Exercise 2.4. *We consider the problem*

$$\begin{cases} -(\alpha u')'(x) + (\beta u')(x) + (\gamma u)(x) = f(x), \quad x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{11}$$

where α, β, and γ are continuous functions on $[0,1]$ with $\alpha(x) \geqslant \alpha_0 > 0$ for all $x \in [0,1]$.
1) Give the weak form of the problem (11).
2) Prove the weak problem admits a unique solution under the following assumption

a. $\beta(x) = 0$, $\gamma \geqslant 0$ for all $x \in [0,1]$;

b. $-\frac{1}{2}\beta' + \gamma \geqslant 0$ for all $x \in [0,1]$;

c. see [Brezis p. 224].

3) Propose a $P1-$FEM for the numerical solution of (11).
4) Carry out an error analysis.

Proof.
1). Let $I = (0,1)$ and $V = H_0^1(I)$. The weak form reads

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ a(u,v) = \mathcal{F}(v), \ \forall v \in V, \end{cases} \tag{1}$$

where the bilinear form $a(u,v) = (\alpha u', v') + (\beta u', v') + (\gamma u, v)$ and $\mathcal{F}(v) = (f, v)$.
2). It is clear that $a(\cdot,\cdot)$ is a bilinear form, and $\mathcal{F}$ is a continuous functional from V to $\mathbb{R}$. By Lax-Milgram Lemma, it remains to show that $a(\cdot,\cdot)$ is continuous and coercive. Continuity is clear for all cases since the coefficients are continuous over $\bar{I}$, i.e.,

$$\begin{aligned} |a(u,v)| &\leqslant \|\alpha\|_\infty \|u'\|_0 \|v'\|_0 + \|\beta\|_\infty \|u'\|_0 \|v\|_0 + \|\gamma\|_\infty \|u\|_0 \|v\|_0 \\ &\leqslant (\|\alpha\|_\infty + \|\beta\|_\infty + \|\gamma\|_\infty) \|u\|_1 \|v\|_1, \quad \forall u, v \in V. \end{aligned}$$

For the coercivity, we consider case by case:

a. By Poincaré inequality on $H_0^1(I)$, there exists $C > 0$ such that

$$a(v,v) = (\alpha v', v') + (\gamma v, v) \geqslant \alpha_0 \|v'\|_0^2 \geqslant C\|v\|_1^2, \quad \forall v \in V.$$

b. Note that for any $v \in V$, we have $(\beta v', v) = (\beta/2, (v^2)') = (-\beta'/2, v^2)$. Thus by Poincaré inequality on $H_0^1(I)$, we have

$$\begin{aligned} a(v,v) &= (\alpha v', v') + (\beta v', v) + (\gamma v, v) = (\alpha v', v') + ((-\beta'/2 + \gamma)v, v) \\ &\geqslant \alpha_0 \|v'\|_0^2 \geqslant C\|v\|_1^2, \quad \forall v \in V. \end{aligned}$$

c. Omitted.

3). Let $0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1$ be a grid on $I = (0,1)$. The space of piecewise linear polynomials is denoted by $X_h^1 = \text{span}\{\varphi_0, \cdots, \varphi_{N+1}\}$. Then $V_h = V \cap X_h^1 = \text{span}\{\varphi_1, \cdots, \varphi_N\}$. Let $u_h = \sum_{j=1}^N u_j \varphi_j(x)$ and

$$\sum_{j=1}^N a(\varphi_j, \varphi_i) u_j = \mathcal{F}(\varphi_i), \quad i = 1, \cdots, N.$$

4). Let u_h be the solution of $P1-$FEM. Note that $a(u - u_h, v_h) = 0$, $\forall v_h \in V_h$. Thus

$$\|u - u_h\|_1^2 \leqslant Ca(u - u_h, u - u_h) = Ca(u - u_h, u - v_h) \leqslant C\|u - u_h\|_1 \|u - v_h\|_1,$$

for any $v_h \in V_h$. It leads to $\|u-u_h\|_1 \leqslant \inf_{v_h\in V_h} \|u-v_h\|_1 \leqslant \|u-u_I\|_1$, where u_I denotes the interpolation of u into V_h. By the Poincaré inequality, we know that $C\|v\|_1 \leqslant \|v'\|_0 \leqslant C\|v\|_1$, then

$$\|u'-u_h'\|_0 \leqslant \|u'-u_I'\|_0 \leqslant Ch\|u''\|_0.$$

To obtain error estimate for $\|u-u_h\|_0$, we use duality argument. Consider the dual problem of (1): given $r \in L^2(I)$,

$$\begin{cases} \text{Find } \varphi(r) \in V \text{ such that} \\ a(v,\varphi(r)) = (r,v), \quad \forall v \in V. \end{cases}$$

The dual problem admits a unique solution $\varphi(r)$ since $a(\cdot,\cdot)$ is continuous and coercive. If we suppose $\varphi(r) \in H^2(I)$, and there exists constant $C>0$ such that $\|\varphi''(r)\|_0 \leqslant C\|r\|_0$.

Thus we denote $\varphi_I(r)$ being the interpolation of $\varphi(r)$ into V_h and obtain

$$\begin{aligned}
\|u-u_h\|_0 &= \sup_{r\in L^2(I),\ r\neq 0} \frac{(r,u-u_h)}{\|r\|_0} = \sup_{r\in L^2(I),\ r\neq 0} \frac{a(u-u_h,\varphi(r))}{\|r\|_0} \\
&= \sup_{r\in L^2(I),\ r\neq 0} \frac{a(u-u_h,\varphi(r)-\varphi_I(r))}{\|r\|_0} \\
&\leqslant C \sup_{r\in L^2(I),\ r\neq 0} \frac{\|u-u_h\|_1 \|\varphi(r)-\varphi_I(r)\|_1}{\|r\|_0} \\
&\leqslant C \sup_{r\in L^2(I),\ r\neq 0} \frac{\|u'-u_h'\|_0 \|\varphi'(r)-\varphi_I'(r)\|_0}{\|r\|_0} \\
&\leqslant Ch\|u'-u_h'\|_0 \sup_{r\in L^2(I),\ r\neq 0} \frac{\|\varphi''(r)\|_0}{\|r\|_0} \\
&\leqslant Ch\|u'-u_h'\|_0.
\end{aligned}$$

□

Exercise 3.1. *Let $\Omega = (a,b)^2$, $f \in L^2(\Omega)$. Consider the Dirichlet elliptic problem*

$$\begin{cases} -\Delta u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \Omega, \\ u\big|_\Gamma = 0. \end{cases}$$

1. Prove the following Poincaré inequality holds: there exists a constant c, depending only on a and b, such that

$$\|v\|_1 \leqslant c|v|_1, \quad \forall v \in H_0^1(\Omega).$$

2. Prove that the Dirichlet elliptic problem admits a unique weak solution in $H_0^1(\Omega)$, and the solution u satisfies

$$\|u\|_1 \leqslant c\|f\|_0,$$

where c is a constant.

Proof. 1. Note that for any $y \in (a,b)$,

$$v(x,y) = \int_a^y \partial_y v(x,y)\mathrm{d}y,$$

we have $|v(x,y)| \leqslant (b-a)^{1/2}\|\partial_y v\|_0$. Similarly we have $|v(x,y)| \leqslant (b-a)^{1/2}\|\partial_x v\|_0$. It leads to

$$\|v\|_0 \leqslant \frac{(b-a)^{3/2}}{\sqrt{2}}|v|_1.$$

Thus $\|v\|_1 = (\|v\|_0^2 + |v|_1^2)^{1/2} \leqslant c|v|_1$.

2. The variational form reads

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ such that} \\ a(u,v) = (f,v), \quad \forall v \in H_0^1(\Omega), \end{cases}$$

where $a(u,v) = (\nabla u, \nabla v)$. It is clear that $a(\cdot,\cdot)$ is a continuous bilinear form. Its coercivity is guaranteed by Poincaré inequality, that is,

$$a(v,v) = |\nabla v|_1^2 \geqslant c\|v\|_1^2, \quad \forall v \in H_0^1(\Omega).$$

Thus by Lax-Milgram lemma, there exists a unique weak solution u such that

$$\|u\|_1 \leqslant c \sup_{v \in H_0^1(\Omega), v \neq 0} \frac{(f,v)}{\|v\|_1} \leqslant c\|f\|_0.$$

$\square$