Notes on Müntz-Jackson Theorem

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1 Introduction

We study Chapter 11 of [Lorentz (1996)]. First, let us take a look at Section 1, where we introduce key concepts and explain some notation.

1.1 From algebraic to Müntz polynomials: denseness

Algebraic polynomials, which are frequently employed in approximation theory, take the form:

$$u(x) = \sum_{k=0}^{n} a_k x^k, \quad a_k \in \mathbb{R},$$

and possess the property of density as characterized by the Weierstrass' Theorem.

Theorem 1.1 (Weierstrass)**.** *Let $f \in C[a, b]$, then for any $\varepsilon > 0$, there exists a algebraic polynomial $u(x) \in \mathbb{P}$ such that*

$$\max_{a \leqslant x \leqslant b} |u(x) - f(x)| < \varepsilon.$$

Remark 1.1. *Why is Weierstrass' Theorem so important?*
- It enables us to utilize algebraic polynomials, which are inherently simple functions, to approximate more intricate functions with arbitrary precision.
- It facilitates the construction of numerical quadrature formulas, which are essential for approximating definite integrals, and it simplifies the process of solving PDEs by allowing us to approximate solutions that may be difficult to obtain analytically.

Several applications of algebraic polynomials in numerical analysis include:

- Projection,

- Interpolation,

- Quadratures,

- Numerical Solution to PDEs (FEM & Spectral Methods, etc.),

- $\cdots$.

Drawbacks of algebraic polynomials: When approximating problems with singularities (where derivatives may become unbounded or non-integrable) or low regularity (such as discontinuities), algebraic polynomials may struggle to provide accurate representations. To address these challenges, two strategies are commonly used:

- Local approximation (piecewise), which is general but **expensive** in computational cost,

- **Global** approximation.

To address the limitations **while retaining the benefits** of polynomials for **global approximation**, a natural approach is to extend the concept of algebraic polynomials. Three common generalizations are encountered in practice:

- Rational polynomials,

$$\frac{u(x)}{v(x)}, \ \forall u, v \in \mathbb{P} \text{ and } v \neq 0,$$

- **Müntz polynomials**,

$$x^\lambda, \ \lambda \in \mathbb{C},$$

- Mapped algebraic polynomials,

$$x^k, \quad x = T(y).$$

Among these, Müntz polynomials are of particular interest and will be the focus of our investigation.

Our goal is to use **Müntz** polynomials as a **global** approximation for certain numerical analysis problems that may exhibit **singularities**, offering superior performance compared to algebraic polynomials.

Let $\Lambda_\infty = \{0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots\}$ be a sequence of real numbers. The functions in the form

$$M(x) = \sum_{k=0}^{n} a_k x^{\lambda_k}, \quad a_k \in \mathbb{R}, \tag{1.1}$$

are called **Müntz polynomials** due to the Müntz's Theorem that characterizes the denseness: We only consider the real coefficients a_k and sequences λ_k.

Theorem 1.2 (Müntz (1914), part of Theorem 1.1 in [Lorentz (1996)]). $\mathcal{M}(\Lambda_\infty) = \mathrm{span}\{x^{\lambda_k} : k = 0, 1, \cdots\}$, the associated **Müntz space** to Λ_∞, is a dense subset of $C[0,1]$ if and only if

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} = \infty. \tag{1.2}$$

We split the Theorem 1.1 in Lorentz (1996) into two parts: one is on Theorem 1.2, the other is the first item in following Remark.

Remark 1.2. Theorem 1.2 can be indeed extended in several ways:

- L_p space. Let $p \in (0, \infty)$, $\mathcal{M}(\Lambda_\infty)$ is dense in $L_p(0,1)$ if and only if $\sum_{k=1}^{\infty} \lambda_k^{-1} = \infty$. (see [Erdélyi (2001), Theorem 1.2]) Sometimes we write $L_p(0,1)$ as $L_p[0,1]$. Note L_∞ will never be considered in studying the denseness property since it is not separable. For the completeness of the spaces, we often consider $p \in [1, \infty)$. Then the convention we will use sometimes

$$L_p[0,1] = \begin{cases} L_p[0,1], & 1 \leqslant p < \infty, \\ C[0,1], & p = \infty. \end{cases}$$

- L_p space (Müntz-Szász). Let $p \in (0, \infty)$, $\Lambda_\infty = (\lambda_k)_{k=0}^{\infty}$ is a sequence of distinct real numbers greater than $-1/p$. Then $\mathcal{M}(\Lambda_\infty)$ is dense in $L_p(0,1)$ if and only if

$$\sum_{k=0}^{\infty} \frac{\lambda_k + 1/p}{(\lambda_k + 1/p)^2 + 1} = \infty.$$

(see [Almira (2007), Theorem 13])

- Complex sequence, known as Müntz-Szász Theorem (see [Agler (2022), Example 1.6]). Let $\Lambda_\infty = \{\lambda_0, \lambda_1, \lambda_2, \cdots\}$ be a sequence of distinct numbers satisfying the real part $\mathrm{Re}(\lambda_k) > -1/2$ for $k = 0, 1, 2, \cdots$. The space $\mathrm{span}\{x^{\lambda_k} : k = 0, 1, 2, \cdots\}$ taken over $\mathbb{C}$, associated with Λ_∞, is a dense subset of $L_2[0,1]$ if and only if

$$\sum_{k=0}^{\infty} \frac{2\,\mathrm{Re}(\lambda_k) + 1}{|\lambda_k + 1|^2} = \infty.$$

- Away from the origin. Let $\Lambda_\infty = \{\lambda_k\}_{k=0}^{\infty} \subset \mathbb{R}$ be a sequence of distinct real numbers, and let $0 < a < b$. Then $\mathcal{M}(\Lambda_\infty)$ is dense in $C[a,b]$ if and only if $\sum_{\lambda_k \neq 0} |\lambda_k|^{-1} = \infty$. (see [Almira (2007), Theorem 29])

Remark 1.3. *With the dense properties, it is anticipated that functions f in $L_p(0,1)$ or $C[0,1]$ can be approximated, which drives the development of numerical techniques for high-efficiency applications, surpassing the capabilities of algebraic polynomials.*

Remark 1.4. *However, the dense properties cannot be applied directly. Our interest lies in examining, for a fixed n, the convergence performance of $\inf_{p \in \mathcal{M}(\Lambda_n)} \|f - p\|$ in certain metrics.* **This is the main topic of this chapter and will be discussed later.**

1.2   Jackson-type Theorem

We will use the notation convention: $n \geqslant 1$ will be fixed and we write $\Lambda := \Lambda_n = \{\lambda_0, \lambda_1, \cdots, \lambda_n\}$.
Jackson Theorem for algebraic polynomials. When $\lambda_k = k$ the Müntz polynomials reduce to algebraic polynomials. We consider in the interval $[a, b]$, then $\forall f \in C[a, b]$, (see [Wang Renhong (1999), p.54])

$$E(f, \Lambda)_\infty := \inf_{v \in \mathbb{P}_n} \|f - v\|_\infty \leqslant K\omega(f, 1/n)_\infty,$$

where K is a constant and only depends on a and b.

$$\omega(f, \delta)_\infty = \sup_{\substack{|x-y| \leqslant \delta \\ x,y \in [a,b]}} \{|f(x) - f(y)|\}, \quad \delta > 0.$$

If $f \in C^1[a, b]$, then $|f(x) - f(y)| \leqslant |x - y|\|f'\|_\infty$ and we have

$$E(f, \Lambda)_\infty \leqslant \frac{K}{n}\|f'\|_\infty.$$

Moreover, if $f \in C^m[a, b]$, then when $n > m$ we have (see [Wang Renhong (1999), p.54])

$$E(f, \Lambda)_\infty \leqslant A_m \left(\frac{1}{n}\right)^m \omega\left(f^{(m)}, \frac{b-a}{2(n-m)}\right),$$

where A_m is a constant and only depends on a, b and m.

Now we consider the general L_p best approximation.
L_p-Best Approximation by Müntz Polynomials. Let $f \in L_p(0,1)$ if $1 \leqslant p < \infty$ (or $C[0,1]$ if $p = \infty$). The error of approximation from $\mathcal{M}(\Lambda)$ to f is

$$E(f, \Lambda)_p := \inf_{M \in \mathcal{M}(\Lambda)} \|f - M\|_{L_p}. \tag{1.3}$$

Müntz-Jackson Theorem: An **estimation of convergent performance** of $E(f, \Lambda)_p$ in terms of the smoothness or the differentiability properties of f.

Moduli of Continulity. Measuring the smoothness of a function by differentiability is too crude for many purpose in approximation. More subtle measurements are provided by the moduli of continuity. For interval $[a, b]$ and $f \in L_p[a, b]$ if $1 \leqslant p < \infty$ (or $f \in C[a, b]$ if $p = \infty$), (see [Lorentz (1993), pp. 40-44])

$$\omega(f, \delta)_p := \sup_{0 \leqslant h \leqslant \delta} \|f(\cdot + h) - f(\cdot)\|_{L_p[a, b-h]}.$$

It is clear that

- if $f \in C^1[a, b]$ then $\omega(f, \delta)_p \leqslant \delta \|f'\|_\infty$.

- If f is $\alpha-$Lipschitz continuous for $0 < \alpha \leqslant 1$, i.e., $|f(x) - f(y)| \leqslant L|x - y|^\alpha$ for some constant L, we have $\omega(f, \delta)_p \leqslant L\delta^\alpha$. The function $f(x) = x^\alpha$ is the simplest case.

Proof of $|x^\alpha - y^\alpha| \leqslant |x - y|^\alpha$ for $x, y \in [a, b]$. In the cases $x = 0$, $x = y$ or $y = 0$, the conclusion holds obviously. Otherwise, suppose that $x > y$,

$$\frac{|x^\alpha - y^\alpha|}{|x - y|^\alpha} = \frac{\left|1 - \left(\frac{y}{x}\right)^\alpha\right|}{\left|1 - \left(\frac{y}{x}\right)\right|^\alpha}.$$

Since $0 < 1 - y/x < 1$ and $0 < \alpha \leqslant 1$, we get

$$\left|1 - \left(\frac{y}{x}\right)\right|^\alpha \geqslant \left|1 - \left(\frac{y}{x}\right)\right| \quad \text{and} \quad \left|1 - \left(\frac{y}{x}\right)^\alpha\right| \leqslant \left|1 - \left(\frac{y}{x}\right)\right|.$$

Hence

$$\frac{|x^\alpha - y^\alpha|}{|x - y|^\alpha} \leqslant 1.$$

□

To facilitate the characterization of the Müntz-Jackson bounds, we introduce the **Blaschke product**

$$B_p(z) := B_p(z, \Lambda) := \prod_{k=1}^{n} \frac{z - \lambda_k - 1/p}{z + \lambda_k + 1/p}, \quad 1 \leqslant p \leqslant \infty, \tag{1.4}$$

and the expression

$$\varepsilon_p := \varepsilon_p(\Lambda) := \max_{y \geqslant 0} \left|\frac{B_p(1 + iy)}{1 + iy}\right|. \tag{1.5}$$

ε_p is called **the index of approximation** of Λ in $C[0, 1]$ or $L_p[0, 1]$.
ε_p is well-defined due to

$$f(y) = \left|\frac{B_p(1 + iy)}{1 + iy}\right|, \quad \text{and} \quad \lim_{y \to \infty} f(y) = 0.$$

In fact, there exist $Y > 0$ such that

$$f(y) < f(0), \quad \forall y > Y,$$

then $f(y)$ achieves its maximum on $[0, Y]$ since the interval is compact.

Now we aim to consider the following theorem.

Theorem 1.3 (Theorem 1.2 in [Lorentz (1996)])**.** *Let $\Lambda = \{0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n\}$. Then*

(i). for $1 \leqslant p \leqslant \infty$ and $f \in W_p^1[0, 1]$,

$$E(f, \Lambda)_p \leqslant A_p \varepsilon_p \|f'\|_p, \tag{1.6}$$

where $A_p < 2^{29}$.

(ii). The inequality (1.6) is false for each $p \geqslant 1$ and each Λ if Λ_p is replaced by $1/600$.

Theorem 1.2 in Lorentz (1996) is the general case for $1 \leqslant p \leqslant \infty$. However, in this chapter, we only consider the special case $2 \leqslant p \leqslant \infty$.

From Theorem 1.2 (i) in Lorentz (1996), we further obtain the error estimations bounded by modulus of continuity $C\omega(f, \varepsilon_p)_p$, as indicated in Theorem 2.7 in Lorentz (1996).

Question 1. *What if we consider the approximation under $0 < p < 1$? For example, the space $L_0(\Omega)$ with the norm*

$$\|f\|_{L_0} = \exp\left[\int_\Omega \log|f(x)| \mathrm{d}x\right].$$

Where the $W_p^1[0, 1]$ space needs to be specified.

$W_p^1[0, 1]$ **space** is defined as

$$W_p^1[0, 1] = \{f(x) \text{ is absolutely continuous on } [0, 1] : f' \in L_p[0, 1]\}.$$

Definition of absolutely continuity. Let $I \subset \mathbb{R}$ be an interval. $f : I \to \mathbb{R}$ is absolutely continuous, if $\forall \varepsilon > 0$, there exists $\delta > 0$, such that whenever a finite (or infinite) sequence of pairwise disjoint sub-intervals (x_k, y_k) of I satisfies

$$\sum_{k=1}^{N} (y_k - x_k) < \delta,$$

then

$$\sum_{k=1}^{N} |f(y_k) - f(x_k)| < \varepsilon.$$

Recall the Sobolev space $W^{k,p}(\Omega)$ for a domain $\Omega \subset \mathbb{R}^m$,

$$W^{k,p}(\Omega) = \{v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega), |\alpha| \leqslant k\},$$

where the weak derivatives are defined in the distribution sense

$$\langle D^\alpha f, v \rangle = (-1)^{|\alpha|} \left(f, v^{(\alpha)}\right), \forall v \in \mathscr{D}(\Omega).$$

Particularly, $W^{1,p}(0, 1) = \{f \in L_p(0, 1) : f' \in L_p(0, 1)\}$. Question: What is the relationship between $W^{1,p}(0, 1)$ and $W_p^1[0, 1]$? Then $W_p^1[0, 1] = W^{1,p}(0, 1)$ if $p > 1$ and $W_1^1[0, 1] \subset W^{1,1}(0, 1)$.

Proof. It is obvious that $W_p^1[0, 1] \subset W^{1,p}(0, 1)$. If $p > 1$, by the embedding theorem $W^{1,p}(0, 1) \hookrightarrow C[0, 1]$, see [Adams (2003), Chapter 5, 5.8], then

$$f(x) = f(0) + \int_0^x f'(t)\mathrm{d}t, \quad \forall x \in [0, 1].$$

It is obvious that f is absolutely continuous on $[0, 1]$. $\qquad\square$

The plan of the Chapter:

- Sec. 2 proves (1.6) for $p \geqslant 2$ and $A_p < 20$, and some corollaries.

- Sec. 3 proves an inverse of Müntz-Jackson Theorem for $p \geqslant 2$ (i.e., Theorem 1.3 (ii)).

- Sec. 4 proves that for special sequences, ε_p can be replaced by a much simpler expression.

- Sec. 5 proves Markov-type inequality for Müntz polynomials.

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