Exercise 1.2. *Prove some alternative forms of the Poincaré inequality:*

$$\|v\|_{L^\infty} \leqslant c_1\|v'\|_0, \quad \forall v \in \{v \in H^1(I),\ v(0) = 0\}.$$
$$\|v\|_0 \leqslant c_2\|v'\|_0, \quad \forall v \in \{v \in H^1(I),\ v(0) = 0\}.$$

Proof. Let $V = \{v \in H^1(I),\ v(0) = 0\}$ and $U = \{v \in C^\infty(I),\ v(0) = 0\}$. Then U is dense in V with respect to $\|\cdot\|_1$, i.e., $\forall v \in V$, there exists $\{v_n\} \subset U$ such that

$$\lim_{n\to\infty} \|v_n - v\|_1 = 0.$$

Thus

$$\begin{aligned}
&\|v - v_n\|_{L^\infty} \leqslant \|v - v_n\|_1 \to 0,\ \text{as } n \to \infty,\ \text{(by Gagliardo-Nirenberg inequality)}\\
&\|v - v_n\|_0 \leqslant \|v - v_n\|_1 \to 0,\ \text{as } n \to \infty,\\
&\|v' - v_n'\|_0 = |v - v_n|_1 \leqslant \|v - v_n\|_1 \to 0,\ \text{as } n \to \infty.
\end{aligned}$$

Therefore, it is sufficient to show that the inequalities hold for any $v \in U$, which is obvious since

$$|v(x)| = \left|\int_0^x v'(x)\mathrm{d}x\right| \leqslant \left(\int_0^x 1^2\ \mathrm{d}t\right)^{\frac{1}{2}} \left(\int_0^x |v_n'(t)|^2\ \mathrm{d}t\right)^{\frac{1}{2}} \leqslant \|v'\|_0.$$

□

Exercise 1.3. *Consider the boundary value problem:*

$$\begin{cases} -u''(x) = f(x), & x \in I := (0,1), \\ u(0) = u'(1) = 0, \end{cases} \tag{2}$$

where f is a given continuous function. Let

$$V = \{v : v \text{ and } v' \text{ are square integrable on } [0,1],\ \text{and } v(0) = 0\}.$$

The corresponding minimization problem of (2) reads: Find $u \in V$, such that

$$\mathcal{F}(u) = \inf_{v\in V} \mathcal{F}(v), \tag{3}$$

where $\mathcal{F}$ is defined as:

$$\mathcal{F}(v) = \frac{1}{2}(v', v') - (f, v), \quad \forall v \in V.$$

The corresponding variational problem of (2): Find $u \in V$, such that

$$(u', v') = (f, v), \quad \forall v \in V. \tag{4}$$

Prove that:
1) All three problems (2), (3) and (4) are equivalent.
2) The problem (2) admits one unique solution.
3) The solution of (4) is unique.

Proof.
1).
• (2) ⇔ (4). Suppose that u is the solution of (2). Then $\forall v \in V$, we have $(-u'', v) = (f, v)$ and

$$(-u'', v) = (u', v') - u'(1)v(1) + u'(0)v(0) = (u', v'),$$

which leads to that u is the solution of (4). Conversely, we suppose that u is the solution of (4). Then we have

$$(u', v') = (f, v), \quad \forall v \in C_0^\infty(I),$$

which leads to

$$(-u'', v) = (f, v), \quad \forall v \in C_0^\infty(I),$$

where u'' is the derivative of u' in the distribution sense. Thus $-u'' = f$ in the distribution sense. For the boundary conditions, $u(0) = 0$ is obvious since $u \in V$, and $u'(1) = 0$ follows from integral by parts, i.e.,

$$(u', v') = (-u'', v) + u'(1)v(1), \quad \forall v \in V.$$

• (3) $\Leftrightarrow$ (4). Suppose that u is the solution of (3). Then $\forall \alpha \in \mathbb{R}$ and $v \in V$, we have $\mathcal{F}(u) \leqslant \mathcal{F}(u + \alpha v)$, which is convex over α and attains its minimum at $\alpha = 0$. Thus

$$\left.\frac{\mathrm{d}\mathcal{F}(u + \alpha v)}{\mathrm{d}\alpha}\right|_{\alpha=0} = 0,$$

which leads to $(u', v') = (f, v)$. Conversely, we suppose that u is the solution of (4). For any $v \in V$, we set $w = v - u$. Then

$$\begin{aligned}
\mathcal{F}(v) = \mathcal{F}(u + w) &= \frac{1}{2}(u' + w', u' + w') - (f, u + w) \\
&= \frac{1}{2}(u', u') - (f, u) + (w', w') + (u', w') - (f, w) \\
&= \mathcal{F}(u) + \|w'\|_0^2 \geqslant \mathcal{F}(u).
\end{aligned}$$

This implies that $\mathcal{F}(u) = \inf_{v \in V} \mathcal{F}(v)$.

2). If we set

$$u(x) = -\int_0^x \int_0^t f(s)\mathrm{d}s\mathrm{d}t + x\int_0^1 f(s)\mathrm{d}s,$$

then u is the solution of (2). Suppose that u_1 and u_2 are two solutions of (2). Let $\tilde{u} = u_1 - u_2$, then

$$\begin{cases} -\tilde{u}'' = 0, \; x \in I, \\ \tilde{u}(0) = \tilde{u}'(1) = 0. \end{cases}$$

Thus $\tilde{u}$ is a bounded harmonic function. By Liouville's theorem, which states any bounded harmonic function is constant, we have $\tilde{u} = 0$.

3). Let $a(u, v) = (u', v')$. By Lax-Milgram Lemma, it is sufficient to check that

(i) $f \in L^2(I)$.

(ii) $a(\cdot, \cdot)$ is a bilinear form.

(iii) $a(\cdot, \cdot)$ is continuous, i.e.,

$$|a(u, v)| = |(u', v')| \leqslant \|u\|_1 \|v\|_1, \quad \forall u, v \in V.$$

(iv) $a(\cdot, \cdot)$ is coercive. By Poincaré inequality:

$$\|v\|_1^2 = \|v\|_0^2 + \|v'\|_0^2 \leqslant (1 + c_p^2)\|v'\|_0^2, \quad \forall v \in V,$$

which leads to

$$a(v, v) = \|v'\|_0^2 \geqslant \frac{1}{1 + c_p^2}\|v\|_1^2.$$

□