Exercise 1. *Let* $\{x_n\}_{n=0}^{N+1}$ *be a grid in the interval* $\Lambda = (0,1)$, *i.e.,* $0 = x_0 < x_1 < x_2 < \cdots < x_N < x_{N+1} = 1$. *Let* $I_n = (x_{n-1}, x_n)$, $h_n = x_n - x_{n-1}$, *and* $h = \max_{1\leqslant n\leqslant N+1} h_n$. *Prove*

$$\{v \in C^0(\Lambda) : v|_{I_n} \in H^1(I_n), n = 1, \cdots, N+1\} \subset H^1(\Lambda).$$

Proof. For any $v \in \{v \in C^0(\Lambda) : v|_{I_n} \in H^1(I_n), n = 1, \cdots, N+1\}$, it is clear that $v \in L^2(\Lambda)$ because continuity implies square integrability on the bounded domain Λ. It remains to show that the weak derivative of v also belongs to $L^2(\Lambda)$. Since $v|_{I_n} \in H^1(I_n)$, we define a piecewise derivative by

$$g|_{I_n}(x) = (v|_{I_n})'(x), \quad x \in I_n,\ n = 1, \cdots, N+1.$$

Obviously, $g \in L^2(\Lambda)$, as each piece $(v|_{I_n})' \in L^2(I_n)$ and the intervals I_n are disjoint and cover Λ. We claim that g is the derivative of v. Indeed, for any test function $\phi(x) \in C_0^\infty(\Lambda)$, we have

$$\begin{aligned}
\int_0^1 g(x)\phi(x)\mathrm{d}x &= \sum_{n=1}^{N+1} \int_{I_n} g|_{I_n}(x)\phi(x)\mathrm{d}x = \sum_{n=1}^{N+1} \int_{I_n} (v|_{I_n})'(x)\phi(x)\mathrm{d}x \\
&= \sum_{n=1}^{N+1} [v(x)\phi(x)]\big|_{x_{n-1}}^{x_n} - \sum_{n=1}^{N+1} \int_{I_n} (v|_{I_n})(x)\phi'(x)\mathrm{d}x \\
&= \sum_{n=1}^{N+1} \big(v(x_n^-)\phi(x_n^-) - v(x_{n-1}^+)\phi(x_{n-1}^+)\big) - \sum_{n=1}^{N+1} \int_{I_n} (v|_{I_n})(x)\phi'(x)\mathrm{d}x.
\end{aligned}$$

Due to the continuity of v across element interfaces, we have $v(x_n^-) = v(x_n^+)$ for $n = 1, \cdots, N$, and since $\phi \in C_0^\infty(\Lambda)$ we have $\phi(x_0) = \phi(x_{N+1}) = 0$. Hence, the sum of boundary terms cancels out, yielding

$$\int_0^1 g(x)\phi(x)\mathrm{d}x = -\int_0^1 v(x)\phi'(x)\mathrm{d}x,$$

which confirms that g is the weak derivative of v. Therefore, $v \in H^1(\Lambda)$. □