le q \le \infty$,

$$\|f - U_{n,r}f\|_q \le \|\varphi_{n,r}\|_q \|f^{(r)}\|_\infty \tag{3.19}$$

(for the significance of this for the widths, see §7 of Chapter 13).

The norms $\|\varphi_{n,r}\|_q$ can be evaluated explicitly for some q. From the definition of $\varphi_{n,r}$ follows that $\|\varphi_{n,r}\|_q = n^{-r}\|\varphi_{1,r}\|_q$. We have

$$\|\varphi_{1,r}\|_\infty = K_r, \quad \|\varphi_{1,r}\|_1 = 4K_{r+1}, \quad \|\varphi_{1,r}\|_2 = 2\sqrt{K_{2r+1}}. \tag{3.20}$$

The value $\|\varphi_{1,r}\|_\infty = K_r n^{-r}$ can be found in [**CA**, p.213]. The other two norms can be evaluated by means of the Fourier series for $\varphi_{1,r}$. Let, for example, r be even. Then

$$\varphi_{1,r}(t) = (-1)^{r/2}\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin(2k+1)t}{(2k+1)^{r+1}} . \tag{3.21}$$

This follows from the Fourier expansion of sign $\sin t$ by repeated integration since the right-hand side of (3.21) has mean value zero. The $\varphi_{1,r}$ is an odd function and by the property (iii) of the $\varphi_{n,r}$ stated earlier in this section, it changes sign on $(-\pi, \pi)$ only at $t = 0$, hence $\|\varphi_{1,r}\|_1 = 2|\int_0^\pi \varphi_{1,r}dt|$ which can be readily computed using (3.21). Similar considerations apply for r odd. For the L_2 norm one can use Parseval's equality.

§ 4. Convergence of Some Spline Operators

In this section we consider two topics introduced by Schoenberg. The Schoenberg operator V_r maps $C(I)$, $I = [a, b]$ into the Schoenberg space $\mathcal{S}_r(T, I)$, $r \ge 2$ of splines on I with the simple knots $T : a < t_1 < \ldots < t_n < b$. In addition we introduce the auxiliary knots $t_{-r+1} = \ldots = t_0 = a$ and $t_{n+1} = \ldots = t_{n+r} = b$. Then we can define the sequence of B-splines (1.2): $N_j(x) := N_{j,r}(x)$, $j \in \Lambda := \{-r+1, \ldots, n\}$, with the supports $[t_j, t_{j+r}]$.

For each $j \in \Lambda$, let

$$\bar{t}_j := \frac{1}{r-1}\sum_{i=j+1}^{j+r-1} t_i. \tag{4.1}$$

Clearly, $\bar{t}_j$ is in the support of N_j and

$$a = \bar{t}_{-r+1} < \bar{t}_{-r+2} < \cdots < \bar{t}_{n-1} < \bar{t}_n = b. \tag{4.2}$$

The *Schoenberg operator* V_r is defined by

$$V_r(f) := \sum_{j=-r+1}^{n} f(\bar{t}_j)N_j . \tag{4.3}$$

The Bernstein polynomials [**CA**, Ch. 10] are a particular case of this. Indeed, let $a = 0$, $b = 1$, $n = 0$, and $t_{-r+1} = \ldots = t_0 = 0$, $t_1 = \ldots = t_r = 1$. We denote by $N_{s,r}$ the B-spline $N(x; 0, \ldots, 0, 1, \ldots, 1)$, with $r + 1 - s$ zeros and s ones, $0 \le s \le r - 1$. The recurrence formula [**CA**, (2.8), p.139] reduces to $N_{s,r} = xN_{s,r-1} + (1 - x)N_{s-1,r-1}$. Induction yields

$$N_{s,r} = \binom{r-1}{s} x^s (1-x)^{r-1-s}.$$

This shows that $V_r(f)$ is the Bernstein polynomial $B_{r-1}(f)$ of f of degree $r - 1$.

With the Bernstein polynomials, V_r shares many shape preserving properties. This operator is in much use in the computer-aided design (see, for example, Barnhill and Riesenfeld [B-1974]). Since the N_j are non-negative functions forming a partition of unity, we have $V_r(f) \ge 0$ on I whenever $f \ge 0$, and $V_r(P_0) = P_0$ for any constant function P_0. The same is true for $P_1(x) = x$. Indeed, by the formula [**CA**, (3.8), p.142], which is valid for any polynomial of degree $\le r - 1$,

$$P = \sum_{j \in \Lambda} c_j(P)\, N_j\,,$$

where, for an arbitrary $\xi_j \in (t_j, t_{j+r}) \cap I$,

$$c_j(P) = \sum_{\nu=0}^{r-1} (-1)^\nu g_j^{(r-\nu-1)}(\xi_j) P^{(\nu)}(\xi_j),$$

with $g_j(x) = (x - t_{j+1}) \cdots (x - t_{j+r-1})/(r-1)!$. Hence, for $P_1(x) = x$,

$$c_j(P_1) = \bar{t}_j = P_1(\bar{t}_j).$$

Thus

1. $V_r(P_1) = P_1$ *for all linear functions* $P_1 = ax + b$.

By (1.9), the number of sign changes of $V_r(f)$ on I is at most equal to the number of sign changes of the sequence $(f(\bar{t}_j))_{j \in \Lambda}$ and hence of f itself. This observation and (4.4) if applied to the difference $f - P$, yield an important variation diminishing property of the Schoenberg operator V_r:

2. *For $f \in C(I)$ and $P \in \mathcal{P}_1$, the number of sign changes of $V_r(f) - P$ on I is at most equal to the number of sign changes of $f - P$.*

A simple consequence of **2** is

Theorem 4.1. *If $f \in C[a, b]$ is monotone increasing (convex) on $[a, b]$, then $V_r(f)$ is also monotone increasing (convex).*

Proof. The convexity of $V_r(f)$ for convex f follows directly from **2**. If f is monotone increasing, then any strictly decreasing line intersects its graph at most once. If, at the same time, $V_r(f)$ is not monotone increasing, then some strictly decreasing line intersects the graph of $V_r(f)$ at least twice, a contradiction to **2**. □

In order to measure the error of uniform approximation of $f \in C[a,b]$ by splines, one uses the maximal distance between the knots:

$$\delta := \delta_T := \max_{0 \le j \le n} (t_{j+1} - t_j).$$

Then one has:

Theorem 4.2. *For each $k = 0, \dots, r-1$ and $f \in C^k(I)$, there exists a spline $S \in \mathcal{S}_r(T, I)$ for which*

$$\|f - S\|_\infty \le C_r \delta^k \omega(f^{(k)}, \delta). \tag{4.4}$$

For $k = 0, 1$, this can be shown with $S = V_r F$. For the general case see [**CA**, Theorem 7.2, p.224].

As an example of *spline approximation with side conditions*, one can prove that (4.4) is valid with monotone or convex S if f has the corresponding property. This is a result of DeVore [1977_1]; it has been simply proved by Beatson [1982] using the Schoenberg operator V_r.

We consider now interpolation by splines. According to a theorem of de Boor and Schoenberg, a spline of order $r = 2m$ with n given simple knots on $I = [-1, 1]$ can be uniquely defined, among all functions $f \in W_2^m(I)$ that interpolate at the knots, by the minimal value of the norm $\|f^{(m)}\|_2$. See [**CA**, Theorem 6.1, p.152].

A related theorem holds for *periodic splines*. The Schoenberg space $\widetilde{\mathcal{S}}_r(X_n)$ $:= \widetilde{\mathcal{S}}_r(X_n, \mathbb{R})$ of 2π-periodic splines with simple knots $X_n : -\pi \le x_1 < \dots < x_n < \pi$ consists of periodic piecewise polynomials on $\mathbb{R}$ of order $\le r$ with breakpoints $(x_i)_1^n$. In §3 we have seen that it has dimension $\dim \widetilde{\mathcal{S}}_r(X_n) = n$: it does not depend on r.

We shall consider the Lagrange interpolation problem

$$f(x_i) = y_i, \quad i = 1, \dots, n, \tag{4.5}$$

with given points $X_n := (x_i)_1^n \subset [-\pi, \pi)$ and data $Y_n := (y_i)_1^n$. We shall say that $f \in \mathcal{I}$ if f satisfies (4.5). In particular, we consider splines $S_m \in \widetilde{\mathcal{S}}_r(X_n) \cap \mathcal{I}$, where r is even: $r = 2m$, $r = 1, 2, \dots$. The smoothness of S_m increases with m. It has been conjectured by Schoenberg and established by him [1972] and v.Golitschek [1972] that for $r \to \infty$ the solution S_m of (4.5) converges to the trigonometric polynomial satisfying (4.5). First an existence theorem:

Theorem 4.3. *For any given $(x_i)_1^n$, $(y_i)_1^n$, $r = 2m$, $m = 1, 2, \dots$ there is a unique 2π-periodic spline $S_m \in \widetilde{\mathcal{S}}_r(X_n)$ that satisfies* (4.5). *This spline is the unique function minimizing the integral*

$$\int_{-\pi}^{\pi} f^{(m)}(x)^2\,dx$$

among all 2π-periodic functions f with $f^{(m)} \in L_2(-\pi,\pi)$ that satisfy (4.5).

Proof. Finding a spline $S \in \widetilde{S}_r(X_n)$ that interpolates the given data is equivalent to solving an $n \times n$ linear system for the coefficients of S in some basis. This system is uniquely solvable for any given Y_n if and only if the conditions $S(x_i) = 0$ for $i = 1, \ldots, n$ imply $S = 0$. So let us assume that some $S \in \widetilde{S}_r(X_n)$ satisfies these conditions. Integrating $m-1$ times by parts we get

$$\int_{-\pi}^{\pi} S^{(m)}(x)^2\,dx = (-1)^{m-1}\int_{-\pi}^{\pi} S^{(2m-1)} S'\,dx.$$

The latter integral is the sum of the integrals over the intervals (x_{i-1}, x_i). Since $S^{(2m-1)}$ is constant on each of these intervals, we have

$$\int_{x_{i-1}}^{x_i} S^{(2m-1)} S'\,dx = \text{const.}(S(x_i-) - S(x_{i-1}+)) = 0, \quad i = 1, \ldots, n.$$

We see that $S_2^{(m)} = 0$, hence, since S vanishes on X_n, $S = 0$. Thus there is one and only one spline $S_m \in \widetilde{S}_r(X_n)$ interpolating the given data.

By a similar argument we prove that $\int_{-\pi}^{\pi}(f^{(m)} - S_m^{(m)})S_m^{(m)}\,dx = 0$ for any function $f \in \mathcal{I}$ with $f^{(m)} \in L_2$. Therefore

$$\|f^{(m)}\|_2^2 = \|f^{(m)} - S_m^{(m)}\|_2^2 + \|S_m^{(m)}\|_2^2 \ge \|S_m^{(m)}\|_2^2,$$

with equality if and only if $f^{(m)} = S_m^{(m)}$ almost everywhere, hence if and only if $f = S_m$. □

The restriction $r = 2m$ in the above theorem is essential since for r odd interpolation at the knots is not always possible, or equivalently, there exists a non-trivial spline $S \in \widetilde{S}_r$ vanishing at the knots. For example, let $r = 3$ and let S be the quadratic spline of period 2, with the simple knots $x_j = j$, $j = 0, \pm 1, \pm 2, \ldots$, defined by $S(x) := x - x^2$ for $0 \le x \le 1$ and by $S(x) := x^2 - 3x + 2$ for $1 \le x \le 2$. Then $S(x_j) = 0$ for all j.

In what follows, n, $(x_i)_1^n$, $(y_i)_1^n$ will be fixed. We shall define a special trigonometric polynomial

$$T^*(x) = \frac{a_0^*}{2} + \sum_{j=1}^{k}(a_j^* \cos jx + b_j^* \sin jx). \tag{4.6}$$

If n is odd, let $n = 2k+1$. Then *we define T^* to be the polynomial* (4.6) *satisfying the interpolation conditions* (4.5). It exists and is unique since $\mathcal{T}_k$ is a Haar space.

If n is even, we put $n = 2k$. The dimension of $\mathcal{T}_k$ is $n+1$. Here the polynomials $\widetilde{T} \in \mathcal{T}_k \cap \mathcal{I}$ depend on a free parameter. We introduce an additional interpolation point by putting, say, $x_0 := (x_1 + x_2)/2$. Let $T_0, T_1 \in \mathcal{T}_k$ be the unique polynomials satisfying

$$\text{(4.7)} \qquad \begin{cases} T_0(x_0) = 1, & T_0(x_i) = 0 \quad i = 1, \dots, n \\ T_1(x_0) = 0, & T_1(x_i) = y_i \quad i = 1, \dots, n. \end{cases}$$

Plainly, the polynomials $\widetilde{T} \in \mathcal{T}_k \cap \mathcal{I}$ have the unique representation

$$\widetilde{T} = \eta T_0 + T_1, \quad \eta \in \mathbb{R}.$$

Let $\tilde{\alpha}_j, \tilde{\beta}_j$, α_j, β_j, a_j, b_j be the Fourier coefficients of $\widetilde{T}$, T_0, T_1, respectively. *For even n, we define T^* to be the polynomial $\widetilde{T}$ with minimal $\tilde{\alpha}_k^2 + \tilde{\beta}_k^2$.*

Now let

$$\text{(4.8)} \qquad \Phi(\eta) = A\eta^2 + 2B\eta + C$$

be a quadratic function. If $A > 0$, it has a unique minimum at $\eta = \eta^*$, $\eta^* := -B/A$. For other values of the argument,

$$\text{(4.9)} \qquad \Phi(\eta^* + \lambda) = \Phi(\eta^*) + A\lambda^2.$$

We want to minimize $\tilde{\alpha}_k^2 + \tilde{\beta}_k^2 = \Phi(\eta)$, with $A = \alpha_k^2 + \beta_k^2$, $B = a_k\alpha_k + b_k\beta_k$, $C = a_k^2 + b_k^2$. Here $A > 0$ (for otherwise $T_0 \in \mathcal{T}_{k-1}$ and (4.7) would provide a contradiction), so that T^* exists and is unique. For a polynomial $T := T^* + \lambda T_0$ (with coefficients a_k, b_k) formula (4.9) yields

$$\text{(4.10)} \qquad a_k^2 + b_k^2 = a_k^{*2} + b_k^{*2} + (\alpha_k^2 + \beta_k^2)\lambda^2 .$$

Convergence $S_m \to T_k^*$ has been proved by v. Golitschek and Schoenberg. The remainder of Theorem 4.4 is due to v. Golitschek.

Theorem 4.4. *The interpolation splines S_m of Theorem 4.3 converge uniformly to the polynomial $T_k^* \in \mathcal{T}_k \cap \mathcal{I}$, and the convergence is geometric:*

$$\text{(4.11)} \qquad \|S_m - T_k^*\|_\infty \le C_0 \rho_n^r,$$

where $C_0 = C_0(\mathcal{I})$ is a constant independent of r, and

$$\text{(4.12)} \qquad \rho_n := \begin{cases} (n-1)/(n+1) & \textit{if } n \textit{ is odd,} \\ \sqrt{n/(n+2)} & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. We shall denote by C different constants which may depend on $\mathcal{I}$ but not on m. For a fixed $m \ge 2$, let $A_j = A_j(m)$ and $B_j = B_j(m)$ be the Fourier coefficients of S_m, that is,

$$S_m(x) = \frac{A_0}{2} + \sum_{j=1}^{\infty} (A_j \cos jx + B_j \sin jx).$$

Since $\|S_m^{(m)}\|_2 \le \|T^{*(m)}\|_2$, one has

$$\sum_{j=1}^{\infty} j^{2m}(A_j^2 + B_j^2) \le \sum_{j=1}^{k} j^{2m}(a_j^{*2} + b_j^{*2}), \tag{4.13}$$

and, for $i = 1, 2, \ldots$,

$$\sum_{j=ik+1}^{ik+k} (A_j^2 + B_j^2) \le (ik+1)^{-2m} \sum_{j=1}^{k} j^{2m}(a_j^{*2} + b_j^{*2}) \le C\left(\frac{k}{ik+1}\right)^{2m}. \tag{4.14}$$

Let $S_m^* \in \mathcal{T}_k$ be the trigonometric polynomial (which depends on k and m)

$$S_m^*(x) := A_0/2 + \sum_{j=1}^{k}(A_j \cos jx + B_j \sin jx).$$

By the Cauchy inequality and (4.14), one has for $i \ge 1$

$$\sum_{j=ik+1}^{ik+k} (|A_j| + |B_j|) \le \sqrt{2k}\left\{\sum_{j=ik+1}^{ik+k} (A_j^2 + B_j^2)\right\}^{1/2} \le C\left(\frac{k}{ik+1}\right)^{m}.$$

This yields

$$\|S_m - S_m^*\| \le C \sum_{i=1}^{\infty} \left(\frac{k}{ik+1}\right)^{m} \le C\left(\frac{k}{k+1}\right)^{m}. \tag{4.15}$$

We now distinguish two cases:

Case $n = 2k+1$. Let I be the operator of Lagrange interpolation for $\mathcal{T}_k$ at the points x_i, $i = 1, \ldots, n$. Then $T^* = I(S_m)$ and since $\rho_n = k/(k+1)$, by (4.15),

$$\|T^* - S_m^*\| = \|I(S_m - S_m)\| \le C\|I\|\left(\frac{k}{k+1}\right)^{m} = C\rho_n^m,$$

and (4.11) follows.

Case $n = 2k$. Let $\delta_m := k^m/(k+1)^m$. For the Fourier coefficients we have by (4.13)

$$A_k^2 + B_k^2 \le k^{-2m} \sum_{j=1}^{k} j^{2m}(A_j^2 + B_j^2) \le k^{-2m} \sum_{j=1}^{k} j^{2m}(a_j^{*2} + b_j^{*2}),$$

which implies, by (4.14) with $i = 1$,

$$A_k^2 + B_k^2 \le a_k^{*2} + b_k^{*2} + C\delta_m. \tag{4.16}$$

In particular, A_k, B_k are uniformly bounded for all m.

Next we define a polynomial $F \in \mathcal{T}_k$ by its values

$$F(x_0) = 0\,, \quad F(x_i) = S_m^*(x_i) - S_m(x_i) = S_m^*(x_i) - T^*(x_i)\,, \tag{4.17}$$

$i = 1, \ldots, n$. For this F and $\lambda := S_m^*(x_0) - T^*(x_0)$ we have the representation $F = S_m^* - T^* - \lambda T_0$, for both sides coincide at $x = x_i$, $i = 0, \ldots, n$. Our purpose is to estimate $|\lambda|$. For the Fourier coefficients a_k, b_k of $T^* + \lambda T_0$ we can use (4.10). The k-th coefficients of F are $A_k - a_k$, $B_k - b_k$. If I is the interpolation operator for $\mathcal{T}_k$ at the points $(x_i)_0^n$, then due to (4.15) and (4.17)

$$\|F\| \le \|I\| \max_{i=0,\ldots,n} |F(x_i)| \le C\|S_m - S_m^*\| \le C\delta_m. \tag{4.18}$$

Therefore $|A_k - a_k| \le C\delta_m$, $|B_k - b_k| \le C\delta_m$, hence $|A_k^2 + B_k^2 - a_k^2 - b_k^2| \le C\delta_m$. Combining this with (4.10) and (4.16) we get $|\lambda| \le C\delta_m^{1/2}$. Now from this and (4.18) we see that $|S_m^*(x_i) - T^*(x_i)| \le C\delta_m^{1/2}$ for $i = 0, \ldots, n$. For even n, $\delta_m = (n/(n+2))^m$, and so

$$\|S^* - T^*\| \le C\delta_m^{r/2} = C\rho_n^m.$$

Together with (4.15) this completes the proof. $\square$

§ 5. Notes

5.1. A *perfect spline* of order r is a spline with simple knots whose $(r-1)$-th derivative takes at most two values, $+c$ or $-c$, for some $c \in \mathbb{R}$. In particular, polynomials of degree $\le r - 1$ are perfect splines of order r.

Let m and r be two natural numbers. For given $0 \le x_1 \le \ldots x_{m+r} \le 1$ and $y = (y_1, \ldots, y_{m+r}) \in \mathbb{R}^{m+r}$, consider the Hermite interpolation conditions

$$f(x_i) = y_i, \quad i = 1, \ldots, m + r, \tag{5.1}$$

with the multiplicity of each knot x_i not exceeding r.

Let $Q_{m,r}$ denote the set of all perfect splines of order $r+1$ on $[0,1]$ with $\le m - 1$ knots. Then for any set of knots $(x_i)_1^{m+r}$ and any $y \in \mathbb{R}^{m+r}$, there is $f_0 \in Q_{m,r}$ satisfying (5.1). Among all the functions f on $[0,1]$ that have an absolutely continuous $(r-1)$-th derivative with $f^{(r)} \in L_\infty$ and satisfy the interpolation conditions (5.1), the perfect spline f_0 has the minimal norm $\|f^{(r)}\|_\infty$. This fact was announced by Karlin [1973] and proved by de Boor [1974]. For generalizations see Goodman [1979]. For simple proofs, see Lorentz, Jetter and Riemenschneider [A-1983, §14.6].

5.2. For $1 \le p \le \infty$, the $\varphi_{n,r}$ of §3 have the least $L_p(\mathbb{T})$ norm among all periodic perfect splines of order $r+1$ whose r-th derivative takes only values ± 1, with $\le 2n$ sign changes (Ligun [1980], Makovoz [1979], Pinkus [1979]).

5.3. Let $\Sigma_{n,r}$ be the set of splines on $[a,b]$ of order r, with $\leq n$ arbitrary knots. Although a best approximation from $\Sigma_{n,r}$ to an arbitrary f is not unique in general, it is unique in the important case of $f = x^{r+1}$: in C or L_p, $1 \leq p < \infty$, there is exactly one monospline $x^{r+1} + S$, $S \in \Sigma_{n,r}$, of minimal norm. This was established by Johnson [1960] for C, by Jetter and Lange [1978] for $p = 2$, by Jetter [1978] and Strauss [1979] for $p = 1$, and by Bojanov [1979] for $1 < p < \infty$.

Chapter 7. Rational Approximation

§ 1. Introduction

In this chapter we study rational approximation of real functions on subsets of $\mathbb{R}$. The approximants for a given function f are taken here from the set $\mathcal{R}_{m,n}$ of functions $R := R_{m,n} := P/Q$, $P \in \mathcal{P}_m$, $Q \in \mathcal{P}_n$. Each function $R = P/Q$ possesses an *irreducible representation*, $R = p/q$, where p, q do not have common polynomial factors. This representation is unique up to constants: if $R = p_1/q_1$ is another one, then $p_1 = Cp$, $q_1 = Cq$, $C \neq 0$. We shall agree that the irreducible form of the zero function R is $0/1$. In the spaces $C[a,b]$, $L_p[a,b]$, $1 \leq p < \infty$, functions R with a pole on $[a,b]$ have infinite norm. To exclude this possibility, one assumes that $Q(x) \neq 0$ on $[a,b]$ and then, for convenience, that $Q(x) > 0$ on $[a,b]$. The set of these R we denote by $\mathcal{R}_{m,n}[a,b]$. Thus, $\mathcal{R}_{m,n}[a,b]$ or $\mathcal{R}_{m,n}$ on $[a,b]$ consists of all fractions P/Q, identified by equality, which are continuous on $[a,b]$. Because P/Q are polynomials, continuity may be replaced by boundedness, or even by $\int_a^b |P/Q|\,dx < \infty$. We shall write $\mathcal{R}_n$ for $\mathcal{R}_{n,n}$.

Since the functions of best approximation from $\mathcal{R}_{m,n}$ may be reducible, the following number d is needed, both for the Chebyshev type Theorems 2.6 and 2.10 below, and for the continuity of the approximation map in Theorem 8.6. A function $R = P/Q$ is a *degenerate member* of $\mathcal{R}_{m,n}$ if $\deg p =: \partial p < m$, $\deg q =: \partial q < n$. This is equivalent to $d \geq 1$ where the *defect* $d =: d(R)$ *of* R *in* $\mathcal{R}_{m,n}$ is defined by

$$(1.1) \qquad d = \begin{cases} d(R) := d(R, \mathcal{R}_{m,n}) := \min(m - \partial p,\, n - \partial q)\ , & R \neq 0 \\ d(0) := n\ , \quad R = 0\ . \end{cases}$$

If X is a normed function space on $[a,b]$, and $f \in X$, we define the *error of rational approximation* by

$$(1.2) \qquad \rho_{m,n}(f)_X = \inf_{R \in \mathcal{R}_{m,n}} \|f - R\|_X \quad ; \quad \rho_n(f)_X = \rho_{n,n}(f)_X\ .$$

In this chapter, we shall usually have $X = C[a,b]$. By Weierstrass' theorem, for $f \in C[a,b]$,

$$(1.3) \qquad \rho_{m,n}(f) \leq \rho_{m,0}(f) = E_m(f) \to 0\ , \qquad m \to \infty\ .$$

If, in addition, $f \in C$ does not vanish on $[a, b]$, then $1/f$ is approximable by polynomials P_n, hence f by $1/P_n$, and we have $\rho_{0,n}(f) \to 0$, $n \to \infty$. Thus, for such f,

$$\rho_{m,n}(f) \to 0 \quad \text{for} \quad m + n \to \infty \,. \tag{1.4}$$

As D.J. Newman once remarked, each branch of approximation theory begins with the approximation of the elementary functions $|x|$, e^x, x^k. Rational approximation is no exception, at its beginning stands one of the main results of the theory, Theorem 3.1 of Newman [1964]. It states that the approximation error of $|x|$ on an interval is of order $q^{\sqrt{n}}$, with appropriate $0 < q < 1$. Not only is this theorem important in itself, but its estimate and the methods of its proof are very useful in many rational approximation problems.

The function e^x one can approximate on a finite interval, for example on $[-1, 1]$. Here $\rho_{m,n}(e^x)$ has been very precisely estimated by Newman [1979_2] and Braess [1984] (Theorem 4.1). The approximation of the exponential e^{-x} on the infinite interval $[0, \infty)$ is a different problem. Many authors gave estimates of the limit $L := \lim_{n\to\infty} \rho_n(e^{-x})^{1/n}$. The full truth was guessed by Magnus [1985] and established by Gonchar and Rakhmanov [1987]: $L = e^{-\pi L_1}$, where L_1 is the quotient of two elliptic integrals, $L := 1/9.2890\ldots$. Finally, for the rational approximation of the functions x^k see Theorem 1.2 below.

For rational approximation of large classes of functions one cannot expect similarly spectacular results, for here the worst functions of the class dominate. For the Lipschitz classes one has no improvement at all in comparison with the polynomial approximation (for example see Theorem 7.5). Freud [1966] was the first to give examples of natural spaces where there is an improvement. The best result is perhaps that of Popov [1977]: for the class of functions $f \in V^1$, for which f' is of bounded variation, $\rho_n(V^1) \le Cn^{-2}$, while $E_n(V^1) \sim n^{-1}$.

We shall continue to discuss properties of rational approximation in Chapters 8, 10 and in parts of Chapter 9. See also the book of Petrushev and Popov [A-1987].

The obvious difficulty that one encounters in rational approximation is the fact that the sets $\mathcal{R}_{m,n}$ are not linear or convex. Their uniformly bounded subsets are not compact. For example, the sequence $R_n = \frac{1}{nx+1} \in \mathcal{R}_{0,1}$, $n = 1, 2, \ldots$ is bounded on $[0, 1]$, but contains no uniformly convergent sequence. The ordinary Bernstein type inequality $\|R_n'\| \le n\|R_n\|$, $R_n \in \mathcal{R}_n$ in $C[-1, 1]$ is also not true. If $R_2(x) = \delta^2/(x^2 + \delta^2)$, $\delta > 0$, then $|R_2(x)| \le 1$ on $[-1, 1]$, while $R_2'(\delta) = -\frac{1}{2}\delta^{-1}$ can be arbitrarily large. What is true, is Pekarskii's [1986_2] inequality (see Theorem 1.1 of Chapter 10), where the norms of R' and of R are taken in different spaces. Its prototype is the inequality of Dolzhenko [1963]:

Theorem 1.1. *If $R_n \in \mathcal{R}_n$, then*

$$\operatorname{Var}_{[a,b]} R_n = \int_a^b |R_n'| \, dx \le 2n\|R_n\|_{C[a,b]} \; ; \tag{1.5}$$

for each $\delta > 0$ and each interval $[a,b]$, there is a subset $e \subset [a,b]$, $|e| < \delta$ for which

$$|R_n'(x)| \le \frac{2n}{\delta} \|R_n\|_{C[a,b]} \ , \quad x \in [a,b] \setminus e \ . \tag{1.6}$$

Proof. The function R_n has a finite number of intervals I of monotonicity on $[a,b]$. If I' are the images of the I under R_n, then $\operatorname{Var} f = \sum |I'|$. The number of solutions of any equation $R_n(x) = c$ in $[a,b]$ is at most n. Let $\|R_n\| = M$, then all intervals I' are contained in $[-M, M]$, and they cover each point $c \in [-M, M]$ at most n times. It follows that $\sum |I'| \le 2Mn$. This is (1.5). From this inequality we can derive (1.6) by taking $e = \{x : |R_n'(x)| > A/\delta\}$, where A is the right-hand side of (1.5). □

An interesting problem is the approximation of powers x^k on $[0,1]$ by rational functions from $\mathcal{R}_n$, with $n < k$. We shall do this by means of inverses $1/S_n$ of polynomials $S_n \in \mathcal{P}_n$, in other words, by means of rational functions in $\mathcal{R}_{0,n}$. With Newman [A-1979] we have

Theorem 1.2. *One has*

$$\rho_{0,n}(x^k) \le \frac{2}{n} \left(\frac{2k-2}{2k+n} \right)^{k-1} \ , \qquad 1 \le n < k \ . \tag{1.7}$$

Proof. Let $S_n \in \mathcal{P}_n$ be the beginning of the Taylor series of x^{-k} around the point 1. We try to approximate x^k by $1/S_n(x)$. By the integral remainder formula,

$$S_n(x) = x^{-k} - \frac{1}{n!} \int_x^1 (t-x)^n \left(-\frac{d}{dt} \right)^{n+1} (t^{-k}) \, dt \ .$$

Differentiating under the integral sign and making the substitution $u = (x/t)^k$ we obtain

$$S_n(x) = x^{-k} [1 - CI(z)] \ , \qquad 0 < z \le 1 \tag{1.8}$$

where $I(z) := \int_z^1 (1 - u^{1/k})^n \, du$, $z = x^k$, and $C > 0$ is some constant. Making $x \to 0$ in (1.8) we see that $C = I(0)^{-1}$. This implies that $S_n(x) > 0$, $0 < x \le 1$ and that $1/S_n(x) - x^k \ge 0$. Therefore we must prove

$$\frac{1}{S_n(x)} - x^k \le \varepsilon \ , \qquad 0 \le x \le 1 \ , \tag{1.9}$$

where $\varepsilon > 0$ is as small as possible. The left-hand side of (1.9) is $z/[1 - \frac{I(z)}{I(0)}] - z$, and (1.9) is equivalent to

$$\phi(z) := (z + \varepsilon) I(z) \le \varepsilon I(0) \ , \qquad 0 \le z \le 1 \ , \tag{1.10}$$

that is, to the statement that $\phi(z)$ attains its maximum at $z = 0$. We shall select $\varepsilon > 0$ as small as possible and so that $\phi(z)$ is convex. Since this function is positive and since $\phi(1) = 0$, this will imply (1.10).

The second derivative of ϕ is

$$\begin{aligned}\phi''(z) &= 2I'(z) + (z+\varepsilon)I''(z)\\ &= -2(1-z^{1/k})^n + (z+\varepsilon)n(1-z^{1/k})^{n-1}\frac{1}{k}z^{(1/k)-1}\\ &= \frac{1}{k}(1-z^{1/k})^{n-1}\{(2k+n)z^{1/k} + \varepsilon n z^{(1/k)-1} - 2k\}\ .\end{aligned}$$

The expression in braces

$$\psi(x) = (2k+n)x + \varepsilon n x^{1-k} - 2k$$

is a function that first decreases on $[0,1]$, later increases, and has exactly one minimum point x_0, $0 < x_0 < 1$ with $\psi'(x_0) = 0$. The solutions x_0, ε of the two equations $\psi(x_0) = 0$, $\psi'(x_0) = 0$ will define the smallest possible ε for which $\phi(z)$ is still convex for $0 \le z \le 1$. The solutions are $x_0 = \frac{2k-2}{2k+n}$, $\varepsilon = \frac{2}{n}x_0^{k-1}$, so that

$$\varepsilon = \frac{2}{n}\left(\frac{2k-2}{2k+n}\right)^{k-1}\ .$$

With this ε, we shall have (1.9) for all $x \in [0,1]$. □

Corollary 1.3. *The powers x^k, $k = 0, 1, \dots$ are uniformly approximable by rational functions from $\mathcal{R}_n$: $\rho_n(x^k) \le \frac{2}{n}$, $k = 0, 1, \dots, n-1$.*

This is not possible with polynomials $P_n \in \mathcal{P}_n$: functions x^k increase arbitrarily rapidly near 1, while the absolute values of the derivatives $|P_n'(x)|$ of the approximants are of order at most n^2.

Very often rational approximation is superior to polynomial approximation, if one compares $\rho_n(f)$ with $E_{2n}(f)$ (for the functions of the classes $\mathcal{R}_n$ and $\mathcal{P}_{2n}$ have approximately the same number of free parameters). This will be one of the main results of this chapter.

In the definition (1.2) of the approximation error $\rho_n(f)$ of a *real valued* function $f \in C[-1,1]$ we have used rational functions $R_n \in \mathcal{R}_n$ with real coefficients. Later in this book, for complex valued functions $f(z)$, $z \in A \subset \mathbb{C}$, we replace $\mathcal{R}_n$ by $\mathcal{R}_n^*$, the set of all rational functions R_n^* of degree $\le n$ with complex coefficients. For $A = [-1,1]$ this leads to an alternative definition of the error,

$$\rho_n^*(f) := \min_{\mathcal{R}_n^*} \|f - R_n^*\|_\infty\ . \tag{1.11}$$

(Notations for this section only.) Clearly, $\rho_n^*(f) \le \rho_n(f)$.

For polynomial approximation, this point of view would produce nothing new, for a real function $f(x)$ cannot be approximated by a polynomial $P_n(x)$ with complex coefficients better than by $\operatorname{Re} P_n(x)$. That this is not so for rational approximation, has been noticed by Gonchar. From Saff and Varga [1978_3] we take the following:

1. For the function $f(x) = x^2$ on $[-1,1]$ one has $\rho_1^*(f) < \rho_1(f)$. Indeed, it follows from Theorem 2.6 of §2 that $\rho_1(f) = \frac{1}{2}$ (with the best approximants $R_1(x) = R_0(x) = \frac{1}{2}$), but a simple calculation yields that

$$\rho_1^*(f) \le \max_{-1\le x\le 1} \left| x^2 - \frac{x+(\sqrt{2}-1)i}{x+i} \right| = \sqrt{2}-1 = 0.41421\ldots < \tfrac{1}{2}\ .$$

2. We always have

$$\rho_{2n}(f) \le \rho_n^*(f) \le \rho_n(f)\ . \tag{1.12}$$

This follows from the fact that for each rational function $R \in \mathcal{R}_n^*$, its real part $\operatorname{Re} R$ belongs to $\mathcal{R}_{2n}$. It is clear that complex approximation is not always unique. For if $\|f - R_n^*\| = \rho_n^*(f) < \rho_n(f)$, then R_n^* and $\overline{R_n^*}$ are two different best approximations to f.

The following theorem of Levin [1986] describes relations between $\rho_n(f)$ and $\rho_n^*(f)$.

Theorem 1.4. *One has*

$$\tfrac{1}{2}\rho_n(f) \le \rho_n^*(f) \le \rho_n(f)\ ; \tag{1.13}$$

for each $n \ge 3$, there is a function $f \notin \mathcal{R}_n$ for which $\rho_n^*(f) = \frac{1}{2}\rho_n(f)$.

Proof. (a) If (1.13) is not true, then for some $f \in C[-1,1]$ and its best approximations R_n, R_n^*,

$$\|f - R_n\| = 1 \quad , \quad \|f - R_n^*\| < \tfrac{1}{2}\ . \tag{1.14}$$

Let $R = p/q$, $R^* = p^*/q^*$ be their irreducible representations with $\mu = \partial p$, $\nu = \partial q$, $\mu^* = \partial p^*$, $\nu^* = \partial q^*$. Then $d = \min(n-\mu, n-\nu)$ is the defect of R. By Theorem 2.6 there are points $-1 \le x_1 < \cdots < x_{2n-d+2} \le 1$ for which $f(x_j) - R(x_j) = \varepsilon(-1)^j$, $j = 1,\ldots,2n-d+2$, $\varepsilon = +1$ or $= -1$. For each $j = 1,\ldots,2n-d+1$ we can find points $y_j, x_j < y_j < x_{j+1}$ with $f(y_j) - R(y_j) = 0$. Then from (1.14), $|R(x_j) - R^*(x_j)| > \frac{1}{2}, j = 1,\ldots,2n-d+2$, $|R(y_j) - R^*(y_j)| < \frac{1}{2}$, $j = 1,\ldots,2n-d+1$. This implies that the equation

$$|R(x) - R^*(x)| = \tfrac{1}{2} \tag{1.15}$$

has at least $2(2n-d+1)$ distinct zeros in $[-1,1]$. The equation (1.15) is equivalent to

$$\big(R(x) - R^*(x)\big)\big(R(x) - \overline{R^*(x)}\,\big) = \tfrac{1}{4}$$

or to $P(x) = 0$, where P is the polynomial

$$P = (pq^* - qp^*)(p\overline{q^*} - q\overline{p^*}) - \tfrac{1}{4}q^2q^*\overline{q^*}\ .$$

Now the degree of P does not exceed

$$2\max(\mu+\nu^*,\nu+\mu^*,\nu+\nu^*)\le 2\max(n+\nu,n+\mu)=4n-2d<2(2n-d+1)\ ,$$

a contradiction.

(b) We define the real function $f:=f_n$, $n\ge 3$ (which actually belongs to $\mathcal{R}_{2n}$) by

$$f(x)=\tfrac{1}{2}\left\{\left(\frac{x-ia}{x+ia}\right)^n+\left(\frac{x+ia}{x-ia}\right)^n\right\}\ ,\qquad a=\tan\frac{\pi}{2n}\ . \tag{1.16}$$

Clearly, $\|f\|\le 1$. Taking $R_n^*(x)=(\frac{x+ia}{x-ia})^n$, we get $\rho_n^*(f)\le\frac{1}{2}$.

On the other hand, at the points

$$x_k=-a\cot\frac{\pi k}{2n}\ ,\qquad k=1,2,\ldots,2n-1\ , \tag{1.17}$$

we have $f(x_k)=\cos\pi k=(-1)^k$, $k=1,\ldots,2n-1$. Since $\tan\frac{\pi}{2n}\cot\frac{\pi k}{2n}\le 1$ for $k=1,\ldots,n$, the points x_k with $k=1,\ldots,n$ lie in $[-1,0]$ and with $k=n,\ldots,2n-1$ in $[0,1]$. The function $R_n(x)\equiv 0$ has defect n in $\mathcal{R}_n$, and by Akhiezer's Theorem 2.6, it is the best approximant to f, since $2n-1\ge n+2$ for $n\ge 3$. Thus, $\rho_n(f)=1$, and from (1.13) we now deduce $\rho_n^*(f)=\frac{1}{2}$. □

Equally interesting is the behavior of the error of the complex approximation, $\rho_{m,n}^*(f):=\min\{\|f-R_{m,n}^*\|:R_{m,n}^*\in\mathcal{R}_{m,n}^*\}$, with $\mathcal{R}_{m,n}^*$ defined in an obvious way. We put

$$\gamma_{m,n}=\inf\{\rho_{m,n}^*(f)/\rho_{m,n}(f):f\in C[-1,1]\setminus\mathcal{R}_{m,n}\}\ ,\qquad m,n\ge 0\ .$$

Then we have: $\gamma_{m,n}=1$ if $n=0$; $\gamma_{m,n}=\frac{1}{2}$ if $n\le m+1$, $n=1,2,\ldots$ (Levin [1986]); $\gamma_{m,n}=\frac{1}{3}$ if $n=m+2$, $m=0,1,\ldots$ (Ruttan and Varga [1989]); $\gamma_{m,n}=0$ if $n\ge m+3$, $m=0,1,\ldots$ (Trefethen and Gutknecht [1983]).

§ 2. Best Rational Approximation

If $f\in X$ is an element of a linear normed space of functions on $A=[a,b]$, then $R\in\mathcal{R}_{m,n}$ is a *best approximation to f from $\mathcal{R}_{m,n}$ in X*, if

$$\rho_{m,n}(f)_X=\|f-R\|_X\ . \tag{2.1}$$

For the approximation, we consider only functions $R\in\mathcal{R}_{m,n}[a,b]$, which means that $R=P/Q$, $Q(x)>0$ on $[a,b]$. The space $Y\subset X$ has the *proximity property* in X, if for each $f\in X$, there is an element $g\in Y$ of best approximation to f.

Let X be a linear normed space of functions on $[-1,1]$ that is symmetric: with f also $f^*(x)=f(-x)$ belongs to X and $\|f^*\|=\|f\|$. We assume that $\mathcal{R}_{m,n}$ has the proximity property in X. The following statements refer to those $f\in X$ for which the best approximation is unique.

1. If $f \in X$ is even, and $R = P/Q$ is an irreducible form of its best approximation from $\mathcal{R}_{m,n}[-1,1]$, then P and Q are even. Indeed, also $R^*(x) = P(-x)/Q(-x)$ is an irreducible best approximation to f. Since irreducible representations are unique up to a constant, $P(-x) = CP(x)$, $Q(-x) = CQ(x)$. Comparing the leading coefficients yields $C = \pm 1$. However, $C = -1$ is impossible, since then P, Q would be odd and have a common factor x.
2. Similarly, if $f \in X$ is odd, then P is odd and Q is even.
3. If n is odd, then in each of the cases **1**, **2**, the unique best approximation to f is degenerate, it belongs to $\mathcal{R}_{m-1,n-1}$: if m is odd, and f is even, we can use **1**, if m is even, f is odd, we use **2**.

In this section we shall discuss the existence, uniqueness and characterization of best approximation. Most of the theorems are similar to those for polynomial approximation, but the proofs are different. The main difference with the linear case is that there is no uniqueness for rational approximation in $L_p[a,b]$ (see §8).

The existence of a best rational approximation can be proved via the following lemma:

Lemma 2.1. *Let $g_k \in \mathcal{R}_{m,n}[a,b]$ be a sequence in one of the spaces $X = L_p[a,b]$, $1 \le p < \infty$ or $C[a,b]$ with bounded norms, $\|g_k\|_X \le C$. Then for a subsequence h_k of the g_k and for some $P/Q \in \mathcal{R}_{m,n}[a,b]$, one has uniformly on each compact interval that has no zeros of Q,*

$$\lim_{k\to\infty} \left[h_k(x) - P(x)/Q(x)\right] = 0 \ . \tag{2.2}$$

Proof. We write $g_k = P_k/Q_k$, normalizing Q_k by $\|Q_k\|_{C[a,b]} = 1$. Then $\|P_k\|_X \le C$. In finite dimensional spaces, all norms are equivalent. It follows that the coefficients of P_k, Q_k are uniformly bounded. Replacing P_k by a proper subsequence, we have for the corresponding P_k, Q_k, $P_k \to P \in \mathcal{P}_m$, $Q_k \to Q \in \mathcal{P}_n$ uniformly on $[a,b]$. This yields (2.2). We have to prove that $R = P/Q \in \mathcal{R}_{m,n}[a,b]$. If $X = L_p$, it follows from (2.2) and Fatou's theorem that $\int_a^b |P/Q|^p \, dx \le C^p$. Thus, P/Q is continuous on $[a,b]$. If $X = C$, then (2.2) implies $|P(x)/Q(x)| \le C$, $x \in [a,b]$, with the same conclusion. □

Theorem 2.2. *In the spaces $L_p[a,b]$, $1 \le p < \infty$ and in $C[a,b]$ the families $\mathcal{R}_{m,n}$ are proximinal.*

Proof. A subset $G \subset X$ is called *approximatively compact*, if for each $f \in X$ each *minimizing sequence* $g_k \in G$ with $\|f - g_k\| \to \operatorname{dist}(f,G)$ contains a subsequence h_k convergent in X to an element $g \in G$. Of course, then $\|f-g\| = \operatorname{dist}(f,G)$. Each such set G is proximinal. If $X = L_p$, $1 \le p < \infty$, we can prove that $\mathcal{R}_{m,n}$ is approximatively compact in X. For $f \in X$, let g_k be a minimizing sequence with $\|f - g_k\|_p \to \Delta := \rho_{m,n}(f)_p$, and let h_k be a subsequence of Lemma 2.1. By Fatou's theorem and (2.2),

$$\Delta \le \|f - P/Q\|_p \le \lim_{k\to\infty} \|f - h_k\|_p \le \Delta \, .$$

In $C[a,b]$, we do not have approximative compactness (see §1 of Chapter 12), but for a minimizing sequence g_k, by Lemma 2.1, for all but finitely many $x \in [a,b]$,

$$|f(x) - P(x)/Q(x)| \le \lim \|f - h_k\|_\infty = \Delta \, . \tag{2.3}$$

Since f is continuous, $P/Q \in \mathcal{R}_{m,n}[a,b]$ and $\|f - P/Q\|_\infty = \Delta$. □

We shall next give the "Chebyshev theorem" of uniform rational approximation on $[a,b]$ (using the approach of Cheney and Loeb [1964]). It characterizes the best approximant R from $\mathcal{R}_{m,n}$ to f in terms of Chebyshev alternations of $f - R$, and was proved by Akhiezer [1930]. An elegant approach to this theorem is by means of Krein's theory (see [**CA**, §9 of Chapter 3]) of zeros of elements of a Haar space.

Our first theorem is valid for very general "rational functions" $\mathcal{R}$.

Let $\mathcal{G}, \mathcal{H}$ be two linear subspaces of $C(A)$, $A = [a,b]$ or $A = \mathbf{T}$, spanned by g_j, $j = 0, \ldots, m$, h_j, $j = 0, \ldots, n$, respectively. We assume that some $Q \in \mathcal{H}$ satisfies $Q(x) > 0$, $x \in A$. The set $\mathcal{R}$ consists of all $R = P/Q$, $P \in \mathcal{G}$, $Q \in \mathcal{H}$, $Q(x) > 0$ on A. For given $f \in C(A)$ and $R \in \mathcal{R}$, let

$$A_0 := \{x \in A : |f(x) - R(x)| = \|f - R\|\} \, . \tag{2.4}$$

A useful tool will be the linear space $\Phi := \Phi(R) := \mathcal{G} + R\mathcal{H}$, which, for a given R, consists of all $P + RQ$, $P \in \mathcal{G}$, $Q \in \mathcal{H}$.

Theorem 2.3. *For $f \in C(A) \setminus \mathcal{R}$, a function $R \in \mathcal{R}$ is a best uniform approximation to f from $\mathcal{R}$ on A if and only if*
(*) *No element $\phi \in \Phi$ has on A_0 the same sign as $\sigma(x) := \operatorname{sign}[f(x) - R(x)]$, that is, satisfies $\phi(x)\sigma(x) > 0$, $x \in A_0$.*

Proof. (a) *Sufficiency.* If R is not a best approximant to f, then there exists $R_0 := P_0/Q_0 \in \mathcal{R}$ with $\|f - R_0\| < \|f - R\|$. Then $\phi_0 := Q_0(R_0 - R) \in \Phi$.

For all $x \in A_0$ one has

$$\sigma(x)\big(f(x) - R_0(x)\big) \le \|f - R_0\| < \|f - R\| = \sigma(x)\big[f(x) - R(x)\big] \, ,$$

hence $\sigma(x)[R_0(x) - R(x)] > 0$ or $\sigma(x)\phi_0(x) > 0$ on A_0. This contradicts (*).

(b) *Necessity.* If (*) is not satisfied for $R = P/Q$, then there exists a $\phi := P_0 + RQ_0 \in \Phi$ satisfying $\sigma(x)\phi(x) > 0$ on A_0. We define $R_\lambda := (P + \lambda P_0)/(Q - \lambda Q_0)$; for all sufficiently small λ, $0 < \lambda < \varepsilon$, R_λ belongs to $\mathcal{R}$, since $Q(x) - \lambda Q_0(x) > 0$ on A. For the functions R_λ, we have $R_\lambda \to R$, $\lambda \to 0$ uniformly on A, moreover $R - R_\lambda = -\frac{\lambda\phi}{Q - \lambda Q_0}$. We take an open neighborhood $G \supset A_0$ in A in such a way that $\sigma(x)\phi(x) \ge \delta > 0$ on G and let $F := A \setminus G$. We want to show that for small $\lambda > 0$, R_λ is a better approximant to f than R. Indeed, on the compact set F, $|f(x) - R(x)| < \Delta := \|f - R\|$, hence also

$$|f(x) - R_\lambda(x)| \le |f(x) - R(x)| + |R(x) - R_\lambda(x)| < \Delta$$

for all small $\lambda > 0$. On the other hand, for $x \in G$, and small λ,

$$\begin{aligned}|f(x) - R_\lambda(x)| &= \sigma(x)\big(f(x) - R(x)\big) + \sigma(x)\big(R(x) - R_\lambda(x)\big) \\ &\le \Delta - \frac{\lambda\sigma(x)\phi(x)}{Q(x) - \lambda Q_0(x)} < \Delta - \frac{\lambda\delta}{\|Q\| + \|Q_0\|} < \Delta ,\end{aligned}$$

and we get $\|f - R_\lambda\| < \Delta$. □

We discuss now the special case of rational functions $R \in \mathcal{R}_{m,n}[a,b]$. Together with the defect $d := d(R)$ of (1.1), an important characteristic of a function $R \in \mathcal{R}_{m,n}$ with an irreducible representation $R = p/q$ is the number

$$\text{(2.5)} \qquad \begin{aligned} N := N(R) : &= N(R, \mathcal{R}_{m,n}) := m + n - d + 1 \\ &= \max(m + \partial p,\ n + \partial q) + 1 \ .\end{aligned}$$

Theorem 2.4 (Cheney and Loeb [1964]). *Let $R = p/q$ be an irreducible representation of $R \in \mathcal{R}_{m,n}$. Then the space*

$$\text{(2.6)} \qquad X_R := q\mathcal{P}_m - p\mathcal{P}_n \subset C[a,b]$$

is a polynomial Haar space of dimension $N(R) = N(R, \mathcal{R}_{m,n})$.

Proof. We first show, with $d = d(R, \mathcal{R}_{m,n})$,

$$\text{(2.7)} \qquad q\mathcal{P}_m \cap p\mathcal{P}_n = pq\mathcal{P}_d \ .$$

Indeed, f belongs to the left-hand side of (2.7) if and only if it has the representations $f = qP_m = pQ_n$, $P_m \in \mathcal{P}_m$, $Q_n \in \mathcal{P}_n$. Then Q_n must be divisible by q and P_m by p. We have $P_m = pP_{m-\partial p}$, $Q_n = qQ_{n-\partial q}$, hence $f = pqP_k$, where $k = \min(m - \partial p, n - \partial q) = d$. Conversely, each f of the form $pq\mathcal{P}_d$ belongs to $q\mathcal{P}_m$ and to $p\mathcal{P}_n$.

For the dimension of X_R we now have

$$\begin{aligned}\dim X_R &= \dim(q\mathcal{P}_m) + \dim(p\mathcal{P}_n) - \dim(q\mathcal{P}_m \cap p\mathcal{P}_n) \\ &= m + 1 + n + 1 - (d + 1) = N(R) \ .\end{aligned}$$

To establish that this will be a Haar system, we show that each element of X_R which has $N(R)$ zeros, vanishes identically. This follows because each $f \in X_R$ is a polynomial of degree

$$\le \max(m + \partial p\ ,\ n + \partial q) = N(R) - 1 \ . \qquad □$$

The functions

$$\text{(2.8)} \qquad qx^k,\ k = 0, \ldots, m\,, \quad px^\ell,\ \ell = 0, \ldots, n$$

span X_R. The theorem implies that it is possible to select $N(R)$ of them which form a basis for X_R. For example, if $d(R) = 0$, then the functions (2.8), with $qx^{\partial p}$ (or $px^{\partial q}$) omitted, span X_R.

Corollary 2.5. *For a given $R = p/q \in \mathcal{R}_{m,n}[a,b]$ also the space $\Phi := \mathcal{P}_m + R\mathcal{P}_n$ is a Haar subspace of $C[a,b]$ of dimension $N(R)$. If $d(R) = 0$, it is spanned by the functions x^k, $k = 0, \dots, m$, Rx^ℓ, $\ell = 0, \dots, n$, $\ell \neq \partial p =: \nu$.*

Our main theorem:

Theorem 2.6 (Akhiezer). *Let $f \in C[a,b] \setminus \mathcal{R}_{m,n}$, then a function $R \in \mathcal{R}_{m,n}$ with the number $N := N(R)$ defined by (2.5) is a best approximation to f from $\mathcal{R}_{m,n}$ if and only if there are $N+1$ points $a \le x_1 < \cdots < x_{N+1} \le b$ where $f(x) - R(x)$ takes the values $\pm\|f - R\|$ with alternating signs.*

Proof. (a) *Necessity.* If the condition is not satisfied, then for some $r \le N$ there are points $a \le y_0 < y_1 < \cdots < y_r \le b$ with the property that each interval (y_k, y_{k+1}), $k = 0, \dots, r-1$ contains points $x \in A_0$ with the same sign of the difference $f(x) - R(x)$, the sign alternating from one interval to the next. Since $r - 1$ is less than the dimension N of the Haar space Φ, by [**CA**, 4, p.70] there is a ϕ that changes sign exactly at $y_1, \dots, y_{r-1}$. Then (replacing ϕ by $-\phi$ if necessary) we have $\sigma(x)\phi(x) > 0$ for all $x \in A_0$, contradicting $(*)$.

(b) *Sufficiency.* Let $a \le x_1 < \cdots < x_{N+1} \le b$ be a sequence of points of A_0 of the theorem. Then no $\phi \in \Phi$ can satisfy $\phi(x_j)[f(x_j) - R(x_j)] > 0$ for all these x_j. For this would imply that ϕ changes sign at least N times, has at least N zeros, which is impossible. By Theorem 2.3, R is best possible. □

We have also the de la Vallée-Poussin theorem for rational approximation:

Theorem 2.7. *If $f \in C[a,b]$, $R_0 = P_0/Q_0 \in \mathcal{R}_{m,n}$, and the points $a \le x_1 < \cdots < x_{N+1} \le b$, $N = N(R_0)$ have the properties*

(2.9)
$$f(x_j) - R_0(x_j) = (-1)^j \eta \Delta_j\,, \quad \eta = +1 \text{ or } \eta = -1\,, \quad \Delta_j > 0\,, \quad j = 1, \dots, N+1$$

then $\rho_{m,n}(f) \ge \Delta = \min_j \Delta_j$.

Proof. For some $R \in \mathcal{R}_{m,n}$, let $\|f - R\| = \rho_{m,n}(f) < \Delta$. Then

$$\phi := Q_0(R_0 - R) = Q_0\big[(f - R) - (f - R_0)\big] \tag{2.10}$$

belongs to $\Phi = \mathcal{P}_m + R\mathcal{P}_n$, and for all j, $\phi(x_j)[f(x_j - R_0(x_j)] < 0$. Again ϕ has at least N zeros, an impossibility. □

A point $x_0 \in A$, $A = [a,b]$ or $A = \mathbb{T}$ is a *double zero* of a function $\phi \in C(A)$ if $\phi(x_0) = 0$, $x_0 \neq a, b$ and if ϕ does not change sign in some neighborhood of x_0. All other zeros of ϕ are *single zeros.* We need

Theorem 2.8 *Let $\phi, g \in C(A)$, with ϕ having only isolated zeros, let $x_1 < \cdots < x_{N+1}$ $(< x_1)$ be points of A. If the signs of $g(x_j)$ alternate (for example, let* sign *$g(x_j) = (-1)^{j-1}$, $j = 1, \dots, N+1$) and if*

$$\phi(x_j) g(x_j) \ge 0\,, \qquad j = 1, \dots, N+1\,, \tag{2.11}$$

then, counting multiplicities, ϕ has at least N zeros on A.

Proof. We prove this by induction on N, beginning with $N = 0$. We can assume that ϕ does not have a zero in $[x_1, x_2)$, for otherwise, in addition to this zero, by the induction hypothesis, it would have $N-1$ zeros in $[x_2, x_{N+1}]$. Therefore we assume that $\phi(x_1) > 0$, $\phi(x_2) = 0$. Let j be the largest index with $\phi(x_2) = \cdots = \phi(x_j) = 0$. We are through if $j = N+1$. Otherwise we can assume that all these zeros are simple and that ϕ has no other zeros in $[x_2, x_j]$. Then ϕ changes sign at x_i from $(-1)^i$ to $(-1)^{i+1}$, $i = 2, \ldots, j$. Thus, ϕ has N zeros: $j-1$ of them in $[x_1, x_j]$, one in (x_j, x_{j+1}), and $N-j$ in $[x_{j+1}, x_{N+1}]$. □

Theorem 2.9 (Uniqueness Theorem). *Each function $f \in C[a,b]$ has a unique function $R \in \mathcal{R}_{m,n}$ of best approximation.*

Proof. Let $f \notin \mathcal{R}_{m,n}$, let R, R_0 be two best approximations to f. Then $\phi := Q(R - R_0) \in \Phi(R_0)$. This time we deduce, for a set $x_1 < \cdots < x_{N+1}$ of alternation points of R, that (see (2.10))

$$\phi(x_j)\left[f(x_j) - R(x_j)\right] \geq 0 , \qquad j = 1, \ldots, N+1 .$$

By the last theorem, ϕ has at least N zeros. Even if double zeros are counted twice, this implies (see [**CA**, **6**, p.71]) that $\phi = 0$, $R = R_0$. □

As an example, from Theorem 2.6 we deduce:

For any non-increasing sequence $(a_k)_0^\infty$ with $\lim a_k = 0$, there exists a function $f \in C[-1,1]$ with the properties

$$\rho_{3^k}(f)_{C[-1,1]} = E_{3^k}(f)_{C[-1,1]} = a_k , \qquad k = 1, 2, \ldots . \tag{2.12}$$

Indeed, for the function

$$f(x) := \sum_{j=1}^{\infty} b_j C_{3^j}(x) , \qquad b_j := a_{j-1} - a_j ,$$

the polynomial $S_k := \sum_{j=1}^{k} b_j C_{3^j}$ is an element of the best approximation to f from the class $\mathcal{P}_{3^k}$ as well as from $\mathcal{R}_{3^k}$, and $\|f - S_k\| = a_{3^k}$ (see also Chapter 1, §1).

We conclude this section with a version of Theorem 2.6 for rational trigonometric functions $U \in \mathcal{U}_{m,n}$, $m, n = 0, 1, \ldots$. These functions are quotients $U = S_m/T_n$, $S_m \in \mathcal{T}_m$, $T_n \in \mathcal{T}_n$, which have no poles on $\mathbb{T}$.

The proof is connected with the divisibility properties in the ring $\mathcal{T} = \cup_0^\infty \mathcal{T}_n$ of all trigonometric polynomials with real coefficients; they are subtly different from those of the ring $\mathcal{P}$ of algebraic polynomials. If $T, T_1 \in \mathcal{T}$, we say that T is divisible by T_1, if $T = T_1 T_2$ for some $T_2 \in \mathcal{T}$.

We shall study the factorizations of the polynomials

$$T_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

with real coefficients, of degree n, so that $a_2^2 + b_n^2 \neq 0$. We let $z = e^{ix}$. Then the polynomial T_n becomes

$$T_n(x) = \sum_{j=-n}^{n} c_j z^j , \tag{2.13}$$

where $c_j = (a_j + ib_j)/2$ for $j = -n, -n+1, \dots, 0$, and $c_j = (a_j - ib_j)/2$ for $j = 1, \dots, n$. The right-hand side of (2.13) in the domain $\mathbb{C} \setminus \{0\}$ is a rational function $R(z) = c_{-n}z^{-n} + c_{-n+1}z^{-n+1} + \dots + c_n z^n$. The relations $\bar{c}_j = c_{-j}$, $j = -n, \dots, n$ imply that $\overline{R(\bar{z})} = R(1/z)$, $z \neq 0$. This shows that the zeros of $R(z)$ are symmetric with respect to the circumference $|z| = 1$. Let $z_1, \dots, z_m$ be the zeros of $R(z)$ in the domain $0 < |z| < 1$, with multiplicities $\nu_1, \dots, \nu_m$. Then the zeros of R in $1 < |z| < \infty$ will be $1/\bar{z}_1, \dots, 1/\bar{z}_m$ with the same multiplicities. Also, let $\zeta_1, \dots, \zeta_\ell$ be zeros of R on the circumference $|\zeta| = 1$, and $\mu_1, \dots, \mu_\ell$ their multiplicities. Since $c_n \neq 0$, the total number of zeros is $2n$, hence

$$2(\nu_1 + \dots + \nu_m) + \mu_1 + \dots + \mu_\ell = 2n . \tag{2.14}$$

Of course, one of the sets $\{z_1, \dots, z_m\}$ and $\{\zeta_1, \dots, \zeta_\ell\}$ may be empty. It follows that with some $a \in \mathbb{C} \setminus \{0\}$ we have the representation

$$R(z) = az^{-n} \prod_{j=1}^{\ell} (z - \zeta_i)^{\mu_j} \prod_{k=1}^{m} \left[(z - z_k)(z - 1/\bar{z}_k)\right]^{\nu_k} . \tag{2.15}$$

We now return to trigonometric polynomials. We put $z = e^{ix}$, $\zeta_j = e^{ix_j}$, $z_k = \rho_k e^{i\varphi_k}$, where $x \in \mathbb{R}$, $x_j \in [0, 2\pi)$, $\varphi_k \in [0, 2\pi)$ and $0 < \rho_k < 1$. From this we derive

$$\begin{aligned} z - \zeta_j = e^{ix} - e^{ix_j} &= 2ie^{i(x+x_j)/2} \sin \frac{x - x_j}{2} , \\ (z - z_k)(z - 1/\bar{z}_k) &= (e^{ix} - \rho_k e^{i\varphi_k})(e^{ix} - \rho_k^{-1} e^{i\varphi_i}) \\ &= -(\rho_k^{-1} + \rho_k) e^{i(x+\varphi_k)} \left(1 - \frac{2\rho_k}{1 + \rho_k^2} \cos(x - \varphi_k)\right) . \end{aligned}$$

From (2.14) and (2.15) we obtain now the following factorization of T_n:

$$T_n(x) = C \prod_{j=1}^{\ell} \sin^{\mu_j} \frac{x - x_j}{2} \prod_{k=1}^{m} \left(1 - q_k \cos(x - \varphi_k)\right)^{\nu_k} , \tag{2.16}$$

with $C \in \mathbb{R} \setminus \{0\}$ and $q_k := 2\rho_k/(1 + \rho_k^2) \in (0, 1)$, and with an even sum $\sum_{j=1}^{\ell} \mu_j$.

In general, (2.16) is not a factorization in $\mathcal{T}$; one can derive from it such factorizations by combining arbitrary pairs $\sin \frac{x-x_j}{2}$, $\sin \frac{x-x_{j_1}}{2}$ in the first product of (2.16). Thus, factorization of polynomials T_n into elementary factors is not unique in $\mathcal{T}$. From (2.16) we derive however:

1. *Let the product of two polynomials T_1T_2 be divisible by a polynomial t. If t has no common zeros with T_1, then it divides T_2.*

We call s/t an *irreducible representation* of a function $U \in \mathcal{U}_{m,n}$ if $s \in \mathcal{T}_m$, $t \in \mathcal{T}_n$ and if s, t have no common zeros (then $t(x) \neq 0$ on $\mathbb{T}$).

2. (i) *Each $U \in \mathcal{U}_{m,n}$ has an irreducible representation;* (ii) *this representation is unique up to a constant factor.*

This is easy to prove. (i) In the process of simplifying S_m/T_n, as long as they have a common zero, they have a common pair of zeros; we can cancel a factor in $\mathcal{T}$ and go on. (ii) If s_1/t_1 is another irreducible representation, then $st_1 = s_1t$. By 1, t divides t_1. Also t_1 divides t, so that $t_1 = Ct$, $s_1 = Cs$, $C \neq 0$.

The *defect* of a $U \in \mathcal{U}_{m,n}$ with an irreducible representation s/t is defined by

$$d := d(U) := \begin{cases} \min(m - \partial s,\ n - \partial t) & \text{if } U \neq 0 \\ n & \text{if } U = 0. \end{cases} \tag{2.17}$$

We also put

$$N(U) := N(U, \mathcal{U}_{m,n}) = 2(m + n - d) + 1 . \tag{2.18}$$

The proof of Theorem 2.4 applies again and yields instead of Corollary 2.5: For each $U \in \mathcal{U}_{m,n}$, the space $\Phi := \mathcal{T}_m + U\mathcal{T}_n$ is a Haar subspace of $C(\mathbb{T})$ of dimension $N(R)$. Also the other proofs apply and we obtain

Theorem 2.10. *Let $f \in C(\mathbb{T}) \setminus \mathcal{U}_{m,n}$, then $U \in \mathcal{U}_{m,n}$ is a best approximation to f from $\mathcal{U}_{m,n}$ if and only if there are $N + 1$ points $x_1 < \cdots < x_{N+1}$ $(< x_1)$ ordered in positive direction on $\mathbb{T}$, where $f - U$ takes the values $\pm\|f - U\|$ with alternating signs; this U is unique.*

§ 3. Rational Approximation of $|x|$

The theory of rational approximation began with the following theorem of Newman, which shows that $|x|$ has excellent approximation by rationals.

Theorem 3.1 (Newman [1964]). *The following estimates hold:*

$$e^{-\pi\sqrt{n+1}} \le \rho_n(|x|)_{C[-1,1]} \le 3e^{-\sqrt{n}} , \qquad n \ge 5 . \tag{3.1}$$

In our proof of the upper estimate (3.1) we shall follow the original paper of Newman [1964]. For given $n \in \mathbb{N}$ we put $a := e^{-1/\sqrt{n}}$ and define

$$N(x) := N_n(x) := \prod_{k=1}^{n-1} (x + a^k) . \tag{3.2}$$

We call the $N_n(x)$ *the Newman polynomials*; they play an important role in several problems of rational approximation.

Lemma 3.2. *The polynomials of Newman $N(x) := N_n(x)$ for $n \geq 5$ and $x \in [e^{-\sqrt{n}}, 1]$ satisfy*

$$\left|\frac{N(-x)}{N(x)}\right| \leq e^{-\sqrt{n}} .$$

Proof. First we show that

$$\prod_{j=1}^{n-1} \frac{1-a^j}{1+a^j} \leq e^{-\sqrt{n}} , \qquad n \geq 5 . \tag{3.3}$$

The function $\varphi(t) = (1+t)e^{-2t} - (1-t)$ satisfies $\varphi(0) = 0$ and $\varphi'(t) > 0$ for $t > 0$. This yields $(1-t)/(1+t) \leq e^{-2t}$ for $t \geq 0$ and

$$\prod_{j=1}^{n-1} \frac{1-a^j}{1+a^j} \leq \exp\left\{-2\sum_{j=1}^{n-1} a^j\right\} = \exp\left\{-2\frac{a-a^n}{1-a}\right\} .$$

But

$$2(a-a^n) \geq 2(e^{-1/\sqrt{5}} - e^{-\sqrt{5}}) > 1 \quad \text{for} \quad n \geq 5 .$$

Also, for all $t \geq 0$, $1 - e^{-t} \leq t$; hence $1/(1-a) \geq \sqrt{n}$, and (3.3) follows.

If $e^{-\sqrt{n}} \leq x \leq 1$, then for some j, $0 \leq j \leq n-1$, $a^{j+1} \leq x \leq a^j$. In this case

$$\begin{aligned}
\left|\frac{N(-x)}{N(x)}\right| &= \prod_{k=1}^{j} \frac{a^k - x}{a^k + x} \cdot \prod_{k=j+1}^{n-1} \frac{x - a^k}{x + a^k} \\
&\leq \prod_{k=1}^{j} \frac{a^k - a^n}{a^k + a^n} \cdot \prod_{k=j+1}^{n-1} \frac{a^j - a^k}{a^j + a^k} \\
&= \prod_{m=n-j}^{n-1} \frac{1-a^m}{1+a^m} \cdot \prod_{m=1}^{n-j-1} \frac{1-a^m}{1+a^m} \\
&= \prod_{m=1}^{n-1} \frac{1-a^m}{1+a^m} \leq e^{-\sqrt{n}} ,
\end{aligned}$$

by (3.3). □

Proof of the Upper Estimate in (3.1). We define

$$R_n(x) := x\,\frac{N(x) - N(-x)}{N(x) + N(-x)} \tag{3.4}$$

and show that

$$\big| |x| - R_n(x) \big| < 3e^{-\sqrt{n}} , \qquad x \in [-1, 1] . \tag{3.5}$$

Since $|x|$ and $R_n(x)$ are both even, it suffices to consider the case when $0 \le x \le 1$. For $0 \le x \le a^n = e^{-\sqrt{n}}$ this is quite trivial, since here $N(x) \ge N(-x) \ge 0$, so that $0 \le x - R_n(x) \le x \le e^{-\sqrt{n}}$. For $x \in (e^{-\sqrt{n}}, 1]$ by Lemma 3.2 we have

$$|x - R_n(x)| \le 2x \left| \frac{N(-x)}{N(x) + N(-x)} \right| \le 2 \left[\left| \frac{N(x)}{N(-x)} \right| - 1 \right]^{-1} \le \frac{2}{e^{\sqrt{n}} - 1} \le 3e^{-\sqrt{n}} .$$

□

The proof of the lower estimate in (3.1) is also based on ideas of Newman with improvements by Bulanov [1969].

Lemma 3.3. *For each polynomial S, $\deg S = n \ge 1$, there exists another polynomial M, $\deg M = n$, with only non-positive zeros for which*

$$\left| \frac{S(-x)}{S(x)} \right| \ge \left| \frac{M(-x)}{M(x)} \right| , \qquad x > 0 .$$

Proof. Let $z = \alpha + i\beta$ be any complex number and $x > 0$. Then

$$\left| \frac{x - z}{x + z} \right|^2 = \frac{(x - \alpha)^2 + \beta^2}{(x + \alpha)^2 + \beta^2} \ge \frac{(x - |\alpha|)^2 + \beta^2}{(x + |\alpha|)^2 + \beta^2} \ge \left(\frac{x - |\alpha|}{x + |\alpha|} \right)^2 . \tag{3.6}$$

If $S(x) = a \prod_{k=1}^n (x - z_k)$, $z_k = \alpha_k + i\beta_k$, is the factorization of S, we take $M(x) = \prod_{k=1}^n (x + |\alpha_k|)$. By (3.6) we get for $x > 0$

$$\left| \frac{S(-x)}{S(x)} \right| = \prod_{k=1}^n \left| \frac{x - z_k}{x + z_k} \right| \ge \prod_{k=1}^n \left| \frac{x - |\alpha_k|}{x + |\alpha_k|} \right| \ge \left| \frac{M(-x)}{M(x)} \right| . \qquad \square$$

Lemma 3.4. *For each polynomial Q, $\deg Q = n \ge 1$, and all $b > a > 0$ we have*

$$\int_a^b \log \left| \frac{Q(-x)}{Q(x)} \right| \frac{dx}{x} \ge -\frac{\pi^2}{2} n .$$

Proof. By the Lemma 3.3 we get

$$\int_a^b \log \left| \frac{Q(-x)}{Q(x)} \right| \frac{dx}{x} \ge n \inf_{\alpha > 0} \int_a^b \log \left| \frac{x - \alpha}{x + \alpha} \right| \frac{dx}{x} .$$

Further, for all $\alpha > 0$ we have

$$\int_a^b \log \left| \frac{x - \alpha}{x + \alpha} \right| \frac{dx}{x} \ge \int_0^\infty \log \left| \frac{x - \alpha}{x + \alpha} \right| \frac{dx}{x} =$$

$$= \int_0^\infty \log \left| \frac{x - 1}{x + 1} \right| \frac{dx}{x} = 2 \int_0^1 \log \frac{1 - t}{1 + t} \frac{dt}{t} = -\frac{\pi^2}{2} .$$

The last integral can be calculated by using the Taylor series for the logarithmic function and the fact that

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8} . \qquad \square$$

Proof of the Lower Estimate in (3.1). Let $R_n \in \mathcal{R}_n$ be the best rational approximation of degree $\le n$ to $|x|$ on $[-1,1]$:

$$(3.7) \qquad \| \, |x| - R_n(x) \|_{C[-1,1]} = \rho_n(|x|)_{C[-1,1]} =: \lambda_n .$$

Then according to **1** of §2, $R_n(x) = P_n(x)/Q_n(x)$, where P_n, Q_n are even polynomials of degree $\le n$ and $Q_n(x) > 0$ on $[-1,1]$. Let $S(x) := xQ_n(x) + P_n(x)$ and let $M(x)$ be the corresponding polynomial, $\deg M = \deg S$, of Lemma 3.3. We show that for $x \in [\delta_n, 1]$, $\delta_n := e^{-\pi\sqrt{n+1}}$, we have

$$(3.8) \qquad \lambda_n \ge x \left| \frac{M(-x)}{M(x)} \right| .$$

Indeed, if for some $x \in [\delta, 1]$, $P_n(x) \ge 0$, then

$$\lambda_n \ge |x - R_n(x)| = \left| \frac{xQ_n(x) - P_n(x)}{Q_n(x)} \right| \ge x \left| \frac{S(-x)}{S(x)} \right|$$

and we can use Lemma 3.3. On the other hand, if $P_n(x) < 0$, then $\lambda_n \ge x$ and it is sufficient to remark that $|M(-x)/M(x)| \le 1$ for $x > 0$, since all zeros of M are real and ≤ 0.

Now (3.8) and Lemma 3.4 imply

$$\begin{aligned}
\int_{\delta_n}^1 \log \lambda_n \frac{dx}{x} &\ge \int_{\delta_n}^1 \log \left| x \frac{M(-x)}{M(x)} \right| \frac{dx}{x} \\
&= \int_{\delta_n}^1 \frac{\log x}{x} \, dx + \int_{\delta_n}^1 \log \left| \frac{M(-x)}{M(x)} \right| \frac{dx}{x} \\
&\ge -\tfrac{1}{2} \log^2(1/\delta_n) - \tfrac{1}{2}\pi^2(n+1) .
\end{aligned}$$

Dividing this by $\log(1/\delta_n) = \pi\sqrt{n+1}$, we get

$$\log \lambda_n \ge -\pi\sqrt{n+1} . \qquad \square$$

The lower estimate (3.1) in Newman's theorem is close to best possible, but the upper estimate can be essentially improved. Actually Vyacheslavov [1975] proved that for some constants $C_1, C_2 > 0$,

$$(3.9) \qquad C_1 e^{-\pi\sqrt{n}} \le \rho_n(|x|)_{C[-1,1]} \le C_2 e^{-\pi\sqrt{n}} .$$

In particular one has the peculiar formula

$$\lim_{n\to\infty} \rho_n(|x|)^{1/\sqrt{n}} = e^{-\pi} \,. \tag{3.10}$$

An interesting relation of Theorem 3.1 to elliptic functions has been found by Gonchar [1967_2]. In an old paper Zolotarev [1877], using elliptic functions, was able to give explicitly best rational approximants to certain functions. An example is sign x on the union of the intervals $[-1,-\alpha]$, $[\alpha,1]$ with $0<\alpha<1$. From this one can derive estimates of types (3.1).

Rational approximation of $\sqrt{x}$ on $[0,1]$ is equivalent to approximation of $|x|$ on $[-1,1]$. More generally, for $\alpha>0$,

$$\rho_n(x^\alpha)_{C[0,1]} = \rho_{2n}(|x|^{2\alpha})_{C[-1,1]} \,. \tag{3.11}$$

This follows from

$$\|x^\alpha - P_n/Q_n\|_{C[0,1]} = \|x^{2\alpha} - P_n(x^2)/Q(x^2)\|_{C[0,1]} \,.$$

Recently, in the hands of Ganelius, Stahl, Varga, Vyacheslavov and others, the quantities $\rho_n(x^\alpha)_{C[0,1]}$ and thus, for $\alpha=\frac{1}{2}$, $\rho_n(|x|)_{C[-1,1]}$ have been determined up to strong equivalences. Together with (3.9), an important intermediate step has been made by Ganelius [1979], who proved that

$$\lim_{n\to\infty} \rho_n(x^\alpha)^{1/\sqrt{n}}_{C[0,1]} = e^{-2\pi\sqrt{\alpha}} \,, \qquad 0<\alpha<1 \tag{3.12}$$

(see also Vyacheslavov [1975]). The complete statements for $0<\alpha<1$, $n\to\infty$ are:

$$\rho_n(|x|)_{C[-1,1]} = 8e^{-\pi\sqrt{n}}\bigl(1+o(1)\bigr) \,, \tag{3.13}$$

$$\rho_n(x^\alpha)_{C[0,1]} = 4^{1+\alpha}|\sin\pi\alpha| e^{-2\pi\sqrt{\alpha n}}\bigl(1+o(1)\bigr) \,. \tag{3.14}$$

First Varga and his collaborators, by means of very precise computations, calculated the errors for many large n, and conjectured (3.13) in Varga, Ruttan and Carpenter [1991], (3.14) in Varga and Carpenter [1992]. Using deep methods: logarithmic potentials, estimates of elliptic integrals, Stahl in [1992] was then able to establish (3.13); in [1993] he also announces (3.14). See Chapter 8. This interrelation between computers and theoretical mathematics is quite remarkable!

§ 4. Approximation of e^x on $[-1,1]$

It has sometimes been said that it is smooth functions with a few singularities that have much better rational than polynomial approximation. This is to some extent confirmed for e^x; from (4.2) and (4.3) it follows that $\rho_n(e^x)_{[-1,1]} \approx \sqrt{\pi n}\, 4^{-n} E_{2n}(e^x)_{[-1,1]}$. (We have used Stirling's strong equivalence $n! \approx \sqrt{2\pi}\, e^{-n} n^{n+1/2}$.)

We can give an exact asymptotic estimate even for $\rho_{m,n}(e^x)$.

The following theorem (conjectured by Meinardus) is due to Braess [1984] and Newman [1979]; Newman found the method of proof, and Braess obtained the exact estimates.

Theorem 4.1. *For $m+n \to \infty$ one has*

$$
\begin{aligned}
r_{m,n} := \rho_{m,n}(e^x)_{[-1,1]} &= \frac{m!n!}{2^{m+n}(m+n)!(m+n+1)!}(1+o(1)) \\
&= \left(\frac{e}{2}\right)^{m+n} \frac{m^{m+1/2}n^{n+1/2}}{(m+n)^{2m+2n+2}}\left(1+o(1)\right) ,
\end{aligned} \tag{4.1}
$$

(the last term for $m>0$, $n>0$), in particular

$$\rho_n(e^x)_{[-1,1]} = \frac{1}{4}\left(\frac{e}{8}\right)^{2n} n^{-2n-1}\left(1+o(1)\right) . \tag{4.2}$$

For comparisons, we recall that [**CA**, Theorem 8.2, p.232]

$$E_n(e^x)_{[-1,1]} = \frac{1}{2^n(n+1)!}\left(1+o(1)\right) . \tag{4.3}$$

The proof is via the complex approximation. For a function $f(z)$ analytic around $z=0$, the rational function $R = P/Q \in \mathcal{R}_{m,n}$ is its m,n-*Padé approximant* at $z=0$, denoted by $[m/n]_f(z)$, if $fQ-P$ has a zero of order $m+n+1$ at $z=0$. (See Chapter 9.)

Theorem 4.2. *The m,n-Padé approximant of e^z at $z=0$ is given by $R = P/Q$, with P,Q from (4.5) and (4.6). For $m+n\to\infty$ we have*

$$
\begin{cases}
e^z - R(z) = A_{m,n} z^{m+n+1} e^{2nz/(m+n)}\left(1+o(1)\right) \\[2ex]
A_{m,n} = (-1)^n \dfrac{m!n!}{(m+n)!(m+n+1)!}
\end{cases} \tag{4.4}
$$

uniformly in each disk $|z| \le M$; moreover, Q does not vanish in $|z|\le M$ for large $m+n$.

Proof. We use the Perron functions

$$P(z) = \int_0^\infty t^n (t+z)^m e^{-t}\,dt \tag{4.5}$$

$$Q(z) = \int_0^\infty t^m (t-z)^n e^{-t}\,dt . \tag{4.6}$$

Here $Q(0) = (m+n)! \ne 0$. We have some freedom in the paths of integration and obtain

$$
\begin{aligned}
e^z Q - P &= \int_0^\infty t^m (t-z)^n e^{z-t}\,dt - \int_0^\infty t^n (t+z)^m e^{-t}\,dt \\
&= \int_0^\infty (t-z)^n t^m e^{z-t}\,dt - \int_z^{z+\infty} (t-z)^n t^m e^{z-t}\,dt \\
&= \int_0^z (t-z)^n t^m e^{z-t}\,dt\,.
\end{aligned}
$$

Therefore

$$
e^z Q(z) - P(z) = (-1)^n z^{m+n+1} \int_0^1 u^n (1-u)^m e^{uz}\,du\,. \tag{4.7}
$$

The last integral is bounded for $|z| \le M$. We get $e^z Q - P = O(z^{m+n+1})$ for $z \to 0$, so that this function has a zero of order $m+n+1$ at $z=0$.

To obtain estimates of this function, we use the formulas, where $u_0 = \frac{n+1}{m+n+2}$,

$$
\int_0^1 u^n (1-u)^m\,du = B(m,n) = \frac{m!n!}{(m+n+1)!}
$$

$$
\int_0^1 u^n (1-u)^m (u-u_0)\,du = 0
$$

$$
\int_0^1 u^n (1-u)^m (u-u_0)^2\,du = \frac{(m+1)(n+1)}{(m+n+2)^2(m+n+3)} \; \frac{m!n!}{(m+n+1)!}\,.
$$

In the integral (4.7) we replace e^{uz} by

$$
e^{uz} = e^{u_0 z} + z(u-u_0)e^{u_0 z} + \rho\,, \qquad |\rho| \le M_1 (u-u_0)^2\,,
$$

with a constant M_1 when $|z| \le M$. Moreover, $e^{u_0 z} = e^{nz/(m+n)}(1+o(1))$ uniformly for $|z| \le M$. We thus obtain

$$
e^z Q - P = (-1)^n z^{m+n+1} \frac{m!n!}{(m+n+1)!}\, e^{nz/(m+n)} \big(1+o(1)\big)\,. \tag{4.8}
$$

Next we show that for $|z| \le M$,

$$
Q(z) = (m+n)!\, e^{-nz/(m+n)} \big(1+o(1)\big)\,. \tag{4.9}
$$

For Q we have the expansion

$$
\begin{aligned}
Q(z) &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} z^k \int_0^\infty t^{m+n-k} e^{-t}\,dt \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (m+n-k)!\, z^k\,.
\end{aligned} \tag{4.10}
$$

Thus we have, for $n > 0$,

(4.11)
$$\frac{Q(z)}{(m+n)!} = \sum_{k=0}^{\infty} \left(\frac{-nz}{m+n}\right)^k \frac{1}{k!}\gamma_k(m,n) ,$$
$$\gamma_k := \gamma_k(m,n) := \frac{(1-\frac{k-1}{n})\cdots(1-\frac{1}{n})}{(1-\frac{k-1}{m+n})\cdots(1-\frac{1}{m+n})} \quad \text{if } k \le n ,$$
$$\gamma_k(m,n) = 0 , \quad \text{if } k > n .$$

This we compare for $z \in D_M := \{z : |z| \le M\}$ with

(4.12)
$$e^{-\frac{nz}{m+n}} = \sum_{k=0}^{\infty} \left(\frac{-nz}{m+n}\right)^k \frac{1}{k!} .$$

Both series for $|z| \le M$ are majorized by $\sum_{k=0}^{\infty} \frac{M^k}{k!}$, so that they converge uniformly in the parameters m, n, $z \in D_M$. Moreover, $|\gamma_k(m,n)| \le 1$, and the function (4.12) is uniformly bounded for all m, n, $z \in D_M$.

Now relation (4.9) needs to be proved only for some subsequence of any sequence of pairs m, n with $m+n \to \infty$. We may assume therefore that n has a finite or infinite limit as $m+n \to \infty$.

Case 1: $n \to \infty$. Here $\gamma_k \to 1$, that is $\gamma_k = 1 + o(1)$ as $m+n \to \infty$ for each fixed k. Thus for $m+n \to \infty$,

$$\begin{aligned}\frac{Q(z)}{(m+n)!} &= e^{-\frac{nz}{m+n}} + \sum_{k=0}^{\infty} \left(-\frac{nz}{m+n}\right)^k \frac{1}{k!}\big(\gamma_k(m,n) - 1\big) \\ &= e^{-\frac{nz}{m+n}} + o(1)\sum_{k=0}^{\infty} \left(-\frac{nz}{m+n}\right)^k \frac{1}{k!} = \big(1+o(1)\big)e^{-\frac{nz}{m+n}} .\end{aligned}$$

Case 2: $n \to n_0$. Here $-\frac{nz}{m+n} \to 0$ uniformly for $m+n \to \infty$. All terms of the series (4.11) converge to zero, except for the zeroth term, which is 1. Also, $e^{-\frac{nz}{m+n}} = 1 + o(1)$ if $m+n \to \infty$. This yields

$$\frac{Q(z)}{(m+n)!} = 1 + o(1) = \big(1+o(1)\big)e^{-\frac{nz}{m+n}} .$$

In both cases we have (4.9). By division we get (4.4) with $R = P/Q$. From (4.9) we see also that $Q(z) \ne 0$ for $|z| \le M$ and large $m+n$. □

Corollary 4.3. *The Padé approximants $[m/n](z)$ of e^z converge uniformly to this function on compact subsets of $\mathbb{C}$.*

For the proof of Theorem 4.1 of special importance is the behavior of $e^z(e^z - R)$ on the circle $|z| = 1/2$.

Lemma 4.4. *There exists a rational function $R^* \in \mathcal{R}_{m,n}$ with the following properties. It has no poles in $|z| \le 1/2$, while $e^z - R^*(z)$ has a single zero of order $m+n+1$ in this disk. Moreover, for $|z| = 1/2$ uniformly for $m+n \to \infty$*

$$|e^z(e^z - R^*(z))| = \frac{m!n!}{2^{m+n+1}(m+n)!(m+n+1)!}\left(1 + o(1)\right) . \tag{4.13}$$

Proof. From (4.4),

$$e^z(e^z - R(z)) = A_{m,n} z^{m+n+1} e^{\alpha z}\left(1 + o(1)\right) , \qquad |z| \le M . \tag{4.14}$$

where $\alpha = 1 + (2n)/(m+n)$ and $1 < \alpha \le 3$. Here, the modulus of $z^{m+n+1}e^{\alpha z}$ is not close to a constant for $|z| = 1/2$, but this can be achieved by a change of variables. We put

$$z_0 = \frac{\alpha}{4(m+n+1)} ,$$

this is a real number and $z_0 \to 0$ for $m + n \to \infty$. In (4.14) we replace z by $z - z_0$. Then, since e^{z_0} and $e^{\alpha z_0}$ are $1 + o(1)$, for the function $R^*(z) = e^{z_0}R(z - z_0) \in \mathcal{R}_{m,n}$ we get

$$e^z(e^z - R^*(z)) = A_{m,n}(z - z_0)^{m+n+1}e^{\alpha z}\left(1 + o(1)\right) .$$

Now we assume that $|z| = 1/2$. Since $1/z = 4\bar{z}$,

$$\begin{aligned} |(z - z_0)^{m+n+1}e^{\alpha z}| &= \frac{1}{2^{m+n+1}}|(1 - 4z_0 z)^{m+n+1}e^{\alpha z}| \\ &= \frac{1}{2^{m+n+1}}\left|\left(1 - \frac{\alpha z}{m+n+1}\right)^{m+n+1} e^{\alpha z}\right| \\ &= \frac{1}{2^{m+n+1}}\left(1 + o(1)\right) . \end{aligned}$$

We have used here the fact that $(1 - (w/N))^N e^w \to 1$ uniformly for $|w| \le M$ and each $M > 0$. □

Proof of Theorem 4.1. This proof will be possible because we shall establish a relation between rational approximation on the interval $[-1, 1]$ and the circle $|z| = 1/2$.

For $x \in [-1, 1]$, we put $z = \frac{1}{2}(x + iy)$, $x^2 + y^2 = 1$. If $P^*(z)$ is a polynomial with real coefficients of degree $\le n$, then $P(x) = P^*(z)P^*(\bar{z})$ is another real polynomial, $P \in \mathcal{P}_n$. Indeed, to a linear factor $z - \alpha$ of $P^*(z)$ there corresponds the linear factor $(z - \alpha)(\bar{z} - \alpha) = \frac{1}{4} + \alpha^2 - \alpha x$ of $P(x)$. Also, $P(x)$ is real valued for $x \in [-1, 1]$. To a function $R^* = P^*/Q^* \in \mathcal{R}_{m,n}$ of Lemma 4.4 we let correspond

$$R(x) = \frac{P^*(z)P^*(\bar{z})}{Q^*(z)Q^*(\bar{z})} \in \mathcal{R}_{n,m} . \tag{4.15}$$

This will be our approximant to $e^x = e^z e^{\bar{z}}$.

For arbitrary complex α, β one has the identity

$$\alpha\bar{\alpha} - \beta\bar{\beta} = 2\operatorname{Re}\big(\bar{\alpha}(\alpha - \beta)\big) - |\alpha - \beta|^2 \,. \tag{4.16}$$

Putting $\alpha = e^z$, $\beta = P^*(z)/Q^*(z)$, we obtain

$$e^x - R(x) = 2\operatorname{Re}\left\{e^{\bar{z}}\big(e^z - R^*(z)\big)\right\} - |e^z - R^*(z)|^2 \,. \tag{4.17}$$

Let

$$\lambda_{m,n} := \lambda := \frac{m!n!}{2^{m+n}(m+n)!(m+n+1)!} \,.$$

By (4.13), the first term in (4.17) does not exceed $\lambda_{m,n}(1 + o(1))$, while the second is $O(\lambda_{m,n}^2) = o(\lambda_{m,n})$. It follows immediately that

$$r_{m,n} \le \max_{-1 \le x \le 1} |e^x - R(x)| \le \lambda_{m,n}\big(1 + o(1)\big) \,.$$

To obtain a similar lower bound for $r_{m,n}$, we study the functions $h(z) = e^z(e^z - R^*)$, $h_1(z) = e^{\bar{z}}(e^z - R^*)$. Obviously, $|h_1(z)| = |h(z)|$. On the other hand, $\arg h_1(z) = \arg h(z) - y$ for $z = \frac{1}{2}(x + iy)$. Since h has a unique zero of order $m+n+1$ in $|z| \le 1/2$ and no poles, $\arg h(z)$ increases by $2\pi(m+n+1)$ as z moves counterclockwise along $|z| = 1/2$. The same is true for $\arg h_1(z)$. Either on the upper or on the lower semi-circle this argument increases by at least $\pi(m+n+1)$. Assuming the first to be the case, we get points $z_k = \frac{1}{2}(x_k + iy_k)$, $k = 1, \ldots, m+n+1$ on this semi-circle with $-1 \le x_1 < \cdots < x_{m+n+1} \le 1$, $y_k \ge 0$ for which $h_1(z_k) = \pm|h_1(z_k)| = \pm|h(z_k)|$ with alternating signs. At these points we have, from (4.14) and (4.13), for some $\sigma = \pm 1$,

$$e^{x_k} - R(x_k) = \sigma(-1)^k\lambda\big(1 + o(1)\big) + O(\lambda^2) = \sigma(-1)^k\lambda\big(1 + o(1)\big) \,, \\ k = 1, \ldots, m+n+1 \,.$$

Finally, de la Vallée-Poussin lemma for rational approximation (Theorem 2.7) yields $r_{m,n} \ge \lambda_{m,n}(1 + o(1))$. □

For the function e^z one has $E_n(e^z, K)^{1/n} \to 0$ for each compact set $K \subset \mathbb{C}$. In other words, the approximation error converges to zero *faster than geometrically*. The same is true for any *entire function*. For a *meromorphic function* f one has instead

$$\lim_{n\to\infty} \rho_n(f, K)^{1/n} = 0 \,. \tag{4.18}$$

Indeed, let $r > 0$, $\lambda > 1$ and let $R \in \mathcal{R}_m$ be the sum of the singular parts of f at all of its poles in the disk $D_{\lambda r} : |z| \le \lambda r$. Then $g = f - R$ is analytic in the disk. For $n > m$ let $P_{n-m} \in \mathcal{P}_{n-m}$ be the sum of the first $n - m + 1$ terms of the Maclaurin expansion of g. Applying Cauchy's inequality for its coefficients, we get $|g(z) - P_{n-m}(z)| \le C\lambda^{-n+m}$ in the disk $D_r : |z| \le r$. Thus $\rho_n(f, D_r)_\infty \le C\lambda^{-n+m}$ and $\limsup_{n\to\infty} \rho_n(f, D_r)^{1/n} \le \lambda^{-1}$, and since $\lambda > 1$ is arbitrary, we get (4.18).

Beyond this, Gonchar [1978] proved that (4.18) is valid for all single valued functions, analytic on $\mathbb{C}$ except for a set of singular points of logarithmic capacity zero.

§ 5. Rational Approximation of e^{-x} on $[0,\infty)$

The history of this problem is very interesting. Already the first workers, Cody, Meinardus and Varga [1969] and Newman [1974₂] found that $\rho_n(e^{-x})_{[0,\infty)}$ decreases geometrically: for some $0 < q_1 < q_2 < 1$, one has

$$q_1^n \le \rho_n(e^{-x})_{[0,\infty)} \le q_2^n \ , \qquad n = 1,2,\ldots \tag{5.1}$$

The problem then became to show that for some q, $0 < q < 1$,

$$\lim \rho_n(e^{-x})_{[0,\infty)}^{1/n} = q \tag{5.2}$$

and to determine this q. A step towards the solution was the result of Schönhage [1973], about approximation of e^{-x} by inverses of polynomials, that is, by rational functions R of the class $\mathcal{R}_{0,n}$. He proved

$$\lim_{n\to\infty} \rho_{0,n}(e^{-x})_{[0,\infty)}^{1/n} = \frac{1}{3} \ . \tag{5.3}$$

Functions $R \in \mathcal{R}_n$ have about twice as many free parameters as the functions $R \in \mathcal{R}_{0,n}$. This gave rise to the "1/9-hypotheses," which postulates that (5.2) holds with $q = 1/9$. This proved to be incorrect: Opitz and Scherer [1985] proved the right inequality (5.1) with $q_2 = 1/9.03$. About the same time, Carpenter, Ruttan and Varga [1984], see also Varga [A-1990], disproved the 1/9-hypothesis computationally. By means of very high precision calculations they obtained for q in (5.2) the empirical value

$$q = \frac{1}{9.28902\ 54919\ 2081} \ . \tag{5.4}$$

Next, Magnus [1985] has guessed the explicit formula for q (see (5.5)). Since his value of q agrees in all 15 digits with (5.4), there was little doubt that this guess was correct. Finally, using ideas of Stahl about Padé approximants, Gonchar and Rakhmanov [1987], in a difficult and beautiful way, proved

Theorem 5.1. *One has*

$$\lim \rho_n(e^{-x})_{[0,\infty)}^{1/n} = \exp(-\pi K/K') \ , \tag{5.5}$$

where K, K' are values of certain elliptic integrals.

We shall prove here only (5.1) and (5.3):

Theorem 5.2. *For some $0 < q_1 < q_2 < 1$ one has* (5.1).

We prove first

Theorem 5.3 (Schönhage). *One has* (5.3), *and more exactly, with some constant* $C > 0$,

$$(5.6) \qquad Cn^{-1/2}3^{-n} \le \rho_{0,n}(e^{-x}) \le \sqrt{2}\,3^{-n} \ .$$

Proof. In this problem, we can replace e^{-x} by $e^{-x/4}$. Assuming this, we shall look for a polynomial P_n that does not vanish for $x \ge 0$, and satisfies

$$(5.7) \qquad \left| e^{-x/4} - \frac{1}{P_n(x)} \right| \le \lambda_n \ , \qquad x \ge 0$$

where λ_n is as small as possible. Now (5.7) is equivalent to

$$(5.8) \qquad |e^{x/4} - P_n(x)| \le \lambda_n |P_n(x)| e^{x/4} \ , \qquad x \ge 0 \ .$$

We replace the problem by a related one, which we can handle by the methods of [**CA**, Chapter 3]. We replace $[0, \infty)$ by $[0, a]$ for some $a > 0$, and replace $|P_n(x)|$ on the right in (5.8) by the function $e^{x/4}$ (which should be close to $|P_n(x)|$). We then face the problem to minimize the norm

$$(5.9) \qquad \|e^{-x/4} - e^{-x/2}P_n(x)\|_{[0,a]} = \|e^{-x/2}(e^{x/4} - P_n(x))\|_{[0,a]} \ .$$

The functions $e^{-x/2}x^k$, $k = 0, \ldots, n$, form a Haar system on $[0, a]$. By [**CA**, Theorem 5.1, p.74], there is a unique minimizing $P_n \in \mathcal{P}_n$, which we denote by Q_n.

For this Q_n, $e^{-x/4} - e^{-x/2}Q_n$ has a Chebyshev alternance on $[0, a]$, hence $d(x) := e^{x/4} - Q_n(x)$ has at least $n+1$ zeros x_i, $0 < x_1 < \cdots < x_{n+1} < a$. We note the following facts:

$$(5.10) \qquad Q_n(x) \le e^{x/4} \ , \qquad x \ge a$$

$$(5.11) \qquad Q_n^{(k)}(x) \ge 0 \ , \quad x \ge a \ , \qquad k = 0, \ldots, n \ .$$

Indeed, if $e^{y/4} - Q_n(y) < 0$ for some $y > a$, then the function $e^{x/4} - Q_n(x)$ would have a zero $x_{n+2} > y$, altogether at least $n+2$ zeros. This would contradict the fact that $1, x, \ldots, x^n, e^{x/4}$ are a Haar system on each finite interval (see [**CA**, p.69]).

Next, we prove (5.11) for $k = 0$. From $d(x_{n+1}) = 0$ it follows that $Q_n(x_{n+1}) > 0$. If $Q_n(y) < 0$ for some $y > a$, there would exist a $y' > x_{n+1}$ with $Q_n'(y') < 0$. Moreover, by Rolle's theorem, there are some x_i', $0 < x_1' < \cdots < x_n' < x_{n+1}$, for which $d'(x_i') = 0$. A repetition of this argument $n+1$ times yields a point $y^{(n+1)}$ with $Q_n^{(n+1)}(y^{(n+1)}) < 0$, which is impossible. Starting this argument with $Q^{(k)}$ instead of Q and with some $n+1-k$ zeros of $d^{(k)}$ in $[0, a]$, we obtain (5.11) for $k = 1, \ldots, n$.

We shall give an exact formula for the degree of approximation of $e^{x/4}$ by polynomials, not in the uniform norm, but instead in the L_2 norm with weight e^{-x} on $[0, \infty)$. The Laguerre polynomials

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(e^{-x}x^n)\,, \qquad n = 0,1,\dots \tag{5.12}$$

are an orthonormal set on $(0,\infty)$ with the weight e^{-x} (see Szegö [B-1975], Freud [B-1969]). The Fourier coefficients in the expansion $e^{x/4} = \sum_0^\infty c_n L_n(x)$ are

$$c_n = \int_0^\infty e^{-x}e^{x/4}L_n(x)\,dx = \frac{1}{n!}\int_0^\infty e^{x/4}(e^{-x}x^n)^{(n)}\,dx\,,$$

and by integration by parts we get

$$c_n = \frac{(-1)^n}{4^n n!}\int_0^\infty e^{-(3/4)x}x^n\,dx = \frac{4(-1)^n}{3^{n+1}}\,.$$

The n-th partial sum $g_n(x)$ of the expansion approximates $e^{x/4}$ with the error

$$\int_0^\infty e^{-x}|e^{x/4} - g_n(x)|^2\,dx = \sum_{k=n+1}^\infty c_k^2 = \frac{2}{3^{2n+2}}\,. \tag{5.13}$$

(a) *The Lower Estimate.* Let $a = 4(n+1)\log 3 + 2\log 2$. We assume that there exists a polynomial P_n that satisfies (5.7) with $\lambda_n < \frac{1}{6\sqrt{a}}3^{-n}$ (this last inequality implies $\lambda_n < \frac{1}{2}e^{-a/4}$), and derive a contradiction. The last inequality (5.6) will then follow, if C is taken so small that $C\frac{1}{\sqrt{n}} \le \frac{1}{6\sqrt{a}}$, $n \ge 1$. Via (5.9) for $0 \le x \le a$

$$|P_n(x)| \le \frac{e^{x/4}}{1 - \lambda_n e^{x/4}} \le 2e^{x/4}\,,$$

and (5.8) becomes

$$|e^{x/4} - P_n(x)| \le 2\lambda_n e^{x/2}\,, \qquad 0 \le x \le a\,. \tag{5.14}$$

This will be preserved if we replace P_n by Q_n which minimizes (5.9). Then Q_n will satisfy (5.14), but also (5.10) and (5.11). Thus $0 \le Q_n(x) \le e^{x/4}$ for $x \ge a$. Using (5.13) we obtain

$$\begin{aligned}\frac{2}{3^{2n+2}} &\le \int_0^\infty e^{-x}\left(e^{x/4} - Q_n(x)\right)^2 dx \\ &\le \int_0^a e^{-x}(2\lambda_n e^{x/2})^2\,dx + \int_a^\infty e^{-x}(e^{x/4})^2\,dx \\ &= 4a\lambda_n^2 + 2e^{-a/2} = 4a\lambda_n^2 + \frac{1}{3^{2n+2}}\end{aligned}$$

and $\lambda_n^2 \ge \frac{1}{4a3^{2n+2}}$, which contradicts our assumption about λ_n.

(b) *The Upper Estimate.* We approximate

$$e^{x/4} = \frac{3}{4}e^x\int_x^\infty e^{-t}e^{t/4}\,dt$$

by the polynomials T_n of degree $\leq n$ defined by

$$T_n(x) = \frac{3}{4} e^x \int_x^\infty e^{-t} g_n(t)\, dt\ .$$

By means of Schwarz' inequality,

$$\begin{aligned}
|e^{x/4} - T_n(x)| &\leq \frac{3}{4} e^x \int_x^\infty e^{-t} |e^{t/4} - g_n(t)|\, dt \\
&\leq \frac{3}{4} e^x \left(\int_x^\infty e^{-t}\, dt \right)^{1/2} \left(\int_x^\infty e^{-t} |e^{t/4} - g_n(t)|^2\, dt \right)^{1/2} \\
&\leq \frac{\sqrt{2}}{4} \frac{1}{3^n} e^{x/2}\ , \qquad x \geq 0\ .
\end{aligned}$$

This time we put $a = 4n \log 3 + 2 \log 2$. For the extremal polynomial Q_n in (5.9) we have

$$(5.15) \qquad |e^{x/4} - Q_n(x)| \leq \frac{\sqrt{2}}{4} \frac{1}{3^n} e^{x/2}\ , \qquad 0 \leq x \leq a$$

also (5.11) with $k = 1$ is valid. For $0 \leq x \leq a$, the right-hand side of (5.15) does not exceed $\frac{1}{2} e^{x/4}$. Consequently, (5.15) implies $Q_n(x) \geq \frac{1}{2} e^{x/4}$, and we obtain

$$(5.16) \qquad |e^{x/4} - Q_n(x)| \leq \frac{1}{\sqrt{2}\, 3^n} e^{x/4} Q_n(x)\ , \qquad 0 \leq x \leq a\ .$$

Since $Q_n'(x) \geq 0$ for $x \geq a$, $Q_n(x) \geq Q_n(a) \geq 0$, $x \geq a$. We now get

$$\left| e^{-x/4} - \frac{1}{Q_n(x)} \right| \leq \frac{\sqrt{2}}{3^n}\ , \qquad 0 \leq x < \infty\ .$$

For $0 \leq x \leq a$ this follows directly from (5.16), for $x \geq a$ because $0 \leq Q_n(x)^{-1} \leq Q_n(a)^{-1} \leq 2e^{-a/4} = \sqrt{2}\, 3^{-n}$. □

Proof of Theorem 5.3. Because of Theorem 5.2, we have to establish only the lower estimate (5.1). In the inequality [**CA**, (2.9), p.101], we can majorize the right-hand side by $2^n M|x|^n$. A linear substitution yields the following simple inequality:

$$(5.17) \qquad |P_n(x)| \leq 4^n \left| \frac{x - c}{b - a} \right|^n \|P_n\|_{[a,b]}\ , \qquad x \notin [a, b]\ ,$$

where $c = \frac{a+b}{2}$ is the middle point of $[a, b]$. Let us assume that, with $Q_n(x) > 0$, $x \in [0, \infty)$,

$$(5.18) \qquad \left\| e^{-x} - \frac{P_n(x)}{Q_n(x)} \right\|_{[0,\infty]} \leq e^{-\lambda n}\ .$$

We shall show that $\lambda < 8$, or that in (5.1) we can take $q_1 = e^{-8} > \frac{1}{3000}$. We shall normalize Q_n by assuming that $\|Q_n\|_{[0,n]} = \max_{0\le x\le n} |Q_n(x)| = 1$. Let this maximum be attained at $x = \xi$, $0 \le \xi \le n$. From (5.18) we obtain

$$P_n(\xi) \ge e^{-n} - e^{-\lambda n} \ . \tag{5.19}$$

Let $I_1 = [0, n]$, $I_2 = [\lambda n, (\lambda + 1)n]$. Using the inequality

$$|P_n(x)| \le (e^{-x} + e^{-\lambda n})|Q_n(x)| \ ,$$

we can estimate $P_n(x)$ on I_2 by means of (5.17) for Q_n with $[a, b] = I_1$. This leads to

$$|P_n(x)| \le 2e^{-\lambda n} 4^n \left(\lambda + \tfrac{1}{2}\right)^n \ , \qquad x \in I_2 \ .$$

From this, using (5.17) with $[a, b] = I_2$,

$$|P_n(\xi)| \le 2e^{-\lambda n} 4^{2n} \left(\lambda + \tfrac{1}{2}\right)^{2n} \ .$$

Comparing this with (5.19) we see that for $\lambda \ge 2$

$$e^{-\lambda n} 4^{2n+1} \left(\lambda + \tfrac{1}{2}\right)^{2n} > e^{-n}$$

so that $\lambda < 3\log 4 + 1 + 2\log(\lambda + \frac{1}{2})$. An easy calculation shows that is not true if $\lambda \ge 8$. □

§ 6. Approximation of Classes of Functions

In this and the next section we shall estimate the error of rational approximation of piecewise analytic functions, and of functions from V^r, W_p^1, $p > 1$. We say that $f \in V^r[a, b]$, $r = 1, 2, \ldots$ if all derivatives $f, \ldots, f^{(r-1)}$ are absolutely continuous on $[a, b]$, and $f^{(r)} \in BV$ is of bounded variation. We shall also discuss two general methods of approximation: one of them uses Theorem 6.3, which allows us to "paste together" several functions, the other is based on the properties of the operator (6.14).

Let $\sigma(x)$ be the jump function, $= 0$ on $(-\infty, 0]$, $= 1$ on $(0, \infty)$. For an interval $[a, b]$, $a < c < b$ we shall approximate $\sigma(x - c)$ by rationals. Let $\lambda = \max(b - c, c - a)$, and $\delta_n := \delta := e^{-\sqrt{n}}$.

Lemma 6.1. *For given $a < c < b$ and each $n \ge 5$ satisfying $a < c - \lambda\delta < c + \lambda\delta < b$, there exists a function $\sigma_n \in \mathcal{R}_n$ for which*

$$|\sigma(x - c) - \sigma_n(x)| < \frac{3}{2} e^{-\sqrt{n}} \quad \text{on} \quad [a, c - \lambda\delta] \cup [c + \lambda\delta, b] \ , \tag{6.1}$$

$$0 \le \sigma_n(x) \le 1 \quad \text{on} \quad [c - \lambda\delta, c + \lambda\delta] \ . \tag{6.2}$$

Proof. (a) If $a = -1$, $c = 0$, $b = 1$, then $\lambda = 1$. We then can put $\sigma_n := R_n$ where

$$R_n(x) := \frac{N_n(x)}{N_n(x) + N_n(-x)}$$

and N_n is given by (3.2). On $(-\delta_n, \delta_n)$, $N_n(\pm x) > 0$, and we have (6.2). From Lemma 3.2 we get (6.1) for $x \in [\delta_n, 1]$:

$$(6.3) \quad |1 - \sigma_n(x)| = \left| \frac{N_n(-x)}{N_n(x) + N_n(-x)} \right| \le \frac{1}{|N_n(x)/N_n(-x)| - 1} \le \frac{3}{2} e^{-\sqrt{n}}$$

and we get (6.1). For $x \in [-1, -\delta_n]$, this inequality follows by symmetry.

(b) In the general case, we put $\sigma_n(x) := R_n(\frac{1}{\lambda}(x - c))$. The transformation $u = (x - c)/\lambda$ maps the interval $[c - \lambda\delta, c + \lambda\delta]$ on $[-\delta, \delta]$, and the intervals $[a, c - \lambda\delta]$, $[c + \lambda\delta, b]$ onto subsets of $[-1, -\delta]$, $[\delta, 1]$, respectively. Thus our statements follow from those of case (a). □

The following lemma serves to convert the *best* approximant R_n of f into a *good* approximant S_n that is, in addition, bounded on $\mathbb{R}$.

Lemma 6.2. *For each $f \in C(I)$, $I = [a, b]$ and each $\eta > 0$, there is a function $S_n \in \mathcal{R}_{2n}$ for which*

$$(6.4) \qquad \|f - S_n\|_I \le \rho_n(f) + 2\sqrt{\eta}\|f\|^2 ,$$

$$(6.5) \qquad \|S_n\|_{\mathbb{R}} \le \frac{1}{2\sqrt{\eta}} .$$

Proof. If $R_n \in \mathcal{R}_n$, $n = 1, 2, \ldots$ is the best approximant to f, then $\|R_n\| \le \|f\| + \rho_n(f) \le 2\|f\|$. We put

$$(6.6) \qquad S_n = \frac{R_n}{1 + \eta R_n^2} .$$

Then $|S_n(x)| \le \max_y \frac{y}{1+\eta y^2} = \frac{1}{2\sqrt{\eta}}$ for all x, and on the other hand

$$\|f - S_n\|_I \le \rho_n(f) + \|R_n - S_n\| = \rho_n(f) + \left\| \frac{\eta R_n^3}{1 + \eta R_n^2} \right\|$$
$$\le \rho_n(f) + \eta \frac{1}{2\sqrt{\eta}} \, 4\|f\|^2 .$$ □

Let $I_1 = [a, c + \varepsilon]$, $I_2 = [c - \varepsilon, b]$, $\varepsilon > 0$ be two overlapping intervals with union $I = [a, b]$. Let f_1, f_2 be defined on I_1, I_2, respectively, and let $f_1(c) = f_2(c)$. This defines on $[a, b]$ the function $f(x) = f_1(x)$, $x \in [a, c]$, $f(x) = f_2(x)$, $x \in [c, b]$. Let λ be defined as in Lemma 6.1.

Theorem 6.3. *For the functions f_1, f_2, f defined above, let $f_i \in \mathrm{Lip}_M 1$ on I_i, $i = 1, 2$. Then for a constant C that depends on M, $\|f\|_I$ and λ, for all large n,*

$$(6.7) \qquad \rho_{5n}(f)_I \le C\{\rho_n(f_1)_{I_1} + \rho_n(f_2)_{I_2} + e^{-\sqrt{n}/2}\} .$$

Proof. Let $\sigma_n \in \mathcal{R}_n$ be the function of Lemma 6.1, and $S_n^{(1)}, S_n^{(2)}$-functions in $\mathcal{R}_{2n}$ of Lemma 6.2, respectively, which correspond to the value $\eta = \delta = e^{-\sqrt{n}}$. It will be sufficient to estimate

$$f - R_n = (1-\sigma_n)(f - S_n^{(1)}) + \sigma_n(f - S_n^{(2)}) \,, \tag{6.8}$$

on $[c, b]$. Let Δ_1 and Δ_2 be the first and the second terms of (6.8), respectively. For large $n \geq 1$, $|\sigma_n(x)| \leq 2$, $|1 - \sigma_n(x)| \leq 2$ on $[a, b]$. By (6.4), for $x \in [c, b]$,

$$|\Delta_2| \leq 2|f_2(x) - S_n^{(2)}(x)| \leq 2\big(\rho_n(f_2)_{I_2} + 2\|f\|^2 e^{-\sqrt{n}/2}\big) \,. \tag{6.9}$$

For large n, $\lambda\delta < \varepsilon$, and then f_1 and $f_2 = f$ are both defined on $[c, c+\lambda\delta]$. On this interval,

$$\begin{aligned} |f(x) - f_1(x)| &\leq |f(x) - f(c)| + |f_1(x) - f(c)| \\ &\leq 2M|x - c| \leq 2M\lambda e^{-\sqrt{n}} \,. \end{aligned}$$

Therefore, by (6.4),

$$\begin{aligned} |\Delta_1| &\leq 2|f(x) - S_n^{(1)}(x)| \leq 4M\lambda e^{-\sqrt{n}} + 2|f_1(x) - S_n^{(1)}(x)| \\ &\leq C\big(\rho_n(f_1)_{I_1} + e^{-\sqrt{n}/2}\big) \,, \quad x \in [c, c+\lambda\delta] \,. \end{aligned} \tag{6.10}$$

On $[c+\lambda\delta, b]$, we can apply (6.1) and (6.5) and get

$$|\Delta_1| \leq \frac{3}{2} e^{-\sqrt{n}}\big(\|f\|_I + \|S_n^{(1)}\|_{I_1}\big) \leq C e^{-\sqrt{n}/2} \,, \qquad x \in [c+\lambda\delta, b] \,. \tag{6.11}$$

We see that $|f(x) - R_n(x)|$ does not exceed on $[c, b]$ the right-hand side of (6.7). The interval $[a, c]$ is treated similarly. □

By iteration, we can extend (6.7) to several overlapping intervals I_q and functions $f_q \in C[I_q]$, $q = 1, \ldots, p$. In particular, for a *piecewise analytic* function f, one can take the f_q to be analytic. Then $\rho_n(f_q)$ and even $E_n(f_q)$ converge to zero geometrically, and we obtain

Theorem 6.4 (Szüsz and Turan [1966]). *For a piecewise analytic function f on $[a, b]$ there exist constants $C > 0$, $c > 0$ such that for all sufficiently large n,*

$$\rho_n(f)_I \leq C e^{-c\sqrt{n}} \,. \tag{6.12}$$

Another general method of approximation is by means of the *DeVore operator* (6.14), which we define below. Like the deBoor-Fix operator of [**CA**, (4.2), p.144], it is not uniquely defined by a function $f \in C[0,1]$, and is highly nonlinear.

Let Δ_i, $i = 1, 2, \ldots, m$, $n \leq m \leq 2n$ be a decomposition of $[0, 1]$ into intervals of length $|\Delta_i| \leq 1/n$ with disjoint interiors. For each i, we select $c_i \in \Delta_i$ and let, for some integer $q \geq 1$

$$(6.13)\qquad \begin{cases} \phi_i(x) = \left[1 + \dfrac{(x-c_i)^2}{|\Delta_i|^2}\right]^{-q}, & i = 1, \dots, m \\ \phi = \displaystyle\sum_1^m \phi_i \quad , \quad R_i = \dfrac{\phi_i}{\phi} . \end{cases}$$

Obviously, $\phi_i \in \mathcal{R}_{2q}$, $\phi \in \mathcal{R}_{2qm}$, $R_i \in \mathcal{R}_{2q(m-1)}$, $i = 1, \dots, m$. For each function of a given class, we select polynomials $P_i(f,x) := P_i(x)$ of degree $\le r \le 2q$ and define the operator

$$(6.14)\qquad R_m(f,x) = \sum_{i=1}^m P_i(f,x) R_i(x) .$$

It is clear that $R_m(f)$ is a rational function of the class $\mathcal{R}_{2qm}$. We note the following properties of the operator $R_m(f)$. We have $R_i(x) > 0$, $\sum R_i(x) = 1$, moreover $\phi_i(x) \ge 2^{-q}$ for $x \in \Delta_i$, hence

$$(6.15)\qquad \phi(x) \ge 2^{-q} , \qquad 0 \le x \le 1$$

$$(6.16)\qquad R_i(x) \le 2^q \left[1 + (x-c_i)^2 |\Delta_i|^{-2}\right]^{-q} , \qquad 0 \le x \le 1 .$$

We need also an estimate of different parts of the sum

$$(6.17)\qquad \Sigma := \sum_1^m \left(1 + \frac{(x-c_i)^2}{|\Delta_i|^2}\right)^{-1} .$$

We divide the integers $i = 1, \dots, m$ into classes according to the size of Δ_i. We say that $i \in K_\nu$, $\nu = 0, 1, \dots$ if

$$(6.18)\qquad 2^{-\nu-1} < n|\Delta_i| \le 2^{-\nu} ,$$

and denote by Σ_ν the part of the sum (6.17) which corresponds to $i \in K_\nu$.

Lemma 6.5. *There are at most $2n$ values of ν with $\Sigma_\nu \ne 0$, and for each of them, for some absolute constant C_0,*

$$(6.19)\qquad \Sigma_\nu \le C_0 .$$

Proof. The first statement is obvious. To prove the second, we have

$$\Sigma_\nu \le \sum_{i \in K_\nu} \left[1 + 4^\nu n^2 |x - c_i|^2\right]^{-1} .$$

For a given $x \in [0,1]$, let c_i^* stand for the c_i to the right of x in this sum: $x \le c_1^* \le c_2^* \le \cdots$. Between any c_j^* and c_{j+2}^* there is at least one interval of class K_ν. Hence $c_{j+2}^* - x \ge 2^{-\nu-1} n^{-1} + (c_j^* - x)$, and further $c_{2k+1}^* - x \ge c_{2k}^* - x \ge 2^{-\nu-1}(k-1)/n$. The corresponding part of the sum Σ_ν is

$$\leq 2\left(1+\sum_{k=1}^{\infty} 4k^{-2}\right) < 16\,.$$

The same applies to the part of $\sum_\nu$ with $c_i < x$. □

Often some of the intervals of a useful partition

$$[0,1] = \bigcup_{i=1}^{m'} \Delta_i' \quad \text{with} \quad m' \leq n \tag{6.20}$$

have lengths $> 1/n$. Then we subdivide them further:

Lemma 6.6. *For each partition* (6.20) *of* $[0,1]$ *there is a finer partition* $[0,1] = \cup_1^m \Delta_i$ *with the properties*

(i) $|\Delta_i| \leq 1/n, \quad i = 1,\dots,m$
(ii) $n \leq m \leq 2n$
(iii) $1/(2n) \leq |\Delta_i| \leq 1/n$ *for each* Δ_i *which is not one of the original* Δ_i'.

Proof. Each interval Δ_i' with $|\Delta_i'| > 1/n$ we subdivide, beginning on the left, into subintervals of length $= 1/n$ and a remainder of length $> 1/n$ and $\leq 2/n$. This remainder we subdivide into two intervals of equal length. This will give the required intervals Δ_i. □

§ 7. Theorems of Popov

In approximating a large class of functions K, we cannot expect a radical improvement of $\rho_n(f)$ as compared to $E_n(f)$. The spectacular theorems of §§3-5 are only possible for very special functions $f \in K$. In some cases, the estimate of the type $E_n(f)_X \leq C\phi(n)\|f^{(r)}\|_X$ cannot be improved at all as it stands for $\rho_n(f)$ instead of $E_n(f)$. What can be often improved, is the space X where the norm on the right is taken, resulting in $\rho_n(f) \leq C\phi(n)\|f^{(r)}\|_Y$, with $\|\cdot\|_Y \leq \|\cdot\|_X$. Thus, we get the same order $\rho_n(f) = O(\phi(n))$, but for a larger class of functions.

The "o-phenomenon" is the possibility to replace this estimate by $\rho_n(f) = o(\phi(n))$ *for each individual* $f \in K$ (but not necessarily uniformly for the class K). A predecessor of Popov was Freud [1967], who proved $\rho_n(f) \leq C\log^2 n/n^2$ for $f \in V^1$, and also showed that this cannot be improved to $\rho_n(f) \leq \varepsilon_n n^{-2}$ for any sequence $\varepsilon_n \to 0$, $\varepsilon_n > 0$. Popov [1977] filled the gap, removing $\log^2 n$ in Freud's estimate. The proof was quite complicated; it has been simplified by DeVore [1983] by using the Hardy-Littlewood maximal functions $Mf(x)$ and the operator (6.14).

The estimate $\rho_n(f) \leq Cn^{-2}$, $f \in V^1$ is much better than the estimate for the polynomial approximation. There we can make conclusions only from the inclusion $f \in V^1 \subset \mathrm{Lip}\,1$, which gives $E_n(f) \le Cn^{-1}$. This is exact for $f(x) = |x|$ (on $[-1,1]$), and this function belongs also to V^1.

The Hardy-Littlewood *maximal function* $Mf(x)$ of any $f \in L_1[0,1]$ is defined by

$$Mf(x) = \sup_{x \in \Delta} \frac{1}{|\Delta|} \int_\Delta |f(t)|\, dt \,, \tag{7.1}$$

and the supremum taken for all intervals Δ with the property $x \in \Delta \subset [0,1]$. From $f \in L_1$ it does not follow that $Mf \in L_1$. We have however (for properties (7.2), (7.11) of Mf see Zygmund [B-1959, vol.1, p.32]),

$$\int_0^1 Mf(x)^p\, dx \le A_p \int_0^1 |f(x)|^p\, dx \,, \qquad f \in L_p \,,\ 1 < p < \infty \,. \tag{7.2}$$

We begin with the following theorem.

Theorem 7.1 (Popov). *For $f \in W_p^1$, $1 < p < \infty$, and $n \ge 1$,*

$$\rho_n(f) \le C_p \frac{\|f'\|_p}{n} \,. \tag{7.3}$$

(For $p = \infty$ we have even $E_n(f) \le C\|f'\|_\infty n^{-1}$.)

Proof. We can assume that $\|f'\|_p = 1$. Then by (7.2), $\int_0^1 (Mf')^p\, dx \le A_p$, and we can decompose $[0,1]$ into n intervals Δ_i', $i = 1, \dots, n$ for which$\int_{\Delta_i'} (Mf')^p\, dx \le A_p/n$. By Lemma 6.6, there is a finer partition of $[0,1]$ into intervals Δ_i, $i = 1, \dots, m$, $n \le m \le 2n$ with the properties

$$\begin{cases} \displaystyle\int_{\Delta_i} (Mf')^p\, dx \le A_p/n \\ |\Delta_i| \le 1/n \end{cases} \qquad i = 1, \dots, m \,. \tag{7.4}$$

We shall use the operator (6.14) with $q = 2$; the points $c_i \in \Delta_i$ we select so that

$$Mf'(c_i) = \min_{x \in \Delta_i} Mf'(x) \,. \tag{7.5}$$

This is possible because the function $Mf'(x)$ is lower semicontinuous and attains its minimum on each compact set. The polynomials P_i will be constants $P_i(x) = f(c_i)$. This completely defines the operator $R_m(f, x) \in \mathcal{R}_{8n}$. With the help of (7.5) and (7.4), we estimate for $x \in [0,1]$,

$$
(7.6)\qquad
\begin{aligned}
|f(x) - P_i(x)| &= \left|\int_{c_i}^{x} f'(t)\,dt\right| = |x - c_i| \left|\frac{1}{x - c_i}\int_{c_i}^{x} f'\,dt\right| \\
&\le |x - c_i| Mf'(c_i) = |x - c_i|\,|\min_{x\in\Delta_i} Mf'(x)| \\
&\le |x - c_i| \left(\frac{1}{|\Delta_i|}\int_{\Delta_i} Mf'(x)^p\,dx\right)^{1/p} \\
&\le \mathrm{Const}\,|x - c_i|(n|\Delta_i|)^{-1/p}\,.
\end{aligned}
$$

Therefore, by (6.16), with $q = 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$,

$$
(7.7)\qquad
\begin{aligned}
|f(x) - R_m f(x)| &\le \sum_{i=1}^{m} |f(x) - P_i(x)| R_i(x) \\
&\le C \sum_{i=1}^{m} |x - c_i|(n|\Delta_i|)^{-1/p} R_i(x) \\
&\le C\frac{1}{n} \sum_{i=1}^{m} \frac{|x - c_i|}{|\Delta_i|}(n|\Delta_i|)^{1/p'} R_i(x) \\
&\le C\frac{1}{n} \sum_{i=1}^{m} n|\Delta_i|^{1/p'} \left[1 + \frac{(x - c_i)^2}{|\Delta_i|^2}\right]^{-1}\,.
\end{aligned}
$$

If i belongs to the class K_ν defined by (6.18), then $(n|\Delta_i|)^{1/p'} \le 2^{-(1/p')\nu}$, and by Lemma 6.5,

$$
(7.8)\qquad |f(x) - R_m(f, x)| \le \frac{C}{n}\sum_{\nu=0}^{\infty} 2^{-(1/p')\nu}\Sigma_\nu \le \frac{C}{n}\,. \qquad \square
$$

The theorem of Popov about V^r is the following.

Theorem 7.2 (Popov). *For each $f \in V^r[0, 1]$, $r = 1, 2, \ldots$ and $n \ge r$,*

$$
(7.9)\qquad \rho_n(f) \le C_r \frac{\mathrm{Var}\, f^{(r)}}{n^{r+1}}\,.
$$

Since W_1^{r+1} is continuously imbedded in V^r, we have, as a special case,

Theorem 7.3. *For each $f \in W_1^{r+1}[0, 1]$, $r = 1, 2, \ldots,$ and $n \ge r$,*

$$
(7.10)\qquad \rho_n(f) \le C_r \frac{\|f^{(r+1)}\|_1}{n^{r+1}}\,.
$$

Proof. We begin with the proof of Theorem 7.3. It is similar to that of Theorem 7.1. Instead of (7.2) we use the property of the maximal function Mf that for $f \in L_1[0,1]$,

$$\left(\int_0^1 Mf(x)^\alpha \, dx\right)^{1/\alpha} \le C_\alpha \int_0^1 |f(x)| \, dx , \qquad 0 < \alpha < 1 , \tag{7.11}$$

with C_α depending only on α. We use the operator (6.14) with $q \ge \frac{1}{2}(r+3)$. We can assume $\|f^{(r+1)}\|_1 = 1$, then from (7.11) and Lemma 6.6 we obtain intervals Δ_i, $i = 1, \dots, m$, $n \le m \le 2n$, for which $\Delta_i \le 1/n$ and

$$\int_{\Delta_i} Mf^{(r+1)}(x)^\alpha \, dx \le C \frac{1}{n} . \tag{7.12}$$

We select the points $c_i \in \Delta_i$ by means of

$$Mf^{(r+1)}(c_i) = \min_{x \in \Delta_i} Mf^{(r+1)}(x) , \tag{7.13}$$

and take $P_i(x) = f(c_i) + (x - c_i)f'(c_i) + \cdots + \frac{1}{r!}(x - c_i)^r f^{(r)}(c_i)$. Then by means of Taylor's formula, by (7.12) and (7.13),

$$\begin{aligned}
|f(x) - P_i(x)| &= \frac{1}{r!} \left| \int_{c_i}^x (x-t)^r f^{(r+1)}(t) \, dt \right| \\
&\le |x - c_i|^{r+1} |Mf^{(r+1)}(c_i)| \\
&\le |x - c_i|^{r+1} \left\{ \frac{1}{|\Delta_i|} \int_{\Delta_i} \left(Mf^{(r+1)}(x)\right)^\alpha dx \right\}^{1/\alpha} \\
&\le C|x - c_i|^{r+1} (n|\Delta_i|)^{-\beta} , \qquad \beta = 1/\alpha .
\end{aligned}$$

It is necessary to assume that $\frac{1}{r+1} < \alpha < \frac{1}{r}$. Then $r < \beta < r+1$, and with $\gamma = r + 1 - \beta > 0$, using (6.16), and $q \ge \frac{r+3}{2}$, we obtain

$$\begin{aligned}
|f(x) - R_m f(x)| &\le \sum_{i=1}^m |f(x) - P_i(x)| R_i(x) \\
&\le \frac{C}{n^{r+1}} \sum_{i=1}^m \left(\frac{|x - c_i|}{|\Delta_i|} \right)^{r+1} (n|\Delta_i|)^\gamma R_i(x) \\
&\le \frac{C}{n^{r+1}} \sum_{i=1}^m \left(\frac{|x - c_i|^2}{|\Delta_i|^2} \right)^{(r+1)/2} (n|\Delta_i|)^\gamma \left[1 + \frac{|x - c_i|^2}{|\Delta_i|^2} \right]^{-q} \\
&\le \frac{C}{n^{r+1}} \sum_{i=1}^m (n|\Delta_i|)^\gamma \left[1 + (x - c_i)^2 |\Delta_i|^{-2}\right]^{-1} .
\end{aligned} \tag{7.14}$$

As in (7.8), the last sum is bounded, and $R_m(f) \in \mathcal{R}_{4qn}$. □

Proof of Theorem 7.2. This is a corollary of what we have just proved. For each $f \in V^r$ and each $\varepsilon > 0$ there is a $g \in W_1^{r+1}$ for which $\|f - g\|_\infty < \varepsilon$, $\|g^{(r+1)}\|_1 = \operatorname{Var} g^{(r)} \le \operatorname{Var} f^{(r)}$. For $r = 1$, one can take g to be the piecewise linear function that interpolates f at the points j/m, $j = 0, \ldots, m$ with a large m. From this, for $r > 1$, one obtains g by integration.

Now if $f \in V^r$, let ε_n be the right-hand side of (7.9). With g_n satisfying $\|f - g_n\|_\infty < \varepsilon_n$, and $\operatorname{Var} g_n^{(r)} \le \operatorname{Var} f^{(r)}$, we get

$$\rho_n(f) \le \varepsilon_n + \rho_n(g) \le 2\varepsilon_n \ . \qquad \square$$

We can show that Theorems 7.1–7.3 are the best possible of their kind. We need

Theorem 7.4 (Lorentz [1960]). *Let K be a convex, bounded and closed subset of a Banach space X and let*

$$N_0(f) := \|f\| \ge N_1(f) \ge \cdots \ge N_n(f) \ge \cdots \tag{7.15}$$

be a sequence of semi-norms on X. Let $\chi_n := \sup_{f \in K} N_n(f)$, $n = 0, 1, \ldots$. Then for each sequence $\varepsilon_n \to 0$, $\varepsilon_n > 0$ there exist an $f_0 \in K$ with

$$N_n(f_0) \ge \varepsilon_n \chi_n \tag{7.16}$$

for infinitely many n. More generally, (7.16) with N_n replaced by $N_{c^2 n}$ is valid if the N_n are homogeneous, satisfy (7.15) and if $N_n(f_1 + f_2) \le N_{cn}(f_1) + N_{cn}(f_2)$ for some integer $c \ge 1$.

Proof. If this is not true, then for each $f \in K$,

$$N_n(f) < \varepsilon_n \chi_n$$

for all sufficiently large n. We select $f_n \in K$ so that $N_n(f_n) \ge \frac{1}{2}\chi_n$. By induction we construct a sequence $b_k > 0$, $k = 1, 2, \ldots$, $\sum_1^\infty b_k \le 1$ and a sequence of natural numbers $n_1 < n_2 < \cdots$ as follows. If the b_i, n_i with $i \le k-1$, $\sum_1^{k-1} b_i < 1$ are known, we first take a $b_k > 0$ so small that

$$b_k < \tfrac{1}{2} b_{k-1} \ , \quad \sum_1^k b_i < 1 \ , \quad b_k < \tfrac{1}{2}\varepsilon_{n_{k-1}} \chi_{n_{k-1}} \ . \tag{7.17}$$

We then take n_k so large that

$$6\varepsilon_{n_k} < b_k \ , \qquad N_{n_k}\left(\sum_1^{k-1} b_i f_{n_i}\right) < \varepsilon_{n_k} \chi_{n_k} \ . \tag{7.18}$$

Let $f_0 := \sum_1^\infty b_k f_{n_k}$. Assuming, without loss of generality, that $\|f\| \le 1$ for all $f \in K$, we have

$$N_{n_k}\left(\sum_{k+1}^{\infty} b_i f_{n_i}\right) \le \sum_{k+1}^{\infty} b_i \|f_{n_i}\| \le \sum_{k+1}^{\infty} b_i \le 2b_{k+1} < \varepsilon_{n_k}\chi_{n_k} \ .$$

Therefore,

$$N_{n_k}(f_0) \ge N_{n_k}(b_k f_{n_k}) - N_{n_k}\left(\sum_{1}^{k-1} b_i f_{n_i}\right) - N_{n_k}\left(\sum_{k+1}^{\infty} b_i f_{n_i}\right)$$
$$\ge \tfrac{1}{2} b_k \chi_{n_k} - 2\varepsilon_{n_k}\chi_{n_k} > \varepsilon_{n_k}\chi_{n_k} \ , \quad k = 1, 2, \ldots \ .$$

This contradicts the assumption and proves the lemma for semi-norms. In the general case, one uses, in the last lines of the proof, the inequality $N_{c^2 n}(f_1 + f_2 + f_3) \ge N_n(f_2) - N_n(f_1) - N_n(f_3)$. □

Theorem 7.5. (i) *For all $n, r = 1, 2, \ldots$ there exists a function $f_n \in C^r(A)$, $A = [0,1]$ for which $\|f_n^{(r)}\|_\infty = 1$ and*

$$\rho_n(f_n) \ge \frac{C(r)}{n^r} \ . \tag{7.19}$$

(ii) *For each $r = 1, 2, \ldots$ and each sequence $\varepsilon_1 \ge \varepsilon_2 \ge \cdots$, $\lim \varepsilon_n = 0$ there is a function $f \in C^r(A)$ which satisfies*

$$\rho_n(f) \ge \frac{\varepsilon_n}{n^r} \tag{7.20}$$

for infinitely many values of n.

Proof. (i) Let $\varphi_r(x) := 2(\sin \pi x)^{2r}$, $x \in \mathbb{R}$; this function satisfies $\varphi_r^{(j)}(k) = 0$, $k \in \mathbb{Z}$, $j = 0, 1, \ldots, r$, and the integers k are zeros of the $\varphi_r^{(j)}$ of order at least r. Also, $\varphi_r(k/2) - 1 = (-1)^{k+1}$, $k \in \mathbb{Z}$. From Theorem 2.6 we get

$$\rho_n(\varphi_r)_{C[0,n]} = 1 \ . \tag{7.21}$$

Therefore $f_n(x) := n^{-r} m_r^{-1} \varphi_r(nx)$, $m_r := \|\varphi_r^{(r)}\|_\infty(\mathbb{R})$, satisfies (7.19) with the constant $C(r) = m_r^{-1}$.

(ii) This follows immediately from Lemma 7.4. □

We have also the o-phenomenon in Theorems 7.1 and 7.3: for each *individual* f the constants C_p and C_r there can be replaced by $o(1)$. More exactly:

Theorem 7.6. *For $f \in W_p^{r+1}[0,1]$, $p \ge 1$, $r = 0, 1, \ldots$, with $p(r+1) > 1$ (that is, with $p > 1$ or with $r \ge 1$) we have*

$$\rho_n(f) \le \frac{C_{p,r}}{n^{r+1}}\, \omega\left(f^{(r+1)}, \frac{1}{n}\right)_p \ , \qquad n \ge r+1 \ . \tag{7.22}$$

Proof. By Theorem 6.2 of [**CA**, Theorem 6.2, p. 219], there is a polynomial $P \in \mathcal{P}_{n+r+1}$ with

$$\|f^{(r+1)} - P^{(r+1)}\|_p \le C\omega\left(f^{(r+1)}, \frac{1}{n}\right)_p .$$

Then we take $R_n \in \mathcal{R}_n$ so that $\|f - P - R_n\|_\infty = \rho_n(f - P)_\infty$ and by Theorem 7.1 or 7.3 obtain (7.22):

$$\rho_{2n+r+1}(f)_\infty \le \|f - P - R_n\|_\infty \le \frac{C}{n^{r+1}} \|f^{(r+1)} - P^{(r+1)}\|_p$$

$$\le \frac{C}{n^{r+1}}\, \omega\left(f^{(r+1)}, \frac{1}{n}\right)_p . \qquad \square$$

As a corollary, we have the positive answer to the *Newman conjecture*: *For each function* $f \in \operatorname{Lip} 1$ *one has* $\rho_n(f) = o(1/n)$. This last statement shows that ε_n can not always be removed in (7.16) of Theorem 7.4.

The following theorem of Dolzhenko (taken from Gonchar [1967_1]) implies that Theorem 7.3 is not valid for $r = 0$, and that the restriction $p > 1$ in Theorem 7.1 is essential.

Theorem 7.7. *For any sequence* $\varepsilon_0 \ge \varepsilon_1 \ge \cdots$, $\lim \varepsilon_n = 0$, *there is a function* $f \in W_1^1(A)$ *so that*

$$\rho_n(f) \ge \varepsilon_n , \qquad n = 0, 1, \dots . \tag{7.23}$$

We shall use the following lemma:

Lemma 7.8 (Gonchar [1967_1]). *If a function* $f \in C[-1, 1]$ *vanishes for* $x \in [-1, -\delta]$, *and is nondecreasing on* $[\delta, 1]$, *for some* $0 < \delta < 1$, *then for* $n = 0, 1, \dots$ *it satisfies the inequality*

$$\rho_n(f) \ge f(\delta)\left[1 + \exp\left(\frac{\pi^2 n}{\log(1/\delta)}\right)\right]^{-1} . \tag{7.24}$$

Proof. Let $\rho_n(f) < f(\delta)$ (otherwise (7.24) is obviously true). For the rational function $R \in \mathcal{R}_n$ of best approximation to f, for all $x \in [\delta, 1]$ we have

$$|R(-x)| \le \rho_n(f)$$
$$R(x) \ge f(x) - \rho_n(f) \ge f(\delta) - \rho_n(f) > 0 .$$

Therefore, for $x \in [\delta, 1]$,

$$\left|\frac{R(-x)}{R(x)}\right| \le \frac{\rho_n(f)}{f(\delta) - \rho_n(f)} =: \lambda_n .$$

Using Lemma 3.4, we derive from this

$$\int_\delta^1 \log \lambda_n \frac{dx}{x} \geq \int_\delta^1 \log \left| \frac{R(-x)}{R(x)} \right| \frac{dx}{x} \geq -\pi^2 n ;$$

$$\lambda_n \geq \exp\left(\frac{-\pi^2 n}{\log(1/\delta)}\right) . \qquad \square$$

Proof of Theorem 7.7. It is convenient to assume here that $A = [-1,1]$. We define $f(x) := 0$, $-1 \leq x \leq 0$, $f(\delta_n) := 4\varepsilon_n$ for $\delta_n := \exp(-\pi^2 n)$, $n = 0, 1, \ldots$ and assume it to be linear on the intervals $[\delta_{n+1}, \delta_n]$. From (7.24) we derive

$$\rho_n(f) \geq \frac{1}{e+1} f(\delta_n) = \frac{4}{e+1}\varepsilon_n > \varepsilon_n , \qquad n = 0, 1, \ldots . \qquad \square$$

§ 8. Properties of the Operator of Best Rational Approximation in C and L_p

In §2 we have seen that the best approximation $\Pi_{\mathcal{R}_{m,n}} f := \Pi f$ from $\mathcal{R}_{m,n}$ (called also the metric projection of f onto $\mathcal{R}_{m,n}$) exists for each $f \in C[a,b]$ and is unique. We shall discuss now the continuity properties of Π. Here, the degeneracy properties and the defect of the function $R = \Pi f$ (see (1.1)) are the key.

We shall call a function $f \in C[a,b]$ *normal with respect to* $\mathcal{R}_{m,n}$ if its best approximation $R = \Pi_{\mathcal{R}_{m,n}}(f) := \Pi(f)$ is not degenerate; the set of all normal functions $\mathcal{N}_{m,n}$ is a subset of $C[a,b]$. A rational function $R \in \mathcal{R}_{m,n}$ belongs to $\mathcal{N}_{m,n}$ if and only if it is non-degenerate.

An obvious remark is that *non-degenerate rational functions are dense in* $\mathcal{R}_{m,n}$ *with respect to the uniform norm.* Indeed, if $d(R) =: d > 0$, then

$$\lim_{c \to +\infty} \left\{ R(x) \left(\frac{x-c}{x-c-1} \right)^d \right\} = R(x) .$$

Lemma 8.1. *Let f be not normal so that $d(R) \geq 1$ for $R = \Pi(f)$. For each $\delta > 0$ there is an $f_1 \in C[a,b]$ with $\|f - f_1\| < \delta$ and with $R_1 = \Pi(f_1)$ satisfying*

$$d(R_1) \leq d(R) - 1 . \tag{8.1}$$

Proof. We take an R_2 with $d(R_2) = 0$, $R_2 \neq R$ so that $\|R - R_2\| < \delta$ and put

$$f_1 = f - R + R_2 . \tag{8.2}$$

If R_2 is the best approximation to f_1, we are through. Otherwise, by Akhiezer's Theorem 2.6, $f_1 - R_2 = f - R$ has an alternation of $N := n + m + 2 - d(R)$ points. Since $\|f_1 - R_1\| < \|f_1 - R_2\|$, where $R_1 = \Pi f_1$, the curve R_2 intersects R_1 in at least $N - 1$ points. Writing $R_2 = p_2/q_2$, $q_2 > 0$, we see that

$$p_2 - R_1 q_2 = q_2(R_2 - R_1) \neq 0$$

has at least $N-1$ zeros. By Corollary 2.5, the dimension of the Haar space Φ defined there is $n+m-d(R_1)+1 \geq N = n+m-d(R)+2$, and we obtain (8.1). □

Theorem 8.2. *The set $\mathcal{N}_{m,n}$ of normal functions is dense and open in $C[a,b]$.*

Proof. (a) *The set is dense.* If $f \in C[a,b]$ is not normal, then $d(R) \geq 1$ for $R = \Pi(f)$. By at most $d(R)$ applications of Lemma 8.1 we obtain a function $f_1 \in \mathcal{N}_{m,n}$ with $\|f - f_1\| < \delta$. (b) *The set is open.* Let $f \in \mathcal{N}_{m,n}$, then

$$\delta := \operatorname{dist}(f, \mathcal{R}_{m-1,n-1}) > \delta_1 := \operatorname{dist}(f, \mathcal{R}_{m,n}) \ .$$

If R is the best approximation to f from $\mathcal{R}_{m,n}$ and $\|f_1 - f\| < \delta - \delta_1$, then $\|f_1 - R\| < \delta$. This yields $f_1 \in \mathcal{N}_{m,n}$. □

We would like to describe points $f \in C[a,b]$ of continuity of the metric projection $\Pi = \Pi_{\mathcal{R}_{m,n}}$. A first remark is that all $f = R \in \mathcal{R}_{m,n}[a,b]$ have this property. Indeed, for any $f_1 \in C[a,b]$ with $R_1 \in \Pi f_1$ we have the Lipschitz inequality

$$\|R - R_1\| \leq \|R - f_1\| + \|f_1 - R_1\| \leq 2\|f - f_1\| \ .$$

The positive part of the following fundamental theorem belongs to Maehly and Witzgall [1960], its negative part jointly to Cheney and Loeb [1964] and to Werner [1964].

Theorem 8.3. *The metric projection Π of $C[a,b]$ onto $\mathcal{R}_{m,n}[a,b]$ is continuous at f if and only if f is normal or belongs to $\mathcal{R}_{m,n}$.*

Proof. (a) Let f be normal with non-degenerate $R = \Pi f$. Let $f_k \to f$ in $C[a,b]$ and $R_k = \Pi f_k$, $k = 1, 2, \ldots$. Since the function $\rho_{m,n}(f)$ is continuous for $f \in C$, we have $\|R_k - f_k\| \to \|R - f\|$. This implies $\|R_k - f\| \to \operatorname{dist}(f, \mathcal{R}_{m,n})$, so that R_k is a minimizing sequence for f. Since $\mathcal{R}_{m,n}$ is approximately compact, $\|R_k - R\| \to 0$.

(b) Let $f \notin \mathcal{N}_{m,n} \cup \mathcal{R}_{m,n}$. Then $R = \Pi f$ is degenerate and $\|f - R\| =: \sigma > 0$. First let all sequences of alternation points of $f - R$ (of amplitude σ) have lengths $\leq m+n+1$. To show that Π is discontinuous at f, we take normal functions $f_k \to f$ with $\Pi f_k = R_k$. Then $R_k \not\to R$, for otherwise alternations of $f_k - R_k$ of lengths $m+n+2$ and amplitudes $\sigma_k = \|f_k - R_k\| \to \sigma$ would in the limit produce an alternation of $f - R$ of amplitude σ of equal length.

We may now assume that $[a,b] = [0,1]$ and that $f - R$ has an alternation set $0 \leq x_1 < \cdots < x_{m+n+2} \leq 1$, of amplitude $\sigma := \|f - R\|$, and with $f(0) - R(0) < 0$. Keeping $\sigma > 0$ fixed, we shall construct, for each $\varepsilon > 0$, a continuous function f^* with the properties

$$\|f - f^*\| < \varepsilon \ , \quad \|R - R^*\| = \sigma \ , \quad R^* = \Pi f^* \ .$$

First, let $x_1 = 0$, when $f(0) - R(0) = -\sigma$. We can find a small pertubation $\tilde{f}$ of f, $\|f - \tilde{f}\| < \varepsilon/2$, changing f only on an interval $[0, \delta]$, $\delta < x_2$, to achieve $\tilde{f}(x) - R(x) > -\sigma$ in $[0, \delta]$, except at a point ξ, $0 < \xi < \delta$, where $\tilde{f}(\xi) - R(\xi) = -\sigma$ and so that $\|f - g\| = \sigma$. Replacing f by $\tilde{f}$, ε by $\varepsilon/2$ and x_1 by ξ, we see that we can eliminate the case $x_1 = 0$.

Thus, we can assume that $x_1 > 0$, $f(0) - R(0) > 0$. We take $0 < \delta < x_1$ so that $f(x) - R(x) > 0$ on $[0, \delta]$. We then define

$$f^*(x) := \begin{cases} f(x) + S(x) & \text{on } [\delta, 1] , \\ f(x) + S(\delta) & \text{on } [0, \delta] , \end{cases}$$

and $R^*(x) = R(x) + S(x)$. If $R = p/q$ with $q(x) > 0$ on $[0,1]$ is an irreducible representation of R, we define $S(x) := Cq(x)^{-1}(Ax + 1)^{-1}$. Clearly, $R^* \in \mathcal{R}_{m,n}$. The number A we take so large that S is decreasing on $[0,1]$. First we select $C = \sigma q(0)$. Then on $[0, \delta]$,

$$f^*(x) - R^*(x) \le f(x) - R(x) \le \delta$$

and

$$f^*(x) - R^*(x) \ge S(\delta) - S(0) > -\sigma .$$

Moreover, $f^*(x) - R^*(x) = f(x) - R(x)$ on $[\delta, 1]$, and the latter difference has $m + n + 2$ alternations on this interval of amplitude $\sigma = \|f - R\|$. Hence $R^* = \Pi f^*$.

Next we select A so large that $S(\delta) < \varepsilon$; this will yield $\|f^* - f\| = S(\delta) < \varepsilon$. Finally, we have $R^*(x) - R(x) = S(x) \ge 0$, with the maximum $S(0) = \sigma$ achieved at $x = 0$. We conclude that $\|R^* - R\| = \sigma$. □

We shall discuss the structure of some of the subsets of $\mathcal{R}_{m,n}$.

Theorem 8.4. *The sets* $\mathcal{D}_d := \{R \in \mathcal{R}_{m,n}[a,b],\ d(R) \le d\}$, $d = 0, 1, \dots$ *are open in* $\mathcal{R}_{m,n}[a,b]$ *in the uniform metric.*

Proof. This can be elegantly proved by means of Theorem 2.4. For each $R_0 \in \mathcal{D}_d$, the space X_{R_0} is a Haar space of dimension $m+n+1-d(R_0) \ge m+n+1-d$. Hence for some $P_m \in \mathcal{P}_m$, $Q_n \in \mathcal{P}_n$ the function $P_m - R_0 Q_n \not\equiv 0$ has $m+n-d$ simple zeros on $[a,b]$. For an arbitrary $R \in \mathcal{R}_{m,n}$,

$$|P_m(x) - R(x)Q_n(x)| \le |P_m(x) - R_0(x)Q_n(x)| + |R(x) - R_0(x)|\,|Q_n(x)| .$$

If $\|R - R_0\|$ is sufficiently small, $P_m - RQ_n$ will also have at least $m + n - d$ zeros in $[a,b]$. By Theorem 2.4, $m + n - d(R) \ge m + n - d$, or $d(R) \le d$. □

We shall also describe and count the components (connected open subsets) of the set $\mathcal{D}_0$, that consists of rational functions in $\mathcal{R}_{m,n}(I)$, $I = [a,b]$ without defect. The information can be used for the selection of an initial approximation in numerical algorithms and also for the counting of the components of $\mathcal{N}_{m,n}(I)$.

Functions $R \in \mathcal{R}_{m,n}$ we shall write in the form

$$R(x) = A\Pi_1^m(x - a_k)'/\Pi_1^n(x - b_\ell)' , \tag{8.3}$$

where A is a constant, and a_k, b_ℓ are zeros of the numerator (N-zeros) and of the denominator (D-zeros) of R, respectively. Values $a_k = \infty$, $b_\ell = \infty$ are allowed; but the accents in (8.3) mean that then the corresponding factors are omitted in the products. An R given by (8.3) belongs to $\mathcal{D}_0(I)$ if and only if $A \neq 0$, if each a_k is different from each b_ℓ, and if the latter avoid the interval $I = [a, b]$.

The D and N zeros of $R \in \mathcal{D}_0$ are contained in the compactified complex plane $\tilde{\mathbb{C}}$. As the distance between $R, R^* \in \mathcal{D}_0$ we take the Hausdorff distance between the sets of zeros a_k, b_ℓ and a_k^*, b_ℓ^*, plus $|A - A^*|$.

The zeros of $R \in \mathcal{D}_0$ are of two kinds: *Free D and N zeros* are complex conjugate pairs, or double zeros, or a pair of single zeros not separated by other zeros. By continuous movement within $\mathcal{D}_0$, these pairs can be transformed into each other and can be moved to any free place on the compactified line $\widetilde{\mathbb{R}}^1$. We shall move a maximal possible number of them as single pairs to the far right of $\widetilde{\mathbb{R}}^1$. What remains after this operation, are the *restricted D and N zeros*. They form a sequence of alternatingly D and N simple zeros. Thus, all zeros of $R \in \mathcal{D}_0$ may be thought to lie on $\widetilde{\mathbb{R}}^1$.

The restricted zeros may be moved along $\mathbb{R}^1$ as long as they do not run over other restricted zeros, but the D zeros must not enter $I = [a, b]$.

A zero may run, with the help of the constant A, over ∞. We explain this for an N factor $A(x - a_1)$, $A \neq 0$, $b < a_1 < \infty$. We first change continuously this into $\frac{A}{a_1}(x - a_1)$, then with variable a, $\frac{A}{a}(x - a)$ across ∞ (with value $-A$ for $a = \infty$) to the negative side of $\mathbb{R}^1$, with $\frac{A}{a_2}(x - a_2)$ for any a_2, $-\infty < a_2 < a$; then lastly this to $-A(x - a_2)$. We can perhaps move across I, obtaining $-A(x - a_1)$, *with a change of sign*. The same holds for D-zeros, except that they cannot run all around $\widetilde{\mathbb{R}}^1$.

We shall place the restricted zeros on the $\mathbb{R}^1$ line after the interval I, and assume that they begin with a D zero (if $n > 0$). As an example of this rearrangement:

$$N_1 D_1 N_2 D_2 D_3 N_3 N_4 D_4 N_5 D_5 N_6$$

becomes

$$D_1 N_4 D_4 N_5 D_5 \| D_2 D_3, N_2 N_3, N_6 N_1 .$$

With K. Bartke (see Berens [A-1972]) we can count the number $C_{m,n}$ of the components of $\mathcal{D}_0$.

Theorem 8.5. *The number of components of $\mathcal{D}_0$ is equal to*

$$C_{m,n} = \begin{cases} \min(m, n) + 1 & \text{if } m, n \text{ are of the same parity, } m > 0 \\ \min(m + 1, n) + 1 & \text{if } m, n \text{ are of opposite parity.} \end{cases} \tag{8.4}$$

Selected elements of each component are given in the following table:

1. *m, n even:* ϕ^*, $DNDN, DNDNDNDN, \ldots$
2. *m, n odd:* $DN, DNDNDN, DNDNDNDNDN, \ldots$
3. *m odd, n even:* N^*, $DND, DNDNDND, \ldots$
4. *m even, n odd:* $D, DNDND, DNDNDNDND, \ldots$

Proof. In this table all possible sequences of restricted zeros are listed, they have to be completed by free pairs, DD or NN, until numbers m, n are completely exhausted. To each sequence obtained there will correspond two R of (8.3), with $A > 0$ and with $A < 0$. An exception are only the sequence N in 3, where one can connect R to $-R$, and the empty set $\emptyset$ in 1 if $m > 0$, where one can rotate the last N of the free pair around $\widetilde{\mathbb{R}}^1$, putting it between b and the first free N.

By direct counting in all 4 cases one obtains the number (8.4) of open subsets of $\mathcal{D}_0$. That they are all different follows because the D zeros cannot cross I. □

We note that $C_{0,n} = 2$ does not satisfy (8.4).

The metric projection Π maps $\mathcal{N}_{m,n}[a, b]$ continuously onto $\mathcal{D}_0$ (with the uniform norm). One can show that the parameters of (8.3) depend continuously upon $R \in \mathcal{D}_0$ in this norm. Thus $\mathcal{N}_{m,n}$ falls apart into $C_{m,n}$ disjoint open sets $\mathcal{G}$ corresponding to the components of $\mathcal{D}_0$. These sets are connected. We can connect $f, f^* \in \mathcal{G}$ as follows: first f to $R = \Pi f$ along a straight segment; then R to $R^* = \Pi f^*$ within the component of $\mathcal{D}_0$; finally, R^* to f^*. This yields that *$\mathcal{N}_{m,n}$ has also $C_{m,n}$ components.*

We complete this section by showing that in the space L_p, $1 < p < \infty$, all functions are normal with respect to the $\mathcal{R}_{m,n}$-approximation.

Theorem 8.6. (Cheney and Goldstein [1967]). *An element of best approximation from $\mathcal{R}_{m,n}$ to a function $f \in L_p[a, b] \setminus \mathcal{R}_{m,n}$, $1 < p < \infty$, cannot be degenerate.*

Proof. We show first that linear combinations of the functions $(x - c)^{-1}$, $c > b$, are dense in $C[a, b]$, hence in all spaces L_p, $1 \le p < \infty$. Polynomials P of all degrees are dense in C. A polynomial $P \in \mathcal{P}_n$ can be approximated arbitrarily closely by $R(x) = P(x) \prod_{k=1}^{n+1} (x - c_k)^{-1}$, with c_k sufficiently large, and then R expanded into a sum of its partial fractions.

Next, let $R = P/Q \in \mathcal{R}_{m-1,n-1}[a, b]$, $Q(x) > 0$, $a \le x \le b$ be the best approximation to $f \in L_p[a, b] \setminus \mathcal{R}_{m,n}$ from $\mathcal{R}_{m,n}$. The function $R_\lambda := R + \lambda(x - c)^{-1}$ belongs to $\mathcal{R}_{m,n}$ for each $c > b$. Since it does not approximate f better than R, the integral

$$\Phi(\lambda) := \int_a^b |f - R_\lambda|^p \, dx$$

attains its maximum at $\lambda = 0$, so that $\Phi'(0) = 0$. Since

$$\Phi'(\lambda) = -p \int_a^b |f - R_\lambda|^{p-1} \operatorname{sign}(f - R_\lambda) \frac{dR_\lambda}{d\lambda} \, dx \ ,$$

we have

$$\int_a^b |f - R|^{p-1} \operatorname{sign}(f - R) \frac{1}{x - c} \, dx = 0 \ .$$

This implies $f - R = 0$ a.e., a contradiction. □

Dunham [1971] remarks that this is not true in L_1. Indeed, let $f_0 \in L_1[-1,1]$, $f_0(x) > 0$ on $(-\frac{1}{2}, \frac{1}{2})$, $f(x) = 0$ elsewhere. Then $\Pi_{\mathcal{R}_{0,1}} f =: R_0 = 0$. Indeed, with R also $|R|$ belongs to $\mathcal{R}_{0,1}[-1,1]$, and since $|R|$ is convex, $\int_{|x| \ge \frac{1}{2}} |R| \, dx \ge \int_{|x| \le \frac{1}{2}} |R| \, dx$. Therefore, if $R \in \mathcal{R}_{0,1}$,

$$\begin{aligned} \|f - R\|_1 \ge \|f - |R|\,\|_1 &\ge \int_{|x| \le \frac{1}{2}} |f| \, dx - \int_{|x| \le \frac{1}{2}} |R| \, dx + \int_{|x| \ge \frac{1}{2}} |R| \, dx \\ &\ge \int_{|x| \le \frac{1}{2}} |f| \, dx = \|f - R_0\|_1 \ . \end{aligned}$$

Theorem 8.7 (Efimov and Stechkin [1958]). *In the space $L_p[a,b]$, $1 \le p < \infty$, the families $\mathcal{R}_{m,n}$ do not have the uniqueness property.*

Proof. (a) *For $1 < p < \infty$, with Braess* [1987] *we prove more.* Let X be an $m+2$ dimensional subspace of L_p with $X \cap \mathcal{R}_{m-1,n-1} = \{0\}$. Then X contains a function with at least two different best approximations from $\mathcal{R}_{m,n}$.

Let $S^{m+1} := \{f \in X, \|f\|_p = 1\}$ be an $m+1$ dimensional sphere in X. Assuming that Πf is single valued on S^{m+1}, we derive a contradiction. Since $f \notin \mathcal{R}_{m-1,n-1}$, by Theorem 8.4, $R = \Pi f$ cannot be degenerate. We write $R(x) = \sum_{j=0}^m a_j x^j / Q_n(x)$, $\|Q_n\|_\infty = 1$, then R determines uniquely the coefficients $\phi(R) := (a_0, a_1, \ldots, a_n)$. This is a continuous function, moreover both ϕ and Π are odd. Hence $T := \phi \circ \Pi$ is a continuous odd map of S^{m+1} into $\mathbb{R}^{m+1}$. By Borsuk's theorem (Appendix 1), for some $f \in S^{m+1}$, $Tf = 0$, that is, $\Pi f = 0/Q$. But then $R \in \mathcal{R}_{m-1,n-1}$, a contradiction.

(b) *The set $\mathcal{R}_{0,2}$ has no uniqueness property in $L_1[-1,1]$.* Indeed, let f be the even function on $[-1,1]$, defined by $f(x) = 1$, $x \in [\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]$, $f(x) = 0$ elsewhere on $[0,1]$, let $0 < \varepsilon < \frac{1}{6}$. Let $R^*(x) = B^2[(x - \frac{1}{2})^2 + B^2]^{-1}$, $0 < B < 1$, then

$$\begin{aligned} \|f - R^*\|_1[-1,1] &\le \int_{-\infty}^{+\infty} R(x) \, dx + 4 \int_0^\varepsilon \Big(1 - R(u + \tfrac{1}{2})\Big) \, du \\ &= B\pi + 4\Big(\varepsilon - \int_0^\varepsilon \frac{B^2 \, du}{u^2 + B^2}\Big) \\ &= 4\varepsilon\Big(1 + \frac{\pi}{4A} - \frac{1}{A} \arctan A\Big) < 4\varepsilon \end{aligned}$$

if $B = \varepsilon/A$, and A is large. On the other hand, if $R \in \mathcal{R}_{0,2}[-1,1]$ is even, it is of the form $R(x) = a(x^2 - b)^{-1}$ with poles outside of $[-1,1]$. It is monotone on $[0,1]$, and we can assume that it is positive. We have, with $\rho := R(\frac{1}{2} + \varepsilon)$

$$\begin{aligned}\|f - R\|_1 &\geq 2\int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} (1 - R(x))\, dx + 2\int_{\frac{1}{2}+\varepsilon}^{1} R(x)\, dx \\ &\geq 4\varepsilon(1-\rho) + 2\Big(\tfrac{1}{2} - \varepsilon\Big)\rho = 4\varepsilon + \rho(1 - 6\varepsilon) > 4\varepsilon\ .\end{aligned}$$

It follows that the best approximant R_0 to f is not even, and we have no uniqueness: with $R_0(x)$ also $R_0(-x)$ is a best approximation. □

§ 9. Approximation by Rational Functions with Arbitrary Powers

The "polynomials" $P_n(x) = \sum_{k=0}^{n} a_k x^{\lambda_k}$ with $\lambda_0 = 0$, and distinct $\lambda_k > 0$, $k = 1, \ldots, n$ are dense in $C[0,1]$ if $\lambda_k \to \infty$ and $\sum_0^\infty \lambda_k^{-1} = \infty$, [**CA**, Theorem 5.1, p.345] or, more generally, if $\sum_1^\infty \lambda_k/(1 + \lambda_k^2) = \infty$ (see Theorem 7.2 of Chapter 11). The subject of this section is the proof of the remarkable fact that no restriction whatsoever is needed for the corresponding "rational functions" $R_n = P_n/Q_n$.

To avoid the possibility that all polynomials vanish at $x = 0$, we shall assume that $\lambda_0 = 0$.

Theorem 9.1. *Let $\lambda_0 = 0$, let $\lambda_k > 0$, $k = 1, 2, \ldots$ be distinct. Then the functions $R_n = P_n/Q_n$, where P_n, Q_n are of the form $\sum_{k=0}^{n} a_k x^{\lambda_k}$ and $Q_n(x) \neq 0$ on $[0,1]$, are dense in $C[0,1]$.*

Case I. Let $\lambda_k \to \infty$ (Somorjai [1976]). We shall treat this case "geometrically". A continuous function z on $[0,1]$ will be called an $\varepsilon_1, \varepsilon_2, \delta, c$-*zoom function*, where $\varepsilon_1, \varepsilon_2, \delta > 0$, $0 \leq c \leq 1$ and $z(x) > 0$ on $[0,1]$ if

$$z(x) \leq \varepsilon_1 \quad \text{for} \quad x < c - \delta \tag{9.1}$$

$$z(x) \geq \frac{1}{\varepsilon_2} \quad \text{for} \quad x > c + \delta\ . \tag{9.2}$$

Lemma 9.2. *If a subspace $S \subset C[0,1]$ contains an $\varepsilon_1, \varepsilon_2, \delta, c$-zoom function for all values of the parameters, then the fractions P/Q, $P, Q \in S$ with $Q(x) > 0$ on $[0,1]$, are dense in $C[0,1]$.*

Proof. We take $n = 1, 2, \ldots$ and construct zoom functions z_k, $k = 0, \ldots, n$, which correspond to $\delta = (2n)^{-1}$, $c_k = \frac{k-(1/2)}{n}$, $k = 1, \ldots, n$. Let $\varepsilon = \delta$. We take $z_0(x) > 0$ arbitrarily on $[0,1]$. If $z_{k-1}(x) > 0$ is already known, we define z_k to satisfy

$$\text{(9.3)} \qquad z_k(x) < \varepsilon z_{k-1}(x) \ \text{ for } \ x \le \frac{k-1}{n}, \qquad k = 1, \ldots, n$$

$$\text{(9.4)} \qquad z_{k-1}(x) < \varepsilon z_k(x) \ \text{ for } \ x \ge \frac{k}{n}, \qquad k = 1, \ldots, n .$$

We then define

$$w_{n,k}(x) = z_k(x)/z(x) \quad , \quad z(x) = \sum_{k=0}^{n} z_k(x) .$$

Clearly, $w_{n,k}(x) > 0$, $\sum_k w_{n,k}(x) = 1$ on $[0,1]$. Moreover,

$$\text{(9.5)} \qquad \sum_{|k/n-x|\ge\delta} w_{n,k}(x) \le 2\delta , \qquad x \in [0,1] .$$

Indeed, for the range $k/n - x \le -\delta$ we have $k < n$. We can use (9.4) with k replaced by $k+1$ and get

$$\sum_{k/n\le x-\delta} z_k(x) \le \delta \sum_{k=0}^{n-1} z_{k+1}(x) \le \delta z(x) .$$

Similarly, for $k/n - x \ge \delta$, by (9.3),

$$\sum_{k/n\ge x+\delta} z_k(x) \le \delta \sum_{k=1}^{n} z_k(x) \le \delta z(x) .$$

This proves (9.5). With the function

$$R = \sum_{0}^{n} f\,(k/m)\, w_{n,k}$$

we now get

$$|f(x) - R(x)| \le \sum_{k=0}^{n} |f(x) - f(k/n)| w_{n,k}(x)$$

$$= \sum_{|k/n-x|<\delta} + \sum_{|k/n-x|\ge\delta} \le 2\omega(f,\delta) + 2\|f\| \cdot 2\delta ,$$

which is arbitrarily small. □

To deal with Case I, we have to construct zoom functions by means of the x^{λ_k}. For $c = 0$, $z(x) = 1/\varepsilon$ is a zoom function, and for $c > 0$, one can take $z(x) = \frac{1}{2}\varepsilon_1 + (x/c)^{\lambda_k}$ with sufficiently large k.

Case II. Let $\lambda_0 = 0$, $\lambda_1 > \lambda_2 > \cdots$, $\lambda_k \to 0$ (Bak and Newman [1978]). By a linear substitution, we can replace $[0,1]$ by $[0,1/e]$. A function $f \in C[0,1/e]$ we can approximate so that

$$\left\| f(x) - \sum_0^n a_k \left(\frac{-1}{\log x}\right)^k \right\|_{[0,1/e]} < \varepsilon \,, \tag{9.6}$$

because we can achieve $\|f(e^{-1/u}) - \sum_0^n a_k u^k\|_{[0,1]} < \varepsilon$, and replace $u = -1/\log x$. We fix an integer q.

Our idea is that the function $\log x$ and its powers, at least approximately, can be obtained by means of differences of the x^{λ_k}. For example $(x^{\lambda_1} - x^{\lambda_2})/(\lambda_1 - \lambda_2) = x^{\mu} \log x$, $\lambda_2 < \mu < \lambda_1$, and this is close to $\log x$ if λ_1 is small. In general, for fixed $x > 0$, we take divided differences of x^{λ} with respect to λ. Thus, $[\lambda_q, \dots, \lambda_{q+k}]x^{\lambda}$ is a linear combination of the x^{λ_i} and is equal to (see [**CA**, (7.4), p.120])

$$P_k(x) := [\lambda_q, \dots, \lambda_{q+k}]x^{\lambda} = \frac{x^{\mu_k}}{k!}(\log x)^k \,, \qquad k = 1, \dots, n$$

with $\lambda_{q+n} < \mu_k < \lambda_q$. Similarly

$$Q_n(x) = [\lambda_{q+n}, \dots, \lambda_{q+2n}]x^{\lambda} = \frac{x^{\sigma_n}}{n!}(\log x)^n \,,$$

$0 < \sigma_n < \lambda_{q+n}$. For the function f of (9.6) we form the rational expression

$$\begin{aligned} R(x) :&= \frac{(-1)^n n!}{Q_n(x)} \sum_{k=1}^n a_{n-k}(-1)^k k! P_k(x) \\ &= \sum_{k=1}^n a_k x^{\xi_k} \left(\frac{-1}{\log x}\right)^k \,, \qquad 0 < \mu_k - \sigma_n = \xi_k < \lambda_q \,. \end{aligned}$$

This R is obviously a quotient of two polynomials of the form $\sum_0^{2n+q} a_k x^{\lambda_k}$. Now for $a > 0$, the function $(x^a - 1)/\log x$ is positive and increasing on $[0,1]$ (for its derivative is a positive multiple of $ax^a \log x - x^a + 1$) and has limit a for $x \to 1$. On $[0, 1/e]$, in addition, $|\log x| \geq 1$. Therefore $(1 - x^a)/|\log x|^k \leq a$ for $0 < x < 1/e$, $k = 1, 2, \dots$.

For the a_k in (9.6), let $A = \sum_1^n |a_k|$. We take q so large that $A\lambda_q < \varepsilon$, then

$$\begin{aligned} \left\| \sum_1^n a_k \left(\frac{-1}{\log x}\right)^k - R(x) \right\| &\leq \sum_1^n |a_k| \left\| \frac{1 - x^{\xi_k}}{(\log x)^k} \right\|_{[0,1/e]} \\ &\leq \lambda_q \sum_1^n |a_k| = A\lambda_q < \varepsilon \,, \end{aligned}$$

and together with (9.6) we obtain $\|f - a_0 - R\| < 2\varepsilon$.

Case III. To complete the proof of the theorem, it remains to study sequences λ_k which do not contain a subsequence of types I or II. Then λ_k contains a subsequence which has a finite limit > 0. Then $\sum \lambda_k^{-1} = \infty$, and (by [**CA**, Theorem 5.1, p.345]) even the polynomials P_n are dense in $[0,1]$. □

§ 10. Problems

10.1. Show that $\sum \rho_n(f)_\infty < +\infty$ implies that f is absolutely continuous (Dolzhenko).

10.2. Prove that $\mathcal{G} + R\mathcal{H}$, $R \in \mathcal{R}_{m,n}$ is not necessarily a Haar space, if $\mathcal{G}, \mathcal{H}$ are Haar spaces.

10.3. Let $R_\lambda := \lambda(\lambda + x)^{-1}$, $\lambda > 0$, let f_λ be a piecewise linear function with $f_\lambda(x_i) = R_\lambda(x_i) + (-1)^i c$, $c > 0$, $x_1 = 0$, $x_2 = \frac{1}{2}$, $x_3 = 1$. From the behavior of R_λ and f_λ deduce that the operator of best rational approximation in $\mathcal{R}_{0,1}[0,1]$ is discontinuous at $f(x) = \lim_{\lambda \to 0} f_\lambda(x)$.

10.4. If $a_n > 0$ decrease and $\lambda_n > 0$ satisfy $\sum \lambda_n < \infty$ and $\sum_{k>n} \lambda_k < \min(a_n, \frac{1}{8}\lambda_n)$, if further $\phi_\varepsilon(x) := \frac{\varepsilon}{x - i\varepsilon}$, $\varepsilon > 0$, then for any sequence $\varepsilon_n \to 0$ the function $f(x) := \sum_{n=1}^\infty \lambda_n \phi_{\varepsilon_n}(x)$ satisfies (1) $\rho_n(f) \le a_n$, $n = 1, 2, \ldots$.

10.5. Let $\omega(t) > 0$, $\omega(t) \to 0$ for $t \to 0+$, let a_n, λ_n and f be as in 10.4. One can select the ε_n inductively so as to have (2) $\limsup_{t \to 0+} [\omega(f,t)/\omega(t)] = \infty$. Thus, there exist functions $f \in C[-1,1]$ which satisfy both (1) and (2) (Gonchar).

10.6. By means of a broken-linear transformation derive from the properties of the quotient $N(x)/N(-x)$ of Newman polynomials a function $g_n(z) \in \mathcal{R}_n$ which has values $|g_n(z)| = 1$ on the circle Γ_h which passes through points $h/2$ and $3/2$ of the real axis, and is perpendicular to it, so that g_n has n zeros α_i on $[h, 1]$, and n poles β_i on $(-\infty, 0)$, symmetric to the zeros with respect to Γ_h.

10.7. If $f(z)$ is analytic on and inside Γ_h, then for a rational function $R_n \in \mathcal{R}_n$ which interpolates f at the α_i and has poles at the β_i,

$$f(z) - R_n(z) = \frac{1}{2\pi i} \int_{\Gamma_h} \frac{g_n(z)(z-1) f(\zeta)\, d\zeta}{g_n(\zeta)(\zeta - 1)(\zeta - z)} .$$

10.8. Let f be analytic on the disk $|z - 1| \le 1$. Using R_n of 10.7, show that

$$\rho_n(f)_{[0,1]} \le C \inf_{1<t<\infty} \left[t e^{-Cn/t} + \omega(f, e^{-t}) \right]$$

(Gonchar, [1968]).

10.9. From 10.8 derive that for any $\alpha > 0$, and some $C := C_\alpha > 0$,

$$\rho_n(x^\alpha)_{[0,1]} = O(e^{-C\sqrt{n}})$$

(Gonchar).

10.10. For any non-increasing sequence $(a_n)_0^\infty$ with $\lim a_n = 0$, there exists a function f analytic on $\overline{\mathbb{C}} \setminus \{0\}$, which is real valued on $\mathbb{R}$, has a limit $\lim_{x\to 0+} f(x)$ and satisfies

$$\rho_n(f)_{C[0,1]} \geq a_n \;, \qquad n = 0, 1, \ldots$$

(Dolzhenko).

10.11. As a corollary of Theorem 9.1 derive: If $\lambda_0 = 0$, $\lim \lambda_k = \infty$, then the rational functions $R_n = P_n/Q_n$ in the x^{λ_k} are dense in the space C_0 of all continuous functions on $[0, \infty)$ that vanish at infinity.

§ 11. Notes

11.1. The Theorem 7.2 of Popov is not the best possible, it can be improved to a o-theorem,

$$\rho_n(f) = o(n^{-r-1}) \;, \qquad f \in V_r[0,1] \;. \tag{11.1}$$

Among the results of Petrushev of this type we shall mention one of his strongest, which depends on the geometry of the curve L: $y = f(x)$. For simplicity, let f be continuous.

On a rectifiable curve L we introduce as a parameter the length of the curve s, and let $x = x(s)$, $y = y(s)$ be the natural equations of L. The derivatives $x'(s)$, $y'(s)$ exist a.e. and satisfy $x'^2 + y'^2 = 1$. The angle $\theta(f)$ of the tangent of L with the x-axis is given by

$$\theta(f,s) = \begin{cases} \arctan\big(y'(s)/x'(s)\big) & \text{if } x'(s) \neq 0 \\ \frac{\pi}{2}\text{sign } y'(s) & \text{if } x'(s) = 0 \;. \end{cases}$$

Petrushev's improvement of Theorem 7.2 is in terms of the modulus of continuity of $\theta(f^{(r)})$:

$$\rho_n(f) \leq C n^{-r-1} \omega\big(\theta(f^{(r)}), 1/n\big)_1 \;, \qquad f \in V^r \;. \tag{11.2}$$

See an exposition of these results in the book Petrushev and Popov [A-1987, section 10.1].

One can show that $\omega(\theta(f^{(r)}))_1 \leq \text{Const}\, \omega(f^{(r+1)})_1$ if $f \in W_1^{(r+1)}$, this gives another proof of the inequality (7.19).

11.2. Some abstract theorems about the density of rational functions are given by Gierz and Shekhtman [1986]. Let A be a compact Hausdorff space. For an arbitrary subspace $E \subset C(A)$ let

$$\mathcal{R} := \mathcal{R}(E) = \big\{g/h : g, h \in E \;,\; h(x) > 0 \;\text{ for all }\; x \in A\big\}$$

be the set of rational functions with respect to E.

Theorem 11.1. *The space $\mathcal{R}(E)$ is dense in $C(A)$ if and only if $\mathcal{R}(F)$ is dense in $C(A)$ for every (not recursively closed) subspace F, $E \subset F$, of codimension ≤ 2.*

Theorem 11.2. *Let E be a linear subspace of $C(A)$; assume that there exists a point $x_0 \in A$ with the property $x_0 \in \operatorname{supp} \mu_+ \cap \operatorname{supp} \mu_-$ for any regular Borel measure on A which is orthogonal to E (that is, which satisfies $\int f \, d\mu = 0$ for all $f \in E$). Then $\mathcal{R}(E)$ is dense in $C(A)$.*

For example, the complete Theorem 9.1 follows in a simple way from Theorem 11.2.

Chapter 8. Stahl's Theorem

§ 1. Introduction and Main Result

This chapter is a continuation of §3 of Chapter 7 which deals with the approximation of the function $|x|$ on $[-1,1]$ by the rational functions $\mathcal{R}_n := \mathcal{R}_{n,n}$. Newman's [1964] famous inequalities

$$(1.1) \qquad e^{-\pi\sqrt{n+1}} \le \rho_n(|x|)_{C[-1,1]} \le 3e^{-\sqrt{n}}, \quad n \ge 5,$$

show that $|x|$ has excellent approximation by rational functions; the error $\rho_n := \rho_n(|x|)_{C[-1,1]}$ is small, even for small degree n. The lower estimate (1.1) in Newman's theorem is close to the best possible, but the upper estimate can be essentially improved. Actually Vyacheslavov [1975] proved that

$$(1.2) \qquad C_1 e^{-\pi\sqrt{n}} \le \rho_n \le C_2 e^{-\pi\sqrt{n}}, \quad n \ge 1,$$

for some constants $C_1, C_2 > 0$. Thus, Vyacheslavov's result gives the exact number π in the exponent of the error estimate. However, it gives no information about the best possible values of the constants C_1 and C_2.

Varga, Ruttan and Carpenter [1991] have done extensive numerical investigations of the asymptotic behavior of the rational error ρ_n. They calculated the numbers ρ_n for even n up to $n = 80$, and then applied Richardson extrapolation to the sequence $(\rho_{2n})_{n=1}^{40}$. The numerical results give strong evidence for the conjecture that the limit

$$(1.3) \qquad \lim_{n\to\infty} e^{\pi\sqrt{n}}\rho_n$$

exists, and that its value is 8. The main purpose of the present chapter is to show that this conjecture is true:

Theorem 1.1 (Stahl [1992]). *The error of best rational approximation of the function $|x|$ in the uniform norm on $[-1,1]$ satisfies*

$$(1.4) \qquad \lim_{n\to\infty} e^{\pi\sqrt{n}}\rho_n(|x|)_{C[-1,1]} = 8.$$

Stahl's proof is based on fresh ideas; it does not use methods of Newman or Vyacheslavov. An outline of the proof is as follows.

By means of several transformations of the rational function $r_n^* \in \mathcal{R}_n$ of best approximation to $|x|$, we arrive in §2 to a function Φ_n. The logarithm $\log|\Phi_n|$ is harmonic in the domain $G_n := H_+ \backslash [1/2, 1/\rho_n]$, $H_+ := \{\mathrm{Re}\ z > 0\}$; it attains the values

$$\log|\Phi_n(x)| = -\log(4x), \quad x \in [1/2, 1/\rho_n]. \tag{1.5}$$

One obtains the representation

$$\log|\Phi_n(z)| = \int_{1/2}^{1/\rho_n} \log\frac{|z-t|}{|z+t|} d\nu_n(t) + \text{ (harmonic function)}, \tag{1.6}$$

where ν_n is a non-negative Borel measure on $[1/2, 1/\rho_n]$ with the norm

$$\|\nu_n\| = \nu_n([1/2, 1/\rho_n]) = n + 1. \tag{1.7}$$

This is the *first approach* to a Dirichlet problem in G_n.

There is a *second approach* to the same Dirichlet problem. As in Gonchar - Rakhmanov [1987], elliptic integrals are involved. Here they appear in the construction of the potentials

$$L_n(z) := \int_{1/2}^{1/\rho_n} \log\frac{|z-t|}{|z+t|} d\mu_n(t), \quad z \in \mathbb{C}, \tag{1.8}$$

(called the negative *Green's potentials of the measures* μ_n) which satisfy (1.5). Here μ_n is a non-negative Borel measure on $[1/2, 1/\rho_n]$ with

$$\|\mu_n\| = \pi^{-2}\log^2(8/\rho_n) + \mathcal{O}(1), \quad n \to \infty. \tag{1.9}$$

A harmonic function v on H_+ and a third non-negative Borel measure ω_n on $[1/2, 1/\rho_n]$ are constructed with $\|\omega_n\| \le 1/2$ so that the function

$$q_n(z) := v(z) + \int_{1/2}^{1/\rho_n} \log\frac{|z-t|}{|z+t|} d\omega_n(t) \tag{1.10}$$

vanishes on $[1/2, 1/\rho_n]$.

By a proper adjustment of the three measures one gets in §4 the relation $\|\mu_n\| = \|\nu_n\| + \mathcal{O}(1)$, and then (1.7) and (1.9) yield immediately

$$\rho_n = \left(8 + \mathcal{O}(n^{-1/2})\right) e^{-\pi\sqrt{n}},$$

which is even better than (1.4).

§ 2. A Dirichlet Problem on $[1/2, 1/\rho_n]$

For $n \ge 2$ let $r_n^* = P_n/Q_n$ be the unique best rational approximation from $\mathcal{R}_n$ of the function $|x|$ in the uniform norm on $[-1, 1]$, where P_n, Q_n are coprime polynomials. Since $|x|$ is an even function, P_n and Q_n are even (see

1 of Chapter 7, §2), and one has $r^*_{2k+1} = r^*_{2k}$. This explains the *restriction to even* $n \geq 2$, which will be adopted througout Chapter 8.

For the error function we write

$$E_n(x) := |x| - P_n(x)/Q_n(x), \quad x \in [-1,1].$$

Hence, $\rho_n = \rho_n(|x|)_{C[-1,1]} = \|E_n\|_{C[-1,1]}$.

We define the rational function $r_n \in \mathcal{R}_{n+1}$ by

$$r_n(z) := \frac{zQ_n(z) - P_n(z)}{zQ_n(z) + P_n(z)}, \quad z \in \mathbb{C}, \tag{2.1}$$

and study some properties of P_n, Q_n and r_n for even n.

Lemma 2.1. (i) *The polynomials P_n and Q_n are even polynomials of degree n. The function $E_n(x)$ attains its extremal values $\pm\rho_n$ at exactly $2n+3$ points $(\xi_j)_{-n-1}^{n+1}$ on $[-1,1]$,*

$$0 = \xi_0 < \xi_1 < \cdots < \xi_{n+1} = 1, \quad \xi_{-j} = -\xi_j, \ j = 1, \ldots, n+1, \tag{2.2}$$

with alternating signs

$$E_n(\xi_j) = (-1)^{j+1}\rho_n, \quad j = -n-1, \ldots, n+1. \tag{2.3}$$

(ii) *The function $r_n(z)$ satisfies*

$$|r_n(iy)| = 1, \quad y \in \mathbb{R}, \tag{2.4}$$

and has its $n+1$ zeros $(\eta_j)_1^{n+1}$ in the interval $(0,1)$,

$$0 = \xi_0 < \eta_1 < \xi_1 < \eta_2 < \cdots < \eta_{n+1} < \xi_{n+1} = 1. \tag{2.5}$$

The $n+1$ poles of $r_n(z)$ are equal to $-\eta_j$, $j = 1, \ldots, n+1$, and one has

$$\eta_1 > \rho_n. \tag{2.6}$$

Proof. (i) We write $P := P_n$, $Q := Q_n$ and denote the maximum points of $|E_n|$ on $[0,1]$ by $0 \leq \xi_0 < \xi_1 < \cdots < \xi_k \leq 1$. Since n, $\deg(P)$, $\deg(Q)$ are even integers, the defect of r_n^*, $d := \min\{n - \deg(P); n - \deg(Q)\}$, is also even. We set

$$M(x) := (xQ(x) - P(x))^2 - \rho_n^2 Q(x)^2, \quad x \in \mathbb{R}.$$

Clearly, $M \in \mathcal{P}_{2n+2-2d}$, $M(x) \leq 0$ on $0 \leq x \leq 1$ and $M(\xi_j) = 0$, $j = 0, \ldots, k$, $M'(\xi_j) = 0$ if $0 < \xi_j < 1$. By Theorem 2.6 of Chapter 7 there exist on $[-1,1]$ at least $2n+2-d$ extremum points of E_n with alternating signs. Hence, by symmetry, $k \geq n+1-d/2$.

In addition to the $2(k-1)$ zeros at $\xi_1, \ldots, \xi_{k-1}$, M has at least two additional ones if $\xi_0 = 0$ and $\xi_k = 1$, and more than two otherwise. From this we deduce that

$$2n+2-d \leq 2k \leq \deg(M) \leq 2n+2-2d,$$

so that $d = 0$, $k = n + 1$, $\deg(M) = 2n + 2$, $\deg(Q) = n$ and that $\xi_0 = 0$ and $\xi_{n+1} = 1$ are simple zeros. Moreover, $1 - P(1)/Q(1) = \rho_n$. (Otherwise, M would have another zero at a point $x > 1$ where $x - P(x)/Q(x) = \rho_n$, which is impossible.) Since $\xi_0 = 0$, $\pm\xi_j$, $j = 1, \ldots, n+1$, are all maximum points of $|E_n|$ on $[-1, 1]$, we have (2.3).

All $2n + 2$ zeros of M lie in $[0, 1]$. This implies that $M^{(q)}(0) \neq 0$ for $q = 1, \ldots, 2n + 2$. In particular, since Q is an even polynomial,

$$0 \neq M^{(2n+1)}(0) = -2\frac{d^{2n+1}}{dx^{2n+1}} \left(xP(x)Q(x)\right)_{x=0},$$

which implies that $\deg(P) = n$.

(ii) Since P and Q are even polynomials, $P(iy)$ and $Q(iy)$ are real numbers and, by (2.1), we have $\overline{r_n(iy)} = 1/r_n(iy)$ and thus (2.4). From (2.3) we deduce that the $n+1$ zeros of r_n, that is, of $x - r_n^*(x)$, satisfy (2.5), and these are all zeros of r_n. Since

$$r_n(z) = r_n(-z)^{-1}, \quad z \in \mathbb{C}, \tag{2.7}$$

the $n + 1$ poles of r_n are equal to $-\eta_j$, $j = 1, \ldots, n + 1$.

It remains to prove (2.6). Since $r_n(\infty) = 1$, we have

$$r_n(z) = \prod_{j=1}^{n+1} \frac{z - \eta_j}{z + \eta_j}, \quad 0 < \eta_1 < \cdots < \eta_{n+1}.$$

If Re $z > 0$, then $|z + \eta_j| > |z - \eta_j|$ for all j; if Re $z < 0$, the opposite holds. It follows that the inverse of (2.4) is also true: $|r_n(z)| = 1$ implies that $z \in i\mathbb{R}$. A point $z \in \mathbb{C}$, $z \neq 0$, is a pole of P/Q if and only if $r_n(z) = -1$, and is a zero of P/Q if and only if $r_n(z) = 1$. This proves that the n zeros $(\zeta_j)_1^n$ of P and $(\pi_j)_1^n$ of Q lie all on the imaginary axis $i\mathbb{R}$.

Since all zeros, η_j, of r_n lie in $H_+ := \{z \in \mathbb{C} : \text{Re } z > 0\}$ and all poles, $-\eta_j$, in $H_- := \{z \in \mathbb{C} : \text{Re } z < 0\}$, $\arg r_n(z)$ grows by $2\pi(n + 1)$ while z runs through $i\mathbb{R}$. As we move on $i\mathbb{R}$ upwards from 0 to ∞, we begin and end up with real values

$$r_n(0) = -1, \quad r_n(\infty) = 1. \tag{2.8}$$

With an appropriate enumeration we have therefore

$$\begin{aligned} &0 < \text{Im}\zeta_1 < \text{Im}\pi_1 < \text{Im}\zeta_2 < \cdots < \text{Im}\zeta_{n/2} < \text{Im}\pi_{n/2}, \\ &\zeta_{j+n/2} = -\zeta_j, \quad \pi_{j+n/2} = -\pi_j, \quad j = 1, \ldots, n/2. \end{aligned} \tag{2.9}$$

From (2.9) we deduce that P/Q can be represented as

$$r_n^*(z) = \frac{P(z)}{Q(z)} = c \prod_{j=1}^{n/2} \frac{z^2 - \zeta_j^2}{z^2 - \pi_j^2}, \quad c > 0.$$

Since $0 < \mathrm{Im}\zeta_j < \mathrm{Im}\pi_j$, $j = 1, \dots, n/2$, r_n^* is monotone increasing in $[0,1]$. From (2.3) we deduce that $0 = \eta_1 - r_n^*(\eta_1) \le \eta_1 - r_n^*(0) = \eta_1 - \rho_n$, which proves (2.6). □

Let S_n be defined by

$$S_n(z) := \frac{4z^2 - 1}{z} r_n(\rho_n z) - \frac{1}{z}. \tag{2.10}$$

Since $r_n(0) = -1$, $S_n \in \mathcal{R}_{n+2,n+1}$. We have $S_n(-1/2) = 2$, $S_n(1/2) = -2$. Moreover, $S_n(0) = -\rho_n r_n'(0)$, and $r_n'(0) = 2/r_n^*(0) = -2/E_n(0) = 2/\rho_n$, so that $S_n(0) = -2$.

By Lemma 2.1(ii), all $n+1$ poles $-\eta_j/\rho_n$, $j = 1, \dots, n+1$, of S_n lie in the interval $(-\infty, -1)$ and the numbers $w_j := \xi_j/\rho_n$, $j = 1, \dots, n+1$, satisfy $1 < w_1 < \cdots < w_{n+1} = 1/\rho_n$. Moreover, from (2.3) we get $r_n^*(\xi_j) = \xi_j + (-1)^j \rho_n$ which implies that

$$r_n(\rho_n w_j) = r_n(\xi_j) = \frac{\xi_j - r_n^*(\xi_j)}{\xi_j + r_n^*(\xi_j)} = \frac{(-1)^{j+1}\rho_n}{2\xi_j + (-1)^j \rho_n}, \quad j = 1, \dots, n+1.$$

From this and the definition of S_n we derive

$$S_n(w_j) = 2(-1)^{j+1}, \quad j = 0, 1, \dots, n+1, \tag{2.11}$$

where we set $w_0 := 1/2$. From $-\rho_n \le x - P_n(x)/Q_n(x) \le \rho_n$ we get the inequalities

$$\frac{-\rho_n}{2x + \rho_n} \le r_n(x) \le \frac{\rho_n}{2x - \rho_n}, \quad \rho_n/2 \le x \le 1,$$

hence $-1/(2x+1) \le r_n(\rho_n x) \le 1/(2x-1)$, $1/2 \le x \le 1/\rho_n$. This implies that

$$-2 \le S_n(x) \le 2 \quad \text{for } 1/2 \le x \le 1/\rho_n \tag{2.12}$$

and that $S_n'(w_j) = 0$, $j = 1, \dots, n$. Since the rational function $S_n(z)^2 - 4$ has at most $2n+4$ zeros, it follows that

(2.13) $-1/2$, 0, $1/2$, $w_{n+1} = 1/\rho_n$ are simple zeros and
$w_1, \dots, w_n$ are double zeros of $S_n(z)^2 - 4$;
$S_n(z)^2 - 4$ has no other zeros;
$S_n(x) > 2$ for $x > 1/\rho_n$;
$S_n(x) < -2$ for $0 < x < 1/2$;
$-2 < S_n(x) < 2$ for $-1/2 < x < 0$.

It also follows from $S_n(-1/2) = 2$, $S_n(0) = -2$ and (2.11) that S_n' satisfies

$$(-1)^j S_n'(x) > 0, \quad w_j < x < w_{j+1}, \ j = 0, \dots, n. \tag{2.14}$$

Indeed, suppose that $S_n'(x_0) = 0$ holds for some $x_0 \in (1/2, 1/\rho_n)$ where $\alpha := |S_n(x_0)| < 2$. Then the rational function $S_n^2 - \alpha^2 \in \mathcal{R}_{2n+4,2n+2}$ would have 2 distinct zeros (if $\alpha > 0$) or a double zero (if $\alpha = 0$) in each of the $n+2$

intervals $(-1/2, 0)$, (w_j, w_{j+1}), $j = 0, \dots, n$. One of these zeros, x_0, would have multiplicity 2 if $\alpha > 0$ and multiplicity 4 if $\alpha = 0$. But this is impossible.

For a function $f(z)$ and $x \in \mathbb{R}$ we will denote by $f(x + i0)$ the limit of $f(z)$ as $z \to x$, $\operatorname{Im} z > 0$; and $f(x - i0)$ is the limit of $f(z)$ as $z \to x$, $\operatorname{Im} z < 0$.

We consider the function

$$\Phi_n(z) := \frac{1}{8z}\left(S_n(z) + \sqrt{S_n(z)^2 - 4}\right), \tag{2.15}$$

where the square root is chosen to be positive for $z > 1/\rho_n$.

Proposition 2.2. (i) *The function $\Phi_n(z)$ is analytic and different from zero in the domain $H_+ \setminus [1/2, 1/\rho_n]$, and $|\Phi_n(z)|$ has continuous boundary values everywhere on $i\mathbb{R} \cup [1/2, 1/\rho_n]$ except at the origin. We have*

$$\log|\Phi_n(x \pm i0)| = -\log(4x), \quad 1/2 \le x \le 1/\rho_n. \tag{2.16}$$

(ii) *On the imaginary axis, for all $y \in \mathbb{R}$,*

$$1 + y^{-2} \ \ge\ |\Phi_n(iy)| \ \ge\ \begin{cases} 1/28 & \text{for } 0 < |y| \le 1 \\ 1 - y^{-2}/2 & \text{for } 1 \le |y| < \infty. \end{cases} \tag{2.17}$$

As a corollary, there exists a constant $c > 0$ (for example, $c = 5$) so that

$$|\log|\Phi_n(iy)|| \le c \log(1 + y^{-2}), \quad y \in \mathbb{R}. \tag{2.18}$$

(iii) *The function $\Phi_n(z)$ converges to 1 for $z \to \infty$, $z \in \mathbb{C}$, and $z\Phi_n(z) \to -1/4$ for $z \to 0$.*

Proof. (i) The square root $\sqrt{S_n(z)^2 - 4}$ in (2.15) is positive for $z > 1/\rho_n$. Since $w_0 = 1/2$ and $w_{n+1} = 1/\rho_n$ are simple zeros and $w_1, \dots, w_n$ are double zeros of $S_n(z)^2 - 4$, since $w_0 < w_1 < \cdots < w_n < w_{n+1}$, the square root $\sqrt{S_n(z)^2 - 4}$ becomes purely imaginary if $z \to x \pm i0$, $x \in (1/2, 1/\rho_n)$. Moreover,

$$\begin{aligned} &\sqrt{S_n(x + i0)^2 - 4} = -\sqrt{S_n(x - i0)^2 - 4} \\ &= i(-1)^j \left|\sqrt{S_n(x + i0)^2 - 4}\right| \quad \text{for } w_j \le x \le w_{j+1}, \ j = 0, \dots, n. \end{aligned} \tag{2.19}$$

It follows from (2.12) and the definition of Φ_n that

$$|\Phi_n(x \pm i0)| = \frac{1}{4|x|}, \quad x \in [1/2, 1/\rho_n]. \tag{2.20}$$

As another consequence, $\Phi_n(z)$ is analytic and single valued in $H_+ \setminus [1/2, 1/\rho_n]$. We have $\Phi_n(z) \ne 0$ for $z \in H_+ \setminus [1/2, 1/\rho_n]$ since $\Phi_n(z) = 0$ implies $S_n(z)^2 = S_n(z)^2 - 4$, which is impossible.

(ii) In order to prove (2.17) we first verify two inequalities involving S_n on $i\mathbb{R}$. From (2.4) and (2.10) we deduce that

$$|S_n(iy)| \le 4|y| + 2|y|^{-1}, \quad y \in \mathbb{R}, \tag{2.21}$$

and

$$|S_n(iy)| \geq |y|^{-1}(4y^2+1) - |y|^{-1} = 4|y|, \quad y \in \mathbb{R}. \tag{2.22}$$

From (2.21) and the definition of Φ_n we obtain

$$\begin{aligned}|\Phi_n(iy)| &\leq \frac{1}{8|y|}\left(4|y| + 2|y|^{-1} + \sqrt{4 + (4|y| + 2|y|^{-1})^2}\right) \\ &= \frac{1}{4}\left(2 + y^{-2} + \sqrt{y^{-2} + (2 + y^{-2})^2}\right) \leq 1 + y^{-2}.\end{aligned}$$

This is the left inequality (2.17).

Let $\pm\alpha(y)$ be the two square roots of $S_n(iy)^2 - 4$, $y \in \mathbb{R}$. They satisfy

$$(S_n(iy) + \alpha(y))(S_n(iy) - \alpha(y)) = 4. \tag{2.23}$$

If $0 < |y| \leq 1$, then

$$\begin{aligned}|S_n(iy) \pm \alpha(y)| &\leq |S_n(iy)| + \sqrt{|S_n(iy)|^2 + 4} \\ &\leq 2|S_n(iy)| + 2 \leq 8|y| + 4|y|^{-1} + 2 \leq 14|y|^{-1}.\end{aligned}$$

This and (2.23) imply that

$$\min\{|S_n(iy) + \alpha(y)|, |S_n(iy) - \alpha(y)|\} \geq \frac{4|y|}{14}.$$

From the definition of Φ_n we therefore get

$$|\Phi_n(iy)| \geq \frac{1}{8|y|}\frac{4|y|}{14} = \frac{1}{28} \quad \text{for } 0 < |y| \leq 1. \tag{2.24}$$

Since the square root in (2.15) has the same sign as S_n near infinity, we have

$$|\Phi_n(iy)| \geq \frac{4|y|}{8|y|}\left(1 + \sqrt{1 - 1/(4y^2)}\right) \geq 1 - y^{-2}/2, \quad |y| \geq 1. \tag{2.25}$$

This and (2.24) conclude the proof of the right inequality (2.17).

(iii) The degree of Q_n is n, therefore $r_n(z) \to 1$ and $S_n(z)/(4z) \to 1$ for $z \to \infty$, and also $\Phi_n(z) \to 1$. Since $S_n(0) = -2$, the definition of Φ_n implies that $z\Phi_n(z) \to -1/4$ for $z \to 0$. □

Theorem 2.3. *There exists a harmonic function $h_n(z)$ in H_+ and a non-negative Borel measure ν_n with*

$$\operatorname{supp}(\nu_n) = [1/2, 1/\rho_n], \quad \|\nu_n\| := \nu_n([1/2, 1/\rho_n]) = n + 1, \tag{2.26}$$

so that

$$h_n(iy) = \log|\Phi_n(iy)|, \quad y \in \mathbb{R}, \tag{2.27}$$

and

$$\log|\Phi_n(z)| = h_n(z) + \int_{1/2}^{1/\rho_n} \log\frac{|z-t|}{|z+t|}\,d\nu_n(t), \quad z \in H_+. \tag{2.28}$$

Proof. By (2.12), $4 - S_n(x)^2$ is non-negative for $w_0 = 1/2 \le x \le 1/\rho_n$; let $\delta_n(x) := \sqrt{4 - S_n(x)^2}$ denote its positive square root. We define the non-negative measure ν_n with continuous density on $(1/2, 1/\rho_n)$ and support $[1/2, 1/\rho_n]$ by

$$d\nu_n(x) := \frac{1}{\pi}\frac{|S_n'(x)|dx}{\delta_n(x)}, \quad 1/2 \le x \le 1/\rho_n. \tag{2.29}$$

From (2.11), (2.14) and (2.29) we deduce that

$$\|\nu_n\| = \frac{1}{\pi}\sum_{j=0}^{n}\int_{w_j}^{w_{j+1}} \frac{(-1)^j S_n'(x)dx}{\delta_n(x)} = \frac{n+1}{\pi}\int_{-2}^{2}\frac{dt}{\sqrt{4-t^2}} = n+1,$$

which proves the second part of (2.26).

We introduce the function

$$f(z) := \frac{S_n'(z)}{\sqrt{S_n(z)^2 - 4}}. \tag{2.30}$$

Because of (2.12) and (2.13), $f(z)$ is analytic in $\mathbb{C} \setminus ((-\infty, 0] \cup [1/2, 1/\rho_n])$. Since $S_n'(z) = \mathcal{O}(1)$ as $z \to \infty$ and since $\deg(Q_n) = n$, we have

$$f(z) = \mathcal{O}(|z|^{-1}) \quad \text{as } z \to \infty. \tag{2.31}$$

Another representation of $f(z)$ follows by elementary calculation from (2.30) and (2.15):

$$f(z) = \frac{d}{dz}\log\left(8z\Phi_n(z)\right). \tag{2.32}$$

We shall relate $f(z)$ to the function

$$f_1(z) := \frac{1}{2\pi i}\int_{1/2}^{1/\rho_n} \frac{f(x+i0) - f(x-i0)}{x-z}dx. \tag{2.33}$$

Clearly, f_1 is analytic on $\mathbb{C}^* \setminus [1/2, 1/\rho_n]$, and from (2.14), (2.19), (2.29) and (2.30) we get

$$f_1(z) = \frac{1}{2\pi i}\int_{1/2}^{1/\rho_n} \frac{2|S_n'(x)|}{i\delta_n(x)}\frac{dx}{x-z} = \int_{1/2}^{1/\rho_n}\frac{d\nu_n(x)}{z-x}. \tag{2.34}$$

Let $C_R^+ = \{z = Re^{it} : -\pi/2 < t < \pi/2\}$ be the half-circle with center 0 and radius R in H_+, For $z \in H_+ \setminus (0, 1/\rho_n]$ and $R > |z|$, $R > 1/\rho_n$, Cauchy's integral formula yields

$$f(z) = \frac{1}{2\pi i}\left(\int_{0+i0}^{1/\rho_n+i0} + \int_{1/\rho_n-i0}^{0-i0} + \int_{0}^{-iR} + \int_{C_R^+} + \int_{iR}^{0}\right)\frac{f(\zeta)d\zeta}{\zeta - z}.$$

Using (2.31) and (2.33) we get for $R \to \infty$

$$f(z) = f_1(z) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{f(iy)dy}{z - iy}. \tag{2.35}$$

This is valid for all $z \in H_+$ since the function

$$f_2(z) := f(z) - f_1(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{f(iy)dy}{z - iy} \tag{2.36}$$

is analytic in H_+. Let F_1 and F_2 be the integrals of f_1, f_2,

$$F_j(z) := \int_{1/4}^{z} f_j(\zeta)d\zeta, \quad j = 1, 2, \tag{2.37}$$

then Re F_1 is harmonic in $\mathbb{C}^* \setminus [1/2, 1/\rho_n]$ and Re F_2 is harmonic in H_+. Integrating (2.36),

$$\log|8z\Phi_n(z)| = \text{Re } F_1(z) + (\text{harmonic function}) \quad \text{for } z \in H_+. \tag{2.38}$$

Applying (2.34) and (2.37) we have

(2.39)

$$\begin{aligned}\text{Re } F_1(z) &= \text{Re } \int_{1/4}^{z}\int_{1/2}^{1/\rho_n} \frac{d\nu_n(x)}{\zeta - x} d\zeta \\ &= \text{Re } \int_{1/2}^{1/\rho_n} \log(z - x)d\nu_n(x) + c = \int_{1/2}^{1/\rho_n} \log|z - x|d\nu_n(x) + c\end{aligned}$$

with a constant c. This implies

$$\begin{aligned}\log|8z\Phi_n(z)| &= \int_{1/2}^{1/\rho_n} \log|z - x|d\nu_n(x) + (\text{harmonic function}) \\ &= \int_{1/2}^{1/\rho_n} \log\frac{|z - x|}{|z + x|}d\nu_n(x) + \phi_n(z) \quad \text{for } z \in H_+ .\end{aligned} \tag{2.40}$$

The logarithmic potential $\int \log|z + x|d\nu_n(x)$ is harmonic in $\mathbb{C}\setminus[-1/\rho_n, -1/2]$ (see Appendix 4), and Green's potential in (2.40) vanishes on $i\mathbb{R}$. Therefore $\phi_n(z)$ is a function harmonic in H_+ and $\phi_n(iy) = \log|8y\Phi_n(iy)|$ for $y \in \mathbb{R}$. The function $h_n(z) := \phi_n(z) - \log|8z|$ has the properties of the theorem. □

§ 3. The Second Approach to the Dirichlet Problem

The main results of this section are Theorems 3.2 and 3.3 about the existence of a certain Green's potential p_a and its properties. For its construction we will use properties of the elliptic integrals

$$
\begin{aligned}
A(r) &:= \int_0^\infty \frac{dt}{\sqrt{(1+t^2)(1+r^2t^2)}},\\
B(r) &:= \int_0^\infty \frac{dt}{\sqrt{(1+t^2)^3(1+r^2t^2)}},\\
C(r) &:= \frac{A(r)}{B(r)} - 1, \qquad (3.1)\\
D(r) &:= \int_{1/r}^\infty \frac{(C(r)+t^2)dt}{\sqrt{(t^2-1)^3(t^2-r^{-2})}},\\
I(r) &:= \int_{1/r}^\infty \frac{dx}{x} \int_x^\infty \frac{(C(r)+t^2)dt}{\sqrt{(t^2-1)^3(t^2-r^{-2})}},
\end{aligned}
$$

$0 < r \le 1$, that are studied in Appendix 2.

In the next lemma we define a number λ_a. It will be shown in Theorem 3.2 that this number is unique for each small $a > 0$. For the integrals (3.1) with the parameter $r = \lambda_a$ we shall write

$$C_a := C(\lambda_a), \quad D_a := D(\lambda_a), \quad I_a := I(\lambda_a). \tag{3.2}$$

Lemma 3.1. *For each sufficiently small number $a > 0$ there exists a number λ_a, $a < \lambda_a < 2a$, for which*

$$I(\lambda_a) = D(\lambda_a)\log(\lambda_a/a). \tag{3.3}$$

Proof. Since $\delta_a(r) := I(r) - D(r)\log(r/a)$ is a continuous function of r for $a \le r < 1$ and $\delta_a(a) = I(a) > 0$, relations (1.8) and (1.9) of Appendix 2 yield

$$
\begin{aligned}
\delta_a(2a) &= I(2a) - D(2a)\log 2\\
&= \left(\frac{\pi}{8}(\log 4 - 1) - \frac{\pi}{4}\log 2\right)(2a)^3 \log(1/2a) + \mathcal{O}(a^3)\\
&= -\pi a^3 \log(1/2a) + \mathcal{O}(a^3)
\end{aligned}
$$

as $a \to 0+$. Hence, $\delta_a(2a) < 0$ for all small $a > 0$, and δ_a has a zero in the interval $(a, 2a)$. □

Theorem 3.2. *For all small numbers $a > 0$ and λ_a of Lemma 3.1, let*

$$b_a := a/\lambda_a. \tag{3.4}$$

Then $1/2 < b_a < 1$, and Green's potential

$$p_a(z) := \int_a^{b_a} \log\frac{|z+x|}{|z-x|}\, d\nu_a(x), \quad z \in \mathbb{C}, \tag{3.5}$$

with the measure ν_a given by its density

$$\frac{d\nu_a(x)}{dx} = \frac{a}{\pi D_a x}\int_x^{b_a} \frac{(a^2C_a + t^2)dt}{\sqrt{(t^2-a^2)^3(b_a^2-t^2)}}, \quad x \in (a, b_a), \tag{3.6}$$

is a continuous function on $\mathbb{C}$ *and satisfies the conditions*

$$
\begin{aligned}
p_a(x) &= -\log x, \quad x \in [a, b_a], \\
p_a(x) &> -\log x, \quad x \in (b_a, \infty).
\end{aligned} \tag{3.7}
$$

The number λ_a *of Lemma* 3.1 *is unique.*

Proof. The density of the measure ν_a is continuous in $(a, b_a]$ and $\sqrt{x-a}\frac{d\nu_a(x)}{dx}$ converges for $x \to a+$. Therefore $p_a(z)$ is a continuous function on $\mathbb{C}$. From $a < \lambda_a < 2a$ it follows that $1/2 < b_a < 1$.

(a) *Another formula for* p_a. The set $G_a := \mathbb{C} \setminus (-\infty, 1/\lambda_a]$ is a simply connected domain. We define the analytic single-valued function Ψ_a on G_a by

$$
\Psi_a(t) := \sqrt{(t^2-1)^3(t^2-\lambda_a^{-2})}, \quad t \in G_a, \tag{3.8}
$$

where the branch of the square root is chosen so that $\Psi_a(t) > 0$ for $t > 1/\lambda_a$. The limits $\Psi_a(x+i0)$, $\Psi_a(x-i0)$ and $\Psi_a(iy)$ have the values

$$
\begin{aligned}
&\Psi_a(x \pm i0) = \pm\ i\ \sqrt{(x^2-1)^3(\lambda_a^{-2}-x^2)}, \quad 1 < x < 1/\lambda_a, \\
&\Psi_a(x \pm i0) = \mp\ i\ \sqrt{(x^2-1)^3(\lambda_a^{-2}-x^2)}, \quad -1/\lambda_a < x < -1, \\
&\Psi_a(x \pm i0) = \sqrt{(1-x^2)^3(\lambda_a^{-2}-x^2)}, \quad -1 < x < 1, \\
&\Psi_a(x \pm i0) = \sqrt{(x^2-1)^3(x^2-\lambda_a^{-2})}, \quad x \geq 1/\lambda_a,\ x \leq -1/\lambda_a, \\
&\Psi_a(iy) = \sqrt{(y^2+1)^3(y^2+\lambda_a^{-2})}, \quad y \in \mathbb{R},
\end{aligned} \tag{3.9}
$$

where the square roots on the right-hand sides in (3.9) are non-negative.

Let Γ_z, $z \in G_a$, be an integration path that connects 0 with z and is fully contained in G_a or one of the sides of the boundary $(-\infty, 1/\lambda_a]$, but does not intersect this boundary, and whose first part lies in $\{\mathrm{Im}\ t \geq 0\}$. For each $z \in G_a$, the integrals

$$
g_a(z) := \int_{\Gamma_z} \frac{(C_a+t^2)dt}{\Psi_a(t)}, \quad z \in G_a,
$$

are independent of the integration path Γ_z. We define

$$
f_a(z) := \frac{1}{D_a}\int_{\Gamma_z} \frac{g_a(\zeta)d\zeta}{\zeta} = \frac{1}{D_a}\int_{\Gamma_z} \frac{d\zeta}{\zeta}\int_{\Gamma_\zeta} \frac{(C_a+t^2)dt}{\Psi_a(t)}, \quad z \in G_a.
$$

Clearly, the functions g_a and f_a are analytic single-valued functions on G_a.

We want to evaluate the difference

$$
\Delta_a(x) := g_a(x-i0) - g_a(x+i0) \quad \text{for all } x \in \mathbb{R}.
$$

From the properties of Ψ_a and the definition of g_a it follows that $\Delta_a(x) = 0$ for $x > 1/\lambda_a$, $x < -1/\lambda_a$.

For circles C_R of radius R and center 0 one has

$$\left|\int_{C_R}\frac{(C_a+t^2)dt}{\Psi_a(t)}\right|\le\frac{2\pi R(C_a+R^2)}{\sqrt{(R^2-1)^3(R^2-\lambda_a^{-2})}}\to 0,\quad R\to+\infty. \tag{3.10}$$

Let $1<x<1/\lambda_a$. If we choose the path of integration along the real axis, first along $\{t+i0 : x\le t\le 1/\lambda_a\}$ then along $\{t-i0 : 1/\lambda_a\ge t\ge x\}$, the definition of g_a and (3.9) yield

$$\begin{aligned}\Delta_a(x)&=\int_x^{1/\lambda_a}\frac{(C_a+t^2)dt}{\Psi_a(t+i0)}+\int_{1/\lambda_a}^{x}\frac{(C_a+t^2)dt}{\Psi_a(t-i0)}\\&=\frac{2}{i}\int_x^{1/\lambda_a}\frac{(C_a+t^2)dt}{\sqrt{(t^2-1)^3(\lambda_a^{-2}-t^2)}},\quad 1<x<1/\lambda_a.\end{aligned}$$

Similarly, for $-1/\lambda_a<x<-1$ and the path of integration $\{t+i0 : x\ge t\ge -1/\lambda_a\}\cup\{t-i0 : -1/\lambda_a\le t\le x\}$, we obtain

$$\begin{aligned}\Delta_a(x)&=\int_x^{-1/\lambda_a}\frac{(C_a+t^2)dt}{\Psi_a(t+i0)}+\int_{-1/\lambda_a}^{x}\frac{(C_a+t^2)dt}{\Psi_a(t-i0)}\\&=\frac{2}{i}\int_{-1/\lambda_a}^{x}\frac{(C_a+t^2)dt}{\sqrt{(t^2-1)^3(\lambda_a^{-2}-t^2)}},\quad -1/\lambda_a<x<-1.\end{aligned}$$

For $x=0$ we integrate from 0 to $i\infty$ to $-\infty$ to x:

$$\begin{aligned}\Delta_a(0)&=\int_0^{+i\infty}\frac{(C_a+t^2)dt}{\Psi_a(t)}+\int_{-\infty}^{0}\frac{(C_a+t^2)dt}{\Psi_a(t)}\\&=2i\int_0^{\infty}\frac{(C_a-t^2)dt}{\sqrt{(1+t^2)^3(\lambda_a^{-2}+t^2)}}\\&=2i\lambda_a\left(C_aB(\lambda_a)-[A(\lambda_a)-B(\lambda_a)]\right)=0,\end{aligned}$$

by the definition (3.1) of the number $C_a=C(\lambda_a)$. We now prove that

$$\Delta_a(x)=0,\quad -1<x<1. \tag{3.11}$$

Indeed, $\Delta_a(x)=g_a(x-i0)-g_a(0-i0)+\Delta_a(0)+g_a(0+i0)-g_a(x+i0)$ implies

$$\Delta_a(x)=\int_0^x\frac{(C_a+t^2)dt}{\Psi_a(t-i0)}+\Delta_a(0)+\int_x^0\frac{(C_a+t^2)dt}{\Psi_a(t+i0)}=\Delta_a(0)$$

for all $-1<x<1$. We conclude that $\Delta_a(x)$ is an even function on $\mathbb{R}$, continuous everywhere on $\mathbb{R}$ except at the points $x=\pm1$, and that the limits $\lim\Delta_a(x)\sqrt{x^2-1}$, $x\to 1+$, $x\to -1-$ are finite.

The derivative of f_a, $f_a'(z)=g_a(z)/(zD_a)$, is analytic and single-valued in the simply connected domain G_a. For each $z\in G_a$, since $|g_a(t)|=\mathcal{O}(|t|^{-1})$ as $|t|\to\infty$, an application of Cauchy's integral formula yields

$$
\begin{aligned}
f_a'(z) &= \frac{1}{2\pi i}\int_{-\infty}^{1/\lambda_a} \frac{[f_a'(x+i0)-f_a'(x-i0)]dx}{x-z} \\
&= \frac{1}{2\pi i D_a}\int_{-\infty}^{1/\lambda_a} \frac{[g_a(x+i0)-g_a(x-i0)]dx}{(x-z)x}.
\end{aligned}
$$

Inserting the values of $\Delta_a(x) = g_a(x-i0) - g_a(x+i0)$ just obtained we get

$$
\begin{aligned}
f_a'(z) &= -\frac{1}{2\pi i D_a}\int_{-\infty}^{1/\lambda_a} \frac{\Delta_a(x)dx}{(x-z)x} \\
&= \frac{1}{\pi D_a}\int_{-1/\lambda_a}^{-1} \frac{dx}{(x-z)x}\int_{-1/\lambda_a}^{x} \frac{(C_a+t^2)dt}{\sqrt{(t^2-1)^3(\lambda_a^{-2}-t^2)}} \\
&\quad + \frac{1}{\pi D_a}\int_{1}^{1/\lambda_a} \frac{dx}{(x-z)x}\int_{x}^{1/\lambda_a} \frac{(C_a+t^2)dt}{\sqrt{(t^2-1)^3(\lambda_a^{-2}-t^2)}} \\
&= \frac{1}{\pi D_a}\int_{1}^{1/\lambda_a} \left(\frac{1}{x+z}+\frac{1}{x-z}\right)\frac{dx}{x}\int_{x}^{1/\lambda_a} \frac{(C_a+t^2)dt}{\sqrt{(t^2-1)^3(\lambda_a^{-2}-t^2)}},
\end{aligned}
$$

for all $z \in G_a$. With the measure (3.6),

$$
\frac{1}{a}f_a'\left(\frac{z}{a}\right) = \int_a^{b_a}\left(\frac{1}{z+x}-\frac{1}{z-x}\right)d\nu_a(x) \quad \text{for all } z \in G_a. \tag{3.12}
$$

The function

$$
P_a(z) := \int_a^{b_a} \log\frac{z+x}{z-x}d\nu_a(x)
$$

is analytic and single-valued in the simply connected domain $\mathbb{C}\setminus(-\infty, b_a]$, with Re $P_a(z) = p_a(z)$ and $P_a'(z) = a^{-1}f_a'(z/a)$. Thus $p_a(z)$ and Re $f_a(z/a)$ differ by a constant. Since $P_a(0+i0) = f_a(0+i0) = 0$,

$$
p_a(z) = \operatorname{Re}\ f_a(z/a), \quad z \in \mathbb{C}\setminus(-\infty, b_a]. \tag{3.13}
$$

(b) *The potential* p_a *satisfies* (3.7). This will be achieved in several steps. By deforming the path of integration in G_a we deduce for $x \in (1, 1/\lambda_a)$

$$
g_a(x+i0) = \left(\int_0^{i\infty} + \int_{\infty}^{1/\lambda_a} + \int_{1/\lambda_a}^{x+i0}\right)\frac{(C_a+t^2)dt}{\Psi_a(t)}.
$$

From (3.9) we see that the first and the third integrals are purely imaginary, while the second is real. With the integral D_a of (3.1),

$$
\operatorname{Re}\ g_a(x+i0) = -D_a, \quad x \in (1, 1/\lambda_a). \tag{3.14}
$$

Similarly,

(3.15)
$$\text{Re } g_a(x) = -\int_x^\infty \frac{(C_a + t^2)dt}{\sqrt{(t^2-1)^3(t^2-\lambda_a^{-2})}} \in (-D_a, 0), \quad x \in (1/\lambda_a, \infty).$$

In the definition of f_a we select the path Γ_x from 0 to $i\infty$ to $+\infty$ to x and obtain, since $g_a(iy)$, $y \in \mathbb{R}$ is purely imaginary,

$$\text{Re } f_a(x) = \frac{1}{D_a}\int_\infty^x \text{Re } g_a(x)\frac{dx}{x}, \quad x > 1/\lambda_a.$$

From this, the first part of (3.15), the definition of I_a and (3.3) it follows that

$$\begin{aligned} \text{Re } f_a(1/\lambda_a) &= \frac{1}{D_a}\int_{1/\lambda_a}^\infty \frac{dx}{x}\int_x^\infty \frac{(C_a+t^2)dt}{\sqrt{(t^2-1)^3(t^2-\lambda_a^{-2})}} \\ &= \frac{I_a}{D_a} = \log(\lambda_a/a). \end{aligned} \tag{3.16}$$

If $1 < x < 1/\lambda_a$, (3.14) and (3.16) imply

$$\begin{aligned} \text{Re } f_a(x+i0) &= \text{Re} f_a(1/\lambda_a) + \frac{1}{D_a}\int_{1/\lambda_a}^x \text{Re } g_a(t+i0)\frac{dt}{t} \\ &= \log(\lambda_a/a) - \int_{1/\lambda_a}^x \frac{dt}{t} = -\log(ax), \quad 1 < x < 1/\lambda_a. \end{aligned} \tag{3.17}$$

Similarly, using (3.15) we get

$$\begin{aligned} \text{Re } f_a(x) &= \text{Re } f_a(1/\lambda_a) + \frac{1}{D_a}\int_{1/\lambda_a}^x \frac{\text{Re } g_a(t)dt}{t} \\ &> -\log(ax), \quad 1/\lambda_a < x < \infty. \end{aligned} \tag{3.18}$$

From (3.13), (3.17) and (3.18) we derive that the potential p_a satisfies (3.7).

(c) *Uniqueness of* λ_a. For fixed a let $a < \lambda_2 < \lambda_1 < 2a$ be two distinct solutions of Lemma 3.1 and let p_1 and p_2 the the Green's potentials with measures ν_1 and ν_2 which correspond to $b_1 = a/\lambda_1$ and $b_2 = a/\lambda_2$, respectively, $1/2 < b_1 < b_2 < 1$. We consider the difference

$$d(z) := p_1(z) - p_2(z) = \int \log\frac{|z+x|}{|z-x|}(d\nu_1 - d\nu_2)(x).$$

This function is continuous on $\mathbb{C}$, harmonic in $\mathbb{C} \setminus ([-b_2, -a] \cup [a, b_2])$ and subharmonic in $H_+ \setminus [a, b_1]$ (see Theorem 1.1 of Appendix 4). From (3.7) we deduce that

$$\begin{aligned} d(z) &= 0, \quad z \in [a, b_1], \\ d(z) &> 0, \quad z \in (b_1, b_2]. \end{aligned} \tag{3.19}$$

Note that $p_j(iy) = 0$ for all $y \in \mathbb{R}$, $j = 1, 2$, and consequently $d(z) = 0$ for all $z \in \partial H_+ = i\mathbb{R}$. Moreover, $d(z) \to 0$ for $z \to \infty$.

From the maximum principle for subharmonic functions (see Theorem 1.2 of Appendix 4) it follows that $d(z)$ has no maximum points in $H_+ \setminus [a, b_1]$. This contradicts the inequality (3.19). □

Theorem 3.3. (i) *The number* b_a *of Theorem* 3.2 *has the property*

$$b_a := \frac{a}{\lambda_a} = \frac{1}{2} + \frac{1}{2} a^2 \log(1/a) + \mathcal{O}(a^2), \quad a \to 0+. \tag{3.20}$$

(ii) *The measure* ν_a *satisfies*

$$\nu_a([1/2, b_a]) = \mathcal{O}(a^2), \quad a \to 0+ \tag{3.21}$$

and

$$\|\nu_a\| := \nu([a, b_a]) = \pi^{-2} \log^2(2/a) + \mathcal{O}(1), \quad a \to 0+. \tag{3.22}$$

Proof. (i) Since $a < \lambda_a < 2a$, an application of (1.8) and (1.9) of Appendix 2 and of (3.3) yields

$$\begin{aligned}
\log(1/b_a) = \frac{I_a}{D_a} &= \frac{\log 2 + \frac{1}{4}(\log 4 - 1)\lambda_a^2 \log(1/\lambda_a) + \mathcal{O}(\lambda_a^2)}{1 + \frac{1}{2}\lambda_a^2 \log(1/\lambda_a) + \mathcal{O}(\lambda_a^2)} \\
&= \log 2 + \left(\frac{1}{4}(\log 4 - 1) - \frac{1}{2}\log 2\right) \lambda_a^2 \log(1/\lambda_a) + \mathcal{O}(a^2) \\
&= \log 2 + \frac{1}{4}\lambda_a^2 \log \lambda_a + \mathcal{O}(a^2).
\end{aligned}$$

This implies that $\log(\lambda_a/a) = \log(1/b_a) = \log 2 + \mathcal{O}(a)$, $\lambda_a = 2a(1 + \mathcal{O}(a))$ and that

$$\begin{aligned}
\log(\lambda_a/a) &= \log 2 + a^2(1 + \mathcal{O}(a)) \log(2a(1 + \mathcal{O}(a))) + \mathcal{O}(a^2) \\
&= \log 2 + a^2 \log a + \mathcal{O}(a^2).
\end{aligned}$$

From this we obtain

$$2b_a = \frac{2a}{\lambda_a} = \exp\left(-a^2 \log a + \mathcal{O}(a^2)\right) = 1 - a^2 \log a + \mathcal{O}(a^2),$$

which completes the proof of (3.20).

(ii) Using (1.6) and (1.8) of Appendix 2, and (3.20) we get

$$\begin{aligned}
C_a &= C(\lambda_a) = \log(1/\lambda_a) + \mathcal{O}(1) = \log(1/a) + \mathcal{O}(1), \\
D_a &= D(\lambda_a) = \frac{\pi \lambda_a}{2} + \mathcal{O}(\lambda_a^2) = \pi a + \mathcal{O}(a^2),
\end{aligned}$$

and therefore, by (3.6),

$$\nu([1/2, b_a]) = \frac{a}{\pi D_a}\int_{1/2}^{b_a}\frac{dx}{x}\int_x^{b_a}\frac{(a^2C_a+t^2)dt}{\sqrt{(t^2-a^2)^3(b_a^2-t^2)}}$$
$$\le c\int_{1/2}^{b_a}dx\int_x^{b_a}\frac{dt}{\sqrt{b_a-t}} = \mathcal{O}\left((b_a-1/2)^{3/2}\right),$$

where $c>0$ is independent of a. Now (3.21) follows from (3.20).

By definition,

$$(3.23)\qquad \|\nu_a\| = \frac{a}{\pi D_a}\int_a^{b_a}\frac{dx}{x}\int_x^{b_a}\frac{(a^2C_a+t^2)dt}{\sqrt{(t^2-a^2)^3(b_a^2-t^2)}} =: \frac{a}{\pi D_a}J^*(a).$$

If we replace b_a in the integral $J^*(a)$ of (3.23) by $1/2$ then we get the integral

$$(3.24)\qquad J(a) := 2\int_a^{1/2}\frac{dx}{x}\int_x^{1/2}\frac{(a^2C_a+t^2)dt}{\sqrt{(t^2-a^2)^3(1-4t^2)}}.$$

For the asymptotic estimation of $J(a)$ we can use the estimates of Appendix 2 (as $a\to 0+$) of $C_a = C(\lambda_a)$, $D_a = D(\lambda_a)$, and of the integrals

$$J_1(a) := \int_a^{1/2}\frac{dx}{x}\int_x^{1/2}\frac{t^2dt}{\sqrt{(t^2-a^2)^3(1-4t^2)}},$$
$$J_2(a) := \int_a^{1/2}\frac{dx}{x}\int_x^{1/2}\frac{dt}{\sqrt{(t^2-a^2)^3(1-4t^2)}}.$$

Indeed, we have

$$J(a) = 2a^2C_aJ_2(a) + 2J_1(a),$$

and therefore

$$J(a) = 2a^2[\log(1/\lambda_a)+\mathcal{O}(1)][a^{-2}\log 2+\mathcal{O}(a^{-1}\log(1/a))]$$
$$+ [\log^2(1/a)+\mathcal{O}(1)]$$
$$= \log^2(1/a) + 2\log 2\log(1/\lambda_a) + \mathcal{O}(1).$$

Since $\log(1/\lambda_a) = \log(1/a)+\mathcal{O}(1)$, we obtain

$$(3.25)\qquad J(a) = \log^2(1/a) + 2\log 2\log(1/a) + \mathcal{O}(1) = \log^2(2/a) + \mathcal{O}(1).$$

We want to compare $J^*(a)$ with $J(a)$. The functions

$$g_a(t) := \frac{a^2C_a+t^2}{\sqrt{(t^2-a^2)^3}}$$

are continuous in $(a, b_a]$. By (1.6) of Appendix 2, $C_a = C(\lambda_a) = \log(1/\lambda_a) + \mathcal{O}(1) = \log(1/a)+\mathcal{O}(1)$. This implies that

$$0 < g_a(t) \le c, \quad 1/4 \le t \le b_a,$$
$$0 < g_a(t) \le c(t-a)^{-3/2}, \quad a < t \le 1/4,$$

for some constant $c > 0$ which is independent of a and t. An application of Lemma 1.5 of Appendix 2 proves that $J^*(a) = J(a) + \mathcal{O}(1)$; and from (3.25) we get

$$J^*(a) = \log^2(2/a) + \mathcal{O}(1). \tag{3.26}$$

From (3.20) we derive that $2a/\lambda_a = 1 + \mathcal{O}(a)$. This and (1.8) of Appendix 2 yield

$$\frac{a}{D_a} = \frac{2a}{\pi\lambda_a + \mathcal{O}(a^2)} = \frac{1}{\pi} + \mathcal{O}(a). \tag{3.27}$$

Finally, from (3.26), (3.27) and the definition of $J^*(a)$ it follows that

$$\|\nu_a\| = \frac{a}{\pi D_a} J^*(a) = \pi^{-2}\log^2(2/a) + \mathcal{O}(1),$$

which is (3.22). □

§ 4. Proof of Theorem 1.1

The two approaches, of §2 and of §3, are still very far away. In particular, the second of them represented by Theorems 3.2 and 3.3, has nothing to do with rational approximation. Therefore we shall reformulate these theorems, by means of the transformation $z \to 1/(4z)$ and $x = 1/(4t)$. We select an a dependent on n: $a := a(n) := \rho_n/4$, then $b_a = b(n)$ will become $\alpha_n := 1/(4b_a)$. By Theorem 3.3,

$$1/4 < \alpha_n < 1/2, \quad \alpha_n \to 1/2, \quad n \to \infty. \tag{4.1}$$

The measure ν_a (with support $[a, b_a]$) will become a non-negative Borel measure μ_n^* with support $[\alpha_n, 1/\rho_n]$. From (3.22) and (3.21) we see that it enjoys the properties

$$\|\mu_n^*\| = \pi^{-2}\log^2(8/\rho_n) + \mathcal{O}(1), \quad \mu_n^*([\alpha_n, 1/2)) = \mathcal{O}(\rho_n^2). \tag{4.2}$$

Finally, after the substitution, the potential $-p_a(1/(4z))$ will be

$$L_n^*(z) := \int_{\alpha_n}^{1/\rho_n} \log\frac{|z-t|}{|z+t|} d\mu_n^*(t), \tag{4.3}$$

with the properties

$$L_n^*(x) = -\log(4x), \quad \alpha_n \le x \le 1/\rho_n; \quad L_n^*(iy) = 0, \quad y \in \mathbb{R}. \tag{4.4}$$

An additional small change of the measure μ_n^* is needed, to make its support equal to the support $[1/2, 1/\rho_n]$ of measure ν_n of Theorem 2.3. This will be achieved by the use of the balayage methods of §4 of Appendix 4. We sweep the measure μ_n^* out of the interval $[\alpha_n, 1/2)$ onto the boundary

$\partial G_n := i\mathbb{R} \cup [1/2, 1/\rho_n]$ of the domain $G_n := H_+ \setminus [1/2, 1/\rho_n]$. We denote the resulting measure by μ_n^{**}. It is supported on ∂G_n. Since $\int_{\alpha_n}^{1/2} \log|z-t| d\mu_n^*(t)$ is continuous on $\mathbb{C}$, we can apply Theorem 4.1 of Appendix 4: Green's potential $L_n^{**}(z)$ of the measure μ_n^{**} is continuous on $\mathbb{C}$ and satisfies

$$(4.5) \qquad L_n^{**}(z) := \int_{\partial G_n} \log \frac{|z-t|}{|z+t|} d\mu_n^{**}(t) = L_n^*(z) \quad \text{for all } z \in \partial G_n.$$

From (4.2) we have $\mu_n^{**}(i\mathbb{R}) \le \mu_n^*([\alpha_n, 1/2)) = \mathcal{O}(\rho_n^2) \to 0$. We denote the restriction of the measure μ_n^{**} onto $[1/2, 1/\rho_n]$ by μ_n. The new measure μ_n is supported on $[1/2, 1/\rho_n]$ and satisfies

$$(4.6) \qquad \|\mu_n\| = \|\mu_n^{**}\| + \mathcal{O}(\rho_n^2) = \|\mu_n^*\| + o(1).$$

Since $\log(|z-t|/|z+t|) = 0$ for $(z,t) \in \mathbb{R} \times i\mathbb{R}$ and for $(z,t) \in i\mathbb{R} \times \mathbb{R}$, $(z,t) \ne (0,0)$, it follows from (4.4) and (4.5) that the potential

$$(4.7) \qquad L_n(z) := \int_{1/2}^{1/\rho_n} \log \frac{|z-t|}{|z+t|} d\mu_n(t)$$

is a continuous function on $\mathbb{C}$ and attains the boundary values

$$(4.8) \qquad L_n(x) = -\log(4x), \quad 1/2 \le x \le 1/\rho_n, \quad L_n(iy) = 0, \quad y \in \mathbb{R}.$$

Assume that we have proved, for the measures μ_n and ν_n,

$$(4.9) \qquad \|\mu_n\| - \|\nu_n\| = \mathcal{O}(1).$$

Then Theorem 1.1 would follow immediately. Indeed, since $\|\nu_n\| = n+1$, we would have

$$(4.10) \qquad \|\mu_n\| = \pi^{-2}\log^2(8/\rho_n) + \mathcal{O}(1) = n + \mathcal{O}(1)$$

and also $\log(8/\rho_n) = \pi\sqrt{n + \mathcal{O}(1)}$. Since $\sqrt{n+\mathcal{O}(1)} = \sqrt{n} + \mathcal{O}(n^{-1/2})$, we would obtain

$$(4.11) \qquad \rho_n e^{\pi\sqrt{n}} = 8 + \mathcal{O}(n^{-1/2}).$$

This is the limit (1.4) with the remainder $\mathcal{O}(n^{-1/2})$.

Next we consider the function

$$(4.12) \qquad f(z) := \frac{1}{2z}(1 + \sqrt{1-4z^2}).$$

We select the branch of the square root with the property $\sqrt{1-4x^2} > 0$ for $-1/2 < x < 1/2$. This function maps $\mathbb{C} \setminus ([-\infty, -1/2] \cup [1/2, \infty])$ twice on the outside of the disk $|z| \le 1$. We shall use the function in order to majorize the difference $\log|\Phi_n(z)| - L_n(z)$, where $\Phi_n(z)$ is given by (2.15), and $L_n(z)$ by (4.7).

Lemma 4.1. *There exists a function v, harmonic in H_+ and continuous on $\overline{H_+} \setminus \{0\}$, and for each $n \geq 1$ a non-negative Borel measure ω_n on $[1/2, 1/\rho_n]$, with $\|\omega_n\| \leq 1/2$, so that the function*

$$q_n(z) := v(z) + \int_{1/2}^{1/\rho_n} \log \frac{|z-t|}{|z+t|} d\omega_n(t) \tag{4.13}$$

attains the boundary values

$$q_n(z) = \begin{cases} \log|f(z)| = 0 & \text{for } z \in [1/2, 1/\rho_n], \\ \log|f(z)| & \text{for } z \in i\mathbb{R}. \end{cases} \tag{4.14}$$

The function $q_n(z)$ is harmonic in $G_n = H_+ \setminus [1/2, 1/\rho_n]$ and continuous on $\overline{H_+} \setminus \{0\}$.

Proof. Let ν be the arcsin-measure $d\nu(x)/dx = \pi^{-1}(1-x^2)^{-1/2}$ on $[-1,1]$. Its negative logarithmic potential is

$$\int_{-1}^{1} \log|z-x| d\nu(x) = \log|z + \sqrt{z^2-1}| - \log 2, \quad z \in \mathbb{C}, \tag{4.15}$$

see Example 3 of Chapter 4. By the transformation $x = 1/(2t)$ we obtain a new measure ω of mass $\|\omega\| = \|\nu\| = 1$, supported on $I := (-\infty, -1/2] \cup [1/2, \infty)$. Replacing z by $1/(2z)$ in (4.15), we have

$$\begin{aligned} \log|f(z)| &= \log|1/(2z) + \sqrt{1/(4z^2) - 1}| \\ &= \log 2 + \int_I \log|1/(2z) - 1/(2t)| d\omega(t) \\ &= \int_I \log|z-t| d\omega(t) - \log|z| - c, \quad z \in \mathbb{C} \setminus I, \end{aligned} \tag{4.16}$$

where $c := \int_I \log|t| d\omega(t)$. Applying (4.15) for $z = 0$ we derive that $c = 0$:

$$c = \int_{-1}^{1} \log \frac{1}{2|x|} d\nu(x) = -\log 2 - \int_{-1}^{1} \log|x| d\nu(x) = 0. \tag{4.17}$$

For the measure ω we have

$$\int_{-\infty}^{-1/2} \log|z-t| d\omega(t) = \int_{1/2}^{\infty} \log|z+t| d\omega(t), \quad z \in \mathbb{C}. \tag{4.18}$$

Therefore,

$$\begin{aligned} \log|f(z)| &= \int_{1/2}^{\infty} (\log|z-t| + \log|z+t|) d\omega(t) - \log|z| \\ &= v(z) + \int_{1/2}^{\infty} \log \frac{|z-t|}{|z+t|} d\omega(t), \quad z \in \overline{H_+} \setminus \{0\}, \end{aligned} \tag{4.19}$$

where

$$(4.20) \qquad v(z) := 2\int_{1/2}^{\infty} \log|z+t|d\omega(t) - \log|z|$$

is harmonic in $\overline{H_+} \setminus \{0\}$.

Let further ω_+ be the restriction of ω to $[1/2, \infty)$. The measure ω_+ has the whole interval $[1/2, \infty)$ as its support, however we want q_n to be harmonic in $G_n = H_+ \setminus [1/2, 1/\rho_n]$. We can have this property without changing the values of $\log|f(z)|$ on ∂G_n by applying the technique of balayage to the measure ω_+. We can apply Theorem 4.1 of Appendix 4 since $\int_{1/\rho_n}^{\infty} \log|z-t|d\omega_+(t)$ is continuous on $\mathbb{C}$ and since $\int_{1/\rho_n}^{\infty} \log(t+1)d\omega_+(t) < \infty$. Let ω_n^* denote the measure of Theorem 4.1 of Appendix 4 that results from sweeping the measure ω_+ out of the domain G_n. The support of ω_n^* is contained in ∂G_n and

$$(4.21) \qquad \int_{\partial G_n} \log\frac{|z-t|}{|z+t|}d\omega_n^*(t) = \int_{1/2}^{\infty} \log\frac{|z-t|}{|z+t|}d\omega_+(t), \quad z \in \partial G_n.$$

Finally, let the measure ω_n of the theorem be the restriction of ω_n^* to $[1/2, 1/\rho_n]$. As required, $\|\omega_n\| \leq \|\omega_n^*\| = \omega([1/2,\infty)) = 1/2$. By Corollary 4.2 of Appendix 4,

$$(4.22) \qquad \int_{1/2}^{1/\rho_n} \log\frac{|z-t|}{|z+t|}d\omega_n(t) = \int_{1/2}^{\infty} \log\frac{|z-t|}{|z+t|}d\omega_+(t), \quad z \in \partial G_n.$$

On $[1/2, \infty)$, $\omega_+ = \omega$. Thus from (4.19) and the definition of q_n we see that

$$(4.23) \qquad \log|f(z)| = q_n(z), \quad z \in \partial G_n.$$

This yields (4.14), and the continuity of the first integral of (4.22) on $\mathbb{C}$ follows from Theorem 4.1 of Appendix 4. □

Corollary 4.2. *The function $q_n(z)$ of Theorem 4.1 has the limits $q_n(z) \to 0$ for $z \to \infty$, and $q_n(z) + \log|z| \to 0$ for $z \to 0$, $z \in \overline{H_+}$.*

Proof. By the definition (4.20) of $v(z)$, $v(z) \to 0$ for $z \to \infty$ since $\omega([1/2,\infty)) = 1/2$. The integral in (4.13) converges also to 0 for $z \to \infty$, hence $q_n(z)$ has the same property.

The integral in (4.13) is continuous on $\mathbb{C}$ and vanishes at $z = 0$, while $v(z) + \log|z|$ converges to zero for $z \to 0$ since the integral in (4.20) vanishes at $z = 0$ by (4.17). This proves the second limit of the corollary. □

As the last step towards Theorem 1.1 we prove:

Theorem 4.3. *The measures ν_n of §2 and μ_n of §4 satisfy*

$$(4.24) \qquad \|\mu_n\| - \|\nu_n\| = \mathcal{O}(1).$$

Proof. We introduce the function

$$d_n(z) := \log|\Phi_n(z)| - L_n(z). \tag{4.25}$$

By the properties of $\Phi_n(z)$ in Proposition 2.2 and Theorem 2.3 and by (4.7) and (4.8), we deduce that $d_n(z)$ is continuous on $\overline{H_+} \setminus \{0\}$ and satisfies

$$|d_n(iy)| = |\log|\Phi_n(iy)|| \le c\log(1+y^{-2}), \quad y \in \mathbb{R}, \tag{4.26}$$

$$d_n(x) = 0, \quad 1/2 \le x \le 1/\rho_n, \tag{4.27}$$

$$d_n(z) = h_n(z) + \int_{1/2}^{1/\rho_n} \log\frac{|z-t|}{|z+t|}(d\nu_n - d\mu_n)(t), \quad z \in H_+, \tag{4.28}$$

where h_n is the harmonic function in H_+ of Theorem 2.3 and where μ_n and ν_n are the measures with supports $[1/2, 1/\rho_n]$.

On the imaginary axis,

$$\log|f(iy)| \ge \frac{1}{2}\log(1+(2y)^{-2}) \ge \frac{1}{8}\log\left(1+y^{-2}\right). \tag{4.29}$$

By (4.26) and (4.29), there exists a constant $c_1 > 0$ such that

$$c_1 \log|f(iy)| \ge |d_n(iy)|, \quad y \in \mathbb{R}, \tag{4.30}$$

that is,

$$c_1 q_n(iy) \pm d_n(iy) \ge 0, \quad y \in \mathbb{R}. \tag{4.31}$$

Since both functions q_n and d_n vanish on $[1/2, 1/\rho_n]$, we have

$$c_1 q_n(x) \pm d_n(x) = 0, \quad x \in [1/2, 1/\rho_n]. \tag{4.32}$$

Both functions $c_1 q_n(z) \pm d_n(z)$ are continuous on $\overline{H_+} \setminus \{0\}$. They are also bounded from below on H_+ for sufficiently large c_1. Indeed, the potential $L_n(z)$ converges to zero for $z \to \infty$ and for $z \to 0$; from Proposition 2.2 (iii) it follows that $\log|\Phi_n(z)| \to 0$ for $z \to \infty$, $\log|\Phi_n(z)| + \log|z| \to \log(1/4)$ for $z \to 0$. This and Corollary 4.2 imply that both functions $c_1 q_n(z) \pm d_n(z)$ converge to zero for $z \to \infty$ and diverge to $+\infty$ for $z \to 0$.

We would like to apply now Theorem 2.11 of Appendix 4, the maximum modulus principle, for the domain H_+, though the boundary $\partial H_+ = i\mathbb{R}$ is not compact. We may do so after we have transformed H_+ by the transformation $w = 1/(z+1)$ into the bounded (and simply connected) domain Ω of Appendix 4, §4. Then the functions $u_\pm(w) := [c_1 q_n \pm d_n](z(w))$ are harmonic in Ω, continuous on $\overline{\Omega}$ except at $w(0) = 1$ and $w(\infty) = 0$, and bounded from below in Ω. Since

$$\liminf_{w \to \xi, w \in \Omega} u_\pm(w) = u_\pm(\xi) \ge 0$$

for all $\xi \in \partial\Omega \setminus \{0; 1\}$, an application of Theorem 2.11 of Appendix 4 yields that $u_\pm(w) \ge 0$ for all $w \in \Omega$, hence

$$c_1 q_n(z) \pm d_n(z) \ge 0 \quad \text{for all } z \in \overline{H_+} \setminus \{0\}. \tag{4.33}$$

By (4.32) and (4.33), the functions $c_1 q_n(z) \pm d_n(z)$ are subharmonic on $[1/2, 1/\rho_n]$, hence subharmonic in H_+. They have the representation

$$c_1 q_n(z) \pm d_n(z) = c_1 v(z) \pm h_n(z) + \int_{1/2}^{1/\rho_n} \log \frac{|z-t|}{|z+t|} (c_1 d\omega_n \pm [d\nu_n - d\mu_n])(t).$$

Moreover, the functions

$$c_1 v(z) \pm h_n(z) - \int_{1/2}^{1/\rho_n} \log |z+t| (c_1 d\omega_n \pm [d\nu_n - d\mu_n])(t)$$

are harmonic in H_+. Therefore, the functions

$$\int_{1/2}^{1/\rho_n} \log |z-t| (c_1 d\omega_n \pm [d\mu_n - d\nu_n])(t) \tag{4.34}$$

are subharmonic in H_+, even subharmonic in $\mathbb{C}$ since they are harmonic in $\mathbb{C} \setminus [1/2, 1/\rho_n]$. From Theorem 1.11 of Appendix 4 it follows that the two measures $c_1\omega_n + [\mu_n - \nu_n]$ and $c_1\omega_n - [\mu_n - \nu_n]$ are non-negative. Since $\|\omega_n\| \le 1/2$, we have $-c_1/2 \le \|\mu_n\| - \|\nu_n\| \le c_1/2$. □

§ 5. Notes

5.1. For even n, Stahl [1994] gives asymptotic formula for the zeros $\zeta_{j,n}$ and the poles $\pi_{j,n}$ of the best rational approximants $r_n^*(x)$ to $|x|$ on $[-1, 1]$. There are exactly n of them in each group, they lie on the imaginary axis, symmetrical about the origin, interlacing each other. For larger n, there is an extreme crowding of the points $\zeta_{j,n}$, $\pi_{j,n}$ to the origin. One of the simpler corollaries of Stahl's estimates is the asymptotic formula, for $0 < c < d < \infty$,

$$\lim_{n\to\infty} n^{-1/2} N_n(c, d) = \frac{1}{\pi} \int_c^d \frac{dt}{t\sqrt{1+t^2}}, \tag{5.1}$$

where $N_n(c, d)$ is the number of $\zeta_{j,n}$ contained in $[ic, id]$.

5.2. In Stahl [1993], the author announces the strong estimate for the rational approximation of x^α on $[0, 1]$, namely

$$\lim_{n\to\infty} e^{2\pi\sqrt{\alpha\pi}} \rho_n(x^\alpha) = 4^{1+\alpha} |\sin \pi\alpha|, \qquad \alpha > 0. \tag{5.2}$$

This is a generalization of Theorem 1.1.

Chapter 9. Padé Approximation

§ 1. The Padé Table

For a function $f(z) = \sum_{j=0}^{\infty} a_j z^j$ with a positive radius of convergence for the series, we have $f(z) = P_m(z) + \mathcal{O}(|z|^{m+1})$ for $z \to 0$, where the polynomial $P_m(z) = \sum_{j=0}^{m} a_j z^j$ depends on $m+1$ parameters a_j. A natural generalization would be a rational function $P_m(z)/Q_n(z) \in \mathcal{R}_{m,n}$ which has $m+n+1$ free parameters: we want P_m/Q_n to have its Maclaurin expansion at $z = 0$, whose first $m+n+1$ coefficients coincide with those of f.

In this section we do not discuss convergence questions (of P_m/Q_n to f). It is therefore possible to replace functions f by *formal* power series $f(z) := \sum_{j=0}^{\infty} a_j z^j$. We then write $f(z) = \mathcal{O}(z^N)$, $N = 1, 2, \ldots$ to mean that $a_0 = \ldots = a_{N-1} = 0$, and $f(z) = \mathcal{O}(z^\infty)$ if $f(z)$ is identically zero. If $f(z)$ is a power series with leading coefficient $a_0 \neq 0$, then also $1/f(z)$ has these properties. If f is of this type, and g an arbitrary power series, then relations $g(z) = \mathcal{O}(z^N)$ and $f(z)g(z) = \mathcal{O}(z^N)$ are equivalent. With Padé [1892], we can define the main subject of this chapter.

A *Padé approximant* $[m/n]_f := [m/n]$ *of a power series* $f(z)$ *of orders* $m, n = 0, 1, \ldots$ is any rational function P/Q, $P \in \mathcal{P}_m$, $Q \in \mathcal{P}_n$ with Q not identically zero which satisfies

$$f(z)Q(z) - P(z) = \mathcal{O}(z^{m+n+1}). \tag{1.1}$$

If $a_0 \neq 0$, then (1.1) and $P(z)/f(z) - Q(z) = \mathcal{O}(z^{m+n+1})$ are equivalent, and we have the symmetry property

$$[m/n]_f = [n/m]_{1/f}, \quad m, n = 0, 1, \ldots \quad . \tag{1.2}$$

Theorem 1.1. *The Padé approximant* $[m/n] = P/Q$ *for a power series* f, *and* $m, n = 0, 1, \ldots$ *exists and is unique. If* P^*/Q^* *is its irreducible representation, then* $Q^*(0) \neq 0$ *and*

$$f(z) - [m/n](z) = \mathcal{O}(z^{m+n+1-d}), \tag{1.3}$$

where d *is the defect of* P/Q:

$$d = \min(m - \deg P^*, n - \deg Q^*). \tag{1.4}$$

Proof. Relation (1.1) is nothing but a system of $m+n+1$ homogeneous linear equations in the $m+n+2$ coefficients of P and Q. So there is a solution (P, Q) with at most one of the P, Q equal to zero. If $Q = 0$, then (1.1) yields $-P(z) = \mathcal{O}(z^{m+n+1})$, so that $P = 0 = Q$, a contradiction. Thus $[m/n] = P/Q$ exists. If also $[m/n] = P_1/Q_1$, then we have

$$QP_1 - PQ_1 = Q(P_1 - fQ_1) + Q_1(fQ - P) = \mathcal{O}(z^{m+n+1}),$$

hence $QP_1 - PQ_1 = 0$, as it is a polynomial of degree $\leq m+n$. So $P/Q = P_1/Q_1$, and $[m/n]$ is unique.

Let $[m/n] = P/Q$, then P, Q have the representations $P(z) = z^r S(z) P^*(z)$, $Q(z) = z^r S(z) Q^*(z)$, where P^*, Q^* have no common factors, r is a nonnegative integer and $S(z)$ is a polynomial with $S(0) \neq 0$. Then

$$z^r S(z)(fQ^* - P^*)(z) = (fQ - P)(z) = \mathcal{O}(z^{m+n+1}),$$

$$(fQ^* - P^*)(z) = \mathcal{O}(z^{m+n+1-r}) \tag{1.5}$$

as $S(0) \neq 0$. If $Q^*(0) = 0$, then also $P^*(0) = 0$, by (1.5), contradicting the fact that P^*, Q^* have no common factors. Hence $Q^*(0) \neq 0$.

Next $m \geq \deg P = r + \deg S + \deg P^* \geq r + \deg P^*$. So $r \leq m - \deg P^*$. Similarly, $r \leq n - \deg Q^*$, hence $r \leq d$, where d is given by (1.4). This yields (1.3). □

It follows from this theorem that $[m/n]_f$ cannot have a pole at $z = 0$. Indeed, the irreducible representation $[m/n]_f = P^*/Q^*$ cannot have $Q^*(0) = 0$.

It may seem more natural to replace the definition of the Padé approximant by the requirement that P/Q is a power series in z (for otherwise it is unbounded for $z \to 0$) and that

$$f(z) - \frac{P(z)}{Q(z)} = \mathcal{O}(z^{m+n+1}). \tag{1.6}$$

But P/Q with these properties does not necessarily exist. For example, for $f(z) = 1 + z^2$, $m = n = 1$, $P(z)/Q(z) = (b_0 + b_1 z)/(c_0 + c_1 z)$. If $c_0 = 0$, then $b_0 = 0$ (otherwise P/Q is not a power series), and (1.6) has no solution. If $c_0 \neq 0$ then (1.6) is equivalent to $(1 + z^2)(c_0 + c_1 z) - (b_0 + b_1 z) = \mathcal{O}(z^3)$, which has no solution.

The *Padé table* of a formal power series $f(z)$ is the array of rational functions

$$\begin{array}{llll} [0/0], & [1/0], & [2/0], & \ldots \\ [0/1], & [1/1], & [2/1], & \ldots \\ [0/2], & [1/2], & [2/2], & \ldots \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots} \end{array}$$

The first two rows of the Padé table can be evaluated explicitly.

1. The elements of the 0-th row of the table are

$$[m/0]_f(z) = \left(\sum_{j=0}^{m} a_j z^j\right)/1, \quad m = 0,1,\dots \quad . \tag{1.7}$$

2. If $a_m \neq 0$ and $q := a_{m+1}/a_m$, then one has

$$\left(\sum_{j=m}^{\infty} a_j z^j\right)(1-qz) - a_m z^m = \mathcal{O}(z^{m+2}),$$

hence

$$[m/1]_f(z) = \sum_{j=0}^{m-1} a_j z^j + \frac{a_m z^m}{1-(a_{m+1}/a_m)z} \quad m = 0,1,\dots \quad . \tag{1.8}$$

If $a_m = 0$, then $[m/1]_f(z) = \sum\limits_{j=0}^{m} a_j z^j$, for one has (1.1) with $P(z) = z\sum\limits_{j=0}^{m} a_j z^j \in \mathcal{P}_m$, $Q(z) = z \in \mathcal{P}_1$.

3. It follows from **2** that if $a_m = 0$ for some m, then the table contains *a square* of four equal approximants:

$$[m-1/0] = [m/0] = [m-1/1] = [m/1]. \tag{1.9}$$

More generally, one can give explicit formulas for $P_m(z) = \sum\limits_{j=0}^{m} b_j z^j$, $Q_n(z) = \sum\limits_{j=0}^{n} c_j z^j$ in (1.1), if certain determinants do not vanish.
Relation (1.1) is equivalent to:

$$\begin{pmatrix} a_m & \cdots & a_{m+n-1} & a_{m+n} \\ \vdots & \ddots & \vdots & \vdots \\ a_{m-n+1} & \cdots & a_m & a_{m+1} \end{pmatrix} \begin{pmatrix} c_n \\ \vdots \\ c_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \tag{1.10}$$

together with

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_m \\ 0 & a_0 & \cdots & a_{m-1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_0 \end{pmatrix} \begin{pmatrix} c_m \\ c_{m-1} \\ \vdots \\ c_0 \end{pmatrix} = \begin{pmatrix} b_m \\ b_{m-1} \\ \vdots \\ b_0 \end{pmatrix}. \tag{1.11}$$

(We put here $a_k = 0$ if $k < 0$.)

If the determinant

$$D := D(m/n) := \begin{vmatrix} a_m & \cdots & a_{m+n-1} \\ \vdots & \ddots & \vdots \\ a_{m-n+1} & \cdots & a_m \end{vmatrix} \tag{1.12}$$

is not zero, (1.10) is solvable for $c_1, \dots, c_n$ for all values of c_0. We put $c_0 = 1$ and solve (1.10) for the remaining c_j using Cramer's rule:

$$c_{n-j} = (-1)^{n-j} D_j / D, \quad j = 0, \dots, n,$$

where the D_j are the appropriate minors of the coefficient matrix in (1.10). We get

$$Q_n(z) = \frac{1}{D} \begin{vmatrix} a_m & a_{m+1} & \cdots & a_{m+n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m-n+1} & a_{m-n+2} & \cdots & a_{m+1} \\ z^n & z^{n-1} & \cdots & 1 \end{vmatrix}. \tag{1.13}$$

With the use of (1.13) and (1.12) we obtain also

$$P_m(z) = \frac{1}{D} \begin{vmatrix} a_m & a_{m+1} & \cdots & a_{m+n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m-n+1} & a_{m-n+2} & \cdots & a_{m+1} \\ \sum_{k=n}^{m} a_{k-n} z^k & \sum_{k=n-1}^{m} a_{k-n+1} z^k & \cdots & \sum_{k=0}^{m} a_k z^k \end{vmatrix}. \tag{1.14}$$

Theorem 1.2 (Padé [1892]). *The Padé table of $f(z)$ consists of square blocks of (not necessarily equal) sizes r, ($1 \le r \le \infty$), with the following properties:*

(i) *All elements in a square block are identical.*

(ii) *Elements of a block are different from all other approximants.*

(iii) *If $[m^*/n^*] = P^*/Q^*$ is the top left-hand corner of an $r \times r$ block, then $\deg P^* = m^*$, $\deg Q^* = n^*$, $Q^*(0) \neq 0$, and*

$$f(z) - [m^*/n^*](z) = \mathcal{O}(z^{m^*+n^*+r}) \tag{1.15}$$

(if $r = \infty$, we take this to mean that $f - [m^*/n^*] = 0$).

Proof. We pick any $[m/n]$ in the table. By Theorem 1.1, we can write $[m/n] = P^*/Q^*$, where P^*, Q^* have no common factors, $Q^*(0) \neq 0$, and

$$f(z) - [m/n](z) = cz^N + \mathcal{O}(z^{N+1}), \tag{1.16}$$

with $c \neq 0$. If d is given by (1.4), then

$$N \ge m + n + 1 - d.$$

Here possibly $N = \infty$. We write

$$m^* := \deg P^*, \quad n^* := \deg Q^*.$$

Case 1: $N = \infty$. Then $(fQ^* - P^*)(z) \equiv 0$, and it follows that

$$[m^* + j/n^* + k] = [m^*/n^*] = P^*/Q^*, \quad j, k \geq 0.$$

We have an $\infty \times \infty$ block with the top left-hand corner $[m^*/n^*]$. Now if $[m_1/n_1]$ is outside this block, then either $m_1 < m^* = \deg P^*$ or $n_1 < n^* = \deg Q^*$, so that

$$[m_1/n_1] \not\equiv [m^*/n^*].$$

Thus, (i), (ii), (iii) are true in this case.

Case 2: $N < \infty$. Here

$$N \geq m + n + 1 - d = 1 + \max(m^* + n, m + n^*).$$

Hence $N = m^* + n^* + r$ with some $r \geq 1$. Let $0 \leq j, k \leq r - 1$ and $s := \min(j, k)$. Then $z^s P^*(z)$, $z^s Q^*(z)$ have degree at most $m^* + j$, $n^* + k$ respectively and

$$\begin{aligned} f(z)(z^s Q^*(z)) - z^s P^*(z) &= \mathcal{O}(z^{N+s}) = \mathcal{O}(z^{m^*+n^*+r+s}) \\ &= \mathcal{O}(z^{(m^*+j)+(n^*+k)+1}), \end{aligned}$$

since $r + s \geq 1 + \max(j, k) + \min(j, k) = 1 + j + k$. So

$$[m^* + j/n^* + k] = z^s P^*(z)/\left(z^s Q^*(z)\right) = [m^*/n^*], \quad 0 \leq j, k \leq r - 1.$$

Thus we have an $r \times r$ block, with the top left-hand corner $[m^*/n^*]$, and in view of (1.16) and the definition of r, we have (1.15). So (i), (iii) are true.

It remains to prove (ii). If $[m_1/n_1]$ is outside of the $r \times r$ block, then for $m_1 < m^*$ or $n_1 < n^*$, we cannot have $[m_1/n_1] = [m^*/n^*]$. So suppose that $[m_1/n_1] = [m^*/n^*]$ for some $m_1 \geq m^*$ and $n_1 \geq n^*$. Then by the argument of Theorem 1.1,

$$f(z) - [m^*/n^*](z) = f(z) - [m_1/n_1](z) = \mathcal{O}(z^{m_1+n_1+1-d_1}),$$

where $d_1 = \min(m_1 - m^*, n_1 - n^*)$. We deduce that

$$\begin{aligned} m^* + n^* + r = N &\geq m_1 + n_1 + 1 - d_1 \\ &\geq 1 + \max(m^* + n_1, n^* + m_1). \end{aligned}$$

It follows that

$$m_1 \leq m^* + r - 1, \quad n_1 \leq n^* + r - 1.$$

Then $[m_1/n_1]$ is contained in our $r \times r$ block. This is a contradiction. □

Corollary 1.3 (Kronecker). *For a power series f, the following assertions are equivalent.*

(i) *$f(z)$ is the Maclaurin series of a rational function $R = P/Q$ where* $\deg P = m$, $\deg Q = n$, *$Q(0) \neq 0$, and P, Q have no common factors.*

(ii) *There is an $\infty \times \infty$ block in the Padé table of f with the top left-hand corner $[m/n]$.*

One sometimes calls a fraction $[m/n]$ of a Padé table *normal*, if it does not belong to any block of side length > 1. For example, all Padé approximants of e^z are normal (see Perron [B-1957, p. 245]).

Corollary 1.4. *A Padé approximant $[m/n]$ is normal if and only if it is equal to an irreducible fraction P/Q, $\deg P = m$, $\deg Q = n$, and if $f(z) - [m/n](z) = \mathcal{O}(z^N)$ is true only for $N = m + n + 1$, but not for any larger N.*

The first condition implies that $[m/n]$ is the top left corner of its square, the while the last means that $r = 1$.

An interesting problem of Trefethen asks to describe all possible patterns (decompositions of a quadrant into square blocks) that can arise in a Padé table.

Among books on Padé approximation, we shall mention Perron [B-1957], Baker [A-1975], and Baker and Graves-Morris [A-1981].

§ 2. Convergence of the Rows of the Padé Table

If $f(z)$ is analytic at $z = 0$, one can hope that $[m/n]_f(z)$ converges for $m+n \to \infty$ in some region G of the complex plane containing zero. If a uniform limit $g(z)$ of a subsequence exists, then $g = f$, for $g - f$ will have all its derivatives at 0 equal to zero.

An example is provided by the function $f(z) = e^z$, when G is the whole complex plane. Indeed, from (4.4) of Chapter 7 it follows that for each $M > 0$,

$$|e^z - [m/n]_f(z)| \le M^{m+n+1} \frac{1}{(m+n)!} e^{2M}(1 + o(1)), \quad m + n \to \infty,$$

uniformly in the disk $|z| \le M$. The convergence of $[m/n]_f(z)$ to e^z can be also derived from Problems 4.2-4.5 of §4.

Padé "approximation" is very closely related to Hermite interpolation at 0 of order $m+n+1$, and like other interpolation processes, it has bad convergence properties. Here is a striking example.

Theorem 2.1 (Perron's Example). *There exist entire functions f with the property that the poles of the approximants $[m/1]_f$, $m = 1, 2, \ldots$ lie everywhere dense in $\mathbb{C}$; consequently, the $[m/1]_f$ cannot converge uniformly on any open portion of $\mathbb{C}$.*

Proof. We choose a sequence $(u_j)_1^\infty$ dense in $\mathbb{C}$, each point repeated infinitely often in the sequence, with $0 < |u_j| \le j, \quad j = 1, 2, \ldots$. Let

$$f(z) = \sum_{j=0}^{\infty} a_j z^j, \quad a_{2m} := \frac{u_m}{(2m+1)!}, \quad a_{2m+1} := \frac{1}{(2m+1)!}, \quad m = 0, 1, \ldots.$$

Then f is entire, and the formula (1.8) shows that $[2m/1]_f$ has a pole at u_m for infinitely many m. □

This illustrates a general fact, which appears in the classical theorem of de Montessus de Ballore. The theorem asserts that $[m/n]_f(z) \to f(z)$ for $m \to \infty$ and a fixed n, if f has exactly n poles. The proof is based on a Hermite type remainder formula for $fQ-P$. In order to emphasise the dependence on m, n of P, Q in $[m/n]_f(z) = P(z)/Q(z)$, we shall now write, for each possible selection of P and Q, $P =: P_{m,n}$, $Q =: Q_{m,n}$.

Theorem 2.2. *Let f be analytic in $|z| \le r$ $(0 < r < \infty)$, except, possibly, for l poles $z_j \neq 0$, $j = 1, \ldots, l$, repeated according to multiplicity. Let*

$$S(z) = \prod_{j=1}^{l} (z - z_j).$$

(If f has no poles, $S := 1$.) Then for $|z| < r$, $n \ge l$ and $m = 1, 2, \ldots$

$$(2.1) \qquad (fQ_{m,n} - P_{m,n})(z)S(z) = \frac{1}{2\pi i} \int_{|t|=r} \frac{(fQ_{m,n}S)(t)}{t - z} \left(\frac{z}{t}\right)^{m+n+1} dt.$$

Proof. The function $(fQ_{m,n} - P_{m,n})(z)S(z)/z^{m+n+1}$ is analytic for $|z| \le r$, so by Cauchy's integral formula

$$\frac{(fQ_{m,n} - P_{m,n})(z)S(z)}{z^{m+n+1}} = \frac{1}{2\pi i} \int_{|t|=r} \frac{(fQ_{m,n} - P_{m,n})(t)S(t)}{(t - z)t^{m+n+1}} dt, \quad |z| < r.$$

For a fixed z, $|z| < r$, $(P_{m,n}S)(t)/((t - z)t^{m+n+1})$ is a rational function of t, analytic for $|t| \ge r$, with the numerator degree $\le m + n$, denominator degree $= m + n + 2$. For $R > r$, Cauchy's integral theorem gives

$$\begin{aligned} I := \int_{|t|=r} \frac{(P_{m,n}S)(t)}{(t - z)t^{m+n+1}} dt &= \int_{|t|=R} \frac{(P_{m,n}S)(t)}{(t - z)t^{m+n+1}} dt \\ &= \mathcal{O}\left(2\pi R \, \frac{1}{R^2}\right) = \mathcal{O}\left(\frac{1}{R}\right), \quad R \to \infty. \end{aligned}$$

So $I = 0$. Hence the result. □

Now let $\max_{|t| \le r} |Q_{m,n}(t)| \le M$. From the integral formula (2.1), we get for $|z| \le \rho$, $\rho < r$:

$$(2.2) \qquad |f(z)S(z)Q_{m,n}(z) - S(z)P_{m,n}(z)| \le CM \left(\frac{\rho}{r}\right)^{m+n+1},$$

where C depends only on the function f and on $r/(r - \rho)$. If we want to estimate $f(z) - [m/n](z)$ on some set K, we have to divide (2.2) by $SQ_{m,n}$, and have then to prove that if the normalization of the $Q_{m,n}$ has been taken properly, this product is bounded from below on K. This leads to the theorem below.

Let $f, f_1, f_2, \dots$ be some analytic functions, with f having zeros (poles) at $z_1, \dots, z_n$ of multiplicities $\mu_1, \dots, \mu_n$. We say that the zeros (poles) of f_ν converge to those of f, if for every $j = 1, \dots, n$ and every sufficiently small circle C_j, centered at z_j, the function $f_\nu(z)$ has exactly μ_j zeros (poles) in C_j, counting multiplicities, as soon as ν is sufficiently large.

Theorem 2.3 (de Montessus de Ballore [1902]). *Let f be analytic for $|z| \le r$ $(0 < r < \infty)$, except, possibly, for the poles z_j, $0 < |z_j| < r$, $j = 1, \dots, n$, repeated according to their multiplicities. Then as $m \to \infty$, the poles of $[m/n]$ converge to the poles of f, and if K is a compact subset of $|z| < \rho$, $\rho < r$, that does not contain any poles of f, then*

$$\limsup_{m\to\infty} \left(\max_{z\in K} |f(z) - [m/n](z)| \right)^{1/m} \le \frac{\rho}{r} < 1. \tag{2.3}$$

Proof. (by Saff). We first take an r_1, $0 < r_1 < r$ with the property that $|z_j| < r_1$, $j = 1, \dots, n$, and show that the zeros of $Q_{m,n}(z)$ converge to those of $S(z) := \prod_1^n (z - z_j)$. We normalize the polynomials $Q_{m,n}(z)$ to have $\max\limits_{|t|=r} |Q_{m,n}(t)| = 1$. Since the $Q_{m,n}(z)$ are polynomials of degree $\le n$ and since they are bounded for $|z| \le r$, they form a compact set and hence contain a uniformly convergent subsequence $Q_{m,n} \to Q \in \mathcal{P}_n$, with $\max\limits_{|t|=r} |Q(t)| = 1$. Now fS is analytic in $|z| \le r$, and uniformly for $|z| \le r_1$, for $m \to \infty$ through the subsequence

$$\lim_{m\to\infty} S(z)P_{m,n}(z) = f(z)S(z)Q(z). \tag{2.4}$$

Since $SP_{m,n}$ are analytic, it follows from (2.4) and a well-known result about convergent sequences of analytic functions that their derivatives $(SP_{m,n})^{(k)}$ converge to $(fSQ)^{(k)}$, $k = 1, 2, \dots$, uniformly on each compact subset of $|z| < r_1$. This implies that for fSQ, the point z_j is a zero of at least the same multiplicity as it is for S. But fS does not vanish at $z = z_j$, hence we have

$$Q = cS, \quad \text{for some } c \ne 0. \tag{2.5}$$

The same argument shows that any subsequence of $(Q_{m,n})_{m=1}^\infty$ contains another subsequence whose zeros converge to those of S. Then it follows that as $m \to \infty$, the zeros of $Q_{m,n}$ converge to those of S.

Let now $Q^*_{m,n}$ be for each m the monic constant multiple of $Q_{m,n}$. It follows from the above that

$$\lim_{m\to\infty} Q^*_{m,n} = S(z),$$

uniformly on compact subsets of $\mathbb{C}$.

For an arbitrary $0 < \rho < r$, let K be a compact subset of $|z| < \rho$ which does not contain any of the z_j. Then $|Q^*_{m,n}(z)| \geq c > 0$ on K for m large enough. Dividing by $SQ_{m,n}$ in (2.2), we get

$$\max_{z\in K} |f(z) - [m/n](z)| \leq C(\rho/r)^m,$$

where C is independent of m. This yields (2.3) □

Another way to escape troubles of Perron's example is to assume with Lubinsky [1987] that the coefficients of the entire function f behave in a certain regular way. This approach depends on asymptotic estimates of determinants (1.12) and (1.13). We first develop some identities for determinants.

For a square matrix A we denote by $A_{r,p}$ (or by $A_{r,s;p,q}$ with $r < s$, $p < q$) the matrix obtained from A by deleting the r-th row and p-th column (respectively, the r-th and s-th rows and p-th and q-th columns).

Theorem 2.4 (Sylvester). *For the determinants of these matrices one has*

$$\det A \ \det A_{r,s;p,q} = \det A_{r,p} \ \det A_{s,q} - \det A_{r,q} \ \det A_{s,p}. \tag{2.6}$$

Proof. Let A be an $(n+2) \times (n+2)$ matrix, let $r = p = n+1$, $s = q = n+2$. If $B = A_{r,s;p,q}$ and if $a, \ldots, h$ are matrices of proper size, we get

$$A = \begin{pmatrix} B & a & b \\ c & d & e \\ f & g & h \end{pmatrix}.$$

For the determinant of the following $(2n+2) \times (2n+2)$ matrix we have

$$\begin{vmatrix} B & a & b & 0 \\ c & d & e & 0 \\ f & g & h & f \\ 0 & 0 & 0 & B \end{vmatrix} = \begin{vmatrix} B & a & b & 0 \\ c & d & e & 0 \\ f & g & h & f \\ B & a & b & B \end{vmatrix} = \begin{vmatrix} B & a & b & 0 \\ c & d & e & 0 \\ 0 & g & h & f \\ 0 & a & b & B \end{vmatrix}$$

$$= \begin{vmatrix} B & a & b & 0 \\ c & d & e & 0 \\ 0 & 0 & h & f \\ 0 & 0 & b & B \end{vmatrix} + \begin{vmatrix} B & 0 & b & 0 \\ c & 0 & e & 0 \\ 0 & g & h & f \\ 0 & a & b & B \end{vmatrix}$$

$$= \begin{vmatrix} B & a \\ c & d \end{vmatrix} \begin{vmatrix} h & f \\ b & B \end{vmatrix} - \begin{vmatrix} B & b \\ c & e \end{vmatrix} \begin{vmatrix} g & f \\ a & B \end{vmatrix}.$$

In other words, we have (2.6). The general case follows by rearrangement of rows and columns. □

For a power series $f(z) = \sum_0^\infty a_j z^j$ we denote by $V_{m,n}(z)$ the determinant (1.13). If the determinant $D(m/n) \neq 0$, we define

$$Q_{m,n}(z) := V_{m,n}(z)/D(m/n). \tag{2.7}$$

This is nothing but the denominator of the fraction $[m/n](z)$, normalized by $Q_{m,n}(0) = 1$.

Lemma 2.5. *One has the identities*

$$D(m/n+1)D(m/n-1) = D(m/n)^2 - D(m+1/n)D(m-1/n), \tag{2.8}$$

and if $D(m-1/n-1)$, $D(m/n-1)$, and $D(m/n)$ do not vanish, then

$$Q_{m,n}(z) = Q_{m,n-1}(z) - zQ_{m-1,n-1}(z)\frac{D(m-1/n-1)D(m+1/n)}{D(m/n-1)D(m/n)}. \tag{2.9}$$

Proof. Applying (2.6) to the $(n+1) \times (n+1)$ matrix

$$\begin{pmatrix} a_m & \cdots & a_{m+n} \\ \vdots & \ddots & \vdots \\ a_{m-n} & \cdots & a_m \end{pmatrix}$$

with $r = p = 1$, $s = q = n+1$ immediately yields (2.8). Applying (2.6) to the $(n+1) \times (n+1)$ matrix of which $V_{m,n}(z)$ is the determinant yields

$$V_{m,n}(z)D(m/n-1) = V_{m,n-1}(z)D(m/n) - zV_{m-1,n-1}(z)D(m+1/n).$$

Dividing by $D(m/n-1)D(m/n)$ then yields (2.9). □

We define the polynomials B_n associated with a number q *inductively* by $B_0(u) = 1$ and

$$B_n(u) := B_{n-1}(u) - uq^{n-1}B_{n-1}(u/q); \tag{2.10}$$

if $q = 0$, we set $B_n(u) := \sum_{j=0}^{n}(-1)^j u^j$.

Theorem 2.6 (Lubinsky [1987]). *Let $f(z) = \sum_0^\infty a_j z^j$ be a power series with $a_j \neq 0$ for large enough j, and for some $q \in \mathbb{C}$,*

$$\lim_{j\to\infty} \frac{a_{j-1}a_{j+1}}{a_j^2} = q. \tag{2.11}$$

If q is a root of unity, that is, if $q^k = 1$ for some $k \in \mathbb{N}$, we assume also that $q_m := a_{m-1}a_{m+1}/a_m^2$ has an asymptotic expansion: For each $L \in \mathbb{N}$,

$$q_m = q\left(1 + \alpha_1 m^{-1} + \alpha_2 m^{-2} + \cdots + \alpha_L m^{-L} + o(m^{-L})\right), \quad m \to \infty, \tag{2.12}$$

where $\alpha_1 \neq 0$, and $\alpha_1, \alpha_2, \ldots \in \mathbb{C}$. Then

(i) *For $n = 1, 2, \ldots,$*

$$\lim_{m\to\infty} D(m/n)/\{a_m^n \prod_{j=1}^{n-1}(1-q_m^j)^{n-j}\} = 1. \tag{2.13}$$

(ii) *For* $n = 1, 2, \ldots,$

$$\lim_{m\to\infty} Q_{m,n}(ua_m/a_{m+1}) = B_n(u), \tag{2.14}$$

uniformly on compact subsets of $\mathbb{C}$.

Proof. (i) Firstly,

$$D(m/1) = a_m,$$
$$D(m/2) = a_m^2(1-q_m),$$

so the assertion (2.13) is trivially true for $n = 1, 2$. Assume now that we have proved (2.13) for $1, \ldots, n$. We take m be so large that $a_m \neq 0$ and $q_m^j \neq 1$, $j = 1, \ldots, n$, and want to prove by induction that (2.13) is valid for $n+1$. We use the notation

$$e_{m,n} := D(m/n)/\{a_m^n \prod_{j=1}^{n-1}(1-q_m^j)^{n-j}\} \;-1.$$

Our induction hypothesis yields

$$e_{m,n-1} \to 0, \quad e_{m,n} \to 0, \text{ as } m \to \infty. \tag{2.15}$$

The identity (2.8) can be written in the form

$$(1+e_{m,n+1})(1+e_{m,n-1})a_m^{2n}(1-q_m^n)\prod_{j=1}^{n-1}(1-q_m^j)^{2n-2j}$$
$$= (1+e_{m,n})^2 a_m^{2n}\prod_{j=1}^{n-1}(1-q_m^j)^{2n-2j}$$
$$-(1+e_{m+1,n})(1+e_{m-1,n})a_{m+1}^n a_{m-1}^n \prod_{j=1}^{n-1}\{(1-q_{m+1}^j)(1-q_{m-1}^j)\}^{n-j}$$

which is equivalent to

$$\begin{aligned}&(1+e_{m,n+1})(1+e_{m,n-1})(1-q_m^n)\\ &= (1+e_{m,n})^2 - (1+e_{m+1,n})(1+e_{m-1,n})q_m^n \Pi_{m,n},\end{aligned} \tag{2.16}$$

where

$$\Pi_{m,n} := \prod_{j=1}^{n-1}\left(\frac{(1-q_{m+1}^j)(1-q_{m-1}^j)}{(1-q_m^j)^2}\right)^{n-j}.$$

If q is not a root of unity, then $q_m^j \to q^j \neq 1$, $j \geq 1$, and we have $\Pi_{m,n} \to 1$ as $m \to \infty$ and thus, by (2.15) and (2.16),

$$\lim_{m\to\infty}(1+e_{m,n+1})(1-q^n) = 1 - \lim_{m\to\infty} q_m^n \Pi_{m,n} = 1-q^n$$

which yields (2.13) for $n+1$.

Next let us treat the more difficult case where q is a root of unity. Given $\ell \geq 1$ and $\{b_m\}_{m=1}^{\infty} \subset \mathbb{C}$, we write $b_m = \mathcal{A}(\ell)$, $m \to \infty$, if b_m has an asymptotic expansion beginning with $m^{-\ell}$: For each $L \geq \ell$,

$$b_m = \beta_\ell m^{-\ell} + \beta_{\ell+1} m^{-\ell-1} + \cdots + \beta_L m^{-L} + o(m^{-L}), \quad m \to \infty.$$

We shall prove by induction on n that as $m \to \infty$,

$$e_{m,n} = \mathcal{A}(1). \tag{2.17}$$

Clearly, the assertion (2.17) is true for $n = 1, 2$. Assume now that we have proved (2.17) for $1, 2, \ldots, n$. We see that for $e_m := e_{m,n}$

$$\begin{aligned}
&\frac{(1+e_{m+1})(1+e_{m-1})}{(1+e_m)^2} \\
&= 1 + \frac{(e_{m+1}+e_{m-1}-2e_m)}{1+e_m} + \frac{(e_{m+1}-e_m)(e_{m-1}-e_m)}{(1+e_m)^2} \\
&= 1 + \mathcal{A}(3).
\end{aligned} \tag{2.18}$$

Moreover, if $q^j \neq 1$, we see that

$$1 - q_m^j = 1 - q^j + \mathcal{A}(1),$$

and if $q^j = 1$, since $\alpha_1 \neq 0$,

$$1 - q_m^j = -j\alpha_1/m + \mathcal{A}(2).$$

Hence for each $j \geq 1$,

$$(1-q_{m+1}^j)(1-q_{m-1}^j)/(1-q_m^j)^2 = 1 + \mathcal{A}(2), \quad m \to \infty$$

and thus $\Pi_{m,n} = 1 + \mathcal{A}(2)$. From this, (2.16) and (2.18) we deduce that

$$\begin{aligned}
&(1+e_{m,n+1})(1+e_{m,n-1})(1-q_m^n) \\
&= (1+e_m)^2\left(1 - \frac{(1+e_{m+1})(1+e_{m-1})q_m^n \Pi_{m,n}}{(1+e_m)^2}\right) \\
&= (1+e_m)^2(1 - q_m^n(1+\mathcal{A}(2))).
\end{aligned}$$

This and (2.12) imply, since $e_m = e_{m,n} = \mathcal{A}(1)$ and $e_{m,n-1} = \mathcal{A}(1)$, that

$$\begin{aligned}
1 + e_{m,n+1} &= (1+\mathcal{A}(1))\frac{1-q_m^n(1+\mathcal{A}(2))}{1-q_m^n} \\
&= (1+\mathcal{A}(1))\left(1 - \frac{\mathcal{A}(2)}{1-q_m^n}\right) = 1 + \mathcal{A}(1)
\end{aligned}$$

and thus that $e_{m,n+1} = \mathcal{A}(1)$.

(ii) First, $Q_{m,0} \equiv 1$, so that $Q_{m,0}(ua_m/a_{m+1}) \equiv B_0(u)$, if m is large enough, while $Q_{m,1}(u) = 1-a_{m+1}u/a_m$, so $Q_{m,1}(ua_m/a_{m+1}) = 1-u = B_1(u)$, if m is large enough. Hence (2.14) is trivially true for $n = 0, 1$. Suppose that we have proved it for $0, 1, \ldots, n-1$, for some $n \geq 2$. By (2.9), and by (i) above,

$$\begin{aligned} Q_{m,n}\left(\frac{ua_m}{a_{m+1}}\right) &= Q_{m,n-1}\left(\frac{ua_m}{a_{m+1}}\right) \\ &- \frac{ua_m}{a_{m+1}} Q_{m-1,n-1}\left(\frac{ua_{m-1}}{a_m q_m}\right) \frac{D(m-1/n-1)D(m+1/n)}{D(m/n-1)D(m/n)}. \end{aligned}$$

Now considering separately the cases where q is, or is not, a root of unity, we see that for each fixed j

$$(1-q_{m\pm 1}^j)/(1-q_m^j) \to 1, \quad m \to \infty,$$

and hence (2.13) shows that

$$\frac{D(m-1/n-1)D(m+1/n)}{D(m/n-1)D(m/n)} = \frac{a_{m+1}}{a_m} q_m^{n-1}(1+o(1)), \quad m \to \infty.$$

Now if $q \neq 0$, then u/q_m will lie in a bounded set, independent of m, if u does, so our induction hypothesis yields

$$\begin{aligned} Q_{m,n}\left(\frac{ua_m}{a_{m+1}}\right) &= B_{n-1}(u) - uB_{n-1}(u/q_m)q_m^{n-1}(1+o(1)) \\ &= B_n(u) + o(1). \end{aligned}$$

If $q = 0$, the last statement is still correct, since $\lim_{q\to 0} q^{n-1}B_{n-1}(u/q) = (-1)^{n-1}u^{n-1}$. □

Theorem 2.7. *Let* $f(z) = \sum_{j=0}^{\infty} a_j z^j$ *be an entire function with* $a_j \neq 0$, *if* j *is large enough, and let*

$$\lim_{j\to\infty} a_{j+1}/a_j = 0, \tag{2.19}$$

while (2.11) *holds for some* $q \in \mathbb{C}$. *If* $q^j = 1$ *for some* $j = 1, \ldots, n$, *we assume in addition* (2.12).
Let $n \geq 1$. *If* K *is any compact subset of* $\mathbb{C}$, *then*

$$\limsup_{m\to\infty} \left(\max_{z\in K} |f(z) - [m/n](z)| \right)^{1/m} = 0. \tag{2.20}$$

Proof. Since $B_n(0) = 1$, (2.14) and (2.19) imply that $\lim_{m\to\infty} Q_{m,n}(u) = 1$, uniformly in compact subsets of $\mathbb{C}$. Now in (2.2) we take $S = 1$. We divide by $Q_{m,n}$ and since r can be arbitrarily large, deduce (2.20). □

Theorem 2.8. *In particular, for some $\lambda > 0$ let*

$$f(z) = \sum_{j=0}^{\infty} z^j/(j!)^\lambda,$$

or let f be the Mittag-Leffler function

$$f(z) = \sum_{j=0}^{\infty} z^j/\Gamma(1 + j/\lambda),$$

then the conclusion of Theorem 2.7 is true.

Proof. One can check all the conditions on a_j (using the Stirling formula). □

Under the hypotheses of Theorem 2.7, Levin and Lubinsky [1990] prove more, namely

$$\lim_{m\to\infty} \frac{f(z) - [m/n](z)}{(-1)^n \frac{D(m+1/n+1)}{D(m/n)} z^{m+n+1}} = 1$$

uniformly on compact subsets of $\mathbb{C}$. From this, using ideas similar to those in the proof of Theorem 4.1 of Chapter 7 they derive for the $\rho_{m,n}(f,r)$ error of the rational approximation of f on the disk $|z| \le r$, the formula

$$\lim_{m\to\infty} \rho_{m,n}(f,r)/\left\{\left|a_{m+1}\left(\frac{a_{m+1}}{a_m}\right)^n \prod_{j=1}^{n}(1 - q_m^j)\right| r^{m+n+1}\right\} = 1.$$

§ 3. The Nuttall-Pommerenke Theorem

Astonishing to a newcomer to Padé approximation could be the fact that the analytic properties of the approximants $[m/n]_f$ do not always reflect the properties of the generating function f. If, for example, f is analytic on an open set G, it can happen that the poles of the $[m/n]_f$ are dense in G. This phenomenon is known as "spurious poles". For the approximants $[m/1]_f$ we have seen this in Theorem 2.1. But it can happen also on the main diagonal of the Padé table:

Theorem 3.1 (Wallin [1974]). *There exists an entire function $f(z)$ so that each point of $\mathbb{C}$ is a limit point of the poles of the sequence $[m/m]_f(z)$.*

Proof. Let $(u_\nu)_1^\infty$, $u_\nu \ne 0$ be any sequence of points dense in $\mathbb{C}$, each point repeated infinitely often in the sequence, let m_ν be the sequence of positive integers given by $m_1 = 1$, $m_\nu = 2m_{\nu-1} + 1$, $\nu = 2, 3, \ldots$. We shall construct a function $f(z) = \sum_{j=0}^{\infty} a_j z^j$ with coefficients $a_j \ne 0$ that are sufficiently small to ensure that f is entire. We construct the a_j by induction. Let $a_0 = 1$; at the ν-th step, when all a_j, $j < m_\nu$ are known, we define a_j, $m_\nu \le j < m_{\nu+1}$.

Let $m := m_\nu$, $P_m(z) := \sum_{j=0}^m b_j z^j$, $Q_m(z) := \sum_{j=0}^m c_j z^j := z - u_\nu$. Then P_m/Q_m will be the $[m_\nu/m_\nu]$ Padé approximant of f if f, P_m, Q_m satisfy the equations (1.10), (1.11) with $m = n = m_\nu$. In addition we want u_ν to be a pole of P_m/Q_m.

Since $c_0 = -u_\nu$, $c_1 = 1$, $c_2 = \cdots = c_m = 0$, the equations (1.10), (1.11) are here

$$a_{m+k-1} - u_\nu a_{m+k} = 0, \quad k = 1, \ldots, n, \tag{3.1}$$

$$b_0 = -u_\nu, \quad b_j = a_{j-1} - u_\nu a_j, \quad j = 1, \ldots, m. \tag{3.2}$$

The system (3.1) has solutions (a_j) with arbitrary small elements $a_j \neq 0$. Let (λ_j) be one of these. We define $a_j := \varepsilon_\nu \lambda_j$, $m_\nu \leq j < m_{\nu+1}$, with $\varepsilon_\nu = +1$ or -1. We first try $\varepsilon_\nu = +1$. If the corresponding P_m satisfies $P_m(u_\nu) \neq 0$, we are through. If $P_m(u_\nu) = 0$, we take $\varepsilon_\nu = -1$. This produces a new polynomial P_m^* with coefficients b_j^*, $j \leq m$. We will have $b_j^* = b_j$, $j < m$, $b_m^* = b_m + 2\lambda_m u_\nu$. Therefore $P_m^*(u_\nu) = 2\lambda_m u_\nu^{m+1} \neq 0$, and P_m^*/Q_m has a pole at u_ν. Taking ε_ν properly, we shall have the $[m_\nu/m_\nu]$ approximant of f with the desired properties; this will be true for any possible choice of the a_j, $j \geq m_{\nu+1}$ at later steps. □

For this f, the sequence $[m/m]_f$ cannot converge uniformly on any open set $G \subset \mathbb{C}$. Actually, Wallin [1974] proved more:

Remark. *The function f of Theorem 3.1 can have the additional property that $[m/m]_f(z)$ diverges at each point $z \neq 0$, and even satisfies*

$$\limsup_{m \to \infty} |[m/m](z)| = \infty, \quad z \in \mathbb{C} \setminus \{0\}. \tag{3.3}$$

For instance, one cannot hope to prove that $[m/m](z)$ converges almost everywhere.

The first to find a way out of this dilemma was Nuttall [1970]: $[m/m](z)$ is close to the meromorphic generating function $f(z)$ if one excludes a small set *which depends on m* (and f): he proved the convergence of $[m/m]$ to f in the two-dimensional Lebesgue measure. This has been refined by Pommerenke [1973] to convergence in logarithmic capacity.

For some $\lambda > 1$ we set up the inequalities

$$\lambda^{-1} \leq m/n \leq \lambda. \tag{3.4}$$

This is what convergence in capacity means in our situation: for all $\varepsilon, \eta > 0$ and $r > 0$ there should exist an n_0 so that for all $m, n \geq n_0$ satisfying (3.4),

$$|f(z) - [m/n](z)| < \varepsilon \text{ for } |z| \leq r, \ z \notin A_{m,n}, \ \gamma(A_{m,n}) < \eta. \tag{3.5}$$

We shall use properties of the logarithmic capacity developed in Appendix 4, but also the following lemma.

Lemma 3.2. *Let $0 < \varepsilon < 1/3$ and $\rho > 0$ be given. Let $g(z)$ be a polynomial of degree $n \geq 1$, and let*

$$\max_{|z| \leq \rho} |g(z)| \geq 1. \tag{3.6}$$

Then the set $B := \{z \in \mathbb{C} : |z| \leq \rho,\ |g(z)| \leq \varepsilon^n\}$ has the logarithmic capacity $\gamma(B) \leq 3\rho\varepsilon$.

Proof. We write

$$g(z) = \alpha \prod_{j=1}^{q} (z - z_j) \prod_{j=q+1}^{n} (z - z_j) \tag{3.7}$$

where z_j, $j = 1, \ldots, q$ are all zeros with $|z_j| \leq 2\rho$ (with $q = 0$ if all zeros of g are outside of $|z| \leq 2\rho$). It follows from (3.6) that

$$1 \leq |\alpha|(3\rho)^q \prod_{j=q+1}^{n} (\rho + |z_j|). \tag{3.8}$$

We consider the polynomial $Q(z) := (z - z_1) \cdots (z - z_q)$ in $\mathcal{P}_q$ if $q \geq 1$ and $Q(z) := 1$ if $q = 0$.

Let $z \in B$. Then $|z| \leq \rho$ and therefore, by (3.7) and (3.8),

$$|Q(z)| = |\alpha|^{-1} |g(z)| \prod_{j=q+1}^{n} \frac{1}{|z - z_j|} \leq \varepsilon^n (3\rho)^q \prod_{j=q+1}^{n} \frac{|z_j| + \rho}{|z_j| - \rho} \leq 3^n \varepsilon^n \rho^q.$$

We must have $q \geq 1$, for otherwise this would imply $1 = |Q(z)| < 1$ for $z \in B$, a contradiction if B is nonempty. Thus $\|Q\|_B^{1/q} \leq 3\rho\varepsilon$. But $Q(z)$ is a monic polynomial of degree q. Hence, by Theorem 2.4 of Appendix 4 and by the definition of the Chebyshev constant $c(B)$,

$$\gamma(B) = c(B) \leq \lim_{l \to \infty} \|Q^l\|_B^{1/(ql)} = \|Q\|_B^{1/q} \leq 3\rho\varepsilon. \qquad \square$$

The following theorem is a generalization of (3.5).

Theorem 3.3. *Let f be a single-valued analytic function on $\mathbb{C} \setminus A$, where A, $0 \notin A$ is a closed set with capacity zero: $\gamma(A) = 0$. Then the Padé approximants $[m/n]$ of f, satisfying (3.4), converge to f in capacity, faster than geometrically, on each disk $D_r : |z| \leq r$, $r > 0$.*

More precisely, for all $\lambda > 1$, $r, \varepsilon, \eta > 0$ there exists an n_0 so that for each pair $m, n \geq n_0$ satisfying (3.4), there is a set $A_{m,n}$ for which

$$|f(z) - [m/n](z)|^{1/n} < \varepsilon, \quad z \in D_r \setminus A_{m,n}, \quad \gamma(A_{m,n}) < \eta. \tag{3.9}$$

Proof. Let $0 < \varepsilon < 1$, $\eta, r > 0$, and $\lambda > 1$ be fixed. We may assume that r is so large that

$$\varepsilon \le 1/(9r^9), \quad r > 2/\eta^2 \tag{3.10}$$

and that f is analytic in the disk $D_{1/(r-1)}$, hence $A^{-1} \subset D_{r-1}$.

Since the set A, which does not contain the origin is not necessarily compact, we perform the inversion $z \to 1/z$; this will produce a compact set $\{0\} \cup A^{-1}$. We enlarge this to $A^* := D_{\delta/4} \cup A^{-1}$, $\delta > 0$. Let Ω_r be the annulus $\{z \in \mathbb{C} : r^{-1} \le |z| \le r^2\}$.

The remainder of the proof is this. We define a polynomial $h \in P_k$ satisfying (3.12), and the integration curve Γ. We then estimate the difference $(Q_n f - P_m)(1/z)$, using a remainder formula similar to (3.12), with a power h^l of h playing the role of S. Then, returning from $1/z$ to z, we prove that $(Q_n f - P_m)(z)$, $z \in \Omega_r$ is small, except for the sets of small capacity (using results of Appendix 4).

We choose a number $\delta > 0$ so small that

$$\delta < \varepsilon^{4\lambda}/(2r)^{2\lambda}. \tag{3.11}$$

This implies that $\delta < \varepsilon$.

The set $A^* := D_{\delta/4} \cup A^{-1}$ is a compact subset of the disk D_{r-1}. Corollaries 2.5 and 2.7 of Appendix 4 give for the logarithmic capacities of the sets $D_{\delta/4}$ and $\{z : \delta/4 \le |z| < \infty,\ z \in A^{-1}\}$ the values $\delta/4$ and zero, respectively, hence $\gamma(A^*) = \delta/4$ by Corollary 2.8(i) of Appendix 4. The Chebyshev constant $c(A^*)$ of A^* is also equal to $\delta/4$ by Theorem 2.4 of Appendix 4. There exist therefore a $k \ge 1$ and a monic polynomial h of degree k with the property

$$\|h\|_{A^*} \le (\delta/2)^k. \tag{3.12}$$

We may assume that all zeros $z_1, \dots, z_k$ of h lie in the disk D_{r-1}. Indeed, let, for example, the z_i, $i = 1, \dots, m$ be the only zeros of h outside of D_{r-1}. Then,

$$h^*(z) := h(z) \prod_{i=1}^{m} \frac{z - z_i^*}{z - z_i}, \quad z_i^* := (r-1) z_i / |z_i|,$$

is also a monic polynomial of degree k, whose zeros lie in D_{r-1}. In addition, h^* satisfies (3.12) since A^* is contained in D_{r-1}. We then replace h by h^*.

We fix $h \in \mathcal{P}_k$ with the required properties and have

$$|h(z)| = \prod_{j=1}^{k} |z - z_j| \ge 1, \quad \text{if } |z| \ge r. \tag{3.13}$$

Next we introduce the set

$$A_0 := \{z \in \mathbb{C} : |h(z)| \le \varepsilon^k\}$$

and the lemniscate

$$\Gamma := \{z \in \mathbb{C} : |h(z)| = \delta^k\}.$$

From (3.13) and $0 < \delta < \varepsilon < 1$ it follows that $A^* \subset A_0 \subset D_r$. In addition, Γ lies in the interior of A_0 and, by (3.12) and (3.13), (3.14) $$\min_{\zeta\in\Gamma}|\zeta| \geq \delta/4, \quad \max_{\zeta\in\Gamma}|\zeta| \leq r.$$

The distance between Γ and the complement of A_0 is positive, that is,

$$\rho := \inf_{z\notin A_0}\min_{\zeta\in\Gamma}|z-\zeta| > 0.$$

Since Γ and A^* do not intersect and are compact, we have

$$M := \max_{\zeta\in\Gamma}|f(1/\zeta)| < \infty.$$

We now consider the Padé approximants $[m/n]$ of f, with $m \geq 2k$, $n \geq 2k$. Let $l \in \mathbb{N}$ be defined by $n/k - 1 < l \leq n/k$, then $h(z)^l$ is a polynomial of degree $kl \leq n$. We normalize $[m/n] = P_m/Q_n$ so that the polynomial $g_{m,n}(z) := z^n Q_n(1/z)$ of degree $\leq n$ satisfies

(3.15) $$\max_{|z|\leq r^2}|g_{m,n}(z)| = 1.$$

Next we want to estimate the function

(3.16) $$F(z) := z^m h(z)^l \left[Q_n(1/z)f(1/z) - P_m(1/z)\right].$$

It follows from (1.1) that, as $z \to \infty$,

$$F(z) = z^m h(z)^l \mathcal{O}(1/z^{m+n+1}) = \mathcal{O}(1/z),$$

so that $F(z)$ is analytic at $z = \infty$ and $F(\infty) = 0$. The function $F(z)$ has the form

(3.17) $$F(z) = z^{m-n}h(z)^l g_{m,n}(z)f(1/z) - z^m h(z)^l P_m(1/z)$$

where the first term on the right-hand side is analytic for $z \notin A^*$ and the second term is a polynomial. With the proper orientation of Γ, the set A^* lies inside of Γ and z lies outside of Γ, and we can apply Cauchy's integral formula for F and z, that is,

(3.18) $$F(z) = \frac{1}{2\pi i}\int_\Gamma \frac{F(\zeta)}{z-\zeta}d\zeta, \quad z \in \Omega_r \setminus A_0.$$

The Cauchy integral over Γ of the polynomial $z^m h(z)^l P_{m,n}(1/z)$ is zero. This leads to

$$F(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\zeta^{m-n}h(\zeta)^l g_{m,n}(\zeta)f(1/\zeta)}{z-\zeta}d\zeta, \quad z \in \Omega_r \setminus A_0.$$

Since the arc length of Γ is finite and since $|h(\zeta)^l| = \delta^{kl} \leq \delta^{n-k}$, $\zeta \in \Gamma$, this yields

(3.19) $$|F(z)| \leq \frac{M|\Gamma|\delta^{kl}}{2\pi\rho}\max_{\zeta\in\Gamma}|\zeta|^{m-n} \leq M_1\delta^n \max_{\zeta\in\Gamma}|\zeta|^{m-n}, \quad z \in \Omega_r \setminus A_0,$$

where M_1 is independent of m, n, z.

Because of (3.15) we can apply Lemma 3.2 for the polynomial $g_{m,n}$ and the disk $|z| \le r^2$: we have

$$|g_{m,n}(z)| \ge \varepsilon^n, \quad |z| \le r^2, \quad z \notin B_{m,n},$$

where the exceptional set $B_{m,n}$ has capacity $\gamma(B_{m,n}) \le 3r^2\varepsilon$. Hence, for all $z \in \Omega_r$, $z \notin B_{m,n} \cup A_0$,

$$(3.20) \quad |z^m h(z)^l Q_n(1/z)| = |z^{m-n} h(z)^l g_{m,n}(z)| \ge |z|^{m-n}\varepsilon^{kl+n} \ge |z|^{m-n}\varepsilon^{2n}.$$

We divide $F(z)$, given in (3.16), by the right-hand side of (3.20). We use inequalities (3.19), (3.20) and the substitution $z \to 1/z$ to arrive at

$$(3.21) \qquad |f(z) - [m/n](z)| \le M_1 \delta^n |z|^{m-n} \varepsilon^{-2n} \max_{\zeta \in \Gamma} |\zeta|^{m-n},$$

for all $z \in \Omega_r^{-1}$, $z \notin (B_{m,n} \cup A_0)^{-1}$, that is, for all points in the disk D_r except for the points z of the union $A_{m,n} := S_0 \cup S_1 \cup S_{m,n}$ of the three sets

$$\begin{aligned} S_0 &:= \{|z| \le r^{-2}\}, \\ S_1 &:= \{z \in \mathbb{C} : \ r^{-2} \le |z| \le r, \ z \in A_0^{-1}\}, \\ S_{m,n} &:= \{z \in \mathbb{C} : \ r^{-2} \le |z| \le r, \ z \in B_{m,n}^{-1}\}. \end{aligned}$$

It follows from (3.21), (3.14) and (3.11), for all $z \in D_r$, $z \notin A_{m,n}$, that

(a) in the case $n \le m \le \lambda n$, since (3.11) implies $\delta \le \varepsilon^4/r^{2\lambda}$,

$$|f(z) - [m/n](z)| \le M_1 \delta^n r^{m-n} \varepsilon^{-2n} r^{m-n} \le M_1 \delta^n r^{2m} \varepsilon^{-2n} \le M_1 \varepsilon^{2n}.$$

(b) in the case $m \le n \le \lambda m$,

$$|f(z) - [m/n](z)| \le M_1 \delta^n r^{2(n-m)} \varepsilon^{-2n} (\delta/4)^{m-n} \le M_1 \delta^m (2r)^{2n} \varepsilon^{-2n}$$

which is also $\le M_1 \varepsilon^{2n}$ because of (3.11). This yields (3.9).

It remains to prove that the capacity of the exceptional set $A_{m,n}$ is less than η. By Corollaries 2.5 and 2.9 of Appendix 4 we have $\gamma(S_0) = 1/r^2$ and $\gamma(A_0) = \varepsilon$. From Corollary 2.7 of Appendix 4 we get

$$\begin{aligned} \gamma(S_1) &\le r^2 \gamma(A_0) = r^2 \varepsilon, \\ \gamma(S_{m,n}) &\le r^2 \gamma(B_{m,n}) \le 3r^4 \varepsilon. \end{aligned}$$

Finally we apply Corollary 2.8(ii) of Appendix 4 twice :

$$\gamma(S_1 \cup S_{m,n}) \le \sqrt{3r^4\varepsilon}\sqrt{2r} \le 3r^{5/2}\sqrt{\varepsilon},$$

and since $3r^{5/2}\sqrt{\varepsilon} \le 1/r^2 = \gamma(S_0)$,

$$\gamma(A_{m,n}) = \gamma(S_0 \cup (S_1 \cup S_{m,n})) \le \sqrt{\gamma(S_0)}\sqrt{2r} = \sqrt{2/r}.$$

This and (3.10) imply $\gamma(A_{m,n}) < \eta$. □

One might, possibly, conjecture that the sets $A_{m,n}$ are contained in a small neighborhood of the set A of singularities of f. The example of Theorem 3.1 shows that this conjecture is not true.

§ 4. Problems

4.1. Using formula (1.8) prove that if f is analytic in $|z| < r$, then at least a subsequence of the approximants $[m/1]$ converge to f, uniformly for $|z| \le \rho$, for each $0 < \rho < r$. (Beardon [1968].)

4.2. For a differentiable function F prove that

$$\begin{aligned}\int_0^1 e^{tx}F(t)dt = \; & e^x\left(\frac{F(1)}{x} - \cdots + (-1)^m \frac{F^{(m)}(1)}{x^{m+1}}\right) \\ - \; & \left(\frac{F(0)}{x} - \cdots + (-1)^m \frac{F^{(m)}(0)}{x^{m+1}}\right).\end{aligned}$$

4.3. Taking $F(t) = t^m(1-t)^n$, prove that

$$\begin{aligned}& e^x\left(F^{(m+n)}(1) - F^{(m+n-1)}(1)x - \cdots + (-1)^m F^{(n)}(1)x^m\right) \\ - \; & \left(F^{(m+n)}(0) - F^{(m+n-1)}(0)x - \cdots + (-1)^n F^{(m)}(0)x^n\right) \\ = \; & (-1)^{m+n+1}x^{m+n+1}\int_0^1 e^{tx}F(t)dt.\end{aligned}$$

4.4. Prove for $f(x) = e^x$ that $[m/n](x) = P_{m,n}(x)/Q_{m,n}(x)$ where

$$\begin{aligned}P_{m,n}(x) = 1 + \frac{n}{m+n}\frac{x}{1!} + \frac{n(n-1)}{(m+n)(m+n-1)}\frac{x^2}{2!} + \cdots \\ + \frac{n(n-1)\cdots 1}{(m+n)(m+n-1)\cdots(m+1)}\frac{x^n}{n!},\end{aligned}$$

$Q_{m,n}(x) = P_{n,m}(-x)$, and that all $[m/n]$ are different.

4.5. Prove that for $f(x) = e^x$ and $m+n \to \infty$, $m/n \to \omega$, one has $P_{m,n}(x) \to e^{x/(1+\omega)}$, $Q_{m,n}(x) \to e^{-\omega x/(1+\omega)}$.

§ 5. Notes

We have only scratched the surface of the Padé approximation theory.

5.1. *Stieltjes Series.* If ϕ is a bounded, nondecreasing function on $[0, \infty)$, which takes infinitely many values, then

$$f(z) = \int_0^\infty \frac{d\phi(x)}{1+zx} \tag{5.1}$$

is the Stieltjes function associated with f, defined for all z in the complex plane $\mathbb{C}$ split by the negative x-axis. If all its moments $f_j := \int_0^\infty x^j d\phi(x)$, $j = 0, 1, \ldots$, are finite, then

$$\sum_{j=0}^{\infty} f_j(-z)^j$$

is called the Stieltjes series of ϕ. There are interesting and important connections between Stieltjes functions, orthogonal polynomials, continued fractions found by Stieltjes in his papers [1889], [1894] (which according to Dieudonné, deal with some "obscure problems").

5.2. *The Padé Table of* e^z. The Padé approximants of this function are known explicitly. Saff and Varga ([1977] and the papers mentioned there) have examined the distribution of zeros of the Padé approximants $[m/n]$. Since the $[m/n]$ approximant of e^{-z} is the $[n/m]$ approximant of e^z, by (1.2), this is equivalent to examining the distribution of the poles of the Padé approximants of the exponential function. Some of their results about the Padé approximants of e^z are:

1. If $n \geq 2$, the region $\cos\theta \geq \frac{m-n+2}{m+n}$ is free of the zeros of the $[m/n](z)$, $z = re^{i\theta}$, $r > 0$.
2. The $[m/n](z)$ approximants to e^{nz}, $z = x + iy$, have no zeros in the parabolic region $y^2 \leq 4(x+1)$.
3. For $m \geq 1$, all zeros of the $[m/n](z)$ lie in the annulus $(m+n)\mu < |z| < m+n+\frac{4}{3}$, where μ is the unique positive root of the equation $\mu e^{1+\mu} = 1$, $\mu = 0.278\ldots$.

5.3. *Subsequences of Rows.* It is easy to prove that a subsequence of the row $[m/1](z)$ converges to $f(z)$ on compact subsets of $\mathbb{C}$ for any entire function $f(z)$. The extension of this by Buslaev, Gonchar and Suetin [1984] is much more difficult: If f has only a finite number of poles in $\mathbb{C}$, then for each $n = 1, 2, \ldots$, the sequence $[m/n](z)$ has a convergent subsequence.

5.4. Very important is the recent work by Stahl; see its review in Stahl [1989]. It is not possible to give an adequate description of it here; for analytic functions with branch points, Stahl sometimes obtains geometric convergence (and not better than geometric convergence, as in the Nuttall-Pommerenke theorem) of multipoint Padé approximants. His work has been used by Gonchar and Rakhmanov [1987] in their determination of the error of rational approximation of e^{-x} on $[0, \infty)$ (see §5 of Chapter 7).

5.5. *Inverse Theory.* Suppose that f is a formal power series, $n \geq 1$, and the poles of $[m/n]$ approach $\xi_1, \xi_2, \ldots, \xi_n$ as $m \to \infty$. What can we say about the analytic properties of f? This *inverse* problem of Padé approximation is important in applications, and has been investigated by the Gonchar school in Moscow. Suetin [1985] showed that $\xi_1, \xi_2, \ldots, \xi_n$ are singularities of f; earlier work of the Gonchar school showed that if $|\xi_j| < r := \max_k |\xi_k|$, then ξ_j is a pole of f, and these are the only singularities of f in $|z| < r$.

Chapter 10. Hardy Space Methods in Rational Approximation

Theorems about the error of rational approximation, $\rho_n(f)_p$, of a function $f \in L_p$ have a different character for $p = \infty$ and $0 < p < \infty$. In the latter case, the errors of rational and of (free knot) spline approximation $\sigma_{n,r}(f)_p$ are closely related (see §6), and one can estimate $\rho_n(f)_p$ by means of the simpler error of spline approximation. In contrast, for the uniform approximation, complex methods are necessary; the approximated functions belong to a Hardy space, and the rôle of the splines as a real analysis component is taken over by the elements of an atomic decomposition (equivalently, by those of a decomposition in a sum of simple functions) of the function f. In this way one can obtain matching direct and inverse theorems, (for functions in Hardy spaces on the disk $D := \{z : |z| < 1\}$, on $\mathbb{T}$ and on an interval) reminiscent of the polynomial approximation theorems in spaces W_∞^r of [**CA**, Chapter 7]. The first who gave matching direct and inverse theorems for rational approximation, was Peller [1980]. He characterized in this way certain Besov spaces of analytic functions in terms of rational approximation in the norm of the space BMOA. Brudnyi [1979], [1980] announced some theorems similar to those of our §6. Most of the results of this chapter are the work of Pekarskii. Results of §5 should be compared to those in Petrushev and Popov [A-1987].

The plan of this chapter is as follows: In §1, we develop the Bernstein-type inequalities for rational functions, in §2 we prove the main approximation theorem. Its proof depends, however, on the technical work of §3 (dealing with sums of simple functions) and of §4 (rational approximation of such sums). In §5, we give several applications of the main theorems, for instance, a new proof of Popov's estimate of $\rho_n(V^r)$. In §6, we treat the case $0 < p < \infty$, and compare the errors $\rho_n(f)_p$ and $\sigma_{n,r}(f)_p$. This is possible because of the results of [**CA**, Chapter 12].

The theory of Hardy spaces is essential throughout the chapter. The reader will find all needed facts of this theory in Appendix 3.

§ 1. Bernstein-Type Inequalities for Rational Functions

The following theorem of Pekarskii [1986$_2$] is a generalization of Dolzhenko's inequality (1.5) of Chapter 7:

Theorem 1.1. *Let $R \in \mathcal{R}_m$, $m = 1, 2, \ldots$ be a rational function without poles on $A := [0, 1]$. Then for $r = 1, 2, \ldots$, $1 < p \le \infty$ and $\gamma := (r + 1/p)^{-1}$ we have*

$$\|R^{(r)}\|_\gamma(A) \le C(p, r) m^r \|R\|_p(A) . \tag{1.1}$$

We shall prove a similar inequality also for functions on $\mathbb{T} := \partial D := \{|z| = 1\}$. We use the notations $D := D_+ := \{z \in \mathbb{C} : |z| < 1\}$, $D_- := \{z \in \mathbb{C} : |z| > 1\}$.

If $(a_k)_{k=1}^{m+n}$, $|a_k| < 1$ are arbitrary points of D, the rational function

$$R(z) = \frac{P(z)}{\prod_{k=1}^{m} (1 - \bar{a}_k z) \prod_{k=m+1}^{m+n} (z - a_k)} , \qquad P \in \mathcal{P}_{m+n} \tag{1.2}$$

belongs to $\mathcal{R}_{m+n}$. Formula (1.2) defines its poles $1/\bar{a}_1, \ldots, 1/\bar{a}_m$ in D_- and $a_{m+1}, \ldots, a_{m+n}$ in D_+ and their order. An expansion of R into elementary fractions yields the unique representation

$$\begin{cases} R(z) = R_+(z) - R_-(z) , \\ R_+(z) := P_+(z) \Big/ \prod_{k=1}^{m} (1 - \bar{a}_k z) , \\ R_-(z) := P_-(z) \Big/ \prod_{k=m+1}^{m+n} (z - a_k) , \end{cases} \tag{1.3}$$

with $P_+ \in \mathcal{P}_m$, $P_- \in \mathcal{P}_{n-1}$. In particular, R_+ is analytic in $\bar{D}_+$, and R_- in $\bar{D}_-$, with $R_-(\infty) = 0$. Using the operators $C^\pm(f)$ of (2.7) of Appendix 3, we see from Cauchy's theorem, that $R_+ = C^+(R)$, $R_- = C^-(R)$.

Theorem 1.2 (Pekarskii [1984]). *Let R be a rational function of the form (1.2), with R_+ and R_- from (1.3), let $1 < p \le \infty$, $r = 1, 2, \ldots$, $\gamma = (r+1/p)^{-1}$. Then*

$$\|R_+^{(r)}\|_\gamma(\partial D) \le C m^r \|R\|_p(\partial D) , \tag{1.4}$$

$$\|R_-^{(r)}\|_\gamma(\partial D) \le C n^r \|R\|_p(\partial D) , \tag{1.5}$$

where C depends only on p and r.

For instance for $p = \infty$ we get

$$\|R_+^{(r)}\|_{1/r} \le C m^r \|R\|_\infty \quad , \quad \|R_-^{(r)}\|_{1/r} \le C n^r \|R\|_\infty . \tag{1.6}$$

We shall first address the question of exactness of our inequalities, for example, of (1.1). The function $R(x) := (x+\delta)^{-1}$ belongs to $\mathcal{R}_1$ for all $\delta > 0$. For $1 < p \le \infty$, $q > (r+1)^{-1}$, and $A := [0,1]$, we have $\|R\|_p \sim \delta^{-1+1/p}$, $\|R^{(r)}\|_q \sim \delta^{-r-1+1/q}$ for $\delta \to 0$. This shows that γ in (1.1) cannot be replaced by any $q > \gamma$. For $p = 1$, $\|R\|_1 \sim \log(1/\delta)$, $\|R^{(r)}\|_{(r+1)^{-1}} \sim (\log(1/\delta))^{r+1}$, $\delta \to 0$, hence the inequality (1.1) is not satisfied for $p = 1$. The example of $R(x) = C_m(2x-1)$, where C_m are the Chebyshev polynomials, shows that the factor m^r, $m \to \infty$, in (1.1) cannot be replaced by a slower increasing function.

Corollary 1.3. *If $R \in \mathcal{R}_n$ has no poles on ∂D, and $\tilde{R}$ is the conjugate function, then for $1 < p \le \infty$, $r = 1, 2, \ldots$ and $\gamma = (r+1/p)^{-1}$,*

$$\|R^{(r)}\|_\gamma(\partial D) \le Cn^r \|R\|_p(\partial D) , \tag{1.7}$$

$$\|\tilde{R}^{(r)}\|_\gamma(\partial D) \le Cn^r \|R\|_p(\partial D) . \tag{1.8}$$

Proof. Inequality (1.7) follows from Theorem 1.2 and (1.3). For (1.8), instead of (1.3), one uses (see (2.11) of Appendix 3) the relation

$$\tilde{R}(z) = -i\big[C^+(R,z) + C^-(R,z)\big] + \text{const} = -i\big[R_+(z) + R_-(z)\big] + \text{const} . \quad \square$$

For $p = \infty$ and $r = 1$, (1.7) has been obtained by Dolzhenko [1966], and (1.8) by Rusak [A-1979].

The proof of the main inequalities will depend on the integral representation of the derivatives $R_\pm^{(r)}(z)$ by means of integral operators. Their kernels depend on Blaschke products, whose properties we discuss now.

A *Möbius transformation* $\mu(z)$ is a conformal mapping of D onto intself. These functions are characterized by a representation

$$\mu(z) = \eta \frac{z-a}{1-\bar{a}z}$$

whose parameters η, a satisfy $|\eta| = 1$, and $a \in D$. Thus, $\mu(z)$ maps also ∂D onto itself, and $|\mu(z)| = 1$, $z \in \partial D$.

A product of m Möbius functions is called a *Blaschke product of order m*:

$$B(z) := B_m(z) := \eta \prod_{k=1}^{m} \frac{z-a_k}{1-\bar{a}_k z} , \tag{1.9}$$

where $a_k \in D$, $k = 1, \ldots, m$ and $|\eta| = 1$. In this section we shall always assume that $\eta = 1$ and often omit the subscript m. Obviously, $|B_m(z)| \le 1$, $z \in D$ and $|B_m(z)| = 1$, $z \in \partial D$. Also the derivatives of the B_m can be estimated.

For $\beta > 0$, $m = 1, 2, \ldots$, we put, with $a_0 = 0$,

$$\lambda(z) := \lambda(z,\beta) := \lambda_m(z,\beta) := \sum_{k=0}^{m} \left| \frac{1-|a_k|}{1-\bar{a}_k z} \right|^\beta \frac{1}{|1-\bar{a}_k z|} . \tag{1.10}$$

Clearly, λ is a decreasing function of β, satisfying $\lambda \ge 1$.

We shall need the following estimates of the $B_m^{(r)}$ and the λ_m:

Lemma 1.4. *Let $m = 1, 2, \ldots$. (i) For $r = 1, 2, \ldots$ and $z \in \partial D$ one has*

$$|B_m^{(r)}(z)| \le 2^r r! \lambda_m \left(z, \frac{1}{r}\right)^r \; ; \tag{1.11}$$

(ii) *For each $\beta > 0$,*

$$\int_{\partial D} \lambda_m(z, \beta)\, |dz| \le C(\beta) m \, . \tag{1.12}$$

Proof. (i) Let $\mu_k(z) := (z - a_k)/(1 - \bar{a}_k z)$. By Leibniz' formula,

$$B^{(r)}(z) = \sum \frac{r!}{k_1! \ldots k_m!} \mu_1^{(k_1)}(z) \ldots \mu_m^{(k_m)}(z) \, , \tag{1.13}$$

where the summation is extended over all selections $k_1, \ldots, k_m$ of non-negative integers that satisfy $k_1 + \cdots + k_m = r$. Now for the function $\mu(z) := (z-a)/(1-\bar{a}z)$, $a \in D$, $\ell = 0, \ldots, r$ one has

$$\begin{aligned} |\mu^{(\ell)}(z)| &= \ell! |a|^{\ell-1}(1 - |a|^2) \left| \frac{1}{1 - \bar{a}z} \right|^{\ell+1} \\ &\le 2^\ell \ell! \left(\left| \frac{1 - |a|}{1 - \bar{a}z} \right|^{1/r} \frac{1}{|1 - \bar{a}z|} \right)^\ell , \qquad z \in \partial D \, , \end{aligned}$$

since $1 - |a| \le |1 - \bar{a}z|$ and $|\mu(z)| = 1$ for $z \in \partial D$. Using this to estimate the sum (1.13) with the help of the extended binomial formula, we obtain (1.11).

(ii) It is sufficient to establish that for each $a \in D$,

$$\int_{\partial D} |1 - \bar{a}z|^{-1-\beta}\, |dz| \le C(\beta)(1 - |a|)^{-\beta} \, . \tag{1.14}$$

For this purpose, let I_k, $k = 0, 1, \ldots$ be the interval of the circle ∂D which belongs to the disk $|z - a| \le 2^k(1 - |a|)$. For $z \in \partial D$ we have $|1 - \bar{a}z| = |z - a|$, hence the integral (1.14) does not exceed

$$\begin{aligned} &\sum_{k \ge 0} |I_{k+1} \setminus I_k| \sup \left\{ |z - a|^{-1-\beta} : z \in I_{k+1} \setminus I_k \right\} \\ &\qquad \le \sum_{k=0}^{\infty} \pi 2^{k+1}(1 - |a|) 2^{-k(\beta+1)}(1 - |a|)^{-1-\beta} = C_\beta (1 - |a|)^{-\beta} \, . \qquad \square \end{aligned}$$

Returning to the rational function R_+ of (1.3), we have by Cauchy's formula

$$R_+^{(r)}(z) = \frac{r!}{2\pi i} \int_{\partial D} \frac{R(\zeta)}{(\zeta - z)^{r+1}}\, d\zeta \, , \qquad z \in D_+ \, . \tag{1.15}$$

As a function of ζ, $R(\zeta)B(\zeta)^{-k}(\zeta - z)^{-r-1}$, $k = 1,2,\dots$ (where B is the Blaschke product (1.9)) is analytic on $\bar{D}_-$, and has at ∞ a zero of multiplicity at least two. Consequently

$$\int_{\partial D} R(\zeta)B(\zeta)^{-k}(\zeta - z)^{-r-1}\,d\zeta = 0\;, \qquad k = 1,2,\dots\;.$$

By the binomial formula, it follows therefore from (1.15) that for $z \in D_+$,

$$R_+^{(r)}(z) = \frac{r!}{2\pi i}\int_{\partial D} R(\zeta)\left(1 - \frac{B(z)}{B(\zeta)}\right)^{r+1} \frac{d\zeta}{(\zeta - z)^{r+1}}\;. \tag{1.16}$$

Let for $\alpha > 0$

$$\begin{cases} Q(z,\zeta,m) := Q(z,\zeta) := \dfrac{B(\zeta) - B(z)}{\zeta - z}\;, \\ K_\alpha(z,\zeta,m) := K_\alpha(z,\zeta) := |Q(z,\zeta)|^{\alpha+1}\;. \end{cases} \tag{1.17}$$

This defines a symmetric positive kernel K_α. A useful remark is that if $\ell \geq \alpha$, $p := (\ell+1)/(\alpha+1)$, then $K_\alpha^p = K_\ell$.

Since $|B(\zeta)| = 1$ on ∂D, from (1.16) we derive the fundamental estimate

$$|R_+^{(r)}(z)| \leq \frac{r!}{2\pi}\int_{\partial D} K_r(z,\zeta)|R(\zeta)|\,|d\zeta|\;, \qquad z \in \bar{D}_+\;. \tag{1.18}$$

For the proof of Theorems 1.2 and 1.1 we have to estimate the norms of some $L_p \to L_\gamma$ integral operators, like the one appearing in (1.18). This is best done in a general form.

Let $L_{\alpha,m}(x,t)$ be a two parameter family of kernels defined for $(x,t) \in A^2$, where A is an interval or $\mathbb{T}$.

Theorem 1.5. *Let $L_{\alpha,m}(x,t) := L_\alpha(x,t)$ be kernels defined for all $\alpha > 0$, $m = 1,2,\dots$ by means of*

$$L_{\alpha,m}(x,t) = Q_m(x,t)^{\alpha+1}\;, \tag{1.19}$$

where Q_m are positive continuous functions on A^2, and let

$$\int_A \left(\int_A L_{\alpha,m}(x,t)\,dt\right)^{1/\alpha} dx \leq C(\alpha)m\;, \qquad \alpha > 0\;,\; m = 1,2,\dots\;. \tag{1.20}$$

Then the operator

$$g(x) = \int_A L_{\alpha,m}(x,t)f(t)\,dt \tag{1.21}$$

maps, for each p, $1 < p \leq \infty$, $L_p(A)$ into $L_\gamma(A)$, $\gamma = (\alpha + 1/p)^{-1}$ with norm $\leq Cm^\alpha$, with C depending only on α, p.

Proof. If $p = \infty$, then $\gamma = 1/\alpha$ and the statement follows directly from (1.20), for we obtain

$$\|g\|_{1/\alpha} \le \left\| \int_A L_{\alpha,m}(\cdot,t)\,dt \right\|_{1/\alpha} \|f\|_\infty = Cm^\alpha \|f\|_\infty\,, \quad \text{with} \quad C = C(\alpha)^\alpha\,.$$

We can assume that $1 < p < \infty$.
(i) First let $\gamma = 1$, when $\alpha = 1 - 1/p$. In this case, by (1.20) and using Hölder's inequality,

$$\|g\|_\gamma = \|g\|_1 \le \int_A \|L_{\alpha,m}(\cdot,t)\|_1 |f(t)|\,dt \le C(\alpha)^\alpha m^\alpha \|f\|_p\,.$$

(ii) The general case can be reduced to (i). We fix some ℓ so that $\max(1,\gamma) < \ell < p$, let ℓ' be its conjugate exponent. We put $\tau := \frac{1}{\ell} - \frac{1}{p'}$, $\sigma := \alpha - \tau$. The definition of ℓ yields $\sigma, \tau > 0$. Factorizing the kernel L_α : $L_\alpha = L_{\ell'\sigma}^{1/\ell'} L_{\ell\tau}^{1/\ell}$, we apply Hölder's inequality to the function g of (1.21), and obtain

$$|g(x)| \le \|L_{\ell'\sigma}(x,\cdot)\|_1^{1/\ell'} \, \|L_{\ell\tau}(x,\cdot)|f(\cdot)|^\ell\|_1^{1/\ell}, \tag{1.22}$$

or $|g| \le g_1 g_2$, where g_1, g_2 are the two factors on the right. From (1.20) with α replaced by $\ell'\sigma$ we obtain

$$\|g_1\|_{1/\sigma} \le C_1 m^\sigma \quad \text{with} \quad C_1 := (C(\ell'\sigma))^\sigma\,. \tag{1.23}$$

On the other hand, to the function

$$g_2(x)^\ell = \int_A L_{\ell\tau}(x,t)|f(t)|^\ell\,dt$$

we apply case (i) with parameters $\bar p := p/\ell$, $\bar\alpha := \ell\tau$ and, because of the selection of τ, $\bar\gamma = 1$. We obtain

$$\|g_2\|_\ell \le C_2 m^{\frac{1}{\ell} - \frac{1}{p}} \|f\|_p\,, \qquad C_2 := (C(\ell\tau))^\tau\,. \tag{1.24}$$

Since $\frac{1}{\gamma} = \sigma + \frac{1}{\ell}$, we can apply Hölder's inequality to obtain

$$\|g\|_\gamma = \|g_1 g_2\|_\gamma \le \|g_1\|_{1/\sigma}\|g_2\|_\ell \le Cm^{\sigma + \frac{1}{\ell} - \frac{1}{p}} \|f\|_p = Cm^\alpha \|f\|_p\,. \qquad \square$$

This explains the following two lemmas.

Lemma 1.6. *For $\ell = 1, 2, \ldots$, and $z \in \partial D$, one has*

$$\begin{aligned} &\frac{(2\ell-1)!}{2\pi} \int_{\partial D} K_{2\ell-1}(z,\zeta)\,|d\zeta| \\ &\quad = z^\ell \sum_{k=1}^{\ell} \binom{2\ell}{\ell-k} (-1)^{\ell-k} B(z)^{-k} \left(B(z)^k z^{\ell-1}\right)^{(2\ell-1)}\,. \end{aligned} \tag{1.25}$$

(For $\ell = 1, 2$ this has been given by Rusak [A-1979].)

Proof. For $|a| = |b| = 1$, we have

$$|a-b|^2 = (a-b)(\bar a - \bar b) = -\frac{(a-b)^2}{ab} .$$

Thus, for $z, \zeta \in \partial D$,

$$|Q(z,\zeta)|^2 = \frac{z\zeta}{B(z)B(\zeta)} Q(z,\zeta)^2 .$$

On ∂D we have $|d\zeta| = \frac{d\zeta}{i\zeta}$. Hence the left-hand side of (1.25) is equal to

$$z^\ell B(z)^{-\ell} \frac{(2\ell-1)!}{2\pi i} \int_{\partial D} Q(z,\zeta)^{2\ell} B(\zeta)^{-\ell} \zeta^{\ell-1}\, dz =: z^\ell B(z)^{-\ell} I_\ell(z) . \tag{1.26}$$

This equation extends the definition of the function $I_\ell(z)$ onto $\bar D$. By means of the binomial formula we derive for $z \in D$ that

$$\begin{cases} I_\ell(z) = \displaystyle\sum_{k=-\ell}^{\ell} \binom{2\ell}{\ell-k} \big(-B(z)\big)^{\ell-k} I_{\ell,k}(z) , \\[2ex] I_{\ell,k}(z) := \dfrac{(2\ell-1)!}{2\pi i} \displaystyle\int_{\partial D} \frac{B(\zeta)^k \zeta^{\ell-1}\, d\zeta}{(\zeta - z)^{2\ell}} . \end{cases} \tag{1.27}$$

The function $B(\zeta)^k \zeta^{\ell-1}$ is analytic on $\bar D_+$ for $k \ge 0$, while for $k < 0$ it is analytic on $\bar D_-$ and has at ∞ a pole of order at most $\ell - 1$. The Cauchy integral formula yields now

$$I_{\ell,k}(z) = \begin{cases} (B(z)^k z^{\ell-1})^{(2\ell-1)} & \text{if } k = 1, \dots, \ell , \\ 0 & \text{if } k = -\ell, \dots, 0 . \end{cases}$$

From (1.26) and (1.27) we derive (1.25). □

The integral of the kernel K_α of (1.17) can be estimated by means of the function λ_m of (1.10).

Lemma 1.7. *For each $\alpha > 0$ and $z \in \partial D$,*

$$\int_{\partial D} K_\alpha(z,\zeta)\, |d\zeta| \le C \lambda_m(z,\beta)^\alpha , \tag{1.28}$$

where $C = C(\alpha)$ and $\beta = (\alpha+2)^{-1}$.

Proof. If α is an odd integer, we estimate the sum in (1.25) by means of Lemma 1.4(i). For $k = 1, 2, \dots, \ell$, $B_m^{(k)}(z) z^{\ell-1}$ is a Blaschke product, whose $(2\ell-1)$-st derivative can be estimated by means of $C(k,\ell)\lambda_m(z, \frac{1}{2\ell-1})^{2\ell-1}$. This yields (1.28), where one can take $\beta = \alpha^{-1}$.

We shall now show that this implies (1.28) for any $\alpha > 0$ with $\beta = (\alpha+2)^{-1}$. Let ℓ be the smallest odd integer $> \alpha$. We put $p := (\ell+1)/(\alpha+1)$ and define

$$S(z) = \{\zeta \in \partial D : |\arg\zeta - \arg z| \le \lambda(z, 1/\ell)^{-1}\} .$$

By applying Hölder's inequality and the case just discussed, we obtain

$$\int_{S(z)} K_\alpha(z,\zeta)\,|d\zeta| \le |S(z)|^{1/p'} \|K_\alpha(z,\cdot)\|_p(S(z))$$
$$\le 2^{1/p'}\lambda(z,1/\ell)^{-1/p'}\|K_\ell(z,\cdot)\|_1^{1/p}(\partial D) \le C\lambda(z,1/\ell)^\alpha .$$

On the other hand, $K_\alpha(z,\zeta) \le 2^{\alpha+1}|\zeta - z|^{-\alpha-1}$ implies that (see also the proof of (1.14))

$$\int_{\partial D\setminus S(z)} K_\alpha(z,\zeta)\,|d\zeta| \le C\lambda(z,1/\ell)^\alpha .$$

The two inequalities yield (1.28) with $\beta = 1/\ell$. It remains to remark that $\ell \le \alpha + 2$. □

Proof of Theorem 1.2. Part (1.4) of this theorem immediately follows from (1.18) and Theorem 1.5, applied to the kernel $L_\alpha = K_\alpha$. The proof of the second part (1.5) is quite similar. Instead of (1.15) we have now, for the component R_- of R in (1.3),

$$R_-^{(r)}(z) = \frac{r!}{2\pi i}\int_{\partial D} \frac{R(\zeta)}{(\zeta - z)^{r+1}}\,d\zeta , \qquad z \in D_- .$$

For the Blaschke product

$$B_-(z) := \prod_{k=m+1}^{m+n} \frac{z - a_k}{1 - \bar{a}_k z}$$

the functions $R(\zeta)B_-(\zeta)^k(\zeta - z)^{-r-1}$, $k = 1, 2, \ldots$ are all analytic functions of ζ on $\bar{D}_+$. As in the proof of (1.18) we get

$$(1.29) \qquad |R_-^{(r)}(z)| \le \frac{r!}{2\pi}\int_{\partial D} K_r^-(z,\zeta)|R(\zeta)|\,|d\zeta| , \qquad z \in \partial D ,$$

where the kernel $K_\alpha^-(z,\zeta)$, $\alpha > 0$, is obtained from $K_\alpha(z,\zeta)$ by replacing the Blaschke product $B(z)$ by $B_-(z)$. Then (1.5) follows from Theorem 1.5 with $L_\alpha = K_\alpha^-$. □

For the proof of Theorem 1.1, it is convenient to replace $[0,1]$ by the interval $[-1,1]$. Let R be a rational function of degree m, without poles on $[-1,1]$. We begin by establishing the estimate (1.31) for $R^{(r)}(x)$, similar to (1.18).

Let Ω be the complement of $[-1,1]$ in $\mathbb{C}$, let Γ be its boundary, which we assume to consist of two copies of $[-1,1]$, the upper interval and the lower one. We take the branch of the function $w(\eta) = \eta + \sqrt{\eta^2 - 1}$, $\eta \in \Omega$, which maps Ω onto D_-. If $\rho > 1$ is so close to one that all poles of R are outside of the ellipse $\Gamma_\rho := \{\eta : |w(\eta)| = \rho\}$, then by Cauchy's formula,

(1.30)
$$R^{(r)}(x) = \frac{r!}{2\pi i}\int_{\Gamma_\rho}\frac{R(\eta)\,d\eta}{(\eta - x)^{r+1}}\,, \qquad x \in \Gamma\,.$$

If $\eta_1, \dots, \eta_m$ are the poles of R (with multiplicities accounted for), then the Blaschke product

$$B\big(w(\eta)\big) := B_m\big(w(\eta)\big) = \prod_{j=1}^{m}\frac{w(\eta) - a_j}{1 - \bar{a}_j w(\eta)}\,, \qquad a_j := 1/\overline{w(\eta_j)}$$

has poles at the η_j. For each $k = 1, 2, \dots$ and $x \in \Gamma$, the function $R(\eta)B(w(\eta))^{-k}$ $(\eta - x)^{-r-1}$ is analytic in Ω, and has at ∞ a zero of order at least two. It follows from (1.30) that, for $x \in \Gamma$,

$$R^{(r)}(x) = \frac{r!}{2\pi i}\int_{\Gamma_\rho}\frac{R(\eta)}{(\eta - x)^{r+1}}\left(1 - \frac{B(w(x))}{B(w(\eta))}\right)^{r+1}\left(1 - \frac{B\big(\overline{w(x)}\big)}{B(w(\eta))}\right)^{r+1}|d\eta|\,.$$

Then, as in the proof of (1.18), and making $\rho \to 1$, we derive from this

(1.31)
$$|R^{(r)}(x)| \le \frac{r!}{2\pi}\int_{\Gamma} G_r(x, \eta; m)|R(\eta)|\,|d\eta|\,, \qquad x \in \Gamma\,,$$

where, for all $\alpha > 0$,

(1.32)
$$\begin{aligned} G_\alpha(x,\eta) &:= G_\alpha(x, \eta; m) \\ &:= \left|\frac{(B(w(\eta)) - B(w(x))(B(w(\eta)) - B\big(\overline{w(x)}\big))}{\eta - x}\right|^{\alpha+1}\,. \end{aligned}$$

Lemma 1.8. *For each $\alpha > 0$, $m = 1, 2, \dots$ and some $C(\alpha)$,*

(1.33)
$$\int_{\Gamma}\left(\int_{\Gamma} G_\alpha(x, \eta; m)\,|d\eta|\right)^{1/\alpha}|dx| \le C(\alpha)m\,.$$

Proof. In the integral (1.33) we perform the substitutions

$$\eta = w^{-1}(\zeta) = \frac{1}{2}\Big(\zeta + \frac{1}{\zeta}\Big)\,, \quad x = \frac{1}{2}\Big(z + \frac{1}{z}\Big)\,, \qquad z, \zeta \in \partial D\,.$$

Then $G_\alpha(x, \eta) = K_\alpha(z, \zeta)K_\alpha(\bar{z}, \zeta)$, $|d\eta| = |1 - \zeta^2|\,|d\zeta|$, $|dx| = |1 - z^2|\,|dz|$. The interior integral $J_\alpha(x)$ in (1.33) becomes

$$J_\alpha(x) = \int_{\partial D} K_\alpha(z, \zeta)K_\alpha(\bar{z}, \zeta)|1 - \zeta^2|\,|d\zeta|\,.$$

We show that

(1.34)
$$J_\alpha(x) \le C|1 - z^2|^{-\alpha}\big(\lambda(z)^\alpha + \lambda(\bar{z})^\alpha\big)\,, \qquad z \in \partial D\,,$$

where $C = C(\alpha)$, $\lambda(\cdot) = \lambda_m(\cdot, \frac{1}{\alpha+2})$.

Indeed, for a fixed α, let S_1 be the arc of ∂D that lies on the same side with z with respect to the horizontal axis. Then

$$\frac{|1-\zeta^2|}{|\zeta-\bar{z}|^{\alpha+1}} \le \frac{2^{\alpha+1}}{|1-z^2|^{\alpha}}\,, \qquad \zeta \in S_1\,. \tag{1.35}$$

Indeed, the numbers $|1-z^2| = |z-\bar{z}|$ and $|1-\zeta^2| = |\zeta-\bar{\zeta}|$ are the lengths of the two bases of the trapezoid with the vertices $z, \bar{z}, \zeta, \bar{\zeta}$, and $|\zeta-\bar{z}|$ is the length of its diagonal. Each base does not exceed twice the diagonal, this gives (1.35). Now, the definition of K_α and (1.35) imply

$$K_\alpha(\bar{z},\zeta) \le \frac{2^{\alpha+1}}{|\zeta-\bar{z}|^{\alpha+1}} \le \frac{4^{\alpha+1}}{|1-\zeta^2|\,|1-z^2|^{\alpha}}\,,$$

and the part of integral (1.34) over S_1 is

$$\le \frac{4^{\alpha+1}}{|1-z^2|^{\alpha}} \int_{S_1} K_\alpha(z,\zeta)\,|d\zeta| \le \frac{C(\alpha)\lambda(z)^{\alpha}}{|1-z^2|^{\alpha}}\,.$$

The part of (1.34) over S_2 — the arc of ∂D on the opposite side of z with respect to the horizontal axis — has a similar estimate with $\lambda(z)$ replaced by $\lambda(\bar{z})$. This establishes the inequality (1.34).

Substituting $J_\alpha(x)$ into the integral (1.33), we get for it by Lemma 1.4(ii) the upper bound

$$C(\alpha)\int_{\partial D} (\lambda(z)+\lambda(\bar{z}))\,|dz| \le C(\alpha)m\,. \qquad \square$$

Proof of Theorem 1.1. This follows by applying Theorem 1.5 for the integral in (1.31). $\square$

§ 2. Uniform Rational Approximation in Hardy Spaces

For a function f analytic on the disk $D := \{z \in \mathbb{C} : |z| < 1\}$ and continuous on its closure $\bar{D}$, we put $\|f\|_{\mathcal{A}} := \max_{z\in\bar{D}} |f(z)|$. This defines the Banach space $\mathcal{A}$. The Hardy spaces H_p, (see Appendix 3, §1) are Banach spaces for $1 \le p \le \infty$, for $0 < p < 1$ they are quasi-normed spaces. In addition, we now define spaces H_p^r, $r = 0, 1, \ldots$, $0 < p \le \infty$ to consist of all functions f, analytic on D, with $f^{(r)} \in H_p$. In particular, $H_p^0 = H_p$. Plainly, we have the inclusions:

$$H_p^{r_0} \subset H_p^{r_1} \quad \text{for} \quad 0 \le r_1 < r_0\,,\ 0 < p \le \infty\,; \tag{2.1}$$

$$H_{p_0}^{r} \subset H_{p_1}^{r} \quad \text{for} \quad r \ge 0\,,\ 0 < p_1 < p_0 \le \infty\,. \tag{2.2}$$

Less obvious are the inclusions

(2.3) $\quad H_{p_0}^{r_0} \subset H_{p_1}^{r_1}$ for $\dfrac{1}{p_0} - \dfrac{1}{p_1} = r_0 - r_1 \geq 1$, $0 < p_0 < p_1 < \infty$

(2.4) $\quad H_{1/r}^r \subset \mathcal{A}$ for $r \geq 1$.

Indeed, (2.3) follows from Theorem 1.5 of Appendix 3 by iteration. In particular, $H_{1/(r+1)}^{r+1} \subset H_{1/r}^r$; this, together with Theorem 1.6 of Appendix 3 implies (2.4).

The following are our main theorems:

Theorem 2.1 (Pekarskii [1987]). *For each $f \in H_{1/r}^r$, $r = 1, 2, \ldots$,*

$$\rho_n(f)_{\mathcal{A}} \leq \frac{C(r)}{n^r} \|f^{(r)}\|_{H_{1/r}} , \qquad n \geq r \tag{2.5}$$

$$\rho_n(f)_{\mathcal{A}} = o\left(\frac{1}{n^r}\right) , \qquad n \to \infty . \tag{2.6}$$

Theorem 2.2. *For each $f \in H_{1/r}^r$, relation* (2.5) *holds also with $\rho_n(f)_{\mathcal{A}}$ replaced by $\rho_n(f)_{C(\partial D)}$.*

Theorem 2.3 (Dolzhenko [1966] for $r = 1$, Pekarskii [1980] for $r \geq 2$). *For $f \in \mathcal{A}$, $r = 1, 2, \ldots$, relation*

$$\left(\sum_{n=0}^{\infty} \rho_n(f)_{\mathcal{A}}^{1/r}\right)^r =: Q < \infty \tag{2.7}$$

implies that $f \in H_{1/r}^r$ and $\|f^{(r)}\|_{H_{1/r}} \leq C(r)Q$.

These results show that for rational approximation in $\mathcal{A}$ or $C(\partial D) = C(\mathbb{T})$, the spaces $H_{1/r}^r$ play a role similar to the Sobolev spaces $W_\infty^r(\mathbb{T})$ for the polynomial approximation in $C(\mathbb{T})$. Another interesting comparison is with the results of Dolzhenko in the space $W_1^1[0,1]$. On one hand, he showed that $f \in C[0,1]$ and $\sum \rho_n(f)_C < \infty$ implies $f \in W_1^1$. His other result (Theorem 7.7 of Chapter 7) asserts that for $f \in W_1^1$, the sequence $(\rho_n(f)_C)$ can decrease to zero arbitrarily slowly. This exhibits considerable difference with rational approximation on the circle.

The proof of Theorem 2.3 in the (quasi-) normed space H_p follows standard inverse theorem methods, once the Bernstein-type inequality (1.6) for rational functions is known. We leave the proof to the reader. □

The proofs of Theorems 2.1 and 2.2 are much more difficult. The second theorem is an obvious corollary of the first, since for all $f \in \mathcal{A}$, $\rho_n(f)_{C(\partial D)} \leq \rho_n(f)_{\mathcal{A}}$. Our plan of proof is as follows. We shall need Theorems 2.5 and 2.6 of the present section (their proofs occupy §§3,4). They will allow us to deduce Theorem 2.2. Finally, properties of Blaschke products will lead from this to

Theorem 2.1. Later, in §5, we shall replace Theorem 2.2 by more general results about approximation of functions definded on $A = \mathbb{T}$ or $A = [-1, 1]$ alone.

It will be convenient to reformulate the atomic decomposition theorems of §3 of Appendix 3 in terms of simple functions. A function φ on $A = \partial D$ or $A = \mathbb{R}$ will be called *an r-simple function*, $r = 1, 2, \dots$, with the *supporting* open finite *interval* $J := J(\varphi)$ if $\varphi \in W_\infty^r(A)$ and $\varphi(\zeta) = 0$ on $A \setminus J$. In the case that $J(\varphi) = \partial D \setminus (\zeta_0)$ for a fixed point $\zeta_0 \in \partial D$, we additionally require that $\varphi(\zeta_0) = \dots = \varphi^{(r-1)}(\zeta_0) = 0$.

There is a close relation between r-atoms $a \in L_\infty(A)$ and r-simple functions. One easily proves by induction on r:

1. *If φ is an r-simple function, then $a := \varphi^{(r)}$ is an r-atom and $J(a) = J(\varphi)$.*

In the opposite direction we have:

2. *If a is an r-atom, and $\zeta_0 \notin J(a)$, then*

$$\varphi(z) := \frac{1}{(r-1)!} \int_{\zeta_0}^{z} (z - \zeta)^{r-1} a(\zeta)\, d\zeta$$

is an r-simple function, and $J(\varphi) = J(a)$.

For a and φ related in this way,

$$\lambda_r(a) := |J(a)|^r \|a\|_\infty = \mu_r(\varphi) := |J(\varphi)|^r \|\varphi^{(r)}\|_\infty .$$

We also note the inequality, for an r-simple function φ,

$$\|\varphi^{(s)}\|_\infty \le |J(\varphi)|^{r-s} \|\varphi^{(r)}\|_\infty , \qquad s = 0, \dots, r . \tag{2.8}$$

The Lemma 3.1 of Coifman of Appendix 3 takes the form:

Lemma 2.4. *For each function f, analytic on $\bar{D}$, and each $r = 1, 2, \dots$, there exists a polynomial $P \in \mathcal{P}_{r-1}$, and a finite or infinite sequence of r-simple functions $(\varphi_k)_{k \ge 1}$ which satisfy:*

(i) $$\sum_{k \ge 1} \mu_r(\varphi_k)^{1/r} \le C \|f^{(r)}\|_{H_{1/r}}^{1/r} , \qquad C = C(r) ;$$

(ii) $$f(\zeta) = P(\zeta) + \sum_{k \ge 1} \varphi_k(\zeta) , \qquad \zeta \in \partial D$$

with uniform convergence on ∂D;

(iii) any two supporting intervals $J(\varphi_k)$, $J(\varphi_{k'})$ are either disjoint, or one contains the other.

Proof. This is obtained by r-tuple integration of the uniformly convergent expansion (ii) of Theorem 3.1, Appendix 3. □

In the following two theorems 1-simple functions on ∂D will be called *simple*; we put $\mu(\varphi) := \mu_1(\varphi)$.

Theorem 2.5. *Let $(\varphi_k)_{k\geq 1}$ be a finite sequence of simple functions, for which any two intervals $J(\varphi_k)$ and $J(\varphi_{k'})$ are either disjoint, or one is contained in the other; let $f := \varphi_1 + \varphi_2 + \cdots$ and $V := \mu(\varphi_1) + \mu(\varphi_2) + \cdots$. Then for each $m = 1, 2, \ldots$ there exist simple functions $(\psi_j)_{j=1}^m$ for which*

$$\Big\| f - \sum_{j=1}^m \psi_j \Big\|_{C(\partial D)} \leq \frac{CV}{m} ,$$

$$\sum_{j=1}^m \mu(\psi_j) \leq CV ,$$

where $C > 0$ is an absolute constant.

For the *proof* see §3.

Theorem 2.6. *Let $(\psi_j)_{j=1}^m$ be simple functions, let $g := \psi_1 + \cdots + \psi_m$. Then there exists a rational function $R \in \mathcal{R}_{8m-4}$ for which*

$$\|g - R\|_{C(\partial D)} \leq \frac{C}{m} \sum_{j=1}^m \mu(\psi_j) ,$$

where $C > 0$ is an absolute constant.

For the *proof* see §4.

The derivatives of the superposition of two functions, $(\varphi \circ w)(t) := \varphi(w(t))$, are given by the formula of Faa di Bruno; under natural assumptions on the functions $y = \varphi(x)$ and $x = w(t)$, one has

(2.9)
$$(\varphi \circ w)^{(r)} = \sum \frac{r!}{s_1! s_2! \ldots s_r!} (\varphi^{(s_1+s_2+\cdots+s_r)} \circ w) \left(\frac{w'}{1!}\right)^{s_1} \left(\frac{w''}{2!}\right)^{s_2} \cdots \left(\frac{w^{(r)}}{r!}\right)^{s_r} ,$$

with the summation extended over all selections of non-negative integers $s_1, s_2, \ldots, s_r$ which satisfy $1s_1 + 2s_2 + \cdots + rs_r = r$.

The reader can easily establish this formula by induction. We may add that the values of the coefficients in (2.9) are of no importance in the sequel.

Before the proof of Theorem 2.2, we interpose two simple lemmas.

Lemma 2.7. *Let φ be an r-simple function on ∂D, $r \geq 2$. Then for each $m = 0, 1, \ldots$ there exists a rational function $R(\cdot) = R_m(\varphi, \cdot) \in \mathcal{R}_m$, without poles on $\bar{D}$, with the property that*

$$\|(C^+\varphi - R)'\|_{H_1} \leq \frac{C(r)}{(m+1)^{r-1}}\, \mu_r(\varphi) . \tag{2.10}$$

Proof. First let $|J(\varphi)| \geq \pi$, then $\mu_r(\varphi) \geq C(r)\|\varphi^{(r)}\|_\infty$. We take $R' = C^+R'_0$, where R'_0 will be determined in a moment. By partial integration of the Cauchy integral we find $(C^+(\varphi - R_0))' = C^+(\varphi - R_0)'$, therefore

$$\|(C^+\varphi - R)'\|_{H_1} \leq \|C^+(\varphi - R_0)'\|_{L_2} \leq \|\varphi' - R'_0\|_{L_2} \,. \tag{2.11}$$

In the case $m \geq 1$ we take R'_0 to be the trigonometric polynomial $\sum_{k=-m+1}^{m-1} c_k\zeta^k \in \mathcal{T}_{m-1} \subset \mathcal{R}_{m-1}$ of best L_2 approximation to φ'. By [**CA**, (2.12), p.205], (2.11) does not exceed $C(m+1)^{r-1}\|\varphi^{(r)}\|_2 \leq C(m+1)^{r-1}\mu_r(\varphi)$, and R has no poles in $\bar{D}$. In the case $m = 0$ we take $R \equiv 0$ and (2.10) is obtained from (2.8).

We assume now that $|J(\varphi)| < \pi$. We can suppose that i is the middle-point of $J(\varphi)$, then its endpoints are $ie^{i\theta}$ and $ie^{-i\theta}$, where $\theta := \frac{1}{2}|J(\varphi)|$. To use the case just discussed, we map $\bar{D}$ onto itself by a Möbius transformation $z = w(\zeta)$, with $w(i) = i$, $w(\pm 1) = ie^{\mp i\theta}$. It is easy to calculate that

$$w(\zeta) = \frac{\zeta + a}{1 - a\zeta}\,, \quad w^{-1}(z) = \frac{z - a}{1 + az}\,, \quad a = i\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

is this map. Then $w^{-1}(J(\varphi)) = \Gamma := \{\zeta \in \partial D : Im\,\zeta > 0\}$ and the function $\varphi_0 := \varphi \circ w$, $\varphi_0(\zeta) = \varphi(w(\zeta))$ is r-simple with $J(\varphi_0) = \Gamma$. We have for $\zeta \in \Gamma$ and $k = 1, 2, \ldots$

$$|w^{(k)}(\zeta)| = \left|\frac{k!a^{k-1}(1 + a^2)}{(1 - a\zeta)^{k+1}}\right| \leq 2k!\theta \,.$$

Therefore by the formula of Faa di Bruno and (2.8) we obtain $|\varphi_0^{(r)}(\zeta)| \leq C\mu_r(\varphi)$ for $\zeta \in \Gamma$. Taking into account that $\varphi_0(\zeta) \equiv 0$ for $\zeta \in \partial D \setminus \Gamma$ we have

$$\mu_r(\varphi_0) \leq C\|\varphi_0^{(r)}\| \leq C\mu_r(\varphi)\,, \qquad C = C(r)\,.$$

We apply (2.11) to the function φ_0, and performing the inverse substitution, obtain (2.10) for $|J(\varphi)| < \pi$. □

Lemma 2.8. *Let $f \in H^r_{1/r}$, $r = 1, 2, \ldots$. There exists then for each $m > r - 1$ a rational function $R \in \mathcal{R}_m$ without poles on $\bar{D}$, which satisfies*

$$\|f' - R'\|_{H_1} \leq \frac{C(r)}{m^{r-1}}\|f^{(r)}\|_{H_{1/r}} \,. \tag{2.12}$$

Proof. Without loss of generality, we can assume that f is analytic on $\bar{D}$. Let P and $(\varphi_k)_{k\geq 1}$ be the functions of Lemma 2.4. For convenience, let $\|f^{(r)}\|_{H_{1/r}} = C_1^{-r}$, where C_1 is the constant in (i), Lemma 2.4, then

$$\sum_{k\geq 1} \mu_r(\varphi_k)^{1/r} \leq 1 \,. \tag{2.13}$$

We put $m_k := [(m - r + 1)\mu_r(\varphi_k)^{1/r}]$, $R_k(z) := R_{m_k}(\varphi_k, z)$, where $R_m(\varphi, \cdot)$ is the rational function of Lemma 2.7. It is defined up to a constant, and we shall assume that $R_k(0) = 0$. We want to show that

$$R(z) := P(z) + \sum_{k\geq 1} R_k(z) \tag{2.14}$$

satisfies the requirements of Lemma 2.8. Since $\mu_r(\varphi_k) \to 0$ for $k \to 0$, we have $m_k = 0$ and therefore $R_k = 0$ for all large k. Then

$$\deg R \leq r - 1 + \sum_{k\geq 1} m_k \leq r - 1 + (m - r + 1) \sum_{k\geq 1} \mu_k(\varphi_k)^{1/r} \leq m \ .$$

On the other hand, from Lemma 2.7,

$$\|f' - R'\|_{H_1} = \|(C^+ f - R)'\|_{H_1} \leq \sum_{k\geq 1} \|(C^+ \varphi_k - R_k)'\|_{H_1}$$

$$\leq C(r) \sum_{k\geq 1} \frac{\mu_r(\varphi_k)}{(m_k + 1)^{r-1}} \leq C(r) \sum_{k\geq 1} \frac{\mu_r(\varphi_k)^{1/r}}{(m - r + 1)^{r-1}} \leq C(r) \frac{1}{m^{r-1}} \ . \qquad \square$$

Proof of Theorem 2.2. (a) For $r = 1$, relation (2.5) follows from Theorems 2.5 and 2.6. Indeed, from Theorem 1.6 of Appendix 3, $\|f(z) - f(0)\|_{\mathcal{A}} \leq C\|f'\|_{H_1}$ and (2.5) is trivially true for small n. We assume that $n \geq 8$. As usual, we can suppose that f is analytic on $\bar{D}$. Lemma 2.4 yields the formula $f = C + \sum_{k\geq 1} \varphi_k$ on ∂D with $V := \sum \mu(\varphi_k) \leq C\|f'\|_{H_1}$. Because of the uniform convergence of $\sum_{k\geq 1} \varphi_k$, we can restrict ourselves to finite sums. We take $m = [n/8]$ in Theorems 2.5 and 2.6 and obtain $\|f - R\|_{C(\partial D)} \leq \frac{C}{n}\|f'\|_{H_1}$ with a rational function $R \in \mathcal{R}_n$.

(b) If $r \geq 2$, we combine (a) with Lemma 2.8 to deduce, for some $R \in \mathcal{R}_n$, that

$$\rho_{2n}(f) \leq \rho_n(f - R) \leq \frac{C}{n}\|f' - R'\|_{H_1} \leq \frac{C(r)}{n^r}\|f^{(r)}\|_{H_{1/r}} \ . \qquad \square$$

To obtain from this Theorem 2.1, we need some properties of Blaschke products from Appendix 3.

Theorem 2.9 (Pekarskii [1982₁]). *For a function $f \in \mathcal{A}$ one has for $n =$ $1, 2, \ldots$*

$$\begin{cases} \rho_n(f)_{C(\partial D)} \leq \rho_n(f)_{\mathcal{A}} \ , \\ \rho_n(f)_{\mathcal{A}} \leq 2\rho_n(f)_{C(\partial D)} \ . \end{cases} \tag{2.15}$$

Proof. The first inequality is obvious. To prove the second, let $R \in \mathcal{R}_n$ be the rational function that satisfies $\|f - R\|_{C(\partial D)} = \rho_n(f)_{C(\partial D)} =: \rho$. There is nothing to prove if R has no poles in D. We assume that the poles of R in D are $z_1, \ldots, z_m$, $1 \leq m \leq n$. This implies that $\rho > 0$.

Since R is continuous on ∂D as a function of the poles, we may assume the z_j to be distinct. Let $B_1(z)$ be the Blaschke product of degree m with zeros at the z_j. The function $g := B_1(f - R)$ belongs to $\mathcal{A}$ with $\|g\|_{\mathcal{A}} = \rho$. By

Theorem 4.2 of Appendix 3 there exists a $q \in (0, \rho]$ and a Blaschke product B_2 of degree at most $m-1$ for which $g(z_j) - qB_2(z_j) = 0$, $j = 1, \dots, m$. Then the function

$$h := \frac{g - qB_2}{B_1} = f - \left(R + q\frac{B_2}{B_1}\right)$$

also belongs to $\mathcal{A}$ and satisfies $\|h\|_{\mathcal{A}} \le 2\rho$. It remains to remark that the rational function $R + qB_2/B_1$ has no poles on $\bar{D}$, so that its degree does not exceed $n-1$. □

Proof of Theorem 2.1. The inequality (2.5) is an immediate consequence of Theorems 2.2 and 2.9.

To prove (2.6), let $P \in \mathcal{P}_m$ be arbitrary, let $m \ge r$. Then from (2.5),

$$\rho_{2m}(f)_{\mathcal{A}} \le \rho_m(f - P)_{\mathcal{A}} \le Cm^{-r}\|f^{(r)} - P^{(r)}\|_{H_{1/r}} \, .$$

Consequently (see Theorem 1.3 of Appendix 3)

$$\rho_{2m}(f)_{\mathcal{A}} \le Cm^{-r}E_{m-r}(f^{(r)})_{H_{1/r}} = o(m^{-r}) \, . \qquad \square$$

§ 3. Approximation by Simple Functions

In this section we shall prove Theorem 2.5, which has been used in the proof of Theorem 2.1. Without loss of generality, we shall assume that the functions φ_k, appearing in Theorem 2.5 are real. First we shall establish an analogue of Theorem 2.5, with all functions on $\mathbb{R}$ (Theorem 3.5 below) and later explain how this implies the required statement for ∂D. All simple functions φ in this section will be of order 1, with $J(\varphi)$, $\mu(\varphi) =: \mu_1(\varphi)$ defined as in §2, approximations will be in the uniform norm $\|\cdot\| = \|\cdot\|_\infty$.

For each fixed $t > 0$, we define the simple function $\Delta_t(x) = \max\{0, 1 - |x|/t\}$, $x \in \mathbb{R}$, with the supporting interval $J(\Delta_t) = (-t, t)$ and with $\mu(\Delta_t) = 2$.

Lemma 3.1. *Let $f(x) := a_1\Delta_{t_1}(x) + \cdots + a_p\Delta_{t_p}(x)$, where $(a_k)_1^p$ and $(t_k)_1^p$ are finite real sequences and $t_1 > \cdots > t_p = 1$. Then for each $m \ge 4$ there exist sequences $(a'_j)_1^m$, $(t'_j)_1^m$ for which $t_1 = t'_1 > \cdots > t'_m = 1$, $|a'_1| + \cdots + |a'_m| = |a_1| + \cdots + |a_p|$ and*

$$f(x) - \sum_{j=1}^m a'_j\Delta_{t'_j}(x) = 0 \quad \textit{for} \quad |x| \le 1 \quad \textit{and} \quad |x| \ge t_1 \, , \tag{3.1}$$

$$\left\| f - \sum_{j=1}^m a'_j\Delta_{t'_j} \right\| \le 4m^{-1}\sum_{k=1}^p |a_k| \, . \tag{3.2}$$

Proof. First let $a_k \geq 0$, $k = 1, \ldots, p$. The function $f(x)$ has then the following properties: it is even on $\mathbb{R}$, convex and non-increasing on $[0, \infty)$, linear on $[0, 1]$, and equal to zero on $[t_1, \infty)$. We put $t'_m := 1$, $t'_1 := t_1$, and determine the numbers $t'_{m-1} < \cdots < t'_2$ from the conditions $f(t'_k) - f(t'_{k-1}) = \frac{1}{m-1} f(1)$. With $f_1(x)$ we denote the even continuous function on $\mathbb{R}$ which satisfies the conditions $f_1(0) = f(0)$, $f_1(t'_k) = f(t'_k)$ for $k = 1, \ldots, m$, $f_1(x) = 0$ on $[t'_1, \infty)$ and which is linear on the intervals $[0, 1]$ and $[t'_k, t'_{k-1}]$ for $k = m, \ldots, 2$. We obtain

$$f_1(x) - f(x) = 0 \quad \text{for} \quad |x| \leq 1 \quad \text{and} \quad |x| \geq t'_1 \ ,$$

$$0 \leq f_1(x) - f(x) \leq \frac{1}{m-1} f(1) \quad \text{for} \quad x \in \mathbb{R} \ ,$$

$$f_1(x) = \sum_{j=1}^{m} a'_j \Delta_{t'_j}(x) \quad \text{for} \quad x \in \mathbb{R} \ ,$$

where the a'_j are some nonnegative numbers. Since $f_1(0) = f(0)$, we get $a_1 + \cdots + a_p = a'_1 + \cdots + a'_m$. This yields the relations (3.1) and (3.2). In the general case, one treats separately the terms $a_k \Delta_{t_k}(x)$ with $a_k \geq 0$ and with $a_k < 0$. □

Lemma 3.2. *Let $(\varphi_k)_{k \geq 1}$ be a finite sequence of simple functions on $\mathbb{R}$ which have disjoint supporting intervals, let $f := \varphi_1 + \varphi_2 + \cdots$, $V := \mu(\varphi_1) + \mu(\varphi_2) + \cdots$. For each $m = 1, 2, \ldots$ there exist numbers $1 \leq k_1 < \cdots < k_d$, $d \leq m$, with the property*

$$\left\| f - \sum_{j=1}^{d} \varphi_{k_j} \right\| \leq \frac{V}{2(m+1)} \ . \tag{3.3}$$

Proof. We can assume that the number of functions φ_k is larger than m (otherwise (3.3) is obvious) and that the sequence $(\mu(\varphi_k))_{k \geq 1}$ is nonincreasing. Then $\mu(\varphi_k) \leq V/k$ for all k. Since the intervals $J(\varphi_k)$ are disjoint, we deduce

$$\left\| f - \sum_{j=1}^{m} \varphi_j \right\| \leq \max_{k \geq m+1} \|\varphi_k\| \leq \frac{V}{2(m+1)} \ . \qquad \square$$

The following is the case of Theorem 3.5 for a special configuration of the J_k.

Lemma 3.3. *Let $(\varphi_k)_{k=1}^{p}$ be a sequence of simple functions on $\mathbb{R}$, let $f := \varphi_1 + \cdots + \varphi_p$, $J_k := J(\varphi_k)$, $\mu_k := \mu(\varphi_k)$ and $V := \mu_1 + \cdots + \mu_p$. If $J_p \subset \cdots \subset J_1$, then for each $m \geq 12$ there exist simple functions $(\psi_j)_{j=1}^{m}$ which satisfy*

$$(a) \qquad f(x) - \sum_{j=1}^{m} \psi_j(x) = 0 \quad for \quad x \in J_p \cup (\mathbb{R} \setminus J_1) \ ,$$

$$(b) \qquad \Big\| f - \sum_{j=1}^{m} \psi_j \Big\| \le \frac{C_1 V}{m} \ ,$$

$$(c) \qquad \sum_{j=1}^{m} \mu(\psi_j) \le C_2 V \ ,$$

where $C_1 > 0$, $C_2 > 0$ are absolute constants.

Proof. Step 1. Without loss of generality, we can assume that $J_k = (-\alpha_k, \beta_k)$, where $\alpha_1 \ge \cdots \ge \alpha_p = 1$ and $\beta_1 \ge \cdots \ge \beta_p = 1$. Let $t_k = \min\{\alpha_k, \beta_k\}$, $\varphi_{0,k}(x) = \varphi_k(0)\Delta_{t_k}(x)$, $\varphi_{1,k}(x) = \varphi_k(x) - \varphi_{0,k}(x)$ and

$$f_s(x) := \sum_{k=1}^{p} \varphi_{s,k}(x) \ , \qquad s = 0, 1 \ .$$

Since $|\varphi_k(0)| \le \mu_k/2$, Lemma 3.1 guarantees the existence of simple functions $(\psi_{0,j})_{j=0}^{q}$, $q := [m/3]$ which satisfy

$$(a_0) \qquad f_0(x) - \sum_{j=1}^{q} \psi_{0,j}(x) = 0 \ \text{ for } \ |x| \ge t_1 \ , \ |x| \le 1 \ ,$$

$$(b_0) \qquad \Big\| f_0 - \sum_{j=1}^{q} \psi_{0,j} \Big\| \le \frac{8V}{m} \ ,$$

$$(c_0) \qquad \sum_{j=1}^{q} \mu(\psi_{0,j}) \le V \ .$$

Step 2. On $\mathbb{R}$ we define the continuous functions $\eta^+(x)$, $\eta^-(x)$ by putting $\eta^+(\pm 2^j) := (1 + (-1)^j)/2$, $j = 0, 1, 2, \ldots$, with a linear extension onto the intervals $(-1, 1)$, $(2^j, 2^{j+1})$, $(-2^{j+1}, -2^j)$ and $\eta^-(x) := 1 - \eta^+(x)$. We define $f_1^\pm(x) := f_1(x)\eta^\pm(x)$. We shall show that there exist two sequences of simple functions $(\psi_{1,j}^\pm)_{j=1}^{q}$ which satisfy, respectively,

$$(a_1^\pm) \qquad f_1^\pm(x) - \sum_{j=1}^{q} \psi_{1,j}^\pm(x) = 0 \ , \quad x \in J_p \cup (\mathbb{R} \setminus J_1) \ ,$$

$$(b_1^\pm) \qquad \Big\| f_1^\pm - \sum_{j=1}^{q} \psi_{1,j}^\pm \Big\| \le \frac{C_3 V}{m} \ ,$$

$$(c_1^\pm) \qquad \sum_{j=1}^{q} \mu(\psi_{1,j}^\pm) \le C_4 V \ .$$

Indeed, since $\varphi_{1,k}(0) = 0$, we have $f_1^+(0) = 0$, so that the function $f_1^+(x)$ is a sum of simple functions $(\theta_s^+(x))_{s\ge 1}$ with the supporting intervals $(-\alpha_1, -2^{k_1})$, $(-2^{k_1}, -2^{k_1-2})$, ..., $(-2^3, -2)$, $(-2, 0)$, $(0, 2)$, $(2, 2^3)$, ..., $(2^{n_1}, \beta_1)$, where k_1 and n_1 are the largest odd integers which satisfy $2^{k_1} < \alpha_1$ and $2^{n_1} < \beta_1$. Here if, for example, $\alpha_1 \le 2$, then $k_1 = -\infty$. Since $J(\varphi_{1,k}) = J_k$, $\mu(\varphi_{1,k}) \le 2\mu_k$, we obtain $\sum_{s\ge 1} \mu(\theta_s^+) \le C_3 V$. Lemma 3.1 yields then the relations $(a_1^+) - (c_1^+)$. In a similar way we get $(a_1^-) - (c_1^-)$.

Step 3. Since $f = f_0 + f_1^+ + f_1^-$, relations $(a_0) - (c_0)$ and $(a_1^\pm) - (c_1^\pm)$ imply $(a) - (c)$. □

Lemma 3.4. *In the situation of Lemma 3.3 for $p \ge 2$, let $1 \le k_1 < \cdots < k_d < p$ be some integers. Then for each $n = 1, 2, \ldots$ there are simple functions $(\psi_j)_{j=1}^m$, $m := n + 12(d+1)$, for which*

$$(A) \qquad f(x) - \sum_{j=1}^m \psi_j(x) = 0 \quad \text{for} \quad x \in J_p \cup (\mathbb{R} \setminus J_1) \cup \bigcup_{s=1}^d (J_{k_s} \setminus J_{k_s+1})$$

$$(B) \qquad \left\| f - \sum_{j=1}^m \psi_j \right\| \le \frac{C_1 V}{n} ,$$

$$(C) \qquad \sum_{j=1}^m \mu(\psi_j) \le C_2 V ,$$

where C_1, C_2 are some absolute constants.

Proof. We let $k_0 = 0$, $k_{d+1} = p$, and can assume that $V > 0$. For $s = 1, \ldots, d+1$ we define

$$f_s(x) := \sum_{k_{s-1}<k\le k_s} \varphi_k(x) , \quad V_s := \sum_{k_{s-1}<k\le k_s} \mu_k , \quad m_s := \left[nV_s/V \right] + 12 .$$

According to Lemma 3.3, there exist sequences of simple functions $(\psi_{s,j})_{j=1}^{m_s}$ which satisfy

$$(a_s) \qquad f_s(x) - \sum_{j=1}^{m_s} \psi_{s,j}(x) = 0 \quad \text{for} \quad x \in J_{k_s} \cup (\mathbb{R} \setminus J_{k_{s-1}+1})$$

$$(b_s) \qquad \left\| f_s - \sum_{j=1}^{m_s} \psi_{s,j} \right\| \le \frac{C_1 V_s}{m_s} \le \frac{C_1 V}{n}$$

$$(c_s) \qquad \sum_{j=1}^{m_s} \mu(\psi_{s,j}) \le C_2 V .$$

The number of the $\psi_{s,j}$ does not exceed $n + 12(d+1)$, consequently these relations imply $(A) - (C)$. □

Theorem 3.5. *Let $(\varphi_k)_{k\geq 1}$ be a finite sequence of simple functions on $\mathbb{R}$, let $f = \varphi_1 + \varphi_2 + \cdots$, $\mu_k = \mu(\varphi_k)$, $J_k = J(\varphi_k)$, $k = 1, 2, \ldots$, $V = \mu_1 + \mu_2 + \cdots$. If any two intervals $J_k, J_{k'}$ are either disjoint, or one of them is contained in the other, then for each $m = 1, 2, \ldots$ there are simple functions $(\psi_j)_{j=1}^m$ which satisfy the conditions $\bigcup_{j\geq 1} J(\psi_j) \subset \bigcup_{k\geq 1} J(\varphi_k)$ and*

$$\left\| f - \sum_{j=1}^{m} \psi_j \right\| \leq \frac{C_1 V}{m} \quad , \quad \sum_{j=1}^{m} \mu(\psi_j) \leq C_2 V \ ,$$

where C_1 and C_2 are positive absolute constants.

Proof. Step 1. Since $\|f\| \leq \frac{1}{2}V$, for $m \leq 29$ the statements of the theorem are obvious, and we can assume that $m \geq 30$. Let $q := [m/15]$; let (x_0, x_{q+1}) stand for the smallest interval containing $\bigcup_{k\geq 1} J_k$. Let x_k, $k = 1, \ldots, q$ be arbitrary points satisfying $x_0 < x_1 < \cdots < x_q < x_{q+1}$. We introduce the sets G_s, $s = 1, \ldots, q+1$, of integers in the following way: $G_1 := \{k : x_1 \in J_k\}$, $G_s := \{k : x_s \in J_k\} \setminus (G_1 \cup \cdots \cup G_{s-1})$ for $s \geq 2$. Thus, $k \in G_{q+1}$ if J_k does not contain any of the points $x_1, \ldots, x_q$. We also define $f_s := \sum_{k\in G_s} \varphi_k$, with $f_s = 0$ if $G_s = \emptyset$. We can assume that all $G_s \neq \emptyset$. The φ_k in the representation of f_s for $s \leq q$, we denote by $(\lambda_{s,i})_{i=1}^{p_s}$. From the construction of the sets G_s and the assumptions of the theorem we see that we may assume that $J(\lambda_{s,p_s}) \subset J(\lambda_{s,p_s-1}) \subset \cdots \subset J(\lambda_{s,1})$. We put $I_s := J(\lambda_{s,1})$. For each s, $1 \leq s \leq q$, we define the sets $E_s \subset I_s$ in the following way. We put $E_s = \emptyset$, if I_s does not contain any interval $I_{s'} \neq I_s$. If, however, intervals $I_{s'}$ of this type exist, then E_s is the union of all such $I_{s'}$. If $E_s \neq \emptyset$, then E_s is the union of some disjoint interval J_k, d_s in number. If $E_s = \emptyset$, then $d_s = 0$. By means of induction we obtain

$$d_1 + \cdots + d_q \leq q \ . \tag{3.4}$$

Step 2. Assuming that $V \neq 0$ (the theorem is obviously true if $V = 0$), we put

$$V_s := \sum_{i=1}^{p_s} \mu(\lambda_{s,i}) \quad , \quad n_s := \left[qV_s/V\right] + 1 \ .$$

According to Lemmas 3.3 and 3.4, one can find simple functions $\psi_{s,j}$, $j = 1, \ldots, m_s$; $m_s := n_s + 12(d_s + 1)$, for which

$$g_s(x) := f_s(x) - \sum_{j=1}^{m_s} \psi_{s,j}(x) = 0 \ \text{ if } \ x \in E_s \cup (\mathbb{R} \setminus I_s) \ , \tag{A_s}$$

$$\|g_s\| \leq \frac{C_1 V_s}{n_s} \leq \frac{C_2 V}{m} \ , \tag{B_s}$$

$$\sum_{j=1}^{m_s} \mu(\psi_{s,j}) \leq C_3 V_s \ . \tag{C_s}$$

Relation (A_s) and the definitions of E_s and f_s imply that for each $x \in \mathbb{R}$, at most one of the values $g_s(x)$, $s = 1, \dots, q$ can be different from zero. Therefore (B_s) implies

$$\left|\sum_{s=1}^{q} g_s(x)\right| \le \max_{1\le s\le q} |g_s(x)| \le \frac{C_2 V}{m}, \qquad x \in \mathbb{R}. \tag{3.5}$$

Step 3. From (3.4), the total number of terms $\psi_{s,j}$ in (3.5) does not exceed

$$\sum_{s=1}^{q}\big(n_s + 12(d_s+1)\big) \le 13q + \sum_{s=1}^{q} n_s \le 15q \le m.$$

From (C_s) we also obtain that

$$\sum_{s=1}^{q}\sum_{j=1}^{m_s} \mu(\psi_{s,j}) \le C_3 V.$$

If we can select the points $x_1, \dots, x_q$ in such a way that

$$\|f_{q+1}\| \le V/(2(q+1)), \tag{3.6}$$

the proof of the theorem will be completed.

We define the function

$$\theta(x) := \sum_{k\ge1} \|\varphi_k'\| \chi_k(x), \qquad \chi_k := \chi_{J_k}.$$

Since $\int_{\mathbb{R}} \theta(x)\,dx = V$, $\theta(x) \ge 0$ and $\theta(x) = 0$ for $x \notin (x_0, x_{q+1})$, we can find points $x_0 < x_1 < \dots < x_q < x_{q+1}$ for which $\int_{x_s}^{x_{s+1}} \theta\,dx = V/(q+1)$, $s = 0, 1, \dots$. In addition, we will have $f_{q+1}(x_0) = \dots = f_{q+1}(x_{q+1}) = 0$, so that

$$\|f_{q+1}\|[x_s, x_{s+1}] \le \tfrac12 \int_{x_s}^{x_{s+1}} |f_{q+1}'|\,dx \le \tfrac12 \int_{x_s}^{x_{s+1}} \theta\,dx \le \frac{V}{2(q+1)}.$$

This completes the proof of (3.6). □

We can now give a *proof of Theorem 2.5*. Let φ be a simple function on ∂D with $J(\varphi) = \{e^{ix} : \alpha < x < \beta,\ 0 < \beta - \alpha \le 2\pi\}$. Then the function $\theta(x) := \varphi(e^{ix})$ on (α, β), $\theta(x) := 0$ outside of this interval is simple on $\mathbb{R}$, and $J(\theta) = (\alpha, \beta)$, $\mu(\theta) = \mu(\varphi)$. This remark allows us to reduce the proof to functions on $\mathbb{R}$ and to use Theorem 3.5. One should treat separately the functions f_0 and f_1, defined by

$$f_s(\zeta) := \sum_{k\in G_s} \varphi_k(\zeta), \qquad s = 0, 1,$$

where $k \in G_0$ if $\zeta_0 \in J_k$ and $k \in G_1$ otherwise, and $\zeta_0 \in \partial D \setminus \bigcup_{k\ge1} J(\varphi_k)$ is some fixed point. □

§ 4. The Jackson-Rusak Operator; Rational Approximation of Sums of Simple Functions

In this section we shall prove Theorem 2.6, which has been used in §2: we shall obtain an estimate of the error of rational approximation of sums of 1-simple functions. For this purpose we shall need a rational operator of Jackson's type, given by Rusak in [A-1979]. It is obtained by a normalization of the operator that appears in (1.18) for $r = 3$:

$$(4.1) \qquad J_m(f,z) := \frac{1}{q_m(z)} \int_{\partial D} K_3(z,\zeta) f(\zeta) |d\zeta| , \qquad f \in C(\partial D) ,$$

where

$$(4.2) \quad q(z) := q_m(z) := \int_{\partial D} K_3(z,\zeta)|d\zeta| , \quad K_3(z,\zeta) = \left| \frac{B_m(\zeta) - B_m(z)}{\zeta - z} \right|^4 .$$

Plainly, $J_m(1,z) = 1$. We prove that J_m maps $C(\partial D)$ into $\mathcal{R}_{4m-4}(\partial D)$. For the proof we introduce the functions h_k on ∂D by:

$$h_1(z) = \frac{\sqrt{1-|a_1|^2}}{1-\bar{a}_1 z} ,$$

$$h_k(z) = \frac{\sqrt{1-|a_k|^2}}{1-\bar{a}_k z} \prod_{j=1}^{k-1} \frac{z-a_j}{1-\bar{a}_j z} , \qquad k \geq 2 .$$

In passing we remark that the h_k form an orthogonal system on ∂D, the so-called Takenaka-Malmquist system.

Important for us is the identity of Dzhrbashyan [1967]:

$$(4.3) \qquad \frac{1 - \overline{B_m(\zeta)} B_m(z)}{1 - \bar{\zeta} z} = \sum_{k=1}^{m} \overline{h_k(\zeta)}\, h_k(z) ,$$

which can be easily proved by induction. Using the fact that $\bar{w} = 1/w$ if $|w| = 1$, one sees that the right side of (4.1) is a rational function of degree $4m - 4$.

Lemma 4.1. *For each $z \in \partial D$ one has*

$$(4.4) \qquad q(z) \geq \pi \left[\sum_{k=1}^{m} \frac{1-|a_k|^2}{|1-\bar{a}_k z|^2} \right]^3 =: M_1 ,$$

$$(4.5) \qquad q(z) \geq 2\pi \sum_{k=1}^{m} \frac{1-|a_k|^4}{|1-\bar{a}_k z|^4} =: M_2 .$$

Proof. Using Lemma 1.6 we derive

$$\frac{3}{\pi}q(z) = z^2\Big\{B(z)^{-2}\big[zB(z)^2\big]''' - 4B(z)^{-1}\big[zB(z)\big]'''\Big\} ,$$

which after some calculations yields

$$\begin{aligned} \frac{3}{\pi}q(z) &= \left\{4\Big[\big(\log B(z)\big)'\Big]^3 - 2\big(\log B(z)\big)'''\right\} - 6z^2\big(\log B(z)\big)''' ; \\ (4.6) \qquad &= 4\left[\sum_{k=1}^{m} \frac{1-|a_k|^2}{|1-\bar a_k z|^2}\right]^3 - 4\sum_{k=1}^{m}\frac{(1-|a_k|^2)^3}{|1-\bar a_k z|^6} + 6\sum_{k=1}^{m}\frac{1-|a_k|^4}{|1-\bar a_k z|^4} . \end{aligned}$$

This immediately implies (4.5). Furthermore, we can write (4.6) in the form
(4.7)

$$\frac{3}{\pi}q(z) = 3\left[\sum_{k=1}^{m} \frac{1-|a_k|^2}{|1-\bar a_k z|^2}\right]^3 + \left\{\left[\sum_{k=1}^{m} \frac{1-|a_k|^2}{|1-\bar a_k z|^2}\right]^3 - \sum_{k=1}^{m}\frac{(1-|a_k|^2)^3}{|1-\bar a_k z|^6}\right\} +$$

$$+ 3\sum_{k=1}^{m}\left\{2\frac{1-|a_k|^4}{|1-\bar a_k z|^4} - \frac{(1-|a_k|^2)^3}{|1-\bar a_k z|^6}\right\} .$$

Here all expressions in the braces are ≥ 0. This yields (4.4). □

Lemma 4.2. *Let $a := \rho e^{it}$ for some $0 \leq \rho < 1$, $0 \leq t < 2\pi$. For each simple function ψ supported on $J := J(\psi) = \{e^{i\theta} : |\theta - t| < \pi(1-\rho)\}$ with $\mu := \mu_1(\psi)$, for all $z \in \partial D$ one has*

$$(4.8) \qquad |\psi(z) - J_m(\psi, z)| \leq C\mu\left\{\frac{1}{q(z)^{1/3}}\,\frac{1-|a|^2}{|1-\bar a z|^2} + \frac{1}{q(z)}\,\frac{1-|a|^4}{|1-\bar a z|^4}\right\} .$$

Proof. We define the sets $E_z := \{\zeta : |\zeta - z| \leq q(z)^{-1/3}\}$ and $E_z' = \partial D \setminus E_z$. Then

$$\begin{aligned} |\psi(z) - J_m(\psi, z)| &\leq \frac{1}{q(z)}\int_{\partial D} |\psi(z) - \psi(\zeta)| K_3(z,\zeta)\,|d\zeta| \\ &= \frac{1}{q(z)}\left(\int_{E_z} + \int_{E_z'}\right) . \end{aligned}$$

Since $\|\psi'\|_\infty = \mu|J|^{-1} = \mu/(2\pi(1-\rho))$, we have for all $z, \zeta \in \partial D$,

$$|\psi(z) - \psi(\zeta)| \leq \frac{\mu\cdot|z-\zeta|}{4(1-\rho)} .$$

Therefore

$$\int_{E_z} \leq \frac{\mu}{4\sqrt[3]{q(z)}\,(1-\rho)}\int_{\partial D} K_3(z,\zeta)|d\zeta| = \frac{\mu q(z)^{2/3}}{4(1-\rho)} ;$$

$$\int_{E_z'} \leq \frac{4\mu}{1-\rho}\int_{E_z'} \frac{|d\zeta|}{|\zeta - z|^3} \leq \frac{18\mu q(z)^{2/3}}{1-\rho} ;$$

$$|\psi(z) - J_m(\psi, z)| \leq \frac{19\mu}{1-\rho}\,q(z)^{-1/3} .$$

Let J^* be the larger interval $J^* := \{e^{i\theta} : |\theta - t| < 2\pi(1-\rho)\}$. From the last inequality, for $z \in J^*$,

$$|\psi(z) - J_m(\psi, z)| \le 133 \frac{\mu}{q(z)^{1/3}} \cdot \frac{1-|a|^2}{|1-\bar{a}z|^2} \,. \tag{4.9}$$

If the set $S := \partial D \setminus J^*$ is not empty, then for $z \in S$, using $\|\psi\|_\infty \le \mu/2$, we obtain

$$|\psi(z) - J_m(\psi, z)| = |J_m(\psi, z)| \le J_m(|\psi|, z) \le$$

$$\le \frac{\mu}{2q(z)} \int_J K_3(z, \zeta)|d\zeta| \le 8 \frac{\mu}{q(z)} \int_J \frac{|d\zeta|}{|\zeta - z|^4} \,;$$

$$|\psi(z) - J_m(\psi, z)| \le 200 \frac{\mu}{q(z)} \frac{1-|a|^4}{|1-\bar{a}z|^4} \,. \tag{4.10}$$

Combining (4.9) and (4.10) we obtain (4.8) with $C = 200$. □

Proof of Theorem 2.6. Let $f = \psi_1 + \cdots + \psi_m$, $\mu_k := \mu(\psi_k)$, $V = \mu_1 + \cdots + \mu_k$. The rational function R of this theorem will be $J_{m_1}(f, z)$ with properly chosen zeros z_j of the Blaschke product $B_m(z)$ in (4.2). Let $J(\psi_k) := \{e^{i\theta} : |\theta - t_k| < \pi(1-\rho_k)\}$ for $t_k \in \mathbb{T}$, $0 < \rho < 1$ be the support of the simple function ψ_k. Let $\sigma_k := [m\mu_k/V] + 1$, then $m \le m_1 := \sum \sigma_k \le 2m$. The zeros of the Blaschke product will be $a_k := \rho_k e^{it_k}$, each repeated σ_k times. Since $\mu_k \le V\sigma_k/m$, we get from (4.8)

$$|f(z) - J_{m_1}(f, z)| \le \frac{CV}{m} \left\{ \frac{1}{q(z)^{1/3}} \sum_{k=1}^m \sigma_k \frac{1-|a_k|^2}{|1-\bar{a}_k z|^2} + \frac{1}{q(z)} \sum_{k=1}^m \sigma_k \frac{1-|a_k|^4}{|1-\bar{a}_k z|^4} \right\}$$

$$= \frac{CV}{m} \left\{ \frac{1}{q(z)^{1/3}} M_1^{1/3} + \frac{1}{q(z)} M_2 \right\} \le \frac{CV}{m} \,,$$

by Lemma 4.2. This establishes Theorem 2.6. □

§ 5. Rational Approximation on $\mathbb{T}$ and on $[-1, 1]$

This section is probably the most important one in Chapter 10, for it follows the principal plan of this book, which deals with approximation of functions of *one real variable.*

The concrete results of the second part of this section are all corollaries of Theorem 2.1 or of its real-variable reformulation, Theorem 5.1. These concrete results are non-trivial in the sense that they give essentially better estimates of the error than those possible for the polynomial approximation. As Theorem 7.5 of Chapter 7 explains, this is not possible for spaces like W_∞^r, but we shall treat instead spaces W_p^1, $p > 1$, V^r, $V_\alpha := (BV) \cap \operatorname{Lip} \alpha$, the class of convex functions. Sometimes we get new natural proofs of theorems of Chapter 7, such as the theorem of Popov (Theorem 5.7). Our proofs should be compared also with those of the book of Petrushev and Popov [A-1987].

We define the spaces of rational functions $\mathcal{R}_n(A)$, where A is the interval $[-1,1]$, or the disk D, or its boundary ∂D, to consist of all functions $R_n = P_n/Q_n$ on A, where $P_n, Q_n \in \mathcal{P}_n$, with the additional assumption that R_n has no poles on A. For $A = \mathbb{T}$, $\mathcal{R}_n(A)$ consists of all trigonometric rational functions $R_n = S_n/T_n$, $S_n, T_n \in \mathcal{T}_n$, $R_n(t) \neq 0$, $t \in \mathbb{T}$. For a continuous function $g \in C(A)$, the error of rational approximations is given by

$$\rho_n(g)_{C(A)} := \inf_{R \in \mathcal{R}_n(A)} \|f - R\|_{C(A)} \, . \tag{5.1}$$

In these definitions, for real valued g on $\mathbb{T}$, ∂D, or $[-1,1]$, we postulate that the infimum is taken only for real valued S, T (or P, Q). (See also Theorem 1.4 of Chapter 7.)

Approximations on $\mathbb{T}$ and on ∂D are closely related. In the identity (where p_k, q_k are arbitrary complex numbers)

$$\frac{\sum_{k=-n}^{n} p_k e^{ikt}}{\sum_{k=-n}^{n} q_k e^{ikt}} = \frac{\sum_{\ell=0}^{2n} p_{\ell-n} \zeta^\ell}{\sum_{\ell=0}^{2n} q_{\ell-n} \zeta^\ell} \, , \qquad \zeta = e^{it} \, , \tag{5.2}$$

the left-hand side is an arbitrary element of $\mathcal{R}_n(\mathbb{T})$, the right-hand side an arbitrary element of $\mathcal{R}_{2n}(\partial D)$. Therefore, if $g \in C(\mathbb{T})$,

$$\rho_n(g)_{C(\mathbb{T})} = \rho_{2n}(g)_{C(\partial D)} \, . \tag{5.3}$$

In particular, if $f \in \mathcal{A}$, $n = 0, 1, \ldots$,

$$\rho_n(\operatorname{Re} f)_{C(\mathbb{T})} = \rho_{2n}(\operatorname{Re} f)_{C(\partial D)} \leq \rho_n(f)_{\mathcal{A}} \, . \tag{5.4}$$

From Appendix 3, §2 we know that if g is real-valued and belongs to $L_1(\mathbb{T})$, then the function

$$f(z) := 2C^+(g, z) - a_0(g) \, , \qquad z \in D \tag{5.5}$$

belongs to H_q for all $0 < q < 1$ and satisfies $\operatorname{Re} f(e^{it}) = g(t)$, $Im\, f(0) = 0$. We say that $g \in \mathcal{H}_p^r$, $r = 1, 2, \ldots$, $0 < p \leq \infty$, if the corresponding function f of (5.5) belongs to H_p^r; and we define

$$\|g\|_{\mathcal{H}_p^r} := \|f^{(r)}\|_{H_p} \, . \tag{5.6}$$

The spaces $\mathcal{H}_{1/r}^r$ we also denote simply by $\mathcal{H}^r$. From (2.4), $H_{1/r}^r \subset \mathcal{A}$, and therefore $\mathcal{H}^r \subset C(\mathbb{T})$.

Theorem 5.1 (Pekarskii [1987]). *For each function $g \in \mathcal{H}^r$, $r = 1, 2, \ldots$ one has*

$$\rho_n(g)_{C(\mathbb{T})} \leq \frac{C_r}{n^r} \|g\|_{\mathcal{H}^r} \, , \qquad n \geq r \tag{5.7}$$

$$\rho_n(g)_{C(\mathbb{T})} = o(n^{-r}) \, , \qquad n \to \infty \, . \tag{5.8}$$

Proof. The assumption $g \in \mathcal{H}^r$ means, that the function $f(z)$ of (5.5) belongs to $H^r_{1/r}$. Since $g(t) = \operatorname{Re} f(e^{it})$ continuous on $\mathbb{T}$, from (2.5) and (5.4) we obtain (5.7):

$$\rho_n(g)_{C(\mathbb{T})} \le \rho_n(f)_{\mathcal{A}} \le \frac{C_r}{n^r}\|f^{(r)}\|_{H_{1/r}} = \frac{C_r}{n^r}\|g\|_{\mathcal{H}^r} .$$

Applying (2.6) instead of (2.5), we obtain (5.8). □

Theorem 5.2 (Sevastyanov [1978] for $r = 1$, Pekarskii [1980] for $r > 1$). *For $g \in C(\mathbb{T})$, and $r = 1, 2, \ldots$, the relation*

$$\left(\sum_{n=0}^{\infty} \sqrt[r]{\rho_n(g)_{C(\mathbb{T})}}\right)^r =: Q < \infty \tag{5.9}$$

implies that $g \in \mathcal{H}^r$ and $\|g\|_{\mathcal{H}^r} \le C_r Q$.

The proof of this theorem is similar to that of [**CA**, Theorem 3.2, p.209] : also the abstract [**CA**, Theorem 5.1, p. 216] can be used. We leave the details to the reader. The necessary theorem of the Bernstein type is as follows:

Theorem 5.3. *If $R \in \mathcal{R}_n(\mathbb{T})$, $n = 1, 2, \ldots$, $r = 1, 2, \ldots$, then*

$$\|R\|_{\mathcal{H}^r} \le C_r n^r \|R\|_{C(\mathbb{T})} . \tag{5.10}$$

Proof. According to (5.2), by substitution of $\zeta = e^{it}$ transforms the function R into $U \in \mathcal{R}_{2n}(\partial D)$ so that $U(e^{it}) = R(t)$ for $t \in \mathbb{T}$. By applying (5.5) and Theorem 1.2 we obtain (5.10):

$$\begin{aligned}\|R\|_{\mathcal{H}^r} &= 2\|(C^+U)^{(r)}\|_{H_{1/r}} = 2^{-r+1}\pi^{-r}\|(C^+U)^{(r)}\|_{1/r}(\partial D)\\ &\le C_r n^r \|U\|_{C(\partial D)} = C_r n^r \|R\|_{C(\mathbb{T})} .\end{aligned}$$

□

To apply our Theorem 5.1, we need sufficient conditions for the requirement $g \in \mathcal{H}^r$. Information of this type is furnished by the two theorems below, which provide real-variable descriptions of the space $\mathcal{H}^r$.

Theorem 5.4. *A function $g \in C(\mathbb{T})$ belongs to the space $\mathcal{H}^1$ if and only if both g and the conjugate function $\tilde{g}$ belong to $W_1^1(\mathbb{T})$. One has also the equivalence of the seminorms $\|g\|_{\mathcal{H}^1}$ and $\|g\|^{(1)}_{\mathcal{H}^1} := \|g'\|_1 + \|(\tilde{g})'\|_1$.*

Proof. If $g \in C(\mathbb{T})$, then, according to the theorem of M. Riesz (Theorem 2.2 of Appendix 3) the function $f(z)$ of (5.5) belongs to H_1. As $f(e^{it}) = g(t) + i\tilde{g}(t)$ a.e. on $\mathbb{T}$, from Theorem 1.6 of Appendix 3 we get the necessary result. In particular, seminorms $\|g\|_{\mathcal{H}^1}$ and $\|g\|^{(1)}_{\mathcal{H}^1}$ satisfy the inequalities

$$2\pi\|g\|_{\mathcal{H}^1} \le \|g\|^{(1)}_{\mathcal{H}^1} \le 2\sqrt{2}\pi\|g\|_{\mathcal{H}^1} .$$

□

Theorem 5.5. *A real-valued function $g \in C(\mathbb{T})$ belongs to $\mathcal{H}^r$, $r = 1,2,\ldots$ if and only if there exist a real valued polynomial $S \in \mathcal{T}_{r-1}$ and r-simple functions φ_k, $k = 1,2,\ldots$ on ∂D which satisfy*

$$\text{(i)} \qquad g(t) = S(t) + \sum_{k\geq 1} \varphi_k(e^{it}) \ , \qquad t \in \mathbb{T} \ ,$$

$$\text{(ii)} \qquad \left(\sum_{k\geq 1} \mu_r(\varphi_k)^{1/r}\right)^r := G < \infty \ .$$

In addition, the quasi-norm $\|g\|_{\mathcal{H}^r}$ is equivalent to

$$\|g\|^{(2)}_{\mathcal{H}^r} := \inf G \ , \tag{5.11}$$

with the infimum taken for all representations (i) *satisfying* (ii).

Proof. (a) *Sufficiency.* We assume (i) and (ii). Because of the definition of the space $\mathcal{H}^r$, we have to show that

$$\|(C^+g)^{(r)}\|_{H_{1/r}} \leq C_r G \ . \tag{5.12}$$

From (2.8), $\|\varphi_k\|_\infty \leq \mu_r(\varphi_k)$. Since (ii) implies that $\sum \mu_r(\varphi_k) \leq G$, the series (i) converges uniformly on ∂D. Therefore (5.12) will follow if we show that for each r-simple function φ on ∂D,

$$\|(C^+\varphi)^{(r)}\|_{H_{1/r}} \leq C(r)\mu_r(\varphi) \ . \tag{5.13}$$

We obtain $(C^+\varphi)^{(r)} = C^+\varphi^{(r)}$ by means of r times partial integration. Thus, $\varphi^{(r)}$ is an $1/r$-atom, and by Lemma 3.3 of Appendix 3,

$$\|C^+\varphi^{(r)}\|_{H_{1/r}} \leq C_r |J(\varphi^{(r)})|^r \|\varphi^{(r)}\|_\infty = C_r\mu_r(\varphi) \ .$$

(b) *Necessity.* Let $g \in \mathcal{H}^r$. Then there is a function $f \in H^r_{1/r}$ with $\operatorname{Re} f(e^{it}) = g(t)$ on $\mathbb{T}$ and with $N := \|f^{(r)}\|_{H_{1/r}} = \|g\|_{\mathcal{H}^r}$. We put $P_0(z) := f(0) + \cdots + \frac{f^{(r-1)}(0)}{(r-1)!} z^{r-1}$ and $f_0(z) := f(z) - P_0(z)$. Let $(\rho_k)_1^\infty$ be a sequence with the properties $0 < \rho_1 < \rho_2 < \cdots$ and $\rho_n \to 1$. For the functions $f_1(z) := f_0(\rho_1 z)$, $f_n(z) := f_0(\rho_n z) - f_0(\rho_{n-1} z)$, $n = 2,3,\ldots$ we clearly have

$$f_0(z) = f_1(z) + f_2(z) + \cdots \ , \qquad z \in \bar{D} \ . \tag{5.14}$$

We can subject the ρ_k to the conditions

$$\|f_n^{(r)}\|_{H_{1/r}} \leq 2^{-n+1} N \ , \qquad n = 1,2,\ldots \ ; \tag{5.15}$$

this is possible because of (1.4) of Appendix 3 and the monotonicity of the means $M_p(f,\rho)$ as functions of ρ.

The functions f_n, $n = 1,2,\ldots$ are analytic on $\bar{D}$ and satisfy $f_n^{(j)}(0) = 0$, $j = 1,\ldots,r-1$. From (5.15) and Lemma 2.3 we derive the existence of polynomials $P_n \in \mathcal{P}_{r-1}$ and of sequences $(\varphi_{n,k})_{k=1}^{\infty}$ of r-simple functions, which for $n = 1, 2, \dots$ have the properties

$$\sum_{k\geq 1} \mu_r(\varphi_{n,k})^{1/r} \leq C_r 2^{-n/r} N^{1/r} \tag{5.16}$$

$$f_n(\zeta) = P_n(\zeta) + \sum_{k\geq 1} \varphi_{n,k}(\zeta) , \qquad \zeta \in \partial D , \tag{5.17}$$

$$\|P_n\|_{\mathcal{A}} \leq C_r 2^{-n} N . \tag{5.18}$$

From this we derive formally

$$\begin{aligned} f(\zeta) &= P(\zeta) + \sum_{n\geq 1} \sum_{k\geq 1} \varphi_{n,k}(\zeta) , \qquad \zeta \in \partial D , \\ P(\zeta) &= P_0(\zeta) + \sum_{n\geq 1} P_n(\zeta) . \end{aligned} \tag{5.19}$$

As $P_n \in \mathcal{P}_{r-1}$, (5.18) implies that $P \in \mathcal{P}_{r-1}$. According to (5.16) we have also

$$\sum_{n\geq 1} \sum_{k\geq 1} \mu_r(\varphi_{n,k})^{1/r} \leq C_r N^{1/r} .$$

It follows that the series (5.19) is uniformly convergent to $f(\zeta)$ with any order of its terms. Since $g(t) = Re\, f(e^{it})$, $t \in \mathbb{T}$, from (5.19) we obtain the desired representation of the function g:

$$g(t) = \operatorname{Re} P(e^{it}) + \sum_{n\geq 1} \sum_{k\geq 1} \operatorname{Re} \varphi_{n,k}(e^{it}) , \qquad t \in \mathbb{T} . \qquad \square$$

If a function $g \in C[-1,1]$ is real-valued, then $g_0(t) := g(\cos t)$ belongs to $C(\mathbb{T})$ and is real-valued and even. The rational trigonometric fraction of best uniform approximation to g_0 is also even. Therefore

$$\rho_n(g)_{C[-1,1]} = \rho_n(g_0)_{C(\mathbb{T})} . \tag{5.20}$$

In what follows, we shall use Theorem 5.1 to obtain several upper estimates of the error of rational approximation for periodic functions. Then one can derive corresponding theorems for the space $C[-1,1]$ by means of the formula (5.20). In many cases, we shall obtain new proofs of theorems of Chapter 7.

Theorem 5.6 (Pekarskii [1982$_2$]). *For $A = [-1,1]$ or $A = \mathbb{T}$, let $g \in W_1^1(A)$. If $g' \in L\log L$, then*

$$\begin{aligned} \rho_n(g)_{C(A)} &\leq \frac{C}{n} \|g'\|_{L\log L} , \qquad n \geq 1, \\ \rho_n(g)_{C(A)} &= o(1/n) , \qquad n \to \infty . \end{aligned}$$

Proof. In the case $A = \mathbb{T}$, the theorem follows from Theorems 5.1, 5.4 and Theorem 2.3 of Zygmund in Appendix 3. The proof of the case $A = [-1, 1]$ is then effected by means of the formula (5.20). □

Remark. If $g \in W_p^1(A)$, $p > 1$, then $g' \in L \log L$. Therefore, as a special case of Theorem 5.6 we obtain Theorem 7.1 of Chapter 7.

For the case $A = [-1, 1]$, statement (5.21) of the following theorem is identical with the theorem of Popov (Theorem 7.2 of Chapter 7), while the statement (5.22) is due to Petrushev [1979].

Theorem 5.7. *Let $A = [-1, 1]$, or $= \mathbb{T}$, and let $g \in V^r(A)$, $r = 1, 2, \ldots$. Then*

$$\rho_n(g)_{C(A)} \le \frac{C_r}{n^{r+1}} \operatorname{Var}_A g^{(r)} , \qquad n \ge r , \tag{5.21}$$

$$\rho_n(g)_{C(A)} = o(n^{-r-1}) , \qquad n \to \infty . \tag{5.22}$$

Proof. We shall restrict ourselves to the case $A = \mathbb{T}$. The case $A = [-1, 1]$ follows from this by using the relation (5.20).

According to Theorem 5.1 and the relation (5.5) we have to show that

$$\|(C^+ g)^{(r+1)}\|_{H_{1/(r+1)}} \le C_r \operatorname{Var}_{\mathbb{T}} g^{(r)} . \tag{5.23}$$

Partial integration, repeated $(r + 1)$-times, yields that $(C^+(g, z))^{(r+1)} = C^+(dg^{(r)}, z)$, $z \in D$. But according to Smirnov's Theorem 2.4 of Appendix 3, for all p, $0 < p < 1$, $\|C^+(dg^{(r)}, z)\|_{H_p} \le C(p) \operatorname{Var}_{\partial D} g^{(r)}$. This is more than (5.23). □

In the proofs of Theorems 5.6 and 5.7 we have established continuous embeddings $W_p^1(\mathbb{T}) \subset \mathcal{H}^1$, $p > 1$ and $V^r(\mathbb{T}) \subset \mathcal{H}^{r+1}$, $r \ge 1$. From this it is possible to derive analogues of Theorem 7.6 of Chapter 7.

In [1966], Freud proved the estimate $\rho_n(g)_C = O(\log^2 n/n)$ for $g \in V_\alpha[-1, 1]$, $0 < \alpha < 1$; the class $V_\alpha[-1, 1]$ is the intersection $\operatorname{Lip} \alpha \cap BV$. He also treated functions with bad moduli of continuity, for example, with $\omega(g, \delta) = O(\log^{-\gamma}(1/\delta))$, $\gamma > 0$, see Lorentz [A-1986]. Estimates from below were obtained by Bulanov [1975]. In particular, he showed that for each $0 < \alpha < 1$, and each decreasing sequence $\varepsilon_n > 0$, $\varepsilon_n \to 0$, there is a function $g \in V_\alpha[-1, 1]$ with the property that $\rho_n(g)_C \ge \varepsilon_n \log n/n$ for infinitely many n. Exact estimates from above for the class V_α have been obtained independently by Petrushev [1977] and Pekarskii [1978] (see the next theorem). The latter showed also that there is no o-phenomenon in this theorem. For further details see the book of Petrushev and Popov [A-1987].

Theorem 5.8. *Let A be the interval $[-1, 1]$ or $\mathbb{T}$, $0 < \alpha < 1$. If $g \in V_\alpha(A)$, $\operatorname{Var}_A g \le V$ and $g \in \operatorname{Lip}_K \alpha$, then*

(5.24) $$\rho_n(g)_{C(A)} \le C_\alpha (V+K)\frac{\log n}{n} .$$

Proof. We shall treat the case $A = \mathbb{T}$; the case $A = [-1,1]$ is reduced to this by means of (5.20). For $h > 0$, let

$$g_h(t) := \frac{1}{h}\int_0^h g(t+x)\,dx .$$

Then $\|g - g_h\|_\infty \le Kh^\alpha$ and $g_h'(t) = (g(t+h)-g(t))/h$. From $g = (g-g_h)+g_h$ and Theorem 5.6 we deduce

(5.25) $$\rho_n(g)_C \le Kh^\alpha + \rho_n(g_h)_C \le Kh^\alpha + \frac{C}{n}\|g_h'\|_{L\log L} .$$

The norm of $f := g_h'$ in $L\log L$ is given by (2.13) of Appendix 3; it is equal to $\inf \lambda$, taken for all $\lambda > 0$ for which the integral $I_\lambda(f)$ below does not exceed 1. We have

$$I_\lambda(f) = \int_{\mathbb{T}} \frac{|f(t)|}{\lambda}\log^+\frac{|f(t)|}{\lambda}\,dt \le \frac{1}{\lambda h}\log^+\left(\frac{Kh^\alpha}{\lambda h}\right)\int_{\mathbb{T}} |g(t+h)-g(t)|\,dt .$$

By [**CA**, Lemma 9.2, p.53], the last integral is $\le \omega(g,h)_1 \le Vh$, so that

$$I_\lambda \le \frac{V}{\lambda}\log^+\frac{Kh^\alpha}{\lambda h} .$$

For $0 < h < 1/e$ let

$$\lambda_0 := \max(V,K)\log\frac{1}{h} ,$$

then

(5.26) $$I_{\lambda_0} \le \frac{1}{\log(1/h)}\log^+\left(\left(\frac{1}{h}\right)^{1-\alpha}\frac{1}{\log(1/h)}\right) \le 1 .$$

From (5.25) and (5.26) we derive for $n \ge 3$, and $h \le 1/e$,

$$\rho_n(g)_C \le Kh^\alpha + C\,C_\alpha\lambda_0/n \le Kh^\alpha + \frac{C(K+V)}{n}\log\frac{1}{h} .$$

It remains to substitute $h = n^{-1/\alpha}$. □

Restricting ourselves to the simpler case $A = \mathbb{T}$, we shall show by means of an example that the estimate (5.24) is the best possible. We shall need the following lemma.

Lemma 5.9. *Let $m = 1,2,\dots$, $\theta := \theta_m := \frac{1}{2}\exp(-2\pi^2 m)$, and let $h := h_m$ be the continuous function on $\mathbb{T}$, which is equal to* 1 *on $[\theta,\pi/2]$, equal to* 0 *on $[-\pi/2,-\theta]$, and is linear on $[-\theta,\theta]$ and on $[\pi/2,3\pi/2]$. Then*

(5.27) $$\rho_m(h)_{C(\mathbb{T})} \ge \frac{1}{8} .$$

Proof. We treat h as a function of $\zeta \in \partial D$, then (5.27) is equivalent to

$$\rho_{2m}(h)_{C(\partial D)} \geq \frac{1}{8}\,. \tag{5.28}$$

The linear fractional map $\zeta = w(x)$, given by $\frac{x+1}{x-1} = i\,\frac{i\zeta-1}{i\zeta+1}$ maps $\mathbb{R}$ onto ∂D, and points $-1, 0, 1$ on $-i, 1, i$ respectively. It also maps some point $\delta := \delta_m \in (0,1)$ onto $e^{i\theta}$. This is found from

$$\frac{1+\delta}{1-\delta} = \tan\frac{1}{2}\left(\theta + \frac{\pi}{2}\right) = \sqrt{\frac{1+\sin\theta}{1-\sin\theta}}\,.$$

Since $\sqrt{\frac{1+\delta/2}{1-\delta/2}} < \frac{1+\delta}{1-\delta}$, if $0 < \delta < 1$, we deduce that $0 < \delta < 2\sin\theta < 2\theta = \exp(-2\pi^2 m)$. Now Lemma 7.8 with $\delta = \delta_m$ of Chapter 7 yields (5.26):

$$\begin{aligned}\rho_{2m}(h)_{C(\partial D)} &= \rho_{2m}(h\circ w)_{C(\mathbb{R})} \geq \rho_{2m}(h\circ w)_{C[-1,1]} \\ &\geq \frac{1}{2(e+1)} > \frac{1}{8}\,.\end{aligned}$$

□

Theorem 5.10. *For each $0 < \alpha < 1$ and $n = 2, 3, \ldots$ there exists a function $g := g_{\alpha,n} \in V_\alpha(\mathbb{T})$ which belongs to* $\mathrm{Lip}_1\,\alpha$, *satisfies* $\mathrm{Var}_{\mathbb{T}}\, g \leq 1$ *and has the property*

$$\rho_n(g)_{C(\mathbb{T})} \geq C_\alpha\,\frac{\log n}{n}\,, \qquad C_\alpha > 0\,. \tag{5.29}$$

Proof. Let θ_m, h_m, $m = 1, 2, \ldots$ be as in Lemma 5.9. For all sufficiently large integers $n \geq n_0(\alpha)$ we put (where $[u]$ stands for the integral part of u)

$$m := \left[\frac{1-\alpha}{2\pi^2\alpha}\log\frac{n}{\log n}\right]\,, \qquad k := \left[\frac{2n}{m}\right] \tag{5.30}$$

$$h_{m,k}(t) := \left(\frac{2\theta_m}{k}\right)^\alpha h_m(kt)\,, \qquad t \in \mathbb{T}\,. \tag{5.31}$$

The function $h_{m,k}(t)$ is $2\pi/k$ periodic and belongs to $\mathrm{Lip}_1\,\alpha$. We have the following weak equivalences: $m \sim \log n$, $k \sim n/\log n$,

$$\theta_m^\alpha = e^{-2\pi^2\alpha m} \sim \left(\frac{\log n}{n}\right)^{1-\alpha}\,.$$

For some $n_0(\alpha)$ and all $n \geq n_0(\alpha)$ it follows that there exist constants $\lambda \geq 1$, $C > 0$, which depend on α, for which

$$\left(\frac{2\theta_m}{k}\right)^\alpha \geq C\frac{\log n}{n}\,, \quad 2k\left(\frac{2\theta_m}{k}\right)^\alpha \leq \lambda\,. \tag{5.32}$$

From the second inequality, $\mathrm{Var}_{\mathbb{T}}\, h_{m,k} \leq \lambda$. We put $g := \lambda^{-1}h_{m,k}$, then $\mathrm{Var}\, g \leq 1$ and $g \in \mathrm{Lip}_1\,\alpha$.

Let $f_1 \in C(\mathbb{T}) \setminus \mathcal{U}_m$ and $f_k(t) := f_1(kt)$. We shall show, for the errors of approximation by spaces $\mathcal{U}_m$, $\mathcal{U}_{mk}$ of rational trigonometric functions of §2, Chapter 5 satisfy

$$\rho_{mk}(f_k) = \rho_m(f_1) \ .$$

If U_1 is the best approximation to f_1 from $\mathcal{U}_m$, and if d is its defect in $\mathcal{U}_m$, then by Theorem 2.9 of Chapter 7, the value $\pm\|f_1 - U_1\|$ is attained, with alternating signs, by $f_1(t) - U_1(t)$ at $p \ge 2(2m-d)+2$ points $0 \le t_1 < \cdots < t_p < 2\pi$ with even p. Then for $U_k(t) := U_1(kt)$ the difference $f_k(t) - U_k(t)$ in absolute value, does not exceed $\|f_1 - U_1\|$ and attains this value, with alternating signs, at the points $0 \le t_1/k < \cdots < t_p/k < (t_1 + 2\pi)/k < \cdots < (t_p + 2\pi(k-1))/k < 2\pi$. Their number is $pk \ge 2(2mk - dk) + 2$, and dk is the defect of U_k. Again by Theorem 2.9 of Chapter 7, U_k is the best approximation to f_k from $\mathcal{U}_{mk}$, and we obtain the required inequality.

From this, together with (5.27) and the first inequality (5.32) we obtain

$$\begin{aligned}\rho_{mk}(h_{m,k}) &= \left(\frac{2\theta_m}{k}\right)^\alpha \rho_{mk}\big(h_m(kt)\big) = \left(\frac{2\theta_m}{k}\right)^\alpha \rho_m(h_m) \\ &\ge \frac{1}{8} C \, \frac{\log n}{n} \ .\end{aligned}$$

Since $mk \ge n$, we get for $n \ge n_0(\alpha)$

$$\rho_n(g) \ge \rho_{mk}(g) \ge \frac{C}{8\lambda} \, \frac{\log n}{n} = C(\alpha) \frac{\log n}{n} \ . \qquad \square$$

Rational approximation of convex functions g on an interval has been studied by Bulanov [1969]. He showed that in this case $\rho_n(g) = O(\log^2 n/n)$ and that for some g_n, with $\|g_n'\|_1 = 1$, one has $\rho_n(g_n)_C \ge C/n$. The final result in this direction is due to Popov and Petrushev [1977]:

Theorem 5.11. *For a continuous convex function g on $[-1,1]$ one has, with an absolute constant C,*

$$\begin{aligned}\rho_n(g) &\le \frac{C}{n}\|g'\|_1 \ , \qquad n \ge 1 \\ \rho_n(g) &= o(n^{-1}) \ , \qquad n \to \infty \ .\end{aligned}$$

Proof. We can restrict ourselves to functions g that are non-decreasing and satisfy $g(-1) = 0$. Let $L := g(1) = \|g'\|_1$. According to Theorem 5.1 and (5.20), it suffices to show that the functions $g_0(t) := g(\cos t)$, $t \in \mathbb{T}$ belongs to $\mathcal{H}^1$ and satisfies

$$\|g_0\|_{\mathcal{H}^1} \le CL \ . \tag{5.33}$$

For this purpose we introduce the truncations

$$g_k(t) := \min\{g_0(t), g(1 - 2^{-k})\} \ , \qquad k = 1, 2, \ldots$$

and the 1-simple functions $\varphi_1(t) = g_1(t)$ and $\varphi_k(t) = g_{k+1}(t) - g_k(t)$, $k \geq 2$, with

$$J(\varphi_k) \subset \{t \in T \ , \ |t| \leq c_k\} \ , \qquad k = 1, 2, \dots \ ,$$

where $c_k := \arccos(1 - 2^{-k}) \leq C2^{-k/2}$, $\sin c_k \leq C2^{-k/2}$. Then $g_0 = \varphi_1 + \varphi_2 + \cdots$ on $\mathbb{T}$. The maximum of $|\varphi_k'(t)|$ is equal to $g_0'(c_{k+1}) \leq C2^{-k/2} g'(1 - 2^{-k-1})$. It follows that $\mu(\varphi_k) = |J(\varphi_k)| \, \|\varphi_k'\|_\infty \leq C2^{-k} g'(1 - 2^{-k-1})$, and we deduce (5.31) from Theorem 5.5:

$$\|g_0\|_{\mathcal{H}^1} \leq C \sum_{k=1}^{\infty} \mu(\varphi_k) \leq C \int_{-1}^{+1} g'(x)\,dx = CL \ . \qquad \square$$

This theorem cannot be improved:

Theorem 5.12. *For each $n = 1, 2, \dots$ there exists a continuous convex function $g_n \in C[-1,1]$ with the properties $\|g_n'\|_1 \leq 1$ and*

$$\rho_n(g_n) \geq \frac{C}{n} \tag{5.34}$$

for some absolute constant $C > 0$.

Proof. Instead of $[-1,1]$, it is more convenient to consider functions on the interval $A := A_n := [2^{-2n}, 2]$. The functions

$$\varphi(x) := \cos\left(\pi \log_2 \frac{1}{x}\right)$$

$$\psi(x) := \sum_{k=1}^{n} \frac{2^{-2k}}{x + 2^{-2k}}$$

are defined on A_n. The first has $2n+2$ alternating extrema there, $\varphi(2^{-k}) = (-1)^k$, $k = 2n, 2n-1, \dots, 0, -1$; the second belongs to $\mathcal{R}_n(A_n)$, is decreasing and non-negative, with a maximum $\psi(2^{-2n}) < \psi(0) = n$. By Theorem 2.6 of Chapter 7,

$$\rho_{2n}(\varphi)_{C(A_n)} = 1 \ . \tag{5.35}$$

We shall prove that

$$\psi''(x) \geq \frac{1}{10x^2} \ , \qquad x \in A_n \ . \tag{5.36}$$

Indeed, for $x \in [2^{-2k}, 2^{-2k+2}]$, $k = 1, \dots, n$,

$$\psi''(x) \geq \frac{2^{-2k+1}}{(x + 2^{-2k})^3} = \frac{2^{-2k+1}}{x + 2^{-2k}} \, \frac{1}{(x + 2^{-2k})^2} \geq \frac{2}{5} \, \frac{1}{(2x)^2} = \frac{1}{10x^2} \ ;$$

similarly, for $x \in [1,2]$, $\psi''(x) \geq 2^{-1}(x + 2^{-2})^3 \geq 1/(10x^2)$. We define, for some $a > 0$,

$$g_n(x) := a\left(10\psi(x) + \frac{1}{57}\varphi(x)\right) .$$

Using $\log 2 > \frac{1}{2}$ we get $|\varphi''(x)| \le 57x^{-2}$, and together with (5.36), $g_n''(x) \ge 0$, $x \in A_n$, so that g_n is convex. We then deduce from (5.35)

$$\rho_n(g_n)_{C(A_n)} \ge \frac{a}{57}\rho_{2n}(\varphi)_{C(A_n)} = \frac{1}{57}a .$$

It remains to remark that for $a := 1/(11n)$ the function g_n satisfies also the condition $\|g_n'\|_1(A_n) \le 1$. □

§ 6. Relations Between Spline and Rational Approximation in the Spaces L_p, $0 < p < \infty$

It has been observed a long time ago that free knot spline approximation and rational approximation are closely related, and that for many important classes of functions they give similar errors of approximation. For instance, for the classes V^r, $r \ge 1$, of functions on an interval with $f^{(r)}$ of bounded variation, both $\sigma_{n,r}(f)$ and $\rho_n(f)$ are of the order $O(n^{-r-1})$ for all $f \in V^r$ (see [**CA**, Theorem 4.5, p.366] and Theorem 7.2 of Chapter 7). This analogy has its limitations. For instance, (see [**CA**, Theorem 4.2, p.365] and Theorem 7.7, Chapter 7) for all $f \in W_1^1$, $\sigma_{n,1}(f) = O(1/n)$, while for $\rho_n(f)$ we can assert, in general, only $\rho_n(f) = o(1)$.

The purpose of this section is to find, in Theorems 6.1 and 6.2, the general facts that govern these relations. As an important application, we shall characterize, in Theorem 6.9, some of the rational *approximation spaces* of order $O(n^{-\alpha})$, $\alpha > 0$. This will follow from Theorem 8.4 for free knot spline approximation, which has been proved in [**CA**, p.388]. The interval of definition will always be $A = [0, 1]$. To simply the arguments and writing, $\widetilde{\Sigma}_{n,r}$ will denote the (non-linear) space of all splines S (piecewise polynomials) of order r and with $n-1$ breakpoints in A. In other words, $S \in \widetilde{\Sigma}_{n,r}$ if for some points $0 = x_0 < x_1 < \cdots < x_n = 1$, S is a polynomial of degree $\le r-1$ on each interval $I_j := [x_{j-1}, x_j]$, $j = 1, \ldots, n$. Let $\sigma_{n,r}(f)_p$ be the error of approximation of $f \in L_p(A)$ from $\widetilde{\Sigma}_{n,r}$.

Theorem 6.1 (Popov [1974] for $p = \infty$, $r = 1$; Pekarskii [1986$_2$] in the general case). *For $f \in L_p(A)$, $1 < p \le \infty$ and $r = 1, 2, \ldots$, let $\gamma := (r + 1/p)^{-1}$ and $n \ge 1$. Then*

$$\sigma_{n,r}(f)_p \le \frac{C(p,r)}{n^r}\left[\sum_{k=1}^{n}\frac{1}{k}\left(k^r\rho_{k-1}(f)_p\right)^\gamma\right]^{1/\gamma} . \tag{6.1}$$

Theorem 6.2 (Pekarskii [1986$_1$], Petrushev [1987]). *For $f \in L_p(A)$, $0 < p < \infty$, let $r = 1, 2, \ldots$, $n \ge r$, $\beta > 0$ and $q = \min(1, p)$. Then*

$$\rho_n(f)_p \le \frac{C(p,r,\beta)}{n^\beta}\left[\sum_{k=1}^{n} \frac{1}{k}\left(k^\beta \sigma_{k,r}(f)_p\right)^q\right]^{1/q} . \tag{6.2}$$

Roughly speaking, these theorems assert that in the space L_p, $1 < p \le \infty$, approximation by splines is at least as good as rational approximation, and that in L_p, $0 < p < \infty$, rational approximation is at least as good as spline approximation.

Our proof of Theorem 6.1 will depend on the Bernstein-type inequalities for rational functions of §1, while that of Theorem 6.2 will use the Newman polynomials of §2, Chapter 7. The key role in the first proof will play the following lemma.

Lemma 6.3. *Let $R \in \mathcal{R}_m(A)$ be a rational function with real coefficients (and without poles on A). Let $r = 1,2,\ldots$, $1 < p \le \infty$, $\gamma = (r + 1/p)^{-1}$ and $n \ge 3(r+2)m$. Then*

$$\sigma_{n,r}(R)_p \le \frac{C(p,r)}{n^r}\|R^{(r)}\|_\gamma . \tag{6.3}$$

Proof. We can assume that R is not constant. Its derivative $R^{(k)}$ is a rational function of degree $\le m(k+1)$, whose denominator does not change sign on A. There exists a decomposition of A by points $0 = x_0 < \cdots < x_\mu = 1$ into $\mu \le m(r+1) + m(r+2) + 1 \le 2(r+2)m$ intervals $I_j = (x_{j-1}, x_j)$, on each of which $R^{(r)}$ is monotone and does not change sign. For convenience, let $\|R^{(r)}\|_\gamma(A) = 1$, and $1 < p < \infty$. (If $p = \infty$, the proof also works with obvious changes.) We put $\delta_j := \|R^{(r)}\|_\gamma^\gamma(I_j)$ and $n_j := [(n - 2m(r+2))\delta_j] + 1$, $j = 1,\ldots,\mu$.

On each I_j, we approximate R by splines. According to [**CA**, Theorem 4.6, p.367], for a function $f \in C^r(I)$ with a monotone derivative $f^{(r)}$ on an interval I,

$$\sigma_{n,r}(f)_p(I) \le C_{r,p} n^{-r}\|f^{(r)}\|_\gamma(I) . \tag{6.4}$$

Since $\delta_1 + \cdots + \delta_\mu = 1$, $n_1 + \cdots + n_\mu \le n$, and $\delta_j n_j^{-1} \le 3n^{-1}$, (6.4) yields

$$\begin{aligned}\sigma_{n,r}(R)_p^p &\le \sum_{j=1}^{\mu} \sigma_{n_j,r}(R)_p(I_j)^p \le C\sum_{j=1}^{\mu} n_j^{-pr}\delta_j^{p/\gamma} \\ &\le \frac{C}{n^{pr}}\sum \delta_j = \frac{C}{n^{pr}} .\end{aligned}$$

□

Proof of Theorem 6.1. This theorem follows, by standard methods, from (6.3) and the Bernstein type inequality for rational functions (1.1). More formally, Theorem 6.1 can be derived from [**CA**, Theorem 5.1 (i) and (ii), p.216], provided one selects the Φ_k differently for the cases (i) and (ii). We always take $X = L_p$, $Y = \mathcal{R}_n$, the latter space equipped with the semiquasinorm $|f|_Y = \|f^{(r)}\|_\gamma$, $\mu = \gamma$, and $K(f,t) := K(f,t;X,Y)$. In case (i), $\Phi_n = \widetilde{\Sigma}_{n,r}$, then $E_n(f)_X = \sigma_{n,r}(f)_p$, and the required inequality (5.4) is (6.3). This yields $\sigma_{n,r}(f)_p \le CK(f,n^{-r})$. In case (ii), $\Phi_k = \mathcal{R}_{k-1}$, $k = 1,\dots,n$, $E_k(f)_X = \rho_{k-1}(f)_p$, $f \in Y$ and the required inequality (5.5) is (1.1). This yields

$$K(f,n^{-r}) \le Cn^{-r}\left\{\sum_{k=1}^{n}\left[k^r\rho_{k-1}(f)_p\right]^\gamma \frac{1}{k}\right\}^{1/\gamma},$$

completing the proof. □

For the proof of Theorem 6.2 we need a lemma about rational L_p-approximation of splines. In a certain sense, this lemma is the dual of Lemma 6.3.

Lemma 6.4. *Let m,n,r be natural numbers, $0 < p < \infty$. There exist then positive numbers a, C, which depend only on p,r, with the property that for each $S \in \widetilde{\Sigma}_{m,r}$ there is a rational function $R \in \mathcal{R}_n$ for which*

$$\|S - R\|_p(A) \le C\exp\left(-a\sqrt{n/m}\right)\|S\|_p(A)\,. \tag{6.5}$$

The idea of the proof is as follows. A spline $S \in \widetilde{\Sigma}_{m,r}$ on A we represent in the form $S = \sum_{j=1}^m \chi_j P_j$, where the P_j are polynomials of degree $\le r-1$, the χ_j are the characteristic functions of the intervals $I_j := [x_{j-1}, x_j]$, $j = 1,\dots,m$, and $0 = x_0 < \cdots < x_m = 1$. We approximate the χ_j on A by rational functions R_j, using Lemma 6.5 below. Then

$$R := \sum_{j=1}^m R_j P_j \tag{6.6}$$

is the rational function of (6.5).

We have to estimate $\sum|(\chi_j - R_j)(x)P_j(x)|$. The size of the polynomials P_j outside of I_j is only moderate (see (6.10)), and for x not too close to the endpoints of I_j, $|\chi_j(x) - R_j(x)|$ is very small. The main difficulty is to estimate these functions and their sums close to x_j. This is achieved in Lemma 6.7, using the functions $\theta_j(x)$.

Lemma 6.5. *For arbitrary natural k, ℓ, $k \ge 5$, there exists a rational function Q, $\deg Q \le 4k + 2\ell$ which satisfies*

$$0 \le Q(t) \le 1 \quad \textit{for} \quad t \in \mathbb{R}\,, \tag{6.7}$$

$$1 - Q(t) \le e^{-2\sqrt{k}} \quad \textit{for} \quad |t| \le 1 - 2e^{-\sqrt{k}}\,, \tag{6.8}$$

$$Q(t) \le |t|^{-2\ell}e^{-2\sqrt{k}} \quad \textit{for} \quad |t| \ge 1\,. \tag{6.9}$$

Proof. With the Newman polynomials $N(x) := N_k(x)$ of degree $k-1$ (see (3.2) of Chapter 5) we define $R(x) := N(-x)/N(x)$. Then the rational function

$$Q(t) := \left[1 + t^{2\ell} R^2\big(w(t)\big)\right]^{-1},$$

$$w(t) := \frac{1-\lambda t^2}{1+\lambda t^2}, \quad \lambda := \frac{1+e^{-\sqrt{k}}}{1-e^{-\sqrt{k}}}$$

has the required properties. Indeed, $\deg Q \le 4k + 2\ell$, and (6.7) is obvious. By Lemma 3.2 of Chapter 7,

$$|R(x)| \le e^{-\sqrt{k}} \quad \text{for} \quad x \in [e^{-\sqrt{k}}, 1]$$

and consequently

$$|R(x)| \ge e^{\sqrt{k}} \quad \text{for} \quad x \in [-1, -e^{-\sqrt{k}}] .$$

To derive from this (6.8) and (6.9) it is sufficient to check the inclusions

$$w\big(\{t : |t| \le 1 - 2e^{-\sqrt{k}}\}\big) \subset [e^{-\sqrt{k}}, 1] ,$$
$$w\big(\{t : |t| > 1\}\big) \subset [-1, -e^{-\sqrt{k}}] ,$$

where $w(S)$ is the image of the set S under w. □

For each polynomial $P \in \mathcal{P}_{r-1}$ for $x \notin I_0 := [-1,1]$ one has $|P(x)| \le (2|x|)^{r-1}\|P\|_\infty(I_0)$ (see [**CA**, Proposition 2.3, p.101]. With the help of the linear transformation of $I = [a,b]$ onto I_0 we can write this in the form

$$|P(x)| \le C_r \theta(x, I)^{-r+1} \|P\|_\infty(I) , \qquad x \in \mathbb{R} , \tag{6.10}$$

where

$$\theta(x, I) := \min\left\{1, \frac{b-a}{|2x-a-b|}\right\} , \qquad x \in \mathbb{R} .$$

On finitely dimensional spaces, all quasi-norms are equivalent. For $P \in \mathcal{P}_{r-1}(I_0)$, equivalent are all quasi-norms $\|P\|_p(I_0)$, $0 < p \le \infty$. By a linear transformation, we get

$$\|P\|_\infty(I) \le C_{p,r} |I|^{-1/p} \|P\|_p(I) , \qquad P \in \mathcal{P}_{r-1}(I) . \tag{6.11}$$

We shall need also properties of the Hardy-Littlewood maximal Mg function (defined for each locally integrable function g on $\mathbb{R}$, see (7.1), Chapter 7), in particular the inequality $\|Mg\|_p \le C(p)\|g\|_p$, for $g \in L_p(\mathbb{R})$, $1 < p < \infty$ (see Bennett and Sharpley [B-1988, p.123]). Another useful inequality is

$$\frac{1}{|J|}\int_J |g(t)|\, dt \le \frac{1}{|I|}\int_I Mg(x)\, dx , \qquad g \in L_1(\mathbb{R}) \tag{6.12}$$

which is valid for any two intervals $I \subset J$. Indeed, for $x \in J$,

$$\frac{1}{|J|}\int_J |g(t)|\,dt \le Mg(x)\ ,$$

and it remains to take the average over I of both sides of this inequality.

The following lemma relates maximal functions to the $\theta(x, I)$.

Lemma 6.6. *Let $\nu > 1$ and let $I := [a, b]$. If the function $\varphi \ge 0$ is integrable on each finite interval, and if $M\varphi$ is integrable on I, then*

$$\int_{\mathbb{R}} \varphi(x)\theta(x, I)^\nu\,dx \le C(\nu)\int_I (M\varphi)(x)\,dx\ . \tag{6.13}$$

Proof. By I_k, $k = 0, 1, \dots$ we denote intervals concentric with I of length $2^k|I|$, and by χ_k their characteristic functions. Since $\theta(x, I) = 1$ for $x \in I$, and $\theta(x, I) \le 2^{-k+1}$ for $x \in I_k \setminus I_{k-1}$, $k \ge 1$, we get

$$\theta(x, I)^\nu \le 2^\nu \sum_{k=0}^{\infty} 2^{-\nu k}\chi_k(x)\ , \qquad x \in \mathbb{R}\ .$$

From this and (6.12) we derive (6.13):

$$\begin{aligned}
\int_{\mathbb{R}} \varphi(x)\theta(x, I)^\nu\,dx &\le 2^\nu \sum_{k=0}^{\infty} 2^{-\nu k}\int_{I_k}\varphi(x)\,dx \\
&= 2^\nu |I| \sum_{k=0}^{\infty} 2^{-(\nu-1)k}\frac{1}{|I_k|}\int_{I_k}\varphi(x)\,dx \\
&\le 2^\nu |I| \sum_{k=0}^{\infty} 2^{-(\nu-1)k}\frac{1}{|I|}\int_I (M\varphi)(x)\,dx \\
&\le C(\nu)\int_I (M\varphi)(x)\,dx\ .
\end{aligned}$$

□

For a sequence of points $0 = x_0 < \dots < x_m = 1$ and the intervals $I_j = (x_{i-1}, x_j)$ we define

$$\theta_j(x) := \theta(x, I_j)\ , \qquad j = 1, \dots, m\ . \tag{6.14}$$

Lemma 6.7. *For $0 < p < \infty$, $\nu > \max(1, 1/p)$ and arbitrary $a_j \ge 0$, $j = 1, \dots, m$ we have*

$$N := \int_{\mathbb{R}} \left(\sum_{j=1}^{m} a_j\theta_j(x)^\nu\right)^p dx \le C\sum_{j=1}^{m} a_j^p |I_j|\ , \tag{6.15}$$

where the constant C depends on p and ν.

Proof. If $0 < p \le 1$, we have

$$N \le \sum_{j=1}^{m} a_j^p \int_{\mathbb{R}} \theta_j^{\nu p}(x)\, dx .$$

Since $\nu p > 1$, an application of Lemma 6.6 with $\varphi = 1$ on $\mathbb{R}$ leads directly to (6.15).

For $1 < p < \infty$ we use duality. There exists a function $\varphi \in L_{p'}(\mathbb{R})$, $\frac{1}{p} + \frac{1}{p'} = 1$, with the properties $\varphi \ge 0$, $\|\varphi\|_{p'} = 1$, for which

$$N^{1/p} = \int_{\mathbb{R}} \varphi(x) \sum_{j=1}^{m} a_j \theta_j(x)^\nu \, dx .$$

We have $M\varphi \in L_{p'}(\mathbb{R})$ and $\|M\varphi\|_{p'} \le C(p)$. Lemma 6.6 and Hölder's inequality then yield (6.15):

$$\begin{aligned} N^{1/p} &\le C(\nu) \sum_{j=1}^{m} a_j \int_{I_j} (M\varphi)(x)\, dx \\ &\le C(\nu) \sum_{j=1}^{m} a_j |I_j|^{1/p} \|M\varphi\|_{p'}(I_j) \\ &\le C(\nu) \Big(\sum_{j=1}^{m} a_j^p |I_j| \Big)^{1/p} \Big(\sum_{j=1}^{m} \|M\varphi\|_{p'}^{p'}(I_j) \Big)^{1/p'} \\ &\le C(p,\nu) \Big(\sum_{j=1}^{m} a_j^p |I_j| \Big)^{1/p} . \end{aligned}$$ □

Proof of Lemma 6.4. For a given spline $S = \sum_{j=1}^{m} \chi_j P_j \in \widetilde{\Sigma}_{m,r}$, we take the rational function R given by (6.6), where

$$R_j(x) := Q\left(\frac{2x - x_{j-1} - x_j}{x_j - x_{j-1}} \right) , \qquad j = 1, \ldots, m \tag{6.16}$$

and Q is the function of Lemma 6.5.

The parameters k, ℓ of this lemma we require to satisfy

$$(2\ell + 1 - r) \min(1,\, 1/p) > 1 \quad \text{and} \quad n \ge (4k + 2\ell)m + r . \tag{6.17}$$

This is possible if $n \ge C_1 m$ with sufficiently large $C_1 = C_1(p, r)$ and even so that $k \ge 5$. (For this purpose we first take the smallest possible ℓ in the first inequality (6.17), then the largest possible k in the second.) We shall assume that $n \ge C_1 m$ is satisfied, otherwise (6.5) is obvious.

With this selection we have $\deg R \le n$. To establish (6.5), we denote by I_j' the interval of length $(1 - 2e^{-\sqrt{k}})|I_j|$, concentric with I_j, $j = 1, \ldots, m$, we let $I_j'' := I_j \setminus I_j'$, further $A'' = \cup I_j''$, $A' = A \setminus A''$. We shall estimate the L_p norm (or quasi-norm) of

$$S - R = \sum_{j=1}^{m} (\chi_j - R_j) P_j$$

on A' and on A''. From Lemma 6.5 and (6.11), for all j,

$$\begin{aligned}\|(\chi_j - R_j)P_j\|_p^p(I_j'') &\le \|P_j\|_p^p(I_j'') \le |I_j''|\, \|P_j\|_\infty^p(I_j) \\ &= 2e^{-\sqrt{k}}|I_j|\, \|P_j\|_\infty^p \le C_{p,r}e^{-\sqrt{k}}\|P_j\|_p^p(I_j) ,\end{aligned}$$

and adding, we obtain

$$\|S - R\|_p^p(A'') \le Ce^{-\sqrt{k}}\|S\|_p^p(A) . \tag{6.18}$$

On the other hand, on A' we can apply either (6.8) or (6.9). Thus, if $x \in I_i$, $i \ne j$, for $t = \frac{2x - x_{j-1} - x_j}{x_j - x_{j-1}}$,

$$|(\chi_j - R_j)(x)| = R_j(x) \le t^{-2\ell}e^{-2\sqrt{k}} = e^{-2\sqrt{k}}\theta(x, I_j)^{2\ell} .$$

This prevails also for $x \in I_j'$, since then $\theta(x, I_j) = 1$. With the help of (6.10) and (6.11) we get for $x \in A'$,

$$|(S - R)(x)| \le Ce^{-2\sqrt{k}} \sum_{j=1}^{m} |I_j|^{-1/p}\|P_j\|_p(I_j)\theta_j(x)^{2\ell - r + 1} .$$

Since $2\ell - r + 1 > \max(1, 1/p)$ we can apply Lemma 6.7:

$$\|S - R\|_p^p(A') \le Ce^{-2p\sqrt{k}} \sum_{j=1}^{m} \|P_j\|_p^p(I_j) = Ce^{-2p\sqrt{k}}\|S\|_p^p . \tag{6.19}$$

Inequalities (6.18) and (6.19) establish the desired result
$\|S - R\|_p \le Ce^{-a\sqrt{k}}\|S\|_p$. □

Proof of Theorem 6.2. If $\beta > 0$, $a > 0$ are given, we can find a $C_1 > 0$ so that $\exp(-a\sqrt{y}) \le C_1 y^{-4\beta}$ for all $y \ge 1$. This shows that in Lemma 6.4 we can replace the inequality (6.5) for $S \in \widetilde{\Sigma}_{m,r}$ by

$$\rho_n(S)_p \le C\left(\frac{m}{n}\right)^{2\beta} \|S\|_p \ \text{ if } \ m \le n , \quad C = C(p, r, \beta) . \tag{6.20}$$

First let $1 \le p < \infty$, $f \in L_p$. For a natural ν and each $j = 1, \dots, \nu$ we select a spline $S_j \in \widetilde{\Sigma}_{4^j,r}$ so that

$$\|f - S_j\|_p \le 2\sigma_{4^j,r}(f)_p =: \delta_j , \qquad j = 1, \dots, \nu . \tag{6.21}$$

If $g_j := S_j - S_{j-1}$, we have

$$f = S_0 + g_1 + \cdots + g_\nu + (f - S_\nu) . \tag{6.22}$$

Here S_0 is a polynomial of degree $< r$, $g_j \in \widetilde{\Sigma}_{4^{j+1},r}$ and $\|g_j\| \le 2\delta_{j-1}$. We select $n := 4^{\nu+1}$, $n_j := 2^{\nu+j}$, then $n_1 + \cdots + n_\nu \le n$. It follows by (6.22), (6.20) and (6.21) that

$$\rho_n(f) \le \|f - S_\nu\|_p + \sum_{j=1}^{\nu} \rho_{n_j}(g_j)_p$$

$$\le \delta_\nu + C \sum_{j=1}^{\nu} \left(\frac{4^j}{2^{\nu+j}}\right)^{2\beta} 4\sigma_{4^{j-1},r}(f) \ .$$

This is equivalent to (6.2) for $p \ge 1$. If $0 < p < 1$, the proof remains the same if we replace $\|\cdot\|_p$ by $\|\cdot\|_p^p$. □

Easy corollaries of Theorems 6.1, 6.2 are:

Example 1. (i) If $\rho_n(f) = O(n^{-\alpha})$, $\alpha > 0$, $1 < p \le \infty$, then $\sigma_{n,r}(f)_p = O(n^{-\alpha})$ for $r > \alpha$, $= O(n^{-r})$ for $r < \alpha$, $= O(n^{-r}\log^{1/\gamma} n)$ for $r = \alpha$.

(ii) If $\sigma_{n,r}(f)_p = O(n^{-\alpha})$, $\alpha > 0$, $0 < p < \infty$, then $\rho_n(f)_p = O(n^{-\alpha})$.

The following examples show that Theorems 6.1 or 6.2 cannot be extended to $p = 1$, or to $p = \infty$, respectively.

Example 2. (i) For each natural n there exists a rational function $R_n \in \mathcal{R}_1$ for which $\|R_n\|_1(A) = 1$ and $\sigma_{n,r}(R_n) \ge C(r) > 0$ for any $r = 1, 2, \ldots$.

(ii) For each natural n there exists a spline $S_n \in \widetilde{\Sigma}_{3,2} \cap C$ for which $\|S_n\|_\infty(A) = 1$ and $\rho_n(S_n)_\infty \ge \frac{1}{2}$.

(i) Let the auxiliary function $f(t) = 1/t$ be defined on $I = [e^{-2n}, 1]$. Linear transformations show that $E_{r-1}(f, I_k)_1 = E_{r-1}(f, I_1)_1$, for $I_k := [e^{-k}e^{-k+1}]$, $k = 1, \ldots, 2n$. Now for each spline $S \in \widetilde{\Sigma}_{n,r}(I)$ there exist n integers k_j, $1 \le k_1 < \cdots < k_n \le 2n$ with the property that none of the intervals I_{k_j} contains knots of the spline S. Then

$$\|f - S\|_1(I) \ge \sum_{j=1}^{n} E_{r-1}(f, I_{k_j})_1 = nE_{r-1}(f, I_1)_1 \ .$$

Thus $\sigma_{n,r}(f, I)_1 \ge nE_{r-1}(f, I_1)_1$. This means that the function $R_n(x) := (1 - e^{-2n})\frac{1}{2n} f(e^{-2n} + (1 - e^{-2n})x)$ has desired properties.

(ii) We define the auxiliary function $f \in C[-1, 1]$ by $f(x) = 0$ for $-1 \le x \le 0$, $f(x) = 1$ for $e^{-\pi^2 n} \le x \le 1$, and let f be linear on $[0, e^{-\pi^2 n}]$. According to Lemma 7.8 of Chapter 7, $\rho_n(f)_{C[-1,1]} \ge \frac{1}{4}$. This means that the function $S(x) := 2f(2x - 1) - 1$ has the desired properties.

As an important application of the main theorems we derive the identity of approximation spaces for the rational and the spline functions. Let $\Phi = \{\Phi_0, \ldots, \Phi_n, \ldots\}$ be increasing sets of functions on A as in [**CA**, (5.2), p.216].

For $f \in L_p$, let $E_{\Phi_n}(f)_p$ be $\inf_{\phi \in \Phi_n} \|f - \phi\|_p$, then for $\alpha > 0$, $q > 0$, the approximation space $A_q^\alpha(L_p, \Phi)$ consists of all f with $\|\mathbf{E}_{\Phi_{2^n}}(f)_p\|_{\alpha,q} < \infty$. Here for a sequence $\mathbf{a} = (a_k)_1^\infty$, $a_k \ge 0$, we define $\|\mathbf{a}\|_{\alpha,q} := (\sum (2^{k\alpha} a_k)^q)^{1/q}$; see [**CA**, §3, Chapter 2]. For example, the space $A_q^\alpha(L_p, \mathcal{R})$ with $\Phi_n = \mathcal{R}_n$, $n = 0, 1, \dots$ consists of all $f \in L_p$ for which

$$(6.23)\quad \sum_{k=1}^{\infty} \left(2^{k\alpha} \rho_{2^k}(f)_p\right)^q < \infty\ , \quad \text{or, equivalently,} \quad \sum_{k=1}^{\infty} \frac{1}{k} \left(k^\alpha \rho_k(f)_p\right)^q < \infty\ .$$

Similarly, for the approximation space $A_q^\alpha(L_p, \widetilde{\Sigma}_r)$ we take $\Phi_n := \widetilde{\Sigma}_{n,r}$, $n = 0, 1, \dots$.

Theorem 6.8. *Let $1 < p < \infty$, $0 < \alpha < r$, $\gamma = (\alpha + 1/p)^{-1}$ and $0 < q \le \infty$. Then the rational and the spline approximation spaces are identical:*

$$(6.24)\qquad A_q^\alpha(L_p, \mathcal{R}) = A_q^\alpha(L_p, \widetilde{\Sigma}_r)\ .$$

Proof. One of the forms of the discrete Hardy inequality is that if two positive sequences $\mathbf{b} = (b_k)_1^\infty$, $\mathbf{a} = (a_k)_1^\infty$ satisfy

$$(6.25)\qquad b_k \le C_0 2^{-k\beta} \left(\sum_{j=1}^{k} (2^{j\beta} a_j)^\mu \right)^{1/\mu}, \qquad \mu, \beta > 0\ ,$$

then for $0 < \alpha < \beta$,

$$(6.26)\qquad \|\mathbf{b}\|_{\alpha,q} \le C_0 C \|\mathbf{a}\|_{\alpha,q}$$

(see [**CA**, Lemma 3.4 and (3.13), pp.27,28]). Now, Theorem 6.2 asserts that $b_k := \rho_{2^k}(f)_p$ and $a_k := \sigma_{2^k,r}(f)_p$ satisfy (6.25). Hence $A_q^\alpha(L_p, \mathcal{R}) \subset A_q^\alpha(L_p, \widetilde{\Sigma}_r)$, and the proof of the opposite inclusion is similar. □

In the special case $q = \gamma$ the spline approximation spaces have been identified by DeVore, Petrushev and Popov in [**CA**, Theorem 8.4, p.388] as certain Besov spaces B^α. Therefore we have:

Theorem 6.9. *A function $f \in L_p(A)$ belongs to $A_\gamma^\alpha(L_p, \mathcal{R})$ if and only if it belongs to $B_\gamma^\alpha(L_\gamma) := B^\alpha$, that is, if and only if*

$$(6.27)\qquad |f|_{B^\alpha} := \left(\int_0^\infty \left[t^{-\alpha} \omega_r(f, t)_\gamma \right]^\gamma \frac{dt}{t} \right)^{1/\gamma} < \infty\ .$$

Using a result of Pekarskii [1985], Peller [1987] also obtained this theorem independently and simultaneously.

§ 7. Problems

7.1. Prove that the best constants $C(p,r)$ for $p=\infty$, $r=1$ in Theorem 1.2 is 2π (Rusak [A-1979], Pekarskii [1982_1]).

7.2. If $g \in C(\mathbb{T})$ and $\sum_{n=1}^{\infty} \rho_n(g)_\infty < \infty$, then the Fourier series of g is absolutely convergent (Sevastyanov [1978]).

7.3. For $\alpha > 0$, $\beta > 0$ we define on $[0,1]$ the function $g_{\alpha,\beta}(x) := x^\alpha \sin(x^{-\beta})$, $0 < x \le 1$, $g_{\alpha,\beta}(0) := 0$. Then for $n \ge 1$

$$E_n(g_{\alpha,\beta}) \sim n^{-2\alpha/(2\beta+1)} \quad , \quad \rho_n(g_{\alpha,\beta}) \sim n^{-\alpha/\beta} \ .$$

7.4. For $\beta > 0$, let $g_\beta(x) := \log^{-\beta}(e/x)$, $x \in (0,1]$, $g_\beta(0) := 0$. If g_β^+ and g_β^- are, respectively, the even and the odd extensions of g_β onto $[-1,1]$, then for $n \ge 1$

$$\rho_n(g_\beta^+)_{C[-1,1]} \sim n^{-\beta-1} \quad , \quad \rho_n(g_\beta^-)_{C[-1,1]} \sim n^{-\beta} \ .$$

7.5. For the even extension g^+ of $g \in C[0,1]$ onto $[-1,1]$ prove that for $n \ge 1$, $r \ge 1$,

$$\rho_n(g^+)_{C[-1,1]} \le \frac{C(r)}{n^r} \left(\sum_{k=0}^{n} \rho_k(g)_{C[0,1]}^{1/r} \right)^r \ .$$

7.6. For each sequence $\varepsilon_1 \ge \varepsilon_2 \ge \cdots$, $\lim \varepsilon_n = 0$, there exists a convex function $g \in C[0,1]$ which satisfies the inequality $\rho_n(g)_\infty \ge \varepsilon_n/n$ for infinitely many values of n (Bulanov [1969]).

7.7. If a real function $g \in C(\mathbb{T})$ is even and convex on $[0,2\pi]$, then $\rho_n(g)_\infty \le \frac{C}{n}\|g'\|_1$. Prove this in two ways: (i) using Theorem 5.1; (ii) using Theorem 2.6.

7.8. If $g \in BV[0,1]$, then for $0 < p < \infty$, $n \ge 1$,

$$\sigma_{n,1}(g)_p \le \frac{C(p)}{n} \mathrm{Var}\, g \quad \text{and} \quad \sigma_{n,2}(g)_p = o(1/n) \ .$$

Prove also similar relations for $\rho_n(g)_p$ (Petrushev for $p=1$; Pekarskii for $p < \infty$. See Petrushev and Popov [A-1987]).

§ 8. Notes

8.1. For the proof of the inequalities (1.7) and (1.8) for $p=\infty$, $r=1$, and $m=n$, Rusak [A-1979] introduced an interpolating formula for the derivative of a rational function, similar to the formula of M. Riesz for the derivative of a trigonometric polynomial. The result of Rusak has been used by Pekarskii [1980] in his first proof of Theorem 1.1 for $p=\infty$, $r \ge 2$ and Theorem 1.2 for $p=\infty$, $r \ge 2$, $n=m$. See also Sevastyanov [1978].

8.2. The differential properties of functions that can be derived from the magnitude of their rational approximation error have been studied in many papers, see for example, Dolzhenko [1962], Gonchar [1968], Sevastyanov [1985]. Several of these results can be derived from Theorem 5.2 and a result of Krotov [1986] about the differential properties of the classes H_p^r.

8.3. In [A-1979], Rusak defined also rational operators of Fejér and of de la Valleé-Poussin type. In particular, he used the latter to obtain rational approximations, in convolution form, to periodic functions.

8.4. A function f, analytic in D, is said to belong to the Carathéodory-Fejér space CF (also denoted by $BMOA$) if $f \in H_2$ and

$$\|f\|_{CF} := \inf\{\|f - g\|_\infty(\partial D) : g \in H_2^-\} < \infty .$$

We denote by $\mathcal{A}_p^\alpha$, $\alpha > 0, p > 0$, the space of all f, analytic in D, which satisfy the condition

$$\|f\|_{\mathcal{A}_p^\alpha} := \left(\iint_D |f^{(r)}(z)|^p (1 - |z|)^{p(r-\alpha)-1}\, dx\, dy\right)^{1/p} < \infty ,$$

where $r := [\alpha] + 1$. According to Peller [1980], for $0 < \alpha \le 1$,

$$A_{1/\alpha}^\alpha(CF, \mathcal{R}) = \mathcal{A}_{1/\alpha}^\alpha . \tag{8.1}$$

This equality is the first example of a complete description of an approximation space defined by the behavior of errors of the rational approximation. The embedding $A_{1/\alpha}^\alpha(CF, \mathcal{R}) \supset \mathcal{A}_{1/\alpha}^\alpha$ has been obtained by Peller for all $\alpha > 0$. Later, Peller [1983], Pekarskii [1984] and Semmes [1984] extended the inverse inclusion to all $\alpha > 1$.

8.5. We shall explain the relation between (8.1) and the Theorems 2.1 and 2.3. Since $\|f\|_{CF} \le \|f\|_{\mathcal{A}}$ for $f \in \mathcal{A}$, we derive from Theorem 2.1 that $f \in H_{1/r}^r$, $r = 1, 2, \ldots$ implies

$$\rho_n(f)_{CF} \le \frac{C(r)}{n^r} \|f^{(r)}\|_{H_{1/r}} , \qquad n \ge r . \tag{8.2}$$

Moreover, (1.6) yields for $R \in \mathcal{R}_m(\bar{D})$

$$\|R^{(r)}\|_{H_{1/r}} \le C(r) m^r \|R\|_{CF} . \tag{8.3}$$

From (8.3) we obtain the following implication

$$\sum_{n=1}^{\infty} \rho_n(f)_{CF}^{1/r} < \infty \Longrightarrow f^{(r)} \in H_{1/r} . \tag{8.4}$$

With the help of interpolation from (8.2) and (8.4) one can derive (8.1) for all $\alpha > 0$.

8.6. A Hankel operator $\mathcal{H}_f : H_2 \to H_2^-$ with the symbol f, $f \in H_2$, is the operator defined by the formula $\mathcal{H}_f g := C^-(\bar{f}g)$, $g \in H_2$. According to the theorem of Nehari, the operator $\mathcal{H}_f$ is bounded if and only if $f \in CF$; then $\|\mathcal{H}_f\| = \|f\|_{CF}$. By a theorem of Hartmann, $\mathcal{H}_f$ is compact if and only if $f = C^+\varphi$ for some $\varphi \in C(\partial D)$. The theorem of Adamjan-Arov-Krein asserts that for a compact operator $\mathcal{H}_f$, one has $\rho_n(f)_{CF} = s_n(\mathcal{H}_f)$, $n = 1, 2, \ldots$, where $s_n(\mathcal{H}_f)$ is the n-th singular number of the operator $\mathcal{H}_f$. This connection of the errors of rational approximation with the operator theory has enabled Peller to prove (8.1). See the papers of Peller [1980], [1983], of Semmes [1984], and of Peller and Khrushchev [1982]. We shall add that, using the theory of Hankel operators, Parfenov [1986] proved the following conjecture of Gonchar (compare [**CA**, Theorem 8.1, p.229]): If g is analytic in the ellipse E_{ρ_0}, $\rho_0 > 1$, with focii ± 1 and the sum of half-axses ρ_0, then

$$\liminf_{n\to\infty} \rho_n(g)_{C[-1,1]}^{1/n} \le \rho_0^{-2} .$$

8.7. Concluding, we shall mention that Pekarskii [1985], [1987] has obtained analogues of the equation (8.1) also for the spaces H_p, $0 < p < \infty$, $\mathcal{A}$, and $C[-1, 1]$.

Chapter 11. Müntz Polynomials

§ 1. Definitions and Simple Properties

Let $\Lambda_\infty = \{0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots\}$, $\lambda_n \to \infty$ be a sequence of real numbers. The famous theorem of Müntz [1914] characterizes those sequences Λ_∞, for which all functions in $C[0,1]$ or $L_p[0,1]$, $1 \le p < \infty$ can be approximated by "Müntz polynomials"

$$M(x) = \sum_{k=0}^{n} a_k x^{\lambda_k} \tag{1.1}$$

or, in other words, for which the sequence x^{λ_k}, $k \ge 0$, is complete in these spaces (see [**CA**, Theorem 5.1, p.345]):

Theorem 1.1. *The functions x^{λ_k} are complete in each of the spaces $C[0,1]$ or $L_p[0,1]$, $1 \le p < \infty$ if and only if*

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} = \infty. \tag{1.2}$$

Condition (1.2) is also necessary and sufficient for the completeness of the x^{λ_k} in each of the spaces $C[a,b]$ or $L_p[a,b]$ if $0 < a < b$ (see [**CA**, Ch.11, §6]). Borwein and Erdélyi (see Note 7.1) have given many interesting theorems about the density of Müntz polynomials.

In this chapter, however, all results will be for the interval $[0,1]$, and $n \ge 1$ will be fixed. We write $\Lambda := \Lambda_n : 0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n$. The *Müntz polynomials* $M \in \mathcal{M}(\Lambda)$ are sums of the type (1.1), with real coefficients a_k. The *error of approximation* from $\mathcal{M}(\Lambda)$ of a real function f in $C[0,1]$ or $L_p[0,1]$, $1 \le p < \infty$ is

$$E(f,\Lambda)_p := \inf_{M \in \mathcal{M}(\Lambda)} \|f - M\|_p. \tag{1.3}$$

An estimation from above of $E(f,\Lambda)_p$ in terms of the smoothness or the differentiability properties of f is called a (direct) Müntz-Jackson theorem. The first theorem of this type has been found by Newman [1965] for $L_2[0,1]$ and integral λ_k satisfying $\lambda_{k+1} - \lambda_k \ge 2$, namely

$$E(f,\Lambda)_2 \le C\omega(f,\varepsilon_2(\Lambda))_2, \quad f \in L_2[0,1],$$

where C is an absolute constant, $\omega(f,h)_2 := \omega(f,h,[0,1])_2$ is the L_2-modulus of continuity of f, (see [**CA**, (6.12), p.44]), and $\varepsilon_2(\Lambda)$ is defined below. Newman's method of proof uses the formulas (3.2) and (3.4), and will be applied in §3 in the proof of the inverse Müntz-Jackson theorem 3.1. His method was extended by Ganelius and Newman [1976] to all spaces $C[0,1]$ and $L_p[0,1]$, $p \ge 1$. In these investigations the Blaschke product

$$B_p(z) := B_p(z,\Lambda) := \prod_{k=1}^{n} \frac{z-\lambda_k-1/p}{z+\lambda_k+1/p}\ , \quad 1 \le p \le \infty, \tag{1.4}$$

and the expression

$$\varepsilon_p := \varepsilon_p(\Lambda) := \max_{y\ge 0} \left| \frac{B_p(1+iy)}{1+iy} \right| \tag{1.5}$$

play an important role. We call ε_p the *index of approximation* of Λ in $C[0,1]$ or $L_p[0,1]$.

Let $W_p^1[0,1]$ be the Sobolev space of all real functions f which are absolutely continuous on $[0,1]$, with $f' \in L_p[0,1]$. Bak, Leviatan, Newman, Tzimbalario [1973] proved the following beautiful result for $2 \le p \le \infty$, and Ganelius and Newman [1976] for all $p \ge 1$:

Theorem 1.2. *Let* $\Lambda = \{0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n\}$. (i) *Then, for* $1 \le p \le \infty$ *and* $f \in W_p^1[0,1]$,

$$E(f,\Lambda)_p \le A_p \varepsilon_p \|f'\|_p, \tag{1.6}$$

where $A_p < 2^{29}$. (ii) *The inequality* (1.6) *is false for each* $p \ge 1$ *and each* Λ *if* A_p *is replaced by* $1/600$.

From this one easily derives the *general Müntz-Jackson theorem* (Theorem 2.7).

Another method of proof for Müntz-Jackson theorems was developed by v. Golitschek [1970] and extended by Leviatan [1974, 1977], Newman [1973, 1974_1] and v. Golitschek [1976_{1-3}]. The new strategy consists in first approximating the function f by an algebraic polynomial $P_m(x) = \sum_{j=0}^{m/2} c_j x^{2j}$ and then approximating each of the x^{2j} by Müntz polynomials. This method yields the inequality (1.6) of Theorem 1.2 with small constants: $A_p < 42$ for all $p \ge 1$ (v. Golitschek [1989, 1991_1]).

The plan of the chapter is as follows. In §2 we prove the inequality (1.6) for $p \ge 2$. From this we derive Müntz-Jackson theorems for all functions $f \in C[0,1]$ if $p = \infty$, $f \in L_p[0,1]$ if $2 \le p < \infty$, and for f having derivatives up to a given order. We do not consider the cases $1 \le p < 2$, which require longer and more complicated proofs.

The most difficult part of the chapter is §3, with a proof of Theorem 1.2(ii) for $p \geq 2$. For special sequences Λ, the index of approximation $\varepsilon_p = \varepsilon_p(\Lambda)$ can be replaced by a much simpler expression. This is the content of §4. In §5 we prove Newman's Markov-type inequality for Müntz polynomials.

Finally, in §7, we point out some additional theorems about Müntz polynomials.

§ 2. Müntz-Jackson Theorems

The exact value of the error of approximation in $L_2[0,1]$, for the monomials x^r, $r > -1/2$, is

$$E(x^r, \Lambda)_2 = \frac{1}{\sqrt{2r+1}} \prod_{k=0}^{n} \frac{|r-\lambda_k|}{r+\lambda_k+1}, \tag{2.1}$$

[**CA**, Theorem 5.4, p.346]. From this formula one can easily derive an upper bound for the uniform norm on $[0,1]$:

$$E(x^r, \Lambda)_\infty \leq \prod_{k=1}^{n} \frac{|r-\lambda_k|}{r+\lambda_k}, \quad r > 0, \tag{2.2}$$

[**CA**, Theorem 5.5, p.347]. Following v. Golitschek [1976_1, 1989] we derive similar estimates for the other values of p.

Lemma 2.1. *Let $1 \leq q < p < \infty$ and let $-1/q < \ell_0 < \ell_1 < \cdots < \ell_n$. For arbitrary real numbers $a_0, \ldots, a_n$ and $b_k := a_k(1+\ell_k+1/p)/(1+1/p)$,*

$$\|x^{1/q-1/p} - \sum_{k=0}^{n} a_k x^{\ell_k+1/q-1/p}\|_p \leq (1+1/p)\|1 - \sum_{k=0}^{n} b_k x^{\ell_k}\|_q . \tag{2.3}$$

Proof. Let $K := 1 + 1/p$ and, for $0 < x \leq 1$, let

$$Q(x) := \sum_{k=0}^{n} b_k x^{\ell_k}, \quad h(x) := x^{1/p}(1 - Q(x)),$$

$$g(x) := K x^{1/q-1-2/p} \int_0^x h(t)dt.$$

One easily verifies that g is the function on the left-hand side of (2.3). Hölder's inequality yields

$$|g(x)| \leq K x^{-2/p} \left(\int_0^x |h(t)|^q dt \right)^{1/q} .$$

With

$$F(x,t) := \begin{cases} x^{-2q/p}|h(t)|^q, & \text{if } \ 0 \le t < x \\ 0, & \text{otherwise,} \end{cases}$$

it follows that

$$|g(x)| \le K \left(\int_0^1 F(x,t)dt \right)^{1/q}, \quad 0 < x \le 1.$$

By the Hölder-Minkowski inequality with the exponent $s := p/q$, we have

$$\begin{aligned} \|g\|_p &\le K \left(\int_0^1 \left(\int_0^1 F(x,t)dt \right)^s dx \right)^{1/p} \\ &\le K \left(\int_0^1 \left(\int_0^1 F(x,t)^s dx \right)^{1/s} dt \right)^{1/q} \\ &= K \left(\int_0^1 |h(t)|^q \left(\int_t^1 x^{-2} dx \right)^{1/s} dt \right)^{1/q} \\ &\le K \left(\int_0^1 |t^{-1/p} h(t)|^q dt \right)^{1/q}, \end{aligned}$$

which is equal to the right-hand side of (2.3). □

Theorem 2.2. *Let $\Lambda = \{\lambda_0 = 0 < \lambda_1 < \cdots < \lambda_n\}$. For $2 < p < \infty$ and $r > -1/p$,*

(2.4) $$E(x^r, \Lambda)_p \le \frac{1 + 1/p}{(2r + 2/p)^{1/p}} \prod_{k=0}^{n} \frac{|r - \lambda_k|}{r + \lambda_k + 2/p} \quad .$$

Proof. Let $\ell_k := (\lambda_k - r)/(2r + 2/p)$. We apply (2.1) to $x^0 = 1$ and the exponents ℓ_k: there exist real numbers b_k so that

$$\|1 - \sum_{k=0}^{n} b_k x^{\ell_k}\|_2 = \prod_{k=0}^{n} \frac{|\ell_k|}{\ell_k + 1} = \prod_{k=0}^{n} \frac{|r - \lambda_k|}{r + \lambda_k + 2/p} \quad .$$

Let $\rho := 2r + 2/p$ and $a_k := b_k(1 + 1/p)/(1 + \ell_k + 1/p)$. Substituting $x = y^\rho$ and applying Lemma 2.1 for $q = 2$, we get

$$\begin{aligned} E(y^r, \Lambda)_p &\le \|y^r - \sum_{k=0}^{n} a_k y^{\lambda_k}\|_p \\ &= \rho^{-1/p} \|x^{1/2-1/p} - \sum_{k=0}^{n} a_k x^{\ell_k + 1/2 - 1/p}\|_p \\ &\le (1 + 1/p)\rho^{-1/p} \|1 - \sum_{k=0}^{n} b_k x^{\ell_k}\|_2 \ , \end{aligned}$$

and thus (2.4). □

We shall compare $E(x^r, \Lambda)_p$ with the index of approximation $\varepsilon_p = \varepsilon_p(\Lambda)$:

Corollary 2.3. *For $2 \le p \le \infty$ and $r \ge 2$,*

$$E(x^r, \Lambda)_p \le 4^{1/p} r^r \varepsilon_p^{r+1/p}. \tag{2.5}$$

Proof. For any $s \ge 1, t \ge 0$ with $st < 1$ we take logarithms on both sides and expand, to obtain

$$\frac{1-st}{1+st} \le \left(\frac{1-t}{1+t}\right)^s .$$

With $t = 2\lambda/(s^2+\lambda^2)$ and $y = \sqrt{s^2-1}$ this implies

$$\left(\frac{s-\lambda}{s+\lambda}\right)^2 \le \left(\frac{y^2+(\lambda-1)^2}{y^2+(\lambda+1)^2}\right)^s$$

for any $\lambda \ge 0$. Using this inequality for $s = r + 1/p$ and $\lambda = \lambda_k + 1/p$, $k = 0, \dots, n$, we get

$$\begin{aligned}\prod_{k=0}^{n} \left(\frac{r-\lambda_k}{r+\lambda_k+2/p}\right)^2 &\le \prod_{k=0}^{n} \left(\frac{y^2+(1-\lambda_k-1/p)^2}{y^2+(1+\lambda_k+1/p)^2}\right)^{r+1/p} \\ &\le |B_p(1+iy)|^{2r+2/p} \le (r+1/p)^{2r+2/p} \varepsilon_p^{2r+2/p}.\end{aligned}$$

Inequality (2.4) now becomes

$$E(x^r, \Lambda)_p \le 2^{-1/p}(1+1/p)\ (r+1/p)^r\ \varepsilon_p^{r+1/p},$$

and (2.5) follows, since $(r+1/p)^r \le e^{1/p} r^r$ and $(e/2)^{1/p}(1+1/p) \le 4^{1/p}$. □

For our proof of Theorem 1.2(i), with a good value for the constant A_p, we need refinements of the theorems on polynomial approximation in [**CA**, Ch.7]. We start with a lemma on the approximation to $g \in W_p^1(\mathbb{T})$ from the space $\mathcal{T}_m$ of trigonometric polynomials of degree $\le m$.

Lemma 2.4. *Let $1 \le p \le \infty$ and $m \ge 1$. Let $T_m \in \mathcal{T}_m$ be the best approximation to $g \in W_p^1(\mathbb{T})$ in $L_p(\mathbb{T})$, then*

$$\|g - T_m\|_{L_p(\mathbb{T})} \ \le\ \frac{\pi}{2(m+1)}\ \|g'\|_{L_p(\mathbb{T})}\ , \tag{2.6}$$

$$\|T_m'\|_{L_\infty(\mathbb{T})}\ \le C_p\ m^{1/p}\ \|g'\|_{L_p(\mathbb{T})}\ , \tag{2.7}$$

where

$$C_p \le (\pi+2)\left(\frac{2r+1}{2\pi}\right)^{1/p} \tag{2.8}$$

and where $r = r(p)$ is the least integer $\ge p/2$.

Proof. The inequality (2.6) is a special case of Favard's theorem, [**CA**, (4.3), p.211]. By [**CA**, Lemma 2.6, p.206], we get

$$\|T_m'\|_{L_p(\mathbb{T})} \le (\pi+2)\|g'\|_{L_p(\mathbb{T})}.$$

We insert this inequality into Nikol'skii's inequality

$$\|T_m'\|_{L_\infty(\mathbb{T})} \le \left(\frac{2mr+1}{2\pi}\right)^{1/p} \|T_m'\|_{L_p(\mathbb{T})}$$

(see [**CA**, (2.15), p.102]) and obtain (2.7). □

Theorem 2.5. *Let $1 \le p \le \infty$ and $m \ge 1$. For all $f \in W_p^1[0,1]$ there exists an even algebraic polynomial $P_m(x) = \sum_{1\le j\le m/2} c_j x^{2j}$ such that*

$$\|f - P_m\|_p \le \frac{\pi}{2(m+1)} \|f'\|_p \tag{2.9}$$

and

$$|c_j| \le \frac{10\cdot 2^j}{(2j)!} m^{2j-1+1/p} \|f'\|_p, \quad j = 1,2,\ldots,m/2. \tag{2.10}$$

Proof. We extend f to an even function on $[-1,1]$ and define $g \in W_p^1(\mathbb{T})$ by $g(t) := f(\cos t)$. Then we have

$$\|g'\|_{L_p(\mathbb{T})} = \left(4\int_0^{\pi/2} |\sin t \; f'(\cos t)|^p dt\right)^{1/p} \le 4^{1/p}\|f'\|_p. \tag{2.11}$$

Let T_m be the trigonometric polynomial of Lemma 2.4. It is even since g is even. The polynomial P_m defined by $T_m(t) = P_m(\cos t)$ is also even since f is even. We obtain (2.9) by using (2.6), (2.11) and

$$\|f - P_m\|_p \le 4^{-1/p}\|g - T_m\|_{L_p(\mathbb{T})}.$$

From (2.7) and (2.11), for $-1 < x < 1$,

$$|P_m'(x)| \le \frac{C_p}{\sqrt{1-x^2}} m^{1/p} \|g'\|_{L_p(\mathbb{T})} \le \frac{4^{1/p}C_p}{\sqrt{1-x^2}} m^{1/p} \|f'\|_p \tag{2.12}$$

We shall prove the inequality $4^{1/p}C_p < 10$, $p \ge 1$. Indeed, if $1 \le p \le 2$ then $r(p) = 1$ and $4^{1/p}C_p \le (\pi+2)(6/\pi)^{1/p} \le 6 + 12/\pi < 10$. If $2 < p \le 4$, then $r(p) = 2$ and $C_p < 5$ hence $4^{1/p}C_p < 10$. The function $(p/\pi)^{1/p}$ attains it maximum at $p = e\pi$. Hence we have $(p/\pi)^{1/p} \le e^{1/(e\pi)}$ for all p. If $p > 4$, then $2r+1 < p+3 < 2p$ and

$$C_p \le (\pi+2)\left(\frac{p}{\pi}\right)^{1/p} \le (\pi+2)e^{1/(e\pi)} < 6,$$

so that $4^{1/p}C_p < 6\cdot 4^{1/4} < 10$.

Now we set $\rho := 1/\sqrt{2}$ and $Q_m(x) := P_m'(\rho x) = \sum_{j=1}^{m/2} 2jc_j(\rho x)^{2j-1}$. This is an odd polynomial of degree $\le m-1$, and (2.12) implies

$$\|Q_m\|_{C[-1,1]} \le 10\sqrt{2}\ m^{1/p}\ \|f'\|_p . \tag{2.13}$$

Let $q := m/2$. The Chebyshev polynomial $C_{2q-1}(x) = \sum_{j=1}^{q} \alpha_{j,q}x^{2j-1}$ has an alternation set on $[-1,1]$ of length $2q$. Hence $C_{2q-1}(\sqrt{x}) = \sum_{j=1}^{q} \alpha_{j,q}x^{j-1/2}$ has an alternation set on $(0,1]$ of length q. For each $j = 1,\ldots,q$, the monomials $\{x^{k-1/2} : k = 1,\ldots,q,\ k \ne j\}$, are a real Haar system on $[\delta,1]$, for any $0 < \delta < 1$. The theorem of Chebyshev [**CA**, Theorem 5.1, p.74] therefore implies that

$$1 = \|C_{2n-1}(\sqrt{x})\|_{C[0,1]} \le \|\alpha_{j,q}x^{j-1/2} + \sum_{k=1,k\ne j}^{q} b_k x^{k-1/2}\|_{C[0,1]}$$

for all $b_k \in \mathbb{R}$, $k \ne j$. Replacing x by x^2, it follows that among all odd algebraic polynomials of degree $\le m-1$ of uniform norm 1 on $[-1,1]$, the Chebyshev polynomial $C_{2q-1}(x)$ has the coefficients of largest absolute value, for each power x^{2j-1}, $j = 1,\ldots,q$. Applying this to Q_m we get

$$2j\rho^{2j-1}|c_j| \le |\alpha_{j,q}|\|Q_m\|_\infty . \tag{2.14}$$

The coefficients of the Chebyshev polynomials satisfy

$$|\alpha_{j,q}| = \frac{2^{2j-2}(2q-1)}{(2j-1)!}\prod_{k=2-j}^{j-1}(q+k-1) \le \frac{m^{2j-1}}{(2j-1)!} .$$

Hence we obtain from (2.14)and (2.13) that

$$|c_j| \le \frac{\sqrt{2}^{2j-1} m^{2j-1}}{(2j)!}\|Q_m\|_{C[-1,1]} \le \frac{10\cdot 2^j}{(2j)!}\ m^{2j-1+1/p}\ \|f'\|_p .$$

□

Following v. Golitschek [1989] we will now prove Theorem 1.2(i) for $p \ge 2$, with a small constant A_p.

Theorem 2.6. *Let $\Lambda : 0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n$ and $2 \le p \le \infty$. Then, for each $f \in W_p^1[0,1]$,*

$$E(f,\Lambda)_p \le A_p\varepsilon_p\|f'\|_p, \tag{2.15}$$

with $A_p < 20$.

Proof. For an arbitrary even polynomial $P_m(x) = \sum_{j=0}^{m/2} c_j x^{2j}$ we get the inequality

$$E(f,\Lambda)_p \le \|f - P_m\|_p + \sum_{j=1}^{m/2}|c_j|\ E(x^{2j},\Lambda)_p . \tag{2.16}$$

Let $h := \varepsilon_p$ and let $m \in \mathbb{N}$ be defined by $hm \leq 1/(3e) < h(m+1)$. Since $0 \in \Lambda$ and $\|f - f(0)\|_p \leq \|f'\|_p$ we may assume that $h < 1/A_p$ and $m \geq 2$. We take the polynomial P_m of Theorem 2.5 and use Corollary 2.3. Then (2.16) yields that $E(f, \Lambda)_p$ satisfies (2.15) with a factor

$$A_p \leq \frac{\pi}{2h(m+1)} + 4^{1/p} 10 \sum_{j=1}^{m/2} \frac{2^j (2j)^{2j} (hm)^{2j-1+1/p}}{(2j)!}.$$

Using Stirling's formula $(2j)! \geq \sqrt{2\pi}(2j)^{2j+1/2} e^{-2j}$ it follows that

$$A_p \leq \frac{3e\pi}{2} + \frac{15e}{\sqrt{\pi}} \sum_{j=1}^{\infty} (2/9)^j = \frac{3e\pi}{2} + \frac{30e}{7\sqrt{\pi}} < 20.$$

This completes the proof of the theorem. □

A convenient way to extend the last theorem to arbitrary functions $f \in C[0,1]$ or $f \in L_p[0,1]$ is to use their modulus of continuity $\omega(f,t)_p$, $1 \leq p \leq \infty$ (see[**CA**, p.40, p.44]), and to use the K-functional defined by

$$K(f, \delta, L_p[0,1], W_p^1[0,1]) := \inf_{g \in W_p^1[0,1]} \{\|f - g\|_p + \delta \|g'\|_p\},$$

([**CA**, (1.11), p.172]). It satisfies

$$K(f, \delta, L_p[0,1], W_p^1[0,1]) \leq C\omega(f, \delta)_p$$

for some positive number C which is independent of $\delta > 0$, f and r ([**CA**, (2.11), p.177]).

Theorem 2.7. *Let $f \in C[0,1]$ if $p = \infty$, $f \in L_p[0,1]$ if $2 \leq p < \infty$. Then,*

$$E(f, \Lambda)_p \leq C_0 \omega(f, \varepsilon_p)_p, \tag{2.17}$$

where $C_0 > 0$ is independent of f, Λ and p.

Proof. We choose $\delta := \varepsilon_p$. For some $g \in W_p^1[0,1]$,

$$\|f - g\|_p + \varepsilon_p \|g'\|_p \leq 2C\omega(f, \varepsilon_p)_p. \tag{2.18}$$

By Theorem 2.6,

$$E(f, \Lambda)_p \leq \|f - g\|_p + E(g, \Lambda)_p \leq \|f - g\|_p + A_p \varepsilon_p \|g'\|_p.$$

This yields (2.17) with $C_0 := 2CA_p < 40C$. □

For $k = 1, 2, \ldots$ we define the finite sequence

$$\Lambda^{(k)} : \ \lambda_0^{(k)} := 0, \ \lambda_1^{(k)} := \lambda_1 - k, \cdots, \ \lambda_n^{(k)} := \lambda_n - k$$

and the indices of approximation $\varepsilon_p^{(k)} := \varepsilon_p(\Lambda^{(k)})$, $k = 1, 2, \ldots$; and $\varepsilon_p^{(0)} := \varepsilon_p$.

For differentiable functions f, we have

Theorem 2.8. *Let $r = 1, 2, \ldots$, $p \geq 2$, and $\Lambda = \{\lambda_0 = 0 < \lambda_1 < \cdots < \lambda_n\}$, $\lambda_1 > r$. For each $f \in W_p^r[0,1]$, there exists a polynomial $P_r \in \mathcal{P}_r$ and a Müntz polynomial $M \in \mathcal{M}(\Lambda)$ so that*

$$(2.19) \qquad \|f - P_r - M\|_p \leq C_r \omega(f^{(r)}, \varepsilon_p^{(r)})_p \prod_{k=0}^{r-1} \varepsilon_p^{(k)},$$

where $C_r > 0$ is independent of f, Λ and p.

Proof. By Theorem 2.7, the theorem is valid for $r = 0$. We assume that it is valid for $r - 1 \geq 0$. Hence, for f' instead of f and the sequence $\Lambda^{(1)}$ there exist $P_{r-1} \in \mathcal{P}_{r-1}$ and $M_1 \in \mathcal{M}(\Lambda^{(1)})$ so that

$$(2.20) \qquad \|f' - P_{r-1} - M_1\|_p \leq C_{r-1} \omega(f^{(r)}, \varepsilon_p^{(r)})_p \prod_{k=0}^{r-2} \varepsilon_p^{(k+1)}.$$

By Theorem 2.6 for the function $F(x) := f(x) - \int_0^x (P_{r-1}(t) + M_1(t))dt$ there exists $M_2 \in \mathcal{M}(\Lambda)$ so that

$$(2.21) \qquad \|F - M_2\|_p \leq A_p \varepsilon_p(\Lambda) \|F'\|_p.$$

Since $F' = f' - P_{r-1} - M_1$, (2.20) and (2.21) yield (2.19). □

§ 3. An Inverse Müntz-Jackson Theorem

In this section we prove the inverse of the Müntz-Jackson Theorem 1.2(ii) for $2 \leq p \leq \infty$. More precisely, we prove

Theorem 3.1. *For $2 \leq p \leq \infty$ and $\Lambda = \{\lambda_0 = 0 < \lambda_1 < \cdots < \lambda_n\}$,*

$$(3.1) \qquad d_p(\Lambda) := \sup\left\{E(f,\Lambda)_p : \ f \in W_p^1[0,1], \|f'\|_p \leq 1\right\} \ \geq \ C\varepsilon_p(\Lambda),$$

where $C > 0$ is independent of p and Λ.

We begin with a simple remark for $f \in L_p$, $1 \leq p \leq \infty$,

$$(3.2) \qquad E(f,\Lambda)_p \geq \sup_{H \in \mathcal{N}_{p'}} \int_0^1 f(x)H(x)dx,$$

where $p' := p/(p-1)$ and where $\mathcal{N}_{p'}$ is the set of functions $H \in L_{p'}[0,1]$ satisfying $\|H\|_{p'} = 1$ and orthogonal to $M(\Lambda)$:

$$(3.3) \qquad \int_0^1 x^{\lambda_k} H(x)dx = 0 \ , \ \ k = 0, 1, \ldots, n.$$

In fact, for any $g \in \mathcal{M}(\Lambda)$ and $1 \leq p \leq \infty$, $|\int_0^1 fH dx| = |\int_0^1 (f-g)H dx| \leq \|f - g\|_p \, \|H\|_{p'} = \|f - g\|_p$.

From (3.2) we derive

Lemma 3.2. *For* $1 \le p \le \infty$

$$d_p(\Lambda) \ge \sup_{H \in \mathcal{N}_{p'}} \left\| \int_x^1 H(y)dy \right\|_{p'} . \tag{3.4}$$

Equivalently, for an arbitrary nonzero function $f_0 \in L_{p'}[0,1]$ *orthogonal to* $\mathcal{M}(\Lambda)$,

$$d_p(\Lambda) \ge \left\| \int_x^1 f_0(y)dy \right\|_{p'} / \|f_0\|_{p'} .$$

Proof. Let Ω be the set of all functions f in (3.1) satisfying $f(0) = 0$. Since $\lambda_0 = 0$, the supremum $d_p(\Lambda)$ in Theorem 3.1 does not change under this additional restriction on f. By partial integration we therefore get

$$\begin{aligned} d_p(\Lambda) &\ge \sup_{f \in \Omega} \sup_{H \in \mathcal{N}_{p'}} \int_0^1 f(x)H(x)dx = \sup_{H \in \mathcal{N}_{p'}} \sup_{f \in \Omega} \int_0^1 f(x)H(x)dx \\ &= \sup_{H \in \mathcal{N}_{p'}} \sup_{f \in \Omega} \int_0^1 f'(x) \left(\int_x^1 H(y)dy \right) dx = \sup_{H \in \mathcal{N}_{p'}} \left\| \int_x^1 H(y)dy \right\|_{p'} . \end{aligned}$$

Indeed, for every $g \in L_{p'}[0,1]$, $1 \le p \le \infty$, $\sup\{\int_0^1 gu\,dx : \|u\|_p = 1\} = \|g\|_{p'}$. □

The proof of (3.1) will depend on the proper choice of a function f_0 of Lemma 3.2, which will yield the desired lower bound.

It will be chosen by means of a point $y_0 \ge 0$ where, for the Blaschke product

$$B(z) := \prod_{k=0}^{n} \frac{z - \lambda_k - 1/p}{z + \lambda_k + 1/p} = \frac{z - 1/p}{z + 1/p} \, B_p(z, \Lambda) \tag{3.5}$$

the quotient $|B(1+iy)|/|1+iy|$ attains its maximum for $y \in \mathbb{R}$. We put $s := \sqrt{1 + y_0^2}$ and define

$$f_0(x) := x^{-1/p'} h(-\log x), \quad 0 < x \le 1, \tag{$*$}$$

where

$$h(t) := \int_{-\infty}^{\infty} e^{ity} F(iy)dy, \quad t \in \mathbb{R}, \tag{$**$}$$

and

$$F(z) := u(z)B(z), \quad u(z) := \frac{z^2}{(z+s)^4(z+1-iy_0)}, \quad z \in \mathbb{C}. \tag{$***$}$$

Since $s \ge 1$ and since $|B(iy)| = 1$ for all $y \in \mathbb{R}$,

$$|F(iy)| = |u(iy)| = \frac{y^2}{(y^2+s^2)^2} \frac{1}{\sqrt{1+(y-y_0)^2}}, \quad y \in \mathbb{R}. \tag{3.6}$$

This implies that $F(iy) \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$. Therefore, the Fourier transform $h(-t)$ of $F(iy)$ is continuous on $\mathbb{R}$ and $h \in C(\mathbb{R}) \cap L_2(\mathbb{R})$.

Lemma 3.3. *The function h is in $L_s(\mathbb{R})$, for all $s \geq 1$, and has the properties*

$$\text{(3.7)} \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} |h(t)|^2 dt = \int_{-\infty}^{\infty} |F(iy)|^2 dt = \int_{-\infty}^{\infty} |u(iy)|^2 dt \leq \frac{\pi}{16s^4},$$

$$\text{(3.8)} \qquad h(t) = 0 \ , \ -\infty < t \leq 0,$$

$$\text{(3.9)} \qquad F(x+iy) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ity-tx} h(t)dt \ , \ x \geq 0, \ y \in \mathbb{R}.$$

Proof. The first part of (3.7) follows from Parseval's identity, the second part from (3.6), and the inequality in (3.7), from

$$\int_{-\infty}^{\infty} |u(iy)|^2 dy \leq \pi \max_{y \geq 0} y^4 (y^2+s^2)^{-4} = \pi s^{-4}/16.$$

Let $\alpha := 1/p$ if $2 \leq p < \infty$, $\alpha := \min\{\lambda_1, 1\}$ if $p = \infty$. The rational function $F(z)$ has no poles in $\operatorname{Re} z > -\alpha$. From Cauchy's theorem for the paths $\Gamma(x) := \{x + iy : -\infty < y < \infty\}$, $x > -\alpha$, it follows that

$$\int_{\Gamma(0)} e^{tz} F(z)dz = \int_{\Gamma(x)} e^{tz} F(z)dz$$

and therefore,

$$\text{(3.10)} \qquad e^{-tx} h(t) = \int_{-\infty}^{\infty} e^{ity} F(x+iy)dy, \quad x > -\alpha, \ t \in \mathbb{R}.$$

Let $x \geq -\alpha/2$. Since $s \geq 1$, $0 < \alpha \leq 1$ and $|B(x+iy)| \leq 1$, $y \in \mathbb{R}$, we have

$$\|u(x+iy)\|_{L_1(\mathbb{R})} \leq \int_{-\infty}^{\infty} \frac{(x^2+y^2)dy}{[(x+s)^2+y^2]^2(1+x)} \leq \int_{-\infty}^{\infty} \frac{2dy}{y^2+1/4} = 8\pi$$

and, using (3.10),

$$\text{(3.11)} \qquad |h(t)| \leq e^{tx}\|F(x+iy)\|_{L_1(\mathbb{R})} \leq e^{tx}\|u(x+iy)\|_{L_1(\mathbb{R})} \leq 8\pi e^{tx},$$

for all $t \in \mathbb{R}$, $x \geq -\alpha/2$. Applying (3.11) for $t < 0$ and $x \to +\infty$, we derive (3.8). For $x = -\alpha/2$, (3.11) yields

$$\text{(3.12)} \qquad |h(t)| \leq 8\pi e^{-t\alpha/2}.$$

This inequality proves that $h \in L_s(\mathbb{R})$ for all $s \geq 1$, and that $\Phi_x(t) := e^{tx}h(-t) \in L_1(\mathbb{R})$ for all $x \geq 0$. In addition, by (3.10), $\Phi_x(t)$ is the Fourier transform of $F(x+iy)$, $x \geq 0$. An application of the inverse Fourier transform yields (3.9). □

The function $e^{-t\alpha/4}h(t)$ is continuous on $\mathbb{R}$ and, by (3.12), tends to zero as $t \to +\infty$. Therefore, the function $f_0(x)$ of (3.6) is continuous in $0 < x \le 1$, $f_0 \in L_{p'}[0,1]$ and $\int_x^1 f_0(y)dy$ is a continuous function for $0 \le x \le 1$.

Corollary 3.4. *The function f_0 defined by* $(*)$ *satisfies* (3.3).

Proof. Substituting $x = e^{-t}$ and applying (3.9),

$$\int_0^1 x^{\lambda_k} f_0(x)dx = \int_0^\infty e^{-(\lambda_k+1/p)t}h(t)dt = F(\lambda_k + 1/p), \quad k = 0, \dots, n.$$

But $F(\lambda_k + 1/p) = 0$ since $B(\lambda_k + 1/p) = 0$. □

Lemma 3.5. *For $2 \le p \le \infty$ in the space $L_{p'}[0,1]$*

$$\left\| \int_x^1 f_0(y)dy \right\|_{p'} \ge \frac{1}{50}\; s^{-2}\; \varepsilon_p(\Lambda)\,. \tag{3.13}$$

Proof. Since the maximum (1.5) is attained at y_0, we have, by (3.5),

$$\frac{|B(1+iy_0)|}{s} \ge \frac{|B(1+iy_0)|}{|1+iy_0|} \ge \frac{s(p-1)}{p+1}\; \varepsilon_p(\Lambda) \ge \frac{s}{3}\; \varepsilon_p(\Lambda). \tag{3.14}$$

The function

$$G(t) := e^{-t/p'} \int_0^t h(s)e^{-s/p}ds\;, \quad t \in \mathbb{R}$$

vanishes for $-\infty < t < 0$. Substituting $x = e^{-t}$, $y = e^{-s}$, we get

$$\left\| \int_x^1 f_0(y)dy \right\|_{p'} = \left(\int_0^\infty \left| \int_0^t h(s)e^{-s/p}ds \right|^{p'} e^{-t}dt \right)^{1/p'} = \|G\|_{L_{p'}(0,\infty)}\,.$$

We shall prove the inequality

$$\|G\|_{L_{p'}(0,\infty)} \ge \frac{3}{50}\; s^{-3}\; |B(1+iy_0)|, \tag{3.15}$$

which together with (3.14) will imply (3.13).

For this purpose we use (3.9). Integration by parts yields

$$\begin{aligned} 2\pi F(1+iy) &= \int_{-\infty}^\infty e^{-ity-t/p'}h(t)e^{-t/p}dt \\ &= (iy + 1/p') \int_{-\infty}^\infty e^{-ity-t/p'} \left(\int_0^t h(\eta)e^{-\eta/p}d\eta \right) dt \\ &= (iy + 1/p') \int_{-\infty}^\infty e^{-iyt}G(t)dt\,. \end{aligned}$$

Hence $2\pi F(1+iy)/(iy+1/p')$ is the Fourier transform of G, and by the inequality of Hausdorff-Young (see Zygmund [B-1959, vol. 2, p. 254]), we have

$$\left\| \frac{2\pi F(1+iy)}{iy+1/p'} \right\|_{L_p(\mathbb{R})} \leq (2\pi)^{1/p} \; \|G\|_{L_{p'}(\mathbb{R})}. \tag{3.16}$$

Since $s^2 = 1 + y_0^2 \geq 1$, we get for $y_0 \leq y \leq y_0 + 1$,

$$\begin{aligned} \left| \frac{s^3 u(1+iy)}{iy+1/p'} \right| &= s^3 \frac{(1+y^2)}{[y^2+(1+s)^2]^2 \; |[iy+1/p'][2+i(y-y_0)]|} \\ &\geq \frac{s^3 \sqrt{1+y^2}}{\sqrt{5} \; [y^2+(1+s)^2]^2}. \end{aligned}$$

The last expression attains its minimum if $y = y_0 + 1$, $s = 1$, $y_0 = 0$, so that

$$\left| \frac{u(1+iy)}{iy+1/p'} \right| \geq \sqrt{10} \; (5s)^{-3}.$$

Since $|B(1+iy)|$ is monotone increasing for $y > 0$, it follows that

$$\begin{aligned} \left\| \frac{F(1+iy)}{iy+1/p'} \right\|_{L_p(\mathbb{R})} &\geq \left(\int_{y_0}^{y_0+1} \left| \frac{F(1+iy)}{iy+1/p'} \right|^p dy \right)^{1/p} \\ &\geq |B(1+iy_0)| \left(\int_{y_0}^{y_0+1} \left| \frac{u(1+iy)}{iy+1/p'} \right|^p dy \right)^{1/p} \\ &\geq |B(1+iy_0)| \sqrt{10} \; (5s)^{-3}. \end{aligned}$$

This and (3.16) yield (3.15). □

It remains to find a small upper bound for $\|f_0\|_{p'}$. We observe that

$$\|f_0\|_{p'} = \left(\int_0^1 x^{-1} |h(-\log x)|^{p'} dx \right)^{1/p'} = \|h\|_{L_{p'}(0,\infty)}. \tag{3.17}$$

We shall use

Lemma 3.6. *For any $c \in \mathbb{R}$,*

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}} \|(t-c)h(t)\|_{L_2(\mathbb{R})} \\ &\leq \|u'(iy)\|_{L_2(\mathbb{R})} + \left\| u(iy) \left(c + \frac{B'(iy)}{B(iy)} \right) \right\|_{L_2(\mathbb{R})}. \end{aligned} \tag{3.18}$$

Moreover, if $a \geq 1/\pi$, then

$$\|h\|^2_{L_{p'}(0,\infty)} \leq \pi \left(a \int_0^\infty |h(t)|^2 dt + \frac{1}{a} \int_0^\infty (t-c)^2 |h(t)|^2 dt \right). \tag{3.19}$$

Proof. Since $F(iy) \to 0$ as $y \to \pm\infty$, integration by parts leads to

$$(t-c)h(t) = \int_{-\infty}^{\infty} (t-c)e^{iy(t-c)}e^{icy}F(iy)dy$$
$$= i\int_{-\infty}^{\infty} e^{iy(t-c)}\frac{d}{dy}\left(F(iy)e^{icy}\right)\ dy\,,$$

and thus, using Parseval's identity and (3.8),

$$\frac{1}{2\pi}\int_0^{\infty}(t-c)^2|h(t)|^2dt = \int_{-\infty}^{\infty}\left|\frac{d}{dy}\left(F(iy)e^{icy}\right)\right|^2 dy\,,$$

which yields (3.18) since $|B(iy)| = 1$, $y \in \mathbb{R}$.

Setting $v(t) = a^2 + (t-c)^2$ and using Hölder's inequality for the exponent $2/p'$ we get

$$\|h\|^2_{L_{p'}(0,\infty)} = \left(\int_0^{\infty} v(t)^{-p'/2}|\sqrt{v(t)}h(t)|^{p'}dt\right)^{2/p'} \leq A\int_0^{\infty} v(t)|h(t)|^2dt\,,$$

where

$$A := \left(\int_0^{\infty} v(t)^{-\rho}dt\right)^{1/\rho}\,, \quad \rho := p'/(2-p')\,.$$

Since $\rho \geq 1$ and $a\pi \geq 1$,

$$A \leq a^{-2+1/\rho}\left(\int_{-\infty}^{\infty}\frac{1}{(1+t^2)^{\rho}}dt\right)^{1/\rho} \leq a^{-2+1/\rho}\pi^{1/\rho} \leq \pi/a\,,$$

we obtain (3.19). □

Lemma 3.7. *For the function* f_0,

$$\|f_0\|_{p'} = \|h\|_{L_{p'}(0,\infty} \leq 4\pi s^{-2}. \tag{3.20}$$

Proof. Since $s \geq 1$,

$$\int_{-\infty}^{\infty}|u(iy)/y|^2dy \leq \pi\max_{y\geq 0} y^2(y^2+s^2)^{-4} = 27\pi s^{-6}/256 \leq \frac{\pi}{9}s^{-6}. \tag{3.21}$$

From

$$\frac{u'(z)}{u(z)} = \frac{2}{z} - \frac{4}{z+s} - \frac{1}{z+1-iy_0} = \frac{2s-2z}{z(z+s)} - \frac{1}{z+1-iy_0}$$

it follows that

$$|u'(iy)| \leq |u(iy)|\left(\left|\frac{2(s-iy)}{y(s+iy)}\right| + 1\right) \leq |u(iy)|\left(\frac{2}{|y|}+1\right), \quad y \in \mathbb{R}.$$

This, (3.7) and (3.21) imply

$$\|u'(iy)\|_{L_2(\mathbb{R})} \le \sqrt{\pi}s^{-2}\left(\frac{2}{3}+\frac{1}{4}\right) = \frac{11\sqrt{\pi}}{12}s^{-2}. \tag{3.22}$$

Furthermore, we have

$$\left\|u(iy)\max\{s^2/y^2; y^2/s^2\}\right\|_{L_2(\mathbb{R})} \le \sqrt{\pi}s^{-2}. \tag{3.23}$$

We set $\ell_k := \lambda_k + 1/p$, $k = 0, \ldots, n$. Since the expression

$$\left|\frac{B(1+iy)}{1+iy}\right|^2 = \frac{1}{1+y^2}\prod_{k=0}^{n}\frac{y^2+(\ell_k-1)^2}{y^2+(\ell_k+1)^2}$$

attains its maximal value at $y = y_0$, we get

$$\lim_{y\to y_0+}\frac{d}{dy}\left|\frac{B(1+iy)}{1+iy}\right|^2 \le 0,$$

with equality if $y_0 > 0$. Hence,

$$\sum_{k=0}^{n}\frac{4\ell_k}{(y_0^2+(\ell_k-1)^2)(y_0^2+(\ell_k+1)^2)} \le \frac{1}{y_0^2+1}.$$

From $s^2 = y_0^2 + 1$ and the inequality

$$(y_0^2+(\ell_k-1)^2)(y_0^2+(\ell_k+1)^2) \le (s^2+\ell_k^2)^2,$$

we get

$$\sum_{k=0}^{n}\frac{4\ell_k}{(s^2+\ell_k^2)^2} \le \frac{1}{s^2}. \tag{3.24}$$

Finally we apply the inequalities (3.18) and (3.19) of Lemma 3.6 for

$$c := \sum_{k=0}^{n}\frac{2\ell_k}{s^2+\ell_k^2}$$

and $a := 6$. We get

$$\left|c+\frac{B'(iy)}{B(iy)}\right| = \sum_{k=0}^{n}\frac{2|s^2-y^2|\ell_k}{(y^2+\ell_k^2)(s^2+\ell_k^2)} \le \frac{1}{2}\max\{s^2/y^2; y^2/s^2\}$$

and using (3.23),

$$\left\|u(iy)\left(c+\frac{B'(iy)}{B(iy)}\right)\right\|_{L_2(\mathbb{R})} \le \sqrt{\pi}s^{-2}/2. \tag{3.25}$$

From (3.18), (3.22) and (3.25) we obtain

$$\|(t-c)h(t)\|_{L_2(0,\infty)} \le \frac{3\pi\sqrt{2}}{2}s^{-2}, \tag{3.26}$$

and from (3.19), (3.7) and (3.26),

$$\|h\|^2_{L_{p'}(0,\infty)} \le s^{-4}\left(a\frac{\pi^2}{8} + \frac{1}{a}\frac{9\pi^2}{2}\right) = \frac{3\pi^3}{2}s^{-4}. \tag{3.27}$$

This yields (3.20). □

Proof of Theorem 3.1. Inequality (3.1) is an immediate consequence of Corollary 3.4 and Lemmas 3.5, 3.7 for the function f_0. □

§ 4. The Index of Approximation

For special sequences Λ, the index of approximation $\varepsilon_p(\Lambda)$ can often be replaced by a much simpler expression. For example, if $p = \infty$ and $\lambda_k = 2k$, $k = 0, 1, \ldots$, we shall derive below that $\varepsilon_\infty(\Lambda) = 1/(2n+1)$. More generally, following Feinerman and Newman [A-1973], one has

Theorem 4.1. *For a sequence $\Lambda = \{0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ and $1 \le p \le \infty$, one has*

(i) *if $\lambda_{k+1} - \lambda_k \ge 2$, $k \ge 0$, then*

$$\varepsilon_p(\Lambda) = |B_p(1,\Lambda)| = \prod_{k=1}^{n} \frac{\lambda_k + 1/p - 1}{\lambda_k + 1/p + 1}; \tag{4.1}$$

(ii) *if $\lambda_{k+1} - \lambda_k \le 2$, $k \ge 0$, and $S_p := S_p(\Lambda) := \sum_{k=1}^{n}(\lambda_k + 1/p) \ge 1/8$, then*

$$\frac{1}{4\sqrt{S_p}} \le \varepsilon_p(\Lambda) \le \frac{3}{\sqrt{S_p}}. \tag{4.2}$$

Proof. We set $\ell_k := \lambda_k + 1/p$, $k = 0, \ldots, n$.

(i) Since $\ell_{k+1} - 1 \ge \ell_k + 1$, we get for $y \in \mathbb{R}$

$$\frac{y^2 + (\ell_{k+1} - 1)^2}{y^2 + (\ell_k + 1)^2} \le \left(\frac{\ell_{k+1} - 1}{\ell_k + 1}\right)^2, \quad k = 1, 2, \ldots,$$

and therefore

$$|B_p(1+iy)|^2 = \prod_{k=1}^{n} \frac{y^2 + (\ell_k - 1)^2}{y^2 + (\ell_k + 1)^2} \le \frac{y^2 + (\ell_1 - 1)^2}{y^2 + (\ell_n + 1)^2} \prod_{k=1}^{n-1} \left(\frac{\ell_{k+1} - 1}{\ell_k + 1}\right)^2.$$

Since $\ell_1 - 1 \ge 1$, we have

$$\frac{y^2 + (\ell_1 - 1)^2(\ell_n + 1)^2}{(1+y^2)(y^2 + (\ell_n + 1)^2)(\ell_1 - 1)^2} \le \frac{y^2 + (\ell_1 - 1)^2}{(1+y^2)(\ell_1 - 1)^2} \le 1,$$

for all $y \geq 0$, and thus

$$\varepsilon_p(\Lambda))^2 = \max_{y\geq 0} \frac{|B_p(1+iy)|^2}{1+y^2} \leq \prod_{k=1}^{n} \left(\frac{\ell_k - 1}{\ell_k + 1}\right)^2 = |B_p(1)|^2.$$

Conversely, $\varepsilon_p(\Lambda) \geq |B_p(1)|$, and (4.1) follows.

(ii) By assumption, $S := S_p = \sum_{k=1}^{n} \ell_k \geq 1/8$. For $y := \sqrt{8S-1}$, we get

$$\varepsilon_p(\Lambda))^2 \geq \left|\frac{B_p(1+iy)}{1+iy}\right|^2 = \frac{1}{8S} \prod_{k=1}^{n} \frac{8S + \ell_k^2 - 2\ell_k}{8S + \ell_k^2 + 2\ell_k}.$$

Since $2S(8S + \ell_k^2 - 2\ell_k) \geq (2S - \ell_k)(8S + \ell_k^2 + 2\ell_k)$, it follows that

$$\varepsilon_p(\Lambda))^2 \geq \frac{1}{8S} \prod_{k=1}^{n} \left(1 - \frac{\ell_k}{2S}\right) \geq \frac{1}{8S} \left(1 - \sum_{k=1}^{n} \frac{\ell_k}{2S}\right) = \frac{1}{16\,S},$$

which yields the lower bound in (4.2).

The proof of the upper bound in (4.2) consists of two steps. First we define an increasing subsequence $\{\sigma_0, \sigma_1, \ldots, \sigma_m\}$ of $\{\ell_k\}_{k=0}^{n}$ by

$$\begin{aligned}
&\sigma_0 = \ell_0 = 1/p, \quad \sigma_m = \ell_n \\
&\sigma_{j+1} - \sigma_j \leq 2, \quad j = 0, \ldots, m-1 \\
&\sigma_{j+2} - \sigma_j > 2, \quad j = 0, \ldots, m-2.
\end{aligned}$$

This is possible since $\ell_{k+1} - \ell_k \leq 2$ for $k \geq 0$. We get $\ell_n = \sigma_m > 2 + \sigma_{m-2} > \ldots > m-1$, $\sigma_j \leq 2j$,

$$\sigma := \sum_{j=1}^{m} \sigma_j \leq \sigma_m + \sum_{j=1}^{m-1} 2j \leq \sigma_m + m(m-1) \leq (1+\ell_n)^2,$$

and, for arbitrary $y \in \mathbb{R}$,

$$\prod_{j=1}^{m} \frac{y^2 + (\sigma_j - 1)^2}{y^2 + (\sigma_j + 1)^2} \leq \frac{y^2 + (\sigma_1 - 1)^2}{y^2 + (\sigma_m + 1)^2} \leq \frac{y^2 + 1}{y^2 + (\ell_n + 1)^2}. \tag{4.3}$$

For the other exponents $L = \Lambda \setminus \{\sigma_j\}_{j=0}^{m}$,

$$\begin{aligned}
&\prod_{\lambda\in L} \frac{y^2 + (\lambda - 1)^2}{y^2 + (\lambda + 1)^2} = \prod_{\lambda\in L} \left(1 - \frac{4\lambda}{y^2 + (\lambda+1)^2}\right) \\
&\leq \exp\left(-\sum_{\lambda\in L} \frac{4\lambda}{y^2 + (\lambda+1)^2}\right) \leq \exp\left(-\frac{4}{y^2 + (\ell_n+1)^2} \sum_{\lambda\in L} \lambda\right) \\
&= \exp\left(-\frac{4(S-\sigma)}{y^2 + (\ell_n+1)^2}\right) \leq e^4 \exp\left(\frac{-4S}{y^2 + (\ell_n+1)^2}\right) \\
&\leq \frac{e^3}{4S}\left(y^2 + (\ell_n+1)^2\right).
\end{aligned}$$

For the last step we have used the inequality $xe^{1-x} \le 1$ if $x > 0$. Finally, from our last inequality and (4.3),

$$\prod_{k=1}^{n} \frac{y^2 + (\ell_k - 1)^2}{y^2 + (\ell_k + 1)^2} \le \frac{e^3(1+y^2)}{4S}\,, \quad y \ge 0\,.$$

This yields the upper bound in (4.2). □

§ 5. Markov-Type Inequality for Müntz Polynomials

In the Markov inequality for the uniform norm on $[0,1]$

$$\|P_n'\|_\infty \le 2n^2 \|P_n\|_\infty, \quad P_n \in \mathcal{P}_n \tag{5.1}$$

the factor 2 in (5.1) is best possible for every $n \ge 1$ since $P_n'(1) = 2n^2\|P_n\|_\infty$ for $P_n(x) = C_n(2x-1)$. From this follows the inequality

$$\|xP_n'(x)\|_\infty \le 2n^2 \|P_n\|_\infty, \quad P_n \in \mathcal{P}_n, \tag{5.2}$$

where the factor 2 is again best possible for all n.

The purpose of this section is to prove Newman's analogue of (5.2) for Müntz polynomials:

Theorem 5.1 (Newman [1976]). *Let $\Lambda : 0 \le \lambda_0 < \lambda_1 < \cdots < \lambda_n$. Then,*

$$\frac{2}{3}\sum_{k=0}^{n} \lambda_k \le \sup_{0 \ne M \in \mathcal{M}(\Lambda)} \frac{\|xM'(x)\|_\infty}{\|M\|_\infty} \le C_1 \sum_{k=0}^{n} \lambda_k. \tag{5.3}$$

One can take $C_1 = 11$.

Proof. By the substitution $x = e^t$, $-\infty < t \le 0$, the Müntz polynomials M of the form (1.1) are transformed into the exponential sums,

$$g(t) := \sum_{k=0}^{n} a_k e^{\lambda_k t}, \quad a_k \in \mathbb{R}, \tag{5.4}$$

and $g'(t) = xM'(x)$. We denote the collection of exponential sums (5.4) by $\mathcal{E}(\Lambda)$. Obviously, (5.3) is equivalent to

$$\frac{2}{3}\sum_{k=0}^{n} \lambda_k \le \sup_{0 \ne g \in \mathcal{E}(\Lambda)} \frac{\|g'\|_{C(-\infty,0]}}{\|g\|_{C(-\infty,0]}} \le C_1 \sum_{k=0}^{n} \lambda_k. \tag{5.5}$$

It is more convenient to prove (5.5) instead of (5.3). By a change of scale, without loss of generality we may assume that

$$\sigma := \sum_{k=0}^{n} \lambda_k = 1. \tag{5.6}$$

Let $\lambda_0 > 0$. We define the function

$$Q_n(t) := \frac{1}{2\pi i} \int_\Gamma \frac{e^{zt}}{B(z)}\, dz \tag{5.7}$$

where Γ is the circle $|z-1| = 1$ and $B(z) := \prod_{k=0}^n (z-\lambda_k)/(z+\lambda_k)$ is the Blaschke product. A direct application of the residue theorem shows that $Q_n \in \mathcal{E}(\Lambda)$.

For each $0 < \lambda \le 1$ and each $z \in \Gamma$ one has

$$\left|\frac{z-\lambda}{z+\lambda}\right| \ge \frac{2-\lambda}{2+\lambda}$$

and therefore

$$|B(z)| \ge B(2) = \prod_{k=0}^n \frac{2-\lambda_k}{2+\lambda_k}.$$

To estimate this product, we note that, for $x > 0$, $y > 0$, $x + y \le 1$

$$\frac{(1-x)(1-y)}{(1+x)(1+y)} \ge \frac{1-(x+y)}{1+(x+y)},$$

and this inequality used repeatedly gives

$$B(2) = \prod_{k=0}^n \frac{1-\lambda_k/2}{1+\lambda_k/2} \ge \frac{1-\sigma/2}{1+\sigma/2} = \frac{1-1/2}{1+1/2} = \frac{1}{3}.$$

This leads to the lower bound

$$|B(z)| \ge \frac{1}{3}, \quad z \in \Gamma. \tag{5.8}$$

Differentiating (5.7) we get

$$Q_n''(t) = \frac{1}{2\pi i} \int_\Gamma \frac{z^2 e^{zt}}{B(z)}\, dz \tag{5.9}$$

and from (5.7) and the substitution $z = 1 + e^{i\phi}$,

$$|Q_n''(t)| \le \frac{3}{2\pi} \int_0^{2\pi} 2(1+\cos\phi) e^{(1+\cos\phi)t} d\phi = \frac{3}{\pi} \int_0^{2\pi} \frac{d}{dt} e^{(1+\cos\phi)t} d\phi$$

and

$$\int_{-\infty}^0 |Q_n''(t)| dt \le \frac{3}{\pi} \int_0^{2\pi} \int_{-\infty}^0 \frac{d}{dt} e^{(1+\cos\phi)t} dt d\phi = \frac{3}{\pi} \int_0^{2\pi} d\phi = 6. \tag{5.10}$$

Next we evaluate the integrals

$$I_\lambda := \int_{-\infty}^0 e^{\lambda t} Q_n''(t)\, dt, \quad 0 < \lambda \le 1.$$

From (5.9) we have

$$I_\lambda = \frac{1}{2\pi i}\int_{-\infty}^{0} e^{\lambda t}\int_\Gamma \frac{z^2 e^{zt}}{B(z)}dz\ dt = \frac{1}{2\pi i}\int_\Gamma \frac{z^2}{B(z)}\int_{-\infty}^{0} e^{(\lambda+z)t}dt\ dz$$

and thus

$$I_\lambda = \frac{1}{2\pi i}\int_\Gamma \frac{z^2}{B(z)(\lambda+z)}\ dz\,, \quad 0<\lambda\le 1\,. \tag{5.11}$$

All poles of the function under the integral in (5.11) are inside Γ. Hence the integral remains the same if we replace Γ by an arbitrary large circle $|z|=r$, hence I_λ is equal to the residue of $\frac{z^2}{B(z)(z+\lambda)}$ at ∞. For the computation of this residue we use that for large z

$$\begin{aligned}\frac{1}{B(z)} &= \prod_{j=0}^{n}\frac{z+\lambda_j}{z-\lambda_j} = \prod_{j=0}^{n}\frac{1+\lambda_j/z}{1-\lambda_j/z}\\ &= 1+\frac{2\sigma}{z}+\frac{2\sigma^2}{z^2}+\ldots = 1+\frac{2}{z}+\frac{2}{z^2}+\ldots\end{aligned}$$

and $z/(z+\lambda) = 1-\lambda/z+\lambda^2/z^2+\ldots$. Then we get

$$\begin{aligned}\frac{z^2}{B(z)(\lambda+z)} &= (1-\frac{\lambda}{z}+\frac{\lambda^2}{z^2}+\ldots)(z+2+\frac{2}{z}+\ldots)\\ &= z+2-\lambda+(\lambda^2-2\lambda+2)\frac{1}{z}+\ldots\,.\end{aligned}$$

The desired residue at ∞ is $\lambda^2-2\lambda+2$ and

$$I_\lambda = \lambda^2-2\lambda+2\,, \quad 0<\lambda\le 1\,. \tag{5.12}$$

With the notation $g_\lambda(t) := e^{\lambda t}$, (5.12) is equivalent to

$$\int_{-\infty}^{0} g_\lambda(t+\tau)Q_n''(t)dt = g_\lambda''(\tau)-2g_\lambda'(\tau)+2g_\lambda(\tau)\,.$$

Taking linear combinations of the $e^{\lambda_k t}, k=0,\ldots,n$, leads to

$$\int_{-\infty}^{0} g(t+\tau)Q_n''(t)dt = g''(\tau)-2g'(\tau)+2g(\tau) \tag{5.13}$$

for an arbitrary exponential sum $g\in\mathcal{E}(\Lambda)$ and any real number τ. Using (5.10) and (5.13) for $\tau\le 0$ we get (in the uniform norm on $(-\infty,0]$)

$$\|g''-2g'+2g\|\le 6\|g\|. \tag{5.14}$$

We take a $g\in\mathcal{E}(\Lambda)$ with $\|g\|=1$ and define $\beta := \|g'\|$, $\gamma := \|g''\|$. It follows from (5.14) that

$$\gamma \le 8+2\beta. \tag{5.15}$$

We may assume that $g'(t_0) = \beta$ at some point $t_0 \le 0$. (Otherwise we would take $-g$.) For an arbitrary $h > 0$, we get

$$g(t_0 - h) - g(t_0) = -hg'(t_0) + \frac{h^2}{2} g''(\xi)$$

for some ξ, $t_0 - h < \xi < t_0$. Hence, $\beta \le 2h^{-1} + h\gamma/2$ and for $h := 2/\sqrt{\gamma}$,

$$\beta \le 2\sqrt{\gamma}. \tag{5.16}$$

Now (5.15) and (5.16) imply $\gamma \le 8 + 4\sqrt{\gamma}$, thus $\sqrt{\gamma} \le 2 + 2\sqrt{3}$ and $\beta \le 2\sqrt{\gamma} \le 4 + 4\sqrt{3} < 11$. This implies the right-hand inequality (5.5).

For the proof of the left-hand inequality(5.5) we consider the exponential sum Q_n of (5.7). Since the Blaschke product B satisfies $|B(z)| \ge 1/3$, $z \in \Gamma$ we see that for $t \le 0$

$$|Q_n(t)| \le \frac{1}{2\pi} \int_\Gamma \frac{|e^{zt}|}{|B(z)|} \, |dz| \le \frac{1}{2\pi} \int_\Gamma 3|dz| = 3,$$

hence $\|Q_n\| \le 3$. On the other hand, at $t = 0$,

$$Q_n'(0) = \frac{1}{2\pi i} \int_\Gamma \frac{z}{B(z)} \, dz. \tag{5.17}$$

There are no poles of the integrand $z/B(z)$ outside of Γ. As shown before, for large z, one has

$$\frac{z}{B(z)} = z + 2 + \frac{2}{z} + \dots.$$

Hence the residue of $z/B(z)$ at ∞ is 2 which together with (5.17) implies that $Q_n'(0) = 2$, and

$$\|Q_n'\| \ge Q_n'(0) \ge \frac{2}{3} \|Q_n\|,$$

for each $n \in \mathbb{N}$. Because of (5.6), this implies the desired inequality.

If $\lambda_0 = 0$, then we apply (5.5) to the exponential sums $e^{\varepsilon t} g(t)$, and the sequence $\{\varepsilon, \lambda_1 + \varepsilon, \dots, \lambda_n + \varepsilon\}$, and then let $\varepsilon \to 0+$.

§ 6. Problems

6.1. Let $1 \le p < 2$ and $r \ge 0$. Prove by using Lemma 2.1 that

$$E(x^r, \Lambda)_p \ge \frac{1}{7} (1 + 2r)^{-1/p} \prod_{k=0}^{n} \frac{|r - \lambda_k|}{r + \lambda_k + 2/p}.$$

6.2. Let $\Lambda = \{\gamma k\}_{k=0}^n$, for some $\gamma > 0$. Show by using Jackson's theorem for algebraic polynomials that for $f \in C[0, 1]$

$$E(f,\Lambda)_\infty \le K\omega(f,n^{-\delta})_\infty$$

where $\delta := 1$ if $0<\gamma\le 2$, $\delta := 2/\gamma$ if $\gamma>2$, and K depends only on γ.

6.3. Show that in the separated case of Theorem 4.1(i),

$$\frac{1}{2}\exp\left(-2\sum_{k=1}^{n}\frac{1}{\lambda_k}\right) \le \varepsilon_\infty(\Lambda) \le \exp\left(-2\sum_{k=1}^{n}\frac{1}{\lambda_k}\right).$$

6.4. Let $\Lambda_n : 0=\lambda_0<\lambda_1<\cdots<\lambda_n$ consist of 0 and the first n prime numbers. Estimate the asymptotic rate of $\varepsilon_\infty(\Lambda_n)$ as $n\to\infty$.

6.5. Let $\Lambda = (\lambda_k)_0^\infty$, $\lambda_0 = 0$, be an increasing sequence of nonnegative integers satisfying the Müntz condition. Show that the Müntz polynomials with integral coefficients are dense in $\{f\in C[0,1] : f(0), f(1)\in\mathbb{Z}\}$ (Ferguson and v. Golitschek [1975]).

6.6. Let $2\le p<\infty$ and let $\Lambda=(\lambda_k)_1^n$ be a finite sequence of complex numbers with real parts exceeding $-1/p$. Prove that the error of approximation from $\mathcal{M}(\Lambda)$ of the monomials x^r, $r>-1/p$, in the L_p-norm on $[0,1]$ satisfies

$$E(x^r,\Lambda)_p \le C\prod_{k=1}^{n}\frac{|r-\lambda_k|}{|r+\bar{\lambda}_k+2/p|}$$

where C depends only on r and p (v. Golitschek [1976_1]).

§ 7. Notes

7.1. Borwein and Erdélyi proved many interesting theorems about Müntz polynomials. See also their book [A-1995]. Via a Remez-type theorem they established in [1995_1]:

Theorem 7.1. *Let $\Lambda_\infty = \{0=\lambda_0<\lambda_1<\cdots\}$ be an infinite sequence, let A be a compact subset of $[0,\infty)$ of positive Lebesgue measure. Then the span of the powers x^{λ_j}, $j=0,1,\ldots$, is dense in $C(A)$ if and only if $\sum_1^\infty \lambda_j^{-1}=\infty$.*

They also prove the "full Müntz theorem" for the spaces $L_p[0,1]$, $p\ge 1$ (for $C[0,1]$ if $p=\infty$):

Theorem 7.2 (Borwein and Erdélyi [1995_2]). *Let $(\lambda_j)_0^\infty$ be a sequence of distinct real numbers greater than $-1/p$. Then the span of the x^{λ_j} is dense in $L_p[0,1]$ (or in $C[0,1]$ if $p=\infty$ and $\lambda_0=0$) if and only if*

$$\sum_{j=1}^{\infty}\frac{\lambda_j+1/p}{(\lambda_j+1/p)^2+1}=\infty.$$

Markov-type inequalities for Müntz polynomials are valid also for $L_p[0,1]$ and for $C[a,b]$, $0 < a < b < \infty$:

Theorem 7.3 (Borwein and Erdélyi [1995₃]). *Let $1 \le p < \infty$. Let $\Lambda = \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$, $\lambda_0 > -1/p$. Then*

$$\|xM'(x)\|_p \le \left(1/p + 12\sum_{k=0}^{n}(\lambda_k + 1/p)\right)\|M\|_p \quad \text{for all } M \in \mathcal{M}(\Lambda).$$

Theorem 7.4 (Borwein and Erdélyi [1995₄]). *Let $\Lambda = \{0 = \lambda_0 < \lambda_1 < \cdots\}$ be an increasing sequence of non-negative real numbers. Suppose there exists a $\delta > 0$ so that $\lambda_j \ge \delta j$ for each j. Then there exists a constant $c(a,b,\delta)$ depending only on a, b, and δ so that*

$$\|M'\|_{C[a,b]} \le c(a,b,\delta)\left(\sum_{j=1}^{n}\lambda_j\right)\|M\|_{C[a,b]}.$$

7.2. A paper by v. Golitschek [1976₂] deals with the approximation from $\mathcal{M}(\Lambda)$ in $C[a,b]$ and $L_p[a,b]$, $1 \le p < \infty$, where $0 < a < b$. A typical result is

Theorem 7.5. *Let $\Lambda = \{0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ satisfy $0 < \lambda_k \le k\delta$, $k = 1, 2, \ldots, n$, for some $\delta > 0$. Then, for any $f \in W_p^1[a,b]$, $1 \le p \le \infty$,*

$$E(f,\Lambda)_{L_p[a,b]} \le K\ n^{-1}\ \|f'\|_{L_p[a,b]} + \mathcal{O}((a/b)^{\gamma n}),$$

where $K > 0$ and $\gamma > 0$ depend only on δ and $[a,b]$.

7.3. Pólya [1931] posed the question: for which sequences $0 < \beta_1 < \beta_2 < \cdots$ are the linear combinations of the functions

$$\cos(\beta_k t),\ \sin(\beta_k t), \quad k = 1, 2, \ldots \tag{7.1}$$

complete in $C[0,2\pi]$? Pólya himself conjectured that the condition

$$\limsup_{k\to\infty}\frac{\beta_k}{k} < 1 \tag{7.2}$$

is sufficient. Szász [1934] proved that this is true and v. Golitschek [1976₃] showed that, under the assumption (7.2), the linear combinations of (7.1) satisfy for $p = \infty$ a Jackson theorem, similar to Theorem 2.7.

Chapter 12. Nonlinear Approximation

§ 1. Definitions and Simple Properties

Approximation in a Banach space X from a subset G is called *linear*, if G is a subspace of X, and *nonlinear* for other sets G. Important examples of the latter are rational approximation, when G is $\mathcal{R}_{m,n}$ or the set of trigonometric rationals $\mathcal{U}_{m,n}$ (see §2 of Chapter 7), approximation by splines with variable knots ([**CA**, §7, Chapter 7] and Chapter 6), approximation by exponential sums $S_n(x) := \sum_{j=1}^{n} a_j e^{\lambda_j x}$ (with variable a_j, λ_j), see §3. The book of Braess [A-1986] and the lecture notes of Berens [A-1977/78] deal with nonlinear approximation.

The *metric projection* $\Pi_G(f)$ of X onto G is the set of all $g \in G$ that are best approximants to $f \in X$; they satisfy $\|f - g\| = \operatorname{dist}(f, G)$. The set G is *proximinal* in X, if $\Pi_G(f)$ is never empty; G is a *uniqueness set*, if $|\Pi_G(f)| \le 1$. Finally, G is a *Chebyshev set*, if it has both these properties. If G is a finite dimensional subspace of X, then $\Pi_G(f)$ is convex and compact. Trivially, a proximinal set is closed.

There are several forms of a sufficient condition for the proximinality of G in X. The set G is *boundedly compact* if the intersection of G with any ball is relatively conditionally compact in X. We call G *approximatively compact* (Efimov and Stechkin [1961]) if each *minimizing sequence* $g_n \in G : \|f - g_n\| \to \operatorname{dist}(f, G)$ has a subsequence which converges to an element g_0 of G. In this case, $\|f - g_0\| = \operatorname{dist}(f, G)$ and we have:

1. *Each approximatively compact set $G \subset X$ is proximinal.*

For instance, *each of the sets $\mathcal{R}_{m,n}[a, b]$ is approximatively compact in L_p, $1 \le p < \infty$.* We have established this in the proof of Theorem 2.2 of Chapter 7.

Examples. (a) For $f(x) := 1 - 2x \in C[0, 1]$, the best $\mathcal{R}_{0,1}$ approximation is $R = 0$, for there are two alternation points, and $\|f - R\| = 1$. The sequence $g_n(x) := (nx + 1)^{-1}$ is a minimizing sequence for f, which does not contain a uniformly convergent subsequence. Thus, $\mathcal{R}_{0,1}$ is not approximatively compact in $C[0, 1]$.

(b) The sequence $\tilde{g}_n := c_n g_n$ with $c_n > 0$ defined by $\|\tilde{g}_n\|_p = 1$, $1 \le p < \infty$, satisfies $\tilde{g}_n(x) \to 0$ on $(0, 1]$, so that $\mathcal{R}_{0,1}$ is not boundedly compact in $L_p(0, 1)$.

2. *A weakly$*$-compact set G in the conjugate space X^* is proximinal.* Indeed, if g_n is a minimizing sequence from G, and g_0 is its weak$*$ limit, then for $x \in X$ we have

$$|\langle f,x\rangle - \langle g_0,x\rangle| = \lim_{n\to\infty} |\langle f,x\rangle - \langle g_n,x\rangle| \le \lim_{n\to\infty} \|f-g_n\|\|x\|$$
$$\le \|x\| \operatorname{dist}(f,G),$$

which implies $\|f - g_0\| = \operatorname{dist}(f,G)$.

3. *A set $G \subset C[a,b]$ is proximinal if each bounded sequence $g_n \in G$ contains a subsequence that converges uniformly to some $g_0 \in G$ on compact subsets of (a,b).*

Indeed, let $g_n \in G$ be a minimizing sequence for $f \in C[a,b]$; we can assume that it has the convergence property of **3**. Then $\|f-g_0\| < |f(x_0)-g_0(x_0)|+\varepsilon$ for each $\varepsilon > 0$, and some $a < x_0 = x_0(\varepsilon) < b$. This implies $\|f - g_0\| < |f(x_0) - g_n(x_0)| + \varepsilon$ for all $n \ge n_0$, so that $\|f-g_0\| \le \|f-g_n\| + \varepsilon$, $n \ge n_0$, and $\|f - g_0\| \le \operatorname{dist}(f,G)$.

Sometimes, for a proximinal set $G \subset X$, there exists a *continuous selection* of Π_G. This is a continuous map $\phi : X \to G$ with the property $\phi(f) \in \Pi_G(f)$.

Example. Let $X = \ell^2_\infty$ with the norm $\|x\| = \|(x_1,x_2)\| = \max(|x_1|,|x_2|)$, let G be the x_1 axis. Then $\Pi_G(x) = [x_1 - |x_2|, x_1 + |x_2|]$, and $\phi(x) = (x_1, 0)$ is a continuous selection.

Theorem 1.1. *The metric projection Π_G of a linear normed space X onto an approximatively compact Chebyshev subset G is continuous.*

Proof. Let $f_n \to f$, $g = \Pi_G f$, $g_n = \Pi_G f_n$. Then

$$\|f-g_n\| \le \|f-f_n\| + \|f_n - g_n\| \le \|f-f_n\| + \|f_n - g\|$$
$$\le \|f-g\| + 2\|f-f_n\| \to \|f-g\| \ .$$

Thus, g_n is a minimizing sequence for f, and for a subsequence of f_n, the corresponding g_n converge to g. We must have $g_n \to g$. □

For nonlinear approximation, of importance are also the *local best approximations* to $f \in X$. An element $g \in G$ has this property, if g is a best approximation in X from $G \cap U_r(g)$, where $U_r(g)$ is some ball in X with center g and radius $r > 0$.

If $g \in \Pi_G(f)$, $f \notin G$ — in this case $\|f-g\| = d > 0$ — then g is also a best approximation to any element of the segment $[g,f]$. Indeed, these elements are of the form $f_\lambda = g + \lambda(f-g)$, $0 \le \lambda \le 1$, and for any $g' \in G$,

$$\|f_\lambda - g'\| \ge \|f-g'\| - \|f-f_\lambda\| \ge \|f-g\| - \|f-f_\lambda\| = \|f_\lambda - g\| \ .$$

This is not necessarily so for all points $f_\lambda = g + \lambda(f-g)$, $\lambda \ge 0$ of the *ray* from g to f. Simplest examples of non-convex sets in the euclidean plane confirm this.

A set $S \subset X$ is called a *sun* (this definition was given by Vlasov [1973]; the original terminology by Efimov and Stechkin [1958] was somewhat different) if for each f with non-empty $\Pi_S(f)$, there is at least one $g \in \Pi_S(f)$ with the property that $g \in \Pi_S(f_\lambda)$ for all $\lambda \ge 0$. The set S is a *strict sun* if each $g \in \Pi_S(f)$ has this property.

Suns are related to local approximations by means of

Proposition 1.2 (Brosowski and Deutsch [1974]). *Each local best approximation to f from a strict sun S is also a global best approximation to f.*

Proof. Let g be a best approximation to f from $S' = S \cap U_r(g)$. If $\lambda > 0$ is small enough, then the distance from f_λ to g will be smaller than its distance to $S \setminus U_r(g)$. Then $g \in \Pi_S(f_\lambda)$. From the sun property we obtain that $g \in \Pi_S(f)$. □

The simplest examples of suns are the linear and the convex sets.

Proposition 1.3. *Each convex set S in a linear normed space is a strict sun.*

Proof. Let $f \notin S$, $g \in \Pi_S(f)$, and $\lambda > 1$. For any $g' \in S$, the element $g'' := g + \frac{1}{\lambda}(g' - g)$ is also in S, since S is convex. Hence $\|f - g''\| \geq \|f - g\|$ and thus $\|f_\lambda - g\| = \lambda\|f - g\| \leq \lambda\|f - g''\| = \|f_\lambda - g'\|$. □

Examples. A sun that is not a strict sun is given in the space ℓ^2_∞ by $S_1 = \{(x_1, x_2) : x_1 \leq 0 \text{ or } x_2 \leq 0\}$. For $f = (2, 1)$, Πf is the set $\{1 \leq x_1 \leq 3,\ x_2 = 0\}$, but the only solar point is $(1, 0)$. This is also an example of a sun that is not convex.

The property of S, stated in Proposition 1.2, is not characteristic for suns. For instance, $S_2 := \{(x_1, x_2) : x_1^2 + x_2^2 \geq 1\} \subset \mathbb{R}^2$, which is not a sun, has it.

The plan of this chapter is as follows. The next §2 is devoted to *varisolvent families* $\mathcal{V}$ of Rice. They admit a characterization of the element $g \in \mathcal{V}$ of best approximation to f by means of the alternation properties of $f - g$; in §3 we apply this to the exponential sums S_n. Theorems of "Vitushkin type" of §4 assert that even very sophisticated means of approximation (if they depend on a comparable number of parameters) do not produce a better approximation than ordinary polynomials. This is true for such spaces as $\mathrm{Lip}(\alpha, L_p)$, $\alpha > 0$, $1 \leq p \leq \infty$. In §5, we study the existence or the nonexistence of continuous selections. The last Section 6 is devoted to "abstract approximation" in a Banach space X, and to properties of suns and of Chebyshev sets.

§ 2. Varisolvent Families

The theory of *varisolvent families* $\mathcal{V}$ of continuous functions on A, $A = [a, b]$ or $\mathbb{T}$, which we discuss in the uniform norm, contains as a special case rational approximation (see Chapter 7) and approximation by exponential sums (in §3). Varisolvent families have been defined and studied by Rice [1961], [1964], who among other things, proved the basic alternation theorem (Theorem 2.4). (See his book, Rice [A-1969].) The corresponding theorems for rational approximation are Theorems 2.6 and 2.7 of Chapter 7. Among the predecessors of Rice we mention Tornheim [1950].

Varisolvent families $\mathcal{V} \subset C(A)$ are described by means of the following properties Z, V, D, D_1:

PROPERTY (Z). *We say that $g \in \mathcal{V}$ has property (Z) with respect to $\mathcal{V}$ if for some integer $N = 1, 2, \dots$ and any $g_1 \in \mathcal{V}$, one has $g_1 = g$, whenever the difference $g - g_1$ has at least N distinct zeros.*

If g satisfies (Z) for some N, then also for each larger N.

The minimal value of N of this type for a given g will be denoted by $N_z(g) := N_z(g, \mathcal{V})$; we put $N_z(g) = +\infty$ if there are no such N.

PROPERTY (V) (Varisolvency property). *We say that $\mathcal{V}$ is locally solvent of order $N \geq 1$ at some $g \in \mathcal{V}$, if, given $\varepsilon > 0$ and N distinct points $x_i \in A$, $i = 1, \dots, N$, there exists a $\delta = \delta(\varepsilon, x_1, \dots, x_N) > 0$ such that for any set of data c_i, $i = 1, \dots, N$ with $|g(x_i) - c_i| \leq \delta$, there is a $g_1 \in \mathcal{V}$ for which*

$$g_1(x_i) = c_i\,, \qquad i = 1, \dots, N \;\; \textit{and} \;\; \|g - g_1\| < \varepsilon\,. \tag{2.1}$$

If g has property (V) with some N, then also with each smaller N.

The supremum of all N of this type (which may be $+\infty$) is denoted by $N_v(g) := N_v(g, \mathcal{V})$.

We note that for any $\mathcal{V}$ with the properties $(Z), (V)$,

$$N_v(g) \leq N_z(g)\,. \tag{2.2}$$

Indeed, $N < N_v(g)$ implies that there are $N + 1$ distinct points $x_i \in A$ for which, with some $\delta > 0$, the equations $g_1(x_i) = g(x_i)$, $i = 1, \dots, N$, $g_1(x_{N+1}) = g(x_{N+1}) + \delta$ are solvable with some $g_1 \in \mathcal{V}$. Then $g_1 - g$ has N zeros without being identically zero, and we deduce that $N < N_z(g)$.

We shall say that $\mathcal{V}$ is *varisolvent at* $g \in \mathcal{V}$, if (Z) and (V) are satisfied for this g with the same finite N, and *varisolvent*, if $\mathcal{V}$ is varisolvent at each $g \in \mathcal{V}$. As a corollary of (2.2), this is the case if and only if

$$N_v(g) = N_z(g) \; < +\infty, \qquad g \in \mathcal{V}\,. \tag{2.3}$$

The number (2.3), *the degree of varisolvency of* $\mathcal{V}$ *at* g, is denoted by $N(g) := N(g, \mathcal{V})$. To prove that $\mathcal{V}$ is varisolvent at g, it is sufficient (a) to find a number N so that $g = g_1$, whenever $g_1 \in \mathcal{V}$ and $g - g_1$ has at least N zeros and (b) to prove that each interpolation problem (2.1) with N points is solvable. Then $N = N(g, \mathcal{V})$. As an example, a Haar space of dimension n, spanned by a Haar system of n functions on A is varisolvent, and $N_v(g) = N_z(g) = n$, $g \in G$. Here we have global, not only local solvability of the equations (2.1).

In the general case, only the condition (Z) has a global character, while (V) is local. In a varisolvent family, the function g_1 satisfying the equations (2.1) with $N = N(g)$ is unique even without the restriction $\|g - g_1\| < \varepsilon$.

In order to prove the basic alternation Theorem 2.4 (due to Rice) for varisolvent families, we need an additional assumption. We shall say that a family $\mathcal{V} \subset C(A)$ has the *density property* if it satisfies

PROPERTY (D). *For each $g \in \mathcal{V}$ and each $\varepsilon > 0$, there are $g_1, g_2 \in \mathcal{V}$, different from g, for which*

$$(2.4) \qquad g_1(x) \le g(x) \le g_2(x)\ , \qquad x \in A\ , \quad \|g_1 - g_2\| < \varepsilon\ .$$

For a linear family G, this property is equivalent to the existence of a $g_0 \in G$, $g_0 \neq 0$, with $g_0(x) \ge 0$ for $x \in A$. For all Haar spaces this is true even with $g_0(x) > 0$, $x \in A$, (see [**CA**, Theorem 9.1, p.80]), so that they satisfy (2.4) with strict inequalities.

A family $\mathcal{V}$ has the *phenomenon of the constant error function* if there exists a function $f \in C(A)$ for which a best approximation g from $\mathcal{V}$ has a constant difference $f(x) - g(x) = c \neq 0$, $x \in A$. Obviously, (D) implies

PROPERTY (D_1). *The family $\mathcal{V}$ has no constant error functions.*

The necessity of assuming (D_1) in the proof of the Theorem 2.4 has been noticed by Dunham [1968].

A continuous function f on $A = [a, b]$ or $= \mathbb{T}$ has a *double zero* at $x_0 \in A$ if x_0 is not one of the endpoints a, b and if f does not change sign in a neighborhood of x_0. All other zeros are *simple zeros* of f. We need the following fact.

Theorem 2.1 (Rice). *For a varisolvent family, the property (Z) remains true even if each double zero of $g - g_1$ is counted twice.*

Proof. We assume that $g, g_1 \in \mathcal{V}$ and that counting multiplicities, $g - g_1$ has $N(g)$ zeros, with at least one double zero. We would like to show that $g = g_1$. Let this be false, and let x_j, $j = 1, \ldots, m$ be the distinct zeros of $g - g_1$ on A. Each double zero x_j we embed into a neighborhood U_j, which does not contain other zeros x_k, $k \neq j$, and on which $g(x) - g_1(x)$, $x \neq x_j$ keeps a constant sign. By $\mathcal{U}^+$ (or $\mathcal{U}^-$) we denote the set of the U_j with $g(x) - g_1(x) > 0$, $x \neq x_j$ (or with $g(x) - g_1(x) < 0$, respectively).

We can assume that the U_j are disjoint. We apply the property (V) to construct a $g_2 \in \mathcal{V}$, $\|g_1 - g_2\| < \varepsilon$, for which

$$(2.5) \qquad \begin{cases} g_2(x_j) = g(x_j) & \text{if } x_j \text{ is a single zero} \\ g_2(x_j) > g(x_j) & \text{if } U_j \in \mathcal{U}^+ \\ g_2(x_j) < g(x_j) & \text{if } U_j \in \mathcal{U}^-\ . \end{cases}$$

If $\varepsilon > 0$ is small enough, then each of the neighborhoods U_j will contain at least two distinct zeros of $g - g_2$. Since $g \neq g_2$, these functions will provide a contradiction to the property (Z) at g. □

Let $A = \mathbb{T}$, let $g, g_1 \in \mathcal{V}$, $g \neq g_1$. Then $g - g_1$ has an even number of zeros (if double zeros are counted twice). Moreover, in this case, *$N_z(g)$ is odd for all $g \in \mathcal{V}$.*

The theorem allows us sometimes to determine the number of changes of sign of $g - g_1$ on A, for $g, g_1 \in \mathcal{V}$, $g \neq g_1$.

Corollary 1. *If the difference $g - g_1 \neq 0$ has $N(g) - 1$ zeros x_i, then all of them are simple and there are no other zeros.*

For otherwise $g - g_1$ would have $\ge N(g)$ zeros.

Corollary 2. *If the difference $g - g_1 \neq 0$ has $N(g) - 2$ zeros x_j and on A it has $N(g) \pmod 2$ changes of sign, then all zeros x_j are simple and there are no other zeros.*

Indeed, the total number of zeros of $g - g_1$, counting multiplicities, is congruent to $N(g)$. It cannot be $N(g) - 1$, so it is $\leq N(g) - 2$. There cannot be zeros other than the x_j, and none of these can be multiple.

We have a de la Vallée-Poussin theorem for a varisolvent family $\mathcal{V}$. We let $v(f)$ denote the *error of approximation of f by the elements of $\mathcal{V}$.*

Theorem 2.2. *Assume that $f \in C(A)$, $g \in \mathcal{V}$ and let $x_1 < \cdots < x_m$ ($< x_1$ if $A = \mathbb{T}$) be points of A numbered in the positive direction on A. If $m = N(g)+1$, if $|f(x_j) - g(x_j)| \geq \delta$ and* $\operatorname{sign}[f(x_j) - g(x_j)] = \sigma(-1)^j$, *$\sigma = +1$ or $= -1$ for $j = 1, \ldots, m$, then*

$$v(f) \geq \delta \ . \tag{2.6}$$

Proof. Otherwise there is a $g_1 \in \mathcal{V}$ for which $|f(x) - g_1(x)| < \delta$, $x \in A$. Then $\operatorname{sign}[g(x_j) - g_1(x_j)] = -\sigma(-1)^j$, $j = 1, \ldots, m$, so that $g - g_1$ changes sign $N(g)$ times, a contradiction. □

The following lemma establishes the existence of an approximating function $g_1 \in \mathcal{V}$, which oscillates around a given $g \in \mathcal{V}$.

Let $A = I := [a, b]$, or $A = \mathbb{T}$, let $A = \bigcup_1^m I_i$ be a decomposition of A into closed intervals with common endpoints, following each other on A in the positive direction. For $A = I$, we allow $m = 1, 2, \ldots$; for $A = \mathbb{T}$, m must be an even integer. An endpoint x of I_i is *interior* if $A = \mathbb{T}$, or if $A = [a, b]$ and $x \neq a$, $x \neq b$.

Lemma 2.3. *Let $A = \bigcup_1^m I_i$ be a decomposition of A, let $g \in \mathcal{V}$, $m \leq N(g)$, and let $c > 0$. Let J_j, $j = 1, \ldots, p, p := [(N(g) - m)/2]$ be some closed disjoint intervals in A, containing no endpoints of the I_i, but otherwise arbitrary. If $\sigma = +1$ or $\sigma = -1$, there exists a function $g_1 \in \mathcal{V}$ with $\|g - g_1\| < c$ with the property*

$$\sigma(-1)^i [g(x) - g_1(x)] \leq 0 \ , \qquad x \in I_i \setminus \bigcup_1^p J_j \ , \quad i = 1, \ldots, m \ , \tag{2.7}$$

and with $g - g_1$ vanishing exactly at the endpoints of the J_j and the interior endpoints of the I_i.

The *proofs* are different for the cases $A = [a, b]$ and $A = \mathbb{T}$.

Case 1. Let $A = I$ and $N(g) \equiv m \pmod 2$. Then

$$N(g) = m + 2p \ .$$

We select the J_j, $j = 1, \ldots, p$, arbitrarily. Let z_k denote the endpoints of the J_j and the interior endpoints of the I_i; there are $2p + m - 1 = N(g) - 1$ of these. We take $\delta > 0$ of the condition (V) for the points a, z_k, and the number c instead of ε. We require

$$\begin{cases} g_1(z_k) = g(z_k) & \text{for all endpoints } z_k \\ g_1(a) = g(a) - \delta\sigma \,, \end{cases} \tag{2.8}$$

and $\|g - g_1\| <$ c. From Corollary 1, the z_k are the only zeros of $g - g_1$, and they are simple, so that this difference changes sign at all z_k. On I_1, due the the second line of (2.8), we have (2.7), and this extends onto $A \setminus \bigcup J_j$.

Case 2. Let $A = I$ and $N(g) \equiv m + 1 \pmod 2$. Then

$$N(g) = m + 1 + 2p \quad \text{for some} \quad p = 0, 1, \ldots \,.$$

We perform a similiar construction, replacing (2.8) by

$$\begin{cases} g_1(z_k) = g(z_k) & \text{at all } z_k \,, \\ g_1(a) = g(a) - \delta\sigma \\ g_1(b) = g(b) - (-1)^{m+1}\delta\sigma \,. \end{cases} \tag{2.9}$$

In this case, we have constructed $N(g) - 2$ zeros z_k of $g - g_1$. In addition, on $[a, b]$ this difference changes sign $m + 1 \pmod 2$ times, since

$$(-1)^{m+1}\big[g(a) - g_1(a)\big]\big[g(b) - g_1(b)\big] > 0 \,.$$

Corollary 2 serves to show that $g - g_1$ changes sign at exactly the z_k; this again leads to (2.7).

Case 3. Let $A = \mathbb{T}$, here $N(g)$ is odd, m is even, so that

$$N(g) = m + 1 + 2p \,.$$

Here we use (2.8), taking a to be an interior point of I_1. The rest is as before. □

The following is the main theorem of Rice.

Theorem 2.4. *Let $\mathcal{V}$ be a varisolvent family on $A = \mathbb{T}$ or $[a, b]$ that satisfies (D_1) or (D). For given $f \in C(A) \setminus \mathcal{V}$, $g \in \mathcal{V}$, the function g is the best approximation to f from $\mathcal{V}$ with $v(f) = \varepsilon := \|f - g\| > 0$ if and only if $f - g$ has an alternation of $N(g) + 1$ points with amplitude ε.*

Proof. The sufficiency of the condition follows from Theorem 2.2.

To prove its necessity, let $g \in \mathcal{V}$ be a best approximation from f to g, with $\|f - g\| = \varepsilon > 0$. Since $f - g$ is not a constant, the set $G := \{x \in A : |f(x) - g(x)| < \varepsilon_1\}$ for some $0 < \varepsilon_1 < \varepsilon$ is a nonempty open set. If the condition is not satisfied, there exists a decomposition $A = \bigcup_1^m I_i$, with even

$m \geq 2$ if $A = \mathbb{T}$, the intervals I_i separated by their interior endpoints where $f - g$ vanishes, so that for some ε_2, $0 < \varepsilon_2 < \varepsilon$ and $\sigma = +1$ or $\sigma = -1$,

$$(2.10) \qquad -\varepsilon_2 \leq \sigma(-1)^i\{f(x) - g(x)\} \leq \varepsilon, \qquad x \in I_i.$$

Let g_1 be the function of Lemma 2.3 for some c with $\varepsilon_1 + c < \varepsilon$, $\varepsilon_2 + c < \varepsilon$. We take $J_j \subset G$ for all j. Then on $\cup J_j$, $|f(x) - g_1(x)| < \varepsilon_1 + c < \varepsilon$, and outside of $\cup J_j$, by (2.7), if x is not an exterior endpoint,

$$(2.11) \qquad -\varepsilon < -(\varepsilon_2 + c) \leq \sigma(-1)^i\{f(x) - g_1(x)\} < \varepsilon.$$

Since $f - g$ vanishes at all interior endpoints of the I_i, we deduce $-\varepsilon < f(x) - g_1(x) < \varepsilon$, $x \in A$, a contradiction. □

Theorem 2.5. *A varisolvent family $\mathcal{V}(A)$ satisfying (D_1) is a sun with the uniqueness property.*

Proof. If g is a best approximation to $f \in C(A) \setminus \mathcal{V}$, it has an alternation of length $N(g) + 1$. Thus, for another best approximation g_1, the difference $g - g_1$ must have $N(g)$ zeros, hence $g = g_1$. Moreover, for $f_\lambda = g + \lambda(f - g)$, $\lambda > 0$ the difference $f_\lambda(x) - g(x) = \lambda(f(x) - g(x))$ also has an alternation of length $N(g)$, so that g is the best approximation to all f_λ, $\lambda > 0$, with $v(f_\lambda) = \lambda v(f)$. □

It should be added that the *proximinality property* does not follow from the assumptions of Theorem 2.4. See the example of the families $\mathcal{E}_n$ in the next section.

It is not quite simple to prove that the families $\mathcal{R}_{m,n}$ are varisolvent (Rice [A-1969]), and we shall omit this proof.

Since the proof of Theorem 2.5 depends only upon the validity of the alternation theorem, we have, however,

Proposition 2.6. *The sets of rational functions $\mathcal{R}_{m,n}$ and $\mathcal{U}_{m,n}$ are suns in $C[a,b]$ with the uniqueness property.*

§ 3. Exponential Sums

We shall discuss here approximation by families $\mathcal{E}_n$ of *exponential sums*

$$(3.1) \qquad g(x) := \sum_{j=1}^{n} a_j e^{\lambda_j x} \ ,$$

where the coefficients a_j and the exponents $\Lambda : \lambda_1 < \cdots < \lambda_n$ are arbitrary; the sum g depends on $2n$ parameters. This problem is not yet sufficiently explored; many known results belong to Braess, who in his book, Braess [A-1986] studies exponential sums and their generalizations. By $\mathcal{E}_n(\Lambda)$ we denote the linear subspace of $\mathcal{E}_n$ of functions (3.1) with fixed Λ. In this case, the substitution $e^x = y$ transforms $g(x)$ into a Müntz polynomial $M(y) = \sum_1^n a_j y^{\lambda_j}$. See Chapter 11; results of that chapter yield useful results about functions $g \in \mathcal{E}_n(\Lambda)$, and conversely.

The families $\mathcal{E}_n$ are not closed in uniform norm on an interval, and hence not proximinal. for example,

$$\lim_{h\to 0} h^{-1}(e^{(\lambda+h)x} - e^{\lambda x}) = xe^{\lambda x} \notin \mathcal{E}_n \; . \tag{3.2}$$

Therefore one often considers the *extended families* $\mathcal{E}_n^*$, consisting of all functions

$$g^*(x) = \sum_{j=1}^{\ell} P_j(x) e^{\lambda_j x} \quad , \quad \sum_{j=1}^{\ell} (m_j + 1) \le n \; , \tag{3.3}$$

where $\ell \le n$, m_j, $j = 1, \dots, \ell$ are non-negative integers with $\sum_{j=1}^{\ell}(m_j+1) \le n$, and P_j are polynomials of degree not exceeding m_j. The subspace of $\mathcal{E}_n^*$ with fixed Λ, ℓ, $(m_j)_1^{\ell}$ is the linear space spanned by the functions

$$e^{\lambda_1 x}, xe^{\lambda_1 x}, \dots, x^{m_1} e^{\lambda_1 x}, \dots, x^{m_\ell} e^{\lambda_\ell x} \; . \tag{3.4}$$

We prove that this is a Haar space in $C[a,b]$. Let $g^* \ne 0$ of (3.3) have n zeros on $[a,b]$, for example, let $P_1 \ne 0$. Dividing, if necesary, by $e^{\lambda_1 x}$, we may assume that $\lambda_1 = 0$. By Rolle's theorem $g^{*(m_1+1)}$ has $n - m_1 - 1$ zeros. This function belongs to the family $\mathcal{E}_{n-m_1-1}^*$ with exponents $\lambda_2, \dots, \lambda_\ell$; the degrees of the polynomial coefficients in this derivative will not change. Repeating this we obtain a polynomial $\widetilde{P}_\ell(x) \ne 0$ of degree $\le m_\ell$ with $m_\ell + 1$ zeros, a contradiction.

Another property of the functions (3.4) is that they are the fundamental solutions of the ordinary differential equation

$$\prod_{j=1}^{\ell} (D - \lambda_j)^{m_j+1} g = 0 \; . \tag{3.5}$$

The set $\mathcal{E}_n$ is a dense subset of $\mathcal{E}_n^*$ on any compact set A of the complex plane: for each $g^* \in \mathcal{E}_n^*$, there is a sequence $g_k \in \mathcal{E}_n$ with $g_k \to g^*$, $g_k^{(r)} \to g^{*(r)}$ uniformly on A. This is a consequence of the fact that each function $x^k e^{\lambda x}$ of (3.4) is the uniform limit of a linear combination of $k+1$ exponentials $e^{\lambda_j x}$ on A.

The family $\mathcal{E}_n$ on $[a,b]$ contains *zoom functions*, that is, strictly increasing functions g satisfying

$$0 \le g(x) \le \varepsilon \; , \quad a \le x \le b - \varepsilon \; , \quad g(b) \ge A \tag{3.6}$$

for arbitrary $\varepsilon > 0$, $A > 0$. An example is given by $g(x) = \varepsilon e^{\lambda(x+\varepsilon-b)}$ where λ is sufficiently large.

This shows that for $\mathcal{E}_n$ there do not exist exact parallels to the Bernstein inequality for polynomials. Inequalities that are true are weaker. The first of them (of Dolzhenko type) is true even for *quotients* of two functions from $\mathcal{E}_n$:

Theorem 3.1. *If $R_n = E_n/F_n$ with $E_n, F_n \in \mathcal{E}_n$, $F_n(x) \neq 0$ on $[a,b]$, then*

$$\int_a^b |R_n'(t)|\,dt \le 2n\|R_n\|_\infty[a,b] \ . \tag{3.7}$$

The proof is the same as that of Theorem 1.1 of Chapter 7.

The following theorem is a weaker form of a result of Borwein and Erdélyi. It gives a Bernstein type estimate of the norm of the derivative of a function $g \in \mathcal{E}_n$. As in (3.7), the norms of g' and g are in different spaces, this time in $C[a+\delta, b-\delta]$ and $C[a,b]$. A forerunner of (3.8), due to Schmidt [1970], had the factor $C(n)\delta^{-1}$ on the right, with a non-specified $C(n)$.

Theorem 3.2. *For any functions $g \in \mathcal{E}_n$ one has, for $0 < \delta < (b-a)/2$, in the uniform norm,*

$$\|g'\|_{[a+\delta,b-\delta]} \le 8n^2\delta^{-1}\|g\|_{[a,b]} \ . \tag{3.8}$$

Proof. We fix Λ, and prove this for a function $g \in \mathcal{E}_n(\Lambda)$. Let $(L_k)_1^n$ be an orthonormal basis for $\mathcal{E}_n(\Lambda)$ on $[-1/2, 1/2]$, with $\int_{-1/2}^{1/2} L_i L_k\,dx = \delta_{i,k}$. Then each $g \in \mathcal{E}_n(\Lambda)$ has the form $g(x) = \sum_1^n a_k L_k(x)$. By Cauchy's inequality and Parseval's theorem, $|g(x)| \le \|g\|_2(-1/2,1/2)(\sum_1^n L_k(x)^2)^{1/2}$, and since $\int \sum_1^n L_k^2\,dx = n$, for some α, $|\alpha| \le 1/2$, which is independent of g in $\mathcal{E}_n(\Lambda)$,

$$|g(\alpha)| \le \sqrt{n}\|g\|_2(-1/2,1/2) \ . \tag{3.9}$$

Applying this to $g_1(x) := g(x-\alpha)$, we derive

$$|g(0)| \le \sqrt{n}\|g\|_2(-1,1) \ . \tag{3.10}$$

Now let

$$C := \sup_{0\neq g\in\mathcal{E}_n} \frac{|g(0)|}{\|g\|_1(-1,1)} = \sup_{0\neq g\in\mathcal{E}_n} \frac{2|g(0)|}{\|g\|_1(-2,2)} \ . \tag{3.11}$$

Let β be an arbitrary point in $[-1,1]$. For $g_2(x) := g(\beta + x)$, one gets

$$|g(\beta)| \le C\|g_2\|_1(-1,1) \le C\|g\|_1(-2,2) \ , \quad -1 \le \beta \le 1. \tag{3.12}$$

Inequalities (3.10), (3.12) yield for any $g \in \mathcal{E}_n(\Delta)$

$$\begin{aligned} |g(0)| &\le \sqrt{n}\|g\|_2(-1,1) \le \sqrt{n}\big(\|g\|_\infty[-1,1]\big)^{1/2}\big(\|g\|_1(-1,1)\big)^{1/2} \\ &\le \sqrt{nC}\|g\|_1(-2,2). \end{aligned} \tag{3.13}$$

Therefore, from the second definition of C in (3.11), $C \le 2\sqrt{nC}$, or $C \le 4n$.

With g, also g' belongs to $\mathcal{E}_n(\Lambda)$, and therefore

$$|g'(0)| \le 2n\|g'\|_1(-2,2) \le 2n \operatorname{Var}_{[-2,2]} g \ .$$

As in the proof of Theorem 1.1 of Chapter 7, we have $\operatorname{Var}_{[-2,2]} g \le 2n\|g\|_\infty[-2,2]$, so that

$$(3.14) \qquad |g'(0)| \le 4n^2\|g\|_\infty[-2,2] \ .$$

A final substitution $x \to y + \frac{1}{2}\delta x$, for a fixed $y \in [a+\delta, b-\delta]$, yields (3.8). □

The *exact* inequality (at least as far as the exponent of n is concerned) is, instead of (3.8),

$$(3.15) \qquad \|g'\|_{[a+\delta,b-\delta]} \le 2n\delta^{-1}\|g\|_{[a,b]} \ , \qquad g \in \mathcal{E}_n \ .$$

See the book of Borwein and Erdélyi [A-1995] for this and many other results.

There are many important corollaries of (3.8) (and (3.15)). The space $\mathcal{E}_n$ is dense in the space $\mathcal{E}_n^*$ on compact subsets of $\mathbb{C}$. Therefore:

Corollary 3.3. *The inequality* (3.8) *is valid also for all* $g \in \mathcal{E}_n^*$.

If $g \in \mathcal{E}_n^*$, then $h(x) := \int_a^x g(t)\,dt$ belongs to $\mathcal{E}_{n+1}^*$. Therefore, from (3.8) and Hölder's inequality we get, if $1/p + 1/p' = 1$,

$$(3.16) \qquad \|g\|_{C[a+\delta,b-\delta]} \le \frac{O(n^2)}{\delta}\|g\|_{L_1[a,b]} \le (b-a)^{1/p'}\frac{O(n^2)}{\delta}\|g\|_{L_p[a,b]} \ .$$

We also have the Nikolskii type inequalities. If $1 \le p, q \le \infty$,

$$(3.17) \qquad \|g\|_{L_q[a+\delta,b-\delta]} \le (b-a)^{1/q+1/p'}\frac{O(n^2)}{\delta}\|g\|_{L_p[a,b]} \ .$$

We shall now discuss the proximinality and the uniqueness properties of the families $\mathcal{E}_n$ and $\mathcal{E}_n^*$. They are not linear, not even convex. A family $\mathcal{E}_n$ is not closed in $C[a,b]$, hence not proximinal. In this section we shall prove that $\mathcal{E}_n$ is varisolvent. In contrast, $\mathcal{E}_n^*$ is proximinal, but not varisolvent, and does not have the uniqueness property.

From Corollary 3.3 it follows that the functions of a bounded (in the uniform norm) subset of $\mathcal{E}_n^*$ on $[a,b]$ as well as all their derivatives, are equicontinuous on each closed subinterval of (a,b). This allows us to prove the following lemma, due to Schmidt [1970]:

Lemma 3.4. *Each sequence of functions* $g_k \in \mathcal{E}_n^*$ *that is uniformly bounded in* $C[a,b]$ *contains a subsequence* g_{k_i}, $i = 1, 2, \ldots$ *that is uniformly convergent on each interval* $[a+\delta, b-\delta]$, $\delta > 0$, *to some* $g_0 \in \mathcal{E}_n^*$.

Proof. We write the elements of the given sequence g_k in the form (3.3), where $P_j, m_j, \ell, \lambda_j$ depend upon k. Passing to a subsequence, we can assume that the

m_j and ℓ are constant; moreover, that $\lambda_{j,k} =: \lambda_j$ have limits for $k \to \infty$, finite or $+\infty$ or $-\infty$. Let $\lim_{k\to\infty} \lambda_{j,k} = \mu_j \in \mathbb{R}$, $j = 1, \dots, p$, and $\lim_{k\to\infty} \lambda_{j,k} = \infty$, $j > p$. The differential equation (3.5) for the g_k can be written in the form

$$\prod_{j=1}^{p} (D - \lambda_{j,k})^{m_j} \prod_{j>p} (\lambda_{j,k}^{-1} D - 1)^{m_j} g_k = 0 \; . \tag{3.18}$$

Applying repeatedly the Ascoli-Arzela theorem and taking subsequences, we obtain a new sequence (denoted again by g_k) that converges uniformly, together with its derivatives of order $\le n+1$, on each interval $[a+\delta, b-\delta]$. If $\lim g_k = g_0$, then by making $k \to \infty$ in (3.18) we obtain

$$\prod_{j=1}^{p} (D - \mu_j)^{m_j} g_0 = 0 \; ,$$

hence $g_0 \in \mathcal{E}_n^*$. □

The lemma has an important corollary.

Theorem 3.5 (Werner [1969]). *For any interval $[a,b]$ the family $\mathcal{E}_n^*$ is closed and proximinal in $C[a,b]$ and also in $L_p[a,b]$, $1 \le p < \infty$.*

Proof. The functions of $\mathcal{E}_n^*$ satisfy on $[a,b]$ the condition of **3** of §1. Let $f \in L_p$, $1 \le p < \infty$, let $g_k \in \mathcal{E}_n^*$ be a minimizing sequence for f: $\|f - g_k\|_p \to \operatorname{dist}(f, \mathcal{E}_n^*)_p$. Then the $\|g_k\|_p$ are bounded. On each interval $[a+\delta, b-\delta]$, the functions g_k are uniformly bounded in $C[a,b]$ because of (3.17). They contain a subsequence that is uniformly convergent on $[a+2\delta, b-2\delta]$. Thus, we can find a subsequence, also denoted by g_k, which is bounded in L_p and convergent on compact subsets of (a,b) to a function $g_0 \in \mathcal{E}_n^*$. By means of the Fatou theorem for L_p, or by means of **3** of §1 for $C[a,b]$, we obtain $\|f - g_0\| = \operatorname{dist}(f, \mathcal{E}_n^*)$. □

We shall now discuss the varisolvency properties of the families $\mathcal{E}_n$, $\mathcal{E}_n^*$. It is clear that all of them have the density property (D) of §2. The following theorems, in a more general setting, have been given by Hobby and Rice [1967], Braess [1973].

For an individual function g in $\mathcal{E}_n^*$ or in $\mathcal{E}_n$, it is useful to know the quantities $\ell(g)$, $k(g)$, defined as follows: $\ell(g)$ is ℓ of (3.3), reduced so that the sum has no zero terms, and $k(g) := \sum_{j=1}^{\ell} (m_j + 1)$, where the m_j are the (exact) degrees of the P_j. We have

Theorem 3.6. *For each function g in $\mathcal{E}_n$ or in $\mathcal{E}_n^*$ on $[a,b]$ one has*

$$N_z(g) \le n + k(g) \tag{3.19}$$

$$N_v(g) \ge n + \ell(g) \tag{3.20}$$

with respect to the corresponding family.

Proof. (a) A function $g \in \mathcal{E}_n^*$ has the representation

$$g(x) = \sum_{k=1}^{\ell} \sum_{\nu=0}^{m_k} a_{k,\nu} x^\nu e^{\lambda_k x} , \qquad \lambda_1 < \cdots < \lambda_\ell , \quad \ell = \ell(g) . \tag{3.21}$$

It belongs to the Haar space spanned by the functions

$$e^{\lambda_1 x}, \ldots, x^{m_1} e^{\lambda_1 x}, \ldots, x^{m_k} e^{\lambda_k x} ,$$

which has dimension $k(g)$. For any $g_1 \in \mathcal{E}_n^*$, $g - g_1$ belongs to a Haar space of similar type of dimension $\le n + k(g)$. If $g - g_1$ has $n + k(g)$ distinct zeros, then $g - g_1$ is identically zero. This proves (3.19).

(b$_1$) First let $g \in \mathcal{E}_n^*$. Then we have (3.21), where $a_{k,m_k} \ne 0$, $k = 1, \ldots, \ell$ and $\sum (m_k + 1) = k(g) \le n$. We shall interpolate values close to the $g(x_i)$ at N points x_i by functions $h \in \mathcal{E}_n^*$,

$$h(u; x) = \sum_{k=1}^{\ell} \sum_{\nu=0}^{m_k} b_{k,\nu} x^\nu e^{\mu_k x} + \sum_{k=\ell+1}^{N-k(g)} b_{k,0} e^{\lambda_k x} , \tag{3.22}$$

where λ_k, $k > \ell$ with $\lambda_\ell < \lambda_{\ell+1} < \lambda_{\ell+2} < \cdots$ are fixed additional exponents and $N := n + \ell$. There are N free parameters b, μ in (3.22). They are coordinates of a point $u \in \mathbb{R}^N$,

$$u = \{u_j\}_{j=1}^N = \{b_{k,\nu}, \mu_k\} .$$

(We order the $b_{k,\nu}$ lexicographically with respect to k, ν, then let follow the μ_k, $k = 1, \ldots, \ell$.) Thus

$$g(x) = h(u_0; x) ,$$

where u_0 is the point with the coordinates $b_{k,\nu} = a_{k,\nu}$ for $k = 1, \ldots, \ell$, $b_{k,0} = 0$ for $k > \ell$ and with $\mu_k = \lambda_k$.

We can prove the solvability of the system (2.1) even with the restriction that g_1 should be a special function h of the form (3.22). These equations are here

$$h(u; x_i) = c_i \quad , \qquad i = 1, \ldots, N , \tag{3.23}$$

with unknowns u, given $c = \{c_i\}$ and fixed x_i.

The point u_0 is an interior point of the open set $G \subset \mathbb{R}^N$ given by

$$G := \left\{ u \in \mathbb{R}^N : \operatorname{sign} b_{k,m_k} = \operatorname{sign} a_{k,m_k} , \; k = 1, \ldots, \ell , \; \mu_1 < \cdots < \mu_\ell < \lambda_{\ell+1} \right\} .$$

We compute the Jacobian determinant J of the system (3.23). Its i-th row is $\{\partial h(u, x_i)/\partial u_j\}_{j=1}^N$, that is, it is

$$\Big\{ e^{\mu_1 x}, \ldots, x^{m_1} e^{\mu_1 x}; \cdots ; e^{\mu_\ell x}, \ldots, x^{m_\ell} e^{\mu_\ell x}; e^{\lambda_{\ell+1} x}, \ldots, e^{\lambda_N x};$$
$$\sum_{\nu=0}^{m_1} b_{1,\nu} x^{\nu+1} e^{\mu_1 x}, \ldots, \sum_{\nu=0}^{m_\ell} b_{\ell,\nu} x^{\nu+1} e^{\mu_\ell x} \Big\} \text{ for } x = x_i .$$

Without changing the value of J, the last sums can be replaced by single terms

$$b_{1,m_1}x^{m_1+1}e^{\mu_1 x}, \ldots, b_{\ell,m_\ell}x^{m_\ell+1}e^{\mu_1 x} .$$

Consequently, $J = (\prod b_{k,m_k})D \neq 0$, where D is the determinant for a Haar system formed by $\sum_k (m_k + 2) = k(g) + \ell(g)$ functions of type $x^\nu e^{\mu x}$. In a neighborhood of the point u_0, we have $u \in G$ and the system (3.23) is uniquely solvable, by the inverse function theorem, with a solution u that is continuous with respect to c. Thus, $\mathcal{E}_n^*$ has at g the property (V), and $N_v(g) \geq N$.

(b$_2$). If $g \in \mathcal{E}_n$, then again all $h \in \mathcal{E}_n$, and we have the same conclusion. □

From this we derive

Theorem 3.7. *The families $\mathcal{E}_n$ are varisolvent and have property (D).*

After this, the alternation and the uniqueness theorems of §2 can be applied to the $\mathcal{E}_n$-approximation. In particular, the families $\mathcal{E}_n$ are strict suns. If the best approximation g to $f \in C[a,b]$ from $\mathcal{E}_n^*$ happens to belong to $\mathcal{E}_n$, then $f - g$ must have an alternation of $N(g, \mathcal{E}_n) + 1 = n + \ell(g) + 1$ points.

The family $\mathcal{E}_n^*$ does not have the uniqueness property:

Example. Let $f \in C[-1,1]$ be even, increasing on $[-1,0]$, with $f(\pm 1) = 0$, $f(0) = 1$. Then f has at least two best approximations from $\mathcal{E}_2^*$.

If $g(x) \in \mathcal{E}_2^*$ is one of them, then also $g(-x)$ is a best approximation. We must show that $g(x) = g(-x)$, $x \in [-1,1]$, $g \in \mathcal{E}_2^*$ leads to a contradiction. But then $g(x) = A(e^{\lambda x} + e^{-\lambda x})$, $A > 0$; $f - g$ can have at most 3 alternations, while $N(g, \mathcal{E}_2) + 1 = 2 + 2 + 1 = 5$.

Theorem 3.8. *The families $\mathcal{E}_n^*$, $n \geq 2$ are not varisolvent. More exactly, for some $g \in \mathcal{E}_n^*$, $N_z(g) = 2n$, $N_v(g) < 2n$ in $\mathcal{E}_n^*$.*

Proof. We can assume that $[a,b] = [-1,1]$. For the function g we can take a polynomial $g \in \mathcal{P}_{n-1}$ with $n-1$ distinct zeros in $(-1,1)$,

$$g(x) = \prod_{k=1}^{n-1} (x - \alpha_k) . \tag{3.24}$$

If $a_1 e^{\lambda_1 x}$ is a zoom function (3.6) with sufficiently large λ_1, then $g(x) - a_1 e^{\lambda_1 x}$ will have $n-1$ zeros close to the α_k, and an additional zero close to 1, altogether n distinct zeros in $(-1,1)$. Repeating this process, we obtain $g_1(x) = \sum_{k=1}^n a_k e^{\lambda_k x} \in \mathcal{E}_n^*$ so that $g - g_1 \neq 0$ has $2n-1$ zeros in $(-1,1)$. Thus the number $N_z(g)$ in $\mathcal{E}_n^*$ is $\geq 2n$, and from (3.19), $N_z(g) = 2n$.

We have now to show that $N_v(g) = 2n$ leads to a contradiction.

We take an arbitrary $\varepsilon > 0$ and $2n$ points $-1 = x_1 < \cdots < x_{2n} = 1$. For all sufficiently small $\delta > 0$, the equations

$$g_1(x_j) = g(x_j) + (-1)^{j-1}\delta , \qquad j = 1, \ldots, 2n \tag{3.25}$$

are solvable with some $g_1 \in \mathcal{E}_n^*$, $\|g - g_1\| < \varepsilon$. We assume that $0 < \varepsilon < g(1)$.

The difference $g - g_1 \neq 0$ has $2n-1$ zeros on $(-1,1)$. This implies that $g_1 \notin \mathcal{P}_n$ and has, therefore, a term with an exponential factor. It follows that $\lim_{x\to+\infty} g_1(x) = -\infty$ or $+\infty$. In the first case, since $g_1(1) > g(1) - \varepsilon > 0$, g_1 has, in addition to $n-1$ zeros close to the α_k, a zero in $(1,\infty)$, altogether n zeros, which is impossible. In the second case, there is an additional zero of $g - g_1$ to the right of 1. In some interval $[-A, A]$, $g - g_1$ has at least $2n$ zeros, and this is also impossible. □

The non-uniqueness of $\mathcal{E}_n^*$-approximation and other facts require the consideration of *local best approximations*. It has been shown by Braess that a function $f \in C[-1,1]$ can have at most $n!$ local best approximations, and Verführt [1982] has reduced this number somewhat.

§ 4. Lower Bounds for Errors of Nonlinear Approximation

The main idea of this section is that for such classes $\mathcal{F} \subset C(A)$ of functions as balls in Sobolev or Lipschitz spaces, even very sophisticated means of uniform for approximation (if they contain a comparable number of free parameters) do not produce better approximation errors than ordinary polynomials or splines. We treat mainly general rational approximation. In some sense, our theorems estimate the "nonlinear widths" of $\mathcal{F}$; they resemble those of §3 of Chapter 14.

The first results by Vitushkin [A-1961] were obtained by means of his theory of "variations of sets" in euclidean spaces. (For a good exposition of this theory see the book of Ivanov [B-1975].) Simple approaches were found later by Lorentz [1960], [1966], Shapiro [1964], Warren [1968].

For functions $f \in C(A)$, where A is a compact metric space, we consider, as an approximation tool, a function $V(x,t)$, $x \in A$, $t := (t_1, \dots, t_n) \in \mathbb{R}^n$, continuous on $A \times \mathbb{R}^n$. We seek to estimate from below the approximation error

$$(4.1) \qquad E_V(\mathcal{F}) := \sup_{f\in\mathcal{F}} E_V(f) \ , \quad E_V(f) := \inf_t \|f(\cdot) - V(\cdot, t)\|_\infty \ .$$

As an example, we can take $V = L$, $L(x,t) = \sum_{j=1}^n t_j\phi_j(x)$, where ϕ_j are some fixed functions from $C(A)$. Polynomials belong here, also splines with fixed knots $a = y_0 < y_1 < \cdots < y_n = b$. If $I_i = (y_{i-1}, y_i)$, $i = 1, \dots, n$, splines of order r and with no continuity restrictions at the knots are given on $A = [a,b]$ by the linear scheme

$$(4.2) \qquad S(x,t) = \sum_{i=1}^{n} \sum_{k=0}^{r-1} t_{i,k}\chi_{I_i}(x)x^k \ ,$$

of length nr. Here, χ_I is the characteristic function of the set I. (In this case the continuity condition on V must be relaxed by omitting a nowhere dense set of exceptional y from A.)

We also consider rational schemes $R(x,t) = L_1(x,t)/L_2(x,t)$, with linear L_1, L_2.

The class $\mathcal{F}$ (with $0 \in \mathcal{F}$) we shall characterize by its *oscillation amplitude* $\Omega_m(\mathcal{F})$ of $\mathcal{F}$, for $m = 1, 2, \ldots$. This is the supremum of all $\varepsilon \geq 0$ with this property: There exist m points $x_1, \ldots, x_m \in A$ such that for every sign vector $\theta := (\theta_1, \ldots, \theta_m)$, $\theta_i = \pm 1$, there is an $f_\theta \in \mathcal{F}$ for which

$$|f_\theta(x_i)| \geq \varepsilon \ , \quad \operatorname{sign} f(x_i) = \theta_i \ , \qquad i = 1, \ldots, m \ . \tag{4.3}$$

Clearly, $\Omega_m(\mathcal{F})$ is a decreasing function of m.

To characterize V, we consider partitions of the space $\mathbb{R}^n$. For points $x_1, \ldots, x_m$ as above (different from the exceptional y, if there are any) the sets $V(x_i, t) = 0$, $i = 1, \ldots, m$ are closed sets in $\mathbb{R}^n$, and the complement G of their union is open. Let $\sigma(V; x_1, \ldots, x_m)$ be the number of (connected) components of G, we define the *component number* $\sigma_m(V)$ of V by

$$\sigma_m(V) := \sup_{x_1, \ldots, x_m} \sigma(V; x_1, \ldots, x_m) \ . \tag{4.4}$$

For some classes of V with fixed n, we can find the *universal bound* $\gamma(m,n)$, which depends only on m and n:

$$\gamma(m,n) = \sup_V \sigma_m(V) \ . \tag{4.5}$$

Theorem 4.1. *For each* $\mathcal{F} \subset C(A)$,

$$E_V(\mathcal{F}) \geq \sup_m \{\Omega_m(\mathcal{F}) : \sigma_m(V) < 2^m\} \geq \sup\{\Omega_m(\mathcal{F}) : \gamma(m,n) < 2^m\} \ . \tag{4.6}$$

Proof. Let $\sigma_m(V) < 2^m$, let (4.3) be valid for some $\varepsilon > 0$ and the points $x_1, \ldots, x_m$. On the components C of the set G, all functions $V(x_j, t)$ are $\neq 0$. They produce $< 2^m$ sign vectors $\operatorname{sign} V(x_j, t)$, $j = 1, \ldots, m$, $t \in C$. There is a $\theta = (\theta_j)_1^m$, $\theta_j = \pm 1$, different from all of them. For the function f_θ, for each $t \in \mathbb{R}^n$, if $t \in C$, there will exist a $j = 1, \ldots, m$ so that $|f_\theta(x_j) - V(x_j, t)| \geq \varepsilon$ for $t \in C$. Thus, $\|f_\theta - V(\cdot, t)\| \geq \varepsilon$ for each $t \in \mathbb{R}^n$. □

Proposition 4.2 (Shapiro [1964]). *The uniform upper bound for all linear schemes* $L(x,t) = \sum_{i=1}^n t_i \phi_i(x)$, *of length* n *satisfies*

$$\gamma_L(m,n) \leq (4em/n)^n \ . \tag{4.7}$$

Proof. Equations $L(x_j, t) = 0$ define hyperplanes in $\mathbb{R}^n$ (passing through the origin), or the whole space $\mathbb{R}^n$. Hence $\gamma_L(m,n)$ does not exceed the largest number $\gamma^0(m,n)$ of components, into which m hyperplanes can split $\mathbb{R}^n$. First of all, we prove

$$\gamma^0(m,n) \leq \gamma^0(m-1,n) + \gamma^0(m-1,n-1) \ , \qquad n \geq 2 \ , \ m \geq 1 \ . \tag{4.8}$$

Indeed, consider a partition of $\mathbb{R}^n$ by some hyperplanes $\pi_1, \dots, \pi_m$. The hyperplanes $\pi_1, \dots, \pi_{m-1}$ divide $\mathbb{R}^n$ into at most $\gamma^0(m-1, n)$ (connected) components. The number of new components created by adding π_m equals the number of existing components intersected by π_m, which number is, in its turn, equal to the number of components in which $\pi_1, \dots, \pi_{m-1}$ partition π_m, so that it cannot exceed $\gamma^0(m-1, n-1)$. This establishes (4.8).

Next, we show, by induction on m, that

$$\gamma^0(m,n) \le 2^{n+1} \sum_{k=0}^{n} 2^k \binom{m}{k} , \qquad n \ge 2 , \ m \ge 1 . \tag{4.9}$$

For this purpose, we use

$$\binom{m-1}{n} + \binom{m-1}{n-1} = \binom{m}{n} , \qquad m, n \ge 1 . \tag{4.10}$$

Assuming that (4.9) is valid with m replaced by $m-1$, we derive (4.9), because

$$\begin{aligned} &\gamma^0(m-1, n) + \gamma^0(m-1, n-1) \\ &\qquad = 2^{n+1} \sum_{k=0}^{n} 2^k \binom{m-1}{k} + 2^n \sum_{k=1}^{n} 2^{k-1} \binom{m-1}{k-1} \\ &\qquad = 2^{n+1} \left\{ \binom{m-1}{0} + \sum_{k=1}^{n} 2^k \left(\binom{m-1}{k} + \frac{1}{4} \binom{m-1}{k-1} \right) \right\} . \end{aligned}$$

From (4.9), since $4/n! > (e/n)^n$, $n \ge 3$, we have, if $m \ge n \ge 3$,

$$\gamma^0(m,n) < 2^{n+1} \sum_{k=0}^{n} 2^k \frac{m^k}{k!} < 4(4m)^n/n! < (4em/n)^n . \qquad \square$$

Proposition 4.3. *For $R(x,t) = L_1(x,t)/L_2(x,t)$, with linear schemes L_i of lengths k_i, $i = 1, 2$ and $N = k_1 + k_2$, one has*

$$\gamma_R(m,n) \le (30m/N)^N . \tag{4.11}$$

Proof. We have

$$\gamma_R(m,n) \le \frac{1}{2} \gamma_{L_1}(m, k_1) \gamma_{L_2}(m, k_2) \le \left(4e\frac{m}{k_1}\right)^{k_1} \left(4e\frac{m}{k_2}\right)^{k_2} .$$

The largest value of $k_1^{-k_1} k_2^{-k_2}$ for $1 \le k_1 < N$ is attained for $k_1 = 1$; it is equal to $(N-1)^{-(N-1)} < eN^{-N+1}$. Since $(eN)^{1/N}$ decreases for $N \ge 1$ and does not exceed e, we deduce $\gamma_R(m,n) \le (4e^2 m/N)^N$. $\square$

We can compare $\gamma_R(m,n)$ of (4.11) with 2^m:

(4.12) $\gamma_R(m,n) < 2^m$ *if* $m = \lambda N$ *and* λ *is sufficiently large.*

Indeed, $\gamma_R(m,n) \le [(30\lambda)^{1/\lambda}]^m$, and $(30\lambda)^{1/\lambda} \to 1$ for $\lambda \to \infty$.

As a concrete example, let $\hat{B}^\alpha_\infty$ be the unit ball of the space $\mathrm{Lip}\,\alpha$, $\alpha > 0$ on $[a, b]$. One sees easily (see also §3, Chapter 14) that $\Omega_m(\hat{B}^\alpha_\infty) \geq Cm^{-\alpha}$. Using Theorem 4.3 and (4.12),

$$E_R(\hat{B}^\alpha_\infty) \geq CN^{-\alpha}, \qquad N = k_1 + k_2 \tag{4.13}$$

for all rational schemes, for instance, for rational approximation by polynomials or splines. This is another form of Theorem 7.5 of Chapter 7. Similarly for classes $\mathrm{Lip}(\alpha, 1)$ on s-dimensional parallelepipeds I_s in $\mathbb{R}^s$. In this case $\Omega_m(\hat{B}^\alpha_\infty) \geq Cm^{-(\alpha/s)}$ (see Lorentz [A-1966, p.136]) and we obtain $E_R(\hat{B}^\alpha_\infty) \geq CN^{-(\alpha/s)}$.

The original results of Vitushkin are much more ambitious. He considered schemes

$$V(x,t) = \frac{P(x,t)}{Q(x,t)}, \tag{4.14}$$

where $x \in I_s$ and P, Q are polynomials in $t = (t_1, \ldots, t_n)$ of degrees $\leq k$. He obtained:

Theorem 4.4. *For each scheme* (4.14), *and* $f \in C^r(I_s)$, $\|f\|_{C^r} \leq 1$,

$$E_V(f) \geq \frac{C}{(n \log(k+1))^{r/s}}. \tag{4.15}$$

(For fixed k this follows from our theorems.) See Vitushkin [A-1961], Ivanov [B-1975, p.322]. There are even much more general theorems involving a "barrier," a notion that we cannot discuss here. According to Ivanov, there exist two different approaches to these general theorems: one of them uses the theory of variations of sets (Ivanov [B-1975]). The other is based on the elementary ideas of this section and the lemma:

Lemma 4.5 (Warren [1968]). *For any polynomials P_j, $j = 1, \ldots, m$ of degrees not exceeding k of n variables $x_1, \ldots, x_n$, the surface $P_1 \ldots P_m = 0$ splits $\mathbb{R}^n$ into at most $(4ekm/n)^n$ connected components.*

Using variations of sets, Ivanov [B-1975, p.324] proves also a variation of Theorem 4.4 in the L_1-metric.

§ 5. Continuous Selections from Metric Projections

Let G be a subset of a linear normed space X. We return to the properties of the metric projection $\Pi_G(f)$ of §1. We shall assume in this section that G is *proximinal*, that is, that $\Pi_G(f)$ is never empty. If G is a subspace of X, then $\Pi_G(f)$ is convex and closed in G.

Sometimes one succeeds in finding a continuous function $\phi(f)$ from X into G with the property

$$\phi(f) \in \Pi_G(f) \quad \text{for all} \quad f \in X \ . \tag{5.1}$$

This is a *continuous selection from* $\Pi_G(f)$.

Originally it was hoped to construct approximation algorithms using selections, and their continuity was needed to ensure the stability of the algorithms. Unfortunately it became clear that, unless G has the uniqueness property (when Π_G becomes the operator of best approximation $\Pi_G(f) = g \in G$), the existence of continuous selections in concrete spaces, such as $C(A)$ or $L_1(A)$ is a rare phenomenon. We shall prove here a few general theorems. For detailed study of continuous selections in the case when $X = C(A)$, see the review article of Deutsch [1983], the books of Singer [A-1974], Nürnberger [A-1989], and Li [A-1991], and for the case $X = L_1$, the book of Pinkus [A-1989].

Example 1 below will show that ϕ does not need to exist even in simple cases.

Continuity properties of set-valued maps $\Pi : X \to K(G)$ will be described in the following general setting. Let X be a metric space, G a finite dimensional normed space, let $K(G)$ be the collection of all nonempty compact subsets of G.

1. The map Π is called *upper semicontinuous* (u.s.c.) at $f_0 \in X$ if for each sequence $f_n \to f_0$, relations $g_n \in \Pi(f_n)$ and $g_n \to g_0$ imply that $g_0 \in \Pi(f_0)$.

2. The map Π is *lower semicontinuous* (l.s.c.) at $f_0 \in X$ if $f_n \in X$, $f_n \to f_0$ and $g_0 \in \Pi(f_0)$ imply the existence of a sequence $g_n \in \Pi(f_n)$ with $g_n \to g_0$.

3. The map Π is continuous at f_0 if it is both u.s.c. and l.s.c. at f_0. Necessary and sufficient for this is that $f_n \to f_0$ implies $\Pi(f_n) \to \Pi(f)$ in the Hausdorff distance of $K(G)$. The Hausdorff distance between two bounded sets A, B in a metric space X is defined by

$$\rho(A, B) := \inf\{\varepsilon > 0 : A \subset U_\varepsilon(B) \ , \ B \subset U_\varepsilon(A)\} \ . \tag{5.2}$$

An example of an u.s.c. map is given by

Theorem 5.1. *The metric projection from a normed linear space X onto its finite dimensional subspace G is u.s.c.*

Proof. Let $f_n \in X$, $f_n \to f_0$, $g_n \in \Pi_G(f_n)$, $g_n \to g_0$. We have, for $n = 1, 2, \ldots$

$$\begin{aligned} \rho(f_0, G) &\le \|f_0 - g_n\| \le \|f_0 - f_n\| + \|f_n - g_n\| \\ &= \|f_0 - f_n\| + \rho(f_n, G) \le 2\|f_0 - f_n\| + \rho(f_0, G) \ . \end{aligned}$$

Since $\|f_0 - f_n\| \to 0$, this implies that $\|f_0 - g_n\| \to \rho(f_0, G)$, and $\|f_0 - g_0\| = \rho(f_0, G)$. In other words, $g_0 \in \Pi_G(f_0)$. □

In the general case of a set-valued map Π from a metric space X into the set of all *convex compact* subsets of a finite dimensional linear space G, Berens and Nürnberger [1990] have suggested a simple rule to construct a selection. Since all finite dimensional linear spaces are isomorphic, we can define on G a strictly convex norm, $\|\cdot\|^*$. We fix an element $h \in G$.

Now, let $f \in X$ be given. Since $\Pi(f)$ is compact and convex, there is a unique element $\phi_h(f) \in \Pi(f)$ of best approximation to h:

$$\|h - \phi_h(f)\|^* = \min_{g \in \Pi(f)} \|h - g\|^* . \tag{5.3}$$

This defines a selection $\phi_h(f) \in \Pi(f)$. We shall discuss its continuity properties.

Theorem 5.2 (Berens and Nürnberger [1990]). *Let X be a metric space, let G be a finite-dimensional normed space, let $h \in G$ be fixed. If the set-valued mapping Π of X into convex and compact subsets of G is both upper and lower semicontinuous at $f_0 \in X$, then the selection ϕ_h defined by* (5.3) *is continuous at f_0.*

Proof. We suppose the contrary: for some sequence $f_n \in X$, $f_n \to f_0$, $\phi_h(f_n) \not\to \phi_h(f_0)$. Since Π is l.s.c. at f_0, there exists a sequence $g_n \in \Pi(f_n)$ for which $g_n \to \phi_h(f_0)$. From the definition of ϕ_h,

$$\|\phi_h(f_n)\|^* - \|h\|^* \le \|h - \phi_h(f_n)\|^* \le \|h - g_n\|^* \le \|h\|^* + \|g_n\|^* . \tag{5.4}$$

We see that both $\|g_n\|^*$ and $\|\phi_h(f_n)\|^*$ are bounded.

Passing to a subsequence, we can assume that $\phi_h(f_n)$ converges: $\phi_h(f_n) \to g_0$, and have $g_0 \ne \phi_h(f_0)$. We use the upper semicontinuity of Π and deduce that $g_0 \in \Pi(f_0)$.

We now make $n \to \infty$ in the inequality

$$\|h - \phi_h(f_n)\|^* \le \|h - g_n\|^* ,$$

and obtain $\|h - g_0\|^* \le \|h - \phi_h(f_0)\|^*$. This is, however, a contradiction, since $g_0 \in \Pi(f_0)$, and $\phi_h(f_0)$ is the unique element of best approximation to f_0 from $\Pi(f_0)$. □

In particular, Theorem 5.2 implies one of the basic theorems about Π_G:

Theorem 5.3 (Michael [1956]). *A metric projection Π_G onto a finite dimensional subspace G that is continuous has a continuous selection.*

Metric projections without lower semicontinuity may have and may lack a continuous selection. This is exhibited in the following examples by A.L. Brown.

The following two spaces are $\mathbb{R}^3$ endowed with a norm defined geometrically by means of a closed, convex and symmetric neighborhoods V, V_1 of the origin, while G is the x_3-axis.

Example 1. We take the two half disks

$$C_+ := \{(x_1, x_2, 1) \in \mathbb{R}^3 : x_1^2 + x_2^2 \le 1 \ , \ x_1 \ge 0\} \ ,$$
$$C_- := \{(x_1, x_2, -1) \in \mathbb{R}^3 : x_1^2 + x_2^2 \le 1 \ , \ x_1 \le 0\} \ ,$$

and let V be the convex hull of C_+ and C_-.

Example 2. We take the two intervals

$$I_\pm := \{(x_1, 0, \pm 1) \in \mathbb{R}^3 : |x_1| \le 1\}$$

and the disk

$$D := \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1\}$$

and let the unit ball V_1 be their convex hull.

Let U be V or V_1. For $z = (x_1, x_2, x_3) \neq 0$, we can find a best approximation as follows. Let $a > 0$ be the unique value so that $z \in a\partial U$; then $0 \in \Pi_G z$ and $\|z - 0\| = a$. This is the unique best approximation to z unless z lies on a vertical interval of $a\partial U$. If $[z - \lambda_2 e_3, z + \lambda_1 e_3]$ is the maximal such interval, then $\Pi_G z = [-\lambda_2 e_3, \lambda_1 e_3] \subset G$.

The only vertical intervals of ∂V and of ∂V_1 are $J_\pm := \{(\pm 1, 0, x_3) : -1 \le x_3 \le 1\}$. We have proved that

$$\Pi_G z = \Pi_G(x_1, x_2, x_3) = (0, 0, x_3) \ \text{ if } \ z \notin J_\pm \ ,$$
$$\Pi_G z = \Pi_G(\pm 1, 0, x_3) = \{(\pm 1, 0, y) : x_3 - 1 \le y \le x_3 + 1\} \ \text{ if } \ z \in J_+ \ \text{ or } \ J_1 \ .$$

From this it easily follows that: (a) $\Pi_G z$ is not l.s.c. in both cases; (b) there is no continuous selection from Π_G in the first example; (c) $\phi(z) = (0, 0, x_3)$ is a continuous and linear selection in the second.

Examples of this kind are not possible in $\mathbb{R}^2$.

We finish with some remarkable theorems for special spaces.

Theorem 5.4 (Lazar, Wulbert, Morris [1969]). *For a nonatomic, σ-additive measure on A, there exists no continuous selection for any finite dimensional subspace G of $L_1(A, \mu)$.*

As a consequence, no G of this type can have the uniqueness property. For otherwise Π_G, interpreted as a map $L_1(A, \mu) \to G$, continuous according to [**CA**, Theorem 1.2, p.60], will provide itself a continuous selection. This is another proof of Krein's [**CA**, Theorem 10.7, p.85] for arbitrary A.

For the Schoenberg space $S_n^r[a, b]$ of splines of order r with n fixed knots, Nürnberger and Sommer [1978] have proved:

Theorem 5.5. *The subspace $S_n^r[a, b]$ of $C[a, b]$ has a continuous selection for the metric projection if and only if $n \le r + 1$.*

For the proof, see also Nürnberger [A-1989].

§ 6. Approximation in Banach Spaces: Suns and Chebyshev Sets

We would like in this section to give the reader at least a glimpse of approximation of elements of a linear normed space X from a nonlinear subset S of X. The most important notions are here those of a metric projection Π_S, of a Chebyshev set and of a sun (see §1). The exact relations among the similar notions of convex sets, of Chebyshev sets and of suns in a Banach space X are not known. In the euclidean space $X = \mathbb{R}^n$ all these notions coincide. For more general spaces, many questions remain open; undecided is, for example, Klee's question, whether in a Hilbert space, each Chebyshev set is convex.

For a set S in a linear normed space X and $f \notin S$, $g \in S$, let $g \in \Pi_S f$. We define $f_\lambda := g + \lambda(f - g)$, $\lambda \geq 0$. One has then $g \in \Pi_S f_\lambda$, $0 \leq \lambda \leq 1$ (see §1). Moreover, if $g \in \Pi_S f_\lambda$ is not true for all $\lambda \geq 0$, then for some $\lambda_0 \geq 1$,

$$g \in \Pi_S f_\lambda \ \text{ for } \ 0 \leq \lambda \leq \lambda_0 \,, \quad g \notin \Pi_S f_\lambda \ \text{ for } \ \lambda > \lambda_0 \,. \tag{6.1}$$

A point $f \notin S$ has the *solar property* if for *some* $g \in \Pi_S f$, also $g \in \Pi_S f_\lambda$ for all $\lambda \geq 0$; S is a *sun*, if this is true for all $f \notin S$ (and a *strict sun*, if this holds for *each* $g \in \Pi_S f$).

The following is a partial inverse of Theorem 1.2:

Theorem 6.1. *Each proximinal sun S in a smooth linear normed space X is convex.*

We call the space X *smooth* if

$$\begin{cases} \text{For each } f \in X,\ f \neq 0, \text{ there is only one linear functional } \ell \\ \text{in the dual space } X^* \text{ for which } \|\ell\| = 1,\ \ell(f) = \|f\| \,. \end{cases} \tag{6.2}$$

(The existence of ℓ follows from the Hahn-Banach theorem.)

Proof. If the theorem is not true, there exist $g_1, g_2 \in S$ and an α, $0 < \alpha < 1$ for which $f := \alpha g_1 + (1-\alpha) g_2 \notin S$. Let g_0 be a best approximation to f with the solar property. Then g_0 is a best approximation also to all $f_\lambda = g_0 + \lambda(f - g_0)$, $\lambda > 0$. Since $g_1 \in S$,

$$d := \|f - g_0\| \leq \lambda^{-1} \|f_\lambda - g_1\| = \|f - ((1 - \lambda^{-1}) g_0 + \lambda^{-1} g_1)\| \,.$$

This means in particular that the segment $[g_0, g_1]$ is outside of the open ball $U_d(f)$. By the Hahn-Banach theorem, a hyperplane separates $U_d(f)$ and this segment; the same applies to $[g_0, g_2]$. We obtain the existence of two linear functionals ℓ_1, ℓ_2 such that $\|\ell_i\| = 1$, $\ell_i(f - g_0) = \|f - g_0\|$, $\ell_i(g_i - g_0) \leq 0$, $i = 1, 2$. Since X is smooth, $\ell_1 = \ell_2$ and we obtain a contradiction

$$\ell_1(f - g_0) = \alpha \ell_1(g_1 - g_0) + (1 - \alpha) \ell_1(g_2 - g_0) \leq 0 \,. \tag{6.3}$$

□

Remark. The space $L_1[a,b]$ with the Lebesgue measure is not smooth. However, the statement (6.2) is true for any function $f \in L_1[a,b]$, $f \neq 0$, that does not vanish on a set of positive measure. This shows that the proof of Theorem 6.1 remains valid for $X = L_1[a,b]$ if the linear hull of S does not contain nontrivial functions that vanish on a set of measure different from zero. *In this case, again, the sun S is convex.*

In §2 we have seen that the sets of rational functions $\mathcal{R}_{m,n}[a,b]$ are suns in $C[a,b]$. By abstract arguments, we can show that this is not true in spaces L_p, $p \neq \infty$.

Proposition 6.2 (Efimov and Stechkin [1961]). *The sets $\mathcal{R}_{m,n}[a,b]$, $n > 0$ are not suns in $L_p[a,b]$, $1 \le p < \infty$.*

Proof. For $1 < p < \infty$, this follows from Theorem 6.1, because the spaces $\mathcal{R}_{m,n}$ are proximinal and not convex, and the spaces L_p, $1 < p < \infty$ are smooth. For $p = 1$, we use the Remark. □

The next theorem, due to Vlasov [1961], depends upon the Schauder fixed-point theorem, which asserts the following. *Let B be a closed, bounded and convex set in a real Banach space. Then each continuous map G of B into itself for which $G(B)$ is relatively compact, has a fixed point $f^* \in B : G(f^*) = f^*$.* (This is a generalization of Brouwer's fixed point theorem, when $X = \mathbb{R}^n$.)

The notions of boundedly compact and of approximatively compact sets (see §1) are related, but not equivalent. We shall see that the parallel theorems of Vlasov (Theorems 6.3 and 6.5) depend on subtle use of these two notions.

Theorem 6.3 (Vlasov [1961]). (i) *Each boundedly compact Chebyshev set S in a real Banach space is a sun.* (ii) *If, in addition, X is smooth, then S is convex.*

Proof. (i) We assume that S is not a sun. Then for some $f \in X \setminus S$ we have (6.1). Renaming f_{λ_0} by f_0 we get for $f_0 \in X \setminus S$,

$$g_0 := \Pi_S f_0 \quad , \quad g_0 \neq \Pi_S(f_{0,\lambda}) \ \text{ for all } \ \lambda > 1 \ . \tag{6.4}$$

We take $0 < \rho < \|f_0 - g_0\|$; then $B_0 := S \cap \overline{U_\rho(g_0)}$ is a compact neighborhood of g_0 in S. By the continuity of $\Pi := \Pi_S$ (Theorem 1.1), it is possible to find $r > 0$ so that the closed neighborhood $B := \overline{U_r(f_0)}$ of f_0 is disjoint with S and that it is mapped by Π into B_0.

We now define a function G to which Schauder's fixed point theorem can be applied. We define the continuous maps

$$F(g) := f_0 + \frac{r}{\|f_0 - g\|}(f_0 - g) \ , \qquad g \in S \ , \tag{6.5}$$

$$G(f) := F(\Pi f) \ , \qquad f \in B \ . \tag{6.6}$$

The function F maps S into B; G maps B into itself. Since B_0 is compact, $G(B)$ is relatively compact in B. There exists therefore a fixed point $f^* \in B$ with $G(f^*) = f^*$.

Solving (6.5) with respect to f_0, we see that f_0 is an interior point of the segment $[g, F(g)]$ for each $g \in S$. In particular, this applies to the segment $[\Pi f^*, f^*]$. Therefore

$$\begin{aligned}\|f^* - \Pi f^*\| &= \|f^* - f_0\| + \|f_0 - \Pi f^*\| \\ &\geq \|f^* - f_0\| + \|f_0 - \Pi f_0\| \geq \|f^* - \Pi f_0\| \; .\end{aligned}$$

Because of the uniqueness property of S, $g_0 = \Pi f_0 = \Pi f^*$. Applying (6.5) to g_0 we see that

$$f^* = f_0 + \frac{r}{\|f_0 - g_0\|}(f_0 - g_0) = g_0 + \lambda(f_0 - g_0)$$

with some $\lambda > 1$. This contradicts (6.4).

(ii) follows now from Theorem 6.1. □

Without additional assumptions, a Chebyshev set is not necessarily a sun:

Example (Dunham [1975]). *The set S of functions on* $[0,1]$

$$F_a(x) = \begin{cases} (1+a)\exp(-x/a) & , a > 0 \\ 0 & , a = 0 \end{cases} \tag{6.7}$$

is a Chebyshev set in $C[0,1]$, *but not a sun.*

This follows from:

(a) *S is proximinal.* Let $E(f) = \inf_a \|f - F_a\| = \lim_{k\to\infty} \|f - F_{a_k}\|$ for some sequence a_k. Since $\|F_a\| \to \infty$ for $a \to \infty$, the sequence a_k is bounded, and we can assume $a_k \to a_0 \geq 0$. If $a_0 > 0$, then $F_{a_k} \to F_{a_0}$ uniformly, and $E(f) = \|f - F_{a_0}\|$. If $a_0 = 0$, then for each $x > 0$, $F_{a_k}(x) \to 0 = F_0(x)$. Therefore for $x > 0$, $|f(x) - F_0(x)| = \lim_{k\to\infty} |f(x) - F_{a_k}(x)| \leq \lim_{k\to\infty} \|f - F_{a_k}\| = E(f)$. By continuity, $\|f - F_0\| = E(f)$, and F_0 is a best approximation to f.

(b) *S has the uniqueness property.* For $0 \leq a < b < c$, $f - F_a < f - F_b < f - F_c$, hence

$$\|f - F_b\| < \max\{\|f - F_a\|, \|f - F_c\|\}$$

so that F_a, F_c cannot be both best approximations to f.

(c) *S is not a sun* by Theorem 1.2, since F_0 is an isolated point of S. Each function $f := F_a$, $a > 0$ has two local best approximations, F_a and F_0.

Theorem 6.5 below is a variation of Theorem 6.3; the proof, however, is quite different. The following lemma is from Vlasov [1973]. Together with $f_\lambda := g + \lambda(f - g)$ of (6.1), we shall use also the notation $f^\lambda := f + \lambda(f - g)$, $\lambda > 0$, $f \notin S$, $g = \Pi_S(f)$.

Lemma 6.4. *Let S be an approximatively compact Chebyshev set in a linear normed space X. If $f \notin S$, $g = \Pi f$, $f^\lambda = f + \lambda(f - g)$ then*

$$\lim_{\lambda \to 0+} \frac{\rho(S, f^\lambda) - \rho(S, f)}{\|f^\lambda - f\|} = 1 . \tag{6.8}$$

Proof. We begin with a simple geometric fact, valid in any two dimensional subset of X.

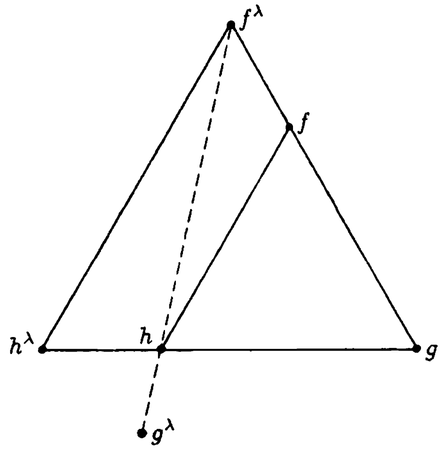


Fig. 6.1. $f^\lambda = F$, $h^\lambda = H$, $h = h(\lambda)$

Let f, h, g be an isosceles triangle in X with $\|f - g\| = \|f - h\|$. If $F = f + \lambda(f - g)$, $\lambda > 0$ is any point on the ray from g through f beyond f, then

$$\frac{\|F - g\| - \|F - h\|}{\|F - f\|} \le \frac{\|h - g\|}{\|f - g\|} . \tag{6.9}$$

Indeed, let H be the auxiliary point $H = h + \lambda(h - g)$ on the ray from g through h. Then the triangles f, h, g and F, H, g are similar, therefore $\|F - g\| = \|F - H\|$ and moreover $\|F - f\| = \lambda\|f - g\|$ and $\|H - h\| = \lambda\|h - g\|$. The left-hand side of (6.9) is

$$= \frac{\|F - H\| - \|F - h\|}{\|F - f\|} \le \frac{\|H - h\|}{\|F - f\|} = \frac{\|h - g\|}{\|f - g\|} .$$

Now let $g^\lambda = \Pi_S f^\lambda$, then $\|f - g^\lambda\| \ge \|f - g\|$. Moreover, for small $\lambda > 0$, $\|f - f^\lambda\| < \|f - g\|$. There exists therefore an $h := h(\lambda) \in [g^\lambda, f^\lambda]$, for which $\|f - g\| = \|f - h\|$. Applying (6.9) for the triangle f, h, g we derive

$$\begin{aligned}
0 &\le 1 - \frac{\rho(S, f^\lambda) - \rho(S, f)}{\|f^\lambda - f\|} = 1 - \frac{\|f^\lambda - g^\lambda\| - \|f - g\|}{\|f^\lambda - f\|} \\
&\le 1 - \frac{\|f^\lambda - h\| - \|f - g\|}{\|f^\lambda - f\|} = \frac{\|f^\lambda - f\| + \|f - g\| - \|f^\lambda - h\|}{\|f^\lambda - f\|} \\
&= \frac{\|f^\lambda - g\| - \|f^\lambda - h\|}{\|f^\lambda - f\|} \le \frac{\|h - g\|}{\|f - g\|} .
\end{aligned} \tag{6.10}$$

It remains to show that $\|h(\lambda) - g\| \to 0$ for $\lambda \to 0+$. By Theorem 1.1, the function Π_S is continuous. This yields $\|g^\lambda - g\| = o(1)$ for $\lambda \to 0+$, and further

$$\begin{aligned}\|h - g\| &= o(1) + \|h - g^\lambda\| = o(1) + \|f^\lambda - g^\lambda\| - \|f^\lambda - h\| \\ &= o(1) + \|f - g\| - \|f - h\| = o(1) . \qquad \square\end{aligned}$$

Theorem 6.5 (Vlasov [1967]). (i) *In a uniformly convex Banach space X each approximatively compact Chebyshev set S is a sun.* (ii) *If, in addition, X is smooth, then S is convex.*

Proof. Only (i) needs to be proved. Let $f \in X \setminus S$, $g = \Pi_S f$, let $r := \|f - g\| > 0$. The open ball $U_r(f)$ in X is disjoint with S. To establish the theorem, it will be sufficient to show that, for $f_2 = g + 2(f - g) = 2f - g$, we have $\rho(S, f_2) = 2r$. Since $\|f_2 - g\| = 2r$, this implies that $\Pi_S f_2 = g$. Replacing f by f_2, r by $2r$ we get $\Pi_S f_4 = g$, then $\Pi_S f_8 = g, \ldots$, which yields $\Pi_S f_\lambda = g$ for all $\lambda > 0$.

Let ε, $0 < \varepsilon < 1$ be fixed. For elements f', f'' of $\overline{U_r(f)}$ we define the partial order $f' \prec f''$ to mean that

$$\rho(S, f') + \varepsilon\|f' - f''\| \le \rho(S, f'') . \tag{6.11}$$

For the subset $M_r(f)$ of $\overline{U_r(f)}$ consisting of all f' satisfying $f \prec f'$ we shall establish the existence of maximal elements by means of the Hausdorff maximal principle (Notes **8.3**). Let (f_α) be a chain in $M_r(f)$, ordered increasingly. For $\beta > \alpha$ we have by (6.10)

$$\varepsilon\|f_\alpha - f_\beta\| \le \rho(S, f_\beta) - \rho(S, f_\alpha) . \tag{6.12}$$

The real numbers $\rho(S, f_\alpha)$ are bounded, therefore there is a cofinal subchain $(f_{\alpha'})$ with converging $\rho(S, f_{\alpha'})$. By (6.12), $f_{\alpha'}$ is a Cauchy chain, it converges $f_{\alpha'} \to f_0$ in X, and f_0 is an upper bound for the chain (f_α).

The Hausdorff maximal principle yields now the existence, for each ε, of a maximal element $F(\varepsilon)$. The fact that $F(\varepsilon)$ is $\succ f$ and that no $F \in M_r(f)$ satisfies $F \succ F(\varepsilon)$, now give, respectively,

$$\rho(S, f) + \varepsilon\|F(\varepsilon) - f\| \le \rho\big(S, F(\varepsilon)\big) \tag{6.13}$$

$$\rho\big(S, F(\varepsilon)\big) + \varepsilon\|F - F(\varepsilon)\| > \rho(S, F) \quad \text{for any} \quad F \in M_r(f) . \tag{6.14}$$

We claim that $F(\varepsilon)$ belongs to the boundary of $\overline{U_r(f)}$. Indeed, otherwise for $F := F(\varepsilon)$ and $G = \Pi_S F$ we would have $F^\lambda = F + \lambda(F - G) \in U_r(f)$ for all sufficiently small $\lambda > 0$, and (6.8) would imply for such λ,

$$\frac{\rho(S, F^\lambda) - \rho(S, F)}{\|F^\lambda - F\|} > \varepsilon ,$$

that is, $F^\lambda \succ F(\varepsilon)$, which is impossible.

From the uniform convexity of X, for elements $f_n \in X$, $f \in X$, with $\|f_n\| = \|f\| = 1$, $n = 1, 2, \ldots$ relation $\|f + f_n\| \to 2$ implies that $f_n \to f$.

We apply this to the $F(\varepsilon)$ for $\varepsilon \to 1$. We have $\|F(\varepsilon) - f\| = \|g - f\| = r$ and

$$\begin{aligned}\|F(\varepsilon) - f + f - g\| = \|F(\varepsilon) - g\| &\geq \rho(S, F(\varepsilon)) \\ &\geq \rho(S, f) + \varepsilon\|F(\varepsilon) - f\| = (1 + \varepsilon)r \to 2r \;.\end{aligned}$$

Thus yields $F(\varepsilon) \to 2f - g = f_2$ for $\varepsilon \to 1$. From (6.12) we now obtain $2r \leq \rho(S, f_2)$, and from (6.13), $2r \geq \rho(S, f_2)$. □

This proof is closely related to *Ekeland variational principle* in nonconvex optimization theory, see Notes **8.4**.

As an example, we derive by abstract means that approximation from $\mathcal{R}_{m,n}$, $n > 0$ in L_p, $1 < p < \infty$ lacks uniqueness. These spaces are smooth and uniformly convex, the sets $\mathcal{R}_{m,n}$, $n > 0$ are proximinal and not convex. By Theorem 6.5(i), they cannot be Chebyshev sets.

We finally give a simple case when all problems of this section have simple answers.

Theorem 6.6. *In a smooth, strictly convex n-dimensional Banach space X the notions of suns, strict suns, Chebyshev sets, convex sets are all equivalent for closed sets S.*

Proof. Theorems 1.3 and 6.1 show that S is a sun (a strict sun) if and only if it is convex. Theorem 6.3 yields that a closed Chebyshev set is a sun. Finally, for a closed convex set S, the proof in [**CA**, Theorem 1.2, p.60] applies and shows that each $f \in X$ has a unique best approximation from S, so that S is a Chebyshev set. □

This theorem, which was proved before Vlasov's theorems, is due essentially to Klee [1953, p.41], who proved that in a finite dimensional linear normed space X a Chebyshev set is a sun. (Earlier, Bunt [1934] established that a Chebyshev set in $\mathbb{R}^n$ is convex.) For a review of literature about suns see Vlasov [1973].

§ 7. Problems

7.1. In the two-dimensional space l_1^2, the curve $S = \{x = (x_1, x_2) : x_2 = x_1^3\}$ is not a sun, and not a set of unicity. But in l_∞^2, S is a sun and a Chebyshev set.

7.2. Each open set is a sun. Prove that there exists non-connected suns by considering the set $S \setminus (0,0)$ of Problem 1 in l_1^2.

7.3. Let X be a real linear normed space, $G \subset X$ and T a linear map of X onto itself with the properties $T(G) = G$ and $\|Tf\| = \|f\|$ for all $f \in X$. Then $g \in G$ is best approximation of f from G whenever $T(g)$ is best approximation of $T(f)$ from G. (Meinardus).

7.4. By means of functions $C_\lambda(x+\lambda)^{-1}$, $\lambda \to 0+$, prove that $\mathcal{R}_{m,n}$, $n > 0$ is not boundedly compact in $L_p[0,1]$, $1 \le p < \infty$ and in $C[0,1]$.
7.5. For $1 \le m \le n$, prove that the maximal number of connected sets into which m hyperplanes through the origin can divide $\mathbb{R}^n$ is exactly 2^m.

§ 8. Notes

8.1. Barrar and Loeb [1970] showed that at each normal function $f_0 \in \mathcal{N}_{m,n}$ (see §8 of Chapter 7), the operator $\Pi := \Pi_{\mathcal{R}_{m,n}}$ is strongly unique: For some $\gamma > 0$, for $R_0 = \Pi f_0$ and all $R \in \mathcal{R}_{m,n}$, one has

$$\|f - R\|_\infty \ge \|f - R_0\|_\infty + \gamma\|R - R_0\|_\infty .$$

8.2. Theorem 6.1 gives a simple sufficient condition which insures convexity of a Chebyshev set. Necessary and sufficient conditions for this are not known. However, for a finite dimensional real Banach space X, Tsarkov [1989] proves that each bounded Chebyshev set A is convex if and only if the extreme points of the unit sphere S^* in the conjugate space X are dense on S^*. At present, this seems to be the only result of this type where boundedness of A is an essential assumption.

8.3. The *Hausdorff maximal principle*, used in the proof of Theorem 6.5, is a general method to prove the existence of maximal elements in a partially ordered set. Let A be nonempty and partially ordered by a transitive and antisymmetric relation $\prec$. A subset $B \subset A$ is called a *chain* if any two elements a, b of B are comparable (either $a \prec b$ or $b \prec a$ must hold). The principle asserts that if each chain B has an *upper bound* in A, then A contains a maximal element a_0, so that no $a_1 \in A$, $a_1 \ne a_0$ satisfies $a_0 \prec a_1$. (See Kelley [B-1955].)

8.4. The *Ekeland principle* (Ekeland [1979], see also Georgiev [1988]) is the following:

Theorem. *Let M be a complete metric space with the distance function ρ. Let Φ be a lower semi-continuous real function on M that is bounded from below, let $\varepsilon > 0$. Then for each $u \in M$ there exists a $v \in M$ with the properties*

(i) $\Phi(v) + \varepsilon\rho(u,v) \le \Phi(u)$

(ii) $\Phi(v) - \varepsilon\rho(w,v) < \Phi(w)$ *for all* $w \ne v$.

Arguments of the proof of Theorem 6.5 can be used to establish this principle. On the other hand, one obtains (6.13) and (6.14) from this principle by taking $M = \overline{U_r(f)}$, $\rho(f', f'') = \|f' - f''\|$ and $\Phi(f') = -\rho(S, f')$.

8.5. In some cases, lower estimates of §5 can be proved to occur infinitely often for some $f \in \mathcal{F}$. (See Theorem 7.6 of Chapter 7 for a result of this type.) One can prove the existence of $f \in \operatorname{Lip}\alpha$ on $[0,1]$ for which, for infinitely many n, $\rho_n(f) \ge Cn^{-\alpha}$ if α is not an integer, and $\rho_n(f) \ge \varepsilon_n n^{-r}$, if $\alpha = r = 1, 2, \ldots$ ($\varepsilon_n > 0$, $\varepsilon_n \to 0$ is an arbitrary sequence).

8.6. In the geometric sense, more natural than the uniform or the L_p-distance between two functions f, g on $[0,1]$ is the *Hausdorff distance between their graphs*, $G(f)$ and $G(g)$. For a continuous f, its graph $G(f)$ is simply the set $D := \{(x,y) : 0 \le x \le 1,\ y = f(x)\}$. For bounded discontinuous functions, $G(f)$ is the smallest closed set containing D whose intersection with any vertical line $x = x_0$, $0 \le x_0 \le 1$ is a point or a closed interval. We define $\rho_h(f,g) := \rho(G(f), G(g))$, where ρ is the Hausdorff distance between two compact sets in the plane. The book of Sendov [A-1979] (see also Chapter 9 of Petrushev and Popov [A-1987]) contains several results about Hausdorff approximation of functions of different classes by polynomials, splines, rational functions. Let $E_n^h(f)$ and $\rho_n^h(f)$ stand for the errors of approximation in the Hausdorff distance of f from $\mathcal{P}_n$, and from $\mathcal{R}_n$, respectively.

1. For each bounded function f on $[0,1]$, one has $E_n^h(f) \le Cn^{-1}\log n$ (Sendov) and $\rho_n^n(f) \le Cn^{-1}\log(e + n\tau(f, n^{-1}))$ (Petrushev [1980]), where C is a constant, and $\tau(f,\delta)$ is the modulus of smoothness of (8.14) of Chapter 2.

2. The following is an analogue of Jackson's theorem:

$$E_n^h(f) \le C\omega(f, n^{-1})\frac{\log(e + n\omega(f, n^{-1}))}{1 + n\omega(f, n^{-1})} \quad \text{for } f \in C[0,1]$$

with the modulus of continuity $\omega(f,\delta)$ (Sendov and Popov).

For a full exposition of this theory, see the book Sendov [A-1990].

8.7. For a general form of Michael's selection theorem see Holmes [B-1975, p.183].

8.8. In [1964] Brown showed that for each subspace G of $X = l_n^p$, $1 \le p \le \infty$, the metric projection Π_G is continuous (that is, u.s.c. and l.s.c.).

8.9. For the space ℓ_n^∞ and its linear subspace G, Rice [1962] defined a continuous selection S_G of Π_G known as the *strict approximation*. It has a simple algorithm. We let $A := \{x_1, \dots, x_n\}$ and interpret ℓ_n^∞ as the space of functions f on A, let G be spanned by functions $g_1, \dots, g_n$, $r < n$ on A. We first determine a best approximation $g = \sum_1^r a_k g_k \in G$, for f, the error $\varepsilon := \|f - g\|_\infty$ and the set A_0 of all $x \in A$ for which

$$(1) \qquad f(x) - g(x) = e(x)\varepsilon\ , \qquad e(x) = \pm 1\ .$$

All other best approximations have the same A_0, and, up to the sign, the same $e(x)$. We next consider the problem of approximating $f(x)$, $x \in A \setminus A_0$, by the linear set spanned by the $g_k \in G$ on $A \setminus A_0$, with the g_k subject to the restriction (1). This is repeated several times until we arrive at a single approximation $S_G(f) := g \in G$. See Descloux [1963], who showed that $S_G(f)$ is the limit of best approximations of f from G in the ℓ_n^p norm, $p \to \infty$; Finzel [1993] proves that $S_G(f)$ is globally Lip 1 continuous.

Chapter 13. Widths I

§ 1. Definitions and Basic Properties

Let K and Q be two subsets of a normed linear space X. The quantity

$$E(K,Q)_X := \sup_{x\in K} \inf_{y\in Q} \|x-y\|$$

is called the *deviation* of K from Q. It shows how well the "worst" elements of K can be approximated by Q. In this chapter Q will be a finite-dimensional subspace of X. One may ask: what subspace X_n of a given dimension n is the best adjusted to K, that is, minimizes the quanity $E(K,X_n)_X$? The number

$$(1.1) \qquad d_n(K,X) := \inf_{X_n} E(K,X_n)_X = \inf_{X_n} \sup_{x\in K} \inf_{y\in X_n} \|x-y\|,$$

where the leftmost infimum is taken over all subspaces $X_n \subset X$ of dimension $\le n$, $n = 0,1,2,\ldots$, is called the *Kolmogorov n-width* of K in X. Instead of $d_n(K,X)$, we shall often write $d_n(K)_X$ or, if $X = L_q$, $d_n(K)_q$. If the infimum is attained by some X_n, then X_n is called an *optimal* subspace.

The best known problem in the theory of widths is the evaluation of the widths of Sobolev classes $B_p^r(A)$, where $A = \mathbb{T}$ or $[a,b]$. By definition (CA, §5, Chapter 2), the Sobolev space $W_p^r(A)$, $r = 1,2,\ldots$, is the collection of functions f for which $f^{(r-1)}$ is absolutely continuous and $f^{(r)} \in L_p(A)$. The *Sobolev class* is the unit ball in the seminorm of the Sobolev space:

$$(1.2) \qquad B_p^r := \{f \in W_p^r : \ \|f^{(r)}\|_p \le 1\}.$$

More general *Lipschitz classes* B_p^α are defined for all $\alpha > 0$. If $\alpha = r+\beta$, where r is an integer, $0 < \beta \le 1$, then $f \in B_p^\alpha$ if and only if $f \in W_p^r$ and $\omega(f^{(r)},t)_p \le t^\beta$ for $t > 0$.

Let l_p^m be the m-dimensional space of vectors $x = (\xi_1,\ldots,\xi_m)$ with the norm

$$\|x\|_p = \Big(|\xi_1|^p + \ldots + |\xi_m|^p\Big)^{1/p}, \ 1 \le p < \infty, \ \ \|x\|_\infty = \max|\xi_k|,$$

and the unit ball $b_p^m := \big\{x \in l_p^m : \|x\|_p \le 1\big\}$. The widths

$$d_n(b_p^m)_q := d_n\left(b_p^m, l_q^m\right), \qquad 1 \le p,\, q \le \infty,$$

are closely related to the widths of the Sobolev and Lipschitz classes. We shall study them extensively in this and the following chapter.

The theory of widths originated with the seminal paper of Kolmogorov [1936], where the widths $d_n(B_2^r)_2$ were determined, both for $A = \mathbb{T}$ and $A = [a, b]$. The subject remained dormant until Tikhomirov's [1960], with exact formulas for the widths $d_n(B_\infty^r)_\infty$; Tikhomirov was the first to recognize the importance of Borsuk's theorem and of Theorem 1.4 for the theory. After this, many Russian mathematicians contributed essentially to the theory of widths, among them Ismagilov, Maiorov and others. Micchelli and Pinkus (see the book of Pinkus [A-1985]) studied widths of classes of functions given by integral operators with totally positive kernels. Optimal subspaces in all these papers are the classic spaces of trigonometric polynomials, of splines, or of their natural generalizations. The important recent paper of Buslaev and Tikhomirov [1990] contains some definitive results on the widths of Sobolev classes B_p^r in L_q for $p \geq q$. For some (p, q) with $p < q$, however, the required approximating tools are non-conventional, and their existence is proved (Kashin, Gluskin) only indirectly, using probabilistic methods.

Consider now approximation by linear operators. Let X and Y be two normed linear spaces and let $L(X, Y)$ be the set of all linear continuous operators from X to Y. We say that an operator $U \in L(X, Y)$ is of *rank* n if its range space, $U(X)$, is of dimension n. For instance, in a Hilbert space, an orthogonal projection onto any n-dimensional subspace is a linear operator of rank n; the operators by Jackson and Fejér, Fourier sums, etc. are finite rank operators from $L_1(\mathbb{T})$ to $C(\mathbb{T})$. We shall denote the set of all linear operators from X to Y of rank $\leq n$ by $L_n(X, Y)$. Every $U \in L_n(X, Y)$ can be represented in the form

$$U(x) = \ell_1(x)y_1 + \ldots + \ell_k(x)y_k, \qquad k \leq n, \tag{1.3}$$

where $\{y_i\}$ is a basis in $U(X)$ and ℓ_i are continuous linear functionals on X. The number

$$\delta_n(K) := \delta_n(K, X) := \inf_{U_n} \sup_{x \in K} \|x - U_n(x)\|, \tag{1.4}$$

where the infimum is taken over all $U_n \in L_n(X, X)$, is called the *linear n-width* of K in X. If the infimum in (1.4) is attained by some U_n, the latter is called an *optimal operator.* Clearly, $\delta_n(K, X) \geq d_n(K, X)$. On the other hand, if X is a Hilbert space, $\delta_n(K) = d_n(K)$ for every K, since in this case the operator of best approximation is linear. We shall see that this equality holds for many important sets in some other spaces. This does not contradict the fact that the best approximation operator is usually non-linear; two operators may act differently but provide the same degree of approximation for the "worst" elements of K.

We say that a subspace $X^n \subset X$ is of *codimension* n if there exist n linearly independent continuous linear functionals $\ell_1, \ldots, \ell_n$ on X such that

$$X^n = \Big\{x \in X : \ \ell_i(x) = 0, \ i = 1, \ldots, n\Big\}, \quad X^0 := X. \tag{1.5}$$

The *Gelfand n-width* of K is defined by

$$d^n(K) := d^n(K, X) := \inf_{X^n} \sup\Big\{\|x\| : x \in K \cap X^n\Big\}, \tag{1.6}$$

where the infimum is taken over all subspaces $X^n \subset X$ of codimension $\leq n$.

Finding an optimal subspace X^n in (1.6) can be interpreted as a problem of optimal (linear) coding. Here by coding we mean any mapping $\ell \in L(X, \mathbb{R}^n)$. For example, a continuous function on a finite interval can be coded by its sampling at a fixed n-point set, or by its first n Fourier coefficients, etc. Usually one cannot expect to recover x from $\ell(x)$ exactly. The uncertainty of information about $x \in K$ contained in the value of $\ell(x)$ can be measured by the quantity

$$\sup\Big\{\|x - y\| : x, y \in K,\ \ell(x) = \ell(y)\Big\}. \tag{1.7}$$

If K is symmetric about zero and convex (as it is in all important cases), the number (1.7) is equal to $2\sup\{\|x\| : x \in K \cap X^n\}$, where X^n is the null space of ℓ.

It follows from (1.3) that if $U \in L_n(X, X)$, then the null space of U has codimension $\leq n$. Therefore we have

Proposition 1.1. *For any set K and all n,*

$$d_n(K) \leq \delta_n(K); \quad d^n(K) \leq \delta_n(K). \tag{1.8}$$

Often all three widths are equal.

Example. Let $a = (a_1, a_2, \ldots)$ be a fixed non-increasing sequence of positive numbers and let E be the ellipsoid in l_2 with half-axes a_i, that is,

$$E := \Big\{x = (\xi_1, \xi_2, \ldots) : \sum_{i=1}^{\infty} \xi_i^2/a_i^2 \leq 1\Big\}.$$

Let H_n denote the subspace of l_2 spanned by the first n coordinate vectors. If P_n is the orthogonal projection operator onto H_n, then for any $x \in E$,

$$\|x - P_n x\|^2 \leq a_{n+1}^2 \sum_{i=n+1}^{\infty} \xi_i^2/a_i^2 \leq a_{n+1}^2,$$

so that $\delta_n(E) \leq a_{n+1}$.

On the other hand, for any subspace X_n of dimension n there is an $x \in H_{n+1}$ orthogonal to X_n, with $\|x\| = a_{n+1}$. Then $x \in E$, since $\sum \xi_i^2/a_i^2 \leq a_{n+1}^{-2} \sum \xi_i^2 = 1$, and $\|x - y\| \geq a_{n+1}$ for any $y \in X_n$. Consequently, $d_n(E) \geq a_{n+1}$. Furthermore, $X^n \cap H_{n+1} \neq \{0\}$ for any subspace X^n of codimension n, and we similarly prove that $d^n(E) \geq a_{n+1}$. Thus,

$$d^n(E) = d_n(E) = \delta_n(E) = a_{n+1}. \tag{1.9}$$

We now derive some elementary properties of widths. When the three widths, Kolmogorov, Gelfand, and linear, are equal or have identical properties, *it will be convenient to denote by* s_n *any of* d_n, d^n *or* δ_n. Sometimes we shall write simply $s_n(K)$ instead of $s_n(K)_X$.

(i) *If* K *is a finite set of* m *elements, then* $s_n(K) = s_n(\mathrm{lin}(K)) = 0$ *for* $n \geq m$.

Indeed, if U is any projection operator onto $\mathrm{lin}(K)$, then $x - U(x) = 0$ for $x \in K$.

(ii) *If* $K_1 \subseteq K$, *then* $s_n(K_1) \leq s_n(K)$.

(iii) *For any scalar* α *and any* K, *one has* $s_n(\alpha K) = |\alpha| s_n(K)$.

(iv) $s_0(K) \geq s_1(K) \geq s_2(K) \geq \dots$.

(v) *If* K *is relatively compact in* X, *then* $d_n(K)_X \to 0$ *and* $d^n(K)_X \to 0$ *as* $n \to \infty$. *If* K *is bounded and* $d_n(K) \to 0$, *then* K *is relatively compact in* X.

Indeed, if K is relatively compact, then for any $\varepsilon > 0$ there is a finite ε-net $\{x_1, \dots, x_n\}$ for K. If $Y = \mathrm{lin}\{x_1, \dots, x_n\}$, then $d_n(K) \leq E(K, Y)_X \leq \varepsilon$. Furthermore, by the Hahn-Banach theorem, there are linear functionals $\ell_1, \dots, \ell_n$ for which $\|\ell_i\| = 1$ and $\ell_i(x_i) = \|x_i\|$ for each i. Let $x \in K$ and $\ell_i(x) = 0$, $i = 1, \dots, n$. Then there is some x_k for which $\|x - x_k\| < \varepsilon$ and

$$\|x_k\| = \ell_k(x_k) = \ell_k(x_k - x) \leq \|\ell_k\| \cdot \|x - x_k\| < \varepsilon,$$

so that $\|x\| < 2\varepsilon$, which proves that $d^n(K) \to 0$.

If K is bounded and $d_n(K) \to 0$, then for any $\varepsilon > 0$ there exists a finite-dimensional subspace Γ for which $E(K, \Gamma) < \varepsilon$. Those elements of Γ that are at a distance $\leq \varepsilon$ from K form a compact subset in Γ. A finite ε-net for this subset is a (2ε)-net for K, hence K is relatively compact.

It is generally not true that $\delta_n(K) \to 0$ *if* K *is compact.* A counter-example can be derived from the paper of Enflo [1973]. He constructed a Banach space X and an operator $V \in L(X, X)$ which maps the unit ball B_X into a compact set while $\|V - U\| \geq \varepsilon_0 > 0$ for some absolute constant ε_0 and every operator U of finite rank. Now if $K = V(B_X)$, then $\delta_n(K) \not\to 0$, since otherwise $\|V - U_n V\| \to 0$ for some sequence $U_n \in L_n(X, Y)$, contradicting the property of V.

(vi) *Let* $K = K_0 + \Gamma_m$, *where* K_0 *is a bounded set and* Γ_m *is a subspace of dimension* m. *If* $E(K, X_n) < \infty$ *for some subspace* X_n, *then* $X_n \supset \Gamma_m$. *Consequently, if* $n < m$, *then* $d_n(K) = \infty$, *hence* $\delta_n(K) = \infty$. *Similarly,* $d^n(K) = \infty$ *if* $n < m$. *Furthermore,* $s_{\nu+m}(K) \leq s_\nu(K_0)$ *for* $\nu = 0, 1, \dots$.

Indeed, if $\Gamma_m \setminus X_n \neq \emptyset$ and $x_0 \in \Gamma_m \setminus X_n$, then the distance from αx_0 to X_n tends to ∞ as $\alpha \to \infty$, so that $E(K, X_n) = \infty$. Similarly, if X^n is a subspace of codimension n, $n < m$, then there exists $x_0 \in \Gamma_m \cap X^n$, $x_0 \neq 0$, and

$$\sup\{\|x\| : x \in K \cap X^n\} \geq \sup_\alpha \|\alpha x_0\| = \infty.$$

Hence $d^n(K) = \infty$ if $n < m$.

The inequality $d_{\nu+m}(K) \le d_\nu(K_0)$ is obvious. In the case of the Gelfand widths, we observe that every $x \in Y := \mathrm{lin}(K_0) + \Gamma_m$ can be uniquely represented in the form $x = x_0 + \sum_{i=1}^k \ell_i(x)x_i$, where $x_0 \in \mathrm{lin}(K_0)$, $k \le m$, and $\{x_i\}$ are some linearly independent elements of Γ_m. The functionals ℓ_i are linear and continouous, and by the Hahn-Banach theorem, they can be extended from $\overline{Y}$ to X. If for some $d > 0$ and some subspace X^ν we have $\|x\| \le d$ for every $x \in K_0 \cap X^\nu$, then $\|x\| \le d$ for $x \in K \cap X^{\nu+k}$, where $X^{\nu+k}$ is the intersection of X^ν with the null space of $\{\ell_1, \dots, \ell_k\}$. Thus $d^{\nu+m}(K) \le d_n(K_0)$. The proof for $\delta_n(K)$ is analogous.

(vii) *For the closure $\overline{K}$ and the convex hull* $\mathrm{co}\, K$*,*

$$\sigma_n\Big(K \cup (-K)\Big) = \sigma_n(\overline{K}) = \sigma_n(\mathrm{co}\, K) = \sigma_n(K),$$

where σ_n stands for d_n or δ_n.

The statement follows from the definitions. For example, suppose that $E(K, X_n) \le \varepsilon$ for some subspace X_n. If $x \in \mathrm{co}\, K$, then $x = \sum \lambda_i x_i$, where $\lambda_i > 0$, $\sum \lambda_i = 1$, $x_i \in K$ for each i. If y_i is a nearest element to x_i in X_n, then

$$\|x - \sum \lambda_i y_i\| \le \sum \lambda_i \|x_i - y_i\| \le \varepsilon.$$

Consequently, $d_n(\mathrm{co}\, K) \le d_n(K)$. The opposite inequality is obvious since $K \subseteq \mathrm{co}\, K$.

Simple examples (see Pinkus [A-1985, p.18]) show that (vii) is not always true for the Gelfand widths.

(viii) *If Y is a subspace of X and $K \subset Y \subseteq X$, then $d^n(K)_X = d^n(K)_Y$, $d_n(K)_X \le d_n(K)_Y$, $\delta_n(K)_X \le \delta_n(K)_Y$.*

The inequality $s_n(K)_X \le s_n(K)_Y$ follows directly from the definitions. On the other hand, every continuous linear functional on Y can be extended to one on X by the Hahn-Banach theorem, which implies the equality for the Gelfand widths. One can show by examples that a strict inequality is possible for the other two widths (see Problem 10.3).

(ix) *If $\widehat{X}$ is a dense subset of X, then in the definition of the widths $d_n(K)_X$ for a bounded set K, one can take supremum over only those subspaces X_n that are spanned by the elements of $\widehat{X}$.*

Indeed, if $X_n = \mathrm{lin}\{x_1, \dots, x_n\}$ is an arbitrary n-dimensional subspace of X and $\sum_{i=1}^n \alpha_i x_i$ is an element in X_n nearest to x, then, since K is a bounded set, for $x \in K$ the coefficients α_i are bounded by a constant independent of x. Therefore, for any given $\varepsilon > 0$ and $x \in K$,

$$\|x - \sum_{i=1}^n \alpha_i \widehat{x}_i\| \le \|x - \sum_{i=1}^n \alpha_i x_i\| + \varepsilon,$$

if $\widehat{x}_i \in \widehat{X}$ are taken sufficiently close to x_i.

An optimal subspace for $d_n(K)$ may not exist. However, if X is a dual space (that is, the space of continuous linear functionals defined on some Banach space), the existence is guaranteed.

Theorem 1.2 (Garkavi [1962]). *If X is a dual space, then for every $K \subset X$ and every $n = 0, 1, 2, \ldots$, there exists an optimal subspace for $d_n(K)_X$.*

We rely upon a well-known lemma.

Lemma 1.3 (Auerbach). *For any n-dimensional normed linear space Γ there exist n elements $x_1, \ldots, x_n$ and n linear functionals $\ell_1, \ldots, \ell_n$ defined on Γ for which $\|x_k\| = \|\ell_k\| = 1$ and $\ell_i(x_k) = 0,\ i \neq k,\ \ell_k(x_k) = 1,\ \ i, k = 1, \ldots, n$.*

Proof. We realize Γ as a space of vectors $x = (\xi_1, \ldots, \xi_n)$ and let

$$x_k := (\xi_{k,1}, \ldots, \xi_{k,n}), \quad k = 1, \ldots, n,$$

be the system of vectors of unit norm for which the determinant $D(x_1, \ldots, x_n)$ $:= \det(\xi_{k,l})$ attains its maximum among all such systems. Then the functionals

$$\ell_i(x) := \frac{D(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n)}{D(x_1, \ldots, x_n)}, \qquad i = 1, \ldots, n,$$

satisfy the requirements of the lemma. □

The proof of Theorem 1.2 is based on the weak* compactness of the unit ball in a dual space. We shall assume for simplicity that the space X is dual to a separable space. Then the unit ball of X is sequentially weakly* compact (see, for example, Rudin [B-1973, 3.15-3.17]). Let $d_n(K) < \infty$, let $\{X_j\}$ be a sequence of subspaces of dimension $\leq n$ for which

$$E(K, X_j)_X \leq d_n(K) + 1/j, \qquad j = 1, 2, \ldots,$$

and let $(x_1^j, \ldots, x_n^j)$ and $(\ell_1^j, \ldots, \ell_n^j)$ be, for each j, the elements and the functionals of Lemma 1.3. We may assume, without loss of generality, that x_k^j converge weakly*, as $j \to \infty$, to some x_k^0, $k = 1, \ldots, n$. Then $\operatorname{lin}(x_1^0, \ldots, x_n^0)$ is an optimal subspace. Indeed, let $x \in K$ and let $\overline{x}^j$ be the nearest element to x in X_j. Then $\|x - \overline{x}^j\| \leq \|x\|$, so that $\|\overline{x}^j\| \leq 2\|x\|$. If $\overline{x}^j = \sum_{k=1}^n \alpha_k^j x_k^j$, then $\alpha_k^j = \ell_k^j(\overline{x}^j)$, hence $|\alpha_k^j| \leq 2\|x\|$ for all k, j. We may assume that the numbers α_k^j converge to some α_k^0 as $j \to \infty$, $k = 1, \ldots, n$. Then the $\overline{x}^j$ converge weakly* to $\overline{x}^0 := \sum_{k=1}^n \alpha_k^0 x_k^0$ and $\|x - \overline{x}^0\| \leq \limsup \|x - \overline{x}^j\| \leq d_n(K)$. □

If $K_1 \subseteq K$ and the widths of K_1 are known, one can use the inequality $s_n(K) \geq s_n(K_1)$ to estimate the widths of K. It is useful therefore to know the widths of some simple sets, say, of the ellipsoids E of Example 1. Likewise, one easily proves that if B_{n+1} is the unit ball of some $(n+1)$-dimensional subspace of a Hilbert space H, then $s_k(B_{n+1})_H = 1,\ \ k = 0, 1, \ldots, n$ (consequently, for a ball of radius r, $s_k(rB_{n+1}) = r$). It turns out that this equality is actually true for any Banach space X.

Theorem 1.4 (Krein, Krasnoselski, Milman [1948]). *If B_{n+1} is a unit ball of some $(n+1)$-dimensional subspace X_{n+1} of a (real or complex) Banach space X, then*

$$s_k(B_{n+1})_X = 1, \qquad k = 0, 1, \dots, n. \tag{1.10}$$

Proof. Let first the space X be real. Obviously, $s_0(B_{n+1})_X = 1$, hence $s_n(B_{n+1})_X \leq 1$ for all $n \geq 1$. The lower estimate for the Gelfand widths (and therefore for the linear widths) can be proved very simply. Indeed, if X^n is an arbitrary subspace of X of codimension n, then the set $B_{n+1} \cap X^n$ contains some $y \neq 0$ and we may assume that $\|y\| = 1$. This shows that $d^n(B_{n+1}) \geq 1$.

A similar inequality for the Kolmogorov widths is a deeper fact. For its proof it will suffice to show that for each n-dimensional subspace X_n of X, there is an element y for which

$$y \in X_{n+1}, \quad \|y\| = 1, \quad \rho(y, X_n) = 1. \tag{1.11}$$

We use the following antipodality theorem of Borsuk (see Appendix 1):

Let Σ_n be the unit sphere of an $(n+1)$-dimensional real Banach space Y_{n+1},

$$\Sigma_n := \{y \in Y_{n+1} : \|y\| = 1\}.$$

If Ψ is a continuous and odd ($\Psi(-y) = -\Psi(y)$) mapping from Σ_n to $\mathbb{R}^n$, then there exists $y_0 \in \Sigma_n$ for which $\Psi(y_0) = 0$.

We consider two cases.

(a) *X is a strictly convex space*, that is, from $x \neq x'$, $\|x\| = \|x'\|$, $\alpha_1, \alpha_2 > 0$, $\alpha_1 + \alpha_2 = 1$ follows $\|\alpha_1 x + \alpha_2 x'\| < 1$. Equivalent to this is the property that for $x, x' \neq 0$, relation $\|x + x'\| = \|x\| + \|x'\|$ implies that one of the x, x' is a positive multiple of the other. In this case, for every fixed finite dimensional subspace X_n of X and every $x \in X$, the nearest element $P(x)$ in X_n is unique and depends continuously on x (see [**CA**, §1, Chapter 3]). Moreover, the mapping P is obviously odd. We now apply Borsuk's theorem to the sphere $\|y\| = 1$ of X_{n+1}. There exists a $y \in X_{n+1}$, $\|y\| = 1$, with $P(y) = 0$, and we obtain (1.11):

$$\rho(y, X_n) = \rho(y, P(y)) = \rho(y, 0) = 1.$$

(b) *If X is not strictly convex*, we assume that it is separable, that is, that it contains a countable dense subset. This is no restriction of generality since in proving (1.11) we may replace X by its smallest subspace containing X_n and X_{n+1} which is finite dimensional and therefore separable.

We now prove that if a Banach space X is separable and not strictly convex, then it can be made strictly convex by an arbitrarily small perturbation of its norm. Let $(x_k)_1^\infty$ be a countable dense set in X. To each x_k there is, by the Hahn-Banach theorem, a $g_k \in B_{X'}$ for which $g_k(x_k) = \|x_k\|$. If $g_i(x) = 0$, $i = 1, 2, \dots$, for some $x \in X$, then $x = 0$. Indeed, for any given $\varepsilon > 0$ one can find an x_k for which $\|x - x_k\| < \varepsilon$ and therefore $\|x_k\| = g_k(x_k) = g_k(x) + g_k(x_k - x) \le 0 + \varepsilon$, so that $\|x\| < 2\varepsilon$. It follows that

$$\|x\|_1 := \left(\sum_{k=1}^{\infty} 2^{-k} g_k(x)^2\right)^{1/2}$$

is a norm on X. From the second definition of strict convexity (and the strict convexity of the Hilbert space) it follows that the norm $\|\cdot\|_1$ enjoys this property. It is not hard to see that the perturbed norm $\|x\|_\delta := \|x\| + \delta\|x\|_1$ is then also strictly convex for any $\delta > 0$. We obviously have $\|x\| \le \|x\|_\delta \le (1+\delta)\|x\|$ for any $x \in X$. Applying (a) to X with the norm $\|x\|_\delta$, we see that there is an element $y_\delta \in X_{n+1}$ for which $(1+\delta)^{-1} \le \rho(y_\delta, X_n) \le 1$. Making $\delta \to 0$ and using the local compactness of $X_n \cup X_{n+1}$, we find an element y that satisfies (1.11). This completes the proof in the case of the real space X.

If X is a complex space, we identify each $x \in X_{n+1}$ with the $(2n+2)$-dimensional real vector formed by the real and imaginary parts of the coordinates of x in some fixed basis of X_{n+1}. The elements of X_n can be likewise identified with $2n$-dimensional real vectors, and the proof proceeds as in the real case. □

The general width theory (for abstract Banach spaces) is not rich, and Theorem 1.4 is its deepest result. The main body of the theory is the computation of widths of some concrete sets K of functions or vectors. Our estimates of the widths $s_n(K)_X$ will be of the following types:

(A) *Weak equivalence* $s_n(K) \sim \phi(n)$, that is, $C_1\phi(n) \le s_n(K) \le C_2\phi(n)$;

(B) *Strong equivalence* $s_n(K) \approx \phi(n)$, that is, $s_n(K) = (1+o(1))\phi(n)$;

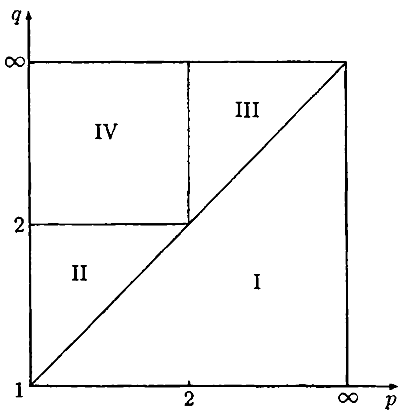
(C) *Exact formulas* $s_n(K) = \phi(n)$.

Here, $\phi(n)$ is some known positive function, $\phi(n) \to 0$ as $n \to \infty$ (often $\phi(n) = n^{-\alpha}$); C_1, C_2, C are positive constants and C in (B) may be known or unknown.

In §3, we give as interesting and useful examples widths of some polyhedra in finite dimensional spaces. Sections 4 and 5 deal with finer estimates of type (C) for Sobolev classes B_p^r in L_q for $q \le p$. Hilbert spaces in §4 have their own methods which yield, in particular, the Kolmogorov case $p = q = 2$. In §5, we use Borsuk's theorem in the form of Theorem 5.1. In §§6,7 we treat the general case of $s_n(B_p^r(A))_q$ for $q \le p$, $A = \mathbb{T}$, with the help of variational problems and differential equations. The results of §8 for $A = [a,b]$ are less striking; they are of type (B). Finally, §9 is devoted to a quite different problem of widths of classes of analytic functions.

Our results about the $s_n(B_p^\alpha)_q$ can be described in terms of Fig.1.1. In Chapter 13 we deal mainly with exact formulas for $A = \mathbb{T}$ for p, q in the region I. Chapter 14, where we obtain only the estimates of type (A), is of different character. We use traditional tools of approximation in either region II or III or in both, depending on the type of the widths. Fresh methods are needed to find asymptotic estimates for the widths in region IV and in the remaining cases in regions II and III. These estimates are due to Kashin for the Kolmogorov and Gelfand widths, and subsequently to Maiorov and Höllig for the linear widths.

Among the books concerned with widths we mention Pinkus [A-1985], Tikhomirov [A-1976], Korneichuk [A-1991, Chapter 8].



I: $1 \le q \le p \le \infty$
II: $1 \le p \le q \le 2$
III: $2 \le p \le q \le \infty$
IV: $1 \le p \le 2 \le q \le \infty$

Fig. 1.1

§ 2. Relations Between Different Widths

As we have already observed in §1, the inequalities

$$d_n(K,X) \le \delta_n(K,X), \quad d^n(K,X) \le \delta_n(K,X)$$

follow for any K, X from the definitions. Also, $d_n(K,H) = \delta_n(K,H)$ in a Hilbert space H. The inequality

$$\delta_n(K,X) \le (\sqrt{n}+1)\, d_n(K,X), \tag{2.1}$$

valid for arbitrary K, X, is based on a deeper fact. For the proof, we take $\varepsilon > 0$ and let X_n be an n-dimensional subspace of X for which $E(K, X_n) \le d_n(K) + \varepsilon$. By [**CA**, Theorem 7.6, p.294], there exists a linear projection operator $P \in L(X,X)$, $P(X) = X_n$ with $\|P\| \le \sqrt{n}$. For any $x \in K$ we can find an $\hat{x} \in X_n$ for which $\|x - \hat{x}\| \le d_n(K) + \varepsilon$. Then

$$\|x - Px\| \le \|x - \hat{x}\| + \|P\| \cdot \|x - \hat{x}\| \le (\sqrt{n}+1)\,(d_n(K) + \varepsilon),$$

and (2.1) follows by letting $\varepsilon \to 0$. In the next section we prove (see (3.5)) that $\delta_n(b_2^m,\, l_\infty^m) = (1 - n/m)^{1/2}$, $n \le m$. On the other hand, in §5 of Chapter 14 we shall establish that $d_n(b_2^m,\, l_\infty^m) \sim n^{-1/2}(\log(em/n))^{1/2}$. Hence

$$\delta_n(b_2^{2n},\, l_\infty^{2n}) \ge \text{const} \cdot d_n(b_2^{2n},\, l_\infty^{2n}) \cdot \sqrt{n},$$

Thus, for general K, X the inequality (2.1) can not be essentially strengthened.

There exist important dualities between different widths. To formulate them, suppose that X and Y are two Banach spaces and $T \in L(X,Y)$. If X', Y' are the dual spaces of X and Y, then the *adjoint operator* $T' \in L(Y',X')$ is uniquely defined by the equality $(T'y')(x) = y'(Tx)$, $x \in X$, $y' \in Y'$ (see, for example, Rudin [B-1973, Ch.4]). The space X is called *reflexive* if X'' is canonically isomorphic to X, that is, if every continuous linear functional on X' is of the form $\ell(x') = x'(x_\ell)$ with some fixed $x_\ell \in X$.

For a Banach space X, let B_X denote its unit ball, $B_X = \{x \in X : \|x\| \le 1\}$. For $x \in X$, $x' \in X'$ we shall write $\langle x, x' \rangle$ instead of $x'(x)$. If G is a subset of X, we write $x' \perp G$ if $\langle x, x' \rangle = 0$ for all $x \in G$.

Theorem 2.1. *Let X and Y be Banach spaces, $T \in L(X,Y)$. If X is reflexive and $T(X)$ is dense in Y, then*

$$d_n\left(T(B_X),\, Y\right) = d^n\left(T'(B_{Y'}),\, X'\right). \tag{2.2}$$

If both X and Y are reflexive, then

$$\delta_n\left(T(B_X), Y\right) = \delta_n\left(T'(B_{Y'}),\, X'\right) \tag{2.3}$$

Proof. We prove (2.3) first. Let $U \in L(Y,Y)$. Then

$$\sup\{\|y - Uy\|_Y \ : y \in T(B_X)\} = \sup\{\langle Tx - UTx,\, y'\rangle \ : x \in B_X,\, y' \in B_{Y'}\} =$$
$$\sup\{\langle x,\, T'y' - T'U'y'\rangle\} = \sup\{\|T'y' - T'U'y'\|_{X'} \ : y' \in B_{Y'}\}.$$

Taking infimum over all U of rank $\le n$ (for these U also rank $(T'U') \le n$) we have

$$\delta_n\Big(T(B_X),\, Y\Big) \ge \delta_n\Big(T'(B_{Y'}),\, X'\Big).$$

By the same argument we also get $\delta_n\Big(T(B_{Y'}),\, X'\Big) \ge \delta_n\Big(T(B_{X''}),\, Y''\Big)$ which completes the proof since, due to reflexivity, X'', Y'', T'' are equivalent, under appropriate canonical isomorphisms, to X, Y, T. For the proof of (2.2) we use [**CA**, Theorem 1.3, p.61] (which is simply a version of the Hahn-Banach theorem). It says that in a Banach space X the distance from every $x \in X$ to a finite dimensional subspace $\Gamma \subset X$ is equal to

$$\sup\Big\{\langle x, x'\rangle \ : x' \in B_{X'},\ x' \perp \Gamma\Big\} \tag{2.4}$$

We may consider, by (ix) of §1, only subspaces of Y of the form $T(X_n)$, where X_n are n-dimensional subspaces of X, and use (2.4):

$$\begin{aligned} d_n\left(T(B_X),\, Y\right) &= \inf_{X_n} \sup\left\{\langle Tx,\, y'\rangle \ : x \in B_X,\, y' \in B_{Y'},\, y' \perp T(X_n)\right\} \\ &= \inf_{X_n} \sup\left\{\langle x, T'y'\rangle : x \in B_X,\, y' \in B_{Y'},\, T'y' \perp X_n\right\} \\ &= \inf_{X_n} \sup\left\{\|T'y'\|_{X'} \ : y' \in B_{Y'},\, T'y' \perp X_n\right\} \end{aligned}$$

It remains to note that the condition $x' \perp X_n$ defines in X' a subspace X'^n of codimension n, and, due to reflexivity, X'^n runs through all such subspaces if X_n runs through all subspaces of X of dimension $\leq n$. □

Taking T to be the identity mapping, we get as an immediate consequence

Theorem 2.2. *For* $1 \leq p$, $q \leq \infty$,

$$d_n(b_p^m,\, l_q^m) = d^n(b_{q'}^m,\, l_{p'}^m), \tag{2.5}$$

$$\delta_n(b_p^m,\, l_q^m) = \delta_n(b_{q'}^m,\, l_{p'}^m), \tag{2.6}$$

where $p' = p/(p-1)$, $q' = q/(q-1)$.

Similar identities exist for the Sobolev classes. Let $B_{p,0}^r, B_{p,1}^r$ be the subclasses of $B_p^r[0,1]$ formed by the functions y satisfying $y(0) = \ldots = y^{(r-1)}(0) = 0$ or $y(1) = \ldots = y^{(r-1)}(1) = 0$, respectively.

Theorem 2.3. *For* $n = 1, 2, \ldots$ *and* $1 \leq p, q \leq \infty$,

$$d_n(B_{p,0}^r)_q = d^n(B_{q',0}^r)_{p'}, \tag{2.7}$$

$$\delta_n(B_{p,0}^r)_q = \delta^n(B_{q',0}^r)_{p'}, \tag{2.8}$$

Proof. We prove (2.7) and (2.8) only for $1 < p, q < \infty$; the rest will follow as a limiting case (see §7, where the continuity of the widths as functions of p and q is discussed in detail). Define $T : L_p \to L_q$ by setting $Tx = y$, where $y^{(r)} = x$, $y(0) = \ldots = y^{(r-1)}(0) = 0$. Then $T' : L_{q'} \to L_{p'}$ is defined by $T'u = v$, where $(-1)^r v^{(r)} = u$, $v(1) = \ldots = v^{(r-1)}(1) = 0$; this is proved by integration by parts. Thus $B_{p,0}^r = T(B_{L_p})$ and (2.7) follows from (2.2) because $T(L_p)$ is dense in L_q, $q > 1$; indeed, every $x \in L_q$ can be approximated by some $g \in C^{(r)}$ with $g(0) = \ldots = g^{(r-1)}(0) = 0$. Similarly, (2.8) follows from (2.3). To complete the proof we observe that the transformation $x(t) \to x(1-t)$ is an isometry in $L_p[0,1]$ that maps $B_{p,1}^r$ into $B_{p,0}^r$. □

So far we have discussed the widths of *sets*. One can also define the widths of *operators* $T \in L(X, Y)$ by setting

$$d_n(T) := d_n(T(B_X),\, Y); \tag{2.9}$$

$$d^n(T) := \inf_{X^n} \sup\{\, \|Tx\|_Y \, : x \in B_X \cap X^n \}. \tag{2.10}$$

The infimum in (2.10) is taken over all subspaces X^n of X of codimension $\leq n$. We also define

$$\delta_n(T) := \inf_{U_n} \|T - U_n\|_{X \to Y} = \inf_{U_n} \sup\{\|Tx - U_n x\|_Y : \, x \in B_X\}, \tag{2.11}$$

where the infimum is taken over all operators $U_n \in L(X, Y)$ of rank $\leq n$.

One may say that $d^n(T)$ is the $d^n(T(B_X), Y)$ in the sense of definition (1.6) but with the infimum taken over only those subspaces of Y of codimension $\le n$ that are images of such subspaces of X. A similar remark applies to $\delta_n(T)$ (the latter is also called the n-th *approximation number* of T). The following theorem is analogous to Theorem 2.1 and has a similar proof.

Theorem 2.4. *For any* $T \in L(X,Y)$, $d^n(T) = d_n(T')$. *If* X *is reflexive, then* $d_n(T) = d^n(T')$ *and* $\delta_n(T) = \delta_n(T')$.

These equalities also hold in some other situations, see Pinkus [A-1985, Ch.2] for details.

§ 3. Widths of Cubes and Octahedra

Here we estimate the widths of the finite dimensional balls: the cube b_∞^m and the octahedron b_1^m. Both are convex polyhedra, that is, intersections of finite families of closed half-spaces. An *extreme point*, or a *vertex*, of a polyhedron Π is any point $x \in \Pi$ that is not the center of some line segment contained in Π. Every convex polyhedron Π has a finite number of vertices; moreover, it is the convex hull of its vertices (see Rudin [B-1973, 3.21]). Throughout the section, e_i^k will denote the standard basis in $\mathbb{R}^k$:

$$e_1^k = (1,0,\dots,0),\ \dots, e_k^k = (0,\dots,0,1).$$

Lemma 3.1 (Lorentz [1960]). *If* Y *is a subspace of* $\mathbb{R}^m$ *of dimension* $\ge k$, *then there exists an* $x^* = (\xi_i)_1^m \in Y$ *for which* $\|x^*\|_\infty = 1$ *and* $|\xi_i| = 1$ *for at least* k *values of* i.

Proof. The set $b_\infty^m \cap Y$ is a polyhedron, therefore it has extreme points. Let x^* be one of them. We claim that at least k coordinates of x^* are equal to ± 1. Indeed, otherwise the magnitudes of some $n - k + 1$ coordinates would be < 1, say, of those indexed $i_1, \dots, i_{n-k+1}$. The intersection of Y with $\operatorname{lin}\{e_{i_1}^m, \dots, e_{i_{n-k+1}}^m\}$ must contain a non-zero vector z. But then we come to a contradiction because $x^* \pm \delta z \in b_\infty^m \cap Y$ for small $\delta > 0$, so that x^* is not an extreme point of $b_\infty^m \cap Y$. □

Theorem 3.2. *For* $1 \le q \le \infty$ *and* $n \le m$,

$$s_n(b_\infty^m)_q = (m-n)^{1/q}. \tag{3.1}$$

Proof. Let U_n be the projection operator $(\xi_1, \dots, \xi_m) \to (\xi_1, \dots, \xi_n, 0, \dots, 0)$ in l_q^m. For $x \in b_\infty^m$ we have $\|x - U_n x\|_q \le (m-n)^{1/q}$ which gives the upper estimate in (3.1). On the other hand, let X^n be an arbitrary subspace of $\mathbb{R}^m$ of codimension n, that is, of dimension $k := m - n$. If x^* is the vector of Lemma

3.1, then $\|x^*\|_q \geq (m-n)^{1/q}$, hence $d^n(b_\infty^m)_q \geq (m-n)^{1/q}$. This establishes (3.1) for $s_n = \delta_n$ and $s_n = d^n$.

It remains to prove the same lower estimate for $d_n(b_\infty^m)_q$ which, due to (2.5), is equivalent, for $p > 1$, to

$$d^n(b_p^m)_1 \geq (m-n)^{1-1/p}. \tag{3.2}$$

For an arbitrary subspace X^n of codimension n, let $x^* = (\xi_1, \ldots, \xi_m)$ be, as before, the extreme point of $b_\infty^m \cap X^n$. We may assume, without loss of generality, that $|\xi_{n+1}| = \ldots = |\xi_m| = 1$. If $y = x^*/\|x^*\|_p$, then $y \in b_p^m \cap X^n$,

$$\|y\|_1 = \frac{(|\xi_1| + \ldots + |\xi_n| + m - n)}{(|\xi_1|^p + \ldots + |\xi_n|^p + m - n)^{1/p}} \geq \frac{(|\xi_1| + \ldots + |\xi_n| + m - n)}{(|\xi_1| + \ldots + |\xi_n| + m - n)^{1/p}},$$

and (3.2) follows, since the last expression is $\geq (m-n)^{1-1/p}$. □

Formula (3.1) was found independently by Pietsch [1974] and Stessin [1975]. They actually proved, by a similar argument, a more general result:

$$s_n(b_p^m)_q = (m-n)^{1/q-1/p}, \qquad 1 \leq q \leq p \leq \infty.$$

In the Euclidean space l_2^m, the linear and the Kolmogorov n-widths of every set K have the same value for every n.

Theorem 3.3 (Stechkin [1954]). *For* $0 \leq n \leq m$,

$$d_n(b_1^m)_2 = \delta_n(b_1^m)_2 = \sqrt{1 - n/m}. \tag{3.3}$$

Proof. Let X_n be an n-dimensional subspace of l_2^m. If $u_1, \ldots, u_{m-n}$ is an orthonormal basis for $X_n^\perp$, then for the distance from e_k^m to X_n we have

$$\rho(e_k^m, X_n)^2 = \sum_{\nu=1}^{m-n} (e_k^m, u_\nu)^2, \qquad k = 1, \ldots, m.$$

Since , for a fixed ν, $\sum_{k=1}^m (e_k^m, u_\nu)^2 = \|u_\nu\|^2 = 1$, we further have

$$\sum_{k=1}^m \rho(e_k^m, X_n)^2 = m - n.$$

It follows that $\rho(e_k^m, X_n) \geq \sqrt{1 - n/m}$ for some k. Since X_n is arbitrary and $e_k^m \in b_1^m$, we obtain the desired lower estimate: $d_n(b_1^m)_2 \geq \sqrt{1 - n/m}$.

An optimal subspace X_n^* can be described explicitly, in terms of the orthonormal vectors $u_\nu^* = (u_{\nu,1}^*, \ldots, u_{\nu,m}^*)$, $\nu = 1, \ldots, m-n$, which span $(X_n^*)^\perp$. We set

$$u_{2j,k}^* := (2/m)^{1/2} \cos(2\pi jk/m), \qquad u_{2j-1,k}^* := (2/m)^{1/2} \sin(2\pi jk/m),$$

with $j = 1, 2, \ldots, (m-n)/2$ if $m-n$ is even or $j = 1, 2, \ldots, (m-n-1)/2$ if $m-n$ is odd; in the latter case we also set $u_{m-n}^* := (m^{-1/2}, \ldots, m^{-1/2})$. It is not hard to verify that the u_ν^*, $\nu = 1, \dots, m-n$ are orthonormal and that $\rho(e_k^m, X_n^*)^2 = \sum_{\nu=1}^{m-n} (u_{\nu,k}^*)^2 = 1 - n/m$ for every k. Since $b_1^m = \mathrm{co}\{\pm e_1^m, \dots, \pm e_m^m\}$, it follows, by (vii) of §1, that $d_n(b_1^m)_2 = \delta_n(b_1^m)_2 \le E(b_1^m, X_n^*) \le \sqrt{1 - n/m}$. □

No explicit formulas for the widths of b_1^m in l_q^m for $q \ne 1, 2$ are known, but one can easily prove that

$$d_n(b_1^m)_X = \delta_n(b_1^m)_X, \tag{3.4}$$

where X denotes $\mathbb{R}^m$ equipped with *any* norm. Indeed, if X_n is an n-dimensional subspace of X, let $g_1, \dots, g_m$ be nearest elements in X_n to $e_1^m, \dots, e_m^m$ in the norm of X and let $U : X \to X$ be the linear operator of rank $\le n$ defined by

$$U(\xi_1 e_1^m + \dots + \xi_m e_m^m) = \xi_1 g_1 + \dots + \xi_m g_m.$$

We have $E(b_1^m, X_n) = \max_k \|e_k - Ue_k\| = \max\{\|x - Ux\| : x \in b_1^m\}$. By taking infimum over all the X_n and corresponding U we get $d_n(b_1^m)_X \ge \delta_n(b_1^m)_X$. The opposite inequality is trivially true.

Using the dualities (2.5),(2.6), we obtain from (3.3)

$$d^n(b_2^m)_\infty = \delta_n(b_2^m)_\infty = \sqrt{1 - n/m} \tag{3.5}$$

As we shall see in the next chapter, estimating $d_n(b_2^m)_\infty$ is much more difficult.

§ 4. Widths in Hilbert Spaces

If H is a Hilbert space, then for any $K \subset H$

$$\delta_n(K, H) = d_n(K, H) \ge d^n(K, H).$$

In §1 we found the widths of the ellipsoid (1.9) in l_2. The same proof yields a formally more general result. Let $g := (g_1, g_2, \dots)$ be an orthonormal system in H and let $a := (a_1, a_2, \dots)$ be a non-increasing sequence of positive numbers. We consider the ellipsoid

$$E := E_{g,a} := \Big\{ x = \sum_k \xi_k g_k : \sum_k \xi_k^2 / a_k^2 \le 1 \Big\}.$$

Then

$$s_n(E) = a_{n+1} \tag{4.1}$$

and $H_n := \mathrm{lin}\{g_1, \dots, g_n\}$ is an optimal subspace. Clearly, if $a_{n-1} = a_n$ or $a_n = a_{n+1}$, there are infinitely many other optimal n-dimensional subspaces. Karlovitz [1976] noted that even if all a_k are different, there are infinitely many optimal subspaces for every $n \ge 1$.

Proposition 4.1. *If U_n is the orthogonal projection onto H_n, then for every $Q \in L(H,H)$ there is $\mu_0 > 0$ such that for every $|\mu| < \mu_0$ the operator $U_n + \mu Q U_n$ is an optimal operator of rank n.*

Proof. For a fixed Q and each $x \in E$ let

$$\mu_1(x) := \sup\{|\mu| : \|x - U_n x - \mu Q U_n x\| \le a_{n+1}\}.$$

If $x \in E$ and $\|x - U_n x\| = a_{n+1}$, then $U_n x = 0$, so that $\mu_1(x) = \infty$. If $\|x - U_n x\| < a_{n+1}$, then $\mu_1(x)$ is positive and depends continuously on x. We now set $\mu_0 := \inf\{\mu_1(x) : x \in E\}$. Since E is compact, μ_0 is attained by some x, hence $\mu_0 > 0$. □

As an immediate corollary of (4.1) we prove

Theorem 4.2 (Kolmogorov [1936]). *For $n, r = 1, 2 \ldots$,*

$$s_{2n-1}(B_2^r(\mathbb{T}))_2 = s_{2n}(B_2^r(\mathbb{T}))_2 = n^{-r}. \tag{4.2}$$

The subspace $\mathcal{T}_{n-1}$ of trigonometric polynomials is an optimal $(2n-1)$-dimensional subspace.

Proof. If $f \in B_2^r(\mathbb{T})$ and

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt),$$

then

$$\|f^{(r)}\|_2^2 = \pi \sum_{k=1}^{\infty} k^{2r}(a_k^2 + b_k^2) \le 1.$$

We see that $B_2^r(\mathbb{T})$ is a direct sum of the one-dimensional space of constants and of the ellipsoid $E_{g,a}$ with

$$g = (\pi^{-1/2}\cos t,\ \pi^{-1/2}\sin t,\ \pi^{-1/2}\cos 2t,\ \pi^{-1/2}\sin 2t \ldots),$$

$$a = (\pi^{1/2}1^{-r},\ \pi^{1/2}1^{-r},\ \pi^{1/2}2^{-r},\ \pi^{1/2}2^{-r}, \ldots),$$

so that (4.2) follows from (4.1). □

By Proposition 4.1, $\mathcal{T}_{n-1}$ is not the only optimal subspace for the $(2n-1)$-widths, a fact overlooked by Kolmogorov. In §7 we shall prove that a very natural spline subspace is also optimal.

Ellipsoids appear in the study of linear self-adjoint operators on H. For every $T \in L(H,H)$ there exists an *adjoint* operator $T' \in L(H,H)$ so that $(Tx, y) = (x, T'y)$ for all $x, y \in H$. If $T' = T$, the operator is called *self-adjoint.* If $T \in L(H,H)$ and $Tx_0 = \lambda x_0$ for some $x_0 \ne 0$ and complex λ, then such λ and x_0 are called an *eigenvalue* and an *associated eigenvector* of T. The eigenvalues of a self-adjoint operator are real since $\lambda(x_0, x_0) = (Tx_0, x_0) =$

$(x_0, Tx_0) = (x_0, \lambda x_0) = \overline{\lambda}(x_0, x_0)$. If, in addition, T is a compact operator (that is, if it maps bounded sets into relatively compact sets), then it has enough eigenvectors to form a basis in H. The following is a classical fact (see, for example, Riesz and Sz.-Nagy [B-1955, Ch.6]).

Proposition 4.3 (Hilbert-Schmidt). *If T is a compact, self-adjoint operator in H, then there exists an orthonormal system $\varphi_1, \varphi_2 \ldots$, of eigenvectors of T so that each $x \in H$ can be represented in the form*

$$x = \psi + \sum_k (x, \varphi_k)\varphi_k,$$

with $T\psi = 0$, $T\varphi_k = \lambda_k \varphi_k$, $|\lambda_1| \geq |\lambda_2| \geq \ldots > 0$.

From now on, T will always be compact. It follows from the above that the norm of T is equal to the magnitude of the largest eigenvalue $\lambda_1(T)$:

$$\|T\| = |\lambda_1(T)|. \tag{4.3}$$

Also, since $\{\varphi_k\}$ is a bounded set, the set $\{\lambda_k \varphi_k\}$ must be precompact which implies that either the set $\{\lambda_k\}$ is finite or $\lim \lambda_k = 0$ and that only a finite number of the φ_k may correspond to any value of $|\lambda_k|$.

If T is a compact operator, it is readily seen that both $T'T$ and TT' are self-adjoint and compact. Furthermore, their eigenvalues are non-negative. Indeed, if, for instance, $T'Tx_0 = \lambda x_0$, $x_0 \neq 0$, then $\lambda = (T'Tx_0, x_0)/(x_0, x_0) = (Tx_0, Tx_0)/(x_0, x_0) \geq 0$. It also follows that for an arbitrary operator

$$\|T\| = \sqrt{\lambda_1(T'T)}. \tag{4.4}$$

Proposition 4.4. *Let T be a linear operator in a Hilbert space H and let E_T be the image of the unit ball in H:*

$$E_T := \left\{x \in H : \ x = Tu, \ \|u\| \leq 1\right\}.$$

Then

$$s_n(E_T) = \sqrt{\lambda_{n+1}(T'T)}. \tag{4.5}$$

Proof. Let $\{\varphi_k\}$ be the system of Proposition 4.3 corresponding to $T'T$. For $\lambda_k > 0$ let $g_k := T\varphi_k/\sqrt{\lambda_k}$. Since $(T\varphi_k, T\varphi_l) = (T'T\varphi_k, \varphi_l) = \lambda_k(\varphi_k, \varphi_l)$, $\{g_k\}$ is an orthonormal system, and $x \in E_T$ if and only if $x = \sum \xi_k g_k$, $\sum \xi_k^2/a_k^2 \leq 1$. Thus $E_T = E_{g,a}$, with $a_k = \sqrt{\lambda_k}$, $k = 1, 2, \ldots$, and (4.5) follows from (4.1). □

In $H = L_2[a, b]$, a classic example of a linear compact operator is given by the formula

$$(4.6) \qquad (Tf)(t) = \int_a^b K(t,\tau)f(\tau)d\tau,$$

where $K(t,\tau)$ is a Hilbert-Schmidt kernel, that is, $K \in L_2$ on $[a,b] \times [a,b]$. The adjoint operator T' corresponds to the kernel $K^*(t,\tau) = K(\tau,t)$. If $K(t,\tau) = K(\tau,t)$, the operator (4.6) is self-adjoint.

It will be instructive now to put Theorem 4.2 into a more general perspective. Consider in $L_2(\mathbb{T})$ the operator D^r or the r-th differentiation defined on those $f(t)$ for which $D^{r-1}f$ is absolutely continuous and $D^r f \in L_2(\mathbb{T})$. If f, g are two such functions, then $(D^r f, g) = (-1)^r(f, D^r g)$. In this sense we say that $(-1)^r D^r$ is the adjoint operator to D^r. Class $B_2^r(\mathbb{T})$ has a simple description in the basis $\{1, \cos t, \sin t, \cos 2t, \sin 2t, \dots\}$ because constants form the null space of D^r whereas all $\cos kt, \sin kt$ are eigenfunctions of $(-1)^r D^r D^r = (-1)^r D^{2r}$, with positive eigenvalues. A similar basis can be constructed in the non-periodic case, for $B_2^r[0,1]$ in $L_2[0,1]$, but it can be described only implicitly.

Proposition 4.5. *Every $f \in L_2[0,1]$ has a unique representation of the form*

$$(4.7) \qquad f = P_{r-1} + \sum_{k=1}^{\infty} c_k \varphi_k$$

where $P_{r-1} \in \mathcal{P}_{r-1}$ and $\varphi_k := \varphi_{k,r}$ are the orthonormal eigenfunctions of the boundary value problem

$$(4.8) \qquad \lambda(-1)^r \varphi^{(2r)} - \varphi = 0,$$

$$(4.9) \qquad \varphi^{(r)}(0) = \varphi^{(r)}(1) = \dots = \varphi^{(2r-1)}(0) = \varphi^{(2r-1)}(1) = 0.$$

All the eigenvalues $\lambda_k := \lambda_{k,r}$ are positive and all the φ_k are orthogonal to $\mathcal{P}_{r-1}$.

Proof. For $f \in L_2$ and $\nu = 1, 2, \dots$, let

$$(4.10) \qquad f_{(\nu)}(t) := \frac{1}{(\nu-1)!}\int_0^t (t-s)^{(\nu-1)} f(s)ds,$$

or, equivalently,

$$f_{(\nu)}^{(\nu)}(t) = f(t), \qquad f_{(\nu)}(0) = \dots = f_{(\nu)}^{(\nu-1)}(0) = 0.$$

Define the operator T in L_2 by the formula $Tf = (-1)^r f_{(2r)}$. It is a Hilbert-Schmidt, and therefore a compact, operator. In the space $\mathcal{P}_{r-1}^{\perp} := \{f \in L_2 : f \perp \mathcal{P}_{r-1}\}$ it is also self-adjoint. Indeed, it follows from (4.10) that if $g \perp \mathcal{P}_{r-1}$ then $g_{(\nu)}(1) = g_{(\nu)}(0) = 0$, $\nu = 1, \dots, r$. If also $f \perp \mathcal{P}_{r-1}$, we integrate by parts r times and get

$$(Tf, g) = (-1)^{r+1}(f_{(2r-1)}, g_{(1)}) = \dots = (f_{(r)}, g_{(r)}),$$

and similiarly $(f, Tg) = (f_{(r)}, g_{(r)})$. By Proposition 4.3 there is an orthogonal basis in $\mathcal{P}_{r-1}^{\perp}$ formed by the eigenfunctions φ_k of T. For every λ_k, φ_k, we have $\lambda_k(\varphi_k, \varphi_k) = (T\varphi_k, \varphi_k) \geq 0$ which shows that $\lambda_k > 0$ (zero is not an eigenvalue since $Tf = 0$ implies $f = 0$). It remains to note that (4.8) follows from $T\varphi = \lambda\varphi$; also, for φ satisfying (4.8), the conditions $\varphi_{(\nu)}(0) = \varphi_{(\nu)}(1) = 0$, $\nu = 1, \ldots, r$, are equivalent to (4.9). □

Theorem 4.6 (Kolmogorov [1936]). *For $B_2^r = B_2^r[0,1]$ in $L_2[0,1]$, $r = 1, 2, \ldots$, the widths $s_n(B_2^r)_2$ are infinite if $n = 1, \ldots, r-1$. For $n \geq r$,*

$$s_n(B_2^r)_2 = \sqrt{\lambda_{n-r+1,r}}\,, \tag{4.11}$$

where $\lambda_{n,r}$ are the eigenvalues of the problem (4.8), (4.9) *arranged in the decreasing order.*

Proof. Let (φ_k) be the functions of Proposition 4.5. Integrating by parts and using (4.9) and (4.8) with $\lambda = \lambda_k$, we have for every $f \in L_2$

$$(f, \varphi_k^{(r)}) = (-1)^r (g, \varphi_k^{(2r)}) = \frac{1}{\lambda_k}(g, \varphi_k), \tag{4.12}$$

where g is any function for which $g^{(r)} = f$. Taking $f = \varphi_{k'}^{(r)}$ in (4.12), we derive from the orthonormality of the (φ_k) that $\psi_k := \varphi_k^{(r)}\sqrt{\lambda_k}$ form an orthonormal system. Furthermore, the system (ψ_k) is complete. Indeed, if $(f, \psi_k) = 0$ for some $f \in L_2$ and some ψ_k, then by (4.12), $(g, \varphi_k) = 0$. If this holds for $k = 1, 2, \ldots$, then it follows from (4.7) and the properties of the φ_k that $g \in \mathcal{P}_{r-1}$, hence $f = 0$. Therefore for every f with $f^{(r)} \in L_2$,

$$\|f^{(r)}\|^2 = \sum_{k=1}^{\infty} (f^{(r)}, \psi_k)^2 = \sum_{k=1}^{\infty} \lambda_k (f^{(r)}, \varphi_k^{(r)})^2 = \sum_{k=1}^{\infty} \lambda_k^{-1} (f, \varphi_k)^2 = \sum_{k=1}^{\infty} c_k^2/\lambda_k,$$

where (c_k) are the coefficients in the representation (4.7). Thus

$$B_2^r = \mathcal{P}_{r-1} \oplus E_{g,a}, \quad g = (\varphi_1, \varphi_2, \ldots), \ a = (\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots),$$

and (4.11) follows from (4.1) and (vi) of §1. □

All the sets studied above were ellipsoids. We now consider a set of a different nature. Let $w \in L_2(\mathbb{T})$ be some fixed function with mean value zero,

$$w(t) = \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt), \tag{4.13}$$

and let

$$B_w := \left\{ x \in C(\mathbb{T}) : \ x = w * \psi, \ \int_{\mathbb{T}} |\psi| d\tau \leq 1 \right\},$$

where

$$(w * \psi)(t) = \int_{\mathbb{T}} w(t - \tau)\psi(\tau) d\tau.$$

It is clear that every $f \in B_w$ can be approximated in L_2 by a convex combination of $\pm w_\tau := \pm w(t-\tau)$, $\tau \in \mathbb{T}$. Conversely, every convex combination of $\pm w_\tau$ can be approximated by a function from B_w. In other words,

$$\overline{B_w} = \overline{\operatorname{co}(\pm w_\tau : \tau \in \mathbb{T})}. \tag{4.14}$$

We estimate the widths of B_w using averaging.

Theorem 4.7 (Ismagilov [1968]). *For $n = 1, 2, \ldots$,*

$$d_{2n}(B_w)_2 = \delta_{2n}(B_w)_2 = \Big(\pi \sum_{k=n+1}^{\infty} c_k^{*2}\Big)^{1/2},$$

where c_k^* denote the numbers $c_k = (a_k^2 + b_k^2)^{1/2}$ arranged in non-increasing order. The subspace X_{2n}^* spanned by the $\cos kt$, $\sin kt$ *corresponding to the n largest c_k is optimal, and the orthogonal projector onto X_{2n}^* is an optimal operator of rank $2n$.*

Proof. Since X_{2n}^* is translation invariant, $(\pi \sum_{n+1}^{\infty} c_k^{*2})^{1/2}$ is the distance from X_{2n}^* not only for w but also for every w_τ, and because of (4.14), it is also equal to $E(B_w, X_{2n}^*)_2$. This gives the upper estimates for the widths.

Let now U be an orthogonal projector onto some $2n$-dimensional subspace of $L_2(\mathbb{T})$ with an orthonormal basis

$$f_j(t) := \sum_{k=0}^{\infty} (a_{k,j} \cos kt + b_{k,j} \sin kt), \quad j = 1, \ldots, 2n.$$

Then

$$\|w_\tau - U w_\tau\|^2 = \|w_\tau\|^2 - \sum_{j=1}^{2n} \Big(w_\tau, f_j\Big)^2 \tag{4.15}$$

But $\|w_\tau\|^2 = \|w\|^2$ and

$$(w_\tau, f_j) = \pi \sum_{k=1}^{\infty} (a_{k,j} a_k + b_{k,j} b_k) \cos k\tau + (b_{k,j} a_k - a_{k,j} b_k) \sin k\tau.$$

Therefore, upon averaging over all $\tau \in \mathbb{T}$, we get

$$\frac{1}{2\pi} \int_{\mathbb{T}} \|w_\tau - U w_\tau\|^2 d\tau = \|w\|^2 - \frac{\pi}{2} \sum_{k=1}^{\infty} c_k^2 \rho_k, \tag{4.16}$$

where $\rho_k := \pi \sum_{j=1}^{2n} (a_{k,j}^2 + b_{k,j}^2)$. We have

$$\sum_{k=1}^{\infty} \rho_k = \sum_{j=1}^{2n} \|f_j\|^2 = 2n.$$

At the same time,

$$\rho_k = (1/\pi)(\|U(\cos kt)\|^2 + \|U(\sin kt)\|^2) \le (1/\pi)(\|(\cos kt)\|^2 + \|(\sin kt)\|^2) = 2.$$

Under these circumstances the sum $\sum_1^\infty c_k^2 \rho_k$ cannot exceed $2\sum_1^{2n} c_k^{*2}$; this value is attained when $\rho_k = 2$ for k corresponding to the n largest c_k, with all the other c_k equal to zero. But then the quantity (4.16) is $\ge \|w\|^2 - \pi \sum_1^n c_k^{*2} = \pi \sum_{n+1}^\infty c_k^{*2}$. It follows that for some τ

$$\|w_\tau - Uw_\tau\| \ge \pi \sum_{k=n+1}^{\infty} c_k^{*2}.$$

Together with (vii) of §1 this gives the lower bound for $d_{2n}(B_w)_2$. □

As an application one may consider the widths $s_{2n}(B_1^r(\mathbb{T}), L_2(\mathbb{T}))$. By [**CA**, (4.14), p.151], every $f \in W_1^r(\mathbb{T})$ can be represented in the form

$$f(x) = \text{const.} + \frac{1}{\pi}\int_{\mathbb{T}} \mathcal{B}_r(x-t) f^{(r)}(t)\,dt,$$

where $\mathcal{B}_r$ is the periodic Bernoulli spline. As a consequence, we have, in the notation of Theorem 4.7, $B_1^r(\mathbb{T}) \subset B_w + \text{const.}$ with $w = \pi^{-1}\mathcal{B}_r$. Using this theorem and the Fourier expansion of $\mathcal{B}_r$ (see [**CA**, (4.13), p.151]), one immediately obtains an upper bound for the widths. A more elaborate argument based on Theorem 4.7 gives a better upper bound and also a lower bound (see Pinkus [A-1985, p.101]).

§ 5. Applications of Borsuk's Theorem

Much effort has been spent to find exact formulas for the widths of the Sobolev balls $B_p^r := B_p^r(\mathbb{T})$ on the circle $\mathbb{T}$. Earlier publications completed this investigation for all (p,q) on the two sides of the lower triangle I of Fig.1.1, that is, for $p = \infty$ and for $q = 1$. It has been established that in these cases, for $r = 1, 2, \ldots$ and $s_m = d_m$, δ_m or d^m, $s_{2n-1}(B_p^r(\mathbb{T}))_q = C(p,q,r)n^{-r}$, with $C(p,q,r)$ evaluated explicitly as norms of certain periodic splines (see (7.24)). Usually Theorems 1.4 or 5.1 had to be used.

Later came striking results for arbitrary (p,q) in the triangle I, for $A = \mathbb{T}$ or $A = [a,b]$. For the diagonal $p = q$ we have Pinkus [1985] and Chen and Li [1992]; for general (p,q), the papers of Buslaev and Tikhomirov [1985], [1990]. See §7 for our exposition of the case $A = \mathbb{T}$, with historical remarks at the end.

The original proofs of (7.24) did not lose, however, their interest because of their relative simplicity. As an example, we prove here the formula

$$s_{2n-1}(B_p^r(\mathbb{T}))_p = K_r n^{-r}, \quad p = 1, \infty, \tag{5.1}$$

due to Tikhomirov [1960] ($p = \infty$) and to Makovoz [1969], Subbotin [1970] ($p = 1$), in which K_r is the constant (3.3) of Chapter 6. By Favard's Theorem

[**CA**, p.213], there is a linear operator U_n from $L_\infty(\mathbb{T})$ into the $(2n-1)$-dimensional subspase of trigonometric polynomials of degree $\leq n-1$ for which $\|f - U_n f\|_\infty \leq K_r n^{-r}$ for every $f \in B^r_\infty(\mathbb{T})$. This gives the upper estimate for the widths (5.1) in the case $p = \infty$. For the lower estimate, we use (3.7) of Chapter 6: $\|S^{(r)}\|_\infty \leq K_r^{-1} n^r \|S\|_\infty$, valid for every S from the $2n$-dimensional spline space $\widetilde{S}_{n,r+1}$. This inequality shows that the L_∞ ball of radius $K_r n^{-r}$ of the space $\widetilde{S}_{n,r+1}$ is contained in $B^r_\infty(\mathbb{T})$, and the desired lower estimate for $p = \infty$ follows from Theorem 1.4. In the case $p = 1$ we use in the same way Nikolskii's theorem [**CA**, p.215] and the inequality (3.8) of Chapter 6.

Here is a generalization of Theorem 1.4 by Makovoz [1972]:

Theorem 5.1. *Let X be a (real or complex) Banach space and let the set $\Omega \subset X$ be of the form*

$$\Omega := Z_\nu + \Phi(\Sigma),$$

where Z_ν is some fixed ν-dimensional subspace of X, Σ is the unit sphere of some $(n+1)$-dimensional (respectively, real or complex) Banach space Y_{n+1}, and Φ is a continuous and odd mapping of Σ into X. Then

$$s_{n+\nu}(\operatorname{co}\Omega)_X \geq \min\{\|x\| : x \in \Omega\} =: \rho. \tag{5.2}$$

Theorem 1.4 is a special case of this statement in which $Z_\nu = \{0\}$, Y_{n+1} is a subspace of X, and Φ is the identity mapping.

Proof. We need the lower estimates only for the Kolmogorov and Gelfand widths since the linear widths are larger. Let first the spaces Y_{n+1} and X be real, and let $e_1, \ldots, e_\nu$ be a basis of Z_ν.

To prove the estimate for the Gelfand widths, consider an arbitrary subspace $X^{n+\nu} \subset X$ of codimension $n+\nu$ defined as the null space of some $n+\nu$ functionals $\ell_k \in X'$, $k = 1, \ldots, n+\nu$. We have to prove that

$$\sup\left\{\|x\| : x \in \operatorname{co}\Omega \cap X^{n+\nu}\right\} \geq \rho. \tag{5.3}$$

Since Φ is an odd mapping, the set $\operatorname{co}\Phi(\Sigma)$ contains a zero element; hence $Z_\nu \subset \operatorname{co}\Omega$. If the rank of the $(n+\nu) \times \nu$ matrix $M := [\ell_i(e_k)]$ is $< \nu$, then there exist $z \in Z_\nu \cap X^{n+\nu}$ with arbitrarily large norms $\|z\|$. In this case the inequality (5.3) holds since its left-hand side is infinite. If $\operatorname{rank}(M) = \nu$, we assume, without loss of generality, that $\det[\ell_i(e_k)]_{i,k=1}^\nu \neq 0$. Then for any $x \in X$ there exists a unique $u = u(x) \in Z_\nu$ for which $\ell_i(x+u) = 0$, $i = 1, \ldots, \nu$. The mapping $x \to u(x)$ is linear and continuous. Next we define $\Psi : \Sigma \to \mathbb{R}^n$ by setting for $y \in \Sigma$,

$$\Psi(y) := (\ell_{\nu+1}(\Phi_1(y)), \ldots, \ell_{n+\nu}(\Phi_1(y))),$$

where $\Phi_1(y) := \Phi(y) + u(\Phi(y))$. Clearly, $\Phi_1(y) \in \Omega$; consequently, $\|\Phi_1(y)\| \geq \rho$ for every $y \in \Sigma$. Since Ψ obviously satisfies the conditions of Borsuk's theorem, there is $y_0 \in \Sigma$ for which $\Psi(y_0) = 0$. If $x_0 := \Phi_1(y_0)$, then $\ell_i(x_0) = 0$ for

$i = \nu + 1, \ldots, n + \nu$; the same is true for $i = 1, \ldots, \nu$ due to the definition of u. Thus $x_0 \in \Omega \cap X^{n+\nu}$, and (5.3) follows.

To estimate $d_{n+\nu}(\operatorname{co} \Omega)) = d_{n+\nu}(\Omega))$ from below, we first assume that the space X is strictly convex, in which case the nearest element to every $x \in X$ in every finite dimensional subspace of X is unique and depends continuously on x. Let $X_{n+\nu}$ be some fixed $(n+\nu)$-dimensional subspace of X. If $Z_\nu \not\subset X_{n+\nu}$, then $E(\Omega, X_{n+\nu}) \geq E(Z_\nu, X_{n+\nu}) = \infty$ (see (vi) of §1). If $Z_\nu \subset X_{n+\nu}$, let $e_1, \ldots, e_{n+\nu}$ be a basis of $X_{n+\nu}$, with $\{e_1, \ldots, e_\nu\} \subset Z_\nu$. For $x \in X$, let $\sum_{i=1}^{n+\nu} \xi_i(x) e_i$ be its best approximation in $X_{n+\nu}$. We define the mapping $\widehat{\Psi} : \Sigma \to \mathbb{R}^n$ as follows. Given $y \in \Sigma$, we first define

$$x_y := \Phi(y) - \sum_{i=1}^{\nu} \xi_i(\Phi(y)) e_i \in \Omega .$$

Then $\xi_1(x_y) = \ldots = \xi_\nu(x_y) = 0$. We set

$$\widehat{\Psi}(y) := (\xi_{\nu+1}(x_y), \ldots, \xi_{n+\nu}(x_y)) .$$

This mapping satisfies the conditions of Borsuk's theorem, hence $\widehat{\Psi}(y_0) = 0$ for some $y_0 \in \Sigma$. For the corresponding x_{y_0}, the best approximation in $X_{n+\nu}$ is zero. But then $E(\Omega, X_{n+\nu}) \geq \|x_{y_0}\| \geq \rho$, and the lower estimate for $d_{n+\nu}(\Omega)$ follows.

If X is not strictly convex, we may assume, without loss of generality, that $X = \operatorname{lin}\{\Omega \cup X_{n+\nu}\}$. Then X is separable, that is, it contains a countable dense subset. Indeed, the spaces $X_{n+\nu}$ and Z_ν are separable because they are finite dimensional, and $\Phi(\Sigma)$ is separable because it is compact in X as a continuous image of a compact set; therefore their linear span is separable. The proof can be now completed as in Theorem 1.4. □

The following theorem is not implied by the results of §7, because it contains also the even-numbered widths d_{2n}. Its proof is elementary in the sense that it does not depend on Borsuk's theorem.

Theorem 5.2 (Tikhomirov [1969]). *On the circle* $\mathbb{T}$, *for* $r, n = 1, 2, \ldots$

$$d_{2n-1}(B_\infty^r(\mathbb{T}))_\infty = d_{2n}(B_\infty^r(\mathbb{T}))_\infty = K_r n^{-r}, \tag{5.4}$$

Proof. We need only the lower estimate for $d_{2n}(B_\infty^r)$. Due to (ix) of §1, in the case of Kolmogorov's widths we may consider only the approximating subspaces formed by continuous functions. For $f \in C(\mathbb{T})$ we define functionals of the form

$$\ell(f) := \sum_{k=1}^{2n} c_k f(k\pi/n + \tau), \quad c_k \in \mathbb{R},\ \tau \in \mathbb{T}.$$

Given a subspace $X_{2n} := \operatorname{lin}(f_1, \ldots f_{2n}) \subset C(\mathbb{T})$, we choose $\tau \in [0, \pi/n]$ for which

$$D(\tau) := \det[f_i(k\pi/n + \tau)]_{i,k=1}^{2n} = 0.$$

This is possible since $D(\pi/n)$ is obtained from $D(0)$ by a cyclic permutation of columns, so that $D(\pi/n) = -D(0)$. Then we find (c_k), from the conditions

$$\ell(f_i) = 0, \quad i = 1, \dots, 2n, \qquad |c_1| + \dots + |c_{2n}| = \|\ell\| = 1.$$

By Theorem 3.2 of Chapter 6, there is a spline $S \in \widetilde{\mathcal{S}}_{n,r+1}$ for which $S(t_{k,r}) = K_r n^{-r} \operatorname{sign} c_k$, for $k = 1, \dots, 2n$, and $f_0(t) := S(t - \tau) \in B^r_\infty(\mathbb{T})$, due to the inequality (3.7) of Chapter 6. For any $g \in X_{2n}$ we now have

$$\|f_0 - g\| \geq |\ell(f_0 - g)| = |\ell(f_0)| = K_r n^{-r},$$

from which the desired lower estimate follows. □

The case of the class $B^r H^\omega_\infty(\mathbb{T})$, the unit ball of the space $W^r H^\omega_\infty(\mathbb{T})$ (in particular, the case of Lipschitz classes B^α_p) is much more difficult (see Korneichuk [A-1991, p.384]). For a given modulus of continuity ω, the class $B^r H^\omega_\infty(\mathbb{T})$ consists of functions $f \in C^r(\mathbb{T})$ with $\omega(f^{(r)}, t) \leq \omega(t)$, $t > 0$.

Theorem 5.3 (Korneichuk). *For $r = 0, 1, \dots$ and each concave modulus of continuity,*

$$d_{2n-1}(B^r H^\omega_\infty(\mathbb{T}))_\infty = d_{2n}(B^r H^\omega_\infty(\mathbb{T}))_\infty = n^{-r-1} \int_0^\pi F_r(t) \omega'(t/n)\, dt,$$

where F_r are the polynomials on $[0, \pi]$ given by the recurrence relation

$$F_r(t) = \frac{1}{2} \int_0^{\pi - t} F_{r-1}(\tau)\, d\tau, \quad r = 1, 2, \dots, \quad F_0(t) = 1/2.$$

All the widths are realized by the subspace of trigonometric polynomials of degree $\leq n - 1$ (see [**CA**, Theorem 4.3, p.344]). The even-numbered widths are also realized by the subspace $\mathcal{S}_{n,r}$ of periodic splines (compare (3.18) of Chapter 6). For $r = 0$, the formula

$$d_{2n-1}(H^\omega_\infty(\mathbb{T}))_\infty = d_{2n}(H^\omega_\infty(\mathbb{T}))_\infty = (1/2)\omega(\pi/n)$$

had been found earlier by Tikhomirov. It is interesting that the linear widths for a general concave ω are not known even in this simplest case.

Our next result is the exact evaluation of the widths $s_n(B^1_p[0,1])_q$ for $1 \leq q \leq p \leq \infty$. We define

$$(5.5) \qquad C_1(p, q) := \sup\left\{\|f\|_q[0,1] : f \in B^1_p[0,1], \quad f(0) = 0\right\}.$$

Theorem 5.4. *For $1 \leq q \leq p \leq \infty$,*

$$(5.6) \qquad s_n(B^1_p[0,1])_q = \frac{1}{2} C_1(p, q) n^{-1}, \qquad n = 1, 2 \dots.$$

The space of step functions on $[0,1]$ with breakpoints $(k/n)_{k=1}^{n-1}$ is an optimal n-dimensional subspace, and the operator U_n of interpolation by these step *functions at the points* $((2k-1)/(2n))_{k=1}^{n}$ *is an optimal linear operator of rank* n.

Proof. We shall consider only the case $1 < q \le p < \infty$ since one can show that both the widths and the constants $C_1(p,q)$ depend continuously on p and q (for a similar situation in the periodic case, see the proofs of Theorems 6.6 and 7.6). From the definition of $C_1(p,q)$ we have, by a linear change of variable,

$$(5.7)\quad \sup\left\{\|f\|_q[a,b] : f \in B_p^1[a,b],\quad f(a)=0\right\} = C_1(p,q)(b-a)^{1-1/p+1/q}.$$

If $f \in W_p^1[0,1]$ and $U_n f$ is the step function equal to $f((2k-1)/(2n))$ on the interval $\Delta_k := [(k-1)/n, k/n]$, $k=1,\dots n$, then from (5.7) we have

$$(5.8)\qquad \|f - U_n f\|_q \le C_1(p,q)(2n)^{-1+1/p-1/q}\left(\sum_{k=1}^{2n}\gamma_k^q\right)^{1/q},$$

where $\gamma_k := \left(\int_{\Delta_k}|f'|^p dt\right)^{1/p}$. If $f \in B_p^1$, $p \ge q$, then by the inequality between means,

$$\left((2n)^{-1}\sum_{k=1}^{2n}\gamma_k^q\right)^{1/q} \le \left((2n)^{-1}\sum_{k=1}^{2n}\gamma_k^p\right)^{1/p} \le (2n)^{-1/p},$$

hence $\|f - U_n f\|_q \le (1/2)C_1(p,q)n^{-1}$.

For the lower estimate we use Theorem 5.1, with $Z_\nu = \{0\}$. In the role of Y_{n+1} we take ℓ_1^{n+1}. To each $y = (\eta_1,\dots,\eta_{n+1})$ with $\|y\|_1 = |\eta_1|+\dots+|\eta_{n+1}| = 1$ we put in correspondence $f_y \in B_p^1$ as follows. We set

$$t_k := |\eta_1| + \dots + |\eta_k|,\quad k=1,\dots,n,$$

and fix some function $w(t) \in W_p^1[0,1]$, satisfying $\|w'\|_p[0,1] = 1$ and $w(t) = w(1-t)$, $0 \le t \le 1$. Then we define for $t \in [0,1]$

$$(5.9)\quad f_y(t) := 2\eta_1 w\left(\frac{t+|\eta_1|}{2|\eta_k|}\right) + \sum_{k=2}^{n}\eta_k w\left(\frac{t-t_{k-1}}{|\eta_k|}\right) + 2\eta_{n+1}w\left(\frac{t-t_n}{2|\eta_k|}\right),$$

We assume that if $\eta_k = 0$ for some k, then the corresponding term in (5.9) is omitted. It is easy to see that the mapping Φ from the unit sphere $\|y\|_1 = 1$ to L_q given by $y \to f_y$ is continuous and odd. By a change of variable, using the symmetry of w, $\|f_y'\|_p = \|w'\|_p = 1$, so that $f_y \in B_p^1$ for every y. Therefore, by Theorem 5.1,

$$s_n(B_p^1) \ge \min\{\|f_y\|_q : \|y\|_1 = 1\}.$$

Now

$$\|f_y\|_q = \|w\|_q\left(2^q|\eta_1|^{q+1} + \sum_{k=2}^{n}|\eta_k|^{q+1} + 2^q|\eta_{n+1}|^{q+1}\right)^{1/q}.$$

Using Lagrange multipliers, we see that the minimum value of the above quantity under the condition $\sum |\eta_k| = 1$ is $\|w\|_q n^{-1}$. It is attained when

$$|\eta_1| = |\eta_{n+1}| = (2n)^{-1}, \quad |\eta_2| = \ldots = |\eta_n| = n^{-1}.$$

By symmetry, since $\|w'\|_p[0,1] = 1$, we have $\|w'\|_p[0,1/2] = 2^{-1/p}$. It follows from (5.7) that $\|w\|_q[0,1/2]$ can be made arbitrarily close to $2^{-1/p}C_1(p,q)$ $2^{-1+1/p-1/q}$. We may therefore assume that $\|w\|_q = (1/2)C_1(p,q) - \varepsilon$, with arbitrarily small $\varepsilon > 0$. This yields the desired lower bound for the widths. □

§ 6. Variational Problems and Spectral Functions

In this and the next section we offer a new approach to the exact determination of the widths $s_n(B_p^r(\mathbb{T}))_q$ for $r = 1, 2, \ldots$ and $1 \le q \le p \le \infty$. This approach is based on the study of the following extremal problem involving norms on $\mathbb{T}$ or $[0,1]$: Find

$$\sup\{\|f\|_q : f \in W_p^r, \ \|f^{(r)}\|_p = 1\}. \tag{6.1}$$

The supremum in (6.1) is, of course, equal to infinity, but it can become finite if some additional (for instance, boundary) conditions are imposed. For $1 < p, q < \infty$, each solution of the extremal problem (6.1) with proper additional conditions, satisfies a certain differential equation, which is known as the Euler-Lagrange equation, with a parameter λ. For example, if $p = q = 2$, it is the equation (4.8) and the boundary conditions are (4.9). In §5 we proved that the problem (4.8)-(4.9) has an infinite sequence (λ_n) of eigenvalues and that $s_n(B_2^r)_2 = \sqrt{\lambda_{n-r+1}}, n \ge r$.

As far back as 1960's, Tikhomirov conjectured that a similar formula must exist for the widths $s_n(B_p^r)_q$ for $1 \le q \le p \le \infty$ (triangle I of Fig. 1.1). (The weak asymptotics for region I, $d_n \sim n^{-r}$, has been given by Lorentz [1960], see Theorem 3.8 of Chapter 14). Since then this conjecture has been confirmed in the cases that correspond to the boundary of triangle I, and for $r = 1$, to the whole triangle (our formula (5.6) can be interpreted in this way). A solution for the whole triangle in the general case $r \ge 1$, for both $A = I$ and $A = \mathbb{T}$, has been announced by Buslaev and Tikhomirov in [1985] and published, with more details but some incomplete proofs, in [1990]. In their paper, Buslaev and Tikhomirov deal mainly with the interval $A = [0,1]$. In contrast, we give here an exposition for $A = \mathbb{T}$, in which case the results can be presented in a more explicit form. The main body of our discussion here and in §7 will be for $1 < q < p < \infty$. By continuity, this extends onto the whole triangle I.

We use the following notations. We shall write $\langle f, g\rangle := \int_{\mathbb{T}} fg\,dt$, and $f \perp g$ will mean $\langle f, g\rangle = 0$. We shall denote by D the differential operator d/dt. For $1 \le p < \infty$, we define Q_p to be the non-linear transformation

$$(Q_p f)(t) := |f(t)|^{p-1} \operatorname{sign} f(t).$$

Since the function $F(y) := |y|^{p-1}\operatorname{sign} y$ is continuous and strictly increasing, $Q_p f$ is continuous if and only if f is. Moreover, since $F(y)$ is uniformly continuous on every compact interval, Q_p is a continuous operator from $C(\mathbb{T})$ to $C(\mathbb{T})$. If $f \in L_p(\mathbb{T})$, $1 < p < \infty$, then $Q_p f \in L_{p'}$, $p' = p/(p-1)$, and $Q_{p'} Q_p f = f$ for every f. Also, for a differentiable f, one has $D(|f(t)|^p) = p(Q_p f)(t) \cdot f'(t)$, $1 < p < \infty$.

Let $\mathcal{F}$ denote the set of functions defined on $\mathbb{T}$ with $f(t+\pi) = -f(t)$ a.e. Note that $f \in \mathcal{F}$ implies $Q_p f \in \mathcal{F}$. We shall replace (6.1) by the following variation of it: Find

$$\text{(6.2)} \qquad C(p,q,r) := \sup\{\|f\|_q : f \in W_p^r \cap \mathcal{F}, \quad \|D^r f\|_p = 1\}.$$

For a function $f \in L_1(\mathbb{T})$, its r-th integral on $\mathbb{T}$, or *r-th periodic integral*, is the function $g(t)$, $t \in \mathbb{T}$, for which $g^{(r)}(t) = f(t)$. This g, defined up to an additive constant for every $r = 1, 2, \ldots$, exists if and only if $f \perp 1$. In particular, it exists for every $f \in \mathcal{F} \cap L_1$. By $I_r f$ we denote the r-th periodic integral with mean value zero, that is, $D^r I_r f = f$, $I_r f \perp 1$. If $f \in \mathcal{F}$ is differentiable, then $f' \in \mathcal{F}$. If $f \in \mathcal{F} \cap L_1$, then also $I_r f \in \mathcal{F}$. (It is enough to prove this for $r = 1$. For the function $f_1(t) := \int_0^t f(s)ds - (1/2)\int_0^\pi f(s)ds$ we have $f_1' = f$, $f_1(\pi + t) = -f_1(t)$, so that $f_1 = I_1 f \in \mathcal{F}$). For $f \in W_p^r \cap \mathcal{F}$ we have $f = I_r D^r f$.

Proposition 6.1. *For $r = 1, 2, \ldots$, the set $B_{p,\perp}^r := \{f \in B_p^r : f \perp 1\}$ is relatively compact in $C(\mathbb{T})$ if $r \geq 2$, $1 \leq p \leq \infty$ or if $r = 1$, $1 \leq p < \infty$. The set $B_{1,\perp}^1$ is relatively compact in every L_q, $1 \leq q < \infty$. Moreover, for $r \geq 1$, $1 < p < \infty$, the set $B_{p,\perp}^r$ is closed, and therefore compact, in $C(\mathbb{T})$.*

Proof. The set $B_{p,\perp}^1$, $1 \leq p \leq \infty$, is bounded in C, hence in every L_q. Indeed, every function $f \in B_{p,\perp}^1$ vanishes at some point t_0, and by Hölder's inequality,

$$|f(t)| = \left|\int_{t_0}^t f'ds\right| \leq |t - t_0|^{1/p'} \|f'\|_p \leq (2\pi)^{1/p'}.$$

On the other hand, it follows from (5.8) that for $(p,q) \neq (1,\infty)$, every $f \in B_p^1(A)$, $A = [0,1]$, can be approximated in L_q by the n-dimensional subspace of step functions with an error tending to zero as $n \to \infty$, so that $d_n(B_{p,\perp}^1(A))_q \to 0$. The same is true, by a change of scale, for $A = [0, 2\pi]$. By (v) of §1, $B_{p,\perp}^1$ is therefore relatively compact in C if $p > 1$ or in every L_q, $q < \infty$, if $p = 1$.

If $f \in B_{p,\perp}^r$ with $r > 1$, we similarly prove that the norms $\|f^{(j)}\|_\infty$ are bounded for $j = r-1, \ldots, 0$ by a constant independent of f, and we can apply the same, or even simpler, argument.

To prove that $B_{p,\perp}^r$, $1 < p < \infty$, is closed in $C(\mathbb{T})$, let us assume that $f_n \in B_{p,\perp}^r$ and $\|f_n - f\|_C \to 0$ for some $f \in C(\mathbb{T})$. Since the unit ball of L_p, $1 < p < \infty$, is weakly compact, we may assume without loss of generality that the functions $w_n := f_n^{(r)}$ converge weakly to some w, $\|w\|_p \leq 1$. Due to the weak convergence, $w_n \perp 1$ implies $w \perp 1$, and $f_n(t) = (I_r w_n)(t) \to (I_r w)(t)$ at every point t, hence $f = I_r w$, so that $f \in B^r_{p,\perp}$. □

Theorem 6.2. *For $1 < p, q < \infty$, $r = 1, 2, \dots$, the supremum (6.2) is attained by some extremal function f. This function satisfies everywhere on $\mathbb{T}$ the differential equation*

$$Q_q f = (-1)^r \lambda D^r Q_p D^r f \tag{6.3}$$

with $\lambda = \|f\|_q^q$ and has the property $Q_p D^r f \in C^r$ (and in particular, $f \in C^r$).

Proof. The supremum in (6.2) is taken over f that belong to a compact subset of $B^r_{p,\perp}$. Since the norm $\|f\|_q$ depends continously on $f \in L_q$, the supremum is attained by some f. For this f and arbitrarily fixed $g \in W^r_p \cap \mathcal{F}$, the function

$$\phi(\eta) := \|f + \eta g\|_q / \|D^r f + \eta D^r g\|_p$$

is defined for sufficiently small $|\eta|$. It is differentiable and has a maximum at $\eta = 0$, hence $\phi'(0) = 0$. We have

$$\frac{d}{d\eta}(\|f + \eta g\|_q) = \left(\int_{\mathbb{T}} |f + \eta g|^q \, dt\right)^{1/q-1} \langle Q_q(f + \eta g), g\rangle,$$

similarly for $\frac{d}{d\eta}(\|D^r f + \eta D^r g\|_p)$. Since $\|D^r f\|_p = 1$, $\phi'(0) = 0$ implies

$$\langle Q_q f, g\rangle - \lambda\langle Q_p D^r f, D^r g\rangle = 0, \quad \lambda = \|f\|_q^q \tag{6.4}$$

Integrating by parts r times in $\langle Q_q f, g\rangle$, we obtain

$$\langle h, D^r g\rangle = 0, \qquad h := I_r Q_q f - \lambda(-1)^r Q_p D^r f. \tag{6.5}$$

Since $f \in \mathcal{F}$, we have $Q_q f \in \mathcal{F}$, $D^r f \in \mathcal{F}$, $I_r Q_q f \in \mathcal{F}$, hence $h \in \mathcal{F}$. Since $D^r f \in L_p$, we also have $Q_p D^r f \in L_{p'}$, therefore $h \in L_{p'}$. If now $g^* := I_r Q_{p'} h$, then $g^* \in W^r_p \cap \mathcal{F}$. With $g = g^*$, (6.5) becomes $\|h\|_{p'}^{p'} = 0$, hence $h(t) = 0$ a.e. This shows first of all that we may take $Q_p D^r f \in C^r(\mathbb{T})$, and (6.3) follows from $h(t) = 0$ by differentiation. □

The rest of this section is devoted to a study of solutions of the non-linear differential equation (6.3).

1. For $r = 1, 2, \dots$, $1 \le q \le p < \infty$, we shall call $f \in W^r_p(\mathbb{T})$ a *spectral function,* $\lambda \in \mathbb{R}$ its *spectral value,* and (f, λ) a *spectral couple,* if

$$\|D^r f\|_p = 1, \quad Q_p D^r f \in C^r(\mathbb{T}), \quad Q_q f = (-1)^r \lambda D^r Q_p D^r f.$$

We do not impose any other restrictions on f and λ. In particular, we do not assume that $f \in \mathcal{F}$. By Theorem 6.2, every extremal function of (6.2) is a spectral function. It will suffice for our purposes to consider only $1 < q < p < \infty$, the interior of the triangle I of Fig. 1.1. However, some of the following properties are actually stated for a wider range of p, q.

2. *If $1 < p,q < \infty$ and (f,λ) is a spectral couple, then $\lambda = \|f\|_q^q$.*

Indeed,

$$\begin{aligned}\|f\|_q^q &= \langle Q_q f, f\rangle = (-1)^r\lambda\langle D^r Q_p D^r f, f\rangle \\ &= (-1)^r\lambda(-1)^r\langle Q_p D^r f, D^r f\rangle = \lambda\|D^r f\|_p^p = \lambda.\end{aligned}$$ □

When $p = q = 2$, the equation (6.3) becomes $f = (-1)^r\lambda D^{2r} f$, and a simple analysis shows that every 2π-periodic solution is given by $f = A\sin(nt+\mathrm{c})$, with arbitrary real A, c and $n = 1, 2, \ldots$.

3. *If (f,λ) is a spectral couple, then also $(g, \lambda n^{-rq})$ is one, with $g(t) := \pm f(nt + \mathrm{c})n^{-r}$, $c \in \mathbb{T}$, $n = 1, 2, \ldots$*

Indeed, with $v := Q_p D^r f$, equation (6.3) becomes $(Q_q f)(t) = (-1)^r \lambda v^{(r)}(t)$ which yields $(Q_q f)(nt+\mathrm{c}) = (-1)^r\lambda v^{(r)}(nt+\mathrm{c})$, equivalently $n^{r(q-1)} Q_q g = (-1)^r\lambda n^{-r} D^r Q_p D^r g$, and for g we have (6.3) with λ replaced by λn^{-rq}.

Our goal is to show (see Theorem 6.4) that all spectral couples can be generated in this way from one single couple. We establish this fact in several steps.

4. *For $1 < p,q < \infty$, if (f,λ) is a spectral couple and $v := Q_p D^r f$, then for $t \in \mathbb{T}$,*

$$\frac{|v(t)|^{p'}}{p'} + \sum_{k=1}^{r-1}(-1)^k v^{(k)}(t) f^{(r-k)}(t) + \frac{|f(t)|^q}{\lambda q} = \frac{r - p^{-1} + q^{-1}}{2\pi}. \tag{6.6}$$

(If $r = 1$, the sum on the left-hand side should be omitted.)

Indeed, let $E(t)$ denote the left-hand side of (6.6). We have $v \in C^r$,

$$\frac{d}{dt}\left(|v(t)|^{p'}\right) = p'(Q_{p'}v)(t)v'(t) = p' f^{(r)}(t) v'(t),$$

and by (6.3),

$$\frac{d}{dt}\left(|f(t)|^q\right) = q(Q_q f)(t) f'(t) = q(-1)^r\lambda v^{(r)}(t) f'(t).$$

From this immediately follows, due to the cancellations of terms, that $E'(t) = 0$, $t \in \mathbb{T}$, so that $E(t) \equiv E$ =const. Also, integrating by parts we have

$$(-1)^k\langle v^{(k)}, f^{(r-k)}\rangle = \langle v, f^{(r)}\rangle = \|D^r f\|_p^p = 1$$

for each k. Furthermore, $\int_{\mathbb{T}} |v|^{p'}\,dt = \int_{\mathbb{T}} |f^{(r)}|^p\,dt = 1$, and by **2**, $\int_{\mathbb{T}} |f|^q\,dt = \lambda$. Hence $2\pi E = \int_{\mathbb{T}} E(t)\,dt = r - p^{-1} + q^{-1}$. □

Most of the following discussion will be based on the count of zeros and sign changes for continuous functions. For $f \in C(\mathbb{T})$, we denote by $Z(f)$ the number of zeros of f, with each zero counted only once. By $S(f)$ we denote the number of sign changes of f, that is, the maximal number k for which there exist k points $t_1 < \ldots < t_k$ with

$$f(t_i)f(t_{i+1}) < 0, \quad i = 1, \ldots, k-1, \quad f(t_k)f(t_1) < 0. \tag{6.7}$$

Both $Z(f), S(f)$ can be infinite, and always $S(f) \leq Z(f)$. If $S(f)$ is finite, it is even. As a functional on $C(\mathbb{T})$, $S(f)$ has the following semi-continuity property: If $S(f) < \infty$, and $\|f - g\|_C$ is sufficiently small, then $S(g) \geq S(f)$.

If $f \neq 0$, $S(f) < \infty$, and $f(z) = 0$, then there exist α, β, $\alpha < z < \beta$, for which $f(\alpha)f(\beta) \neq 0$ and f does not change sign on (α, z) and on (z, β). We say that z is a zero *with* or *without sign change* if $f(\alpha)f(\beta) < 0$ or $f(\alpha)f(\beta) > 0$, respectively. We shall need the following variations of Rolle's theorem.

Lemma 6.3. (i) *If $f \in C^1(\mathbb{T})$ has no zero intervals, then $S(f) \leq Z(f) \leq S(f')$. Moreover, if $Z(f) < \infty$ and f has a multiple zero, then $Z(f) < Z(f')$.*

(ii) *If $f \in C^1(\mathbb{T})$, $f \neq 0$, $S(f) < \infty$, and f has a zero without sign change, then $S(f) < S(f')$.*

Proof. (i) If $(z_i)_1^k$ are the zeros of f, then for $i = 1, \ldots, k$, in each open interval (z_i, z_{i+1}), $z_{k+1} = z_1 + 2\pi$, there are points s_i for which $\operatorname{sign} f'(s_i) = (-1)^i$, hence $S(f') \geq Z(f)$. Moreover, f' has zeros in all these intervals. If one of the z_i is a multiple zero of f, then it is a zero of f', in addition to the k zeros inside the intervals, hence $Z(f') > Z(f)$.

(ii) Let f have k sign changes at the points (t_i), as in (6.7), and a zero z without sign change, with the corresponding α, β. Then for some $i = i_0$ the function f does not change sign between z an t_{i_0}. Replacing t_{i_0} by the three points α, z, β, one obtains a new sequence $\bar{t}_1, \ldots, \bar{t}_{k+2}$ of length $k+2$, with alternating signs of the differences

$$f(\bar{t}_2) - f(\bar{t}_1), \ldots, f(\bar{t}_{k+2}) - f(\bar{t}_{k+1}), \; f(\bar{t}_1) - f(\bar{t}_{k+2}).$$

This yields $S(f') \geq k + 2$. □

5. *For $1 < p, q < \infty$, every spectral function f has a finite (even) number of zeros. Moreover, $Z(f) = Z(D^j f) = Z(v) = Z(D^k v)$, where $v := Q_p D^r f$, $j, k = 1, \ldots, r$. All zeros of f and v are simple.*

Indeed, f must have zeros since $Q_q f \perp 1$ because of (6.3). If there were infinitely many zeros, they would have a limit point t_0 for which $f(t_0) = f'(t_0) = \ldots = f^{(r)}(t_0) = v(t_0) = 0$ contradicting (6.6). Thus, $Z(f) < \infty$. By (6.3), $Q_q f$ is a constant multiple of $v^{(r)}$. Therefore by Lemma 6.3 (i),

$$\begin{aligned} Z(f) &\leq Z(f') \leq \ldots \leq Z(f^{(r)}) = Z(v) \\ &\leq Z(v') \leq \ldots \leq Z(v^{(r)}) = Z(Q_q f) = Z(f). \end{aligned}$$

If f or v had multiple zeros, one would have $Z(f') > Z(f)$ or $Z(v') > Z(v)$ leading to a contradiction: $Z(f) < Z(f)$. □

Although a sum of two spectral functions is not necessarily a spectral function, sums $f_1 + f_2$ still have properties similar to those of spectral functions.

6. (a) *If* $1 < q < p < \infty$ *and* f_1, f_2 *are two spectral functions, then*

$$S(f_1 + f_2) \le \max\{S(f_1), S(f_2)\} < \infty, \tag{6.8}$$

and similarly $S(f_1^{(j)} + f_2^{(j)}) < \infty$, $j = 1, \dots, r$.

(b) *If* f_1, f_2 *correspond to the same spectral value and* $f_2 \ne -f_1$, *then all the zeros of* $f_1 + f_2$, $f_1' + f_2'$, $\dots$, $f_1^{(r)} + f_2^{(r)}$ *are with sign changes.*

Suppose that (f_1, λ_1), (f_2, λ_2) are spectral couples and, say, $0 < \lambda_1 \le \lambda_2$. For $\varepsilon > 0$, let $\sigma(\varepsilon) := S(f_1 + \varepsilon f_2)$. For all sufficiently small ε we have $\sigma(\varepsilon) = S(f_1) = Z(f_1) =: N$. Indeed, if $t_1, \dots, t_N$ are the zeros of f_1, then $f_1'(t_i) \ne 0$ by 5, and by continuity, there exist neighborhoods $V_{t_1}, \dots, V_{t_N}$ in which $f_1' + \varepsilon f_2' \ne 0$ for all small ε, so that $f_1 + \varepsilon f_2$ has exactly one zero in each V_{t_i}. On the other hand, $f_1(t) + \varepsilon f_2(t) \ne 0$ if $t \in \mathbb{T} \setminus \bigcup_i V_{t_i}$ and $\varepsilon > 0$ is sufficiently small.

In the following estimate (6.9) we use Lemma 6.3 (i), equation (6.3), and the identity $\operatorname{sign}(a + b) = \operatorname{sign}(|a|^{q-1}\operatorname{sign} a + |b|^{q-1}\operatorname{sign} b)$. We have

$$\begin{aligned}
\sigma(\varepsilon) &\le S(Df_1 + \varepsilon Df_2) \le \dots \le S(D^r f_1 + \varepsilon D^r f_2) \\
&= S(Q_p D^r f_1 + Q_p(\varepsilon D^r f_2)) = S(Q_p D^r f_1 + \varepsilon^{p-1} Q_p D^r f_2) \\
&\le S(D^r Q_p D^r f_1 + \varepsilon^{p-1} D^r Q_p D^r f_2) = S((-1)^r \lambda_1^{-1} Q_q f_1 + (-1)^r \lambda_2^{-1} \varepsilon^{p-1} Q_q f_2) \\
&= S(Q_q f_1 + Q_q((\lambda_1 \lambda_2^{-1} \varepsilon^{p-1})^{1/(q-1)} f_2)) \\
&= S(f_1 + \varepsilon^{(p-1)/(q-1)} (\lambda_1/\lambda_2)^{1/(q-1)} f_2) = \sigma(\varepsilon^{(p-1)/(q-1)} (\lambda_1/\lambda_2)^{1/(q-1)}).
\end{aligned} \tag{6.9}$$

Iterating this inequality for $0 < \varepsilon < 1$, we obtain $\sigma(\varepsilon) \le \sigma(\varepsilon_0)$, where ε_0 can be made arbitrarily close to zero, so that we may assume that $\sigma(\varepsilon_0) = N$. Consequently, $\sigma(\varepsilon) \le N$ for $0 < \varepsilon < 1$. But then also $\sigma(1) = S(f_1 + f_2) \le N$ for otherwise one could choose $\varepsilon < 1$ so close to 1 that $\sigma(\varepsilon) > N$.

Using $\sigma_1(\varepsilon) := S(f'_1 + \varepsilon f'_2)$, we similarly prove that $S(Df_1 + Df_2) < \infty$; the same is true for the derivatives of higher orders.

To prove (b), we take in (6.9) $\lambda_1 = \lambda_2$, $\varepsilon = 1$. If $f_1 + f_2$ or, for some j, $D^j f_1 + D^j f_2$ had a zero without sign change, then by Lemma 6.3 (ii) the corresponding inequality in (6.9) would be strict: $S(D^j f_1 + D^j f_2) < S(D^{j+1} f_1 + D^{j+2} f_2)$ for $0 \le j \le r-1$ or $S(Q_p D^r f_1 + Q_p D^r f_2) < S(DQ_p D^r f_1 + DQ_p D^r f_2)$ for $j = r$. Hence a contradiction: $\sigma(1) < \sigma(1)$. □

For a spectral function f, let $t_1 < \dots < t_m$ be all its zeros on $\mathbb{T}$, and let $s_k := (1/2)(t_k + t_{k+1})$, $k = 1, \dots, m$, $t_{m+1} = t_1 + 2\pi$ be the midpoints of the intervals between them.

Theorem 6.4. *For* $1 < q < p < \infty$, *a spectral function* f *is odd with respect to each of its zeros* t_k, *that is,* $f(t_k - t) = -f(t_k + t)$, *and is even with respect to each* s_k. *The same applies to the derivatives* $f', \dots, f^{(r)}$ *and their zeros.*

Moreover, the number of zeros is even, $m = 2n$, and the points t_k are evenly spaced on $\mathbb{T}$. The zeros of $f^{(j)}$ are s_k if j is odd, and t_k if j is even. All functions $f, f', \ldots, f^{(r)}$ are $2\pi/n$-periodic on $\mathbb{T}$.

Proof. We need to consider only f and f'. For each k, with $f(t)$ also $f(t_k \pm t)$ is a spectral function, with the same λ. Then $f(t_k - t) + f(t_k + t)$ has a zero at $t = 0$ without sign change. By **6**, this function must be zero.

The zeros of f' are also evenly spaced, with the same distance $2\pi/n$ between them. They cannot coincide with the simple zeros t_k of f. So each interval (t_k, t_{k+1}) contains exactly one zero s_k^* of f'. Then the next interval contains the zero $s_k^* + 2\pi/n$ and also (due to the fact that f is odd at t_{k+1}) the zero $2t_{k+1} - s_k^*$. This yields $s_k^* + 2\pi/n = 2t_{k+1} - s_k^*$, or $s_k^* = s_k$. □

We shall call a function $f \in C(\mathbb{T})$ *waveshaped on* $\mathbb{T}$ *with zeros* $t_0 < \ldots < t_{2n-1}$, $t_{k+1} - t_k = \pi/n$, if it is odd with respect to each t_k, even with respect to the midpoints s_k. As we have proved, all spectral functions are waveshaped. Every waveshaped function is $2\pi/n$-periodic and has mean value zero. For a waveshaped f, the function $Q_q f$, $1 < q < \infty$, is also waveshaped and satisfies $Q_q f \perp 1$ on $\mathbb{T}$. Moreover,

7. *If f is waveshaped on $\mathbb{T}$ with zeros $(t_k)_0^{2n-1}$, then the mean value zero integrals $I_r f$ are also waveshaped, with zeros $t_{k,r} = t_k$ if r is even, $t_{k,r} = s_k$, if r is odd.*

Indeed, the function $g(t) := \int_{s_1}^t f(s)ds$ vanishes at the midpoints s_k and is odd with respect to each s_k, even with respect to each t_k. This means, in particular, that $g \perp 1$, so that $g = I_1 f$ and $I_1 f$ is waveshaped. By induction on r, the same applies to all $I_r f$.

We are now ready to state the main theorem of this section.

Theorem 6.5 (Buslaev and Tikhomirov [1990]). *For $1 < q < p < \infty$, $r = 1, 2, \ldots$, there exists a unique spectral function f^* on $\mathbb{T}$ with exactly two zeros $0, \pi$ on $\mathbb{T}$ and with $f^{*\prime}(0) > 0$. All the solutions of the extremal problem (6.2) are given by the functions $\pm f^*(t + c)$, and $\|f^*\|_q = C(p, q, r)$. All the spectral functions corresponding to this problem are of the form $f_{n,c}(t) = \pm n^{-r} f^*(nt + c)$, with some $n = 1, 2, \ldots$ and some $c \in \mathbb{T}$.*

Proof. A function f^* exists. Indeed, let f be some spectral function and let $2n$ be its number of zeros. By **3**, $f^*(t) = \pm n^r f((t - c)/n)$, $c \in \mathbb{T}$, is a spectral function with only two zeros. We can select c and the sign $\pm$ so that the zeros are $0, \pi$ and $f^{*\prime}(0) > 0$.

The function f^* is unique. Let f_1, f_2 be two spectral functions with zeros $0, \pi$ and with $f_i'(0) > 0$ (and $\|D^r f_i\|_p = 1$), $i = 1, 2$. Let $f_1 \neq f_2$. Then also $D^r f_1 \neq D^r f_2$. For $D^r f_1 = D^r f_2$ would imply $f_1 = f_2$, by integration and due to the fact that all $D^j f_i$, $j = 0, \ldots, r - 1$ have zeros on $\mathbb{T}$. From $f_i'(0) > 0$ it follows that f_1, f_2 are of the same sign on $\mathbb{T}$. On one of the intervals $(a, a + \pi)$, hence by symmetry on both of them, $D^r f_1 \neq D^r f_2$. Since the L_p norm of these functions on each of the intervals is the same, this implies that on each interval $D^r f_1 - D^r f_2$ changes sign at least twice, on $\mathbb{T}$ at least four times, in contradiction to (6.8).

From f^*, we can derive all other spectral functions by means of the formula $f(t) = \pm n^{-r} f^*(nt+c)$. Since $\|f\|_q = n^{-r}\|f^*\|_q$, extremal functions are those with $n = 1$. □

We shall establish some properties of the constants $C(p,q,r)$.

Theorem 6.6. *The constants $C(p,q,r)$, $r = 1,2,\ldots$, are defined and continuous in the region I of Fig. 1.1 ($1 \le q \le p \le \infty$) and have the following duality property:*

$$C(p,q,r) = C(q',p',r), \quad 1 \le p,q \le \infty. \tag{6.10}$$

Proof. That the constants are finite for (p,q) in I we know from Theorem 6.2. We shall compare the constants C and C^* that result when W_p^r in (6.2) is replaced by W and W^*, where W and $W^* \subset W$ are some linear function spaces. Assume that for some $\varepsilon > 0$ for each $f \in W \cap \mathcal{F}$ there is an $f^* \in W^* \cap \mathcal{F}$ so that

$$\|f - f^*\|_q \le \varepsilon, \quad \|D^r f - D^r f^*\|_p \le \varepsilon. \tag{6.11}$$

Then

$$C^* \le C \le \varepsilon + (1+\varepsilon)^{-1} C^*. \tag{6.12}$$

In fact, if $\|D^r f\|_p \le 1$, then $\|D^r f^*\|_p \le 1$ and $\|f\|_q \le \|f^*\|_q + \varepsilon$, and therefore

$$\begin{aligned} C &\le \sup\{\|f^*\|_q + \varepsilon : f \in W \cap \mathcal{F},\ \|D^r f^*\|_p \le 1+\varepsilon\} \\ &\le \sup\left\{(1+\varepsilon)^{-1}\|f^*\|_q + \varepsilon : f^* \in W \cap \mathcal{F},\ \|D^r f^*\|_p \le 1\right\}. \\ &= \varepsilon + (1+\varepsilon)^{-1} C^* \end{aligned} \tag{6.13}$$

We apply this to $W = W_p^r$, $W^* = \mathcal{T}_n$, with $C^* = C^n(p,q,r) =: C^n$. Functions $f \in W_p^r \cap \mathcal{F}$, as well as their derivatives $f', \ldots, f^{(r-1)}$, all have zeros on $\mathbb{T}$. Therefore the inequality $\|f^{(r)}\|_p \le 1$ implies that all f belong to the same class $\mathrm{Lip}(\alpha,p)$, with an absolute constant. The properties of the simultaneous trigonometric approximation (see [**CA**, Theorem 2.7, p.207, and (2.17), p.206]) yield then, for each $\varepsilon > 0$ and $f \in W_p^r \cap \mathcal{F}$, a trigonometric polynomial T_n for which $\|f - T_n\|_q \le \varepsilon$, $\|f^{(r)} - T_n^{(r)}\|_p \le \varepsilon$. We can reformulate (6.12) in this case by saying that for some n

$$|C - C^n| \le \varepsilon. \tag{6.14}$$

To compare C^n with $C_1^n := C^n(p_1,q_1,r)$ we need the inequalities, for $p \le p_1$

$$(6.15)\qquad \left(\frac{1}{2\pi}\right)^{1/p-1/p_1}\|T_n\|_p \le \|T_n\|_{p_1} \le \left(\frac{2nr+1}{2\pi}\right)^{1/p-1/p_1}\|T_n\|_p.$$

The first relation is a consequence of Hölder's inequality, the second is the inequality of Nikolskii [**CA**, (2.15), p.102]. We derive from this that for a fixed n and any $\varepsilon > 0$ there is a $\delta > 0$ so that $|1/p - 1/p_1| < \delta$ implies $1-\varepsilon \le \|T_n\|_p/\|T_n\|_{p_1} \le 1+\varepsilon$. For C^n, C_1^n this yields, similarly to (6.13),

$$(6.16)\qquad (1-\varepsilon)^2 \le C_1^n/C^n \le (1+\varepsilon)^2.$$

From (6.14), a similar relation for $C_1 := C(p_1, q_1, r)$ and C_1^n, and (6.16), we derive that $C_1 \to C$ if $p_1 \to p$, $q_1 \to q$. □

For some combinations (p,q,r), the constants $C(p,q,r)$ can be evaluated explicitly. For $p = q = 2$, if a_k, b_k are the Fourier coefficients of some $f \in W_2^r \cap \mathcal{F}$, then $a_0 = 0$ and $\sum_{k=1}^{\infty}(a_k^2 + b_k^2) \le \sum_{k=1}^{\infty} k^{2r}(a_k^2+b_k^2)$, equivalent to $\|f\|_2 \le \|f^{(r)}\|_2$. Therefore, $C(2,2,r) = 1$ for all r, and the extremal functions are $A\cos t + B\sin t$.

Proposition 6.7. *For $r \ge 1$, $1 \le p, q \le \infty$,*

$$(6.17)\qquad C(\infty, q, r) = \|\varphi_{1,r}\|_q, \qquad C(p,1,r) = \|\varphi_{1,r}\|_{p'},$$

where $\varphi_{1,r} := I_r(\text{sign}\sin t)$ are the splines of §3 of Chapter 6.

Proof. By (3.19) of Chapter 6, for $f \in W_\infty^r$, $1 \le q \le \infty$, we have $\|f - U_{1,r}f\|_q \le \|\varphi_{1,r}\|_q\|f^{(r)}\|_\infty$. It follows from the definition of $\mathcal{F}$ that every continuous function $f \in \mathcal{F}$ vanishes at some two equidistant points $t_0, t_1 := t_0 + \pi$ on $\mathbb{T}$ which implies $U_{1,r}f_1 = 0$ for some translation f_1 of f because $U_{1,r}f_1$ is a spline interpolating f_1 at two equidistant points. Since the norms of f and $f^{(r)}$ do not change under translation, we have $\|f\|_q \le \|\varphi_{1,r}\|_q\|f^{(r)}\|_\infty$ for every $f \in W_\infty^r \cap \mathcal{F}$, with the equality for $f = \varphi_{1,r}$. This proves the first formula (6.17), and the second follows by duality. □

For the values of some of the norms (6.17), see (3.20) of Chapter 6. In particular, $C(\infty,\infty,r) = C(1,1,r) = K_r$, where K_r is the Favard constant.

We can find an explicit formula for $C(p,q,1)$ for all $1 \le q \le p \le \infty$.

Proposition 6.8. *For $1 \le q \le p \le \infty$, $E := (1 - 1/p + 1/q)/2\pi$,*

$$(6.18)\qquad C(p,q,1) = (\pi/2)E^{1/p-1/q}(p')^{1/p}q^{-1/q}\left(\int_0^1 \frac{dx}{(1-x^q)^{1/p}}\right)^{-1}.$$

Proof. If f is an extremal function for (6.2) with $r = 1$, then by (6.6)

$$(6.19)\qquad \frac{|f'|^p}{p'} + \frac{|f|^q}{\lambda q} = E, \qquad \lambda = \|f\|_q^q = C(p,q,1)^q.$$

We may assume that $f(0) = 0$, $f'(0) > 0$. Then by Theorems 6.4 and 6.5, $f'(t) > 0$ on $(-\pi/2, \pi/2)$. On this interval we obtain from (6.19) the differential equation for $y = f(t)$:

$$y' = (E\lambda q - |y|^q)^{1/p} \left(\frac{p'}{\lambda q}\right)^{1/p}. \tag{6.20}$$

In addition to this, we have $f(0) = 0$, and, from (6.19), since $f'(\pi/2) = 0$ by Theorem 6.4, also $f(\pi/2) = (E\lambda q)^{1/q}$. This leads to the relation

$$\int_0^{(E\lambda q)^{1/q}} \frac{dy}{(E\lambda q - y^q)^{1/p}} = \frac{\pi}{2}\left(\frac{p'}{\lambda q}\right)^{1/p},$$

from which (6.18) easily follows, since $\lambda^{1/q} = C(p,q,1)$. □

The values of $C(\infty, q, 1)$ and of $C(p, 1, 1)$ are contained in (6.17); alternatively, they can be derived from (6.18) as limit cases. The values of $C(1, q, 1)$ and of $C(p, \infty, 1)$ also follow from (6.18) (see Problem 10.7).

§ 7. Results of Buslaev and Tikhomirov

Here we evaluate the widths $s_n(B_p^r(\mathbb{T}))$ for $1 \le q \le p \le \infty$ (region I of Fig.1.1, §1) with the help of the spectral functions and constants $C(p,q,r)$ of §6. Another essential tool will be the spline interpolation operators $U_{n,r}$ of §3 of Chapter 6. For $f \in C(\mathbb{T})$, $U_{n,r}f$ is the spline $S \in \widetilde{\mathcal{S}}_{n,r}$ (that is, a periodic spline of order r, with simple knots $(k\pi/n)_{k=0}^{2n-1}$) interpolating f at the points $t_k := t_{k,r} := (k - \beta_r)\pi/n$, where $\beta_r = 0$ if r is even, $\beta_r = 1/2$ if r is odd. For the error of approximation one has the integral representation of Zhensykbaev and Korneichuk:

$$f(t) - U_{n,r}(f,t) = \int_{\mathbb{T}} G(t,s) f^{(r)}(s)\, ds. \tag{7.1}$$

where the kernel $G(t,s) := G_{n,r}(t,s)$, with $r \ge 2$, has the properties (see Theorem 3.5 of Chapter 6):

(i) $G_{n,r}(t,s) = 0$ for $t = t_k$ and for $s = k\pi/n$, $k = 0, \dots, 2n-1$;
(ii) $\theta_r \sin(nt + \beta_r\pi) G_{n,r}(t,s) \sin ns \ge 0$, $\theta_r = \pm 1$, $t, s \in \mathbb{R}$;
(iii) $G_{n,r}(t,s) = (-1)^r G_{n,r}(s - \beta_r\pi/n,\ t - \beta_r\pi/n)$;
(iv) $G_{n,r}(t,s)$ belongs to $C^1(\mathbb{T})$ in each variable, $r \ge 3$, and for $r = 2$ it has one-sided derivatives at each point.

In the proof of the following theorem (not given in Buslaev and Tikhomirov [1990]) we use for $r \ge 2$ some ideas of Pinkus [1985] and Chen and Li [1992]. For $r = 1$ we give a simple direct proof.

Theorem 7.1 (Upper Estimate). *For $f \in B_p^r$, $r \ge 1$, $1 < q < p < \infty$, one has*

$$\|f - U_{n,r}f\|_q \le C(p,q,r) n^{-r}. \tag{7.2}$$

Proof. (a) For $r = 1$, by Theorem 5.4, every function $f \in B_p^1[0,1]$ can be approximated in $L_q[0,1]$ by a step function on $2n$ equal subintervals, with an error $\le (1/2)C_1(p,q)(2n)^{-1}$, where $C_1(p,q)$ is the constant (5.5). By a change of scale, this gives for every $f \in B_p^1[0,2\pi]$

$$\|f - U_{n,1}f\|_q[0,2\pi] \le \frac{1}{2}C_1^*(p,q)(2n)^{-1}, \tag{7.3}$$

where

$$C_1^*(p,q) := \sup\left\{\|f\|_q[0,2\pi] : f \in B_p^1[0,2\pi],\ f(0) = 0\right\}.$$

There is a simple relation between $C_1^*(p,q)$ and $C(p,q,1)$. To show this, consider an arbitrary $f \in B_p^1[0,2\pi]$ with $f(0) = 0$. We extend f to $[2\pi, 4\pi]$ by setting $f(2\pi + t) = f(2\pi - t)$, then to $[-4\pi, 0]$ by $f(-t) = -f(t)$, and finally to $\mathbb{R}$ as a 8π-periodic function. If now $g(t) := (1/4)f(4t)$, then g is 2π-periodic, $g \in \mathcal{F}$, and $\int_0^{2\pi} |g'|^p dt = 4^{-1}\int_0^{8\pi} |f'|^p = 1$. Therefore, by the definition of $C(p,q,1)$, we have $\|g\|_q(\mathbb{T}) \le C(p,q,1)$. On the other hand, $\|g\|_q(\mathbb{T}) = 4^{-1}\|f\|_q(\mathbb{T})$, hence $\|f\|_q(\mathbb{T}) \le 4C(p,q,1)$. Passing to supremum over f yields $C_1^*(p,q) \le 4C(p,q,1)$. Together with (7.3), this gives the desired upper estimate for $r = 1$.

(b) For $r \ge 2$, we introduce the operator

$$L_n(f,t) := L_{n,r}(f,t) := \int_{\mathbb{T}} G(t,s)f(s)\,ds$$

(it is related to $U_{n,r}$: $L_{n,r}g = I_r g - U_{n,r}I_r g$.) The proof will consist in comparing a function $u \in L_p(\mathbb{T})$, $\|u\|_p = 1$, realizing the norm

$$\gamma := \|L_n\|_{p\to q} := \sup_{\|u\|_p=1} \|L_n u\|_q\,,$$

with a spectral function of the problem (6.2).

The functions $f = L_n u$, $\|u\|_p \le 1$ are uniformly bounded in $C(\mathbb{T})$ since $\|f\|_C \le \max_{t,s} G(t,s)\|u\|_1$ and $\|u\|_1 \le (2\pi)^{1/p'}\|u\|_p \le 2\pi$. These functions are also equicontinuos, due to the uniform continuity of $G(t,s)$ on $\mathbb{T}\times\mathbb{T}$. By Arzela's theorem, they form a relatively compact set in $C(\mathbb{T})$. Moreover, they form a closed, and therefore compact, set; this can be established using the weak compactness of the ball $\|u\|_p = 1$ as in the proof of Proposition 6.1. It follows that for $1 < p < \infty$ the norm $\|L_n\|_{p\to q}$ is attained on some $u \in L_p$. For this u and arbitrary $v \in L_p$, the function

$$\psi(\eta) := \|L_n(u + \eta v)\|_q / \|u + \eta v\|_p$$

is defined for small $|\eta|$ and has a maximum at $\eta = 0$, hence $\psi'(0) = 0$, and, as in the proof of Theorem 6.2,

$$\langle Q_q L_n u, L_n v\rangle - \gamma^q\langle Q_p u, v\rangle = 0.$$

The first term is an iterated integral of a function continuous on $\mathbb{T}\times\mathbb{T}$. Changing the order of integration, we have

$$(L_n^* Q_q L_n u, v) - \gamma^q (Q_p u, v) = 0,$$

where L_n^* is the integral operator with the transposed kernel $G^*(t,s) := G(s,t)$. Since $v \in L_p$ is arbitrary, we get the equation for the extremal function u:

$$L_n^* Q_q L_n u = \gamma^q Q_p u. \tag{7.4}$$

It follows from (7.4) that $u \in C(\mathbb{T})$. We may further assume that for all t

$$u(t) \sin nt \geq 0. \tag{7.5}$$

Indeed, due to the property (ii) of the kernel $G(t,s)$, the norm $\|L_n u\|_q$ can only increase if $u(t)$ is replaced by $|u(t)|\text{sign } \sin nt$, while $\|u\|_p$ does not change. Thus $u(k\pi/n) = 0,\ k = 0, 1, \ldots, 2n-1$.

Now let f be a spectral function for the chosen p, q, r, with period $2\pi/n$ and zeros $t_k = (k + \beta_r)\pi/n$. By **3** of §6 and Theorem 6.5, such an f exists, is waveshaped, and

$$\|f\|_q = C(p,q,r)n^{-r} =: \alpha.$$

We may further assume that $f(t)\sin(nt + \beta_r \pi) \geq 0$. The function f belongs to $C^r(\mathbb{T})$ and $f \perp 1$. If $w := D^r f$, then $f = I_r w$, where I_r is the periodic integral with mean value zero. The equation (6.3), where $\lambda = \|f\|_q^q = \alpha^q$ now takes the form

$$I_r Q_q I_r w = (-1)^r \alpha^q Q_p w. \tag{7.6}$$

The left-hand side of (7.6) can be reset in a different form. Since $U_{n,r}$ interpolates at the points t_k and $f = I_r w$ has them as zeros,

$$L_n w = I_r w - U_{n,r} I_r w = I_r w \tag{7.7}$$

On the other hand, let T be the translation operator $(Tg)(t) := g(t + \beta_r \pi/n)$, $g \in C(\mathbb{T})$. Because $G^*(t,s) = G(s,t) = (-1)^r G(t - \beta_r\pi/n,\ s - \beta_r \pi/n)$, we have for $g \in C(\mathbb{T})$

$$(-1)^r L_n^* g = T^{-1} L_n T = T^{-1} I_r T g - T^{-1} U_{n,r} I_r T g. \tag{7.8}$$

We take $g = Q_q f (= Q_q I_r w)$. With f, also $Q_q f$ is waveshaped with zeros (t_k); also $I_r T(Q_q f)$ has these points as zeros, and the last term in (7.8) vanishes. As for $T^{-1} I_r T g$, it has mean value zero, and its r-th derivative is $T^{-1} T g = g$, so it reduces to $I_r g$. Hence $L_n^* g = (-1)^r I_r g$, and with (7.7), relation (7.6) becomes

$$L_n^* Q_q L_n w = \alpha^q Q_p w, \tag{7.9}$$

in complete analogy to (7.4).

From (7.4), (7.9) it follows that $Q_p u$, $Q_p w$ are continuously differentiable. Plainly, the points $k\pi/n$ are the only zeros of $Q_p w = |w|^{p-1} \text{sign}\, w$, and they are zeros of $Q_p u$. We shall prove that for some $0 < \rho < \infty$,

$$(7.10) \qquad |u(t)| \le \rho |w(t)|, \qquad t \in \mathbb{T}.$$

This is equivalent to the boundedness of $Q_p(u,t)/Q_p(w,t)$. Around the points $k\pi/n$, this follows from the L'Hospital rule applied to this quotient (the zeros of $Q_p w = Q_p D^r f$ are simple by **4** of §6), and can then be extended to the whole of $\mathbb{T}$.

Let $\rho = \rho(n)$ be the minimal constant for which (7.10) is valid. Since $\|u\|_p = \|w\|_p = 1$, from (7.10) follows that $\rho \ge 1$. For each fixed t, the sign pattern of $G(t,s)$ is the same as, or opposite to, that of $u(s)$, $w(s)$. From this fact and (7.10), we have $|(L_n u)(t))| \le \rho |(L_n w)(t)|$ for all t. Moreover, due to the properties of $G(t,s)$, the three functions: $(L_n u)(s)$, $(L_n w)(s)$, and, for a fixed t, $G^*(t,s)$, the kernel of the integral operator L_n^*, change sign exactly at the points (t_k), so that

$$|(L_n^* Q_q L_n u)(t)| \le \rho^{q-1} |(L_n^* Q_q L_n w)(t)|, \quad t \in \mathbb{T}.$$

From (7.4), (7.9) now follows

$$|u(t)| \le \rho_1 |w(t)|, \qquad \rho_1 := (\alpha/\gamma)^{\frac{q}{p-1}} \rho^{\frac{q-1}{p-1}}.$$

Since ρ is the minimal constant in (7.10), we have $\rho_1 \ge \rho$. With $q < p$ and $\rho \ge 1$, this yields $\gamma \le \alpha$, leading to the desired upper estimate (7.2). □

Theorem 7.2 (Buslaev and Tikhomirov [1990]). *For $r \ge 1$, $1 < q < p < \infty$,*

$$(7.11) \qquad s_{2n}(B_p^r)_q \ge C(p,q,r) n^{-r}, \quad n = 1, 2, \ldots$$

For a function $u \in L_1(\mathbb{T})$, $u \perp 1$, $1 < p < \infty$, we denote by $J_{r,p}u$ the r-th periodic integral of u with the minimal L_p norm. If u_r is some fixed r-th periodic integral of u, then $J_{r,p}u = u_r - c$, where c is the best L_p-approximation to u_r from the one-dimensional subspace of constants. From the properties of the space L_p, this approximant is unique, and from the properties of any Banach space, it depends continuously on u_r in the L_p norm (see [**CA**, p.59, and Theorem 1.2, p.60]). This defines $J_{r,p}u$ uniquely. It is characterized by the property

$$(7.12) \qquad Q_p J_{r,p} u \perp 1 \quad \text{for every } u \in L_1, \quad 1 < p < \infty,$$

which follows from the condition $z'(0) = 0$ with $z(c) := \|J_{r,p}u + c\|_p^p$.

Our method of obtaining the lower estimate for the widths is based on some iteration process that starts with an arbitrary function $u \in L_1(\mathbb{T})$ with mean value zero and produces a sequence of functions w_k. A subsequence of their integrals f_k converges to a spectral function f. Its properties enable us to employ Theorem 5.1, a variant of Borsuk's theorem.

The iterative process we use is related to the equation (6.3). We take some $w_0 \in L_1$, $w_0 \ne 0$, $w_0 \perp 1$, and for $k = 0, 1, 2, \ldots$ define w_k with $w_k \perp 1$ inductively:

(7.13) $$Q_p w_{k+1} := (-1)^r \mu_k J_{r,p'} Q_q J_{r,q} w_k, \quad p' = p/(p-1),$$

with $\mu_k > 0$ chosen from the condition $\|w_{k+1}\|_p = 1$. This defines all w_k with $w_k \perp 1$. In fact, if w_k is defined, then (7.12) ensures that the right-hand side of (7.13) is defined and orthogonal to 1. Then $w_{n+1} \perp 1$ is also defined.

Lemma 7.3. *Let $f_k := J_{r,q} w_k$. Then*

(7.14) $$\|f_k\|_q^q \le \mu_k^{-1} \le \|f_{k+1}\|_q^q, \quad k = 1, 2, \ldots.$$

Proof. We use Hölder's inequality, (7.13), integration by parts, (7.12), and the identity $\|Q_p g\|_{p'} = \|g\|_p^{p-1}$ to derive for $k \ge 1$

$$\begin{aligned} 1 &= \|w_{k+1}\|_p^{p-1} \|w_k\|_p \ge \langle Q_p w_{k+1}, w_k \rangle \\ &= (-1)^r \mu_k \langle J_{r,p'} Q_q f_k, w_k \rangle = \mu_k \langle Q_q f_k, f_k \rangle = \mu_k \|f_k\|_q^q, \end{aligned}$$

which proves the first inequality (7.14). We now use this first inequality and similarly prove the second inequality:

$$\begin{aligned} 1 = \|w_{k+1}\|_p^p &= \langle Q_p w_{k+1}, w_{k+1} \rangle = (-1)^r \mu_k \langle J_{r,p'} Q_q f_k, w_{k+1} \rangle \\ &= \mu_k \langle Q_q f_k, J_{r,q} w_{k+1} \rangle = \mu_k \langle Q_q f_k, f_{k+1} \rangle \le \mu_k \|f_k\|_q^{q-1} \|f_{k+1}\|_q \\ &= \|f_{k+1}\|_q (\mu_k \|f_k\|_q^q)^{(1-1/q)} \mu_k^{1/q} \le \mu_k^{1/q} \|f_{k+1}\|_q. \end{aligned}$$

□

Since $\|w_k\|_p = 1$ and since, due to (7.12), the function $(J_{r,q} w_k)(t)$ has zeros, we have , as in the proof of Proposition 6.1, $\|f_k\|_\infty = \|J_{r,q} w_k\|_\infty \le M$, with M independent of k. From this fact and (7.14) we obtain the existence of the limit

(7.15) $$\mu := \lim_{k\to\infty} \mu_k = \lim_{k\to\infty} \|f_k\|_q^{-q} > 0.$$

We now establish a relation between the iterative process (7.13) and the spectral functions. By **1** of §6, a couple (f, λ) is spectral for the set p, q, r if

(7.16) $$\|D^r f\|_p = 1, \quad Q_q D^r f \in C(\mathbb{T}), \quad Q_q f = (-1)^r \lambda D^r Q_p D^r f.$$

Lemma 7.4. *For each starting function $w_0 \ne 0$, $w_0 \perp 1$, the sequence (w_k) of (7.13) contains a subsequence (w_{k_i}) for which $f_{k_i} := J_{r,q} w_{k_i}$ converge uniformly to a spectral function (with a spectral value $\lambda = \mu^{-1}$) having at most as many changes of sign as w_0 has.*

Proof. Using the weak compactness of the unit ball in L_p, $1 < p < \infty$, and the compactness, by Proposition 6.1, of the set $B_{p,\perp}^r := \{f \in B_p^r : f \perp 1\}$ in $C(\mathbb{T})$, one can select a subsequence w_{k_i} converging weakly to some w, $\|w\|_p = 1$, with $I_r w_{k_i}$ converging uniformly to $I_r w$. But then also the f_{k_i} converge uniformly to $f := J_{r,q} w$ because $J_{r,q} u = I_r u - c$ for every $u \perp 1$, with c depending continuously on $I_r u$. The operators Q_p, $1 < p < \infty$, preserve uniform convergence, and so do the operators $J_{r,p}$. It follows from (7.13) that $Q_p w_{k_i+1}$ converge uniformly. Consequently, $Q_{p'} Q_p w_{k_i+1} = w_{k_i+1}$ converge uniformly to some v, $\|v\|_p = 1$. We let $k \to \infty$ in (7.13) and with μ of (7.15) obtain

$$Q_p v = (-1)^r \mu J_{r,p'} Q_q J_{r,q} w. \tag{7.17}$$

We now prove that (f, μ^{-1}) is a spectral couple. For $k_i \to \infty$,

$$\begin{aligned} \langle Q_p w_{k_i+1}, w_{k_i} \rangle &= (-1)^r \mu_{k_i} \langle J_{r,p'} Q_q J_{r,q} w_{k_i}, w_{k_i} \rangle \\ &= \mu_{k_i} \langle Q_q f_{k_i}, f_{k_i} \rangle = \mu_{k_i} \|f_{k_i}\|_q^q \to \mu \cdot \mu^{-1} = 1. \end{aligned}$$

Since $Q_p w_{k_i+1} \to Q_p v$ uniformly and $w_{k_i} \to w$ weakly in L_p, this implies $\langle Q_p v, w \rangle = 1$. On the other hand, since $\|v\|_p = \|w\|_p = 1$, by Hölder's inequality $\langle Q_p v, w \rangle \le \|v\|_p^{p-1} \|w\|_p = 1$. We have, therefore, the case of equality which can occur only if $|Q_p v|^{p'} = |w|^p$, $\operatorname{sign} Q_p v = \operatorname{sign} w$ a.e., or, equivalently, if $w = v$. We have proved that $\|w\|_p = 1$, from (7.17) we see that $Q_q w \in C^r(\mathbb{T})$, and the last condition (7.16), with $\lambda = \mu^{-1}$, follows from (7.17) by differentiation.

Finally, the number of sign changes remains invariant if Q_p or Q_q are applied, and by Rolle's theorem does not increase upon application of $J_{r,1}$ or $J_{r,p'}$. If w_0 has $2m$ sign changes on $\mathbb{T}$, then due to (7.13), all the w_k and all f_k have $\le m$ of them, and this extends also to $f = \lim f_{k_i}$. □

We also need a simple lemma of a rather general nature.

Lemma 7.5. *Let $(F_k)_1^\infty$ be a sequence of real-valued functions defined and continuous on some compact A. If $(F_k(y))$ is a monotone non-decreasing sequence for each $y \in A$, then*

$$\sup_k \min_{y \in A} F_k(y) = \inf_{y \in A} \sup_k F_k(y). \tag{7.18}$$

Proof. Let L and R be the left and the right side of (7.18). We have to prove only that $R \le L$ since obviously $L \le R$ for any sequence (F_k) ("maxmin $\le$ minmax"). For the proof we note that the sets $A_k := \{y \in A : F_k(y) \le L\}$ are non-empty for all k. Since the functions F_k are continuous, the A_k are closed, and $A_1 \supset A_2 \supset \dots$ due to the monotonicity of the F_k. Therefore the set $\cap A_k$ is non-empty. If $y^* \in \cap A_k$, then $\sup_k F_k(y^*) \le L$, which implies $R \le L$. □

Proof of Theorem 7.2. To obtain (7.11) we use Theorem 5.1. We take $X := L_q(\mathbb{T})$. For the space Y we take the $2n$-dimensional subspace $\eta_1 + \dots + \eta_{2n+1} = 0$ of $y = (\eta_1, \dots, \eta_{2n+1}) \in \mathbb{R}^{2n+1}$ equipped with the norm $\|y\| = (2\pi)^{-1} \sum_{k=1}^{2n+1} |\eta_k|$. Moreover, Z_1 will be the one-dimensional subspace of constants of L_q. The unit sphere Σ of Y is given by

$$\eta_1 + \dots + \eta_{2n+1} = 0, \quad |\eta_1| + \dots + |\eta_{2n+1}| = 2\pi.$$

For $y \in \Sigma$, we set

$$t_0 = 0, \quad t_j = |\eta_1| + \ldots + |\eta_j|, \quad j = 1, \ldots, 2n+1.$$

We define the function $w_0 := w_0^y$ by $w_0^y(t) = \operatorname{sign} \eta_j$ on each non-empty interval (t_{j-1}, t_j). Then $w_0 \perp 1$, $\|w_0\|_p \le 1$. The inductive procedure (7.13) defines $w_k := w_k^y$ and $f_k^y := J_{r,q} w_k^y$.

For a fixed $k \ge 1$, the map $\Phi : \Sigma \to L_q(\mathbb{T})$ shall be $\Phi(y) := I_r w_k^y$. Plainly, Φ is continuous, odd: $\Phi(-y) = -\Phi(y)$, and satisfies $\Phi(y) + c \in B_p^r(\mathbb{T})$ for any constant c. For the set $\Omega := \{\Phi(y) + c : y \in Y, c \in \mathbb{R}\}$ Theorem 5.1 yields

$$s_{2n}(B_p^r(\mathbb{T}))_q \ge s_{2n}(\operatorname{co} \Omega)_q \ge \min_{y,c} \|\Phi(y) + c\|_q = \min_y \|f_k^y\|_q.$$

The above inequalities hold for every k, so we can take maximum with respect to k in the last term. By (7.18), with $F_k(y) := \|f_k^y\|_q$, for the spectral functions $f^y = \lim_{i\to\infty} f_{k_i}^y$,

$$s_{2n}(B_p^r(\mathbb{T}))_q \ge \max_k \min_y \|f_k^y\|_q = \min_y \|f^y\|_q.$$

Now w_0 has at most $2n$ sign changes, so by Lemma 7.4, f^y has $2m$, $m \le n$ of them. By Theorem 6.5,

$$\|f^y\|_q = C(p,q,r)m^{-r} \ge C(p,q,r)n^{-r}.$$

This establishes (7.11). □

We now prove the main theorem of this section.

Theorem 7.6. *For* $1 \le q \le p \le \infty$, $r, n = 1, 2, \ldots$, *for all three types of widths,*

$$s_{2n}\left(B_p^r(\mathbb{T}), L_q(\mathbb{T})\right) = C(p,q,r)n^{-r}, \qquad 1 \le q \le p \le \infty, \tag{7.19}$$

where $C(p,q,r)$ *is the constant* (6.2). *The space* $\widetilde{S}_{n,r}$ *is an optimal* $2n$-*dimensional subspace, the interpolation operator* $U_{n,r}$ *is an optimal linear operator of rank* $2n$, *and sampling at* $2n$ *equidistant points is an optimal set of linear functionals for the Gelfand widths.*

Proof. The validity of (7.19) for $1 < q < p < \infty$, the interior of triangle I, follows from Theorems 7.1 and 7.2, and we shall extend it by continuity to the boundary of the triangle. We shall prove that the widths $s_n(B_p^r)_q$ depend continuously on p for every fixed q. The proof is similar to that of Proposition 6.5, where we established the continuity of constants $C(p,q,r)$.

Let $p_1 \ge p$. Then for any given $\varepsilon > 0$ the following relations hold for the linear widths:

$$(2\pi)^{1/p-1/p_1} \delta_n(B_{p_1}^r)_q \le \delta_n(B_p^r)_q \le \varepsilon + c_m(p,p_1)\delta_n(B_{p_1}^r)_q, \tag{7.20}$$

where $c_m(p,p_1) := \left(\frac{2mr+1}{2\pi}\right)^{1/p-1/p_1}$, with m depending on ε. The first inequality (7.20) follows from the inequality $\|D^r f\|_p \le (2\pi)^{1/p-1/p_1}\|D^r f\|_{p_1}$.

To prove the second inequality (7.20), we use the Fejér operators σ_m. We have [**CA**, p.268] for $1 \le p \le \infty$: $\|\sigma_m\|_{p\to p} = 1$, $\sigma_m(f) \to f$ in L_p. Note also that σ_m commutes with the differentiation operator D. If $f \in B_p^r$, then for any m, $\|D^r\sigma_m(f)\|_p = \|\sigma_m(D^r f)\|_p \le 1$, hence by Nikolskii's inequality (6.15), $\|D^r\sigma_m(f)\|_{p_1} \le c_m(p,p_1)$, so that $\sigma_m(f) \in c_m(p,p_1)B_{p_1}^r$. If now $U_n : L_q \to L_q$ is an operator of rank $\le n$, then so is the operator $U_n\sigma_m$, and

$$\|f - U_n\sigma_m(f)\|_q \le \|f - \sigma_m(f)\|_q + \|\sigma_m(f) - U_n\sigma_m(f)\|_q. \tag{7.21}$$

Due to the relative compactness of B_p^r in L_q, there is an m for which $\|f - \sigma_m(f)\|_q < \varepsilon$ for all $f \in B_p^r$. With this m we have

$$\sup_{f\in B_p^r} \|f - U_n\sigma_m(f)\|_q < \varepsilon + c_m(p,p_1) \sup_{g\in B_{p_1}^r} \|g - U_n g\|_q.$$

From this we obtain (7.20) by taking the infimum over all U_n of rank $\le n$. The continuity of the linear widths with respect to p follows from (7.20).

In the case of the Kolmogorov widths, we take instead of $U_n f$, a best approximation to f from an n-dimensional subspace; the rest of the proof remains the same. Similar arguments apply to the Gelfand widths.

Once the continuity of the widths with respect to p has been established, we use the continuity of $C(p,q,r)$ to extend (7.19) from the inside of triangle I of Fig. 1.1 to $1 < q = p < \infty$ and to $p = \infty$, $1 < q < \infty$. The operator $U_{n,r}$ remains optimal for $p = \infty$ due to the inclusion $B_\infty^r \subset (2\pi)^{1/p} B_p^r$. Since the L_q norm depends continuously on q, we can extend the upper estimate for $\|f - U_{n,r}f\|_q$ to $1 < p \le \infty$, $q = 1$ and to $p = q = \infty$.

The lower estimate in the case $q = 1$, $1 < p \le \infty$ is needed only for the Kolmogorov and Gelfand widths since the linear widths are larger. Both estimates are obtained analogously, so we consider the Kolmogorov widths. By (ix) and (vi) of §1, we may consider only approximating spaces consisting of continuous functions and containing constants. Let X_n be a near-optimal subspace for $d := d_n(B_p^r)_1$, that is, $E(B_p^r, X_n)_1 < d + \varepsilon$, for some small $\varepsilon > 0$. Since $B_{p,\perp}^r$ is relatively compact in C, it has in C a finite ε-net $B_\varepsilon \subset B_{p,\perp}^r$. Due to the continuity of the L_q norm with respect to q, we have for the distance from X_n, $\sup_{g\in B_\varepsilon} \rho(g, X_n)_q < d + \varepsilon$ if q is sufficiently close to 1. Since an ε-net in C is an ε-net in every L_q, this implies $E(B_{p,\perp}^r, X_n)_q < d + 2\varepsilon$, and since X_n contains constants, we can replace here $B_{p,\perp}^r$ by B_p^r. It follows that for arbitrary $\varepsilon > 0$ and q sufficiently close to one,

$$d_n(B_p^r)_1 \ge d_n(B_p^r)_q - 2\varepsilon = C(p,q,r)n^{-r} - 2\varepsilon.$$

Due to the continuity of $C(p,q,r)$, letting $q \to 1$ yields $d_n(B_p^r)_1 \ge C(p,1,r)n^{-r}$. The lower estimate for $(p,q) = (\infty,\infty)$ can be obtained similarly.

Formula (7.19) in the remaining case $p = q = 1$ now follows from the continuity with respect to p. □

Buslaev and Tikhomirov also state that for $p \ge q$

$$s_{2n-1}\left(B_p^r(\mathbb{T}), L_q(\mathbb{T})\right) = C(p,q,r)n^{-r}, \tag{7.22}$$

that is, that the $(2n-1)$-widths coincide with the $2n$-widths. We have already noted this fact in §5 for the Kolmogorov widths in the case $p=q=\infty$; an optimal $(2n-1)$-dimensional subspace was the space $\mathcal{T}_{n-1}$ of trigonometric polynomials of degree $\le n-1$. This subspace is a likely candidate for an optimal subspace for all other combinations (p,q,r), $p\ge q$, but our knowledge here is incomplete. The most general result belongs to Taikov [1967] who proved that for $1\le p\le\infty$

$$E_{n-1}(B_p^r(\mathbb{T}))_1=\|\varphi_{1,r}\|_{p'}n^{-r},\qquad \varphi_{1,r}(t):=I_r(\operatorname{sign}\sin t).$$

This agrees with (7.22) due to (6.16), but only for the Kolmogorov widths, since it is not clear from Taikov's proof that there is, for $r\ne 1$, a linear method with the same error of approximation.

Buslaev amd Tikhomirov claim that there exists an optimal *spline* operator of rank $2n-1$ which realizes (7.22). In the general situation their outlined proof seems to be involved; for $r=1$, see Problem 10.8.

We conclude this section with a very condensed historical sketch of the study of widths $s_n(B_p^r(A))_q$, $1\le q\le p\le\infty$ (triangle I), for $A=\mathbb{T}$ and $A=[a,b]$. After the seminal paper of Kolmogorov [1936], there has been a lull until Stechkin's [1954]. In Lorentz [1960], the *weak equivalence*

$$d_n(B_p^\alpha)_q \quad\text{and}\quad \delta_n(B_p^\alpha)_q\sim n^{-\alpha},\quad p\le q, \tag{7.23}$$

is proved for the unit ball B_p^α of the Lipschitz space $\operatorname{Lip}(\alpha,p)$, $\alpha>0$, $1\le q\le p\le\infty$. The elementary method applies to a variety of cases, for $A=\mathbb{T}$ or $[a,b]$, $[a,b]^m$. In the same year, Tikhomirov [1960] found for $A=\mathbb{T}$ the *exact value* $d_{2n-1}(B_\infty^r)_\infty=K_rn^{-r}$, $r=1,2,\dots$, and Makovoz [1969], Subbotin [1970] added the same for $p=q=1$.

In [1971] Korneichuk evaluated, for concave moduli of continuity, the widths $d_{2n-1}(B^rH_\infty^\omega(\mathbb{T}))_\infty$ (see §5). Since then, most authors discuss the three widths s_n at once, with methods that require integer $\alpha=r$. A major step has been achieved in the papers Ligun [1980], Makovoz [1979], Pinkus [1979], published almost simultaneously. For $A=\mathbb{T}$, they treat the vertical ($p=\infty$) and the horizontal sides of I:

$$s_{2n-1}(B_\infty^r)_q=s_{2n}=\|\varphi_{n,r}\|_q=\|\varphi_{1,r}\|_q\,n^{-r} \tag{7.24}$$

for $1\le q\le\infty$. The same formulas yield $s_{2n-1}(B_p^r)_1$ if one replaces $\|\varphi_{n,r}\|_q$ by $\|\varphi_{n,r}\|_{p'}$. The estimates of these widths from below were obtained by the method of Makovoz [1972], where the case $p=\infty$, $q=1$ had been considered.

Splines do not work for (p,q) in the interior of I. New approximation methods are necessary. Meanwhile in the important paper of Tikhomirov [1965], the extremal problem (6.1), the differential equation (6.3) and the spectral functions of 1 of §6 appear for the first time, and in Tikhomirov [1969] they are used for evaluation of the widths in the non-periodic case $p=q=\infty$. A more general result, for the diagonal $1\le p=q\le\infty$ in the non-periodic case, was obtained by Pinkus [1985]. See also Micchelli and Pinkus [1978] for the $A = [a, b]$ version of (7.24).

The paper of Buslaev and Tikhomirov [1990], announced in [1985], gives a proof of Theorem 7.6 and sketches the proof of its non-periodic analogue. In [1992], independently of Buslaev and Tikhomirov [1990], Chen and Li consider the diagonal case $p = q$ for $A = \mathbb{T}$. Their proof is almost sufficient for the upper estimate in the case $q < p$; we have used some of their ideas for our Theorem 7.1.

One should add that in this generality it is not clear which of the cases $A = [a, b]$ or $A = \mathbb{T}$ is simpler.

§ 8. Classes of Differentiable Functions on an Interval

The theory of widths of the Sobolev classes $B_p^r := B_p^r(I)$, $I := [0, 1]$ is in many ways parallel to that of $B_p^r(\mathbb{T})$, but the results are less definitive, and optimal subspaces and operators are usually described only implicitly (see Tikhomirov [1969] and Buslaev, Tikhomirov [1990]). A notable exception is the case $r = 1, p \geq q$ for which the widths have been found in §5. Optimal subspaces in this case are step functions with equidistant breakpoints. It turnes out that spaces of splines of order r are optimal for all r, but positions of the breakpoints depend on p, q, and the widths can be effectively computed only for small r and some exceptional p, q (see Problem 10.9). However, if we content ourselves with asymptotic equivalences, then the formulas for $B_p^r(I)$ can be derived from those for $B_p^r(\mathbb{T})$ by a fairly elementary argument.

We define the subspaces $W_{p,0}^r$, $W_{p,1}^r$, $\widetilde{W_p^r}$, formed by all functions f of the Sobolev space $W_p^r = W_p^r(I)$ satisfying, respectively, the conditions $f^{(k)}(0) = 0$, or $f^{(k)}(1) = 0$, or $f^{(k)}(0) = f^{(k)}(1)$, $k = 0, \ldots, r-1$. We also define $\widehat{W_p^r} := W_{p,0}^r \cap W_{p,1}^r$. We shall denote $B_{p,0}^r := W_{p,0}^r \cap B_p^r$, and similarly in the other cases. All these classes are relatively compact in L_q if $(p, q, r) \neq (1, \infty, 1)$. More generally, by the argument used in the proof of Proposition 6.1 we obtain:

The class $B_{p,*}^r[a, b]$ of all functions $f \in B_p^r[a, b]$ for which $f^{(j)}(x_j) = 0$ for some $x_j \in [a, b]$, $j = 0, 1, \ldots, r-1$, is relatively compact in $L_q[a, b]$ if $(p, q, r) \neq (1, \infty, 1)$.

If $f \in \widetilde{B_p^r}$ and $g(t) := (2\pi)^{-1/p} f(t/(2\pi))$, then $g \in B_p^r(\mathbb{T})$. Therefore,

$$(8.1) \qquad s_n(B_p^r(\mathbb{T}))_q = (2\pi)^{r-1/p+1/q} s_n(\widetilde{B_p^r})_q, \quad 1 \leq p, q \leq \infty.$$

Clearly, $B_p^r = B_{p,0}^r + \mathcal{P}_{r-1}$. From this and (ii), (vi) of §1 follows

$$(8.2) \qquad s_{n+r}(B_p^r)_q \leq s_n(B_{p,0}^r)_q \leq s_n(B_p^r)_q.$$

Similarly, we prove

Proposition 8.1. *For $1 \le p, q \le \infty$ and $n, r = 1, 2, \ldots$,*

$$d^{n+r}(B_p^r(I))_q \le d^n(\widetilde{B_p^r})_q \le d^n(B_p^r(I))_q. \tag{8.3}$$

Proof. The second inequality is obvious. To prove the first inequality, we observe that the class $\widetilde{B_p^r}$ is obtained from B_p^r by imposing r linear conditions

$$\ell_k(f) := f^{(k)}(1) - f^{(k)}(0) = 0, \ k = 0, \ldots, r-1.$$

The functionals $\ell_k(f)$ are defined not for all $f \in L_q$, so we introduce $\ell_k^h(f) := \ell_k(f_h)$, where $f_h(t) := (2h)^{-1} \int_{-h}^h f(t+s)\,ds$. Defining f_h, we assume here that f is extended from [0,1] to $(-\infty, 0)$, $(0, +\infty)$ by its Taylor's polynomials of degree $r-1$ at the points $t = 0$ or $t = 1$. We have $f_h^{(r)}(t) := (2h)^{-1} \int_{-h}^h f^{(r)}(t+s)\,ds$, and by the integral version of Minkowski's inequality (Hardy, Litttlewood, Pólya [B-1964, Theorem 202]), $\|f_h^{(r)}\|_p \le \|f^{(r)}\|_p$. Thus $f_h \in B_p^r$ if $f \in B_p^r$. Moreover, $\|f - f_h\|_q \to 0$, $f \in L_q$, if $h \to 0$ (see [**CA**, (4.19), p.34]).

Let Y_h^r be the subspace of $f \in L_q$ satisfying $\ell_k^h(f) = 0$, $k = 0, 1, \ldots, r-1$. Then for $(p, q, r) \ne (1, \infty, 1)$

$$\sup_{B_p^r \cap Y_h^r} \|f - f_h\|_q \to 0 \quad \text{as} \quad h \to 0. \tag{8.4}$$

Indeed, since $f_h = f$ for $f = \text{const.}$, we may take the supremum in (8.4) over $f \in B_p^r$ satisfying $f_h^{(j)}(0) = f_h^{(j)}(1) = 0$, and (8.4) follows since for $h < 1$ these f belong to the relatively compact class $B_{p,*}^r[0, 2]$.

Let now X^n be a near-optimal subspace of L_q of codimension $\le n$ for which

$$\sup\left\{\|f\|_q : f \in \widetilde{B_p^r} \cap X^n\right\} \le d^n(\widetilde{B_p^r})_q + \varepsilon,$$

and let $Z^{n+r} := X^n \cap Y_h^r$. If $f \in B_p^r \cap Z^{n+r}$, then $\|f_h^{(r)}\| \le 1$, hence $f_h \in \widetilde{B_p^r} \cap X^n$, so that $\|f_h\|_q \le d^n(\widetilde{B_p^r})_q + \varepsilon$, and the first inequality now follows from (8.4) by letting $h \to 0$. □

Proposition 8.2. *For $1 \le p, q \le \infty$, $r \ge 1$,*

$$s_n(B_p^r(I))_q = \hat{C}(p, q, r) n^{-r} + O(n^{-r-1}), \tag{8.5}$$

with $\hat{C}(p, q, r) := 2^r (2\pi)^{-r+1/p-1/q} C(p, q, r)$, where $C(p, q, r)$ is the constant (6.2), and s_n stands for d_n or d^n.

Proof. For the Gelfand widths, (8.5) follows from (8.3), (7.19), and (8.1). From (8.2) we obtain the same formula for $d^n(B_{p,0}^r(I))_q$. By the duality (2.7), the formula extends to $d_n(B_{p,0}^r(I))_q$, with the constant $\hat{C}(q', p', r)$ which is equal, by (6.10), to $\hat{C}(p, q, r)$. Again by (8.2), the formula now extends to $d_n(B_p^r(I))_q$. □

It is instructive to compare (8.5) with (4.11). Since $C(2,2,r) = 1$, we obtain an asymptotic formula for the eigenvalues of the problem (4.8)-(4.9):

$$\lambda_{n,r} = \pi^{-2r} + O(n^{-2r-1}).$$

Formula (8.5) can be extended to more general classes of functions. For $y \in W_p^r(I)$, let $A(y)$ be the differential expression

$$A(y) = y^{(r)} + a_1(t)y^{(r-1)} + \ldots + a_r(t)y, \tag{8.6}$$

where $a_k \in C^{r-k}(I)$, $k = 1, \ldots, r$. Let B_p^A be the set of all $y \in W_p^r$ for which $\|A(y)\|_p \le 1$. It turns out that the asymptotic behavior of the widths of B_p^A does not depend essentially on $a_1(t), \ldots, a_r(t)$:

Theorem 8.3 (Makovoz [1983]). *For $1 \le q \le p \le \infty$ and $A(y)$ of the form (8.6),*

$$s_n(B_p^A) = \hat{C}(p,q,r)n^{-r} + O(n^{-r-1}),$$

where s_n stands for d_n or d^n.

§ 9. Classes of Analytic Functions

Let G be a domain (an open, connected set) in $\mathbb{C}$, let $H_\infty(G)$ be the Hardy space of bounded analytic functions on G, equipped with the norm $\|f\| := \sup_{z\in G} |f(z)|$. Let $\mathcal{A} := \mathcal{A}_G$ be the subset of $H_\infty(G)$ with $\|f\| \le 1$. For a compact subset $K \subset G$, $\mathcal{A}_G$ is compact in $C(K)$ by Arzela's theorem. We would like to find the width of $\mathcal{A}_G$ in the norm $\|f\|_{C(K)}$.

Our main result is Theorem 9.1 by Fisher and Micchelli [1980] in which G is the unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$. We prove that the widths of $\mathcal{A}_D$ are equal to the norms of some extremal Blaschke products.

A Blaschke product of degree n is (see Appendix 3) a function of the form

$$B(z) = \eta \prod_{j=1}^{n} (z - a_j)/(1 - \bar{a}_j z), \quad |a_j| < 1, \quad |\eta| = 1. \tag{9.1}$$

If $|z| = 1$, then $|1 - \bar{a}_j z| = |1 - a_j z^{-1}| = |z - a_j|$, so that $|B(z)| = 1$; hence, by the maximum principle, $\|B\| = 1$. The set of all Blaschke products of degree $\le n$ is denoted $\mathcal{B}_n$, and $\mathcal{B}_0$ means the set of all constants η, $|\eta| = 1$.

Theorem 9.1. *For $n = 1, 2, \ldots$,*

$$s_n(\mathcal{A})_{C(K)} = \inf\left\{\|B\|_{C(K)} : B \in \mathcal{B}_n\right\}, \tag{9.2}$$

where s_n stands for any of d_n, d^n, or δ_n.

Proof. We consider the integral

$$\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{B(z)}{B(\zeta)}\cdot\frac{1-|z|^2}{1-\zeta\bar{z}}\cdot\frac{f(\zeta)}{\zeta-z}\,d\zeta \tag{9.3}$$

with $z\in D$, $f\in\mathcal{A}$ and some $B(z)$ of the form (9.1). The integrand has a simple pole at $\zeta=z$ and poles at the zeros a_j of $B(z)$. Using residues one can represent (9.3) in the form

$$f(z)-\sum_{j=1}^{n}u_j(z)f^{(\nu_j)}(a_j)=:f(z)-(Tf)(z),$$

where $u_j(z)$ are some functions continuous on D (they depend on $a_1,\dots,a_n$ as parameters). If the a_j do not coalesce, then all $\nu_j=0$; if some a_j has multiplicity r, then the corresponding ν_j runs through $0,1,\dots r-1$. Thus defined, Tf is a linear operator of rank $\le n$. We have, for $f\in\mathcal{A}_D$,

$$|f(z)-(Tf)(z)|\le\frac{1}{2\pi}|B(z)|\int_{|\zeta|=1}\frac{1-|z|^2}{|\zeta-z|^2}|d\zeta|.$$

The last integral is equal to 2π since the integrand is the familiar Poisson kernel. Indeed, with $z=re^{i\phi},\zeta=e^{i\theta}$, one has $|\zeta-z|^2=1+r^2-2r\cos(\phi-\theta)$, $|d\zeta|=d\theta$. Hence $\|f-Tf\|_K\le\|B\|_K$. Taking infimum over all $B\in\mathcal{B}_n$ we obtain the desired upper bound for $\delta_n(\mathcal{A})$.

For the lower estimate we use Borsuk's theorem in the form of Theorem 5.1. Let $z_0,\dots,z_n$ be some arbitrarily fixed distinct points in D and let $w=(w_0,\dots,w_n)\in\mathbb{C}^{n+1}$, $w\ne 0$. The Pick-Nevanlinna Theorem 4.2 of Appendix 3 asserts that the infimum

$$\rho(w):=\inf\{\|f\|:f\in H_\infty(D),\quad f(z_j)=w_j,\quad j=0,\dots,n\}$$

is attained for unique $f:=f_w$ of the form $f=\rho(w)B_w$, $B_w\in\mathcal{B}_n$.

The map $\Psi:w\to f$, as a map from $\mathbb{C}^{n+1}\setminus\{0\}$ to $C(K)$, is continuous. To prove this, we first note that $\rho(w)$ is a continuous function of w. Indeed, $\rho(0)=0$, and if $P(z)$ is a polynomial of degree n for which $P(z_j)=w_j$, $j=0,\dots,n$, then $\rho(w)\le\|P\|\to 0$ as $w\to 0$. This means that $\rho(w)$ is continuous at $w=0$ and, consequently, at every other point w since obviously $|\rho(w+\Delta w)-\rho(w)|\le\rho(\Delta w)$. Suppose now that $w^k=(w_0^k,\dots,w_n^k)\to w^*=(w_0^*,\dots,w_n^*)\ne 0$ while for some $\varepsilon>0$ and all k we have $\|\Psi(w^k)-\Psi(w^*)\|\ge\varepsilon$. By Montel's theorem we may assume that $\Psi(w^k)$ converge to some f. Then $f(z_j)=w_j^*$ for each j and $\|f\|=\lim\|\Psi(w^k)\|=\rho(w^*)$, while $f\ne\Psi(w^*)$, a contradiction to the uniqueness of f. Thus Ψ maps $\mathbb{C}^{n+1}\setminus\{0\}$ continuously into $C(K)$ and so does $\Phi(w):=B_w=\Psi(w)/\rho(w)$. The function Φ is odd, that is, $\Phi(-w)=\Phi(w)$.

We now apply Theorem 5.1. We put $X:=C(K)$, $Z_\nu=\{0\}$. We let Σ to be the unit ball $|w_0|^2+\dots+|w_n|^2=1$ of $\mathbb{C}^{n+1}=:Y_{n+1}$. With $\Omega=\Phi(\Sigma)\subset\mathcal{A}$, also $\operatorname{co}\Omega\subset\mathcal{A}$. From (5.2),

$$s_n(\mathcal{A})\ge s_n(\operatorname{co}\Omega)\ge\inf\left\{\|B_w\|_{C(K)}:w\in\Sigma\right\}\ge\inf\left\{\|B\|_{C(K)}:B\in\mathcal{B}_n\right\},$$

which is the desired lower bound for $s_n(\mathcal{A})$. □

It follows from the above proof that the infimum in (9.2) is attained by some $B^* \in \mathcal{B}_n$. Such B^* must have exactly n zeros in D, counting multiplicities, for otherwise $zB^*(z) \in \mathcal{B}_n$ and $\|zB^*(z)\|_{C(K)} < \|B^*\|_{C(K)}$.

Corollary 9.2. *For* $0 < r < 1,\ n = 1, 2, \ldots,$

$$s_n(\mathcal{A})_{C(\overline{D}_r)} = r^n. \tag{9.4}$$

Proof. With the help of Jensen's formula, for a function $g(z)$, $g(0) \neq 0$, analytic and with zeros a_k, $1 \le k \le n - s$, $0 \le s \le n$ in $\overline{D}_r$, we have

$$\begin{aligned}\log \|z^s g(z)\|_{C(\overline{D}_r)} &\ge \frac{1}{2\pi} \int_{|z|=r} \log |z^s g(z)|\, |dz| \\ &= \log r^s + \log |g(0)| - \sum \log |a_k|.\end{aligned}$$

For a Blaschke product $B \in \mathcal{B}_n$ this yields $\|B\|_{C(\overline{D}_r)} \ge r^n$, with the infimum attained for $B(z) = z^n$. □

Similar results hold for much more general domains G. With the help of their generalized Blaschke products, with very weak restrictions on G, Fisher and Micchelli [1980] give a new proof for the theorem of Widom [1972]:

Theorem 9.3. *If the boundary ∂D of a domain G has positive logarithmic capacity, and if $\mathbb{C} \setminus G$ is a countable union of disjoint connected sets, then for each compact $K \subset G$,*

$$\lim_{n\to\infty} \left[s_n(\mathcal{A}_G)_{C(K)}\right]^{1/n} = \exp(-c(K,G)^{-1}). \tag{9.5}$$

Here $c(K, G)$ is the condenser capacity (see §2, Appendix 4).

§ 10. Problems

10.1. Let $A = \{(x,y) : 0 < x, |y| < 1\} \cup (0,0)$, $B = \{(x,y) : |x| < 1, y = \pm x\}$. Then

$$d^1(A, l^2_\infty) = 0 < d^1(\overline{A}, l^2_\infty) = 1, \quad d^1(B, l^2_\infty) = 0 < d^1(\mathrm{co}B, l^2_\infty) = 1.$$

(Pinkus)

10.2. Prove that $d^1(b^3_1, l^3_2) = \sqrt{2/3} > d_1(b^3_1, l^3_2) = \sqrt{1/2}$. It follows by duality that $d_1(b^3_2, l^3_\infty) > d^1(b^3_2, l^3_\infty)$, so there is no apriori inequality between the Kolmogorov and Gelfand widths.

10.3. Let l_∞ be the space of real vectors $x = (\xi_1, \xi_2, \ldots)$, with the norm $\|x\| = \sup |\xi_i| < \infty$ and let c_0 be the subspace of l_∞ formed by all vectors x for which $\xi_i(x) \to 0$. If $B := \left\{x : \sum_i |\xi_i| \le 1\right\}$, then (Hutton [1974]) for $n \ge 1$,

$$d_n(B, c_0) = 1, \ d_n(B, l_\infty) = \delta_n(B, l_\infty) \ = 1/2.$$

10.4. There are infinitely many optimal subspaces for $d_n(b_1^m)_2$ (Ismagilov).

10.5. Let $\Pi \subset \mathbb{R}^m$ be a convex polyhedron with the vertices $a_k := (a_{k,1}, \dots, a_{k,m})$, $k = 1, \dots, N$. For the matrix $A := [a_{k,\ell}]$ and the transposed matrix A', let

$$\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_r > 0 = \lambda_{r+1} = \lambda_{r+2} = \dots = \lambda_N$$

be the eigenvalues of AA', and let $v_k := (v_{k,1}, \dots, v_{k,N})$, $k = 1, \dots, N$ be the corresponding orthonormal eigenvectors. Then for $n \ge r$, $s_n(\Pi, l_2^m) = 0$. For $0 \le n < r$,

$$\sqrt{N^{-1} \sum_{k=n+1}^{r} \lambda_k} \le d_n(\Pi, l_2^m) = \delta_n(\Pi, l_2^m) \le \max_{1 \le \nu \le N} \sqrt{\sum_{k=n+1}^{r} \lambda_k v_{k,\nu}^2} \, .$$

This is a generalization of (3.3).

10.6. For $n = 1, 2, \dots$,

$$d_n(B_2^1[0,1])_2 = 1/(\pi n), \qquad d_n(B_2^2[0,1])_2 = 1/(4\mu_n),$$

where μ_n are the non-zero roots of the equation $|\tan\sqrt{\mu}| = \tanh\sqrt{\mu}$ (Tikhomirov).

10.7. Prove that the formula (6.17) for $C(p,q,1)$ is valid for $1 < p, q < \infty$. The integral in (6.17) can be expressed through the Euler B-function:

$$\int_0^1 \frac{dz}{(1-z^q)^{1/p}} = (1/q)B(1/q, 1-1/p), \qquad 1 < p, q < \infty.$$

Derive from this:

$$C(\infty, q, 1) = \frac{\pi}{2}\left(\frac{2\pi}{1+q}\right)^{1/q}, \qquad C(p, 1, 1) = \frac{\pi}{2}\left(\frac{2\pi}{1+p'}\right)^{1/p'} .$$

10.8. For $f \in W_p^1(\mathbb{T})$, let $\tilde{U}_{n,1}f$ be the step function

$$(\tilde{U}_{n,1}f)(t) := (1/2)\left(f((k-1)\pi/2) + f(k\pi/2)\right), \quad t \in ((k-1)\pi/2, k\pi/2).$$

Then $\|f - \tilde{U}_{n,1}f\|_q \le C(p,q,1)n^{-1}\|f^{(r)}\|_p$. The same estimate is valid for the interpolation operator $U_{n,1}$ of Theorem 7.1. However, the rank of $\tilde{U}_{n,1}$ is $2n-1$ while the rank of $U_{n,1}$ is $2n$.

10.9. One can prove that for $n = 0, 1, \dots, r = 1, 2, \dots$ there exists a perfect spline $\psi_{n,r}$ of order $r+1$ on [-1,1], with $\le n$ breakpoints, that attains its $C[-1,1]$ norm at $n+r+1$ points with alternating signs. This spline is related to the widths by the equality

$$s_n(B^r_\infty[-1,1])_C = \|\psi_{n-r,r}\|, \quad n \geq r$$

Prove that

$$\psi_{1,2}(t) = (\sqrt{2}-1)t - (t^2/2)\operatorname{sgn} t, \quad \psi_{1,3}(t) = -|t|^3/6 + t^2/6 - 1/81.$$

Another fact (Tikhomirov [A-1976]): in the $L_\infty[-1,1]$ norm,

$$\|\psi_{n,2}\| = 2^{-1}(n+\sqrt{2})^{-2}; \qquad \|\psi_{n,3}\| = (3(n+2)^3)^{-1}.$$

§ 11. Notes

11.1. Several other quantities related to the three widths of this chapter are sometimes considered. For instance, if in the definition (1.4) of linear widths one takes the infimum only over linear projection operators U_n, the corresponding quanity is called *the projection n-width* of K. Or one may allow U_n to run over all continuous (not necessarily linear) mappings of K onto compacts of dimension $\leq n$; the corresponding quantity is called *the Aleksandrov n*-width of K. Similar modifications of $d^n(K)$ are also possible (see Ioffe, Tikhomirov [1968]). The so-called *s-numbers* of operators, a concept more universal than the widths, proved to be useful in functional analysis (see Pietsch [B-1987]).

11.2. Tikhomirov [1966] showed that for every sequence $\alpha_0 \geq \ldots \geq \alpha_n \geq \ldots$, $\alpha_n \to 0$, there exists a Banach space X and a compact set $K \subset X$ such that $d_n(K, X) = \alpha_n$, $n = 0, 1, \ldots$. The same is true for $d^n(K)$ and $\delta_n(K)$.

11.3. Optimal subspaces for the Kolmogorov widths may not exist even for some natural classes of functions. Ruban [1975] demonstrates this in the case of classes H^ω_∞ in C, with some special moduli of continuity ω.

11.4. Ligun [1973] proved that in the periodic case not only the space $\mathcal{S}_{n,r}$ is optimal for the class $B^r_\infty(\mathbb{T})$ in L_∞ but also all spaces $\mathcal{S}_{n,m}$, $m > r$ of higher smoothness. Velikin [1984] showed that the optimality of the trigonometric space $\mathcal{T}_{n-1}$ is a limiting form of this result.

11.5. One may try to extend the results concerning the n-widths of $B^r_p(A)$ to more general classes of functions f of the form

$$f(t) := Kf + \varphi := \int_A K(t,\tau)g(\tau)d\tau + \varphi(t), \quad \|g\|_p \leq 1, \quad \varphi \in \Gamma_r,$$

where $K(t,\tau)$ is continuous on $A \times A$ and Γ_r is some fixed r-dimensional space ($B^r_p[0,1]$ corresponds to $K(t,\tau) = (t-\tau)^{r-1}_+/(r-1)!$, $\Gamma_n = \mathcal{P}_{r-1}$).

Functions f of this form arise naturally as solutions of linear differential equations. It turns out that the essential property of the kernel is its *total positivity*, that is, the fact that $\det(K(t_i, \tau_j))_{i,j=1}^m \geq 0$ for all choices $0 \leq t_1 \leq t_2 \leq \ldots \leq t_m = 1$, $0 \leq \tau_1 \leq \tau_2 \leq \ldots \leq \tau_m = 1$ and $m = 1, 2, \ldots$. Such kernels are *sign variation diminishing*: the number of sign changes of Kg does not exceed that of g. In the case of $(t-\tau)_+^{r-1}$ it is simply the Rolle theorem which has been often used in this chapter. An excellent exposition of total positivity can be found in Karlin's book [A-1968]. The monograph of Pinkus [A-1985] treats this concept systematically, both for $A = [0,1]$ and $A = \mathbb{T}$, in its relation to the widths.

11.6. Let A^q be the restriction of the unit ball of the Hardy space H^q to a compact subset E of the open unit disk. In the case when E is not very large (in terms of the hyperbolic radius), Fisher and Stessin [1991] determine the widths of A^q with the help of Blaschke products.

Chapter 14. Widths II: Weak Asymptotics for Lipschitz Balls, Random Approximants

§ 1. Introduction

The determination of widths s_n of Lipschitz balls B_p^α, $\alpha > 0$, $1 \le p \le \infty$ (for the definition see §1 of Chapter 13) in different spaces L_q, $1 \le q \le \infty$ is a famous problem of the theory of widths. The seminal paper of Kolmogorov [1936] is the case $p = q = 2$, $\alpha = 1, 2, \dots$ of it.

The purpose of this chapter is to evaluate the widths $s_n(B_p^\alpha)_q$ for all $\alpha > 0$ and all $1 \le p, q \le \infty$. At present we can achieve this only by means of weak equivalences $s_n(B_p^\alpha)_q \sim \phi(n)$. However, some of the results of Chapter 14 are valid with *equality* instead of *weak equivalence*. This is the case for the *lower triangle* I: $1 \le q \le p \le \infty$ of Fig.1.1, where $\alpha = r$ is an integer, and $A = \mathbb{T}$ (the case $A = [a, b]$ is not quite clear); see §7 of Chapter 13. For I, our results here are much weaker, but the simple proofs apply uniformly for all $\alpha > 0$, and for $A = [a, b]$.

Much more difficult is the case of the *upper triangle* $1 \le p < q \le \infty$. No *strong equivalences* are known here, with the sole exception of $(p, q) = (1, 2)$ (see §4 of Chapter 13).

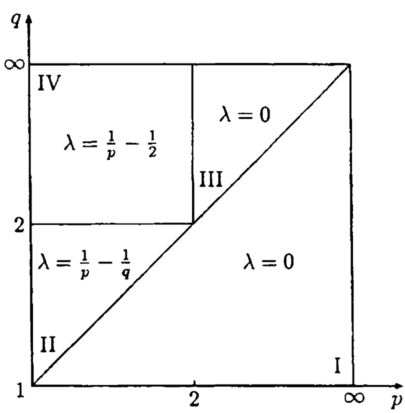
With the knowledge of the basic properties of widths, this chapter can be read independently of Chapter 13. Some authors, instead of B_p^α, $\alpha > 0$, study the widths of balls F_p^α of a Sobolev space W_p^α with fractional α. For definitions of these spaces, see Adams [B-1975], where one finds the embedding fact $F_p^\alpha \subset B_p^\alpha$.

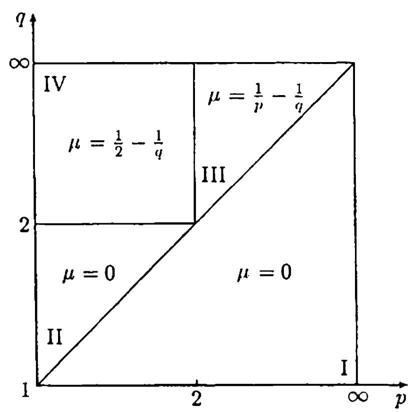
The simplest approach to weak asymptotic estimates is as follows. From [**CA**] we know many Jackson-type theorems about approximation of functions by n-dimensional subspaces of polynomials, of splines, and so on. This produces upper estimates for the d_n and d^n widths. (For the balls F_p^α one can use the approximation theorems of Sun Yongshen [1959].) If these approximations are realized by linear operators on L_q, we get also upper bounds for the δ_n. It remains then to find lower estimates for the $s_n(B_p^\alpha)_q$.

This latter problem will be completely solved in §3 of this chapter for values of α, p, q with compact B_p^α. If, up to a constant, the upper bound matches the lower bound, we are through. That this is the case for all points in the p, q square of Figures 1.1, 1.2, except for the regions III and IV for d_n, regions II and IV for d^n and the region IV for δ_n (see Fig. 8.1 in §8).

Historically, there has been a stalemate at this point, with a breakthrough achieved by Kashin. The first sign was an example of Ismagilov [1974]. Using number theoretic methods he showed that $d_n(B_1^2)_\infty \le Cn^{-6/5}\log n$, while conventional methods yield only an upper bound Cn^{-1}. Then in [1977], Kashin established the existence of non-standard subspaces of L_q with upper estimates matching the lower estimates of §3 in the critical cases for d_n, d^n. This has been achieved by using probabilistic (or measure theoretic) arguments in $\mathbb{R}^n$. They allow to find the Kolmogorov and Gelfand widths of the unit ball b_p^m of the space l_p^m in l_q^m. It remains to use the discretization formulas of §2. They reduce the computation of $s_n(B_p^\alpha, L_q)$ to those of $s_n(b_p^m, l_q^m)$.

Later, independently, Maiorov [1978] and Höllig [1979] used similar ideas for the linear widths δ_n. The results are summed up in Theorem 1.1 and Figures 1.1 and 1.2 below.





I: $1 \le q \le p \le \infty$, II: $1 \le p \le q \le 2$, III: $2 \le p \le q \le \infty$, IV: $1 \le p \le 2 \le q \le \infty$

Fig. 1.1. $d_n(B_p^\alpha)_q \sim n^{-\alpha+\lambda}$ **Fig. 1.2.** $d^n(B_p^\alpha)_q \sim n^{-\alpha+\mu}$

Theorem 1.1. *For* $1 \le p, q \le \infty$, *with proper restrictions for* α *(see Theorems* 3.8, 5.4, 5.5, 8.1*) one has*

$$(1.1) \qquad d_n(B_p^\alpha)_q \sim n^{-\alpha+\lambda}, \quad d^n(B_p^\alpha)_q \sim n^{-\alpha+\mu}, \quad \delta_n(B_p^\alpha)_q \sim n^{-\alpha+\nu},$$

where $\nu = \mu \ge \lambda$ *if* $1/p + 1/q \ge 1$, $\nu = \lambda \ge \mu$ *if* $1/p + 1/q \le 1$. *The values of* λ, μ *are seen on Figs.* 1.1, 1.2.

Restrictions for $\alpha > 0$ *are as follows: none in* I; *for* d_n: $\alpha > 1/p - 1/q$ *in* II, $\alpha > 1/p$ *in* III *and* IV; *for* d^n: $\alpha > 1/p - 1/q$ *in* III, $\alpha > 1 - 1/q$ *in* II *and* IV; *for* δ_n: $\alpha > 1/p - 1/q$ *in* II *and* III, $\alpha > 1/p$ *in* IVa *(=* IV *with* $p' \ge q$*),* $\alpha > 1 - 1/q$ *in* IVb.

The theorems of this chapter are about the most classical Lipschitz and Sobolev spaces of one variable. Kashin [1981] treats widths of Sobolev classes "of low smoothness." Spaces of analytic functions have been treated in §9 of Chapter 13. Other authors have discussed widths in a large variety of other function spaces. See in particular Edmunds and Triebel [1989], [1992], also Linde [1986] and the book of Triebel [B-1983].

We describe here widths by the finiteness of the norm of the sequence $(n^\sigma s_n(B))_1^\infty$ in the space l_∞, with a properly chosen $\sigma > 0$. One can replace l_∞ by other sequence spaces, see for example Theorem 3.1(iii).

The plan of this chapter is as follows. After the discretization theorems of §2, needed later, we give in §3 an "elementary" approach to the weak equivalence theorems. After this, there are two parallel roads leading to Kashin's and to Maiorov-Höllig theorems, respectively. Two different kinds of measures in $\mathbb{R}^n$ are developed in §4 and §6. Then in §5 and §7 follow two finite dimensional theorems estimating $d_n(b_2^m, l_\infty^m)$ and $\delta_n(b_p^m, \ell_q^m)$. Using this, we can finally, in §5 and §8, complete our proof of Theorem 1.1.

§ 2. Discretization

Discretization theorems are inequalities for $s_n(B_p^\alpha)_q$, that estimate these widths from above or below by means of widths $s_n(b_p^m, l_q^m)$ of the unit balls b_p^m of the spaces l_p^m. See Maiorov [1975], Höllig [1980].

Let A be $[0,1]$ or $\mathbb{T}$. We use the spline spaces $\mathcal{S}_{n,r}^*$ and operators $U_{n,r}$ of §1 and of Proposition 1.3 of Chapter 6. In the following theorems, s_n is d_n, d^n or δ_n.

Theorem 2.1. *Let $1 \le p \le q \le \infty$, $\alpha > 1/p - 1/q$. For any $n > \alpha$ and any non-negative integers $n_0, n_1, \ldots$, satisfying $n = n_0 + n_1 + \cdots$, one has*

$$(2.1) \qquad s_n(B_p^\alpha, L_q) \le C(p,q,\alpha) \sum_{k=0}^{\infty} 2^{-k(\alpha - 1/p + 1/q)} s_{n_k}(b_p^{m_k}, l_q^{m_k}) \, ,$$

where $m_k := 2^k(r+1)$ and r is the largest integer $< \alpha$.

(Here, for a set $B \subset l_q^m$, we put $s_0(B, l_q^m) = \sup_{c \in B} \|c\|_q$.)

Proof. (a) We shall use the abbreviations $\delta_k := \delta_{n_k}(b_p^{m_k}, l_q^{m_k})$, $\mathcal{S}_k^* := \mathcal{S}_{2^k, r+1}^*$ and prove the existence of operators G_k, $k = 0, 1, \ldots$ of rank $\le n_k$ (of course, $G_k = 0$ if $n_k = 0$) that map $\mathcal{S}_k^*$ into itself so that for some $C := C(p,q,r)$ and $1/s = 1/p - 1/q$,

$$(2.2) \qquad \|S - G_k S\|_q \le C 2^{k/s} \delta_k \|S\|_p \, , \qquad S \in \mathcal{S}_k^* \, .$$

(If $n_k \ge m_k$, the identity map provides this operator.) By the definition of δ_k there exists a linear map Q_k of $l_q^{m_k}$ into itself of rank $\le n_k$ with $\|c - Q_k c\|_q := \|c - Q_k c\|_{l_q^{m_k}} \le 2\delta_k$ for all $c \in b_p^{m_k}$; then

$$(2.3) \qquad \|c - Q_k c\|_q \le 2\delta_k \|c\|_p \, , \qquad c \in l_p^{m_k} \, .$$

We shall use the isomorphism I of $\mathbb{R}^{m_k}$ onto $\mathcal{S}_k^*$ defined in **1** of §1, Chapter 6. If these spaces are equipped with the norms of $l_q^{m_k}$ and L_q, respectively, then

(1.5) of Chapter 6 gives $\|I\| \sim m_k^{-1/q}$ and $\|I^{-1}\| \sim m_k^{1/q}$. For arbitrary $S \in \mathcal{S}_k^*$, let $c := I^{-1}(S) \in l_q^{m_k}$, then

$$\begin{aligned}\|S - IQ_kI^{-1}(S)\|_{L_q} &= \|I(I^{-1} - Q_kI^{-1})S\|_q \\ &\le Cm_k^{-1/q}\|c - Q_kc\|_q \le Cm_k^{-1/q}\delta_k\|c\|_p \\ &\le Cm_k^{-1/q}\delta_k\|I^{-1}(S)\|_{l_p^{m_k}} \le Cm_k^{1/p-1/q}\delta_k\|S\|_{L_p}\ .\end{aligned} \tag{2.4}$$

This establishes (2.2), with $G_k := IQ_kI^{-1}$.

Let now $U_{2^k,r}$, $k = 0, 1, \ldots$, be the sequence of operators of Proposition 1.3 of Chapter 6, mapping L_1 into $\mathcal{S}_k^*$. Let

$$V_0 := U_{1,r+1}\ , \quad V_k := U_{2^k,r+1} - U_{2^{k-1},r+1}\ , \qquad k = 1, 2, \ldots\ .$$

Then we have the representation, convergent in L_p and L_q,

$$f = \sum_{k=0}^{\infty} V_kf\ , \quad f \in B_p^\alpha\ , \tag{2.5}$$

with $\|V_kf\|_p \le C2^{-k\alpha}$. Since $V_kf \in \mathcal{S}_k^*$,

$$\|V_kf - G_kV_kf\|_q \le C2^{-k(\alpha-1/s)}\delta_k\ , \qquad f \in B_p^\alpha\ . \tag{2.6}$$

The rank of the operator G_kV_k does not exceed that of G_k. Thus the spline operator $G := \sum_{k=0}^{\infty} G_kV_k$ (with actually a finite sum) has rank $\le n$. We find that $\|f - Gf\|_q$ does not exceed the right-hand side of (2.1) with $s_n = \delta_n$.

(b) Similarly, for the Kolmogorov widths, there is a subspace X_k of $\mathcal{S}_{2^k,r+1}^*$ of dimension $\le n_k$, $n_k \ge 0$, so that for each $S \in \mathcal{S}_{2^k,r+1}$ there is some $S' \in X_k$ for which

$$\|S - S'\|_q \le C2^{k(1/p-1/q)}d_{n_k}(b_p^{m_k})_q\|S\|_p\ .$$

In particular, we replace S by V_kf for some $f \in B_p^\alpha$, and obtain an analogue of (2.6). Summation yields (2.1) for $s_n = d_n$.

(c) For the Gelfand widths, arguing as above, on each $\mathcal{S}_{2^k,r+1}^*$, $k = 0, 1, \ldots$ one can define $n_k \ge 0$ linear functionals $\ell_{k,j}$ so that the conditions $\ell_{k,j}(S) = 0$, $j = 1, \ldots, n_k$, for $S \in \mathcal{S}_k^*$ imply $\|S\|_q \le 2d^{n_k}(b_p^{m_k})_q\|S\|_p$. Now if $f \in B_p^\alpha$ and $\ell_{k,j}(V_kf) = 0$ for all k and all $j = 1, \ldots, n_k$, then for $n > \alpha$ the norm $\|f\|_q$ does not exceed the right-hand side of (2.1) with $s_n = d^n$. □

Theorem 2.2. *For any integers n, $N = 1, 2, \ldots$, $1 \le p, q \le \infty$,*

$$s_n(B_p^\alpha, L_q) \ge CN^{-\alpha+1/p-1/q}s_n(b_p^N, l_p^N)\ , \tag{2.7}$$

where $s_n = d_n$ or $s_n = d^n$.

This is a special case of Lemma 3.5 of §3.

§ 3. Weak Equivalences for Widths. Elementary Methods

The main problem of this section is to supplement the known approximation theorems, which provide upper bounds for widths, by estimates of widths from below. We will be satisfied in this section with estimates which contain an unspecified constant. In particular, we shall discuss widths of sets in Lipschitz spaces $\mathrm{Lip}^*(\alpha, L_p)$ and $\mathrm{Lip}(\alpha, L_p)$, $\alpha > 0$, $1 \le p \le \infty$, $A = \mathbb{T}$ or $A = [a, b]$. For their properties see [**CA**, §9, Chapter 2].

Let r be the smallest integer satisfying $r > \alpha$, that is, let $r := [\alpha] + 1$. The first space $\mathrm{Lip}^*(\alpha, L_p)$ consists of all $f \in L_p(A)$ (for $p = \infty$, we replace L_∞ by C) with the modulus of smoothness that satisfies, for some $M > 0$,

$$\omega_r(f, t)_p \le M t^\alpha , \qquad t > 0 . \tag{3.1}$$

For instance, $\mathrm{Lip}^*(1, C)$ is the Zygmund space.

Now let $\alpha = r + \beta$, where r is an integer, $0 < \beta \le 1$. For the second space, $f \in \mathrm{Lip}(\alpha, L_p)$ if and only if $f, \ldots, f^{(r-1)}$ are absolutely continuous, $f^{(r)} \in L_p$, and

$$\omega(f^{(r)}, t)_p \le M t^\beta . \tag{3.2}$$

The two spaces are identical if α is not an integer. The smallest M in (3.2) is the semi-norm $|f|$ in $\mathrm{Lip}^*(\alpha, L_p)$; we set $B_p^{*\alpha} := \{f \in L_p : |f| \le 1\}$, in analogy to B_p^α.

The L_p modulus of continuity $\omega(g, h)_p$, $g \in L_p$, $h \ge 0$, used in these definitions, for $A = [a, b]$ is given by

$$\omega(g, h)_p := \max_{0<t\le h} \|\Delta_t(g, \cdot)\|_p = \max_{0<t\le h} \left(\int_a^{b-h} |g(x+t) - g(x)|^p \, dx \right)^{1/p} \tag{3.3}$$

A generalization of $\mathrm{Lip}(\alpha, L_p)$ is $W^r H_p^\omega(A)$, where ω is a fixed modulus of continuity $\omega(t)$, $0 \le t \le |A|$. It is defined by the requirement that $\omega(f^{(r)}, t)_p \le M\omega(t)$; the set $B^r H_p^\omega$ consists of all f for which the smallest possible M is ≤ 1; this value is the semi-norm $|f|_{W^r H_p^\omega}$.

Our first approach applies to *full approximation sets* in an arbitrary Banach space X. Let $\Phi := (\phi_n)_1^\infty$ be a sequence of linearly independent elements of X, and let X_n be the n-dimensional space spanned by $\phi_1, \ldots, \phi_n$. Let Λ be a non-increasing sequence of real numbers $\lambda_n > 0$, $n = 0, 1, \ldots$, $\lambda_n \to 0$ as $n \to \infty$. We allow some of the λ_n to be ∞.

The *full approximation set* $K := K(\Phi, \Lambda)$ consists of all elements $f \in X$ for which the error of approximation $E_n(f) := E(f, X_n)_X$ (we put $E_0(f) = \|f\|$) satisfies

$$E(f, X_n)_X \le \lambda_n , \qquad n = 0, 1, \ldots . \tag{3.4}$$

Often, instead of (3.4), one knows the equivalence $E(f, X_n)_X \sim \lambda_n$, $f \in K$, $n \to \infty$. We interpret this to mean that for some fixed constants $0 < C_1 \le C_2 < \infty$,

$$C_1K(\Phi,\Lambda) \subset K \subset C_2K(\Phi,\Lambda) \ . \tag{3.5}$$

A full approximation set $K_q := K_q(\Phi,\Lambda)$ of another type can be defined for $0 < q < \infty$ as the set of all $f \in K$ for which

$$\sum_{j=1}^{\infty} E_j(f)^q/\lambda_j \le 1 \ . \tag{3.6}$$

(In analogy to the (θ, q)-norms of sequences in [**CA**, §3, Chapter 2].)

Theorem 3.1. (i) *If $K \subset X$ is given by* (3.4), *then $d_n(K)_X = \lambda_n$ and the subspace X_n is extremal for each $n = 0, 1, \ldots$; also $d^n(K)_X \ge \lambda_n$.*
(ii) *If K is given by* (3.5), *then $d_n(K)_X \sim \lambda_n$, $n \to \infty$.*
(iii) *If K_q is given by* (3.6) *and $\sum \lambda_n^{-1} = \infty$, then*
$d_n(K_q)_X \ge (\sum_1^n \lambda_j^{-1})^{-1/q} =: \mu_n$ and $d^n(K_q) \ge \mu_n$.

Proof. We have $d_n(K) \le E(f, X_n) \le \lambda_n$. On the other hand, let B_n be the ball $\|f\| \le \lambda_n$ in X_{n+1}. For $f \in B_n$ we then have $E(f, X_k) \le \lambda_n \le \lambda_k$, $k = 0, 1, \ldots, n$, and $E(f, X_k) = 0$, $k = n+1, \ldots$. Therefore $B_n \subset K$ and by Theorem 1.4 of Chapter 13,

$$s_n(K)_X \ge s_n(B_n)_X = \lambda_n \ .$$

Moreover, $E(K, X_n) = d_n(K)$. This establishes (i). Now (ii) follows from (i) and (3.5). To prove (iii), we observe that for each $f \in K_q$

$$E_n(f)^q \sum_1^n 1/\lambda_j = \sum_1^n E_n(f)^q/\lambda_j \le \sum_1^n E_j(f)^q/\lambda_j \le 1 \ .$$

Hence $d_n(K_q) \le \sup_{f \in K_q} E_n(f) \le \mu_n$.

For the lower estimates in (iii) we consider the ball

$$B_n^* := \{f \in X_{n+1} : \|f\| \le \mu_n\} \ .$$

Then $B_n^* \subset K_q$. Indeed, if $f \in B_n^*$ and $j \le n$, then $E_j(f) \le \|f\| \le \mu_n$, and for $j > n$, $E_j(f) = 0$. Therefore

$$\sum_{j=1}^{\infty} E_j(f)^q/\lambda_j = \sum_{j=1}^{n} E_j(f)^q/\lambda_j \le \mu_n^q \sum_{j=1}^{n} \lambda_j^{-1} = 1 \ ,$$

that is, $f \in K_q$. Now the desired lower estimate follows from Theorem 1.4 of Chapter 13.

By [**CA**, Theorem 3.3, p.210], $f \in \mathrm{Lip}^*(\alpha, L_p)$, $A = \mathbb{T}$ is equivalent to the property $E_n(f)_p = O(n^{-\alpha})$ of the trigonometric approximation; an asymptotically optimal linear operator also exists. This leads to

Example. For the unit ball of the space $\mathrm{Lip}^*(\alpha, L_p)$ on $A = \mathbb{T}$, for $1 \le p \le \infty$, $\alpha > 0$, $n \ge r$,

$$(3.7) \qquad d_n(B_p^{*\alpha})_p \sim \delta_n(B_p^{*\alpha})_p \sim n^{-\alpha} , \qquad n \to \infty .$$

This statement holds also on $A = [a, b]$, but with a somewhat different definition of the Lipschitz spaces which uses the Ditzian-Totik moduli of smoothness (compare [**CA**, Theorem 7.7, p.265]).

The Besov spaces $B_q^\alpha(L_p)$ defined in [**CA**, §10, Chapter 2] can be characterized by trigonometric approximations. By [**CA**, Theorem 9.2, p.235], $f \in B_q^\alpha(L_p)$ if and only if $f \in L_p$ and

$$(3.8) \qquad \sum_{k=1}^{\infty} k^{\alpha q - 1} E_k(f)_p^q < \infty .$$

A ball $B_{q,\alpha,p}$ of radius $M > 0$ of the space $B_q^\alpha(L_p)$ can be described as the set of all $f \in L_p$ for which the sum in (3.8) is $\le M$. We take $M = 1$. Then, by (iii) of Theorem 3.1, it follows:

$$(3.9) \qquad d_n(B_{q,\alpha,p})_p = \left(\sum_{k=1}^{n} k^{\alpha q - 1} \right)^{-1/q} \sim (n^{\alpha q})^{-1/q} = n^{-\alpha} .$$

For example, the space $B_1^1(L_p)$ is a subset of those $f \in L_p(\mathbb{T})$ for which

$$\int_0^\infty \omega_2(f, t)_p t^{-2} \, dt < \infty .$$

A function $f \in L_p(\mathbb{T})$ belongs to $B_1^1(L_p)$ if and only if $\sum_1^\infty E_n(f)_p < \infty$. By (3.9), the width of the unit ball of $B_1^1(L_p)$ in the L_p norm is $d_n(B) \sim n^{-1}$.

One of the most popular problems in the theory of widths has been the determination of the widths $s_n(B_p^\alpha)_q$ in the norm of the space L_q. Up to weak equivalence, we shall calculate here the $s_n(B_p^\alpha)_q$ for many combinations of p, q, α. Our elementary methods have the advantage of wide applicability. One can take A to be $\mathbb{T}$, $[a, b]$, or even $[a, b]^s$. This could be a useful introduction to the results of §§4-7 of Chapter 13 which aim at exact equalities or strong equivalences but have restrictions (for example, that $A = \mathbb{T}$ or that n is odd).

We note that $B^r H_p^\omega \subset B^r H_{p_1}^\omega$ for $p_1 \le p$. The space L_{q_1} is continuously imbedded in L_q for $q \le q_1$. Therefore

$$(3.10) \qquad s_n(B^r H_p^\omega)_q \le C s_n(B^r H_{p_1}^\omega)_{q_1} \quad \text{if} \quad p \ge p_1 \; , \; q \le q_1 \; ;$$

General lower estimates of widths of Lipschitz balls (with the method exemplified by Theorem 3.2, Corollary 3.4) started with Lorentz [1960] who gave the lower estimate $d_n(B_p^\alpha)_p \ge C n^{-\alpha}$ and, by a combinatorial lemma, obtained $d_n(B_p^\alpha)_1 \ge C n^{-\alpha}$. From this and (3.10) it follows that

$$d_n(B_p^\alpha)_q \ge C n^{-\alpha} \quad \text{for} \quad 1 \le p, q \le \infty \; , \; \alpha > 0 \; .$$

However, in some regions of the (p, q)-diagram (Fig. 1.1) one can get better estimates. For this purpose, the *method of discretization* is needed (see Lemma 3.5). Both methods appear also in Chapter 13. In what follows, we shall restrict A to $[0, 1]$. We start with a simple general observation.

Given a set K in a Banach space X, we choose N linearly independent linear functionals $\ell_1, \ldots, \ell_N$ defined on X. If for every linear combination $\ell = \sum_1^N c_j \ell_j$ with $\|\ell\| = 1$ there is an $f_0 \in K$ for which $|\ell(f_0)| \geq \varepsilon$, then $d_{N-1}(K) \geq \varepsilon$. Indeed, if X_{N-1} is an $(N-1)$-dimensional subspace of X with the basis $\phi_1, \ldots, \phi_{N-1}$, then the system

$$\ell(\phi_k) = \sum_{j=1}^{N} c_j \ell_j(\phi_k) = 0 , \qquad k = 1, \ldots, N-1 ,$$

has a nontrivial solution $(c_i)_1^N$ which can be normalized by the condition $\|\ell\| = 1$. For these (c_j) and the corresponding ℓ and f_0 one has for every $x \in X_{N-1}$

$$\|f_0 - x\| \geq |\ell(f_0 - x)| = |\ell(f_0)| \geq \varepsilon .$$

Hence $d_{N-1}(K)_X \geq \varepsilon$. As an example, we have:

Theorem 3.2. *Let $\varepsilon, \delta > 0$ be given and let ϕ_j, $j = 1, \ldots, N$, be elements of $L_p[0,1]$, $1 \leq p \leq \infty$, with disjoint supports Δ_j, $\|\phi_j\|_p = \delta$, $j = 1, \ldots, N$. If for each $a = (a_j)_1^N$, $\|a\|_p \leq \delta^{-1}$, the function $g_a := \varepsilon \sum_1^N a_j \phi_j$ belongs to $K \subset L_p$, then $d_{N-1}(K) \geq \varepsilon$.*

Proof. We assume that $1 \leq p < \infty$; the case $p = \infty$ is simpler. To apply our method, we choose $h_k \in L_{p'}$, $1/p + 1/p' = 1$ for which the functionals $\ell_k(f) := \int_{\Delta_k} f h_k \, dx$ satisfy the conditions

$$\|\ell_k\| = 1 , \quad \ell_k(\phi_j) = \delta \delta_{k,j} , \qquad k, j = 1, \ldots, N$$

(where $\delta_{k,j}$ is the Kronecker delta). Then $\ell := \sum_{k=1}^N c_k \ell_k$ is also a functional on L_p for any $c := (c_k)_1^N$, and $\|\ell\| = \|c\|_{p'}$. Indeed,

$$\|\ell\|^{p'} = \int_0^1 |\sum c_k h_k|^{p'} \, dx = \sum_{k=1}^N |c_k|^{p'} \int_{\Delta_k} |h_k|^{p'} \, dx = \|c\|_{p'}^{p'} .$$

For any ℓ with $\|\ell\| = 1$ we give a function $g \in K$ with $\ell(g) \geq \varepsilon$. For this purpose we take $g = g_a = \sum_{j=1}^N a_j \phi_j$, where the a_j are selected so that $\|a\|_p = \delta^{-1}$, $\sum_{k=1}^N a_k c_k = \delta^{-1}$. Then

$$\ell(g_a) = \varepsilon \sum a_k \ell(\phi_k) = \varepsilon \sum_{k,j=1}^{N} a_k c_j \ell_j(\phi_k) = \varepsilon \sum \delta a_k c_k = \varepsilon . \qquad \square$$

Let $\phi \in C^{r+1}(\mathbb{R})$ be an arbitrary fixed function that is zero outside of $[0, 1]$ and > 0 on $(0, 1)$, normalized by $\|\phi^{(r+1)}\|_\infty = 1$. With its help we define functions $\phi_j(x) := \phi(Nx-j+1)$ on $[0,1]$ with supports $\Delta_j = ((j-1)/N, j/N)$, $j = 1, \dots, N$, and put

$$g_a := \sum_{k=1}^{N} a_k \phi_k \ , \qquad a := (a_k)_1^N \in \mathbb{R}^N \ . \tag{3.11}$$

Lemma 3.3. (i) *For all functions g_a,*

$$\|g_a\|_q = C_q N^{-1/q} \|a\|_q \ , \qquad C_q := \|\phi\|_q \ , \quad 1 \le q \le \infty \ . \tag{3.12}$$

(ii) *For each modulus of continuity ω there is a constant C which depends only on p, $1 \le p \le \infty$ (and ϕ), so that*

$$\omega(g_a^{(r)}, h)_p \le C \frac{N^{r-1/p}}{\omega(1/N)} \|a\|_p \omega(h) \qquad h > 0 \ . \tag{3.13}$$

(iii) *For properly chosen $C > 0$, all functions g_a with $\|a\|_p \le CN^{-r+1/p} \cdot \omega(1/N)$ belong to $B^r H_p^\omega$.*

Proof. (i) follows because the Δ_k are disjoint, and (iii) is a corollary of (ii) which we shall now prove. For $h > 0$, we first compare $\omega(h)$ with $\omega(\phi_k^{(r)}, h)_\infty$ and establish

$$\omega(\phi_k^{(r)}, h)_\infty \le C\rho\omega(h) \ , \qquad \rho := N^r / \omega(1/N) \ . \tag{3.14}$$

Indeed, if $0 < h \le 1/N$, then

$$\omega(\phi_k^{(r)}, h)_\infty \le \|\phi_k^{(r+1)}\|_\infty h = N^{r+1} h \le 2\rho\omega(h) \ .$$

We have used here the fact that

$$\omega(1/N) = \omega\left(\frac{1}{Nh} h\right) \le \left(\frac{1}{Nh} + 1\right)\omega(h) \le \frac{2}{Nh}\omega(h) \ .$$

On the other hand, if $h \ge 1/N$, we have

$$\omega(\phi_k^{(r)}, h)_\infty \le \|\phi_k^{(r)}\|_\infty C N^r \ ,$$

while the left-hand side of (3.14) is at least CN^r. Fixing $h > 0$, we put $\psi(x) := \phi^{(r)}(x+h) - \phi^{(r)}(x)$ and get from the definition (3.3),

$$\omega(g_a^{(r)}, h)_p^p \le \int_0^1 \Big| \sum_{k=1}^{N} a_k \psi_k(x) \Big|^p dx = \sum_{j=1}^{N} \int_{\Delta_j} \Big| \sum_{k=1}^{N} a_k \psi_k(x) \Big|^p dx \ .$$

On the interval Δ_j, at most three terms of the sum can be different from zero, namely, when $x \in \Delta_j$ or $x + h \in \Delta_j$. In view of (3.14), this gives, with some fixed integer $j_0 := j_0(h)$,

$$\omega(g_a^{(r)}, h)_p^p \le C\rho^p \omega(h)^p N^{-1} \sum_{j=1}^{n} (|a_j| + |a_{j-j_0}| + |a_{j-j_0-1}|)^p$$

$$\le C\rho^p \omega(h)^p N^{-1} \|a\|^p . \qquad \square$$

(A variation of this lemma may be found in Lorentz [A-1966, p.134].)

Example for (iii): for a proper constant C, for $N = 1, 2, \dots$, all functions g_a with $\|a\|_p \le N^{1/p}$, satisfy $CN^{-r}\omega(1/N)g_a \in W^r H_p^\omega$, $1 \le p \le \infty$. In particular, this is the case when a is a *sign vector*: $a_j = \pm 1$, $j = 1, \dots, N$. From Theorem 3.3, taking $\delta = n^{-1/p}$, $\varepsilon = Cn^{-r}\omega(1/n)$, we have

Corollary 3.4. *For* $n = 1, 2, \dots$,

$$d_n(W^r H_p^\omega)_p \ge Cn^{-r}\omega(1/n) . \tag{3.15}$$

In what follows, ω is a modulus of continuity, $1 \le p, q \le \infty$, $r = 0, 1, \dots$, and $A = [0, 1]$ (but the results are valid for $A = [a, b]$, $A = \mathbb{T}$ as well). We distinguish the regions I–IV of Fig.1.1:

$$\mathrm{I} : q \le p ; \quad \mathrm{II} : p \le q \le 2 ; \quad \mathrm{III} : 2 \le p \le q ; \quad \mathrm{IV} : p \le 2 \le q . \tag{3.16}$$

We need a discretization lemma (Maiorov [1975]), a slight generalization of Theorem 2.2.

Lemma 3.5. *For arbitrary integers* n, $N = 1, 2, \dots$ *and for* $s_n = d_n$ *or* $s_n = d^n$,

$$s_n(B^r H_p^\omega)_q \ge CN^{-r+1/p-1/q}\omega(1/N)s_n(b_p^N)_q . \tag{3.17}$$

Proof. Let $G = G_N$ be the N-dimensional subspace of L_q formed by all the functions g_a given by (3.11). By (3.12) G is isomorphic, up to the factor $C_q N^{-1/q}$, to l_q^N. It follows from (iii) of Lemma 3.3 that if $a \in \varepsilon b_p^N$, $\varepsilon := \omega(1/N)N^{-r+1/p}$, then $g_a \in B^r H_p^\omega$. The inequality (3.17) for the Gelfand widths follows now from (viii) of §1 of Chapter 13 since

$$d^n(B^r H_p^\omega, L_q) \ge d^n(B^r H_p^\omega \cap G, G) \ge d^n(\varepsilon b_p^N)_q C_q N^{-1/q} .$$

For the Kolmogorov widths, (3.17) follows similarly from the inequality

$$d_n(B^r H_p^\omega, L_q) \ge C' d_n(B^r H_p^\omega \cap G, G) , \tag{3.18}$$

where C' does not depend on n. To prove (3.18), we consider the projection operator $P : L_q \to G$ defined by

$$Pg := \sum_{k=1}^{N} c_k(g)\phi_k , \qquad c_k(g) := \frac{1}{\|\phi_k\|_2^2} \int_A g\phi_k \, dt .$$

By Hölder's inequality, $|c_k(g)|^q \le (\|\phi_k\|_{q'}^q/\|\phi_k\|_2^{2q}) \int_{\Delta_k} |g|^q\,dt$, $k = 1,\dots,N$, where $q' = q/(q-1)$. This yields the estimate

$$\|Pg\|_q^q \le \sum_{k=1}^N M_k^q \int_{\Delta_k} |g|^q\,dt\,, \qquad M_k := (\|\phi_k\|_{q'}\|\phi_k\|_q)/\|\phi_k\|_2^2\,.$$

By a change of variable one can easily see that ϕ_k in the definition of the M_k can be replaced by ϕ, so that $M_k = M$, independent of k, hence $\|Pg\|_q \le M\|g\|_q$. For $g \in G$ and arbitrary $f \in L_q$, one has $\|g - Pf\|_q = \|Pg - Pf\|_q \le M\|g - f\|_q$. Therefore $d_n(K, L_q) \ge M^{-1} d_n(K \cap G, G)$ for any set $K \subset L_q$, and (3.18) follows. □

Theorem 3.6 (Lower Estimates of d_n, d^n). *With a constant C dependent only on p, q, ω, we have*

$$\text{(3.19)} \quad d_n(B^r H_p^\omega)_q \ge C n^{-r+\lambda}\omega(1/n) \quad ; \quad d^n(B^r H_p^\omega)_q \ge C n^{-r+\mu}\omega(1/n)\,,$$

with the following values of λ, μ in the regions I–IV: $\lambda = \mu = 0$ *in* I; $\lambda = 1/p - 1/q$, $\mu = 0$ *in* II; $\lambda = 0$, $\mu = 1/p - 1/q$ *in* III; $\lambda = 1/p - 1/2$, $\mu = 1/2 - 1/q$ *in* IV.

Proof. Similarly to (3.10) we have:

$$\text{(3.20)} \qquad s_n(b_p^N)_q \le s_n(b_{p_1})_{q_1} \quad \text{if} \quad p \le p_1\,,\ q \ge q_1\,.$$

In the following four cases, K and G stand for the Kolmogorov and the Gelfand widths, respectively.

1) *Region I, K and G.* By (3.10), $s_n(B^r H_p^\omega)_q \ge C s_n(B^r H_\infty^\omega)_1$. We set $N = 2n$, $p = \infty$, $q = 1$ on the right-hand side of (3.17) and use (3.1) of Chapter 13, namely $s_n(b_\infty^{2n})_1 = n$. See Lorentz [1960].

2) *Region III, K and region II, G.* Here the inequality

$$s_n(B^r H_p^\omega)_q \ge C s_n(B^r H_p^\omega)_p \ge n^{-r}\omega(1/n)$$

(by 1)) is enough.

In the remaining two cases we estimate only the values of $s_n(b_p^N)_q$ for $N = 2n$; their substitution into (3.17) gives the desired result. The two cases are due to Ismagilov [1974], Scholz [1976], with earlier partial results by Rudin [1952], Stechkin [1954], Solomyak and Tikhomirov [1967].

3) *Region II, K and region III, G.* We need here the results (3.6) and (3.8) of Chapter 13, namely

$$d_n(b_1^{2n})_2 = d^n(b_2^{2n})_\infty = 1/\sqrt{2}\,.$$

From (3.20) we obtain $d_n(b_p^{2n})_q \ge d_n(b_1^{2n})_2 = 1/\sqrt{2}$ and $d^n(b_p^{2n})_q \ge d^n(b_2^{2n})_\infty = 1/\sqrt{2}$.

4) *Region III, K and G.* We use two related facts:

$$\|a\|_q \le N^{1/q-1/q_1}\|a\|_{q_1}\ , \qquad a \in \mathbb{R}^n\ ,\ q \le q_1\ ; \tag{3.21}$$

$$b_{q_1}^N \subset N^{1/q_1-1/q} b_q^N\ , \qquad q \le q_1\ . \tag{3.22}$$

For $p \le 2 \le q$, using (3.20) and these relations, we obtain

$$d_n(b_p^{2n})_q \ge d_n(b_1^{2n})_q \ge (2n)^{1/q-1/2} d_n(b_1^{2n})_2 = Cn^{1/q-1/2}$$

and

$$d^n(b_p^{2n})_q \ge d^n(b_p^{2n})_\infty \ge (2n)^{1/2-1/p} d^n(b_2^{2n})_\infty = Cn^{1/2-1/p}\ . \qquad \square$$

Using the inequality $\delta_n(B) \ge \max(d_n(B), d^n(B))$, we prove:

Theorem 3.7. *One has*

$$\delta_n(B^r H_p^\omega)_q \ge Cn^{-r+\nu}\omega(1/n)\ , \tag{3.23}$$

where $\nu = 0$ *in* I; $\nu = 1/p - 1/q$ *in* II; $\nu = \max\{1/p - 1/2, 1/2 - 1/q\}$ *in* III; $\nu = 1/p - 1/q$ *in* IV.

For the B_p^α, (3.18) and (3.23) are exactly the lower estimates for the three widths implied by Theorem 1.1.

Upper estimates follow from [**CA**, Chapter 7]. For trigonometric approximation one can use the Jackson type operators, for spline approximation the quasi-interpolant formulas. Even simpler is to use Proposition 1.3 of Chapter 6 (adjusted for $h^r\omega(h)$ instead of h^α, with some restrictions for ω). In this way we get for some operators $U_{n,r}$ of rank n,

$$\delta_n(B^r H_p^\omega)_q \le \sup_{f \in B^r H_p^\omega} \|f - U_{n,r} f\|_q \le Cn^{-r+(1/p-1/q)_+}\omega(1/n)\ . \tag{3.24}$$

These are *the upper estimates* corresponding to Theorems 3.6, 3.7. Returning to B_p^α, we list the weak equivalences for $s_n(B_p^\alpha)$, proved in this section.

Theorem 3.8. *Let* $\alpha > (1/p - 1/q)_+$. *Then* $s_n(B_p^\alpha)_q \sim n^{-\alpha}$ *for* (p,q) *in* I. *For* (p,q) *in* II,

$$d_n(B_p^\alpha)_q \sim \delta_n(B_p^\alpha)_q \sim n^{-\alpha+1/p-1/q}\ . \tag{3.25}$$

For (p,q) *in* III,

$$d^n(B_p^\alpha)_q \sim \delta_n(B_p^\alpha)_q \sim n^{-\alpha+1/p-1/q}\ . \tag{3.26}$$

Finally, we show that the restriction $\alpha > (1/p - 1/q)_+$ in this theorem is necessary. This is a consequence of the compactness properties of underlying sets. Since the set $B_p^\alpha(A)$, $p \le q$, $A = [0,1]$ or $A = \mathbb{T}$, is not bounded in L_q, we consider its bounded subset $B_{p,0}^\alpha(A)$ which consists of $f \in B_p^\alpha(A)$ satisfying $f(0) = \cdots = f^{(r)}(0) = 0$ for $A = [0,1]$ or $f(0) = 0$ for $A = \mathbb{T}$.

The polynomial approximation properties of $B_p^\alpha(A)$ and $B_{p,0}^\alpha(A)$ are identical. If $\alpha > 1/p - 1/q$, then $B_{p,0}^\alpha$ is relatively compact due to (3.25) and (v) of §1 of Chapter 13. On the other hand, if $\alpha \le 1/p - 1/q$, for a fixed N we take the functions $g_k := N^{1/q}\phi_k$, $k = 1, \ldots, N$ with the ϕ_k of (3.11). Then $\|g_k - g_l\|_q = 2\|g_k\|_q = 2C_q > 0$ and Lemma 3.3 yields $Cg_k \in B_{p,0}^\alpha(A)$ for some $C > 0$. This implies that $B_{p,0}^\alpha(A)$ does not contain a finite ε-net for small ε. Thus:

Proposition 3.9. *The set $B_{p,0}^\alpha(A)$ is relatively compact in $L_q(A)$ if and only if $\alpha > (1/p - 1/q)_+$. Theorem* 3.9 *does not hold if $\alpha \le (1/p - 1/q)_+$.*

For the Sobolev classes, when $\alpha = r$ is a positive integer, the only case when $B_{p,0}^r$ is *not* precompact in L_q is when $(p, q, r) = (1, \infty, 1)$.

§ 4. Distribution of Scalar Products of Unit Vectors

In this section we derive some auxiliary facts about the scalar products (x, y) of vectors from the unit sphere

$$S^{m-1} := \left\{x = (\xi_1, \ldots, \xi_m) : \xi_1^2 + \cdots + \xi_m^2 = 1\right\} .$$

This will be needed for the proof of Kashin's theorem in §5.

We denote by μ the unique rotation-invariant Haar measure on S^{m-1} (see Halmos [B-1974, Ch.11]) with $\mu(S^{m-1}) = 1$. It differs by a constant factor from the Euclidean surface area on S^{m-1}:

$$\mu(A) = \frac{\text{Area}(A)}{\text{Area}(S^{m-1})} , \qquad A \subset S^{m-1} . \tag{4.1}$$

For example, let D be a bounded domain in $\mathbb{R}^{m-1}$ and let $f : D \to \mathbb{R}$ be a continuously differentiable function. The equation $\xi_m = f(\xi_1, \ldots, \xi_{m-1})$ defines a surface in the space of vectors $(\xi_1, \ldots, \xi_m)$; its area is given by

$$\int_D \sqrt{1 + f_{\xi_1}^2 + \cdots + f_{\xi_{m-1}}^2}\, d\xi_1 \ldots d\xi_{m-1} . \tag{4.2}$$

For $m \ge 2$, we denote by $S_{\alpha,\beta}^{m-1}$ the spherical belt

$$S_{\alpha,\beta}^{m-1} := \{x \in S^{m-1} : \alpha \le \xi_m \le \beta\} , \qquad 0 \le \alpha < \beta \le 1 .$$

Applying (4.2) with $f = \sqrt{1 - \xi_1^2 - \cdots - \xi_{m-1}^2}$ we have

$$\text{Area}(S_{\alpha,\beta}^{m-1}) = \int_D (1 - \xi_1^2 - \cdots - \xi_{m-1}^2)^{-1/2} d\xi_1 \ldots d\xi_{m-1} , \tag{4.3}$$

where $D \subset \mathbb{R}^{m-1}$ is defined by the inequalities

$$a := \sqrt{1-\beta^2} \le r := \sqrt{\xi_1^2 + \cdots + \xi_{m-1}^2} \le \sqrt{1-\alpha^2} =: b \,.$$

We shall need the following remark. For each function $f \in L_1(0,\infty)$ and for a domain $D_{a,b} \subset \mathbb{R}^{m-1}$ given by $a \le r \le b$ one has

(4.4)
$$\int_{D_{a,b}} f(r)\, d\xi_1 \ldots \xi_{m-1} = \int_a^b f(r)\, dV(r) = (m-1)V(1) \int_a^b f(r) r^{m-2}\, dr \,,$$

where $V(r) := V_{m-1}(r)$ is the volume of the ball of radius r in $\mathbb{R}^{m-1}$. Indeed, the first equality holds for each step function f with finitely many steps, hence for any $f \in L_1$. In particular,

$$\begin{aligned}\mathrm{Area}(S^{m-1}_{\alpha,\beta}) &= (m-1)V(1) \int_{\sqrt{1-\beta^2}}^{\sqrt{1-\alpha^2}} r^{m-2}(1-r^2)^{-1/2} dr \\ &= (m-1)V(1) \int_\alpha^\beta (1-u^2)^{\frac{m-3}{2}}\, du \,.\end{aligned}$$

Also, $\mathrm{Area}(S^{m-1}) = 2(m-1)V(1)I_m$, where

$$I_m := \int_0^1 r^{m-2}(1-r^2)^{-1/2} dr = \int_0^{\pi/2} \cos^{m-2} t\, dt \,, \tag{4.5}$$

and using (4.1), we get

$$\mu(S^{m-1}_{\alpha,\beta}) = \frac{1}{2I_m} \int_\alpha^\beta (1-u^2)^{\frac{m-3}{2}} du. \tag{4.6}$$

Let $N := m-2$ if m is even, $N := m-1$ if m is odd. It is useful to observe that

$$I_m \ge \int_0^{\pi/2} \cos^N t\, dt = 2^{-N-2} \int_0^{2\pi} (e^{it} + e^{-it})^N\, dt = \pi 2^{-N-1} \binom{N}{N/2} \,.$$

Using Stirling's formula this implies

$$I_m \ge \pi 2^{-N-1} \frac{N!}{(N/2)!} \ge \frac{1}{\sqrt{N}} \ge \frac{1}{\sqrt{m}} \,. \tag{4.7}$$

For a function $\phi \in L_1(S^{m-1})$ the integral $E(\phi) := \int_{S^{m-1}} \phi\, d\mu$ is the *mean value* of ϕ. For example, if $\xi_i := \xi_i(x)$ is the i-th coordinate of $x \in S^{m-1}$, then

$$E(\xi_1^2) = \ldots = E(\xi_m^2) = m^{-1} E(\xi_1^2 + \cdots + \xi_m^2) = m^{-1} \,.$$

It follows that m^{-1} is the mean value of $(x,y)^2$, where x is fixed and $y \in S^{m-1}$ (one can assume that $x = (1,0,\ldots,0)$). Hence $|(x,y)|$ is, typically, $m^{-1/2}$.

Let now $\mathbf{y} := (y_1, \ldots, y_n)$ be a multivector , with $y_1, \ldots, y_n \in S^{m-1}$ and let

$$F(x,\mathbf{y}) := F_{m,n}(x,\mathbf{y}) := n^{-1}[|(x,y_1)| + \cdots + |(x,y_n)|] \,. \tag{4.8}$$

The following lemma shows that for a fixed $x \in S^{m-1}$ and large n, $F(x,\mathbf{y})$ is approximately $m^{-1/2}$ for "most choices" of $\mathbf{y}$. More exactly, let Σ be the product of n spheres S^{m-1},

$$\Sigma := \Sigma_m^n := (S^{m-1}) \times \cdots \times (S^{m-1}) \quad (n \text{ times}) \ ,$$

One can define the product measure P on Σ by setting $P(A) = \mu(A_1)\cdots\mu(A_n)$ for $A = A_1 \times \ldots \times A_n$, with the unique extension onto all Borel subsets of Σ (see Halmos [B-1974, Chapter 7]). Clearly, $P(\Sigma) = 1$.

Lemma 4.1. *For any $x \in S^{m-1}$ and $n, m \geq 2$*

$$P\{\mathbf{y} : (1/8)m^{-1/2} \leq F(x,\mathbf{y}) \leq 3m^{-1/2}\} > \begin{cases} 1 - e^{-n}, & n > 2 \\ 1/2, & n = 2 \ . \end{cases} \tag{4.9}$$

(the left-hand side does not depend on x).

Proof. For a fixed $x \in S^{m-1}$, we shall estimate the function of $t \in \mathbb{R}$,

$$E_n(t) := E(\exp(tF(x,\mathbf{y}))) = \int_\Sigma \exp(tF(x,\mathbf{y}))dP(\mathbf{y}) \ .$$

This function does not depend on x. Indeed, every $x \in S^{m-1}$ can be transformed into $(1, 0, \ldots, 0)$ by an appropriate rotation. If the same rotation is applied to each y_i in (4.8), then $F(x,\mathbf{y})$ will not change; at the same time every rotation of S^{m-1} induces a transformation of Σ under which the product measure P remains invariant. If $t > 0$, then $E_n(t) \geq \exp(bt) \cdot P\{\mathbf{y} : F(x,\mathbf{y}) \geq b\}$ for any $b > 0$, so that

$$P\{\mathbf{y} : F(x,\mathbf{y}) > b\} \leq E_n(t)\exp(-bt) \ , \qquad t > 0 \ . \tag{4.10}$$

Similarly, for any $a > 0$, $t < 0$,

$$P\{\mathbf{y} : F(x,\mathbf{y}) < a\} \leq E_n(t)\exp(-at) \ , \qquad t < 0 \ . \tag{4.11}$$

By the definition of P, since $\exp(tF(x,\mathbf{y})) = \exp(t(x,y_1)/n)\cdots\exp(t(x,y_n)/n)$,

$$E_n(t) = (E_1(t/n))^n \ . \tag{4.12}$$

To compute $E_1(t)$ we take $x = (1, 0, \ldots, 0)$, and with $y_1 = (\eta_1, \ldots, \eta_m)$, obtain

$$E_1(t) = \int_{S^{m-1}} \exp(t|\eta_1|)d\mu(y_1) = 2\int_{S_{0,1}^{m-1}} \exp(t\eta_1)d\mu(y_1) \ .$$

Now (4.6) allows us to write $E_1(t)$ as a single integral

$$E_1(t) = (I_m)^{-1}\int_0^1 \exp(tu)(1-u^2)^{\frac{m-3}{2}}\,du \ .$$

Due to (4.7) and the inequality $1 - u^2 \leq e^{-u^2}$, we get

$$E_1(t) \le m^{1/2} \int_0^1 \exp(tu - mu^2/2 + 3u^2/2)\,du$$
$$\le e^{3/2} m^{1/2} \int_0^1 \exp(tu - mu^2/2)\,du\ .$$

Extending integration to $[0,\infty)$ and substituting $v = m^{1/2}(u - t/m)$, we obtain

$$E_1(t) \le e^{3/2} \exp(t^2/2m) \int_{-t/\sqrt{m}}^{\infty} \exp(-v^2/2)\,dv\ . \tag{4.13}$$

First let $t > 0$. The integral in (4.13) is less than $\sqrt{2\pi}$. Combining (4.10), (4.12) and (4.13) and setting $t = bmn$, we have

$$P\{\mathrm{y} : F(x,\mathrm{y}) > b\} < (2\pi e^3)^{n/2} \exp(-b^2 mn/2)\ . \tag{4.14}$$

In particular, for $b = 3/\sqrt{m}$,

$$P\{\mathrm{y} : F(x,\mathrm{y}) > 3/\sqrt{m}\} < (2\pi e^{-6})^{n/2} < e^{-2n}\ . \tag{4.15}$$

For $t < 0$, we use different estimates. If $m \ge 3$,

$$E_1(t) \le I_m^{-1} \int_0^1 \exp(tu)\,du \le -\sqrt{m}/t\ .$$

(This is also true if $m = 2$ for then $I_m = \pi/2$ and

$$E_1(t) = \int_0^{\pi/2} \exp(t \sin u)\,du \le \int_0^{\pi/2} \exp(2tu/\pi)\,du \le -1/t\ .)$$

For $a := (1/8)m^{-1/2}$, $t := -8nm^{1/2}$ we therefore have

$$P\{\mathrm{y} : F(x,\mathrm{y}) < a\} \le e^{-at}(1/8)^n = (e/8)^n\ . \tag{4.16}$$

The estimate (4.9) follows from (4.14) and (4.16). □

A particular case of (4.14) when $n = 1$ is worth noting:

$$\mu\{y \in S^{m-1} : |(x,y)| > b\} < C \exp(-mb^2/2)\ , \tag{4.17}$$

where $C = (2\pi e^3)^{1/2} = 11.23\ldots$. A closer analysis shows that one can take $C = 2$. It follows from (4.17) that when $x \in S^{m-1}$ and m is large, $|(x,y)|$ is small for most $y \in S^{m-1}$ ("on a planet of high dimension most people live near the equator").

§ 5. Kashin's Theorems

The main result of this section is Kashin's Theorem 5.4. The key fact in its proof is the upper estimate in the following theorem.

Theorem 5.1 (Kashin, improved by Gluskin). *For $1 \le n \le m$,*

$$C_1 \min\left(\sqrt{\frac{\log(em/n)}{n}}, 1\right) \le d_n(b_2^m, l_\infty^m) \le C_2 \sqrt{\frac{\log(em/n)}{n}}\ , \tag{5.1}$$

where C_1, C_2 are absolute constants (one may take $C_1 = 1/6$, $C_2 = 51$).

The original upper estimate, which had $\log^3(1+m/n)$ instead of $\log(em/n)$ under the square root, was also sufficient for Kashin's purposes. His proof (Kashin [1977]) relied on difficult probabilistic arguments. Afterwards, Gluskin [1983] and Garnaev and Gluskin [1984] simplified the proof using an isoperimetric inequality in $\mathbb{R}^m$ instead and added the lower estimate. Our final simplification, due to Makovoz [1988_1], is based on Lemma 4.1. We relegate the proof of the lower estimate (5.1) to Problems, as it will not be used below.

Lemma 5.2. *For a fixed $\ell = 1, 2, \cdots, m$, let $V_{m,\ell}$ be the set of all vectors in $\mathbb{R}^m$ having some ℓ coordinates equal to $\pm 1/\ell$ and the other coordinates equal to zero. Then*

$$W := b_1^m \cap (1/\ell) b_\infty^m = \mathrm{co}(V_{m,\ell}) \ .$$

Proof. The set W is compact and convex, so it is spanned by its extreme points. Since $V_{m,\ell} \subset W$, it is sufficient to prove that they are identical with the points $x \in V_{m,\ell}$. To show this, suppose that $x \in V_{m,\ell}$ and $x = (x' + x'')/2$, where $x', x'' \in W$. Then for some ℓ coordinates $\xi_j(x) = \pm 1/\ell$, and then also both coordinates $\xi_j(x')$ and $\xi_j(x'')$ are equal to this. Since $x', x'' \in b_1^m$, the other coordinates of these points are equal to zero. Thus $x' = x'' = x$, and x is an extreme point. On the other hand, each point $x \in W \setminus V_{m,\ell}$ is not extreme. Indeed, if for such a point $\|x\|_1 < 1$, we set $x_1 = x + \varepsilon e_{j_0}$, $x_2 = x - \varepsilon e_{j_0}$, where j_0 is any index for which $|\xi_0| < 1/\ell$. If $\|x\|_1 = 1$, then x has more than ℓ non-zero coordinates and $0 < |\xi_{j_1}|, |\xi_{j_2}| < 1/\ell$ for at least two different j_1, j_2. We then set $x_1 = x + w$, $x_2 = x - w$, where $w := (\varepsilon \operatorname{sign} \xi_1) e_{j_1} - (\varepsilon \operatorname{sign} \xi_2) e_{j_2}$. In both cases $x = (x_1 + x_2)/2$, $x_1 \ne x_2$, and $x_1, x_2 \in W$ for small $\varepsilon > 0$, so that x is not an extreme point. □

We now show that for $n \le m$ there exists a multivector $\mathbf{y}^*$ so that $F(x, \mathbf{y}^*)$ of (4.8) is approximately $\|x\|_2/\sqrt{m}$ for many points $x \in b_1^m$.

Lemma 5.3. *For natural ℓ, m, $1 \le \ell \le m$, let $b_{m,\ell}$ be the subset of all vectors in b_1^m whose coordinates are of the form k/ℓ, $k \in \mathbb{Z}$. Then for any natural $n \le m$ with*

(5.2) $$\ell := \left[\frac{n}{4\log(em/n)}\right] \geq 1 ,$$

there exists a multivector $\mathbf{y}^* = (y_1^*, \ldots, y_n^*)$ *such that for all* $x \in b_{m,\ell}$,

(5.3) $$(1/8)m^{-1/2}\|x\|_2 \leq F(x, \mathbf{y}^*) \leq 3m^{-1/2}\|x\|_2 .$$

Proof. We begin with estimating the number $|b_{m,\ell}|$ of elements of the set $b_{m,\ell}$. Obviously, this number is equal to the number $N(m,l)$ of solutions of the inequality $|\nu_1| + \cdots + |\nu_m| \leq \ell$ for integers $\nu_i \in \mathbb{Z}$. Let $N^*(m,\ell)$ be the number of non-negative integer solutions $(\nu_1, \ldots, \nu_m)$ of the inequality $\nu_1 + \cdots + \nu_m \leq \ell$. In each such solution there are at most ℓ non-zero ν_k, hence $|b_{m,\ell}| = N(m,l) \leq 2^\ell N^*(m,\ell)$. Now $N^*(m,\ell)$ is also the number of non-negative integer solutions of the equation $\nu_0 + \cdots + \nu_m = \ell$. To find $N^*(m,\ell)$, consider all "words" of length $m+\ell$ formed by ℓ copies of letter A and m copies of letter B. Then ν_0 can be interpreted as the number of A's before the first entry of B, ν_1 as the number of A's between the first and the second entry of B, and so on. It follows that $N^*(m,\ell)$ is the number of ways to place ℓ copies of A in $m+\ell$ available places, that is, $N^*(m,\ell) = \binom{m+\ell}{\ell}$. Since $\ell! > (\ell/e)^\ell$,

(5.4) $$|b_{m,\ell}| = N(m,l) =\leq \frac{2^\ell(m+\ell)^\ell}{\ell!} \leq \left(\frac{2e(m+\ell)}{\ell}\right)^\ell .$$

For ℓ given by (5.2) we have $1 \leq \ell \leq n/4 \leq m/4$, hence

$$|b_{m,\ell}| \leq \left(\frac{5em}{2\ell}\right)^\ell \leq \left(\frac{e^2m}{\ell}\right)^\ell .$$

To prove the lemma, we show that the measure P_0 of the set of all $\mathbf{y}$ for which (5.3) does not hold is less than one. By Lemma 4.1, applied to $x/\|x\|_2$, this measure is less than $e^{-n}|b_{m,\ell}| \leq e^{-n}(e^2m/\ell)^\ell$, hence $\log P_0 \leq \ell(2+\log(m/\ell)) - n$. Since $t\log(m/t)$, $1 \leq t \leq m$, is an increasing function of t, it follows that for ℓ defined by (5.2)

$$\begin{aligned}
\log P_0 &< \frac{n}{4\log(em/n)}(2 + \log[(4m/n)\log(em/n)]) - n \\
&< \frac{n}{4\log(em/n)}(4 + \log(m/n) + \log\log(em/n) - 4\log(em/n)) \\
&= \frac{n}{4\log(em/n)}(\log\log(em/n) - 3\log(m/n)) < 0 ,
\end{aligned}$$

so that $P_0 < 1$. □

Proof of the Upper Estimate (5.1). If ℓ is the integer (5.2) and $1 \leq \ell \leq 99$, then $n^{-1}\log(em/n) > 1/400$ and the upper estimate in (5.1) is trivial since $d_n(b_2^m)_\infty \leq 1$. So let $\ell \geq 100$. By the duality Theorem 2.2 of Chapter 13, $d_n(b_2^m)_\infty = d^n(b_1^m)_2$, and we shall estimate the latter quantity. We have to show that there exists a subspace X^n of $\mathbb{R}^m$ of codimension $\le n$ so that for each point $x \in X^n \cap b_1^m$ we have $\|x\|_2 \le 25/\sqrt{\ell}$. For X^n we take

$$X^n := \{x \in \mathbb{R}^n : F(x, \mathbf{y}^*) = 0\} ,$$

where $\mathbf{y}^*$ is one of the multivectors of Lemma 5.3. Let $x \in b_1^m \cap X^n$ and let $x' \in b_{m,\ell}$ be a closest point to x in the l_∞^m norm. Then $x'' := x - x' \in b_1^m \cap (1/\ell) b_\infty^m = \operatorname{co}(V_{m,\ell})$. We have

$$\|x''\|_2 \le (\|x''\|_1 \cdot \|x''\|_\infty)^{1/2} \le \ell^{-1/2} , \tag{5.5}$$

We compare the values of $F(x, \mathbf{y}^*)$ in different points. Since $x' \in b_{m,\ell}$, by (5.3),

$$F(x', \mathbf{y}^*) \ge (1/8) m^{-1/2} \|x'\|_2 \tag{5.6}$$

For each $z \in b_{m,\ell}$, again by (5.3), $F(z, \mathbf{y}^*) \le 3m^{-1/2}\|z\|_2$. In particular, on $V_{m,\ell}$ one has $F(z, \mathbf{y}^*) \le 3/\sqrt{m\ell}$, and since F is subadditive, this holds also for $z = x''$, hence

$$F(x', \mathbf{y}^*) \le F(x, \mathbf{y}^*) + F(x'', \mathbf{y}^*) = F(x'', \mathbf{y}^*) \le 3/\sqrt{m\ell} .$$

By (5.6), $\|x'\|_2 \le 24/\sqrt{\ell}$, and together with (5.5), $\|x\|_2 \le 25/\sqrt{\ell}$. □

We are now ready to prove Kashin's theorem on Lipschitz classes.

Theorem 5.4 (Kashin [1977]). *Let $\alpha > 1/p$. Then in the region* III*: $2 \le p \le q \le \infty$ of Fig.* 1.1,

$$d_n(B_p^\alpha)_q \sim n^{-\alpha} , \tag{5.7}$$

and in the region IV*: $1 \le p \le 2 \le q$,*

$$d_n(B_p^\alpha)_q \sim n^{-\alpha + 1/p - 1/2} . \tag{5.8}$$

It should be noted that the restriction $\alpha > 1/p$ is essential (see Theorem 8.2).

Proof. We have only to find the upper bounds for the widths in (5.7), (5.8) since the lower bounds are given by Theorem 3.6.

Let first $1 \le p \le 2$, $q = \infty$. We fix a real number ρ satisfying

$$0 < \rho < 2(\alpha - 1/p) . \tag{5.9}$$

It will be convenient to assume that ρ^{-1} is an integer. It will suffice to establish the desired estimates only for some subsequence $n = n(\nu)$, $\nu = 0, 1, \ldots$, for which $n(\nu) \sim 2^\nu$. We shall use the discretization Theorem 2.1. For (2.1), we define

$$
(5.10) \qquad n_k := n_k(\nu) := \begin{cases} 2^k(r+1)\ , & \text{if } 0 \le k \le \nu \\ \left[2^{\nu(1+\rho)-k\rho}(r+1)\right]\ , & \text{if } \nu < k < (1+\rho^{-1})\nu \\ 0\ , & \text{if } k \ge (1+\rho^{-1})\nu\ , \end{cases}
$$

and then set $n := n(\nu) := n_0 + n_1 + \cdots$. The dimensions $m_k := 2^k(r+1)$ of the balls in (2.1) are independent of ν. One immediately verifies that $n(\nu) \sim 2^\nu$ as $\nu \to \infty$. Applying Theorem 2.1 we use the inequality $d_n(b_p^m)_\infty \le d_n(b_2^m)_\infty$. For $k \le \nu$ we have $n_k = m_k$, hence $d_{n_k} := d_{n_k}(b_2^{m_k}, l_\infty^{m_k}) = 0$. For $k \ge (1+\rho^{-1})\nu$, we have $n_k = 0$, hence $d_{n_k} = 1$. For $\nu < k < (1+\rho^{-1})\nu$ we apply (5.1):

$$
d_{n_k} \le C n_k^{-1/2} (\log(e m_k/n_k))^{1/2} \le C 2^{-\nu/2} \cdot 2^{(k-\nu)\rho/2} \cdot (k-\nu)^{1/2}\ .
$$

Using these estimates in Theorem 2.1 we get

$$
d_n(B_p^\alpha)_\infty \le C \left[\sum_1 2^{k(1/p-\alpha)} \cdot 2^{-\nu/2+(k-\nu)\rho/2}(k-\nu)^{1/2} + \sum_2 2^{k(1/p-\alpha)} \right]\ .
$$

The sum $\sum_1$, over $k = \nu+1, \ldots, (1+\rho^{-1})\nu - 1$, is less than

$$
2^{\nu(-\alpha+1/p-1/2)} \sum_{k=\nu+1}^{\infty} 2^{(k-\nu)(-\alpha+1/p+\rho/2)}(k-\nu)^{1/2} \sim n^{-\alpha+1/p-1/2}\ .
$$

(The series converges due to (5.9)). The sum $\sum_2$, over $k \ge (1+\rho^{-1})\nu$, does not exceed $C\, 2^{(1+\rho^{-1})(1/p-\alpha)\nu} \sim n^{(1+\rho^{-1})(1/p-\alpha)} < n^{-\alpha+1/p-1/2}$ (the last inequality is again due to (5.9)). This completes the proof of (5.8) for $q = \infty$.

For $q = 2$, (5.8) is contained in Theorem 3.8. For $2 < q < \infty$, (5.8) follows from the obvious inequalities

$$
d_n(B_p^\alpha)_2 \le d_n(B_p^\alpha)_q \le d_n(B_p^\alpha)_\infty\ .
$$

Similarly, (5.7) follows from the inequalities

$$
d_n(B_p^\alpha)_p \le d_n(B_p^\alpha)_q \le d_n(B_2^\alpha)_\infty\ , \qquad 2 \le p \le q \le \infty\ ,
$$

in which both extreme terms are $O(n^{-\alpha})$. □

We also have a theorem on the Gelfand widths.

Theorem 5.5. *Let $\alpha > 1 - 1/q$. In the region* II *of Fig. 1.2: $1 \le p \le q \le 2$, one has*

$$
(5.11) \qquad d^n(B_p^\alpha)_q \sim n^{-\alpha}\ ,
$$

and in the region IV*: $1 \le p \le 2 \le q \le \infty$,*

$$
(5.12) \qquad d^n(B_p^\alpha)_q \sim n^{-\alpha+1/2-1/q}\ .
$$

The proof is analogous to that of Theorem 5.4, so we leave it to the reader. When $\alpha = r$ is an integer, it also follows from Theorem 5.4 by the duality (2.7) of Chapter 13. This completely establishes the facts illustrated on Figures 1.1, 1.2.

§ 6. Gaussian Measures

The Gaussian measure P^* studied here will be needed for an evaluation of linear widths in finite dimensional Banach spaces of §7. It has also independent interest as a tool in the modern theory of such spaces (compare, for example, Milman and Schechtman [B-1986] or Pisier [B-1989]). We need the following quadratic functional

$$F^*(x, \mathbf{y}) := F^*_{m,n}(x, \mathbf{y}) := n^{-1}[(x, y_1)^2 + \cdots + (x, y_n)^2]$$

analogous to the $F(x, \mathbf{y})$ of (4.8). This time we shall assume that each vector $y_i := (\eta_{i,1}, \ldots, \eta_{i,m})$, $i = 1, \ldots, n$, changes over the whole space $\mathbb{R}^m$ endowed with the euclidean metric. The multivector $\mathbf{y} = (y_1, \ldots, y_n)$ will be treated as an element of $\mathbb{R}^{mn}$ with the coordinates $\eta_{i,k}$, $i = 1, \ldots, n; k = 1, \ldots, m$. The measure P^* in the set of all multivectors $\mathbf{y}$ is introduced by setting

$$dP^* := (2\pi)^{-mn/2} \exp\left(-\tfrac{1}{2} \sum_{i,k} \eta_{i,k}^2\right) d\eta ,$$

where $d\eta = \prod_{i,k} d\eta_{i,k}$ is the Lebesgue measure in $\mathbb{R}^{mn}$. Since $\int_{-\infty}^{+\infty} e^{-u^2/2}\, du = \sqrt{2\pi}$, P^* is a probability measure on $\mathbb{R}^{mn}$. It is also rotation-invariant. More exactly, if $U : \mathbb{R}^m \to \mathbb{R}^m$ is any rotation about zero and U^* is the corresponding transformation $(y_1, \ldots, y_n) \to (Uy_1, \ldots, Uy_n)$ of multivectors, then for any set Ω of multivectors $P^*(U^*\Omega) = P^*(\Omega)$. This follows from the invariance of the sum $\sum_{i,k} \eta_{i,k}^2 = \|y_1\|^2 + \cdots + \|y_n\|^2$ under rotations. Furthermore, for all fixed i, k and $-\infty \le \alpha \le \beta \le \infty$,

$$P^*\{\mathbf{y} : \alpha \le \eta_{i,k} \le \beta\} = (2\pi)^{-1/2} \int_\alpha^\beta \exp(-u^2/2)du . \tag{6.1}$$

(In the probabilistic language one says that the $\eta_{i,k}$ are independent Gaussian variables, with mean zero and variance one.) By direct computation,

$$E(\|y_i\|^2) = \sum_{k=1}^m E(\eta_{i,k}^2) = m , \qquad i = 1, \ldots, n ,$$

so that $\|y_i\|$ is, typically, $\sqrt{m}$.

There is a relation between the measure P^* and the measure μ on S^{m-1} defined in §4. The latter induces, for any fixed $r > 0$, the measure μ^* on the sphere rS^{m-1} by setting $\mu^*(\Omega) := \mu(r^{-1}\Omega)$ for $\Omega \subset rS^{m-1}$. In particular, if

$\eta_k(y)$ is the k-th coordinate of y, then $\mu^*\{y \in \sqrt{m}S^{m-1} : \alpha \le \eta_k(y) \le \beta\}$ equals

$$\mu\left\{y \in S^{m-1} : \frac{\alpha}{\sqrt{m}} \le \eta_k(y) \le \frac{\beta}{\sqrt{m}}\right\} = \frac{1}{2I_m\sqrt{m}} \int_\alpha^\beta (1 - v^2/m)^{(m-3)/2} dv ,$$

where I_m is given by (4.5). One can show that this tends to the right-hand side of (6.1) when $m \to \infty$, so that P^* in (6.1) can be viewed as a limit version of μ^*.

We prove two lemmas on the distribution of values of $F^*(x, \mathrm{y})$.

Lemma 6.1. *For any fixed $x \in S^{m-1}$ and $0 \le \alpha \le \beta \le \infty$,*

$$P^*\{\mathrm{y} : \alpha \le F^*(x, \mathrm{y}) \le \beta\} = (n/2)^{n/2} \Gamma(n/2)^{-1} \int_\alpha^\beta t^{n/2-1} e^{-nt/2} dt . \tag{6.2}$$

Proof. As in the proof of Lemma 4.1, we may assume that $x = (1, 0, \dots, 0)$. Then

$$P^*\{\mathrm{y} : \alpha \le F^*(x, \mathrm{y}) \le \beta\} = (2\pi)^{-mn/2} \int_D \exp\left(-\tfrac{1}{2}\sum_{i,k} \eta_{ik}^2\right) d\eta ,$$

where D is the domain in $\mathbb{R}^{mn}$ defined, in the coordinates $\eta_{i,k}$, by $\alpha n \le \eta_{1,1}^2 + \dots + \eta_{n,1}^2 \le \beta n$. Integrating first over all the coordinates $\eta_{i,k}$, $k = 2, 3, \dots, m$, we reduce the latter integral to

$$(2\pi)^{-n/2} \int_\Delta \exp\left(-\tfrac{1}{2}\sum_{i=1}^n u_i^2\right) du_1 \dots du_n ,$$

where $\Delta \subset \mathbb{R}^n$ is the domain $\sqrt{\alpha n} \le r \le \sqrt{\beta n}$, $r^2 = u_1^2 + \dots + u_n^2$. By (4.4) this is equal to

$$(2\pi)^{-n/2} n V_n(1) \int_{\sqrt{\alpha n}}^{\sqrt{\beta n}} \exp(-r^2/2) r^{n-1}\, dr .$$

Since $V(1) = \pi^{n/2}/\Gamma(n/2+1)$, the substitution $r = \sqrt{nt}$ yields (6.2). □

The next lemma shows that large deviations of $F^*(x, \mathrm{y})$ from its average (equal to one for every fixed $x \in S^{m-1}$) are unlikely.

Lemma 6.2. *For every $x \in S^{m-1}$, $n \ge 1, m \ge 2$, and $0 < \varepsilon < 1$,*

$$P^*\{\mathrm{y} : |F^*(x, \mathrm{y}) - 1| > \varepsilon\} \le \frac{2}{\varepsilon\sqrt{n}} \exp(-n\varepsilon^2/8). \tag{6.3}$$

Proof. With $k := n/2$, we have

$$P^*\{\mathbf{y} : F^*(x, \mathbf{y}) > 1 + \varepsilon\} = (k^k/\Gamma(k)) \int_{1+\varepsilon}^{\infty} t^{k-1} e^{-kt}\, dt . \tag{6.4}$$

For $s \in \mathbb{R}$, $0 \le s \le k$, let $I_s := \int_{1+\varepsilon}^{\infty} t^s e^{-kt}\, dt$. By partial integration,

$$I_s = (1+\varepsilon)^s I_0 + (s/k) I_{s-1} \le (1+\varepsilon)^s I_0 + I_{s-1} ,$$

where $I_0 := (1/k)e^{-k(1+\varepsilon)}$. With $k_0 := [k - 1/2]$ this implies

$$I_{k-1} \le I_{k_0} \le I_0 \sum_{j=0}^{k_0} (1+\varepsilon)^j \le \frac{1}{k\varepsilon} e^{-k(1+\varepsilon)} (1+\varepsilon)^{k+1/2} . \tag{6.5}$$

By the Stirling formula $\Gamma(k+1) > \sqrt{2\pi k} k^k e^{-k}$, so that

$$k^k/\Gamma(k) < e^k \sqrt{k/(2\pi)} . \tag{6.6}$$

Since $1 + \varepsilon < \exp(\varepsilon - \varepsilon^2/4)$ for $0 \le \varepsilon \le 1$, it follows from (6.4), (6.5) , (6.6) that

$$P^*\{\mathbf{y} : F^*(x, \mathbf{y}) \le 1 + \varepsilon\} = (k^k/\Gamma(k)) I_{k-1} < \frac{1}{\varepsilon\sqrt{n}} e^{-n\varepsilon^2/8} . \tag{6.7}$$

Similarly,

$$P^*\{\mathbf{y} : F^*(x, \mathbf{y}) > 1 - \varepsilon\} = (k^k/\Gamma(k)) \int_0^{1-\varepsilon} t^{k-1} e^{-kt}\, dt .$$

Setting $J_s := \int_0^{1-\varepsilon} t^{s-1} e^{-kt}\, dt$, $s \ge k$, it follows by partial integration that

$$J_s = s^{-1}(1-\varepsilon)^s e^{-k(1-\varepsilon)} + (k/s) J_{s+1} .$$

Since $J_s \to 0$ as $s \to \infty$, this implies that

$$J_s < e^{-k(1-\varepsilon)} \sum_{s=k}^{\infty} s^{-1} (1-\varepsilon)^s < (k\varepsilon)^{-1} e^{-k(1-\varepsilon)} (1-\varepsilon)^k .$$

Using (6.6) and the inequality $1 - \varepsilon < \exp(-\varepsilon - \varepsilon^2/2)$, we have

$$P^*\{\mathbf{y} : F^*(x, \mathbf{y}) \le 1 - \varepsilon\} \le \frac{k^k}{\Gamma(k)} J_k < \frac{1}{\varepsilon\sqrt{n}} e^{-n\varepsilon^2/4} . \tag{6.8}$$

Now (6.3) follows from (6.7) and (6.8). □

For $x, y \in \mathbb{R}^m$ and a multivector $\mathbf{v} = (v_1, \dots, v_n)$ let

$$\Phi(x, y, \mathbf{v}) := n^{-1} \sum_{i=1}^{n} (x, v_i)(y, v_i) - (x, y) .$$

Lemma 6.3. *For any* $x, y \in \mathbb{R}^m$,

$$P^*\{\mathbf{v} : |\Phi(x,y,\mathbf{v})| > \varepsilon\} \le \frac{8\|x\|\,\|y\|}{\varepsilon\sqrt{n}} \exp(-n\varepsilon^2/(32\|x\|^2\|y\|^2)) \; . \tag{6.9}$$

(We interpret the right-hand side to be zero if $x = 0$ or $y = 0$.)

Proof. It suffices to prove (6.9) only for $\|x\| = \|y\| = 1$, $x \neq y$, in which case it follows from Lemma 6.2 and the relation

$$\Phi(x,y,\mathbf{v}) = F^*\left(\frac{x+y}{2}, \mathbf{v}\right) - F^*\left(\frac{x-y}{2}, \mathbf{v}\right) - (x,y) \; .$$

Indeed, defining the two functions F_+, F_- by

$$F_\pm(x,y,\mathbf{v}) := F^*\left(\frac{x \pm y}{\|x \pm y\|}, \mathbf{v}\right) - 1$$

we have

$$\Phi(x,y,\mathbf{v}) = \frac{\|x+y\|^2}{4} F_+ - \frac{\|x-y\|^2}{4} F_- \le |F_+| + |F_-| \; .$$

Therefore

$$P^*\{|\Phi| > \varepsilon\} \le P^*\{|F_+| > \varepsilon/2\} + P^*\{|F_-| > \varepsilon/2\} \le \frac{8}{\varepsilon\sqrt{n}} \exp(-n\varepsilon^2/32) \; . \; \square$$

§ 7. Linear Widths of Finite Dimensional Balls

Our approach to the linear widths of the sets B_p^α in L_q will be via the estimates of widths of sets in the finite dimensional space l_p^n. For this purpose we shall prove Theorem 7.1 of Gluskin, which is also of considerable independent interest. Thus, our proof of Theorem 8.1 will parallel that of Kashin's theorems in §4,5. An independent, shorter approach to Theorem 8.1, bypassing §6, is outlined in the Note 10.2.

We begin with some known facts. From the inequalities (3.1), (3.3), (3.5) of Chapter 13 we get for $1 \le n \le m$, $1 \le q \le \infty$

$$\delta_n(b_\infty^m)_q = (m-n)^{1/q} \; ; \quad \delta(b_1^m)_2 = \delta_n(b_2^m)_\infty = \sqrt{1 - n/m} \; . \tag{7.1}$$

One should compare the latter quantity with the estimate (5.1) for $d_n(b_2^m)_\infty$: it is a remarkable example of $\delta_n \neq d_n$. Furthermore, since $b_1^m \subset b_2^m$ and since $\delta_n(b_1^m)_X = d_n(b_1^m)_X$ by (3.4) of Chapter 13, one gets an immediate consequence of Theorem 5.1 for $n \le m$:

$$\delta_n(b_1^m)_\infty \le C\sqrt{\log(em/n)/n} \; . \tag{7.2}$$

To this one may add the equality $\delta_n(b_p^m) = 1$, $1 \le n \le m-1$, $1 \le p \le \infty$, which follows immediately from Theorem 1.4 of Chapter 13.

Theorem 7.1 (Gluskin [1983]). *For $1 < p \le 2 \le q < \infty$, $1 \le n \le m$, and some constant $C(p,q)$ depending only on p and q,*

$$\delta_n(b_p^m)_q \le \begin{cases} C(p,q)m^{1/q}n^{-1/2} & \text{if } p^{-1}+q^{-1} \ge 1 \\ C(p,q)m^{1-1/p}n^{-1/2} & \text{if } p^{-1}+q^{-1} \le 1 \, . \end{cases} \tag{7.3}$$

(Actually, this is only the most important part: $p \le 2 \le q$, $m/n \ge C > 0$, of the complete Gluskin's estimate; see Note 8.1.)

We note that the case $p^{-1}+q^{-1} \le 1$ in (7.3) follows from the case $p^{-1}+q^{-1} \ge 1$ by duality (2.6) of Chapter 13. The case $p = q = 2$ is trivially valid, with $C(2,2) = 1$. Moreover, since $b_p^m \subset b_{p_1}^m$ for $p < p_1$, it suffices to establish (7.3) only for $p^{-1}+q^{-1} = 1$ with $1 < p < 2$. In other words, we have only to prove that for all n, m, $1 \le n \le m$,

$$\delta_n(b_p^m)_{p'} \le C_1(p)m^{1/p'}n^{-1/2} \, , \qquad p' = p/(p-1) \qquad 1 < p < 2 \, . \tag{7.4}$$

The proof of (7.4), in which we follow Makovoz (1988_2), bears some analogy to the proof of (5.1); we use a lemma about random vectors (this time, Gaussian vectors and Lemma 6.3) and the geometric Lemmas 7.2, 7.3. We shall use various norms in $\mathbb{R}^m$; in particular, $\|\cdot\|_p$ will mean the l_p^m norm and $\|\cdot\|$ will stand for $\|\cdot\|_2$.

We start with the following operator T on $\mathbb{R}^m$ of rank $\le n$ given by a multivector $\mathbf{v} = (v_1, \dots, v_n)$, $v_i \in \mathbb{R}^m$, $i = 1, 2, \dots, n$:

$$Tx := n^{-1} \sum_{i=1}^{n} (x, v_i)v_i \, , \qquad x \in \mathbb{R}^m \, . \tag{7.5}$$

According to the definition of linear widths, we have, for $1 \le n \le m$, $1 < p < 2$, and each $\mathbf{v}$,

$$\delta(b_p^m)_{p'} \le \max_{x \in b_p^m} \|Tx - x\|_{p'} = \max_{x,y \in b_p^m} |\Phi(x, y, \mathbf{v})| \, , \tag{7.6}$$

where Φ is the function of Lemma 6.3:

$$\Phi(x, y, \mathbf{v}) := n^{-1} \sum_{i=1}^{n} (x, v_i)(y, v_i) - (x, y) \, . \tag{7.7}$$

We shall show that there is a $\mathbf{v}$ with the required smallness of the right-hand side of (7.6). This will be derived from the properties of Φ. Our main idea will be to embed b_p^m into a convex set spanned by finitely many vectors in l_2^m of bounded length $\|x\| \le C(p)$, the majority of which are much shorter than $C(p)$.

We select an integer k_0, $0 \le k_0 \le m$, and numbers λ_k, $k_0 < \lambda_k \le m$, and define the sets of vectors G_k, $k_0 \le k \le m$. The set G_{k_0} consists of all vectors of b_2^m whose coordinates are taken from the set $\{\pm i/m_0\}_{i=0}^{m}$, where $m_0 := [4\sqrt{k_0}] + 1$. The sets G_k, $k_0 < k \le m$, consist of all vectors with some k

coordinates equal to $\pm\lambda_k$, with all other coordinates equal to zero. We assume that $\sum_{k_0<k\le m} k^{-p'}\lambda_k^{-p'} \le C(p)$, for some constant $C(p)>0$, and specify k_0 and $\lambda_k = \lambda_{k,m,p}$ later. Let $G := \bigcup_{k=k_0}^m G_k$.

Lemma 7.2. *For* $1<p\le 2$,

$$b_p^m \subset A(p)\,\mathrm{co}(G)\ , \tag{7.8}$$

where $A(p)>0$ *depends only on* p.

Proof. Let X be the space $\mathbb{R}^m$ equipped with the norm $\|\cdot\|_X$ for which $A(p)\,\mathrm{co}(G)$ is the unit ball, and let X' be its conjugate. Then (7.8) is equivalent to $\|x\|_X \le \|x\|_p$ for all $x\in\mathbb{R}^m$, or to $\|y\|_{p'} \le \|y\|_{X'}$, or to $\max_{g\in A(p)\mathrm{co}(G)}(g,y) \ge \|y\|_{p'}$ for all $y\in\mathbb{R}^m$, or finally to

$$\max_{g\in G}(g,y) \ge A(p)^{-1} \ \text{ for all } \ y, \qquad \|y\|_{p'}=1\ . \tag{7.9}$$

To prove (7.9), we take $y=(\eta_1,\dots,\eta_m)$ with $\|y\|_{p'=1}$ and let $\eta_1^*,\dots,\eta_m^*$ be the decreasing rearrangement of $|\eta_1|,\dots,|\eta_m|$. If $\sum_1^{k_0}(\eta_i^*)^{p'} \ge 1/4$, we denote by $\overline{y}$ the vector obtained from y by retaining k_0 coordinates with the largest absolute values and replacing all other coordinates by zeros. Let g be the nearest, in the l_2^m norm, element to $\overline{y}$ in G_{k_0}. Then $\|\overline{y}-g\| \le (k_0(4\sqrt{k_0})^{-2})^{1/2} \le 1/4$ and $\|\overline{y}\|^2 \ge \|\overline{y}\|_{p'}^{p'} \ge 1/4$, so that $\|g\| \ge \|\overline{y}\| - \|\overline{y}-g\| \ge 1/4$, hence

$$(y,g) = (\overline{y},g) = (1/2)(\|\overline{y}\|^2 + \|g\|^2 - \|\overline{y}-g\|^2) \ge 1/8\ .$$

Now let $\sum_1^{k_0}(\eta_i^*)^{p'} \le 1/4$. If we assume that $\eta_k^* \le B_1 k^{-1}\lambda_k^{-1}$ for some B_1 and all $k>k_0$, then

$$1 = \sum_1^m (\eta_k^*)^{p'} < 1/4 + B_1^{p'} \sum_{k=1}^m k^{-p'}\lambda_k^{-p'} \le 1/4 + B_1^{p'} C(p)\ ,$$

hence $B_1 > (3/4)^{1/p'} C(p)^{-1/p'} =: B(p)$. Consequently, there is a $k_1>k_0$ so that $\eta_{k_1}^* \ge B(p)k_1^{-1}\lambda_{k_1}^{-1}$. If g is an element of G_{k_1} whose coordinates are of the same sign as the corresponding coordinates of y, then $(g,y) \ge k_1\lambda_{k_1}\eta_{k_1} = B(p)$. Thus we may set $A(p) = \min(1/8,\ B(p))$. □

We now specify k_0 and the λ_k, $k_0<k\le m$, by setting

$$k_0 := [m^{2/p'}/\log(em)]\ , \quad \lambda_k := m^{1/p'}k^{-1}/\sqrt{\log(em/k)}\ . \tag{7.10}$$

Then we have

$$\sum_{k_0<k\le m} k^{-p'}\lambda_k^{-p'} \le \sum_{k=1}^m m^{-1}(\log(em/k))^{p'/2} < \int_0^1 (\log(e/t))^{p'/2}\,dt =: C(p)\ ,$$

as required for the validity of Lemma 7.2. In what follows, $C, C_1, C_2, \ldots$ denote positive constants depending only on p; the same C_k may stand for different quantities in different places.

Lemma 7.3. *With the choice* (7.10) *we have for* $1 < p < 2$

$$G = \bigcup_{k=k_0}^{m} G_k \subset C_1 b_2^m \ , \tag{7.11}$$

$$\log |G_k| \le C_2 k \log(em/k) \ , \qquad k = k_0, \ldots, m \ . \tag{7.12}$$

Proof. If $x \in G_{k_0}$, then $\|x\| \le 1$. If $x \in G_k$, $k > k_0$, then $\|x\|^2 = k\lambda_k^2 = m^{2/p'-1}/\phi(k/m)$, where $\phi(t) := t\log(e/t)$. Since $\phi(t)$ increases for $0 < t \le 1$, for $k \ge k_0$ we have $\|x\|^2 \le m^{2/p'-1}/\phi(k_0/m)$. This quantity depends only on m and p. When $m \to \infty$, it has a finite limit depending on p. Therefore, for all m it remains bounded by a constant depending only on p. Thus, $\|x\| \le C_1$ for all $x \in G$.

To prove (7.12), we can assume that $1 \le m_0^2 \le m/2$. For $k > k_0$, the formula $|G_k| = \binom{m}{k} 2^k$ and the inequality $\binom{m}{k} \le (em/k)^k$ yield $\log |G_k| \le C_2 k \log(em/k)$. This estimate extends to $k = k_0$ because every vector in G_{k_0} has $\le m_1 := m_0^2$ non-zero coordinates, hence

$$|G_{k_0}| \le \binom{m}{m_1} (2m_0)^{m_1} \le \exp\bigl(C_3 m_1 \log(em)\bigr) \le \exp\bigl(C_4 k_0 \log(em/k_0)\bigr) \ . \ \square$$

Proof of Theorem 7.1. In view of (7.5), relation (7.3) will follow from the existence of $\mathbf{v}$ and C satisfying

$$\max_{x,y \in b_p^m} |\Phi(x, y, \mathbf{v})| \le C m^{1/p'} n^{-1/2} \tag{7.13}$$

for $1 \le n \le m$. By Lemma 7.2, one can replace b_p^m in (7.13) by the set $\mathrm{co}(G)$. Since $\Phi(x, y, \mathbf{v})$ is linear in x and linear in y, the set $\mathrm{co}(G)$ may be replaced by G. Thus it will suffice to prove that for some $\beta > 0$ and some $\mathbf{v}$

$$\max_{x,y \in G} |\Phi(x, y, \mathbf{v})| < \varepsilon \ , \qquad \varepsilon := \beta m^{1/p'} n^{-1/2} \ . \tag{7.14}$$

With the probability measure P^* of §6, it will be enough to establish that

$$P_0 := P^*\bigl(\mathbf{v} : \max_{x,y \in G} |\Phi(x, y, \mathbf{v})| \ge \varepsilon\bigr) < 1 \ . \tag{7.15}$$

We set

$$P_{k,k'} := P^*\Bigl(\mathbf{v} : \max_{x \in G_k,\, y \in G_{k'}} |\Phi(x, y, \mathbf{v})| > \varepsilon\Bigr)$$

and have $P_0 \le \sum_{k,k'=k_0}^{m} P_{k,k'}$. To estimate $P_{k,k'}$, we apply Lemma 6.3. Taking $\beta \ge 1$ in the definition of ε, we have $\varepsilon\sqrt{n} = \beta m^{1/p'} > 1$. If $x, y \in G$, then

$8\|x\|\|y\|(\varepsilon\sqrt{n})^{-1} \le 8C_1^2 =: C_2$. If $x \in G_k$, $y \in G_{k'}$ and either k or k' is not equal to k_0, say, $k \ge k'$, then $\|x\| = \lambda_k\sqrt{k}$, $\|y\| \le C_1$, and by (6.9),

$$\begin{aligned} P^*(\mathrm{v} : |\Phi(x,y,\mathrm{v})| > \varepsilon) &\le C_2 \exp(-C_3 n\varepsilon^2 k^{-1}\lambda_k^{-2}) \\ &= C_2 \exp\bigl(-C_3\beta^2 k \log(em/k)\bigr) \ . \end{aligned}$$

This inequality remains valid when $k = k' = k_0$ in which case we use (6.9) and the estimates $\|x\| \le C_1$, $\|y\| \le C_1$.

For $k \ge k'$, using (7.12), we get

$$\begin{aligned} P_{k,k'} &\le |G_k| \cdot |G_{k'}| \cdot C_2 \exp\bigl(-C_3\beta^2 k \log(em/k)\bigr) \\ &\le \exp\bigl(2C_4 k \log(em/k)\bigr) \cdot C_2 \exp\bigl(-C_3\beta^2 k \log(em/k)\bigr) \ . \end{aligned}$$

We now choose β sufficiently large so that $2C_4 - C_3\beta^2 =: -\gamma^2 < 0$. Then for all $k, k' \ge k_0$, with $\phi(t) = t\log(e/t)$,

$$P_{k,k'} \le C_2 \exp(-\gamma^2 m\phi(k/m)) \le C_2 \exp(-\gamma^2 m\phi(k_0/m)) \le C_2 \exp(-C_5\gamma^2 m^{2/p'}) \ .$$

Since there are $\le m^2$ combinations (k,k'), we have $P_0 \le m^2 C_2 \exp(-C_5\gamma^2 m^{2/p'})$. It is clear now that $P_0 < 1$ for all $m = 1, 2, \ldots$ if $\beta = \beta(p)$ is sufficiently large.

□

We shall give here some additional estimates of widths of finite-dimensional balls. In Note 8.2 we shall sketch the proof of the inequality

$$\delta_n(b_1^m)_\infty \le C n^{-1/2} \log m / \log n \ . \tag{7.16}$$

As a corollary, we have

$$\delta_n(b_1^m)_q \le C(q) m^{1/q} n^{-1/2} \ , \qquad 1 \le q < \infty \ . \tag{7.17}$$

Indeed, let $C(q) := \max\{1, Cq/2\}$, where C is the constant of (7.16). If $m \ge n^{q/2}$, then (7.16) follows from the trivial inequality $\delta_n(b_1^m)_q \le 1$. On the other hand, since $\|x\|_q \le m^{1/q}\|x\|_\infty$ for $x \in \mathbb{R}^m$, we have $\delta_n(b_1^m)_q \le m^{1/q}\delta_n(b_1^m)_\infty$ and for $m \le n^{q/2}$ (7.17) follows from (7.16). □

We shall now prove a lower estimate, obtained by Kashin [1980] and Gluskin [1981], which matches (7.17).

Theorem 7.4. *For* $n = 1, 2, \ldots,\ m \ge 2n,\ 2 < q < \infty$,

$$\delta_n(b_1^m)_q = d_n(b_1^m)_q \ge (1/4)\min\{1,\ m^{1/q} n^{-1/2}\} \ . \tag{7.18}$$

Let $e_1^m := (1, 0, \ldots, 0)$, $e_2^m := (0, 1, \ldots, 0), \ldots,$ $(e_m^m) := (0, 0, \ldots, 1)$ be the standard basis for $\mathbb{R}^m$. By $\rho(x, Y)_q$ we denote the distance from $x \in l_q^m$ to a subspace Y of l_q^m. With Kashin, we first prove

Lemma 7.5. *Let $(u_i)_1^{2n}$ and $(v_i)_1^{2n}$ be two sequences of vectors from $\mathbb{R}^n$ for which $(u_i, v_i) = 1$, $i = 1, \ldots, 2n$. Then*

$$\sum_{i \neq j} (u_i, v_j)^2 \geq n \; . \tag{7.19}$$

Proof. We consider the matrix A with $2n$ columns and n rows, with i-th column being the vector u_i. Let L be the n-dimensional subspace of $\mathbb{R}^{2n}$ spanned by the rows of A. The vector

$$a := (a_1, \ldots, a_{2n}) \; , \quad a_1 = (u_1, v_1) = 1 \; , \quad a_2 = (u_2, v_1), \ldots, a_{2n} = (u_{2n}, v_1) \; ,$$

is equal to $\sum_{k=1}^n v_{1,k} y_k$, where y_k are the rows of A and $v_{1,k}$ are the coordinates of v_1. Thus, $a \in L$, hence

$$\rho(e_1^{2n}, L)_2^2 \leq \|e_1^{2n} - a\|_2^2 = \sum_{i=2}^{2n} (u_i, v_1)^2 \; .$$

Approximating $e_2^{2n}, \ldots, e_{2n}^{2n}$ similarly, one obtains

$$\sum_{k=1}^{2n} \rho(e_k^{2n}, L)_2^2 \leq \sum_{i \neq j} (u_i, v_j)^2 \; .$$

On the other hand, if $g_1, \ldots, g_n$ is an orthonormal basis for L, then by Parceval's identity $\rho(e_k^{2n}, L)_2^2 = 1 - \sum_{j=1}^n (e_k^{2n}, g_j)^2$, $k = 1, \ldots, 2n$. Therefore

$$\sum_{k=1}^{2n} \rho(e_k^{2n}, L)_2^2 = 2n - \sum_{j=1}^{n} \sum_{k=1}^{2n} (e_k^{2n}, g_j)^2 = 2n - n = n \; ,$$

and (7.19) follows. □

We now prove (7.18). We may assume that $d := d_n(b_1^m)_q \leq 1/4$ for otherwise (7.18) is trivially valid. Let X_n be an optimal n-dimensional subspace for this width and let $x_1, \ldots, x_n$ be a basis for X_n. We consider the $n \times m$ matrix with columns $x_1, \ldots, x_n$ and rows $y_1, \ldots, y_m$. Since $e_i^m \in b_1^m$, $i = 1, \ldots m$, there exist linear combinations $x_i^* := \sum_{j=1}^n v_{i,j}^* x_j$, $v_{i,j}^* \in \mathbb{R}$, for which $\|e_i^m - x_i^*\|_q \leq d \leq 1/4$, $i = 1, \ldots, m$. For the vectors $v_i^* := (v_{i,1}^*, \ldots, v_{i,n}^*)$, $i = 1, \ldots, n$, we have for the j-th coordinate of x_i^*

$$(x_i^*)_j = (v_i^*, y_j) \; , \qquad j = 1, \ldots m \; .$$

It follows that for $i = 1, \ldots, n$

$$(1/4)^q \geq d^q \geq \|e_i^m - x_i^*\|_q^q = |1 - (v_i^*, y_j)|^q + \sum_{j : j \neq i} |(v_i^*, y_j)|^q \; ,$$

hence $3/4 \leq r_i := (v_i^*, y_i) \leq 5/4$ and

$$\sum_{j:j\neq i} |(v_i^*, y_j)|^q \leq d^q , \qquad 1 \leq i \leq m .$$

For the vectors $v_i := (1/r_i)v_i^*$, we have

$$(v_i, y_i) = 1 , \qquad \sum_{j:j\neq i} |(v_i, y_j)|^q \leq (4/3)^q d^q , \qquad 1 \leq i \leq m .$$

From this we shall derive (7.18). First of all,

$$S := \sum_{\substack{1\leq i,j\leq m \\ i\neq j}} |(v_i, y_j)|^q \leq m(4/3)^q d^q . \tag{7.20}$$

Let $T := \sum_\Omega \sum_{i,j\in\Omega,\, i\neq j} |(v_i, y_j)|^q$, where the outer summation is extended to all $(2n)$-element subsets Ω of the set $\{1, \dots, m\}$. Each term $|(v_i, y_j)|^q$, $i \neq j$, belongs to an Ω if and only if $i, j \in \Omega$, the other elements of Ω being arbitrary. Thus, each term appears $\binom{m-2}{2n-2}$ times in T, hence $T = \binom{m-2}{2n-2} S$. On the other hand, since the total number of the sets Ω is $\binom{m}{2n}$, there is an Ω_0 for which $P := \sum_{i,j\in\Omega_0,\, i\neq j} |(v_i, y_j)|^q \leq T\binom{m}{2n}^{-1}$, so that

$$S \geq \binom{m}{2n}\binom{m-2}{2n-2}^{-1} P \geq m(m-1)(4n^2)^{-1} P .$$

From this and (7.20),

$$d \geq (3/4)(m-1)^{1/q}(4n^2)^{-1/q} P^{1/q} . \tag{7.21}$$

The sum P contains $\leq 4n^2$ terms. Therefore, by the inequality between the l_q and the l_2 means and since $(v_i, y_i) = 1$ for all i, Lemma 7.7 yields

$$P^{1/q} \geq (4n^2)^{1/q-1/2} \left(\sum_{i,j\in\Omega_0,\, i\neq j} (v_i, y_j)^2 \right)^{1/2} \geq (4n^2)^{1/q-1/2} n^{1/2} .$$

Combining this with (7.21), we get

$$d \geq (3/8)(m-1)^{1/q} n^{-1/2} \geq (1/4) m^{1/q} n^{-1/2} . \qquad \square$$

§ 8. Linear Widths of the Lipschitz Classes

We study here the asymptotic behavior of the linear widths of the Lipschitz classes B_p^α for (p, q) in III, the only region not covered by Theorem 3.8. The following theorem of Maiorov [1978] and Höllig [1979] is based on the estimates for the widths of finite dimensional balls of §7.

Region IV is divided in two parts by the line $1/p + 1/q = 1$ (or, what is the same, by $q = p'$), with IVa: $p' \geq q$, IVb: $p' \leq q$.

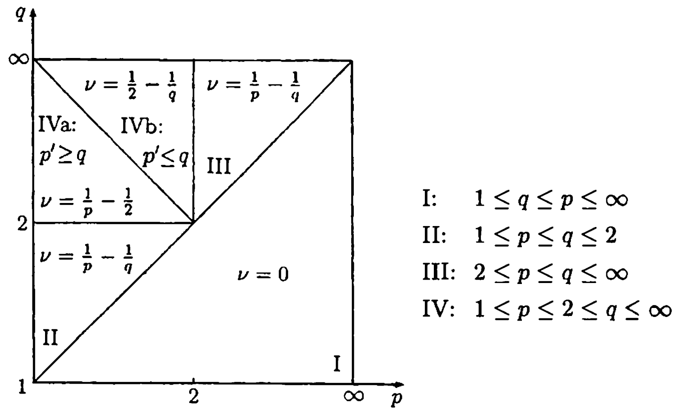


Fig. 8.1. $\delta_n(B_p^\alpha)_q \sim n^{-\alpha+\nu}$

Theorem 8.1. *For* $1 \leq p \leq 2 \leq q \leq \infty$, $p' = p/(p-1)$,

$$\delta_n(B_p^\alpha)_q \sim \begin{cases} n^{-\alpha+1/p-1/2}\ , & \textit{if}\ p' \geq q\ ,\ \alpha > 1/p\ , \\ n^{-\alpha+1/2-1/q}\ , & \textit{if}\ p' \leq q\ ,\ \alpha > 1 - 1/q\ . \end{cases} \tag{8.1}$$

We need to prove only the estimates from above, since those from below appear in (3.19). Moreover, it suffices to establish the first estimate from above only for $q = p'$. Indeed, for $q \leq p'$ this estimate then follows trivially since $\delta_n(B_p^\alpha)_q \leq \delta_n(B_p^\alpha)_{p'}$. Once this is known, for $q \geq p'$ we have $q' \leq p$, hence for $\alpha > 1/q' = 1 - 1/q$,

$$\delta_n(B_p^\alpha)_q \leq \delta_n(B_{q'}^\alpha)_q \sim n^{-\alpha+1/q'-1/2} = n^{-\alpha+1/2-1/q}\ .$$

Proof. Let $q = p'$. If $p = 1, q = \infty$, the proof of (8.1) for δ_n is the same as the proof of (5.8) for the Kolmogorov widths d_n since one can use the estimate (7.1) for the $\delta_n(b_1^m)_\infty$. If $p > 1$ (so that $q = p' < \infty$), the proof differs from that for d_n only slightly. We define the $n_k(\nu)$ by (5.10) with $0 < \rho < 2(\alpha - 1/p)$ as in (5.9). Using Theorem 2.1 and (7.2) we come to the inequality, with $n \sim 2^\nu$,

$$\delta_n(B_p^\alpha)_{p'} \leq C\left[\sum_1 2^{k(-\alpha+1/p-1/p')}.2^{k/p'}.2^{-(\nu(1+\rho)-k\rho)/2} + \sum_2 2^{k(-\alpha+1/p-1/p')}\right],$$

with $\sum_1$ over $\nu < k < (1+\rho^{-1})\nu$ and $\sum_2$ over $k \geq (1+\rho^{-1})\nu$. We have

$$\sum_1 \leq 2^{-\nu(1+\rho)/2} \sum_{k=\nu+1}^{\infty} 2^{k(-\alpha+1/p+\rho/2)} \sim n^{-\alpha+1/p-1/2}\ .$$

The second sum is $\sim n^{(1+\rho^{-1})(-\alpha+1/p-1/p')}$ with $p' \geq 2$ which is $o(n^{-\alpha+1/p-1/2})$ if we take ρ close enough to zero. □

We have completed the main result of this chapter. Theorem 1.1 now follows by summing up Theorems 3.8, 5.4, 5.5 and 8.1.

The restriction $\alpha > 1/p$ (or $\alpha > 1 - 1/q$)) in Theorem 8.1, and earlier in Theorems 5.4, 5.5 is essential: one can prove that for $p \le q$ and small α the orders of decay of the widths $s_n(B_p^\alpha)(A)$, $A = [0,1]$ or $\mathbb{T}$, are given by different formulas. We demonstrate this in the case $p = 1$, $2 < q < \infty$. We assume that $\alpha > 1 - 1/q$, since otherwise $d_n(B_1^\alpha)_q \not\to 0$ (see Proposition 3.9).

Theorem 8.2. *For* $2 < q < \infty$, $1 - 1/q < \alpha < 1$,

$$\delta_n(B_1^\alpha)_q \sim d_n(B_1^\alpha)_q \sim n^{(q/2)(-\alpha+1-1/q)} . \tag{8.2}$$

Proof.

Lower Estimate. From (2.7) we have for arbitrary $n, N = 1, 2, \ldots$

$$d_n(B_1^\alpha)_q \ge C N^{-\alpha+1-1/q} d_n(b_1^N)_q \, .$$

We choose $N = [n^{q/2}]$ and use the lower estimate (7.18) for $d_n(b_1^N)_q$.

Upper Estimate. We again use Theorem 2.1, but with different n_k. For a given n, we set $n_k := 0$ if $k > k_0 := [(q/2)\log_2 n] + 1$ and $n_k := [\gamma n 2^{-(k_0-k)(1-\alpha)}]$, $\gamma := (1 - 2^{\alpha-1})^{-1}$, if $0 \le k \le k_0$. We then have $\gamma \sum_{j=0}^\infty 2^{-j(1-\alpha)} = 1$, hence $\sum_{k=0}^\infty n_k \le n$. By (2.1), with $m_k = 2^k$,

$$s_n(B_1^\alpha)_q \le C \sum_{k=0}^\infty 2^{-k(\alpha-1+1/q)} s_{n_k}(b_1^{m_k})_q = C\,(\Sigma_1 + \Sigma_2) \,,$$

with the first sum extended to $0 \le k \le k_0$ and the second to $k > k_0$. In Σ_2 we use the estimate $s_{n_k}(b_1^{m_k})_q \le 1$ and get

$$\Sigma_2 \le C \sum_{k_0+1}^\infty 2^{-k(\alpha-1+1/q)} \sim 2^{-k_0(\alpha-1+1/q)} \sim n^{(q/2)(-\alpha+1-1/q)}.$$

In Σ_1 we have, by (7.2), $s_{n_k}(b_1^{m_k})_q \le C m_k^{1/q} n_k^{-1/2}$, so that

$$\Sigma_1 \le C \sum_{k=0}^{k_0} 2^{-k(\alpha-1+1/q)} n^{-1/2} 2^{k/q} 2^{-(1/2)(k_0-k)(1-\alpha)}$$

$$= C n^{-1/2} 2^{-(1-\alpha)(k_0/2)} \sum_{k=0}^{k_0} 2^{(1-\alpha)(3k/2)} \,,$$

which does not exceed $C n^{-1/2} 2^{k_0(1-\alpha)} \sim n^{(q/2)(-\alpha+1-1/q)}$. □

For $\delta_n(B_1^1)_q$, $q < \infty$, we similarly obtain an upper estimate of the form $C n^{-1/2} \log n$. Exact orders of these widths are unknown but one can prove that the logarithmic factor can not be dropped (Kulanin [1983₁].

§ 9. Problems

9.1. Prove the following refinement of (4.17): for any fixed $x \in S^{m-1}$ and $0 < b < 1$,

$$\mu(y \in S^{m-1} : |(x,y)| > b) \le 2\exp(-mb^2/2) .$$

Hint: $\{y \in S^{m-1} : (x,y) > b\} \subset \{y \in \mathbb{R}^m : \|y - bx\|_2 \le \sqrt{1-b^2}\}$.

9.2. Let $\hat{S}_m$ be the subset of S^{m-1} consisting of all vectors with coordinates $\pm m^{-1/2}$ and let $\hat{\mu}$ be the atomic measure equal to 2^{-m} on each vector of $\hat{S}_m$. Then for each $x \in \hat{S}_m$ and $0 < b < 1$,

$$\hat{\mu}(y \in \hat{S}_m : |(x,y)| > b) \le 2\exp(-mb^2/2) .$$

Hint: Prove the inequality $E(\exp t(x,y)) \le \exp(t^2/2m)$, $t > 0$.

9.3. (a) Prove the inequality $d_n(b_2^m)_\infty \ge (n+1)^{-1/2}$, $1 \le n \le m-1$.
(b) Let k be an integer, $1 \le k \le m$. If $X_n \subset l_\infty^m$ is a subspace of dimension $\le n$ with $E(b_2^m, X_n) \le 1/(3\sqrt{k})$, then the ball $2k^{-1/2}(b_\infty^m \cap X_n)$ contains $N = 2^k \binom{m}{k}$ disjoint balls $z_i + (1/6)k^{-1/2}(b_\infty^m \cap X_n)$, $z_i \in X_n$, $i = 1, \dots, N$. Comparing the volumes, prove the lower estimate (5.1) (Gluskin). *Hint*: Consider the vectors of b_2^m having exactly k non-zero coordinates equal to $\pm 1/\sqrt{k}$.

9.4. Using (4.17) or (8.1) prove the following slightly weakened version of (7.1):

$$\delta_n(b_1^m)_\infty \le C\sqrt{(\log m)/n} .$$

§ 10. Notes

10.1. We have established in §5 and §7 only those estimates of the widths $s_n(b_p^m)_q$ that are needed for applications to Lipschitz classes. More general (and more accurate) estimates, due to Gluskin [1983], are as follows: Let $1 \le p < q < \infty$, $n < m$. Then

$$0 < C_1 \le d_n(b_p^m)_q / \Psi(m,n,p,q) \le C_2 ,$$

where C_1, C_2 depend only on p and q, and for $2 \le p < q < \infty$,

$$\Psi(m,n,p,q) := \left(\min\{1,\ m^{1/q} n^{-1/2}\}\right)^{(1/p-1/q)/(1/2-1/q)} ;$$

for $1 \le p < 2 \le q < \infty$,

$$\Psi(m,n,p,q) := \max\left\{m^{1/q-1/p},\ (1-n/m)^{1/2} \cdot \min\{1, m^{1/q} n^{-1/2}\}\right\} ;$$

for $1 \le p < q \le 2$,

$$\Psi(m,n,p,q) := \max\left\{m^{1/q-1/p},\ (1-n/m)^{(1/p-1/q)/(2/p-1)}\right\} .$$

It should be noted that the case $q=\infty$ is not covered by the above estimates. Similarly, for the linear widths, if $1 \le p < q \le \infty$, $(p,q) \ne (1,\infty)$, $n < m$, then

$$C_1 \le \delta_n(b_p^m)_q/\Psi^*(m,n,p,q) \le C_2 ,$$

where $\Psi^*(m,n,p,q) := \Psi(m,n,p,q)$ if $1 \le p < q \le p'$, $\Psi^*(m,n,p,q) := \Psi(m,n,q',p')$ if $q > \max\{p,p'\}$.

10.2. *Other Proofs of Theorem 8.1.* Two vectors $x,y \in \mathbb{R}^n$ are *ε-orthonormal* if $\|x\|_2 = \|y\|_2 = 1$, $|(x,y)| \le \varepsilon$. Let $y_1^*,\dots,y_m^*$ be vectors in $\mathbb{R}^n$, columns of an $n\times m$ matrix, let $y_1,\dots,y_n \in \mathbb{R}^m$ be its rows. For $n \le m$, the formula $Tx := \sum_{k=1}^n (x,y_k)y_k$ defines an operator on $\mathbb{R}^m$. One easily shows that

$$\|x - Tx\|_\infty \le \varepsilon\|x\|_\infty , \tag{10.1}$$

if the y_i^* are pairwise ε-orthonormal. This yields $\delta_n(b_1^m)_\infty \le \varepsilon$. To obtain concrete estimates, Höllig [1979] uses polynomials

$$\varphi(x) := a_1x + \cdots + a_\lambda x^\lambda , \tag{10.2}$$

where a_k, $1 \le k \le \lambda$, are integers mod p, for a prime p. By a standard number theoretic lemma, φ has $\le \lambda$ zeros mod p, if at least one coefficient $a_k \not\equiv 0$ mod p.

To obtain many ε-orthonormal vectors, we enumerate somehow the $n = p^2$ points (x,y) of the set $Q := \{1,2,\dots,p\}^2$. To each φ of (10.2) we assign a vector $y^* := y^* = (v_{x,y}/\sqrt{p}) \in \mathbb{R}^n$, where $v_{x,y} = 1$ if $y = \varphi(x)$, $v(x,y) = 0$ otherwise. There are $n^4 = p^2$ of these vectors, and there are $m = p^\lambda - 1$ of the φ. Since any two graphs $y = \varphi(x)$, $y = \varphi_1(x)$, $\varphi \not\equiv \varphi_1$ (mod p), have at most λ common points, the vectors y^* are ε-orthonormal for $\varepsilon = \lambda/p$. For these y^* and the corresponding vectors y, the operator (10.1) yields

$$\delta_n(b_1^m)_\infty \le Cn^{-1/2}\log m/\log n ,$$

first for $n = p^2$, $m = p^\lambda - 1$, and then, by extension, for arbitrary n,m, $1 \le n \le m$.

Moreover, a modification of this construction yields an operator T satisfying, in addition, $\|x - Tx\|_2 \le C\frac{m}{n}\|x\|_2$. By interpolation, one obtains from this an estimate for $\delta_n(b_p^m)_p$, which is sufficient for the proof of Theorem 8.1; for this last step see Pinkus [A-1985, p.220].

Maiorov [1978] uses polynomials (10.2) to construct vectors y_i^* in a different way. His main tool is the following theorem of A. Weil: If a polynomial (10.2) has at least one coefficient $\not\equiv 0$ mod p, then

$$\left|\sum_{k=1}^p \exp(2\pi i\varphi(k)/p)\right| \le (\lambda-1)\sqrt{p} .$$

10.3. Theorem 1.1 has been extended to the Besov spaces [**CA**, §10 of Ch.2]. The unit ball $\widehat{B}^{\alpha_1}_{q_1}(L_{p_1})$ of the space $B^{\alpha_1}_{q_1}(L_{p_1})$ is a relatively compact subset of the space $B^{\alpha_2}_{q_2}(L_{p_2})$ if and only if $\omega := \alpha_1 - \alpha_2 - (1/p_1 - 1/p_2)_+ > 0$. Here is a typical result of Edmunds and Triebel [1989], [1992]: For $p_1 \le p_2 \le 2$, $2 \le p_1 \le p_2 \le \infty$ or $p_2 \le p_1 \le \infty$,

$$\delta_n(\widehat{B}^{\alpha_1}_{q_1}(L_{p_1}), B^{\alpha_2}_{q_2}(L_{p_2})) \sim n^{-\omega} .$$

10.4. Several authors studied widths of intersections of standard classes. DeVore [1984] considers, for $0 < \alpha \le 1$, the class

$$V_\alpha := \{f \in C[0,1] : \ f \in \operatorname{Lip}\alpha, \operatorname{Var}(f) \le 1\} .$$

He proves that $d_n(V_\alpha)_\infty \sim n^{-(\alpha+1)/2}$ (while $d_n(\operatorname{Lip}\alpha)_\infty \sim n^{-\alpha}$). Galeev [1985] gives estimates for the widths of intersections $\cap_\nu B^{r_\nu}_{p_\nu}$ of Sobolev classes, for finite or infinite sets $\{\nu\}$. Makovoz [1991] considers, for a given sequence of positive numbers $\lambda_1, \lambda_2, \ldots$, intersections of the form $\cap_r \lambda_r B^r_p$ in L_q, $1 \le p, q \le \infty$.

10.5. For $\alpha > 0$, we say that the function f defined on $\mathbb{T}$ belongs to the class F^α_p if it can be represented by the convolution

$$f = \text{const} + \mathcal{B}_\alpha * \psi = \text{const} + \frac{1}{\pi}\int_{\mathbb{T}} \mathcal{B}_\alpha(x-t)\psi(t)\,dt ,$$

where $\mathcal{B}_\alpha(x)$ is the Bernoulli function

$$\mathcal{B}_\alpha(x) := \sum_{k=1}^{\infty} k^{-\alpha}\cos(kx - \alpha\pi/2) ,$$

and $\psi \in L_p$, $\|\psi\|_p \le 1$. For integer $\alpha = 1, 2, \ldots$, the sets F^α_p and B^α_p are identical (see [**CA**, Theorem 5.1, p.151]), and their widths behave similarly also for fractional α. In particular, Kashin [1981] established an analogue of Theorem 8.2 for $p = 1$, $1 - 1/q < \alpha < 1$. Kulanin [1983_1], [1983_2] studied the widths $d_n(F^\alpha_p)_q$ for small α systematically. He found that the behavior of these widths changes essentially when α passes a certain critical value α_0, where $\alpha_0 = 1/p$ for $p \le 2$ (compare Theorem 8.2), $\alpha_0 = \frac{1}{2}(1/p - 1/q)/(1/2 - 1/q)$ for $p \ge 2$.

10.6. *Trigonometric widths* $\tilde{d}_n(A)_X$ are obtained by replacing in Kolmogorov's definition arbitrary subspaces X_n by n-dimensional subspaces spanned by n harmonics $\cos(k_\nu x + \alpha_\nu)$, $\nu = 1, \ldots, n$. Statements of Kashin's theorems are still valid, for many sets p, q, α, for these widths. See Makovoz [1984], Maiorov [1986], Belinskii [1987].

Chapter 15. Entropy

§ 1. Entropy and Capacity

The Kolmogorov widths $d_n(K)$ of a set K in a Banach space characterize the error of approximation of K by *finite dimensional* sets: if $d_n(K)$ is small, then K is "almost n-dimensional." We shall now introduce another geometric concept by Kolmogorov, the entropy of K. It shows how well one can approximate K by *finite* sets.

We begin by recalling some well-known definitions. Let K be a subset of a metric space X, and let $\varepsilon > 0$ be given.

(a) A set $\widehat{K} \subset X$ is called an *ε-net* for K if for each $x \in K$ there is at least one $y \in \widehat{K}$ at a distance from x not exceeding $\varepsilon : \rho(x, y) \le \varepsilon$.

(b) Points $y_1, \ldots, y_m$ of X are called *ε-distinguishable* if the distances between any two of them exceed $\varepsilon : \rho(y_i, y_k) > \varepsilon$ for all $i \neq k$.

If K is a compact set, then it contains a finite ε-net for each $\varepsilon > 0$. Moreover, a compact set K can contain only finitely many ε-distinguishable points.

From now on, we shall suppose that K is *compact*. For a given K and $\varepsilon > 0$, let $N_\varepsilon(K)$ be the minimal value of n for which there exists an ε-net for K consisting of n points. The (natural) logarithm

$$H_\varepsilon(K) := H_\varepsilon(K)_X := \log N_\varepsilon(K) \tag{1.1}$$

is the *entropy* (or *ε-entropy*) *of the set K in X*. It is also called *metric entropy*, in contradistinction to entropy in thermodynamics and in information theory. The idea to characterize the "massiveness" of a set K by means of the function (1.1) was conceived by Kolmogorov [1956]. We are interested in the asymptotic behavior of $H_\varepsilon(K)$ for $\varepsilon \to 0$; in general, $H_\varepsilon(K)$ will increase rapidly to infinity.

Similarly, for a given K and $\varepsilon > 0$, one may consider a finite subset of K consisting of m ε-distinguishable points. Let $M_\varepsilon(K) := \max m$. The number

$$C_\varepsilon(K) := \log M_\varepsilon(K) \tag{1.2}$$

is called the *capacity* (or *ε-capacity*) of K.

An ε-net consisting of $N_\varepsilon(K)$ points and an ε-distinguishable set consisting of $M_\varepsilon(K)$ points are called a *minimal* ε-net and a *maximal* ε-distinguishable set.

By definition, $C_\varepsilon(K)$ depends only on the compact metric space K itself (and not on the larger space X in which K may be contained), but $H_\varepsilon(K)_X$ may depend on X. Both functions are monotone increasing in K. As functions of $\varepsilon > 0$, they are decreasing step functions; $C_\varepsilon(K)$ is continuous on the left, and $H_\varepsilon(K)$ on the right.

The logarithms in (1.1) and (1.2) can be explained as follows. Given a finite set of cardinality n, one can number its elements by $1, 2, \ldots, n$. It takes $\sim \log_2 n$ digits to represent these numbers in the binary system. One may say that the elements of the set can be coded by "codewords" of length $\sim \log_2 n$. Similarly, if a set K admits an ε-net of n elements, then codewords of length $\sim \log_2 n$ are sufficient to recover $x \in K$ with an error $\le \varepsilon$.

Proposition 1.1. *For each compact set K and each $\varepsilon > 0$,*

$$C_{2\varepsilon}(K) \le H_\varepsilon(K) \le C_\varepsilon(K) \ . \tag{1.3}$$

Proof. If $y_1, \ldots, y_m$ is a maximal ε-distinguishable subset of K, then every $x \in K$ is at a distance $\le \varepsilon$ from some y_k, so that $\{y_k\}$ is an ε-net for K; hence $N_\varepsilon(K) \le M_\varepsilon(K)$. Furthermore, if $x, x' \in K$ and $\rho(x, x') > 2\varepsilon$, then x, x' can not be approximated with an error $\le \varepsilon$ by the same element of an ε-net; hence $M_{2\varepsilon}(K) \le N_\varepsilon(K)$. □

For example, if K is the interval $[a, b]$ with the usual metric, then $N_\varepsilon(K)$ is the smallest n with $n \ge \ell/(2\varepsilon)$, $\ell := b - a$, and $M_\varepsilon(K)$ is the largest m with $m < \ell/\varepsilon$. Therefore,

$$N_\varepsilon(K) = \ell/(2\varepsilon) + O(1) \quad , \quad M_\varepsilon(K) = \ell/\varepsilon + O(1) \ . \tag{1.4}$$

In most cases of interest we cannot determine the entropy of K with similar precision; we may be content with finding it only up to a strong or weak equivalence. For this purpose, (1.3) is used in the following way: one seeks an upper bound for $H_\varepsilon(K)$ and a lower bound for $C_{2\varepsilon}(K)$; if the bounds are close to each other, a good estimate for both entropy and capacity results.

The entropy of a cartesian product can be estimated if the entropies of the factors are known. Assume, for example, that $K = \prod_1^n K_i$ is a subset of $X = \mathbb{R}^n$ and that each K_i is a compact subset of the i-th coordinate axis. Let $\varepsilon > 0$ be given and let $\varepsilon_1 := \varepsilon/\sqrt{n}$. For each i, let $\widehat{K}_i$ be a minimal ε_1-net for K_i. Then the set $\prod_1^n \widehat{K}_i$ is an ε-net for K; its cardinality is the product of all the $N_{\varepsilon_1}(K_i)$. Thus,

$$H_\varepsilon(K) \le \sum_{i=1}^n H_{\varepsilon/\sqrt{n}}(K_i) \ .$$

In the same way we obtain

$$C_\varepsilon(K) \ge \sum_{i=1}^n C_\varepsilon(K_i) \ .$$

In particular, if each K_i is an interval, then by (1.4) both $C_\varepsilon(K_i)$ and $H_\varepsilon(K_i)$ are $\log(1/\varepsilon) + O(1)$, and we obtain for each fixed n and any n-dimensional parallelepiped K in $\mathbb{R}^n$

$$C_\varepsilon(K) \text{ and } H_\varepsilon(K) = n\log(1/\varepsilon) + O(1) , \qquad \varepsilon \to 0 . \tag{1.5}$$

Because of the monotony of our set functions, we have, more generally:

Proposition 1.2. *Relation* (1.5) *holds for each bounded subset K of $\mathbb{R}^n$ with interior points.*

If X_n is an arbitrary n-dimensional Banach space and $\varphi_1, \dots, \varphi_n$ is some fixed basis in X_n, then the points $x = \xi_1\varphi_1 + \dots + \xi_n\varphi_n$ of X_n can be identified with the points $\xi = (\xi_1, \dots, \xi_n)$ of $\mathbb{R}^n$. To certain subsets $B \subset X_n$ we assign their volume, $\mathrm{vol}(B)$, by identifying it with the euclidean volume of the image of B under the map $x \to \xi$. We shall constantly use the following fact: if $K \subset \mathbb{R}^n$ and $K_1 = x_0 + \lambda K$, then $\mathrm{vol}(K_1) = |\lambda|^n \mathrm{vol}(K)$.

Proposition 1.3. *For the ball $B_r := \{x \in X_n : \|x\| \le r\}$ of X_n and $0 \le \varepsilon \le r$,*

$$2^{-n}(r/\varepsilon)^n \le N_\varepsilon(B_r) \le 3^n(r/\varepsilon)^n . \tag{1.6}$$

Proof. Let $y_1, \dots, y_m$, $m = M_\varepsilon(B_r)$, be a maximal set of ε-distinguishable points in B_r. The closed balls with centers y_i and radii ε cover B_r. Comparing the volumes we see that

$$\mathrm{vol}(B_1)\varepsilon^n M_\varepsilon(B_r) \ge \mathrm{vol}(B_1)r^n .$$

On the other hand, balls with centers y_i and radii $\varepsilon/2$ are disjoint and contained in the ball $B_{r+\varepsilon/2} \subset B_{3r/2}$. Therefore

$$\mathrm{vol}(B_1)(\varepsilon/2)^n M_\varepsilon(B_r) \le \mathrm{vol}(B_1)(3r/2)^n .$$

Thus, $(r/\varepsilon)^n \le M_\varepsilon(B_r) \le 3^n(r/\varepsilon)^n$, and (1.6) follows from the inequalities $M_{2\varepsilon}(K) \le N_\varepsilon(K) \le M_\varepsilon(K)$. □

A sum $\sum_i K_i$ (finite or countable) of sets $K_i \subset X$ consists of all elements $g = \sum_i g_i$ (with convergence in X) with $g_i \in X_i$.

Proposition 1.4. *If K is contained in a sum of finitely or countably many sets $K_i \subset X$, if $\varepsilon = \sum \varepsilon_i$, $\varepsilon_i > 0$ and if the series $\sum H_{\varepsilon_i}(K_i)$ converges (so that only finitely many of its terms are not zero), then*

$$H_\varepsilon(K)_X \le \sum_i H_{\varepsilon_i}(K_i)_X . \tag{1.7}$$

Proof. Let $H_{\varepsilon_i}(K_i) = 0$, $i \ge k$. Then each of these K_i has an ε_i-net consisting of a single $g_i \in X$, and the set $K' := \sum_{i \ge k} K_i$ has the net consisting of one element $g' := \sum_{i \geq k} g_i$. This reduces the proposition to the finite case $K \subset K_1 + \cdots + K_k$; inequality (1.7) follows from $N_\varepsilon(K) \leq \prod_{i=1}^k N_{\varepsilon_i}(K_i)$. □

Simple useful remarks (valid for C_ε as well as for H_ε):

(1.8) $$H_\varepsilon(\lambda A)_X = H_{\varepsilon/|\lambda|}(A)_X \ , \qquad \lambda \neq 0 \ ;$$

if T is a bounded linear map from X to Y and $A \subset X$, then

(1.9) $$H_\varepsilon\big(T(A)\big)_Y \leq H_{\varepsilon/\|T\|}(A)_X \ .$$

Some facts concerning entropy are better formulated in terms of the so-called *entropy numbers* $e_n(K)$. For a set K in a Banach space Xwe define

(1.10) $$e_k(K) := \inf\big\{\varepsilon : \text{ there exist } 2^k \text{ closed balls in } X \text{ of radius } \varepsilon \text{ that cover } K\big\} \ .$$

The $e_k(K)$ is, in some sense, an inverse function to $H_\varepsilon(K)$. It follows directly from the definitions that $e_k(K) = \varepsilon$ is equivalent to $H_{\varepsilon+}(K) \leq k \log 2 < H_\varepsilon(K)$.

One can also define entropy numbers of operators T acting between two Banach spaces, X and Y (compare §2 of Chapter 13 where the widths of operators were defined). For $T \in L(X,Y)$ we set $e_k(T) := e_k(T(B_X))$, where B_X is the unit ball of X. In other words

(1.11) $$e_k(T) := \inf\big\{\varepsilon : \text{ there is an } \varepsilon\text{-net for } T(B_X) \text{ in } Y \text{ consisting of } 2^k \text{ elements}\big\} \ .$$

We shall study the entropy numbers of operators in §7.

Metric entropy has been used to describe non-linear approximation properties of sets in function spaces; it has been computed for many such sets. Comparison of entropy with other concepts of approximation has been quite fruitful. This is the main subject of the present chapter. In last years, however, entropy has been used with success in the general theory of Banach spaces and of operators. We give a few examples of this development in §7. For further results, we must refer the reader to works in functional analysis (for example, Pisier [B-1989], Carl and Stephani [B-1991]).

First works on entropy, by Kolmogorov [1958], Kolmogorov and Tikhomirov [1959], Vitushkin [A-1961] served to determine the entropy of some concrete spaces of functions of several variables in the uniform norm: of Lipschitz spaces, of some spaces of analytic functions. Special properties of these spaces were used. For an exposition of this approach, see the book Lorentz [A-1966, Chapter 10]. Here, however, we shall follow a different route. Concrete results will appear as corollaries of estimates of entropy in arbitrary Banach spaces. In §3 this will be done by means of "approximation sets," while in §4 we shall use widths. In this way, also some sets of analytic functions can be treated. Somewhat isolated remains the theorem of Birman and Solomyak of §6, which deals with sets equipped with two different L_p-norms.

Our concrete applications refer only to standard sets in Lipschitz, Sobolev spaces, spaces of analytic functions. Richer sets of examples can be found in Kolmogorov and Tikhomirov [1959], Edmunds and Triebel [1987] and [1992].

§ 2. Elementary Estimates

For the estimates of entropy from below we can use the techniques that have been employed in §3 of Chapter 14.

We have two variants, for subspaces K of $C(A)$ and of $L_1(A,\mu)$ (here A is a compact metric space, possibly equipped with a measure μ). We shall use the set $S := S(n)$ of all sign vectors $\theta := (\theta_j)_1^n$, $\theta_j = \pm 1$.

Proposition 2.1. *Let $f_j \in C(A)$, $j = 1, \dots, n$ be functions on A with disjoint supports, and for some $\varepsilon > 0$,*

$$\|f_j\|_\infty > \varepsilon \, , \qquad j = 1, \dots, n \, . \tag{2.1}$$

If for each $\theta \in S$, the function $f_\theta := \sum_{j=1}^n \theta_j f_j$ belongs to K, then

$$C_{2\varepsilon}(K)_\infty \geq (\log 2) n \tag{2.2}$$

Proof. If $\theta \neq \theta'$, we have $\|f_\theta - f_{\theta'}\| > 2\varepsilon$, so that the f_θ are 2ε-distinguishable points in K, 2^n in number. Thus, $M_{2\varepsilon}(K) \geq 2^n$. □

For $L_1(A)$ we need the following combinatorial lemma.

Lemma 2.2. *For each sufficiently large natural n there exists a set $S_0 \subset S$ which consists of $\geq (4/3)^n$ sign vectors $\theta = (\theta_j)_1^n$, so that any two $\theta \neq \theta'$ in S_0 are different in more than $[n/8]$ places.*

Proof. For any $\theta \in S$, let $U(\theta)$ be the set of all θ' which are different from θ in $\leq [n/8]$ places. Then, with $\ell = [n/8]$ we get

$$|U(\theta)| \leq 1 + n + \dots + \binom{n}{\ell} \leq (\ell + 1)\binom{n}{\ell} \, .$$

Since

$$\binom{n}{\ell} \leq \left(\frac{ne}{\ell}\right)^\ell \leq (8e)^{[n/8]} < (1.47)^n \, ,$$

we have $|U(\theta)| < (3/2)^n$, if n is sufficiently large. We now construct θ^k, $k = 1, \dots, m$ as follows. Element $\theta^1 \in S$ is arbitrary, then θ^2 is taken arbitrarily in $S \setminus U(\theta^1)$, then θ^3 arbitrarily in $S \setminus (U(\theta^1) \cup U(\theta^2))$, and so on, until S is exhausted. The set $S_0 = (\theta^k)_1^m$ has the property that θ^k, θ^{k_1} for $k \neq k_1$ are distinct in $> [n/8]$ places, while $2^n = |S| \leq \sum |U(\theta^k)| \leq m(3/2)^n$ implies that $m \geq (4/3)^n$. □

Proposition 2.3. *Let $f_j \in L_1(A)$, $j = 1,\dots,n$, be functions on A with disjoint supports, and for some $\varepsilon > 0$,*

$$\|f_j\|_1 \ge \varepsilon/n \ , \qquad j = 1,\dots,n \ . \tag{2.3}$$

If for each $\theta \in S$, $f_\theta := \sum_{j=1}^n \theta_j f_j \in K$, then for some $C > 0$,

$$C_\varepsilon(K)_1 \ge Cn \ . \tag{2.4}$$

Proof. With the θ^k of the lemma, the functions f_{θ^k} are $\ge (4/3)^n$ in number. For $k \ne k_1$, the coordinates of θ^k and θ^{k_1} are distinct in $\ge [n/8]$ places, so that

$$\|f_{\theta_k} - f_{\theta_{k_1}}\|_1 \ge 2[n/8]\varepsilon/n > C\varepsilon \ ,$$

and therefore $M_{C\varepsilon}(K)_1 \ge \log(4/3)^n$. □

The unit ball $\bar{B}^\alpha_\infty$ of the space $\mathrm{Lip}\,\alpha := \mathrm{Lip}(\alpha, L_\infty)$ on $[0,1]$ is different from the set B^α_∞ of the two preceding chapters by the additional assumption that $\|f\|_\infty \le 1$ for $f \in \bar{B}^\alpha_\infty$. If $\alpha = r + \beta$, where $r = [\alpha]$ and $0 \le \beta < 1$, then we require also that the derivatives $f^{(r-1)}$ are absolutely continuous and that $|f^{(r)}(x) - f^{(r)}(x')| \le |x - x'|^\beta$, $x, x' \in [0,1]$ if $\beta > 0$ or that $\|f^{(r)}\|_\infty \le 1$ if $\beta = 0$.

Corollary 2.4. *Let $1 \le p, q \le \infty$. The Lipschitz ball $\bar{B}^\alpha_p[0,1]$ has, in the space $L_q[0,1]$, the capacity*

$$C_\varepsilon(\bar{B}^\alpha_p)_q \ge C(1/\varepsilon)^{1/\alpha} \ , \qquad \varepsilon > 0 \ . \tag{2.5}$$

Proof. For each set B, $C_\varepsilon(B)_q$ decreases with q, so it is sufficient to prove (2.5) when $q = 1$. We use notations and statements of Lemma 3.3 of Chapter 14. For a given $0 < \varepsilon < 1$, and a function $\phi \in C^{r+1}(\mathbb{R})$ of the lemma with support $[0,1]$, we set $f_j(x) := Cn^{-\alpha}\phi(nx - j + 1)$, $j = 1,\dots,n$, with $n := [\varepsilon^{-1/\alpha}]$. Clearly $\|f_j\| \ge \varepsilon/n$ for some sufficiently large $C > 0$ independent of ε. If $a := (a_k)_1^n$, $a_k := Cn^{-\alpha}\theta_k$, $\theta_k := \pm 1$, $k = 1,\dots,n$, the function g_a of (3.11) of Chapter 14, with the f_j instead of ϕ_j, becomes f_θ. Since $\|a\|_p \le Cn^{-\alpha+1/p}$, part (iii) of the lemma and Proposition 2.3 yield (2.5). □

In §6 we obtain an upper bound for the capacity (2.5) of the same order $(1/\varepsilon)^{1/\alpha}$.

The following lemma compares two elements of $\bar{B}^\alpha_\infty$.

Lemma 2.5. *Let $x_k := k/n$, $k = 0,\dots,n$, and for $k \le n - r$, let $I := I_{k,r} := [x_k, x_{k+r}]$. Then $f, g \in \bar{B}^\alpha_\infty$ and*

$$|f(x_k) - g(x_k)| \le \varepsilon \ , \qquad k = k_0,\dots,k_0 + r - 1$$

for some $0 \le k_0 \le n - r$ and $\varepsilon > 0$ imply that for some constant $C(\alpha)$,

$$(2.6) \qquad |f(x) - g(x)| \le C(\alpha)(\varepsilon + n^{-\alpha}) , \qquad x \in I_{k_0,r} .$$

Proof. Let $P \in \mathcal{P}_{r-1}$ be the polynomial interpolating $f - g$ at the points $x_{k_0}, \dots, x_{k_0+r-1}$, let $\varphi := f - g - P$. By Rolle's theorem, each derivative $\varphi^{(s)}$, $0 \le s < r$ has a zero on I, while $\varphi^{(r)} = f^{(r)} - g^{(r)}$ and (assuming that $\beta > 0$), $|\varphi^{(r)}(x) - \varphi^{(r)}(x')| \le 2|x - x'|^\beta$, $x, x' \in I$. Since $\varphi^{(r-1)}$ has a zero on I and $|I| = r/n$, we get $|\varphi^{(r-1)}(x)| \le (r/n)2(r/n)^\beta = Cn^{-1-\beta}$. In the same way, $|\varphi^{(r-2)}(x)| \le Cn^{-2-\beta}, \dots, |\varphi(x)| \le Cn^{-\alpha}$, $x \in I$. On the other hand, since all norms on $\mathcal{P}_{r-1}[0,1]$ are equivalent,

$$(2.7) \qquad \max_{0 \le x \le 1} |Q(x)| \le C \max_{0 \le j \le r-1} |Q(j/r)| .$$

By a change of variables, from $|P(x_k)| \le \varepsilon$, $k = k_0, \dots, k_0 + r - 1$ we get $|P(x)| \le C(r)\varepsilon$, $x \in I$, and (2.6) follows. □

Theorem 2.6 (Kolmogorov [1958]). *For each $\alpha > 0$ we have the weak equivalences (with constants that depend only on α):*

$$(2.8) \qquad C_\varepsilon(\bar{B}^\alpha_\infty)_\infty \sim H_\varepsilon(\bar{B}^\alpha_\infty)_\infty \sim \varepsilon^{-1/\alpha} , \qquad \varepsilon \to 0 .$$

Proof. In view of (2.5), we need to establish only the estimates from above. For a given $\varepsilon > 0$, we define $n := [1/\varepsilon]^{1/\alpha}$. We cover $\bar{B}^\alpha_\infty$ by sets U, each of them defined by a set of integers $\nu_0, \dots, \nu_n$. We postulate $f \in U$ if for each k, $(\nu_k - 1)\varepsilon \le f(x_k) \le \nu_k\varepsilon$. From the definition of n and Lemma 2.5 it follows that if f, g belong to the same U, then $\|f - g\| \le C\varepsilon$. Let U_0 be the nonempty among the sets $U \cap \bar{B}^\alpha_\infty$. From each U_0 we select a $g \in U_0$, these g form a $C\varepsilon$-net for $\bar{B}^\alpha_\infty$. It remains to estimate the number N of the U_0. Since $g \in U_0$ satisfy $\|g\|_\infty \le 1$, the number of all possible integers $\nu_0, \dots, \nu_{r-1}$ for the sets U_0 does not exceed $C(1/\varepsilon)^r$. For given $\nu_0, \dots, \nu_{r-1}$, by (2.2), the number of possible ν_r is $\le 2(C(\alpha)+2) =: C_1(\alpha)$. So the number N of the sets U_0 satisfies $N \le C(1/\varepsilon)^\alpha C_1(\alpha)^n$, and $H_\varepsilon(\bar{B}^\alpha_\infty) \le \log N \le Cn \le C(1/\varepsilon)^{1/\alpha}$. □

Similar estimates are valid for classes $\bar{B}^\alpha_\infty(I_s)$ of functions of s variables defined on a parallelpiped $I_s \subset \mathbb{R}^s$. With a slightly different proof (Taylor expansions replacing interpolation, see Lorentz [A-1966]) one gets

Theorem 2.7 (Kolmogorov). *With constants depending on $s = 1, 2, \dots$ and $\alpha > 0$ one has*

$$(2.9) \qquad C_\varepsilon\big(\bar{B}^\alpha_\infty(I_s)\big) \sim H_\varepsilon\big(\bar{B}^\alpha_\infty(I_s)\big) \sim (1/\varepsilon)^{s/\alpha} .$$

This can be even extended to classes $\bar{B}^{r,\omega}_\infty(I_s)$, where the Lipschitz condition for the $D^r f$ is replaced by $|D^r f(x) - D^r f(x')| \le \omega(|x - x'|)$, and ω is a non-trivial modulus of continuity.

§ 3. Linear Approximation and Entropy

In this section, in a Banach space X, we compute the entropy of sets A with known errors of approximation. The theory presented in this section is due to Lorentz [1966].

Let $\Phi := \{\varphi_1, \varphi_2, \dots\}$ be a *fundamental sequence* of linearly independent elements in a Banach space X, that is, a sequence of elements whose linear combinations are dense in X. Let $X_0 = \{0\}$,

$$X_n := \operatorname{lin}\{\varphi_1, \dots, \varphi_n\} , \qquad n = 1, 2, \dots ,$$

and for $x \in X$ let $E_n(x) := \inf\{\|x - y\| : y \in X_n\}$. Let $\Delta := \{\delta_0, \delta_1, \dots\}$ be a sequence of numbers for which $\delta_n > 0$, $\delta_0 \ge \delta_1 \ge \cdots$, $\delta_n \to 0$. We shall call the set

$$A(\Delta) := A(\Delta, \Phi) := \{x \in X : E_n(x) \le \delta_n , \quad n = 0, 1, \dots\} \tag{3.1}$$

the *approximation set* associated with Δ, Φ (in the terminology of [**CA**, §5, Chapter 7], $A(\Delta)$ is the ball of an approximation space). We obtain some upper and lower estimates for the entropy of the sets $A(\Delta, \Phi)$. It is remarkable that these estimates depend only on Δ. Neither the structure of the sequence Φ, nor the properties of the space X are important for the final results.

Theorem 3.1. (i) *Let* $0 = n_0 \le n_1 \le n_2 \le \cdots \le n_j$, *be some integers and* $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_j$, *some positive numbers satisfying, for a given sequence* Δ,

$$\varepsilon_i \le 9\delta_{n_i} , \qquad i = 0, \dots, j - 1 , \ \varepsilon_j \ge \delta_{n_j} . \tag{3.2}$$

Then for $\varepsilon := \varepsilon_0 + \cdots + \varepsilon_j$, $\Delta n_i := n_{i+1} - n_i$,

$$H_\varepsilon(A(\Delta)) \le \theta \sum_{i=0}^{j-1} \Delta n_i \log(9\delta_{n_i}/\varepsilon_i) , \tag{3.3}$$

where $\theta = 1$ *if the space* X *is real and* $\theta = 2$ *if it is complex.*

(ii) *For every natural* n *and* $\varepsilon > 0$

$$C_\varepsilon(A(\Delta)) \ge \theta \sum_{i=0}^{n} \log(\delta_i/\varepsilon) , \tag{3.4}$$

Proof. We first consider the real case. In the proof of (3.3) we may assume that all the n_i are distinct. If $\widehat{A}$ is the image of $A := A(\Delta)$ under the canonical mapping from X onto the factor space $\widehat{X} := X/X_{n_1}$, then $\widehat{A}$ is an approximation set in $\widehat{X}$ corresponding to the sequence $(\delta_{n_1+1}, \delta_{n_1+2}, \dots)$. Let $\hat{x}_1, \dots, \hat{x}_n$, $n := N_{\varepsilon-\varepsilon_0}(\widehat{A})$, be a minimal $(\varepsilon - \varepsilon_0)$-net for $\widehat{A}$ in $\widehat{X}$. Suppose first that $\varepsilon - \varepsilon_0 \le \delta_{n_1+1}$. In each class $\hat{x}_\nu$ we take an element $x_\nu \in \hat{x}_\nu$ for which $\|x_\nu\| = \|\hat{x}_\nu\| \le \delta_{n_1+1}$. Then to every $x \in A$ there exist x_ν and some $y \in X_1$ such that $\|x - x_\nu - y\| \le \varepsilon - \varepsilon_0$. We have

$\|y\| \le \|x\| + \|x_\nu\| + \varepsilon - \varepsilon_0 \le \delta_0 + 2\delta_{n_1+1} \le 3\delta_0$, that is, $y \in B_r$, where B_r is the ball of X_{n_1} of radius $r = 3\delta_0$. By (1.8) there is an ε_0-net $y_1, \dots, y_q$ for B_r with $q = N_{\varepsilon_0}(B_r) \le (9\delta_0/\varepsilon_0)^{n_1}$. The sums $\{x_i + y_k\}$, nq in number, form an ε-net for A. Thus

$$N_\varepsilon(A) \le (9\delta_0/\varepsilon_0)^{n_1} N_{\varepsilon-\varepsilon_0}(\widehat{A}) \ . \tag{3.5}$$

If $\varepsilon - \varepsilon_0 > \delta_{n_1}$, (3.5) is also valid since in this case $\widehat{A}$ is contained in the ball of radius δ_{n_1} of the space $\widehat{X}$ so that $n = 1$. To estimate $N_{\varepsilon-\varepsilon_0}(\widehat{A})$, we apply a similar argument. After several steps we arrive at

$$N_\varepsilon(A) \le (9\delta_0/\varepsilon_0)^{\Delta n_0} \cdots (9\delta_{j-1}/\varepsilon_{j-1})^{\Delta n_{j-1}} N_{\varepsilon_j}(A^{(j)}) \ ,$$

where $A^{(j)}$ is an approximation set corresponding to the sequence $(\delta_{n_j+1}, \delta_{n_j+2}, \dots)$ in the j-th factor space. But $N_{\varepsilon_j}(A^{(j)}) = 1$ since $\delta_{n_j} \le \varepsilon_j$, and (3.3) follows by passing to logarithms.

For the proof of (3.4) we may assume that $\delta_0 > \varepsilon$. We fix $n \ge 1$. Let $A_n := A \cap X_n$ and let

$$A_{n,k} := \{x \in X_n : E_i(x) \le \delta_i \ , \quad i = 0, 1, \dots, k\} \ , \qquad k = 0, 1, \dots, n-1 \ ,$$

so that $A_{n,0} = \delta_0 U_n = \delta_0(U \cap X_n)$ (where U_n, U are the unit balls of X_n, X, respectively), and $A_{n,n} = A_n$. We shall identify the points $\xi_1\varphi_1 + \cdots + \xi_n\varphi_n$ of X_n with their coordinate vectors $(\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. To certain subsets B of X_n we shall assign euclidean volumes of the corresponding sets in $\mathbb{R}^n$. We shall also consider the k-dimensional volumes, $\mathrm{vol}_k(B)$, of sets B in X_k and in the planes parallel to X_k, $k \le n$.

The *projection* of $A_{n,k-1}$ onto the $(n-k)$-dimensional plane spanned by $\varphi_{k+1}, \dots, \varphi_n$ is the set

$$\begin{aligned} D_k := \{(\xi^0_{k+1}, \dots, \xi^0_n) : {}& \text{there exists } x = (\xi_i)_1^n \in A_{n,k-1} \\ & \text{for which } \xi_i = \xi_i^0 \ , \ i = k+1, \dots, n\} \ . \end{aligned} \tag{3.6}$$

One sees that the sets $A_{n,k}$, D_k are all convex and symmetric about the origin. For $(\xi_i)_{k+1}^i \in D_k$, the set of all $(\xi_i)_1^k$ for which $(\xi_i)_1^n \in A_{n,k+1}$ is the *cross-section* $A_{n,k+1}(\xi_{k+1}, \dots, \xi_n)$ of $A_{n,k+1}$. We have

$$\mathrm{vol}(A_{n,k-1}) = \int_{D_k} \mathrm{vol}_k\, A_{n,k-1}(\xi_{k+1}, \dots, \xi_n)\, d\xi_{k+1} \dots d\xi_n \ . \tag{3.7}$$

We now prove that

$$A_{n,k}(\xi_{k+1}, \dots, \xi_n) = A_{n,k-1}(\xi_{k+1}, \dots, \xi_n) \text{ on } \lambda D_k \ , \tag{3.8}$$

if $\lambda := \delta_k/\delta_{k-1}$. Clearly, $0 < \lambda \le 1$.

Since $A_{n,k} \subset A_{n,k-1}$ and $E_i(x) \le \delta_i$, $i = 1, \dots, k-1$ it is sufficient to show that $x = (\xi_i)_1^n \in A_{n,k-1}$ and $(\xi_{k+1}, \dots, \xi_n) \in \lambda D_k$ imply that $E_k(x) \le \delta_k$ (and hence that $x \in A_{n,k}$). By the definition of D_k, there exist $\eta_1, \dots, \eta_k$ with the property

$$x' := \left(\frac{\eta_1}{\lambda}, \dots, \frac{\eta_k}{\lambda}, \frac{\xi_{k+1}}{\lambda}, \dots, \frac{\xi_n}{\lambda}\right) \in A_{n,k-1} \ .$$

This implies that $E_{k-1}(x') \le \delta_{k-1}$. On the other hand, since the value of $E_k(x)$ does not depend on the first k coordinates of x,

$$E_k(x) = E_k(\lambda x') = \lambda E_k(x') \le \lambda E_{k-1}(x') \le \lambda \delta_{k-1} = \delta_k \ .$$

The Brunn-Minkowski inequality (Theorem 2.3 of Appendix 1) yields that for each $0 < \lambda \le 1$,

$$\text{(3.9)} \qquad \operatorname{vol}_k A_{n,k-1}(\xi_{k+1}, \dots, \xi_n) \le \operatorname{vol}_k A_{n,k-1}(\lambda\xi_{k+1}, \dots, \lambda\xi_n) \ .$$

From (3.8) and (3.9) we conclude that

$$\begin{aligned} \operatorname{vol} A_{n,k} &\ge \int_{\lambda D} \operatorname{vol}_k A_{n,k}(\xi_{k+1}, \dots, \xi_n)\, d\xi_{k+1} \dots d\xi_n \\ &= \lambda^{n-k} \int_D \operatorname{vol}_k A_{n,k}(\lambda\eta_{k+1}, \dots, \lambda\eta_n)\, d\eta_{k+1} \dots d\eta_n \\ &\ge \lambda^{n-k} \int_D \operatorname{vol} A_{n,k-1}(\eta_{k+1}, \dots \eta_n)\, d\eta_{k+1} \dots d\eta_n \ . \end{aligned}$$

From this and (3.7) we now get

$$\operatorname{vol}(A_{n,k}) \ge (\delta_k/\delta_{k-1})^{n-k} \operatorname{vol}(A_{n,k-1}) \ , \qquad k = 1, \dots, n-1 \ .$$

Iterating this inequality and using the relation $\operatorname{vol}(A_{n,0}) = \delta_0^n \operatorname{vol}(U_n)$, we further obtain

$$\text{(3.10)} \qquad \operatorname{vol}(A_n) \ge \delta_0 \cdots \delta_{n-1} \operatorname{vol}(U_n) \ .$$

If now $y_1, \dots, y_m$ is a maximal ε-distinguishable set in A, then the balls $\{y_j + \varepsilon U\}$, where U is the unit ball of X, cover A, therefore the sets $S_j := (y_j + \varepsilon U) \cap X_n$ cover A_n. Hence, $\sum \operatorname{vol}(S_j) \ge \operatorname{vol}(A_n)$. Again by the Brunn-Minkowski theorem,

$$\operatorname{vol}(S_j) \le \operatorname{vol}(\varepsilon U_n) = \varepsilon^n \operatorname{vol}(U_n)$$

for each j, so that by (3.10)

$$m\varepsilon^n \operatorname{vol}(U_n) \ge \delta_0 \cdots \delta_{n-1} \operatorname{vol}(U_n) \ ,$$

and (3.4) follows (with n replaced by $n-1$).

The case of a complex space X can be deduced from that of a real space. Indeed, linear combinations $\sum_{k=1}^n a_k\varphi_k$ with complex a_k may be viewed as linear combinations of $2n$ elements φ_k, $i\varphi_k$ with real coefficients. Accordingly, if $\Psi = (\psi_j)_0^\infty$ consists of the functions φ_k, $i\varphi_k$, $k = 0, 1, \dots$ and if $\Delta' = (\delta'_j)_0^\infty = (\delta_0, \delta_0, \delta_1, \delta_1, \delta_2, \dots)$, then $A(\Phi, \Delta) = A(\Psi, \Delta')$ in the real space X. The corresponding N'_j are

$$\begin{aligned} N'_i = \min\{k : \delta'_k \le e^{-i}\} &= \min\{k : \delta_{[k/2]} \le e^{-i}\} = \min\{2j : \delta_j \le e^{-i}\} \\ &= 2\min\{j : \delta_j \le e^{-i}\} = 2N_i \ . \end{aligned}$$

Thus the bounds for $H_\varepsilon(A)$, $C_\varepsilon(A)$ in the complex case must be doubled. □

To obtain good estimates for the entropy from (3.3) or (3.4) one has to choose the numbers j, n_i, ε_i judiciously. One can see that the choice below minimizes the upper bound in (3.3).

Theorem 3.2. *Let $N_0 := 0$ and for $i = 1, 2, \ldots$,*

$$N_i := \min\{k : \delta_k \le e^{-i}\} \ . \tag{3.11}$$

For a given ε, $0 < \varepsilon \le 1$, let j be defined by

$$e^{-(j-1)} < \varepsilon \le e^{-(j-2)} \ . \tag{3.12}$$

Then (i)

$$\begin{aligned} \theta^{-1} H_\varepsilon(A(\Delta)) \le\ & N_1 + \cdots + N_j + 2N_j \\ & + \sum_{i=0}^{j-1} \Delta N_i \log(N_j/\Delta N_i) + N_1 \log \delta_0 \ , \end{aligned} \tag{3.13}$$

where $\Delta N_i \log(N_j/\Delta N_i)$ is interpreted as zero when $\Delta N_i = N_i - N_{i-1} = 0$, and (ii)

$$\theta^{-1} C_\varepsilon(A) \ge N_1 + \cdots + N_{j-3} \ . \tag{3.14}$$

Proof. (i) We take in (3.3) $n_i = N_i$ and $\varepsilon_i = (1-1/e)\varepsilon \Delta N_i/N_j$, $i = 0, \ldots, j-1$; $\varepsilon_j = \varepsilon/e$. If for some i, $0 \le i < j$, $\Delta N_i \ne 0$, then

$$9\delta_{N_i}/\varepsilon \le \frac{9}{e-1} \delta_{N_i} N_j e^j / \Delta N_i$$

and since $\delta_{N_i} \le e^{-i}$,

$$\log(9\delta_{N_i}/\varepsilon_i) \le \log\left(\frac{9}{e-1} N_j e^{j-i}/\Delta N_i\right) < 2 + (j-i) + \log(N_j/\Delta N_i) \ ,$$

and by (3.3) we get

$$\theta^{-1} H_\varepsilon(A) \le N_1 \log \delta_0 + 2\sum_{i=0}^{j-1} \Delta N_i + \sum_{i=0}^{j-1} (j-i)\Delta N_i + \sum_{i=0}^{j-1} \Delta N_i \log(N_j/\Delta N_i) \ ,$$

from which (3.11) follows.

(ii) We define the N_i and j as before. From the definition of N_{i+1}, $\delta_k > e^{-(i+1)}$ if $k < N_{i+1}$, and we derive from (3.4):

$$\begin{aligned} \theta^{-1} C_\varepsilon(A) &\ge \sum_{i=1}^{j-2} \sum_{N_i \le k < N_{i+1}} \log(\delta_k/\varepsilon) \ge \sum_{i=1}^{j-2} \Delta N_i \log(e^{-i-1} e^{j-1}) \\ &= \sum_{i=1}^{j-3} (j-i-3)\Delta N_i = N_1 + \cdots + N_{j-3} \ . \end{aligned}$$

□

It is sometimes useful to write (3.13) in the form

$$\begin{aligned}\theta^{-1}H_\varepsilon(A) \le\ & N_1+\cdots+N_j+N_j\log N_j+2N_j \\ & -\sum_{i=0}^{j-1}\Delta N_i\log\Delta N_i+N_1\log\delta_0\ .\end{aligned} \tag{3.15}$$

If N_j is increasing to infinity, the terms in (3.15) follow in order of decreasing magnitude.

If the δ_n decrease to zero as $n^{-\alpha}$, $\alpha>0$, or faster, the upper estimate (3.13) comes very close to the lower estimate (3.14). More exactly, if N_i and j are defined as before, the following is true.

Theorem 3.3. (i) *If $\delta_{2n}\le c\delta_n$ for some c, $0<c<1$, and $n=0,1,\ldots$, then*

$$C_\varepsilon\big(A(\Delta)\big)\sim H_\varepsilon\big(A(\Delta)\big)\sim N_1+\cdots+N_j\ . \tag{3.16}$$

(ii) *If $\delta_{\rho n}/\delta_n\to 0$ as $n\to\infty$ for each $\rho>1$, then*

$$C_\varepsilon\big(A(\Delta)\big)\approx H_\varepsilon\big(A(\Delta)\big)\approx\theta[N_1+\cdots+N_j]\ . \tag{3.17}$$

(We assume here that δ_λ is defined for all $\lambda>0$ as a decreasing function.)

We need two lemmas.

Lemma 3.4. *For $j=1,2,\ldots$, let $\sigma_j:=N_1+\cdots+N_j$. Then*

$$S_j:=\sum_{i=0}^{j-1}\Delta N_i\log(N_j/\Delta N_i)\le\sigma_j\ . \tag{3.18}$$

Proof. Let $S_0:=0$. Then for $k=1,2,\ldots$,

$$\begin{aligned}S_k-S_{k-1}&=\Delta N_{k-1}\log(N_k/\Delta N_{k-1})+\sum_{i=0}^{k-2}\Delta N_i\log(N_k/N_{k-1})\\ &=(N_k-N_{k-1})\log\big(N_k/(N_k-N_{k-1})\big)+N_{k-1}\log(N_k/N_{k-1})\ .\end{aligned}$$

Now the function

$$f(x)=(b-x)\log\big(b/(b-x)\big)+x\log(b/x)$$

has its maximum on $[0,b]$ equal to $b\log 2<b$. This proves $S_k=S_{k-1}\le N_k$, hence the inequality (3.18). □

Lemma 3.5. (i) *If $0<c<1$ and $\delta_{2n}\le c\delta_n$, $n=0,1,\ldots$, then $N_{i+1}\le$ const N_i.*

(ii) *If $\delta_{\rho n}/\delta_n\to 0$ for each $\rho>1$, then $N_{i+1}/N_i\to 1$ and $S_j=o(N_1+\cdots+N_j)$.*

Proof. (i) We have $N_i = \min\{k : \delta_k \le e^{-i}\} \ge \min\{k : \delta_{2k} \le ce^{-i}\} = (1/2)\min\{2k : \delta_{2k} \le ce^{-i}\} \ge (1/2)\min\{k : \delta_k \le ce^{-i}\}$. Repeating this p times with p satisfying $c^p \le e^{-1}$ we obtain

$$N_i \ge 2^{-p}\min\{k : \delta_k \le c^p e^{-i}\} \le 2^{-p}N_{i+1} .$$

(ii) Let $\varepsilon > 0$ be given. For all sufficiently large i, the inequality $\delta_k \le e^{-i}$ implies that $\delta_{\rho k}/\delta_k \le e^{-1}$. For all such i

$$N_i = \min\{k : e^{-1}\delta_k \le e^{-i-1}\} \ge \min\{k : \delta_{\rho k} \le e^{-i-1}\} \ge \rho^{-1}N_{i+1} .$$

Hence for sufficiently large i, $1 \le N_{i+1}/N_i \le 1+\varepsilon$, and the relation $N_{i+1}/N_i \to 1$ is established.

As in the proof of Lemma 3.4, we have, with $x_k = N_{k-1}/N_k$,

$$S_k - S_{k-1} = -N_k\{(1-x_k)\log(1-x_k) + x_k \log x_k\} .$$

Since $x_k \to 1$ when $k \to \infty$ we have $S_k - S_{k-1} = o(N_k)$, hence $S_j = o(\sigma_j)$. □

Proof of Theorem 3.3. This follows immediately from the inequalities $C_{2\varepsilon}(A) \le H_\varepsilon(A) \le C_\varepsilon(A)$ and the lemmas. Indeed, if ε is replaced by 2ε in the definition of j, this number will change by at most one unit. □

As an application, one calculates, using the approximation theorems of [**CA**, Chapter 7] the entropy of the unit ball $\bar{B}_p^{*\alpha}$ of the space $\mathrm{Lip}^*(\alpha, L_p)$, $\alpha > 0$, (we recall that $\mathrm{Lip}^*(1, L_\infty)$ is the Zygmund space Z):

Proposition 3.6. *For the unit ball of the Lipschitz space* $\mathrm{Lip}^*(\alpha, L_p)$ *on* $\mathbb{T}$ *or on* $[a,b]$ *(with* L_∞ *interpreted as* C*),*

$$C_\varepsilon(\bar{B}_p^{*\alpha})_p \sim H_\varepsilon(\bar{B}_p^{*\alpha})_p \sim (1/\varepsilon)^{1/\alpha} . \tag{3.19}$$

This is the extension of Kolmogorov's Theorem 2.3 to the L_p metric. (In particular, it follows from this and Corollary 2.4 that $\bar{B}_\infty^{\alpha}$ and $\bar{B}_p^{*\alpha}$ have asymptotically the same entropy in L_p.)

§ 4. Relations Between Entropy and Widths

In this section we derive estimates for the entropy $H_\varepsilon(K)$ of a set K in the Banach space X in terms of the Kolmogorov widths $d_n(K)_X$. If one can find a sequence of decreasing optimal subspaces X_n for the widths, then K is contained in an approximation set $A(\Delta)$, $\Delta = (d_n(K))_0^\infty$ and one can then use the upper estimates for $H_\varepsilon(K)$ of §3. The general case requires fresh arguments. The upper estimates derived below are generally weaker than those of §3. Our lower estimates do not compare simply with those for approximation sets.

We shall formulate our first result in terms of the entropy numbers $e_n(K)$ defined in §1. It is interesting that for a good estimate of $e_n(K)$ one needs not just $d_n(K)$ but the whole sequence $d_0(K), \dots, d_n(K)$.

Theorem 4.1 (Carl [1981]). *For every $\alpha > 0$ there is a $C(\alpha) > 0$ such that for any set K in a Banach space X*

$$e_n(K) \le C(\alpha) n^{-\alpha} M(n) , \qquad n = 1, 2, \dots , \tag{4.1}$$

where

$$M(n) := \max_{1 \le i \le n} \left(i^\alpha d_{i-1}(K) \right) .$$

Proof. For some fixed natural number ν, let $M := M(2^\nu)$. Then for every $i = 1, \dots, \nu$ there is a subspace $X_i \subset X$, $\dim X_i \le 2^{i-1}$ such that $\operatorname{dist}(K, X_i) \le 2^\alpha M 2^{-i\alpha}$. For each i, we select $x_i \in X_i$ with $\|x - x_i\| \le 2^\alpha M 2^{-i\alpha}$. In particular, since $d_0(K) \le M$, we have $\|x_1\| \le 2M$.

For each i, $\|x_{i+1} - x_i\| \le r_i := 2^{1+\alpha} M 2^{-i\alpha}$. Therefore

$$x_\nu = x_1 + (x_2 - x_1) + \cdots + (x_\nu - x_{\nu-1}) \in \sum_{i=1}^{\nu} B_i ,$$

where B_i is the ball $\|y\| \le r_i$ of the space $X_i + X_{i+1}$ (the dimension of this space is $\le 2^{i+1}$).

We define $\varepsilon := 2^\alpha M 2^{-\nu\alpha}$. It follows that $\operatorname{dist}(K, \sum_{i=1}^{\nu} B_i) \le \varepsilon$. Let

$$\varepsilon_i = M 2^\lambda , \quad \lambda = \alpha(-2\nu + i + 1 + 2\sigma) + 3 ,$$

where σ is a large integer. If D_i are minimal ε_i-nets for B_i in X_i, then $D := \sum_{i=1}^{\nu} D_i$ is a $\sum_1^\nu \varepsilon_i \le C_1\varepsilon$-net for $\sum_1^\nu B_i$ in X, and therefore a $C\varepsilon$-net for K in X, $C = 1 + C_1$.

It remains to estimate the cardinality of D. According to (1.6), if $\varepsilon_i < r_i$, then for the space $X_i + X_{i+1}$, $|D_i| \le (3r_i/\varepsilon_i)^{2^{i+1}}$; if $\varepsilon_i = r_i$, $|D_i| = 1$. We have

$$\log_2(3r_i/\varepsilon_i) = \log_2 3 + 1 + \alpha - i\alpha - \lambda < 2\alpha(\nu - i - \sigma) ,$$

$$\log |D| \le \sum_{i=1}^{\nu} \log_2 |D_i| < 4\alpha \sum_{i=1}^{\nu-\sigma} (\nu - i - \sigma) 2^i < 4\alpha 2^{\nu-\sigma} \sum_{j=0}^{\infty} j 2^{-j} < 2^\nu ,$$

if σ is sufficiently large. This proves (4.1) for $n = 2^\nu$, $\nu = 1, 2, \dots$, from which it is easily extended to all n. □

From the relation between $e_n(K)$ and $H_\varepsilon(K)$ we derive

Corollary 4.2. *If for some C, $\alpha > 0$, $d_{n-1}(K) \le C n^{-\alpha}$, $n = 1, 2, \dots$, then*

$$H_\varepsilon(K) \le \operatorname{const}(1/\varepsilon)^{1/\alpha} .$$

Thus, in the case of $d_n(K) \sim n^{-\alpha}$ we obtain the same upper estimate of the entropy as the one given by (3.12) for an approximation set $A(\Delta)$ with $\delta_n \sim n^{-\alpha}$.

We now state similar results for exponentially decaying widths. This is the case for classes of analytic functions which we explore in §6. With Levin and Tikhomirov [1968] we prove

Theorem 4.3. *If K is a compact set in a real Banach space and $d_n(K) \le Ce^{-rn}$, $n = 0, 1, \ldots$, for some positive C, r, then*

$$\limsup_{\varepsilon \to 0} \bigl(H_\varepsilon(K)/\log^2(1/\varepsilon)\bigr) \le 1/(2r) . \tag{4.2}$$

We introduce the closed α-neighborhood K_α of K:

$$K_\alpha := \bigl\{y \in X : \rho(y, K) \le \alpha\bigr\} , \qquad \alpha \ge 0 .$$

Lemma 4.4. *Let $d_n(K) < \varepsilon/2$. Then for arbitrary $\alpha > 0$*

$$N_{\varepsilon+\alpha}(K_\alpha) \le \Bigl(4\bigl(d_0(K) + \varepsilon\bigr)/\varepsilon\Bigr)^n . \tag{4.3}$$

Moreover, for arbitrary $\alpha \ge 0$, $\delta > 0$,

$$N_{\varepsilon+\alpha}(K_\alpha) \le N_{\varepsilon+\alpha+\delta}(K_{\varepsilon+\alpha})\bigl(4(\varepsilon + \alpha + \delta)/\varepsilon\bigr)^n . \tag{4.4}$$

Proof. There exists a subspace X_n of dimension n for which $\operatorname{dist}(K, X_n) < \varepsilon/2$. A maximal $(\varepsilon/2)$-distinguishable set $z_1, \ldots, z_M$ for $K_\varepsilon \cap X_n$ is an ε-net for K and an $(\varepsilon + \alpha)$-net for K_α. Let U be the ball $\|x\| \le 1$ of X. The balls

$$V_i := z_i + (\varepsilon/4)U \cap X_n$$

are disjoint and all contained in the ball $(d_0(K) + \varepsilon)U \cap X_n$. Now (4.3) follows from a comparison of volumes.

If $w_1, \ldots, w_N$ is an $(\varepsilon + \alpha + \delta)$-net for $K_{\varepsilon+\alpha}$, then the sets

$$Q_k := \bigl[w_k + (\varepsilon + \alpha + \delta)U\bigr] \cap X_n , \qquad k = 1, \ldots, N ,$$

form a covering of $K_{\varepsilon+\alpha} \cap X_n$ while the balls V_i are all contained in $K_\varepsilon \cap X_n \subset K_{\varepsilon+\alpha} \cap X_n$. By the Brunn-Minkowski theorem (Theorem 2.3 of Appendix 1), $\operatorname{vol}(Q_k) \le \operatorname{vol}\left((\varepsilon + \alpha + \delta)U \cap X_n\right)$ for each k. Therefore,

$$\begin{gathered} M(\varepsilon/4)^n \operatorname{vol}(U \cap X_n) = \operatorname{vol}(\cup_{i=1}^M V_i) \\ \le \operatorname{vol}(K_{\varepsilon+\alpha} \cap X_n) \le \operatorname{vol}(\cup_{k=1}^N Q_k) \le N(\varepsilon + \alpha + \delta)^n \operatorname{vol}(U \cap X_n) . \end{gathered}$$

and (4.4) follows. □

Lemma 4.5. *If K is a compact set in a real Banach space X and $d_{n_i}(K) < \varepsilon_i/2$, $i = 1, \ldots, s$, then*

$$(4.5) \qquad N_{\varepsilon_1}(K) \le \left(\frac{4(d_0(K)+\varepsilon_s)}{\varepsilon_s}\right)^{n_s} \prod_{i=2}^{s} \left(\frac{4(\varepsilon_1+\cdots+\varepsilon_i)}{\varepsilon_{i-1}}\right)^{n_{i-1}} .$$

Proof. Applying (4.4) several times, we obtain

$$N_{\varepsilon_1}(K) \le N_{\varepsilon_1+\varepsilon_2}([K]_{\varepsilon_1}) \left(\frac{4(\varepsilon_1+\varepsilon_2)}{\varepsilon_1}\right)^{n_1}$$
$$\le N_{\varepsilon_1+\varepsilon_2+\varepsilon_3}([K]_{\varepsilon_1+\varepsilon_2}) \left(\frac{4(\varepsilon_1+\varepsilon_2)}{\varepsilon_1}\right)^{n_1} \left(\frac{4(\varepsilon_1+\varepsilon_2+\varepsilon_3)}{\varepsilon_2}\right)^{n_2}$$
$$\cdots \le N_{\varepsilon_1+\cdots+\varepsilon_s}([K]_{\varepsilon_1+\cdots+\varepsilon_{s-1}}) \left(\frac{4(\varepsilon_1+\varepsilon_2)}{\varepsilon_1}\right)^{n_1} \cdot \ldots \cdot \left(\frac{4(\varepsilon_1+\cdots+\varepsilon_s)}{\varepsilon_{s-1}}\right)^{n_{s-1}} .$$

To prove (4.5), we now use (4.3) with $\varepsilon = \varepsilon_s$ and $\alpha = \varepsilon_1 + \cdots + \varepsilon_{s-1}$. □

Proof of Theorem 4.3. We take $\varepsilon > 0$ and some natural number s. Then we define positive numbers n, $\varepsilon_1, \ldots, \varepsilon_s$ by

$$\varepsilon_1 := \varepsilon := 4Ce^{-rn} , \quad \varepsilon_2 := 4Ce^{-rn(s-1)/s}, \ldots, \quad \varepsilon_s := 4Ce^{-rn/s} .$$

Let $n_1 := [n]+1$, $n_2 := [n(s-1)/s]+1, \ldots$, $n_s := [n/s]+1$. If ε is sufficiently small, then for each i, $\varepsilon_1 + \cdots + \varepsilon_i \le 8C\varepsilon_i$. Using (4.5) and the obvious inequality $n_{i-1} - n_i \ge [n/s]$ we obtain for small ε

$$\begin{aligned} N_\varepsilon(K) &\le (32C)^{n_1+\cdots+n_s} (\varepsilon_2/\varepsilon_1)^{n_1} \ldots (\varepsilon_s/\varepsilon_{s-1})^{n_{s-1}} \big((d_0+\varepsilon_s)/\varepsilon_s\big)^{n_s} \\ &\le C^{-n_s}(32C)^{n_1+\cdots+n_s} \varepsilon_1^{-n_1} \varepsilon_2^{n_1-n_2} \ldots \varepsilon_s^{n_{s-1}-n_s} (d_0+1)^{n_s} \\ &\le C^{-n_s} (32)^{n(s+1)/2+s} \big(4C(d_0+1)\big)^{n/s+1} \exp\big(rn^2 - rn[n/s]s(s-1)/2\big) \\ &= \exp\left\{\lambda_n + rn^2 - rn[n/s]\frac{s(s-1)}{2}\right\} , \qquad \lambda_n = O(n) . \end{aligned}$$

For fixed s, with $\varepsilon \to 0$ and $n \to \infty$ we now have

$$\limsup_{\varepsilon \to 0+} \frac{H_\varepsilon(K)}{\log^2(1/\varepsilon)} \le \lim_{n\to\infty} \frac{\frac{s+1}{2s} rn^2 + O(n)}{(rn - \log(4C))^2} = \frac{s+1}{s}\,\frac{1}{2r} .$$

From this (4.2) follows since s can be arbitrarily large. □

We now establish a lower estimate for the capacity $C_\varepsilon(K)$ in terms of the $d_k(K)$.

Theorem 4.6 (Mityagin [1961]). *If, in a real Banach space, K is a compact set with central symmetry (that is, if $x \in K$ implies $-x \in K$), then for any natural n*

$$(4.6) \qquad C_\varepsilon(K) \ge \sum_{k=1}^{n} \log\big(d_{k-1}/(k\varepsilon)\big) , \qquad d_k =: d_k(K) .$$

The best result in (4.6) is obtained if the summation is extended only to those k for which $d_{k-1} \ge k\varepsilon$. For the *proof* we select a sequence of points

$x_k \in K$. We take $x_1 \in K$ with $a_1 := \|x_1\| = d_0$. If $x_1, \ldots, x_k$ are already known, we consider the linear space X_k spanned by these points. Let x_{k+1} be a point of K at the maximal distance from X_k. Then $a_{k+1} := \rho(x_{k+1}, X_k) \geq d_k$. We put $y_k := x_k / a_k$. The y_k span the same spaces X_k and have the property

$$\rho(y_{k+1}, X_k) = 1\ , \qquad k = 0, 1, \ldots\ . \tag{4.7}$$

Because K is convex and symmetric with respect to the origin, it contains for each n, together with the points $a_k y_k = x_k$, the octahedron

$$\Omega_n := \left\{ y : y = \sum_1^n \lambda_k a_k y_k\ ,\ \sum_1^n |\lambda_k| \leq 1 \right\} . \tag{4.8}$$

Clearly, $C_\varepsilon(K) \geq C_\varepsilon(\Omega_n)$.

In Ω_n we find many points that are ε-distinguishable. Let $\varepsilon' > \varepsilon$; we consider the points

$$y = \sum_1^n m_k \varepsilon' y_k\ , \tag{4.9}$$

where the m_k are integers such that the corresponding y belong to Ω_n. Two different points y', y'' of this type are ε-distinguishable. Indeed, if k is the largest index for which they have different coefficients in (4.9), then by (4.7), $\|y' - y''\| \geq \rho(\varepsilon' y_k, X_{k-1}) = \varepsilon' > \varepsilon$.

We estimate the number N of different points y in (4.9). For this purpose we use euclidean volumes in X_n identifying the points $\xi_1 y_1 + \cdots + \xi_n y_n$ with the vectors $(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. The octahedron Ω_n has the volume $2^n (1/n!) a_1 \cdot \ldots \cdot a_n$. The n-dimensional cubes with centers y and sides $2\varepsilon'$ have volumes $(2\varepsilon')^n$. They cover Ω_n; hence $a_1 \cdots a_n / n! \leq N \varepsilon'^n$. From this we obtain

$$C_\varepsilon(\Omega_n) \geq \log N \geq \sum_{k=1}^n \log \frac{a_k}{k\varepsilon'} \geq \sum_{k=1}^n \log \frac{d_{k-1}}{k\varepsilon'}\ ,$$

upon which we pass to the limit $\varepsilon' \to \varepsilon$. □

To use Theorems 4.3, 4.6 in the complex cases of §5, we indicate the necessary changes. For the first theorem, as in the proof of Theorem 3.1, we convert the space X into a space with real scalars. If $(\varphi_k)_{k=0}^n$ are the elements spanning an optimal subspace for $d_n(K)$ in X, we take $\Psi = (\psi_j)_0^{2n+1} = (\varphi_0, i\varphi_0, \varphi_1, i\varphi_1, \ldots)$. The condition $d_n(K) \leq Ce^{-rn}$, $n = 0, 1, \ldots$ becomes $d_n'(K) \leq Ce^{-r[(n+1)/2]} \leq Ce^{-(r/2)n}$. The real case result then yields (4.2) with $1/r$ instead of $1/(2r)$ on the right. The second theorem is replaced by:

Theorem 4.7. *If K is a compact set in a complex Banach space, with the property that $x \in K$ implies $\lambda x \in K$ for every complex λ, $|\lambda| \leq 1$, then for every natural n,*

$$C_\varepsilon(K) \geq 2n\log(1/\varepsilon) + 2\sum_{k=0}^{n-1}\log d_k - \log\big((2n)!\big)\ . \tag{4.10}$$

The *proof* is similar to that of Theorem 4.6. With the same notations, we now identify vectors $\xi_1 y_1 + \cdots + \xi_n y_n$, $\xi_i \in \mathbb{C}$, with vectors $(\operatorname{Re}\xi_1, \operatorname{Im}\xi_1, \ldots, \operatorname{Re}\xi_n, \operatorname{Im}\xi_n) \in \mathbb{R}^{2n}$. The set Ω_n defined by (4.8) (with complex λ_k) contains the octahedron Ω'_n defined by (4.8) with

$$\sum_{k=1}^{n}\big(|\operatorname{Re}\lambda_k| + |\operatorname{Im}\lambda_k|\big) \leq 1\ .$$

The $2n$-dimensional volume of Ω'_n is $2^{2n}((2n!))^{-1}a_1^2\cdots a_n^2$. Instead of points (4.9) we now take

$$y = \sum_{k=1}^{n}(m_k + in_k)\varepsilon' y_k\ ,$$

with integers m_k, n_k for which $y \in \Omega'_n$. It remains to compare the volumes in $\mathbb{R}^{2n}$. □

§ 5. Entropy of Classes of Analytic Functions

We continue our policy of using general Banach space results (Theorems 3.2, 3.3, 4.3, 4.7) for concrete function classes.

Let G be a domain (connected open set in $\mathbb{C}$), let K be an infinite compact subset of G. Then $\mathcal{A} := \mathcal{A}(K,G)$ is the set of analytic functions $f(z)$, $z \in G$, which are equipped with the uniform norm on K, and are bounded, $|f(z)| \leq 1$, on G. For functions $f(z) = f(z_1, \ldots, z_s)$ of several complex variables, $z = (z_1, \ldots, z_s) \in G \subset \mathbb{C}^s$, and a compact set $K \subset G$, $\mathcal{A}^s := \mathcal{A}(K,G)$ is similarly defined. The problem is to find the entropy of these sets.

We begin with a special case. For any $r > 0$, let D_r be the closed disk $|z| \leq r$ in $\mathbb{C}$. We consider $\mathcal{A}_r := \mathcal{A}(D_1, D_r)$ for $r > 1$. Let $E_n(f) := \inf_{P\in\mathcal{P}_{n-1}}\|f - P\|$, $n = 0, 1, \ldots$.

Lemma 5.1. (i) *For functions $f \in \mathcal{A}_r = \mathcal{A}(D_1, D_r)$, the error of uniform approximation on D_1 satisfies*

$$E_n(f) \leq \frac{1}{r-1}\,\frac{1}{r^n} =: \delta_n\ , \qquad n = 0, 1, \ldots\ ; \tag{5.1}$$

(ii) *Conversely, if for an analytic function f*

$$E_n(f) \leq \frac{1}{4(n+1)^2}\,\frac{1}{r^n} =: \delta'_n\ , \qquad n = 0, 1, \ldots \tag{5.2}$$

then $f \in \mathcal{A}_r$.

Proof. (i) From $|f(z)| \le 1$ for $|z| \le r$ we have for the coefficients c_k of the representation $f(z) = \sum_0^\infty c_k z^*$ the inequalities $|c_k| \le r^k$, $k = 0, 1, \ldots$. Consequently

$$E_n(f) \le \|\sum_{n+1}^{\infty} c_k z^k\| \le \sum_{n+1}^{\infty} r^{-k} = \frac{1}{r-1} r^{-n} .$$

(ii) We first note that if a polynomial $P_n(z)$ of degree $\le n$ satisfies $|P_n(z)| \le M$ on D_1, then

$$|P_n(z)| \le M r^n \quad \text{for} \quad |z| \le r , \qquad r > 1 . \tag{5.3}$$

Indeed, $Q_n(z) = z^n P_n(1/z)$ is also a polynomial of degree $\le n$, and $|Q_n(z)| \le M$ for $|z| = 1$. Hence, $|Q_n(z)| \le M$ for $|z| = 1/r$, and this gives (5.3).

From (5.2) we derive the existence of polynomials $P_n(z)$ with the property $|f(z) - P_n(z)| \le (4(n+1)^2 r)^{-1} r^{-n}$, $|z| \le 1$. Then $|P_n(z) - P_{n+1}(z)| \le (2(n+1)^2 r)^{-1} r^{-n}$, $|z| \le 1$; and $|P_n(z) - P_{n+1}(z)| \le (2(n+1)^2)^{-1}$, for $|z| \le r$. Thus, the series $f = P_0 + \sum_1^\infty (P_{n+1} - P_n)$ converges uniformly on D_r and yields $|f(z)| \le 1$. □

Theorem 5.2. (Vitushkin [A-1959]). *For the set $\mathcal{A}(r)$ one has*

$$C_\varepsilon(\mathcal{A}_r) \text{ and } H_\varepsilon(\mathcal{A}_r) = \frac{1}{\log r} \log^2 \frac{1}{\varepsilon} + O\Big(\log \frac{1}{\varepsilon} \log\log \frac{1}{\varepsilon}\Big) . \tag{5.4}$$

Proof. We let $\Phi := \{1, z, z^2, \ldots\}$, and $\Delta := (\delta_n)$, $\Delta' := (\delta_n')$, where δ_n, δ_n' are given by (5.1) and (5.2). Then, in the notation of §3,

$$A(\Phi, \Delta') \subset \mathcal{A}_r \subset A(\Phi, \Delta) . \tag{5.5}$$

Applying Theorem 3.2 to the sequence Δ, we see that the number N_i is the smallest k for which $\log(1/\delta_k) = k \log r + \log(r-1) \ge i$. This gives the value $N_i = i/\log r + O(1)$, and $2(N_1 + \cdots + N_j) = \frac{j^2}{\log r} + O(j) = \frac{1}{\log r} \log^2 \frac{1}{\varepsilon} + O(\log \frac{1}{\varepsilon})$. Moreover, the next largest term in (3.15) is $N_j \log N_j = O(\log \frac{1}{\varepsilon} \log\log \frac{1}{\varepsilon})$. This yields an upper estimate for $H_\varepsilon(\mathcal{A}_r)$. In a similar way, (3.14) yields $C_\varepsilon(\mathcal{A}_r) \ge \frac{1}{r} \log^2 \frac{1}{\varepsilon} + O(\log \frac{1}{\varepsilon})$. □

The entropy of more general sets $\mathcal{A}(K, G)$ is also known, but with less precision. The desired key formula here is the strong equivalence for $\varepsilon \to 0$

$$H_\varepsilon(A) \approx c(K, G) \log^2 \frac{1}{\varepsilon} , \tag{5.6}$$

where $c(K, G)$ is the condenser capacity of the pair K, G to G (see §2 of Appendix 4). Several authors established asymptotic properties of $E_n(\mathcal{A})$ or of $d_n(\mathcal{A})$ from which (5.6) follows. We shall use the theorem of Widom [1972] (Theorem 9.3 of Chapter 13) which asserts that

$$\lim_{n\to\infty} d_n(\mathcal{A})^{1/n} = \exp\big(-V(K, G)\big), \quad V(K, G) := c(K, G)^{-1} . \tag{5.7}$$

The assumptions here are quite weak, they postulate that the boundary of G has positive logarithmic capacity, that G has a complement with countably many (connected) components, and that $K \subset G$ is compact. Using Theorems 4.3 and 4.7 we prove:

Theorem 5.3. *Relation* (5.6) *holds for each pair of sets* K, G *with a valid Widom's relation* (5.7).

Proof. Let $\delta > 0$. From (5.7) we derive the existence of two constants $C_1, C_2 > 0$ (which depend on δ) so that for $n = 0, 1, \ldots$

$$C_1 \exp\big((-V-\delta)n\big) \le d_n(\mathcal{A}) \le C_2 \exp\big((-V+\delta)n\big), \quad V := V(K,G).$$

From the upper estimate, using the complex form of Theorem 4.3, we derive

$$\limsup_{\varepsilon \to 0} \big(H_\varepsilon(\mathcal{A}) \log^{-2}(1/\varepsilon)\big) \le (V-\delta)^{-1} . \tag{5.8}$$

To estimate $C_\varepsilon(\mathcal{A})$ from below we use (4.10), substituting $n = [\frac{1}{V} \log \frac{1}{\varepsilon}]$ and obtain

$$\begin{aligned} C_\varepsilon(\mathcal{A}) &\ge 2n \log \frac{1}{\varepsilon} + 2(-V-\delta) \sum_{k=1}^{n-1} k - 2n \log(2n) \\ &\ge 2n \log \frac{1}{\varepsilon} - (V+\delta)n^2 + O(n \log n) \\ &\ge \left(\frac{2}{V} - \frac{V+\delta}{V^2}\right) \log^2 \frac{1}{\varepsilon} + O\left(\log \frac{1}{\varepsilon} \log\log \frac{1}{\varepsilon}\right) , \end{aligned}$$

so that

$$\liminf_{\varepsilon \to 0} \left(C_\varepsilon(\mathcal{A}) \log^{-2} \frac{1}{\varepsilon}\right) \ge \frac{2}{V} - \frac{V+\delta}{V^2} . \tag{5.9}$$

Making $\delta \to 0$, we obtain (5.6). □

We complete this section by an estimation of entropy of classes of analytic functions of several complex variables $z = (z_1, \ldots, z_n)$: the set $\mathcal{A}^s := \mathcal{A}^s(K,G)$ consists of functions analytic in a domain (connected open set) $G \subset \mathbb{C}^s$, with $|f(z)| \le 1$ in G; K is a compact subset of G; we endow $\mathcal{A}^s$ with the uniform norm on K. Here only a weak equivalence is known:

Theorem 5.4 (Kolmogorov [1958]). *Under the above conditions, if G is bounded and K has interior points,*

$$H_\varepsilon(\mathcal{A}^s) \sim \log^{s+1}(1/\varepsilon) . \tag{5.10}$$

A *polydisk* $D_r^s(\mathrm{c})$ in $\mathbb{C}^s$ with center c is defined by s numbers $r := (r_1, \ldots, r_s)$, $r_j > 0$, $j = 1, \ldots, s$ and a point $c = (c_1, \ldots, c_s) \in \mathbb{C}^s$. It consists of all $z = (z_1, \ldots, z_s)$ with $|z_j - c_j| < r_j$, $j = 1, \ldots, s$. Analytic functions in $D_r^s := D_r^s(0)$ have the representation

(5.11) $$f(z) = \sum_k c_k z^k = \sum_{\substack{k_j \geq 0 \\ j=1,\dots,s}} c_{k_1,\dots,k_s} z_1^{k_1} \cdots z_s^{k_s} .$$

A polynomial $P \in \mathcal{P}_n^s$ of s variables and of coordinate degree not more than n is given by

(5.12) $$P_n(z) := \sum_{\substack{0 \leq k_j \leq n \\ j=1,\dots,s}} c_{k_1,\dots,k_s} z_1^{k_1} \cdots z_s^{k_s} ;$$

this sum has $(s+1)^n$ terms. We need a lemma:

Lemma 5.5. *Relation* (5.10) *holds for each class* $\mathcal{A}(D_{r'}^s, D_r^s)$, $r' < r$.

The *proof* is similar to that of Theorem 5.2. We first prove that the statements of Lemma 5.1 remain true for $f \in \mathcal{A}^s$ if conditions (5.1) and (5.2) are replaced respectively by

(5.13) $$E_n(f) := \inf_{P_n \in \mathcal{P}_n^s} \|f - P_n\| \leq \prod_{j=1}^s \frac{1}{r_j - 1} r_j^{-n}$$

and

(5.14) $$E_n(f) \leq \prod_{j=1}^s \frac{1}{4(n+1) r_j} r_j^{-n} .$$

The proof is the same as in Lemma 5.1, but elementary facts about analytic functions of several complex variables have to be used: the maximum modulus principle and the estimate $|c_{k_1,\dots,k_s}| \leq M r_1^{-k_1} \cdots r_s^{-k_s}$ of the coefficients in (5.11), if the function satisfies $|f(z)| \leq M$ in D_r^s.

Then we can apply Theorem 3.3. All possible powers $z_1^{k_1} \cdots z_s^{k_2}$ we order into a sequence Φ which begins with the term 1, then follow $2^s - 1$ powers of $\mathcal{P}_1^s$ but not 1, then all $3^s - 2^s$ powers present in $\mathcal{P}_2^s$ but not in $\mathcal{P}_1^s, \dots$. The sequence $\Delta = (\delta_n)_0^\infty$ we define by $\delta_{n^s} := A\rho^{-n}$, $A := \prod_1^s \frac{1}{r_j - 1}$, $\rho = \prod_1^s r_j$, $n = 1, 2, \dots$, and for arbitrary k by $\delta_k := A\rho^{-k^{1/s}}$, $k = 0, 1, \dots$. We get then $\mathcal{A}^s \subset A(\Phi, \Delta)$. The number N_i is the smallest k with $\log \frac{1}{\delta_k} \geq i$. This yields $N_i = Ci^s + o(i^s)$, $N_1 + \cdots + N_j \sim j^{s+1}$ and $H_\varepsilon(\mathcal{A}^s) \leq C \log^{s+1}(1/\varepsilon)$. The estimate of $C_\varepsilon(\mathcal{A}^s)$ from below we derive from (ii) of Theorem 3.3 and (5.14). □

Proof of Theorem 5.4. If c is an interior point of K, we can find concentric polydisks $D_{r_1}^s(c)$, $D_{r_2}^s(c)$ for which $D_{r_1}^s(c) \subset K \subset G \subset D_{r_2}^s(c)$. Then $\mathcal{A}' := \mathcal{A}(D_{r_1}^s(c), D_{r_2}^s(c)) \subset \mathcal{A}$, and the distances in $\mathcal{A}'$ do not exceed distances in $\mathcal{A}$. Therefore

$$H_\varepsilon(\mathcal{A}) \geq H_\varepsilon(\mathcal{A}') \sim \log^{s+1} \frac{1}{\varepsilon} .$$

On the other hand, we cover K by a finite number of polydisks D_j contained with their closure in G, and find concentric polydisks D_j' strictly containing the D_j with $D_j \subset D'_j \subset G$. Then $\mathcal{A}(D_j, G) \subset \mathcal{A}(D_j, D'_j)$. Let $\varepsilon > 0$. By Lemma 5.5, for each γ, $\mathcal{A}^s(D_j, G)$ can be covered by N_j balls $B_{j,i}$ with radii $\le \varepsilon$ in the metric of $C(D_j)$, where $\log N_j \le C \log^{s+1}(1/\varepsilon)$. The sets $B = \bigcap_{j=1}^m B_{j,i_j}$, are $N := \prod N_j$ in number, they cover $\mathcal{A}(K, G)$, and each B is contained in a ball of radius $\le \varepsilon$ in each of the metrics of $C(D_j)$, $j = 1, \dots, m$, hence also in the $C(K)$ metric. This shows that

$$H_\varepsilon(\mathcal{A}) \le \log N \le C \log^{s+1}(1/\varepsilon) . \qquad \square$$

§ 6. The Birman-Solomyak Theorem

In this section we consider, as has been done in Chapter 14 in the case of the widths, a *two-norm problem* of estimating the entropy of the ball $\bar{B}_p^\alpha[0,1]$ of the space $\mathrm{Lip}(\alpha, L_p)$ in the norm of $L_q[0,1]$. For $\alpha = r + \beta$, where $r = 0, 1, \dots,$ and $0 < \beta \le 1$, $\bar{B}_p^\alpha$ is the class of functions f for which

$$\omega(f^{(r)}, t)_p := \sup_{0<h\le t} \|f^{(r)}(\cdot + h) - f^{(r)}(\cdot)\|_{L_p(0,1-h)} \le t^\beta , \qquad t > 0 ,$$

and $\|f\|_p \le 1$. We assume that $\alpha > 1/p - 1/q$, for otherwise the entropy is infinite for small $\varepsilon > 0$ since $\bar{B}_p^\alpha$ is not relatively compact in L_q (see Proposition 3.9 of Chapter 14). Our main result is

Theorem 6.1 (Birman-Solomyak [1967]). *For $1 \le p, q \le \infty$ and $\alpha > 1/p - 1/q$,*

$$H_\varepsilon(\bar{B}_p^\alpha)_q \sim (1/\varepsilon)^{1/\alpha} , \qquad \varepsilon > 0 . \tag{6.1}$$

We see that, unlike the widths, the entropy has essentially the same asymptotic behavior for all p, q. This fact has been discovered a decade before the asymptotic theory of widths $d_n(B_p^\alpha)_q$ has been completed.

The method of proof, like that of Chapter 14, is discretization. We derive inequalities for $H_\varepsilon(\bar{B}_p^\alpha)_q$ from the corresponding estimates (6.2) of the entropy of the unit ball b_p^n of l_p^n, in the l_q-norm. The reduction is achieved by using appropriate approximation means for Lipschitz functions on $[0,1]$. We select for this purpose the splines of $\mathcal{S}_{n,r}^*[0,1]$ of §1 of Chapter 6.

We need the following theorem, given for special p, q by Höllig [1980], see Schütt [1984] for a generalization; the latter author also showed that (6.2) is unimprovable.

Theorem 6.2. *For $1 \le p < q \le \infty$, $0 < \varepsilon < 1$, $s := (p^{-1} - q^{-1})^{-1}$ and some $C(p) > 0$,*

$$H_\varepsilon(b_p^n)_q \le \begin{cases} C(p)\varepsilon^{-s} \log(2n\varepsilon^s) , & \text{if } \varepsilon \ge n^{-1/s} \\ C(p) n \log(2/(n\varepsilon^s)) , & \text{if } \varepsilon \le n^{-1/s} . \end{cases} \tag{6.2}$$

Proof. Let first $q = \infty$, so that $s = p$. For the given $0 < \varepsilon \le 1$ we define a natural number k by the condition $(k+1)^{-1} < \varepsilon \le k^{-1}$. Consider the subset $\hat{b}_p^n \subset b_p^n$ of all vectors $z = (\zeta_1, \dots, \zeta_n) \in b_p^n$ whose coordinates ζ_i are of the form ν/k, $\nu = 0, \pm 1, \dots, \pm k$. Obviously, $\hat{b}_p^n$ is an ε-net for b_p^n in the l_∞^n metric. The cardinality $|\hat{b}_p^n|$ is equal to the number of integer solutions $(\nu_1, \dots, \nu_n)$ of the inequality

$$|\nu_1|^p + \cdots + |\nu_n|^p \le k^p ,$$

which, in turn, does not exceed the number of integer solutions of

$$(6.3) \qquad |\nu_1| + \cdots + |\nu_n| \le \ell , \qquad \ell = k^p .$$

According to (5.4) of Chapter 14, the number of solutions of (6.3) for $\ell, n \ge 2$ does not exceed $(2e(n+\ell)/\ell)^\ell$.

If $\varepsilon \ge n^{-1/s} = n^{-1/p}$, then $(2\varepsilon)^{-p} \le \ell \le \varepsilon^{-p} \le n$, and we get

$$(6.4) \qquad H_\varepsilon(b_p^n)_\infty \le \log|\hat{b}_p^n| \le \varepsilon^{-p}\log(2^p 4en\varepsilon^p) \le C\varepsilon^{-p}\log(2n\varepsilon^p) .$$

In particular, we can put $\varepsilon = n^{-1/p}$ and get $H_{n^{-1/p}}(b_p^n)_\infty \le Cn$. For a set A in the n-dimensional normed linear space l_∞^n (whose balls are euclidean cubes), if $\lambda \ge 1$, $\lambda' := [\lambda] + 1$, $\varepsilon_0 > 0$,

(6.5)

$$H_{\varepsilon_0/\lambda}(A)_\infty \le H_{\varepsilon_0/\lambda'}(A)_\infty \le \log(\lambda'^n) + H_{\varepsilon_0}(A)_\infty \le n\log(2\lambda) + H_{\varepsilon_0}(A)_\infty .$$

For $\varepsilon \le n^{-1/p}$, it follows from this with $\varepsilon_0 = n^{-1/p}$, $\lambda = \varepsilon^{-1}n^{-1/p}$

$$(6.6) \qquad H_\varepsilon(b_p^n)_\infty \le n\log(2\varepsilon^{-1}n^{-1/p}) + Cn \le Cn\log\big(2/(n\varepsilon^p)\big) .$$

Together (6.4) and (6.6) establish (6.2) for $q = \infty$.

Let now $1 \le p \le q < \infty$ and suppose that for a given $\eta > 0$, there is an η-net of cardinality N for b_p^n in l_∞^n. Then, obviously, there exist some $x_1, \dots, x_N$ in b_p^n itself forming a 2η-net in the l_∞^n metric. If $x \in b_p^n$ and $\|x - x_i\|_\infty \le 2\eta$ for some i, then

$$\|x - x_i\|_q \le \|x - x_i\|_\infty^{1-p/q}\|x - x_i\|_p^{p/q} \le (2\eta)^{1-p/q}2^{p/q} = 2\eta^{1-p/q} ,$$

so that the x_i form a $(2\eta^{1-p/q})$-net for b_p^n in the l_p^n metric. Thus, with $\eta = \varepsilon^{q/(q-p)}$, we get

$$H_{2\varepsilon}(b_p^n)_q \le H_{\varepsilon^{q/(q-p)}}(b_p^n)_\infty = H_{\varepsilon^{s/p}}(b_p^n)_\infty .$$

From this, (6.4) and (6.6) we again obtain (6.2). □

Proof of Theorem 6.1. We have to prove only the upper estimate for the entropy in (6.1); the lower estimate follows from Corollary 2.4 and (1.3).

We shall use splines of the Schoenberg space $\mathcal{S}_{n,r+1}^* := \mathcal{S}_{n,r+1}^*[0,1]$ of §1 of Chapter 6, with no restrictions for splines at the knots $t_k := k/n$, $k = 1, \dots, n-1$. By Proposition 1.3 of Chapter 6 with $p = q$, for every $f \in \bar{B}_p^\alpha$, there is an $S \in \mathcal{S}_{n,r+1}^*$, for which $\|f - S\|_p \le Cn^{-\alpha}$. Since $\dim \mathcal{S}_{n,r+1}^* =$

$n(r+1)$, this implies that $d_n(\bar{B}_p^\alpha)_p \le Cn^{-\alpha}$, and Corollary 4.2 yields $H_\varepsilon(\bar{B}_p^\alpha)_p \le C(1/\varepsilon)^{1/\alpha}$. Thus, if $p \ge q$, from $\bar{B}_p^\alpha[0,1] \subset \bar{B}_q^\alpha[0,1]$ we derive $H_\varepsilon(\bar{B}_p^\alpha)_q \le C(1/\varepsilon)^{1/\alpha}$.

More difficult is the case $1 \le p \le q \le \infty$. Due to Proposition 1.3 of Chapter 6, for any $f \in \bar{B}_p^\alpha$, we can find a sequence of splines $S_k \in \mathcal{S}^*_{2^k,r+1}$, for which $\|f - S_k\|_p \le C2^{-k\alpha}$, $k = 1, 2, \ldots$. It follows that f has a representation $f = g_1 + g_2 + \cdots$, where $g_1 := S_1$, $g_k := S_k - S_{k-1}$, $k = 2, \ldots$. If $B_{p,k}^\alpha$ is the intersection of the ball $\|g\|_p \le C2^{-k\alpha}$ with $\bar{B}_p^\alpha$, we have

$$\bar{B}_p^\alpha \subset \sum_{k=1}^{\infty} B_{p,k}^\alpha \, . \tag{6.7}$$

Each series $\sum \bar{g}_k$, $\bar{g}_k \in B_{p,k}^\alpha$, $k = 1, 2, \ldots$ converges also in the L_q norm, since by **2** of §1 of Chapter 6,

$$\|\bar{g}_k\|_q \le C2^{-k\alpha}2^{k(1/p-1/q)}, \quad k = 1, 2, \ldots \, .$$

The series (6.7) converges in L_q in the sense of Proposition 1.4, and we get

$$H_\varepsilon(\bar{B}_p^\alpha, L_q) \le \sum_{k=1}^{\infty} H_{\varepsilon_k}(B_{p,k}^\alpha, L_q) \, , \qquad \varepsilon = \varepsilon_1 + \varepsilon_2 + \cdots \, . \tag{6.8}$$

In this sum we would like to replace $B_{p,k}^\alpha$ by $b_p^{m_k}$, L_q by $l_q^{m_k}$, $m_k := 2^k(r+1)$. For this purpose we use the isomorphism I of §1 of Chapter 6, which maps l_p^{nr} onto $\mathcal{S}^*_{n,r}$ (equipped with the L_q-norm). For this isomorphism

$$C_1 n^{-1/p}\|c\|_p \le \|S\|_p \le C_2 n^{-1/p}\|c\|_p \, , \qquad S = I(c) \, . \tag{6.9}$$

The set $B_{p,k}^\alpha$ is contained in the ball $\|S\|_p \le C2^{-k\alpha}$, while according to (6.9) with $n = 2^k$, the image $I(b_p^{m_k})$ of the unit ball $b_p^{m_k}$ covers the ball $\|S\|_p \le C2^{-k/p}$. It follows that $B_{p,k}^\alpha \subset \lambda I(b_p^{m_k})$, if $\lambda := C2^{k/p-k\alpha}$. Thus, using (1.8),

$$H_{\varepsilon_k}(B_{p,k}^\alpha)_q \le H_{\varepsilon_k}\big(\lambda I(b_p^{m_k})\big)_q = H_{\varepsilon_k/\lambda}\big(I(b_p^{m_k}), L_q\big) \, .$$

To the last term we apply (1.9), taking $X := l_p^{m_k}$, $Y := \mathcal{S}^*_{2^k,p+1}$ (with the L_q topology), $T := I$. The norm of I is now $\sim 2^{-k/q}$, and (1.9) yields, with $\eta_k := C(\varepsilon_k/\lambda)2^{k/q} = C\varepsilon_k 2^{k(\alpha-1/s)}$,

$$H_{\varepsilon_k}(B_{p,k}^\alpha)_q \le H_{\eta_k}(b_p^{m_k}, l_p^{m_k}) \, .$$

Inequality (6.8) now becomes

$$H_\varepsilon(\bar{B}_p^\alpha, L_q) \le \sum_{k=1}^{\infty} H_{\eta_k}(b_p^{m_k}, l_p^{m_k}) \, , \qquad \eta_k := C\varepsilon_k 2^{k(\alpha-1/s)} \, . \tag{6.10}$$

Let now a small $\varepsilon > 0$ be given. We define

$$k_0 := \big[(1/\alpha)\log_2(1/\varepsilon)\big] \, , \quad \varepsilon_k := 2^{-\alpha k_0 - A|k-k_0| + B} \, , \quad k = 1, 2, \ldots \tag{6.11}$$

where A is any fixed number satisfying $0 < A < \alpha - 1/s$ and B will be chosen to facilitate the application of the estimate (6.2) as follows. We observe that $\log_2(2^k \eta_k^s) = s(k\alpha - k_0\alpha - A|k-k_0| + B) + C_0$ and set $B := -C_0/s$. Then

$$\log_2(2^k \eta_k^s) = s\big[\alpha(k-k_0) - A|k-k_0|\big] .$$

This logarithm is negative if $k < k_0$, so that the second estimate (6.2) applies:

$$\sum_{k=0}^{k_0-1} H_{\eta_k}(b_p^{2^k})_q \le C \sum_{k=0}^{k_0} 2^k (k_0 - k) \le C 2^{k_0} \sum_{j=0}^{\infty} j 2^{-j} \sim (1/\varepsilon)^{1/\alpha} .$$

For $k \ge k_0$, we similarly use the first estimate (6.2) with $\beta := s(A - \alpha + 1/s) < 0$:

$$\begin{aligned}\sum_{k=k_0}^{\infty} H_{\eta_k}(b_p^{2^k})_q &\le C \sum_{k=k_0}^{\infty} \eta_k^{-s}(k - k_0 + 1) \\ &\le C 2^{k_0} \sum_{k=k_0}^{\infty} 2^{s(A-\alpha+1/s)(k-k_0)}(k - k_0 + 1) \\ &\le C 2^{k_0} \sum_{j=0}^{\infty} 2^{\beta j}(j+1) \sim (1/\varepsilon)^{1/\alpha} .\end{aligned}$$

For the ε_k we have $\sum \varepsilon_k \le C 2^{-\alpha k_0} \le C\varepsilon$ so it follows now from (6.10) that $H_{C\varepsilon}(\bar{B}_p^\alpha)_q \le \mathrm{const}(1/\varepsilon)^{1/\alpha}$ which is equivalent to the desired upper estimate. □

Theorem 6.1 can be extended to functions of several variables. For example, let $\bar{B}_p^r(I^N)$, denote the Sobolev class of real-valued functions f defined on the cube $I^N = [0,1]^N$ which have all partial derivatives $D^j f$ of orders $j = 1, \dots, r$, with each rth partial derivative belonging to $L_p(I^N)$ and with $\|D^r f\|_p \le 1$, $\|f\|_p \le 1$. Then for $1 \le p, q \le \infty$ and $r > N(1/p - 1/q)$,

$$H_\varepsilon\big(\bar{B}_p^r(I^N)\big)_q \sim (1/\varepsilon)^{N/r} . \tag{6.12}$$

The proof proceeds along the same lines although obtaining the upper estimate is considerably more difficult.

§ 7. Entropy Numbers of Operators

In §1 we defined the entropy numbers $e_n(T)$ of a linear operator T acting from a real or complex Banach space X into a Banach space Y. For every T,

$$\|T\| = e_0(T) \ge e_1(T) \ge \cdots .$$

We have $e_n(T) \to 0$ for $n \to \infty$ if and only if the operator T is compact (that is, if the image $T(B_X)$ of the unit ball is a relatively compact set in Y). For this reason, we shall consider in this section only compact operators.

For all $T_1, T_2 \in L(X,Y)$ and arbitrary n, m one has

$$e_{n+m}(T_1 + T_2) \le e_n(T_1) + e_m(T_2) \ . \tag{7.1}$$

If $T \in L(X,Y)$, $S \in L(Y,Z)$, then

$$e_{n+m}(ST) \le e_n(T) e_m(S) \ . \tag{7.2}$$

In particular,

$$e_n(ST) \le \|S\| e_n(T) \ , \quad e_n(ST) \le e_n(S)\|T\| \ . \tag{7.3}$$

All this follows directly from definitions.

In this section we establish some relations between entropy numbers, widths and eigenvalues of operators. A number λ (which can be complex for a complex Banach space X) is an *eigenvalue* and $x \in X$ is an *eigenvector* of $T \in L(X,X)$ if $Tx = \lambda x$, $x \ne 0$. The eigenvalue λ has multiplicity n if the set of its eigenvectors is an n-dimensional subspace of X. In 1917 F. Riesz proved (see, for example, Riesz and Sz.-Nagy [B-1955, No. 77–80, 89]) that in a Banach space every compact operator T has at most a countable set of eigenvalues, each of finite multiplicity, which, arranged in a sequence $(\lambda_n(T))$, tend to zero. We shall repeat each $\lambda_n(T)$ according to its multiplicity and order them so that $\|T\| \ge |\lambda_1(T)| \ge |\lambda_2(T)| \ge \cdots$. According to Riesz, for each n there is an n-dimensional subspace $X_n \subset X$ such that T maps X_n into itself and that the restriction of T to X_n has exactly the eigenvalues $\lambda_1(T), \dots, \lambda_n(T)$.

In some situations one can estimate the rate of decay of the $\lambda_k(T)$. For example, let X be the real space $L_2[a,b]$ and let

$$(Tf)(t) := \int_a^b K(t,s) f(s)\, ds \ ,$$

with a symmetric kernel $K \in L_2([a,b]^2)$. Then (see §4 of Chapter 13) the eigenfunctions φ_k can be made orthonormal in $L_2[a,b]$. From the equality

$$\int_a^b K(t,s)\varphi_k(s)\, ds = \lambda_k \varphi(t)$$

we derive that for every t and $N = 1, 2, \dots$,

$$\int_a^b K(t,s)^2\, ds \ge \sum_{k=1}^N \lambda_k^2 \varphi_k(t)^2 \ .$$

Integrating this we obtain

$$\int_a^b \int_a^b K(t,s)^2\, dt\, ds \ge \sum_{k=1}^N \lambda_k^2 \ ,$$

so that $\sum \lambda_k^2 < \infty$.

In the general setting, what properties of T may assure certain rate of decay of the $\lambda_k(T)$, say, $\sum |\lambda_k(T)|^q < \infty$ for some $q > 0$? Our answer is, roughly, as follows: the eigenvalues are dominated by the entropy numbers which, in turn, are dominated by the Kolmogorov widths $d_k(T)$. As a corollary of Theorem 4.1 we have

Theorem 7.1 (Carl [1981]). *For each $q > 0$, there is a constant $C(q) > 0$ with the property that for each operator T on X*

$$\sum_n e_n(T)^q \le C(q) \sum_n d_n(T)^q \,. \tag{7.4}$$

Proof. Let $d_n := d_n(T)$, $e_n := e_n(T)$. We fix $\alpha > 1/q$ and define for natural ν

$$\rho(\nu) := 2^{-\nu\alpha} \max_{0 \le i \le \nu} (2^{i\alpha} d_{2^i}) \,.$$

Then

$$\rho(\nu)^q \le 2^{-\nu\alpha q} \sum_{i=0}^{\nu} 2^{i\alpha q} d_{2^i}^q \,.$$

From (4.1) we have $e_{2^\nu} \le C_1(\alpha)\rho(\nu)$. Hence

$$\sum_n e_n^q \le C_2 \sum_\nu 2^\nu e_{2^\nu}^q \le C_3(\alpha) \sum_\nu 2^{\nu(1-\alpha q)} \sum_{i=1}^{\nu} 2^{i\alpha q} d_{2^i}^q$$

$$= C_4(\alpha, q) \sum_{i=1}^{\infty} 2^i d_{2^i}^q \le C_5(\alpha, q) \sum_{n=1}^{\infty} d_n^q \,. \qquad \square$$

We now turn to the relations between $\lambda_k(T)$ and $e_k(T)$. In the proof we shall deal with the euclidean volumes of certain convex sets in $\mathbb{R}^n$ and $\mathbb{C}^n$. For a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$, we have the formula for the volumes

$$\operatorname{vol}\big(T(\Omega)\big) = \operatorname{vol}(\Omega) \cdot |\det(T)| \,, \tag{7.5}$$

where $\det(T)$ is the determinant of the matrix generated by T. In the complex case, vectors $x = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ can be identified with the real vectors

$$(\operatorname{Re}\xi_1, \operatorname{Im}\xi_1, \dots, \operatorname{Re}\xi_n, \operatorname{Im}\xi_n) \in \mathbb{R}^{2n} \,,$$

and accordingly, every linear operator $T : \mathbb{C}^n \to \mathbb{C}^n$ can be treated as an operator from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}$. For example, the operator $T : \mathbb{C}^1 \to \mathbb{C}^1$ defined by $Tx = \lambda x$, $\lambda \in \mathbb{C}$, can be treated as the linear operator taking the vector $(\operatorname{Re} x, \operatorname{Im} x) \in \mathbb{R}^2$ to the vector $(\operatorname{Re}(Tx), \operatorname{Im}(Tx)) \in \mathbb{R}^2$, with the matrix

$$A(\lambda) = \begin{pmatrix} \operatorname{Re}\lambda\,, & -\operatorname{Im}\lambda \\ \operatorname{Im}\lambda\,, & \operatorname{Re}\lambda \end{pmatrix} .$$

By volumes in $\mathbb{C}^n$ we understand corresponding volumes in $\mathbb{R}^{2n}$. By Schur's lemma (see, for example, Bellman [B-1960, Ch.11]), for every $T : \mathbb{C}^n \to \mathbb{C}^n$ there is an orthonormal basis in $\mathbb{C}^n$ in which T is represented by an upper triangular matrix, with the eigenvalues $\lambda_1, \ldots, \lambda_n$ on the main diagonal. Under the above identification, T can be treated as the operator $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ whose matrix A is built of 2×2 blocks, with the blocks $A(\lambda_1), \ldots, A(\lambda_n)$ on the main diagonal and zero blocks below it. Since $\det(A) = \det(A_1) \cdots \det(A_n) = |\lambda_1|^2 \cdots |\lambda_n|^2$, from (7.5) we have for $\Omega \subset \mathbb{C}^n$

$$\text{(7.6)} \qquad \operatorname{vol}(T(\Omega)) = \operatorname{vol}(\Omega) \cdot \prod_{i=1}^{n} |\lambda_i(T)|^2 .$$

The following theorem is due to Carl [1981].

Theorem 7.2 *Let X be a complex Banach space and let $T \in L(X, X)$ be a compact operator. Then for n, $k = 1, 2, \ldots$,*

$$\text{(7.7)} \qquad \prod_{i=1}^{n} |\lambda_i(T)|^{1/n} \le 2^{k/(2n)} e_k(T) .$$

Proof. We may assume that $\lambda_n(T) \neq 0$. Let X_n denote the n-dimensional space for which $T(X_n) \subset X_n$ and the restriction of T to X_n has exactly the eigenvalues $\lambda_1(T), \ldots, \lambda_n(T)$. Let B_{X_n} be the unit ball of X_n. If $e_k(T) = \varepsilon$ then for every $\delta > 0$ there are $\le 2^k$ closed balls in X of radii $(\varepsilon + \delta)$ whose intersections with X_n cover $T(B_{X_n})$. Each such intersection is of diameter $\le 2(\varepsilon + \delta)$, and therefore is contained in some ball in X_n of radius $2(\varepsilon + \delta)$. Comparing the volumes we have

$$\text{(7.8)} \qquad 2^k \big(2(\varepsilon + \delta)\big)^{2n} \operatorname{vol}(B_{X_n}) \ge \operatorname{vol}(T(B_{X_n})) .$$

Since δ is arbitrary, (7.8) and (7.6) (with $\Omega = B_{X_n}$) yield (7.7) with the additional factor 2 on the right. To obtain (7.7) we have to remove this factor. To this end we take some natural number N and replace T by T^N and k by Nk. We have $\lambda_i(T^N) = \lambda_i(T)^N$ while by (7.2), $e_{Nk}(T^N) \le e_k(T)^N$, and 2 is removed by taking the $1/N$ power and passing to the limit as $N \to \infty$. □

An interesting corollary of (7.7) is

$$\text{(7.9)} \qquad |\lambda_n(T)| \le \sqrt{2}\, e_n(T) , \qquad n = 1, 2, \ldots .$$

In the rest of this section we shall deal with operators on the complex Hilbert space H. In our exposition we follow Pisier [B-1989]. For the general facts about operators on a Hilbert space see Riesz and Sz.-Nagy [B-1955, Ch.6].

An operator $S \in L(H, H)$ is called *positive* if the scalar product satisfies $(Sx, x) \ge 0$ for all $x \in H$. The eigenvalues of a positive operator are clearly non-negative. If $T \in L(H, H)$ is arbitrary and T' is the adjoint operator, then

$T'T$ is positive since $(T'Tx, x) = (Tx, Tx) \geq 0$. Every positive operator S has a *square root*, that is, a positive self-adjoint operator $S^{1/2}$ uniquely defined by $(S^{1/2})^2 = S$. If we denote $|T| := (T'T)^{1/2}$, then every T can be represented in the *polar form*: there is some $V \in L(H, H)$, with $\|V\| = \|V'\| = 1$, for which

$$T = V|T| \ , \quad |T| = V'T \ .$$

As an immediate corollary of the polar representation and (7.3) we have for all k

$$e_k(T) = e_k(T') = e_k(|T|) \ . \tag{7.10}$$

Theorem 7.3 *If $T \in L(H, H)$ is a compact operator and $q > 0$, then $(e_k(T)) \in \ell_q$ if and only if $(\lambda_k(|T|)) \in \ell_q$.*

Proof. By Proposition 4.4 of Chapter 13, in the Hilbert space

$$d_k(T) = a_k(T) = \sqrt{\lambda_{k+1}(T'T)} = \lambda_{k+1}(|T|) \ .$$

By (7.4) in any Banach space from $(d_k(T)) \in \ell_q$ follows $e_k(T) \in \ell_q$. Conversely, from (7.10) and (7.9) we obtain

$$\lambda_k(|T|) \leq \sqrt{2}\, e_k(|T|) = \sqrt{2}\, e_k(T) \ ,$$

so that $(e_k(T)) \in \ell_q$ implies $(\lambda_k(|T|)) \in \ell_q$. □

Theorem 7.4 *Let $T \in L(H, H)$ be a compact operator and let*

$$c_k(T) := \sup_{n \geq 1} 2^{-k/(2n)} \prod_{i=1}^{n} \lambda_k(|T|)^{1/n} \ .$$

Then

$$c_k(T) \leq e_k(T) \leq 6c_k(T) \ , \qquad k = 1, 2, \ldots \ . \tag{7.11}$$

Proof. The first inequality in (7.11) follows from (7.10) and (7.7). To derive the second inequality, we need, for the operator $|T|$, the existence in H of the orthonormal system (φ_k) of Proposition 4.3 of Chapter 13. Applying $|T|$ to the series of the proposition, we get

$$|T|x = \sum_{k=1}^{\infty} (x, \varphi_k)\lambda_k \varphi_k \ , \qquad \lambda_k := \lambda_k(|T|) \ , \quad x \in H \ .$$

Obviously, the norm of this operator equals λ_1, the largest eigenvalue. If $\lambda_1 \leq \varepsilon := 2c_k(T)$, then $e_k(T) \leq e_0(T) = \|T\| \leq \varepsilon$ for all k, which agrees with (7.11). So we assume that $\lambda_1 > \varepsilon$ and define m by the condition $\lambda_m > \varepsilon \geq \lambda_{m+1}$.

Let B be the unit ball of H, $H_m := \mathrm{lin}\{\varphi_1, \ldots, \varphi_m\}$, $B_m := B \cap H_m$, and let $T_m \in L(H, H_m)$ be defined by

$$T_m x := \sum_{k=1}^{m} (x, \varphi_k)\lambda_k \varphi_k \ .$$

The set $T_m(B) = T_m(B_m)$ is an ellipsoid in H_m whose semi-axes $\lambda_1, \dots, \lambda_m$ are $> \varepsilon$; hence $\varepsilon B_m \subset T_m(B_m)$. If $y_1, \dots, y_p$ is a maximal 2ε-distinguishable set in $T_m(B)$, then it is also a 2ε-net for $T_m(B)$ and a 3ε-net for $|T|(B)$ (because $\| \, |T| - T_m \| \le |\lambda_{m+1}| \le \varepsilon$). The balls $y_i + \varepsilon B_m$ are disjoint and all contained in $T_m(B_m) + \varepsilon B_m \subset 2T_m(B_m)$. Comparing the volumes (H_m should be identified with $\mathbb{R}^{2m}$), we have

$$p \operatorname{vol}(B_m)\varepsilon^{2m} \le \operatorname{vol}\big(2T_m(B_m)\big) = 2^{2m} \operatorname{vol}\big(T_m(B_m)\big) \ . \tag{7.12}$$

On the other hand,

$$\operatorname{vol}\big(T_m(B_m)\big) / \operatorname{vol}(B_m) \le \prod_1^m \lambda_i^2 \le \big(c_k(T) 2^{k/(2m)}\big)^{2m} = (\varepsilon/2)^{2m} 2^k \ . \tag{7.13}$$

From (7.12) and (7.13) follows that $p \le 2^k$. Thus $e_k(T) = e_k(|T|) \le 3\varepsilon = 6c_k(T)$. □

§ 8. Notes

8.1. Many natural classes F of functions defined on $\mathbb{R}$ are not compact in $C(\mathbb{R})$. For them one can introduce, instead of the usual entropy, the *average entropy per unit of length* $\bar{H}_\varepsilon(F)$. If F_a is the class formed by restrictions of $f \in F$ to the interval $[-a, a]$, then

$$\bar{H}_\varepsilon(F) := \lim_{a \to \infty} \frac{1}{2a} H_\varepsilon\big(F_a, C[-a, a]\big) \ ,$$

assuming that the limit exists.

Let, for example, $\mathcal{B}_\sigma$ be the class of functions f defined on $\mathbb{R}$ that can be extended to the complex plane as entire functions satisfying the inequality $|f(z)| \le e^{\sigma |\operatorname{Im} z|}$. Then

$$\bar{H}_\varepsilon(\mathcal{B}_\sigma) \sim \frac{2\sigma}{\pi} \log \frac{1}{\varepsilon} \ .$$

This result (Kolmogorov and Tikhomirov [1959]) was motivated by the sampling theorem in signal processing. See also Buslaev and Vitushkin [1974].

8.2. By means of a somewhat more precise calculation, with approximation by polynomials of *total degree* $\le n$ (instead of coordinate degree $\le n$ for $\mathcal{P}_n^s$) one obtains (Vitushkin [A-1959], Lorentz [A-1966]) for $\mathcal{A} := \mathcal{A}(D_1^s, D_r^s)$, $r > 1$:

$$H_\varepsilon(\mathcal{A}) \text{ and } C_\varepsilon(\mathcal{A}) = \frac{1}{(s+1)! \log(\prod_{j=1}^s r_j)} \log^{s+1}(1/\varepsilon) + O\big(\log^s(1/\varepsilon) \log\log(1/\varepsilon)\big) \ . \tag{8.1}$$

Vitushkin computes also the entropy of the classes of entire functions. For the family F_s^σ of entire functions $f(z)$, $z \in \mathbb{C}$, of order σ and class s, defined by $|f(z)| \le \exp(\sigma |z|^s)$, in the uniform norm on $|z| \le 1$, he gets

(8.2)
$$H_\varepsilon(F_s^\sigma) = \left(\log \frac{1}{\varepsilon}\right)^2 / \log\log\log \frac{1}{\varepsilon} + O\left(\left(\log \frac{1}{\varepsilon}\right)^2 \log\log \frac{1}{\varepsilon} / \log\log\log \frac{1}{\varepsilon}\right) .$$

8.3. The Birman-Solomyak estimate (5.10) has been extended to more general spaces of functions. For the unit balls of the Besov spaces one has

$$H_\varepsilon\big(\bar{B}_{q_1}^{\alpha_1}(L_{p_1}), B_{q_2}^{\alpha_2}(L_{p_2})\big) \sim (1/\varepsilon)^{\frac{N}{\alpha_1-\alpha_2}} . \tag{8.3}$$

Here the Besov spaces consist of functions defined on an arbitrary domain $A \subset \mathbb{R}^N$ with the sufficiently smooth boundary; the definitions extend those given in [**CA**, §10, Chapter 2] for $A \subset \mathbb{R}^1$. It is assumed that $p_1, p_2, q_1, q_2 \in (0, \infty]$, $0 < \alpha_1 < \alpha_2 < \infty$, and that $\alpha_1 - \alpha_2 - N(1/p_1 - 1/p_2)_+ > 0$. In this generality (8.2) is due to Edmunds and Triebel [1987]. In this paper and in Triebel [B-1978] the reader will find references to many related results on entropy and widths, with more general Besov-type spaces, Sobolev-type spaces of fractional smoothness, Orlicz spaces.

8.4. A number of authors studied entropy and widths of classes of functions with bounded mixed derivatives or differences. Here is a typical result (from Temlyakov [1989]). Let W_{p,r_1,r_2} be the class of functions $f(x,y)$ defined on $[0,1]^2$ and having partial derivatives for which $\|D_x^{\theta_1 r_1} D_y^{\theta_2 r_2} f\|_p \le 1$ for each of the four combinations of $\theta_1, \theta_2 = 0, 1$. Then for $1 \le p \le \infty$, $1 \le q < \infty$ and $r := \min\{r_1, r_2\} > 1$,

$$H_\varepsilon(W_{p,r_1,r_2}) \sim (1/\varepsilon)^{1/r} \big(\log(1/\varepsilon)\big)^{\nu(1+1/(2r))} ,$$

where $\nu = 0$ if $r_1 \ne r_2$, $\nu = 1$ if $r_1 = r_2$.

8.5. We shall briefly discuss the Kolmogorov complexity of a real continuous function f on $[0,1]$. It is defined by reduction to the complexity of Boolean functions, that is, functions ϕ transforming binary sequences of length m into binary sequences of length n, where m and n depend on ϕ. Every Boolean function ϕ can be represented in many ways as a composition of a certain number N of the 20 elementary functions for which $m \le 2$, $n = 1$. The *complexity* of ϕ, $K(\phi)$, is the minimal possible N in such representations of ϕ. For a continuous $f : [0,1] \to [-1,1]$, a function ϕ transforming binary numbers of the form $x = .\xi_1, \dots, \xi_m$ into binary numbers of the form $y = \pm.\eta_1, \dots, \eta_n$ is called an *ε-representation* of f if $\omega(f, 2^{-m}) < \varepsilon/2$, and $|f(x) - y| < \varepsilon/2$ for all x. Any such ϕ can be treated as a Boolean function and the *ε-complexity* of f, $K_\varepsilon(f)$, is defined as the minimal complexity of its ε-representations. For a class W, $K_\varepsilon(W) := \sup_{f \in W} K_\varepsilon(f)$.

There is a general relation between ε-complexity and ε-capacity: for every $W \subset C[0,1]$,

$$K_\varepsilon(W) \ge \text{const}\, C_{2\varepsilon}(W)/\log C_{2\varepsilon}(W)\ . \tag{8.4}$$

For functions of finite smoothness, the formula

$$K_\varepsilon(\bar{B}_p^r) \sim (1/\varepsilon)^{1/r}/\log(1/\varepsilon)\ , \qquad r = 1, 2, \ldots$$

was proved by Kolmogorov and Ofman for $p = \infty$ and by Makovoz [1986] for $1 \le p \le \infty$. The lower estimate here follows from (8.4) and (6.1). A survey of known results on Kolmogorov's complexity, with proofs, is given in Asarin [1984].

Chapter 16. Convergence of Sequences of Operators

§ 1. Introduction

The simplest theorem here is due to Korovkin [1953], [A-1960]. It concerns sequences T_n of positive linear operators on the space $C[a,b]$: *For the functions* $g_k(x) = x^k$, $k = 0,1,2$, *relations* $T_n(g_k) \to g_k$, $k = 0,1,2$ *in* $C[a,b]$ *imply convergence* $T_n(f) \to f$ *for any* $f \in C[a,b]$. We say that the functions $1, x, x^2$ are a *Korovkin set*, or a *convergence dominating* (or simply *dominating*) *set* for positive operators on $C[a,b]$. In [1953], Korovkin proved also that there are no sets of this type that consist of only two functions, and that a set of three continuous functions is dominating for $C[a,b]$ if and only if it is a Haar set.

For positive operators on $C(A)$, where A is a compact metric set, a useful sufficient condition has been given in [**CA**, Theorem 3.1, p.8]. Later, dominating sets have been discussed for different spaces, for example, for the L_p-spaces. This chapter consists of three different parts. In §§2–4, we discuss the beautiful *geometric theory*, due mainly to Shashkin ([1962], [1969]), but also to Wulbert [1968], Berens and Lorentz [1975], which tries to describe dominating convergence sets $S := \{g_0, g_1, \ldots, g_m\}$ on $C(A)$ by means of properties of related convex sets in $\mathbb{R}^{m+1}$. Sections §§5–6 deal with the *analytic theory* for positive operators, which has been developed simultaneously and independently by Bauer [1973] and Berens and Lorentz [1973] for *continuous functions*. In the second of these papers, this leads to characterization of dominating sets of continuous functions in *Banach lattices* X. Continuity is a desirable property for elements of S in any space $X \supset C(A)$. Going further, Donner [A-1982] describes dominating sets S in L_p that consist of arbitrary elements of this space. The last section §8 deals with dominating sets of contractions and of positive contractions on X. Sections §§2–4 and §§5–7 can be read independently. Another possible shortcut for the reader is to study §§2–4 only for the positive operators $\mathcal{T}^+$. He then could omit, for example, Propositions 3.2, 3.8, 3.10, Theorems 3.3, 3.9, 3.11.

An abstract formulation of the problems before us is as follows. Let X be a Banach space or a Banach lattice. Let $\mathcal{T}$ be a fixed subset of the set of all continuous linear operators on X, S a fixed subset of X. The *convergence set* $C(T_n)_1^\infty$ for a sequence $T_n \in \mathcal{T}$ is the set $\{f \in X : T_n f \to f\}$. The *shadow* $\Sigma(S)$ is the set of all $f \in X$ with the property that for each sequence $T_n \in \mathcal{T}$, the set of relations $T_n g \to g$, $g \in S$ implies $T_n f \to f$. In other words,

$$\Sigma(S) = \bigcap_{\substack{T_n \in \mathcal{T} \\ T_n g \to g,\, g \in S}} C(T_n)_1^\infty . \tag{1.1}$$

We shall study three classes of operators: $\mathcal{T} = \mathcal{T}^+$ are *positive operators* $T \geq 0$ on X; $\mathcal{T} = \mathcal{T}_1$ are *contractions* T with $\|T\| \leq 1$; $\mathcal{T} = \mathcal{T}_1^+$ are *positive contractions*. Many theorems will be formulated and proved for the three cases at once. Thus, $\Sigma(S)$ could stand for the three types of shadows, $\Sigma^+(S)$, $\Sigma_1(S)$, $\Sigma_1^+(S)$.

Our problem is to find $\Sigma(S)$ for given S; if $\Sigma(S) = X$, then S is a *convergence dominating set*. In the next three sections, X will be the space $C(A)$ of all real continuous functions on a compact metric space A. For positive linear operators $T \in \mathcal{T}^+$ one has $|Tf| \leq T(|f|)$; moreover, if 1_A is the constant function equal to 1 on A,

$$\|T\| = \|T1_A\| . \tag{1.2}$$

Similar remarks apply to positive linear functionals.

By $\mathcal{M}^*(A)$, we denote the space of all real regular Borel measures μ on A. The norm of $\mu \in \mathcal{M}^*$ is its total variation $\int |d\mu|$ on A. To $\mu \in \mathcal{M}^*(A)$ corresponds the functional

$$\mu(f) = \int f \, d\mu .$$

According to the Riesz representation theorem, this relation establishes a linear isomorphism between $\mathcal{M}^*(A)$ and the dual space of $C(A)$. We denote by $\mathcal{M}(A)$ the space of all probability measures μ on A, that is of positive μ satisfying $\mu(A) = 1$. Equivalently, they are characterized by the relation $\mu(1_A) = 1 = \|\mu\|$. Corresponding to the classes of operators $\mathcal{T}$, we define: $\mathcal{L}^+$ as the cone of positive measures in $\mathcal{M}^*$, $\mathcal{L}_1$ as the unit ball in $\mathcal{M}^*$, and $\mathcal{L}_1^+ := \mathcal{L}^+ \cap \mathcal{L}_1$. $\mathcal{L}$ will stand for one of the three classes.

For given A and $\mathcal{T}$, we denote by $M(A, \mathcal{T})$ the smallest cardinal number of a dominating set $S \subset C(A)$ with respect to $\mathcal{T}$. One proves without difficulty:

Theorem 1.1 *Let Φ be a homeomorphic mapping of A onto another compact metric space A'. If S is a convergence dominating set for $C(A)$ and the class $\mathcal{T}$, then also $S' = \{g' = g \circ \Phi^{-1} : g \in S\}$ is a dominating set for $C(A')$ and the same class $\mathcal{T}$.*

Hence, $M(A, \mathcal{T})$ is a topological invariant of A.

§ 2. Simple Necessary and Sufficient Conditions

For a set $S \subset C(A)$, we shall denote by $G := \operatorname{lin} S$ its linear hull in $C(A)$, by $\overline{G} := \overline{\operatorname{lin}}\, S$ its closed linear hull, and by G^* the dual space of G.

To each of the three possible sets $\mathcal{L}$ and each $x \in A$ we make correspond the set of functionals

$$\mathcal{L}_x(S) := \{\mu \in \mathcal{L} : \mu(g) = g(x),\ g \in S\} .$$

The evaluation functional ε_x, defined by $\varepsilon_x(f) := f(x)$, obviously belongs to $\mathcal{L}_x$, but this set may contain other functionals. Let ℓ_x be the restriction of ε_x to G. Then $\mathcal{L}_x$ consists exactly of all those functionals $\mu \in \mathcal{L}$ that are extensions of ℓ_x. Considering separately $\mathcal{L}^+, \mathcal{L}_1, \mathcal{L}_1^+$, we easily prove:

$$\text{(2.1)} \qquad \begin{cases} \textit{If } 1_A \in S, \textit{ then for each } x \in A, \\ \mathcal{L}_x = \mathcal{M}_x(S) := \{\mu \in \mathcal{M} : \mu(g) = g(x)\,,\ g \in S\} . \end{cases}$$

We shall need some lemmas about the class $\mathcal{L}^+$.

Lemma 2.1 *The linear hull G of $S \subset C(A)$ contains a strictly positive function exactly when the zero measure is the only measure in $\mathcal{L}^+$ that annihilates G.*

Proof. The condition is clearly necessary. To prove its sufficiency, assume that G does not contain a strictly positive function. The representation of functions from $G_1 := G + \mathbb{R}1_A$ in the form $g + \lambda 1_A$ is unique. It follows that the functional

$$\ell(g + \lambda 1_A) := \lambda$$

is well-defined on G_1. It annihilates G and is positive, since $g + \lambda 1_A \ge 0$ implies $\lambda \ge 0$. We extend ℓ by means of the Hahn-Banach theorem to a measure μ on A, and have $|\mu| = \|\ell\| = \ell(1_A) = 1$, so that $\mu(A) = 1$ or, equivalently, $\mu \in \mathcal{L}^+$ and $\mu \ne 0$. □

Lemma 2.2 *Let S be a subset of $C(A)$ that satisfies for some $x \in A$ the condition that the measures $\mathcal{L}_x^+$ consist only of the evaluation functional ε_x:*

$$\text{(2.2)} \qquad \mathcal{L}_x^+(S) = \{\varepsilon_x\} .$$

Then the linear hull G of S contains a strictly positive function.

Proof. Otherwise, according to Lemma 2.1, there exists a measure $\mu_0 \in \mathcal{L}^+$, $\mu_0 \ne 0$ that annihilates G. Then $\varepsilon_x \ne \varepsilon_x + \mu_0 \in \mathcal{L}_x^+(S)$, contradicting (2.2). (We shall see in §4 that this is not necessarily true for $\mathcal{T}_1$ and for $\mathcal{T}_1^+$.)

Lemma 2.3 *Let (2.2) be satisfied for some $x \in A$, and let $(\mu_n : n = 1, 2, \ldots)$ be a sequence in $\mathcal{L}^+$ for which $\lim_n \mu_n(g) = g(x)$ for all $g \in S$. Then the norms $\|\mu_n\|$ are bounded.*

Proof. Let g_0 be a strictly positive function in G, and let $g_0(x) \ge c > 0$ for all $x \in A$. Then

$$\|\mu_n\| = \mu_n(1_A) \le (1/c)\mu_n(g_0) \to (1/c)g_0(x) . \qquad \square$$

The following statement is true for all three cases, $\mathcal{T}^+, \mathcal{T}_1$ and $\mathcal{T}_1^+$:

Theorem 2.4 *Let $S \subset C(A)$ and let $x \in A$ be given. Then the condition*

$$\mathcal{L}_x(S) = \{\varepsilon_x\} \tag{2.3}$$

is necessary and sufficient in order that for each sequence $(\mu_n : n = 1, 2, \ldots)$ in $\mathcal{L}$, relations

$$\lim_n \mu_n(g) = g(x) \ , \qquad g \in S \tag{2.4}$$

should imply

$$\lim_n \mu_n(f) = f(x) \ , \qquad f \in C(A) \ . \tag{2.5}$$

Proof. The necessity of the condition is obvious: If $\mu_0 \in \mathcal{L}_x$, $\mu_0 \neq \varepsilon_x$, then the consideration of the sequence $(\mu_n := \mu_0 : n = 1, 2, \ldots)$ leads to a contradition.

To prove the sufficiency, let (2.3) and (2.4) be satisfied. We establish (2.5) by using the weak $*$ topology in $\mathcal{M}^*(A)$. Let $(\mu_{n_k} : k = 1, 2, \ldots)$ be an arbitrary subsequence of (μ_n). The sequence of norms $\|\mu_{n_k}\|$ is bounded; for the case $\mathcal{L} = \mathcal{L}^+$ this follows from Lemma 2.3. The weak $*$ compactness of balls in $\mathcal{M}^*(A)$ and the separability of $C(A)$ imply the existence of a subsequence $n'_k \to \infty$ and of an element $\mu_0 \in \mathcal{M}^*(A)$ for which $\mu'_{n_k} \to \mu_0$ in the weak $*$ topology. Since $\mathcal{L}$ is weak $*$ closed, $\mu_0 \in \mathcal{L}$. Now (2.4) implies $\mu_0(g) = g(x)$, $g \in S$, or $\mu_0 \in \mathcal{L}_x$. By (2.3) we have $\mu_0 = \varepsilon_x$. □

The main theorem of this section is as follows:

Theorem 2.5 *In each of the three cases, $\mathcal{T}^+$, $\mathcal{T}_1$, $\mathcal{T}_1^+$, a set $S \subset C(A)$ is convergence dominating if and only if*

$$\mathcal{L}_x(S) = \{\varepsilon_x\} \ \text{ for all } \ x \in A \ . \tag{2.6}$$

More exactly, (i) *if* (2.6) *is satisfied, then for a sequence (T_n) in $\mathcal{T}$, the relations*

$$\lim_{n \to \infty} T_n(g) = g \ , \qquad g \in S \tag{2.7}$$

imply

$$\lim_{n \to \infty} T_n(f) = f \ , \qquad f \in C(A) \ . \tag{2.8}$$

(ii) *If* (2.6) *is not satisfied, then even the pointwise convergence*

$$\lim_{n \to \infty} T_n(f)(x) = f(x) \ , \qquad x \in A \tag{2.9}$$

is not implied by (2.7) *for some function $f_0 \in C(A)$.*

Proof. (i) Assume that there is a sequence $T_n \in \mathcal{T}$ with the properties $T_n g \to g$ for $g \in S$ and $T_n f_0 \not\to f_0$ for some $f_0 \in C(A)$. Then there exists an $\varepsilon > 0$, a sequence $(n_k : k = 1, 2, \ldots)$ and a sequence of points $(x_k : k = 1, 2, \ldots)$ in A for which

$$\varepsilon \leq |(T_{n_k} f_0)(x_k) - f_0(x_k)| , \qquad k = 1, 2, \ldots \ .$$

Since A is compact, we can assume that x_k converges, say, to $x_0 \in A$. We define the sequence (μ_k) of functionals by means of the formula

$$\mu_k(f) = T_{n_k} f(x_k) \ .$$

The functionals μ_k belong to $\mathcal{L}$ and satisfy

$$\lim_k \mu_k(g) = g(x_0) \ , \qquad g \in S \ .$$

According to Theorem 2.4, $\mu_k \to \varepsilon_{x_0}$ weak $*$ for $k \to \infty$. In particular, $\mu_k(f_0) \to f_0(x_0)$ for $k \to \infty$, and this is a contradiction.

(ii) We assume that there exists a point $x_0 \in A$ and a functional $\mu_0 \in \mathcal{L}_{x_0}(S)$, $\mu_0 \neq \varepsilon_{x_0}$.

Let ρ be the metric on A. For each $n = 1, 2, \ldots$ we select a function $\phi_n \in C(A)$ with the properties $0 \leq \phi_n(x) \leq 1$, $\phi_n(x_0) = 1$, and $\phi_n(x) = 0$ for $\rho(x, x_0) \geq 1/n$. We define

$$T_n f(x) = \mu_0(f) \cdot \phi_n(x) + f(x) \cdot \left[1 - \phi_n(x)\right] , \qquad n = 1, 2, \ldots \ .$$

This sequence (T_n) belongs to $\mathcal{T}$, and for each $g \in S$

$$\lim_n T_n g(x) = g(x) \ \text{ uniformly on } \ A \ .$$

On the other hand, there exists a function $f_0 \in C(A)$, for which $\mu_0(f_0) \neq f_0(x_0)$. For this function we have

$$T_n f_0(x_0) = \mu_0(f_0) \neq f_0(x_0) \ , \qquad n = 1, 2, \ldots \ . \qquad \square$$

From this theorem it follows that a convergence dominating set S with respect to $\mathcal{T}^+$, $\mathcal{T}_1$, $\mathcal{T}_1^+$ must satisfy:

(2.10) *S must separate points of* A: for each pair $x, x' \in A$, there is a function $g \in S$ for which $g(x) \neq g(x')$.

(2.11) *S must not vanish on* A: for each $x \in A$, there must exist a function $g \in S$ with $g(x) \neq 0$.

To prove (2.10), we compare ε_x and $\varepsilon_{x'}$; to prove (2.11), we do this for ε_x and 0. Sets S satisfying (2.10) and (2.11) we shall call *admissible*.

With the help of (2.1) we have:

Corollary 2.6 *If* $1_A \in S$, *then condition* (2.6) *in Theorem* 2.5 *can be replaced by*

(2.12) $$\mathcal{M}_x(S) = \{\varepsilon_x\} \ \textit{for all} \ \ x \in A \ .$$

We write S_0 instead of S if it is known that this set contains the function 1_A. In the case of $\mathcal{T}^+$, this can be assumed without loss of generality.

Lemma 2.7 *If S is a convergence dominating set with respect to $\mathcal{T}^+$ in $C(A)$, then the same holds for the $S' := \{g' = g_0 g : g \in S\}$, where g_0 is an arbitrary strictly positive function in $C(A)$.*

Proof. We select a point $x \in A$ and have to show that $\mathcal{L}_x^+(S') = \{\varepsilon_x\}$. Let $\mu_0 \in \mathcal{L}_x^+(S')$. Then

$$\mu_0(g_0 g) = g_0(x) g(x) \ , \qquad g \in S \ .$$

The measure γ, defined by

$$\gamma(f) = \frac{1}{g_0(x)} \mu_0(g_0 f)$$

is positive and belongs to $\mathcal{L}_x^+(S)$. But according to Theorem 2.5 the last set consists only of the functional ε_x. Thus, $\gamma = \varepsilon_x$ and, consequently, $\mu_0 = \varepsilon_x$. □

Theorem 2.8 *If S is a convergence dominating set of order m with respect to $\mathcal{T}^+$ in $C(A)$, then there exists another such set S which contains 1_A and has order not exceeding m.*

Proof. According to Theorem 2.5 and Lemma 2.2, the set $G = \operatorname{lin} S$ contains a strictly positive function g_0. By Lemma 2.7, the set $S' = \{g' = g/g_0 : g \in S\}$ is also a dominating set. The dimension of $G' = \operatorname{lin} S'$ is the same as that of G, hence at most $m + 1$, and $1_A \in G'$. We can select for G' a basis S_0 of at most $m + 1$ functions, which contains 1_A. This will be the required set. □

Example 1. For each of the three cases, $\mathcal{T}^+, \mathcal{T}_1, \mathcal{T}_1^+$ on $C(A)$ the restriction of a convergence dominating set S to a compact subset $A_1 \subset A$ is also dominating. — For otherwise there would exist two measures $\nu_1 \neq \nu_2$ on A_1 which represent the functional ε_{x_1} for some $x_1 \in A_1$ on G. Then the measures $\mu_i := \nu_i$ on A_1, and $:= 0$ on $A \setminus A_1$ would do the same for G on A.

Example 2. For $\mathcal{T}^+$ on $C[a, b]$ there does not exist a dominating set of only two functions g_0, g_1. — Assume it does. Without loss of generality, let $g_0 = 1$, then g_1 is not constant, and there are three points with $g_1(x_1) < g_1(x_2) < g_1(x_3)$. For some $\alpha, \beta > 0$, $\alpha + \beta = 1$, $g_1(x_2) = \alpha g_1(x_1) + \beta g_1(x_3)$, and then $\alpha \varepsilon_{x_1} + \beta \varepsilon_{x_3} = \varepsilon_{x_2}$ on G.

Example 3. The subspace $C_1 \subset C[a, b]$ consisting of all piecewise linear functions is dense in $C[a, b]$, but does not possess a finite dimensional dominating subset G. — Indeed, on some subintervals of $[a, b]$, G would be two-dimensional.

§ 3. Geometric Properties of Dominating Sets

We shall discuss some geometric properties of subsets $S \subset C(A)$ that are characteristic for their convergence dominating property. We first direct our attention to sets S_0 that separate points in the sense of (2.10) and contain the function 1_A, which are, therefore, admissible. We shall see that in this case the same conditions and constructions work for all three possible classes $\mathcal{T}$ of operators. For general S, however, the conditions will be slightly different in all three cases.

As before, let $G = \operatorname{lin} S_0$, $\overline{G} = \overline{\operatorname{lin}} S_0$, and let $\ell_x(g) = g(x)$, $g \in G$. Of basic importance will be the *Shashkin map*

$$\Phi : x \to \ell_x \,, \qquad x \in A \tag{3.1}$$

which sends points x of A into functionals $\ell_x \in G^*$. We denote by A^* the image of A under Φ in G^*. If G^* is equipped with the weak $*$ topology, Φ is continuous, hence A^* is weak $*$ compact as the image of the compact set A. If G separates points, then this map is one-to-one, and Φ is a homeomorphism.

Let

$$K_0 = \overline{\mathrm{co}} A^* \tag{3.2}$$

be the weak $*$ closed convex hull of A^* in G^*. Also K_0 is weak $*$ compact, moreover we have

Lemma 3.1 *If G separates points and contains the function 1_A, then $K_0 = \{\ell \in G^* : \ell(1_A) = 1 = \|\ell\|\}$.*

Proof. If K_1 is the right-hand side of the equation, then obviously $K_0 \subset K_1$. Assume that there exists an $\ell_0 \in K_1 \setminus K_0$, then ℓ_0 and K_0 can be separated by an element $g \in G$, so that

$$\sup\{g(x) : x \in A\} \le \sup\{\ell(g) : \ell \in K_0\} < \ell_0(g) \,. \tag{3.3}$$

This inequality is not destroyed if a constant is added to g. Selecting this constant properly, we will have $\|g\| = \sup\{g(x) : x \in A\}$, and then (3.3) is a contradiction since $\ell_0(g) \le \|g\|$. □

A point $x_0 \in K$ of a compact convex set K in a Banach space X, is called an *extremal point*, if x_0 is not an interior point of an interval contained in K, that is, if the relation

$$\alpha x + \beta x' = x_0 \,, \quad \alpha, \beta > 0 \,, \quad \alpha + \beta = 1 \,, \quad x, x' \in K \,, \quad x, x' \ne x_0 \tag{3.4}$$

is impossible. A variation of this are *exposed points* $x_0 \in K$ which have the property that some continuous linear functional ℓ on X attains its strict maximum on K at x_0:

$$\ell(x_0) = a \,, \quad \ell(x) < a \,, \quad x \in K \,, \quad x \ne x_0 \,. \tag{3.5}$$

A point with this property cannot satisfy (3.4): each exposed point is an extremal point,

$$\exp K \subset \operatorname{ext} K \ . \tag{3.6}$$

According to the theorem of Krein-Milman (see Royden [B-1968, p.207]), K is the convex hull of its extremal points; in addition, if $K = \overline{\text{co}} M$, then each extremal point of K belongs to the closure of M.

We shall use the following proposition of Phelps [B-1966, p.38]:

Proposition 3.2 *Let $x \in A$, then the functional ℓ_x is an extremal point of K_0 if and only if*

$$\mathcal{M}_x(S_0) = \{\varepsilon_x\} \ . \tag{3.7}$$

Proof. First assume that (3.7) is violated. Then the functional ℓ_x has an extension $\mu \in \mathcal{M}_x$ that is not identical with ε_x. Because the measure μ is regular, there exists a compact set $D \subset A \setminus \{x\}$, for which $\mu(D) > 0$. Since D is compact, there exists a point $x_0 \in A$ and a decreasing sequence $(D_k : k = 1, 2, \ldots)$ of compact sets for which

$$\bigcap_{k=1}^{\infty} D_k = \{x_0\} \quad \text{and} \quad \lambda_k = \mu(D_k) > 0 \ , \qquad k = 1, 2, \ldots \ .$$

If we had $\lambda_k = 1$ for all k, then it would follow that $\mu = \varepsilon_{x_0}$, hence $\ell_k = \ell_{x_0}$, which contradicts (2.10). Hence, $\lambda_k < 1$ for all large k.

We define the sequence of probability measures

$$\mu_k(B) = \lambda_k^{-1} \cdot \mu(B \cap D_k) \ , \qquad k = 1, 2, \ldots , \quad B \in \mathcal{B} \ ,$$

where $\mathcal{B}$ is the class of Borel subsets of A. This sequence converges weak $*$ in $\mathcal{M}^*(A)$ to ε_{x_0}. The restriction ℓ_k of μ_k to G belongs to K_0 and satisfies

$$\ell_k \to \ell_{x_0} \neq \ell_x \ \text{ for } \ k \to \infty \ ,$$

weak $*$ in G^*. We fix a k for which $\ell_k \neq \ell_x$, $\lambda_k < 1$, and define

$$\mu_k'(B) = (1 - \lambda_k)^{-1} \mu(B \cap D_k') \qquad k = 1, 2, \ldots , \quad B \in \mathcal{B} \ ,$$

where D_k' is the complement of D_k in A. The relation

$$\mu(B) = \mu(B \cap D_k) + \mu(B \cap D_k') = \lambda_k \mu_k(B) + (1 - \lambda_k)\mu_k'(B)$$

yields

$$\ell_x = \lambda_k \ell_k + (1 - \lambda_k)\ell_k' \ ,$$

with $\ell_k, \ell_k' \in K_0$, $0 < \lambda_k < 1$, and $\ell_k \neq \ell_x$. This means that ℓ_x is not an extremal point of K_0.

Conversely, let ℓ_x not be an extremal point of K_0, then $\ell_x = \lambda \ell_1 + (1-\lambda)\ell_2$ for $0 < \lambda < 1$ and some $\ell_1, \ell_2 \in K_0$, $\ell_1 \neq \ell_x$. Let $\mu_1, \mu_2 \in \mathcal{M}$ be any two extensions of ℓ_1, ℓ_2, respectively, and let $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$. Since $\mu_1 \neq \varepsilon_x$, we have $\mu_1(\{x\}) < 1$, hence also $\mu(\{x\}) < 1$. Thus we obtained a measure μ in $\mathcal{M}_x(S_0)$ that is different from ε_x. □

Let S_0 be as above, and let $G = \operatorname{lin} S_0$. The set of points $x \in A$ for which ℓ_x is an extremal point of K_0, is called the *Choquet boundary*, $\partial_{\mathrm{ch}} G$, of G.

With this terminology we obtain from Corollary 2.6 and Proposition 3.2 the definitive description of dominating sets S_0. With finite S_0 it has been given by Shashkin [1962] for $\mathcal{T} = \mathcal{T}^+$; by Wulbert [1968] and Shaskin [1969] for $\mathcal{T} = \mathcal{T}_1$; for arbitrary S_0, also for $\mathcal{T} = \mathcal{T}_1^+$ by Berens and Lorentz [1975].

Theorem 3.3 *Let S_0 be a subset of $C(A)$ that separates points of A and contains the function 1_A. Then S_0 is a convergence dominating set for $\mathcal{T} = \mathcal{T}^+, \mathcal{T}_1$ or $\mathcal{T}_1^+$ exactly when*

$$\partial_{\mathrm{ch}} G = A\ , \qquad G = \operatorname{lin} S_0 \tag{3.8}$$

or, equivalently, when (3.7) *holds for all $x \in A$.*

Following Bishop, we call a point $x_1 \in A$ a *peak point* of G if there exists a $g_1 \in G$ for which $g_1(x_1) = \|g_1\|$ and $|g_1(x)| < \|g_1\|$ for $x \neq x_1$. This implies that for each $\mu \in \mathcal{M}_{x_1}(G)$, $\mu(g_1) = \int g_1\, d\mu = \|g_1\|$, hence $\mu = \varepsilon_{x_1}$, and by Proposition 3.2, $x_1 \in \partial_{\mathrm{ch}} G$. For the set $p(G)$ of all peak points of G we have therefore

$$p(G) \subset \partial_{\mathrm{ch}} G\ . \tag{3.9}$$

We shall now discuss arbitrary sets S; we can assume that S is admissible, that is, it satisfies both conditions (2.10), (2.11). Let

$$\begin{aligned} K^+ &:= \overline{\operatorname{cone}}(A^*)\ , \quad K_1 := \overline{\operatorname{co}}\{A^* \cup (-A^*)\} \\ K_1^+ &:= \overline{\operatorname{co}}\{A^* \cup \{0\}\} \end{aligned} \tag{3.10}$$

be the weak$*$ closed convex cone or set generated by A^*, $A^* \cup (-A^*)$ or $A^* \cup \{0\}$ in G, respectively, and let K stand for any of these sets.

Theorem 3.4 (i) *We have*

$$\begin{aligned} K^+ &= \{\ell \in G^* : \ell(g) \geq 0 \text{ whenever } g(x) \geq 0 \text{ on } A\}\ , \\ K_1 &= \{\ell \in G^* : |\ell(g)| \leq 1 \text{ whenever } |g(x)| \leq 1 \text{ on } A\}\ . \end{aligned}$$

(ii) *In all three cases K consists of the restrictions to G of all functionals in $\mathcal{L}$.*

Proof. To establish (i), we denote the set of $\ell \in G^*$, described in (i), by K_0'. Clearly, K_0' is weak $*$ closed and convex, and contains K. Assume that there exists an $\ell_0 \in K_0' \setminus K$. Then ℓ_0 can be separated from K by means of a hyperplane in G^*, given by some $g_0 \in G$. This means that

$$\sup\{\ell(g_0) : \ell \in K\} < \ell_0(g_0) \ . \tag{3.11}$$

This leads to a contradiction. In the case $\mathcal{T}^+$, (3.11) means that $\lambda g_0(x) < \ell_0(g_0)$ for all $\lambda \geq 0$ and all $x \in A$. Consequently, $g_0(x) \leq 0$ for all x and $\ell_0(g_0) \leq 0 < \ell_0(g_0)$. In case $\mathcal{T}_1$, one gets $|g_0(x)| < \ell_0(g_0)$ for all x, or $\|g_0\| < \ell_0(g_0) \leq \|g_0\|$.

Statement (ii) for K_1 and K^+ follows from (i). For K_1^+ we note that K_1^+ is contained in $\mathcal{L}_1^+$ restricted to G. If the inclusion would be proper, there would exist a $\mu_0 \in \mathcal{L}_1^+$ and a $g_0 \in G$ so that $g_0^+(x) \leq \mu_0(g_0) \leq \mu_0(g_0^+) \leq \|g_0^+\|$, a contradiction. □

Example. For K_1^+, the analogy to (i) breaks down: one cannot define this set to be equal to

$$\{\ell \in G^* : 0 \leq \ell(g) \leq 1 \text{ whenever } 0 \leq g(x) \leq 1 \text{ on } A\} = K^+ \cap K_1 \ .$$

Indeed, let $A := [-1, -1/4] \cup [1/4, 1/2]$, $g_0(x) := x$, $x \in A$, and $G = \mathrm{lin}\{g_0\}$. Plainly, G is admissible. If ℓ_x in G^* denotes the evaluation functional at $x \in A$, then $K^+ = G^*$, $K_1 = \{\lambda \ell_{-1} : -1 \leq \lambda \leq 1\}$, while $K_1^+ = \{\lambda \ell_{-1} + (1-\lambda)\ell_{1/2} : 0 \leq \lambda \leq 1\}$.

From now on, we shall treat the three cases separately. Here are some simple necessary conditions that must be satisfied by S in the case $\mathcal{T}^+$. Let R_x be the ray in K^+ generated by $x \in A$: $R_x = \{\lambda \ell_x : \lambda \geq 0\}$. From condition (2.10) it follows, as a necessary condition for a dominating set S:

$$R_x \neq R_{x'} \ \text{ if } \ x \neq x' \ , \quad x, x' \in A \ . \tag{3.12}$$

Another simple necessary condition is given by

Lemma 3.5 *The cone K^+ is acute, that is, it satisfies $K^+ \cap (-K^+) = \{0\}$ if and only if G contains a strictly positive function g_0.*

Proof. It is clear that the condition is sufficient. To prove its necessity, let J be the intersection of K^+ with the unit sphere in A^*, and let K be its weak $*$ closed convex hull; K is weak $*$ compact. The assumption is equivalent to the statement that 0 is an extreme point of K^+. Then 0 does not belong to K. For otherwise it would be an extreme point of K, which is impossible, since by Krein-Milman's theorem all extreme points of K belong to A^*. Then we can separate 0 and K by a hyperplane in G^* with an equation $\ell(g_0) = 1$, $\ell \in G^*$. The function g_0 is obviously strictly positive. □

A ray R of a cone K with vertex in 0 is called *extreme*, if whenever $\ell \in R$ and $\ell = \lambda \ell_1 + (1-\lambda)\ell_2, \ell_1, \ell_2 \in K$, then $\ell_1, \ell_2 \in R$. We have:

Proposition 3.6 *Assume that the cone K^+ is acute and that* (3.12) *is satisfied. Then a ray R_x is an extreme ray of K^+ exactly when*

$$\mathcal{L}_x^+(S) = \{\varepsilon_x\} \ . \tag{3.13}$$

Proof. The proof is similar to that of Proposition 3.2, and we shall omit the details. First let (3.13) be violated. Then there exists a measure $\mu \in \mathcal{L}_x^+$, $\mu \neq \varepsilon_x$. If the mass of μ is concentrated at x, then $\mu = \alpha\varepsilon_x$, and applying this to a $g_0 \in G$, $g_0(x) \neq 0$, we obtain $\alpha = 1$. Hence, there exists a point $x_0 \neq x$ and a decreasing sequence $(D_k : k = 1, 2, \ldots)$ of compact subsets, not containing x, whose intersection is $\{x_0\}$, with the property that $\lambda_k = \mu(D_k) > 0$ for all k. If for all k, λ_k is the constant $\lambda_0 = \mu(A) > 0$, then $\mu = \lambda_0\varepsilon_{x_0}$. Restriction to G would give $\ell_x = \lambda_0\ell_{x_0}$, contrary to the assumption (3.12). Hence, $\lambda_k < \lambda_0$ for all large k. We define the measures in $\mathcal{L}^+$,

$$\mu_k(B) = \lambda_k^{-1}\mu(B \cap D_k) \ , \qquad \mu_k'(B) = (\lambda_0 - \lambda_k)^{-1}\mu(B \cap D_k')$$

for any Borel measurable set B, and denote the restriction of the corresponding functionals by ℓ_k, ℓ_k'. Since $\mu_k \to \varepsilon_{x_0}$ weak $*$ in $\mathcal{M}^*(A)$, and consequently $\ell_k \to \ell_{x_0} \notin R_x$ weak $*$ in G^*, we have $\ell_k \notin R_x$ for all large k. The relation between μ_k, μ_k', μ yields

$$\ell_x = \frac{\lambda_k}{\lambda_0}\lambda_0\ell_k + \left(1 - \frac{\lambda_k}{\lambda_0}\right)\lambda_0\ell_k' \ , \qquad 0 < \frac{\lambda_k}{\lambda_0} < 1 \ .$$

This means that ℓ_x is an interior point of the interval $[\lambda_0\ell_k, \lambda_0\ell_k']$ with end points in K^+, which is not contained in R_x. Thus, R_x is not an extremal ray of K^+. □

From Theorem 2.5 and the necessity of conditions (i) and (ii) below, we obtain

Theorem 3.7 *An admissible set S in $C(X)$ is a convergence dominating set with respect to $\mathcal{T}^+$ exactly when the following conditions are satisfied:*

$$(B^+) \qquad \begin{array}{ll} \text{(i)} & R_x \neq R_{x'} \ \textit{for} \ x \neq x', x, x' \in A \\ \text{(ii)} & K^+ \cap (-K^+) = \{0\} \\ \text{(iii)} & \textit{for each } x \in A,\ R_x \textit{ is an extremal ray of } K^+ . \end{array}$$

We turn our attention to the case $\mathcal{T}_1$. A necessary condition for $S \subset C(X)$ to be a dominating set is

$$(3.14) \qquad A^* \cap (-A^*) = \emptyset \ .$$

For otherwise $\ell_x = -\ell_{x'}$ for some $x \neq x'$, and then $\mathcal{L}_{1,x}$ contains the element $-\varepsilon_{x'}$, different from ε_x.

The transformation $\ell \to -\ell$ is an isomorphism of K_1 onto itself; it maps A^* onto $-A^*$. It follows that ℓ_x and $-\ell_x$ are at the same time extreme points of K_1 or they are not.

Proposition 3.8 *Let* (3.14) *be satisfied. For a point $x \in A$,*

$$(3.15) \qquad \mathcal{L}_{1,x}(S) = \{\varepsilon_x\}$$

holds if and only if ℓ_x is an extreme point of K_1.

Proof. (Similar to the proofs of Propositions 3.2 and 3.6.) Assume that there exists a $\mu \in \mathcal{L}_{1,x}$, $\mu \neq \varepsilon_x$. We have $\|\mu\| = 1$, $|\mu|(A) = 1$. If the total mass of μ is concentrated at x, then $|\mu| = \varepsilon_x$; hence, $\mu = \pm\varepsilon_x$ (even if all $g \in G$ vanish at x). Hence, there exists a compact set $D_0 \subset A \setminus \{x\}$ with $\mu(D_0) \neq 0$. By Hahn's decomposition theorem, we can find a compact subset $D \subset D_0$, $\mu(D) \neq 0$, on which μ is positive (or negative). Using (3.14) we can prove (in a way similar to the proof of Proposition 3.2) that ℓ_x is a nontrivial convex combination of points of K_1. The proof of the inverse is easy. □

From this and Theorem 2.5 we obtain:

Theorem 3.9 *An admissible set S in $C(A)$ is a Korovkin set with respect to $\mathcal{T}_1$ if and only if*

(B_1) *it satisfies* (3.14) *and in addition all points of A^* are extreme points of K_1.*

By Krein-Milman's theorem, the last condition is equivalent to

$$\operatorname{ext} K_1 = A^* \cup (-A^*) \ . \tag{3.16}$$

We formulate the results for $\mathcal{T}_1^+$ without proof.

Proposition 3.10 *An element $\ell_x \in A^*$ is an extreme point of K_1^+ exactly when*

$$\mathcal{L}_{1,x}^+(S) = \{\varepsilon_x\} \ .$$

Theorem 3.11 *An admissible subset S of $C(A)$ is a dominating subset for $\mathcal{T}_1^+$ exactly when each point of A^* is an extreme point of K_1^+, that is,*

$$\operatorname{ext} K_1^+ \supset A^* \ . \tag{B_1^+}$$

§ 4. Strict Dominating Systems; Minimal Systems; Examples

If we do not assume that $1_A \in G$, the definition of the boundary $\partial_{\mathrm{ch}}G$ needed in our theory should be slightly different from $\partial_{\mathrm{ch}}G$. In each of the three cases, $\mathcal{T}^+, \mathcal{T}_1, \mathcal{T}_1^+$, we define the *boundary* (or *general boundary*) of G by

$$\partial G = \{x \in A : \mathcal{L}_x = \{\varepsilon_x\}\} \ , \tag{4.1}$$

obtaining $\partial^+ G$, $\partial_1 G$ and $\partial_1^+ G$ with the proper choice of $\mathcal{L}_x$. (If $1_A \in G$, property (2.1) implies that they are all identical and equal to the Choquet boundary $\partial_{\mathrm{ch}}G$.) Thus, Theorems 3.7, 3.9 and 3.11 can be summed up by saying that the set S is convergence dominating if and only if one has

$$\partial G = A \ , \tag{4.2}$$

and if some additional modest assumptions are satisfied; namely: for $\mathcal{T}^+$, G must satisfy (2.10) and (2.11), while for $\mathcal{T}_1$, there should not exist two points $x, x' \in A$ for which $g(x) = -g(x')$ for all $g \in S$.

Definitions of peak points are also different in the three cases. We say that $x_0 \in A$ is a *peak point for G and the class $\mathcal{T}$* if there is a function $g_0 \in G$ with the following properties:

(P^+) In case of $\mathcal{T}^+$, g_0 must satisfy $g_0(x_0) = 0$, $g_0(x) > 0$, $x \neq x_0$ (here, x_0 is a *zero minimum* point).

(P_1) For $\mathcal{T}_1$, the condition is $|g_0(x_0)| = 1$ and $|g_0(x)| < 1$, $x \neq x_0$ (a *maximum modulus* point).

(P_1^+) For $\mathcal{T}_1^+$, we require that $g_0(x_0) > 0$, $g_0(x) < g_0(x_0)$ for $x \neq x_0$ (a *positive maximum* point).

We leave it to the reader to prove:

Theorem 4.1 *If x_0 is a peak point for $\mathcal{T}$, then $\mathcal{L}_{x_0}(S) = \{\varepsilon_{x_0}\}$. The three definitions of peak points and their definition in §3 coincide if $1_A \in G$.*

If each point $x_0 \in A$ is a peak point, we automatically have all properties stated after (4.2). Therefore:

Theorem 4.2 *The set S is a convergence dominating set for $\mathcal{T}$ if for the corresponding set $p(G)$ of peak points,*

$$p(G) = A \ . \tag{4.3}$$

Sometimes the following is useful in determining the boundary.

Proposition 4.3 *Let $g_k \in G$, $k = 1, 2, \ldots$, and let A_k be the set where g_k vanishes. Assume that g_n is nonnegative on the set $A_1 \cap \cdots \cap A_{n-1}$ (for $n = 1$, this means $g_1 \geq 0$). If $\bigcap_k A_k = \{x_0\}$, then $x_0 \in \partial^+ G$.*

Proof. By induction one shows that if $\mu \in \mathcal{L}_{x_0}^+(G)$, then μ is concentrated on A_n. □

Here is an interesting application:

Example 4. Each Haar system S (with more than three functions) on $A = [a, b]$ or $A = \mathbb{T}$ is a $\mathcal{T}^+$-dominating set; moreover, $p^+(S) \supset (a, b)$ in the first case, $p^+(S) = \mathbb{T}$ in the second.

Proof. Let n be the number of functions in S. By a theorem of Krein [**CA**, Theorem 9.1, p.80] for given $x_0 \in A$, there exists a $g^0 \in G$ that vanishes at x_0 and satisfies $g^0(x) > 0$ for $x \neq x_0$. The only exception is when $A = [a, b]$,

$x_0 = a$ (or $x_0 = b$) and n is odd; in this case, the function g^0 could vanish also at the other end point of $[a, b]$. In this exceptional case, there is a function $g^{(1)} \in G$ which vanishes at x_0, and is > 0 at the other end point. Then Proposition 4.3 (with one or two sets A_k) gives the proof at once. □

Dominating sets with the property (4.3) are called *strict convergence dominating sets*. We shall give many examples of such sets.

Of main interest for us are *finite* dominating subsets of $C(A)$ of order m:

$$S = \{g_0, g_1, \dots, g_m\} . \tag{4.4}$$

We assume that the set A contains at least $m+1$ points and that the g_k are linearily independent. By means of the correspondence

$$g = \sum_{k=0}^{m} a_k g \leftrightarrow (a_0, a_1, \dots, a_m)$$

$$\ell(g) = \sum_{k=0}^{m} a_k \ell_k \leftrightarrow (\ell_0, \ell_1, \dots, \ell_m)$$

defined for $g \in G = \operatorname{lin} S$ and $\ell \in G^*$, we see that G and G^* are both isomorphic to the $m+1$-dimensional euclidean space $\mathbb{R}^{m+1}$. Hence, the evaluation map (3.1) becomes

$$x \to \ell_x \ , \quad \ell_x(g) = \sum_{k=0}^{m} a_k g_k(x) \ , \qquad g \in G \ ; \tag{4.5}$$

it can be identified with the map

$$x \to \Phi(x) = \big(g_0(x), g_1(x), \dots, g_m(x)\big) \tag{4.6}$$

of A into $\mathbb{R}^{m+1}$, with $A^* = \Phi(A)$. If S is admissible, then Φ is a homeomorphism, and $0 \notin A^*$.

For a set S_0, A^* lies in the hyperplane of $\mathbb{R}^{m+1}$, consisting of points with first coordinate $= 1$. We then replace Φ by

$$x \to \Phi_0(x) = \big(g_1(x), \dots, g_m(x)\big) \in \mathbb{R}^m \ , \tag{4.7}$$

and write $A_0^* = \Phi_0(A)$.

In a finite dimensional space, the convex hull of a closed set is closed. Hence, our definitions of the sets K of §3 reduce to:

(4.8) K^+ is the convex cone in $\mathbb{R}^{m+1}$ with the vertex in the origin, generated by A^*;

$$K_1 = \operatorname{co}\big(A^* \cup (-A^*)\big) \ , \quad K_1^+ = \operatorname{co}\big(A^* \cup \{0\}\big) \ , \tag{4.9}$$

$$K_0 = \operatorname{co} A_0^* \ . \tag{4.10}$$

With $\mathcal{T}$ standing for $\mathcal{T}^+, \mathcal{T}_1, \mathcal{T}_1^+$, K for the set K^+, K_1, K_1^+ and (B) for the conditions $(B^+), (B_1), (B_1^+)$ of §3, we have (Shashkin [1969]): A finite subset S of $C(A)$ is a dominating set for $\mathcal{T}$ if and only if the corresponding set K satisfies condition (B). A finite set S_0 is a dominating set for $\mathcal{T}$ if and only if

$$(B_0) \qquad \operatorname{ext} K_0 = A_0^* \ , \text{ or, equivalently, } \operatorname{ext} K_0 \supset A_0^* \ ,$$

The logical relations between these notions are as follows. Each dominating set for $\mathcal{T}^+$, or for $\mathcal{T}_1$, is also (trivially) a dominating set for $\mathcal{T}_1^+$. There are no other relations; this follows from Examples 5 and 8 below.

A point ℓ_0 of a convex set K in $\mathbb{R}^{m+1}$ is an exposed point if and only if there exists a hyperplane (supporting K) for which $H \cap K = \{\ell_0\}$. Similarly, a ray R of a convex cone K in $\mathbb{R}^{m+1}$ with vertex 0 is an *exposed ray* of K if for some hyperplane H passing through 0, $H \cap K = R$. With these notions, we shall interpret different conditions (P).

For a given $x_0 \in A$, condition (P^+) means that there is a point $\tilde{g} = \sum_0^m a_k g_k$ in G for which $\ell_{x_0}(\tilde{g}) = 0$, and $\ell(\tilde{g}) > 0$ for all $\ell \in A^*$, $\ell \neq \ell_{x_0}$. Then also $\ell(\tilde{g}) = 0$, $\ell \in R_{x_0}$, and $\ell(\tilde{g}) > 0$, for $\ell \in K^+ \setminus R_{x_0}$. Hyperplanes through 0 in $\mathbb{R}^{m+1} = G^*$ are given by equations $\ell(g) = 0$ with fixed g. Hence, (P^+) means that the ray R_{x_0} is an exposed ray of K^+. Similarly, (P_1) and (P_1^+) mean that ℓ_{x_0} is an exposed point of K_1 or K_1^+, respectively. This leads to the following geometric characterization: *A set $S \subset C(A)$ is a strict dominating set for $\mathcal{T}$ if and only if for each $x \in A$, R_x is an exposed ray (or ℓ_x is an exposed point) of K^+ (or of K_1, or of K_1^+) with respect to $\mathcal{T}$,*

For $A = [a, 1]$, $S = \{x, x^2\}$, the set A^* is the parabolic arc $x_2 = x_1^2$ in $\mathbb{R}^2$. Examining the sets (4.8), (4.9) we obtain:

Example 5. For $A = [a, 1]$, $0 \le a < 1$, $S = \{x, x^2\}$, the set S is not a dominating set for $\mathcal{T}^+$. It is a dominating set for $\mathcal{T}_1$ if and only if $a \ge a_0 := \sqrt{2} - 1$. For $a > a_0$, it is also a strict dominating set, with the function g_0 of (P_1) which corresponds to x_0, $a \le x_0 \le 1$ given by $g_0(x) = (x/x_0)(2 - x/x_0)$. For $a = a_0$ we have $p_1(G) = (a_0, 1]$. For $\mathcal{T}_1^+$, S is a strict dominating set for each $a > 0$.

Example 6. Let $f_1, f_2, \ldots, f_r$ be finitely many continuous functions on A, which separate points. Then

$$(4.11) \qquad S_0 = \{1, f_1, \ldots, f_r, f_1^2 + \cdots + f_r^2\}$$

is a strict dominating set for each of the classes $\mathcal{T}$ on A. With the help of the function

$$(4.12) \qquad g_0(x_0) = \sum_{k=1}^{r} \big(f_k(x) - f_k(x_0)\big)^2$$

one sees that each point $x_0 \in A$ is a peak point for $\mathcal{T}$. *In particular: For an arbitrary compact subset A of $\mathbb{R}^r$, the system of functions*

$$S_0 = \{1, x_1, \ldots, x_r, x_1^2 + \cdots + x_r^2\}$$

is a strict dominating system for each of the classes $\mathcal{T}$.

This leads to the following beautiful fact:

Theorem 4.4 *Let $\mathbf{S}_{r-1}$ be the unit sphere in $\mathbb{R}^r$, given by the equation $x_1^2 + \cdots + x_r^2 = 1$, and let A be a closed subset of $\mathbf{S}_{r-1}$. Then (i) $S_0 = \{1, x_1, \ldots, x_r\}$ is a strict dominating set for $\mathcal{T}^+$. (ii) For $\mathcal{T}_1^+$, already the set of the coordinates $S = \{x_1, \ldots, x_r\}$ is a strict dominating set, (iii) for $\mathcal{T}_1$ this is true precisely when $A \cap (-A) = \emptyset$, that is, when A has no antipodal points.*

Proof. (i) follows from example 6. The necessity of the condition in (iii) is (3.14). If this holds for $\mathcal{T}_1$ and if $\sum a_i x_i = 1$ is the equation of the supporting hyperplane to $\mathbf{S}_{r-1}$ at $x_0 \in A$, then $g_0(x) := \sum a_i x_i$ is the function required for (P_1) and (P_1^+). □

The following example gives a dominating system which is not strict.

Example 7. Let A be the curve formed by the arcs of the circles $x_1^2 + x_2^2 = 4$, $(x_1 \pm 1)^2 + x_2^2 = 1$, as shown on Fig. 4.1. Then the set of functions $S_0 = \{1, x_1, x_2\}$ is a dominating set for $\mathcal{T}$, but not a strict dominating set.

Indeed, the map Φ_0 transforms A into itself; all points of $K_0 = \operatorname{co} A$ are extreme points, but the points $(-1,-1)$ and $(1,-1)$ are not exposed points of K_0.

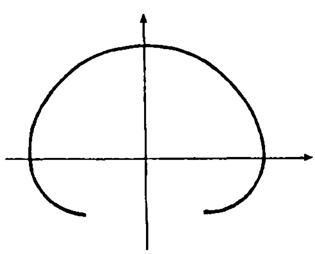


Fig. 4.1

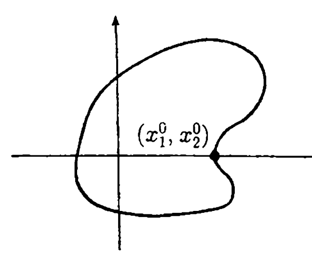


Fig. 4.2

Example 8. Let K^+ be the cone $x_3^2 = x_1^2 + x_2^2$ in $\mathbb{R}^3$. Let A be any closed bounded star-shaped curve in the x_1, x_2-plane, so that each ray from the origin intersects A exactly once; let A be the curve on K^+ whose projection is A. Then $S = \{x_1, x_2, x_3\}$ is a strict $\mathcal{T}^+$ dominating set on A, but not a $\mathcal{T}_1$ dominating set, if A is shaped as shown on Fig. 4.2, for then the point of A corresponding to (x_1^0, x_2^0) is interior to K_1.

In Theorem 4.4, we have seen that there are dominating sets for $\mathcal{T}_1^+$, for which $G = \operatorname{lin} S$ does not contain a strictly positive function. For $\mathcal{T}_1$ there is a similar example:

Example 9. Let A be any closed set in the unit upper semisphere of $\mathbb{R}^3$, which contains only the three points $(-1, 0, 0)$ and $(1/2^{1/2}, \pm 1/2^{1/2}, 0)$ in the equatorial plane $x_3 = 0$. Then $S = \{x_1, x_2, x_3\}$ is a dominating set in $C(A)$ for $\mathcal{T}_1$, but if g is any nonnegative function in $G = \operatorname{lin} S$, then g vanishes at each of the three points.

We shall discuss Haar sets from a slightly different point of view, comparing them with dominating (Korovkin) sets.

A finite set of continuous functions $S = \{g_0, g_1, \dots, g_m\}$ on A we shall call *k-independent (or positively k-independent) if no point*

$$\{g_0(x), g_1(x), \dots, g_m(x)\} \tag{4.13}$$

is a non-trivial linear combination (or: non-trivial linear combination with positive coefficients) of any k points

$$\{g_0(x_j), g_1(x_j), \dots, g_m(x_j)\}\ , \qquad x_j \in A\ ,\ j = 1, \dots, k\ . \tag{4.14}$$

The first assumption holds if and only if each matrix $(g_i(x_j))_{i=0,}^{m}{}_{j=1}^{k+1}$ with distinct x_j has full rank $k+1$. Thus, S is a Haar set if and only if it is m-independent. The second is equivalent to the statement that the point (4.13) is no non-trivial positive multiple of a convex combination of the points (4.14). (This is related to the last requirement of (B^+).)

We shall assume that S is admissible. This is necessary for a $\mathcal{T}^+$-dominating set and also, by [**CA**, Theorem 4.3, p.72], for instance, for a Haar set on an interval or on $\mathbb{T}$. The set $A^* = \Phi(A)$ lies in $\mathbb{R}^{m+1}$ and is *connected* whenever A is. A convex hull of A^* in $\mathbb{R}^{m+1}$ consists of convex combinations of any $m+2$ points of A^* (Carathéodory) or of any $m+1$ points if A^* is connected (Fenchel, see Eggleston [B-1958]). We have therefore

Proposition 4.5 *A set $S = \{g_0, \dots, g_m\}$ of the described kind is a $\mathcal{T}^+$-dominating set if and only if it is $(m+1)$-positively independent, and if A is connected, if and only if it is m-positively independent.*

Corollary 4.6 *Each Haar set S on a compact and connected A is a $\mathcal{T}^+$-dominating set.*

This follows from (4.8)–(4.10).

Corollary 4.7 *A set $S = \{g_0, g_1, g_2\}$ on $[a, b]$ or $\mathbb{T}$ is $\mathcal{T}^+$-dominating if and only if it is a Haar set.*

To prove this, we have only to show that a $\mathcal{T}^+$ dominating set which is 2-dependent is 2-positively dependent. By Lemma 2.2 we can assume that $g_0 > 0$. Let $y = \alpha y_1 + \beta y_2$, where y is a point (4.13) in $\mathbb{R}^3$ and y_1, y_2 are two points (4.14). Then α, β cannot be both negative, we can assume that $\alpha > 0$, $\beta < 0$, and have then $y_1 = (1/\alpha)y + (-\beta/\alpha)y_2$. □

The minimal order of a dominating set with respect to $\mathcal{T}$, which can exist on a given metric compact A, is a topological invariant $m(A, \mathcal{T})$. We are able to determine it.

Theorem 4.8 (Shashkin [1969], Berens and Lorentz [1975]). *Let r_0, r be the minimal dimension of the sphere $\mathbf{S}_{r_0}$ or of the euclidean space $\mathbb{R}^r$, respectively,* *into which A can be topologically embedded. Then*

$$m(A,\mathcal{T}^+) = r_0 + 1\ , \quad m(A,\mathcal{T}_1) = r\ , \quad m(A,\mathcal{T}_1^+) = r_0\ . \tag{4.15}$$

Proof. Case $\mathcal{T}^+$. Let $S = \{g_0, g_1, \dots, g_m\}$ be a dominating set of minimal order carried by A. Then the functions are linearly independent, and without loss of generality we may assume that $g_0 = 1$.

We map A homeomorphically by means of $x \to (g_1(x), \dots, g_m(x))$ into the space $\mathbb{R}^m$. If A_0^* is the image of A and $K_0 = \operatorname{co} A_0^*$, then K_0 is a convex compact set with interior points in $\mathbb{R}^m$. Otherwise K_0 would be contained in a hyperplane of $\mathbb{R}^m$, and the functions g_k would be linearly dependent.

Let $y^{(0)}$ be an interior point of K_0, and let $\mathbf{S}_{m-1}$ be the unit sphere in $\mathbb{R}^m$ with center $y^{(0)}$. Theorem 3.3 shows that A_0^* is situated on the boundary of K_0, and the central projection from $y^{(0)}$ maps A_0^* homeomorphically into $\mathbf{S}_{m-1}$. Thus, also A is embeddable in $\mathbf{S}_{m-1}$. This proves that $m(A,\mathcal{T}^+) \geq r_0+1$. The inverse inequality follows from Theorem 4.4, (i): Each subset of $\mathbf{S}_{r_0}$ carries a $\mathcal{T}^+$ dominating set of order $r_0 + 1$.

Case $\mathcal{T}_1$. Again let $S = \{g_0, g_1, \dots, g_m\}$ be a dominating set of minimal order. We consider the homeomorphic embedding Φ of A into $\mathbb{R}^{m+1}$ with image A^*. From the properties of the set K_1 mentioned in Theorem 3.2, it follows that A^* and $-A^*$ are disjoint and that each ray through the origin meets A^* at most once. The central projection of A^* from the origin onto the unit sphere $\mathbf{S}_m$ with center 0 defines an embedding of A into $\mathbf{S}_m$, which contains no antipodal points. But then A is homeomorphic to a subset of $\mathbb{R}^m$. We obtain $m \geq r$. The inverse inequality follows again from Theorem 4.4, (ii).

Case $\mathcal{T}_1^+$ can be treated in similar fashion, and is left to the reader.

This proof gives also:

Corollary 4.9 *If A carries a dominating set of order m, then also a strict dominating set of the same order.*

Example 10. Let $\mathbb{T}_r$, $r = 2, 3, \dots$ be the r-dimensional torus. Then

$$m(\mathbb{T}_r, \mathcal{T}^+) = r + 2\ , \quad m(\mathbb{T}_r, \mathcal{T}_1) = m(\mathbb{T}_r, \mathcal{T}_1^+) = r + 1\ .$$

In fact, $\mathbb{T}_r$ is embeddable in $\mathbb{R}^{r+1}$, but not in $\mathbf{S}_r$.

We shall conclude this section by describing with Berens and Lorentz [1976] *all minimal $\mathcal{T}^+$-dominating sets S on the unit sphere* $\mathbf{S}_{m-1}$ of the m-dimensional euclidean space $\mathbb{R}^m$. By Theorem 4.8, a set S of this type consists of $m+1$ linearly independent functions, and for Proposition 4.5, it is 2-independent. These are the necessary and sufficient conditions:

Theorem 4.10 *A set $S = \{g_0, \dots, g_m\}$ on $\mathbf{S}_{m-1}$ is a $\mathcal{T}^+$-dominating set if and only if it is 2-independent. Each such set is strictly dominating.*

We have to prove only the sufficiency. The map $\Phi : x \to \{g_0(x), g_1(x), \dots, g_m(x)\}$ is a homeomorphism of $\mathbf{S}_{m-1}$ into $\mathbb{R}^{m+1}$; and S is strictly dominating on $\mathbf{S}_{m-1}$ if and only if the set of coordinates $\{y_0, y_1, \dots, y_m\}$ has this property on the image $Y = \Phi(\mathbf{S}_{m-1})$. Now the set Y will also be 2-independent (this means that no three points of Y lie on a 2-dimensional plane through the origin). We see that our theorem is implied by the following purely geometric result.

Theorem 4.11 *Let Y be a topological image of $\mathbf{S}_{m-1}$ in $\mathbb{R}^{m+1}$, which is 2-independent. Then at each point $y^{(0)}$ of Y there exists a strict supporting hyperplane for Y, passing through the origin.*

Proof. Each two-dimensional plane through the origin intersects Y at most twice. In particular, $0 \notin Y$, and each straight line through the origin has at most one point on Y. Let Σ be the unit sphere of $\mathbb{R}^{m+1}$ with center 0. The sets Y and $-Y$ are disjoint, they are projected onto Σ by rays through the origin into two topological images of $\mathbf{S}_{m-1}$, say Σ' and $-\Sigma'$. The sphere Σ' divides Σ into two closed regions. Let B be the one of them that contains no points of $-\Sigma'$. By symmetry, $-\Sigma'$ is the boundary of $-B$, disjoint with B.

Let now $y^{(0)}$ be an arbitrary interior point of B. We consider any two-dimensional plane through $y^{(0)}$ and 0, and the one-dimensional circle σ, intersection of the plane with Σ. We wish to prove that $\sigma \cap B$ is a circular arc $y^{(1)}y^{(2)}$ of an opening $< \pi$, with $y^{(1)}, y^{(2)} \in \Sigma'$, all other points being interior points of B.

In fact, there are on σ interior points of $\sigma \cap B$, for example, $y^{(0)}$, and points not belonging to B, for example, $-y^{(0)}$. Hence there are at least two, and since Σ' is 2-independent, exactly two points $y^{(1)}, y^{(2)}$ of Σ' on σ. Let $y^{(1)}y^{(2)}$ be the arc containing $y^{(0)}$. Then it coincides with $\sigma \cap B$, and since $y^{(1)}, y^{(2)} \notin -B$, has an opening $< \pi$ (see Fig. 4.3).

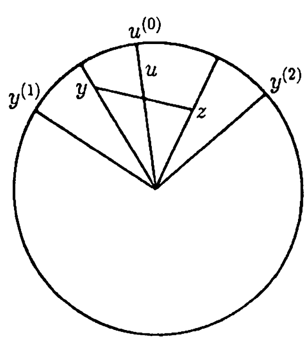


Fig. 4.3

Let K be the closed cone formed by the rays connecting 0 with all points of B. It is easy to prove that the boundary of K consists of all rays connecting 0 with Y (or with Σ'). We show that K is convex.

Assume that $y, z \in K$ and that $u = \alpha y + \beta z$, $\alpha, \beta > 0$, $\alpha + \beta = 1$. Without loss of generality, we can assume that $y \neq 0$, $z \neq 0$ and that they are not on the same ray through 0. Then they define a 2-dimensional plane through 0; its intersection with Σ is given on Fig. 4.3. The points y, z belong to the convex circular sector $0y^{(1)}y^{(2)}$. Hence u belongs to it, and therefore to K.

Any point $y \in Y$ belongs to the boundary of K, and there is a supporting hyperplane H to K at y, which passes through the origin.

Assume that some other point $z \in Y$, $z \neq y$ belongs to H. Then the segment yz is contained in $K \cap H$, hence it lies in the boundary of K. If $u = \frac{1}{2}(y+z)$, there is a point $u^0 \in Y$ on the ray $0u$. This is a contradiction: the plane through y, z and 0 contains three distinct points of Y. Thus, H strictly supports Y. □

§ 5. Shadows of Sets of Continuous Functions

Examination of shadows in spaces of continuous or integrable functions requires more function analytic tools than those developed in §§1-4. Our results will be more general, but refer only to positive operators (to the class $\mathcal{T}^+$ of §1).

Let X be a Banach function space (see [**CA**, p.19]) on a compact metric space A, and let P be a lattice homomorphism of $C(A)$ into X. This means that P is a linear operator, which preserves the lattice relations:

$$P(f \vee g) = P(f) \vee P(g) , \qquad f, g \in C(A) . \tag{5.1}$$

The dual formula, with $\vee$ replaced by $\wedge$, follows from this. For simplicity, we assume that $Pf = 0$ in X implies $f = 0$ in $C(A)$ (for the general case, see Berens and Lorentz [1973]). A simple example of a P is the identity map $P = I$, if $C(A)$ has a natural imbedding into X.

Let S be a subset of $C(A)$. The *shadow* $\mathcal{S}(S, X, P)$ of S in X with respect to P is the set of all $f \in C(A)$ which satisfy the following:

If T_n is a sequence of positive linear operators mapping $C(A)$ into X for which

$$\lim T_n g = Pg \quad \textit{for all} \quad g \in S , \tag{5.2}$$

then

$$\lim T_n f = Pf . \tag{5.3}$$

In particular, S is a *dominating* (*or a Korovkin*) *set if* $\mathcal{S}(S, X, P) = C(A)$. Relation (5.2) holds also for all $g \in G := \operatorname{lin} S$. Plainly, $\mathcal{S}(S, X, P)$ is a linear subspace of $C(A)$.

With the set A we associate classes of measures. The class $\mathcal{M}^*(A)$ consists of all regular Borel measures μ on A; for example, Riesz' representation theorem is that each linear bounded functional $\ell(f)$, $f \in C(A)$, has the unique representation $\ell(f) = \int_A f\, d\mu$, $\|\ell\| = \operatorname{Var} \mu$. Further, $\mathcal{M}^+(A)$ will stand for the subset of $\mathcal{M}^*(A)$ of positive measures, and $\mathcal{M}(A)$ for that of probability measures (with $\mu(A) = 1$).

About the set S *we shall always assume that it contains a strictly positive element*, that is, a function $g_0(x) > 0$, $x \in A$. For $G := \operatorname{lin} S$, we define the (*generalized*) *boundary* $\partial G = \widehat{G}_A$ of G in $C(A)$, and, more generally, a set $\widehat{G}_B$ for each non-empty compact set $B \subset A$, in the following way.

For each $x \in A$, the set $\mathcal{M}_x^+(S)$ consists of all measures $\mu \in \mathcal{M}^+$ for which

$$\mu(g) := \int_A g\, d\mu = g(x) =: \varepsilon_x(g)\ , \qquad g \in S\ , \tag{5.4}$$

where ε_x is the evaluation functional. Thus, $\mathcal{M}_x^+$ is not empty: $\varepsilon_x \in \mathcal{M}_x^+$. Of course, $\mathcal{M}_x^+$ may contain other measures. Now $\widehat{G}_B$ consists of all $f \in C(A)$ with the property

$$\mu(f) = f(x) \ \text{ for all } \ \mu \in \mathcal{M}_x^+ \ \text{ and all } \ x \in B\ . \tag{5.5}$$

For example, if $S = \{1\}$, then $\mathcal{M}_x^+ = \mathcal{M}^+$ for all $x \in A$, and $\widehat{G}_B = G$ consists of constants for each B. In general, $\widehat{G}_B \supset G$ is a closed linear subspace of $C(A)$.

Let ℓ_x be the restriction of the functional ε_x to G. Formula (5.4) describes all functionals $\mu = \ell_x$ on G. Then (5.5) means that all these ℓ_x are necessarily $= \varepsilon_x$ on f. Hence, $\widehat{G}_B$ is the largest closed linear set onto which all ℓ_x, $x \in B$ have a unique extension $(= \varepsilon_x)$.

The subset of $C(A) : \partial G = \widehat{G}_A$ we call the *generalized Choquet boundary* of G. This is not quite the terminology of the Choquet boundary specialists (see Phelps [B-1966], the nice review of Bauer [1978], and papers of Choquet mentioned there): they require S to be "admissible", which means that S should separate points of A and contain the function 1 (see §2). These assumptions are not useful for the description of dominating sets and shadows, and we will not make them in general. For this reason, it appears that our theory and Choquet theory are parallel theories, which share some basic notions, but otherwise do not influence each other.

We shall give two other descriptions of the sets $\widehat{G}$.

Let $G = \operatorname{lin} S$, $\emptyset \neq S \subset C(A)$, $g_0 \in G$, $g_0(x) \geq C > 0$, $x \in A$. Relations

$$\bar{f}(x) := \inf\{g(x) : f \leq g\ ,\ g \in G\}\ , \tag{5.6}$$

$$\underline{f}(x) := \sup\{g(x) : g \leq f\ ,\ g \in G\}\ . \tag{5.7}$$

define the *upper* and the *lower* G-*envelopes* of $f \in C(A)$, respectively. Since $g_0 \in G$, $\bar{f}$ and $\underline{f}$ are well defined. Obviously $\underline{f} = -\overline{(-f)}$, and $\bar{f}$ is an upper semi-continuous function on A. Also, for $f_1, f_2 \in C(A)$, $f_1 \leq f_2$ implies $\bar{f}_1 \leq \bar{f}_2$, moreover one has $\overline{f_1 + f_2} \leq \bar{f}_1 + \bar{f}_2$, and $\overline{\lambda f} = \lambda \bar{f}$ in case $\lambda \geq 0$, further $\overline{f+g} = \bar{f} + g$ for each $g \in G$.

The following lemma goes back to Lorentz [1948] for special A:

Lemma 5.1 *For a function $f \in C(A)$ and a compact set $B \subset A$, $B \neq \emptyset$, the following statements are equivalent:*

(i) $f \in \widehat{G}_B$;
(ii) $\underline{f}(x) = f(x) = \bar{f}(x)$ *for all* $x \in B$,
(iii) *given any* $\delta > 0$, *there exist finitely many* $g_1', g_2', \dots, g_m', g_1'', \dots, g_m''$ *in* G *such that for the elements*

$$h' := \bigwedge_{i=1}^{m} g_i' \quad , \quad h'' := \bigvee_{i=1}^{m} g_i''$$

one has $h' \le f \le h''$ *and* $h''(x) - h'(x) < \delta$, $x \in B$.

We first establish the following two facts.

(5.8) *For all* $x \in A$, $\mu \in \mathcal{M}_x^+(S)$ *and* $f \in C(A)$, *one has* $\underline{f}(x) \le \mu(f) \le \bar{f}(x)$.

Indeed, if $f \le g$, then $\mu(f) \le \mu(g) = g(x)$ for all such g, hence $\mu(f) \le \bar{f}(x)$. We have also:

(5.9) *If for a real* λ, $f_0 \in C(A)$ *and some* $x \in A$ $\underline{f}_0(x) \le \lambda \le \bar{f}_0(x)$, *then for some* $\mu \in \mathcal{M}_x^+(S)$, $\mu(f_0) = \lambda$.

This follows from the standard proof of the abstract form of the Hahn-Banach theorem. (This theorem deals with an extension of a linear functional $\mu(f)$ which is majorized by a subadditive functional $p(f)$. We take $p(f) := \bar{f}(x)$.)

Proof of Lemma. The equivalence of (i) and (ii) follows quickly from (5.8) and (5.9) To prove the implication (ii) $\Rightarrow$ (iii), let $\delta > 0$ be given. For each $x \in B$, by (ii) there exist g_x' and g_x'' in G such that

$$g_x' \le f \le g_x'' \quad \text{and} \quad g_x''(x) - g_x'(x) < \delta \ .$$

By continuity, $g_x''(y) - g_x'(y) < \delta$ for some open neighborhood U_x of x. The family $\{U_x : x \in B\}$ forms an open cover of B, and since B is compact, there is a finite subcover, say $\{U_{x_i} : i = 1, 2, \dots, m\}$. The associated functions g_{x_i}' and g_{x_i}'' $(i = 1, 2, \dots, m)$, in simplified notation g_i' and g_i'', have the desired properties. Finally, let us show that (iii) $\Rightarrow$ (i). Let $f \in C(A)$ be such that (iii) holds. For an $x \in B$ and $\mu \in \mathcal{M}_x^+(S)$

$$g_i'(x) \le \int f \, d\mu \le g_i''(x) \ , \qquad i = 1, 2, \dots, m \ .$$

Hence $h'(x) \le \mu(f) \le h''(x)$. Since $h'(x) \le f(x) \le h''(x)$ for all $x \in A$, $|f(x) - \mu(f)| \le h''(x) - h'(x) \le \delta$ for each $x \in B$, proving the implication. □

Another description of the boundary $\partial G = \widehat{G}_A$ is by means of *quasi-peak points* x_0 of a space $G \subset C(A)$. This is a point $x_0 \in A$ with the property that for each ε, $0 < \varepsilon < 1$ there is a neighborhood U of x_0 and a $g \in G$ for which

$$(5.10) \qquad g(x) \ge 0 \ , \ x \in A \ ; \ g(x_0) < \varepsilon \ ; \ g(x) \ge 1 \ , \ x \in A \setminus U \ .$$

(For a *peak-point* one assumes here $g(x_0) = 0$.) A simple consequence of Lemma 5.1 is

Lemma 5.2 (Berens and Lorentz [1976]). *If G contains a strictly positive function g_0, then ∂G is equal to the set of all quasi-peak points of G in A.*

Proof. We apply Lemma 5.1 to $B = \{x_0\}$ and all $f \in C(A)$. It follows that $x_0 \in \partial G$ if and only if one of the two equivalent statements (i) and (ii) holds, where

(i) $f \in \hat{G}_{x_0}$ *for all* $f \in C(A)$, *that is, the evaluation functional* ε_{x_0} *has a unique (equivalently, unique positive) linear extension from* G *to* $C(X)$,

(ii) $\underline{f}(x_0) = \bar{f}(x_0)$ *for all* $f \in C(A)$.

This is equivalent to

(iv) x_0 *is a quasi-peak point of* G.

Indeed, if (ii) holds we apply this to a function $f \geq 0$ which satisfies $f(x_0) = 0$, $f(x) > 1$, $x \in G \setminus U$. The existence of a $g \in G$ required in (5.10) follows from the definition (5.6) of $\bar{f}$. We get (iv).

Conversely, we derive (i) from (iv). Let $\ell \geq 0$ be a linear functional coinciding with ε_{x_0} on G, let μ be the measure so that $\ell(f) = \int_A f\, d\mu$. If g satisfies (5.10), then

$$\mu(A \setminus U) \leq \int_A g\, d\mu = \ell(g) = g(x_0) < \varepsilon \ .$$

Thus, $\mu(A \setminus U) = 0$, and since U is arbitrary, the whole mass of μ is concentrated at x_0. Hence

$$\ell(f) = \int_A f\, d\mu = f(x_0) \ . \qquad \square$$

The main result of this section is an estimation of the shadow from below, obtained independently by Bauer [1973] and by Berens and Lorentz [1973]:

Theorem 5.3 *Let S be a Banach function space on A, and $C(A)$ the space of continuous functions, let P be a linear homomorphism of $C(A)$ into X. If $Pf = 0$ implies $f = 0$, and if $S \subset C(A)$ contains a strictly positive function g_0, then the boundary $\widehat{G}_A$ of $G = \operatorname{lin} S$ is contained in the shadow $\mathcal{S}(S, C(A), P)$:*

$$\partial G = \widehat{G}_A \subset \mathcal{S}(S, C(A), P) \ . \tag{5.11}$$

Proof. Let f be a function in $\widehat{G}_A$. Since A is compact, by Lemma 5.1 (iii) for any given $\delta > 0$ there exist

$$h'' = \bigwedge_{i=1}^{m} g_i'' \quad \text{and} \quad h' = \bigvee_{i=1}^{m} g_i'$$

such that

$$h' \leq f \leq h'' \quad \text{and} \quad |h''(x) - h'(x)| < \delta \quad \text{for} \quad x \in \operatorname{supp} P \ .$$

Hence

$$T_n h' - Ph'' \le T_n f - Pf \le T_n h'' - Ph'$$

and therefore

$$T_n h' - Ph' - \delta P1 \le T_n f - Pf \le T_n h'' - Ph'' + \delta P1 \, .$$

Furthermore,

$$T_n h'' \le \bigwedge_{i=1}^{n} T_n g_i'' \text{ and } T_n h' \ge \bigvee_{i=1}^{n} T_n g_i' \, ,$$

and

$$T_n h'' - Ph'' \le \sum_{i=1}^{m} |T_n g_i'' - P g_i''| \text{ and } T_n h' - Ph' \ge -\sum_{i=1}^{m} |T_n g_i' - P g_i'| \, ,$$

giving

$$|T_n f - Pf| \le \sum_{i=1}^{m} |T_n g_i'' - P g_i''| + \sum_{i=1}^{m} |T_n g_i' - P g_i'| + \delta P1 \, .$$

Consequently,

$$\lim_{n\to\infty} \|T_n f - Pf\| \le \delta \|P1\|$$

for any $\delta > 0$, proving (5.11). □

In general, inclusion (5.11) is proper, as we shall see in §6. However, if $X = C(A)$ and $P = I$, we have the inverse inclusion. Only in the following proof do we need that A is a metric space, all results earlier applied to any compact Hausdorff space A.

Theorem 5.4 (Berens and Lorentz [1973]). *If $S \subset C(A)$ contains a strictly positive function g_0, then the shadow of S in $C(A)$ under I is equal to ∂G:*

(5.12) $$\mathcal{S}(S, C, I) = \partial G \, ,$$

and equal to the set of all quasi-peak points of A.

Proof. We only have to prove that $\mathcal{S}(S, C, I) \subset \widehat{G}_A$. Let f_0 be a function in $C(A)$ which does not belong to $\widehat{G}_A$. We shall construct a sequence T_n, $n = 1, 2, \ldots$, in $\mathcal{T}^+(C)$ such that $T_n g \to g$ in C for all $g \in S$, while $T_n f_0 \not\to f_0$ in C. Since $f_0 \notin \widehat{G}_A$, by Lemma 5.1(ii) there exists a point x_0 in A such that either $\underline{f}_0(x_0) < f_0(x_0)$ or $f_0(x_0) < \bar{f}_0(x_0)$. Let us assume the latter inequality is true. If $d(\cdot,\cdot)$ is the distance function on A, we define functions $\varphi_n \in C(X)$, $n = 1, 2, \ldots$, as follows: $\varphi_n(x) = 1$ when $d(x, x_0) \le 1/2n$, $\varphi_n(x) = 0$ when $d(x, x_0) \ge 1/n$, and $0 \le \varphi_n(x) \le 1$ elsewhere. The sequence of operators in $\mathcal{T}^+(C)$

(5.13) $$T_n f(x) = \varphi_n(x)\mu_0(f) + \{1 - \varphi_n(x)\} f(x) \, , \qquad n = 1, 2, \ldots$$

where $\mu_0 \in \mathcal{M}_{x_0}(S)$ is such that $\mu_0(f_0) = \bar{f}_0(x_0)$, has the desired properties. The existence of such a μ_0 is guaranteed by property (5.9). Indeed, for any $g \in S$

$$T_n g(x) - g(x) = \varphi_n(x)\{g(x_0) - g(x)\} ,$$

which converges uniformly to zero as $n \to \infty$. On the other hand, for the function f_0

$$\lim_{n\to\infty} T_n f_0(x_0) = \bar{f}_0(x_0) \neq f_0(x_0) ,$$

i.e., $T_n f_0(x)$ does not even converge pointwise to $f_0(x)$. □

In particular, S containing a g_0 is a convergence dominating set in $C(A)$ for $\mathcal{T}^+$ if and only if each $x \in A$ is a quasi-peak point.

Comparing Theorems 5.3 and 5.4 with those of §4 we see that they estimate shadows, not just dominating sets, and that they deal with operators from $C(A)$ to X, not those of $C(A)$ into itself.

§ 6. Shadows in Banach Function Spaces

Our Theorem 5.3 is of importance also because its method of proof yields good information about shadows in specific Banach function spaces X. This is so at least in the most important case when S consists of continuous functions. The general case has been treated by Donner [A-1982]; it is considerably more difficult.

We assume that elements $f(x)$ of X are functions on a compact metric space A equipped with a regular Borel measure ν. For the theory of Banach function spaces, see the book [B-1988] of Bennett and Sharpley. In addition, a result from the book of Vulich [B-1967] will be needed.

We assume that the natural imbedding I:

$$C(A) \subset X \tag{6.1}$$

is continuous. From §5, the support of the homomorphism I is equal to $B =: B(\nu) := \operatorname{supp}\nu$. This set B is the largest subset of A with the property that if $x \in B$, then for each neighborhood U of x, $\nu(U \cap B) > 0$.

As a measure-theoretic analogue of the set $\widehat{G}_A = \partial G$ of §5 we define the set $\widetilde{G}_A$, which consists of all $f \in C(A)$ for which the condition (i) of Lemma 5.1 is valid ν-a.e., that is

(i*) *For the function f we have $f(x) = \int f\, d\mu$, $\mu \in \mathcal{M}_x^+(G)$ for ν-almost all $x \in A$* .

Plainly, $\widetilde{G}_A$ is a closed linear subspace of $C(A)$, and

$$\widehat{G}_A \subset \widehat{G}_{\operatorname{supp}\nu} \subset \widetilde{G}_A \subset C(A) . \tag{6.2}$$

The following corresponds to Lemma 5.1:

Lemma 6.1 *If $G = \operatorname{lin} S$ contains a strictly positive function g_0, then for $f \in C(A)$ the following three statements are equivalent:* (i*) *and*

(ii*) $$\underline{f}(x) = f(x) = \bar{f}(x) \quad \nu\text{-a.e.}$$

(iii*) *given any ε, $0 < \varepsilon < 1$, and $\delta > 0$, there is a compact set $K \subset A$ such that $\nu(A \setminus K) < \varepsilon$ and there are functions $g_1', \dots, g_m'$, $g_1'', \dots, g_m''$ in G such that*

$$\bigvee_{i=1}^{m} g_i' =: h' \le f \le h'' := \bigwedge_{i=1}^{m} g_i'' \quad \text{and} \quad h''(x) - h'(x) < \delta \,, \qquad x \in K \,.$$

Proof. We modify the statements (i*) and (ii*), requiring that for each $\varepsilon > 0$, there should exist a compact set $K \subset A$, $\nu(A \setminus K) < \varepsilon$, with the respective conditions satisfied for all $x \in K$. The modified statements are equivalent by Lemma 5.1. Then we note that because of the regularity of ν, a subset of A has ν-measure zero if and only if it is contained in sets $A \setminus K$ of arbitrary small measure. □

Lemma 6.2 *If G contains a strictly positive function g_0, and if $f_0 \in C(A) \setminus \widetilde{G}_A$, then there exists a positive linear operator T which maps $C(A)$ into $L_\infty(A)$ and satisfies*

(6.3) $$T(g) = g \,, \quad g \in S \,,$$

(6.4) $$T(f_0, x) \ne f_0(x) \quad \textit{on a set of positive measure.}$$

Proof. Since $f_0 \notin \widetilde{G}_A$, there exists a set of positive measure on which $f_0(x) < \bar{f}_0(x)$ (if not, we replace f_0 by $-f_0$). Let G_0 be the set of all functions $g_0 = g + \lambda f_0$, $\lambda \in \mathbb{R}$, $g \in G$. We define

(6.5) $$T_0(g + \lambda f_0) := g + \lambda \bar{f}_0 \,, \qquad g \in G \,, \ \lambda \in \mathbb{R} \,.$$

Since $\bar{f}_0$ is upper semi-continuous, and since $\bar{f}_0(x)$ is bounded: $f_0(x) \le \bar{f}_0(x) \le \frac{1}{C}\|f_0\| g_0(x)$, T_0 is a linear map of G_0 into L_∞. Moreover, T_0 is positive, since $g + \lambda f_0 \ge 0$ implies $g + \lambda \bar{f}_0 \ge 0$ (both when $\lambda > 0$ and when $\lambda < 0$), and restricted to G, T_0 is the identity, $T_0 g = g$ ($g \in G$).

The rest of the proof follows from an extension theorem of Kantorovich (see Vulikh [B-1967, Theorem X,3.1]): Let F be a vector lattice and F_0 a subspace of F which majorizes F (that is, for each $f \in F$ there is an $f_0 \in F_0$ such that $|f| \le f_0$). Then each positive linear transformation T_0 defined on F_0 and with values in an order complete Banach function space X, has a positive linear extension on F. Hereby, X is order complete if, whenever A, B are non-empty subsets of X and $a \le b$, that is, $(b - a)(x) \ge 0$ a.e. for all $a \in A$, $b \in B$, then there exists a function $f \in X$ with the property $a \le f \le b$ for all $a \in A$, $b \in B$.

Because of the function g_0, the subspace G_0 majorizes $C(A)$, and L_∞ is obviously order complete. By the theorem of Kantorovich, T_0 has a positive linear extension T onto $C(A)$, satisfying (6.3) and (6.4). □

The following theorems of Berens and Lorentz [1973] describe shadows in different Banach function spaces. In all of them we assume that $G = \operatorname{lin} S$ contains a strictly positive function g_0. For simplicity, we write $\mathcal{S}(S,X) := \mathcal{S}(S,X,I)$.

Theorem 6.3 *If $C(A)$ is normally imbedded in X, then*

$$\widehat{G}_{\operatorname{supp}\nu} \subset \mathcal{S}(S,X) \subset \widetilde{G}_A \ . \tag{6.6}$$

Proof. The first inclusion follows directly from Theorem 5.3. The second inclusion is implied by Lemma 6.2: if $f \notin \widetilde{G}_A$, then there exists a positive linear operator T from $C(A)$ into X (even into L_∞) for which $T_g = g$, $g \in G$ and $Tf \neq f$. □

A Banach function space X has *absolutely continuous norm* if

$$\textit{For any } f \in X \ , \ \|f\chi_e\| \to 0 \textit{ if } \nu e \to 0 \ . \tag{6.7}$$

An example are the spaces $L_p(A)$, $1 \le p < \infty$, and the Lorentz spaces $\Lambda(\phi,p)$ on $[0,1]$ (see [**CA**, p.23]). The latter have the norm

$$\|f\|_{\Lambda(\phi,p)} := \left\{ \int_0^1 \phi f^{*p}\, dx \right\}^{1/p}$$

where $\phi \ge 0$ is decreasing and integrable, $1 \le p < \infty$, and f^* is the decreasing rearrangement of $|f|$.

Theorem 6.4 *For a space X with absolutely continuous norm, one has*

$$\mathcal{S}(S,X) = \widetilde{G}_A \ . \tag{6.8}$$

Proof. We only have to prove the inclusion $\widetilde{G}_A \subset \mathcal{S}(S,X)$. Let $f \in \widetilde{G}_A$. We may assume that $|f| \le g_0$, where g_0 is a strictly positive function in G. Given $0 < \varepsilon < 1$ and $\delta > 0$, we shall use (iii*) of Lemma 6.1, where we can assume that $|h'|, |h''| \le g_0$.

with $\Phi := \delta + 2g_0\chi_{A\setminus K}$ that

$$h'' \le h' + \delta + (h'' - h')\chi_{A\setminus K} \le h' + \Phi \ . \tag{6.9}$$

Let T_n be any sequence of positive linear operators that map X into itself and satisfy $T_n g \to g$, $g \in G$. As in the proof of Theorem 5.3 it follows from (6.9) that

$$T_n h' - h' - \Phi \le T_n f - f \le T_n h'' - h'' + \Phi \ .$$

We can then derive that

$$|T_n f - f| \le \sum_{i=1}^{m} \{|T_n g_i' - g_i'| + |T_n g_i'' - g_i''|\} + \Phi \,. \tag{6.10}$$

Consequently,

$$\limsup_{n\to\infty} \|T_n f - f\|_X \le \delta\|1\|_X + 2\|g_0 \cdot \chi_{A\setminus K}\|_X \,. \tag{6.11}$$

Since the norm is absolutely continuous, the last term in (6.11) is $< \delta\|1\|_X$, if ε is small enough. This shows that $T_n f \to f$. □

There exist spaces X with the property

(6.12) If $f(x) \ge \varepsilon > 0$, $f \in X$, on some subset of A of measure > 0, then $\|f\|_X \ge C(f)\varepsilon$.

The space $L_\infty(A)$ is one of them. Also spaces $M(\phi, q)$ on $[0,1]$, $1 \le q < \infty$ (see [**CA**, p.23]) with the norm

$$\|f\|_{M(\phi,q)} = \sup_{c>0}\left\{\int_0^c f^{*q}\,dx \Big/ \int_0^c \phi\,dx\right\}^{1/q}$$

have this property.

Theorem 6.5 *If X has the property (6.12), then*

$$\mathcal{S}(S, X) = \widehat{G}_{\operatorname{supp}\nu} \,. \tag{6.13}$$

Proof. We have to show that $f_0 \notin \widehat{G}_B$, $B = \operatorname{supp}\nu$, implies that f_0 does not belong to the shadow. We can assume that there is a point $x_0 \in B$ with the property $\bar{f}_0(x_0) - f_0(x_0) = \varepsilon > 0$. Then each neighborhood $U_n := \{x \in A : d(x, x_0) \le \frac{1}{n}\}$ has positive measure, $\nu(U_n) > 0$. We define a sequence of positive linear operators $T_n : C(A) \to X$ by means of the formula (5.15) of Theorem 5.4. This implies

$$T_n f_0(x) - f_0(x) = \varphi_n(x)\left[\bar{f}_0(x_0) - f_0(x)\right] .$$

On the set U_n this is $= \bar{f}_0(x_0) - f_0(x) = \varepsilon + [f_0(x_0) - f_0(x)] \ge \frac{1}{2}\varepsilon$ for all large n. For these n, $\|T_n f_0 - f_0\|_X \ge C\frac{1}{2}\varepsilon$, so that $T_n f_0 \not\to f_0$. □

Until now, we have studied sequences of operators (T_n) mapping $C(A)$ into X of the class $\mathcal{T}^+(C(A), X)$. They give valuable information about operators on X, if we assume that $C(A)$ is dense in X. By $\mathcal{B}^+(X)$ we denote the class of all sequences $(\widetilde{T}_n)_1^\infty$ of positive linear operators, each with uniformly bounded norms, which map X into itself. Then the operators $T_n := \widetilde{T}_n I$, $n = 1, 2, \ldots$ map $C(A)$ into X and we have, for example:

1. *If S of Theorems* 6.3, 6.4 *is a dominating convergence set for the class $\mathcal{T}^+(C(A), X)$, then S has this property for all sequences $(\widetilde{T}_n) \in \mathcal{B}^+(X)$.*

2. *In particular, this is true if the $\widetilde{T}_n \in \mathcal{T}_1^+(X, X)$ are positive contractions of X.*

§ 7. Positive Contractions

The theory of shadows and of dominated convergence sets for sequences of operators (T_n) that are contractions, is quite different from what we have had in §§5–6. The case of positive contractions is particularly simple.

A *contraction* of a Banach space (or a Banach lattice) X is a linear operator T of X into itself with $\|T\| \le 1$. The first theorems of Korovkin type for contractions appear in Shashkin [1969] and Wulbert [1968]. We treat here and in the next section this theory for Banach lattices, in particular for spaces $L_p(A,\mu,\mathcal{B})$, $1 \le p < +\infty$, where $(A,\mu,\mathcal{B})$ is a separable measure space. Here μ is a measure on A, and $\mathcal{B}$ is the class of all μ-measurable sets.

Let S be a fixed closed subspace of X which is a closed line or span of the "test elements" $g \in S$. A *convergence set* $C_{\mathbf{T}}$ of a sequence of contractions $\mathbf{T} = (T_n)$ is the set $\{f \in X : T_n f \to f\}$. The *shadow* $\Sigma_1(S)$ is the set of all $f \in X$, for which $T_n g \to g$ for all $g \in S$ for a sequence of contractions T_n implies $T_n f \to f$.

In other words,

$$(7.1) \qquad \Sigma_1(S) = \bigcap_{C_{\mathbf{T}} \supset S} C_{\mathbf{T}} \ , \quad \text{for all sequences of contractions } \ \mathbf{T} \subset \mathcal{T}_1 \ .$$

Using *positive contractions* of a Banach lattice X instead, we obtain the shadow $\Sigma_1^+(S)$. We have the following properties, where Σ means either of Σ_1 or Σ_1^+ and $C_{\mathbf{T}}$ is the convergence set for contractions of the corresponding class $\mathcal{T}_1$ or $\mathcal{T}_1^+$:

(a) $\Sigma(S) \supset S$;
(b) $S_1 \subset S_2 \Rightarrow \Sigma(S_1) \subset \Sigma(S_2)$
(c) $\Sigma(C_{\mathbf{T}}) = C_{\mathbf{T}}$
(d) $\Sigma(\Sigma(S)) = \Sigma(S)$, by the definition of Σ, by (a) and (b);
(e) $\Sigma_1(S) \subset \Sigma_1^+(S)$.

There is an important estimate of the shadow from above. For a contraction T, let I_T be the linear closed space of the fixed points

$$I_T = \{f : Tf = f\} \ .$$

In particular, if T is a *projection* of X onto a linear space R (the *range* of the projection T), then $I_T = R$.

(f) *If for some contraction T, $I_T \supset S$, then $\Sigma_1(S) \subset I_T$.*

Indeed, I_T is the set of convergence for the constant sequence $T_n = T$, $n = 1, 2, \ldots$. A similar statement holds for $\Sigma_1^+(S)$ and positive contractions T.

To find a bound for $\Sigma_1^+(S)$ from below, we assume that X has *uniformly monotone norm*. This means (Birkhoff [B-1973, p.371]) that the modulus of monotonicity of X,

$$(7.2) \qquad \delta_m(\varepsilon) = \inf\left\{1 - \|f^*\| : 0 \le f^* \le f \ , \ \|f\| = 1 \ , \ \|f - f^*\| \ge \varepsilon\right\}$$

which is defined for $0 < \varepsilon \le 1$, satisfies $\delta_m(\varepsilon) > 0$. Or equivalently, for each $0 < \varepsilon \le 1$ there is a $\delta > 0$ with the property that $\|f^*\| > 1 - \delta$, $0 \le f^* \le f$, $\|f\| = 1$ imply $\|f - f^*\| < \varepsilon$. This in turn is equivalent to

(7.3) *If* $f_n, f_n^* \in X$, $n = 1, 2, \ldots$, *satisfy* $0 \le f_n^* \le f_n$ *and if* $M \ge \|f_n\| \ge \|f_n^*\| \to M$, $n \to \infty$ *for some* $M \ge 0$, *then* $\|f_n - f_n^*\| \to 0$.

The space X is *uniformly convex*, if $\delta_c(\varepsilon) > 0$, $0 < \varepsilon < 2$, where $\delta_c(\varepsilon)$ is the modulus of convexity,

$$\delta_c(\varepsilon) = \inf\left\{1 - \left\|\frac{f+f^*}{2}\right\| : \|f\| = \|f^*\| = 1\ ,\ \|f - f^*\| \ge \varepsilon\right\} . \tag{7.4}$$

This condition implies $\delta_m(\varepsilon) > 0$. Indeed, in the definition (7.4) one can replace the conditions $\|f\| = \|f^*\| = 1$ by $\|f^*\| \le \|f\| = 1$. If f, f^* satisfy $0 \le f^* \le f$, then $\|f^*\| \le \frac{1}{2}\|f + f^*\|$. Hence

$$\delta_c(\varepsilon) \le \delta_m(\varepsilon)\ , \qquad 0 < \varepsilon \le 1\ . \tag{7.5}$$

The spaces L_p, $1 < p < \infty$ are uniformly convex (Clarkson [1936]). For the space $L_1(A, \mu, \mathcal{B})$ one has immediately $\delta_m(\varepsilon) = \varepsilon$. These remarks and (7.5) show that all spaces L_p, $1 \le p < \infty$ have uniformly monotone norm.

Theorem 7.1 (Douglas [1965]). *For a Banach lattice X with uniformly monotone norm:*
(i) *each convergence set $C_{\mathbf{T}}$ of positive contractions is a lattice;*
(ii) *for $S \subset X$, and the closed linear lattice $\widehat{S}$ generated by S,*

$$\widehat{S} \subset \Sigma_1^+(S)\ . \tag{7.6}$$

Proof. It is sufficient to prove (i). We show that $f \in C_{\mathbf{T}}$ implies $|f| \in C_{\mathbf{T}}$:

$$\begin{aligned} T_n|f| - |f| &= (T_n|f| - |T_nf|) + (|T_nf| - |f|) \\ &\le (T_n|f| - |T_nf|) + |T_nf - f|\ . \end{aligned}$$

This last term converges to zero. Also, if $f_n^* = |T_nf|$, $f_n = T_n|f|$, then $0 \le f_n^* \le f_n$ and $\|f\| \ge \|f_n\| \ge \|f_n^*\| \to \|f\|$. By (7.3), $\|T_n|f| - |T_nf|\,\| \to 0$. □

In particular, if $\widehat{S} = X$, *then S is a dominated convergence set (or a Korovkin set) for positive contractions.*

For the spaces $L_p(A, \mu, \mathcal{B})$ we can use a theorem of Douglas and Andô (see Lacey [B-1974, p.146]):

Theorem 7.2 *Each closed linear lattice of L_p, $1 \le p < +\infty$ is the range of a positive contractive projection.*

As a corollary of (f) and Theorem 7.2, we have

Theorem 7.3 (Berens and Lorentz [1974]). *For a space $L_p(A, \mu, \mathcal{B})$, $1 \le p < \infty$, and each $S \subset L_p$,*

$$\Sigma_1^+(S) = \widehat{S} \ . \tag{7.7}$$

In the special case when S consists of just two functions $1, g$, one can describe the condition $\widehat{S} = X$ in another way. We shall assume that X is a Banach function space of $(A, \mu, \mathcal{B})$-measurable functions with separable measure μ, which is spanned by the characteristic functions χ_B, $B = [x : a \le g(x) \le b]$, $\mu B < +\infty$, and that X has uniformly monotone norm. *If the closed Boolean algebra generated by the sets B is identical with $\mathcal{B}$, we have $\widehat{S} = X$.*

Examples. 1. For $A = [0, 1]$ and the Lebesgue measure λ, functions $1, x$ form a Korovkin set for $X(A, \lambda, \mathcal{B})$.

2. For $A = [0, \infty)$, we have the Korovkin set $1, (1 + x^2)^{-1}$.

Proposition 7.4 *For a separable measure space $(A, \mu, \mathcal{B})$ with non-atomic measure there exists a measurable bounded function g, for which $S = \{1, g\}$ is a Korovkin set for any lattice X with the above properties, associated with $(A, \mu, \mathcal{B})$.*

Proof. By a theorem of Carathéodory, the measure space $(A, \mu, \mathcal{B})$ is isomorphic to a subspace of the Lebesgue measure space $(A_0, \lambda, \mathcal{B})$ where $A_0 = [0, 1]$ or $A_0 = [0, \infty)$. The lattice X is isomorphic to a Banach function space X on $(A_0, \lambda, \mathcal{B})$ and the functions of Examples 1,2 are transferable to A. □

§ 8. Contractions

For the shadows $\Sigma_1(S)$ for arbitrary contractions, there is no such simple results as Theorem 7.1. That the problem is more difficult is shown by the special position of the space $L_2(A, \mu, \mathcal{B})$ among all other spaces L_p. For each closed subspace $S \subset L_2$, there exists a projection of norm one of L_2 onto S, consequently

$$\Sigma_1(S) = S \quad , \quad X = L_2(A, \mu, \mathcal{B}) \ . \tag{8.1}$$

The situation for L_p, $p \ne 2$ is very much different. We need

Lemma 8.1 *Let X be a Banach function space over $(A, \mu, \mathcal{B})$ and let A be the union of a sequence of increasing sets $B \in \mathcal{B}$, $\mu B < \infty$. Then each closed subspace S of X has a function of maximal support.*

The *proof* depends on the following remark. *Let $f_1, f_2 \in S$, with supports $D_i = \operatorname{supp} f_i$, $i = 1, 2$. Then for all but at most countably many α, modulo sets of measure zero,*

$$B_\alpha := \operatorname{supp}(f_1 + \alpha f_2) = D_1 \cup D_2 \ . \tag{8.2}$$

Clearly, $B_\alpha \subset D_1 \cup D_2$ for all α. If x belongs to the complements $B_\alpha^\sim$ of the sets B_α for two different values of α, then $f_1(x) + \alpha f_2(x) = 0$ for these α, hence $f_1(x) = f_2(x) = 0$, $x \notin D_1 \cup D_2$. It follows that the intersections $B_\alpha^\sim \cap (D_1 \cup D_2)$ are disjoint, and at most countably many of them have measure > 0. For all other α we have (8.2).

For $D = \bigcup_{f \in S} \operatorname{supp} f$, there exists a sequence $f_k \in S$ with $D = \bigcup_{k=1}^\infty \operatorname{supp} f_k$. We put $g_0 = \sum_{k=1}^\infty \varepsilon_k f_k$ with $\varepsilon_k > 0$ so small that $\sum \varepsilon_k \|f_k\| < +\infty$. Then $\sum \varepsilon_k f_k(x)$ converges a.e., and also $\sum \varepsilon_k f_k(x)$ converges to some $g_0 \in X$. Because of the remark made earlier, we can require that all functions $g_n = \sum_1^n \varepsilon_k f_k$ satisfy

$$D_n := \operatorname{supp} g_n = \bigcup_{k=1}^n \operatorname{supp} f_k \ .$$

Let U_n be an increasing sequence of sets of finite measure with $\cup U_n = A$. In addition to the above restrictions on ε_n, we require that for each n, the set e_n of points $x \in D_k \cap U_k$ for which one of the inequalities

$$\varepsilon_n |f_n(x)| < \frac{1}{2^{n-k+1}} |g_k(x)| \ , \qquad k = 1, \ldots, n-1 \tag{8.3}$$

is violated, has measure $\mu e_n < 2^{-n}$. This can be done because $g_k(x) \neq 0$ on D_k.

Let $e_n^* = \bigcup_{k=n+1}^\infty e_k$. Then for each k, each $x \in (D_k \cap U_k) \setminus e_k^*$, and $n > k$

$$\begin{aligned} |g_n(x)| &= |g_k(x) + \varepsilon_{k+1} f_{k+1}(x) + \cdots + \varepsilon_n f_n(x)| \\ &\geq |g_k(x)| \left(1 - \frac{1}{2^2} - \cdots - \frac{1}{2^{n-k+1}}\right) \geq \frac{1}{2} |g_k(x)| \ . \end{aligned}$$

This means that on this set, $g_0(x) \neq 0$. Since this is also true with k replaced by n, $g_0(x) \neq 0$ for $x \in (D_k \cap U_k) \setminus e_n^*$, $n > k$. It follows $g_0(x) \neq 0$ a.e. on D. □

The following considerations depend essentially on a function $g_0 \in S$ of maximal support, which we assume to be fixed. We put

$$\phi(x) = \begin{cases} 1 & \text{if } g_0(x) \geq 0 \\ -1 & \text{if } g_0(x) < 0 \ . \end{cases} \tag{8.4}$$

The following theorem for $p = 1$ is due to Berens and Lorentz [1974]; for $p \neq 1, 2$, Bernau [1974] gave a different characterization of the shadow. Very essential in our proof is Theorem 8.4.

Theorem 8.2 *Let* $1 \le p < +\infty$, $p \neq 2$. *For* $L_p(A, \mu, \mathcal{B})$, *the shadow* $\Sigma_1(S)$ *of a subspace* S *is* (i) *the space* $\widehat{S}$ *if* $\phi(x) \ge 0$ *a.e.;* (ii) *it is* $\phi(\phi S)^\wedge$ *in the general case.*

Proof of Theorem 8.2 *for* $p = 1$. Let $D = \operatorname{supp} g_0$; the set $L_1(D)$ consisting of functions $f\chi_D$, $f \in L_1$ is the range of the positive contractive projection given by $f \to f\chi_D$.

To prove (i), we need the special property of an L_1-space: *for each linear operator* T *on* L_1, *there exists the absolute value* $|T|$ *of* T, *which is a linear operator with*

$$(8.5) \qquad |Tf| \le |T|\,|f| \ , \ f \in L_1 \quad ; \quad \|\,|T|\,\| = \|T\| \ .$$

(See Vulikh [B-1967].)

Lemma 8.3 *If* $g_0 > 0$, *if* T_n *is a sequence of contractions of* L_1 *and if* $T_n g_0 \to g_0$, *then*

$$(8.6) \qquad |T_n| f - T_n f \to 0 \ , \qquad f \in L_1(D) \ .$$

Proof. For $g_n = |T_n g_0|$ and $f_n = |T_n| g_0$ we have $0 \le g_n \le f_n$, $\|g_n\| \to \|g_0\|$ and $\|g_n\| \le \|f_n\| \le \|g_0\|$, hence $|T_n| g_0 - |T_n g_0| \to 0$. Because the absolute value is continuous, $|T_n g_0| \to g_0$, and we get $|T_n| g_0 - T_n g_0 \to 0$, that is, we have proved (8.6) for $f = g_0$.

Now let $0 \le |f| \le g_0$; then since $|T_n| - T_n$ is positive, $0 \le |(|T_n| - T_n) f| \le (|T_n| - T_n)|f| \le (|T_n| - T_n) g_0 \to 0$. Since functions λf, $\lambda \in \mathbb{R}$ with $|f| \le g_0$ are dense in $L_1(A)$, and since $\|\,|T_n| - T_n\| \le 2$, we have the complete statement (8.6). □

To prove statement (i), we use Theorem 7.3 for $p = 1$ and show that $\Sigma_1(S) = \Sigma_1^+(S)$. To begin with,

$$(8.7) \qquad \Sigma_1(S) \subset \Sigma_1^+(S) \subset L_1(A) \ .$$

Let $f \in \Sigma_1^+(S)$. Let T_n be a sequence of contractions with the property $T_n g \to g$ for all $g \in S$. In particular, $T_n g_0 \to g_0$, hence by the lemma, $|T_n| f - T_n f \to 0$ for all $f \in L_1(A)$. In particular, for $f = g$ we get $|T_n| g \to g$, $g \in S$. Since $|T_n|$ are positive contractions, this implies $|T_n| f \to f$ for all $f \in L_1(A)$, hence also $T_n f \to f$.

(ii) Let $\{T_n\}$ be a sequence of contractions for which $T_n g \to g$ for $g \in S$. We put $S^* := \phi S$ and $T_n^* := \phi T_n \phi$, $n = 1, 2, \ldots$. The operators T_n^* are also contractions; for $g^* \in S^*$ we have $T_n^* g^* \to g^*$. Also, $T_n^* f \to f$ means that $T_n \phi f \to \phi f$. The function ϕg_0 is ≥ 0, and case (i) yields $\Sigma_1(S^*) = \widehat{S}^*$, or $\Sigma_1(S) = \phi(\phi S)^\wedge$. □

For $p \neq 1$ we need the following result by Andô [1969] and Bernau [1974]:

Theorem 8.4 *For a sequence of contractions $\mathbf{T} = (T_n)$ of L_p, $p \neq 1, 2, \infty$, the convergence set $C := C_{\mathbf{T}}$ has the exchange property: if $f, g \in C_{\mathbf{T}}$, then $|f| \operatorname{sign} g \in C_{\mathbf{T}}$.*

Proof. With the sequence of operators T_n we associate the sequence $\mathbf{T}^* = (T_n^*)$ of their conjugates, also contractions, which map L_q, $1/p + 1/q = 1$, into itself. Let $C^* := C_{\mathbf{T}^*}$.

First we prove: (a) *A function $f \in L_p$ belongs to C if and only if $F = |f|^{p-1} \operatorname{sign} f$ belongs to C^*.* Let the first be the case. We have $\|F\|_q = \|f\|_p^{p/q}$, so that $F \in L_q$. The sequence $T_n^* F$ is bounded in L_q, hence relatively weakly compact. Let h be one of its weak limits: $T_{n_k}^* F \to h$. Then

$$\begin{aligned} \int f(h - T_{n_k}^* F)\, d\mu &= \int fh - \int f T_{n_k}^* F \\ &= \int fh - \int F T_{n_k} f \to \int fh - \int fF . \end{aligned}$$

Hence

$$\int fh = \int fF = \|f\|_p^p = \|f\|_p \|F\|_q .$$

Since $\|h\|_q \leq \underline{\lim} \|T_{n_k}^* F\| \leq \|F\|_q$, this is possible only if $h = |f|^{p-1} \operatorname{sign} f = F$. The weak limit of a subsequence of $T_n^* F$ is unique, hence $T_n^* F \to F$ weakly. But from this and $\|T_n^* F\| \leq \|F\|$ we have norm convergence. So we get $g \in C^*$. The second part of (a) follows by symmetry.

(b) The derivative of the function $|a + b\lambda|$ of the real variable λ exists and is equal to $b \operatorname{sign}(a + b\lambda)$, provided $a + b\lambda \neq 0$. Similarly, the derivative of $|a + b\lambda|^{p-1}$ for $p > 2$ is equal to $(p-1)|a + b\lambda|^{p-2} b \operatorname{sign}(a + b\lambda)$, and this is true even for all λ.

Assume that $p > 2$ and that $f, g \in C$. For real λ, $f + \lambda g \in C$, hence by (a) the function

$$h_\lambda(t) = |f(t) + \lambda g(t)|^{p-1} \operatorname{sign}\bigl[f(t) + \lambda g(t)\bigr]$$

belongs to C^*. Its derivative with respect to λ at $\lambda = 0$ is a.e.

$$(8.8) \quad (p-1)|f|^{p-2} g \operatorname{sign}^2 f = (p-1)|f|^{p-2} g \leq (p-1)(|f|^{p-1} + |g|^{p-1}) \in L_q .$$

But we need also to know that the convergence of $(h_\lambda - h_0)/\lambda$ for $\lambda \to 0$ to the function (8.8) is dominated in L_q. By Lagrange's formula, this quotient is equal to $(0 \leq \theta \leq 1)$,

$$h'_{\theta\lambda} = (p-1)|f + \theta\lambda g|^{p-2} g \operatorname{sign}^2(f + \theta\lambda g) ,$$

and for $|\lambda| \leq 1$,

$$|(h_\lambda - h_0)/\lambda| \le (p-1)(|f|+|g|)^{p-2}|g|$$
$$\le \text{Const } (|g|\,|f|^{p-2} + |g|^{p-1}) \in L_q \ .$$

This establishes that for $f, g \in C$, we have $|f|^{p-2}g \in C^*$. Applying (a), we get

$$g_1 = |f|^{(p-2)(q-1)}|g|^{(q-1)} \operatorname{sign} g = |f|^{1-r}|g|^r \operatorname{sign} g \in C \ , \tag{8.9}$$

where $r = q - 1$, $0 < r < 1$.

We repeat this argument. Combining the function g_1 with f we get

$$g_2 = |f|^{1-r}|g_1|^r \operatorname{sign} g = |f|^{1-r^2}|g|^{r^2} \operatorname{sign} g \in C \ .$$

In general, $g_n = |f|^{1-r^n}|g|^{r^n} \operatorname{sign} g \in C$. The function $|g_n|$ is dominated by $\max(|f|, |g|)$ and converges to $|f| \operatorname{sign} g$. We are done if $p > 2$. If $p < 2$, then $q > 2$, our statement is true for C^*, and applying (a), we deduce it also for C. □

Proof of Theorem 8.2 *for* $p \neq 1, 2, \infty$. Instead of (8.7), we have $\Sigma_1(S) \subset \Sigma_1^+(S) \subset L_p(A)$; a function $f \in \Sigma_1(S)$ is zero outside D. In case (i), $|f| \operatorname{sign} g_0 = |f| \in C$, and C is a closed linear lattice. So is $\Sigma_1(S)$, and from Theorem 7.3, $p > 1$ we see that it is $\hat{S}$. For (ii), we can follow the argument in the discussion of the case $p = 1$. □

This chapter gives only an incomplete exposition of the "Korovkin Theory." Compare also Altomare and Campiti [A-1995].

§ 9. Notes

9.1. For a lattice homeomorphism P of $C(A)$ into a Banach lattice X, let N_f, $f \in C(A)$ be the set of all zeros of f, $N_F := \{x \in A : f(x) = 0\}$, and let $N(P)$ be the null space of P. *The support* $\operatorname{supp} P$ of P is defined as the intersection

$$\operatorname{supp} P := \bigcap_{f \in N(P)} N_f \ .$$

Proposition 9.1 (Berens and Lorentz, [1973]). *A function* $f \in C(A)$ *belongs to* $N(P)$ *if and only if* f *vanishes on* $\operatorname{supp} P$.

With these notions, Theorem 5.3 can be improved *by replacing in it* $\widehat{G}_A$ *by the larger set* $\widehat{G}_{\operatorname{supp} P}$.

Chapter 17. Representation of Functions by Superpositions

§ 1. The Theorem of Kolmogorov

We shall discuss here exact representation of functions by *superpositions*, that is, by functions of functions. For example,

$$f(x,y,z) = F\Big(g(x,y), h\big(\phi(x), \psi(x,z)\big)\Big)$$

is a superposition of functions of one and two variables. Functions of several variables we would like to represent by superpositions of functions of fewer variables, as above.

Even a good beginning student may notice that the textbooks of calculus do not exhibit genuine functions of two or three variables. He meets the function $x + y$, but other functions reduce to this one and the functions of one variable, for example, $xy = e^{\log x + \log y}$, or $x+y+z = (x+y)+z$. The question is then: are there genuine functions of two or more variables, except the trivial one $x + y$? The astonishing answer of Kolmogorov is "no", for continuous functions.

Let $I = [0,1]$, let $I^n = [0,1]^n$ be the n-dimensional unit cube.

Theorem 1.1. *There exist n constants $\lambda_p > 0$, $p = 1, \ldots, n$, $\sum \lambda_p \le 1$, and $2n+1$ continuous strictly increasing functions ϕ_q, $q = 1, \ldots, 2n+1$ which map I into itself, with the property that each function $f \in C(I^n)$ has a representation*

$$f(x_1, \ldots, x_n) = \sum_{q=1}^{2n+1} g\big(\lambda_1 \phi_q(x_1) + \cdots + \lambda_n \phi_q(x_n)\big) \tag{1.1}$$

with some $g \in C(I)$, depending on f.

Remark. As we shall see later, one can assume $\phi_q \in \mathrm{Lip}_1 1$, $q = 1, \ldots, 2n+1$. (We write $\mathrm{Lip}_M \alpha$ for the subset of $\mathrm{Lip}(\alpha, L_\infty)$ with the Lipschitz constant $\le M$.)

This formula reduces the function f to sums and superpositions of functions of one variable, g, $\lambda_p \phi_q$. The function g depends on f, but the $n(2n+1)$ functions $\lambda_p \phi_q$ do not. It will be seen that for $n = 2$ we can take $\lambda_1 = 1$,

$\lambda_2 = \lambda$. We have, therefore, for each continuous function $f(x, y)$, $0 \le x, y \le 1$ a representation

$$f(x,y) = \sum_{q=1}^{5} g\big(\phi_q(x) + \lambda\phi_q(y)\big) .$$

The history of this theorem is very interesting. In his famous lecture at the International Conference of Mathematicians in Paris, 1900, Hilbert formulated 23 problems, which in his opinion were important for the further development of mathematics. They have since attracted the attention of many outstanding mathematicians. The thirteenth of these problems contained (implicitly) the conjecture that not all continuous functions of three variables are representable as superpositions of continuous functions of two variables. This conjecture was refuted in 1957 by Kolmogorov and his pupil, Arnold. In the last of their three papers on this subject, Kolmogorov [1957] proved Theorem 1.1. It is clear that Theorem 1.1 refutes Hilbert's conjecture since sums of several terms that appear in the formula (1.1) can be built up of sums of two terms. Kolmogorov's original formulation had the representation

$$f(x_1, \ldots, x_n) = \sum_{q=1}^{2n+1} g_q\left(\sum_{p=1}^{n} \phi_{p,q}(x_p)\right) \tag{1.2}$$

instead of (1.1). It has been improved to its present form by Sprecher, Lorentz and Fridman.

We can give a geometric interpretation of Theorem 1.1. Consider the continuous map of I^n into $\mathbb{R}^{2n+1}$ given by

$$z_q = \lambda_1\phi_q(x_1) + \cdots + \lambda_n\phi_q(x_n) , \qquad q = 1, \ldots, 2n+1 . \tag{1.3}$$

This map is one-to-one. For otherwise there would exist two points of I^n which are not distinguished by the family of functions $\lambda_1\phi_q(x_1) + \cdots + \lambda_n\phi_q(x_n)$, $q = 1, \ldots, 2n+1$. Then all f representable by the sum (1.1) would coincide at these two points, and the representation (1.1) would be impossible for some functions $f \in C(I^n)$.

Since I^n is compact, its image T under (1.3) is also compact, and (1.3) is a homeomorphism between I^n and T. It follows that there is a one-to-one correspondence between all continuous functions on I^n and all continuous functions $F(z_1, \ldots, z_{2n+1})$ on T. Therefore, Theorem 1.1 is equivalent to the following. There exists a homeomorphic imbedding of I^n into $\mathbb{R}^{2n+1}$, of the special form (1.3), so that each continuous function F on the image of I^n has the form

$$F(z_1, \ldots, z_{2n+1}) = \sum_{q=1}^{2n+1} g(z_q) . \tag{1.4}$$

Let S be the rectifiable curve in $\mathbb{R}^{2n+1}$ given parametrically by

$$z_q = \phi_q(u) , \qquad 0 \le u \le 1 , \quad q = 1, \ldots, 2n+1 . \tag{1.5}$$

Another equivalent form of Theorem 1.1 is the following. There exists a curve S given by (1.5), so that for n copies S_p, $p = 1, \dots, n$ of it, the set $T = \lambda_1 S_1 + \dots + \lambda_n S_n$ has the property described by (1.4). Among all representations (1.5), the most natural one is when the parameter u is arc length along S. Then the functions ϕ_q belong to the class $\mathrm{Lip}_1\, 1$. This justifies the Remark to Theorem 1.1.

One can give an "honest" proof of Theorem 1.1 with a construction of the functions ϕ_q, see Lorentz [A-1966]. Simpler is the idea of Kahane [1975] to use the Baire category theorem.

We consider the complete metric space X of all increasing continuous functions ϕ on $[0,1]$, for which $0 \le \phi(x) \le 1$, with the metric $\rho(\phi, \phi') = \max_x |\phi(x) - \phi'(x)|$. Let X^{2n+1} be the product of $2n+1$ copies of X with the metric $\max_{q=1,\dots,2n+1} \rho(\phi_q, \phi'_q)$.

A subset of first category in a metric space Y is a countable union of nowhere dense sets. The complements of sets of first category are precisely the countable intersections of dense sets in Y. According to Baire's category theorem, an intersection of this type is not empty if Y is complete. We shall say that a property holds "quasi-everywhere" in Y if it holds at all points of Y except for some set of first category.

As an example, we can show that for quasi all $(\phi_1, \dots, \phi_{2n+1}) \in X^{2n+1}$, each ϕ_q is strictly increasing. It is sufficient to show that quasi all $\phi \in X$ have this property. But each $\phi \in X$ which is not strictly increasing is constant on an interval $I' = [r, r']$ with rational r, r', $0 < r < r' < 1$, and the set of $\phi \in X$ constant on I' is a closed nowhere dense subset of X.

Theorem 1.1 will follow from the following statement:

Theorem 1.2 (Kahane). *For a given n and rationally independent $\lambda_p > 0$, $\sum_{p=1}^n \lambda_p \le 1$, quasi all $(\phi_1, \dots, \phi_{2n+1}) \in X^{2n+1}$ have the property that for each $f \in C(I^n)$ there exists a $g \in C(I)$ satisfying* (1.1).

Numbers λ_p are called *rationally independent*, if an equation $\sum_{p=1}^n r_p \lambda_p = 0$ with rational r_p is possible only if all $r_p = 0$.

§ 2. Proof of the Theorems

Let $(\lambda_1, \dots, \lambda_n)$ be a fixed sequence of rationally independent numbers. In what follows, we will have to consider several sums $\sum_{p=1}^n \lambda_p y_{j_p}$, and it is important that they can be made all different.

Lemma 2.1. *Let $\mathbb{R}^M$ be the euclidean space with points $y = (y_1, \dots, y_M)$, and let $c_1, \dots, c_m$ be a finite set of real numbers. Except for a nowhere dense set of $y \in \mathbb{R}^M$, all sums $\sum_{p=1}^n \lambda_p y_{j(p)}$, where $j(p)$, $p = 1, \dots, n$, is any function with values in the set $1, \dots, M$, are different from each other and different from $c_1, \dots, c_m$.*

Proof. Each of the equations

$$\sum_{p=1}^{n} \lambda_p y_{j(p)} = \sum_{p=1}^{n} \lambda_p y_{j'(p)} , \tag{2.1}$$

where j, j' are two functions of the described type, either defines a hyperplane in $\mathbb{R}^M$, or is an identity. We have to show that the latter cannot happen unless the functions $j(p)$, $j'(p)$ are identical. Let $j'(p_0) \neq j(p_0) = j$. Let P and P' be sets of $p = 1, \ldots, n$, for which $j(p) = j$ and $j'(p) = j$, respectively. Then $P \neq P'$. The coefficients of y_j in the sums (2.1) are $\sum_{p \in P} \lambda_p$ and $\sum_{p \in P'} \lambda_p$. These sums cannot be equal, since the λ_p are rationally independent. Thus, the nowhere dense set of the lemma is a union of finitely many hyperplanes. □

The next lemma is the key to the proof; it is basic for Kolmogorov's construction.

Lemma 2.2. *Let $0 < \varepsilon < 1$, $f \in C(I^n)$, $f \neq 0$, and let $\Omega(f)$ be the collection of all $(\phi_1, \ldots, \phi_{2n+1}) \in X^{2n+1}$ for which there exists a function $h \in C(I)$ with the properties*

$$\|h\| \le \|f\| , \tag{2.2}$$

$$\left\| f - \sum_{q=1}^{2n+1} h(\Phi_q) \right\| < (1 - \varepsilon)\|f\| , \tag{2.3}$$

where

$$\Phi_q(x_1, \ldots, x_n) = \sum_{p=1}^{n} \lambda_p \phi_q(x_p) . \tag{2.4}$$

Then the set $\Omega(f)$ for sufficiently small ε is dense in X^{2n+1}.

Proof. To prove that $\Omega(f)$ is dense we take an arbitrary point $(\phi_1^0, \ldots, \phi_{2n+1}^0) \in X^{2n+1}$, a neighborhood U of this point and construct $(\phi_1, \ldots, \phi_{2n+1}) \in U$ that belongs to $\Omega(f)$. We take $0 < \varepsilon < 1/(4n+2)$.

Let $\delta > 0$. We consider the system S_1 of closed intervals $[(2n+1)j\delta, (2n+1)j\delta + 2n\delta]$, $j = \cdots, -1, 0, 1, \cdots$. Each interval has length $2n\delta$, they together cover the line except for small gaps of length δ between them. Their translations form $2n+1$ systems S_q, $q = 1, \ldots, 2n+1$, of intervals

$$I_q(j) = \left[(q-1)\delta + (2n+1)j\delta, (q-1)\delta + (2n+1)j\delta + 2n\delta\right] , \tag{2.5}$$

which we restrict to values of j for which $I_q(j)$ intersects $[0,1]$. If M is the number of intervals in S_1, the other systems will have M, $M+1$ or $M-1$ intervals. From intervals $I_q(j)$ we form, for each q, a set of disjoint cubes

$$C_q := C_q(j_1, \ldots, j_n) = I_q(j_1) \times \cdots \times I_q(j_n) ,$$

with intervals $I_q(j)$ in the system S_q. We take δ so small that the oscillation of f on each cube C_q is $< \frac{1}{2n+2}\|f\|$.

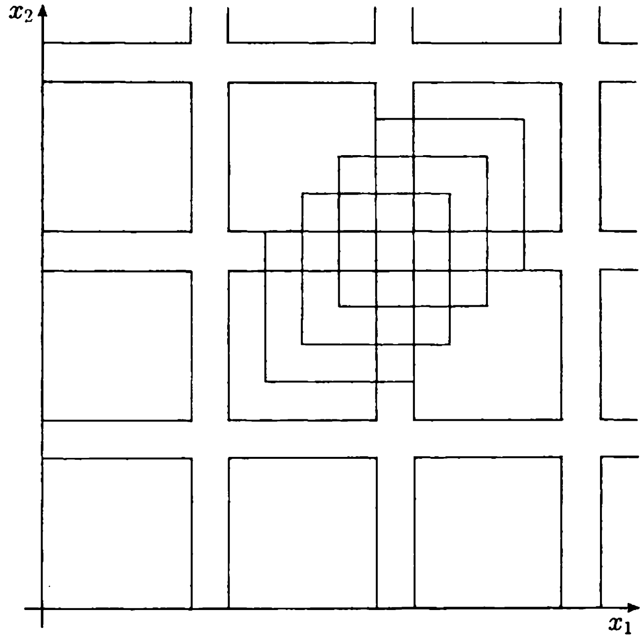


Fig. 2.1. $n = 2$: squares C_1, and one each of C_q, $q = 2, \dots, 5$.

The intervals of any fixed system S_q cover $[0,1]$ with small gaps, but the intervals of all the $2n+1$ systems S_q combined cover $[0,1]$ completely. Each $x \in [0,1]$ is covered by the intervals of the systems at least $2n$ times, not covered by at most one system, for one value of q. Consequently, each $x = (x_1, \dots, x_n) \in I^n$ is covered by a cube C_q for at least $n+1$ different values of q, and fails to be covered by at most n values of q.

We shall now define the continuous increasing functions ϕ_q. On each interval $I_q(j)$, $\phi_q(x)$ is a constant $y_{q,j}$; between two adjacent intervals of S_q, ϕ_q is linear. If this fails to define ϕ_q near 0 or 1, we take it to be constant on the small remaining intervals.

If $\delta > 0$ is sufficiently small, and if we take $y_{q,j}$ close to the value of ϕ_q^0 at the left endpoint of $I_q(j)$, the set $(\phi_1, \dots, \phi_{2n+1})$ will be in the neighborhood U.

On the cube $C_q = C_q(j_1, \dots, j_n)$, the function Φ_q of (2.4) will have the constant value

$$a_{q,j_1,\dots,j_n} := \Phi_q(C_q) = \sum_{p=1}^{n} \lambda_p y_{q,j_p} \ . \tag{2.6}$$

Invoking Lemma 2.1 with M equal to the number of intervals in S_1, we can make all $a_{1,j_1,\dots,j_n}$ different, and then, by repeated application of the lemma, we can even achieve that all the numbers (2.6) are different.

Now let $b_{q,j_1,\dots,j_n}$ be the value of f at the center of $C_q(j_1, \dots, j_n)$. For each point x in this cube, (2.7) $$f(x) = b_{q,j_1,\dots,j_n} + \rho\,, \qquad |\rho| < \frac{1}{2n+2}\|f\|\,.$$

We define the function $h \in C(I)$ by putting

$$h(a_{q,j_1,\dots,j_n}) = \frac{1}{2n+1} b_{q,j_1,\dots,j_n}$$

and by extending it continuously onto $[0,1]$ so that

$$\|h\| \le \frac{1}{2n+1}\|f\|\,.$$

Then for each $x \in I^n$,

(2.8) $$\left|\frac{1}{2n+1} f(x) - h(\Phi_q(x))\right| \le \frac{2}{2n+1}\|f\|\,.$$

Let Q be the set of $(n+1)$ values of q for which some cubes C_q cover x. Then

$$\begin{aligned}\left|f(x) - \sum_{q=1}^{2n+1} h(\Phi_q(x))\right| &\le \sum_{q\in Q}\left|\frac{1}{2n+1} f(x) - h(\Phi_q(x))\right| + \sum_{q\notin Q}\\ &\le \frac{n+1}{2n+1}\,\frac{1}{2n+2}\|f\| + \frac{2n}{2n+1}\|f\|\\ &= \frac{2n+\frac{1}{2}}{2n+1}\|f\| < (1-\varepsilon)\|f\|\,,\end{aligned}$$

by (2.7) and (2.8). □

Proof of Theorem 1.2. The statement of the lemma is much weaker than that of Theorem 1.2 but nevertheless implies it easily.

Let $\bar F = \{\bar f\}$ be a countable dense subset of $C(I^n)$. We shall show that (1.1) holds for all

$$(\phi_1,\dots,\phi_{2n+1}) \in \bigcap_{\bar f\in\bar F} \Omega(\bar f)\,.$$

Let $f_0 \in C(I^n)$ be arbitrary, let $\varepsilon > 0$ be as in Lemma 2.2. If $f_0 \ne 0$, we select $\bar f \in \bar F$, for which

$$0 < \|\bar f\| \le \|f_0\|\,, \qquad \|\bar f - f_0\| < \tfrac{1}{2}\varepsilon\|f_0\|\,.$$

Then for $\bar f$ we get, according to the lemma (with ε replaced by $\frac{1}{2}\varepsilon$) an $h_0 \in C(I)$ and have $\|h_0\| \le \|f_0\|$, $\|f_1\| \le (1-\varepsilon)\|f_0\|$ with $f_1 = f_0 - \sum_{q=1}^{2n+1} h_0(\Phi_q)$. If $f_0 = 0$, we take $h_0 = 0$.

We continue this process, for f_1 we construct an h_1 using Lemma 2.2, obtain the remainder f_2, and so on. In general, for $i = 0, 1, \dots,$ we shall have

$$\|h_i\| \leq \|f_i\| \quad , \quad \|f_{i+1}\| \leq (1-\varepsilon)\|f_i\| \leq (1-\varepsilon)^{i+1}\|f_0\| \, ,$$

$$f_{i+1} = f_i - \sum_{q=1}^{2n+1} h_i(\Phi_q) \, . \tag{2.9}$$

Then $\sum \|f_i\| < +\infty$, $\sum \|h_i\| < +\infty$ and for $g = \sum_0^\infty h_i$ we obtain, using (2.9),

$$0 = f_0(x) - \sum_{q=1}^{2n+1} g\big(\Phi_q(x)\big) \, . \qquad \square$$

§ 3. Functions Not Representable by Superpositions

Theorem 1.1 disproves what we called the "Hilbert conjecture". But it does not solve Hilbert's 13-th problem completely. For one has to examine whether this conjecture is true in other important situations. That this is indeed so has been shown by Vitushkin [1954]:

Theorem 3.1. *Let $r \geq 1$. There exist for $n \geq 2$, r times continuously differentiable functions of n variables, not representable by r-times continuously differentiable functions of fewer variables. There exist r-times continuously differentiable functions of two variables, not representable by sums and r-times continuously differentiable functions of one variable.*

Hilbert's conjecture was based on a sound principle, namely that not all "bad" functions f can be superpositions of "good" functions. But the number n of variables is not a reasonable measure for the "badness" of f (nor is $1/n$ one for the "goodness" of f). Reasonable is the following characteristic $\chi(f)$. Let $\Lambda_n^\beta := \Lambda^\beta(I^n)$, $\beta = r + \alpha$, $r = 0, 1, \ldots$, $0 < \alpha \leq 1$ be the space of all functions on I^n with continuous derivatives of orders $\leq r$, and with all derivatives of order r belonging to $\mathrm{Lip}\,\alpha$. In other words, Λ_n^β is the Lipschitz space $\mathrm{Lip}(\alpha, L_\infty)$ on I^n. We say that f has characteristic $\chi(f) \geq \beta/n$ if $f \in \Lambda_n^\beta$. We say that $\chi(f) > \chi_0$ if $\chi(f) \geq \beta/n$, with $\beta/n > \chi_0$.

Theorem 3.2 (Kolmogorov and Tikhomirov [1959]). *Let β_0/n_0 be given, $\chi_0 = \beta_0/n_0$. Not all functions $f \in \Lambda_{n_0}^{\beta_0}$ can be represented by superpositions with characteristics $\chi = (r+\alpha)/n > \chi_0$ and with $r \geq 1$.*

Functions $f \in C^r(I^n)$ have characteristic $\chi(f) \geq r/n$, hence Theorem 3.1 is an immediate corollary of this.

Proof of Theorem 3.2. First we must introduce some new notions. They will help us describe all possible superpositions. Consider, for example, the function $g(h(x,y),z)$. We can write this as $g(y_1, y_2)$, where $y_1 = h(y_{1,1}, y_{1,2})$, $y_2 = z$ and $y_{1,1} = x$, $y_{1,2} = y$. In this example, the basic variable z is reached in one step, while two steps are needed to reach the basic variables x and y. We can make the number of steps equal by introducing intermediate variables $y_{2,1} = y_2$, $z = y_{2,1}$. This leads to the following definitions. A *scheme* S is a table of natural numbers of the following type:

$$S : \begin{cases} n, p \\ m \\ m_{k_1} \ , \ k_1 = 1, \ldots, m \\ m_{k_1,k_2} \ , \ k_1 = 1, \ldots, m \ ; \ k_2 = 1, \ldots, m_{k_1} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ m_{k_1,\ldots,k_p} \ , \ k_1 = 1, \ldots, m \ ; \ \ldots \ ; \ k_p = 1, \ldots, m_{k_1,\ldots,k_{p-1}} \\ \text{each } m_{k_1,\ldots,k_p} \text{ is one of the numbers } 1, \ldots, n \ . \end{cases}$$

With S we associate *admissible sets of subscripts*. These are all sets of natural numbers $k_1, \ldots, k_r$, $0 \le r \le p$, which appear as subscripts of the integers m in S. The value $r = 0$ is not excluded and gives the empty set of subscripts.

If a scheme S is given, with each admissible set of subscripts we associate a function $g_{k_1,\ldots,k_r}$ of $m_{k_1,\ldots,k_r}$ variables. We set up the formulas:

$$(3.1) \quad \begin{cases} f(x_1, \ldots, x_n) = g(y_1, \ldots, y_m) \ ; \\ \qquad y_{k_1} = g_{k_1}(y_{k_1,1}, \ldots, y_{k_1,m_{k_1}}) \ , \\ \qquad \cdots\cdots\cdots\cdots \\ y_{k_1,\ldots,k_{p-1}} = g_{k_1,\ldots,k_{p-1}}(y_{k_1,\ldots,k_{p-1}1}, \ldots, y_{k_1,\ldots,k_{p-1},m_{k_1,\ldots,k_{p-1}}}) \ , \\ \quad y_{k_1,\ldots,k_p} = x_{m_{k_1,\ldots,k_p}} \ . \end{cases}$$

The subscripts of the y's in (3.1) are precisely all nonempty admissible sets of subscripts of the scheme S.

We shall say that f is a *superposition of the functions* g (with admissible sets of subscripts), if all functions g are defined on the unit cubes of the corresponding spaces and if all these functions (except perhaps the function g without subscripts) have values that satisfy $0 \le g_{k_1,\ldots,k_r} \le 1$. Clearly, f is defined on the unit cube $I^n : 0 \le x_k \le 1$, $k = 1, \ldots, n$.

We need the notion of a *type* T *of superpositions*. T is given by a scheme S and by an assignment of a class

$$\Lambda := \Lambda_{k_1,\ldots,k_r} = \Lambda^\beta(M_0, \ldots, M_r/I^{m_*}) \subset \mathrm{Lip}_M 1$$

for each admissible set of subscripts; the number of variables must be $m_* := m_{k_1,\ldots,k_r}$. A function g belongs to this class, if $g \in \Lambda^\beta(I^{m_*})$ and if the partial derivatives of g of orders $s = 0, \ldots, r$ do not exceed M_s in absolute value. Under these assumptions, T consists of all superpositions f that can be formed, according to (3.1), with functions $g_{k_1,\ldots,k_r} \in \Lambda_{k_1,\ldots,k_r}$. We see that T is a set of continuous functions f on I^s. With each type T, we associate the integer p – the *height* of T, and the number $M \ge 0$ – the maximum of all M corresponding to the different classes Λ.

Lemma 3.3. *The entropy of each type T satisfies*

$$(3.2) \qquad H_{\varepsilon_1}(T) \le \sum H_\varepsilon(\Lambda_{k_1,\ldots,k_r}) \;, \qquad \varepsilon_1 = (M+1)^p \varepsilon \;,$$

where the sum is extended over all admissible sets of subscripts of the scheme S.

Proof. It depends on an estimation of entropy of multivariate Lipschitz balls, similar to (3.19) of Chapter 15. Let T_k, $k = 1, \ldots, m$, be the types of height $p-1$ that are obtained from T in the following way: We remove the first row in (3.1) and fix the first subscript $k_1 = k$ in all remaining rows. In other words, the type T_k consists of all possible functions $f_k(x_1, \ldots, x_s) = g_k(y_{k,1}, \ldots, y_{k,m_k})$, obtainable by (3.1), with functions $g_{k_1,\ldots,k_r}$ in the classes $\Lambda_{k_1,\ldots,k_r}$, and defined on I^n. For all $f \in T$,

$$f = g(f_1, \ldots, f_m) \;, \quad g \in \Lambda \;, \quad f_k \in T_k \;, \quad k = 1, \ldots, m \;.$$

Let $\varepsilon_2 = (M+1)^{p-1}\varepsilon$. Let U be the sets of some minimal ε-covering of Λ, $U^{(k)}$ the sets of minimal ε_2-coverings of T_k, $k = 1, \ldots, m$. If $f = g(f_1, \ldots, f_m)$, $f' = g'(f_1', \ldots, f_m')$, where g and g' belong to the same U, and where for each k, f_k and f_k' belong to the same $U^{(k)}$, then, by definition of the class $\mathrm{Lip}_M\, 1$,

$$\begin{aligned}
|f(x_1, \ldots, x_s) - f'(x_1, \ldots, x_s)| &\le |g(f_1, \ldots, f_m) - g'(f_1, \ldots, f_m)| \\
&\quad + |g'(f_1, \ldots, f_m) - g'(f_1', \ldots, f_m')| \le \|g - g'\| \\
&\quad + M \max_k \{|f_k(x_1, \ldots, x_n) - f_k'(x_1, \ldots, x_n)|\} \le 2\varepsilon + M \cdot 2\varepsilon_2 \le 2\varepsilon_1 \;.
\end{aligned}$$

In this way we obtain an ε_1-covering of T, which consists of $N_\varepsilon(\Lambda) \prod_{k=1}^m N_{\varepsilon_2}(T_k)$ sets. Hence,

$$H_{\varepsilon_1}(T) \le H_\varepsilon(\Lambda) + \sum_{k=1}^m H_{\varepsilon_2}(T_k) \;.$$

From this, (3.2) follows by iteration. □

Returning to the proof of the theorem, we consider the Banach space $X = \Lambda^{\beta_0}(I^{n_0})$. A ball U of radius ρ in X is a translation of a ball with the center in the origin, and this ball is $\Lambda^{\beta_0}(\rho, \ldots, \rho; I^{n_0})$. The entropy of U in the uniform norm is estimated by Lorentz [A-1986,(3), p.153]. This gives $H_\varepsilon(U) \ge C_1 \varepsilon^{-\beta_0/n_0}$. Similarly, for each of the classes Λ with $\beta/n > \beta_0/n_0$, $H_\varepsilon(\Lambda) \le C_2 \varepsilon^{-\beta/n}$. Hence

$$H_\varepsilon(\Lambda)/H_\varepsilon(U) \le \mathrm{const}\, \varepsilon^{\beta_0/n_0 - \beta/n} \to 0 \quad \text{for} \quad \varepsilon \to 0 \;.$$

If the type T is formed with classes Λ with this property, then (3.2) implies that $H_\varepsilon(T)/H_\varepsilon(U) \to 0$. It follows that T does not contain any ball U of X.

Since the classes Λ are compact in the uniform topology, T is also compact and hence closed. Since convergence in the norm of X implies uniform convergence, $X \cap T$ is closed in the space X. It follows that $X \cap T$ is *nowhere dense* in X.

Classes $\Lambda = \Lambda^{\beta}(M_0, \ldots, M_{p-1}; I^n)$ with $\beta/n > \chi_0$ we can restrict to countably many of them, taking the M_i rational. There are countably many schemes S. This produces countably many types T. For these T, the set $X \cap \cup T$ is of the first category in X; hence, by Baire's theorem, $U \backslash \cup T$ is not empty for each ball U in X. In particular, there exists a function $f \in \Lambda^{\beta_0}(I^{n_0}) \backslash \cup T$, which is not representable by superpositions of functions of $\cup T$. □

§ 4. Linear Superpositions

Theorem 1.1 is best possible in the sense that one cannot take the functions ϕ_q to be *continuously differentiable.*

More generally, this is true for *linear superpositions*, which are given by a representation

$$f(x_1, \ldots, x_n) = \sum_{k=1}^{N} p_k(x_1, \ldots, x_n) g_k(\phi_{k,1}, \ldots, \phi_{k,n-1}) \,. \tag{4.1}$$

Here, the p_k, $\phi_{k,i}$ are fixed functions from $C(I^n)$, the $\phi_{k,i}$ are continuously differentiable, and the g_k are free continuous functions on $\mathbb{R}^{n-1}$. For example, Kolmogorov's formulas (1.2) are linear superpositions. Without loss of generality, we shall assume that g_k are supported on compact sets.

First deep results are due to Vitushkin and Khenkin [1967]. They showed that for a fixed N, all functions

$$\sum_{k=1}^{N} p_k(x_1, \ldots, x_n) g_k(\phi_k(x_1, \ldots, x_n)) \tag{4.2}$$

form a *nowhere dense set* in $C(I^n)$. They have even indicated concrete polynomials (for $n = 2$) which are not representable in form (4.2). Their proofs are very difficult. Later Fridman [1972] obtained results of similar type for some superpositions of form (4.1) in the L_2 metric. Instead of this, we give, with Kaufman [1974] (see also Lorentz [1976]) a simple proof of

Theorem 4.1. *For a given continuous p_k, continuously differentiable $\phi_{k,i}$ and arbitrary continuous g_k, the functions (4.1) are a set of first category in $C(I^n)$.*

We can fix N. For each k, let J_k be the Jacobian matrix

$$J_k = \left(\frac{\partial \phi_{k,i}}{\partial x_j} \right)_{i=1,j=1}^{n-1,n} \,. \tag{4.3}$$

There exists a point $x^0 = (x_1^0, \ldots, x_n^0) \in I^n$ at which the rank of each of the J_k attains a local maximum, say $\ell_k - 1$. By $U \subset I^n$ we denote a compact neighborhood of x^0 so that the J_k have rank $\ell_k - 1$ on U. We can assume that it is the first $\ell_k - 1$ columns of J_k that are independent for $x \in U$.

By A_M we denote the subset of the superpositions (4.1) for which $\|g_k\|_\infty \le M$, $k = 1, \dots, N$. It is sufficient to prove that each A_M is nowhere dense in $C(I^n)$. This will follow if we prove that the functions $f \in A_M$ are nowhere dense in $C(U)$.

Lemma 4.2. *For each $\varepsilon > 0$ and $M > 0$ there exists a measurable function $h(x_1, \dots, x_n)$ on I^n with the property that the linear continuous functional*

$$L_\varepsilon(f) = \int \cdots \int_U f(x_1, \dots, x_n) h(x_1, \dots, x_n)\, dx_1 \dots dx_n \;, \tag{4.4}$$

defined for $f \in C(U)$, satisfies $\|L_\varepsilon\| \ge 1$ and $|L_\varepsilon(f)| < \varepsilon$ for all $f \in A_M$.

Proof. We take $h := r(\alpha_1 x_1 + \dots + \alpha_n x_n)$, where r is the Rademacher function $r(y) = \text{sign } (\sin \lambda y)$. Here $\lambda > 0$ is a parameter, and the α_k are so chosen that the row $\alpha_1, \dots, \alpha_n$ is independent of any $\ell_k - 1$ rows of the matrix J_k at $x = x^0$. One can then assume that this will be true throughout U. With $\ell = \ell_k$ we will then have

$$D_k = \begin{vmatrix} \alpha_1 & , \dots , & \alpha_\ell \\ \dfrac{\partial \phi_{k,1}}{\partial x_1} & , \dots , & \dfrac{\partial \phi_{k,\ell}}{\partial x_1} \\ \cdots\cdots & \cdots & \cdots\cdots \\ \dfrac{\partial \phi_{k,1}}{\partial x_{\ell-1}} & , \dots , & \dfrac{\partial \phi_{k,\ell}}{\partial x_{\ell-1}} \end{vmatrix} \neq 0 \,, \qquad x \in U \,.$$

We shall estimate the integral

$$I_k = \int \cdots \int_U r(\alpha_1 x_1 + \dots + \alpha_n x_n) p_k g_k(\phi_{k,1}, \dots, \phi_{k,n-1})\, dx_1 \dots dx_n$$

suppressing in what follows the subscript k for simplicity. We make the change of variables

$$\left\{ \begin{aligned} y_1 &= \alpha_1 x_1 + \dots + \alpha_n x_n \\ y_2 &= \phi_1(x_1, \dots, x_n) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ y_\ell &= \phi_{\ell-1}(x_1, \dots, x_n) \\ y_{\ell+1} &= x_{\ell+1} \\ y_n &= x_n \end{aligned} \right. \tag{4.5}$$

Since $\phi_\ell, \dots, \phi_{n-1}$ are expressible in terms of $\phi_1, \dots, \phi_{\ell-1}$ on U, for some continuous g^*,

$$g(\phi_1, \dots, \phi_{n-1}) = g^*(\phi_1, \dots, \phi_{\ell-1}) \,.$$

Hence

$$I_k = \int \cdots \int_V h(y_1) p g^*(\phi_1, \ldots, \phi_{\ell-1}) \frac{1}{|D|} \, dy_1 \ldots dy_n$$

$$= \int \cdots \int g^*(y_2, \ldots, y_\ell) \, dy_2 \ldots dy_n \int_B h(y_1) \frac{p}{|D|} \, dy_1 \, , \qquad (4.6)$$

where V is the 1–1 image of U under (4.5), and B is a cross-section of V. The functions $p/|D|$ of y_1 depend on the parameters $y_2, \ldots, y_n$ and k, but they are uniformly bounded and equicontinuous. Thus, the interior integral converges to zero for $\lambda \to \infty$, and we obtain $|I_k| \le \text{const}\, M\varepsilon$, where $\varepsilon > 0$ is arbitrarily small. The norm of the functional (4.4) on $C(U)$ is $\int \cdots \int_U |h| \, dx_1 \ldots dx_n = |U|$. We obtain the lemma by adjusting the constants. □

Proof of Theorem 4.1. Let $f_0 \in X := C(U)$, let $\varepsilon > 0$ be given and $L := L_{\varepsilon/3}$. For $g \in X$ with $\|g\| < 1$, $L(f_0 + \varepsilon g) - L(f_0) = \varepsilon L(g)$, and the right-hand side can be made arbitrarily close to ε. Taking $f_1 = f_0 + \varepsilon g$ or $f_1 = f_0$ we obtain in the ε-neighborhood of f_0 an element f_1, which satisfies $|L(f_1)| > \frac{1}{3}\varepsilon$. There is a neighborhood of f_1, where $|L(f)| > \frac{1}{3}\varepsilon$. By the lemma, this neighborhood is disjoint with A_M. □

§ 5. Notes

5.1 The fact that the number of terms in the different formulations of Kolmogorov's theorem (1.1) and (1.2) must be at least $2n+1$ has been established only recently by Sternfeld [1985]. The number of terms can not be reduced even in much more general representations,

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{2n+1} g_i(\phi_i(x_1, \ldots, x_n))$$

with given $\phi_i \in C(I^n)$ and some $g_i \in C(\mathbb{R})$. The matter depends on some fundamental topological notions: dimension of the topological space, embeddings into euclidean spaces (see Hurewicz and Wallman [B-1948]). One of the classic theorems of n-dimensional topology (by Nöbeling) asserts that each n-dimensional topological space X can be embedded into the $(2n+1)$-dimensional euclidean space. Kolmogorov's theorem is a special statement of this type.

Let X, Y_i be compact metric spaces. A family $F = \{\varphi_i\}_{i=1}^k$ of continuous functions $X \to Y_i$, $i = 1, \ldots, k$ is said to be a *basic family* for X, if every $f \in C(X)$ admits a representation

$$f(x) = \sum_{i=1}^{k} g_i(\varphi_i(x)) \, , \qquad x \in X$$

with some $g_i \in C(Y_i)$, $i = 1, \ldots, k$. Thus, for $X = I^n$, $k = 2n+1$, $Y_i = I$, Kolmogorov proved that there exists a basic family $\{\varphi_i\}_{i=1}^n \subset C(X)$, and even one of the special form $\varphi_i(x,\dots,x_n) = \sum_{j=1}^{n} \varphi_{i,j}(x_j)$, $i = 1,\dots,n$. Ostrand [1965] extended this to arbitrary compact metric spaces X, of dimension $\le n$. Using topological as well as combinatorial arguments, Sternfeld [1985] established

Theorem 5.1. *A compact metric space X has dimension $\dim X \le n$ if and only if it possesses a basic family $F \subset C(X)$ consisting of $\le 2n+1$ functions.*

Another formulation of this necessary and sufficient condition is that the algebra $C(X)$ should be an algebraic sum of $2n+1$ subalgebras, each of which contains the constants and is generated by one element.

The main result of Sternfeld is

Theorem 5.2. *For a compact metric space X with $\dim X = n$, each basic family $F \subset C(X)$ contains at least $2n+1$ functions.*

(The original proof has been essentially shortened by Levin [1990].)

If one is prepared to replace the intervals I, on which the $\varphi_{i,j}$ are defined, by some one-dimensional spaces (which depend on X), one can reduce the number of functions in Kolmogorov's representation, see Sternfeld [1983] and [1985].

5.2 Approximation of functions of two variables $f(x,y)$, $x \in A$, $y \in B$ by means of tensor products $\sum_{j=1}^{n} g_j(x)h_j(y)$ is discussed in the book of Light and Cheney [A-1985]. Best results are for the simplest cases. In what follows, let A, B be compact Hausdorff spaces. We define the operators $U : C(A\times B) \to C(A)$ and $V : C(A,B) \to C(B)$ by means of

$$(5.1) \qquad \begin{cases} (Uf)(x) = \frac{1}{2}\left\{\max\limits_{y\in B} f(x,y) + \min\limits_{y\in B} f(x,y)\right\} \\[2ex] (Vf)(y) = \frac{1}{2}\left\{\max\limits_{x\in A} f(x,y) + \min\limits_{x\in A} f(x,y)\right\} . \end{cases}$$

For an arbitrary $f \in C(A \times B)$ we define $f_0 = f$,

$$(5.2) \qquad f_{2n} = f_{2n-1} - Uf_{2n-1} \quad , \quad f_{2n+1} = f_{2n} - Vf_{2n} \, .$$

If S is the set of all sums $g+h$, $g \in C(A)$, $h \in C(B)$, we have (Diliberto-Straus and Aumann):

Theorem 5.3. *For each $f \in C(A \times B)$, the algorithm* (5.2) *converges uniformly to a member of S which is a best approximation to f from S.*

This is far from obvious, because S is infinitely dimensional. Using an essentially different algorithm, v. Golitschek proved this even for the space S_1 of all functions $F[h_0(y)g(x)+g_0(x)h(y)]$, where $g_0 > 0$, $h_0 > 0$ and F are fixed continuous functions on $A, B, \mathbb{R}$, respectively, and F is strictly increasing. For details see Light and Cheney [A-1985, Chapter 6].

Appendix 1. Theorems of Borsuk and of Brunn-Minkowski

§ 1. Borsuk's Theorem

1.1. Introduction; Different Forms of the Theorem. The remarkable *antipodality theorem of Borsuk* [1933] has extensive applications in Analysis. Like Brower's fixed point theorem to which it is related, Borsuk's theorem belongs to the elementary topology of $\mathbb{R}^n$. It can be easily proved using advanced topological means (homotopy, cohomology), or using the notion of the degree of a mapping (see Dugundji [B-1972]). Instead, with DeVore, Kierstead, Lorentz [1988] we shall give here a direct, elementary proof which uses combinatorial properties of triangulations of $\mathbb{R}^n$ and is based on ideas of Tucker [1945]; a similar proof is given in Weiss [1989].

Let $\Sigma_n := \{x = (x_1, \dots, x_n) : \|x\|_X = 1\}$ be the unit sphere of an n-dimensional Banach space X. Thus, Σ_n can be identified with the boundary of any convex compact symmetric neighborhood of zero in the euclidean space $\mathbb{R}^n$. Let P be a mapping of Σ_n into an $(n-1)$-dimensional Banach space X_{n-1} (In other words, P is a $(n-1)$-dimensional vector field on Σ_n.) A mapping $P : \Sigma_n \to \mathbb{R}^n$ is *odd* if $P(-x) = -P(x)$ for all $x \in \Sigma_n$.

Theorem 1.1 (Borsuk's theorem). *An odd continuous mapping P of Σ_n into X_{n-1} must vanish: $P(x) = 0$ for some $x \in \Sigma_n$.*

It is sufficient to establish this for the euclidean unit sphere. Indeed, for any two unit Banach spheres in $\mathbb{R}^n$, the rays emanating from the origin establish a one-to-one correspondence which is itself an odd and continuous mapping. Similarly, since all spaces X_{n-1} are isomorphic, we can substitute for X_{n-1} one of them, $\mathbb{R}^{n-1}$.

We consider mappings R of the n-dimensional cube $Q_0 := [0,1]^n$ into its boundary B_n. If $x \in B_n$, its *antipodal point* x^*, which also belongs to B_n, is symmetric to x with respect to the center of Q_0. In other words, it is defined by $x^* := e - x$, where $e := (1, 1, \dots, 1)$. More generally, if $A \subset B_n$, we define $A^* := \{x^* : x \in A\}$. A mapping R of Q_0 into B_n is *antipodal* if

$$P(x^*) = P(x)^* \quad \text{for all} \quad x \in B_n. \tag{1.1}$$

Theorem 1.2. *For $n = 1, 2, \dots$, there does not exist a continuous antipodal mapping R of Q_0 into its boundary B_n.*

We shall prove this theorem in **1.4**. Here we show that Borsuk's Theorem 1.1 follows easily from this. Indeed, Theorem 1.2 remains true for any n-dimensional cube, for example, for the cube $[-1,1]^n$. It is also true for antipodal mappings of the n-dimensional ball $U_n := \{x = (x_1, \ldots, x_n) : \|x\| \le 1\}$ into its boundary Σ_n. To see this, let S be the mapping which assigns to $x \in U_n$, $\|x\| = r$, its projection $S(x)$ onto the boundary of the cube $[-r, r]^n$ by the ray emanating from the origin and passing through x. Both S and S^{-1} are continuous and preserve antipodality. If there existed a continuous antipodal mapping R of U_n into Σ_n, then SRS^{-1} would be a continuous antipodal mapping of $[-1,1]^n$ into its boundary, a contradiction.

We derive Borsuk's theorem from Theorem 1.2. If the former were not true, there would exist a continuous odd mapping P of Σ_{n+1}, $n \ge 1$, into $\mathbb{R}^n$ that does not vanish on Σ_{n+1}. Now if $y = (y_1, \ldots, y_n) \in U_n$, the point $z := (y_1, \ldots, y_n, \sqrt{1 - \|y\|^2})$ is on Σ_{n+1}. We define a mapping R of U_n into Σ_n by putting

$$R(y) = P(z)/\|P(z)\|, \quad y \in U_n. \tag{1.2}$$

This R is well defined and continuous. If $y \in \Sigma_n$, then $z = (y_1, \ldots, y_n, 0)$. Since P is odd, R is antipodal. This would contradict Theorem 1.2. Thus, P must vanish.

1.2. Properties of the "Equators" B_k. We return to the cube Q_0. Its *k-dimensional facets*, $k = 0, 1, \ldots, n-1$, are defined as the intersections of Q_0 with some $n-k$ hyperplanes $x_{i_j} = c_j$, $j = 1, \ldots, n-k$, where $c_j = 0$ or 1, and $I := I_{n-k} := \{1 \le i_1 < \ldots < i_{n-k} \le n\} \in \mathcal{I}_{n-k}$ is a fixed set, and $\mathcal{I}_k$, $k = 0, \ldots, n$, stands for the set of all subsets of $\{1, 2, \ldots, n\}$ of cardinality k. Facets of dimension $n-1$ are faces of Q_0, those of dimensions 1 and 0 are edges and vertices, respectively.

We shall need *special k-dimensional facets* $F_{I,k} := \{x \in Q_0 : x_{i_1} = 0, x_{i_2} = 1, x_{i_3} = 0, \ldots\}$, $I \in \mathcal{I}_{n-k}$, with alternating values of $x_{i_1}, x_{i_2}, \ldots$, and also their antipodal sets $F^*_{I,k} := \{x \in Q_0 : x_{i_1} = 1, x_{i_2} = 0, \ldots\}$.

With the help of the $F_{I,k}$ and $F^*_{I,k}$ we construct something like a k-dimensional equator of Q_0 which separates its south pole $(0, \ldots, 0)$ from the north pole $(1, \ldots, 1)$. For $k = 0, \ldots, n-1$, let

$$H_k := \bigcup_{I \in \mathcal{I}_{n-k}} F_{I,k}, \quad H^*_k := \bigcup_{I \in \mathcal{I}_{n-k}} F^*_{I,k}, \quad B_k := H_k \cap H^*_k. \tag{1.3}$$

For example, H_0 consists of the single point $(0, 1, 0, 1, \ldots)$, and H^*_0, of the point $(1, 0, 1, 0, \ldots)$. We let $H_n := H^*_n := B_n$ be the boundary of Q_0. We call two k-dimensional polyhedrons *disjoint* if their interiors sre disjoint. Thus, the facets F, F^* in (1.3) are all disjoint. Hence B_k is $(k-1)$-dimensional.

Lemma 1.3. *We have* (i)

$$B_k = H_{k-1} \cup H^*_{k-1}, \quad k = 1, \ldots, n-1;$$

(ii) *If F is a facet of dimension $k-1$ contained on H_k, $k \le n-1$, then either* (a) *F is a face of exatly two facets $\widetilde{F}$ of H_k, and F and B_k are disjoint, or* (b) *F is a face of exactly one such $\widetilde{F}$, and $F \subset B_k$.*

Proof. (i) For $x \in Q_0$, we examine x_j as a function of j, $j = 1, \ldots, n$. Thus, $x \in H_k$ (or $x \in H_k^*$), $k = 0, \ldots, n-1$, means that either x_j contains an alternating sequence of 0's and 1's of length $> n-k$, or that it has a sequence of length $n-k$ beginning with 0 (or, correspondingly, with 1). Therefore $x \in H_k \cup H_k^*$ if and only if x has an alternating sequence of length $\ge n-k$. And $x \in H_k \cap H_k^* = B_k$ means that x has an alternating sequence of length $\ge n-k+1$. Thus we have (1.4).

(ii) Here, F is a face of some $\widetilde{F} := F_{I,k}$ from H_k, with $I \in \mathcal{I}_{n-k}$. The set $\widetilde{F}$ is described by the equations $x_{i_j} = c_j$, $j = 1, \ldots, n-k$, where the c_j are alternatively 0's or 1's, the remaining x_j being arbitrary. The face F is obtained by adding an equation $x_i = c$, $c = 0$ or $c = 1$, $i \notin I$. Let $c = 0$. If $i < i_{n-k}$, or $i > i_{n-k}$ and $c_{n-k} = 0$, the new sequence of c's, of length $n-k+1$, would have two adjacent zeros. Thus F would be a face of exactly two facets from H_k obtained by omitting one of the zeros, and would not belong to any of the $F^*_{I,k}$. But if the last $c_{n-k} = 1$ and $i > i_{n-k}$, then F would be the face of the original $F_{I,k}$ and of a facet from H_k^* obtained by omitting $c_{i_1} = 0$, and would not belong to other facets. Moreover, $F \subset B_k$. The case $c = 1$ is similar. □

Another formulation of (ii) is that B_k is the common boundary of H_k and of H_k^* in B_{k+1}.

1.3. Partition and Triangulation of the Cube Q_0. For the proof of Theorem 1.2 we need a decomposition of the B_k into simplices, not cubes, and they must be sufficiently small. The last aim is achieved by selecting a large integer N and by decomposing Q_0 into N^n small cubes Q of side length $h = N^{-1}$. Each of the small cubes Q consists of points $y_Q + hx$, where y_Q is the smallest vector in Q, and x is an arbitrary point of Q_0.

A k-dimensional facet $\widetilde{R}$ of Q is obtained by setting $n-k$ of the coordinates of x to 0 or 1. A face R of $\widetilde{R}$ has an additional coordinate set to 0 or 1. For example, the $\widetilde{R}$ contained in $\widetilde{F} := F_{I,k}$, $I = \{i_1, \ldots, i_{n-k}\}$, are obtained by setting $x_{i_1} = 0$, $x_{i_2} = 1, \ldots$. Hence,

(1.6) if R is interior to $\widetilde{F}$, then R appears as a face of exatly two $\widetilde{R}$.

For example, if R is obtained from $\widetilde{R}$ by setting $x_j = 0$, then R is also a face of the k-dimensional facet $\widetilde{R}'$ of Q', $y_{Q'} = y_Q - he_j$ obtained by setting $x_j = 1$. (By e_i, $i = 1, \ldots, n$, we denote the i-th unit vector of $\mathbb{R}^n$, with i-th coordinate equal to one and all other coordinates equal to zero.)

We further partition each Q into simplices. We use the following procedure (known as the *Kuhn triangulation*). Let $\sigma := (\sigma(1), \ldots, \sigma(n))$ be any permutation of the integers $1, \ldots, n$. For each σ, the simplex $T_\sigma(Q)$ has the vertices $y_Q + hx$, where y_Q is the smallest vector in Q, and x is one of the vectors

$$0, e_{\sigma(1)}, e_{\sigma(1)} + e_{\sigma(2)}, \ldots, e_{\sigma(1)} + \cdots + e_{\sigma(n)} = (1, \ldots, 1).$$

Equivalently, $T_\sigma(Q)$ consists of all points $y_Q + hx$ with $x = (x_1, \ldots, x_n)$ satisfying

$$0 \le x_{\sigma(n)} \le \ldots \le x_{\sigma(1)} \le 1; \tag{1.7}$$

and y is in the interior of T_σ if and only if all these inequalities are strict. Since for any x with distinct coordinates x_i there is a unique decreasing rearrangement of the x_i, the T_σ has disjoint interiors, thus $Q = \bigcup_\sigma T_\sigma(Q)$ is a disjoint decomposition of Q.

The faces T of $T_\sigma(Q)$ are obtained by changing one of the inequalities of (1.7) to equality. Changing an outer inequality gives a face contained in the boundary (in one of the faces) of Q. Such a T is contained in only one $T_\sigma(Q)$. Changing an interior inequality $x_{\sigma(i)} < x_{\sigma(i+1)}$ to equality produces an interior face T which is shared by one other simplex $T_{\sigma'}(Q)$; one gets σ' by interchanging $\sigma(i)$ and $\sigma(i+1)$.

The Kuhn triangulation of Q induces also a Kuhn triangulation of the facets F of Q of dimensions $1 \le k \le n-1$. It is given by those facets of $T_\sigma(Q)$ of dimension k that are contained in F. For example, let F be the face of $Q = \{y_Q + hx\}$ with $x_i = 0$. Then a T_σ has a face contained in F precisely when $x_{\sigma(n)} = 0$ and $\sigma(n) = i$ in (1.6). Since $y_F = y_Q$, this face of T_σ is given by the relations

$$0 = x_{\sigma(n)} \le x_{\sigma(n-1)} \le \ldots \le x_{\sigma(1)} \le 1. \tag{1.8}$$

The faces of the T_σ with this property produce the Kuhn triangulation of F. The same is true if F is given by $x_i = 1$, but then $y_F = y_Q + e_i$. This proof also applies to facets of Q of lower dimensions:

We let $\mathcal{T}$ denote the set of all faces T of all $T_\sigma(Q)$, for all small cubes $Q \subset Q_0$.

Lemma 1.4. *Let $T \in \mathcal{T}$ be a $(k-1)$-dimensional simplex in $\mathcal{T}$ such that $T \subset H_k$, $1 \le k \le n-1$. Then either* (A) *T is a face of exactly two k-dimensional simplices $\widetilde{T} \subset H_k$, and T is disjoint with B_k, or* (B) *T is a face of exactly one such $\widetilde{T}$, and $T \subset B_k$.*

Proof. Let $T \subset \widetilde{F} := F_{I,k}$, then T is contained in some k-dimensional cube $\widetilde{R}$ produced by the h partition of Q_0, and is a face of some $\widetilde{T} \in \mathcal{T}$ with $\widetilde{T} \subset \widetilde{R}$. It can happen that T is interior to $\widetilde{R}$, then by what was said above, T is the face of exactly two $\widetilde{T} \in \mathcal{T}$, $\widetilde{T} \subset H_k$, and we have (A). Another possible situation is that T is contained in a face R of $\widetilde{R}$ which is interior to $\widetilde{F}$. Then, by (1.6), exactly two $\widetilde{R}$ contain R as a face, and hence T is a face of exactly two $\widetilde{T} \in \mathcal{T}$, $\widetilde{T} \subset H_k$, and again we have (A).

It remains to consider the case when T is contained in a face R of $\widetilde{R}$ and $R \subset F$, where F is a face of $\widetilde{F}$. In case (a) of Lemma 1.3, T is contained in two different facets of H_k, and by (a) of the lemma, we again have (A).

However, in case (b) of Lemma 1.3, T is contained in exactly one $\widetilde{T} \subset \widetilde{F}$ and $T \subset \widetilde{F} \subset B_k$. This yields (B). □

1.4. Proof of Theorem 1.2. We assume that R of Theorem 1.2 exists and derive a contradiction. For $x \in Q_0$ and $y = R(x)$, let i be the smallest integer such that $y_i = 0$ or $y_i = 1$. We assign to x the "color" i if $y_i = 0$, the "color" $-i$ if $y_i = 1$. In this way, each $x \in Q_0$ is assigned one of the colors $\pm 1, \ldots, \pm n$. Antipodal elements are assigned opposite colors.

We assume that h is so small that no two points of a simplex $T \in \mathcal{T}$ are mapped onto opposite faces of Q_0. This guarantees that no two vertices of T get opposite colors. As a consequence, the colors of the vertices of any k-dimensional $T \in \mathcal{T}$ can be uniquely written as a sequence $c_0, \ldots, c_k$ with

$$1 \le |c_0| \le \ldots \le |c_k| \le n;$$

there is strict inequality $|c_i| < |c_{i+1}|$ if c_i, c_{i+1} are of opposite signs. With this ordering, we call $C = (c_0, \ldots, c_k)$ *the color of* T. We also need *special colors* $\widehat{C}$ of order k (and their set C_k) whose components satisfy

$$1 \le |c_0| < \ldots < |c_k| \le n, \quad \operatorname{sign} c_i = (-1)^i.$$

Then also $-\widehat{C} = (-c_0, \ldots, -c_k)$ is a color with alternating signs. For example, if $k = 0$, T is a single point of some color c_0, then $C = (c_0)$, and either C or $-C$ is special.

By $N_k(C)$ we denote the number of k-dimensional $T \in \mathcal{T}$ contained in H_k that have color C.

Let $A_k(C)$ be the number of incidences of k-dimensional $T \in \mathcal{T}$ of color C as a face of $(k+1)$-dimensional $\widetilde{T} \in \mathcal{T}$ with $\widetilde{T} \subset H_{k+1}$. From Lemma 1.4, if $T \subset B_{k+1}$, then T appears exactly once as a face of a $\widetilde{T}$. Otherwise, T appears in two $\widetilde{T}$. Thus to compute $A_k(\widehat{C})$, modulo 2, we have to count the number of T of color $\widehat{C}$ contained in $B_{k+1} = H_k \cup H_k^*$. Either $T \subset H_k$ and has color $\widehat{C}$, or $T \subset H_k^*$ and then $T^* \subset H_k$ has color $-\widehat{C}$. Thus $A_k(\widehat{C}) \equiv N_k(\widehat{C}) + N_k(-\widehat{C})$, mod 2. We therefore obtain, summing over all possible $\widehat{C} \in C_k$, for $k = 0, \ldots, n-1$,

$$\sigma_k := \sum_{\widehat{C} \in C_k} A_k(\widehat{C}) \equiv \sum [N_k(\widehat{C}) + N_k(-\widehat{C})], \quad \text{mod } 2. \tag{1.9}$$

For example, $\sigma_n = 0$, for there are no special colors of order n. Also σ_0 can be easily computed. Since H_0 is a single point with a definite color, the sum on the right in (1.9) reduces to a single term 1, so that $\sigma_0 \equiv 1$, mod 2.

We can count σ_k in a different way. Namely, if $\widetilde{T} \subset H_{k+1}$ is a simplex of dimension $k+1$ which contributes to σ_k, and $\widetilde{T}$ has color $D = (d_0, \ldots, d_{k+1})$, then the color of T is a subsequence of D, so that D has either k or $k+1$ changes of sign. In the first case, for some i we have $|d_0| < \ldots < |d_i| \le |d_{i+1}| < \ldots < |d_{k+1}|$ and $\operatorname{sign} d_i = \operatorname{sign} d_{i+1}$. Then exactly two faces of $\widetilde{T}$ contribute to

σ_k. On the other hand, if D has $k+1$ changes of sign, then *exactly one* face of $\widetilde{T}$ has a color from C_k. This happens when $D \in C_{k+1}$ or $-D \in C_{k+1}$. Hence for $0 \leq k < n$,

$$\sigma_k \equiv \sum_{\widehat{C} \in C_{k+1}} [N_{k+1}(\widehat{C}) + N_{k+1}(-\widehat{C})], \quad \text{mod } 2.$$

It follows that $\sigma_k \equiv \sigma_{k+1}$, $k = 0, \ldots, n-1$. However, this is a contradiction since $\sigma_n = 0$ and $\sigma_0 \equiv 1$. □

§ 2. The Brunn-Minkowski Inequality

The main purpose of this section is to establish Theorem 2.3 for convex bodies. This proves to be a corollary to Theorem 2.1 of Brunn-Minkowski. A convex body K in $\mathbb{R}^n$ is the closure of a non-empty convex bounded open set. The Lebesgue measures of convex bodies are usually called volumes, thus $\text{vol}\, K := \text{vol}_n\, K > 0$.

Boxes in $\mathbb{R}^n$ are products $I_1 \times \ldots \times I_n$ of closed intervals on each of the coordinate axes. An elementary set A is a bounded set in $\mathbb{R}^n$ that is a union of finitely many boxes without common interior points, and $\text{vol}\, A$ is the sum of the volumes of the boxes. The volume of a convex body is the supremum of the volumes of elementary sets contained in K:

$$\text{vol}\, K = \sup \{\text{vol}\, A : A \subset K\}. \tag{2.1}$$

We also have

$$\text{vol}\,(\lambda K) = |\lambda|^n \text{vol}\, K, \quad \text{vol}\,(K + x_0) = \text{vol}\, K. \tag{2.2}$$

An intersection of a convex body K in $\mathbb{R}^n$ with an m-dimensional plane, $1 \leq m < n$ (that is, with $X_m + x_0$, where X_m is an m-dimensional subspace of $\mathbb{R}^n$) is an m-dimensional convex body, K', with the m-dimensional volume $\text{vol}_m\, K'$.

If A, B are non-empty subsets of $\mathbb{R}^n$, we write $A + B := \{a + b : a \in A, b \in B\}$. If A and B are convex bodies in $\mathbb{R}^n$, so is $A + B$.

Theorem 2.1 (Brunn -Minkowski). *If K_1, K_2 are convex bodies in $\mathbb{R}^n$, then*

$$(\text{vol}\,(K_1 + K_2))^{1/n} \geq (\text{vol}\, K_1)^{1/n} + (\text{vol}\, K_2)^{1/n}. \tag{2.3}$$

This inequality was first established in 1887, in the dissertation of Brunn. A refined proof, including the description of all cases of equality in (2.3), was given in 1897 by Minkowski. Later the inequality was proved for the Lebesgue measures of arbitrary compact sets in $\mathbb{R}^n$, not necessarily convex bodies.

Proof. We first prove (2.3) for the case when $K_1 = A$, $K_2 = B$ are elementary sets. We use induction on the total number k of boxes in A and B. If $k = 2$, that is, if A and B are single boxes with side lengths $(c_1, \dots, c_n)$, $(d_1, \dots, d_n)$, respectively, then (2.3) becomes

$$\prod_{i=1}^{n}(c_i + d_i)^{1/n} \geq \prod_{i=1}^{n} c_i^{1/n} + \prod_{i=1}^{n} d_i^{1/n}. \tag{2.4}$$

To prove (2.4), we use the inequality between means

$$\left(\prod_{k=1}^{n} a_k\right)^{1/n} \leq \frac{1}{n}\sum_{k+1}^{n} a_k, \quad a_k \geq 0,\ k = 1, \dots, n$$

(see, for example, Hardy, Littlewood, Pólya [B-1988, (2.5.2)]), which yields

$$\left(\prod_{i=1}^{n} \frac{c_i}{c_i + d_i}\right)^{1/n} + \left(\prod_{i=1}^{n} \frac{d_i}{c_i + d_i}\right)^{1/n} \leq \frac{1}{n}\sum_{i=1}^{n} \frac{c_i}{c_i + d_i} + \frac{1}{n}\sum_{i=1}^{n} \frac{d_i}{c_i + d_i} = 1.$$

Suppose now that (2.3) is true for all elementary sets $K_1 = A$, $K_2 = B$ with the total number of boxes $\leq k - 1$. If $k \geq 3$, then one of the sets, say A, consists of at least two boxes. Fixing in A arbitrarily two boxes, $P = I_1 \times \dots \times I_n$ and $P' = I_1' \times \dots \times I_n'$, one can find a hyperplane Π perpendicular to one of the coordinate axes and separating these two boxes. Indeed, since P and P' are disjoint, I_k and I_k' must be disjoint for at least one k, and one can put $\Pi = \{x = (\xi_1, \dots, \xi_n) : \xi_k = c\}$, where c is any number separating I_k from I_k'. If A', A'' are the parts of A contained in the two half-spaces R' and R'', defined by the plane Π, then each of them has fewer boxes than A.

Let $\lambda := (\operatorname{vol} A'/\operatorname{vol} A)^{1/n}$. By a parallel translation one can place B in a position in which it is divided by Π into sets B', B'', with $(\operatorname{vol} B'/\operatorname{vol} B)^{1/n} = \lambda$. Translations do not change $\operatorname{vol} B$ and $\operatorname{vol}(A + B)$. The sets B', B'' are non-empty elementary sets, with the number of boxes in each of them not exceeding that of B. Hence the total number of boxes in each couple A', A'' and B', B'' does not exceed $k-1$. The sets $A' + B'$ and $A'' + B''$ are separated by the plane Π. If $A_1, A_2 \subset R'$, then also $A_1 + A_2 \subset R'$. Therefore using the induction hypothesis, one has

$$\begin{aligned}
\operatorname{vol}(A + B) &\geq \operatorname{vol}(A' + B') + \operatorname{vol}(A'' + B'') \\
&\geq [(\operatorname{vol} A')^{1/n} + (\operatorname{vol} B')^{1/n}]^n + [(\operatorname{vol} A'')^{1/n} + (\operatorname{vol} B'')^{1/n}]^n \\
&= \lambda[(\operatorname{vol} A)^{1/n} + (\operatorname{vol} B)^{1/n}]^n + (1 - \lambda)[(\operatorname{vol} A)^{1/n} + (\operatorname{vol} B)^{1/n}]^n \\
&= [(\operatorname{vol} A)^{1/n} + (\operatorname{vol} B)^{1/n}]^n.
\end{aligned}$$

For two convex bodies K_1, K_2, we now select elementary sets $A_1 \subset K_1$, $A_2 \subset K_2$ and have $A_1 + A_2 \subset K_1 + K_2$, hence

$$\operatorname{vol}(K_1 + K_2)^{1/n} \geq \operatorname{vol}(A_1 + A_2)^{1/n} \geq (\operatorname{vol} A_1)^{1/n} + (\operatorname{vol} A_2)^{1/n},$$

and an application of (2.1) leads to (2.3). □

From (2.2) and (2.3) immediately follows

Theorem 2.2. *Let K_0, K_1 be two convex bodies in $\mathbb{R}^n$, and for $0 \le t \le 1$, let $K_t := tK_1 + (1-t)K_0$. Then the function $v(t) := (\operatorname{vol} K_t)^{1/n}$ is convex:*

$$v(t) \ge (1-t)v(0) + tv(t). \tag{2.5}$$

Theorem 2.3. *Let $K, X \subset \mathbb{R}^n$ be, respectively, a compact convex body symmetric about the origin and an m-dimensional subspace, $1 \le m < n$. For a fixed vector x_0 and a real number λ, let $X_\lambda := X + \lambda x_0$. Then $\operatorname{vol}_m (K \cap X_\lambda)$ is a monotone non-increasing function of $|\lambda|$.*

Proof. Let $|\lambda_2| > |\lambda_1|$. We may assume that $\lambda_2 > \lambda_1 \ge 0$ and that $S := K \cup X_{\lambda_2}$ is non-empty. Let S', S'' be, respectively, the image of S under the translation $x \to x + (\lambda_1 - \lambda_2)x_0$ and the image of $-S$ under the translation $x \to x + (\lambda_1 + \lambda_2)x_0$. Then both S' and S'' are in the plane X_{λ_1}, and $\operatorname{vol}_m S' = \operatorname{vol}_m S'' = \operatorname{vol}_m S$. The sets S' and S'' are not necessarily contained in K, but their convex combination $tS' + (1-t)S''$, with $t := (\lambda_1 + \lambda_2)/(2\lambda_2)$, is. Indeed, if $x' \in S'$, $x'' \in S''$, then $x' = y_1 + (\lambda_1 - \lambda_2)x_0$, $x'' = y_2 + (\lambda_1 + \lambda_2)x_0$, where $y_1 \in S$, $y_2 \in -S$. One can easily verify that $tx' + (1-t)x'' = ty_1 + (1-t)y_2$. Since $y_1, y_2 \in K$, this implies $tx' + (1-t)x'' \in K$. Thus,

$$tS' + (1-t)S'' \subset K \cup X_{\lambda_1},$$

so that by (2.5),

$$\operatorname{vol}_m (K \cup X_{\lambda_1}) \ge \left[t(\operatorname{vol}_m S')^{1/n} + (1-t)(\operatorname{vol}_m S'')^{1/n}\right]^n = \operatorname{vol}_m S. \qquad \square$$

Appendix 2. Estimates of Some Elliptic Integrals

We discuss properties of some elliptic integrals, which are required for the construction of potentials in Chapter 8. The expressions $A(r)$, $B(r)$, $C(r)$, $D(r)$ and $I(r)$, $0 < r \leq 1$, are defined by

$$A(r) := \int_0^\infty \frac{dt}{\sqrt{(1+t^2)(1+r^2t^2)}}, \tag{1.1}$$

$$B(r) := \int_0^\infty \frac{dt}{\sqrt{(1+t^2)^3(1+r^2t^2)}}, \tag{1.2}$$

$$C(r) := \frac{A(r)}{B(r)} - 1, \tag{1.3}$$

$$D(r) := \int_{1/r}^\infty \frac{(C(r)+t^2)dt}{\sqrt{(t^2-1)^3(t^2-r^{-2})}}, \tag{1.4}$$

$$I(r) := \int_{1/r}^\infty \frac{dx}{x} \int_x^\infty \frac{(C(r)+t^2)dt}{\sqrt{(t^2-1)^3(t^2-r^{-2})}}. \tag{1.5}$$

In particular, we have to study the behavior of $C(r)$, $D(r)$ and $I(r)$ if $r \to 0+$ and $r \to 1-$.

Lemma 1.1. *The function $C(r)$ is positive and continuous on $0 < r \leq 1$ and satisfies $C(1) = 1$,*

$$C(r) = \log(1/r) + \mathcal{O}(1), \quad r \to 0+. \tag{1.6}$$

Proof. The positivity and continuity of C on $(0,1]$ is obvious. For $r = 1$,

$$A(1) - B(1) = \int_0^\infty \frac{t^2 dt}{(1+t^2)^2} = B(1),$$

hence $C(1) = 1$. In addition,

$$B(0) = \int_0^\infty \frac{dt}{\sqrt{(1+t^2)^3}} = \left[\frac{t}{\sqrt{1+t^2}}\right]_0^\infty = 1$$

and, since $1 \le \sqrt{1+r^2t^2} \le 1+rt$,

$$0 \le B(0) - B(r) \le \int_0^\infty \frac{dt}{\sqrt{(1+t^2)^3}} \left(1 - \frac{1}{1+rt}\right) = \mathcal{O}(r).$$

This implies that

$$B(r) = 1 + \mathcal{O}(r), \quad r \to 0+. \tag{1.7}$$

Elementary estimates yield

$$A(r) = \int_1^\infty \frac{dt}{\sqrt{(1+t^2)(1+r^2t^2)}} + \mathcal{O}(1) = \int_r^1 \frac{dy}{\sqrt{(r^2+y^2)(1+y^2)}} + \mathcal{O}(1).$$

To the last integral we apply the inequalities $y^2 \le (r^2+y^2)(1+y^2) \le (y+r)^2(1-y)^{-2}$ for $r \le y < 1$, to get $A(r) = \log(1/r) + \mathcal{O}(1)$. This and (1.7) yield (1.6). □

Lemma 1.2. *The function $D(r)$ satisfies*

$$D(r) = \frac{\pi r}{2} + \frac{\pi}{4} r^3 \log(1/r) + \mathcal{O}(r^3), \quad r \to 0+. \tag{1.8}$$

Proof. As $r \to 0+$,

$$\begin{aligned}\int_{1/r}^\infty \frac{t^2 dt}{\sqrt{(t^2-1)^3(t^2-r^{-2})}} &= \int_1^\infty \frac{rx^2 dx}{\sqrt{(x^2-r^2)^3(x^2-1)}} \\ &= r(1+\mathcal{O}(r^2)) \int_1^\infty \frac{dx}{x\sqrt{x^2-1}} = \frac{r\pi}{2} + \mathcal{O}(r^3).\end{aligned}$$

Similarly,

$$\begin{aligned}&\int_{1/r}^\infty \frac{dt}{\sqrt{(t^2-1)^3(t^2-r^{-2})}} = \int_1^\infty \frac{r^3 dx}{\sqrt{(x^2-r^2)^3(x^2-1)}} \\ &= r^3(1+\mathcal{O}(r^2)) \int_1^\infty \frac{dx}{x^3\sqrt{x^2-1}} = \frac{\pi}{4} r^3(1+\mathcal{O}(r^2)).\end{aligned}$$

From (1.4) and (1.6) we therefore get (1.8). □

Lemma 1.3. *For the function $I(r)$ we have*

$$I(r) = \frac{\pi r \log 2}{2} + \frac{\pi}{8}(\log 4 - 1) r^3 \log(1/r) + \mathcal{O}(r^3), \quad r \to 0+. \tag{1.9}$$

Proof. We first note that

$$\int_{1/r}^\infty \frac{dx}{x} \arcsin \frac{1}{xr} = \int_0^1 \frac{dy}{y} \arcsin y = \int_0^{\pi/2} \frac{x \cos x}{\sin x} dx = \frac{\pi \log 2}{2}. \tag{1.10}$$

Let $0 < r \le 1/2$, $x \ge 1/r$. Since

$$\arcsin \frac{1}{xr} = \int_{xr}^{\infty} \frac{dy}{y\sqrt{y^2-1}} = \int_x^{\infty} \frac{dt}{t\sqrt{r^2t^2-1}},$$

we have

$$\begin{aligned}
0 &< \int_x^{\infty} \frac{t^2dt}{\sqrt{(t^2-1)^3(t^2-r^{-2})}} - r\arcsin\frac{1}{xr} \\
&= r\int_x^{\infty} \frac{(t^3-\sqrt{(t^2-1)^3})dt}{t\sqrt{(t^2-1)^3(r^2t^2-1)}} \\
&\le 3r\int_x^{\infty} \frac{dt}{\sqrt{(t^2-1)^3(r^2t^2-1)}} \le \frac{12r^2}{x}.
\end{aligned}$$

Hence,

$$\begin{aligned}
0 &< \int_{1/r}^{\infty} \frac{dx}{x}\left(\int_x^{\infty} \frac{t^2dt}{\sqrt{(t^2-1)^3(t^2-r^{-2})}} - r\arcsin\frac{1}{xr}\right) \\
&\le 12r^2\int_{1/r}^{\infty} \frac{dx}{x^2} = \mathcal{O}(r^3) \quad \text{as } r \to 0+.
\end{aligned}$$

We now estimate another component of $I(r)$. For $x \ge 1/r$ and $0 < r \le 1/2$, we have

$$\int_x^{\infty} \frac{dt}{\sqrt{(t^2-1)^3(t^2-r^{-2})}} = r^3\int_0^{1/(xr)} \frac{t^2dt}{\sqrt{(1-r^2t^2)^3(1-t^2)}},$$

and also

$$\begin{aligned}
0 &< \int_0^{1/(xr)} \frac{t^2dt}{\sqrt{(1-r^2t^2)^3(1-t^2)}} - \int_0^{1/(xr)} \frac{t^2dt}{\sqrt{1-t^2}} \\
&= \int_0^{1/(xr)} \frac{t^2dt}{\sqrt{1-t^2}}\left(\frac{1}{\sqrt{(1-r^2t^2)^3}} - 1\right) \\
&\le \int_0^{1/(xr)} \frac{dt}{\sqrt{1-t^2}}\left(\frac{1}{(1-r^2t^2)^2} - 1\right) \\
&\le \int_0^{1/(xr)} \frac{dt}{\sqrt{1-t^2}}\frac{2r^2t^2}{(1-t^2/4)^2} = \mathcal{O}(r^2).
\end{aligned}$$

From this we deduce that

$$\int_x^{\infty} \frac{dt}{\sqrt{(t^2-1)^3(t^2-r^{-2})}} = r^3(1+\mathcal{O}(r^2))\int_0^{1/(xr)} \frac{t^2dt}{\sqrt{1-t^2}}.$$

This implies that

$$
\begin{aligned}
&\int_{1/r}^{\infty} \frac{dx}{x} \int_{x}^{\infty} \frac{dt}{\sqrt{(t^2-1)^3(t^2-r^{-2})}} \\
&= r^3(1+\mathcal{O}(r^2)) \int_{1/r}^{\infty} \frac{dx}{x} \int_{0}^{1/(xr)} \frac{t^2 dt}{\sqrt{1-t^2}} \\
&= r^3(1+\mathcal{O}(r^2)) \int_{0}^{1} \frac{dy}{y} \int_{0}^{y} \frac{t^2 dt}{\sqrt{1-t^2}} \\
&= \frac{1}{2} r^3(1+\mathcal{O}(r^2)) \int_{0}^{1} \frac{dy}{y} \left(\arcsin y - y\sqrt{1-y^2} \right) \\
&= \frac{1}{2} r^3(1+\mathcal{O}(r^2)) \left(\int_{0}^{\pi/2} \frac{t \cos t}{\sin t} dt - \int_{0}^{1} \sqrt{1-y^2} dy \right) \\
&= \frac{\pi}{8} r^3 (\log 4 - 1)(1+\mathcal{O}(r^2)).
\end{aligned}
$$

Using the definition (1.5) of $I(r)$ and (1.6), we get

$$
\begin{aligned}
I(r) &= \int_{1/r}^{\infty} \frac{dx}{x} \int_{x}^{\infty} \frac{(C(r)+t^2)dt}{\sqrt{(t^2-1)^3(t^2-r^{-2})}} \\
&= \frac{\pi}{8} r^3 \log(1/r)(\log 4 - 1) + r \int_{1/r}^{\infty} \frac{dx}{x} \arcsin \frac{1}{xr} + \mathcal{O}(r^3).
\end{aligned}
$$

This and (1.10) yield (1.9). □

Lemma 1.4. *As $a \to 0+$,*

$$
\begin{aligned}
J_1(a) &:= \int_{a}^{1/2} \frac{dx}{x} \int_{x}^{1/2} \frac{t^2 dt}{\sqrt{(t^2-a^2)^3(1-4t^2)}} \\
&= \frac{1}{2} \log^2 a + \mathcal{O}(1), \\
J_2(a) &:= \int_{a}^{1/2} \frac{dx}{x} \int_{x}^{1/2} \frac{dt}{\sqrt{(t^2-a^2)^3(1-4t^2)}} \\
&= a^{-2} \log 2 + \mathcal{O}(a^{-1} \log(1/a)).
\end{aligned}
$$

Proof. We use the following elementary identities:

$$
\begin{aligned}
&\int_{x}^{1/2} \frac{dt}{t\sqrt{1-4t^2}} = \log \frac{1+\sqrt{1-4x^2}}{2x}, \quad 0 < x < 1/2, \\
&\int_{x}^{1/2} \frac{dt}{t\sqrt{1-4t^2}} = \log(1/x) + \mathcal{O}(a), \quad \sqrt{a} \le x < 1/2,
\end{aligned}
$$

$$\int_x^{\sqrt{a}} \frac{t^2 dt}{\sqrt{(t^2-a^2)^3}} = \frac{x}{\sqrt{x^2-a^2}} - \log(x+\sqrt{x^2-a^2}) - 1 + \log(2\sqrt{a}) + \mathcal{O}(a), \quad a < x < \sqrt{a},$$

$$\int_x^{\sqrt{a}} \frac{dt}{\sqrt{(t^2-a^2)^3}} = a^{-2}\left(\frac{x}{\sqrt{x^2-a^2}} - 1 + \mathcal{O}(a)\right), \quad a < x \le \sqrt{a},$$

$$\int_x^{1/2} \frac{dt}{t^3\sqrt{(1-4t^2}} = \frac{\sqrt{1-4x^2}}{2x^2} + 2\log\frac{1+\sqrt{1-4x^2}}{2x} = \mathcal{O}(a^{-1}), \quad \sqrt{a} \le x < 1/2.$$

Then we get

$$J_1(a) = \int_a^{\sqrt{a}} \frac{dx}{x}\left(\int_x^{\sqrt{a}} \frac{t^2 dt}{\sqrt{(t^2-a^2)^3}} + \int_{\sqrt{a}}^{1/2} \frac{dt}{t\sqrt{1-4t^2}}\right) + \int_{\sqrt{a}}^{1/2} \frac{dx}{x}\int_x^{1/2} \frac{dt}{t\sqrt{1-4t^2}} + \mathcal{O}(1)$$

and therefore

$$\begin{aligned} J_1(a) &= \int_a^{\sqrt{a}} \frac{dx}{x}\left(\frac{x}{\sqrt{x^2-a^2}} - \log(x+\sqrt{x^2-a^2}) - 1 + \log(2\sqrt{a})\right) \\ &\quad + \int_a^{\sqrt{a}} \frac{\log(1/\sqrt{a})dx}{x} + \int_{\sqrt{a}}^{1/2} \frac{\log(1/x)dx}{x} + \mathcal{O}(1) \\ &= \left[\log(x+\sqrt{x^2-a^2}) - \frac{1}{2}\log^2 x - \log x + \log\sqrt{a}\log x\right]_a^{\sqrt{a}} \\ &\quad + \log(1/\sqrt{a})\left[\log x\right]_a^{\sqrt{a}} - \frac{1}{2}\left[\log^2 x\right]_{\sqrt{a}}^{1/2} + \mathcal{O}(1) \\ &= \frac{1}{2}\log^2 a + \mathcal{O}(1), \end{aligned}$$

which proves the estimation for $J_1(a)$. Since

$$J_2(a) = \int_a^{\sqrt{a}} \frac{dx}{x}\left(\int_x^{\sqrt{a}} \frac{dt}{\sqrt{(t^2-a^2)^3}} + \int_{\sqrt{a}}^{1/2} \frac{dt}{t^3\sqrt{1-4t^2}}\right)(1+\mathcal{O}(a)) + \int_{\sqrt{a}}^{1/2} \frac{dx}{x}\int_x^{1/2} \frac{dt}{t^3\sqrt{1-4t^2}}(1+\mathcal{O}(a))$$

we get

$$\begin{aligned} J_2(a) &= \int_a^{\sqrt{a}} \frac{dx}{x}a^{-2}\left(\frac{x}{\sqrt{x^2-a^2}} - 1 + \mathcal{O}(a)\right)(1+\mathcal{O}(a)) + \int_{\sqrt{a}}^{1/2} \frac{dx}{x}\mathcal{O}(a^{-1}) \\ &= a^{-2}\log 2 + \mathcal{O}(a^{-1}\log(1/a)), \end{aligned}$$

which yields the estimation for $J_2(a)$. □

Lemma 1.5. *For $0 < a < 1/4$, let $b_a > 1/2$ and functions $g_a \in C(a, b_a]$ be given which satisfy*

$$b_a = \frac{1}{2} + \mathcal{O}(a), \quad a \to 0+,$$
$$0 < g_a(t) \le 1, \quad 1/4 \le t \le b_a, \tag{1.11}$$
$$0 < g_a(t) \le (t-a)^{-3/2}, \quad a < t \le 1/4.$$

Then the functions

$$J^*(a) := \int_a^{b_a} \frac{dx}{x} \int_x^{b_a} \frac{g_a(t)dt}{\sqrt{b_a^2 - t^2}}, \quad J(a) := \int_a^{1/2} \frac{dx}{x} \int_x^{1/2} \frac{g_a(t)dt}{\sqrt{1/4 - t^2}} \tag{1.12}$$

satisfy

$$J^*(a) = J(a) + \mathcal{O}(1), \quad a \to 0+. \tag{1.13}$$

Proof. For $1/2 \le b \le 1$,

$$I_1(b) = \int_{1/4}^{b} \frac{dx}{x} \int_x^{b} \frac{g_a(t)dt}{\sqrt{b^2 - t^2}} \le (4b-1) \int_{1/4}^{b} \frac{dt}{\sqrt{b^2 - t^2}} \le c_1$$

for some absolute constant $c_1 > 0$. Moreover,

$$I_2 := \left| \int_{1/4}^{b_a} \frac{dt}{\sqrt{b_a^2 - t^2}} - \int_{1/4}^{1/2} \frac{dt}{\sqrt{1/4 - t^2}} \right| = \arcsin\frac{1}{2} - \arcsin\frac{1}{4b_a},$$

hence $I_2 = \mathcal{O}(b_a - 1/2)$; and from

$$I_3 := \left| \int_{1/4}^{b_a} \frac{g_a(t)dt}{\sqrt{b_a^2 - t^2}} - \int_{1/4}^{1/2} \frac{g_a(t)dt}{\sqrt{1/4 - t^2}} \right| \le 2 \int_{1/2}^{b_a} \frac{dt}{\sqrt{b_a^2 - t^2}} + I_2$$

it follows that $I_3 = \mathcal{O}(\sqrt{b_a - 1/2})$. For $a < x \le 1/4$, we get

$$\begin{aligned} I_4(x) &:= \left| \int_x^{1/4} \frac{g_a(t)dt}{\sqrt{b_a^2 - t^2}} - \int_x^{1/4} \frac{g_a(t)dt}{\sqrt{1/4 - t^2}} \right| \\ &\le \int_x^{1/4} \frac{dt}{(t-a)^{3/2}} \left(\frac{1}{\sqrt{1/4 - t^2}} - \frac{1}{\sqrt{b_a^2 - t^2}} \right) \le \frac{c_2(b_a - 1/2)}{\sqrt{x-a}}, \end{aligned}$$

for some absolute constant $c_2 > 0$. From the above inequalities we deduce that

$$\begin{aligned} |J^*(a) - J(a)| &\le I_1(1/2) + I_1(b_a) + \int_a^{1/4} \frac{dx}{x}(I_3 + I_4(x)) \\ &\le \mathcal{O}(1) + I_3 \log(1/a) + \mathcal{O}(b_a - 1/2) \int_a^{1/4} \frac{dx}{x\sqrt{x-a}}. \end{aligned}$$

This proves (1.13) since $\mathcal{O}(b_a - 1/2) = \mathcal{O}(a)$. □

Appendix 3. Hardy Spaces and Blaschke Products

§ 1. Hardy Spaces

The Hardy spaces H_p are a counterpart, for analytic functions in the disk $D := \{z : |z| < 1\}$, of the L_p spaces on $\mathbb{T}$. Their theory started with the famous early (1906) theorem of Fatou, who proved that the bounded analytic functions on D (they form the space H_∞) have radial limits almost everywhere on the boundary $\partial D = \mathbb{T}$. For $0 < r < 1$, $0 < p \le \infty$, and an analytic function f on D, let

$$M_p(f,r) := \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_{\mathbb{T}} |f(re^{it})|^p\,dt\right)^{1/p}, & \text{if } 0 < p < \infty \\ \max\limits_{|z|=r} |f(z)|, & \text{if } p = \infty . \end{cases} \tag{1.1}$$

This expression is an increasing function of r. In order to generalize spaces H_∞, Hardy and F. Riesz defined the space $H_p := H_p(D)$ on the disk D to consist of f, analytic on D, for which

$$\|f\|_{H_p} := \sup_{0<r<1} M_p(f,r) < \infty . \tag{1.2}$$

For the theory of Hardy spaces, one can recommend the excellent books of Duren [B-1970], Garnett [B-1981], Zygmund [B-1959].

In §§1-2 of Appendix 3 we shall list some facts about the H_p spaces and the conjugate functions, needed in Chapter 10. In the last section, we will discuss the atomic decomposition of functions $f \in H_p$, due to Coifman [1974]. To the theory of spaces $H_p(D)$, there is a parallel and very similar theory of Hardy spaces $H_p(U)$ on the upper half plane U, with a similar definition, where (1.1) is replaced by the expression

$$M_p(f,y) = \left(\int_{-\infty}^{+\infty} |f(x+iy)|^p\,dx\right)^{1/p}, \qquad y > 0 .$$

For $0 < \alpha < \pi$, $\zeta \in \mathbb{T}$, let $\Omega_\alpha(\zeta)$ be the drop-shaped region, contained in the disk D, bounded by the two tangents to the circle $|z| = r_0$, $0 < r_0 < 1$, which emanate from ζ, and by the larger of the two enclosed arcs of that circle. The region $\Omega_\alpha(\zeta)$ is completely described by the angle α between the two tangents.

If we select $r_0 = 1/\sqrt{2}$, we obtain the region $\Omega(\zeta) := \Omega_{\pi/2}(\zeta)$ with the right angle $\alpha = \pi/2$. The basic property of the spaces H_p is the existence of non-tangential boundary values $g(\zeta)$ of $f \in H_p$.

Theorem 1.1. *For each $f \in H_p$, $0 < p \le \infty$, there exists for almost all $\zeta \in \mathbb{T}$ and all α, $0 < \alpha < \pi$ the limit*

$$g(\zeta) = \lim_{\substack{z \to \zeta \\ z \in \Omega_\alpha(\zeta)}} f(z) . \tag{1.3}$$

The boundary function g belongs to $L_p(\mathbb{T})$ and one has

$$\|f(re^{it}) - g(e^{it})\|_{L_p} \to 0 , \qquad r \to 1 . \tag{1.4}$$

If the boundary function g vanishes on a set of positive measure, then both f and g are zero identically. There is therefore a 1–1 correspondence between functions $f \in H_p(D)$ and their boundary functions g, which form a proper subset of $L_p(\mathbb{T})$. It is customary to identify f with g on $\mathbb{T}$ and write $f(e^{it}) := g(e^{it})$, $t \in \mathbb{T}$.

Theorem 1.2. *The expression* (1.2) *is a norm if $1 \le p \le \infty$, a quasi-norm if $0 < p < 1$; the space H_p is complete. If $f \in H_p$ and $f(e^{it}) \in L_q$, $q > p$, then $f \in H_q$.*

For a given $f \in H_p$ we shall often use functions $f_r(z) := f(rz)$, defined for $0 \le r < 1$. They have the following properties: f_r is analytic on $\bar{D}$, $\|f_r\|_{H_p} \le \|f\|_{H_p}$ — because the functional $M_p(f,r)$ is increasing, and (see (1.4)) for $p \ne \infty$,

$$\lim_{r \to 1-} \|f - f_r\|_{H_p} = 0 . \tag{1.5}$$

The boundary functions $f(e^{it})$ of $f \in H_p$ can be characterized in the following way:

Theorem 1.3. *The boundary values $f(z)$, $z \in \mathbb{T}$ of the $f \in H_p$ are precisely the L_p-limits on $\mathbb{T}$ of polynomials $a_0 + \cdots + a_n z^n$. If $1 \le p \le \infty$, they are precisely the functions $f(e^{it}) \in L_p(\mathbb{T})$ with the Fourier series $f(e^{it}) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}$ satisfying $c_k = 0$, $k < 0$.*

One of the deeper facts of the theory are the properties of the Hardy-Littlewood majorants $\mathcal{M}(f) := \mathcal{M}(f,\zeta)$ of analytic functions. In connection with the regions $\Omega_\alpha(\zeta)$ we define

$$\mathcal{M}(f,\zeta) := \sup_{z \in \Omega_\alpha(\zeta)} |f(z)| , \qquad \zeta \in \mathbb{T} . \tag{1.6}$$

Theorem 1.4. *With a constant C that depends only on p and α, we have*

$$\|\mathcal{M}(f)\|_{L_p} \le C\|f\|_{H_p}\,, \qquad f \in H_p\,,\ 0 < p \le \infty\,. \tag{1.7}$$

(For the proof see Garnett [B-1981], where it is carried out for the $H_p(U)$, and Zygmund [B-1959].)

We write $D^+ := D$, $H_p^+ := H_p(D^+)$. Similar theorems prevail for Hardy spaces H_p^- of functions analytic on $D^- := \{z : |z| > 1\}$ (D^- includes the point ∞). They are based on functionals (1.1) with $r > 1$. These results are obtainable from those given above by the transformation $z \to 1/z$.

Here are two important special theorems of Hardy and Littlewood.

Theorem 1.5. *If f is analytic on D and satisfies $f(0) = 0$ and $f' \in H_p$ for some $0 < p < 1$, then $f \in H_q$, $q := p/(1-p)$; moreover*

$$\|f\|_{H_q} \le C\|f'\|_{H_p}\,, \qquad C = C(p)\,. \tag{1.8}$$

Proof. It is easy to see that for each x, $0 \le x < 1$, $\zeta \in \partial D$, the region $\Omega_{\pi/2}(\zeta e^{i(1-x)})$ contains the point $x\zeta$. Hence $|f'(x\zeta)| \le \mathcal{M}(f', \zeta e^{i(1-x)})$, and from (1.7) we get $\int_0^1 |f'(x\zeta)|^p\,dx \le CM^p$, where $M := \|f'\|_{H_p}$. This allows us to estimate $|f(\zeta)|$:

$$\begin{aligned}|f(\zeta)| = |f(\zeta) - f(0)| &\le \int_0^1 |f'(x\zeta)|\,dx \\ &\le \mathcal{M}(f',\zeta)^{1-p}\int_0^1 |f'(x\zeta)|^p\,dx \le C\,\mathcal{M}(f',\zeta)^{1-p}M^p\,.\end{aligned}$$

We apply (1.7) again:

$$\int_{\partial D} |f(\zeta)|^q|d\zeta| \le C\int_{\partial D} \mathcal{M}(f',\zeta)^p\,|d\zeta|\; M^{pq} \le CM^{p+pq} = CM^q\,. \qquad \square$$

Theorem 1.6. *A function $f \in H_1$ coincides a.e. on ∂D with some function of bounded variation if and only if $f' \in H_1$. In this case, f is continuous on $\bar{D}$ and absolutely continuous on ∂D. If, in addition, $f(0) = 0$, then*

$$\|f\|_{C(D)} \le \|f\|_{W_1^1(\partial D)} = 2\pi\|f'\|_{H_1}\,. \tag{1.9}$$

Proof. Let f a.e. on ∂D be equal to a function of bounded variation. For each $n = 1, 2, \dots$ we define $t_k := t_{k,n} := \pi k/n$, $k = 0, \dots, 2n-1$. Then for some $V \ge 0$, we have

$$\sum_{k=0}^{2n-1} |f(e^{i(t+t_k)}) - f(e^{i(t+t_{k-1})})| \le V \ \text{ a.e. on } \mathbb{T}\,.$$

Putting $f_k(z) := f(ze^{it_k}) - f(ze^{it_{k-1}})$ and integrating, we get

$$\sum_{k=0}^{2n-1} M_1(f_k, 1) = \sum_{k=0}^{2n-1} \frac{1}{2\pi} \int_{\mathbf{T}} |f(e^{i(t+t_k)}) - f(e^{i(t+t_{k-1})})| \, dt \le V$$

and since $M_1(f_k, r) \le M_1(f_k, 1)$ for $0 < r < 1$,

$$\sum_{k=0}^{2n-1} \frac{1}{2\pi} \int_{\mathbf{T}} |f(re^{i(t+t_k)}) - f(re^{i(t+t_{k-1})})| \, dt \le V \, . \tag{1.10}$$

For a fixed $0 < r < 1$ we can exchange sum and integration and then make $n \to \infty$. This yields $r \int_{\mathbf{T}} |f'(re^{it})| \, dt \le V$, that is, $f' \in H_1$.

To establish the remaining part, let $f' \in H_1$ and $f(0) = 0$. First let f be analytic on $\bar{D}$. We show that f satisfies (1.9).

Indeed, since $f(0) = 0$, the mean values of the harmonic functions $u := \operatorname{Re} f$, $v = \operatorname{Im} f$ on ∂D are zero, and they must have zeros there. From the principle of the maximum modulus for f we get (1.9):

$$\begin{aligned} \|f\|_{C(\bar{D})} &= \|f\|_{C(\partial D)} \le \|u\|_{C(\partial D)} + \|v\|_{C(\partial D)} \\ &\le \tfrac{1}{2} \mathrm{Var}_{\partial D} u + \tfrac{1}{2} \mathrm{Var}_{\partial D} v \le \int_{\partial D} |f'(z)| \, |dz| \\ &= 2\pi \|f'\|_{H_1} \, . \end{aligned}$$

In the general case, we use the sequence $f_j(z) := f(r_j z)$ for $0 < r_1 < \cdots$, $r_j \to 1$. Since $f' \in H_1$ by (1.5), the sequence (f_j') converges to f' in H_1. It follows that (f_j) is a Cauchy sequence in both $W_1^1(\partial D)$ and $C(\bar{D})$. An appeal to the completeness of the spaces completes the proof. □

§ 2. Conjugate Functions and Cauchy Integrals

For $z = re^{it}$, $r \in [0, 1)$ and $t \in \mathbb{T}$, we define the *Poisson kernel*

$$P(r, t) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{int} = \frac{1 - r^2}{1 - 2r \cos t + r^2}$$

and the *conjugate Poisson kernel*

$$Q(r, t) = -i \sum_{n=-\infty}^{\infty} (\operatorname{sign} n) r^{|n|} e^{int} = \frac{2r \sin t}{1 - 2r \cos t + r^2} \, .$$

With their help, for real-valued functions $g \in L_1(\mathbb{T})$, we obtain the *Poisson integral*

$$u(z) = \frac{1}{2\pi} \int_{\mathbf{T}} P(r, t - \theta) g(\theta) \, d\theta = \sum_{n=-\infty}^{\infty} r^{|n|} g_n e^{int} \tag{2.1}$$

and the *conjugate Poisson integral*

$$v(z) = \frac{1}{2\pi}\int_{\mathbb{T}} Q(r,t-\theta)g(\theta)\,d\theta = -i\sum_{n=-\infty}^{\infty}(\operatorname{sign} n)r^{|n|}g_n e^{int}\,, \tag{2.2}$$

where $g_n := (2\pi)^{-1}\int_{\mathbb{T}} g(t)e^{-int}\,dt$ are the Fourier coefficients of the function g.

The series (2.1) and (2.2) converge uniformly on compact subsets of D, the functions $r^{|n|}e^{int}$, $n = 0, \pm 1, \pm 2, \dots$, are harmonic on D. Therefore also $u(z)$ and $v(z)$ are harmonic on D. The function

$$f(z) := u(z) + iv(z) = g_0 + 2\sum_{n=1}^{\infty} g_n z^n \tag{2.3}$$

is analytic on D. Thus, $u(z)$ and $v(z)$ are conjugate harmonic functions.

Arguments similar to those in the proof of [**CA**, Theorem 2.4, p.6] show that if $g \in L_1(\mathbb{T})$, then for each $\alpha \in (0,1)$ and almost all $x \in \mathbb{T}$ one has

$$\lim u(z) = g(x)$$

as $z \to e^{ix}$, and $z \in \Omega_\alpha(e^{ix})$.

Also for the function v, if $\alpha \in (0,1)$, for almost all $x \in \mathbb{T}$ there exists the limit

$$\lim v(z) =: \tilde{g}(x)\,, \qquad z \to e^{ix}\,, \quad z \in \Omega_\alpha(e^{ix})\,.$$

The function $\tilde{g}$ is called *the conjugate function* of g.

Theorem 2.1. *If $g \in L_1(\mathbb{T})$, then for almost all $x \in \mathbb{T}$ the integral*

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{g(x+t)}{2\tan\frac{t}{2}}\,dt := \lim_{\delta\to+0}\frac{1}{\pi}\left(\int_{\delta}^{\pi} + \int_{-\pi}^{-\delta}\right)\frac{g(x+t)}{2\tan\frac{t}{2}}\,dt \tag{2.4}$$

exists and is equal to $\tilde{g}(x)$.

Integral (2.4) is sometimes taken as a definition of the conjugate function.

In what follows, functions $g \in L_1(\mathbb{T})$ will be treated as functions of the complex variable $\zeta = e^{it}$. We introduce the two Cauchy integrals, $C^+(g,\cdot)$ and $C^-(g,\cdot)$:

$$C^{\pm}(g,z) = \frac{1}{2\pi i}\int_{\partial D}\frac{g(\zeta)}{\zeta - z}\,d\zeta\,, \qquad z \in D_{\pm}\,. \tag{2.5}$$

If $\zeta \in \partial D$ and $z \in D_+$, then $(\zeta - z)^{-1} = \zeta^{-1}\sum_{n=0}^{\infty} z^n\zeta^{-n}$, and we obtain

$$C^+(g,z) = \sum_{n=0}^{\infty} g_n z^n\,, \qquad z \in D_+\,. \tag{2.6}$$

In a similar way it follows that

$$C^-(g,z) = -\sum_{n=-1}^{-\infty} g_n z^n\,, \qquad z \in D_-\,. \tag{2.7}$$

From the relations (2.1), (2.2), (2.6) and (2.7) we get, if $0 \le r < 1$ and $t \in \mathbb{T}$,

$$C^+(g, re^{it}) - C^-\left(g, \frac{1}{r}e^{it}\right) = u(re^{it})\,,$$

$$C^+(g, re^{it}) + C^-\left(g, \frac{1}{r}e^{it}\right) = iv(re^{it}) + g_0\,.$$

This yields the existence of non-tangential limits of $C^{\pm}(g,\cdot)$ a.e. on $\mathbb{T}$. We denote these by $C^{\pm}(g,\zeta)$ and have a.e.

$$C^+(g,\zeta) - C^-(g,\zeta) = g(\zeta)\,, \qquad \zeta \in \partial D\,, \tag{2.8}$$

$$C^+(g,\zeta) + C^-(g,\zeta) = i\tilde{g}(\zeta) + g_0\,, \qquad \zeta \in \partial D\,, \tag{2.9}$$

$$f(z) = 2C^+(g,z) - g_0\,, \qquad z \in D\,. \tag{2.10}$$

If, in particular, $g \in L_2(\mathbb{T})$, then $\tilde{g} \in L_2(\mathbb{T})$, the functions $C^{\pm}(g,\cdot)$ belong to $H_2^{\pm}$ and one has

$$\|C^{\pm}(g,\cdot)\|_{H_2^{\pm}} \le (2\pi)^{-1/2}\|g\|_{L_2(\mathbb{T})}\,, \tag{2.11}$$

$$\|\tilde{g}\|_{L_2(\mathbb{T})} \le \|g\|_{L_2(\mathbb{T})}\,. \tag{2.12}$$

Indeed, if g is a trigonometric polynomial, then (2.11) follows from (2.6) and (2.7); to obtain (2.12) it is necessary to use also (2.9). Because the trignonometric polynomials are dense in H_2 and L_2, inequalities (2.11) and (2.12) hold also in the general case.

The following result generalizes this to $p \neq 2$:

Theorem 2.2 (M. Riesz). *If $g \in L_p(\mathbb{T})$, $1 < p < \infty$, then $\tilde{g} \in L_p(\mathbb{T})$, and*

$$\|\tilde{g}\|_p(\mathbb{T}) \le C_p\|g\|_p(\mathbb{T})\,.$$

Moreover, one has $C^{\pm}(g,\cdot) \in H_p^{\pm}$ and

$$\|C^{\pm}(g,\cdot)\|_{H_p} \le C_p\|g\|_p\,.$$

This theorem is not valid for $p = 1$ and $p = \infty$. For $p = 1$, one replaces L_1 by the Orlicz space $L\log L$. This space consists of all measurable functions g on $\mathbb{T}$ with the property $|g|\log^+|g| \in L_1$, where $\log^+ a$, $a \ge 0$, is defined by $\log^+ a = \log a$ if $a \ge 1$, $\log^+ a = 0$ if $0 < a < 1$. The norm of $L\log L$ (which makes it a Banach space) is given by

$$\|g\|_{L\log L} := \inf\left\{\lambda > 0 : \int_T \frac{|g(t)|}{\lambda}\log^+\frac{|g(t)|}{\lambda}\,dt \le 1\right\}\,. \tag{2.13}$$

Theorem 2.3 (Zygmund). *If $g \in L \log L$ on $\mathbb{T}$, then $\tilde{g} \in L_1(\mathbb{T})$, $C^{\pm}(g, \cdot) \in H_1^{\pm}$ and one has the inequalities*

$$\|\tilde{g}\|_1 \le C\|g\|_{L \log L} ,$$
$$\|C^{\pm}(g, \cdot)\|_{H_1^{\pm}} \le C\|g\|_{L \log L} .$$

In (2.5) we replace the measure $g(\zeta)\, d\zeta$ by any regular complex-valued Borel measure $d\mu$ on ∂D. This defines a Cauchy-Stieltjes integral $C^{\pm}(d\mu, z)$ for $z \in D_{\pm}$.

Theorem 2.4 (Smirnov). *Each Cauchy-Stieltjes integral $C^{\pm}(d\mu, z)$ belongs to $H_p^{\pm}$ for $0 < p < 1$ and satisfies*

$$\|C^{\pm}(d\mu, \cdot)\|_{H_p^{\pm}} \le C_p \operatorname{Var}_{\partial D} \mu .$$

§ 3. Atomic Decompositions in Hardy Spaces

Atomic decompositions are representations on ∂D of functions $g \in H_p$, $0 < p \le 1$, by means of sums of simple functions. We begin with Theorem 3.1 of Coifman [1974] which we state for functions analytic on $\overline{D}$. In this form the theorem is needed in Chapter 10 (usually when $1/p$ is an integer).

A function $a \in L_\infty(A)$, $A = \partial D$ or $A = \mathbb{R}$, and an associated open interval $J \subset A$ is a p-atom, $0 < p \le 1$, on A if with $r = [1/p]$,

$$\text{(3.1)} \qquad \begin{cases} \text{(i)} \quad a(z) = 0 , \quad z \in A \setminus J \\ \text{(ii)} \quad \displaystyle\int_A z^k a(z)\, dz = 0 , \quad k = 0, \dots, r-1 . \end{cases}$$

Each possible $J := J(a)$, we will call a *supporting interval* of the atom a. For $A = \partial D$ we do not exclude the case $J(a) = \partial D \setminus \{z_0\}$ for a single point $z_0 \in \partial D$, or $J(a) = \partial D$.

Theorem 3.1 (Coifman's decomposition lemma). *Let $0 < p \le 1$ and $A = \mathbb{T}$ $(= \partial D)$. There exists a constant $C := C(p)$ with the property that for each function g, analytic on $\overline{D}$, there is a finite or infinite sequence of p-atoms $(a_k)_{k \ge 1}$ satisfying*

$$\text{(i)} \qquad \sum_{k \ge 1} |J(a_k)|\, \|a_k\|_\infty^p \le C\|g\|_{H_p}^p ,$$

$$\text{(ii)} \qquad g(z) = \sum_{k \ge 1} a_k(z) , \qquad z \in \partial D ,$$

(iii) *any two intervals* $J(a_k)$, $J(a_{k'})$ *are either disjoint, or one of them is contained in the other.*

Proof. We can assume that (for $\alpha = \pi/2$ in (1.6))

$$\|\mathcal{M}g\|_p(\partial D) = 1 . \tag{3.2}$$

(a) First some elementary properties of orthonormal polynomials q_k of degree $k = 0, \dots, r-1$ with weight 1 on an interval $[\alpha, \beta] \subset \mathbb{C}$, with the orthogonality relation $\int_\alpha^\beta q_k q_\ell \, dz = 0$. If g is continuous on $[\alpha, \beta]$ and if h_r is the r-th sum of its orthogonal expansion, then $g - h_r$ is orthogonal to the q_k, $k = 0, \dots, r-1$, hence also to the powers z^k:

$$\int_\alpha^\beta z^k g(z)\, dz = \int_\alpha^\beta z^k h_r(z)\, dz , \qquad k = 0, \dots, r-1 . \tag{3.3}$$

The partial sum h_r has the representation $h_r(z) = \int_\alpha^\beta K(z,\zeta) g(\zeta)\, d\zeta$ with the kernel $K(z,\zeta) = \sum_{k=0}^{r-1} q_k(z) q_k(\zeta)$. If $d = |\beta - \alpha|$ is the length of the interval $[\alpha, \beta]$, we have $|K(z,\zeta)| \le Cd^{-1}$, $C = C(p)$ for $z, \zeta \in G$. Here $G =: G(\alpha, \beta)$ is the closed disk with center $\frac{1}{2}(\alpha + \beta)$ and radius $\frac{1}{2}d$. Indeed, if $\alpha = -1$, $\beta = 1$, then $G = \overline{D}$, $\|K\|_2([-1,1]\times[-1,1]) = \sqrt{r}$ and, therefore, $\|K\|_\infty(G \times G) = C = C(p)$. (On finitely dimensional linear spaces, all norms are weakly equivalent.) By a linear transformation, we get our inequality in the general case.

(b) We define the open subsets of ∂D by

$$A_\ell := \{z \in \partial D \ , \ \mathcal{M}g(z) > 2^\ell\} , \qquad \ell = 0, 1, \dots .$$

Since $\mathcal{M}g$ is continuous on ∂D, $A_\ell = \emptyset$ for all large ℓ. If $A_0 = \emptyset$, we can take the single atom $a_1 = g$ with $J(a_1) = \partial D$. Then $\|g\|_\infty \le 1$ and we have (i) because of (3.2) and Theorem 1.4. Each set $A_\ell \ne \emptyset$ is the union of disjoint open intervals $J_{\ell,j} : A_\ell = \bigcup_{j \ge 1} J_{\ell,j}$, and (iii) is satisfied.

(c) To construct the atoms $a_{\ell,j}$, we define for $\ell = 0, 1, \dots$

$$g_\ell(z) := \begin{cases} g(z) & \text{for } z \in \partial D \setminus A_\ell \\ h_{\ell,j}(z) & \text{for } z \in J_{\ell,j} , \end{cases}$$

where $h_{\ell,j}$ is a polynomial of degree $\le r-1$, namely the r-th partial sum of the orthogonal expansion of g on $J_{\ell,j}$. From (3.3),

$$\int_{J_{\ell,j}} z^k \big(g(z) - h_{\ell,j}(z)\big)\, dz = 0 , \qquad k = 0, \dots, r-1 . \tag{3.4}$$

For all $z \in \partial D$,

$$g(z) = g_0(z) + \sum_{\ell \ge 0} [g_{\ell+1}(z) - g_\ell(z)] .$$

If $\chi_{\ell,j}$ is the characteristic function of the set $J_{\ell,j}$, we put $a_0 := g_0$ and for $\ell > 0$

$$a_{\ell,j} := \chi_{\ell,j}(g_{\ell+1} - g_\ell) . \tag{3.5}$$

For the sets $J_{0,j}$ we use the property (3.3) for the $h_{0,j}$, Cauchy's theorem, and get

$$\int_{\partial D} z^k g_0(z)\,dz = \int_{\partial D} z^k g(z)\,dz = 0\ , \qquad k = 0,\dots,r-1\ ,$$

so that a_0 is an r-atom. Likewise, for $a_{\ell,j}$, $\ell > 0$ we use (3.5) on $J_{\ell,j}$ and (3.4) on all the intervals $J_{\ell+1,j_1} \subset J_{\ell,j}$ to obtain for $k = 0,\dots,r-1$

$$\int_{\partial D} z^k a_{\ell,j}(z)\,dz = \int_{J_{\ell,j}} z^k a_{\ell,j}\,dz = \int_{J_{\ell,j}} z^k g_{\ell+1}\,dz -$$

$$- \int_{J_{\ell,j}} z^k g_\ell\,dz = \int_{J_{\ell,j}} z^k g\,dz - \int_{J_{\ell,j}} z^k g\,dz = 0\ .$$

(d) To prove (i) of the theorem, we have to establish for all ℓ, j

$$|h_{\ell,j}(z)| \le C2^\ell\ , \qquad z \in J_{\ell,j}\ , \quad C = C(r)\ . \tag{3.6}$$

From (3.2) and the inequality $|\mathcal{M}g(z)| \ge 1$ on A_ℓ, $\ell = 0,1,\dots$, we obtain $|J_{\ell,j}| \le 1$ for all ℓ, j. Let α, β be the endpoints of $J_{\ell,j}$, let $K(z,\zeta)$ be the corresponding kernel of (a), then $h_{\ell,j}(z) = \int_\alpha^\beta K(z,\zeta)g(\zeta)\,d\zeta$. By Γ we denote a line connecting α and β which consists of intervals of the boundaries of the regions $\Omega(\alpha)$, $\Omega(\beta)$ with the opening angle $\pi/2$, which are contained in the disk $G = G(\alpha,\beta)$ of (a). The intevals meet on the boundary of G. Since Kg is an analytic function of ζ for $|\zeta| \le 1$, we also have $h_{\ell,j}(z) = \int_\Gamma K(z,\zeta)g(\zeta)\,d\zeta$. Now $\mathcal{M}g(z) = 2^\ell$ for $z = \alpha, \beta$, consequently $|g(\zeta)| \le 2^\ell$ on Γ, the length of Γ does not exceed $2d$. From (a) we get for $z \in J_{\ell,j}$

$$\begin{aligned} |h_{\ell,j}(z)| &\le \|K\|_\infty(G \times G)\|g\|_\infty(\Gamma)|\Gamma| \\ &\le \frac{C}{d} 2^\ell 2d = C2^{\ell+1}\ . \end{aligned}$$

Now we see from (3.5) that on $J_{\ell,j}$,

$$|a_{\ell,j}(z)| \le C2^\ell\ , \qquad \ell, j \ge 1\ , \quad |a_0(z)| \le C\ .$$

Using Theorem 1.4 we derive

$$\begin{aligned} \|a_0\|_\infty + \sum_{\ell,j\ge1} |J_{\ell,j}|\ \|a_{\ell,j}\|_\infty^p &\le C \sum_{\ell\ge0\,,\,j\ge1} 2^{\ell p}|J_{\ell,j}| \\ &\le C \sum 2^{\ell p}(|A_\ell| - |A_{\ell+1}|) \le C \int_{\partial D} (\mathcal{M}g(z))^p |dz| \\ &\le C\|g\|_{H_p}\ . \end{aligned}$$

□

To illustrate the usefulness of Theorem 3.1, we shall establish the standard atomic decomposition theorem for the space H_1. We put $\lambda(a) := |J(a)|\ \|a\|_\infty$.

Theorem 3.2. *Every function $g \in H_1(D)$ can be represented by 1-atoms (a_k) so that*

$$\text{(i)} \qquad \sum_{k\geq 1} \lambda(a_k) \leq C\|g\|_{H_1} ,$$

$$\text{(ii)} \qquad g(z) = \sum_{k\geq 1} a_k(z) , \qquad z \in \partial D ,$$

$$\text{(iii)} \qquad g(z) = \sum_{k\geq 1} C^+(a_k, z) , \qquad z \in D ,$$

where C is an absolute constant; the series (ii) *converges a.e. and in the L_1-norm, and* (iii) *converges uniformly on compact subsets of D.*

Proof. We take functions g_j $(= g_{\rho_j}, \rho_j < 1)$, $j = 1, 2, \ldots, g_{-1} = 0$ of (1.5) in such a way that $\|g - g_j\|_1 \leq 2^{-j-1}\|g\|_1$. The $g_j - g_{j-1}$ are analytic on $\bar{D}$, and for each of them we have a decomposition $g_j - g_{j-1} = \sum_{k\geq 1} a_{j,k}$ of Theorem 3.1 (where we take $r = 1$) with $\sum_{k\geq 1} \lambda(a_{j,k}) \leq C\|g_j - g_{j-1}\|_1 \leq 2^{-j}C\|g\|_1$. For a 1-atom a, $\|a\|_1 \leq \lambda(a)$. Thus, for the sequence (a_k) consisting of all the $a_{j,k}$ we shall have (i) and (ii), while (iii) will follow from (ii) by integration. □

We shall need some properties of p-atoms. For a p-atom a we write

$$\lambda_p(a) := |J| \, \|a\|_\infty^p . \tag{3.7}$$

Lemma 3.3. *The Cauchy transform $C^+(a, z)$ of each p-atom a belongs to H_q for all $q < \infty$. Its norm in H_p, $0 < p \leq 1$, satisfies*

$$\|C^+(a)\|_{H_p} \leq C_p \lambda_p(a)^{1/p} . \tag{3.8}$$

Proof. Since a is bounded on ∂D, it belongs to all spaces L_q, $q < \infty$. By Theorem 2.2, $C^+(a, z)$ is defined on D, and belongs to H_q. To prove (3.8) we note (see inequality (2.11)) that for a p-atom a, $\|C^+(a)\|_{L_2(\partial D)} \leq \|a\|_{L_2(\partial D)}$ and that $\|a\|_{L_2(\partial D)} \leq |J|^{1/2}\|a\|_\infty$. Let J_1 be the interval concentric with J of length $2|J|$ if $|J| \leq \pi$, otherwise let $J_1 = \partial D$. Using Hölder's inequality and the facts stated, we see that

$$\begin{aligned} \|C^+(a)\|_{L_p(J_1)} &\leq |J_1|^{\frac{1}{p}-\frac{1}{2}} \|C^+(a)\|_{L_2(\partial D)} \\ &\leq 2^{1/p}|J|^{1/p}\|a\|_\infty . \end{aligned} \tag{3.9}$$

This already proves (3.8) if $|J| \geq \pi$. If $|J| < \pi$, let z_0 be the middle point of the arc J. For the p-atom a, we can use (ii) of (3.1) with $r := [1/p]$ and obtain for $z \in \partial D \setminus J_1$

$$
\begin{aligned}
C^+(a,z) &= \frac{1}{2\pi i}\int_J \frac{a(\zeta)d\zeta}{\zeta - z} \\
&= \frac{1}{2\pi i}\int_J \frac{d(\zeta)}{\zeta - z}\left(1 - \frac{\zeta - z}{z_0 - z}\right)^r d\zeta \\
&= \frac{1}{2\pi i}\cdot\frac{1}{(z_0 - z)^r}\cdot\int_J \frac{a(\zeta)(z_0-\zeta)^r}{\zeta - z}\,d\zeta\,.
\end{aligned}
$$

In the integral, $|z_0 - \zeta| \le \frac{1}{2}|J|$ and $|\zeta - z| \ge \frac{1}{2}|z - z_0|$, so that

$$
|C^+(a,z)| \le \frac{1}{2^r\pi}|J|^{r+1}\frac{\|a\|_\infty}{|z - z_0|^{r+1}}\,, \qquad z \in \partial D \setminus J_1\,.
$$

Integrating, we obtain

$$
\|C^+(a)\|_{L_p(\partial D\setminus J_1)} \le C_p|J|^{1/p}\|a\|_\infty\,. \qquad \square
$$

§ 4. Blaschke Products

A *Möbius transformation* $\mu(z)$ is a conformal mapping of $D = \{z : |z| < 1\}$ onto itself. Every such function is given by a formula

$$
\mu(z) = \eta\frac{z-a}{1-\bar{a}z}
$$

where parameters η, and a satisfy $|\eta| = 1$ and $a \in D$. The function $\mu(z)$ is analytic on $\bar{D}$ and satisfies $|\mu(z)| = 1$ for $|z| = 1$. With μ also its inverse μ^{-1} is a Möbius transformation.

A product of n Möbius transformations is called the *Blaschke product of degree n*:

$$
B(z) = \eta\prod_{j=1}^{n}\frac{z-a_j}{1-\bar{a}_j z}\,, \tag{4.1}
$$

where $a_j \in D$, $j = 1,\dots,n$ and $|\eta| = 1$. The Blaschke product of degree 0 will mean the constant η with $|\eta| = 1$. The set of all Blaschke products of degree $\le n$ will be denoted by $\mathcal{B}_n$. Every Blaschke product B is (a) analytic on $\bar{D}$; (b) satisfies $|B(z)| = 1$ for $|z| = 1$. Conversely, every function f satisfying (a), (b) is a Blaschke product. Indeed, such f has only a finite number of zeros on D. Let B_0 be the Blaschke product having the same zeros as f on D. Then both f/B_0 and B_0/f are analytic and have no zeros on D. Moreover they have absolute value $= 1$ on ∂D. By the maximum modulus principle, both functions are constants with absolute value equal to one. Thus $f = \eta_0 B_0$ for some $|\eta_0| = 1$.

Proposition 4.1. (i) *If μ is a Möbius transformation and $B \in \mathcal{B}_n$, then $\mu(B) \in \mathcal{B}_n$.* (ii) *If two Blaschke products of degrees not higher than n coincide at $n+1$ distinct points of the disk D, then they coincide identically.*

Proof. The function $\mu(B(z))$ is a rational fraction of degree at most n. Moreover, it is analytic on $\bar{D}$ and satisfies $|\mu(B(z))| = 1$ for $|z| = 1$. This proves (i).

Statement (ii) we prove by induction. For $n = 0$ it is obvious. Let (ii) be correct for some $n = k \geq 0$, let $B_1, B_2 \in \mathcal{B}_{k+1}$ and let

$$B_1(z_j) = B_2(z_j) \quad \text{for} \quad j = 0, 1, \ldots, k+1 \ . \tag{4.2}$$

If $|B_1(z_{k+1})| = 1$, then by the maximum modulus principle, $B_1, B_2 \in \mathcal{B}_0$, and from (4.2) we obtain (ii). We can assume that $a := B_1(z_{k+1}) \in D$. Then we introduce the Möbius transformation $\mu(z) = (z - a)/(1 - \bar{a}z)$. By (i), the functions $B_s^*(z) := \mu(B_s(z))$, $s = 1, 2$, belong to $\mathcal{B}_{k+1}$. From (4.2) we also deduce that $B_1^*(z_j) = B_2^*(z_j)$ for $j = 0, 1, \ldots, k$ and $B_1^*(z_{k+1}) = B_2^*(z_{k+1}) = 0$. We conclude that the equality

$$B_1^*(z) : \frac{z - z_{k+1}}{1 - \bar{z}_{k+1} z} = B_2^*(z) : \frac{z - z_{k+1}}{1 - \bar{z}_{k+1} z} \tag{4.3}$$

holds for $z = z_0, z_1, \ldots, z_k$. By (i) both sides of (4.3) are functions of the class $\mathcal{B}_k$. Therefore, by the induction hypothesis, (4.3) holds for all $z \in D$. Hence, $B_1^* = B_2^*$. Let μ^{-1} be the inverse Möbius transformation to μ. Applying μ^{-1} to the last equality we obtain $B_1 = B_2$. □

Blaschke products have some interesting interpolation properties. Suppose that $z_0, \ldots, z_n$ are some fixed distinct points of the disk D. Then for any set $w = \{w_0, w_1, \ldots, w_n\}$ of complex numbers, there are functions $f \in H_\infty$ (for example, polynomials) satisfying the interpolation conditions

$$f(z_j) = w_j \ , \qquad j = 0, 1, \ldots, n \ . \tag{4.4}$$

It turns out that among all such functions the minimum of $\|f\|$ is attained by a properly normalized Blaschke product. The only exception is when $w = 0$, that is when all $w_j = 0$. Then the obvious unique solution is $f = 0$. Let $w \neq 0$ and let

$$\rho(w) := \inf\{\|f\| : f \in H_\infty(D) \ , \ f(z_j) = w_j \ , \ j = 0, \ldots, n\} \ . \tag{4.5}$$

We first note that this infimum is attained. Indeed, from the sequence $f_N \in H_\infty$ satisfying (4.4) and with $\|f_N\| \leq \rho(w)(1 + 1/N)$, by Montel's theorem we can extract a subsequence convergent on compact subsets of D. The limit function f will be the desired extremal function.

Theorem 4.2 (Pick-Nevanlinna). *For each $w \neq 0$ there is a function $f \in H_\infty(D)$ satisfying (4.4) with $\|f\| = \rho(w)$. This f is unique and has the form $f = \rho(w)B$, $B \in \mathcal{B}_n$.*

Proof. We note that $w \neq 0$ implies $\rho(w) \geq \|w\| := \max_{0 \leq j \leq n} |w_j| > 0$. If w' is obtained from w by omitting one of the terms w_j and the corresponding z_j, then $\rho(w') \leq \rho(w)$. We shall prove that for the extremal function f we have

$f = \rho(w)B$, $B \in \mathcal{B}_n$. If $n = 0$, this immediately follows from the maximum modulus principle. Suppose that this holds for some $n = k > 0$, we prove it for $n = k + 1$. Since $\rho(\lambda w) = |\lambda|\rho(w)$ for each $\lambda \in \mathbb{C}$, we can assume that $\rho(w) = 1$ for the problem with $z_0, \dots, z_{k+1}; w_0, \dots, w_{k+1}$.

Case (a). Let $|w_{k+1}| = 1$. Then for the reduced problem with z_0, w_0 omitted, we have $1 \le \rho(w') \le \rho(w) = 1$, hence $\rho(w') = 1$. We have $f = B \in \mathcal{B}_k \subset \mathcal{B}_{k+1}$ by the induction hypothesis.

Case (b). Let $|w_{k+1}| < 1$. Then $f(z_{k+1}) = w_{k+1} \in D$. We can use the function

$$g(z) := \frac{f(z) - f(z_{k+1})}{1 - \overline{f(z_{k+1})}\, f(z)} : \frac{z - z_{k+1}}{1 - \overline{z_{k+1}}\, z} \,. \tag{4.6}$$

Plainly, $|g(z)| \le 1$, $z \in \partial D$, with equality in at least one point $z \in \partial D$. Hence $g \in H_\infty(D)$ and $\|g\| = 1$.

Let $w_j^* := g(z_j)$, $j = 1, \dots, k$, then $\rho(w^*) \le \|g\| = 1$. We shall prove that $\rho(w^*) = 1$. Otherwise $\rho(w^*) = q$, $0 \le q < 1$. For an extremal function g_1 with values $g_1(z_j) = w_j^*$, $j = 0, \dots, k$ we have $\|g_1\| = q < 1$. Let h be defined by

$$g_1(z) = \frac{h(z) - f(z_{k+1})}{1 - \overline{f(z_{k+1})}\, h(z)} : \frac{z - z_{k+1}}{1 - \bar{z}_{k+1} z} \,. \tag{4.7}$$

Then $\|h\| \le \|g_1\| < 1$, and $h(z_{k+1}) = f(z_{k+1})$. Moreover, since the right-hand sides of (4.6) and (4.7) are equal for $z = z_j$, $j = 0, 1, \dots, k$, we have $h(z_j) = f(z_j)$. This is a contradiction, because it implies that f, with $\|f\| = 1$ is not an extremal function.

Thus $\rho(w^*) = 1$. From the inductive hypothesis we derive $g \in B_1 = \mathcal{B}_k$. Now (4.6) yields, with some Möbius transformation μ,

$$f = \mu(B_2) \quad , \quad B_2(z) = B_1(z) \frac{z - z_{k+1}}{1 - \bar{z}_{k+1} z} \in \mathcal{B}_{k+1} \,.$$

This completes the induction proof. The uniqueness of f follows from Proposition 4.1(ii). □

Appendix 4. Potential Theory and Logarithmic Capacity

§ 1. Logarithmic Potentials

This appendix contains only those aspects of potential theory which are used in the main text. The reader can be recommended to turn to the texts by Hille [B-1962], Landkof [B-1972], Tsuji [B-1959], to the appendix of Stahl and Totik [B-1992] and to the book by Halmos [B-1974] on measure theory.

Let $A \subset \mathbb{C}$ be a compact set. A non-negative Borel measure μ (called regular Borel measure in Halmos [B-1974]) defined on all Borel subsets of A is called a *probability measure* or a probability distribution if $\mu(A) = 1$. The set of all probability measures on A is denoted $\mathcal{M}(A)$.

The *support* of μ, denoted $\operatorname{supp}(\mu)$, is the set of all points $z_0 \in A$ for which $\mu(\mathcal{U} \cap A) > 0$ for each open neighborhood $\mathcal{U}$ of z_0. A point $z_0 \in \operatorname{supp}(\mu)$ is called an *atom* of μ if $\mu(\{z_0\}) > 0$.

A sequence $(\mu_n)_1^\infty \subset \mathcal{M}(A)$ is *weakly*-convergent* to a measure μ, $\mu_n \rightarrow \mu$, if for all $f \in C(A)$,

$$\lim_{n\to\infty} \int_A f(t)d\mu_n(t) = \int_A f(t)d\mu.$$

The set $\mathcal{M}(A)$ is *weakly*-compact*: each sequence $(\mu_n)_1^\infty \subset \mathcal{M}(A)$ contains a subsequence that is weakly*-convergent to an element of $\mathcal{M}(A)$.

If for two sequences of measures $\mu_n \rightarrow \mu$, $\nu_n \rightarrow \nu$, then for all $f \in C(A \times A)$,

$$\lim_{n\to\infty} \int_{A\times A} f(x,t)d\mu_n(x)d\nu_n(t) = \int_{A\times A} f(x,t)d\mu(x)d\nu(t).$$

Indeed, by Fubini's theorem, this is so for each f of the form $f(x,t) = g(x)h(t)$, and the gh span a dense subspace of $C(A \times A)$.

The *logarithmic potential* of a measure $\mu \in \mathcal{M}(A)$ with support $\operatorname{supp}(\mu) =: S$ is the function

$$(1.1) \qquad u_\mu(z) = -\int_A \log|z-t|d\mu(t) = -\int_S \log|z-t|d\mu(t), \quad z \in \mathbb{C}.$$

This integral exists and represents a continuous function for $z \in \mathbb{C} \setminus S$, and $u_\mu(\infty) = -\infty$. On S, the integral can take the value $+\infty$ (for example, at z_0 if $\mu = \delta_{z_0}$ is the Dirac measure), but since $|z-t|$ is bounded for $z, t \in A$, it does not take the value $-\infty$. Often $u_\mu(z)$ is continuous in $\mathbb{C}$, for example, if $g \in C(A)$ and $d\mu(t) = g(t)dt$.

We prove below that $u_\mu(z)$ is harmonic in $\mathbb{C} \setminus S$ and superharmonic in the whole plane $\mathbb{C}$. We first recall the definitions and basic facts concerning harmonic and superharmonic functions.

Let G be an open set in the extended complex plane $\mathbb{C}^* := \mathbb{C} \cup \{\infty\}$. We shall call G a *domain* if it is also connected. By $\overline{G}$, ∂G we denote, respectively, the closure and the boundary of G. A continuous real-valued function $v(x, y)$ is called *harmonic* in an open subset of $\mathbb{R}^2$ if it is twice continuously differentiable and satisfies the Laplace equation $\Delta v = v_{xx} + v_{yy} = 0$. For $G \subset \mathbb{C}$ we say that $u(z)$, $z \in G$, is harmonic if $v(x, y) = u(x + iy)$ is harmonic. If ∞ is an interior point of G, then u is called harmonic at ∞ if $u(1/z)$ is harmonic at $z = 0$.

Real and imaginary parts of analytic functions are harmonic. Conversely, if $G \subset \mathbb{C}$ is open and simply connected then any function u harmonic in G is the real part of an analytic function.

If u is harmonic in the disk $D_R : |z - z_0| < R$ and continuous on $\overline{D}_R$, then the *mean-value theorem* states that

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\phi})d\phi \ , \quad 0 < r \le R. \tag{1.2}$$

To prove (1.2) we take the analytic function f for which $u = \operatorname{Re} f$ and take real parts on both sides of Cauchy's formula for $C_r : |z - z_0| = r$,

$$f(z_0) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\phi})d\phi.$$

The mean-value theorem implies, by a standard argument, the *Maximum Modulus principle*: If u is a nonconstant harmonic function in a domain $G \subset \mathbb{C}^*$, then u has no maximum or minimum points in G.

A one-dimensional analogue of harmonic functions are linear functions. Superharmonic (subharmonic) functions are two-dimensional analogues of concave (convex) functions. They don't need to be continuous, only semi-continuous. We say that a function $u(z)$ defined in a vicinity of a point $z_0 \in \mathbb{C}$ is *lower semi-continuous* at z_0 if u is not identically $+\infty$ and if

$$\liminf_{z \to z_0} u(z) \ge u(z_0).$$

If u is lower semi-continuous at every point of an open set G, we call it *lower semi-continuous* in G. We say that u is *upper semi-continuous* if $-u$ is lower semi-continuous.

We call $u : G \to (-\infty, +\infty]$ *superharmonic* in an open set G if

(i) u is lower semi-continuous in G;

(ii) for any $z_0 \in G$ and all sufficiently small $r > 0$,

$$u(z_0) \ge \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\phi})d\phi. \tag{1.3}$$

(More precisely, for every z_0 there exists $r_0 = r_0(z_0) > 0$ for which the disk $|z - z_0| \leq r_0$ is contained in G and (1.3) is valid for all $r \leq r_0$.)

By integrating (1.3) with respect to r we derive the following property of superharmonic functions which is equivalent to (ii):

(ii′) For any $z_0 \in G$ and all sufficiently small $r > 0$,

$$u(z_0) \geq \frac{1}{\pi r^2} \int_{D_r} u(t)dm(t),$$

where $D_r : |z - z_0| \leq r$ and $dm(t)$ is the planar Lebesgue measure.

We say that u is *subharmonic* in G if $-u$ is superharmonic in G.

If $v(x,y) = u(x+iy)$ has continuous partial derivatives of the second order in G, one can prove that u is superharmonic (subharmonic) in a domain G if and only if $\Delta v \leq 0$ ($\Delta v \geq 0$) in G. Consequently, $u(z)$ is harmonic if it is both super- and subharmonic.

If $f(z)$ is analytic in G, then $u(z) = \log|f(z)|$ is subharmonic. Indeed, if $z_0 \in G$ is not a zero of f, then $\log f$ admits an analytic branch in a vicinity of z_0. The function $u(z) = \log|f(z)|$ is the real part of this branch. It is therefore harmonic, so that the equality (1.2) is valid. If $f(z_0) = 0$, then $u(z_0) = -\infty$, and the desired inequality (the inverse of (1.3)) is fulfilled trivially.

One can similarly prove that $|z|^q|f(z)|^p$, $p > 0$, $q \geq 0$, are also subharmonic.

It follows directly from the definition that if $u_k(z)$ are superharmonic (subharmonic), then all finite linear combinations $\sum c_k u_k$, $c_k > 0$, have this property. Moreover, if $A \subset \mathbb{C}$ is compact, $\mu \in \mathcal{M}(A)$, and if for each fixed $t \in A$ the function $u(z,t)$ is superharmonic (subharmonic) for $z \in G$, then so is

$$\tilde{u}(z) := \int_A u(z,t)d\mu(t) = \int_{\mathrm{supp}(\mu)} u(z,t)d\mu(t).$$

This leads to the following important conclusion.

Theorem 1.1. *The logarithmic potential u_μ defined by* (1.1) *is harmonic in* $\mathbb{C} \setminus \mathrm{supp}(\mu)$ *and superharmonic in the whole complex plane* $\mathbb{C}$.

Indeed, each function $u(z,t) = -\log|z-t|$, $t \in \mathrm{supp}(\mu)$, is superharmonic, as a function of z, in $\mathbb{C}$ and harmonic in $\mathbb{C} \setminus \mathrm{supp}(\mu)$. □

Theorem 1.2 (The Maximum Principle for Subharmonic Functions).
(i) *If u is subharmonic in an open set $G \subset \mathbb{C}$ and attains a local maximum in a point $z_0 \in G$, then $u(z) = u(z_0)$ in a neighborhood of z_0.*
(ii) *If u is a non-constant subharmonic function in a domain G, then u has no maximum points in G.*

The proof of (i) is essentially the same as the standard proof of the maximum principle for harmonic functions. To derive (ii) from (i), we assume that both sets $\Omega := \{z : u(z) = u(z_0)\}$, $\Omega' := \{z : u(z) < u(z_0)\}$, are not empty. By (i) and the upper semi-continuity of u, Ω and Ω' are open disjoint sets, contradicting the connectedness of $G = \Omega \cup \Omega'$. $\square$

The following lemma about the logarithmic potential u_μ will be useful.

Lemma 1.3. *If $z_0 \in \operatorname{supp}(\mu) =: S$ is not an atom of μ, if $z_n \in \mathbb{C}$ is a sequence with $z_n \to z_0$ and if $t_n \in S$ have the property $|t_n - z_n| = \min_{t \in S} |t - z_n|$, then*

$$\limsup_{n\to\infty} u_\mu(z_n) \le \limsup_{n\to\infty} u_\mu(t_n). \tag{1.4}$$

Proof. For $r > 0$ let S_r be the intersection of S with the disk $|z - z_0| \le r$. Since z_0 is not an atom of μ, for a given $\varepsilon > 0$, one can take $r > 0$ so small that $\mu(S_r) < \varepsilon$. For all $t \in S$, $|t - t_n| \le |t - z_n| + |z_n - t_n| \le 2|t - z_n|$. Hence $|z_0 - t_n| \le 2|z_0 - z_n| \to 0$ and

$$\int_{S_r} \log \left| \frac{t_n - t}{z_n - t} \right| d\mu(t) < \mu(S_r) \log 2 < \varepsilon \log 2.$$

Since $|z_n - t_n| \to 0$, we have $\log(|t_n - t|/|z_n - t|) < \varepsilon$ for $t \in S \setminus S_r$ and all large n. Therefore,

$$u_\mu(z_n) - u_\mu(t_n) = \int_S \log \left| \frac{t_n - t}{z_n - t} \right| d\mu(t) < \varepsilon(1 + \log 2),$$

and (1.4) follows. $\square$

Theorem 1.4. *Let $A \subset \mathbb{C}$ be compact and let $\mu \in \mathcal{M}(A)$.*
(i) *(*Maximum Principle for Logarithmic Potentials.*) If $u_\mu(z) \le M$ on the support* $\operatorname{supp}(\mu) =: S$ *of μ, then $u_\mu(z) \le M$ for all $z \in \mathbb{C}$.*
(ii) *If $u_\mu(z)$ is continuous at $z_0 \in S$ as a function on S, then it has the same property as a function on $\mathbb{C}$.*

Proof. (i) Since u_μ is bounded from above on S, μ has no atoms. Let $M_1 := \sup_{z\in\mathbb{C}} u_\mu(z)$. There is a point z_0 with the property that each neighborhood of z_0 contains points z with values of $u_\mu(z)$ arbitrarily close to M_1. If $z_0 \notin S$, then u_μ is harmonic at z_0, $u_\mu(z_0) = M_1$, and by Theorem 1.2, $u_\mu(z) = M_1$ in the component of $\mathbb{C} \setminus S$ which contains the point z_0. Hence, $M_1 = M$ by Lemma 1.3. If $z_0 \in S$, Lemma 1.3 yields again the same conclusion.

(ii) In this case, z_0 is not an atom of μ. We have $\limsup u_\mu(z) \le u_\mu(z_0)$, $z \to z_0$, $z \in \mathbb{C}$ because of (1.4) and the continuity of u_μ at z_0 along S, while the lower semi-continuity of μ_w yields $\liminf u_\mu(z) \ge u_\mu(z_0)$ as $z \to \infty$. $\square$

The next two theorems show that a measure μ can be recovered from the logarithmic potential u_μ it generates, even if the potential is only known up to a harmonic summand.

Theorem 1.5. *Let μ_1, μ_2 be non-negative Borel measures with compact supports. Let u_{μ_1}, u_{μ_2} be their logarithmic potentials, let h be harmonic.* (i) *If for some $z_0 \in \mathbb{C}$ and $R > 0$, for $v := u_{\mu_1} - u_{\mu_2} + h$ the integrals*

$$\int_0^{2\pi} v(z_0 + re^{i\phi})d\phi =: d(z_0), \quad 0 < r < R, \tag{1.5}$$

exist and take the same value $d(z_0)$ for all r, $0 < r < R$, then $\mu_1(D_r) = \mu_2(D_r)$ for all disks $D_r : |z - z_0| \le r$, $0 < r < R$. (ii) *If for a domain G containing the union S of the supports of μ_1, μ_2, the assumption holds for each $\overline{D}_R \subset G$, then $\mu_1 = \mu_2$.*

Proof. (i) We shall use the identity

$$\frac{1}{2\pi}\int_0^{2\pi} \log|re^{i\phi} - t|d\phi = \max\{\log r; \log|t|\}, \tag{1.6}$$

valid for any $r > 0$ and any $t \in \mathbb{C}$. If $|t| > r$, (1.6) follows from (1.2) with $u(z) = \log|z|$. If $|t| < r$ we transform the integral to $\frac{1}{2\pi}\int_0^{2\pi} \log|r - te^{i\phi}|d\phi$ and again use (1.2); $|t| = r$ is the limit case.

Let $\nu := \mu_1 - \mu_2$ and let r be fixed. We may assume that $z_0 = 0$ so that $D_r : |z| \le r$. Using (1.6) we get for each ρ, $0 < \rho < R$,

$$\begin{aligned} d(z_0) &= \int_0^{2\pi} v(\rho e^{i\phi})d\phi = \int_0^{2\pi}\int_{\mathbb{C}} \log|\rho e^{i\phi} - t|d\nu(t)d\phi \\ &= \int_{\mathbb{C}}\int_0^{2\pi} \log|\rho e^{i\phi} - t|d\phi d\nu(t) = 2\pi\int_{\mathbb{C}} \max\{\log\rho; \log|t|\}d\nu(t) \\ &= 2\pi\nu(D_\rho)\log\rho + 2\pi\int_{\mathbb{C}\setminus D_\rho} \log|t|d\nu(t). \end{aligned}$$

Since $d(z_0)$ is independent of ρ, we have

$$\nu(D_r)\log r + \int_{\mathbb{C}\setminus D_r} \log|t|d\nu(t) = \nu(D_\rho)\log\rho + \int_{\mathbb{C}\setminus D_\rho} \log|t|d\nu(t). \tag{1.7}$$

Hence, if $r < \rho < R$, we get

$$\nu(D_\rho)\log\rho - \nu(D_r)\log r = \int_{D_\rho\setminus D_r} \log|t|d\nu(t)$$

and, since μ_1 and μ_2 are non-negative,

$$\nu(D_\rho)\log\rho - \nu(D_r)\log r \le \mu_1(D_\rho \setminus D_r)\log\rho - \mu_2(D_\rho \setminus D_r)\log r.$$

This implies

$$(\mu_1(D_r) - \mu_2(D_\rho))\log\rho \le (\mu_1(D_r) - \mu_2(D_\rho))\log r$$

and thus $\mu_1(D_r) \le \mu_2(D_\rho)$. For the compact set D_r we have $\lim \mu_2(D_\rho) = \mu_2(D_r)$ as $\rho \to r$, $\rho > r$. We conclude that $\mu_1(D_r) \le \mu_2(D_r)$. Similarly, if we exchange μ_1 and μ_2, we get $\mu_2(D_r) \le \mu_1(D_r)$.

(ii) For continuous functions f, g on $\mathbb{C}$ with compact support we define the convolution

$$f * g(z) := \int_{\mathbb{C}} f(t)g(t-z)d\lambda(t) = \int_{\mathbb{C}} f(t+z)g(t)d\lambda(t),$$

where $d\lambda$ is the two-dimensional Lebesgue measure. Let for $\varepsilon > 0$,

$$g_\varepsilon(z) := \begin{cases} (\pi\varepsilon^2)^{-1} & \text{for } |z| \le \varepsilon \\ 0 & \text{for } |z| > \varepsilon. \end{cases}$$

For $\varepsilon \to 0$, the measures $g_\varepsilon(z)d\lambda(z)$ converge to the Dirac measure. For a function $f \in C(S)$, let f stand also for its extensions, with compact support, onto $\mathbb{C}$. We have

$$\begin{aligned}\int_{\mathbb{C}} f * g_\varepsilon d\mu_1 &= \int_{\mathbb{C}} f(t)d\lambda(t) \int_{\mathbb{C}} g_\varepsilon(t-z)d\mu_1(z) \\ &= \frac{1}{\pi\varepsilon^2}\int_{\mathbb{C}} f(t)\mu_1\{z : |z-t| \le \varepsilon\}d\lambda(t) = \int_{\mathbb{C}} f * g_\varepsilon d\mu_2.\end{aligned}$$

On the other hand, for $\varepsilon \to 0$,

$$\int_{\mathbb{C}} f * g_\varepsilon d\mu_1 = \int_{\mathbb{C}} d\mu_1(z) \int_{\mathbb{C}} f(z+t)g_\varepsilon(t)d\lambda(t) \to \int_{\mathbb{C}} f(z)d\mu_1(z).$$

Hence $\int_{\mathbb{C}} f d\mu_1 = \int_{\mathbb{C}} f d\mu_2$, $\mu_1 = \mu_2$. □

From this we obtain

Theorem 1.6. *If the logarithmic potentials of measures μ_1, μ_2 of compact support are identical (or differ by a harmonic function) in a domain containing the supports, then $\mu_1 = \mu_2$.*

The following three results about logarithmic potentials with compact *real* support have been used in Chapter 4.

Lemma 1.7. *Let $A \subset \mathbb{R}$ be a closed set, let μ be a non-negative Borel measure with compact support S.* (i) *The logarithmic potential u_μ satisfies*

(1.8) $$u_\mu(x+iy) \le u_\mu(x) \quad \textit{for all } x, y \in \mathbb{R}$$

and

(1.9) $$\lim_{y \to 0, y \in \mathbb{R}} u_\mu(x+iy) = u_\mu(x) \quad \textit{for all } x \in \mathbb{R}.$$

(ii) *Let $I := (a, b)$ be a bounded subinterval of $\mathbb{R} \setminus S$. If $u_\mu(a) < \infty$, then $\lim u_\mu(x) = u_\mu(a)$, $x \to a$, $x \in I$. If $u_\mu(b) < \infty$ then $\lim u_\mu(x) = u_\mu(b)$, $x \to b$, $x \in I$.*

(iii) *If $x_0 \in S$ is not an atom of μ,*

$$\limsup_{z \to x_0, z \notin S} u_\mu(z) = \limsup_{x \to x_0, x \in S} u_\mu(x). \tag{1.10}$$

Proof. (i) Clearly, (1.8) follows from the definition (1.1) of u_μ since $|x+iy-t| \geq |x-t|$ for all $x, y, t \in \mathbb{R}$, and (1.9) follows from (1.8) and the lower semi-continuity of u_μ.

(ii) If $a \notin S$ then u_μ is continuous in a and we have $\lim u_\mu(z) = u_\mu(a)$ as $z \to a$. Let $a \in S$ and $u_\mu(a) < \infty$. Hence a is no atom of μ. For all $a < x < (b-a)/2$, a is the nearest point from S to x. Hence by Lemma 1.3, $\limsup u_\mu(x) \leq u_\mu(a)$, $x \to a$, $x \in I$. Conversely, the lower semi-continuity of u_μ implies that $\liminf u_\mu(x) \geq u_\mu(a)$, $x \to a$. The proof for the point b is similar.

(iii) Let M_1 and M_2 be the limits on the left-hand side and right-hand side of (1.10), respectively. Then we deducc the inequality $M_1 \geq M_2$ from (1.9) and the opposite inequality from Lemma 1.3. □

Theorem 1.8. *Let μ, S and u_μ be as in Lemma 1.7, and let $u_\mu(x) < +\infty$ for all $x \in S$. If a polynomial P_n of degree $\leq n$ satisfies*

$$\frac{1}{n} \log |P_n(x)| \leq -u_\mu(x) + \lambda, \quad x \in S, \tag{1.12}$$

for some $\lambda \in \mathbb{R}$, then (1.12) is valid in the whole complex plane $\mathbb{C}$.

Proof. Let Z be the zeros of P_n. The function

$$v(z) := \frac{1}{n} \log |P_n(z)| + u_\mu(z), \quad z \in \mathbb{C}$$

is harmonic on $D := \mathbb{C} \setminus \{S \cup Z\}$ and $-\infty$ in Z. At $z = \infty$, v is harmonic if deg $P_n = n$ and $v(\infty) = -\infty$ if deg $P_n < n$. If $v(z) = \text{const}$ for $z \in D$, then deg $P_n = n$ and $v(z)$ is constant in $\mathbb{C} \setminus Z$ by (1.9) which is impossible.

Let $M := \sup v(z)$, $z \in \mathbb{C}$, and let $z_k \in \mathbb{C}$ be a sequence so that $\lim v(z_k) = M$ as $k \to \infty$. Since v is continuous in D and cannot attain its maximum in D, we may assume that the z_k converge to some $z_0 \in S$. Then z_0 is not an atom of μ since $u_\mu(z_0) < \infty$. Hence, by Lemma 1.3 and the continuity of $\log |P_n(z)|$ at z_0,

$$M = \lim_{k \to \infty} v(z_k) \leq \limsup_{x \to z_0, x \in S} v(x) \leq \lambda,$$

which yields (1.12). □

Theorem 1.9. *Let μ_1, μ_2 be non-negative Borel measures whose supports are contained in some real compact interval $B = [\alpha, \beta]$. The measures are identical if their logarithmic potentials coincide in the interval (β, ∞).*

Proof. The domain $G := \mathbb{C}\setminus(-\infty, \beta]$ is simply connected, the function $v(z) := u_{\mu_1}(z) - u_{\mu_2}(z)$ is harmonic in G and vanishes on $(\beta, \infty) = G \cap \mathbb{R}$. Since G is simply connected, there exists an analytic function $f(z) = v_0(z) + iv(z)$ on G with imaginary part $v(z)$. Since G is symmetric with respect to the real axis and $f(z)$ is real for real $z \in G$, it follows from the Schwarz reflection principle that $f(z) = \overline{f}(\overline{z})$, $z \in G$. Hence, $v_0(z) + iv(z) = v_0(\overline{z}) - iv(\overline{z})$ and $v(z) = -v(\overline{z})$. Since the supports of μ_1 and μ_2 are real, $v(z) = v(\overline{z})$ and thus $v(z) = 0$ for all $z \in G$. The integrals (1.5) exist and are equal to zero, for all $z_0 \in \mathbb{C}$ and all $r > 0$. Hence $\mu_1 = \mu_2$ by Theorems 1.5 and 1.6. □

We shall give now a theorem about the representation of subharmonic functions by potentials. Its proof can be found in Landkof [B-1972, Theorem 1.22, p.101].

Theorem 1.10. *Suppose that $f(z)$ is a subharmonic function on $\mathbb{C}$. Then for any bounded domain Ω there is a non-negative Borel measure $\mu := \mu_\Omega$ supported on Ω, and a function $h := h_\Omega$ harmonic in Ω, such that*

$$f(z) = \int_\Omega \log|z-t|\,d\mu(t) + h(z), \quad z \in \Omega. \tag{1.13}$$

Moreover, given Ω, the measure μ and the function h are unique.

We shall apply Theorem 1.10 in the proof of the next theorem which is needed in Chapter 8.

Theorem 1.11. *Let μ_1 and μ_2 be non-negative Borel measures with compact supports. If the difference $f(z) := \int \log|z-t|(d\mu_1(t) - d\mu_2(t))$ of their logarithmic potentials is subharmonic in $\mathbb{C}$, then $\mu := \mu_1 - \mu_2$ is a non-negative Borel measure.*

Proof. Let Ω be an open disk with center 0 which contains the supports of μ_1 and μ_2. By Theorem 1.10 there exists a non-negative Borel measure μ, $\operatorname{supp}(\mu) \subset \Omega$, and a function h harmonic in Ω, for which (1.13) is satisfied. This implies that

$$h(z) = \int_\Omega \log|z-t|(d\mu_1 - d[\mu_2 + \mu](t)), \quad z \in \Omega. \tag{1.14}$$

Since the supports of the three measures are contained in Ω, the potential (1.14) is even harmonic in the whole complex plane $\mathbb{C}$. Therefore, by Theorem 1.5, $[\mu_2 + \mu](D_r) = \mu_1(D_r)$ for all disks D_r. From Theorem 1.6 it follows that $\mu_2 + \mu = \mu_1$, hence $\mu_1 - \mu_2 = \mu \geq 0$. □

§ 2. Equilibrium Distribution and Logarithmic Capacity

For a compact set $A \subset \mathbb{C}$ and $\mu \in \mathcal{M}(A)$, we call

$$I(\mu) := -\int_{A\times A} \log|z-t|d\mu(t)d\mu(z) = \int_A u_\mu(z)d\mu(z)$$

the *energy integral* of μ. Obviously, $-\infty < I(\mu) \leq \infty$. If the infimum

$$V := V(A) := \inf_\mu I(\mu) \quad , \quad \mu \in \mathcal{M}(A),$$

is finite, then, as a consequence of the weak$*$-compactness of $\mathcal{M}(A)$, there exists an extreme measure $\mu^* \in \mathcal{M}(A)$ for which

$$V = I(\mu^*). \tag{2.1}$$

We call μ^* the *equilibrium distribution (equilibrium measure)* of A. The corresponding logarithmic potential

$$u^*(z) := -\int_A \log|z-t|d\mu^*(t) \tag{2.2}$$

is called the *conductor potential* of A. The *uniqueness* of the equilibrium distribution μ^* is far from obvious and will be established in Theorem 2.12.

The number

$$\gamma(A) := e^{-V(A)}$$

is called the *logarithmic capacity* of A. For an arbitrary set $S \subset \mathbb{C}$ we put

$$\gamma(S) := \sup\{\gamma(A) : \ A \subseteq S, \quad A \ \text{ compact }\}. \tag{2.3}$$

For example, $\gamma(\mathbb{C}) = +\infty$ and, if S is a single point, then $\gamma(S) = 0$.

The equilibrium distribution μ^* of the unit circle $C_1 := \{|z| = 1\}$ is $\mu^* = \frac{1}{2\pi}m_1$, where m_1 is the Lebesgue measure on C_1; the conductor potential is

$$u^*(z) = -\frac{1}{2\pi}\int_0^{2\pi} \log|z-e^{i\phi}|d\phi = \begin{cases} 0, & |z| \leq 1 \\ -\log|z|, & |z| > 1. \end{cases}$$

It takes the value $u^*(z) = V(C_1) = 0$ for all $z \in C_1$. Hence, $\gamma(C_1) = 1$.

Indeed, if μ^* is an equilibrium distribution of C_1, then for each $\delta > 0$ the measure μ_δ defined by $\mu_\delta(S) := \mu^*(S_\delta)$, $S_\delta := \{z : ze^{i\delta} \in S\}$ is also an equilibrium distribution of C_1. The uniqueness of μ^* implies that $\mu^* = \frac{1}{2\pi}m_1$.

Using the standard substitution $x = \cos\phi$, $-1 \leq x \leq 1$, $0 \leq \phi \leq \pi$, one can easily derive from the above results that the logarithmic capacity of the interval $[-1,1]$ is $\gamma([-1,1]) = 1/2$ and the equilibrium distribution for $[-1,1]$ is $d\mu^*(x) = dx/(\pi\sqrt{1-x^2})$.

About capacities we prove the following simple facts.

1. (i) *If* $S_1 \subset S_2$ *then* $\gamma(S_1) \leq \gamma(S_2)$.

(ii) *If S' is obtained from S by the linear transformation $z' = az + b$, then* $\gamma(S') = |a|\gamma(S)$.
(iii) *If A is compact and if $B := \operatorname{supp}(\mu^*)$ is the support of the equilibrium distribution μ^* of A, then $B \subset A$ and $\gamma(B) = \gamma(A)$.*
(iv) *If $S_1, S_2, \ldots$ are sets of capacity zero and if $S := \bigcup_1^\infty S_n$, then $\gamma(S) = 0$.*

Proof. The statements (i) and (ii) follow easily from the definition of the capacity.

(iii) We can assume that $\gamma(A) > 0$. Then there exists an equilibrium distribution μ^* on A, with $-\log\gamma(A) = I(\mu^*)$. Since μ^* is also a probability measure on its support B, we have

$$I(\mu^*) \geq \inf_{\mu \in \mathcal{M}(B)} I(\mu) = -\log\gamma(B).$$

It follows that $\gamma(A) \leq \gamma(B)$. Since $B \subset A$, we also have $\gamma(B) \leq \gamma(A)$.

(iv) Because of (ii) we may assume that S is contained in $|z| \leq 1/2$ so that $-\log|z-t| \geq 0$ for $z, t \in S$. Suppose that $\gamma(S) > 0$. Then, by definition (2.3), there exists a compact subset A of S with $\gamma(A) > 0$, so that the equilibrium distribution μ^* of A has a finite energy integral. We denote $A_n := S_n \cap A$, $n = 1, 2, \ldots$. Since $A = \cup A_n$ and since $1 = \mu^*(A) \leq \sum \mu^*(A_n)$, there exists an index k so that $\mu^*(A_k) > 0$. As A is contained in $|z| \leq 1/2$, we have

$$-\int_{A_k \times A_k} \log|z-t| d\mu^*(z) d\mu^*(t) \leq -\int_{A \times A} \log|z-t| d\mu^*(z) d\mu^*(t) < \infty.$$

This shows that A_k and thus S_k have positive capacity, a contradiction to the assumption that $\gamma(S_k) = 0$. □

As an example, the logarithmic capacity of a finite or countable set is zero.

2. *For each compact set $A \subset \mathbb{C}$, there exists a sequence O_n, $n = 1, 2, \ldots$, of open sets each of which is a finite union of open disks, that satisfy $A \subset O_{n+1} \subset O_n$, $n \geq 1$, and $\cap_1^\infty O_n = A$. In this situation, for the closures $\overline{O}_n$ of the O_n,*

$$\lim_{n\to\infty} \gamma(\overline{O}_n) = \gamma(A).$$

Proof. We put $A_n := \overline{O}_n$. Because of **1**(ii) we may assume that all A_n lie in $|z| \leq 1/2$. Since $A \subset A_{n+1} \subset A_n$, $\gamma(A) \leq \gamma(A_{n+1}) \leq \gamma(A_n)$ and $\gamma(A) \leq \lim\gamma(A_n)$.

We first suppose that $\gamma(A) > 0$. Let μ^* be the equilibrium distribution of A and μ_n be that of A_n, then $\mu_n(A_n) = 1$ and

$$V(A_n) = -\log\gamma(A_n) = -\int_{A_n \times A_n} \log|z-t| d\mu_n(z) d\mu_n(t).$$

The μ_n, $n \geq 1$, are probability measures on A_1. Hence they contain a subsequence (again denoted by μ_n) which converges weakly$*$ to some probability measure ν on A_1. Since $\cap_1^\infty A_n = A$, ν is a probablity measure on A. Since $-\log|z-t| \geq 0$, $z, t \in A_1$, we have by Fatou's lemma and the definition of $V(A)$,

$$
\begin{aligned}
-\log\gamma(A) = V(A) &\leq -\int_{A\times A} \log|z-t|d\nu(z)d\nu(t) \\
&\leq -\lim \int_{A_n\times A_n} \log|z-t|d\mu_n(z)d\mu_n(t) \\
&= \lim V(A_n) = -\lim\log\gamma(A_n)
\end{aligned}
$$

so that $\gamma(A) \geq \lim\gamma(A_n)$, thus $\gamma(A) = \lim\gamma(A_n)$.

If $\gamma(A) = 0$, then it follows easily that $\lim\gamma(A_n) = 0$. □

3. *Let $A \subset E$ be compact subsets of $\mathbb{C}$. If ν is a finite positive measure on E with finite energy $I(\nu) < \infty$, then $\nu(A) > 0$ implies $\gamma(A) > 0$.*

Indeed, with $c = \nu(A)^{-1}$, we define a probability measure ν_1 on A by $\nu_1(S) := c\nu(S)$, $S \subset A$. This is a probability measure on A, with $I(\nu_1) = c^2 I(\nu) < \infty$. Then $I(\mu^*) \leq I(\nu_1)$ is also finite, so that $\gamma(A) > 0$. □

Conversely, $\gamma(A) = 0$ implies $\nu(A) = 0$. For the planar Lebesgue measure m on $\mathbb{C}$, one has a stronger relation (see Golusin [B-1957; Chapter 7]):

$$\gamma(A) \geq \sqrt{m(A)/\pi}. \tag{2.4}$$

4. *If $\gamma(A) = 0$, $A \subset \mathbb{C}$ then $mA = 0$ (2-dimensional Lebesgue measure). If $\gamma(A) = 0$, $A \subset C_r$ then $m_1 A = 0$ (Lebesgue measure on the circle C_r).*

Sets of zero logarithmic capacity are small sets of the potential theory. We say that a property takes place *quasi everywhere* on A (or q.e. on A) if it holds for all $z \in A$, except for a set of capacity zero.

Theorem 2.1 (Frostman). *If $A \subset \mathbb{C}$ is a compact set of positive capacity, then for the conductor potential u^* of A,*

(i) $\quad u^*(z) \leq V(A)$ *for all* $z \in \mathbb{C}$,

(ii) $\quad u^*(z) = V(A)$ *quasi everywhere on* A.

The proof can be found in Tsuji [B-1959 ; Theorem III.12]. It is very similar to the proof of Theorem 3.2 in Chapter 4, therefore we omit it here.

From (i) and the lower semi-continuity of u^* we have

Corollary 2.2. *If $u^*(z_0) = V(A)$, then $u^*(z)$ is continuous at z_0.*

Corollary 2.3. *Let $A \subset \mathbb{C}$ be a compact set of positive capacity and let ν be a probability measure on A. Then its logarithmic potential satisfies*

$$\inf_{z\in A} u_\nu(z) \leq V(A) \leq \sup_{z\in A} u_\nu(z). \tag{2.5}$$

Proof. We may assume that A lies in $|z| \leq 1/2$, hence $-\log|z-t| \geq 0$ if $z, t \in A$. Changing the order of integration in the double integral we get

$$\int_A u_\nu(z)d\mu^*(z) = \int_A u^*(z)d\nu(z). \tag{2.6}$$

Using Theorem 2.1(i) we get $\int_A u_\nu(z)d\mu^*(z) \leq V(A)$, and the left inequality in (2.5) follows.

Next we assume that the supremum in (2.5) is finite. Then $V(A) < \infty$. By Theorem 2.1(ii), $u^*(A) = V(A)$, $z \in A$ q.e. By **3**, the same is true except for a set of ν measure zero. Thus the second integral in (2.6) is equal to $V(A)$, and from $V(A) = \int_A u_\nu d\mu^*$ we derive the right-hand side of (2.5). □

The logarithmic capacity is not the only important measure of a compact set $A \subset \mathbb{C}$ in the potential theory. Other measures are the transfinite diameter of Fekete

$$\begin{aligned} \tau(A) &:= \lim_{n\to\infty} \tau_n(A), \\ \tau_n(A) &:= \max_{z_1,\ldots,z_n\in A} \Bigg(\prod_{1\leq j<k\leq n} |z_k - z_j| \Bigg)^{2/n(n-1)} \end{aligned} \tag{2.7}$$

and the Chebyshev constant

$$\mathrm{c}(A) := \lim_{n\to\infty} E_n(A)^{1/n}, \quad E_n(A) := \min_{p\in\mathcal{P}_{n-1}} \|z^n - p(z)\|_A. \tag{2.8}$$

According to the beautiful theorem of Fekete [1923] and Szegö [1924], the quantities $\gamma(A), \mathrm{c}(A), \tau(A)$ are equal.

The existence of the limits $\tau(A)$ and $\mathrm{c}(A)$ in (2.7) and (2.8) will be a consequence of the proof of the next theorem.

Theorem 2.4. *For each compact set $A \subset \mathbb{C}$,*

$$\gamma(A) = \mathrm{c}(A) = \tau(A).$$

Proof. (i) For $n \in \mathbb{N}$ let $\zeta_1, \ldots, \zeta_n \in \mathbb{C}$ be taken so that the minimum $E_n(A) = \|\prod_{j=1}^n (z - \zeta_j)\|_A$ is attained. We define the compact set A_n by

$$A_n := \{z \in \mathbb{C} : \ v_n(z) := -n^{-1} \sum_{j=1}^n \log|z - \zeta_j| \geq -n^{-1} \log E_n(A)\}.$$

Then, $A \subset A_n$ and $\zeta_j \in A_n$, $j = 1, \ldots, n$. Let ν_n be the probability measure on A_n with mass $1/n$ at the points ζ_j, then $v_n(z)$ can be written in the form

$$v_n(z) = -\int_{A_n} \log|z - t| d\nu_n(t).$$

By Corollary 2.3,

$$-n^{-1}\log E_n(A) = \inf_{z\in A_n} v_n(z) \le V(A_n) = -\log\gamma(A_n).$$

Hence, $\gamma(A_n) \le E_n(A)^{1/n}$. Since $A \subset A_n$, $\gamma(A) \le \gamma(A_n)$ by **1**, and

$$\gamma(A) \le E_n(A)^{1/n}. \tag{2.9}$$

(ii) Here and in the proof of Evans' theorem (Theorem 2.10) we need the numbers

$$m_n(A) := \min_{a_1,\dots,a_n\in A} \|P_n\|_A, \quad P_n(z) := \prod_{j=1}^{n}(z-a_j).$$

Clearly, $E_n(A) \le m_n(A)$ for all n. Let $z_1,\dots,z_{n+1}$ be $n+1$ distinct points on A so that the maximum $\tau_{n+1} := \tau_{n+1}(A)$ is attained in (2.7), that is,

$$\tau_{n+1}^{n(n+1)/2} = \prod_{1\le j<k\le n+1} |z_k - z_j|.$$

For $r = 1,2,\dots,n+1$ let $\Omega_r := \{(j,k) : 1 \le j < k \le n+1, j \ne r, k \ne r\}$ and $\delta_r := \prod_{\Omega_r} |z_k - z_j|$. Hence, by the definitions of τ_{n+1} and $m_n(A)$,

$$\prod_{j=1,j\ne r}^{n+1} |z_r - z_j| = \max_{z\in A} \prod_{j=1,j\ne r}^{n+1} |z - z_j| \ge m_n(A),$$

which yields

$$\tau_{n+1}^{n(n+1)/2} = \delta_r \prod_{j=1,j\ne r}^{n+1} |z_r - z_j| \ge \delta_r m_n(A). \tag{2.10}$$

Moreover, we have

$$\prod_{r=1}^{n+1}\delta_r = \prod_{1\le j<k\le n+1} |z_k - z_j|^{n-1} = \tau_{n+1}^{n(n+1)(n-1)/2}$$

and thus, taking the product in (2.10) for $r = 1,\dots,n+1$,

$$m_n(A)^{n+1} \le \tau_{n+1}^{n(n+1)(n+1)/2} \prod_{r=1}^{n+1}(1/\delta_r) = \tau_{n+1}^{n(n+1)}.$$

Hence,

$$E_n(A)^{1/n} \le m_n(A)^{1/n} \le \tau_{n+1}(A). \tag{2.11}$$

(iii) Let $\varepsilon > 0$ be fixed. By **2** there exists a finite union O of open disks which contains A and satisfies $\gamma(\overline{O}) < \gamma(A) + \varepsilon$.

For $n \in \mathbb{N}$ we choose n distinct points $z_j \in A$, $j = 1,\dots,n$, for which the maximum τ_n is attained, and we take n so large that the n disks Δ_j :

$|z - z_j| \le \delta$, $\delta := 1/\sqrt{\pi n}$, lie in O. Next we define the probability measure μ_n on $A_n := \cup_j \Delta_j$ as follows: if $\chi_j(z)$ is the characteristic function of Δ_j, then

$$d\mu_n(z) := \sum_{j=1}^{n} \chi_j(z) dm(z),$$

where $dm(z)$ is the planar Lebesgue measure.

We shall use now that for fixed $z \in \mathbb{C}$, $\log|z-t|$ is a subharmonic function of t, hence, for $z \neq z_k$,

$$\log|z - z_k| \le \frac{1}{m(\Delta_k)} \int_{\Delta_k} \log|z-t| dm(t) = n \int_{\Delta_k} \log|z-t| dm(t). \tag{2.12}$$

The energy integral of μ_n satisfies therefore

$$\begin{aligned} I(\mu_n) &= -\int_{A_n} \left(\int_{A_n} \log|z-t| d\mu_n(t) \right) d\mu_n(z) \\ &= -\sum_{j=1}^{n} \int_{\Delta_j} \left(\sum_{k=1}^{n} \int_{\Delta_k} \log|z-t| dm(t) \right) dm(z) \\ &\le -\frac{1}{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \int_{\Delta_j} \log|z - z_k| dm(z). \end{aligned}$$

If $j \neq k$, by (2.12),

$$n \int_{\Delta_j} \log|z - z_k| dm(z) \ge \log|z_j - z_k|,$$

and if $j = k$,

$$\int_{\Delta_j} \log|z - z_j| dm(z) = \pi\delta^2 \log\delta - \pi\delta^2/2 = -\frac{1 + \log(\pi n)}{2n}.$$

Therefore,

$$\begin{aligned} I(\mu_n) &\le -\frac{1}{n^2} \sum_{j=1}^{n} \left(\sum_{k=1, k\neq j}^{n} \log|z_j - z_k| \right) + \frac{1 + \log(\pi n)}{2n} \\ &= -\frac{2}{n^2} \sum_{1 \le j < k \le n} \log|z_j - z_k| + \frac{1 + \log(\pi n)}{2n} \\ &\le -\frac{n-1}{n} \log \tau_n + \frac{1 + \log(\pi n)}{2n}. \end{aligned}$$

Since μ_n is a probability measure on A_n and also on $\overline{O}$, we have $V(\overline{O}) \le I(\mu_n)$ and thus

$$\gamma(A) + \varepsilon > \gamma(\overline{O}) \ge \exp(-I(\mu_n)) \ge \tau_n^{(n-1)/n} + o(1).$$

Replacing n by $n+1$, we get for all sufficiently large n

$$\tau_{n+1}(A) \leq \gamma(A) + 2\varepsilon. \tag{2.13}$$

Since $\varepsilon > 0$ is arbitrary, it follows from (2.9), (2.10) and (2.13) that the limits $\tau(A)$ and $c(A)$ in (2.7) and (2.8) exist and are equal to $\gamma(A)$. □

Corollary 2.5. *The logarithmic capacity of the disks $D_r : |z| \leq r$ and the circles $C_r : |z| = r$, $r > 0$, is $\gamma(D_r) = \gamma(C_r) = r$.*

Proof. We have shown earlier that $\gamma(C_1) = 1$, hence $c(C_1) = 1$. By the maximum principle for analytic functions, $c(D_r) = c(C_r)$ and thus, by the last theorem and **1**(ii), $c(D_r) = c(C_r) = \gamma(D_r) = \gamma(C_r) = r\gamma(C_1) = r$. □

Corollary 2.6. *Let $A \subset \mathbb{C}$ be a real interval or a line segment of length L in $\mathbb{C}$, then $\gamma(A) = c(A) = \tau(A) = L/4$.*

Proof. This follows from the last theorem and **1**(ii) since $\gamma([-1,1]) = 1/2$. Instead, we could also use the fact that the Chebyshev polynomial of the interval $[-1,1]$, $C_n(x) = \cos(n \arccos x)$, has the leading coefficient 2^{1-n}, so that $E_n([-1,1]) = 2^{1-n}$ and $c([-1,1]) = 1/2$. □

The next three corollaries are used in Chapter 9 on Padé approximation.

Corollary 2.7. *Let $A \subset \mathbb{C} \setminus \{0\}$ be compact. The set $S := \{z \in \mathbb{C} : 1/z \in A\}$ is also compact, and*

$$\gamma(S) \leq \gamma(A) \max_{z \in S} |z|^2. \tag{2.14}$$

Proof. Since A is compact and $0 \notin A$, $A \subset \{|z| \geq r\}$ for some $r > 0$, hence $S \subset \{|z| \leq 1/r\}$ is compact. The boundedness of A implies that $0 \notin S$. For a fixed $n \geq 1$ let $z_1^*, \ldots, z_n^*$ be distinct points in S, for which the maximum $\tau_n(S)$ is attained. Then $z_j := 1/z_j^* \in A$ and

$$\tau_n(S) = \left(\prod_{1 \leq j < k \leq n} |z_k^* z_j^*| |z_j - z_k| \right)^{2/n(n-1)} \leq \tau_n(A) \max_S |z|^2.$$

Taking the limit we get $\tau(S) \leq \tau(A) \max_S |z|^2$ and thus (2.14). □

Corollary 2.8. *Let A_1, $A_2 \subset \mathbb{C}$ be compact sets.*
(i) *If $\gamma(A_2) = 0$ then $\gamma(A_1 \cup A_2) = \gamma(A_1)$.*
(ii) *If $0 < \gamma(A_1) \leq \alpha$ and $0 < \gamma(A_2) \leq \alpha$ for some $\alpha > 0$ then*

$$\gamma(A_1 \cup A_2) \leq \sqrt{\alpha}\sqrt{\operatorname{diam}(A_1 \cup A_2)}. \tag{2.15}$$

Proof. We assume that $0 \le \gamma(A_2) \le \gamma(A_1) = \alpha$, hence $0 \le \tau(A_2) \le \tau(A_1) = \alpha$. The set $A := A_1 \cup A_2$ is compact. Let $d := \operatorname{diam}(A)$ be the diameter of A. By (2.7), $\tau_n(A_j) \le \tau_n(A) \le d$, $j = 1, 2$, and thus $\tau(A_j) \le \tau(A) \le d$.

For $n \ge 2$ let $z_1, \dots, z_n \in A$ be distinct points in A, for which the maximum $\tau_n(A)$ is attained. We may assume that $z_1, \dots, z_m$ lie in A_2 and $z_{m+1}, \dots, z_n$ lie in $A \setminus A_2$, for some $0 \le m \le n$; and $m = 0$ means that $\tau_n(A_1) = \tau_n(A)$, while $m = n$ means that $\tau_n(A_2) = \tau_n(A)$. If $1 \le m \le n-1$, then we have

$$\begin{aligned}
\tau_n(A)^{n(n-1)/2} &= \prod_{1 \le j < k \le n} |z_k - z_j| \\
&\le d^{m(n-m)} \prod_{1 \le j < k \le m} |z_k - z_j| \prod_{m+1 \le j < k \le n} |z_k - z_j| \\
&\le d^{m(n-m)} \tau_m(A_2)^{m(m-1)/2} \tau_{n-m}(A_1)^{(n-m)(n-m-1)/2}
\end{aligned}$$

and

$$\tau_n(A) \le d^{\frac{2m(n-m)}{n(n-1)}} \tau_m(A_2)^{\frac{m(m-1)}{n(n-1)}} \tau_{n-m}(A_1)^{\frac{(n-m)(n-m-1)}{n(n-1)}}. \tag{2.16}$$

We consider a subsequence $n \to \infty$ for which m/n converge to some $0 \le \beta \le 1$.

(i) If $\tau(A_2) = 0$, then (2.16) implies $\tau(A) = 0$ if $0 < \beta \le 1$ and $\tau(A) \le \tau(A_1)$ if $\beta = 0$. In both cases, $\tau(A) = \tau(A_1)$.

(ii) Let $0 < \tau(A_2) \le \tau(A_1) \le \alpha$. If $\beta = 0$ or $\beta = 1$, then (2.16) implies $\tau(A) \le \tau(A_1)$ or $\tau(A) \le \tau(A_2)$, respectively, hence $\tau(A) = \tau(A_1) \le \sqrt{\alpha d}$.

If $0 < \beta < 1$, then

$$\begin{aligned}
\tau(A) &\le d^{2\beta(1-\beta)} \tau_m(A_2)^{\beta^2} \tau(A_1)^{(1-\beta)^2} \\
&\le \left(\frac{\tau(A_1)}{d}\right)^{2\beta^2 - 2\beta + 1/2} \sqrt{\tau(A_1) d} \le \sqrt{\alpha d}.
\end{aligned}$$

We have used that $\tau(A_1) \le d$ and $2\beta^2 - 2\beta^2 + 1/2 \ge 0$ for all $0 \le \beta \le 1$. □

Corollary 2.9. *Let $k \ge 1$, $\varepsilon > 0$, and let h be a monic polynomial of degree k. The set*

$$A := \{z \in \mathbb{C} : |h(z)| \le \varepsilon^k\}$$

has the capacity $\gamma(A) = \varepsilon$.

Proof. The set A is compact. By the definition (2.8) of the Chebyshev constant $c(A)$,

$$\gamma(A) = c(A) \le \lim_{m \to \infty} \|h^m\|_A^{1/(km)} = \|h\|_A^{1/k} = \varepsilon.$$

Suppose now that $c(A) < \varepsilon$. By (2.8), there exist some $m \ge 1$ and a monic polynomial Q_n of degree $n := km$ so that $\|Q_n\|_A^{1/n} < \varepsilon$. Let Γ be the boundary curve of A, that is,

$$\Gamma := \{z \in \mathbb{C} : |h(z)| = \varepsilon^k\}.$$

Since $\|Q_n\|_\Gamma < \varepsilon^n = \|h^m\|_\Gamma$, it follows from Rouché's theorem that the polynomials h^m and $h^m - Q_n$ have the same number of zeros inside of Γ. By definition, all $n = km$ zeros of h^m lie inside of Γ, but $h^m - Q_n \not\equiv 0$ is a polynomial of degree $\leq n-1$ and cannot have n zeros, a contradiction. $\square$

In the proof of Theorem 2.11 we shall need the following

Theorem 2.10 (Evans). *Let $S \subset \mathbb{C}$ be a compact set of capacity zero. Then there exists a probability measure μ on S whose logarithmic potential $v(z) = -\int_S \log|z-t|d\mu(t)$ tends to $+\infty$, when z tends to any point of S.*

Proof. Let $m_n := m_n(S)$ be defined as in the proof of Theorem 2.4, that is,

$$m_n := \min_{a_1,\ldots,a_n \in S} \|P_n\|_S \ , \quad P_n(z) := \prod_{j=1}^{n}(z-a_j).$$

Since $m_n^{1/n} \leq \tau_{n+1}(S)$ by (2.11), and since $\lim \tau_{n+1}(S) = \tau(S) = 0$, it follows that $m_n^{1/n} \to 0$ as $n \to \infty$. Let $P_n(z) = (z-z_1)\cdots(z-z_n)$, $z_j \in S$, be a monic polynomial satisfying $m_n = \|P_n\|_S$, and let μ_n be the probability measure with mass $1/n$ at the points z_j, $j = 1,\ldots,n$. The logarithmic potential of μ_n is

$$u_n(z) = -\frac{1}{n}\sum_{j=1}^{n}\log|z-z_j| = -\frac{1}{n}\log|P_n(z)|.$$

It follows from the definition of m_n that

$$u_n(z) \geq -\frac{1}{n}\log m_n =: M_n \quad z \in S.$$

Since $\lim M_n = \infty$, we can choose the integers n_k, $k = 1,2,\ldots$, so large that $M_{n_k} \geq 2^k$, hence if we put $v_k(z) := 2^{-k}u_{n_k}(z)$, then $v_k(z) \geq 1$ on S and

$$v(z) := \sum_{k=1}^{\infty} v_k(z)$$

is a logarithmic potential of a probability measure on S, and $v(z) = +\infty$ on S. Since it is lower semi-continuous, $\lim v(z) = +\infty$, whenever $z \in \mathbb{C}$ tends to some point of S. $\square$

Theorem 2.11 (Maximum Modulus Principle with Capacity for Harmonic Functions). *Let $G \subset \mathbb{C}^*$ be a domain with a compact boundary $\Gamma \subset \mathbb{C}$ and let S be a subset of Γ of capacity zero. Let $u(z)$ be a harmonic function in G, bounded from above in G, which satisfies*

$$\limsup_{z\to\xi, z\in G} u(z) \leq M$$

for all $\xi \in \Gamma \setminus S$. Then, $u(z) \leq M$ in G.

Proof. If $S = \emptyset$, the above statement is the standard maximum modulus principle for harmonic functions (see also Theorem 1.2). We may assume therefore that $S \neq \emptyset$, and we may also assume that $\Gamma \subset \{|z| \leq 1/4\}$. Otherwise we consider $u(\rho z)$ instead of $u(z)$ for a large $\rho > 1$.

Let M_0 be the supremum of the $u(z)$ in G, and let M_1 be the supremum of the $u(z)$ in $|z| \geq 1/2$. By the maximum principle for harmonic functions, $M_1 < M_0$. Suppose now that $M_0 > M$. We choose $\varepsilon := \min\{(M_0 - M_1)/2, (M_0 - M)/2\}$ and define the set $G_0 := \{z \in G : u(z) > M_0 - \varepsilon\}$. The boundary Γ_0 of G_0 consists of the two disjoint sets $\Gamma_1 := \Gamma_0 \cap G$ and $\Gamma_2 := \Gamma_0 \cap \Gamma$. Since Γ_0 and Γ are closed sets, Γ_2 is also closed. In addition, Γ_2 is a subset of S since for each $\xi \in \Gamma_2$, for $z \to \xi$, $z \in G_0$, $\liminf u(z) \geq M_0 - \varepsilon > M$. As a closed subset of S, Γ_2 has capacity $\gamma(\Gamma_2) = 0$.

Let $v(z) = -\int_{\Gamma_2} \log|z - t| d\mu(t)$ be the Evans function of Γ_2, then $v(z) > 0$ for $z \in G_0$ since $M_1 < M_0 - \varepsilon$ hence $G_0 \subset \{|z| < 1/2\}$. For a fixed $\eta > 0$, let $w(z) := u(z) - \eta v(z)$. Since u is bounded from above in G_0, we have $w(z) \to -\infty$ as $z \in G_0$ tends to any point in Γ_2. If $z \in G_0$ tends to some point $\xi \in \Gamma_1$, then u and w are continuous in ξ and, since $u(\xi) = M_0 - \varepsilon$, the limit of $w(z)$ is

$$w(\xi) = u(\xi) - \eta v(\xi) = M_0 - \varepsilon - \eta v(\xi) \leq M_0 - \varepsilon.$$

We conclude by the standard maximum modulus principle for harmonic functions that $w(z) \leq M_0 - \varepsilon$ for all $z \in G_0$. Since $\eta > 0$ is arbitrary, $u(z) \leq M_0 - \varepsilon$ in G_0, which is absurd. □

Theorem 2.12. *The equilibrium distribution μ^* of a compact set $A \subset \mathbb{C}$ of positive capacity is unique.*

Proof. Let μ_1 and μ_2 be two equilibrium distributions for A. By Theorem 2.1 their logarithmic potentials u_1 and u_2 are bounded from above by $V := V(A)$ on $\mathbb{C}$, and satisfy $u_1(z) = u_2(z) = V$, $z \in A \setminus S$, where $\gamma(S) = 0$. Since A is compact, they are bounded from below. By Corollary 2.2, $v(z) := u_1(z) - u_2(z) \to 0$ if $z \to \xi \in A \setminus S$. By Theorem 2.11, $v(z) \leq 0$ on $\mathbb{C} \setminus S$. Similarly, $v(z) \geq 0$, $\mathbb{C} \setminus S$, so that $v(z) = 0$ q.e. By 4, $v(z) = 0$ a.e. (in the Lebesgue measure) on each circle C_r. Therefore, condition (1.5) is satisfied by v for each $R > 0$ and each $z_0 \in \mathbb{C}$, with $d(z_0) = 0$. Theorem 1.6 now yields $\mu_1 = \mu_2$. □

We prove now that the mass of the equilibrium distribution μ^* is concentrated at the boundary of A.

Theorem 2.13. *Let A be a compact set of positive capacity.* (i) *At each interior point z_0 of A, the conductor potential satisfies $u^*(z_0) = V(A)$.*
(ii) *If A^o is the interior of A, the equilibrium distribution μ^* satisfies $\mu^*(A^o) = 0$, and therefore, for the boundary $\partial A = A \setminus A^o$,*

$$\mu^*(\partial A) = \mu^*(A), \quad \gamma(\partial A) = \gamma(A). \tag{2.17}$$

Proof. (i) Let $V := V(A)$. By Theorem 2.1, $u^*(z) \le V$ and $u^*(z) = V$ q.e. on A. On each circle C_r with center z_0 contained in A^o, by **4**, a.e. $u^* = V$. Since u^* is superharmonic,

$$u^*(z_0) \ge \frac{1}{2\pi} \int_0^{2\pi} u^*(z_0 + re^{i\phi})d\phi = V.$$

We get $u^*(z_0) = V$.

(ii) We now have that $u^*(z) = V$ is harmonic on A^0. Let μ_1 be any probability measure with $\text{supp}(\mu_2) \cap A = \emptyset$. The logarithmic potential u_1 of μ_1 is harmonic in A^o. From Theorem 1.5 it follows that $\mu^*(D_r) = \mu_1(D_r) = 0$ for each compact disk $D_r \subset A^o$. This shows that $B :=$ supp (μ^*) is disjoint with A^o. Since $B \subset A$, by **1** (i) and (iii),

$$\gamma(B) \le \gamma(A \setminus A^o) \le \gamma(A) = \gamma(B),$$

and the second identity in (2.17) follows. □

For a compact set A of positive capacity let G_∞ be the component of $\mathbb{C}^* \setminus A$ which contains the point ∞. Let G_∞ be simply connected. Then there exists a conformal mapping $w = w(z)$ of the form

$$w(z) = z + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots, \tag{2.18}$$

which maps G_∞ onto $|w| > R$.

Theorem 2.14. *In this situation,* $R = \gamma(A)$.

Proof. Let u^* be the conductor potential of A. The function

$$u(z) := u^*(z) + \log|w(z)|$$

is bounded and harmonic in G_∞, and $u(\infty) = 0$. The boundary Γ_∞ of G_∞ is a compact subset of A. By Theorem 2.1 and Corollary 2.2, $u^*(z) \to V(A)$, $z \to \xi \in A \setminus S$, where $\gamma(S) = 0$. Since $|w(z)| \to R$ if $z \in G_\infty$ tends to any point $\xi \in \Gamma_\infty$, we have

$$\lim_{z \to \xi} u(z) = V(A) + \log R, \quad z \in G_\infty, \ \xi \in \Gamma_\infty \setminus S.$$

By Theorem 2.11, $u(z) \le V(A) + \log R$ on G_∞. Similarly, $u(z) \ge V(A) + \log R$ on G_∞, so that $u(z) = V(A) + \log R$ on G_∞. Since $u(\infty) = 0$, the constant $V + \log R$ is zero. Hence, $R = \exp(-V) = \gamma(A)$. □

In some questions of complex approximation, there appears the notion of *condenser capacity.* Let $K \subset G \subset \mathbb{C}$, with compact K, be given. For two probability measures μ_1, μ_2 on K and $\mathbb{C} \setminus G$, respectively and $\sigma := \mu_1 - \mu_2$, we define the energy integral

$$I(\sigma) := -\int\int \log|z-t| d\sigma(z) d\sigma(t).$$

Then $V(G, K) := \inf I(\sigma)$, for all possible μ_1, μ_2 is an analogue of $V(A)$ of (2.1). The condensor capacity,

$$\mathrm{c}(G,K) := V(G,K)^{-1},$$

will be used in §9 of Chapter 13, §5 of Chapter 15.

§3. The Dirichlet Problem and Green's Function

1. Let G be a subdomain of $\mathbb{C}^*$ with compact boundary $K \subset \mathbb{C}$ and suppose that K is of positive capacity $\gamma(K) > 0$. Let f be a continuous real valued function on K. The Dirichlet problem consists of finding a (unique) function $F(z)$, $z \in \overline{G} = G \cup K$, which is continuous on $\overline{G}$, harmonic in G and on K satisfies $F(z) = f(z)$. Because of the maximum principle for harmonic functions, there exists at most one solution F. But such a function may not exist. For example, there is no $F(z)$ that is harmonic in the unit disk without its center and for which $F(0) = 1$, $F(z) = 0$, $|z| = 1$.

We say that G is *regular with respect to the Dirichlet problem* if the above F exists for every $f \in C(K)$. We also say that K is *regular* (with respect to the Dirichlet problem in G).

2. Let $G := G_\infty$ with compact boundary $K \subset \mathbb{C}$ be the component of $\mathbb{C}^* \setminus K$ which contains the point ∞. Then *Green's function* of G is the function $g(z) = g(z,\infty)$ with the following properties.

(i) g is nonnegative, subharmonic in $\mathbb{C}$, and harmonic in $G \setminus \{\infty\}$;
(ii) $g(z) - \log|z|$ remains bounded as $z \to \infty$;
(iii) $g(z) = 0$ for every $z \in \mathbb{C} \setminus G$.

If K is regular (with respect to the Dirichlet problem in G), then $g(z)$ exists and is unique (Helms [B-1969, Corollary 10.12]). It turns out that (ii) takes the form

(ii′) $g(z) - \log|z| = -\log(\gamma(\mathbb{C} \setminus G)) + o(1)$ as $z \to \infty$.

Sometimes condition (iii) is replaced by

(iii′) $g(z) = 0$ for quasi every $z \in \mathbb{C} \setminus G$.

This generalized Green's function exists if and only if $\gamma(K) > 0$. For further details see Stahl and Totik [B-1992, Appendix V, VI].

3. For certain G the solution of the Dirichlet problem can be written explicitly. Let $C: \ |z| = 1$ and let $f(z)$ be continuous on C. Then

$$F_0(z) := \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

is analytic in the open disk $D_1 : |z| < 1$, continuous in $\overline{D}_1$ and $F_0(z) = f(z)$ for $z \in C$. The real part $H(z) = \operatorname{Re} F_0(z)$ is the solution of the Dirichlet problem for D_1, f. It can be represented by the *Poisson integral*

$$H(re^{it}) = \frac{1-r^2}{2\pi}\int_0^{2\pi} \frac{f(e^{i\phi})}{1 - 2r\cos(\phi - t) + r^2}\,d\phi. \tag{3.1}$$

If $[c, d] \subset \mathbb{R}$ is a compact interval and $G := \mathbb{C}^* \setminus [c, d]$, then the function

$$\Psi(u) = \frac{1}{2}(c+d) + \frac{1}{4}(d-c)(u+u^{-1}), \quad 0 \le |u| \le 1 \tag{3.2}$$

maps the open disk $D_1 : |u| < 1$ onto $G := \mathbb{C}^* \setminus [c, d]$. In particular, (3.2) maps the circle $|u| = 1$ onto $[c, d]$, covered twice, and 0 onto ∞.

Therefore, the function

$$F(z) := H(\Psi^{-1}(z)) \tag{3.3}$$

is the solution of the Dirichlet problem for the domain $G = \mathbb{C}^* \setminus [c, d]$ and the boundary values $f(\Psi^{-1}(x))$ on the boundary $[c, d]$, with

$$\Psi^{-1}(z) = \frac{2}{d-c}\left(z - \frac{c+d}{2} + \sqrt{\left(z - \frac{c+d}{2}\right)^2 - \frac{(d-c)^2}{4}}\right). \tag{3.4}$$

In (3.4) we have to select the branch of the square root which is positive for $z > d$.

Green's function for $G = \mathbb{C}^* \setminus [c, d]$ is given by

$$g(z, \infty) = \log\left|z - \frac{c+d}{2} + \sqrt{\left(z - \frac{c+d}{2}\right)^2 - \frac{(d-c)^2}{4}}\right| - \log\frac{d-c}{2}. \tag{3.5}$$

Indeed, this function is harmonic in $\mathbb{C} \setminus [c, d]$, continuous on $\mathbb{C}$, vanishes on $[c, d]$ and

$$g(z, \infty) = \log|z| - \log\frac{d-c}{4} + o(1), \quad z \to \infty. \tag{3.6}$$

As a corollary,

Proposition 3.1. *Compact intervals $[c, d]$ are regular. The function*

$$\Lambda(z) := H(\Psi^{-1}(z)) + g(z, \infty) \tag{3.7}$$

solves the Dirichlet problem of the domain $\mathbb{C}^* \setminus [c, d]$ and the boundary values $f(\Psi^{-1}(x))$, $x \in [c, d]$, that is Λ is continuous on $\mathbb{C}$, harmonic in $\mathbb{C} \setminus [c, d]$, attains the values $q(x) := f(\Psi^{-1}(x))$, $x \in [c, d]$, and

$$\Lambda(z) = \log|z| + \text{const} + o(1), \quad z \to \infty. \tag{3.8}$$

The union of finitely many compact intervals is also regular.

§ 4. Balayage Methods

Let $G \subset \mathbb{C}$ be a domain with compact boundary ∂G and let μ be a non-negative Borel measure of finite mass $\|\mu\|$ and $\text{supp}(\mu) \subset \overline{G} = G \cup \partial G$. The problem of *balayage* (or *sweeping out*) consists of finding a new non-negative Borel measure μ^* supported on ∂G so that $\|\mu^*\| = \|\mu\|$ and

$$\int \log|z-t|d\mu^*(t) = \int \log|z-t|d\mu(t) \quad \text{for quasi all } z \notin G. \tag{4.1}$$

For bounded domains G such a measure always exists (see Landkof [1972, Chapter 4, Section 2], and also the Appendix of Stahl and Totik [1992]).

If the domain G is unbounded then (4.1) must be replaced by

$$\int \log|z-t|d\mu^*(t) = \int \log|z-t|d\mu(t) + c \quad \text{for quasi all } z \notin G, \tag{4.2}$$

where the constant c is equal to

$$-\int_G g(t,\infty)d\mu(t), \tag{4.3}$$

where $g(t,\infty)$ is Green's function of G (see Landkof [1972],(4.2.6)). If the domain G is regular with respect to the Dirichlet problem, then in (4.1) and (4.2) one has equality for all $z \notin G$.

In Chapter 8 the balayage technique has been applied to the unbounded domains G of the form

$$G := H_+ \setminus [a,b], \quad H_+ := \{z \in \mathbb{C} : \text{Re } z > 0\}, \quad 0 < a < b < \infty, \tag{4.4}$$

whose boundary $\partial G = i\mathbb{R} \cup [a,b]$ is unbounded.

Theorem 4.1. *Let G be the domain* (4.4) *and μ be a non-negative Borel measure with (not necessarily bounded) support $S := supp(\mu) \subset (0,\infty)$ and finite mass $\|\mu\| := \mu(S)$. Moreover, let the potential $L(z) := \int_0^\infty \log|z-t|d\mu(t)$ be continuous on $\mathbb{C}$ and let $d := \int_0^\infty \log(t+1)d\mu(t) < \infty$. Then there exists a non-negative Borel measure μ^* supported on $\partial G = i\mathbb{R} \cup [a,b]$ with the following properties:*

(i) $\|\mu^*\| = \|\mu\|$;

(ii) *the measure μ^* is symmetric with respect to the real axis, that is,*

$$\mu^*(S) = \mu^*(\{z \in \mathbb{C} : \overline{z} \in S\}) \quad \textit{for each Borel set } S \textit{ in } \mathbb{C};$$

(iii) *Green's potentials of the measures μ and μ^* satisfy*

$$\int_{\partial G} \log\frac{|z-t|}{|z+t|}d\mu^*(t) = \int_0^\infty \log\frac{|z-t|}{|z+t|}d\mu(t) \quad \textit{for all } z \in \partial G; \tag{4.5}$$

(iv) *both Green's potentials in* (4.5) *are continuous functions of z on $\mathbb{C}$.*

Proof. The mapping $w = 1/(z+1)$ transforms H_+ onto the simply connected bounded domain $\Omega \subset \mathbb{C}$ with the compact boundary

$$\partial\Omega = \left\{w \in \mathbb{C} : w = \frac{1}{1+iy} = \frac{1-iy}{1+y^2}, \quad -\infty < y < \infty\right\},$$

the interval $[a, b]$ onto the interval $[\alpha, \beta] \subset \Omega$, $\alpha := 1/(b+1)$, $\beta := 1/(a+1)$, and the domain G onto $D := \Omega \setminus [\alpha, \beta]$ with the boundary $\partial D = \partial\Omega \cup [\alpha, \beta]$. The set D is a bounded domain contained in H_+. Let ν be the image of the measure μ under the mapping $w = 1/(z+1)$. Clearly, ν is a non-negative Borel measure of mass $\|\nu\| = \|\mu\|$, whose support is contained in $[0, 1)$. Its negative logarithmic potential is

$$\begin{aligned} P(w) &:= \int_0^1 \log|w - s| d\nu(s) = \int_0^\infty \log\left|\frac{1}{z+1} - \frac{1}{t+1}\right| d\mu(t) \\ &= L(z) - \|\mu\| \log|z+1| - d, \quad w = \frac{1}{z+1}, \end{aligned} \tag{4.6}$$

where $d := \int_0^\infty \log(t+1) d\mu(t)$ is assumed to be finite. Since $L(z) := \int_0^\infty \log|z - t| d\mu(t)$ is continuous on $\mathbb{C}$ by assumption, $P(w)$ is a continuous function in $\mathbb{C} \setminus \{0\}$.

Applying the balayage technique to ν and D we find a new measure ν^* supported on ∂D so that $\|\nu^*\| = \|\nu\|$ and

$$P^*(w) := \int_{\partial D} \log|w - s| d\nu^*(s) = P(w) \quad \text{for quasi all } w \notin D. \tag{4.7}$$

The measure ν^* has no atoms on $\partial D \setminus \{0\}$ since P is continuous there.

We shall prove next that $P^*(w)$ is continuous on $[\alpha, \beta]$, as a function on $\mathbb{C}$, which implies that (4.7) is valid for *all* $w \in [\alpha, \beta]$ since P is continuous. If $\rho := \nu^*([\alpha, \beta]) = 0$, nothing has to be proved. Let $\rho > 0$ and define the functions

$$q(x) := P(x) - \int_{\partial\Omega} \log|x - s| d\nu^*(s), \quad x \in [\alpha, \beta],$$

and

$$Q(w) := \int_\alpha^\beta \log|w - s| d\nu^*(s), \quad w \in \mathbb{C}.$$

Then $q \in C[\alpha, \beta]$ and by (4.7), $Q(x) = q(x)$ for quasi all $x \in [\alpha, \beta]$. The interval $[\alpha, \beta]$ is a regular set (§3). Therefore, there exists a function F, harmonic on $\mathbb{C} \setminus [\alpha, \beta]$, continuous on $\mathbb{C}$, with $F(x) = \rho^{-1} q(x)$ for all $x \in [\alpha, \beta]$, and where $F(w) - \log|w|$ is harmonic at $w = \infty$. The difference $v(w) := \rho^{-1} Q(w) - F(w)$ is harmonic in $\mathbb{C}^* \setminus [\alpha, \beta]$, upper semi-continuous on $\mathbb{C}$, bounded from above in $\mathbb{C}^* \setminus [\alpha, \beta]$ and equal to zero at quasi every point of $[\alpha, \beta]$. By Theorem 2.11, $v(w) \le 0$ for all $w \in \mathbb{C} \setminus [\alpha, \beta]$. This and (1.9) of Lemma 1.7 imply that

$$v(x) = \lim_{y \to 0+} v(x + iy) \le 0, \quad \text{for all } x \in [\alpha, \beta].$$

Thus, $v(x) = 0$ for all $x \in [\alpha, \beta]$, since v has this property quasi everywhere on $[\alpha, \beta]$ and is upper semi-continuous. From (1.10) of Lemma 1.7 we deduce that v is continuous on $[\alpha, \beta]$ as function on $\mathbb{C}$, and so are Q and P^*.

From $\log|w - s| > \log|s|$, $w < 0$, $\operatorname{Re} s \ge 0$, and the upper semi-continuity of P and P^* it follows that

$$P(0) = \lim_{w \to 0, w<0} P(w), \quad P^*(0) = \lim_{w \to 0, w<0} P^*(w). \tag{4.8}$$

From (4.6) and (4.8) we deduce that

$$\begin{aligned} P(0) &= \lim_{x \to -\infty} (L(x) - \|\mu\| \log|x+1|) - d \\ &= \lim_{x \to +\infty} \int_0^\infty \log(1 + t/x) d\mu(t) - d = -d. \end{aligned}$$

Since $D \subset H_+$, (4.7) and (4.8) imply $P^*(0) = P(0) = -d$ and $\nu^*(\{0\}) = 0$.

The inverse of the mapping $w = 1/(z+1)$ transforms ν^* into another measure, denoted by μ^*, which is supported on $\partial G = i\mathbb{R} \cup [a,b]$, with $\|\mu^*\| = \|\nu^*\| = \|\mu\|$. By (4.6) and (4.7),

$$\int_{\partial G} \log\left|\frac{1}{z+1} - \frac{1}{t+1}\right| d\mu^*(t) = L(z) - \|\mu\| \log|z+1| - d,$$

for quasi all $z \notin G$. Since $\|\mu^*\| = \|\mu\|$, it follows that

$$\Lambda(z) := \int_{\partial G} \log\left|\frac{z-t}{t+1}\right| d\mu^*(t) = L(z) - d \quad \text{for quasi all } z \notin G, \tag{4.9}$$

that is, for quasi all $z \in \partial G = i\mathbb{R} \cup [a,b]$ and quasi all $z \in H_- := \{z \in \mathbb{C} : \text{Re } z < 0\}$. Both functions in (4.9) are continuous in H_-, therefore (4.9) is valid for *all* $z \in H_-$. From

$$P^*(w) = \Lambda(z) - \|\mu^*\| \log|z+1|, \quad w = \frac{1}{z+1}, \quad z \in \mathbb{C} \setminus \{-1\}, \tag{4.10}$$

and the upper semi-continuity of P^* we derive that $\Lambda(z)$ is upper semi-continuous on $\mathbb{C} \setminus \{-1\}$. We do not know if $d^* := \int_{\partial G} \log|t+1| d\mu^*(t)$ is finite, but we know that $\Lambda(iy - \delta) > \Lambda(iy)$ for all $y \in \mathbb{R}$ and all $\delta > 0$. By the upper semi-continuity of Λ we therefore get

$$\Lambda(iy) = \lim_{\delta \to 0+} \Lambda(iy - \delta) = \lim_{\delta \to 0+} L(iy - \delta) - d = L(iy) - d, \tag{4.11}$$

for all $y \in \mathbb{R}$. This and the continuity of L imply that Λ is a continuous function on $i\mathbb{R}$ as a function on $i\mathbb{R}$, hence P^* is continuous on $\partial\Omega \setminus \{0\}$ as a function on $\partial\Omega$. By Theorem 1.4, P^* is continuous on $\partial\Omega \setminus \{0\}$ as a function on $\mathbb{C}$. Therefore, $\Lambda(z)$ is continuous on $i\mathbb{R}$, as a function on $\mathbb{C}$, and (4.9) is valid for all $z \in i\mathbb{R}$. Moreover, since $P^*(w)$ is continuous on $[\alpha, \beta]$, $\Lambda(z)$ is continuous on $[a,b]$ and (4.9) is valid for all $z \in [a,b]$.

From this it follows that the left potential (4.5), which is eqaul to $\Lambda(z) - \Lambda(-z)$, is continuous on $\mathbb{C}$ and that $\Lambda(z) - \Lambda(-z) = L(z) - L(-z)$ for all $z \in \partial G$, which proves (4.5). This concludes the proof of (i), (iii) and (iv).

It remains to prove that there exists a measure μ^* with the additional property (ii): we define another non-negative Borel measure, μ', by

$$\mu'(S) := \mu^*(\{z \in \mathbb{C} : \overline{z} \in S\}) \quad \text{for each Borel set in } \mathbb{C}.$$

Since μ is supported on $(0, \infty)$ and since ∂G is symmetric with respect to the real axis, the measure μ' has also the properties (i), (iii), (iv). Therefore, the measure $\frac{1}{2}(\mu^* + \mu')$ satisfies (i)-(iv). □

Corollary 4.2. *For the measure* μ^* *of Theorem* 4.1,

$$\int_{i\mathbb{R}} \log \frac{|z-t|}{|z+t|} d\mu^*(t) = 0 \quad \text{for all } z \in \mathbb{C} \tag{4.12}$$

and

$$\int_a^b \log \frac{|z-t|}{|z+t|} d\mu^*(t) = \int_0^\infty \log \frac{|z-t|}{|z+t|} d\mu(t) \quad \text{for all } z \in \partial G. \tag{4.13}$$

Proof. Indeed, (4.12) follows from (ii) of Theorem 4.1, and (4.13) from (4.5) and (4.12). □

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Abbreviations: JAT = Journal of Approximation Theory; CA = Constructive Approximation; BAMS or PAMS or TAMS = Bulletin or Proceedings or Transactions of the American Mathematical Society; Dokl. = Doklady Akad. Nauk SSSR; Izv. = Izvestiya Akad. Nauk SSSR, ser. mat.; SIAM Num. = SIAM J. Num. Anal.; Mat. Sb. = Matematicheskii Sbornik, N.S.; Uspekhi = Uspekhi Matematicheskikh Nauk

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Author Index

Ahlberg, J.H. 621
Akhiezer, N.I. 23–27, 29, 32, 38, 212, 221, 621, 625, 626
Altomare, F. 551, 621
Andô, T. 546, 550, 626
Arnold, V.I. 554
Asarin, E.A. 516, 626
Auerbach, G. 394
Aumann, G. 565
Aziz, A. 84, 626

Bak, J. 249, 346, 626
Baker, G.A., Jr. 282, 621
Barrar, R.B. 396, 626
Bartke, K. 245, 621, 626
Bauer, H. 517, 537, 539, 626
Beardon, A.F. 296, 626
Beatson, R.K. 82, 198, 626
Belinskii, E.S. 483, 626
Berens, H. 388, 517, 525, 533, 536, 539, 543, 547, 548, 551, 621, 626
Bernau, S.J. 548, 550, 626
Bernstein, S.N. 20, 27–29, 32, 38, 90, 621, 626
Birkhoff, G. 178
Birman, M. 488, 506, 626
Blaschke, W. 591
Blatt, H.-P. 8, 12, 626
Boas, R.P. 621
Bochkarev, S.V. 162, 627
Bojanic, R. 77, 627
Bojanov, B.D. 181, 203, 621, 627
de Boor, C. 177, 201, 202, 621, 627
Borsuk, K. 405, 418, 567, 627
Borwein, P. 12, 64, 67, 73, 83, 133, 345, 366, 367, 378, 621, 627
Bourgain, J. 165
Braess, D. 189, 206, 222, 380, 383, 621, 627
Brosowski, B. 371, 627
Brown, A.L. 388, 397, 627
Brudnyi, Yu.A. 299, 627
Brunn, H. 572
Buck, R.C. 621
Bulanov, A.P. 219, 327, 330, 341, 627
Bunt, L.N. 395, 628
Buslaev, A.P. 400, 418, 423, 429, 432, 435, 440, 441, 628
Buslaev, V.I. 297, 514, 628
Butzer, P.L. 36, 621, 628

Campiti, M. 551, 621
Carl, B. 498, 511, 512, 628
Carleman, T. 21, 628
Carpenter, A.J. 221, 227, 255, 628, 640
Chebyshev, P.L. 2
Chen, H. 418, 432, 441, 628
Cheney, E.W. 213, 243, 246, 565, 621, 622, 628
Choquet, G. 525
Clarkson, J.A. 546, 628
Cody, N.J. 227, 628
Coifman, R.R. 150, 581, 587, 628

Dawood, Q.M. 84, 626
Deutsch, F. 371, 387, 627
Descloux, J. 397, 628
DeVore, R.A. 49, 77, 82, 198, 235, 340, 483, 567, 621, 627, 628
Diliberto, S.P. 565
Dieudonné 297
Ditzian, Z. 455, 621
Dolzhenko, E.P. 206, 241, 251, 252, 309, 629
Donner, K. 541, 621
Doronin, V.G. 76, 622
Douglas, R.G. 546, 629
Dunham, C.B. 247, 371, 392, 629
Dzhrbashyan, M.M. 320, 629
Dzyadyk, V.K. 622

Edmunds, D.E. 450, 483, 489, 515, 629
Efimov, N.V. 247, 369, 370, 391, 629

Ekeland, I. 395, 396, 629
Enflo, P. 402, 629
Erdélyi, T. 64, 73, 75, 81, 83, 345, 366, 367, 378, 621, 627, 629
Erdös, P. 64, 629
Evgrafov, M.A. 622

Faber, G. 162, 629
Favard, J. 190
Feinerman, R.P. 360, 622
Fekete, M. 49, 51, 55, 606, 629
Fenchel, W. 533
Ferguson, L.B.O. 358, 630
Finzel, M. 397, 630
Fisher, S.D. 443, 445, 448, 630
Freud, G. 78, 105, 142, 206, 235, 327, 630
Fridman, B.L. 554, 562, 630
Frostman, O. 120, 605

Gaier, D. 622
Galeev, E.M. 483, 630
Galkin, P.V. 188, 630
Ganelius, T. 78, 221, 346, 622, 630
Garkavi, A.L. 404, 630
Garnaev, A.Yu. 465, 630
Gelfand, I.M. 391
Gelfond, A.O. 622
Georgiev, P.G. 396, 630
Gierz, G. 252, 630
Gluskin, E.D. 400, 465, 472, 473, 476, 481, 630
Goldstein, A.A. 246, 628
Golitschek, M. v. 39, 86, 93, 98, 103, 104, 106, 132, 133, 136, 198, 200, 346, 347, 351, 366, 367, 565, 630, 631
Gonchar, A.A. 206, 209, 227, 241, 251, 256, 297, 343, 631
Goodman, T.N.T. 202, 631
Graves-Morris, P. 282, 621
Gutknecht, M.N. 210, 640

Hakopian, H.A. 181, 621
Hall, T. 38, 631
Hayman, W.K. 622
He, X. 132, 631
Hewitt, E. 49, 55, 59, 631
Hilbert, D. 554
Hobby, C.R. 380, 631
Höllig, K. 407, 450, 451, 478, 482, 506, 631
Hu, Y. 49, 631
Hutton, C.V. 446, 631

Ibragimov, I.I. 622
Iliev, G.L. 82, 631
Ioffe, A.D. 447, 631
Ismagilov, R.S. 400, 417, 446, 459, 632
Ivanov, K.G. 38, 632

Jackson, D. 188, 622
Jetter, K. 181, 202, 203, 622, 632
Jentsch, K. 12
Johnson, R.S. 203, 632

Kadec, M.I. 6, 632
Kahane, J.P. 555, 632
Kakeya, S. 50
Kantorovich, L.V. 50, 542
Karlin, S. 178, 202, 448, 622, 632
Karlovitz, L.A. 412, 632
Kashin, B.S. 400, 407, 439, 440, 449–451, 465, 467, 476, 483, 632
Kaufman, R. 562, 632
Kemperman, J.H.B. 87, 632
Khenkin, G.M. 562, 640
Khinchin, A.Ya. 56, 632
Khrushchev, S.V. 343, 636
Kierstead, H. 567, 628
Kilgore, T. 173, 632
Kitahara, K. 622
Klee, V.L. 395, 632
Knopfmacher, A. 143, 632
Kolmogorov, A.N. 400, 413, 416, 440, 449, 485, 488–491, 504, 514–516, 554, 559, 632
Korneichuk, N.P. 76, 181, 194, 407, 421, 432, 440, 622, 632
Korovkin, P.P. 517, 622
Krasnoselski, M.A. 405, 632
Krein, M.G. 405, 632
Kronecker, L. 59, 281, 633
Kroó, A. 83, 633
Krotov, V.G. 342, 633
Kuhn, H. 559
Kulanin, E.D. 480, 483, 633

Lange, G. 203, 632
Lazar, A.J. 389, 633
Lebedev, N.A. 623
Leviatan, D. 39, 49, 82, 346, 626, 631, 633
Levin, A.L. 209, 210, 290, 435, 499, 633
Levin, M. 565, 633
Li, C. 418, 432, 441, 628
Li, W. 622
Li, X. 132, 631
Light, W.A. 555, 622

Ligun, A.A. 76, 202, 440, 447, 622, 633
Linde, R. 450, 633
Loeb, H.L. 212, 243, 396, 626, 628
Lorentz, G.G. 4, 9, 43, 47, 64, 85–87, 93, 98, 100, 103, 106, 109, 133, 178, 179, 181, 233, 383, 386, 410, 423, 455, 458, 459, 488, 492, 517, 525, 533, 536, 537, 539, 543, 547, 548, 551, 554, 555, 562, 567, 622, 626, 628, 631–634
Lorentz, R.A. 43, 151, 164, 170, 622, 634
Lubinsky, D.S. 105, 109, 134, 136, 140, 142, 143, 285, 286, 290, 622, 632–634

Madych, W. 173, 634
Maehly, H.J. 243, 634
Magnus, A. 143, 206, 227, 634
Maiorov, V.E. 400, 407, 450, 451, 458, 478, 482, 483, 634
Makovoz, Y. 106, 133, 202, 418, 419, 440, 443, 465, 473, 483, 516, 634, 631, 635
Mallat, S. 150, 635
Malozemov, V.N. 622
Máté, A. 83, 629
Meinardus, G. 222, 227, 395, 622, 628
Mergelyan, S.N. 29, 38, 635
Meyer, Y. 150, 628
Mhaskar, H.N. 105, 116, 118, 131, 142, 143, 634, 635
Micchelli, C.A. 400, 441, 443, 445, 630, 635
Michael, E. 388, 397, 635
Milman, D.P. 405, 632
Milovanovic, G.V. 622
Minkowski, H. 572
Mitrinovic, D.S. 622
Mityagin, B.S. 500, 635
Morris, P.D. 389, 633
de Montessus de Ballore, R. 284, 635
Müntz, C.H. 345, 635

Nachbin, L. 35, 622
Natanson, I.P. 622
Nessel, R.J. 621
Nevai, P. 83, 143, 629, 632
Nevanlinna, R. 592
Newman, D.J. 37, 82, 206, 207, 217, 222, 227, 248, 255, 345, 346, 360, 362, 622, 623, 626, 630, 635
Nikishin, E.M. 623
Nikolskii, S.M. 623
Nilson, E.N. 621
Nöbeling, G. 564
Nürnberger, G. 181, 184, 186, 388, 389, 623, 626, 635
Nuttall, J. 291, 635

Obreschkov, N. 623
Offin, D. 165, 635
Ofman, Yu. 516
Okada, Y. 50, 635
Opitz, H.U. 227, 636
Oskolkov, K. 165, 635
Ostrand, P.A. 565, 636

Padé, H. 277, 280, 636
Parfenov, O.G. 343, 636
Pekarskii, A.A. 206, 300, 309, 313, 323, 324, 327, 340–343, 636
Peller, V.V. 299, 340, 342, 343, 636
Perron, O. 222, 282
Petrushev, P.P. 181, 252, 327, 330, 340, 341, 397, 623, 636, 637
Phelps, R.R. 524
Pick, G. 592
Pietsch, A. 411, 636
Pinkus, A. 202, 400, 403, 407, 410, 418, 432, 441, 445, 448, 482, 623, 635–637
Plonka, G. 173, 637
Pollard, H. 29, 31, 637
Pólya, G. 367
Pommerenke, C. 291, 637
Popov, V.A. 78, 80, 181, 206, 237, 291, 327, 330, 340, 324, 332, 397, 637, 623
Powell, M.J.D. 235, 623
Prestin, J. 173, 632
Privalov, A.A. 162, 165, 637
Prolla, J.B. 623

Quake, S. 150, 628

Rahman, Q.I. 623
Rakhmanov, E.A. 142, 206, 227, 240, 256, 297, 631, 637
Rassias, Th.M. 622
Remez, E.J. 73, 637
Rice, J.R. 183, 371, 373, 375, 380, 397, 623, 631, 637
Riemenschneider, S.D. 181, 202, 622
Riesz, M. 586
Rivlin, T.J. 37, 623, 635
Roulier, J.A. 39, 637

Ron, A. 172, 627
Ruban, V.I. 447, 637
Rudin, W. 459, 637
Rusak, V.N. 301, 304, 320, 341, 342, 623
Ruttan, A. 210, 221, 227, 255, 628, 637, 640

Saff, E.B. 9, 12, 37, 38, 86, 90, 91, 104, 105, 109, 116, 118, 129, 131-133, 136, 140-143, 209, 284, 297, 622, 626, 627, 632, 634, 635, 637, 638
Sahakian, A.A. 164, 170, 181, 621, 634
Sard, A. 623
Schauder, J.P. 160, 161, 638
Scheinberg, S. 21, 638
Scherer, K. 36, 227, 628, 636
Schmeisser, G. 623
Schmidt, E. 379, 638
Schoenberg, I.J. 196, 198, 200, 623, 638
Scholz, R. 459, 638
Schönhage, A. 227, 638
Schumaker, L.L. 181, 183, 189, 623, 637
Schütt, C. 506, 637
Selig, K. 173, 632
Semmes, S. 334, 638
Sendov, B. 78, 80, 397, 623
Sevastyanov, E.A. 324, 341, 638
Shapiro, H.S. 383, 623, 638
Shashkin, Ya.A. 517, 523, 525, 531, 533, 535, 638
Shekhtman, B. 252, 630
Shevchuk, I.A. 82, 638
Shvedov, A.S. 49, 82, 638
Simkani, M. 12, 626
Singer, I. 184, 623, 635
Smirnov, V.I. 587, 623
Solomyak, M.Z. 459, 488, 506, 626, 638
Sommer, M. 389, 635
Somorjai, G. 248, 638
Sprecher, D. 554
Stafney, J.D. 39, 638
Stahl, H. 221, 227, 255, 276, 297, 638
Stechkin, S.B. 247, 369, 370, 391, 411, 440, 459, 623, 629, 639
Stepanets, A.I. 623
Sternfeld, Y. 564, 565, 639
Stessin, M.I. 411, 448, 639, 628
Stieltjes, T.J. 297, 639
Strauss, H. 188, 203, 639
Straus, S.P. 565
Studden, W.J. 622
Subbotin, Yu.N. 191, 418, 440, 623, 639
Suetin, S.P. 297, 628, 639
Sun Yongshen 449, 639
Sylvester, J.J. 285
Szabados, J. 66, 83, 623, 639
Szász, O. 367, 639
Szegö, G. 51, 606, 639
Szüsz, P. 233, 639

Taikov, L.V. 440, 639
Tamrazov, P.M. 623
Tasche, E. 173, 637
Tashev, Sp. 6, 639
Temlyakov, V.N. 515, 639
Tikhomirov, V.M. 191, 400, 407, 418-423, 429, 432, 435, 439-441, 446, 447, 459, 488, 489, 499, 514, 559, 623, 624, 628, 631, 632, 637, 638
Timan, A.F. 625
Tornheim, L. 371, 631
Totik, V. 9, 38, 109, 134, 140, 142, 143, 455, 621, 624, 632, 634, 637, 640
Trefethen, L.N. 210, 282, 640
Triebel, H. 450, 483, 489, 515, 629
Tsarkov, I.G. 396, 640
Tucker, A.W. 567, 640
Turan, P. 233, 639
Turetskii, A.Kh. 624
Tzimbalario, J. 346, 626

Ullman, J.L. 104, 129, 141, 637

de la Vallée-Poussin, C. 624
Varga, R.S. 19, 37, 85, 86, 90, 91, 104, 129, 132, 141, 209, 210, 221, 227, 255, 297, 624, 628, 637, 640
Velikin, V.L. 447, 640
Vértesi, P. 623
Verfürth, R. 374, 640
Vitushkin, A.G. 374, 377, 488, 503, 514, 559, 562, 624, 628, 640
Vlasov, L.P. 380, 385, 386, 640
Vyacheslavov, N.S. 221, 255, 640

Wallin, H. 290, 291, 640
Walsh, J.L. 624
Warren, H.E. 383, 640
Weil, H. 472
Weiss, B. 567, 640
Werner, H. 243, 380, 640
Wickerhauser, M.V. 146, 628
Widom, H. 445, 503, 640

Witzgall, C. 243, 634
Wojtaszchyk, P. 165, 640
Woźniakowski, K. 161, 640
Wulbert, D.E. 184, 389, 517, 525, 535, 633, 640

Yu, X.M. 49, 82, 628, 631, 640

Zeller, K.L. 3, 43, 47, 634
Zhensykbaev, A. A. 194, 432, 641
Ziegler 178
Zielke, R. 624
Zolotarev, E.I. 221, 641
Zuckerman, H.S. 49, 55, 59, 631
Zygmund, A. 577

Subject Index

Alternation points of approximation by polynomials 4, 7
– – – – – –, incomplete 90
– – – – – –, weighted 109, 133
– – – – – splines 82
Approximation by entire functions 21
– – exponential sums 376
– – functions $\sum A_i(x - c_i)^{-1}$ 23
– – polynomials 1
– – –, convex 49
– – –, monotone 43
– – –, Müntz 345
– – –, one-sided, two-sided 76
– – –, weighted 28
– – – with integral coefficients 49
– – – – – – –, characteristic sets 55
– – rational functions 205
– – – –, Padé 222, 277
– – simple functions 314
– – splines 175
– – –, monotone 198
Approximation set 492
– spaces, comparison 340
Atomic decomposition 310, 587

Bases, orthonormal 145, 164
–, polynomial 162, 165
–, Schauder 160
–, wavelet 145
–, – packet 151, 170
Blaschke product 591
Balayage 271, 616
Bernstein, weight problem of 28, 38
Best approximation, polynomial 1
– –, –, distribution of zeros 11, 19
– –, strong uniqueness 184, 186
– –, rational 208, 241, 246
– –, –, uniqueness 215, 247
– –, –, trigonometric 215
Boundary of a function set 528
–, Choquet 525
–, –, generalized 537

Capacity, condenser 445, 503, 613
– in metric spaces 485, 492, 496
– – – – and widths 500
–, logarithmic 105, 291, 603
Chebyshev constant 606
Complexity of a function 515
Contractions 518, 547
–, positive 517, 545
Contractive projection 546

Degree of varisolvency 372, 380
Density of families of functions 28, 39, 49, 131, 134, 243, 248, 252, 345, 366
Deviation $E(K, X)$ 399
Dirichlet problem 124, 256, 614
Discretization for widths 451, 468, 479

Entropy (metric) 485
– of analytic functions 502
– of approximation sets 492
– of a sum 487
– of balls 487
– of Lipschitz balls 491, 506
–, relation to widths 498
Entropy numbers 488, 509
Error of approximation, in Banach spaces 20
– – –, non-linear 383
– – –, polynomial, of x^k 41, 347
– – –, –, of $(x - a)^{-1}$ 2
– – –, rational, best 211, 214, 217
– – –, –, in C^r 240
– – –, –, in Hausdorff distance 397
– – –, –, in Hardy spaces 299, 323
– – –, –, in $L \log L$ 326
– – –, –, in V^r 237, 327
– – –, –, in $V^1 \cap \mathrm{Lip}\alpha$ 327
– – –, –, in W_p^1 237, 346
– – –, –, of convex functions 330
– – –, –, of e^x on $[-1, 1]$ 222
– – –, –, of e^{-x} on $[0, \infty)$ 227
– – –, –, of piecewise analytic functions 233

Error of approximation, rational
– – –, –, of x^k 207
– – –, –, of x^α 221
– – –, –, of $|x|$ 217, 255
– – –, –, with complex coefficients 208
– – –, spline 180, 195, 324, 421, 432
– – –, –, comparison with rational 332
Euclidean algorithm 54
Equilibrium distribution 603

Faa di Bruno, formula 311
Factorization of trigonometric polynomial 216
Family of functions, Haar 77
– – –, proximinal 210, 369, 380
– – –, varisolvent 371, 382
– – –, weak Haar 178, 182
– – –, with uniqueness property 247
Favard numbers 79
Feiér means 103
Freud's conjecture 142
Function, conjugate 585
–, Green's 124
–, maximal 236, 335
–, normal 245
–, simple, r-simple 310
–, Stieltjes 297
–, sub-, superharmonic 596
–, wave-shaped 429
–, zoom 248

Grey code 173

Hilbert's 13-th problem 554

Integral, elliptic 270, 575
–, energy 117, 603
–, Jackson 45
–, Poisson 126, 614
Inequalities of Bernstein-Markov type for exponential sums 378
– – – – – Müntz polynomials 362
– – – – – polynomials 64
– – – – – rational functions 206, 300
– – Nikolskii 350
– – Remez 73
Incomplete polynomials 85
– –, trigonometric 92
Interpolation, Birkhoff 43
–, Hermite 109, 282
–, Lagrange 59
– by splines 178, 190
– – –, interlacing conditions of 179

K-functional 352

Korovkin theorems 517

Lattice homomorphism 551

Measure 18
–, Chebyshev ν_w 112
–, equilibrium μ_w 117, 603
–, Gaussian probability 469
–, rotation invariant 461
Metric projection, continuity of 243, 369
Modulus of smoothness, averaged 79
Möbius transformation 591
Multiresolution 145
–, complete 155
–, upper 146

Norm, absolutely continuous 543
–, uniformly convex 545
–, uniformly monotone 545

Operator, DeVore 233
–, Jackson-Rusak 320
–, of best rational approximation 242

Periodization 156
Point, exposed 523
–, extremal 523
–, peak 529
–, quasi peak 538
Poisson formula 157
Polynomials, Bernstein 42
–, Chebyshev 50
–, cyclotomic 61
–, incomplete 85
–, –, Chebyshev 89
–,–, trigonometric 85
–, Müntz 39, 345
–, Newman 218
–, symmetric 56
–, weighted 105
–, –, Chebyshev 109
–, with many real zeros 85
–, with integral coefficients 49
–, with positive coefficients 64
Potential, conductor 603
–, logarithmic 108, 595
Principle, Ekeland, variational 396
–, Hausdorff, maximal 396
–, Maximal modulus 597, 598, 611

Rational functions 205
– –, defect of 205
– –, degenerate 205, 246
– –, with arbitrary powers 248
Refinement equation 147

Scaling function 146, 165
– –, with monotone majorant 152
Selection, continuous 387, 388
Set, approximatively, boundedly compact 369
–, Chebyshev 369
–, dominating (Korovkin) 517
–, –, examples of 529, 532
–,–, strict 528
–, essential, minimal essential 106, 107
–, full approximation 453
–, proximinal 210, 369, 380
–, regular 124, 614
–, Weierstrass, maximal Weierstrass 131
Shadow of a set 517, 536, 541
Space, Besov 455, 483
–, Hardy 104, 308, 581
–, Lipschitz 453
–, of approximation theory 33
–, rearrangement invariant 34
–, Schoenberg 175
–, Sobolev 399
Spectral function, value, couple 425
Sun 370, 390, 394
Superposition 553
–, linear 562
–, scheme of 560

Transfinite diameter 606
Triangulation, Kuhn 569

Variational principle of Eceland 396
Varisolvent family 372

Wavelet 145, 165
Weight, Freud, Jacobi 105, 117, 116
Weight function 28
Widths, Kolmogorov 399
–, Gelfand 401
–, duality of 408
–, linear 400
– of ellipsoids 401, 412
– – finite dimensional balls 410, 465, 472
– – Lipschitz balls 450, 453, 479
–, trigonometric 483

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