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ON INTERPOLATION APPROXIMATION: CONVERGENCE RATES FOR POLYNOMIAL INTERPOLATION FOR FUNCTIONS OF LIMITED REGULARITY*

SHUHUANG XIANG†

Abstract. The convergence rates on polynomial interpolation in most cases are estimated by Lebesgue constants. These estimates may be overestimated for some special points of sets for functions of limited regularities. In this paper, by applying the Peano kernel theorem and Wainerman's lemma, new formulas on the convergence rates are considered. Based upon these new estimates, it shows that the interpolation at strongly normal point systems can achieve the optimal convergence rates, the same as the best polynomial approximation. Furthermore, by using the asymptotics on Jacobi polynomials, the convergence rates are established for Gauss–Jacobi, Jacobi–Gauss–Lobatto, or Jacobi–Gauss–Radau point systems. From these results, we see that the interpolations at the Gauss–Legendre, Legendre–Gauss–Lobatto point systems, or at strongly normal point systems, have essentially the same approximation accuracy compared with those at the two Chebyshev point systems, which also illustrates the equal accuracy of the Gauss and Clenshaw–Curtis quadratures. In addition, numerical examples illustrate the perfect coincidence with the estimates, which means the convergence rates are optimal.

Key words. polynomial interpolation, Peano kernel, convergence rate, limited regularity, strongly normal point system, Gauss–Jacobi point, Jacobi–Gauss–Lobatto point, Chebyshev point

AMS subject classifications. 65D05, 65D25

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1. Introduction.

A central problem in approximation theory is the construction of simple functions that are easily implemented on computers and approximate well a given set of functions.

There exist many investigations for the behavior of continuous functions approximated by polynomials. Weierstrass [75] in 1885 proved the well-known result that every continuous function $f(x)$ in $[-1,1]$ can be uniformly approximated as closely as desired by a polynomial function. This result has both practical and theoretical relevance, especially in polynomial interpolation.

Polynomial interpolation is a fundamental tool in many areas of scientific computing. Lagrange interpolation is a classical technique for approximation of continuous functions. Let us denote by

$$-1 \le x_n^{(n)} < x_{n-1}^{(n)} < \cdots < x_2^{(n)} < x_1^{(n)} \le 1 \tag{1.1}$$

the n distinct points in the interval $[-1,1]$, and let $f(x)$ be a function defined in the same interval. The nth Lagrange interpolation polynomial of $f(x)$ is unique and given by the formula

$$L_n[f] = \sum_{k=1}^{n} f(x_k^{(n)})\ell_k^{(n)}(x), \quad \ell_k^{(n)}(x) = \frac{\omega_n(x)}{\omega_n'(x_k^{(n)})(x - x_k^{(n)})}, \tag{1.2}$$

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http://www.siam.org/journals/sinum/54-4/M102528.html
†School of Mathematics and Statistics, Central South University, Changsha, Hunan 410083, People's Republic of China (xiangsh@mail.csu.edu.cn).

where $\omega_n(x) = (x - x_1^{(n)})(x - x_2^{(n)}) \cdots (x - x_n^{(n)})$.

There is a well-developed theory that quantifies the convergence or divergence of the Lagrange interpolation polynomials (Brutman [9, 10] and Trefethen [61]). Two key notions for interpolation in a given set of points are that of the *Lebesgue function*

$$\lambda_n(x) = \sum_{k=1}^{n} \left|\ell_k^{(n)}(x)\right| \tag{1.3}$$

and the *Lebesgue constant*

$$\Lambda_n = \max_{x\in[-1,1]} \lambda_n(x), \tag{1.4}$$

which are of fundamental importance (Cheney [11], Davis [14], and Szegö [57]). The Lebesgue constant can also be interpreted as the ∞-norm of the projection operator $L_n : C([-1,1]) \to \mathcal{P}_{n-1}$,

$$\Lambda_n = \sup_{f\neq 0} \frac{\|L_n[f]\|_\infty}{\|f\|_\infty},$$

where $\mathcal{P}_{n-1}$ is the set of polynomials of degree less than or equal to $n-1$.

Based upon the Lebesgue constant, the interpolation error can be estimated by

$$\left\|L_n[f] - f\right\|_\infty \le (1+\Lambda_n)\left\|p_{n-1}^* - f\right\|_\infty, \tag{1.5}$$

where p_{n-1}^* is the best polynomial approximation of degree $n-1$. Thus, the Lebesgue constant Λ_n indicates how good the interpolant $L_n[f]$ is in comparison with the best polynomial approximation p_{n-1}^*.

The study of the Lebesgue constant Λ_n originated more than 100 years ago. Comprehensive reviews can be found in, e.g., Brutman [10], Lubinsky [43], and Trefethen [61, Chapter 15]. For an arbitrarily given system of points $\{x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)}\}_{n=1}^\infty$, Bernstein [3] and Faber [21] in 1914 obtained that

$$\Lambda_n \ge \frac{1}{12}\log n,$$

which, together with the boundedness principle, implies that there exists a continuous function $f(x)$ in $[-1,1]$ for which the sequence $L_n[f]$ $(n = 1, 2, \ldots)$ is not uniformly convergent to f in $[-1,1]$.[1] More precisely, Erdös [18] and Brutman [9] proved that

$$\Lambda_n \ge \frac{2}{\pi}\log n + C \ \text{ for some constant } C \quad [18]; \quad \Lambda_n \ge \frac{2}{\pi}\left(\gamma_0 + \log\frac{4}{\pi}\right) + \frac{2}{\pi}\log n \quad [9], \tag{1.6}$$

where $\gamma_0 = 0.577\ldots$ is Euler's constant. In particular, for the equidistant point system

$$\left\{x_k^{(n)} = -1 + \frac{2k}{n-1}\right\}_{k=0}^{n-1},$$

[1]Grünwald [28] in 1935 and Marcinkiewicz [44] in 1937 independently showed that even for the Chebyshev points of the first kind,

$$x_k^{(n)} = \cos\left(\frac{2k-1}{2n}\pi\right), \quad k = 1, 2, \ldots, n, \quad n = 1, 2, \ldots,$$

there is a continuous function $f(x)$ in $[-1,1]$ for which the sequence $L_n[f]$ is divergent everywhere in $[-1,1]$.

Schönhage [54] showed that

$$\Lambda_n \sim \frac{2^n}{e(\log(n-1)+\gamma_0)(n-1)}, \quad n \to \infty.$$

Additionally, Trefethen and Weideman [63] established that

$$\frac{2^{n-3}}{(n-1)^2} \le \Lambda_n \le \frac{2^{n+2}}{n-1}, \quad n \ge 0.$$

Then, generally, the set of equally spaced points is a bad choice for Lagrange interpolation (see Runge [51]).

However, for well-chosen sets of points, the growth of Λ_n may be extremely slow as $n \to \infty$:

- Chebyshev point system of the first kind $T_n = \left\{x_k^{(n)} = \cos\left(\frac{2k-1}{2n}\pi\right)\right\}_{k=1}^n$: An asymptotic estimate of $\Lambda_n(T_n)$ was given by Bernstein [2] as

$$\Lambda_n(T_n) \sim \frac{2}{\pi}\log n, \quad n \to \infty, \tag{1.7}$$

  which is improved by Ehlich and Zeller [17], Rivlin [49], and Brutman [9] as

$$\frac{2}{\pi}\left(\gamma_0 + \log\frac{4}{\pi}\right) + \frac{2}{\pi}\log n < \Lambda_n(T_n) \le 1 + \frac{2}{\pi}\log n, \quad n = 1, 2, \ldots.$$

- Chebyshev point system of the second kind $U_n = \left\{x_k^{(n)} = \cos\left(\frac{k}{n-1}\pi\right)\right\}_{k=0}^{n-1}$ (also called Chebyshev extreme or Clenshaw–Curtis points [60]): Ehlich and Zeller [17] proved that

$$\Lambda_n(U_n) = \begin{cases} \Lambda_{n-1}(T_{n-1}), & n = 2, 4, 6, \ldots, \\ \Lambda_{n-1}(T_{n-1}) - \alpha_n, \quad 0 < \alpha_n < \frac{1}{(n-1)^2}, & n = 3, 5, 7, \ldots. \end{cases} \tag{1.8}$$

- The roots of Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ $(\alpha, \beta > -1)$: The asymptotic estimate of $\Lambda_n(J_n)$ was found by Szegö [57] as

$$\Lambda_n(J_n) = \begin{cases} O(n^{\gamma+\frac{1}{2}}), & \gamma > -\frac{1}{2}, \\ O(\log n), & \gamma \le -\frac{1}{2}, \end{cases} \quad \gamma = \max\{\alpha, \beta\}. \tag{1.9}$$

Comparing (1.7), (1.8), and (1.9) with (1.6), we see that the two Chebyshev point systems and the Jacobi point system with $\gamma \le -\frac{1}{2}$ are nearly optimal and of order $O(\log n)$.

Nevertheless, it is worth noting that if $f(x)$ has an absolutely continuous $(k-1)$st derivative $f^{(k-1)}$ on $[-1, 1]$ for some $k \ge 1$ and its kth derivative $f^{(k)}$ is of bounded variation $\mathrm{Var}(f^{(k)}) < \infty$, Mastroianni and Szabados [45], Trefethen [61], and Xiang, Chen, and Wang [78] proved that

$$\|f - L_n[f]\|_\infty = O(n^{-k}), \tag{1.10}$$

where $L_n[f]$ is at the n Chebyshev points of the first or second kind, which has the same asymptotic order as $\|f - p_{n-1}^*\|_\infty$ for the best approximation p_{n-1}^*, following de la Vallée Poussin [15]. In particular, for $f(x) = |x|$, the error on the $L_n[f]$ at the above two Chebyshev point systems satisfies

$$\|f - L_n[f]\|_\infty \le \frac{4}{\pi(n-1)}$$

(see [61, 78]), while

$$\|f - p_{n-1}^*\|_\infty \sim \frac{\beta}{n-1}, \quad 0.2801685 < \beta < 0.2801734$$

(see Bernstein [4] and Varga and Carpenter [64]). Thus, the error convergence rate about the order on n (1.5) by using the Lebesgue constant may be overestimated for some special points of sets for functions of limited regularities.

Moreover, it has been observed, by Clenshaw and Curtis [12] and O'Hara and Smith [46], that n-point Gauss quadrature and n-point Clenshaw–Curtis quadrature have essentially the same accuracy, which has been shown recently by Trefethen [60, 61], Brass and Petras [8], and Xiang and Bornemann [77]. Both of these two quadratures are derived from the interpolation polynomial $L_n[f]$ by

$$Q_n[f] = \int_{-1}^{1} L_n[f](x)dx,$$

based on the n Gauss–Legendre and Clenshaw–Curtis points, respectively. From this observation, we may conclude that the corresponding interpolation $L_n[f]$ based on these two point systems may have the same convergence rate. However, it cannot be derived from (1.5).

In this paper, we present new convergence rates of the interpolation polynomials for functions of limited regularities, based upon the famous Peano kernel theorem [47] and applying an interesting lemma by Wainerman [74]. Suppose $f(x)$ has an absolutely continuous $(r-1)$st derivative $f^{(r-1)}$ on $[-1,1]$, and its rth derivative $f^{(r)}$ is of bounded variation $\mathrm{Var}(f^{(r)}) < \infty$. We will show that

$$\|f - L_n[f]\|_\infty \le \frac{\pi^r \mathrm{Var}(f^{(r)})}{(n-1)(n-2)\cdots(n-r)} \max_{1\le j\le n} \|\ell_j^{(n)}\|_\infty, \tag{1.11}$$

which leads to

$$\|f - L_n[f]\|_\infty = O(n^{-r} \max_{1\le j\le n} \|\ell_j^{(n)}\|_\infty). \tag{1.12}$$

The Lebesgue constant $\Lambda_n = \max_{x\in[-1,1]} \sum_{k=1}^n \left|\ell_k^{(n)}(x)\right|$ is replaced by $\max_{1\le j\le n} \|\ell_j^{(n)}\|_\infty$ in some sense since $\|f - p_{n-1}^*\|_\infty = O(n^{-r})$ [15].

Particularly, from (1.12), it directly follows that the interpolation $L_n[f]$ at a strongly normal point system (see Fejér [22]) can achieve the optimal convergence rate as $O(\|f - p_{n-1}^*\|_\infty)$.

Furthermore, $\|\ell_j\|_\infty$ can be explicitly estimated for Gauss–Jacobi, Jacobi–Gauss–Lobatto, or Jacobi–Gauss–Radau point systems by using the asymptotics on Jacobi polynomials given by Szegö [57] and some results given in, e.g., Kelzon [37, 38], Vértesi [68, 70], Sun [56], Prestin [48], Kvernadze [41], Vecchia, Mastroianni, and Vértesi [65] as follows:

- For the n Gauss–Jacobi points,

$$\max_{1\le j\le n} \|\ell_j^{(n)}\|_\infty = O(n^{\max\{\gamma-\frac{1}{2},0\}}), \quad \gamma = \max\{\alpha,\beta\}.$$

- For the n Jacobi–Gauss–Lobatto points (the roots of $(1-x^2)P_{n-2}^{(\alpha,\beta)}(x)=0$),

$$\max_{1\le j\le n}\|\ell_j^{(n)}\|_\infty=\begin{cases} O\left(n^{-\min\{0,\alpha+\frac12,\beta+\frac12\}}\right), & -1<\alpha,\beta\le\frac32,\\ O\left(n^{-\min\{0,\alpha+\frac12,2+\alpha-\beta,\frac52-\beta\}}\right), & -1<\alpha\le\frac32,\beta>\frac32,\\ O\left(n^{-\min\{0,\beta+\frac12,2+\beta-\alpha,\frac52-\alpha\}}\right), & \alpha>\frac32,-1<\beta\le\frac32,\\ O\left(n^{-\min\{0,2+\alpha-\beta,2+\beta-\alpha,\frac52-\alpha,\frac52-\beta\}}\right), & \alpha,\beta>\frac32. \end{cases}$$

- For the n Jacobi–Gauss–Radau points $(1-x)P_{n-1}^{(\alpha,\beta)}(x)$,

$$\max_{1\le j\le n}\|\ell_j^{(n)}\|_\infty=\begin{cases} O\left(n^{-\min\{0,\alpha+\frac12,\alpha-\beta\}}\right), & -1<\alpha\le\frac12,\\ O\left(n^{-\min\{0,\frac12-\beta,\frac52-\alpha,\alpha-\beta\}}\right), & \alpha>\frac12. \end{cases}$$

- For the n Jacobi–Gauss–Radau points $(1+x)P_{n-1}^{(\alpha,\beta)}(x)$,

$$\max_{1\le j\le n}\|\ell_j^{(n)}\|_\infty=\begin{cases} O\left(n^{-\min\{0,\beta+\frac12,\beta-\alpha\}}\right), & -1<\beta\le\frac12,\\ O\left(n^{-\min\{0,\frac12-\alpha,\frac52-\beta,\beta-\alpha\}}\right), & \beta>\frac12. \end{cases}$$

From the above estimates, we see that the interpolation at the Gauss–Legendre or at the Legendre–Gauss–Lobatto point system has essentially the same approximation accuracy compared with those at the two Chebyshev point systems. All of them satisfy that $\max_{1\le j\le n}\|\ell_j^{(n)}\|_\infty=O(1)$ (for more general cases see Figure 1). In addition, the convergence rate is attainable illustrated by some functions of limited regularities.

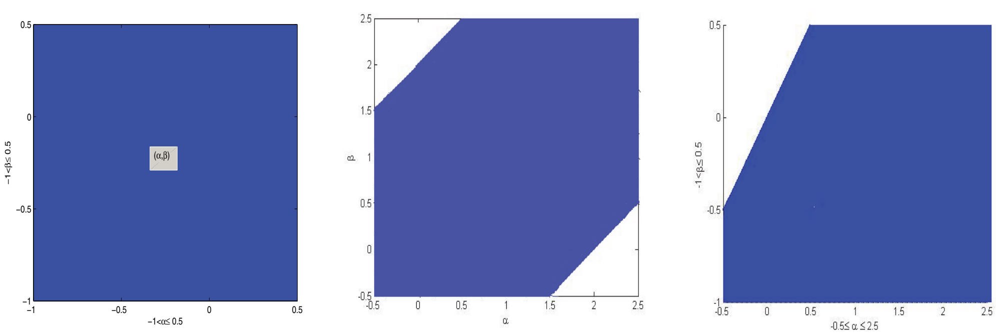


Fig. 1. *The neighborhood on* (α,β) *such that* $\|f-L_n[f]\|_\infty=O(n^{-r})$ *for the Gauss–Jacobi point systems (left), Jacobi–Gauss–Lobatto point systems (middle), and Jacobi–Gauss–Radau* ($(1-x)P_{n-1}^{(\alpha,\beta)}(x)=0$) *point systems (right).* $\mathrm{Var}(f^{(r)})<\infty$.

Thus, the best approximation polynomial is challenged by the interpolation polynomials at the special point systems shown in Figure 1. Furthermore, we will see that the interpolation polynomials at the special point systems perform much better than the best approximation polynomial for approximating the derivatives f' and f'' by $L_n'[f]$, $L_n''[f]$, $[p_{n-1}^*]'$, and $[p_{n-1}^*]''$, respectively, illustrated by numerical examples in the final section.

It is worth mentioning that the interpolation polynomial $L_n[f]$, at the Gauss–Jacobi, Jacobi–Gauss–Lobatto, or Jacobi–Gauss–Radau point system, can be efficiently evaluated by applying the second barycentric formula

$$L_n[f](x) = \frac{\sum_{j=1}^{n} \frac{\lambda_j}{x-x_j} f(x_j)}{\sum_{j=1}^{n} \frac{\lambda_j}{x-x_j}},$$

which is robust in the presence of rounding errors [36] and costs overall computational complexity $O(n)$ [5], where the nodes x_j and the barycentric weights λ_j are computed by **jacpts** and the formulas given in [32, 73, 72], respectively. A MATLAB routine **jacpts**, which uses the algorithm in [30] for the computation of these nodes and weights, can be found in the CHEBFUN system [62]. For more details on this topic, see Salzer [52], Henrici [33], Berrut and Trefethen [5], Higham [35, 36], Glaser, Liu, and Rokhlin [27], Wang and Xiang [73], Bogaert, Michiels, and Fostier [6], Hale and Trefethen [32], Hale and Townsend [30], Trefethen [61], and Wang, Huybrechs, and Vandewalle [72]. MATLAB routines can be found in the CHEBFUN system [62] and Xiang and He [79].

The paper is organized as follows: In section 2, we present the error of $f(x) - L_n[f](x)$ for each fixed $x \in [-1,1]$ by using the Peano representation and the bounded variation. In section 3, we introduce the interesting Wainerman's lemma and deduce the error bound on $\|f - L_n[f]\|_\infty$ by $\max_{1\le j\le n} \|\ell_j^{(n)}\|_\infty$. We consider in section 4 the estimates of $\|\ell_j^{(n)}\|_\infty$ and derive the convergence rates for the interpolation polynomial at strongly normal point systems, Gauss–Jacobi, Jacobi–Gauss–Lobatto, and Jacobi–Gauss–Radau point systems, respectively, where the convergence rates and attainability are illustrated by numerical experiments.

Throughout this paper, $A \sim B$ means that there exist positive constants C_1 and C_2 such that

$$C_1 B \le A \le C_2 B.$$

For simplicity, in the following we abbreviate $x_k^{(n)}$ as x_k and $\ell_k^{(n)}(x)$ as $\ell_k(x)$.

All the numerical results in this paper are carried out using MATLAB R2012a on a desktop (2.8 GB RAM, 2 Core2 (32 bit) processors at 2.80 GHz) with Windows XP operating system.

2. The Peano kernel theorem.

There are two general methods for deriving strict error bounds (Dahlquist and Björck [13]). One applies the norms and distance formula together with the Lebesgue constants, which often overestimates the error. The other is due to the Peano kernel theorem.

Suppose $\mathcal{L}$ is a continuously linear functional that maps functions $f \in C([-1,1])$ to R satisfying $\mathcal{L}(f_1 + f_2) = \mathcal{L}f_1 + \mathcal{L}f_2$ for any $f_1, f_2 \in C([-1,1])$ and $\mathcal{L}(\alpha f) = \alpha\mathcal{L}f$ for any scalar α. In addition, we assume $\mathcal{L}[\mathcal{P}_{r-1}] = \{0\}$ for some $r \in \{1, 2, \ldots\}$, where $\mathcal{P}_{r-1}$ denotes the set of polynomials with degree less than or equal to $r-1$.

The Peano kernel theorem (Peano [47]; see also Kowalewski [39], Schmidt [53], and von Mises [71]) is the identity

$$\mathcal{L}[f] = \int_{-1}^{1} f^{(r)}(t) K_r(x,t)\,dt \tag{2.1}$$

holding for all such functions $f \in C^r([-1,1])$, where $K_r(x,t) = \frac{1}{(r-1)!}\mathcal{L}[(x-t)_+^{r-1}]$

and

(2.2)
$$(x-t)_+^{r-1} = \begin{cases} (x-t)^{r-1}, & x \ge t, \\ 0, & x < t \end{cases} \quad (r \ge 2), \quad (x-t)_+^0 = \begin{cases} 1, & x \ge t, \\ 0, & x < t \end{cases} \quad (r = 1).$$

For simplicity and when no confusion can arise, we denote $K_r(t)$ instead of $K_r(x,t)$ in the following.

For each fixed $x \in [-1,1]$, we consider the special functional $\mathcal{L} = E_n$, where $E_n[f](x)$ is defined for all $f \in C([-1,1])$ by

$$E_n[f](x) = f(x) - \sum_{j=1}^{n} f(x_j)\ell_j(x) = f(x) - L_n[f](x),$$

with $-1 \le x_n < x_{n-1} < \cdots < x_2 < x_1 \le 1$. $E_n[f]$ is a continuously linear functional since $|E_n[f](x) - E_n[g](x)| \le (1+\Lambda_n)\|f-g\|_\infty$ for arbitrary $f, g \in C([-1,1])$, and then by the Peano theorem [47] $E_n[f]$ can be represented if $f \in C^r([-1,1])$ for $n \ge r$ as

$$E_n[f](x) = \int_{-1}^{1} f^{(r)}(t)K_r(t)dt, \tag{2.3}$$

with

$$K_r(t) = \frac{1}{(r-1)!}(x-t)_+^{r-1} - \frac{1}{(r-1)!}\sum_{j=1}^{n}(x_j-t)_+^{r-1}\ell_j(x). \tag{2.4}$$

Particularly, from (2.3) this implies that

$$|E_n[f](x)| \le \|f^{(r)}\|_\infty \int_{-1}^{1} |K_r(t)|dt \le 2\|f^{(r)}\|_\infty \|K_r\|_\infty.$$

Similarly to the Peano kernel for quadrature [8], the kernel for interpolation satisfies the following proposition.

Proposition 2.1 (Peano representation). *Let*

$$K_s(t) = \frac{1}{(s-1)!}(x-t)_+^{s-1} - \frac{1}{(s-1)!}\sum_{j=1}^{n}(x_j-t)_+^{s-1}\ell_j(x), \quad s = 1, 2, \ldots. \tag{2.5}$$

Then for $s \ge 2$ the Peano kernel can be rewritten as

$$K_s(u) = \int_{u}^{1} K_{s-1}(t)dt, \quad s = 2, 3, \ldots, \tag{2.6}$$

and satisfies $K_s(1) = 0$ for $s = 2, 3, \ldots$ and $K_s(-1) = 0$ for $1 \le s \le n$.

Proof. From the definition of K_s in (2.5), it is easy to verify that $K_s(1) = 0$ for $x \in [-1,1]$ and $s = 2, 3, \ldots$. The identity $K_s(-1) = 0$ for $1 \le s \le n$ follows from $\sum_{j=1}^{n}(x_j+1)^{s-1}\ell_j(x) \equiv (x+1)^{s-1}$ for $x \in [-1,1]$.

Furthermore, from (2.5) we find that for $s > 2$

$$\frac{\partial K_s(u)}{\partial u} = -K_{s-1}(u) \tag{2.7}$$

since K_s is differentiable in this case, which yields (2.6). For $s = 2$, formula (2.6) is still valid since K_2 is piecewise differentiable in u. ∎

In the remainder of the article, functions of limited regularities will be considered as follows:

(2.8)

Suppose that $f(t)$ has an absolutely continuous $(r-1)$st derivative $f^{(r-1)}$ on $[-1,1]$ for some $r \geq 1$ with $f^{(r-1)}(t) = f^{(r-1)}(-1) + \int_{-1}^{t} g(y)dy$, where g is absolutely integrable and of bounded variation $\mathrm{Var}(g) < \infty$ on $[-1,1]$.

From Stein and Shakarchi [55, p. 130] and Tao [58, pp. 143–145], we see that a function $G : [-1,1] \to R$ is absolutely continuous if and only if it takes the form $G(t) = \int_{-1}^{t} g(y)dy + C$ for some absolutely integrable $g : [-1,1] \to R$ and a constant C. It is obvious that such g is not unique with respect to measure-0 sets. Then in this paper we suppose $f(t)$ satisfies (2.8) and define

$$V_r = \inf\left\{\mathrm{Var}(g)\,\middle|\, \begin{array}{l} f^{(r-1)}(t) = f^{(r-1)}(-1) + \int_{-1}^{t} g(y)dy \text{ for all } t \in [-1,1] \text{ with } g \\ \text{being absolutely integrable and of bounded variation} \end{array}\right\}.$$

Remark 1. Here, we use the condition "$f^{(r-1)}(t) = f^{(r-1)}(-1)+\int_{-1}^{t} g(y)dy$, where g is absolutely integrable and of bounded variation $\mathrm{Var}(g) < \infty$" instead of "$f^{(r)}$ is of bounded variation $V_r = \mathrm{Var}(f^{(r)}) < \infty$" in [60, 61]. If $f^{(r)}$ is of bounded variation, then $f^{(r+1)}$ exists almost everywhere and $f^{(r+1)} \in L^1([-1,1])$ (see Lang [42] and Rudin [50]). On the other hand, $f^{(r)}$ in [60, 61] denotes an equivalent representation in the sense of almost everywhere. An example for $f(t) = |t|$ is given in [60, 61], where $f(t)$ is not differentiable at $t = 0$, but f' can be chosen as

$$f'(t) = \begin{cases} 1, & t > 0, \\ c, & t = 0, \\ -1, & t < 0; \end{cases}$$

then $\mathrm{Var}(f') = \left\{\begin{smallmatrix} 2, & |c|\leq 1, \\ |1+c|+|1-c| & \text{otherwise} \end{smallmatrix}\right.$. Using the new condition, we see that $|t|$ can be represented as $|t| = 1 + \int_{-1}^{t} g(y)dy$ with

$$g(y) = \begin{cases} 1, & y > 0, \\ c, & y = 0, \\ -1, & y < 0, \end{cases}$$

and $V_1 = 2$ is unique.

THEOREM 2.2. *Suppose $f(t)$ satisfies* (2.8)*; then for $n \geq r$ we have*

$$\|E_n[f]\|_\infty \leq V_r \|K_{r+1}\|_\infty. \tag{2.9}$$

Proof. For each fixed $x \in [-1,1]$, applying the Taylor expansion of f at -1 for $0 \leq s \leq r-1$,

$$\begin{aligned} f(x) = f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2!}(x+1)^2 + \cdots + \frac{f^{(s)}(-1)}{s!}(x+1)^s \\ + \frac{1}{s!}\int_{-1}^{x} f^{(s+1)}(t)(x-t)^s dt, \end{aligned}$$

we have

$$E_n[f](x) = \frac{1}{s!}\int_{-1}^{x} f^{(s+1)}(t)(x-t)^s dt - \sum_{j=1}^{n}\frac{1}{s!}\int_{-1}^{x_j} f^{(s+1)}(t)(x_j-t)^s dt\ell_j(x)$$
$$= \int_{-1}^{1} f^{(s+1)}(t)K_{s+1}(t)dt,$$

and then

$$E_n[f](x) = \int_{-1}^{1} g(t)K_r(t)dt.$$

Directly following Brass and Petras [8], g can be written as $g = g_1 - g_2$, where g_1 and g_2 are monotonically increasing, and $\mathrm{Var}(g) = \mathrm{Var}(g_1) + \mathrm{Var}(g_2)$ (see Lang [42, pp. 280–281]). Without loss of generality, assume g is monotonically increasing. Then by the second mean value theorem of integral calculus, it follows from $K_{r+1}(-1) = \int_{-1}^{1} K_r(t)dt = 0$ that there exists a $\xi \in [-1,1]$ such that

$$E_n[f](x) = g(-1)\int_{-1}^{\xi} K_r(t)dt + g(1)\int_{\xi}^{1} K_r(t)dt = (g(1)-g(-1))K_{r+1}(\xi)$$
$$= \mathrm{Var}(g)K_{r+1}(\xi),$$

which leads to the desired result. $\square$

Lemma 2.3 (see [8, Lemma 5.7.1]). *Assume that*

$$\sup_{-1\le t\le 1} w(t)\sqrt{1-t^2} < \infty, \quad z_u(y) = \begin{cases} 0, & y < u, \\ 1, & y \ge u, \end{cases}$$

where $w(x)$ satisfies $w(x) \ge 0$ and $\int_{-1}^{1} w(x)dx > 0$, and the values of $w(t)\sqrt{1-t^2}$ at $t = \pm 1$ are defined by the right limit $\lim_{t\to -1^+} w(t)\sqrt{1-t^2}$ and the left limit $\lim_{t\to 1^-} w(t)\sqrt{1-t^2}$, respectively. Then, for every positive integer ℓ and every $u \in [-1,1]$, there is a $q_u \in \mathcal{P}_\ell$ satisfying

$$q_u(y) \ge z_u(y) \quad \text{for all } y \in [-1,1]$$

and

$$\int_{-1}^{1} [z_u(y) - q_u(y)]\, w(y)dy \ge -\frac{\pi}{\ell+1}\sup_{-1\le t\le 1} w(t)\sqrt{1-t^2}.$$

Lemma 2.4.

$$|K_{s+1}(u)| \le \frac{\pi}{n-s}\sup_{-1\le t\le 1}|K_s(t)|. \tag{2.10}$$

Proof. In Lemma 2.3, letting $\ell = n-s-1$, $w(t) \equiv 1$, representing q_u as $q_u(t) = p_{n-1}^{(s)}(t)$, and noting that $E_n[\mathcal{P}_{n-1}] = 0$, by Theorem 2.2 we have

$$0 = E_n[p_{n-1}] = \int_{-1}^{1} p_{n-1}^{(s)}(t)K_s(t)dt = \int_{-1}^{1} q_u(t)K_s(t)dt.$$

Consequently, by Lemma 2.3 we get that

$$\Big|K_{s+1}(u)\Big| = \Big|\int_{u}^{1} K_s(t)dt\Big| = \Big|\int_{-1}^{1} K_s(t)z_u(t)dt\Big| = \Big|\int_{-1}^{1} K_s(t)\,[z_u(t)-q_u(t)]\,dt\Big|$$
$$\le \frac{\pi}{n-s}\sup_{-1\le t\le 1}|K_s(t)|. \qquad \square$$

From Theorem 2.2 and Lemma 2.4 we obtain the following.

THEOREM 2.5. *Suppose* $f(t)$ *satisfies* (2.8)*; then for* $n \geq r+1$

$$\|E_n[f]\|_\infty \leq \frac{\pi^r V_r}{(n-1)(n-2)\cdots(n-r)}\|K_1\|_\infty. \tag{2.11}$$

3. Wainerman's lemma. In the following, we shall focus on the estimate of $\|K_1\|_\infty$.

Notice that $\sum_{j=1}^n \ell_j(t) \equiv 1$ for $t \in [-1,1]$ and

$$K_1(u) = (x-u)_+^0 - \sum_{j=1}^n (x_j-u)_+^0 \ell_j(x). \tag{3.1}$$

If $x_1 < u \leq 1$, we have $K_1(u) = 1$ for $u \leq x$, and $K_1(u) = 0$ for $u > x$, while for $-1 < u \leq x_n$, we have $K_1(u) = 0$ for $u \leq x$, and $K_1(u) = -1$ for $u > x$. Thus, in these cases we obtain

$$|K_1(u)| \leq 1 \leq \max_{1\leq j\leq n} \|\ell_j\|_\infty \tag{3.2}$$

since $\ell_j(x_j) = 1$ for $j = 1, 2, \ldots, n$.

Suppose that $x_{m+1} < u \leq x_m$ for some positive integer m; then for $u \leq x$ we get

$$K_1(u) = 1 - \sum_{j=1}^n (x_j-u)_+^0 \ell_j(x) = 1 - \sum_{j=1}^m \ell_j(x) = \sum_{j=m+1}^n \ell_j(x), \tag{3.3}$$

while for $u > x$ we have

$$K_1(u) = -\sum_{j=1}^n (x_j-u)_+^0 \ell_j(x) = -\sum_{j=1}^m \ell_j(x). \tag{3.4}$$

LEMMA 3.1 (Wainerman's lemma [74]). *Suppose* $x_{m+1} < u \leq x_m$ *for some positive integer* m*, and let*

$$a_k(u) = \begin{cases} \displaystyle\sum_{j=1}^k \ell_j(u), & k = 1, 2, \ldots, m, \\ \displaystyle\sum_{j=k}^n \ell_j(u), & k = m+1, m+2, \ldots, n, \end{cases}$$

and $a_0(u) = a_{n+1}(u) \equiv 0$*. Then it follows for* $x_{m+1} < u < x_m$ *that*

$$\operatorname{sgn}(a_k(u)) = \operatorname{sgn}(\ell_k(u)) = \begin{cases} (-1)^{m-k}, & k = 1, 2, \ldots, m, \\ (-1)^{k-m-1}, & k = m+1, m+2, \ldots, n, \end{cases} \tag{3.5}$$

and for $x_{m+1} < u \leq x_m$ *that*

$$|a_k(u)| \leq |\ell_k(u)|, \quad k = 1, 2, \ldots, n, \tag{3.6}$$

where sgn *denotes the sign function.*

Proof. The interesting result and its proof are published in Russian in [74]. For convenience and completeness, we present the proof here.

For $x_{m+1} < u < x_m$, from the definition of $\ell_k(t)$ we see that

$$\begin{aligned}\operatorname{sgn}(\ell_k(u)) &= \operatorname{sgn}\left(\frac{(u-x_1)\cdots(u-x_{k-1})(u-x_{k+1})\cdots(u-x_n)}{(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}\right)\\ &= (-1)^{1-k}\operatorname{sgn}\left((u-x_1)\cdots(u-x_{k-1})(u-x_{k+1})\cdots(u-x_n)\right),\end{aligned}$$

which directly leads to the desired result (3.5) for $\operatorname{sgn}(\ell_k(u))$ based on $k \le m$ or $k > m$, respectively.

In the following, we will show that $\operatorname{sgn}(a_k(u))$ also satisfies (3.5).

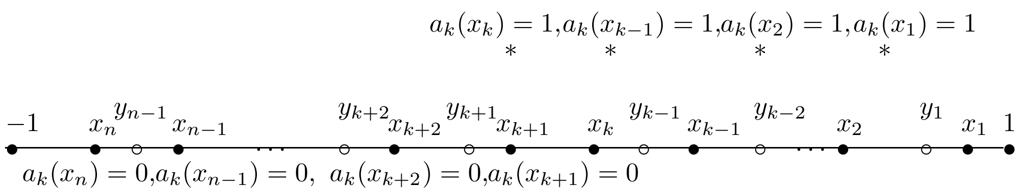
In the case $k \le m$: Since

$$a_k(x_j) = \sum_{i=1}^{k} \ell_i(x_j) = \begin{cases} 1, & j = 1, 2, \ldots, k,\\ 0, & j = k+1, k+2, \ldots, n,\end{cases}$$

then by Rolle's theorem it follows that

$$a_k'(y_j) = 0$$

for some y_j satisfying $x_{j+1} < y_j < x_j$ for $j = 1, \ldots, k-1, k+1, \ldots, n-1$.



Note that $a_k(t)$ is a polynomial of degree $n-1$; then $a_k'(t)$ is a polynomial of degree $n-2$, which implies that y_j are the exact zeros of $a_k'(t)$, and then $a_k'(t)$ has the form

$$a_k'(t) = C(t-y_1)\cdots(t-y_{k-1})(t-y_{k+1})\cdots(t-y_{n-1}) \tag{3.7}$$

for some nonzero constant C. In addition, from (3.7) $a_k'(t)$ has an alternative sign between these roots. Then, by $a_k(x_{k+1}) = 0$ and $a_k(x_k) = 1$, this yields

$$a_k'(t) > 0, \quad t \in (y_{k+1}, y_{k-1}),$$

and

$$\operatorname{sgn}(a_k(t)) = 1, \quad t \in (x_{k+1}, x_k) \subset (y_{k+1}, y_{k-1}),$$

since $a(t)$ is strictly increasing in (x_{k+1}, x_k) and $a_k(x_{k+1}) = 0$.

By the alternative property of $a_k'(t)$ between these roots, it deduces that $\operatorname{sgn}(a_k'(t)) = (-1)^{j-k}$ for $t \in (y_{j+1}, y_j)$ and $j > k$, particularly,

$$\operatorname{sgn}(a_k'(t)) = 1, \quad t \in (y_{k+1}, x_{k+1}) \subset (y_{k+1}, y_{k-1}),$$

and

$$\operatorname{sgn}(a_k'(t)) = -1, \quad t \in (x_{k+2}, y_{k+1}) \subset (y_{k+2}, y_{k+1}),$$

which, together with $a_k(x_{k+1}) = a_k(x_{k+2}) = 0$, derives $\operatorname{sgn}(a_k(t)) = -1$ for $t \in (x_{k+2}, x_{k+1})$. Similarly, applying

$$\operatorname{sgn}(a_k'(t)) = (-1)^{j-k},\ t \in (y_{j+1}, x_{j+1}), \qquad \operatorname{sgn}(a_k'(t)) = (-1)^{j-k+1},\ t \in (x_{j+2}, y_{j+1}),$$

together with $a_k(x_{j+2}) = a_k(x_{j+1}) = 0$ for $j > k$, derives $\operatorname{sgn}(a_k(t)) = (-1)^{j-k+1}$ for $j > k$ and $t \in (x_{j+2}, x_{j+1})$ by induction. So we get $a_k(u) = (-1)^{m-k}$.

In the case $k > m$: By

$$a_k(x_j) = \sum_{i=k}^{n} \ell_i(x_j) = \begin{cases} 0, & j = 1, 2, \ldots, k-1, \\ 1, & j = k, k+1, \ldots, n, \end{cases}$$

applying similar arguments derives $a_k(u) = (-1)^{k-m-1}$ for $k > m$.

Furthermore, from (3.5) and the definition of $a_k(t)$, we see that immediately: for $k \le m$ and $x_{m+1} < u < x_m$,

$$|a_k(u)| = |\ell_k(u) + a_{k-1}(u)| = |\ell_k(u)| - |a_{k-1}(u)| \le |\ell_k(u)|,$$

and for $k > m$

$$|a_k(u)| = |\ell_k(u) + a_{k+1}(u)| = |\ell_k(u)| - |a_{k+1}(u)| \le |\ell_k(u)|.$$

The special case of (3.6) for $u = x_m$ directly follows from $|a_k(x_m)| = |\ell_k(x_m)|$ by the definitions of $a_k(u)$ and $\ell_k(u)$. ∎

Theorem 2.5 together with (3.2), (3.3), (3.4), and (3.6) leads to the following estimate.

THEOREM 3.2. *Suppose $f(t)$ satisfies* (2.8)*; then for $n \ge r+1$*

$$\|E_n[f]\|_\infty \le \frac{\pi^r V_r}{(n-1)(n-2)\cdots(n-r)} \max_{1 \le j \le n} \|\ell_j\|_\infty. \tag{3.8}$$

In the next section, we shall focus on estimates of $\max_{1\le j\le n} \|\ell_j\|_\infty$ for special points of sets.

4. Estimates of $\|\ell_j\|_\infty$ and convergence rates on $\|f - L_n[f]\|_\infty$. For any convergent quadrature derived from polynomial interpolation at the grid points (1.1) for

$$\int_{-1}^{1} f(x)w(x)dx = \int_{-1}^{1} f(x)d\sigma(x)$$

for each $\sigma(x)$ of bounded variation and any analytic function $f(x)$ on $[-1, 1]$, the clustering of the n points has a limiting Chebyshev distribution

$$\mu(t) = \frac{1}{\pi}\int_{-1}^{t} \frac{1}{\sqrt{1-x^2}}dx$$

(see Krylov [40, Theorem 7, p. 263]); that is, the clustering will be asymptotically the same: on $[-1, 1]$, n points will be distributed with density

$$\frac{n}{\pi\sqrt{1-x^2}}$$

as n tends to infinity (see Hale and Trefethen [31] and Trefethen [60]).

Moreover, the clustering of optimal point systems for polynomial interpolation implies near endpoints ± 1 (see Ditzian and Totik [16] and [60]). (The Gauss–Jacobi-type point systems have this proposition.) The density of the zeros of orthogonal polynomials has been extensively studied in Erdös and Turán [19, 20], Gatteschi [25], and Szegö [57].

4.1. Strongly normal point systems. One of the proofs of Weierstrass's approximation theorem using interpolation polynomials was first presented by Fejér [22] in 1916 based on the Chebyshev point system of first kind $\left\{x_k = \cos\left(\frac{2k-1}{2n}\pi\right)\right\}_{k=1}^{n}$: If $f \in C([-1,1])$, then there is a unique polynomial $H_{2n-1}(f,t)$ of degree at most $2n-1$ such that $\lim_{n\to\infty} \|H_{2n-1}(f) - f\|_\infty = 0$, where $H_{2n-1}(f,t)$ is determined by

$$H_{2n-1}(f, x_k) = f(x_k), \quad H'_{2n-1}(f, x_k) = 0, \quad k = 1, 2, \ldots, n. \tag{4.1}$$

This polynomial is known as the Hermite–Fejér interpolation polynomial.

The convergence result has been extended to general Hermite–Fejér interpolation of $f(x)$ at nodes (1.1) by Grünwald [29] in 1942, upon strongly normal point systems introduced in Fejér [23]: Given, respectively, the function values $f(x_1)$, $f(x_2)$, $\ldots$, $f(x_n)$ and derivatives d_1, d_2, $\ldots$, d_n at these grids, the general Hermite–Fejér interpolation polynomial $H_{2n-1}(f)$ has the form

$$H_{2n-1}(f,t) = \sum_{k=1}^{n} f(x_k) h_k(t) + \sum_{k=1}^{n} d_k b_k(t), \tag{4.2}$$

where $h_k(t) = v_k(t)\left(\ell_k(t)\right)^2$, $b_k(t) = (t - x_k)\left(\ell_k(t)\right)^2$, and

$$v_k(t) = 1 - (t - x_k)\frac{\omega''_n(x_k)}{\omega'_n(x_k)} \quad \text{(see Fejér [24])}. \tag{4.3}$$

The point system (1.1) is called strongly normal if for all n

$$v_k(t) \geq c > 0, \quad k = 1, 2, \ldots, n, \quad t \in [-1, 1], \tag{4.4}$$

for some positive constant c. The point system (1.1) is called normal if for all n

$$v_k(t) \geq 0, \quad k = 1, 2, \ldots, n, \quad t \in [-1, 1]. \tag{4.5}$$

Fejér [23] (also see Szegö [57, p. 339]) showed that for the zeros of Jacobi polynomial $P_n^{(\alpha,\beta)}(t)$ of degree n ($\alpha > -1$, $\beta > -1$)

$$v_k(t) \geq \min\{-\alpha, -\beta\} \quad \text{for } -1 < \alpha \leq 0,\ -1 < \beta \leq 0,\ k = 1, 2, \ldots, n, \text{ and } t \in [-1, 1], \tag{4.6}$$

while for the Legendre–Gauss–Lobatto point system (the roots of $(1-t^2)P_{n-2}^{(1,1)}(t) = 0$),

$$v_k(t) \geq 1, \quad k = 1, 2, \ldots, n, \quad t \in [-1, 1] \quad \text{(Fejér [24])}. \tag{4.7}$$

These results have been extended to the Jacobi–Gauss–Lobatto point system (the roots of $(1-t^2)P_{n-2}^{(\alpha,\beta)}(t) = 0$) and the Jacobi–Gauss–Radau point system (the roots of $(1-t)P_{n-1}^{(\alpha,\beta)}(t) = 0$ or $(1+t)P_{n-1}^{(\alpha,\beta)}(t) = 0$) by Vértesi [66, 67]: for all k and $t \in [-1, 1]$,

$$v_k(t) \geq \min\{2-\alpha, 2-\beta\} \quad \text{for } \{x_k\} \bigcup \{-1, 1\} \text{ with } 1 \leq \alpha \leq 2 \text{ and } 1 \leq \beta \leq 2, \tag{4.8}$$

$$v_k(t) \geq \min\{2-\alpha, -\beta\} \quad \text{for } \{x_k\} \bigcup \{1\} \text{ with } 1 \leq \alpha \leq 2 \text{ and } -1 < \beta \leq 0, \tag{4.9}$$

$$v_k(t) \geq \min\{-\alpha, 2-\beta\} \quad \text{for } \{x_k\} \bigcup \{-1\} \text{ with } -1 < \alpha \leq 0 \text{ and } 1 \leq \beta \leq 2. \tag{4.10}$$

PROPOSITION 4.1. (i) [23, 57]. *The Gauss–Jacobi point system is strongly normal if and only if* $\max\{\alpha,\beta\}<0$.

(ii) [66, 67]. *The Jacobi–Gauss–Lobatto point system is strongly normal if and only if* $1\le\alpha<2$ *and* $1\le\beta<2$.

(iii) [66, 67]. *The Jacobi–Gauss–Radau point system including* $x_1=1$ *is strongly normal if and only if* $1\le\alpha<2$ *and* $-1\le\beta<0$*, and the Jacobi–Gauss–Radau point system including* $x_n=-1$ *is strongly normal if and only if* $-1<\alpha<0$ *and* $1\le\beta<2$.

It is worth noting that if the point system is strongly normal, then it implies $v_i(t)\ge c>0$ for all $i=1,2,\dots,n$ and $t\in[-1,1]$, and

$$1\equiv\sum_{i=1}^{n}h_i(t)=\sum_{i=1}^{n}v_i(t)\ell_i^2(t)\ge c\sum_{i=1}^{n}\ell_i^2(t) \tag{4.11}$$

(see [23]), and then

$$\|\ell_i\|_\infty\le\frac{1}{\sqrt{c}},\quad i=1,2,\dots,n.$$

THEOREM 4.2. *Suppose* $f(t)$ *satisfies* (2.8) *and* $\{x_j\}_{j=1}^n$ *is a strongly normal point system; then for* $n\ge r+1$

$$\|E_n[f]\|_\infty\le\frac{\pi^rV_r}{\sqrt{c}(n-1)(n-2)\cdots(n-r)}. \tag{4.12}$$

Following de la Vallée Poussin [15], the error bound indicates that $\|f-L_n[f]\|_\infty$ has the same asymptotic order as the estimate of $\|f-p_{n-1}^*\|_\infty$ for the interpolant at a strongly normal point system for functions of limited regularity with $V_r<\infty$ for some $r\ge1$.

To check the error bounds in Theorem 4.2 numerically, we consider two limited regularity functions: $f(x)=|x|$ ($V_1<\infty$) and $f(x)=|x|^3$ ($V_3<\infty$). All (α,β) are generated by $rand(1,2)$ except for $(\alpha,\beta)=(-0.5,-0.5)$, $(\alpha,\beta)=(0,0)$, $(\alpha,\beta)=(1,1)$, or $(\alpha,\beta)=(1.5,1.5)$.[2] Particularly, we used $-rand(1,2)$ in Figures 2–3 for strongly normal Gauss–Jacobi point systems, while $rand(1,2)+1$ in Figures 4–5 is used for strongly normal Jacobi–Gauss–Lobatto point systems. In Figures 6–7, we used $(rand(1)+1,-rand(1))$ (first row) and $(-rand(1),rand(1)+1)$ (second row) for strongly normal Jacobi–Gauss–Radau point systems, respectively.

From Figures 2–7, we see that these convergence rates conform to the estimates and are attainable.

4.2. General Gauss–Jacobi point systems. In this subsection, we will consider convergence rates for general Gauss–Jacobi point systems, which include the corresponding strongly normal point systems ($-1<\alpha,\beta<0$) as special cases.

Let $\{x_k\}_{k=1}^n$ be the roots of the Jacobi polynomial $P_n^{(\alpha,\beta)}(t)$ ($\alpha,\beta>-1$) and $x_k=\cos(\theta_k)$. Then from Szegö [57], it follows that

$$P_n^{(\alpha,\beta)}(t)=(-1)^nP_n^{(\beta,\alpha)}(-t)\qquad [57, (4.1.3)], \tag{4.13}$$

(4.14)

$$\max_{-1\le t\le1}\left|P_n^{(\alpha,\beta)}(t)\right|=\begin{cases}\begin{pmatrix}n+q\\ n\end{pmatrix}\sim n^q, & q=\max\{\alpha,\beta\}\ge-\frac12,\\ \left|P_n^{(\alpha,\beta)}(t')\right|\sim n^{-\frac12}, & q=\max\{\alpha,\beta\}<-\frac12,\end{cases}\qquad [57, (7.32.2)],$$

[2] $rand$(m, n) returns an m-by-n matrix containing pseudorandom values drawn from the standard uniform distribution on the open interval $(0,1)$.

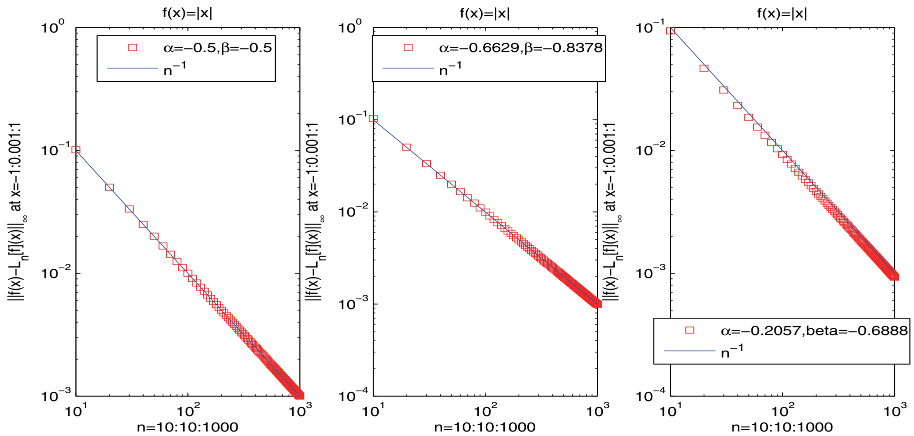


FIG. 2. $\max_{x=-1:0.001:1} |f(x) - L_n[f](x)|$ *with* $n = 10 : 10 : 1000$ *at the strongly normal Gauss–Jacobi point systems for* $f(x) = |x|$.

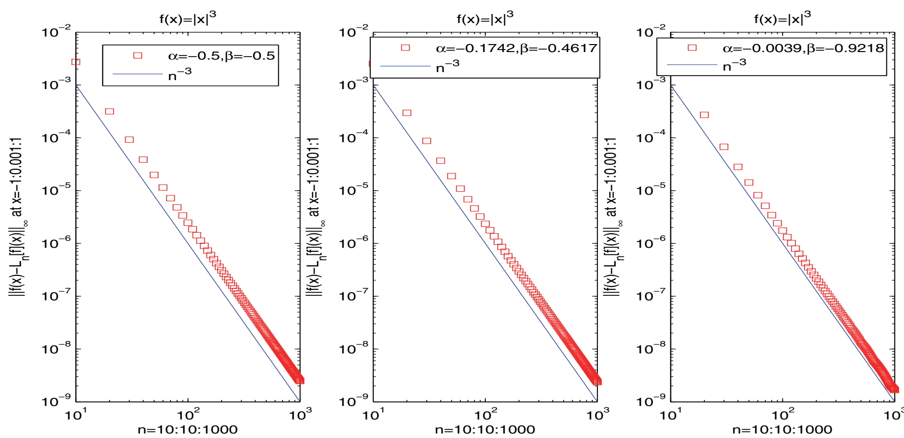


FIG. 3. $\max_{x=-1:0.001:1} |f(x) - L_n[f](x)|$ *with* $n = 10 : 10 : 1000$ *at the strongly normal Gauss–Jacobi point systems for* $f(x) = |x|^3$.

where t' is one of the two maximum points, and for $t = \cos(\theta)$ and any fixed constant c with $0 < c < 1$,

(4.15)
$$P_n^{(\alpha,\beta)}(\cos(\theta)) = \begin{cases} O\left(n^{\alpha}\right), & 0 \le \theta \le cn^{-1}, \\ \theta^{-\alpha-\frac{1}{2}} O\left(n^{-\frac{1}{2}}\right), & cn^{-1} \le \theta \le \frac{\pi}{2}, \end{cases} \qquad [57, \text{Theorem } 7.32.2],$$

$$\theta_k = n^{-1}\left[k\pi + O(1)\right] \qquad [57, (8.9.1)], \tag{4.16}$$

$$\left|P_n^{(\alpha,\beta)\prime}(\cos(\theta_k))\right| \sim k^{-\alpha-\frac{3}{2}} n^{\alpha+2}, \quad 0 < \theta_k \le \frac{\pi}{2} \qquad [57, (8.9.2)]. \tag{4.17}$$

Moreover, expression (4.17) can be extended to

$$\left|P_n^{(\alpha,\beta)\prime}(\cos(\theta_k))\right| \sim k^{-\alpha-\frac{3}{2}} n^{\alpha+2}, \quad 0 < \theta_k \le c_1\pi, \tag{4.18}$$

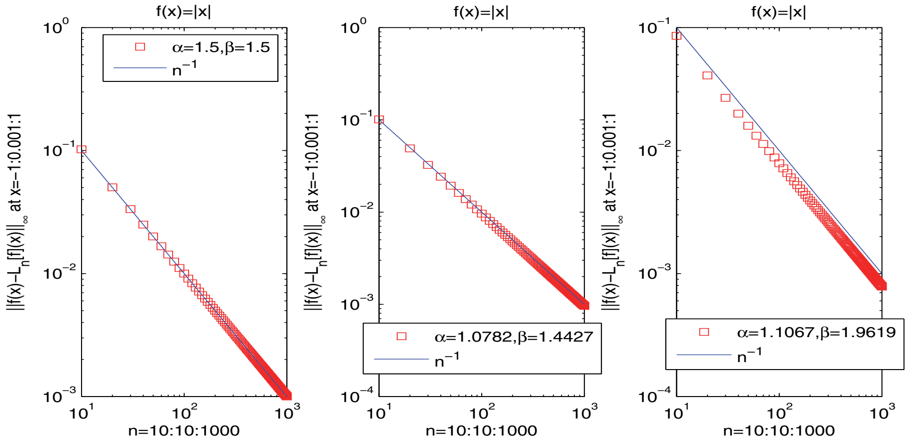


FIG. 4. $\max_{x=-1:0.001:1} |f(x) - L_n[f](x)|$ *with* $n = 10 : 10 : 1000$ *at the strongly normal Jacobi–Gauss–Lobatto point systems for* $f(x) = |x|$.

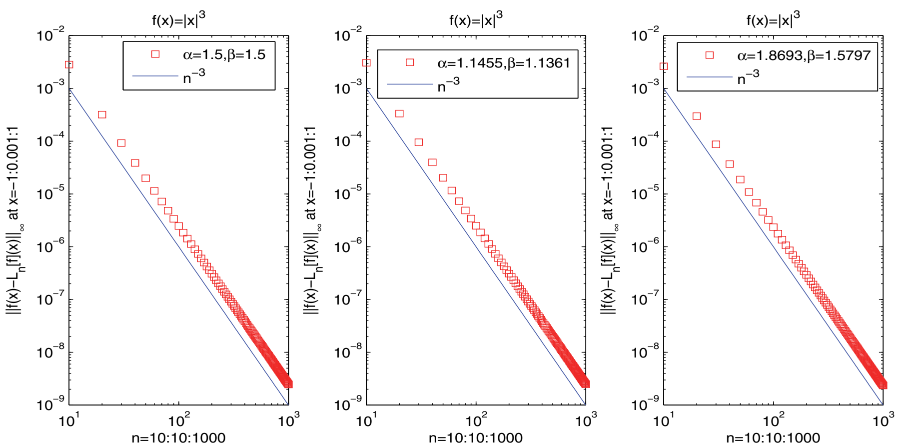


FIG. 5. $\max_{x=-1:0.001:1} |f(x) - L_n[f](x)|$ *with* $n = 10 : 10 : 1000$ *at the strongly normal Jacobi–Gauss–Lobatto point systems for* $f(x) = |x|^3$.

for any fixed c_1 with $0 < c_1 < 1$ [69, (4.6)].

Based on these identities, the estimates on $\ell_k(t) = \frac{P_n^{(\alpha,\beta)}(t)}{P_n^{(\alpha,\beta)\prime}(x_k)(t-x_k)}$ have been extensively studied in, e.g., Kelzon [37, 38], Vértesi [68, 70], Sun [56], Prestin [48], Kvernadze [41], and Vecchia, Mastroianni, and Vértesi [65].

LEMMA 4.3 (see [56]; also see [41]). *For* $t \in [-1, 1]$, *let* x_m *be the root of the Jacobi polynomial* $P_n^{(\alpha,\beta)}$ *which is closest to* t. *Then we have*

$$\ell_k(t) = \begin{cases} O\left(|k-m|^{-1} + |k-m|^{\gamma - \frac{1}{2}}\right), & k \neq m, \\ O(1), & k = m, \end{cases} \qquad \gamma = \max\{\alpha, \beta\}, \tag{4.19}$$

for $k = 1, 2, \ldots, n$.

Proof. In [56], the proof of Lemma 4.3 is given only for $0 \leq \theta_k \leq \frac{\pi}{2}$ or $k = m$.

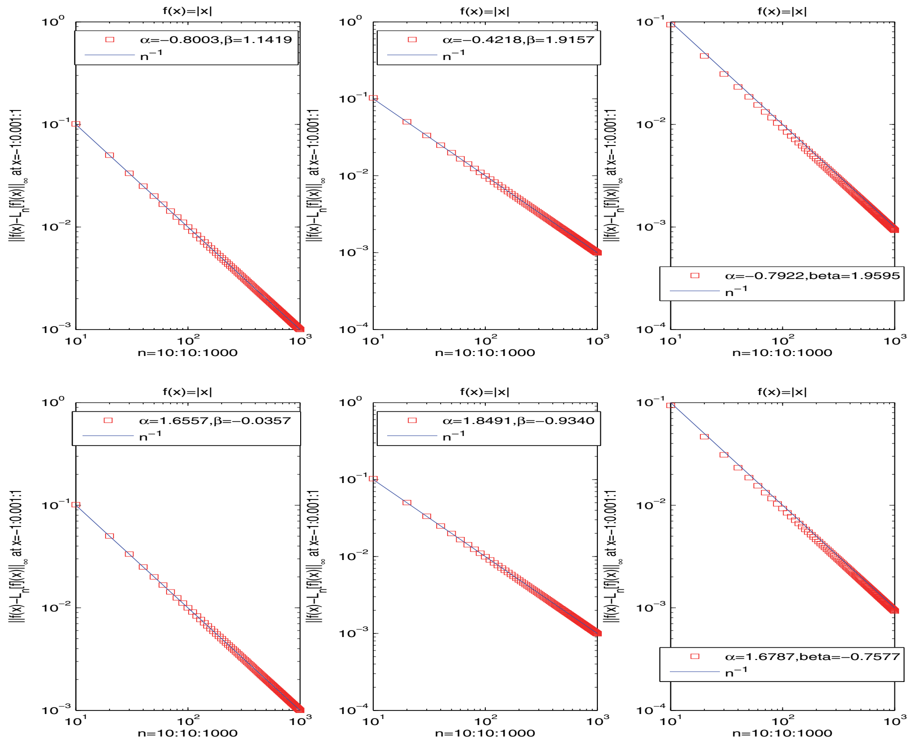


FIG. 6. $\max_{x=-1:0.001:1} |f(x) - L_n[f](x)|$ *with* $n = 10 : 10 : 1000$ *at the strongly normal Jacobi–Gauss–Radau point systems including* -1 *(first row) and* 1 *(second row) for* $f(x) = |x|$.

That proof can be readily extended to $0 \le \theta_k \le \frac{2\pi}{3}$ due to (4.18). We complement the proof for $\frac{2\pi}{3} < \theta_k < \pi$ and $k \ne m$ next.

From (4.13) and (4.18), we see that

$$\left|P_n^{(\alpha,\beta)\prime}(\cos(\theta_k))\right| \sim (n-k+1)^{-\beta-\frac{3}{2}} n^{\beta+2}, \quad \frac{2\pi}{3} < \theta_k < \pi \qquad [48, (9)]. \tag{4.20}$$

Then for $0 \le t = \cos(\theta) \le 1$ with $0 \le \theta \le cn^{-1}$ and $\frac{2\pi}{3} < \theta_k < \pi$, it follows by (4.15) and (4.20) that

$$\ell_k(t) = O\left(\frac{n^\alpha}{(n-k+1)^{-\beta-\frac{3}{2}} n^{\beta+2}}\right) = O\left(\frac{(n-k+1)^{\beta+\frac{3}{2}}}{n^{\beta+2-\alpha}}\right) = O\left(n^{\alpha-\frac{1}{2}}\right),$$

while for $cn^{-1} \le \theta \le \frac{\pi}{2}$ and $\frac{2\pi}{3} < \theta_k < \pi$, it follows by (4.15)–(4.18) and (4.20) that

$$\ell_k(t) = O\left(\frac{(m\pi/n)^{-\alpha-\frac{1}{2}} n^{-\frac{1}{2}}}{(n-k+1)^{-\beta-\frac{3}{2}} n^{\beta+2}}\right) = O\left(\frac{1}{m^{\frac{1}{2}+\alpha} n^{\frac{1}{2}-\alpha}}\right) = \begin{cases} O\left(n^{\alpha-\frac{1}{2}}\right), & \alpha > -\frac{1}{2}, \\ O\left(n^{-1}\right), & -1 < \alpha \le -\frac{1}{2}. \end{cases}$$

Thus for $0 \le t \le 1$, we have $\ell_k(t) = O\big(n^{-1} + n^{\alpha-\frac{1}{2}}\big)$ for $k \ne m$, which leads to the desired result because $k - m \sim n$ in the case $\frac{2\pi}{3} < \theta_k < \pi$.

Similarly, by (4.13) together with the above analysis, we get for $-1 \le t \le 0$ that

$$\ell_k(t) = O\left(|k-m|^{-1} + |k-m|^{\beta-\frac{1}{2}}\right), \quad k \ne m.$$

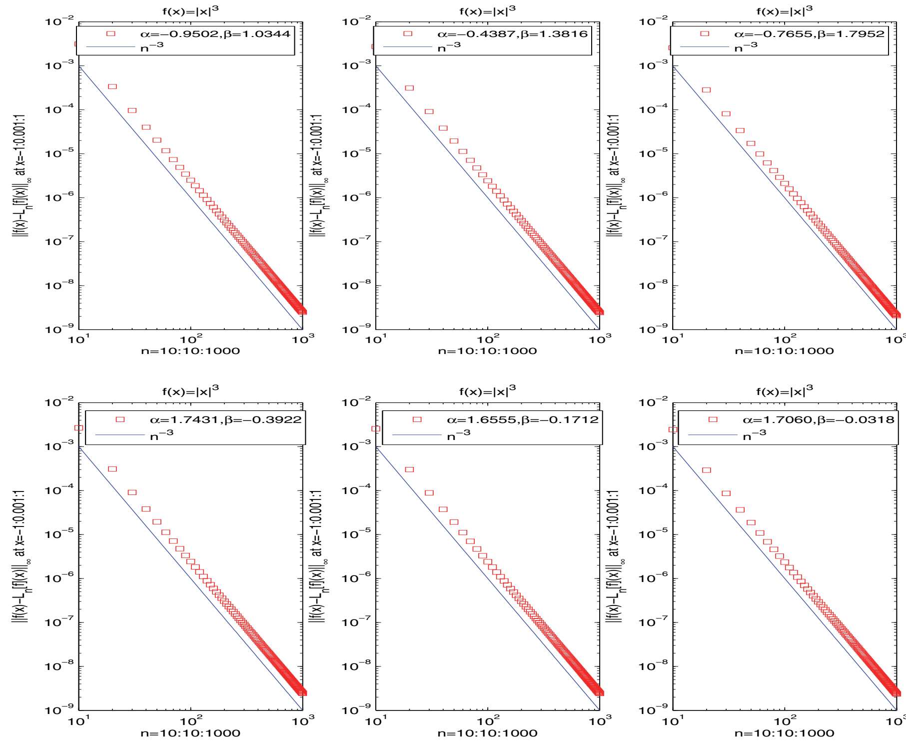


FIG. 7. $\max_{x=-1:0.001:1} |f(x) - L_n[f](x)|$ *with* $n = 10 : 10 : 1000$ *at the strongly normal Jacobi–Gauss–Radau point systems including* -1 *(first row) and* 1 *(second row) for* $f(x) = |x|^3$.

These together lead to the desired result (4.19) for $k \neq m$. ∎

THEOREM 4.4. *Suppose* $f(t)$ *satisfies* (2.8) *and* $\{x_j\}_{j=1}^n$ *are the roots of the Jacobi polynomial* $P_n^{(\alpha,\beta)}(t)$*; then for* $n \geq r+1$

$$\|E_n[f]\|_\infty = O\left(n^{-r+\max\left\{0,\gamma-\frac{1}{2}\right\}}\right), \quad \gamma = \max\{\alpha,\beta\}. \tag{4.21}$$

Proof. From Lemma 4.3, we see that $\max_{1\leq j\leq n} \|\ell_j\|_\infty = O\big(n^{\max\left\{0,\gamma-\frac{1}{2}\right\}}\big)$, which together with Theorem 3.2 yields the desired result. ∎

Remark 2. Theorem 4.4 implies that $\|f - L_n[f]\|_\infty$ has the same asymptotic order as $\|f - p_{n-1}^*\|_\infty$ [15] at the roots of the Jacobi polynomial $P_n^{(\alpha,\beta)}(t)$ for $-1 < \alpha, \beta \leq \frac{1}{2}$. Then the interpolations at the n-point Gauss–Legendre points and at the n-point Chebyshev points of first kind or second kind have essentially the same accuracy. All of them can achieve the optimal convergence rate $O(\|f - p_{n-1}^*\|_\infty)$. Consequently, the corresponding quadrature Gauss, Clenshaw–Curtis, and Fejér first rules have essentially the same accuracy [76].

Here, we used Figures 8–9 to illustrate the convergence rates for general Gauss–Jacobi point systems, where (α, β) are obtained by $rand(1,2)$ (first row) and $mrand(1,2)$ with $m\|rand(1,2)\|_\infty > m-1$ for $m = 2,3,4$ (second row), respectively. From these figures, we see that the convergence rates are attainable too, which are in accordance with the estimates. Then the convergence rates at the Gauss–Jacobi point systems are optimal.

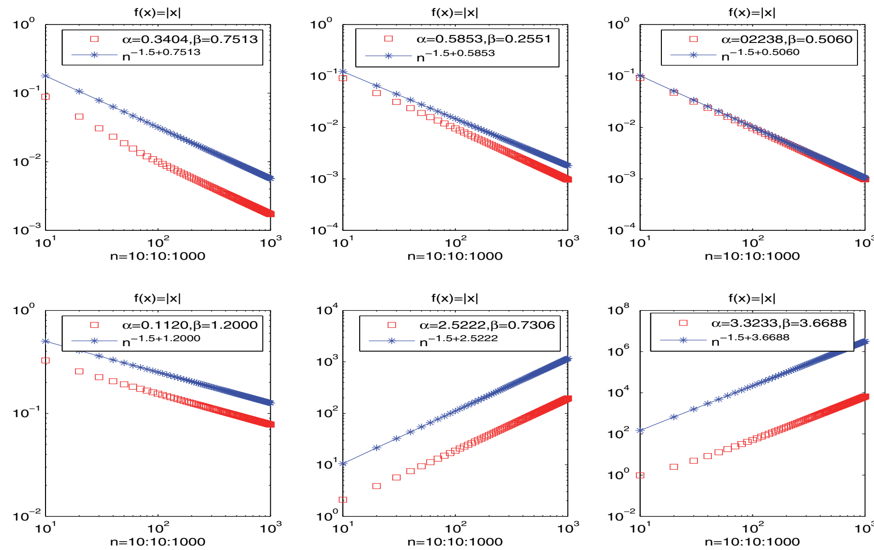


FIG. 8. $\max_{x=-1:0.001:1} |f(x) - L_n[f](x)|$ *with* $n = 10 : 10 : 1000$ *at the Gauss–Jacobi point systems for* $f(x) = |x|$.

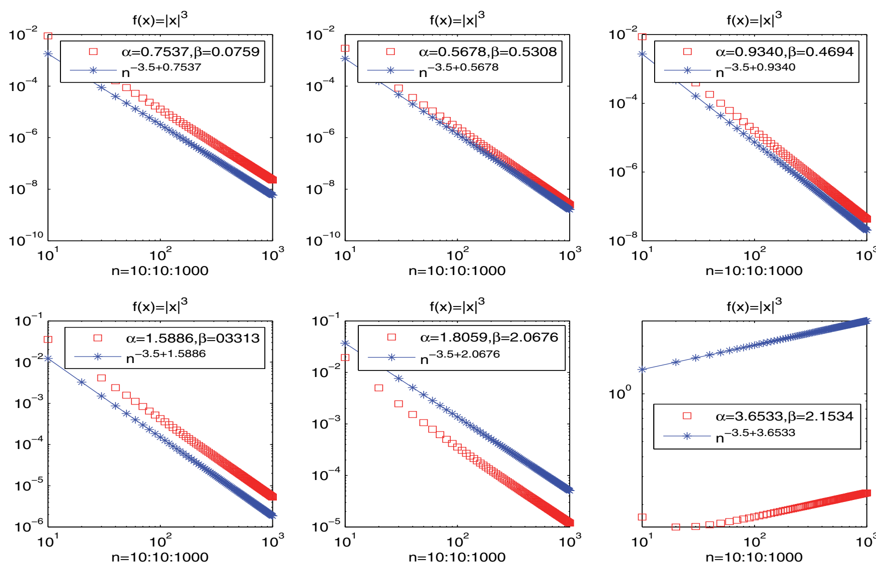


FIG. 9. $\max_{x=-1:0.001:1} |f(x) - L_n[f](x)|$ *with* $n = 10 : 10 : 1000$ *at the Gauss–Jacobi point systems for* $f(x) = |x|^3$.

Remark 3. In Figures 8–9, it is of particular relevance in the cases where the polynomial interpolations are divergent if $r - \max\left\{0, \gamma - \frac{1}{2}\right\} \le 0$ to note that the divergence rate is also controlled by the order $O\left(n^{-r+\max\left\{0,\gamma-\frac{1}{2}\right\}}\right)$ and is obtainable for some function f of finite regularity $V_r < \infty$.

4.3. General Jacobi–Gauss–Lobatto point systems. Let

$$-1 = x_{n+1} < x_n < x_{n-1} < \cdots < x_2 < x_1 < x_0 = 1 \tag{4.22}$$

be the roots of $(1-t^2)P_n^{(\alpha,\beta)}(t) = 0$ $(\alpha, \beta > -1)$, $x_k = \cos(\theta_k)$, and

$$\omega(t) = (t-x_0)(t-x_1)\cdots(t-x_n)(t-x_{n+1}), \quad \ell_k(t) = \frac{\omega(t)}{(t-x_k)\omega'(x_k)}.$$

Then

$$\ell_0(t) = \frac{1+t}{2}\cdot\frac{P_n^{(\alpha,\beta)}(t)}{P_n^{(\alpha,\beta)}(1)}, \quad \ell_{n+1}(t) = \frac{1-t}{2}\cdot\frac{P_n^{(\alpha,\beta)}(t)}{P_n^{(\alpha,\beta)}(-1)}, \tag{4.23}$$

and

$$\ell_k(t) = \frac{(1-t^2)P_n^{(\alpha,\beta)}(t)}{(t-x_k)(1-x_k^2){P_n^{(\alpha,\beta)}}'(x_k)}, \quad k = 1, 2, \ldots, n. \tag{4.24}$$

Next, we shall concentrate on estimates of $\ell_k(t)$ for $k = 0, 1, 2, \ldots, n+1$.

- On the estimate of $\ell_0(t)$:
  (i) In the case $0 \le t \le 1$, setting $t = \cos\theta$ for $0 \le \theta \le \frac{\pi}{2}$, and using

  $$P_n^{(\alpha,\beta)}(1) = \begin{pmatrix} n+\alpha \\ n \end{pmatrix} \sim n^\alpha \qquad [57, (4.1.1), (7.32.2)],$$

  we find that from (4.15) and (4.23) for $0 \le \theta \le \frac{\pi}{2}$,

  $$\ell_0(t) = \begin{cases} O(1), & 0 \le \theta \le cn^{-1}, \\ O\left(\theta^{-\alpha-\frac{1}{2}}n^{-\frac{1}{2}}n^{-\alpha}\right) = O\left((n\theta)^{-\alpha-\frac{1}{2}}\right) & \\ \quad = O\left(n^{-\min\{0,\alpha+\frac{1}{2}\}}\right), & cn^{-1} \le \theta \le \frac{\pi}{2}. \end{cases}$$

  (ii) In the case $-1 \le t \le 0$, letting $t = -\cos(\theta)$ for $0 \le \theta \le \frac{\pi}{2}$ and applying $P_n^{(\alpha,\beta)}(-\cos(\theta)) = (-1)^n P_n^{(\beta,\alpha)}(\cos(\theta))$ and $1-\cos(\theta) = 2\sin^2\left(\frac{\theta}{2}\right)$ and $\frac{2}{\pi}(\theta) \le \sin(\theta) \le \theta$, together with (4.15) and (4.23), we have

  $$\begin{aligned} \ell_0(t) &= O\left(\theta^2 P_n^{(\beta,\alpha)}(\cos(\theta))n^{-\alpha}\right) \\ &= \begin{cases} O\left(\frac{1}{n^{2+\alpha-\beta}}\right), & 0 \le \theta \le cn^{-1}, \\ O\left(\theta^{-\beta+\frac{3}{2}}n^{-\frac{1}{2}}n^{-\alpha}\right) = O\left(n^{-\min\{2+\alpha-\beta,\alpha+\frac{1}{2}\}}\right), & cn^{-1} \le \theta \le \frac{\pi}{2}. \end{cases} \end{aligned}$$

  Together these yield

  $$\|\ell_0\|_\infty = O\left(\frac{1}{n^{\min\left\{0,2+\alpha-\beta,\alpha+\frac{1}{2}\right\}}}\right). \tag{4.25}$$

- Similarly, we have

  $$\|\ell_{n+1}\|_\infty = O\left(\frac{1}{n^{\min\left\{0,2+\beta-\alpha,\beta+\frac{1}{2}\right\}}}\right). \tag{4.26}$$

- For $k = 1, 2, \dots, n$, let x_m be the nearest to $t \in [0,1]$ and $t = \cos(\theta)$. From (4.24), we have for $k \neq m$ that

$$\ell_k(t) = \frac{\sin^2\theta P_n^{(\alpha,\beta)}(\cos\theta)}{(\cos\theta - \cos\theta_k)\sin^2\theta_k P_n^{(\alpha,\beta)'}(\cos\theta_k)} = \frac{-\sin^2\theta P_n^{(\alpha,\beta)}(\cos\theta)}{2\sin\left(\frac{\theta-\theta_k}{2}\right)\sin\left(\frac{\theta+\theta_k}{2}\right)\sin^2\theta_k P_n^{(\alpha,\beta)'}(\cos\theta_k)}. \tag{4.27}$$

In the case $0 \le \theta \le cn^{-1}$ *and* $0 \le \theta_k \le \frac{2\pi}{3}$: From (4.15)–(4.18), it follows that

$$\ell_k(\cos\theta) = O\left(\frac{n^{-2}n^{\alpha}}{|k-m||k+m|n^{-2}k^2n^{-2}k^{-\alpha-\frac{3}{2}}n^{\alpha+2}}\right) = O\left(\frac{k^{\alpha-\frac{1}{2}}}{|k-m||k+m|}\right). \tag{4.28}$$

Define

$$h_1(u) = \frac{u^{\alpha-\frac{1}{2}}}{u^2-m^2} \text{ for } m+1 \le u \le n, \quad h_2(u) = -\frac{u^{\alpha-\frac{1}{2}}}{u^2-m^2} \text{ for } 1 \le u \le m-1.$$

Then by an elementary proof and noting that $m \le c_1 n$ for $0 < c_1 < 1$, we get

$$\max_{m+1\le u\le n} h_1(u) = \begin{cases} h_1(m+1) = O\left(m^{\alpha-\frac{3}{2}}\right), & -1 < \alpha \le \frac{5}{2}, \\ \max\{h_1(m+1), h_1(n)\} \\ \quad = O\left(\max\left\{m^{\alpha-\frac{3}{2}}, n^{\alpha-\frac{5}{2}}\right\}\right), & \alpha > \frac{5}{2}, \end{cases}$$

and

$$\max_{1\le u\le m-1} h_2(u) = \begin{cases} \max\{h_2(1), h_2(m-1)\} \\ \quad = O\left(\max\left\{m^{-2}, m^{\alpha-\frac{5}{2}}\right\}\right), & -1 < \alpha \le \frac{1}{2}, \\ h_2(m-1) = O\left(m^{\alpha-\frac{3}{2}}\right), & \alpha > \frac{1}{2}, \end{cases}$$

which, together with $m \sim 1$ under the assumption, establishes that

$$\ell_k(\cos\theta) = \begin{cases} O(1), & -1 < \alpha \le \frac{5}{2}, \\ O\left(n^{\alpha-\frac{5}{2}}\right), & \alpha > \frac{5}{2}. \end{cases} \tag{4.29}$$

In the case $0 \le \theta \le cn^{-1}$ *and* $\frac{2\pi}{3} < \theta_k < \pi$: Similarly, from (4.15) and (4.20) we have

$$\begin{aligned} \ell_k(\cos\theta) &= O\left(\frac{n^{-2}n^{\alpha}}{(n-k+1)^2n^{-2}(n-k+1)^{-\beta-\frac{3}{2}}n^{\beta+2}}\right) \\ &= O\left(\frac{(n-k+1)^{\beta-\frac{1}{2}}}{n^{2+\beta-\alpha}}\right) \\ &= O\left(n^{-\min\{2+\beta-\alpha, \frac{5}{2}-\alpha\}}\right). \end{aligned} \tag{4.30}$$

In the case $cn^{-1} \le \theta \le \frac{\pi}{2}$ *and* $0 \le \theta_k \le \frac{2\pi}{3}$: By (4.27), together with (4.15)–(4.18), we obtain

$$\ell_k(\cos\theta) = O\left(\frac{m^{\frac{3}{2}-\alpha}k^{\alpha-\frac{1}{2}}}{|k-m||k+m|}\right) = m^{\frac{3}{2}-\alpha}O\left(\frac{k^{\alpha-\frac{1}{2}}}{|k-m||k+m|}\right), \tag{4.31}$$

which establishes that by applying the estimates to $h_1(u)$ and $h_2(u)$

$$\ell_k(\cos\theta) = \begin{cases} O\left(n^{-\alpha-\frac{1}{2}}\right), & -1<\alpha<-\frac{1}{2}, \\ O(1), & -\frac{1}{2}\le\alpha\le\frac{5}{2}, \\ O\left(n^{\alpha-\frac{5}{2}}\right), & \alpha>\frac{5}{2}. \end{cases} \tag{4.32}$$

In the case $cn^{-1}\le\theta\le\frac{\pi}{2}$ *and* $\frac{2\pi}{3}<\theta_k<\pi$: From (4.15)–(4.18), (4.20), and (4.27), we find that

$$\begin{aligned} \ell_k(\cos\theta) &= m^{\frac{3}{2}-\alpha} O\left(\frac{(n-k+1)^{\beta-\frac{1}{2}}}{n^{2+\beta-\alpha}}\right) \\ &= \begin{cases} O\left(n^{-\min\{0,\beta+\frac{1}{2}\}}\right), & -1<\alpha\le\frac{3}{2}, \\ O\left(n^{-\min\{2+\beta-\alpha,\frac{5}{2}-\alpha\}}\right), & \alpha>\frac{3}{2}. \end{cases} \end{aligned} \tag{4.33}$$

Thus for $t\in[0,1]$, we get

$$\|\ell_k\|_\infty = \begin{cases} O\left(n^{-\min\{0,\beta+\frac{1}{2},\alpha+\frac{1}{2}\}}\right), & -1<\alpha\le\frac{3}{2}, \\ O\left(n^{-\min\{0,2+\beta-\alpha,\frac{5}{2}-\alpha\}}\right), & \alpha>\frac{3}{2}. \end{cases} \tag{4.34}$$

For $t\in[-1,0]$, by $P_n^{(\beta,\alpha)}(-t)=(-1)^n P_n^{(\alpha,\beta)}(t)$, setting $t=-\cos\theta$ and $y_k=-x_{n-k+1}=\cos\overline{\theta}_k$ for $k=1,2,\ldots,n$, we see that y_k are the roots of $P_n^{(\alpha,\beta)}(-t)=(-1)^n P_n^{(\beta,\alpha)}(t)$; then (4.24) can be represented for $k=1,2,\ldots,n$ as

$$\ell_{n-k+1}(t) = \frac{(1-\cos^2\theta)(-1)^n P_n^{(\beta,\alpha)}(\cos\theta)}{-(\cos\theta-y_k)(1-y_k^2){P_n^{(\alpha,\beta)}}'(-y_k)} = \frac{\sin^2\theta P_n^{(\beta,\alpha)}(\cos\theta)}{(\cos\theta-y_k)(1-y_k^2){P_n^{(\beta,\alpha)}}'(y_k)}.$$

By

$$\left[P_n^{(\alpha,\beta)}(t)\right]' = \frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(t) \quad [57,(4.21.7)],$$

similarly we get that

$$\|\ell_k\|_\infty = \begin{cases} O\left(n^{-\min\{0,\alpha+\frac{1}{2},\beta+\frac{1}{2}\}}\right), & -1<\beta\le\frac{3}{2}, \\ O\left(n^{-\min\{0,2+\alpha-\beta,\frac{5}{2}-\beta\}}\right), & \beta>\frac{3}{2}, \end{cases} \tag{4.35}$$

which together with (4.34) leads to the following: for $t\in[-1,1]$

$$\|\ell_k\|_\infty = \begin{cases} O\left(n^{-\min\{0,\alpha+\frac{1}{2},\beta+\frac{1}{2}\}}\right), & -1<\alpha,\beta\le\frac{3}{2}, \\ O\left(n^{-\min\{0,\alpha+\frac{1}{2},2+\alpha-\beta,\frac{5}{2}-\beta\}}\right), & -1<\alpha\le\frac{3}{2},\beta>\frac{3}{2}, \\ O\left(n^{-\min\{0,\beta+\frac{1}{2},2+\beta-\alpha,\frac{5}{2}-\alpha\}}\right), & \alpha>\frac{3}{2},-1<\beta\le\frac{3}{2}, \\ O\left(n^{-\min\{0,2+\alpha-\beta,2+\beta-\alpha,\frac{5}{2}-\alpha,\frac{5}{2}-\beta\}}\right), & \alpha,\beta>\frac{3}{2}. \end{cases} \tag{4.36}$$

THEOREM 4.5. *Suppose $f(t)$ satisfies (2.8) and $\{x_j\}_{j=0}^{n+1}$ are the roots of $(1-t^2)P_n^{(\alpha,\beta)}(t)$; then for $n \geq r+1$*

$$E_n[f] = n^{-r} \cdot \begin{cases} O\left(n^{-\min\{0,\alpha+\frac{1}{2},\beta+\frac{1}{2}\}}\right), & -1 < \alpha,\beta \leq \frac{3}{2}, \\ O\left(n^{-\min\{0,\alpha+\frac{1}{2},2+\alpha-\beta,\frac{5}{2}-\beta\}}\right), & -1 < \alpha \leq \frac{3}{2}, \beta > \frac{3}{2}, \\ O\left(n^{-\min\{0,\beta+\frac{1}{2},2+\beta-\alpha,\frac{5}{2}-\alpha\}}\right), & \alpha > \frac{3}{2}, -1 < \beta \leq \frac{3}{2}, \\ O\left(n^{-\min\{0,2+\alpha-\beta,2+\beta-\alpha,\frac{5}{2}-\alpha,\frac{5}{2}-\beta\}}\right), & \alpha,\beta > \frac{3}{2}. \end{cases} \tag{4.37}$$

Particularly, we have for $(\alpha,\beta) \in S$

$$\|E_n[f]\|_\infty = O\left(\frac{1}{n^r}\right), \tag{4.38}$$

where $S := \left[-\frac{1}{2},\frac{5}{2}\right] \times \left[-\frac{1}{2},\frac{5}{2}\right] - \{(\alpha,\beta) : -\frac{1}{2} \leq \alpha \leq \frac{1}{2},\ 2+\alpha < \beta \leq \frac{5}{2}\} \cup \{(\alpha,\beta) : \frac{3}{2} \leq \alpha \leq \frac{5}{2},\ -\frac{1}{2} \leq \beta < 2-\alpha\}$.

Remark 4. Theorem 4.5 implies that $\|f - L_n[f]\|_\infty$ has the same asymptotic order as $\|f - p_{n-1}^*\|_\infty$ [15] at the roots of the Jacobi polynomial $(1-t^2)P_n^{(\alpha,\beta)}(t)$ for $(\alpha,\beta) \in S$, which includes the corresponding strongly normal point systems as special cases.

Figures 10–11 show the convergence rates for $f(x) = |x|$ or $f(x) = |x|^3$ at the Jacobi–Gauss–Lobatto point systems, respectively, where each (α,β) is generated by $2rand(1,2) - 0.5$.

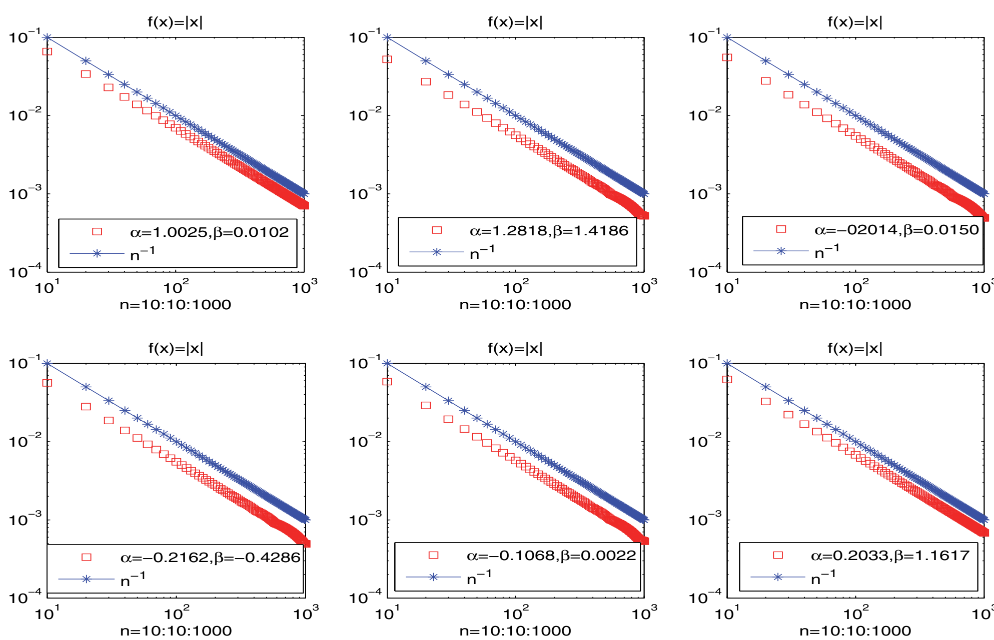


FIG. 10. $\max_{x=-1:0.001:1} |f(x) - L_n[f](x)|$ *with* $n = 10 : 10 : 1000$ *at the Jacobi–Gauss–Lobatto point systems for* $f(x) = |x|$.

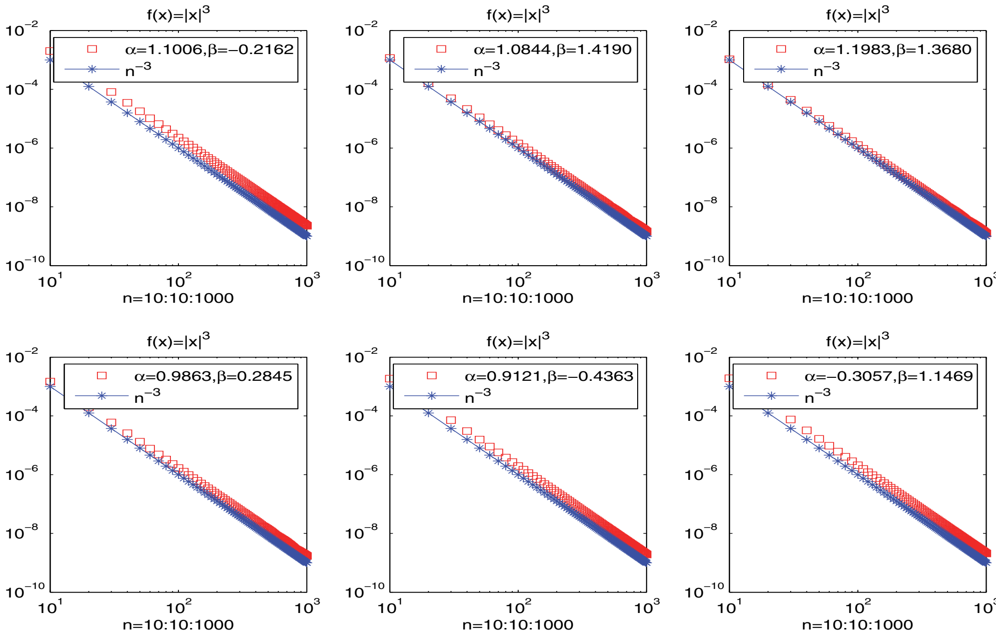


FIG. 11. $\max_{x=-1:0.001:1} |f(x) - L_n[f](x)|$ *with* $n = 10 : 10 : 1000$ *at the Jacobi–Gauss–Lobatto point systems for* $f(x) = |x|^3$.

4.4. General Jacobi–Gauss–Radau point systems. Let

$$-1 < x_n < x_{n-1} < \cdots < x_2 < x_1 < x_0 = 1 \tag{4.39}$$

be the roots of $(1-t)P_n^{(\alpha,\beta)}(t) = 0$ $(\alpha, \beta > -1)$, $x_k = \cos(\theta_k)$, and

$$\omega(t) = (t - x_0)(t - x_1)\cdots(t - x_n), \quad \ell_k(t) = \frac{\omega(t)}{(t - x_k)\omega'(x_k)}.$$

Then

$$\ell_0(t) = \frac{P_n^{(\alpha,\beta)}(t)}{P_n^{(\alpha,\beta)}(1)}, \qquad \ell_k(t) = \frac{(1-t)P_n^{(\alpha,\beta)}(t)}{(t - x_k)(1 - x_k)P_n^{(\alpha,\beta)'}(x_k)}, \quad k = 1, 2, \ldots, n. \tag{4.40}$$

Additionally, for $t = \cos\theta \in [0, 1]$, we have

$$\ell_0(t) = \begin{cases} O\left(\frac{n^\alpha}{n^\alpha}\right), & 0 \le \theta \le cn^{-1}, \\ O\left(\theta^{-\alpha-\frac{1}{2}} n^{-\alpha-\frac{1}{2}}\right), & cn^{-1} \le \theta < \frac{\pi}{2} \end{cases} = O\left(n^{-\min\{0,\alpha+\frac{1}{2}\}}\right)$$

and

$$\begin{aligned} \ell_k(t) &= -\frac{2\sin^2(\theta/2)P_n^{(\alpha,\beta)}(\cos\theta)}{2\sin^2(\theta_k/2)P_n^{(\alpha,\beta)'}(\cos\theta_k)2\sin((\theta-\theta_k)/2)\sin((\theta+\theta_k)/2)} \\ &= \begin{cases} O\left(n^{-\min\{0,\alpha+\frac{1}{2}\}}\right), & -1 < \alpha \le -\frac{1}{2}, \\ O\left(n^{-\min\{0,\frac{5}{2}-\alpha\}}\right), & \alpha > -\frac{1}{2}, \end{cases} \end{aligned} \tag{4.41}$$

following (4.27) similarly.

Similarly, for $t \in [-1,0]$, by $P_n^{(\beta,\alpha)}(-t) = (-1)^n P_n^{(\alpha,\beta)}(t)$, setting $t = -\cos\theta$ and $y_k = -x_{n-k+1} = \cos\overline{\theta}_k$ for $k = 1,2,\dots,n$, we obtain $\ell_0(t) = O\big(n^{-\min\{\alpha+\frac{1}{2},\alpha-\beta\}}\big)$ and

$$\ell_k(t) = \begin{cases} O\left(n^{-\min\{0,\alpha+\frac{1}{2},\alpha-\beta\}}\right), & -1<\alpha<\frac{1}{2},\\ O\left(n^{-\min\{0,\frac{1}{2}-\beta\}}\right), & \alpha \geq \frac{1}{2}. \end{cases} \tag{4.42}$$

Thus for $t \in [-1,1]$, we get

$$\|\ell_k\|_\infty = \begin{cases} O\left(n^{-\min\{0,\alpha+\frac{1}{2},\alpha-\beta\}}\right), & -1<\alpha\leq\frac{1}{2},\\ O\left(n^{-\min\{0,\frac{1}{2}-\beta,\frac{5}{2}-\alpha,\alpha-\beta\}}\right), & \alpha > \frac{1}{2}, \end{cases} \tag{4.43}$$

for $k = 0,1,2,\dots,n$.

THEOREM 4.6. *Suppose $f(t)$ satisfies* (2.8) *and $\{x_j\}_{j=0}^n$ are the roots of $(1-t)P_n^{(\alpha,\beta)}(t)$; then for $n \geq r+1$*

$$E_n[f] = n^{-r}\cdot\begin{cases} O\left(n^{-\min\{0,\alpha+\frac{1}{2},\alpha-\beta\}}\right), & -1<\alpha\leq\frac{1}{2},\\ O\left(n^{-\min\{0,\frac{1}{2}-\beta,\frac{5}{2}-\alpha,\alpha-\beta\}}\right), & \alpha > \frac{1}{2}. \end{cases} \tag{4.44}$$

Particularly, we have for $(\alpha,\beta) \in \overline{S}$

$$\|E_n[f]\|_\infty = O\left(\frac{1}{n^r}\right), \tag{4.45}$$

where $\overline{S} := \left[-\frac{1}{2},\frac{5}{2}\right] \times \left(-1,\frac{1}{2}\right] - \left\{(\alpha,\beta) : -\frac{1}{2} \leq \alpha \leq \frac{1}{2},\, \alpha < \beta \leq \frac{1}{2}\right\}$.

Similarly we have the following.

THEOREM 4.7. *Suppose $f(t)$ satisfies* (2.8) *and $\{x_j\}_{j=0}^n$ are the roots of $(1+t)P_n^{(\alpha,\beta)}(t)$; then for $n \geq r+1$*

$$E_n[f] = n^{-r}\cdot\begin{cases} O\left(n^{-\min\{0,\beta+\frac{1}{2},\beta-\alpha\}}\right), & -1<\beta\leq\frac{1}{2},\\ O\left(n^{-\min\{0,\frac{1}{2}-\alpha,\frac{5}{2}-\beta,\beta-\alpha\}}\right), & \beta > \frac{1}{2}. \end{cases} \tag{4.46}$$

Particularly, we have for $(\alpha,\beta) \in \widehat{S}$

$$\|E_n[f]\|_\infty = O\left(\frac{1}{n^r}\right), \tag{4.47}$$

where $\widehat{S} := \left(-1,\frac{1}{2}\right] \times \left[-\frac{1}{2},\frac{5}{2}\right] - \left\{(\alpha,\beta) : -\frac{1}{2} \leq \beta \leq \frac{1}{2},\, \beta < \alpha \leq \frac{1}{2}\right\}$.

Figure 12 shows the convergence rates for $f(x) = |x|$ at the Jacobi–Gauss–Radau point systems, where each $(\alpha,\beta) \in \overline{S}$ or $(\alpha,\beta) \in \widehat{S}$.

5. Final remarks. The results in section 4 indicate the fact that the interpolations, for functions of limited regularities, at strongly normal point systems, Gauss–Jacobi point systems with $-1 < \alpha, \beta \leq \frac{1}{2}$, Jacobi–Gauss–Lobatto point systems with $(\alpha,\beta) \in S$, or Jacobi–Gauss–Radau point systems $\overline{S}$ or $\widehat{S}$, have the same convergence order compared with the best polynomial approximation of the same degree. Numerical experiments confirm the estimates.

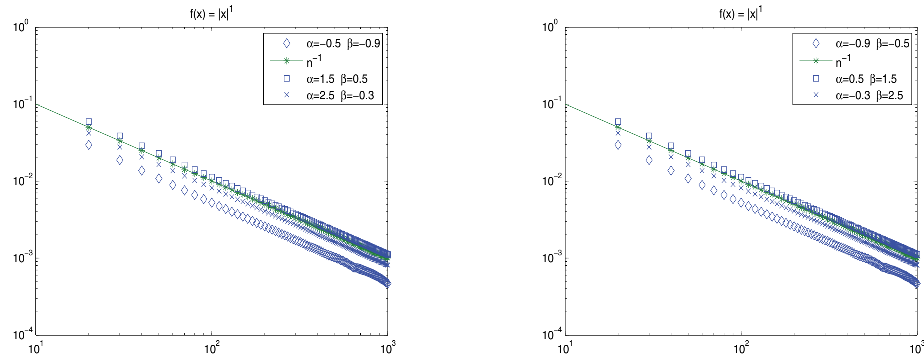


FIG. 12. *The absolute errors of* $\max_{x=-1:0.001:1} |f(x)-L_n[f](x)|$ *for Jacobi–Gauss–Radau point systems: the roots of* $(1-t)P_n^{(\alpha,\beta)}(t)$ *(left) and the roots of* $(1+t)P_n^{(\alpha,\beta)}(t)$ *(right) for* $f(x)=|x|$.

In addition, numerical experiments also show that the same occurs for analytic or smooth functions. Here we illustrate the phenomena by entire function $f(x)=e^x$, i.e., analytic throughout the complex plane, $f(x)=1/(1+25x^2)$, which is analytic in a neighborhood of $[-1,1]$ but not throughout the complex plane, and $f(x)=e^{-1/x^2}$, which is not analytic in a neighborhood of $[-1,1]$ but is infinitely differentiable in $[-1,1]$.

In Figures 13–15, the left columns are computed by zeros of Gauss–Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, the middle columns by Jacobi–Gauss–Lobatto $(1-x^2)P_{n-2}^{(\alpha,\beta)}(x)$, and the right columns by Jacobi–Gauss–Radau $(1-x)P_{n-1}^{(\alpha,\beta)}(x)$ (the first three cases) or $(1+x)P_{n-1}^{(\alpha,\beta)}(x)$ (the last three cases), respectively. From these figures, we see that the interpolations at these point systems, including Gauss–Legendre and Legendre–Gauss–Lobatto, achieve essentially the same approximation accuracy compared with those at the two Chebyshev point systems too.

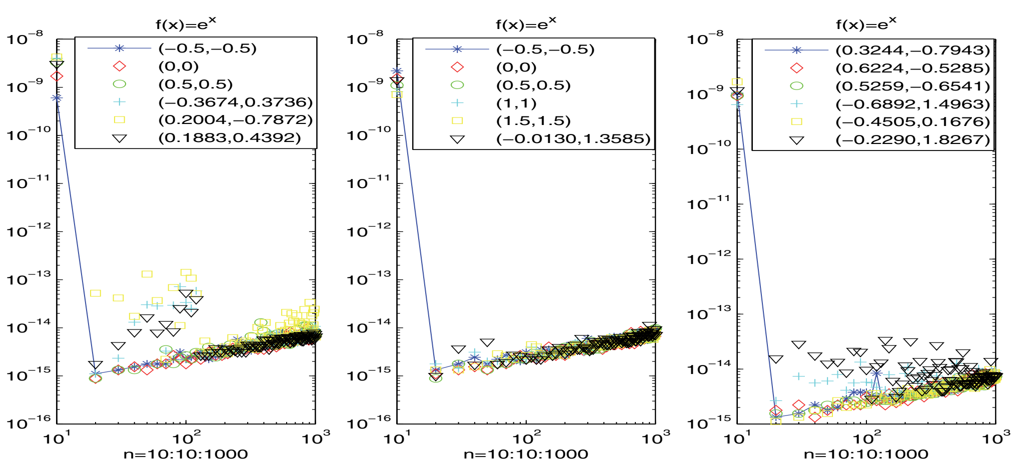


FIG. 13. $\max_{x=-1:0.001:1} |f(x)-L_n[f](x)|$ *with* $n=10:10:1000$ *at Gauss–Jacobi (left), Jacobi–Gauss–Lobatto (middle), and Jacobi–Gauss–Radau (right) point systems for* $f(x)=e^x$.

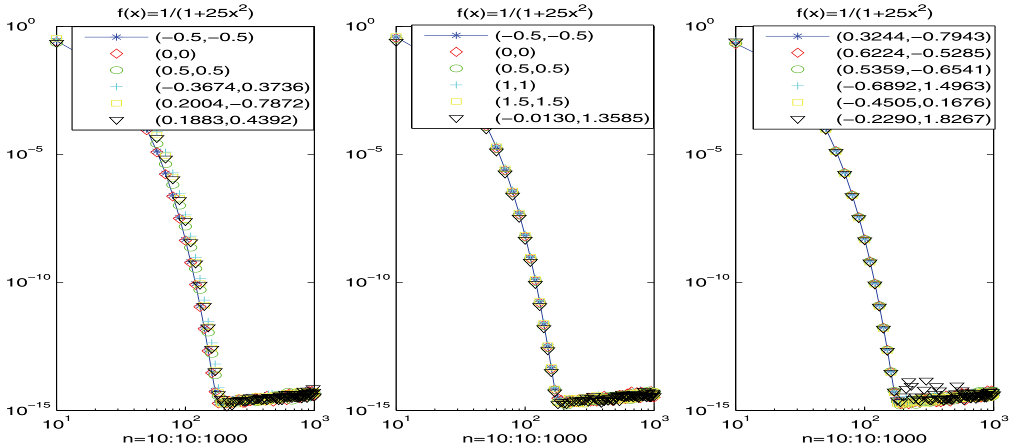


FIG. 14. $\max_{x=-1:0.001:1} |f(x) - L_n[f](x)|$ *with* $n = 10 : 10 : 1000$ *at Gauss–Jacobi (left), Jacobi–Gauss–Lobatto (middle), and Jacobi–Gauss–Radau (right) point systems for* $f(x) = \frac{1}{1+25x^2}$.

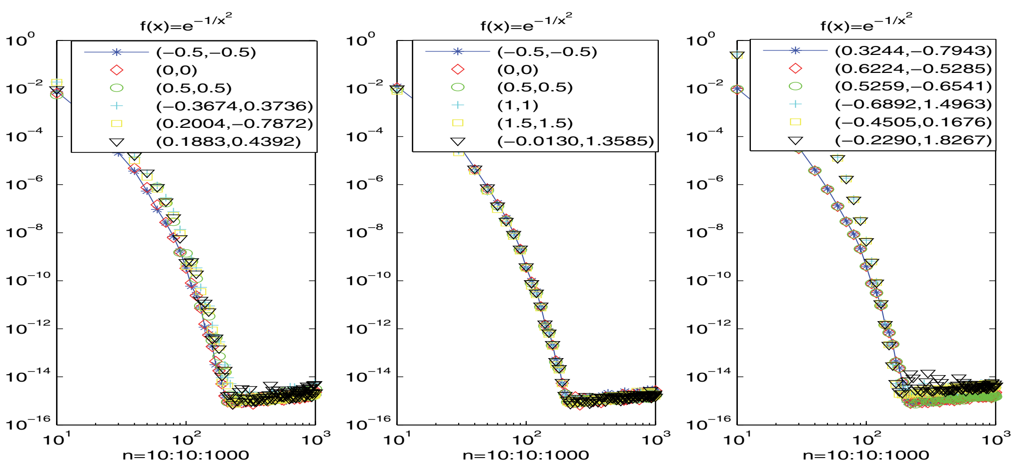


FIG. 15. $\max_{x=-1:0.001:1} |f(x) - L_n[f](x)|$ *with* $n = 10 : 10 : 1000$ *at Gauss–Jacobi (left), Jacobi–Gauss–Lobatto (middle), and Jacobi–Gauss–Radau (right) point systems for for* $f(x) = e^{-1/x^2}$.

It is interesting to note that the interpolation approximation polynomial $L_n[f]$ challenges the best approximation polynomial p_{n-1}^* of f at the above nice point systems not only on the equally asymptotic order on the convergence rate, but also on the faster convergence on the first or second derivative approximation by the polynomials, which has plenty of applications in spectral methods [34, 59].

Figure 16 illustrates that the convergence rates $\max_{x=-1:0.001:1} |f'(x) - [p_{n-1}^*]'(x)|$ and $\max_{x=-1:0.001:1} |f''(x) - [p_{n-1}^*]''(x)|$ reduce to 2-order and 3-order, respectively, compared with $\max_{x=-1:0.001:1} |f(x) - p_{n-1}^*(x)|$ for $f(x) = |x|^5$ or $|x|^7$. Here, the best approximation polynomial p_{n-1}^* is obtained by the **remez** algorithm in the CHEBFUN system [62].

However, the interpolation polynomial $L_n[f]$ at the above nice point systems performs much better than p_{n-1}^* for approximations f' and f'' by $L_n'[f]$, $L_n''[f]$, $[p_{n-1}^*]'$,

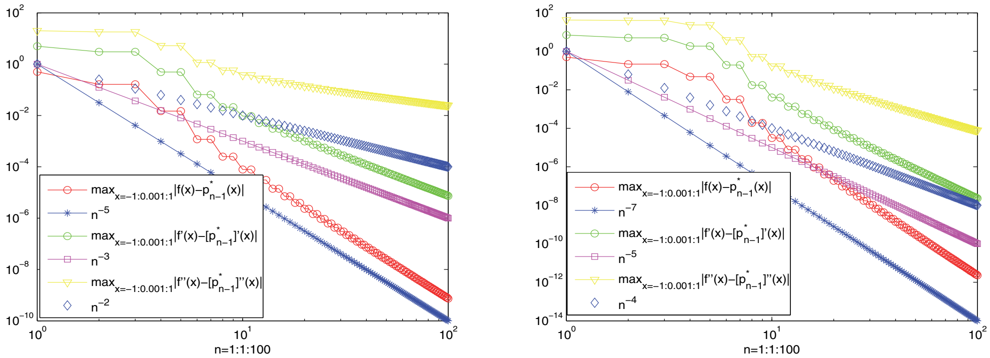


FIG. 16. $\max_{x=-1:0.001:1} |f^{(m)}(x) - [p_{n-1}^*]^{(m)}(x)|$ *with* $n = 10 : 1 : 100$ *for* $f(x) = |x|^5$ *(left),* $f(x) = |x|^7$ *(right) and* $m = 0, 1, 2$.

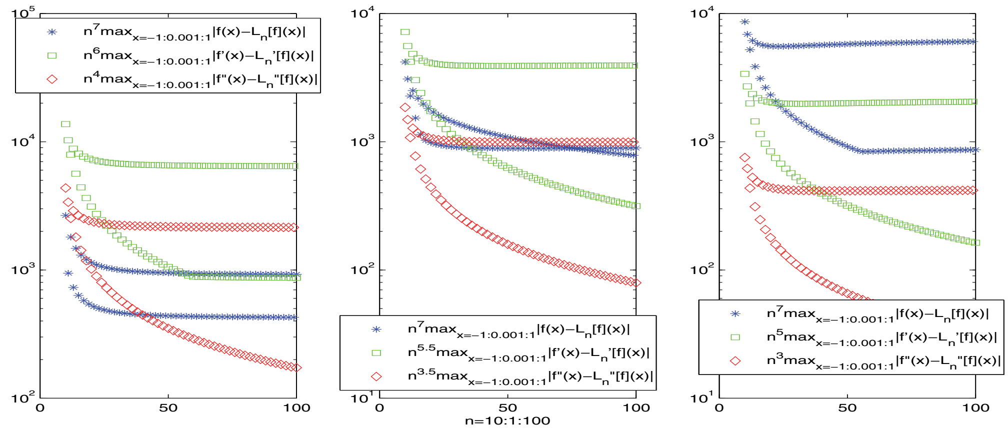


FIG. 17. $n^{s(m)} \max_{x=-1:0.001:1} |f^{(m)}(x) - L_n^{(m)}[f](x)|$ *with* $n = 10 : 1 : 100$ *at Gauss–Jacobi point systems for* $f(x) = |x|^7$ *and* $m = 0, 1, 2$, *and* $(-0.5, -0.5)$ *(left),* $(0, 0)$ *(middle),* $(0.5, 0.5)$ *(right).*

and $[p_{n-1}^*]''$, respectively. Here, we use the usual point systems to show the performance (see Figures 17–19).

Figure 20 shows the convergence rates $\max_{x=-1:0.001:1} |f^{(m)}(x) - [p_{n-1}^*]^{(m)}(x)|$ and $\max_{x=-1:0.001:1} |f^{(m)}(x) - L_n^{(m)}[f](x)|$ for $f(x) = |x|^5$ and $|x|^7$, respectively, where $L_n[f]$ is the interpolation at Jacobi–Gauss–Lobatto points $\{x_k = \cos\left(\frac{k\pi}{n-1}\right)\}_{k=0}^{n-1}$.

It is worth noting that comparing (1.5) $\left(1 + \frac{2}{\pi}\log n\right)\|f - p_{n-1}^*\|_\infty$ with (1.11) $\frac{\pi^r}{\prod_{j=1}^r (n-j)} \max_{1\le j\le n} \|\ell_j\|_\infty \mathrm{Var}(f^{(r)})$ in an optimal Lebesgue constant estimate (essentially achieved for the Chebyshev nodes), we see that the π^r term looks especially bad for modest values of r compared with $\frac{2}{\pi}\log(n)$ if n is not extremely large. Even for $n = 10^{66}$, $2 + \frac{2}{\pi}\log(10^{66}) \approx 98.7475$ (see [61, p. 120]). Here we illustrate using an example in Ganzburg [26] for $f_\lambda(x) = |x|^\lambda$ with $\lambda = 1$ and $\lambda = 3$, respectively: Note that

$$\lim_{n\to\infty} n^\lambda \|f_\lambda - p_n^*\|_\infty = B_{\lambda,\infty},$$

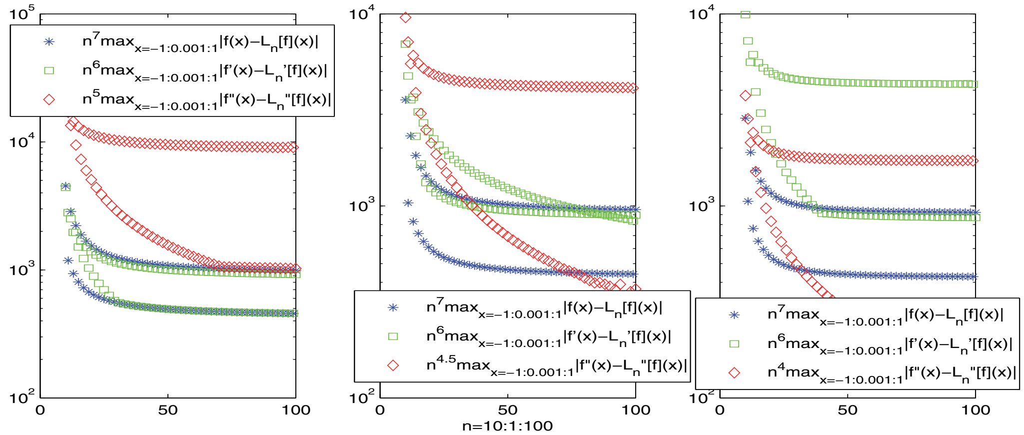


FIG. 18. $n^{s(m)} \max_{x=-1:0.001:1} |f^{(m)}(x) - L_n^{(m)}[f](x)|$ *with* $n = 10 : 1 : 100$ *at Jacobi–Gauss–Lobatto point systems for* $f(x) = |x|^7$ *and* $m = 0, 1, 2,$ *and* $(0.5, 0.5)$ *(left),* $(1, 1)$ *(middle),* $(1.5, 1.5)$ *(right).*

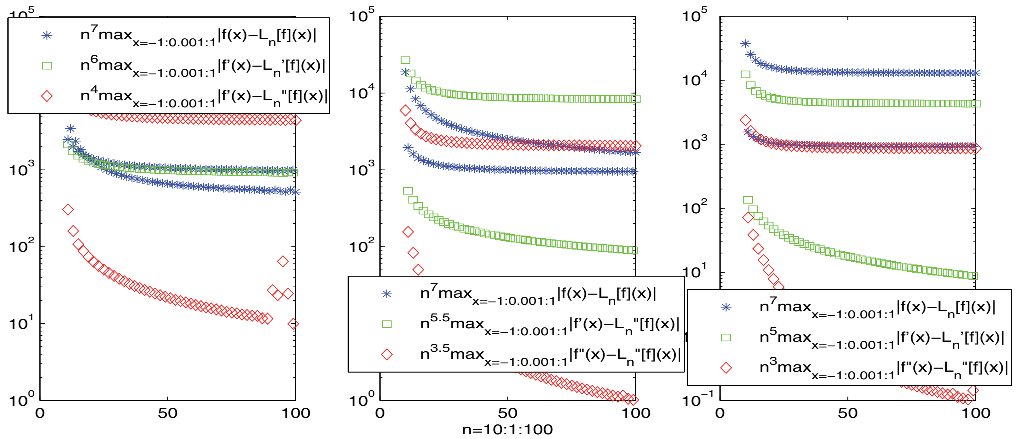


FIG. 19. $n^{s(m)} \max_{x=-1:0.001:1} |f^{(m)}(x) - L_n^{(m)}[f](x)|$ *with* $n = 10 : 1 : 100$ *at Jacobi–Gauss–Radau point systems with* $x_0 = 1$ *for* $f(x) = |x|^7$ *and* $m = 0, 1, 2,$ *and* $(-0.5, -0.5)$ *(left),* $(0, 0)$ *(middle),* $(0.5, 0.5)$ *(right).*

with

$$B_{\lambda,\infty} \le C_{\lambda,\infty} = \min\{|C_1(\lambda)|/2, |C_2(\lambda)|\},$$

where

$$C_1(\lambda) = \frac{4}{\pi} \sin\left(\frac{\pi\lambda}{2}\right) \int_0^\infty \frac{y^{\lambda-1}}{e^y + e^{-y}} dy = \frac{4}{\pi} \sin\left(\frac{\pi\lambda}{2}\right) \Gamma(\lambda) \sum_{j=0}^{\infty} (-1)^j (2j+1)^{-\lambda}$$

and

$$C_2(\lambda) = \frac{4}{\pi} \sin\left(\frac{\pi\lambda}{2}\right) \int_0^\infty \frac{y^{\lambda}}{e^y - e^{-y}} dy = \frac{4}{\pi} \sin\left(\frac{\pi\lambda}{2}\right) \Gamma(\lambda+1) \sum_{j=0}^{\infty} (2j+1)^{-\lambda-1}.$$

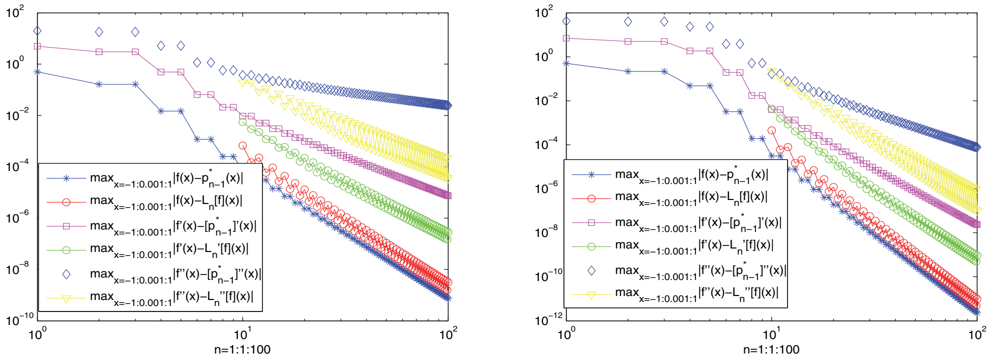


FIG. 20. *Convergence rates* $\max_{x=-1:0.001:1} |f^{(m)}(x) - [p_{n-1}^*]^{(m)}(x)|$ *and* $\max_{x=-1:0.001:1} |f^{(m)}(x) - L_n^{(m)}[f](x)|$ *for* $f(x) = |x|^5$ *(left) and* $|x|^7$ *(right) with* $m = 0, 1, 2$*, where* $L_n[f]$ *is the interpolation at Jacobi–Gauss–Lobatto points* $\{x_k = \cos\left(\frac{k\pi}{n-1}\right)\}_{k=0}^{n-1}$.

Moreover, from (1.11), (4.6), and (4.12), it follows that

$$\|f - p_n\|_\infty \le \frac{\sqrt{2}\pi^r V_r}{n(n-1)\cdots(n-r+1)}$$

at the Chebyshev points $x_k = \cos\left(\frac{(2k+1)\pi}{2n+2}\right)$ for $k = 0, 1, \ldots, n$ with $V_1 = \mathrm{Var}(f^{(1)}) = 2$ and $V_3 = \mathrm{Var}(f^{(3)}) = 12$. Figure 21 shows the Lebesgue constant multiplied by $C_{\lambda,\infty}$ compared with $\pi^\lambda V_\lambda$ with $\lambda = 1$ and 3, respectively, where $C_1(1) = 1$, $C_2(1) = \frac{\pi}{2}$, $C_1(3) \approx -2.46740110027233965460$, and $C_2(3) = -\frac{\pi^3}{4}$.

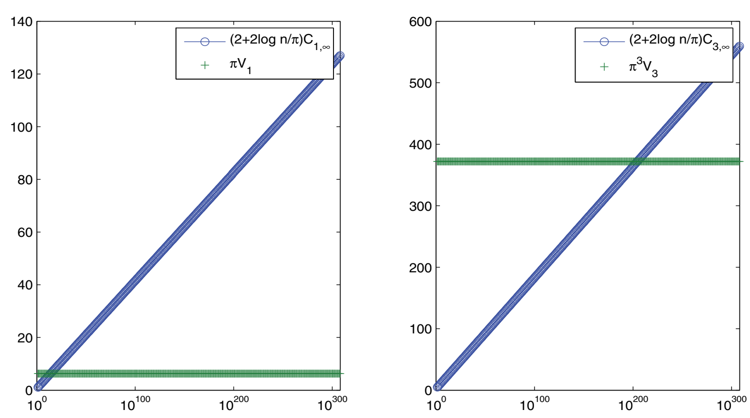


FIG. 21. *The Lebesgue constant multiplied by* $C_{\lambda,\infty}$ *compared with* $\pi^\lambda V_\lambda$ *with* $\lambda = 1$ *and* 3.

In particular, the high dimensional interpolation with nontensor-type points and sparse interpolation with L_1 minimization have been extensively studied in Bos et al. [7], Xu and Zhou [80], and Adcock [1]. In future work, we will extend the above classical interpolation problems to some modern interpolation methods.

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