Exercise 2.15. *Consider the elliptic problem*

$$-u_{xx} = f, \quad \forall x \in (a,b),$$
$$u(a) = 0, \; u'(b) = \beta,$$

and its finite difference schema

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = f_i, \quad \forall i = 1, \cdots, N-1,$$
$$u_0 = 0,$$
$$\frac{u_N - u_{N-1}}{h} = \beta,$$

in an uniform mesh $\{x_i\}_{i=0}^N$, $x_i = a + ih$, $h = (b-a)/N$.
1) Derive an estimate for the truncation errors:

$$R_i^{(1)} = L_h[u(x_i)] - [Lu](x_i) \text{ for } i = 1, \cdots, N-1, \; R^{(2)} = \frac{u(x_N) - u(x_{N-1})}{h} - u'(x_N).$$

2) Rewrite the discrete problem under matrix form.
3) Establish an a priori estimate for $\|u_h\|_1$.
4) Derive an error estimate for $\|e_h\|_1$, *where* $e_i = u(x_i) - u_i$.

Solution. 1). Let the operator $Lu = -u_{xx}$ and the discrete operator L_h on $\{u_i\}_{i=1}^{N-1}$ as

$$L_h u_i = -\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}.$$

By the Tylor development:

$$u(x_{i+1}) = u(x_i) + hu'(x_i) + \frac{h^2}{2}u''(x_i) + \frac{h^3}{3!}u^{(3)}(x_i) + \frac{h^4}{4!}u^{(4)}(\xi_i), \text{ for some } \xi_i \in (x_i, x_{i+1}),$$

$$u(x_{i-1}) = u(x_i) - hu'(x_i) + \frac{h^2}{2}u''(x_i) - \frac{h^3}{3!}u^{(3)}(x_i) + \frac{h^4}{4!}u^{(4)}(\eta_i), \text{ for some } \eta_i \in (x_{i-1}, x_i),$$

we obtain that $R_i^{(1)} = L_h[u(x_i)] - [Lu](x_i) = O(h^2)$, while $R^{(2)} = O(h)$ as $h \to 0$.
2).

$$\begin{bmatrix} \frac{2}{h^2} & -\frac{1}{h^2} & & & \\ -\frac{1}{h^2} & \frac{2}{h^2} & -\frac{1}{h^2} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{1}{h^2} & \frac{2}{h^2} & -\frac{1}{h^2} \\ & & & -\frac{1}{h} & \frac{1}{h} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N-1} \\ \beta \end{bmatrix},$$

or

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} h^2 f_1 \\ h^2 f_2 \\ \vdots \\ h^2 f_{N-1} \\ h\beta \end{bmatrix}.$$

3). Note that $L_h u_i = -\left((u_i)_{\bar{x}}\right)_{\hat{x}}$, then multiplying both sides of the finite difference schema $L_h u_i = f_i$ by $u_i h_i$ yields

$$-\left((u_i)_{\bar{x}}\right)_{\hat{x}} u_i h_i = f_i u_i h_i, \quad \forall i = 1, \cdots, N-1.$$

Summing in i gives

$$-\left(((u_h)_{\bar{x}})_{\hat{x}}, u_h\right)_{I_h} = (f_h, u_h)_{I_h}.$$

In virtue of discrete Green formula (9) and the fact that $u_0 = 0$, we have

$$-\left(((u_h)_{\bar{x}})_{\hat{x}}, u_h\right)_{I_h} = ((u_h)_{\bar{x}}, (u_h)_{\bar{x}})_{I_h^+} - (u_N)_{\bar{x}} u_N.$$

Note that $(u_N)_{\bar{x}} = \beta$, and

$$u_N = \sum_{i=1}^{N} (u_i)_{\bar{x}} h \leqslant \left(\sum_{i=1}^{N} h\right)^{1/2} \left(\sum_{i=1}^{N} (u_i)_{\bar{x}}^2 h\right)^{1/2} = \sqrt{b-a}|u_h|_1.$$

We have

$$|u_h|_1^2 = ((u_h)_{\bar{x}}, (u_h)_{\bar{x}})_{I_h^+} = \beta u_N + (f_h, u_h)_{I_h}.$$

By discrete Cauchy-Schwarz inequality:

$$(f_h, u_h)_{I_h} \leqslant \left(\sum_{i=1}^{N-1} f_i^2 h\right)^{1/2} \left(\sum_{i=1}^{N-1} u_i^2 h\right)^{1/2} \leqslant \|f_h\|_0 \|u_h\|_0.$$

By discrete Poincaré inequality: $\|u_h\|_0 \leqslant C|u_h|_1$, we obtain

$$|u_h|_1 \leqslant C(\|f_h\|_0 + |\beta|),$$

where C represents a constant depending only on a and b, and it may different in different scenarios. Thus by using discrete Poincaré again, we obtain

$$\|u_h\|_1 = \left(\|u_h\|_0^2 + |u_h|_1^2\right)^{1/2} \leqslant C|u_h|_1 \leqslant C(\|f_h\|_0 + |\beta|).$$

4). It is obvious that

$$\begin{cases} L_h e_i = R_i, \quad \forall i = 1, 2, \cdots, N-1, \\ e_0 = 0, \\ \dfrac{e_N - e_{N-1}}{h} = R^{(2)}. \end{cases}$$

By 1) and 3) we have

$$\|e_h\|_1 \leqslant C(\|R_h^{(1)}\|_0 + |R^{(2)}|) = O(h) \text{ as } h \to 0.$$

□

Exercise 3.1. *Derive an estimate for the truncation error of the 9-point schema.*

Solution. Consider the 2D problem:

$$\begin{cases} Lu(\mathbf{x}) = f(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega, \\ \quad u(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in \partial\Omega, \end{cases}$$

where $\Omega = (a, b)^2$ and $Lu = -\Delta u$. Let $\{\mathbf{x}_{i,j} : i = 0, \cdots, N_x, j = 0, \cdots, N_y\}$ be the discrete mesh in Ω equispaced of $h_x = (b-a)/N_x$ and $h_y = (b-a)/N_y$ over x and y axis respectively. Then the coordinate of $\mathbf{x}_{i,j}$ is $(a + ih_x, a + jh_y)$.

The 9-point schema is

$$\begin{cases} \bar{L}_h u_{i,j} = \bar{f}(\mathbf{x}_{i,j}), \quad \forall \mathbf{x}_{i,j} \in \Omega, \\ \quad u_{i,j} = 0, \quad \forall \mathbf{x}_{i,j} \in \partial\Omega, \end{cases}$$

where

$$\begin{aligned} \bar{L}_h u_{i,j} = L_h u_{i,j} - \frac{h_x^2 + h_y^2}{12 h_x^2 h_y^2} [&u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} \\ &- 2u_{i+1,j} + 4u_{i,j} - 2u_{i-1,j} \\ &+ u_{i+1,j-1} - 2u_{i,j-1} + u_{i-1,j-1}], \end{aligned} \tag{1}$$

$$L_h u_{i,j} = -\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h_x^2} - \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h_y^2}, \tag{2}$$

and $\bar{f}(\mathbf{x}_{i,j})$ is given by

$$\bar{f}(\mathbf{x}_{i,j}) = f(\mathbf{x}_{i,j}) + \frac{1}{12}\left(h_x^2 \frac{\partial^2 f}{\partial x^2} + h_y^2 \frac{\partial^2 f}{\partial y^2}\right)(\mathbf{x}_{i,j}). \tag{3}$$

The truncation error $\bar{R}_{i,j} = \bar{L}_h[u(\mathbf{x}_{i,j})] - \bar{f}(\mathbf{x}_{i,j})$. By the Tylor development, we have

$$u(\mathbf{x}_{i+1,j}) = u(\mathbf{x}_{i,j}) + h_x \frac{\partial u}{\partial x}(\mathbf{x}_{i,j}) + \frac{h_x^2}{2}\frac{\partial^2 u}{\partial x^2}(\mathbf{x}_{i,j}) + \frac{h_x^3}{3!}\frac{\partial^3 u}{\partial x^3}(\mathbf{x}_{i,j}) + \frac{h_x^4}{4!}\frac{\partial^4 u}{\partial x^4}(\mathbf{x}_{i,j}) + O(h_x^4),$$

$$u(\mathbf{x}_{i-1,j}) = u(\mathbf{x}_{i,j}) - h_x \frac{\partial u}{\partial x}(\mathbf{x}_{i,j}) + \frac{h_x^2}{2}\frac{\partial^2 u}{\partial x^2}(\mathbf{x}_{i,j}) - \frac{h_x^3}{3!}\frac{\partial^3 u}{\partial x^3}(\mathbf{x}_{i,j}) + \frac{h_x^4}{4!}\frac{\partial^4 u}{\partial x^4}(\mathbf{x}_{i,j}) + O(h_x^4),$$

$$u(\mathbf{x}_{i,j+1}) = u(\mathbf{x}_{i,j}) + h_y \frac{\partial u}{\partial y}(\mathbf{x}_{i,j}) + \frac{h_y^2}{2}\frac{\partial^2 u}{\partial y^2}(\mathbf{x}_{i,j}) + \frac{h_y^3}{3!}\frac{\partial^3 u}{\partial y^3}(\mathbf{x}_{i,j}) + \frac{h_y^4}{4!}\frac{\partial^4 u}{\partial y^4}(\mathbf{x}_{i,j}) + O(h_y^4),$$

$$u(\mathbf{x}_{i,j-1}) = u(\mathbf{x}_{i,j}) - h_y \frac{\partial u}{\partial y}(\mathbf{x}_{i,j}) + \frac{h_y^2}{2}\frac{\partial^2 u}{\partial y^2}(\mathbf{x}_{i,j}) - \frac{h_y^3}{3!}\frac{\partial^3 u}{\partial y^3}(\mathbf{x}_{i,j}) + \frac{h_y^4}{4!}\frac{\partial^4 u}{\partial y^4}(\mathbf{x}_{i,j}) + O(h_y^4).$$

Replacing $u_{i,j}$ with $u(\mathbf{x}_{i,j})$ in (2) and inserting those above formulae into it, we obtain

$$L_h[u(\mathbf{x}_{i,j})] = -\frac{\partial^2 u}{\partial x^2}(\mathbf{x}_{i,j}) - \frac{\partial^2 u}{\partial y^2}(\mathbf{x}_{i,j}) - \frac{1}{12}\left(h_x^2 \frac{\partial^4 u}{\partial x^4}(\mathbf{x}_{i,j}) + h_y^2 \frac{\partial^4 u}{\partial y^4}(\mathbf{x}_{i,j})\right) + O(h_x^2 + h_y^2). \tag{4}$$

Similarly,

$$u(\mathbf{x}_{i+1,j+1}) - 2u(\mathbf{x}_{i,j+1}) + u(\mathbf{x}_{i-1,j+1}) = h_x^2 \frac{\partial^2 u}{\partial x^2}(\mathbf{x}_{i,j+1}) + \frac{h_x^4}{12}\frac{\partial^4 u}{\partial x^4}(\mathbf{x}_{i,j+1}) + O(h_x^4),$$

$$-2u(\mathbf{x}_{i+1,j}) + 4u(\mathbf{x}_{i,j}) - 2u(\mathbf{x}_{i-1,j}) = -2h_x^2 \frac{\partial^2 u}{\partial x^2}(\mathbf{x}_{i,j}) - \frac{h_x^4}{6}\frac{\partial^4 u}{\partial x^4}(\mathbf{x}_{i,j}) + O(h_x^4),$$

$$u(\mathbf{x}_{i+1,j-1}) - 2u(\mathbf{x}_{i,j-1}) + u(\mathbf{x}_{i-1,j-1}) = h_x^2 \frac{\partial^2 u}{\partial x^2}(\mathbf{x}_{i,j-1}) + \frac{h_x^4}{12}\frac{\partial^4 u}{\partial x^4}(\mathbf{x}_{i,j-1}) + O(h_x^4).$$

By Tylor development, we have

$$\frac{\partial^2 u}{\partial x^2}(\mathbf{x}_{i,j+1}) = \frac{\partial^2 u}{\partial x^2}(\mathbf{x}_{i,j}) + h_y \frac{\partial^3 u}{\partial y \partial x^2}(\mathbf{x}_{i,j}) + \frac{h_y^2}{2}\frac{\partial^4 u}{\partial y^2 \partial x^2}(\mathbf{x}_{i,j}) + \frac{h_y^3}{3!}\frac{\partial^5 u}{\partial y^3 \partial x^2}(\mathbf{x}_{i,j}) + \frac{h_y^4}{4!}\frac{\partial^6 u}{\partial y^4 \partial x^2}(\mathbf{x}_{i,j}) + O(h_y^4),$$

$$\frac{\partial^2 u}{\partial x^2}(\mathbf{x}_{i,j-1}) = \frac{\partial^2 u}{\partial x^2}(\mathbf{x}_{i,j}) - h_y \frac{\partial^3 u}{\partial y \partial x^2}(\mathbf{x}_{i,j}) + \frac{h_y^2}{2}\frac{\partial^4 u}{\partial y^2 \partial x^2}(\mathbf{x}_{i,j}) - \frac{h_y^3}{3!}\frac{\partial^5 u}{\partial y^3 \partial x^2}(\mathbf{x}_{i,j}) + \frac{h_y^4}{4!}\frac{\partial^6 u}{\partial y^4 \partial x^2}(\mathbf{x}_{i,j}) + O(h_y^4),$$

$$\frac{\partial^4 u}{\partial x^4}(\mathbf{x}_{i,j+1}) = \frac{\partial^4 u}{\partial x^4}(\mathbf{x}_{i,j}) + h_y \frac{\partial^5 u}{\partial y \partial x^4}(\mathbf{x}_{i,j}) + \frac{h_y^2}{2}\frac{\partial^6 u}{\partial y^2 \partial x^4}(\mathbf{x}_{i,j}) + O(h_y^2),$$

$$\frac{\partial^4 u}{\partial x^4}(\mathbf{x}_{i,j-1}) = \frac{\partial^4 u}{\partial x^4}(\mathbf{x}_{i,j}) - h_y \frac{\partial^5 u}{\partial y \partial x^4}(\mathbf{x}_{i,j}) + \frac{h_y^2}{2}\frac{\partial^6 u}{\partial y^2 \partial x^4}(\mathbf{x}_{i,j}) + O(h_y^2).$$

Then

$$\begin{aligned} &u(\mathbf{x}_{i+1,j+1}) - 2u(\mathbf{x}_{i,j+1}) + u(\mathbf{x}_{i-1,j+1}) - 2u(\mathbf{x}_{i+1,j}) + 4u(\mathbf{x}_{i,j}) - 2u(\mathbf{x}_{i-1,j}) + \\ &\qquad u(\mathbf{x}_{i+1,j-1}) - 2u(\mathbf{x}_{i,j-1}) + u(\mathbf{x}_{i-1,j-1}) \\ &= h_x^2 h_y^2 \frac{\partial^4 u}{\partial y^2 \partial x^2}(\mathbf{x}_{i,j}) + \frac{1}{12} h_x^2 h_y^4 \frac{\partial^6 u}{\partial y^4 \partial x^2}(\mathbf{x}_{i,j}) + \frac{1}{12} h_x^4 h_y^2 \frac{\partial^6 u}{\partial y^2 \partial x^2}(\mathbf{x}_{i,j}) + O(h_x^2 h_y^4) + O(h_x^4 h_y^2). \end{aligned} \tag{5}$$

Combining (4) with (5), we have

$$\bar{L}_h[u(\mathbf{x}_{i,j})] = -\frac{\partial^2 u}{\partial x^2}(\mathbf{x}_{i,j}) - \frac{\partial^2 u}{\partial y^2}(\mathbf{x}_{i,j}) - \frac{1}{12}\left(h_x^2\frac{\partial^2}{\partial x^2} + h_y^2\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)(\mathbf{x}_{i,j}) + O((h_x^2+h_y^2)^2).$$

It remains to determine $\bar{f}(\mathbf{x}_{i,j})$. We note that $f(\mathbf{x}_{i,j}) = -[\Delta u](\mathbf{x}_{i,j})$, and

$$\frac{1}{12}\left(h_x^2\frac{\partial^2 f}{\partial x^2} + h_y^2\frac{\partial^2 f}{\partial y^2}\right)(\mathbf{x}_{i,j}) = -\frac{1}{12}\left(h_x^2\frac{\partial^2}{\partial x^2} + h_y^2\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)(\mathbf{x}_{i,j}),$$

which leads to the desired result: $\bar{R}_{i,j} = O(h_x^4 + h_y^4)$ if $h_x = O(h_y)$. □

Appendix: Notations for Discrete Representation

Let $I=[a,b]$. We define the discrete grid points as

$$a=x_0<x_1<\cdots<x_N=b.$$

We introduce the following sets:

$$I_h=\{x_1,\cdots,x_{N-1}\},\ \bar{I}_h=\{x_0,x_1,\cdots,x_N\},\ I_h^+=\{x_1,\cdots,x_N\}.$$

The grid spacing is defined as

$$h_i=x_i-x_{i-1},\quad i=1,\cdots,N.$$

Additionally, we define the averaged grid spacing:

$$\begin{aligned}\bar{h}_i&=\frac{1}{2}(h_i+h_{i+1}),\ i=1,\cdots,N-1,\\ \bar{h}_0&=\frac{1}{2}h_1,\quad \bar{h}_N=\frac{1}{2}h_N.\end{aligned}$$

A discrete function defined on $\bar{I}_h$ is denoted as

$$v_h=[v_0,v_1,\cdots,v_N]^{\mathrm{T}}.$$

We define the following difference operators:

$$\begin{aligned}(v_i)_{\bar{x}}&:=v_{i,\bar{x}}:=\frac{v_i-v_{i-1}}{h_i},\ i=1,\cdots,N,\\ (v_i)_x&:=v_{i,x}:=\frac{v_{i+1}-v_i}{h_{i+1}},\ i=0,\cdots,N-1,\\ (v_i)_{\hat{x}}&:=v_{i,\hat{x}}:=\frac{v_{i+1}-v_i}{\bar{h}_i},\ i=0,\cdots,N-1.\end{aligned}$$

The discrete inner products are given by

$$(u_h,v_h)_{I_h}=\sum_{i=1}^{N-1}u_iv_i\bar{h}_i,\ (u_h,v_h)_{\bar{I}_h}=\sum_{i=0}^{N}u_iv_i\bar{h}_i,\ (u_h,v_h)_{I_h^+}=\sum_{i=1}^{N}u_iv_ih_i. \tag{6}$$

We define the discrete norms as follows:

$$\begin{aligned}&\|v_h\|_c:=\max_{\bar{I}_h}|v_i|,\ \|v_h\|_0:=(v_h,v_h)_{\bar{I}_h}^{1/2},\\ &|v_h|_1:=((v_h)_{\bar{x}},(v_h)_{\bar{x}})_{I_h^+}^{1/2},\ \|v_h\|_1^2=\|v_h\|_0^2+|v_h|_1^2.\end{aligned} \tag{7}$$

The discrete integral by parts:

$$\sum_{i=m+1}^{n}v_i(w_i)_{\bar{x}}h_i=-\sum_{i=m}^{n-1}(v_i)_xw_ih_{i+1}+v_nw_n-v_mw_m,\ \text{for some } 0\leqslant m<n\leqslant N. \tag{8}$$

The discrete Green formula:

$$\sum_{i=m+1}^{n-1}((u_i)_{\bar{x}})_{\hat{x}}\,v_i\bar{h}_i=-\sum_{i=m+1}^{n}(u_i)_{\bar{x}}(v_i)_{\bar{x}}h_i+(u_n)_{\bar{x}}v_n-(u_m)_xv_m,\ \text{for some } 0\leqslant m<n\leqslant N. \tag{9}$$

The discrete Cauchy-Schwarz inequality states that

$$|(u_h,v_h)_{\bar{I}_h}|\leqslant(u_h,u_h)_{\bar{I}_h}^{1/2}(v_h,v_h)_{\bar{I}_h}^{1/2}. \tag{10}$$

If $v_0=0$ (or $v_N=0$ or $v_0=v_N=0$), the discrete Poincaré inequality holds:

$$\|v_h\|_c\leqslant C|v_h|_1,\quad \|v_h\|_0\leqslant C|v_h|_1, \tag{11}$$

where C is a constant depending only on a and b.