Exercise 3.2. *Consider the heat conduction problem:*

$$\begin{cases} u_t - u_{xx} = f, & \forall t \in (0,T], \forall x \in (a,b), \\ u(x,0) = u_0(x), & \forall x \in (a,b), \\ u(a,t) = u(b,t) = 0, & \forall t \in (0,T]. \end{cases}$$

The Forward Euler/centered schema:

$$L_h u_i^n := \frac{u_i^{n+1} - u_i^n}{k} - \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} = f_i^{n+1}, \quad n = 0, 1, \cdots, M, i = 1, 2, \cdots, N-1,$$

where u_i^n is an approximation of $u(x_i, t^n)$. Prove that the truncation error $R_i^n = L_h u(x_i, t^n) - [Lu](x_i, t^n) = O(k + h^2)$.

Proof. By Tylor development, we have

$$\begin{aligned} &u(x_i, t^{n+1}) - u(x_i, t^n) = k u_t(x_i, t^n) + O(k^2), \\ &u(x_{i+1}, t^n) - 2u(x_i, t^n) + u(x_{i-1}, t^n) = h^2 u_{xx}(x_i, t^n) + O(h^4). \end{aligned}$$

Thus $R_i^n = O(k + h^2)$. $\square$

Exercise 3.3. *Consider the transport-diffusion problem*

$$\begin{aligned} u_t - u_{xx} + v u_x = 0, &\quad \forall x \in (a,b),\ t \in (0,T), \\ u(a,t) = u(b,t) = 0, &\quad t \in (0,T), \\ u(x,0) = u_0(x), &\quad \forall x \in (a,b), \end{aligned}$$

where v is a constant. Derive estimates for the truncation error and global error of the following schema, and prove that

$$\|u_h^n\|_0 \leqslant \|u_h^0\|_0, \quad \forall n = 0, 1, \ldots,$$

- *If $v \geqslant 0$,*

$$\begin{aligned} \frac{u_i^{n+1} - u_i^n}{k} - \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} + v\frac{u_i^{n+1} - u_{i-1}^{n+1}}{h} &= 0, \quad \forall i = 1, \cdots, N-1, \\ u_0^{n+1} = u_N^{n+1} &= 0, \\ u^0 &= u_0, \end{aligned}$$

- *if $v \leqslant 0$,*

$$\begin{aligned} \frac{u_i^{n+1} - u_i^n}{k} - \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} + v\frac{u_{i+1}^{n+1} - u_i^{n+1}}{h} &= 0, \quad \forall i = 1, \cdots, N-1, \\ u_0^{n+1} = u_N^{n+1} &= 0, \\ u^0 &= u_0, \end{aligned}$$

in an uniform mesh $\{x_i\}_{i=0}^N$, $x_i = a + ih$, $h = (b-a)/N$, $\{t^n\}_{n=0}^M$, $t^n = nk$, $k = T/M$.

Solution.
• Truncation Error.
Let $Lu = u_t - u_{xx} + v u_x$ and

$$L_h u_i^n = \begin{cases} \dfrac{u_i^{n+1} - u_i^n}{k} - \dfrac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} + v\dfrac{u_i^{n+1} - u_{i-1}^{n+1}}{h}, & \text{if } v \geqslant 0, \\ \dfrac{u_i^{n+1} - u_i^n}{k} - \dfrac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} + v\dfrac{u_{i+1}^{n+1} - u_i^{n+1}}{h}, & \text{if } v \leqslant 0. \end{cases}$$

Then $R_i^n = L_h u(x_i, t^n) - [Lu](x_i, t^n)$. If $v \geqslant 0$, by Tylor developments:

$$\begin{aligned}
&u(x_i, t^{n+1}) - u(x_i, t^n) = k u_t(x_i, t^n) + O(k^2),\\
&u(x_{i+1}, t^{n+1}) - 2u(x_i, t^{n+1}) + u(x_{i-1}, t^{n+1}) = h^2 u_{xx}(x_i, t^{n+1}) + O(h^4),\\
&u(x_i, t^{n+1}) - u(x_{i-1}, t^{n+1}) = h u_x(x_i, t^{n+1}) + O(h^2),
\end{aligned}$$

we have $R_i^n = O(k+h)$. The similar result can also be obtained for $v \leqslant 0$.

• Global Error.
Let $e_i^n = u(x_i, t^n) - u_i^n$. Then $L_h e_i^n = R_i^n,\ \forall i = 1, \cdots, N-1$.
- If $v \geqslant 0$, we have

$$\left(1 + \frac{2k}{h^2} + v\frac{k}{h}\right) e_i^{n+1} = \frac{k}{h^2} e_{i+1}^{n+1} + \left(\frac{k}{h^2} + v\frac{k}{h}\right) e_{i-1}^{n+1} + e_i^n + kR_i^n.$$

Multiplying both sides of the above formula by $e_i^{n+1} h$ and summing in i from 0 to N gives

$$\left(1 + \frac{2k}{h^2} + v\frac{k}{h}\right) \|e_h^{n+1}\|_0^2 = \frac{k}{h^2} \sum_{i=0}^{N-1} e_{i+1}^{n+1} e_i^{n+1} h + \left(\frac{k}{h^2} + v\frac{k}{h}\right) \sum_{i=1}^{N} e_{i-1}^{n+1} e_i^{n+1} h + \sum_{i=0}^{N} (e_i^n + kR_i^n) e_i^{n+1} h.$$

By Cauchy-Schwarz inequality, we have

$$\left(1 + \frac{2k}{h^2} + v\frac{k}{h}\right) \|e_h^{n+1}\|_0^2 \leqslant \left(\frac{2k}{h^2} + v\frac{k}{h}\right) \|e_h^{n+1}\|_0^2 + (\|e_h^n\|_0 + k\|R_h^n\|_0)\|e_h^{n+1}\|_0.$$

Thus

$$\|e_h^{n+1}\|_0 \leqslant \|e_h^n\|_0 + k\|R_h^n\|_0 \leqslant \cdots \leqslant \|e_h^0\|_0 + k\sum_{j=0}^{n} \|R_h^j\|_0 \leqslant T \max_j \|R_h^j\|_0 = O(k+h).$$

The similar result can be obtained for $v \leqslant 0$.
• Stability.
If $v \geqslant 0$, multiplying both sides of $L_h u_i^n = 0$ by $u_i^{n+1} h$ yields

$$\frac{u_i^{n+1} - u_i^n}{k} u_i^{n+1} h - \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h} u_i^{n+1} + v(u_i^{n+1} - u_{i-1}^{n+1}) u_i^{n+1} = 0.$$

Summing in i from 1 to $N-1$ gives

$$\frac{h}{k} \sum_{i=1}^{N-1} (u_i^{n+1} - u_i^n) u_i^{n+1} - \frac{1}{h} \sum_{i=1}^{N-1} (u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}) u_i^{n+1} + v \sum_{i=1}^{N-1} (u_i^{n+1} - u_{i-1}^{n+1}) u_i^{n+1} = 0.$$

The first term

$$\begin{aligned}
\sum_{i=1}^{N-1} (u_i^{n+1} - u_i^n) u_i^{n+1} &= \frac{1}{2} \sum_{i=1}^{N-1} (u_i^{n+1} - u_i^n)(u_i^{n+1} - u_i^n + u_i^{n+1} + u_i^n)\\
&\geqslant \frac{1}{2} \sum_{i=1}^{N-1} (u_i^{n+1} - u_i^n)(u_i^{n+1} + u_i^n)\\
&= \frac{1}{2h} \left(\|u_h^{n+1}\|_0^2 - \|u_h^n\|_0^2\right).
\end{aligned}$$

The second term

$$\begin{aligned}
-\frac{1}{h} \sum_{i=1}^{N-1} (u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}) u_i^{n+1} &= \frac{1}{h} \sum_{i=1}^{N-1} \left[2(u_{i+1}^{n+1})^2 - (u_{i+1}^{n+1} + u_{i-1}^{n+1}) u_i^{n+1}\right]\\
&= \frac{1}{h} \left[(u_{N-1}^{n+1})^2 + \sum_{i=1}^{N-1} \left(u_i^{n+1} - u_{i-1}^{n+1}\right)^2\right] \geqslant 0,
\end{aligned}$$

where we note that $\sum_{i=1}^{N-1} u_{i+1}^{n+1} u_i^{n+1} = \sum_{i=1}^{N-1} u_i^{n+1} u_{i-1}^{n+1}$ and

$$\sum_{i=1}^{N-1} (u_i^{n+1})^2 = (u_{N-1}^{n+1})^2 + \sum_{i=1}^{N-1} (u_{i-1}^{n+1})^2.$$

The third term

$$\begin{aligned} v \sum_{i=1}^{N-1} (u_i^{n+1} - u_{i-1}^{n+1}) u_i^{n+1} &= \frac{v}{2} \sum_{i=1}^{N-1} (u_i^{n+1} - u_{i-1}^{n+1})(u_i^{n+1} - u_{i-1}^{n+1} + u_i^{n+1} + u_{i-1}^{n+1}) \\ &\geqslant \frac{v}{2} \sum_{i=1}^{N-1} \left[(u_i^{n+1})^2 - (u_{i-1}^{n+1})^2 \right] = \frac{v}{2} (u_{N-1}^{n+1})^2 \geqslant 0. \end{aligned}$$

Thus we obtain $\|u_h^{n+1}\|_0 \leqslant \|u_h^n\|_0$, which leads to $\|u_h^n\|_0 \leqslant \|u_h^0\|_0$. For $v \leqslant 0$, a similar approach can be applied to obtain the desired result, except for the treatment of the third term:

$$\begin{aligned} v \sum_{i=1}^{N-1} (u_{i+1}^{n+1} - u_i^{n+1}) u_i^{n+1} &= \frac{-v}{2} \sum_{i=1}^{N-1} (u_i^{n+1} - u_{i+1}^{n+1})(u_i^{n+1} - u_{i+1}^{n+1} + u_i^{n+1} + u_{i+1}^{n+1}) \\ &\geqslant \frac{-v}{2} \sum_{i=1}^{N-1} \left[(u_i^{n+1})^2 - (u_{i+1}^{n+1})^2 \right] = \frac{-v}{2} (u_1^{n+1})^2 \geqslant 0. \end{aligned}$$

$\square$