Müntz Polynomials, Müntz Gauss-type Quadratures, and Applications

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Outline

Corner Singularity

Singularity occurs in many scientific problems:

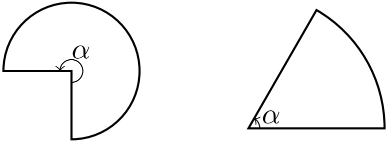
- Laplace equation:

$$\begin{cases} -\Delta u(x,y) = 0, & (x,y) \in \Omega, \\ u(x,y) = f(x,y), & (x,y) \in \partial\Omega, \end{cases}$$

- Helmholtz equation:

$$\begin{cases} -\Delta u(x,y) + k^2 u(x,y) = 0, & (x,y) \in \Omega, \\ u(x,y) = f(x,y), & (x,y) \in \partial\Omega, \end{cases}$$

where Ω is a bounded connected domain with corner $\alpha = \eta\pi$ and $0 < \eta < 2$. A typical domain is the sector:



Corner Singularity

Theorem 1 (Wasow)

[1] *The harmonic function $u(x,y)$ in Ω can be written in the form:*

$$u(x,y) = u_1(x,y) + u_2(x,y),$$

where all partial derivatives of u_1 remain bounded as x, y approach the corner. If r denotes the distance from the corner, then u_2 has the order of magnitude

$$u_2(x,y) = \begin{cases} O\left(r^{1/\eta}\right), & \text{if } \eta \neq \frac{1}{m}, \quad \text{for some } m \text{ an integer}, \\ O\left(r^{1/\eta}\log r\right), & \text{if } \eta = \frac{1}{m}, \quad \text{for some } m \text{ an integer}. \end{cases}$$

[1] W. Wasow, Asymptotic development of the solution of Dirichlet's problem at analytic corners, 1957.

Approximations

Let S_n be a finite-dimensional space. The error of the best approximation to a function f by elements of S_n is defined as

$$E(f, S_n) := \inf_{g \in S_n} \|f - g\|_{L^\infty(0,1)}.$$

• Algebraic Polynomials. Let $\mathbb{P}_n$ denote the space of algebraic polynomials of degree at most n. By Jackson's theorem [2]

$$E(x^r, \mathbb{P}_n) \leqslant Cn^{-r}, \quad 0 < r < 1.$$

• Rational Polynomials. Let $\mathcal{R}_{nn} = \left\{ \frac{p}{q} : p, q \in \mathbb{P}_n, \ q \neq 0 \right\}$. Then, by Stahl [3]

$$\lim_{n \to \infty} e^{2\pi\sqrt{rn}} E\left(x^r, \mathcal{R}_{nn}\right) = 4^{1+r} |\sin \pi r|, \quad r > 0.$$

Approximations

- Fractional Polynomials (Müntz Polynomials). Let $M_n = \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \cdots, x^{\lambda_n}\}$, where $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n$. Then, by Lorentz et al.[4],

$$E\left(x^r, M_n\right) \leqslant \prod_{k=1}^{n} \frac{|r-\lambda_k|}{r+\lambda_k}, \quad r > 0.$$

∗ Müntz polynomials are particularly well-suited for approximating functions that exhibit (weakly) singular behavior, which often arise in the solutions of PDEs.

∗ Their flexibility in non-integer exponents makes them a powerful tool for capturing non-smooth features that standard algebraic polynomials may fail to approximate effectively.

[2] DeVore R A, Lorentz G G. Constructive approximation, 1993.

[3] H. R. Stahl, Best uniform rational approximation of x^α on $[0, 1]$, 2003.

[4] Lorentz G G, von Golitschek M, Makovoz Y. Constructive approximation: advanced problems, 1996.

Outline

Müntz polynomials

Let $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \dots\}$ be a real **Müntz sequence** consisting of distinct real numbers. Its first $N+1$ elements is denoted by

$$\Lambda_N = \{\lambda_0, \lambda_1, \dots, \lambda_N\}.$$

The linear space over the field of real numbers generated by the Müntz sequence, denoted by $M(\Lambda_N) = \mathrm{span}\left\{x^{\lambda_0}, x^{\lambda_1}, \cdots, x^{\lambda_N}\right\}$, is called the **Müntz space**. That is, the Müntz space is the collection of all **Müntz polynomials**

$$p(x) = \sum_{i=0}^{N} a_k x^{\lambda_k}, \quad a_k \in \mathbb{R}.$$

∗ Müntz polynomials generalize algebraic polynomials by allowing arbitrary power indices, which necessitates the redevelopment of classical results originally established for algebraic polynomials.

Müntz polynomials

Theorem 2 (Müntz-Szász Theorem)

[5] *Let* $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \cdots\}$ *be a sequence of distinct numbers satisfying the real part* $\lambda_k > -1/p$ *for* $k = 0, 1, 2, \cdots$. *The Müntz space* $M(\Lambda)$ *is dense in* $L^p(0,1)$ *if* $1 \leqslant p < \infty$, *or* $C[0,1]$ *if* $p = \infty$, *if and only if*

$$\sum_{k=0}^{\infty} \frac{\lambda_k + 1/p}{(\lambda_k + 1/p)^2 + 1} = \infty.$$

$*$ The Müntz-type theorem can be extended to complex Müntz sequences as well as to a general interval (a, b), where $0 < a < b$.

$*$ For improved applicability and numerical stability, it is necessary to apply orthogonalization to Müntz polynomials.

[5] Almira J M. Muntz type theorems I, 2007.

Orthogonal Müntz polynomials

Theorem 3 (Müntz-Legendre polynomials)

[6] *Let Müntz sequence* Λ_N *satisfy* $\lambda_k > -1/2$, $k = 0, 1, \cdots, N$, *and*

$$L_n(x; \Lambda_n) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_n} W_n(t) x^t \mathrm{d}t, \;\; W_n(t) = \prod_{k=0}^{n-1} \frac{t + \lambda_k + 1}{t - \lambda_k} \frac{1}{t - \lambda_n},$$

where $n = 0, 1, 2, \cdots, N$, $\Lambda_n = \{\lambda_0, \lambda_1, \cdots, \lambda_n\}$, Γ_n *is a simple contour on the complex plane that encircles* $\lambda_0, \cdots, \lambda_n$. *Then*

$$\int_0^1 L_n(x) L_m(x) \mathrm{d}x = \frac{\delta_{n,m}}{1 + 2\lambda_n}, \quad n, m = 0, 1, \cdots, N, \tag{1}$$

where $\delta_{n,m}$ *is the Kronecker-Delta symbol.*

Orthogonal Müntz polynomials

$*$ Since the definition of L_n does not require the exponents λ_k to be distinct, we define the generalized Müntz space as

$$\hat{M}(\Lambda_N) = \mathrm{span}\{L_0, L_1, \cdots, L_N\}.$$

Clearly, $M(\Lambda_N) \subset \hat{M}(\Lambda_N)$. More specifically, we have the following:

Lemma 4

[7] *Let* $\Lambda_N = \{\lambda_0, \cdots, \lambda_0, \cdots, \lambda_s, \cdots, \lambda_s\}$ *consist of* r_k *copies of* λ_k, $r_k \geq 1$, $0 \leq k \leq s \leq N$, *and* $\sum_{k=0}^{s} r_k = N + 1$. *Hence*

$$\hat{M}(\Lambda_N) = \mathrm{span}\left\{x^{\lambda_0}, \cdots, x^{\lambda_0} \log^{r_0 - 1} x, \cdots, x^{\lambda_s}, \cdots, x^{\lambda_s} \log^{r_s - 1} x\right\}.$$

Orthogonal Müntz polynomials

Let $\omega(x) = x^{\beta}$, where $\beta \in \mathbb{R}$ satisfies

$$\lambda_n + \beta/2 > -1/2, \quad n = 0, 1, \cdots, N.$$

Putting $\lambda_k + \beta/2$ instead of λ_k, $k = 0, 1, \cdots, N$, in the Müntz sequence Λ_N, we can define a kind of **Müntz-Jacobi polynomials** by $L_n^{\beta}(x; \Lambda_n) = x^{-\beta/2} L_n(x; \Lambda_n + \beta/2)$. It follows that

$$\int_0^1 L_n^{\beta}(x) L_m^{\beta}(x) x^{\beta} \mathrm{d}x = \frac{\delta_{n,m}}{\lambda_n + \lambda_n + \beta + 1}, \quad n, m = 0, 1, \cdots, N.$$

$*$ The computation of L_n or L_n^{β} is nontrivial[8]; however, their moments can be evaluated in a straightforward manner.

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[6] P. Borwein, T. Erdélyi, J. Zhang, Müntz systems and orthogonal Müntz-Legendre polynomials, 1994.

[7] H. Wang, C. Xu, On recurrence formulae of müntz polynomials and applications, 2025.

[8] Gradimir V Milovanović. Müntz orthogonal polynomials and their numerical evaluation, 1999.

Moments Recurrence

Let

$$\chi^{\lambda,\mu}(x) = x^{\lambda}(-\log x)^{\mu}, \quad \lambda > -1 \text{ and } \mu \in \mathbb{N}. \tag{2}$$

The integrals of the Müntz-Legendre polynomials L_n with respect to $\chi^{\lambda,\mu}$ are called the generalized moments, or simply moments, with respect to $\chi^{\lambda,\mu}$, denoted by

$$\sigma_n(\lambda,\mu;\Lambda_n) = \int_0^1 L_n(x)\chi^{\lambda,\mu}(x)\mathrm{d}x. \tag{3}$$

∗ $\sigma_n(\lambda,\mu;\Lambda_n)$ satisfies a recurrence relation, and its computation is numerically stable with a complexity of $O(n(\mu+1)^2)$.

Moments Recurrence

Theorem 5

[9] *Suppose that Müntz sequence* $\Lambda_n = \{\lambda_0, \lambda_1, \cdots, \lambda_n\}$, *and* $\lambda \in \mathbb{R}$, $\mu \in \mathbb{N}$, *such that* $\lambda + \lambda_v > -1$ *for* $v = 0, 1, \cdots, n$. *Then for any* $n \geq 1$, *we have*

$$\sigma_n(\lambda, \mu) = \frac{\lambda - \lambda_{n-1}}{\lambda_n + \lambda + 1} \sigma_{n-1}(\lambda, \mu) - \sum_{j=0}^{\mu-1} \frac{\mu!}{j!} \frac{\lambda_n + \lambda_{n-1} + 1}{(\lambda_n + \lambda + 1)^{\mu-j+1}} \sigma_{n-1}(\lambda, j), \tag{4}$$

and the results of $\sigma_0(\lambda, j)$ *for* $j = 0, 1, \cdots, \mu$ *are given as*

$$\sigma_0(\lambda, j) = \int_0^1 x^{\lambda_0 + \lambda} (-\log x)^j \mathrm{d}x = \frac{j!}{(\lambda_0 + \lambda + 1)^{j+1}}. \tag{5}$$

[9] H. Wang, C. Xu, On recurrence formulae of müntz polynomials and applications, 2025.

Recurrence formula of generalized Legendre polynomials

Suppose the Müntz sequence

$$\Lambda_{2N+1} = \{\lambda_{2r} = \lambda_{2r+1} = r : \ r = 0, 1, \cdots, N\}. \tag{6}$$

We define the corresponding Müntz-Legendre polynomials as generalized Legendre polynomials

$$\mathcal{G}_n(x) := L_n(x; \Lambda_n) = \frac{1}{2\pi i} \int_{\Gamma_n} W_n(t) x^t \mathrm{d}t, \quad n = 2r, 2r+1.$$

The first few generalized Legendre polynomials are

$$\begin{aligned}
\mathcal{G}_0(x) &= 1, \\
\mathcal{G}_1(x) &= 1 + \log x, \\
\mathcal{G}_2(x) &= -3 + 4x - \log x, \\
\mathcal{G}_3(x) &= 9 - 8x + 2(1 + 6x) \log x.
\end{aligned}$$

Recurrence formula of generalized Legendre polynomials

$\mathcal{G}_n$ generalize the Legendre polynomials in the sense of multiple orthogonal polynomials[10], that is,

$$\int_0^1 \mathcal{G}_n(x)x^j \mathrm{d}x = 0, \quad j = 0, 1, \cdots, \lfloor (n-1)/2 \rfloor,$$

and

$$\int_0^1 \mathcal{G}_n(x)x^j(-\log x)\mathrm{d}x = 0, \quad j = 0, 1, \cdots, \lfloor (n-2)/2 \rfloor,$$

where $\lfloor x \rfloor$ denotes the floor function and

$$\lfloor (n-1)/2 \rfloor + \lfloor (n-2)/2 \rfloor + 2 = n.$$

Recurrence formula of generalized Legendre polynomials

Theorem 6

[11] *Let* Λ_{2N+1} *satisfy* (6). $\mathcal{G}_n(x)$ *is the generalized Legendre polynomial for* $n = 0, 1, \cdots, 2N+1$. *Let*

$$\begin{aligned} &\mathcal{G}_0(x) = 1, \\ &\mathcal{G}_1(x) = 1 + \log x, \\ &\mathcal{G}_{-1}(x) = 0, \quad \mathcal{G}_{-2}(x) = 0, \quad \mathcal{G}_{-3}(x) = 0. \end{aligned} \tag{7}$$

Then for $n \geq 2$, *we have*

$$\mathcal{G}_n = (\alpha_n^{(n)} x + \alpha_{n-2}^{(n)})\mathcal{G}_{n-2} + \alpha_{n-1}^{(n)}\mathcal{G}_{n-1} + \alpha_{n-3}^{(n)}\mathcal{G}_{n-3} + \alpha_{n-4}^{(n)}\mathcal{G}_{n-4}.$$

[10] AI Aptekarev. Multiple orthogonal polynomials, 1998.

[11] H. Wang, C. Xu, On recurrence formulae of müntz polynomials and applications, 2025.

The coefficients in Theorem 6 are given as

$$\alpha_n^{(n)} = \begin{cases} \dfrac{16n(n-2)}{(n-1)^2}, & n = \text{odd}, \\ \dfrac{16(n-1)^2}{n^2}, & n = \text{even}, \end{cases} \qquad \alpha_{n-1}^{(n)} = \begin{cases} \dfrac{2(1+4n-2n^2)}{(n-1)(n-2)}, & n = \text{odd}, \\ \dfrac{-4(n-1)^2}{n^2}, & n = \text{even}, \end{cases}$$

$$\alpha_{n-2}^{(n)} = \begin{cases} \dfrac{2n(7-3n)}{(n-1)^2}, & n = \text{odd}, \\ \dfrac{2(1-n)(3n-2)}{n^2}, & n = \text{even}, \end{cases} \qquad \alpha_{n-4}^{(n)} = \begin{cases} \dfrac{-n(n-3)^2}{(n-1)^2(n-2)}, & n = \text{odd}, \\ \dfrac{(1-n)(n-2)^2}{n^2(n-3)}, & n = \text{even}, \end{cases}$$

$$\alpha_{n-3}^{(n)} = \begin{cases} \dfrac{4n(2-n)}{(n-1)^2}, & n = \text{odd}, \\ \dfrac{2(2-n)(2n^2-8n+5)}{n^2(n-3)}, & n = \text{even}. \end{cases}$$

$*$ This result can be extended to $(2n+1)-$term recurrence formula with n repeated indices ($n \geqslant 3$).

Outline

Müntz Gauss-type Quadratures

Let $N, K \in \mathbb{Z}^+$, Λ_K be the Müntz sequence, and $\omega(x) = x^\beta$ be a weight function. Assume that β and Λ_K satisfy one of the following conditions:

- Case 1: $\lambda_k + \beta > -1, \quad k = 0, 1, \cdots, K,$
- Case 2: $\beta > -1$ and $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_K$.

Our goal is to determine the Gauss-type nodes $\{x_k\}_{k=0}^N$ and weights $\{\omega_k\}_{k=0}^N$ such that

$$\sum_{k=0}^{N} p(x_k)\omega_k = \int_0^1 p(x)\omega(x)\mathrm{d}x, \quad \forall p \in \hat{M}(\Lambda_K), \tag{8}$$

where the quadrature type is specified as follows:

- Gauss quadrature: $K = 2N + 1$, and β, Λ_K satisfy Case 1.

Müntz Gauss-type Quadratures

- Gauss-Radau quadrature:
  - If $x_0 = 0$ is fixed, then $K = 2N$, and β, Λ_K satisfy Case 2.
  - If $x_N = 1$ is fixed, then $K = 2N$, and β, Λ_K satisfy Case 1.
- Gauss-Lobatto quadrature: $K = 2N - 1$, and β, Λ_K satisfy Case 2.

The problem in Eq. (8) has been previously studied in works such as [MRW96][12] and [MC05][13].

- The algorithm proposed in [MRW96] is more general but often suffers from ill-conditioning (see [MRW96, Remark 6.2]).
- In contrast, the algorithm in [MC05] is specifically designed for Müntz systems and is directly applicable only when $\beta = \lambda_0$ (see [MC05, Eq. (3.2)]).
- The Gauss-Radau and Gauss-Lobatto quadrature rules for Müntz systems have not been studied previously.

[12] J Ma, V Rokhlin, and Stephen Wandzura. Generalized Gaussian quadrature rules for systems of arbitrary functions, 1996.

[13] Gradimir V Milovanović and Aleksandar S Cvetković. Gaussian-type quadrature rules for Müntz systems, 2005.

Existence and Uniqueness

We propose a method that deals with these three quadratures in a framework and is valid for any value of β in an efficient manner.

Recall the existence and uniqueness result:

Definition 7 (Chebyshev System)

A finite set of functions $\varphi_0, \varphi_1, \cdots, \varphi_n$ defined on the interval $[a, b]$ is called a Chebyshev system if and only if $\varphi_j \in C[a, b]$, $j = 0, 1, \cdots, n$, and the determinant of $\mathbf{\Phi}$ is non-zero, where

$$\mathbf{\Phi} = \begin{bmatrix} \varphi_0(x_0) & \varphi_0(x_1) & \cdots & \varphi_0(x_n) \\ \varphi_1(x_0) & \varphi_1(x_1) & \cdots & \varphi_1(x_n) \\ \vdots & \vdots & & \vdots \\ \varphi_n(x_0) & \varphi_n(x_1) & \cdots & \varphi_n(x_n) \end{bmatrix},$$

and $x_0, x_1, \cdots, x_n$ are any distinct points in the interval $[a, b]$.

Existence and Uniqueness

Theorem 8

[14] *If any basis of $\hat{M}(\Lambda_K)$ forms a Chebyshev system on $[0,1]$, then there exists a unique quadrature rule satisfying Eq. (8) for any weight function. Moreover, all nodes x_k lie in $(0,1)$, except for possible boundary nodes, and all weights ω_k are positive.*

* If $0 = \lambda_0 < \lambda_1 \leqslant \cdots \leqslant \lambda_K$, then it can be readily verified that any basis of $\hat{M}(\Lambda_K)$ forms a Chebyshev system on $[0,1]$.
* For Gauss and Gauss-Radau (with $x_N = 1$ fixed) quadratures , it is straightforward to extend Müntz polynomials to handle left-endpoint singularities, i.e., to allow negative and repeated indices in the Müntz sequence.

[14] S. Karlin and W. J. Studden, Tchebycheff systems: With applications in analysis and statistics, 1966.

Numerical Construction

We consider the Gauss quadrature case with $K = 2N + 1$ and $\omega(x) = x^\beta$. Then Eq. (8) is equivalent to

$$\sum_{k=0}^{N} L_n^\beta(x_k; \Lambda_n)\omega_k = m_n, \quad m_n = \int_0^1 L_n^\beta(x; \Lambda_n)\omega(x)\mathrm{d}x, \tag{9}$$

where $n = 0, 1, \cdots, 2N + 1$. Then, Eq. (9) can be written in the matrix form as

$$\begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix} \operatorname{diag}(\mathbf{x}^{-\frac{\beta}{2}})\boldsymbol{\omega} = \begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix}, \tag{10}$$

where $\operatorname{diag}(\mathbf{x})$ is a diagonal matrix with $\mathbf{x}$ as its entries, and we define $\mathbf{c} = [m_0, \cdots, m_N]^{\mathrm{T}}$, $\mathbf{d} = [m_{N+1}, \cdots, m_{2N+1}]^{\mathrm{T}}$,

Numerical Construction

$\mathbf{m} = [\mathbf{c}^{\mathrm{T}}, \mathbf{d}^{\mathrm{T}}]^{\mathrm{T}}$, $\boldsymbol{\omega} = [\omega_0, \cdots, \omega_N]^{\mathrm{T}}$, $\mathbf{x}^{-\frac{\beta}{2}} = [x_0^{-\frac{\beta}{2}}, \cdots, x_N^{-\frac{\beta}{2}}]^{\mathrm{T}}$,
and the matrices $\mathbf{U}$ and $\mathbf{V}$

$$\mathbf{U} = \begin{bmatrix} L_0(x_0;\Lambda_0+\beta/2), & \cdots, & L_0(x_N;\Lambda_0+\beta/2) \\ L_1(x_0;\Lambda_1+\beta/2), & \cdots, & L_1(x_N;\Lambda_1+\beta/2) \\ \vdots & & \vdots \\ L_N(x_0;\Lambda_N+\beta/2), & \cdots, & L_N(x_N;\Lambda_N+\beta/2) \end{bmatrix},$$

$$\mathbf{V} = \begin{bmatrix} L_{N+1}(x_0;\Lambda_{N+1}+\beta/2), & \cdots, & L_{N+1}(x_N;\Lambda_{N+1}+\beta/2) \\ L_{N+2}(x_0;\Lambda_{N+2}+\beta/2), & \cdots, & L_{N+2}(x_N;\Lambda_{N+2}+\beta/2) \\ \vdots & & \vdots \\ L_{2N+1}(x_0;\Lambda_{2N+1}+\beta/2), & \cdots, & L_{2N+1}(x_N;\Lambda_{2N+1}+\beta/2) \end{bmatrix}.$$

Numerical Construction

Let $\mathbf{\Psi} = [\mathbf{U}^{\mathrm{T}}, \mathbf{V}^{\mathrm{T}}]^{\mathrm{T}}$ and $\mathbf{z} = [\mathbf{x}^{\mathrm{T}}, \boldsymbol{\omega}^{\mathrm{T}}]^{\mathrm{T}}$. We construct a mapping $\mathbf{F} : \mathbb{R}^{K+1} \to \mathbb{R}^{K+1}$ such that

$$\mathbf{F}(\mathbf{z}) = \mathbf{\Psi}\mathrm{diag}(\mathbf{x}^{-\frac{\beta}{2}})\boldsymbol{\omega} - \mathbf{m}. \tag{11}$$

Then, the equation $\mathbf{F}(\mathbf{z}) = \mathbf{0}$ has a unique solution $\mathbf{z}^*$, which corresponds to the Gaussian nodes $\mathbf{x}^*$ and weights $\boldsymbol{\omega}^*$.

$*$ The computation of $\mathbf{\Psi}$ depends on L_n, and the vector $\mathbf{m}$ can be efficiently evaluated using the recurrence:

$$\int_0^1 L_n^\beta x^\beta \mathrm{d}x = \frac{-\lambda_{n-1}}{1+\lambda_n+\beta} \int_0^1 L_{n-1}^\beta x^\beta \mathrm{d}x. \tag{12}$$

$*$ For the Gauss-Radau and Gauss-Lobatto cases, one can also construct the mapping $\mathbf{F} : \mathbb{R}^{K+1} \to \mathbb{R}^{K+1}$, where $K = 2N$ and $K = 2N - 1$, respectively.

Newton's Method

We employ Newton's method to solve $\mathbf{F}(\mathbf{z}) = \mathbf{0}$.
The Jacobian of $\mathbf{F}$ is

$$\mathbf{J}(\mathbf{z}) = \tilde{\mathbf{J}}(\mathbf{z}) \begin{bmatrix} \mathrm{diag}(\mathbf{x}^{-\frac{\beta}{2}-1}) & \\ & \mathrm{diag}(\mathbf{x}^{-\frac{\beta}{2}}) \end{bmatrix} \begin{bmatrix} \mathrm{diag}(\boldsymbol{\omega}) & \\ & \mathrm{diag}(\mathbf{1}) \end{bmatrix},$$

where

$$\tilde{\mathbf{J}}(\mathbf{z}) = \begin{bmatrix} -\frac{\beta}{2}\mathbf{U} + \mathbf{U}'\mathrm{diag}(\mathbf{x}) & \mathbf{U} \\ -\frac{\beta}{2}\mathbf{V} + \mathbf{V}'\mathrm{diag}(\mathbf{x}) & \mathbf{V} \end{bmatrix},$$

$$\mathbf{U}' = \begin{bmatrix} L_0'(x_0; \Lambda_0 + \beta/2), & \cdots, & L_0'(x_N; \Lambda_0 + \beta/2) \\ L_1'(x_0; \Lambda_1 + \beta/2), & \cdots, & L_1'(x_N; \Lambda_1 + \beta/2) \\ \vdots & & \vdots \\ L_N'(x_0; \Lambda_N + \beta/2), & \cdots, & L_N'(x_N; \Lambda_N + \beta/2) \end{bmatrix},$$

Newton's Method

and

$$\mathbf{V}' = \begin{bmatrix} L'_{N+1}(x_0;\Lambda_{N+1}+\beta/2), & \cdots, & L'_{N+1}(x_N;\Lambda_{N+1}+\beta/2) \\ L'_{N+2}(x_0;\Lambda_{N+2}+\beta/2), & \cdots, & L'_{N+2}(x_N;\Lambda_{N+2}+\beta/2) \\ \vdots & & \vdots \\ L'_{2N+1}(x_0;\Lambda_{2N+1}+\beta/2), & \cdots, & L'_{2N+1}(x_N;\Lambda_{2N+1}+\beta/2) \end{bmatrix}.$$

$*$ We obtain $\tilde{\mathbf{J}}(\mathbf{z})$ by computing both $\mathbf{U}'\mathrm{diag}(\mathbf{x})$ and $\mathbf{V}'\mathrm{diag}(\mathbf{x})$ in a stable and accurate manner

$$xL'_n(x;\Lambda_n+\beta/2) = xL'_{n-1}(x;\Lambda_n+\beta/2) + \left(\lambda_n+\frac{\beta}{2}\right)L_n(x;\Lambda+\beta/2) + \left(1+\lambda_{n-1}+\frac{\beta}{2}\right)L_{n-1}(x;\Lambda_{n-1}+\beta/2).$$

Newton's Method

Theorem 9

[15] *For any distinct* $x_k \in (0,1)$ *and* $\omega_k > 0$, $k = 0, 1, \cdots, N$, $\mathbf{J}(\mathbf{z})$ *is nonsingular.*

The iterative update rule with initial guess $\mathbf{z}^{(0)}$ is given by

$$\mathbf{z}^{(k+1)} = \mathbf{z}^{(k)} - \mathbf{J}(\mathbf{z}^{(k)})^{-1}\mathbf{F}(\mathbf{z}^{(k)}), \quad k = 0, 1, 2, \cdots. \tag{13}$$

Theorem 10 (Local Convergence)

There exists some $\varepsilon > 0$ *such that the Jacobian* $\mathbf{J}(\mathbf{z})$ *is Lipschitz continuous in a neighborhood of* $\mathbf{z}^*$ *with radius* ε. *Therefore, provided that the starting values are sufficiently good, the Newton's method solving* $\mathbf{F}(\mathbf{z}) = \mathbf{0}$ *is convergent quadratically.*

Newton's Method

∗ For the Gauss-Radau and Gauss-Lobatto cases, a similar Newton iteration rule as in Eq. (13) can be constructed, with the difference being the Jacobian, which remains nonsingular. The method retains local quadratic convergence.

∗ In terms of the local convergence property of the Newton's method, it is imperative to select an optimal initial guess. As a result, we resort to the continuation method as a means of obtaining the initial guess.

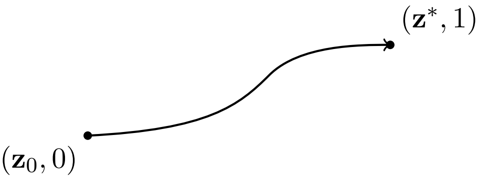
[15] H. Wang, C. Xu, On weighted generalized Gauss quadratures for Müntz systems, 2023.

Continuation Method

For the mapping $\mathbf{F} : \mathbb{R}^{K+1} \to \mathbb{R}^{K+1}$, we define a homotopy mapping $\mathbf{H} : \mathbb{R}^{K+1} \times \mathbb{R} \to \mathbb{R}^{K+1}$ such that

$$\mathbf{H}(\mathbf{z}, 0) = \mathbf{G}(\mathbf{z}), \quad \mathbf{H}(\mathbf{z}, 1) = \mathbf{F}(\mathbf{z}),$$

where $\mathbf{G}(\mathbf{z}) = \mathbf{0}$ is a trivial mapping, so that the solution $\mathbf{z}_0$ of $\mathbf{G}(\mathbf{z}) = \mathbf{0}$ is unique and known. Then it is possible to trace the implicitly defined curve $\mathcal{C} \in \mathbf{H}^{-1}(\mathbf{0})$ from a starting point $(\mathbf{z}_0, 0)$ to the desired point $(\mathbf{z}^*, 1)$ by a sufficiently small stepsize.



Continuation on Sequences

Let $\alpha \in [0,1]$, and $\Lambda_K^{(\alpha)} = \left\{\lambda_0^{(\alpha)}, \lambda_1^{(\alpha)}, \cdots, \lambda_K^{(\alpha)}\right\}$, where

$$\lambda_k^{(\alpha)} = \alpha\lambda_k + (1-\alpha)k, \quad k = 0, 1, \cdots, K.$$

• We define the homotopy mapping as $\mathbf{H}(\mathbf{z}, \alpha) := \mathbf{F}(\mathbf{z}, \alpha)$, where $\mathbf{F}(\mathbf{z}, \alpha)$ is derived from (11) by replacing Λ_K with $\Lambda_K^{(\alpha)}$.
• For each value of α, the Gauss-type nodes and weights $\mathbf{z}(\alpha)$ can be computed by solving the nonlinear system $\mathbf{F}(\mathbf{z}(\alpha), \alpha) = \mathbf{0}$ using Newton's method.
• In particular, when $\alpha = 0$, $\mathbf{z}(0)$ corresponds to the Jacobi Gauss-type quadrature, while $\alpha = 1$ gives $\mathbf{z}(1)$, which represents our desired quadrature rule.

Continuation on Sequences

Theorem 11

[16] *For any* $\alpha \in [0,1]$, *the Gauss nodes and weights* $x_k(\alpha)$ *and* $\omega_k(\alpha)$, $k = 0, 1, \cdots, N$, *are continuous with respect to* α.

- By continuity, if $\mathbf{z}(\alpha)$ is known, then for sufficiently small $\Delta\alpha$, the solution $\mathbf{z}(\alpha + \Delta\alpha)$ can be obtained using Newton's method with $\mathbf{z}(\alpha)$ as the initial guess.
- The continuation method is carried out by repeatedly applying this procedure while incrementally advancing α from 0 to 1.

[16] H. Wang, C. Xu, On weighted generalized Gauss quadratures for Müntz systems, 2023.

Continuation on Sequences

Algorithm 1 Gauss-type quadrature: Continuation on sequences

Input: Λ_{2N+1} and $\beta \in \mathbb{R}$.

  Compute $\mathbf{z}(0)$ and set $\alpha \leftarrow \Delta\alpha$.

  while $\alpha \leq 1$ **do**

    Compute moments $\mathbf{m}(\alpha)$ by Eq. (12).

    Compute $\mathbf{z}(\alpha)$ by Newton's method with the initial $\mathbf{z}(\alpha - \Delta\alpha)$.

    Update $\alpha \leftarrow \alpha + \Delta\alpha$.

  end while

  return $\mathbf{z}(1)$.

Continuation on Moments

Assume that Gauss-type quadrature $\{\bar{x}_k, \bar{\omega}_k\}_{k=0}^{N}$ have already been determined for $\omega(x) = x^\beta$, and we want to find $\{\tilde{x}_k, \tilde{\omega}_k\}_{k=0}^{N}$ for the general weight function $\tilde{\omega}(x)$.
Suppose that the moments can be accurately computed:

$$\tilde{m}_n = \int_0^1 L_n^\beta(x)\tilde{\omega}(x)\mathrm{d}x, \quad \forall n = 0, 1, \cdots, K.$$

Let

$$\hat{m}_n(\tau) = (1-\tau)m_n + \tau\tilde{m}_n, \quad 0 \leqslant \tau \leqslant 1, \tag{14}$$

where m_n is the moment for x^β.
• We define mapping $\mathbf{H}(\mathbf{z}, \tau) := \mathbf{F}(\mathbf{z}, \tau)$, where $\mathbf{F}(\mathbf{z}, \tau)$ is derived from (11) by replacing $\mathbf{m}$ with $\hat{\mathbf{m}}(\tau) = [\hat{m}_0(\tau), \cdots, \hat{m}_K(\tau)]^{\mathrm{T}}$.

Continuation on Moments

- For each τ, the Gauss-type nodes and weights $\mathbf{z}(\tau)$ is **unique**, and can be computed by solving the nonlinear system $\mathbf{F}(\mathbf{z}(\tau), \tau) = \mathbf{0}$ using Newton's method.
- In particular, when $\tau = 0$, $\mathbf{z}(\tau)$ corresponds to the quadrature rule for $\omega(x)$, while $\tau = 1$ gives $\mathbf{z}(\tau)$, which represents our desired quadrature rule for $\tilde{\omega}(x)$.

Theorem 12

For any $\tau \in [0, 1]$, the Gauss nodes and weights $x_k(\tau)$ and $\omega_k(\tau)$, $k = 0, 1, \cdots, N$, are continuous with respect to τ.

- By continuity, if $\mathbf{z}(\tau)$ is known, then for sufficiently small $\Delta\tau$, the solution $\mathbf{z}(\tau + \Delta\tau)$ can be obtained using Newton's method with $\mathbf{z}(\tau)$ as the initial guess.

Continuation on Moments

- The continuation method is carried out by repeatedly applying this procedure while incrementally advancing τ from 0 to 1.

Algorithm 2 Gauss-type quadrature: Continuation on moments

Input: Λ_{2N+1} and weight function $\tilde{\omega}(x)$.
  Compute $\mathbf{z}(0)$ in Algorithm 1 for some β and set $\tau \leftarrow \Delta\tau$.
  while $\tau \leq 1$ **do**
    Compute moments $\hat{\mathbf{m}}(\tau)$ by Eq. (14).
    Compute $\mathbf{z}(\tau)$ by Newton's method with the initial $\mathbf{z}(\tau - \Delta\tau)$.
    Update $\tau \leftarrow \tau + \Delta\tau$.
  end while
  return $\mathbf{z}(1)$.

Continuation on Moments

When Müntz polynomials reduce to algebraic polynomials, the computation of Algorithm 2 can be carried out explicitly, known as **Modified Chebyshev Algorithm** due to W. Gautschi[17]. This method leverages the three-term recurrence relation of orthogonal polynomials and has a computational cost of only $O(N^2)$[18].

For a specific class of weight functions, exact calculation of $\tilde{m}_n$ is possible. For example, for $\tilde{\omega}(x) = x^\beta(-\log x)^\mu$ with $\mu \in \mathbb{N}$, $\tilde{m}_n$ can be obtained by

$$\tilde{m}_n = \int_0^1 L_n^\beta(x)x^\beta(-\log x)^\mu \mathrm{d}x = \frac{-\lambda_{n-1}}{\lambda_n+\beta+1}\int_0^1 L_{n-1}^\beta(x)x^\beta(-\log x)^\mu \mathrm{d}x - \sum_{j=0}^{\mu-1}\frac{\mu!}{j!}\frac{\lambda_n+\lambda_{n-1}+\beta+1}{(\lambda_n+\beta+1)^{\mu-j+1}}\int_0^1 L_{n-1}^\beta x^\beta(-\log x)^j \mathrm{d}x.$$

[17] W. Gautschi, On generating orthogonal polynomials

[18] H. Wang, C. Xu, On recurrence formulae of müntz polynomials and applications, 2025.

Condition Number of The Jacobian

The efficiency of the update rule (13) depends on the condition number of $\mathbf{J}$. We consider a special case of Müntz polynomials.

Theorem 13

Let $\Lambda_{2N+1} = \{\lambda k : 0 \leqslant k \leqslant 2N+1\}$ *for* $\lambda \geqslant 1$, *and* $\beta = 0$. *The condition number of* $\mathbf{J}$ *at the Gauss nodes and weights satisfies the following estimate:*

$$\|\mathbf{J}\|_{\mathrm{F}} \|\mathbf{J}^{-1}\|_{\mathrm{F}} \leqslant \begin{cases} C_1 N^{3/2}\sqrt{\log N}, & 1 \leqslant \lambda \leqslant 2, \\ C_2 N^{3/2} \log N, & 2 < \lambda \leqslant 4, \\ C_3 N^{2-2/\lambda} \log N, & \lambda > 4, \end{cases}$$

where C_1, C_2 *and* C_3 *are positive constants independent of both* N *and* λ.

Condition Number of The Jacobian

∗ For $\lambda \geqslant 1$, the condition number exhibits moderate growth as N increases.
∗ When $\lambda > 0$ is very small, the condition number grows significantly as N increases. There is a way to mitigate this issue. Specifically, let $\kappa > 1$ be a scaling factor. The Gauss-type quadrature rule is equivalent to solve

$$\int_0^1 p(x)\mathrm{d}x = \sum_{j=0}^{N} p(y_j)A_j, \quad \forall p \in \hat{M}(\kappa\Lambda_K + \kappa - 1),$$

with the change of variables $x_j = y_j^{\kappa}$ and corresponding weights $\omega_j = \kappa A_j y_j^{\kappa-1}$. Its computation remains in our framework.

Error Estimate

We consider the Müntz sequence

$$\Lambda_{2N+1} = \{\lambda_{2r} = \lambda_{2r+1} = r : \ r = 0, 1, \cdots, N\}, \qquad (15)$$

and denote the exact integral of f and its Gauss quadrature by

$$I[f] = \int_0^1 f(x) x^{\beta} \mathrm{d}x, \quad Q_N[f] = \sum_{j=0}^{N} f(x_j)\omega_j.$$

Theorem 14

[19] *Let* Λ_{2N+1} *be defined as in* (15)*. For any* $u, v \in C^{N+1}[0,1]$*, we set* $f(x) = u(x) + v(x)\log x$*, then*

$$|I[f] - Q_N[f]| \leqslant \frac{1}{N!}\left(\frac{1}{1+\beta}\|u^{(N+1)}\|_\infty + \frac{1}{(1+\beta)^2}\|v^{(N+1)}\|_\infty\right).$$

[19] H. Wang, C. Xu, On weighted generalized Gauss quadratures for Müntz systems, 2023.

Generally, let $T \in \mathbb{N}^+$, we consider Müntz sequence as

$$\Lambda_{T(N+1)} = \left\{ \lambda_v = \left\lfloor \frac{v}{T} \right\rfloor : v = 0, 1, \cdots, T(N+1) \right\},$$

where $\lfloor \cdot \rfloor$ denotes the floor function.

Theorem 15

[20] *For any $u_j \in C^{N+1}[0,1]$, we set $f(x) = \sum_{j=0}^{T-1} u_j(x) \log^j x$, then*

$$|I[f] - Q_S[f]| \leqslant \frac{1}{N!} \sum_{j=0}^{T-1} \frac{\Gamma(j+1)}{(1+\beta)^{j+1}} \|u_j^{(N+1)}\|_\infty, \qquad (16)$$

where Γ is the Gamma function, $S = \left\lceil \frac{T(N+1)}{2} \right\rceil$ and $\lceil \cdot \rceil$ denotes the ceil function.

$*$ It is similar to consider the general $x^\lambda \log^\mu x$ singularities.

[20] H. Wang, C. Xu, On weighted generalized Gauss quadratures for Müntz systems, 2023.

Numerical Example

Example 1

Compute the integral:

$$I(a) = \int_0^1 \frac{x - \log x}{1 + \sqrt{2}ax + a^2x^2} \mathrm{d}x, \quad 0 < |a| \leqslant 1.$$

It is known that[21] $I(a) = \frac{\sqrt{2}}{a} \sum_{k=1}^{\infty} \left(\frac{1}{1+k} + \frac{1}{k^2} \right) a^k \sin\left(\frac{3k\pi}{4}\right)$.

When $a = \pm 1$, the expansion converges slowly, and there are the alternative formulae:

$$I(1) = -\frac{\mathrm{i}\sqrt{2}}{2} \left[-\mathrm{Li}_2\left(e^{-\frac{3\pi \mathrm{i}}{4}}\right) + \mathrm{Li}_2\left(e^{\frac{3\pi \mathrm{i}}{4}}\right) + (-1)^{\frac{3}{4}} \log\left(1 + (-1)^{\frac{1}{4}}\right) + (-1)^{\frac{1}{4}} \log\left(1 - (-1)^{\frac{3}{4}}\right) \right],$$

$$I(-1) = -\frac{\mathrm{i}\sqrt{2}}{2} \left[-\mathrm{Li}_2\left(e^{-\frac{\pi \mathrm{i}}{4}}\right) + \mathrm{Li}_2\left(e^{\frac{\pi \mathrm{i}}{4}}\right) + (-1)^{\frac{3}{4}} \log\left(1 - (-1)^{\frac{1}{4}}\right) + (-1)^{\frac{1}{4}} \log\left(1 + (-1)^{\frac{3}{4}}\right) \right],$$

where $\mathrm{Li}_s(z)$ is the polylogarithm function of order s.

[21] I. S. Gradshteyn, I. M. Ryzhik, Table of integrals, series, and products, 2014.

we choose $\beta = 0$ and Müntz sequences Λ_{2N+1} that are categorized into three cases for comparison. For $k = 0, 1, \cdots, 2N+1$, the cases are as follows:

- Case 1: $\lambda_k = k$;
- Case 2: $\lambda_k = \lfloor k/2 \rfloor$;
- Case 3: $\lambda_k = \lfloor k/3 \rfloor$.

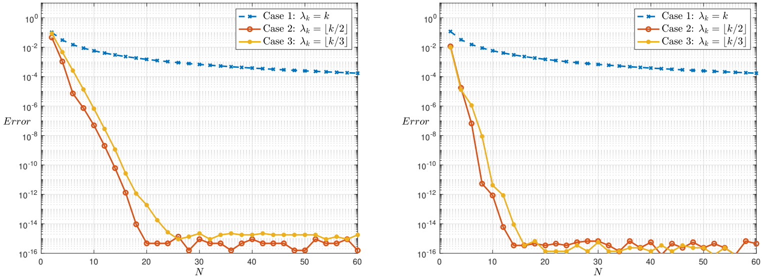


Figure: Errors of Gauss quadrature rules for Example 1: approximation to $I(-1)$ (left) and $I(1)$ (right).

Outline

Best Approximation by Müntz polynomials

Let Λ_N be the Müntz sequence. The minimax, or best approximation, problem is to find an approximant on $\hat{M}(\Lambda_N)$ to a function $f \in C[0,1]$ that minimizes the error in $\infty-$norm:

$$\begin{cases} \text{Find } p^* \in \hat{M}(\Lambda_N) \text{ such that} \\ \|f - p^*\|_\infty \leqslant \|f - p\|_\infty, \quad \forall p \in \hat{M}(\Lambda_N). \end{cases} \tag{17}$$

The Remez algorithm[22] to solve minimax problem is to set an initial **reference** $\{\xi_i^{(0)}\}_{i=0}^{N+1}$ and to solve (17) iteratively follows two main steps:

1. At kth step, from the trial reference set $\{\xi_i^{(k)}\}_{i=0}^{N+1}$, to obtain the trial approximation $p^{(k)}$.
2. Update $\{\xi_i^{(k+1)}\}_{i=0}^{N+1}$ based on $p^{(k)}$ and $\{\xi_i^{(k)}\}_{i=0}^{N+1}$, according to the exchange strategy.

[22] M. J. D. Powell, et al., Approximation theory and methods, 1981.

Unlike the classical minimax problem for algebraic polynomials, which typically uses **Chebyshev Gauss-Lobatto** nodes as the initial reference, we adopt the **Müntz Gauss-Lobatto** nodes as the starting point for the Müntz minimax problem to achieve improved results.

Example 2

Minimax approximation on $\hat{M}(\Lambda_N)$ to $f_i(x)$ that are listed in Table 1, where

$$\lambda_k = \left\lfloor \frac{k+1}{2} \right\rfloor, \ k = 0, 1, \ldots, 20.$$

i	$f_i(x)$	niter	$\Vert f_i - p^{(0)} \Vert_\infty$	$\Vert f_i - p^* \Vert_\infty$
1	$\tanh(2x - 1/2) - \tanh(2x - 3/2)$	7	0.000000002948499	0.000000001852034
2	$\sin(\exp(2x - 1))$	6	0.000000015309231	0.000000010575408
3	$\sqrt{2x}$	8	0.002074565070495	0.000809747539229
4	$\sqrt{\lvert 2x - 1.1 \rvert}$	9	0.230593159635940	0.088185954888281
5	$1 - \sin(5\lvert 2x - 3/2 \rvert)$	7	0.140002892942718	0.084982255340392
6	$\min\{\mathrm{sech}(3\sin(20x - 10))), \sin(18x - 9)\}$	6	0.504493784477548	0.261997272487487
7	$\max\{\sin(40x - 20), \exp(2x - 2)\}$	9	0.315849213829298	0.174340502069304
8	$\mathrm{sech}(10(x - 0.2))^2 +$ $\mathrm{sech}(100(x - 0.4))^4 +$ $\mathrm{sech}(1000(x - 0.6))^6$	12	0.963322816087389	0.499415728919011
9	$\log(2x + 0.0001)$	9	0.038024274676468	0.021314198706278

Table: Errors of minimax approximation to nine functions in $[0, 1]$ by $\hat{M}(\Lambda_{20})$ in Example 2.

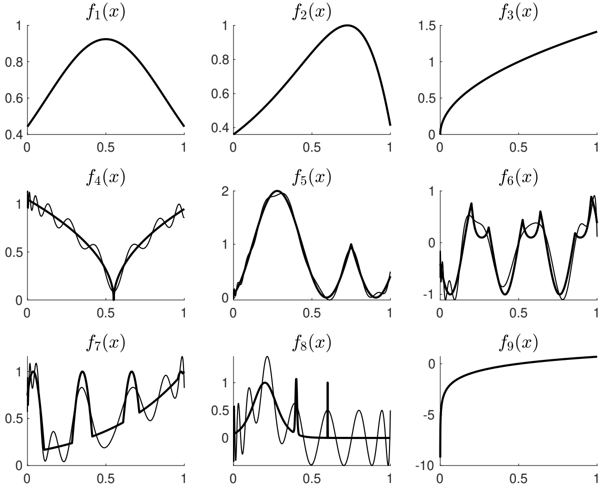


Figure: Best approximations of the functions in Example 2: the function f_i (bold line) and its approximant p^* (thin line) are presented.

Müntz Collocation Method

Let us consider the two-dimensional Poisson equation with the homogeneous Dirichlet boundary condition

$$\begin{cases} -\Delta u = f(x,y), \ (x,y) \in \mathcal{D}, \\ u|_{\partial\mathcal{D}} = 0, \end{cases} \tag{18}$$

where $f \in L^2(\mathcal{D})$, $\mathcal{D} = \{(r\cos\theta, r\sin\theta) : 0 < r < 1, 0 < \theta < \alpha\}$ for some $0 < \alpha < 2\pi$, distributed as a sector with corner.

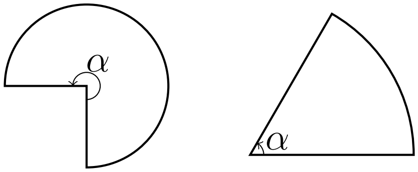


Figure: Two-dimensional sectors.

The solution of (18) often exhibits a weak singularity[23] as r is close to 0. Note that the polar coordinate transformation from (x, y) to (r, θ) transforms the Laplace operator into the form $\Delta = \partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta}$. Thus, the problem is equivalent to:

$$\begin{cases} -\dfrac{q^2}{r^2}\dfrac{\partial^2 v}{\partial \theta^2} - \dfrac{1}{r}\dfrac{\partial v}{\partial r} - \dfrac{\partial^2 v}{\partial r^2} = g(r, \theta), \quad (r, \theta) \in \Omega, \\ v|_{\partial\Omega} = 0, \end{cases} \tag{19}$$

where for convenience, we use the same notation for certain variables. The domain is $\Omega = \{(r, \theta) : 0 < r < 1, 0 < \theta < 1\}$, and $q = 1/\alpha$ where α refers to the angle of the sector $\mathcal{D}$ in (18).

- Let

$$\Lambda^{(1)}_{2N_1-1} = \{\lambda^{(1)}_0, \cdots, \lambda^{(1)}_{2N_1-1}\}, \quad \Lambda^{(2)}_{2N_2-1} = \{\lambda^{(2)}_0, \cdots, \lambda^{(2)}_{2N_2-1}\}$$

be two Müntz sequences, and their corresponding Müntz Gauss-Lobatto nodes are respectively denoted as r_i and θ_j, where $0 \leqslant i \leqslant N_1$ and $0 \leqslant j \leqslant N_2$.

- Define the $(N_1+1)(N_2+1)$ dimensional Müntz space:

$$W_{N_1,N_2} = \hat{M}(\Lambda^{(1)}_{N_1}) \times \hat{M}(\Lambda^{(2)}_{N_2}) = \{p_1(r)p_2(\theta) : p_1(r) \in \hat{M}(\Lambda^{(1)}_{N_1}) \text{ and } p_2(\theta) \in \hat{M}(\Lambda^{(2)}_{N_2})\}.$$

- The collocation method we propose consists in finding Q_{N_1,N_2} in W_{N_1,N_2} such that it satisfies (19) at (r_i, θ_j), i.e.,

$$\begin{cases} \text{Find } Q_{N_1,N_2}(r,\theta) \in W_{N_1,N_2} \text{ such that} \\ \text{for } \quad 1 \leqslant i \leqslant N_1 - 1, 1 \leqslant j \leqslant N_2 - 1, \\ \qquad -\dfrac{q^2}{r_i^2}\dfrac{\partial^2}{\partial\theta^2} Q_{N_1,N_2}(r_i,\theta_j) - \dfrac{1}{r_i}\dfrac{\partial}{\partial r} Q_{N_1,N_2}(r_i,\theta_j) - \dfrac{\partial^2}{\partial r^2} Q_{N_1,N_2}(r_i,\theta_j) = g(r_i,\theta_j), \\ \text{and for } \quad 0 \leqslant i \leqslant N_1, 0 \leqslant j \leqslant N_2, \\ \qquad Q_{N_1,N_2}(r_0,\theta_j) = 0, Q_{N_1,N_2}(r_{N_1},\theta_j) = 0, Q_{N_1,N_2}(r_i,\theta_0) = 0, Q_{N_1,N_2}(r_i,\theta_N) = 0. \end{cases}$$

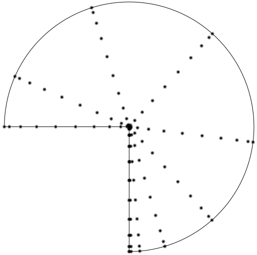
[23] Z. Li, T. Lu, Singularities and treatments of elliptic boundary value problems, 2000.

Example 3

Solving the partial differential equation (18) with $\alpha = 3\pi/2$ and f is chosen such that $u(r,\theta) = (r^{0.8} - r^{1.6})(\theta^{0.6} - \theta^{3.6})$.

We chose $\beta_1 = \beta_2 = 0$, $N_1 = N_2 = N$, $\Lambda_N^{(1)}$ and $\Lambda_N^{(2)}$ with

$$\lambda_i^{(1)} = \ell_1 i,\ \ \lambda_j^{(2)} = \ell_2 j, \quad 0 \leqslant i, j \leqslant N.$$



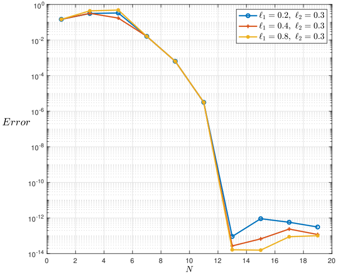


Figure: Numerical results for Example 3: collocation nodes with $\ell_1 = 0.8$, $\ell_2 = 0.3$, and $N = 10$ (left), and the numerical errors measured in the $\infty-$norm (right).

Conclusion and Future Work

• We investigate the theoretical properties of Müntz polynomials and derive several recurrence relations associated with them. Building upon this foundation, we develop efficient numerical algorithms for computing generalized weighted Gauss quadrature rules, including their Gauss-Radau and Gauss-Lobatto variants, tailored to Müntz systems with arbitrary index sequences.

• Furthermore, we rigorously establish the convergence of the proposed methods and demonstrate favorable conditioning for a specific class of Müntz sequences. We also analyze a class of integrals whose integrands exhibit singular or weakly singular behavior, and prove that Müntz-based Gauss quadrature achieves rapid convergence in such settings.

• As practical applications, we employ Müntz Gauss-Lobatto nodes to address minimax approximation problems involving Müntz polynomials and apply the Müntz collocation method to the numerical solution of partial differential equations.

Thank you for your attention!