Chapter 1 Abstract Integration

1, Does there exist an infinite $\sigma-$algebra which has only countably many members?

Solution: No. There exists no such a infinite $\sigma-$algebra $\mathfrak{M}$ which has only countably members.

Proof: Suppose $\mathfrak{M}$ is a $\sigma-$algebra of subsets in a set X which has only infinite countably many members. So we can pick from $\mathfrak{M}$ a countably many members $A_1, A_2, \cdots$. Without loss of generality, $\{A_i\}_{i=1}^{\infty}$ are mutually pairwise disjoint, for otherwise letting

$$\bar{A}_1 = A_1,\ \bar{A}_2 = A_2 - A_1,\ \cdots,\ \bar{A}_k = A_k - \bigcup_{i=1}^{k-1} A_i,\ \cdots$$

then clearly $\bar{A}_i \in \mathfrak{M}$ and $\bar{A}_i \bigcap \bar{A}_j = \emptyset$ if $i \neq j$.

Let 2^N be the collection of all subsets of the natural numbers N, then it is well known that the cardinal number of 2^N is $\overline{\overline{2^N}} = \overline{\overline{[0,1]}} = \mathcal{C}$. (See p25, exercise 3 in the book of Jiang zejian).

For any given $Y \in 2^N$, since Y is a countable set and $\mathfrak{M}$ is a $\sigma-$algebra, $\bigcup_{n\in Y} A_n \in \mathfrak{M}$. Let $f : 2^N \to \mathfrak{M}$ is given by $f(Y) = \bigcup_{n\in Y} A_n$. Then f is well defined and it is one to one, i.e. if $Y, Z \in 2^N,\ Y \neq Z$, then $f(Y) \neq f(Z)$.

In fact, $Y \neq Z$ implies that there exists an $n_0 \in Y \bigcap Z^c \subset N$. Since $A_i \bigcap A_j = \emptyset$ if $i \neq j$ and $n_0 \in Z^c$, by the definition of f, $A_{n_0} \bigcap f(Z) = A_{n_0} \bigcap (\bigcup_{n\in Z} A_n) = \emptyset$. So, $A_{n_0} \in f(Z)^c$ while $A_{n_0} \in f(Y)$, i.e. $f(Y) \neq f(Z)$. Therefore $f : 2^N \to \mathfrak{M}$ is one-to-one, which implies that $\mathcal{C} = \overline{\overline{2^N}} \leq \overline{\overline{\mathfrak{M}}} \leq \overline{\overline{N}} = \mathcal{C}_0$ is a contradiction. So, the contradiction shows that $\mathfrak{M}$ can not has only countable many members.

We give the proof that $\overline{\overline{2^N}} = \overline{\overline{[0,1]}} = \mathcal{C}$ here. For any given $A \subset N$, set $\varphi_A(n) = \{ \begin{array}{ll} 1, & \text{if } n \in A; \\ 0, & \text{if } n \notin A. \end{array}$ Let $g : A \to (0,1)$ is given by $g(A) = 0.\ \varphi_A(1)\varphi_A(2)\cdot\cdot\cdot$, it is obvious that g is one-to-one. So, $\overline{\overline{2^N}} \leq \overline{\overline{[0,1]}}$.

On the other hand, for any $x \in (0,1)$, set $A_x = \{r;\ r \leq x, r \in \mathcal{R}\}$, where $\mathcal{R}$ is the set of all the rational in $(0,1)$. If $x, y \in (0,1)$ and $x \neq y$, by Berstein's theorem, the density of rational in R, $A_x \neq A_y$. So, $\overline{\overline{(0,1)}} \leq \overline{\overline{\mathcal{R}}} = \overline{\overline{2^N}}$.

So, $\overline{\overline{2^N}} = \overline{\overline{[0,1]}} = \mathcal{C}$. □

2, Prove an analogue of Theorem 1.8 for n functions.

Solution: We have the following result.

Theorem 1.8$'$: Let $u_1,\ u_2,\ ...,\ u_n$ be real measurable functions on a measurable space X, let Φ be a continuous mapping of R^N onto a topological space Y, and define

$$h(x) = \Phi(u_1(x), ..., u_n(x))$$

for $x \in X$. Then $h : X \to Y$ is measurable.

Proof: Since $h = \Phi \circ f$, where $f(x) = (u_1(x),\ ...,\ u_n(x))$ and f maps X into R^n. Theorem 1.7 shows that it is enough to prove the measurability of f.

If R is any open rectangle in R^n with sides parallel to the coordinate super plane, then $R = I_1 \times I_2 \times \cdots \times I_n$, where $I_i = (a_i, b_i) \subset R^1$ $(i = 1, 2, ..., n)$ are open segments and

$$f^{-1}(R) = u_1^{-1}(I_1) \bigcap u_2^{-1}(I_2) \bigcap \cdots \bigcap u_n^{-1}(I_n).$$

In fact, for any given $x \in f^{-1}(R)$, we have $f(x) = (u_1(x), ..., u_n(x)) \in R = I_1 \times I_2 \times \cdots \times I_n$ and $u_i(x) \in I_i$, then $x \in u_i^{-1}(I_i)$ for $i = 1, 2, ..., n$, that is $x \in \bigcap_{i=1}^n u_i^{-1}(I_i)$. On the other hand, for any $x \in \bigcap_{i=1}^n u_i^{-1}(I_i)$, we have $x_i \in u_i^{-1}(i)$, $u_i(x) \in I_i$ and $f(x) = (u_1(x), ..., u_n(x)) \in I_1 \times I_2 \times \cdots \times I_n$. So $f^{-1}(R)$ is measurable for each open rectangle interval $R \subset R^n$ as u_i are measurable on X.

By Lindelif's Theorem, every open set V is a countable union of open interval with sides parallel to the coordinate super-plane, $V = \bigcup R_i$, So $f^{-1}(V) = \bigcup_{i=1}^n f^{-1}(R_i)$ is measurable as well. □

3, Prove that if f is a real function on a measurable space X such that $\{x; f(x) > r\}$ is measurable for every rational r, then f if measurable.

Proof: For any $\alpha \in R^1$, there is a sequence of rational $\{r_n;\ n = 1, 2, ...\}$ such that $r_1 > r_2 > r_3 > ... > r_n > ... > \alpha$ with $\lim\limits_{n\to\infty} r_n = \alpha$. So

$$\{x; f(x) > \alpha\} = \bigcup_{n=1}^{+\infty} \{x; f(x) \geq r_n\}.$$

In fact, for any $x \in \{x; f(x) > \alpha\}$, there is a n_0 such that $\alpha < r_{n_0} \leq f(x)$, therefore $x \in \{x; f(x) > \alpha\}$. On the other hand, $r_n > \alpha$ implies that

$$\bigcup_{n=1}^{+\infty} \{x; f(x) \geq r_n\} \subset \{x; f(x) > \alpha\}.$$

By the condition, for any given n, $\{x; f(x) \geq r_n\}$ is measurable, we see that $\{x; f(x) > \alpha\}$ is measurable for any $\alpha \in R^1$. By Theorem 1.12, f is measurable. □

4, Let $\{a_n\}$ and $\{b_n\}$ be sequence in $[-\infty, +\infty]$, prove the following assertions:

(a) $\limsup\limits_{n\to\infty}(-a_n) = -\liminf\limits_{n\to\infty} a_n$;

(b) $\limsup\limits_{n\to\infty}(a_n + b_n) \leq \limsup\limits_{n\to\infty} a_n + \limsup\limits_{n\to\infty} b_n$ provided none of the sums is of the form $\infty - \infty$;

(c) If $a_n \leq b_n$ for all n, then

$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n$$

show by an example that strict inequality can hold in (b).

Proof:

$$\begin{aligned} \text{(a)}\ \limsup_{n\to\infty}(-a_n) &= \inf_{k\geq 1} \sup\{-a_k, -a_{k+1}, ...\} = \inf_{k\geq 1}\{-\inf\{-a_k, -a_{k+1}, ...\}\} \\ &= -\sup_{k\geq 1} \inf\{a_k, a_{k+1}, ...\} = \liminf_{n\to\infty} a_n. \end{aligned}$$

$$\begin{aligned} \text{(b)}\ \limsup_{n\to\infty}(a_n + b_n) &= \inf_{k\geq 1}[\sup\{a_k + b_k, a_{k+1} + b_{k+1}, ...\}] \\ &= \liminf_{k\geq 1}[\sup\{a_k + b_k, a_{k+1} + b_{k+1}, ...\}] \\ &\leq \liminf_{k\to\infty}[\sup_{n\geq k}\{a_n\} + \sup_{n\geq k}\{b_n\}] \\ &= \liminf_{n\to\infty}[\sup\{a_n\}] + \liminf_{n\to\infty}[\sup\{b_n\}] \\ &= \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n. \end{aligned}$$

(c) $\liminf_{n\to\infty} a_n = \lim_{k\to infty} \inf_{n\geq k}\{a_n\} \leq \inf_{k\to\infty}\inf_{n\geq k}\{a_k\} = \liminf_{n\to\infty} a_n$.

In (b), the strict inequality can hold. For example, let $a_n = (-1)^n$, $b_n = -(-1)^n$, then $a_n + b_n = 0$, $\limsup_{n\to\infty} a_n = 1$, $\limsup_{n\to\infty} b_n = 1$, but $\limsup_{n\to\infty}(a_n + b_n) = 0 < 2 = 1 + 1 = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.□

5, (a) Suppose $f : X \to [-\infty, \infty]$ and $g : X \to [-\infty, \infty]$ are measurable. Prove that the sets $\{x;\ f(x) < g(x)\}$, $\{x;\ f(x) = g(x)\}$ are measurable.

(b) Prove that the set of points at which a sequence of measurable real-valued functions converges(to a finite limit) is measurable.

Proof: (a1) Suppose f, g are measurable on X, then $\{x;\ f(x) < g(x)\}$ is measurable. Suppose $\{r_n\}$ is the set of rational in R^1, we claim that

$$\{x;\ f(x) < g(x)\} = \bigcup_{n=1}^{\infty}[\{x;\ f(x) < r_n\}\bigcap\{x;\ r_n < g(x)\}]$$

In fact,

$$\begin{aligned} x_0 \in \{x;\ f(x) < g(x)\} &\iff f(x_0) < g(x_0) \\ &\iff \exists r_{n_0},\ \text{such that } f(x_0) < r_{n_0} < g(x_0) \\ &\iff x_0 \in \textstyle\bigcup_{n=1}^{\infty}[\{x;\ f(x) < r_n\}\bigcap\{x;\ r_n < g(x)\}]. \end{aligned}$$

For any $r \in R$, $\{x;\ f(x) < r\}$ and $\{x;\ r < g(x)\}$ are measurable since f, g are measurable on X. So, by the claim we have proved above, $\{x;\ f(x) < g(x)\}$ is measurable.

(a2) By (a1), $\{x;\ f(x) = g(x)\} = X - [\{x;\ f(x) < g(x)\}\bigcup\{x;\ f(x) > g(x)\}]$ is measurable.

(b) Suppose $\{f_n\}$ is a sequence of real-valued measurable functions on $(X, \Re)$(a measurable space), then by Theorem 1.9(c), for any $n, m \in N$, $|f_n(x) - f_m(x)|$ is a measurable function as $|\cdot|$ is a continuous function on $R^1 \times R^1$. So, $\{x;\ |f_n(x) - f_m(x)| < a\}$ is a measurable set for any $a \in R^1$.

Clearly, the set where $\{f_n(x)\}$ has finite limit is given by

$$A = \bigcap_{k=1}^{+\infty}\bigcup_{N=1}^{+\infty}\bigcap_{n,m\geq N}\{x;\ |f_n(x) - f_m(x)| < \frac{1}{k}\}$$

by Cauchy's criterion for convergence.

A is a measurable set because for any $n, m \in N$ and $k \in N$, $\{x;\ |f_n(x) - f_m(x)| < \frac{1}{k}\} \in \mathfrak{M}$ and $\mathfrak{M}$ is a $\sigma-$algebra. □

6, Let X be an uncountable set, let $\mathfrak{M}$ be the collection of all sets $E \subset X$ such that either E or E^c is at most countable, and define $\mu(E) = 0$ in the first case, $\mu(E) = 1$ in the second. Prove that $\mathfrak{M}$ is a $\sigma-$algebra in X and that μ is a measure on $\mathfrak{M}$. Describe the corresponding measurable functions and their integrals.

Proof: 1.) Since $X^c = \emptyset$ is at most countable, we see that $X \in \mathfrak{M}$.

If $A \in \mathfrak{M}$, then either A or A^c is at most countable, i.e. either A^c or $(A^c)^c = A$ is at most countable, so $A^c \in \mathfrak{M}$ as well.

If $A_n \in \mathfrak{M}$ for $n = 1, 2, 3...$, then $\bigcup_{n=1}^{+\infty} A_n \in \mathfrak{M}$, for if $\bigcup_{n=1}^{+\infty} A_n$ is at most countable, we have $\bigcup_{n=1}^{+\infty} A_n \in \mathfrak{M}$ and if $\bigcup_{n=1}^{+\infty} A_n$ is not at most countable, then there exists n_0, such that $\bigcup_{n=1}^{+\infty} A_{n_0}$

is not countable by the fact that a countable union of a sequence of countable sets is countable, so as $A_{n_0} \in \mathfrak{M}$, we must have $A_{n_0}^c$ is at most countable. hence for

$$(\bigcup_{n=1}^{+\infty} A_n)^c = \bigcap_{n=1}^{+\infty} A_n^c \subset A_{n_0}^c,$$

we see that $(\bigcup_{n=1}^{+\infty} A_n)^c$ is at most countable.

Thus $\mathfrak{M}$ is a $\sigma-$algebra in X.

2.) By the definition, $\mu : 2^X \to [0, +\infty]$. Suppose $A_i \in \mathfrak{M}$ for $i = 1, 2, 3, ...,$ $A_i + A_j = \emptyset$ if $i \neq j$, then either A_i or A_i^c is at most countable.
If $\bigcup_{n=1}^{+\infty} A_n$ is countable, then for any $n \in N$, A_n is also countable.

$$\mu(\bigcup_{n=1}^{+\infty} A_n) = 0 = \sum_{n=1}^{+\infty} \mu(A_n).$$

If $\bigcup_{n=1}^{+\infty} A_n$ is not countable, then there exists $n_0 \in N$, A_0 is uncountable and $A_{n_0}^c$ is at most countable. Since $\{A_n\}$ are mutually disjoint, we have

$$\bigcup_{n\neq n_0} A_i = \bigcup_{n=1}^{+\infty} A_n - A_{n_0} \in \mathfrak{M}.$$

if $\bigcup_{n\neq n_0} A_i$ is uncountable, $(\bigcup_{n\neq n_0} A_n)^c$ is at most countable, this contradicts to $A_{n_0} \subset (\bigcup_{n\neq n_0} A_n)^c$ and A_{n_0} is uncountable, then $\bigcup_{n\neq n_0} A_n$ is at most countable. So

$$\mu(\bigcup_{n=1}^{+\infty} A_n) = 1 = 0 + 1 = \sum_{n\neq n_0} \mu(A_n) + \mu(A_{n_0}) = \sum_{n=1}^{+\infty} \mu(A_n).$$

3.) Since for any given measurable function $f(x) \in \mu(X)$, $X = \{x;\ f(x) = r\} \bigcup \{x;\ f(x) < r\} \bigcup \{x;\ f(x) > r\} \widehat{=} E_1 \bigcup E_2 \bigcup E_3 \in \mathfrak{M}$ for any $r \in R^1$. $X \in \mathfrak{M}$, $\mu(X) = 1$ implies that there exists a unique uncountable set E of E_i, $i \in \{1, 2, 3\}$.

In fact, since X is uncountable, there exits at least one of E_{i_0} is uncountable, then $\mu(E_{i_0}) = 1$ and $\mu(\bigcup_{i\neq i_0} E_i) = 0$, and there will be a contradiction if there exist at least two uncountable E_i.

If $i_0 = 1$, then $f(x) = r$ a.e. $[\mu]$.

If $i_0 = 2$ or 3, one can take $\underline{r} = \sup\limits_{\mu(E)=0} \inf\limits_{X\setminus E} |f(x)|$ and $\bar{r} = \inf\limits_{\mu(E)=0} \sup\limits_{X\setminus E} |f(x)|$.

For the case $i_0 = 2$, let $r \to -\infty$, one will get that there exists a $r_1 \in [\underline{r}, r)$, such that $\mu\{x;\ f(x) = r_1\} = 1$, i.e. $f(x) = r_1$ a.e. $[\mu]$;

For the case $i_0 = 3$, let $r \to +\infty$, one will get that there exists a $r_2 \in (r, \bar{r}]$, such that $\mu\{x;\ f(x) = r_2\} = 1$, i.e. $f(x) = r_2$ a.e. $[\mu]$.

From above all, there exists $r \in [\underline{r}, \bar{r}]$, such that $\mu\{x;\ f(x) = r\} = 1$, i.e. $f(x) = r$ a.e. $[\mu]$. □

7, Suppose $f_n : X \to [0, \infty]$ is measurable for $n = 1, 2, 3, \cdots$, $f_1 \geq f_2 \geq \cdots \geq 0$, $f_n(x) \to f(x)$ as $n \to \infty$, for every $x \in X$, and $f_1 \in L^1(\mu)$. Prove that then

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$$

and show that this conclusion does *not* follow if the condition "$f_1 \in L^1(\mu)$" is omitted.

Proof: Since $f_1 \in L^1(\mu)$, $\{x; f_1(x) = +\infty\}$ is measurable and $\mu\{x;\ f_1(x) = +\infty\} = 0$, i.e. $f(x) < +\infty$ a.e. $[\mu]$. Since $f_1 \geq f_2 \geq ... \geq f_n \geq 0$ and $f_1 \in L^1(\mu)$, $f_n \in L^1(\mu)$ and f is measurable.

Let $A = \{x;\ f_1(x) = +\infty\}$ and $A_n = \{x;\ f_n(x) = +\infty\}$, then $\mu(A_n) \leq \mu(A) = 0$. Let $g_n(x) = f_1 - f_n$ if $x \in A^c$ and 0 if $x \in A$, then $0 \leq g_1 \leq g_2 \leq ... \leq g_n \leq ...$ and g_n is measurable.

By Fatou's lemma, $0 \leq \int_X f d\mu = \int_X \lim_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu \leq \int_X f_1 d\mu < +\infty$. By monotone convergence theorem,

$$\int_X f_1 d\mu - \lim_{n\to\infty}\int_X f_n d\mu = \lim_{n\to\infty}\int_X g_n d\mu = \int_X \lim_{n\to\infty} g_n d\mu = \int_X (f_1 - f_n) d\mu = \int_X f_1 d\mu - \int_X f_n d\mu.$$

i.e. $-\lim_{n\to\infty}\int_X f_n d\mu = -\int_X f d\mu$, hence $\lim_{n\to\infty}\int_X f_n d\mu = \int_X f d\mu$.

Counterexample: $f_n(x) = \chi_{[n,+\infty]}$ and μ is a counting measure on R^1(see p17 for the definition of counting measure), then $f_1 \geq f_2 \geq ... \geq f_n \geq 0$.

For any given $x \in R^1$, $f_n(x) : \chi_{[n,+\infty]}(x) = 0$ if $n \geq N(x)$ if $N(x)$ is large enough, so $f_n(x) \to f(x) = 0$ and $\int_X f d\mu = 0$.

On the other hand, since $\int_X f_n d\mu = \mu([n, +\infty]) = +\infty$, $\lim_{n\to\infty}\int_X f_n d\mu = +\infty \neq 0 = \int_X f d\mu$. □

8, (E is a measurable set in $(X, \mathfrak{M})$) Put $f_n = \chi_E$ if n is odd, $f_n = 1 - \chi_E$ if n is even. What is the relevance of this example to Fatou's lemma?

Solution: Suppose $\mu(X) = 1$, $0 < \mu(E) < \frac{1}{2}$. Since $f_{2n} = 1 - \chi_E$ and $f_{2n+1} = \chi_E$, $\int_X f_{2n} dx = \mu(X) - \mu(E) = 1 - \mu(E)$ and $\int_X f_{2n+1} dx = \mu(E)$, then

$$\int_X \liminf_{n\to\infty} f_n = \int_E \liminf_{n\to\infty} f_n + \int_{E^c} \liminf_{n\to\infty} f_n = \int_E (1-\chi_E) d\mu + \int_{E^c} \chi_E d\mu < \mu(E) = \liminf_{n\to\infty}\int_X f_n d\mu.$$

So, the strict inequality in Fatou's lemma holds sometimes. □

9. Suppose μ is a positive measure on X, $f : X \to [0, +\infty]$ is measurable, $\int_X f d\mu = c$ where $0 < c < \infty$, and α is a constant. Prove that

$$\lim_{n\to+\infty}\int_X n\log[1 + (\frac{f}{n})^\alpha] d\mu = \begin{cases} \infty, & \text{if } 0 < \alpha < 1, \\ c, & \text{if } \alpha = 1, \\ 0, & \text{if } 1 < \alpha < \infty. \end{cases}$$

Hint: If $\alpha \geq 1$,then integrands are dominated by αf.If $\alpha < 1$, Fatou Lemma can be applied.

proof: Let $g(x) = \alpha x - n\log[1 + (\frac{x}{n})^\alpha]$,$(\alpha \geq 1)$. then $g(0) = 0$ and

$$\begin{aligned} g'(x) &= \alpha - n\frac{\alpha(\frac{x}{n})^{\alpha-1}\frac{1}{n}}{1 + (\frac{x}{n})^\alpha} = \alpha[1 - \frac{(\frac{x}{n})^{\alpha-1}}{1 + (\frac{x}{n})^\alpha}] \\ &= \frac{\alpha}{1 + (\frac{x}{n})^\alpha}[1 + (\frac{x}{n})^{\alpha-1}(\frac{x}{n} - 1)] \end{aligned}$$

If $x \geq n$, it is clear that $g'(x) \geq \dfrac{\alpha(1 + (\frac{x}{n})^{\alpha-1})}{1 + (\frac{x}{n})^\alpha} \geq 1$.

If $0 < x < n$,$\alpha \geq 1$ implies that $(\dfrac{x}{n})^\alpha \leq (\dfrac{x}{n}) \leq (\dfrac{x}{n})\dfrac{1}{1 - \frac{x}{n}}$,i.e.

$$(\frac{x}{n})^{\alpha-1}(1 - \frac{x}{n}) \leq 1$$

$$(\frac{x}{n})^{\alpha-1}(\frac{x}{n}-1) \geq -1$$
$$1+(\frac{x}{n})^{\alpha-1}(\frac{x}{n}-1) \geq 0$$

hence

$$g'(x) = \frac{\alpha[1+(\frac{x}{n})^{\alpha-1}(\frac{x}{n}-1)]}{1+(\frac{x}{n})^{\alpha}} \geq 0$$
$$g'(0) = \alpha > 0$$

So $g'(x) \geq 0$, for $x \in (0, +\infty]$. We thus have $g(x) \geq 0$,i.e. $n \log(1+(\frac{f}{n})^{\alpha}) \leq \alpha f \in L^1(\mu)$, since $\int f d\mu = c < +\infty$. By dominated convergence theorem

$$\lim_{n\to+\infty} \int_X n \log[1+(\frac{f}{n})^{\alpha}] d\mu = \int_X \lim_{n\to+\infty} n \log[1+(\frac{f}{n})^{\alpha}] d\mu$$
$$= \begin{cases} \int_X f d\mu = c, & \text{if } \alpha = 1, \\ 0. & \text{if } 1 < \alpha < \infty. \end{cases}$$

This is because when $\alpha = 1$,we have

$$\lim_{x\to 0+} \frac{\log(1+ax)}{x} = a$$

for $a \geq 0$, and when $\alpha > 1$,

$$\lim_{x\to 0+} \frac{\log(1+(ax)^{\alpha})}{x} = \lim_{x\to 0+} \frac{\log(1+(ax)^{\alpha})a^{\alpha}x^{\alpha-1}}{(ax)^{\alpha}}$$
$$= \lim_{x\to 0+} a^{\alpha}x^{\alpha-1} \lim_{y\to 0+} \frac{\log(1+y)}{y} = 0$$

If $0 < \alpha < 1$, $a > 0$ then

$$\lim_{x\to 0+} \frac{\log(1+(ax)^{\alpha})}{x} = \lim_{x\to 0+} \frac{\log(1+(ax)^{\alpha})a^{\alpha}x^{\alpha-1}}{(ax)^{\alpha}}$$
$$= \lim_{x\to 0+} a^{\alpha}x^{\alpha-1} \lim_{y\to 0+} \frac{\log(1+y)}{y} = 1 \cdot \infty = \infty$$

As $\int f d\mu = c > 0$, we see that $\mu\{x : f(x) > 0\} > c$. Fatou's Lemma implies that

$$\liminf_{n\to+\infty} \int_X n \log[1+(\frac{f}{n})^{\alpha}] d\mu \geq \int_X \liminf_{n\to+\infty} n \log[1+(\frac{f}{n})^{\alpha}] d\mu$$
$$\geq \int_{\{x:f(x)>0\}} \liminf_{n\to+\infty} n \log[1+(\frac{f}{n})^{\alpha}] d\mu$$
$$= \int_{\{x:f(x)>0\}} \infty d\mu = \infty \cdot \mu\{x; f(x) > 0\}$$
$$= +\infty$$

$\square$

10. Suppose $\mu(X) < \infty$, $\{f_n\}$ is a sequence of bounded complex measurable functons on X, and $f_n \to f$ uniformly on X. Prove that

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$$

and show that the hypothesis $\mu(X) < \infty$ can not be omitted.

Proof: $\forall\ n, |f_n| \leq C_n < +\infty$ on X. So when $\mu(X) < \infty$ we have

$$\int_X f_n d\mu \leq C_n \mu(X) < +\infty,$$

which implies that $f_n \in L^1(\mu)$. Since f_n converges to f uniformly on X, which implies that f is measurable, then there exits N, s.t. $|f_N(x) - f(x)| < 1$ for $x \in X$. Thus

$$|f| \leq |f_N - f| + |f_N| \leq 1 + C_N < +\infty$$

and $f \in L^1(\mu)$. If $\mu(X) = 0$, then clearly $0 = \int_X f_n d\mu = \int_X f d\mu$ and $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$. If $0 < \mu(X) < +\infty$, as f_n converges to f uniformly on X, then $\forall\ \varepsilon > 0$, $\exists\ N = N(\varepsilon) > 0$, s.t. $n \geq N$ we have $|f_n(x) - f(x)| < \dfrac{\varepsilon}{\mu(X)}$ for $x \in X$. Thus

$$\begin{aligned}
&|\int_X f_n d\mu - \int_X f d\mu| = |\int_X (f_n - f) d\mu| \\
&\leq \int_X |f_n - f| d\mu \\
&\leq \frac{\varepsilon}{\mu(X)} \mu(X) = \varepsilon.
\end{aligned}$$

So $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$.

Now we prove that the condition $\mu(X) < \infty$ can not be omitted. We can construct a example as follows: set $X = \{1, 2, ..., N, ...\}$, μ is the counting measure and f_n is defined as: $f_n = \dfrac{1}{n}$, $\forall\ x \in X$, $f \equiv 0$. It is obvious that both f_n and f are complex measurable and $f_n \to f$ uniformly on X. But

$$\int_X f_n d\mu = \frac{1}{n}\mu(X) = +\infty, \quad \int_X f d\mu = 0 \cdot \mu(X) = 0$$

then

$$\lim_{n\to+\infty} \int_X f_n d\mu = +\infty \neq 0 = \int_X f d\mu$$

This implies that the condition $\mu(X) < \infty$ can not be omitted. □

11. Show that

$$A = \bigcap_{n=1}^{+\infty} \bigcup_{k=n}^{+\infty} E_k$$

in Theorem 1.41, and hence prove the theorem without any reference to integration.

Solution: Theorem 1.41: Let $\{E_k\}$ be a sequence of measurable sets in X, such that

$$\sum_{k=1}^{+\infty} \mu(E_k) < \infty \tag{1}$$

Then almost all $x \in X$ lie in at most finitely many of the sets E_k. If A is the set of all x which lie in infinitely many E_k, we have to show that $\mu(A) = 0$. We show

$$A = \bigcap_{n=1}^{+\infty} \bigcup_{k=n}^{+\infty} E_k$$

On one hand, if $x \in \bigcap_{n=1}^{+\infty}\bigcup_{k=n}^{+\infty} E_k$, then $\forall\ n$, $\exists\ k_n \geq n$, s.t. $x \in E_k$, we may also assume that

$$k_1 < k_2 < ... < k_n < ...$$

hence x belongs to infinitely many of the sets E_k(We can first take k_1 and then take $n_2 > k_1$,and take $k_2 > n_2, ...$), then $x \in A$.
On the other hand, if $x \in A$,then there are $k_1 < k_2 < ... < k_n$, s.t.

$$x \in E_{k_1}, \quad x \in E_{k_2}, \quad ... \quad x \in E_{k_n}.$$

so $x \in \bigcap_{i=1}^{+\infty} E_{k_i}$,$\forall\ n$, $\exists\ k_{n_1} \geq n$, s.t. $x \in E_{k_n}$, we can get $x \in \bigcup_{k=n}^{+\infty} E_k$ and $x \in \bigcap_{n=1}^{+\infty}\bigcup_{k=n}^{+\infty} E_k$. Thus $A = \bigcap_{n=1}^{+\infty}\bigcup_{k=n}^{+\infty} E_k, \quad \forall\ n$.

$$\begin{aligned}\mu(A) = \mu(\bigcap_{n=1}^{+\infty}\bigcup_{k=n}^{+\infty} E_k) &\leq \mu(\bigcup_{k=n}^{+\infty} E_k)\\ &\leq \sum_{k=n}^{+\infty}\mu(E_k) \to 0, \quad n \to +\infty.\end{aligned}$$

since $\sum_{k=n}^{+\infty}\mu(E_k) < \infty$. □

12. Suppose $f \in L^1(\mu)$. Prove that for each $\varepsilon > 0$, there is a $\delta > 0$ such that $\int_E |f| d\mu < \varepsilon$ whenever $\mu(E) < \delta$.
Proof: $f \in L^1(\mu) \Rightarrow \int_X |f| d\mu < +\infty$. According to Theorem 1.17 (P15), since $|f|$ is nonnegative measurable on X, there exist a sequence of infinite valued nonnegative simple functions $S_N = \sum_{i=1}^{m_N} a_i^N \chi_{E_i^N}$ s.t. $S_1 \leq ... \leq S_{N-1} \leq S_N \leq ...$, and $S_N(x) \to |f(x)|$ for $\forall\ x \in X$. By monotone convergence theorem , we get

$$\lim_{N\to+\infty}\int_X S_N d\mu = \int_X |f| d\mu$$

But $\int_X |f| d\mu < +\infty$, so for $\varepsilon > 0$, $\exists\ N = N(\varepsilon)$, s.t.

$$0 \leq \int_X |f| d\mu - \int_X S_N d\mu < \frac{\varepsilon}{3}$$

For above fixed $S_N = \sum_{i=1}^{m_N} a_i^N \chi_{E_i^N}$, $\{a_i^N\}_{i=1,...,m_N}$ are nonnegative finite real number. Set $k_N = \max_{1\leq i\leq m_N}\{a_i^N\} + 1$, then obvious $0 < k_N < +\infty$. Take $\delta(\varepsilon) = \delta(\varepsilon, N(\varepsilon)) = \dfrac{\varepsilon}{3k_N m_N}$, then when $E \in \mathfrak{M}$ and $\mu(E) < \delta$, we get

$$\begin{aligned}\int_E S_N d\mu = \sum_{i=1}^{m_N} a_i^N \mu(E_i^N \cap E) &\leq \sum_{i=1}^{m_N} k_N \mu(E)\\ &\leq m_N k_N \mu(E) < \frac{m_N k_N \varepsilon}{3 m_N k_N} = \frac{\varepsilon}{3}\end{aligned}$$

So when $\mu(E) < \delta$ we have

$$\begin{aligned}\int_E |f| d\mu \leq \int_E |f| d\mu - \int_E S_N d\mu + \int_E S_N d\mu &\leq \int_E (|f| - S_N) d\mu + \frac{\varepsilon}{3}\\ &\leq \int_X (|f| - S_N) d\mu + \frac{\varepsilon}{3} = \int_X |f| d\mu - \int_X S_N d\mu + \frac{\varepsilon}{3}\\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon.\end{aligned}$$

□

13. Show that proposition 1.24(c) is also true when $c = +\infty$.

Proof: We want to prove that if $f \geq 0$ and $c = +\infty$ (f is measurable), then

$$\int_E cf d\mu = c \int_E f d\mu$$

In fact, since f is nonnegative measurable, so $0 \leq \int_X f d\mu$ always exists. If $\int_X f d\mu = 0$, then $f \in L^1(\mu)$. According to Theorem 1.39 and $\forall\ E \in \mathfrak{M}$, we have

$$0 \leq \int_E f d\mu \leq \int_X f d\mu = 0$$

it follows that $f = 0$ a.e on X, then $\infty f(x) = 0$ a.e on X. Thanks to Theorem 1.24(d), $\exists\ A \in \mathfrak{M}$ s.t. $\mu(A) = 0$, $\infty f(x) = 0$ on $X - A$, and $\int_{X-A} \infty f(x) d\mu = 0$.

Note that by Theorem 1.24(e), one has $\int_A \infty \cdot f(x) d\mu = \int_A f(x) \infty d\mu = 0$. Then,

$$\int_X \infty \cdot f(x) d\mu = \int_{X-A} \infty \cdot f(x) d\mu + \int_A \infty \cdot f(x) d\mu = 0 + 0 = 0 = \infty \int_X f(x) d\mu$$

If $0 < \int_X f d\mu$, using the facts $\{x : f(x) > 0\} = \bigcup_{n=1}^{+\infty} \{x : f(x) \geq \frac{1}{n}\}$ and $\mu\{x : f(x) > 0\} > 0$, there $\exists\ n$ s.t. $\mu\{x : f(x) \geq \frac{1}{n}\} > 0$, so

$$\int_X \infty \cdot f(x) d\mu \geq \int_{\{x : f(x) \geq \frac{1}{n}\}} \infty \cdot f(x) d\mu = \infty \mu\{x : f(x) \geq \frac{1}{n}\} = \infty.$$

Moreover, $\infty \cdot \int_X f d\mu = \infty$. Hence we get

$$\infty \cdot \int_X f d\mu = \int_X \infty \cdot f d\mu.$$

We obtain the desired conclusion. $\square$

Chapter 2 Positive Borel Measures

1. Let $\{f_n\}$ be a sequence of real nonnegative functions on R^1, and considering the following four statements:

(a) If f_1 and f_2 are upper semicontinuous, then $f_1 + f_2$ is upper semicontinuous.

(b) If f_1 and f_2 are lower semicontinuous, then $f_1 + f_2$ is lower semicontinuous.

(c) If each f_n are upper semicontinuous, then $\sum_1^\infty f_n$ is upper semicontinuous.

(d) If each f_n are lower semicontinuous, then $\sum_1^\infty f_n$ is lower semicontinuous.

Show that three of these are true and that one is false. What happens if the word nonnegative is omitted? Is the truth of the statements affected if R^1 is replaced by a general topological space?

Proof: We first state a proposition and give its proof.

Proposition : f is lower semicontinuous on $X \Leftrightarrow \forall\ x_0 \in X$, $\forall\ \varepsilon > 0$, s.t. $\{x : f(x) > f(x_0) - \varepsilon\}$ is open set $\Leftrightarrow \forall\ x_0 \in X$, $\forall\ \varepsilon > 0$, s.t. open set $B(x_0) \subset \{x : f(x) > f(x_0) - \varepsilon\}$, $x_0 \in B(x_0)$.

Proof of Proposition: We begin to prove the first $\Leftrightarrow$. In fact, if f is lower semicontinuous on X, then by definition $\forall\ a \in R^1$, $\{x : f(x) > a\}$ is open, especially one can take $a = f(x_0) - \varepsilon \in R^1$, then $\{x : f(x) > f(x_0) - \varepsilon\}$ is open set. On the other hand, $\forall\ a \in R^1$, if $\{x : f(x) > a\} = \phi$, then $\{x : f(x) > a\}$ is open. If $\{x : f(x) > a\} \neq \phi$, we know $\exists\ x_0 \in \{x : f(x) > a\}$, obviously $f(x_0) - a = \varepsilon > 0$, note also

$$\{x : f(x) > a\} = \{x : f(x) > f(x_0) - (f(x_0) - a)\}$$

is open, we get f is lower semicontinuous on X.

As for the the second $\Leftrightarrow$, the $\Rightarrow$ is obvious and we only need to prove the part $\Leftarrow$. If $y_0 \in \{x : f(x) > f(x_0) - \varepsilon\}$, using the condition, $\exists$ open set $B(y_0) \subset \{x : f(x) > f(y_0) - [f(y_0) - (f(x_0) - \varepsilon)]\}$, i.e. $\exists$ open set $B(y_0) \subset \{x : f(x) > f(x_0) - \varepsilon\}$, so $\{x : f(x) > f(x_0) - \varepsilon\}$ is open set, according to the above result we get f is lower semicontinuous on X.

(b) is true. In fact, if f_1 and f_2 are l.s.c, then $\forall\ x_0 \in X$, $\forall\ \varepsilon > 0$, $\exists$ open sets V and W, $x_0 \in V \cap W$, s.t. $V \subset \{x : f_1(x) > f_1(x_0) - \frac{\varepsilon}{2}\}$, $W \subset \{x : f_2(x) > f_2(x_0) - \frac{\varepsilon}{2}\}$. Note $V \cap W$ is also open, $x_0 \in V \cap W$, then

$$\begin{aligned} V \cap W &\subset \{x : f_1(x) > f_1(x_0) - \frac{\varepsilon}{2}\} \cap \{x : f_2(x) > f_2(x_0) - \frac{\varepsilon}{2}\} \\ &\subset \{x : f_1(x) + f_2(x) > f_1(x_0) + f_2(x_0) - \varepsilon\} \end{aligned}$$

According to the above proposition, we conclude that $f_1 + f_2$ is l.s.c. In an analogous way, we can prove that (a) is true. Moreover we must notice that (a) and (b) hold without the hypothesis that f_1 and f_2 are nonnegative.

(d) is true. Set $S_n(x) = \sum_{i=1}^n f_i$, since f_i is l.s.c, according to (b), S_n is l.s.c on X, so $\forall\ a \in R^1$, $\{x : S_n(x) \leq a\} = X - \{x : S_n(x) \geq a\}$ is closed set, then $g(x) = \sum_{i=1}^\infty f_i(x) = \lim_{n\to\infty} S_n(x)$ always exists. The following is well known: $g(x) : X \to [0, \infty]$, $g(x)$ l.s.c on $X \Leftrightarrow \forall\ a \in R^1$, $\{x : g(x) \leq a\}$ is a closed set. We will show that

$$\{x : g(x) \leq a\} = \bigcap_{n=1}^{+\infty} \{x : S_n(x) \leq a\}$$

On one hand, $\forall\ x_0 \in \{x : g(x) \leq a\}$, then $a \geq g(x_0) = \lim_{n\to\infty} S_n(x_0) \geq S_m(x_0)$, $\forall\ m \in N$, since $f_i \geq 0$. So $\forall\ m \in N$, one has $x_0 \in \{x : S_m(x) \leq a\}$, then $x_0 \in \bigcap_{n=1}^{+\infty}\{x : S_n(x) \leq a\}$.

On the other hand, if $x_0 \in \bigcap_{n=1}^{+\infty}\{x : S_n(x) \leq a\}$, we get $\forall\ n$, $S_n(x_0) \leq a$, hence $g(x_0) = \lim_{n\to\infty} S_n(x_0) \leq a$, $x_0 \in \{x : g(x) \leq a\}$. Thus $\{x : g(x) \leq a\} = \bigcap_{n=1}^{+\infty}\{x : S_n(x) \leq a\}$. Note that $\{x : S_n(x) \leq a\}$ is closed, then $\{x : g(x) \leq a\}$ is closed, which entails $\sum_1^\infty f_n$ is lower semicontinuous.

If $f_n \leq 0$, u.s.c on X, then $-f_n \geq 0$ l.s.c on X. hence $\sum_{n=1}^{+\infty}(-f_n) = -\sum_{n=1}^{+\infty} f_n$ l.s.c, then $\sum_{n=1}^{+\infty} f_n$ u.s.c.

(c) is false. If $f_n \geq 0$ u.s.c, $g(x) = \sum_{n=1}^{+\infty} f_n(x)$. Note that $g(x)$ u.s.c $\Leftrightarrow \forall\ a \in R^1$, $\{x : g(x) \geq a\}$ is a closed set. It is readily to show that $f : R^n \to R^1$ u.s.c $\Leftrightarrow \forall\ x_0 \in R^n$, $\limsup_{x\to x_0} f(x) \leq f(x_0)$, where

$$\limsup_{x\to x_0} f(x) = \inf_{\delta>0} \sup_{0<|x-x_0|<\delta} f(x) = \lim_{\delta>0} \sup_{0<|x-x_0|<\delta} f(x)$$

If $\sum_{n=1}^{+\infty} f_n(x)$ convergences uniformly on R^1, then $\forall\ x_0 \in R^1$, $x_m \to x_0$ but $x_m \neq x_0$, we want to prove that

$$\limsup_{m\to+\infty} \sum_{n=1}^{+\infty} f_n(x_m) \leq \sum_{n=1}^{+\infty} \limsup_{m\to+\infty} f_n(x_m) \leq \sum_{n=1}^{+\infty} f_n(x_0)$$

Set $\{x_m\} \subset \{x : g(x) \geq a\}$, $x_m \to x_0$. If for some m, $x_m = x_0$ holds, then $x_0 \in \{x : g(x) \geq a\}$. If $x_m \neq x_0$, $\forall\ m$, then $a \leq g(x_m)$ and hence

$$a \leq \limsup_{m\to+\infty} g(x_m) \leq \sum_{n=1}^{+\infty} \limsup_{m\to+\infty} f_n(x_m) \leq \sum_{n=1}^{+\infty} f_n(x_0) = g(x_0)$$

since f_n u.s.c, we see $x_0 \in \{x : g(x) \geq a\}$, so $\{x : g(x) \geq a\}$ is closed and g is u.s.c.

Let $f_n(x) = \chi_{[\frac{1}{n+1},\frac{1}{n}]}(x)$, $n = 1, 2, ...$, then $[\frac{1}{n+1}, \frac{1}{n}]$ is the closed set in R^1 and f_n is u.s.c. Note that

$$g(x) = \sum_{n=1}^{+\infty} f_n(x) = \begin{cases} 2, & \text{if } x = \frac{1}{k}, \quad k = 1, 2, 3, .., \\ 1, & \text{if } \frac{1}{k+1} < x < \frac{1}{k}, \quad k = 1, 2, 3, .., \\ 0, & \text{otherwise.} \end{cases}$$

we have

$$\begin{aligned} \{x : g(x) \geq 2\} = \{x : g(x) = 2\} &= \bigcup_{k=1}^{+\infty} \{\frac{1}{k}\} \\ &= \{1, \frac{1}{2}, \frac{1}{3}, ...\} \triangleq A \end{aligned}$$

It is obvious that A is not closed beacuse the limit point 0 is not in A. So $\sum_{n=1}^{+\infty} f_n(x)$ is not u.s.c and we conclude that (c) is false in general.

Note (a) (b) (d) are true if R^1 is replaced by a general topological space X. Moreover, (a) and (b) also hold without the condition that f_1 and f_2 are nonnegative. Since f l.s.c $\Leftrightarrow -f$ u.s.c, we can explain the result in (d) is not true without the hypothesis $f_i \geq 0$ by the inverse example given in (c). In fact, set $f_n(x) = -\chi_{[\frac{1}{n+1},\frac{1}{n}]}(x)$, $n = 1, 2, ...$, then $\chi_{[\frac{1}{n+1},\frac{1}{n}]}$ u.s.c $\Rightarrow f_n$ l.s.c. If

$$\sum_{n=1}^{+\infty} f_n(x) = -\sum_{n=1}^{+\infty} \chi_{[\frac{1}{n+1},\frac{1}{n}]}(x) = -g(x)$$

(see the notation in (c)) is l.s.c , then $g(x)$ is u.s.c, according to the inverse example in (c), g is not u.s.c, hence $\sum_{n=1}^{+\infty} f_n(x)$ is not l.s.c. $\square$

2. Let f be arbitrary complex function on R^1, and define

$$\varphi(x, \delta) = \sup\{|f(s) - f(t)| : s, t \in (x - \delta, x + \delta)\},$$

$$\varphi(x) = \inf\{\varphi(x, \delta) : \delta > 0\}$$

Prove that φ is upper semicontinuous, that f is continuous at a point x if and only if $\varphi(x) = 0$, and hence that the set of points of continuity of an arbitrary complex function is a G_δ.

Formulate and prove an analogous statement for general topological spaces in place of R^1.

Proof: We prove this for an arbitrary function on metric space (X, d) first. Let $f : X \to Y$ ((Y, ρ) is metric space), $\varphi(x, \delta) = \sup\{\rho(f(s), f(t)), s, t \in B_\delta(x)\}$, where $B_\delta(x) = \{y \in X : d(x, y) < \delta\}$, $\varphi(x) = \inf\{\varphi(x, \delta) : \delta > 0\}$.

(1) $\varphi(x)$ is upper semicontinous on X.

$\forall\ a \in R^1$, we show that $\{x : \varphi(x) < a\}$ is an open set. In fact, $\forall\ x_0 \in \{x : \varphi(x) < a\}$, $\exists$ $\delta_0 = \delta(x_0, a) > 0$ s.t. $\varphi(x_0, \delta_0) < a$. For any $y \in B(x_0, \frac{\delta_0}{2})$, $B(y, \frac{\delta_0}{2}) \subset B(x_0, \delta_0)$, by triangle inequality we have $\varphi(y) \le \varphi(y, \frac{\delta_0}{2}) \le \varphi(x_0, \delta_0) < a$ and $B(y, \frac{\delta_0}{2}) \subset \{x : \varphi(x) < a\}$, which implies that $\{x : \varphi(x) < a\}$ is open and φ is upper semicontinous on X.

(2) f(x) is continuous at x_0 if and only if $\varphi(x_0) = 0$.

On one hand, if f(x) is continuous at x_0, then $\forall\ \varepsilon > 0\ \exists\ \delta = \delta(x_0, \varepsilon)$, s.t. $\forall\ x, y \in B(x_0, \delta)$, $\rho(f(x), f(x_0)) < \frac{\varepsilon}{2}$, $\rho(f(y), f(x_0)) < \frac{\varepsilon}{2}$, it follows that

$$\begin{aligned}\rho(f(x), f(y)) &\le \rho(f(x), f(x_0)) + \rho(f(x_0), f(y))\\ &= \rho(f(x), f(x_0)) + \rho(f(y), f(x_0))\\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

So $\varphi(x_0, \delta) \le \varepsilon$, $\varphi(x_0) \le \varphi(x_0, \delta) \le \varepsilon$, then $\varphi(x_0) = 0$ because the $\varepsilon > 0$ is arbitrary.

On the other hand, if $\varphi(x_0) = 0$, then for any $\varepsilon > 0$, there is a $\delta > 0$ such that $\varphi(x_0, \delta) < \varepsilon$. Therefore for any $x \in B(x_0, \delta)$, $\rho(f(x), f(x_0)) \le \varphi(x_0, \delta) < \varepsilon$, which shows the continuity of $f(x)$ at x_0.

(3) For any $f : X \to Y$ ((X, d) (Y, ρ) are metric space). We know from the above (2) that f is continuous at x if and only if $\varphi(x) = 0$, so the points where f in not continuous is

$$\{x : \varphi(x) > 0\} = \bigcup_{n=1}^{+\infty}\{x : \varphi(x) \ge \frac{1}{n}\}$$

We see from the upper semicontinuity of $\varphi(x)$ that $\{x : \varphi(x) \ge \frac{1}{n}\}$ is closed for each $n \in N$, hence $\{x : \varphi(x) > 0\}$ is a F_δ and $\{x : \varphi(x) = 0\} = \{x : \varphi(x) > 0\}^c = \bigcap_{n=1}^{+\infty}\{x : \varphi(x) < \frac{1}{n}\}$ is a G_δ.

General case: Let X be a topological space and Y be a metric space with metric ρ. $f : X \to Y$ be a map. For any $x \in X$ and any neighborhood V, define $\varphi(x, V) = \sup\{\rho(f(s), f(t)), s, t \in V\}$, $\varphi(x) = \inf\{\varphi(x, V)\}$, V is neighborhood of x.

Claim: (1) $\varphi(x)$ is upper semicontinuous.

(2) f is continuous at x if and only if $\varphi(x) = 0$ and hence the set of points of continuity of f is a G_δ.

Proof of claim : (1) $\forall\ a \in R^1$, $\forall\ x_0 \in \{x : \varphi(x) < a\}$, then there is an open set V containing x_0 such that $\varphi(x, V) < a$. For any $y \in V$, V is also a neighborhood of y. So

$$\varphi(y) \le \varphi(y, V) = \varphi(x, V) < a$$

hence $x_0 \in V \subset \{x : \varphi(x) < a\}$, which implies that $\{x : \varphi(x) < a\}$ is an open set. It follows that φ is u.s.c on X.

(2) If $\varphi(x_0) = 0$, then $\forall\ \varepsilon > 0\ \exists$ open set $V \ni x_0$, $\varphi(x_0, V) < \varepsilon$, so for any $y \in V$,

$$\rho(f(y), f(x_0)) \le \varphi(x_0, V) < \varepsilon$$

which shows that $f(x)$ is continuous at $x = x_0$. On the other hand, if f is continuous at x_0, then for any $\varepsilon > 0$, $\exists$ open set $V \ni x_0$ such that $\forall\ y, x \in V$, we have $\rho(f(y), f(x_0)) < \frac{\varepsilon}{2}$ and $\rho(f(x), f(x_0)) < \frac{\varepsilon}{2}$, so $\rho(f(y), f(x)) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, $\varphi(x_0, V) \leq \varepsilon$, then $\varphi(x_0) \leq \varphi(x_0, V) \leq \varepsilon$, it follows that $\varphi(x_0) = 0$ since the $\varepsilon > 0$ is arbitrary. $\square$

3. Let X be a metric space, with metric ρ. For any nonempty $E \subset X$, define

$$\rho_E(x) = \inf\{\rho(x, y) : y \in E\}.$$

Show that ρ_E is a uniformly continuous function on X. If A and B are disjoint nonempty closed subsets of X, examine the relevance of the function

$$f(x) = \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}$$

to Urysohn's lemma.

proof: (1) ρ_E is a uniformly continuous on X. First, $\rho_E \geq 0$ is apparently well defined and finite for any $x \in X$. $\forall\ x, y \in X$,

$$\rho_E(x) \leq \rho(x, y) + \rho_E(y) \tag{2}$$

In fact , $\forall z \in X$,

$$\rho(x, z) \leq \rho(x, y) + \rho(y, z)$$

$$\begin{aligned} \rho_E(x) = \inf\{\rho(x, z) : z \in E\} &\leq \rho(x, y) + \inf\{\rho(y, z) : z \in E\} \\ &= \rho(x, y) + \rho_E(y) \end{aligned}$$

By (1), we have , for any $\forall\ x, y \in E$,

$$\rho_E(x) \leq \rho(x, y) + \rho_E(y)$$

$$\rho_E(y) \leq \rho(x, y) + \rho_E(x)$$

hence

$$|\rho_E(x) - \rho_E(y)| \leq \rho(x, y)$$

which clearly shows that ρ_E is Lipschitz on X,hence uniformly continuous. If A and B are disjoint nonempty closed subsets of X, then

$$f(x) = \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}$$

is continuous on X.

It is enough to show that (by (1))

$$\rho_A(x) + \rho_B(x) \neq 0$$

for any $\forall x \in X$. In fact, if $\rho_A(x) + \rho_B(x) \neq 0$ then

$$\rho_A(x) = \rho_B(x) = 0$$

hence there exist $\{y_n\} \subset A$, $\{z_n\} \subset B$ such that

$$\rho(x, y_n) \longrightarrow 0 \qquad \rho(x, z_n) \longrightarrow 0$$

i.e.

$$y_n \longrightarrow x \qquad z_n \longrightarrow x$$

Since A and B are closed, $x \in A$, $x \in B$, thus $x \in A \bigcap B = \emptyset$, a contradiction. So f is continuous.

If $K \subset V$ and $K \neq \emptyset$ is compact, V is an open set, $K \cap V^c = \emptyset$

$$f(x) = \frac{\rho_{V^c}(x)}{\rho_K(x) + \rho_{V^c}(x)}$$

then $f(x)$ is continuous on X, and $\forall x \in K$, $\rho_{V^c}(x) > 0$. (for if $\rho_{V^c}(x) = 0$ and since $x \in K$ we have $\rho_K(x) = \rho_{V^c}(x) = 0$, it follows that $x \in K \cap V^c = \emptyset$ by the result proved above)

So

$$f(x) = \frac{\rho_{V^c}(x)}{0 + \rho_{V^c}(x)} = 1, \ if \ x \in K.$$

also,

$$\forall x \in V^c, \ f(x) = \frac{0}{1+0} = 0.$$

But it is not clear that $f \in C_c(X)$!

Suppose X is a locally compact metric space. $K \neq \emptyset$ is compact ,V is open, $K \subset V$, $\forall x \in K$, there exist open set V_x satisfying $\overline{V_x}$ is compact, $\overline{V_x} \subset V$ with $x \in V_x$. So $K \subset \bigcup\limits_{x \in K} V_x$. The compactness implies that there exits $n \in N$, $x_1, x_2, \cdots, x_n \in K$

$$K \subset \bigcup_{i=1}^{n} V_{x_i} \doteq W,$$

W is open and

$$\overline{W} = \overline{\bigcup_{i=1}^{n} V_{x_i}} \subset \bigcup_{i=1}^{n} \overline{V_{x_i}} \doteq Y$$

Y is compact. Since X is metric space, it follows that X is Hausdorff. So the closed set $\overline{W} \subset Y$ is compact (P36 corollary).

$$f(x) = \frac{\rho_{W^c}(x)}{\rho_K(x) + \rho_{W^c}(x)}$$

is continuous. and $f(x) = 1$ on K, $f(x) = 0$ on W^c. Hence

$$suppf \subset \overline{(W^c)^c} = \overline{W} \subset V$$

So $f \in C_c(X)$. □

4. Examine the proof of the Riesz theorem and prove the following tow statements:

(a) If $E_1 \subset V_1$ and $E_2 \subset V_2$, where V_1 and V_2 are disjoint open sets, then $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$, even if E_1 and E_2 are not in $\mathfrak{M}$.

(b) If $E \in \mathfrak{M}_F$, then $E = N \cup K_1 \cup K_2 \cup \cdots$, where $\{K_i\}$ is a disjoint countable collection of compact sets and $\mu(N) = 0$.

Proof: (a) By definition (P42 (2)) for any $E \subset X$ one defines

$$\mu(E) = \inf\{\mu(V) : E \subset V, V \text{ open}\}.$$

It is clear that $\mu \geq 0$. We firstly show that $\forall \{E_i\}_{i=1}^{\infty} \subset X$,

$$\mu(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu(E_i). \text{ (subadditivity)}$$

If $\exists\, i$ s.t. $\mu(E_i) = \infty$, we are done. So assume that $\forall\, i \;\; \mu(E_i) < \infty$.

$\forall \varepsilon > 0, \exists$ open set V_i s.t. $E_i \subset V_i$

$$\mu(E_i) \leq \mu(V_i) < \mu(E_i) + \frac{\varepsilon}{2^i}$$

So

$$\mu(\bigcup_{i=1}^{\infty} E_i) \leq \mu(\bigcup_{i=1}^{\infty} V_i) \leq \sum_{i=1}^{\infty} \mu(V_i) \leq \sum_{i=1}^{\infty} \mu(E_i) + \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \sum_{i=1}^{\infty} \mu(E_i) + \varepsilon.$$

for $V_i \in \mathfrak{M}$, subadditivity holds. In fact, this is the step I in P42.

So

$$\mu(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu(E_i) \quad \textit{holds.}$$

The step I in P42 is proved. Hence

$$\mu(E_1 \cup E_2) \leq \mu(E_1) + \mu(E_2)$$

Let V_1 and V_2 are open sets, $E_1 \subset V_1$ and $E_2 \subset V_2$, $V_1 \cap V_2 = \emptyset$. If $\mu(E_1 \cup E_2) = \infty$, then clearly

$$\mu(E_1) + \mu(E_2) \leq \infty = \mu(E_1 \cup E_2)$$

If $\mu(E_1 \cup E_2) < \infty$, then $\exists$ open V s.t. $E_1 \cup E_2 \subset V$ and

$$\mu(E_1 \cup E_2) \leq \mu(V) \leq \mu(E_1 \cup E_2) + \varepsilon$$

Since $E_1 \subset V \cap V_1$, $E_2 \subset V \cap V_2$, and μ is monotone by definition

$$\begin{aligned}
\mu(E_1) + \mu(E_2) &\leq \mu(V \cap V_1) + \mu(V \cap V_2) \\
&= \mu((V \cap V_1) \cup (V \cap V_2)) \\
&= \mu(V \cap (V_1 \cap V_2)) \\
&\leq \mu(V) \\
&\leq \mu(E_1 \cup E_2) + \varepsilon
\end{aligned}$$

Letting $\varepsilon \longrightarrow 0$, we have

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2).$$

This proves (a).

(b) By the proof of step VIII (p45). $\mathfrak{M}_F$ consists of precisely those sets $E \in \mathfrak{M}$, for which $\mu(E) < \infty$. So $\forall\, E \in \mathfrak{M}_F$, and $A \subset E$, with $A \in \mathfrak{M}$, we have

$$\mu(E - A) = \mu(E) - \mu(A).$$

by

$$\mu(E) = \mu((E - A) \cup A) = \mu(E - A) + \mu(A)$$

and

$$\mu(A) \leq \mu(E) < \infty.$$

Since

$$\mu(E) = \inf\{\mu(K) : K \subset E, K \text{ compact}\}.$$

If $E \neq \emptyset$, (If $E = \emptyset$, let $N = K_i = \emptyset$, $\cdots$ OK) there exists compact $K_1 \subset E$, s.t.

$$\mu(E) \leq \mu(K_1) + 1.$$

As $E \in \mathfrak{M}_F$, K_1 compact $\Longrightarrow E$, $K_1 \in \mathfrak{M}$, $\mu(E - K_1) = \mu(E) - \mu(K_1) < \infty$ and $E - K_1 \in \mathfrak{M}_F$.

So there is a compact set $K_2 \subset E - K_1$, with $(K_1 \cap K_2 = \emptyset)$

$$\mu(E - K_1) \leq \mu(K_2) + \frac{1}{2},$$

then

$$\mu(E - K_1 - K_2) = \mu(E - K_1) - \mu(K_2) \leq \frac{1}{2}.$$

Thus we obtain a countable collection of compact sets $\{K_i\}_{i=1}^{\infty}$ pairwise disjoint, with

$$K_n \subset E - \bigcup_{i=1}^{n-1} K_i \in \mathfrak{M}_F$$

and

$$\mu(E - \bigcup_{i=1}^{n} K_i) = \mu(E - \bigcup_{i=1}^{n-1} K_i) - \mu(K_n) \leq \frac{1}{n}, \ (n = 1, 2, \cdots)$$

Let $N = E - \bigcup\limits_{i=1}^{\infty} K_i$, then since $E \in \mathfrak{M}$, $K_i \in \mathfrak{M}$, $N \in \mathfrak{M}$, and

$$\mu(N) = \mu(E - \bigcup_{i=1}^{\infty} K_i) \leq \mu(E - \bigcup_{i=1}^{n} K_i) \leq \frac{1}{n},$$

Letting $n \longrightarrow \infty$, we have $\mu(N) = 0$, and

$$\begin{aligned} E &= (E - \bigcup_{i=1}^{\infty} K_i) \cup (\bigcup_{i=1}^{\infty} K_i) \\ &= N \cup K_1 \cup K_2 \cup \cdots \qquad \square \end{aligned}$$

$\square$

In Exercise 5 to 8 , m stands for Legesgue measure on $\mathbb{R}^1$.

5. Let E be Cantor's familiar "middle thirds" set. Show that $m(E) = 0$, even though E and $\mathbb{R}^1$ have the same cardinality.

Proof: E is obtained by remove countably many open intervals I_i from the closed interval $[0, 1]$,

$$E = [0, 1] - \bigcup_{i=1}^{\infty} I_i$$

and

$$\bigcup_{i=1}^{\infty} I_i = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{2^{n-1}} I_i^{(n)}$$

where each $I_i^{(n)}$ is an open interval in $[0,1]$ pairwise disjoint, $|I_i^{(n)}| = \frac{1}{3^n}$, $\forall\, i = 1, 2, \cdots, 2^{n-1}$.

$$m(\bigcup_{i=1}^{2^{n-1}} I_i^{(n)}) = \sum_{i=1}^{2^{n-1}} |I_i^{(n)}| = \frac{2^{n-1}}{3^n}$$

$$\begin{aligned} m(\bigcup_{n=1}^{\infty} \bigcup_{i=1}^{2^{n-1}} I_i^{(n)}) &= \sum_{n=1}^{\infty} \sum_{i=1}^{2^{n-1}} |I_i^{(n)}| = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} \\ &= \frac{1}{2} \sum_{n=1}^{\infty} (\frac{2}{3})^n = \frac{1}{2} \frac{\frac{2}{3}}{1-\frac{2}{3}} = 1 \end{aligned}$$

So

$$m(E) = m([0,1]) - m(\bigcup_{n=1}^{\infty} \bigcup_{i=1}^{2^{n-1}} I_i^{(n)}) = 1 - 1 = 0.$$

Cantor's middle set E has the cardinality $c = \overline{\overline{R}} = \overline{\overline{(0,1]}}$.

Proof: It is well known that if we use the 3-decimal expression to express the points in E, then the points in E are the decimals in the $[0,1]$ which have no 1 in 3-decimal expression and two points 0 and 1.

Let

$$A = \left\{ \sum_{i=1}^{\infty} \frac{a_i}{3^i},\ a_i = 2 \textit{ or } 0, \text{ and with infinite many } a_i \text{ such that } a_i \neq 0 \right\}$$

then $A \subset E$. And, when we use the 3-decimal expression to express the numbers in $(0,1]$, and for the numbers with the form $\frac{m}{2^k}$, $0 < m < 2^k$ we use the first expression forever. Then the case that the all decimal digits from some decimal place are 0 isn't appear.

$\forall\, x \in A$, $x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$, let

$$\varphi(x) = \sum_{i=1}^{\infty} \frac{1}{2} \frac{a_i}{2^i},$$

then $\forall\, i$, $\frac{a_i}{2} = 1 \textit{ or } 0$ and there are infinite many a_i satisfying $a_i \neq 0$. Consequently, $\varphi(x)$ is a binary digit in the interval $(0,1]$. Now let us prove the mapping φ is injective and surjective.

① If $x = \sum\limits_{i=1}^{\infty} \frac{a_i}{3^i}$, $\widetilde{x} = \sum\limits_{i=1}^{\infty} \frac{\widetilde{a_i}}{3^i} \in A$ and $\varphi(x) = \varphi(\widetilde{x})$, i.e. $\sum\limits_{i=1}^{\infty} \frac{a_i}{2^i} = \sum\limits_{i=1}^{\infty} \frac{\widetilde{a_i}}{2^i}$. If $a_1 \neq \widetilde{a_1}$, without the generality, we can assume $a_1 = 2$ and $\widetilde{a_1} = 0$, then by infinite many $a_i > 0$

$$\begin{aligned} \frac{1}{2} &< \frac{1}{2} + \sum_{i=2}^{\infty} \frac{a_i}{2} \frac{1}{2^i} = \varphi(x) = \varphi(\widetilde{x}) = \sum_{i=2}^{\infty} \frac{\widetilde{a_i}}{2 \cdot 2^i} \\ &= \sum_{i=2}^{\infty} \frac{\widetilde{a_i}}{2} \frac{1}{2^i} \leq \sum_{i=2}^{\infty} \frac{2}{2} \frac{1}{2^i} = \sum_{i=2}^{\infty} \frac{1}{2^i} = \frac{1}{2} \end{aligned}$$

thus $a_1 = \widetilde{a_1}$. We can prove $a_i = \widetilde{a_i}$ by the same way. Hence $x = \widetilde{x}$. Consequently φ is injective.

② φ is surjective. $\forall\, z \in (0,1]$, and $z = \sum\limits_{i=1}^{\infty} \frac{a_i}{2^i}$, $a_i = 0 \textit{ or } 1$, $\exists$ infinite many i s.t. $a_i \neq 0$. Let

$$x = \sum_{i=1}^{\infty} \frac{2a_i}{3^i},$$

then $2a_i = 0\, or\, 2$, and there are infinite many i such that $2a_i \neq 0$. So $x \in A$, $\varphi(x) = z$. Consequently, φ is ont to one and from A onto $(0,1]$. So

$$\overline{\overline{E}} \geq \overline{\overline{A}} = \overline{\overline{(0,1]}} \geq \overline{\overline{E}}$$

Hence

$$\overline{\overline{E}} = c = \overline{\overline{(0,1]}} = \overline{\overline{\mathbb{R}^1}} \qquad .$$

□

6. Construct a totally disconnected compact set $K \subset \mathbb{R}^1$ such that $m(K) = 0$. (K is to have no connected subset consisting of more than one point.)

If v is lower semi continuous and $v \leq \chi_K$, show that actually $v \leq 0$. Hence χ_K cannot be approximated from below by lower semi continuous functions, in the sense of the Vitali-Carathéodory theorem.

Solution: Let a_1, a_2, a_3, $\cdots$, be a sequence of positive numbers, with $a_1 > a_2 > a_3 > \cdots$,

$$\sum_{i=1}^{\infty} a_i = A \, < \, 1.$$

Delete an open interval of length a_1 centered at the middle point of [0,1], to get two closed intervals; then delete from the two closed intervals two open intervals centered at the middle points of length $\frac{a_2}{2}$; And then delete from the remained four closed intervals four open intervals , each of them has the length $\frac{a_3}{2^2}$, $\cdots$, and, denoting the all deleted open intervals by $\{I_i\}_{i=1}^{+\infty}$, they are disjoint. Then $K = [0,1] - \bigcup\limits_{i=1}^{+\infty} I_i$ is a closed set, $K \subset [0,1]$, hence K is compact. So

$$\begin{aligned} m(\bigcup_{i=1}^{+\infty} I_i) &= \sum_{i=1}^{+\infty} m(I_i) = a_1 + 2\frac{a_2}{2} + 2^2\frac{a_3}{2^2} + \cdots + 2^n\frac{a_{n+1}}{2^n} + \cdots \\ &= a_1 + a_2 + a_3 + \cdots = \, A \, < 1. \end{aligned}$$

Thus

$$m(K) = m([0,1]) - m(\bigcup_{i=1}^{+\infty} I_i) = 1 - A \, > \, 0.$$

Just like what one does for the Cantor's set,one can show that K is nowhere dense and perfect.

We notice that after the nth step deleting, there are 2^n closed intervals each has the length

$$\frac{1 - \sum\limits_{i=1}^{n} a_i}{2^n}$$

For $x \in K$, if (α, β) is an open interval containing x, let $\delta = \min(x - \alpha, \beta - x)$, then $\delta > 0$. So if $n_0 > 0$ is large enough, $\dfrac{1 - \sum\limits_{i=1}^{n_0} a_i}{2^{n_0}} < \delta$. And $x \in K$ implies that x is in some closed interval $I_{n_0}^i$, where $I_{n_0}^i$ is one of the remaining closed intervals after the n_0th deleting. Thus $I_{n_0}^i \subset (\alpha, \beta)$, for $\forall\ y \in\ I_{n_0}^i$

$$|y - x| < \frac{1 - \sum\limits_{i=1}^{n_0} a_i}{2^{n_0}} < \delta.$$

So the two end points of $I_{n_0}^i$ also belong to (α, β), thus (α, β) possesses at least one point different from x to belong to K. (each end point of I_n^i is a point of K, they could not be delete any time!) This shows that any neighborhood of x contains a point $y \in K \backslash \{x\}$, hence $x \in K'$ (x is a limit point of K). So K is perfect. Also from the above process, for any open interval (α, β) with $x \in (\alpha, \beta)$, there is an entire interval $I_{n_0}^i \subset (\alpha, \beta)$. So the n_0th-deleting would result an open interval $I \subset I_{n_0}^i \subset (\alpha, \beta)$ which is not in K. So K is nowhere dense, or totally disconnected.

If v is a lower semi continuous function on $\mathbb{R}^1$ and $v \leq \chi_K$, then $v \leq 0$ on $\mathbb{R}^1$!

In fact, If these were $x_0 \in \mathbb{R}^1$ with $0 < v(x_0)$, then by the definition of l.s.c. there is an open interval (α, β) such that $x_0 \in (\alpha, \beta)$ and $v(x) > 0$ for all $x \in (\alpha, \beta)$. Since $v(x) \leq \chi_K(x)$, we have $\chi_K(x) > 0$ for all $x \in (\alpha, \beta)$. But

$$\chi_K(x) > 0 \Longleftrightarrow x \in K,$$

this shows that $(\alpha, \beta) \subset K$, contradicts to the fact the K is nowhere dense. So $v(x) \leq 0$ on $\mathbb{R}^1$.

Clearly $0 < m(K) < 1$, $\chi_K \in L^1(\mathbb{R}^1)$, if there is a lower semi continuous function v such that $v \leq \chi_K$, and

$$\int_{\mathbb{R}^1} (\chi_K - v)\, \mathrm{d}x < \varepsilon.$$

But the above argument showed $v \leq 0$, then

$$\varepsilon > \int_{\mathbb{R}^1} (\chi_K - v)\, \mathrm{d}x \geq \int_{\mathbb{R}^1} \chi_K\, \mathrm{d}x = m(K) > 0.$$

So if $\varepsilon < m(K)$, it is impossible to have a l.s.c. v such that

$$\int_{\mathbb{R}^1} (\chi_K - v)\, \mathrm{d}x < \varepsilon. \qquad .$$

□

7. If $0 < \varepsilon < 1$, construct an open set $E \subset [0, 1]$ which is dense in $[0, 1]$, such that $m(E) = \varepsilon$. (To say that A is dense in B means that the closure of A contains B.)

Solution: Look at again the compact set K constructed in Exercise 6 of this chapter, we actually proved that for any $0 < \varepsilon < 1$, there exists an perfect set

$$K = [0, 1] - \bigcup_{i=1}^{+\infty} I_i$$

and $m(K) = 1 - \varepsilon$. Let $E = \bigcup_{i=1}^{+\infty} I_i$, then E is an open subset in [0,1], and

$$m(E) = m([0, 1] - K) = m([0, 1]) - m(K) = 1 - (1 - \varepsilon) = \varepsilon.$$

If $\exists\, x \in K$, and $\exists$ open interval (α, β) s.t. $x \in (\alpha, \beta)$ and $(\alpha, \beta) \cap E = \emptyset$, then $(\alpha, \beta) \subset K$, this contradicts to the fact K is nowhere dense. So for $\forall\, x \in K$ and $\forall$ open interval (α, β) s.t. $x \in (\alpha, \beta)$, we have $(\alpha, \beta) \cap E \neq \emptyset$. And clearly $x \in (\alpha, \beta) \backslash E$, consequently $K \subset \overline{E}$. Then $\overline{E} = [0, 1]$ □

9. Construct a sequence of continuous function f_n on [0,1] such that $0 \leq f_n \leq 1$, such that

$$\lim_{n \to \infty} \int_0^1 f_n(x)\, \mathrm{d}x = 0.$$

but such that the sequence $\{f_n(x)\}$ converges for no $x \in [0,1]$.

Solution: We define for $n = 2^k + j,\ k = 0,1,2,\cdots,\ j = 0,1,2,\cdots,2^k - 1$, $(\forall\, n\, \exists!\, k, j,\ n = 2^k + j)$

$$\widetilde{f_n}(x) = \begin{cases} 2^{k+1}[x - (\frac{j}{2^k} - \frac{1}{2^{k+1}})], & \text{if } x \in [\frac{j}{2^k} - \frac{1}{2^{k+1}}, \frac{j}{2^k}], \\ 1, & \text{if } x \in [\frac{j}{2^k}, \frac{j+1}{2^k}], \\ -2^{k+1}[x - (\frac{j+1}{2^k} + \frac{1}{2^{k+1}})], & \text{if } x \in [\frac{j+1}{2^k}, \frac{j+1}{2^k} + \frac{1}{2^{k+1}}], \\ 0, & \text{otherwise.} \end{cases}$$

then $\widetilde{f_n}$ is continuous on $\mathbb{R}^1$. Let $f_n = \widetilde{f_n}\mid_{[0,1]}$, then f_n is continuous on [0,1] with $0 \le f_n \le 1$ and

$$\begin{aligned}
\int_0^1 f_n(x)\,\mathrm{d}x &\le \int_{-\infty}^{\infty} f_n(x)\,\mathrm{dx} \\
&= 2^{k+1}\int_{\frac{j}{2^k}-\frac{1}{2^{k+1}}}^{\frac{j}{2^k}} x\,\mathrm{d}x \;-\; 2^{k+1}(\frac{j}{2^k} - \frac{1}{2^{k+1}})\cdot(\frac{j}{2^k} - \frac{j}{2^k} + \frac{1}{2^{k+1}}) \\
&\quad - 2^{k+1}\int_{\frac{j+1}{2^k}}^{\frac{j+1}{2^k}+\frac{1}{2^{k+1}}} x\,\mathrm{d}x \;+\; 2^{k+1}(\frac{j+1}{2^k} + \frac{1}{2^{k+1}})(\frac{j+1}{2^k} + \frac{1}{2^{k+1}} - \frac{j+1}{2^k}) \\
&= 2^k(\frac{j}{2^k} + \frac{j}{2^k} - \frac{1}{2^{k+1}})(\frac{1}{2^{k+1}}) \;-\; (\frac{j}{2^k}) + \frac{1}{2^{k+1}} \\
&\quad - 2^k[\frac{1}{2^{k+1}}(\frac{j+1}{2^k} + \frac{j+1}{2^k} + \frac{1}{2^{k+1}})] + \frac{j+1}{2^k} + \frac{1}{2^{k+1}} \\
&= \frac{1}{2}(\frac{j}{2^{k-1}} - \frac{1}{2^{k+1}}) - \frac{j}{2^k} + \frac{1}{2^{k+1}} \\
&\quad - \frac{1}{2}(\frac{j+1}{2^{k-1}} + \frac{1}{2^{k+1}}) + \frac{j+1}{2^k} + \frac{1}{2^{k+1}} \\
&= -\frac{1}{2}\frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} \\
&\quad - \frac{1}{2}\frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} \\
&= \frac{1}{2^{k+1}}.
\end{aligned}$$

Since let $n \longrightarrow +\infty$, it must be $k \longrightarrow +\infty$. Consequently

$$\int_0^1 f_n(x)\,\mathrm{dx} \le \frac{1}{2^{\mathrm{k}+1}} \longrightarrow 0.$$

But for $\forall\, x \in [0,1]$, $\{f_n(x)\}$ contains an infinite number of $0's$ and $1's$. Thus $f_n(x)$ converges nowhere in $[0,1]$. □

11, Let μ be a regular Borel measure on a compact Hausdorff space X, assume $\mu(X) = 1$. Prove that there is a compact set $K \subset X$ (the carrier or support of μ) such that $\mu(K) = 1$ but $\mu(H) < 1$ for every proper compact subset H of K.

Hint: Let K be the intersection of all Compact K_α with $\mu(K_\alpha) = 1$; show that every open set V which contains K also contains some K_α. Regularity of μ is needed; compare Exercise 18. Show that K^c is the largest open set in X whose measure is 0 .

Proof: (1) If K_1, K_2 are compact subsets in X with $\mu(K_1) = \mu(K_2) = 1$, then $\mu(K_1 \bigcap K_2) = 1$. In fact, Since $\mu(X) = 1$, $\mu(K_i) = 1$ implies that $\mu(K_i^c) = \mu(X - K_i) = \mu(X) - \mu(K_i) = 0$
1=$\mu(K_1) = \mu((K_1 \bigcap K_2) \bigcup (K_1 \bigcap K_2^c)) = \mu(K_1 \bigcap K_2) + \mu(K_1 \bigcap K_2^c) = \mu(K_1 \bigcap K_2) \therefore \mu(K_1 \bigcap K_2) =$

1 also $K_1 \bigcap K_2$ is compact. Hence if $K_1, \cdots, K_n$ are compact sets in X, with $\mu(K_i) = 1$ $(i = 1, \cdots, n)$. then $\bigcap\limits_{i=1}^{n} K_i$ is compact and (by the fact that X is Hausdorff Space and corollaries in p36) $\mu(\bigcap\limits_{i=1}^{n} K_i) = 1$.

(2) Let $K = \bigcap\{K_\alpha : K_\alpha compact, \mu(K_\alpha) = 1\}$ then since $\mu(X) = 1$ and X is compact. i.e. $K = \bigcap\limits_{\alpha\in I} K_\alpha$ $\mu(K_\alpha) = 1$, K_α compact. then $K \neq \phi$, in fact, if $K = \bigcap\limits_{\alpha\in I} K_\alpha = \phi$ by the Th2.6(finite times) $\exists$ compact sets $K_1, \cdots, K_n$ $\mu(K_i) = 1 (1 \leq i \leq n)$ s.t. $\bigcap\limits_{i=1}^{n} K_1 = \phi$ contradicting to (1). as $\mu(\bigcap\limits_{i=1}^{n} K_i) = 1 \Rightarrow \bigcap\limits_{i=1}^{n} K_i \neq \phi$ so $K \neq \phi$.

(3) $\mu(K) = 1$. If not, then by the regularity of μ, there is an open set V with $K \subset V$, $\mu(V) < 1$. we claim that $\exists$ compact set $K_\alpha \subset X$ s.t. $\mu(K_\alpha) = 1$ and $K_\alpha \subset V$. In fact, $K \subset V$ implies $K \bigcap V^c = \phi$ i.e

$$\bigcap_{\alpha\in I} (K_\alpha \bigcap V^c) = \phi$$

Since $K_\alpha \bigcap V^c$ is closed and K_α is compact. Corollary in p36 implies $k_\alpha \bigcap V^c$ is compact. By Th2.6, $\exists K_1, \cdots K_n$ (compact sets) $\mu(k_i) = 1$ $1 \leq i \leq n$ $\phi = \bigcap\limits_{i=1}^{n} (K_i \bigcap V^c) = (\bigcap\limits_{i=1}^{n} K_i) \bigcap V^c$ which gives that $K_\alpha = \bigcap\limits_{i=1}^{n} \subset V$ and $\mu(K_\alpha) = 1$ by (1). Thus

$$1 = \mu(K_\alpha) \leq \mu(V) < 1$$

a contradiction.

(4) If $H \subsetneqq K$ is a compact set. then $\mu(H) < 1$. Indeed, If $\mu(H) = 1$, then $K = \bigcap\limits_{\alpha\in I} K_\alpha$ implies $K \subset H \subsetneqq K$

(5) K^c is the largest open set such that $\mu(K^c) = 1$ If V is an open set with $\mu(V) = 0$ then $V^c$is closed in the compact Hausdorff space hence V^c is compact and

$$\mu(V^c) = \mu(X) - \mu(V) = 1 - 0 = 1$$

So $K \subset V^c$, hence $V \subset K^c$ also $\mu(K^c) = \mu(X) - \mu(K) = 1 - 1 = 0$, So K^c is the largest open set with zero measure. □

12, Show that every compact subset of R^1 is the support of a Borel measure.

Proof: $\forall$ compact set $K \subset R^1$.

Case 1. K is finite. $K = \{a_1, a_2, \cdots, a_n\}$ Define for any Borel set $E \in \mathscr{B}$ $\mu(E) = \sum\limits_{i=1}^{n} \delta_{a_i}(E)$ where

$$\delta_{a_i}(E) = \begin{cases} 1, a_i \in E \\ 0, a_i \overline{\in} E \end{cases}$$

Then as $\forall i$ $\delta_{a_i}(E)$ defines a measure(p17 1.20 (b) Unit mass) μ is a Borel measure with $\mu(R^1) = 1$ $\mu(K) = 1$. $\forall$ compact $H \subsetneq K$, $\mu(H) < 1$. (while since for any bounded Borel measure must be regular (which will be proved later), μ is regular) So in this case, the conclusion is true.

Case 2. K is infinite. Since R^1 is separable. $\exists$ countable set $A = \{a_1, a_2, \cdots, a_n, \cdots.\}$ s.t. $\bar{A} = K$

here $\bar{A}$ is the closure of A. For any $E \in \mathscr{B}$. define

$$\mu(E) = \sum_{i=1}^{\infty} \frac{1}{2^i} \delta_{a_i}(E).$$

It is easy to see that $\mu(E)$ is a Borel measure. $\mu(R^1) = 1$(hence μ is regular) $\mu(K) = \sum_{i=1}^{\infty} \frac{1}{2^i} = 1$. For any proper compact set $H \subset K$ $H \neq K$.$\exists x_0 \in K$, $x_0 \overline{\in} H$ Since H is closed, $R^1 - H$ is open, As $(x_0 - \delta,\ x_0 + \delta) \subset R^1 - H$. But $\bar{A} = K$, so $\exists i_0$ s.t. $|a_{i_0} - x_0| < \frac{\delta}{2}$. i.e. $a_{i_0} \in K,\ a_{i_0} \in R^1 - H$ $a_{i_0} \overline{\in} H$ so

$$\mu(H) \leq \sum_{i \neq i_0} \frac{1}{2^i} < 1 = \mu(K)$$

A compact set K is called the support of a Borel measure μ of $(X, \mathscr{B}, \mu)$, if $\mu(K) = \mu(X)$. and $\forall$ proper compact set $H \subsetneq K$ $\mu(H) < \mu(K)$. $\square$

14,Let f be a real-valued Lebesgue measurable function on R^k. Prove that there exist Borel functions g and h such that $g(x) = h(x)$a.e.$[m]$, and $g(x) \leq f(x) \leq h(x)$ for every $x \in R^k$.

Proof: Firstly, we assume f is a nonnegative real-valued Lebesgue measurable function on R^k. so $\forall n \in N$, and $1 \leq i \leq n2^n$, define $E_{n,i} = f^{-1}([\frac{i-1}{2^n}, \frac{i}{2^n}))$ and $F_n = f^{-1}([n, \infty))$ $S_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \Xi_{E_{n,i}} + n\Xi_{F_n}$ then S_n increases to f on R^k as $n \to +\infty (s_1 \leq s_2 \leq s_3 \leq \cdots \leq f)$.and $\lim_{n \to +\infty} S_n = \sup_n S_n = f$. $\forall x \in R^k$. Since $E_{n,i},\ F_n$ are Lebesgue measurable sets. by Th2.20(p50)(b) there are F_σ sets $A_{n,i},\ B_n$, G_σ sets $C_{n,i}$ and D_n s.t. $A_{n,i} \subset E_{n,i} \subset C_{n,i}$, $B_n \subset F_n \subset D_n$ with $m(C_{n,i} - A_{n,i}) = m(F_n - B_n) = 0$ for all $n,\ i$ Denote $g_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \Xi_{A_{n,i}} + n\Xi_{B_n}$ Then g_n is Borel and $g_n \leq S_n (\forall x \in R^k)$ $g_n(x) = S_n(x)$ a.e. in R^k. Let $g(x) = \sup_n S_n$ then g is Borel. (p14 Th1.14). and $g(x) = \sup_n S_n(x)$ a.e.in R^k. $g(x) \leq \sup_n S_n(x) = f(x)$ for any $x \in R^k$.

Now assume $\exists M < +\infty$. $0 \leq f \leq M$ on R^k. then for n large enough. $F_n = \phi$ therefore $S_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \Xi_{E_{n,i}}$. Set $\bar{S_n} = \sum_{i=1}^{n2^n} \frac{i}{2^n} \Xi_{E_{n,i}}$. then $\bar{S_n} \geq f$ and $\bar{S_n}$ decreases to f. as $n \to +\infty$. Set $h_n(x) = \bar{S_n}(x)$ a.e.in R^k. Let $h(x) = \inf_n h_n(x)$ So $\forall x \in R^k$ $h(x) \geq \inf \bar{S_n}(x) = f(x)$ and $h(x) = f(x)$ a.e. in R^k. h is Borel because h_n is as Th1.14.

Since f is real-valued. $\{x \in R^k, f(x) = +\infty\} = \phi$. So $\lim_{N \to +\infty} f_N(x) = f(x)$ $x \in R^k$ where

$$f_N(x) = \begin{cases} f(x) & if\ f(x) < N \\ N & if\ f(x) \geq N. \end{cases}$$

for any nonnegative measurable function f. As $0 \leq f_N \leq 1$. By the preceding step there is a Borel function h_N s.t. $h_N(x) \geq f_N(x), \forall x$, and $h_N(x) = f_N(x)$ a.e. in R^k. Denote $h = \inf_N h_N$. Then $h(x) \geq f(x)$ $\forall x$ and $h(x) = f(x) a.e.$ in R^k.

In conclusion, we have proved that , for any nonnegative real-valued function f. $\exists$ borel functions g and h s.t. $g(x) \leq f(x) \leq h(x) \forall x \in R^k$ and $g(x) = h(x) a.e.[m]$.

Now for general real-valued function $f = f^+ - f^-$. where $f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$ Applying the Preceding results for $f^\pm$, to abtain Borel functions $g_i,\ h_i,\ i = 1, 2$. s.t. $g_1 \leq f^+ \leq h_1$,

$g_2 \le f^- \le h_2$ $\forall x \in R^k$ and $g_1 = h_2 a.e.$ $[m]$ $g_2 = h_2 a.e.$ $[m]$ let $g(x) = g_1 - h_2,\ h(x) = h_1 - g_2$. then $g(x),\ h(x)$ are Borel. and

$$g \le g_1 - h_2 \le f^+ - f^- \le h_1 - g_2 = h$$

and $g = ha.e.$ $[m]$ on R^k. □

15,It is easy to guess the limits of $\int_0^n (1-\frac{x}{n})^n e^{\frac{x}{2}} dx$ and $\int_0^n (1+\frac{x}{n})^n e^{-2x} dx$ as $n \to +\infty$. Prove that your guesses are correct.

Proof:

$$\int_0^n (1-\frac{x}{n})^n e^{\frac{x}{2}} dx = \int_0^{+\infty} \Xi_{[0,n]}(x)(1-\frac{x}{n})^n e^{\frac{x}{2}} dx.$$

$$0 \le \Xi_{[0,n]}(x)(1-\frac{x}{n})^n e^{\frac{x}{2}} \to e^{-x} e^{\frac{x}{2}} = e^{-\frac{x}{2}}.$$

We claim that

$$0 \le (1-\frac{x}{n})^n \le e^{-x}\ for\ x \in [0,n].$$

To this end , Let $f(x) = (1+\frac{1}{x})^x$. $g(x) = \frac{f'(x)}{f(x)} = ln(1+\frac{1}{x}) + \frac{x}{1+\frac{1}{x}}(-\frac{1}{x^2}) = ln(1+\frac{1}{x}) - \frac{1}{x+1}$. If $-1 < x < 0$, then

$$\begin{aligned} g'(x) &= \frac{1}{1+\frac{1}{x}}(-\frac{1}{x^2}) + \frac{1}{(x+1)^2} = -\frac{1}{x(x+1)} + \frac{1}{(x+1)^2} \\ &= \frac{1}{x+1}(-\frac{1}{x} + \frac{1}{x+1}) = \frac{1}{x+1}\left[\frac{x-(x+1)}{x(x+1)})\right] \\ &= \frac{-1}{x(x+1)^2} > 0 \end{aligned}$$

So g(x) is monotone decreasing on $(-1,0)$ and $\lim_{x\to 0^-} g(x) < 0$. $\therefore$ $g(x) < 0$ on $(-1,0)$. Hence $f'(x) = g(x)f(x) < 0$ on $(-1,0)$. and $f(x)$ is monotone decreasing on $(-1,0)$. $\forall$ n_1 if $x \in (0,n)$, $-1 < -\frac{x}{n} < 0$ $f(-\frac{n}{x}) \ge f(-\frac{n+1}{x})$ $f(-\frac{n}{x})^{-x} \le f(-\frac{n+1}{x})^{-x}$ $x \in (0,n)$ i.e.

$$\begin{aligned} 0 &\le f_n(x) \triangleq (1-\frac{x}{n})^n = \left[\left(1-\frac{1}{\frac{n}{x}}\right)^{-\frac{n}{x}}\right]^{-x} \\ &= f(-\frac{n}{x})^{-x} \le f(-\frac{n+1}{x})^{-x} = f_{n+1}(x) \\ &\to e^{-x}.\ x \in (0,n) \end{aligned}$$

$\therefore |f_n(x)| \le e^{-x}\ a.e[m]on[0,n].$

$$|\Xi_{[0,n]}(1-\frac{x}{n})^n e^{\frac{x}{2}}| \le e^{-x} e^{\frac{x}{2}} = e^{-\frac{x}{2}}\ a.e.on[0,+\infty)$$

Clearly, $0 < e^{-\frac{x}{2}}$ is continuous and hence measurable. By Levi monotone convergence theorem

$$\begin{aligned} \int_0^{+\infty} e^{-\frac{x}{2}} dx &= \lim_{n\to+\infty} \int_0^{+\infty} \Xi_{[0,n]} e^{-\frac{x}{2}} dx \\ &= \lim_{n\to+\infty} \int_0^{+\infty} e^{-\frac{x}{2}} dx \\ &= \lim_{n\to+\infty} (R)(-2) \int_0^{+\infty} e^{-\frac{x}{2}} d(-\frac{x}{2}) \\ &= -2 \lim_{n\to+\infty} \left[e^{-\frac{n}{2}} - e^0\right] = 2 \end{aligned}$$

$\therefore\ e^{-\frac{x}{2}} \in L^1(0,+\infty)$. Dominated convergence theorem implies that

$$\lim_{n\to+\infty}\int_0^n (1-\frac{x}{n})^n e^{\frac{x}{2}}dx = \int_0^{+\infty} e^{-x}e^{\frac{x}{2}}dx = \int_0^{+\infty} e^{-\frac{x}{2}}dx = 2.$$

Clearly for $x \in [0,n]$. $(1+\frac{x}{n})^n$ increases to e^x.

$$|\Xi_{[0,n]}(1+\frac{x}{n})^n e^{-2x}| \le e^{-x}\ x \in [0,+\infty)$$

$$\Xi_{[0,n]}(1+\frac{x}{n})^n e^{-2x} \to e^x e^{-2x} = e^{-x}\ a.e.on(0,+\infty)$$

Dominated convergence theorem implies that

$$\lim_{n\to+\infty}\int_0^n (1+\frac{x}{n})^n e^{-2x}dx = \lim_{n\to+\infty}\int_0^{+\infty} \Xi_{[0,n]}(1+\frac{x}{n})^n e^{-2x}dx = \int_0^{+\infty} e^{-x}dx = 1.$$

□

16,Why is $m(Y)=0$ in the proof of Theorem 2.20(e)?

Proof: We need to prove that If $Y \subset R^k$ is a $l-dim(l<k)$ subspace of R^k then $m(Y)=0$ where m is the Lebesgue measure on R^k. First, we note that any two $l-dim$ linear subspace Y, Z are isometric. i.e.$\forall$ linear transformation $S:\ Y \to R^l$. $\|Sx\| = \|x\|$. In fact. WLOG $Y = span\{\eta_1,\cdots,\eta_l\}$ $(\eta_1,\cdots,\eta_l)$ is an orthonormal basis of Y. Extend $(\eta_1,\cdots,\eta_l)$ to get a orthonormalized basis $(\eta_1,\cdots,\eta_l,\eta_{l+1},\cdots,\eta_k)$ of R^k.

Let $S(\eta_i) = e_i = (0,\cdots,1,0,\cdots,0)$ $i=1,\cdots,k$; Clearly. S is an orthogonal linear transformation

$$S(Y) = R^l = span\{e_1,\cdots,e_l\}.$$

$$m(Y) = m(S^{-1}(R^l)) =\triangle\ (S^{-1})m(R^l).$$

So it is enough to show that $m(R^l)=0$.

$$R^l \subset \bigcup_{i,j\in X, j=1,\cdots,l} W_{i_1,\cdots,i_l}$$

where $W_{i_1,\cdots,i_l} = \{x \in R^k | x = (\xi_1,\cdots,\xi_l,\xi_{l+1},\cdots,\xi_k) i_j - 1 \le \xi_j \le i_j,\ 1 \le k \le l, 0 \le \xi_j \le i_{l+1} \le j \le K(??)\}$ is a k-cell. $m(W_{i_1,\cdots,i_l}) = 0$ by Th2.20(a) which was proved before 2-20(e) is touched. So $m(R^l)=0$ is proved. □

17, Define the distance between points (x_1,y_1) and (x_2,y_2) in the plane to be

$$|y_1-y_2|\ if\ x_1 = x_2,\ 1+|y_1-y_2|\ if\ x_1 \ne x_2.$$

Show that this is indeed a metric, and that the resulting metric space X is locally compact.

If $f \in C_c(X)$, let $x_1,\cdots,x_n$ be those values of x for which $f(x,y) \ne 0$ for at least one y(there are only finitly many such x!)and define

$$\bigwedge f = \sum_{j=1}^n \int_{-\infty}^{\infty} f(x_j,y)dy.$$

Let μ be the measure associated with this $\bigwedge$ by Theorem 2.14. If E is the $x-axis$, show that $\mu(E)=\infty$ although $\mu(K)=0$ for every compact $K \subset E$.

Proof: Firstly, we prove the distance given in the problem is indeed a distance.
1.$\forall$ (x_1, y_1) (x_2, y_2) in the plane R^2, trivially $\rho((x_1, y_1), (x_2, y_2)) \geq 0$ If $(x_1, y_1) = (x_2, y_2)$ then $\rho((x_1, y_1), (x_2, y_2)) = |y_1 - y_2| = 0$ when $x_1 = x_2,\ y_1 = y_2$ i.e.$(x_1, y_1) = (x_2, y_2)$.
2.$\forall (x_i, y_i) \in R^2 i = 1, 2.\rho((x_1, y_1), (x_2, y_2)) = \rho((x_2, y_2), (x_1, y_1))$ by definition
3.$\forall (x_i, y_i) \in R^2 i = 1, 2, 3.$
i) If $x_1 = x_2 = x_3$. then

$$\rho((x_1, y_1), (x_2, y_2)) = |y_1 - y_2| \leq |y_1 - y_3| + |y_3 - y_2| = \rho((x_1, y_1), (x_3, y_3)) + \rho((x_3, y_3), (x_2, y_2)).$$

ii) If $x_1 = x_2 \neq x_3$, then

$$\rho((x_1, y_1), (x_2, y_2)) = |y_1 - y_2| \leq |y_1 - y_3| + |y_3 - y_2| + 2 = \rho((x_1, y_1), (x_3, y_3)) + \rho((x_2, y_2), (x_3, y_3))$$

iii) If $x_1 = x_3 \neq x_2$, or $x_1 \neq x_2 = x_3$ then

$$\rho((x_1, y_1)(x_2, y_2)) = 1 + |y_1 - y_2| \leq 1 + |y_1 - y_3| + |y_3 - y_2| = \rho((x_1, y_1), (x_3, y_3)) + \rho((x_3, y_3), (x_2, y_2)).$$

iv) if $\{x_i\}_{i=1}^3$ are different, then

$$\rho((x_1, y_1)(x_2, y_2)) = 1 + |y_1 - y_2| \leq 1 + |y_1 - y_3| + |y_3 - y2| + ? = \rho((x_1, y_1), (x_3, y_3)) + \rho((x_3, y_3), (x_2, y_2)).$$

So (X, ρ) is metric space (X=place)
Secondly, we prove (X, ρ) is locally compact. Indeed ,$\forall (x_0, y_0) \in X$ the open ball

$$B \triangleq \{(x, y) : \rho((x, y), (x_0, y_0)) < \frac{1}{2}\} = \{x_0\} \times (y_0 - \frac{1}{2}, y_0 + \frac{1}{2})$$

because when $x \neq x_0,\ \rho((x, y), (x_0, y_0)) = 1 + |y - y_0| \geq \frac{1}{2}$ and the closure of B (w.r.t. the top is deduced by the metric ρ)is $\bar{B} = \{x_0\} \times [y_0 - \frac{1}{2}, y_0 + \frac{1}{2}]$. $(\forall (x_0, y_n) \in B$ if $\rho((x_0, y_n), (x_0, z)) \to 0$ $z \in [y_0 - \frac{1}{2}, y_0 + \frac{1}{2}]$ On the other hand, $\forall z \in [y_0 - \frac{1}{2}, y_0 + \frac{1}{2}]$ $\exists y_n \in [y_0 - \frac{1}{2}, y_0 + \frac{1}{2}]$ $(x_0, y_n) \to (x_0, z))$
We only need to show that $\bar{B}$ is compact. we characterize the compact sets in X now.
If K is a compact set in X, then $K \subset \bigcup_{x \in K} B(x, \frac{1}{2})$ where $B(x, \frac{1}{2}) = \{y \in X, \rho(y, x) < \frac{1}{2}\}$ then $\exists$ finite number $n \in N$, s.t.$K \subset \bigcup_{i=1}^{n} B(X_i, \frac{1}{2})$ while $x_i \in K$. However, $B(X_i, \frac{1}{2}) = \{X_{i1}\} \times (X_{i2} - \frac{1}{2}, X_{i2} + \frac{1}{2})$ where $X_i = (X_{i1}, X_{i2})$ Therefore K can be written as $K = \bigcup_{i=1}^{m} \{x_i\} \times B_i$ where $x_i \in R^1$, and $B_1 \subset R^1$
Furthermore, B_i are bounded closed in R^1 in usual toplogy. In fact, if $\{V_\alpha^i\}_{\alpha \in \Gamma_i}$ be an open cover of B_i in R^1, by the structure of open sets in R^1, $V_\delta^i = \bigcup_{j \in J_{i\alpha}} I_\alpha^{ij}$ where $J_{i\alpha}$ is at most countable, $I_\alpha^{ij} = (y_\alpha^{ij} - r_\alpha^{ij}, y_\alpha^{ij} + r_\alpha^{ij})$ $0 < r_\alpha^{ij} < \frac{1}{2}$ so $x_i \times I_\alpha^{ij} = B((x_i, y_\alpha^{ij}), r_\alpha^{ij})$ is an open ball in (X, ρ) with center (x_i, y_0^{ij}) radius $r_\alpha^{ij} < \frac{1}{2}$. so $\{\{x_i\} \times I_\alpha^{ij}\}_{i \leq i \leq m \alpha \in \Gamma_\alpha, j \in J_{i\alpha}}$ is an open cover of $K = \bigcup_{i=1}^{m} \{x_i\} \times B_i$.
Since K is compact $\exists$ finite many $\{x_i\} \times I_\alpha^{ij}$.covering K, Hence, $\forall i$, $\exists$ finite many I_α^{ij} covering B_i,hence finite many V_α^i covering B_i. So B_i is compact in R^1. in the usual topology. i.e. B_i is bounded closed set in R^1.

On the other hand, if $x_i \in R^1$, $B_i \subset R^1$ is compact in R^1. (e.g $B_i = [x_0 - \frac{1}{2}, x_0 + \frac{1}{2}]$) then if m is finite. then $K = \bigcup_{i=1}^{m} \{x_i\} \times B_i$ is compact in (X, ρ). In fact, consider the topological basis $\{B(x,r).x \in X,\ 0 < r < 1\}$ of (X, ρ). the open cover of K by balls in the basis, deduces an open cover of $B_i's$ in R^1(whose elements are in basis of R^1),($K \subset \bigcup_{\alpha \in \Gamma} \{x_\alpha\} \times (y_\alpha - r, y_\alpha + r) \Rightarrow \ \forall i,\ B_i \subset \bigcup_{\alpha \in \Gamma} (y_\alpha - r, y_\alpha + r)$)
So K is compact. This shows that $\bar{B} = \{x_0\} \times [y_0 - \frac{1}{2}, y_0 + \frac{1}{2}]$ is compact in (X, ρ) for any $(x_0, y_0) \in X$.

If E is the $x-axis$.$\forall$ open set $E \subset V$, then $\forall x \in R^1$, $\{x\} \times R = \bigcup_{y \in R^1} \{x\} \times (y - \frac{1}{2}, y + \frac{1}{2}) = \bigcup_{y \in R^1} B((x,y), \frac{1}{2}))$ is an open set in X, and hence $E \subset \{x\} \times R \bigcap V$ and $(\{x\} \times R) \bigcap V$ is open as well.
As $(x, 0) \in V$ and V is open. $\forall b_x > 0$ small enough s.t. $\overline{B((x,0), b_x)} \subset V$ i.e.$\{x\} \times [-b_x, b_x] \subset V$
We claim that $\exists C > 0$ and $\{x_i\}_{i=1}^{\infty} \subset R$ s.t.$b_{x_i} \geq c$ for all $i \in N$. Indeed, let $B \triangleq \{b_x : x \in R\}$, then $B \subset (0, +\infty]$
1. If B is at most countable , then $\exists c \in B$.s.t.$\{x \in R : b_x = c\}$ is uncountable.otherwise,$\forall\ c \in B$ $\{x \in R :\ \ b_x = c\}$ is countable .implies that R is countable. which is impossible.(From map: $f : R \to B,\ b(x) = b_x$. $R = f^{-1}(B) = \bigcup_{c \in B} f^{-1}(c)\ \ f^{-1}(c) = \{x : b_x = c\}$. if $f^{-1}(c)$ is countable, B finite $\Rightarrow \bigcup_{c \in B} b_{-1}(c)$ is countable.)
So $\exists c \in B,\ b^{-1}(c) = \{x \in R : b_x = c\}$ is uncountable, so $\exists \{x_i\}_{i=1}^{+\infty} \subset f^{-1}(c)$. i.e. $b_{x_i} = c\ i \in N$.
2. If B is uncountable, let $B_n = \{b_x : x \in R^1,\ b_x > \frac{1}{n}\}$ then $B = \bigcup_{n=1}^{\infty} B_n$. So $\exists N \in N$,s.t. B_N is uncountable, set $c = \frac{1}{N}$, $\exists \{x_i\}_{i=1}^{\infty} \subset B_N$, $b_{x_i} > \frac{1}{N} = c$:
For the $\{x_i\}_{i=1}^{+\infty}$, given above, let

$$K = \bigcup_{i=1}^{n} \{x_i\} \times [-b_{x_i}, b_{x_i}]$$

which is compact, and $K \subset V$ By Urysohn's Lemma, $\exists f \in C_c(X)$ s.t. $K \prec f \prec V$ Notice taht for any $f \in C_c(X)$.$Suppf = K$ is compact. So by the above analysis we know $\exists n \in N\ x_i \in R^1,\ B_i \subset R^1$ B_i compact in R^1, s.t.

$$Suppf = \bigcup_{i=1}^{m} \{\tilde{x_i}\} \times B_i$$

$\forall (x,y) \bar{\in} Suppf,\ f(x,y) \equiv 0$. possibly So only finite many $\{\tilde{x_i}\}_{i=1}^{m}$ satisfy $f(x_i, y) \neq 0$ for at least one y. If $f \in C_c(X),\ f \neq 0$ for some (x,y) then $\exists x'$ s.t. $\exists$ at least one $y \in R^1$ with $f(x'y) \neq 0$ If $f \equiv 0$ define $\bigwedge f = 0$ If $f \in C_c(X)$ $(f(x,y) \neq 0$ for some $(x,y))$ Let $x_1, \cdots, x_n$ be those values of x for which $f(x,y) \neq 0$ for at least one y .

$$\bigwedge f = \sum_{j=1}^{n} \int_{-\infty}^{+\infty} f(x_j, y) dy$$

For the $\{x_i\}_{i=1}^{+\infty}$, b_{x_i} C chosen above. $\forall\ n$. let $K = \bigcup_{i=1}^{n} \{K_i\} \times [-b_{x_i}, b_{x_i}]$ then K is compact, and by Urysohn's Lemma $\exists\ f_n \in C_c(X)$,

$$K \prec f \prec V$$

. as $f_n = 1$ an K, $\forall i$, $\forall y \in [-b_{x_i}, b_{x_i}]$. $f_n(x_i, y) = 1$. So

$$\begin{aligned} \bigwedge f_n &\geq \sum_{i=1}^{n} \int_{-\infty}^{+\infty} f_n(x_i, y)dy = \sum_{i=1}^{n} \int_{-b_{x_i}}^{b_{x_i}} f_n(x_i, y)dy \\ &= \sum_{i=1}^{n} 2b_{x_i} \geq 2nc. \end{aligned}$$

Letting $n \to +\infty$, we get $\bigwedge f_n \to +\infty$. So $\mu(V) = \sup\{\bigwedge f : f \prec V\} = +\infty$. hence

$$\mu(E) = \inf\{\mu(V), E \subset V, V\ open\} = +\infty$$

For each compact set K in R^1, We know from above that $\exists x_i \in R^1$ $B_i \subset R^1$, B_i compact. s.t. $K = \bigcup_{i=1}^{n} \{x_i\} \times B_i$ If in addition. $K \subset E$ then

$$K = \bigcup_{i=1}^{n} \{x_i\} \times \{O\} = \bigcup_{i=1}^{n} (x_i, O)$$

We want to show that $\mu(K) = 0$ It is enough to show that $\forall\ x_0 \in R^1$, $\mu(\{(x_0, 0)\}) = 0$. $\forall 0 < r < \frac{1}{2}$, $\{x\} \times (-r, r) = B((x, 0), r)$ is an open neighborhood of $(x, 0) \in X$ hs Th 2.14(d) $\mu(\{x_0\} \times (-r, r)) = \sup\{\bigwedge f : f \prec \{x_0\} \times (-r, r)\} \leqq 2r$
In fact, if $f \prec \{x_0\} \times (-r, r)$ then only x_0 satisfy $f(x_0, y) \neq 0$ for some $y \in R_1$, so

$$\begin{aligned} \bigwedge f &= \int_{-\infty}^{+\infty} f(x_0, y)dy = \int_{-r}^{r} f(x_0, y)dy \\ &\leq \int_{-r}^{r} dy \leq 2r \end{aligned}$$

letting $r \to 0$ we see that

$$\mu\{(x_0, 0)\} = \inf\{\mu(V)\ (x_0, 0) \subset V \leq \mu\{(x_0) \times (-r, r)\} \leq 2r \to 0$$

$\therefore \mu\{(x_0, 0)\} = 0$ therefore $\mu(k) = 0$ for any compact $K \subset E$. $\square$

20, Find continuous functions $f_n : [0, 1] \to [0, \infty)$, such that $f_n(x) \to 0$ for all $x \in [0, 1]$. as $n \to \infty$, $\int_0^1 f_n(x)\ dx \to 0$, but $\sup_n$ is not in L^1. (This shows that the conclusion of the dominated convergence theorem may hold even when part of its hypothesis is violated.)
Solution: For all u, define $f_n : [0, 1] \to [0, \infty)$ as follows

$$f_n(x) = \begin{cases} 0 & x \in [0, \frac{1}{2^n}] \cup [\frac{1}{2^{n-1}}, 1]; \\ \frac{4^{n+1}}{n}(x - \frac{1}{2^n}) & x \in [\frac{1}{2^n}, \frac{1}{2^n} + \frac{1}{2^{n+1}}]; \\ -\frac{4^{n+1}}{n}(x - \frac{1}{2^{n-1}}) & x \in [\frac{1}{2^n} + \frac{1}{2^{n+1}}, \frac{1}{2^{n-1}}]. \end{cases}$$

then $f_n \in C^0[0, 1]$, $f_n \geq 0$. Obviously, for $x = 0$, $f_n(x) = 0$, for $x = 1$, $f_n(x) = 0$, so we have for all $x \in [0,1]$, $\lim\limits_{n\to\infty} f_n(x) = 0$.

$$\begin{aligned}
0 &\leq \int_0^1 f_n(x)\,dx = \int_{\frac{1}{2^n}}^{\frac{1}{2^n}+\frac{1}{2^{n+1}}} \frac{4^{n+1}}{n}(x-\frac{1}{2^n})\,dx + \int_{\frac{1}{2^n}+\frac{1}{2^{n+1}}}^{\frac{1}{2^{n-1}}} -\frac{4^{n+1}}{n}(x-\frac{1}{2^{n-1}})\,dx \\
&\leq \frac{4^{n+1}}{n}(\frac{1}{2^n}+\frac{1}{2^{n+1}}-\frac{1}{2^n})(\frac{1}{2^n}+\frac{1}{2^{n+1}}-\frac{1}{2^n}) + \frac{4^{n+1}}{n}(\frac{1}{2^{n-1}}-\frac{1}{2^n}-\frac{1}{2^{n+1}})(\frac{1}{2^{n-1}}-\frac{1}{2^n}-\frac{1}{2^{n+1}}) \\
&= \frac{4^{n+1}}{n}\cdot\frac{1}{2^{n+1}}\cdot\frac{1}{2^{n+1}} + \frac{4^{n+1}}{n}\cdot\frac{1}{2^{n+1}}\cdot\frac{1}{2^{n+1}} \\
&= \frac{2}{n}(\to 0).
\end{aligned}$$

Moreover, after computation we may obtain $\int_0^1 f_n(x)\,dx = \frac{1}{n}$.

For all $x \in [0,1]$, if there exists $n \geq 0$ such that $x = \frac{1}{2^n}$, then $f_n(x) = 0$, and for all $m \in \mathbb{N}$, $f_m(x) = 0$. At this moment, $\sup\limits_n f_n(x) = 0 = \sum\limits_{n=1}^{+\infty} f_n(x)$. if there exists $n \geq 0$ such that $\frac{1}{2^n} < x < \frac{1}{2^{n-1}}$, then $f_n(x) \geq 0$, and for all $m \neq n$, $x \notin [\frac{1}{2^m}, \frac{1}{2^{m-1}}]$, $f_m(x) = 0$. So $\sup\limits_k f_k(x) = f_n(x) = \sum\limits_{k=1}^{+\infty} f_k(x)$. Thus for all $x \in [0,1]$, $\sup\limits_n f_n(x) = \sum\limits_{n=1}^{+\infty} f_n(x)$, $\int_0^1 \sup\limits_n f_n(x)\,dx = \sum\limits_{n=1}^{+\infty} \int_0^1 f_n(x)\,dx = \sum\limits_{n=1}^{+\infty} \frac{1}{n} = +\infty$. Then we have $\sup\limits_n f_n(x) \notin L^1([0,1], m)$. $\square$

21, If X is compact and $f : X \to (-\infty, +\infty)$ is upper semicontinuous, prove that f attains its maximum at some point of X.

Proof: First, we show that f is bounded from above. Since $f : X \to (-\infty, +\infty)$ is upper semicontinuous, for all $x_0 \in X$ for $\varepsilon = 1$, $f(x_0) < f(x_0) + 1$. So $x_0 \in \{x : f(x) < f(x_0) + 1\}$ which is open, and hence there exists a neighborhood V_{x_0} of x_0 such that $V_{x_0} \subset \{x : f(x) < f(x_0) + 1\}$, $X = \bigcup\limits_{x\in X} V_{x_0}$.$X$ is compact implies that there exist $x_1, \cdots, x_n \in X$ such that $X = \bigcup\limits_{i=1}^{n} V_{x_i}$. Let $M = \max\{f(x_1), \cdots, f(x_n)\} + 1$, for all $x \in X$ we know there exists i such that $x \in V_{x_i}$, $f(x) \leq f(x_i) + 1 \leq M$. Then M is bounded from above.

Let $\sup\limits_{x\in X} f(x) = a$, we will show that there exists $x_0 \in X$ such that $f(x_0) = a$ to finish the proof. Just suppose that f can not attain its maximum, that is to say for all $x \in X$, $f(x) < \sup\limits_{x\in X} f(x) = a$. Define $g : X \to (-\infty, +\infty)$ as follows:

$$g(x) = \frac{1}{a - f(x)}, \quad \forall x \in X.$$

Since $a - f(x) > 0$ for each $x \in X$, we have $g(x) > 0$ for all $x \in X$. For any $\alpha > 0$, $\{x \in X, g(x) < \alpha\} = \{x, f(x) < a - \frac{1}{\alpha}\}$ is open because that f is upper semicontinuous. For $\alpha \leq 0$, $\{x \in X, g(x) < \alpha\} = \emptyset$ is open too. So $g(x)$ is upper semicontinuous in X as well, thus $g(x)$ is bounded from above, there exists $K > 0$ such that $g(x) < K$ for all $x \in X$. Hence $f(x) < a - \frac{1}{K}$ or all $x \in X$. Which implies $a = \sup\limits_{x\in X} f(x) \leq a - \frac{1}{K} < a$, a contradiction. So there exists $x_0 \in X$, $f(x_0) = \sup\limits_{x\in X} f(x)$. $\square$

22, Suppose that X is a metric space with metric d, and that $f : X \to [0, +\infty)$ is lower semicontinuous, $f(p) < \infty$ for at least one $p \in X$. For $n = 1, 2, 3, \cdots$, $x \in X$, define $g_n(x) = \inf\{f(p) + n\,d(x,p) : p \in X\}$ and prove that

(i) $|g_n(x) - g_n(y)| \leq n\,d(x,y)$;

(ii) $0 \le g_1 \le g_2 \le \cdots \le f$;

(iii)$g_n(x) \to f(x)$ as $n \to \infty$, for all $x \in X$.

Thus f is the pointwise limit of an increasing sequence of continuous functions. (Note that the converse is almost trivial. If $f = \lim_{n\to\infty} f_n$, f_n is lower semicontinuous, and f_n is increasing, then f is lower semicontinuous. Indeed, for all $a \in \mathbb{R}$, $\{x : f(x) > a\} = \bigcup_{n=1}^{+\infty} \{x : f_n(x) > a\}$. Since f_n is lower semicontinuous, we have $\{x : f_n(x) > a\}$ is open, and $\{x : f(x) > a\}$ is open.)

Proof: (i) For any $p,\ x,\ y \in X,\ n \in \mathbb{N}$, we have $f(p) + n\,d(y,p) + n\,d(x,y)$,

$$g_n(x) = \inf\{f(q) + n\,d(x,q) : q \in X\} \le f(p) + n\,d(x,p) \le f(p) + n\,d(y,p) + n\,d(x,y)$$

and then we get

$$g_n(x) \le \inf\{f(p) + n\,d(y,p) : p \in X\} + n\,d(x,y) = g_n(y) + n\,d(x,y),$$

hence $g_n(x) - g_n(y) \le n\,d(x,y)$ for all $x,\ y \in X$. Similarly, $g_n(y) - g_n(x) \le n\,d(x,y)$ for all $x,\ y \in X$. So $|g_n(x) - g_n(y)| \le n\,d(x,y)$. This implies that g_n is continuous on X.

(ii) Clearly, $g_n(x) \ge 0$ for $x \in X$ and $n \in \mathbb{N}$ as $f \ge 0$. For any $x \in X$, since $f(p) + n\,d(x,p) \le f(p) + (n+1)d(x,p)$, we have

$$g_n(x) = \inf\{f(p) + n\,d(x,p) : p \in X\} \le \inf\{f(p) + (n+1)d(x,p) : p \in X\} = g_{n+1}(x).$$

By definition, $g_n(x) \le f(x) + n\,d(x,x) = f(x)$, therefore $0 \le g_1 \le g_2 \le \cdots \le f$.

(iii) By (ii), $0 \le g_1 \le g_2 \le \cdots \le f$, $\lim_{n\to\infty} g_n(x) \in [0, +\infty]$ exists for each $x \in X$.

Claim: If $0 \le f \le +\infty$ lower semicontinuous on X, and there exists $p_n \in X,\ f(p_n) < +\infty$, then for all $x \in X$, if $d(p_n, x) \to 0$ then $\liminf_{n\to\infty} f(p_n) \ge f(x)$.

Indeed, if $0 \le f(x) < \infty$, for all $\varepsilon > 0,\ x \in \{y : f(y) > f(x) - \varepsilon\}$, since f is lower semicontinuous, there exists a open set $V = B(x,\delta)$, $(\delta > 0)$, for all $y \in B(x,\delta),\ f(y) > f(x) - \varepsilon$. If $p_n \to x$, then there exists N, for $n \ge N$, we have $p_n \in B(x,\delta)$. Hence $f(p_n) > f(x) - \varepsilon$, $\liminf_{n\to\infty} f(p_n) \ge f(x) - \varepsilon$. Then as $\varepsilon \to 0$, we have $\liminf_{n\to\infty} f(p_n) \ge f(x)$. If $f(x) = +\infty$, $p_n \to x$, Just suppose that $0 \le \liminf_{n\to\infty} f(p_n) = M < f(x)$, then $+\infty = f(x) > 2M$, $x \in \{y : f(y) > 2M\}$. There exists $\delta > 0$ such that $B(x,\delta) \subset \{y : f(y) > 2M\}$. If N is large, $p_n \in B(x,\delta)$, $f(p_n) > 2M$, $M = \liminf f(p_n) \ge 2M > M$, a contradiction. So $\liminf f(p_n) \ge f(x)$ as well.

Now we prove that $\lim_{n\to\infty} g_n(x) = f(x)$ for any $x \in X$. We should distinguish two cases: (case a) $f(x) = +\infty$; (case b) $0 \le f(x) < +\infty$.

In (case a), By the definition of $g_n(x)$, for all $\varepsilon > 0$, there exist $p_n \in X$ such that $g_n(x) \le f(p_n) + n\,d(x,p_n) \le g_n(x) + \varepsilon$. If $\limsup_{n\to\infty} d(x,p_n) = c > 0$, then there exists subsequence $\{p_{n_k}\}$ such that $d(x, p_{n_k}) \to c > 0$. $n_k d(x, p_{n_k}) \le g_{n_k}(x) + \varepsilon$. Letting $k \to \infty$ we have $+\infty \le \lim_{n\to\infty} g_{n_k}(x) + \varepsilon = \lim_{n\to\infty} g_n(x) + \varepsilon$. Since g_n is increasing, $\lim_{n\to\infty} g_n(x) = f(x) = +\infty$. If $\limsup_{n\to\infty} d(x,p_n) = 0$, then $d(x,p_n) \to 0$. $f(p_n) \le f(p_n) + n\,d(x,p_n) \le g_n(x) + \varepsilon$, the Claim above gives that $+\infty = f(x) \le \liminf_{n\to\infty} f(p_n) \le \liminf_{n\to\infty} g_n(x) + \varepsilon = \lim_{n\to\infty} g_n(x) + \varepsilon$. Thus $\lim_{n\to\infty} g_n(x) = f(x) = +\infty$.

In (case b), for all $\varepsilon > 0$, there exist $p_n \in X$ such that

$$f(x) + \varepsilon \ge g_n(x) + \varepsilon \ge f(p_n) + n\,d(x,p_n) \ge n\,d(x,p_n),$$

then $\limsup\limits_{n\to\infty} n\, d(x,p_n) \le \limsup\limits_{n\to\infty} \frac{f(x)+\varepsilon}{n} = 0,\ d(x,p_n)\to 0$. The Claim implies that

$$f(x) \le \liminf_{n\to\infty} f(p_n) \le \liminf_{n\to\infty}(f(p_n)+n\,d(x,p_n)) \le \liminf_{n\to\infty}(g_n(x)+\varepsilon) = \lim_{n\to\infty} g_n(x)+\varepsilon.$$

Let $\varepsilon\to 0$, we get $f(x)\le \lim\limits_{n\to\infty} g_n(x) \le f(x)$, then $\lim\limits_{n\to\infty} g_n(x) = f(x)$. □

23, Suppose V is open in $\mathbb{R}^k$ and μ is a finite positive Borel measure on $\mathbb{R}^k$. Is the function that sends x to $\mu(V+x)$ necessarily continuous ? lower semicontinuous ? upper semicontinuous ?

Answer: the function $x\mapsto \mu(V+x)$ is not necessarily continuous, nor upper semicontinuous, but lower semicontinuous.

For example, let $\mathscr{B}$ be the collection of all Borel sets in $\mathbb{R}^k$. Denote $B(a,r)=\{x\in\mathbb{R}^k, |x-a|<r\}$. Now for any fixed $a\in\mathbb{R}^k$, for any $A\in\mathscr{B}$, define

$$\mu(A)=\begin{cases} 1 & a\in A;\\ 0 & a\notin A.\end{cases}$$

($\mu(A)$ is the unit mass concentrated at a.)
Clearly, μ is a finite positive Borel measure. Choose $V=B(0,1)$ then

$$\mu(V+x)=\mu(B(x,1))=\begin{cases} 1 & a\in B(x,1);\\ 0 & a\notin B(x,1).\end{cases}=\begin{cases} 1 & x\in B(a,1);\\ 0 & x\notin B(a,1).\end{cases}=\chi_{B(a,1)}(x).$$

We know that as the characteristic function of an open ball $B(a,1)$, $\mu(V+x)$ is not continuous nor upper semicontinuous, but lower semicontinuous. We prove that for any finite positive Borel measure on $\mathbb{R}^k$, $x\mapsto\mu(V+x)$ is lower semicontinuous.

Claim 1: X is metric space, if for all $x\in X$, and for all $\{x_n\}\subset X$ with $x_n\to x$ we have $f(x)\le\liminf f(x_n)$, then f is lower semicontinuous on $\mathbb{R}^k$.

Indeed, for all $\alpha\in\mathbb{R}$, we want to show that $\{x: f(x)\le\alpha\}$ is closed. For all $\{x_n\}$, $x_0\in X$, $x_n\to x_0$ and $x_n\in\{x: f(x)\le\alpha\}$, we have $f(x_n)\le\alpha$, $f(x_0)\le\liminf f(x_n)\le\alpha$, $x_0\in\{x: f(x)\le\alpha\}$, $\{x: f(x)\le\alpha\}$ is closed.

Claim 2: Let x_0, $\{x_n\}\subset\mathbb{R}^k$, $x_n\to x_0$, $V_0=x_0+V$, where V is an open set. $V_n=x_n+V$, then $\bigcap\limits_{m=1}^{+\infty}\bigcup\limits_{n=m}^{+\infty}(V_0\backslash V_n)=\emptyset$.

Indeed, if $\bigcap\limits_{m=1}^{+\infty}\bigcup\limits_{n=m}^{+\infty}(V_0\backslash V_n)\ne\emptyset$, then there exists $y\in\bigcap\limits_{m=1}^{+\infty}\bigcup\limits_{n=m}^{+\infty}(V_0\backslash V_n)$, i.e. for all $m\in\mathbb{N}$, $y\in\bigcap\limits_{m=1}^{+\infty}\bigcup\limits_{n=m}^{+\infty}(V_0\backslash V_n)$. So there exist $\{n_k\}_{k=1}^{+\infty}\subset\mathbb{N}$ such that n_k is increasing as a function of k, and $n_k\to+\infty$ as $k\to\infty$, $y\in V_0\backslash V_{n_k}$. Since $y\in V_0$ is open, there exists $\delta>0$ such that $B(y,\delta)\subset V_0$. This implies that $B(y+(x_{n_k}-x_0),\delta)\subset V_{n_k}$. In fact, for all $z\in B(y+(x_{n_k}-x_0),\delta)$ we have $z-x_{n_k}+x_0\in B(y,\delta)\subset V_0=x_0+V$, then $z-x_{n_k}\in V$, $z\in x_{n_k}+V=V_{n_k}$. As $x_{n_k}\to x_0$, for k large enough, we have $|x_{n_k}-x_0|<\delta/2$. Therefore $B(y,\delta/2)\subset B(y+(x_{n_k}-x_0),\delta)\subset V_{n_k}$. So $y\in B(y,\delta/2)\subset V_{n_k}$, contradicting to the fact $y\in V_0\backslash V_n$.

Claim 3: $\mu(V+x)$ is lower semicontinuous.

Indeed, for any $\{x_n\}\subset\mathbb{R}^k$, $x_0\in\mathbb{R}^k$ with $x_n\to x_0$, we need only to show that $\liminf\limits_{n\to\infty}\mu(V+$

$x_n) \geq \mu(V+x_0)$ by Claim 1. Since $\mu(\mathbb{R}^k) < +\infty$, we have

$$\begin{aligned}
\mu(V+x_n) - \mu(V+x_0) &= \mu(V_n) - \mu(V_0) \\
&= \mu((V_n \backslash V_0) \cup (V_n \cap V_0)) - \mu((V_0 \backslash V_n) \cup (V_0 \cap V_n)) \\
&= \mu(V_n \backslash V_0) + \mu(V_n \cap V_0) - \mu(V_0 \backslash V_n) - \mu(V_0 \cap V_n) \\
&= \mu(V_n \backslash V_0) - \mu(V_0 \backslash V_n) \geq -\mu(V_0 \backslash V_n).
\end{aligned}$$

Let $A_m = \bigcup\limits_{n=m}^{+\infty} (V_0 \backslash V_n)$, then Claim 2 shows that $\bigcap\limits_{m=1}^{+\infty} A_m = \emptyset$, $\mu(\bigcap\limits_{m=1}^{+\infty} A_m) = \mu(\emptyset) = 0$, and clearly

$$A_1 \supset A_2 \supset A_3 \supset \cdots, \ \mu(A_1) \leq \mu(\mathbb{R}^k) < +\infty.$$

So Theorem 1.19 gives that

$$\lim_{m \to +\infty} \mu(A_m) = \mu(\bigcap_{m=1}^{+\infty} A_m) = 0, i.e. \lim_{m \to +\infty} \mu(\bigcup_{n=m}^{+\infty} (V_0 \backslash V_n)) = 0.$$

As $\mu(V_0 \backslash V_m) \leq \mu(\bigcup\limits_{n=m}^{+\infty} (V_0 \backslash V_n))$, we have $\lim\limits_{m \to +\infty} \mu(V_0 \backslash V_m) = 0$. Thus

$$\liminf_{n \to +\infty} (\mu(V+x_n) - \mu(V+x_0)) \geq \liminf_{n \to +\infty} (-\mu(V_0 \backslash V_n)) = 0, \ \liminf_{n \to +\infty} \mu(V+x_n) \geq \liminf_{n \to +\infty} \mu(V+x_0).$$

So $\mu(V+x)$ is lower semicontinuous on $\mathbb{R}^k$. □

24, A step function is, by definition, a finite linear combination of characteristic functions of bounded intervals in $\mathbb{R}^1$. Assume $f \in L^1(\mathbb{R}^1)$, and prove that there is a sequence $\{g_n\}$ of step functions so that $\lim\limits_{n \to +\infty} \int_{-\infty}^{+\infty} |f(x) - g_n(x)| \, dx = 0$

Proof: It is enough to prove the conclusion for real $f \in L^1(\mathbb{R}^1)$. By Theorem 3.13, 3.14 (p69 Ch.3), $C_c(\mathbb{R}^1)$ is dense in $L^1(\mathbb{R}^1)$. $\forall \varepsilon > 0$, there exists $s \in C_c(\mathbb{R}^1)$ such that s is real and $\int_{-\infty}^{+\infty} |f(x) - s(x)| \, dx < \varepsilon/2$. Let supp$s = [a,b]$, then s is Riemanical integrable on $[a,b]$, so there exists partition $a = x_0 < x_1 < \cdots < x_m = b$, $I_i = (x_i, x_{i+1})$, $\Delta x_i = x_{i+1} - x_i$, with $m_i = \min\limits_{x_i \leq x \leq x_{i+1}} s(x)$, $i = 0, 1, \cdots m-1$, such that $|\int_a^b s(x) \, dx - \sum\limits_{i=1}^{m-1} m_i \Delta x_i| < \varepsilon$. i.e. $|\int_a^b s(x) \, dx - \int_a^b \sum\limits_{i=1}^{m-1} m_i \chi_{I_i}(x) \, dx| < \varepsilon/2$. Since $s(x) - \sum\limits_{i=1}^{m-1} m_i \chi_{I_i}(x) \geq 0$ on $[a,b]$, we have

$$\int_a^b |s(x) - \sum_{i=1}^{m-1} m_i \chi_{I_i}(x)| \, dx = |\int_a^b (s(x) \, dx - \int_a^b \sum_{i=1}^{m-1} m_i \chi_{I_i}(x) \, dx| < \varepsilon/2.$$

So

$$\begin{aligned}
\int_{-\infty}^{+\infty} |f(x) - \sum_{i=1}^{m-1} m_i \chi_{I_i}(x)| \, dx &\leq \int_{-\infty}^{+\infty} |f(x) - s(x)| \, dx + \int_{-\infty}^{+\infty} |s(x) - \sum_{i=1}^{m-1} m_i \chi_{I_i}(x)| \, dx \\
&< \varepsilon/2 + \int_a^b |s(x) - \sum_{i=1}^{m-1} m_i \chi_{I_i}(x)| \, dx \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

$\sum\limits_{i=1}^{m-1} m_i \chi_{I_i}(x)$ is a step function, so the conclusion is proved. □

25, (i) Find the smallest constant c such that $\log(1+e^t) < c+t$ $(0 < t < +\infty)$.
(ii) Does $\lim\limits_{n\to\infty} \frac{1}{n}\int_0^1 \log(1+e^{nf(x)})\, dx$ exists for every $f \in L^1$? If it exists, what is its solution ?
Solution: (i) $\log(1+e^t) < c+t$ $(0 < t < +\infty)$ is equivalent to $1+e^t < e^c e^t$, $1+e^{-t} < e^c$, $\ln(1+e^t) < c$, then $\lim\limits_{n\to\infty} \ln(1+e^t) \le c$, i.e. $\ln 2 \le c$, so the smallest c is $\ln 2$.

(ii) If real $f \in L^1((0,1))$, then

$$\begin{aligned} 0 \ &\le \ |\frac{\ln(1+e^{nf(x)})}{n}| = \frac{\ln(1+e^{nf(x)})}{n} \le \frac{\ln(1+e^{n|f(x)|})}{n} \\ &\le \ \frac{c+n|f(x)|}{n} \le 1+|f(x)|. \quad (if\ c \le n) \end{aligned}$$

Clearly, $1+f(x) \in L^1((0,1))$ as $f \in L^1((0,1))$, so for all n, $f_n(x) = \frac{\ln(1+e^{nf(x)})}{n} \in L^1((0,1))$

$$f_n(x) \to \begin{cases} 0 & f(x) = 0; \\ f(x) & f(x) > 0; \\ 0 & f(x) < 0. \end{cases}$$

Then $f_n(x) \to f^+(x)$ as $n \to +\infty$ for each $x \in L^1((0,1))$. So dominated convergence theorem can be applied to get $\lim\limits_{n\to\infty} \frac{1}{n}\int_0^1 \ln(1+e^{nf(x)})\, dx = \int_0^1 f^+(x)\, dx$. □

13, Is it true that every compact subset of $\mathbb{R}^1$ is the support of a continuous function? If not, can you describe the class of all compact sets in $\mathbb{R}^1$ which are supports of continuous functions? Is you description valid in other topological space ?
Solution: Let C be the Cantor set in $[0,1]$, then C^c is dense in $[0,1]$. As for any open interval I in $[0,1]$, $I \cap C \neq \emptyset$, $I \cap C^c \neq \emptyset$. If there exists $f \in C([0,1], C)$, $C = \text{supp} f = \overline{\{x : f(x) \neq 0\}}$, then $f \equiv 0$ on C^c, hence $f \equiv 0$ on $[0,1]$. So $\{x : f(x) \neq 0\} = \emptyset$, $C = \emptyset$, a contradiction. So not every compact set is the support of a continuous function.

Claim: A compact set $K \subset \mathbb{R}^1$ is the support of a continuous iff $K = \overline{V}$ for some bounded open set $V \subset \mathbb{R}^1$.

Indeed, If $K = \text{supp} f$, for some continuous function f on $\mathbb{R}^1$, then $V = \{x : f(x) \neq 0\}$ is open and $K = \overline{V}$, V is bounded. Now suppose $K = \overline{V}$ for some bounded open set $V \subset \mathbb{R}^1$. Clearly, V is a bounded open set, and $V = \bigcup\limits_{i=1}^{n} (a_i, b_i)$ for $n \in \mathbb{N}$ or $n = +\infty$. $a_i < b_i$, $1 \le i \le n$, $a_i, b_i \in V^c$, $\{(a_i, b_i)\}$ are pairwise disjoint. Let

$$f(x) = \begin{cases} 0\ if\ x \in V^c; \\ continuous\ and\ strictly\ increasing\ in (a_i, a_i + \frac{b_i - a_i}{2}]; \\ stricly\ decreasing\ in\ [a_i + \frac{b_i - a_i}{2}, b_i); \\ \lim\limits_{x \to a_i+} f(x) = \lim\limits_{x \to b_i-} f(x) = 0,\ f > 0\ on\ (a_i, b_i). \end{cases}$$

Then f is continuous on $\mathbb{R}^1$, and $\{x : f \neq 0\} = V$, $K = \overline{V}$. Here the fact that $\{(a_i, b_i)\}_{i=i}^{n}$ are pairwise disjoint is used to make f well-defined and continuous. □

18, Let $\mathcal{Z}$ This exercise requires more set-theory skill than the preceding ones. Let X be a well-ordered uncountable set which has a last element ω_1, such that every predecessor of ω_1 has at most countable many predecessor ("construction": Take any well-ordered set which has elements with uncountable many predecessors and let ω_1 be the first of these; ω_1 is called the first uncountable ordinal.) For $\alpha \in \mathcal{Z}$, let P_α $[S_\alpha]$ be the set of all predecessor (successors) of α, and call a subset of $\mathcal{Z}$ open if it is a P_α or an S_β or a $P_\alpha \cap S_\beta$ or a union of such sets. Prove that $\mathcal{Z}$ is then a compact Hausdorff space (Hint: No well-ordered set contains an infinite decreasing sequence.)

Prove that the complement of the poiint ω_1 is an open set which is not σ-compact.

Prove that to every $f \in C(\mathcal{Z})$ there corresponds an $\alpha \neq \omega_1$ such that f is constant on S_α.

Prove that the intersection of every countable collection $\{K_\alpha\}$ of uncountable compact subsets of $\mathcal{Z}$ is uncountable. (Hint: Consider limits of increasing countable sequences in $\mathcal{Z}$ which intersects each K_α is infinitely many points.)

Let m be the collection of all $E \subset \mathcal{Z}$ such htat either $E \cup \{\omega_1\}$ or $E^c \cup \{\omega_1\}$ contains an uncountable compact set; in the first case, define $\lambda(E) = 1$; in the second case, define $\lambda(E) = c$. Prove that m is a σ-algebra which contains all Borel sets in $\mathcal{Z}$, that λ is a measure on m which in NOT regular (every nighborhood of ω_1 has measure 1), and that

$$f(\omega_1) = \int_{\mathcal{Z}} f \, d\lambda,$$

for every $f \in C(\mathcal{Z})$. Describe the regular μ which Th 2.14 associates with this linear function.

Proof: A well-ordered set $\mathcal{Z}$ in a set $\mathcal{Z}$ with a totalls-ordered relation " $\leq$ " such that any $\phi \neq A \subset \mathcal{Z}$?? element $a \in A$ such that $a \leq x$ for aby $x \in A$. $x < y \Leftarrow x \leq y, x \neq y$.

$$\forall \alpha \in \mathcal{Z},\ P_\alpha = \{x \in \mathcal{Z}, x < \alpha\}, \quad S_\alpha = \{x \in \mathcal{Z}; \alpha < x\}.$$

Firstly, we must show that the "open sets" τ given in the problem indeed give a topology.

(i) $\mathcal{Z} \in \tau$. Since $\mathcal{Z}$ is well-oredered, there is a smallest element $a_0 \in \mathcal{Z}$ s.t $a_0 \leq x$, $\forall x \in \mathcal{Z}$. Since $\mathcal{Z}$ is uncountable, there is a $\beta \in \mathcal{Z}$, $a_0 < \beta$, then $a_0 \in P_\beta$, $\mathcal{Z} = S_{a_0} \cup P_\beta$. $\phi \in \tau$, $\phi = P_{a_0} \in \tau$.

(ii) Clearly, $\tau = \{all\ P_\alpha, S-\beta,\ P_\alpha \cap S_\beta,\ \alpha \in \mathcal{Z},\ \beta \in \mathcal{Z},\ and\ a\ union\ of\ such\ sets.\}$ So union of sets in τ belongs to τ.

(iii) Finite intersection of sets in τ belongs to τ. Let

$$\Omega = (\cup_{\alpha \in A}) \cap (\cup_{\beta \in B} S_\beta) \cap (\cup_{\alpha' \in \tilde{A}, \beta' \in \tilde{B}} P_{\alpha'} \cap S_{\beta'}),$$

where $A, B, \tilde{A}, \tilde{B} \subset \mathcal{Z}$. Then B has a smallest element β_0 as $\mathcal{Z}$ is well-ordered, and hence

$$\cup_{\beta \in B} S_\beta = S_{\beta_0}.$$

$$\begin{aligned}
\Omega &= (\cup_{\alpha \in A} P_\alpha) \cap S_{\beta_0} \cap (\cup_{\alpha' \in \tilde{A}, \beta' \in \tilde{B}} P_{\alpha'} \cap S_{\beta'}) \\
&= (\cup_{\alpha \in A} (P_\alpha \cap S_{\beta_0})) \cap (\cup_{\alpha' \in \tilde{A}, \beta' \in \tilde{B}} P_{\alpha'} \cap S_{\beta'}) \\
&= (\cup_{\alpha \in A, \beta \in B, \alpha' \in \tilde{A}, \beta' \in \tilde{B}}) P_\alpha \cap P_{\alpha'} S_{\beta_0} \cap \cap S_{\beta'}) \\
&= (\cup_{\alpha \in A, \beta \in B, \alpha' \in \tilde{A}, \beta' \in \tilde{B}}) P_{\min \alpha, \alpha'} \cap S_{\max \gamma, \beta_0} \in \tau,
\end{aligned}$$

where

$$\min(\alpha, \alpha') = \begin{cases} \alpha & if\ \alpha \leq \alpha', \\ \alpha' & if\ \alpha \geq \alpha', \end{cases}$$

and

$$\max(\beta, \beta_0) = \begin{cases} \beta & if\ \beta \geq \beta_0, \\ \beta_0 & if \beta \leq \beta_0. \end{cases}$$

In fact, we actually proved that intersection (not necessary finite) of sets in τ is still in τ. $(\mathcal{Z},\tau)$ is indeed a topology space.
We prove $(\mathcal{Z},\tau)$ is a Hausdorff.
For any $\alpha,\beta\in\mathcal{Z}$ if $\alpha\neq\beta$, $WLOG$ $\alpha<\beta$. Then if there is a $\alpha_0\in\mathcal{Z}$ with $\alpha<\alpha_0<\beta$, then $\alpha\in P_{\alpha_0}$, $\beta\in S_{\alpha_0}$, $P_{\alpha_0}\cap S_{\alpha_0}=\emptyset$.
If there is no $a\in\mathcal{Z}$ with $\alpha<a<\beta$, then $P_\beta\cap S_\alpha=\emptyset$ and $\alpha\in P_\beta$, $\beta\in S_\alpha$. So $\mathcal{Z}$ is Hausdorff.

I doubt $\mathcal{Z}$ that the conclusion which $\mathcal{Z}$ is compact may not be correct.
If $\mathcal{Z}$ does't have a maximal member M such that $\forall x\in\mathcal{Z}$, $M\geq x$, then $\forall x\in\mathcal{Z},\exists y_x\in\mathcal{Z},x<y_x$, $x\in P_{y_x}$, $\mathcal{Z}=\cup_{x\in\mathcal{Z}}P_{y_x}$ and P_{y_x} open.
If $\mathcal{Z}$ is compact, then exists finite member x_i, such that $\mathcal{Z}=\cup_{i=1}^{+\infty}P_{y_x}$. Set $y_i=\max\{y_{x_1},\cdots,y_{x_m}\}$, then $\mathcal{Z}=P_{y_0}$ i.e $\forall x\in\mathcal{Z},x<y_0$, it is contradict on $\mathcal{Z}$ is not bounded from above. Thus $\mathcal{Z}$ may not be compact set. $\square$

Chapter 3

1, Prove that to supermum of any collection of convex function on (a,b) is convex on (a,b) (if it is finie) and that pointwise limits of a sequence of convex functions are convex. What can you say about upper and lower limits of sequences of convex functions?
Proof: Let $\{\varphi_\tau\}_{\tau\in I}$ be a collection of convex functions on (a,b). $\forall x,y\in(a,b),0\leq\lambda\leq1$, we have

$$\varphi_\tau(\lambda x+(1-\lambda)y)\leq\lambda\varphi_\tau(x)+(1-\lambda)\varphi_\tau(y),$$

$$\begin{aligned}\sup_{\tau\in I}\varphi_\tau(\lambda x+(1-\lambda)y)&\leq\sup_{\tau\in I}(\lambda\varphi_\tau(x)+(1-\lambda)\varphi_\tau(y))\\&\leq\sup_{\tau\in I}(\lambda\varphi_\tau(x))+\sup_{\tau\in I}((1-\lambda)\varphi_\tau(y))\\&=\lambda\sup_{\tau\in I}\varphi_\tau(x)+(1-\lambda)\sup_{\tau\in I}\varphi_\tau(y).\end{aligned}$$

Thus $(\sup\limits_{\tau\in I}\varphi_\tau)(x)$ is convex on (a,b).
If $\{\varphi_n(x)\}_{n=1}^{+\infty}$ is a sequence of convex functions on (a,b), $\lim\limits_{n\to\infty}\varphi_n(x)=\varphi(x)$ exists, then $\forall x,y\in(a,b),0\leq\lambda\leq1$,

$$\varphi_n(\lambda x+(1-\lambda)y)\leq\lambda\varphi_n(x)+(1-\lambda)\varphi_n(y),$$

$$\lim_{n\to\infty}\varphi_n(\lambda x+(1-\lambda)y)\leq\lambda\lim_{n\to\infty}\varphi_n(x)+(1-\lambda)\lim_{n\to\infty}\varphi_n(y),$$

therefore $\lim\limits_{n\to\infty}\varphi_n(x)$ is convex on (a,b).
$\varphi_\tau,(\tau\in I)$ is convex $\Rightarrow\inf\limits_{\tau\in I}\varphi_\tau$ is convex.???// $\square$

2, If φ is convex on (a,b) and if ψ is convex and nondecreasing on the range of φ, prove that $\psi\cdot\varphi$ is convex on (a,b). For $\varphi>0$, show that the convexity of $\log\varphi$ implies the convexity of φ, but not vice versa.

Proof: Since ψ is nondecreasing on range φ, $\forall\lambda \in [0,1]$ and x_1, $x_2 \in (a,b)$, by the convexity of φ and ψ on (a,b), we have

$$\varphi(\lambda x_1 + (1-\lambda)x_2) \leq \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2),$$

$$\begin{aligned} \psi\cdot\varphi(\lambda x_1 + (1-\lambda)\varphi_2) &\leq \psi(\lambda\varphi(x_1) + (1-\lambda)\varphi(x_2)) \\ &\leq \lambda\psi\cdot\varphi(x_1) + (1-\lambda)\psi\cdot\varphi(x_2). \end{aligned}$$

So $\psi\cdot\varphi$ is convex on (a,b).
For $\varphi > 0$ if $\log\varphi$ is convex, then $\forall\lambda \in [0,1]$, $\forall x_1$, $x_2 \in (a,b)$, we have

$$\begin{aligned} \log\varphi(\lambda x_1 + (1-\lambda)x_2) &\leq \lambda\log\varphi(x) + (1-\lambda)\log\varphi(x_2) \\ &= \log(\varphi(x_1)^\lambda\varphi(x_2)^{(1-\lambda)}), \end{aligned}$$

then

$$\varphi(\lambda x_1 + (1-\lambda)x_2) \leq \varphi(x_1)^\lambda\varphi(x_2)^{(1-\lambda)}.$$

By young's inequality, set $\lambda = \frac{1}{p}$, $(1-\lambda) = \frac{1}{\epsilon}$, then $\frac{1}{p} + \frac{1}{\epsilon} = 1$, and

$$\begin{aligned} \varphi(x_1)^\lambda\varphi(x_2)^{(1-\lambda)} &= \varphi(x_1)^{\frac{1}{p}}\varphi(x_2)^{\frac{1}{\epsilon}} \\ &\leq \tfrac{1}{p}[\varphi(x_1)^{\frac{1}{p}}]^p + \tfrac{1}{\epsilon}(\varphi(x_2)^{\frac{1}{\epsilon}})^\epsilon \\ &= \tfrac{1}{p}\varphi(x_1) + \tfrac{1}{\epsilon}\varphi(x_2) = \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2). \end{aligned}$$

So

$$\varphi(\lambda x_1 + (1-\lambda)x_2) \leq \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2),$$

and φ is convex on (a,b).
But the fact that $\varphi > 0$ is convex does't always imply that $\log\varphi$ is convex. For example $\varphi(x) = x^2, x \in (1,3)$ is convex but $\log\varphi(x) = 2\log x$ is convex on $(1,2)$, hence not convex. $\square$

3, Assume that φ is a continuous real function on (a,b) such that

$$\varphi(\frac{x+y}{2}) \leq \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(y),$$

for all $x,y \in (a,b)$. Prove that φ is convex. (The conlusion does not follow if continuity is omitted from the hypotheses.)
Proof: We want to show that $\forall x,y \in (a,b)$, $\forall\lambda \in [0,1]$, there holds

$$\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y). \tag{3}$$

Denote $S_n = \{\Sigma_{k=1}^n \frac{a_k}{2^k},\, a_k = 0, or, 1\}, S = \cup_{n=1}^\infty S_n$.
Claim: $\forall\lambda \in S$, (**??**) holds for all $x,y \in (a,b)$, we use indution to prove the claim.
Step1: $\forall\lambda \in S_1$, (**??**) holds for all $x,y \in (a,b)$ if $\lambda \in S_1$, then $\lambda = 0, or\ \frac{1}{2}$, then

$$\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y)$$

due to the assumption. ($\lambda = 0$ is trivial, $\lambda = \frac{1}{2}$ are assumption)
Step2: Suppose $\forall\lambda \in S_n, \forall x,y \in (a,b)$, (**??**) is true.

Now if $\lambda \in S_{n+1}$, then $\lambda = \lambda' + \frac{a_{n+1}}{2^{n+1}}$, $a_{n+1} = 0$ *or* $\lambda' \in S_n$. If $a_{n+1} = 0$, then $\lambda = \lambda\lambda' \in S_n$, (**??**) holds by the induction assumption; If $a_{n+1} = 1$, then $\lambda = \lambda' + \frac{1}{2^{n+1}} = \frac{1}{2}((\lambda' + \frac{1}{2^n}) + \lambda')$. Let $\lambda' = \sum\limits_{i=1}^{n} \frac{a_i}{2^i}$. If $a_n = 0$, then $\lambda' + \frac{1}{2^n} = \sum\limits_{i=1} n - 1\frac{a^i}{2^i} + \frac{1}{2^n} \in S_n$.

If $a_n = 1$, and $a_{n-1} = 0$, $\lambda' + \frac{1}{2^n} \in S_{n-1} \subset S_n$.

If $a_n = 1, a_{n-1} = 1, a_{n-2} = 0$, $\lambda' + \frac{1}{2^n} \in S_{n-2} \subset S_n$.

If $a_n = a_{n-1} = \cdots = a_1 = 1$, then

$$\begin{aligned} \lambda' + \tfrac{1}{2^n} &= \tfrac{1}{2} + \tfrac{1}{2^2} + \cdots + \tfrac{1}{2^n} + \tfrac{1}{2^n} \\ &= \frac{\frac{1}{2}(1-\frac{1}{2^n})}{1-\frac{1}{2}} + \tfrac{1}{2^n} = 1 - \tfrac{1}{2^n} + \tfrac{1}{2^n} = 1. \end{aligned}$$

So only two posibilities to $\lambda' + \frac{1}{2^n}$. Either $\lambda' + \frac{1}{2^n} = 1$ or $\lambda' + \frac{1}{2^n} \in S_n$.

If $\lambda' + \frac{1}{2^n} = 1$, then $1 - \lambda' = \frac{1}{2^n} \in S_n$, let $\lambda = \frac{1}{2}(1 + \lambda')$, so by the assumption,

$$\begin{aligned} &\varphi(\lambda x + (1-\lambda)y) = \varphi(\tfrac{1}{2}(1+\lambda')x + (1 - \tfrac{1+\lambda'}{2})y) \\ &= \varphi(\tfrac{1}{2}x + \tfrac{1}{2}(\lambda' x + (1-\lambda')y)) \\ &\le \tfrac{1}{2}\varphi(x) + \tfrac{1}{2}\lambda'\varphi(x) + \tfrac{1}{2}(1-\lambda')\varphi(y) \\ &= \tfrac{1}{2}(1+\lambda')\varphi(x) + \tfrac{1}{2}(1-\lambda')\varphi(y) \\ &= \lambda\varphi(x) + (1-\lambda)\varphi(y). \end{aligned}$$

Here used the fact that $\lambda' = 2\lambda - 1$, $\frac{1-\lambda'}{2} = \frac{1}{2}(1 - 2\lambda + 1) = 1 - \lambda$.

If $\mu = \lambda' + \frac{1}{2^n} \in S_n$, then by induction assumption

$$\begin{aligned} &\varphi(\lambda x + (1-\lambda)y) = \varphi(\tfrac{1}{2}(\mu+\lambda')x + (1 - \tfrac{\mu+\lambda'}{2})y) \\ &= \varphi(\tfrac{1}{2}(\mu x + (1-\mu)y) + \tfrac{1}{2}\varphi(\lambda' x + (1-\lambda')y)) \\ &\le \tfrac{1}{2}\varphi(\mu x + (1-\mu)y) + \tfrac{1}{2}\varphi(\lambda' x + (1-\lambda')y) \\ &\le \tfrac{1}{2}[\mu\varphi(x) + (1-\mu)\varphi(y)] + \tfrac{1}{2}[\lambda'\varphi(x) + (1-\lambda')\varphi(y)] \\ &= \lambda\varphi(x) + (1-\lambda)\varphi(y). \end{aligned}$$

So (**??**) is true for $\lambda \in S_{n+1}$, hence (**??**) holds for $\lambda \in S_m$, claim is proved.

Now, $\forall \lambda \in [0,1]$, there is $\{\lambda_n\} \subset S$ s.t, $\lambda_n \to \lambda,\ as\ n \to \infty$. By the continuity of φ and the claim, $\forall x, y \in (a,b)$

$$\varphi(\lambda_n x + (1-\lambda_n)y) \le \lambda_n\varphi(x) + (1-\lambda_n)\varphi(y),$$

the continuity of φ shows that as $n \to \infty$,

$$\varphi(\lambda x + (1-\lambda)y) \le \lambda\varphi(x) + (1-\lambda)\varphi(y),$$

so φ is convex. □

If the assumption of continuity of φ on (a,b) is dropped, the conclusion is not true.

Indeed, we have following collary:

Collary If $f(x)$ is upper bounded for all bounded set on domain I, and s.t

$$f(\frac{x_1 + x_2}{2}) \le \frac{f(x_1) + f(x_2)}{2},\ x_1,\, x_2 \in I.$$

In other words, $f(x)$ is convex on I.

Proof: By the contrary, there exist $x_1 \in I$, $x_2 \in I$, $0 < t < \frac{1}{2}$, s.t

$$f(tx_1 + (1-t)x_2) - tf(x_1) - (1-t)f(x_2) = \alpha > 0,$$

otherwise $\forall\lambda \neq \{\frac{1}{2}, 0, 1\}$, there exists one number between $(1-\lambda)$ and λ in $(0, \frac{1}{2})$. Thus $f(\lambda x_1 + (1-\lambda)x_2) - \lambda f(x) - (1-\lambda)f(x_2) \leq 0$, which contradict on $f(x)$ is convex.

Let

$$F(x) = f(x) - f(x_1) - \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x_1 - x_2).$$

Obviously, for all closed subinterval in I, $F(x)$ and $f(x)$ are all bounded or unbounded, and we can prove that

$$F(\frac{u_1 + u_2}{2}) \leq \frac{F(u_1) + F(u_2)}{2},\ u_1,\ u_2 \in I,$$

and $F(x_1) = F(x_2) = 0$,

$$F(tx_1 + (1-t)x_2) = f(tx_1 + (1-t)x_2) - tf(x_2) - (1-t)f(x_2) = \alpha > 0.$$

Thus

$$\begin{aligned} 2\alpha = 2F(tx_1 + (1-t)x_2) &= 2F[\tfrac{2tx_1 + (1-2t)x_2 + x_2}{2}] \\ &\leq F[2tx_1 + (1-2t)x_2] + F(x_2) = F[2tx_1 + (1-2t)x_2]. \end{aligned}$$

Let $t' = 2t$, we have

$$F(t'x_1 + (1-t')x_2] \geq 2\alpha > 0.$$

If $t' < \frac{1}{2}$, repeating this process as above. If $t' > \frac{1}{2}$, then $t_1 = 1 - t'$, $x_1' = x_2$, $x_2' = x_1$, and

$$F(t_1'x_1' + (1-t_1')x_2') \geq 2\alpha > 0,$$

then repeating the process as above. Thus there exist t'', t''', $\cdots$, t^n such that

$$F(t^n x_1 + (1-t^n)x_2) \geq 2^n\alpha.$$

when $n \to +\infty$, $\Rightarrow$ F is unbounded on $[x_1, x_2]$, contradiction! Indeed F is unbounded $\Leftrightarrow$ f is unbounded on $[x_1, x_2]$.

Collary, If f is a real Lehesgue measurable function on $[a, b]$ and such that

$$f(\frac{x+y}{2}) \leq \frac{f(x) + f(y)}{2},\ x, y \in (a, b).$$

Then f is a convex function on $[a, b]$. (particular if f is continuous).

Proof: We can only prove that f is bounded on all bound closed subinterval of $[a, b]$. From Heine-Borel finite covering Theorem, we only need to prove f is bounded on $\forall x_0 \in [a, b]$????.

?? By contrary, there exists $x_0 \in (a, b)$, and assume $x_0 = 0 \in (a, b)$, f is unbounded on all open neighborhood of (a, b), then there exists $t_n \to 0$, such that $|f(t_n)| \to +\infty$. We can assume that $f(t_n) \to +\infty, (n \to +\infty)$. By Lusin Theorem, there exists closed set $E \subset (-\frac{1}{2}, \frac{1}{2})$, $m(E) > \frac{4}{5}$, for f is continuous on E, therefore there exists $M > 0$, $f < M$ on E. Choosing $t_0 \in (-\frac{1}{4}, \frac{1}{4})$, such that $f(t_0) > M$. We can also assume $t_0 > 0$, and $\tilde{E} = \{y = 2t_0 - x, x \in E\}$, then $\tilde{E} \subset (-\frac{1}{2}, 1)$, $E \cup \tilde{E} \subset (-\frac{1}{2}, 1)$ by the fact for all $y \in \tilde{E}, x = 1$?????????????????????????????? $\square$

4, Suppose f is a complex measurable function on $\mathcal{Z}$, μ is a positive measure on $\mathcal{Z}$, and

$$\varphi(p) = \int_{\mathcal{Z}} |f|^p \, d\mu = \|f\|_p^p, \ (0 < p < \infty).$$

Let $E = \{p : \varphi(p) < \infty\}$, assume $\|f\|_\infty > 0$,
(a) If $r < p < s, r \in E$ and $s \in E$, prove that $p \in E$.
(b) Prove that $\phi \cdot \varphi$ is convex in the interior of E and that φ is continuous on E.
(c) By (a) E is connected, if E necessarily open? closed? Can E consist of a single point? Can E be any connected subset of $(0, \infty)$?
(d) If $r < p < s$, prove that $\|f\|_p \leq \max(\|f\|_r, \|f\|_s)$ show that this implies the conclusion $L^r(\mu) \cap L^s(\mu) \subset L^p(\mu)$.
(e) Assume that $\|f\|_r < \infty$ for some $r < \infty$ and prove that

$$\|f\|_p \mapsto \|f\|_\infty \ as \ p \to \infty.$$

Proof: (a) $p = \frac{(p-r)}{s-r}s + \frac{s-p}{s-r}r$ where $\frac{p-r}{s-r} + \frac{s-p}{s-r} = 1$. If $r, s \in E$, then

$$\begin{aligned}
&\textstyle\int_{\mathcal{Z}} |f|^p \, d\mu = \int_{\mathcal{Z}} |f|^{\frac{p-r}{s-r}s} |f|^{\frac{s-p}{s-r}r} \, d\mu \\
&\leq (\textstyle\int_{\mathcal{Z}} |f|^{\frac{p-r}{s-r}s\frac{s-r}{p-r}} \, d\mu)^{\frac{p-r}{s-r}} (\int_{\mathcal{Z}} |f|^{\frac{s-p}{s-r}r\frac{s-r}{s-p}} \, d\mu)^{\frac{s-p}{s-r}} \\
&= (\textstyle\int_{\mathcal{Z}} |f|^s \, d\mu)^{\frac{p-r}{s-r}} (\int_{\mathcal{Z}} |f|^r \, d\mu)^{\frac{s-p}{s-r}} \\
&= \varphi(s)^{\frac{p-r}{s-r}} \varphi(r)^{\frac{s-p}{s-r}} < +\infty.
\end{aligned}$$

Therefore $p \in E$.
(b) In (a), we proved in fact that if $r < s, r, s \in E$, $0 < \alpha < 1$, then

$$\varphi(\alpha s + (1-\alpha)r) \leq \varphi(s)^\alpha \varphi(r)^{1-\alpha}.$$

Since $\|f\|_\infty > 0$, we have $\varphi(p) > 0$, $\forall p > 0$. So if $r, s \in E$, $0 < \alpha < 1$, then

$$\log \varphi(\alpha s + (1-\alpha)r) \leq \log \varphi(s) \alpha \varphi(r)^{1-\alpha} = \alpha \log \varphi(s) + (1-\alpha) \log \varphi(r).$$

Note that hs (a) if $s, r \in E$, then $(s, r) \subset E$ (if $s < r$), so E is a convex set, the interior $E^0$of E is an open set and connected. So E^0 is an interval in $\mathbb{R}^1$, and E must be an interval in $(0, +\infty)$. Say $E = [p, q), p > 0$, $E^0 = (p, q)$ as $\log \varphi$ is convex in E^0, it is continuous in (p, q). We need only show that φ is continuous at p to show that φ is continuous on E. Indeed, let $p_n \searrow p$, we assume $p < p_n < p_1$ $(n \geq 2)$, so by Holder's inequality

$$\varphi(p_n) = \textstyle\int_{\mathcal{Z}} |f|^{p_n} \, d\mu = \int_{\mathcal{Z}} |f|^\alpha |f|^\beta \, d\mu \leq (\int_{\mathcal{Z}} |f|^{\alpha\frac{p}{\alpha}})^{\frac{\alpha}{p}} (\int_{\mathcal{Z}} |f|^{\beta\frac{p_1}{\beta}} \, d\mu)^{\frac{\beta}{p_1}} = \varphi(p)^{\frac{\alpha}{p}} \varphi(p_1)^{\frac{\beta}{p_1}},$$

where $\alpha = \frac{p(p_1-p_n)}{p_1-p}$, $\beta = \frac{p_1(p_n-p)}{p_1-p}$, let $n \to +\infty$, we have

$$\overline{\lim}_{n\to\infty} \varphi(p_n) \leq \overline{\lim}_{n\to\infty} (\varphi(p)^{\frac{p_1-p_n}{p_1-p}} \varphi(p_1)^{\frac{p_n-p}{p_1-p}}) = \varphi(p).$$

On the other hand, by Fatou's Lemma

$$\varphi(p) = \int_{\mathcal{Z}} |f|^p \, d\mu \leq \underline{\lim}_{n\to+\infty} \int_{\mathcal{Z}} |f|^{p_n} \, d\mu = \underline{\lim}_{n\to+\infty} \varphi(p_n) \leq \overline{\lim}_{n\to+\infty} \varphi(p_n) \leq (p),$$

thus $\lim\limits_{n\to+\infty}\varphi(p_n)=\varphi(p)$. Similar if $E=(p,q]$ or $E=[p,q]$, we can prove if $p_n\to q,$, $p_n\in E$, $\lim\limits_{n\to\infty}\varphi(p_n)=\varphi(q)$ as well, therefore φ is continuous on E.
(c) E is connected, but E may not be open, neither be closed. For example, let μ be Lebesgue's measure in $\mathbb{R}^1$, $f(x)=\sum\limits_{n=1}^{+\infty}\frac{1}{n}\chi_{[n,n+1)}$, therefore

$$\int_{\mathcal{Z}}|f|^p\ d\mu=\sum_{n=1}^{+\infty}\frac{1}{n^p}.$$

It is easy to see that

$$\int_{\mathcal{Z}}|f|^p\ d\mu=\begin{cases}+\infty & if\ p\le 1,\\ <+\infty & if\ p>1.\end{cases}$$

So $E=(1,+\infty)$, E is not closed, E can also consist of a single point. To see this, let μ be the Lebesgue measure on $\mathbb{R}^1$ too. Let

$$f(x)=\frac{1}{x(1+|\ln x|)^2},$$

since

$$\begin{aligned}&\textstyle\int_0^{+\infty}\frac{dx}{x(1+|\log x|)^2}=(\int_0^1+\int_1^{+\infty})\frac{dx}{x(1+|\log x|)^2}\\ &=\textstyle\int_{-\infty}^0\frac{dt}{(1+|t|)^2}+\int_0^{+\infty}\frac{dt}{(1+|t|)^2}=2\int_0^{+\infty}\frac{dt}{(1+|t|)^2}=1,\end{aligned}$$

which implies $\int_0^{+\infty}|f|\ d\mu<+\infty$, and $f\in L^1(\mu)$.
It is easy to see that $\forall\alpha>0,\exists N=N(\alpha)>1$ s.t $1+\ln x\le x^\alpha$ for $x\le N$, so if $xp>1$, then

$$\textstyle\int_0^1|f|^p\ dx=\int_0^1\frac{dx}{x^p(1+|\ln x|)^{2P}}\ge\int_0^{\frac{1}{N(\alpha)}}\frac{dx}{x^p(1+\ln\frac{1}{x})^{2P}}\ge\int_0^{\frac{1}{N(\alpha)}}\frac{dx}{x^px^{-2\alpha p}}=\int_0^{\frac{1}{N(\alpha)}}\frac{dx}{x^{p(1-2\alpha)}}.$$

Choose $\alpha=\frac{p-1}{4p}>0$, then $\exists\bar{N}=N(\alpha)>1$, s.t

$$\int_0^1|f|^p\ dx\ge\int_0^{\frac{1}{N}}\frac{dx}{x^{\frac{1+p}{2}}}\ge\int_0^{\frac{1}{N}}\frac{dx}{x}=+\infty.$$

If $0<p<1$, then $\int_1^{+\infty}|f|^p\ dx\ge\int_{N(\alpha)}^{+\infty}\frac{dx}{x^px^{2p\alpha}}=\int_{N(\alpha)}^{+\infty}\frac{dx}{x^{p(1+2\alpha)}}$. Choose $\alpha=\frac{1-p}{4p}>0$, then $\exists\bar{N}>1$ s.t

$$\int_0^{+\infty}|f|^p\ d\mu\ge\int_N^{+\infty}\frac{dx}{x^{\frac{1+p}{2}}\ge\int_N}^{+\infty}\frac{dx}{x}=+\infty.$$

Thus $E=\{1\}$ is not open.
Also, E can be any connected subset of $(0,+\infty)$. Let μ be the Lebesgue measure on $\mathbb{R}$, let

$$f_1(X)=\frac{1}{x}\chi_{[0,1]}(x),\quad f_2(x)=\frac{1}{x}\chi_{[1,+\infty)}(x),$$

$$f_3(x)=\frac{1}{x(1+|\ln x|)^2}\chi_{[0,1]}(x),\quad f_4(x)=\frac{1}{x(1+|\ln x|)^2}\chi_{[1,+\infty}(x)).$$

And let $E(f)=\{p|\|f\|_p=\int_0^{+\infty}|f|^p\ d\mu<+\infty,\ 0<p<+\infty\}$, then

$$E(f_1)=(0,1),\ E(f_2)=(1,+\infty),\ E(f_3)=(0,1],\ E(f_4)=[1,+\infty).$$

Indeed, $p\in E(f_1)\Leftrightarrow\int_0^1\frac{dx}{x^p}<+\infty\Leftrightarrow 0<p<1$,
$p\in E(f_2)\Leftrightarrow\int_1^{+\infty}\frac{dx}{x^p}<+\infty\Leftrightarrow 1<p<+\infty$,

$p \in E(f_3) \Leftrightarrow \int_0^1 \frac{dx}{x^p(1+|\ln x|)^{2p}} < +\infty \Leftrightarrow 0 < p < 1$,
$p \in E(f_4) \Leftrightarrow \int_1^{+\infty} \frac{dx}{x^p(1+|\ln x|)^{2p}} < +\infty \Leftrightarrow 1 < p < +\infty$. So all kinds of continuous connected subsets of $(0, +\infty)$ can be expressed by f_1, f_2, f_3, f_4. For example, consider $(p, q], (p < q)$, then

$$(p, q] = E(f_2^{\frac{1}{p}} + f_3^{\frac{1}{q}}) = E(f_2^{\frac{1}{p}}) \cap E(f_3^{\frac{1}{q}}) = (pE(f_2)) \cap (qE(f_3)) = (p, +\infty) \cap (0, q] = (p, q],$$

$$(p, q) = (p, +\infty) \cap (0, q) - (p(1, +\infty)) \cap (q(0, 1)) = (pE(f_2)) \cap (qE(f_1)) = E(f_2^{\frac{1}{p}}) \cap E(f_1^{\frac{1}{q}}) = E(f_2^{\frac{1}{p}} + f_1^{\frac{1}{q}}),$$

$$[p, q] = [p, +\infty) \cap [0, q] = (pE(f_4)) \cap (qE(f_3)) = E(f_4^{\frac{1}{p}}) \cap E(f_3^{\frac{1}{q}}) = E(f_4^{\frac{1}{p}} + f_3^{\frac{1}{q}}),$$

$$[p, q) = [p, +\infty) \cap (0, q) = (p(1, +\infty)) \cap (q(0, 1)) = (pE(f_4)) \cap (qE(f_1)) = E(f_4^{\frac{1}{p}} + f_1^{\frac{1}{q}}),$$

and

$$[p, +\infty) = E(f_4^{\frac{1}{p}}), \quad (0, q] = E(f_3^{\frac{1}{q}}),$$

$\cdots$, and so on.
(d) If $r < p < s$, then $\exists \lambda \in (0, 1)$, $p = \lambda x + (1 - \lambda)s$, just as what we did in proving (a)

$$\begin{aligned} \|f\|_p^p &\leq \|f\|_r^{\lambda r} \|f\|_s^{(1-\lambda)s} \leq (\max(\|f\|_r, \|f\|_s))^{\lambda r + (1-\lambda)s} \\ &= \max(\|f\|_r, \|f\|_s)^p, \ i.e \ \|f\|_p \leq \max(\|f\|_r, \|f\|_s). \end{aligned}$$

So if $f \in L^r(\mu) \cap L^s(\mu)$, i.e $\|f\|_r, \|f\|_s < +\infty$, then $\|f\|_p < +\infty$, i.e $f \in L^p(\mu)$, therefore $L^r(\mu) \cap L^s(\mu) \subset L^p(\mu), r < p < s$.
(e) If $\|f\|_\infty = \infty$, then $\forall n \in N$, $\mu(E_n) > 0$ where $E_n = \{x \in Z : |f(x)| > n\}$, by the continuity of $\| \cdot \|_\infty$, so $\|f\|_p = (\int_x |f|^p \, d\mu)^{\frac{1}{p}} \geq (\int_{E_n} |f|^p \, d\mu)^{\frac{1}{p}} \geq n(\mu(E_n))^{\frac{1}{p}}$. For fixed $n \in N$,

$$\underline{\lim} \|f\|_p \geq \underline{\lim}_{p \to \infty} (n\mu(E_n)^{\frac{1}{p}}) = n\mu(E_n)^0 = n.$$

Thus

$$\underline{\lim}_{p \to \infty} \|f\|_p = +\infty, \ \lim_{p \to \infty} \|f\|_p = +\infty.$$

If $\|f\|_\infty \in (0, +\infty)$, then for $p > r$ (where $\|f\|_r < +\infty$ by assumption), we have

$$\int_{\mathcal{Z}} |f|^p \, d\mu)^{\frac{1}{p}} = (\int_{\mathcal{Z}} |f|^r |f|^{p-r} \, d\mu)^{\frac{1}{p}} \leq (\|f\|_\infty^{p-r} \int_{\mathcal{Z}} f^r \, d\mu)^{\frac{1}{p}} = \|f\|_\infty^{\frac{p-r}{p}} \|f\|_r^{\frac{r}{p}}.$$

Since $\|f\|_\infty > 0$, we have $\|f\|_r > 0$, so

$$\overline{\lim}_{p \to \infty} \|f\|_p \leq \overline{\lim}_{p \to \infty} (\|f\|_\infty^{\frac{p-r}{p}} \|f\|_r^{\frac{r}{p}}) = \|f\|_\infty.$$

On the other hand, $\forall q > 0$, denote $E = \|x \in \mathcal{Z}, s.t, \ \|f\|_\infty - q > 0, |f(x)| \geq \|f\|_\infty - q\}$, then $\mu(E) > 0$, by the definition of $\|f\|$, as $\|f\|_r < +\infty$, we have $\mu(E) < +\infty$. Then

$$\|f\|_p = (\int_{\mathcal{Z}} |f|^p \, d\mu)^{\frac{1}{p}} \geq (\int_E |f|^p \, d\mu)^{\frac{1}{p}} \geq (\|f\|_\infty - q)(\mu(E))^{\frac{1}{p}}.$$

So $\underline{\lim}_{p \to \infty} \|f\|_p \geq \|f\|_\infty - q$. Letting $q \to 0$, we have

$$\|f\|_\infty \leq \underline{\lim}_{p \to \infty} \|f\|_p \leq \overline{\lim}_{p \to \infty} \|f\|_p \leq \|f\|_\infty.$$

So $\lim\limits_{p \to \infty} \|f\|_p = \|f\|_\infty$.

Question? IF $\forall r, \|f\|_r = +\infty$, we can get $\|f\|_\infty = +\infty$ or not?

Example: we define

$$f(x) = \begin{cases} 1 & x \in A, \\ 0 & x \in \mathbb{R}^1 \setminus A, \end{cases}$$

where $A = \cup_{n=1}^{\infty}[n, n+\frac{1}{n}]$, then for $p < +\infty$, we have

$$\|f\_L^p(\mathbb{R}^1) = \{\int_A dx\}^{\frac{1}{p}} = (\mu(A))^{\frac{1}{p}} = (\sum_{n=1}^{+\infty} \frac{1}{n})^{\frac{1}{p}} = +\infty.$$

But we know $\|f\|_\infty = 1$, so $\lim \|f\|_{L^p(\mathbb{R}^1)} \neq \|f\|_{L^\infty(\mathbb{R}^1)}$. Therefore the condition $\exists r, \|f\|_r < +\infty$ in (e) is necessary.

5, Assume in addition to the hypotheses of Exercise 4 that $\mu(Z) = 1$.

(a) Prove that $\|f\|_r \leq \|f\|_s$ if $0 < r < s \leq \infty$;

(b) Under what conditions does it happen that $0 < r < s \leq \infty$ and $\|f\|_r =\leq \|f\|_s < +\infty$;

(c) Prove that $L^s(\mu) \subset L^r(\mu)$ if $0 < r < s$, under what condition do these two spaces contain the same functions?

(d) Assume that $\|f\|_r < +\infty$ for some $r > 0$ and prove that

$$\lim_{\rho \to 0} \|f\|_\rho = exp(\int_Z log|f|\, d\mu)$$

if $exp(-\infty)$ is defined to be 0.

Proof: (a) Since $\mu(Z) = 1$, by Hölder's inequality

$$\int_Z |f|^r\, d\mu \leq (\int_Z |f|^{r\cdot\frac{s}{r}}\, d\mu)^{\frac{r}{s}} (\int_Z 1\, d\mu)^{1-\frac{r}{s}} = (\int_Z |f|^s\, d\mu)^{\frac{r}{s}}, \quad 0 < r < s < \infty.$$

So $\|f\|_r \leq \|f\|_s$ if $0 < r < s < +\infty$. If $s = \infty$,

$$\|f\|_r = (\int_Z |f|^r\, d\mu)^{\frac{1}{r}} \leq (\|f\|_\infty^r \int_Z 1\, d\mu)^{\frac{1}{r}} = \|f\|_\infty \mu(Z)^{\frac{1}{r}} = \|f\|_\infty.$$

(b) From (a), $\|f\|_r \leq \|f\|_s$ and the equality hold $\Leftrightarrow \int_Z |f|^r\, d\mu = (\int_Z |f|^{r\cdot\frac{s}{r}}\, d\mu)^{\frac{r}{s}} (\int_Z 1\, d\mu)^{1-\frac{r}{s}}$, hence $\frac{\|f\|^r}{|Z|^{\frac{r}{s}}} = const$, by the condition of the equality for Hölder's inequality$\Leftrightarrow$ $|f|$ must be a constant.

(c) By (a), if $f \in L^s(\mu)$, then $\|f\|_r \leq \|f\|_s < +\infty$ $(0 < r < s)$, hence $f \in L^r(\mu)$, then $L^s(\mu) \subset L^r(\mu)$.

Let (Z, m, μ) be a measure space and assume that, $\exists\ \alpha > 0$ s.t. $\forall A \in m$, $\mu A = 0$ or $\mu A \geq \alpha$ (e.s. contins measure), then $L^r(\mu) = L^s(\mu)$, $(0 < r < s)$.

In fact, we already have $L^r(\mu) \supset L^s(\mu)$ $(0 < r < s)$, we need only to show that $L^r(\mu) \subset L^s(\mu)$.

$\forall f \in L^r(\mu)$ and for any simple function $h = \sum\limits_{i=1}^{n} a_i \chi_{E_i}$ with $o \leq h \leq |f|^s$, which means $0 \leq h^{\frac{r}{s}} \leq |f|^r$, we have

$$E_i \cap E_j = \emptyset \text{ if } i \neq j, \quad \sum_{i=1}^{n} a_i^{\frac{r}{s}} \mu(E_i) \leq \int_Z |f|^r, d\mu < +\infty.$$

We may assume $\forall\ i,\ \mu(E_i) \neq 0$ have $\mu(E_i) \geq \alpha$, by assumption on μ, then

$$\alpha \sum_{i=1}^{n} a_i^{\frac{r}{s}} \leq \int_Z |f|^r, d\mu,$$

therefore $0 \le a_i^{\frac{r}{s}} \le \frac{1}{\alpha}\int_Z |f|^r\,d\mu$, that is to say

$$0 \le a_i \le (\frac{\int_Z |f|^r\,d\mu}{\alpha})^{\frac{s}{r}}.$$

Then

$$\begin{aligned}\int_Z h\,d\mu &= \sum_{i=1}^{n} a_i\mu(E_i) = \sum_{i=1}^{n} a_i^{(1-\frac{r}{s})} a_i^{\frac{r}{s}}\mu(E_i) \le (\frac{\int_Z |f|^r\,d\mu}{\alpha})^{\frac{s}{r}(1-\frac{s}{r})}\sum_{i=1}^{n} a_i^{\frac{r}{s}}\mu(E_i)\\ &\le (\frac{\int_Z |f|^r\,d\mu}{\alpha})^{\frac{s}{r}-1}\int_Z |f|^r\,d\mu = \frac{(\int_Z |f|^r\,d\mu)^{\frac{s}{r}}}{\alpha^{\frac{s}{r}-1}} < +\infty.\end{aligned}$$

So

$$\int_Z |f|^s\,d\mu = sup\int_Z h\,d\mu \le \frac{(\int_Z |f|^r\,d\mu)^{\frac{s}{r}}}{\alpha^{\frac{s}{r}-1}} < +\infty.$$

Then $f \in L^s(\mu)$ and $L^r(\mu) \subset L^s(\mu)$ for $0 < r < s$. Then $L^s(\mu) = L^r(\mu)$.

(d) Since $\|f\|_r < +\infty$ for some $r > 0$, by (a), $\forall 0 < p < r$, $f \in L^p(\mu)$. Let

$$g(x) = \frac{a^x - 1}{x}, \qquad \text{where } a > 0,\ r > 0.$$

Then $\lim_{x\to 0^+} g(x) = lna$ and $g(x)$ is increasing. To see this,

$$g'(x) = \frac{1}{x^2}(xa^x lna - (a^x - 1)) = \frac{1}{x^2}(a^x(xlna - 1) + 1) = \frac{a^x}{x^2}(xlna - 1 + a^{-x})$$

It is easy to see that $lnt + \frac{1}{t} - 1 \ge 0$ for $t > 0$. In fact, the function $lnt + \frac{1}{t} - 1$ takes the minimum in $(0, +\infty)$ at $t = 1$.

So $g''(x) \ge 0$ for $x > 0$ and $g(x)$ is increasing.

$\forall p_n \to 0^+$, $0 < p_n < r$, $p_{n+1} \le p_n$, then

$$\frac{|f|^{p_{n+1}} - 1}{p_{n+1}} \le \frac{|f|^{p_n} - 1}{p_n}, \quad \text{if } |f| \ne 0.$$

If $|f| = 0$, then

$$\frac{|f|^{p_{n+1}} - 1}{p_{n+1}} = -\frac{1}{p_{n+1}} \le -\frac{1}{p_n} = \frac{|f|^{p_n} - 1}{p_n}.$$

Let $g_n(t) = \frac{|f|^{p_n}-1}{p_n}$, then we have

$$g_{n+1}(t) \le g_n(t) \le \cdots \le g_1(t).$$

Using Monotone Convergence Theorem, by Exercise 7 in CH 1 (note $g_n \ge 0$ was not needed in that exercise), we have

$$\lim_{n\to\infty}\int_Z g_n(x)\,dx = \int_Z \lim_{n\to\infty} g_n(x)\,dx = \int_Z ln|f|\,d\mu.$$

(Indeed, if $|f| > 0$, then

$$\lim_{n\to\infty} g_n(t) = \lim_{n\to\infty}\frac{|f|^{p_n} - 1}{p_n} = ln|f|,$$

if $|f| = 0$, then $\lim_{n\to\infty} g_n(t) = -\infty = ln|f|$.)

From this, we conclude that

$$\lim_{p\to 0^+}\int_Z \frac{|f|^p-1}{p}\,d\mu = \int_Z ln|f|\,d\mu. \tag{4}$$

Otherwise, $\exists \epsilon > 0$, $\{p_n\}\to 0^+$, such that

$$\int_Z \frac{|f|^{p_n}-1}{p_n}\,d\mu - \int_Z ln|f|\,d\mu \geq \epsilon.$$

Take decreasing subsequence p_{n_k} from p_n, we must have

$$\int_Z \frac{|f|^{p_{n_k}}-1}{p_{n_k}}\,d\mu - \int_Z ln|f|\,d\mu \geq \epsilon.$$

Using the above argument, we have

$$\lim_{n\to\infty}\int_Z \frac{|f|^{p_{n_k}}-1}{p_{n_k}}\,d\mu = \int_Z ln|f|\,d\mu \geq \epsilon.$$

So we have a contradiction.

Since

$$x-1 \geq lnx \quad \text{for all } x>0, \tag{5}$$

we have

If $\|f\|_{p_0} = 0$ for some $p_0 = 0$, then $f = 0$ a.e. in Z, and $\lim\limits_{n\to\infty}\|f\|_p = 0 = exp(-\infty) = exp(\int_Z ln|f|\,d\mu)$. So we assume in the following that $\|f\|_p > 0, \forall p \in (0,\infty)$, and we have

$$\begin{aligned}\int_Z \frac{|f|^p-1}{p}\,d\mu &= \frac{1}{p}(\|f\|_p^p - \mu(Z)) = \frac{1}{p}(\|f\|_p^p - 1)\\ &\geq \frac{1}{p}ln\|f\|_p^p = \frac{1}{p}\int_Z ln|f|^p\,d\mu. \end{aligned}\tag{6}$$

If $0<p<r$, then $f\in L^p(\mu)$, $ln|f|^p \in L^1(\mu)$. Let $x = \frac{|f|^P}{\int_Z |f|^p\,d\mu}$ in (2), then integrate over Z, we have

$$\begin{aligned}0 &= \frac{1}{\int_Z |f|^p\,d\mu}\int_Z |f|^p\,d\mu - \mu(Z) \geq \int_Z ln\frac{|f|^P}{\int_Z |f|^p\,d\mu}\\ &= \int_Z ln|f|^p - ln\int_Z |f|^p\,d\mu,\end{aligned}$$

thus

$$\int_Z ln|f|^p\,d\mu \geq ln\int_Z |f|^p\,d\mu, \tag{7}$$

By $(1)-(4)$, we have

$$\begin{aligned}\int_Z ln|f|\,d\mu &= \lim_{p\to 0^+}\int_Z \frac{|f|^p-1}{p}\,d\mu \geq \overline{\lim_{p\to 0^+}}\frac{1}{p}ln\int_Z |f|^P\,d\mu\\ &\geq \liminf_{p\to 0^+}\frac{1}{p}ln\int_Z |f|^p\,d\mu \geq \liminf_{p\to 0^+}\frac{1}{p}\int_Z ln|f|^p\,d\mu\\ &= \liminf_{p\to 0^+}\int_Z ln|f|\,d\mu = \int_Z ln|f|\,d\mu.\end{aligned}$$

So

$$\lim_{p\to 0^+}\frac{1}{p}ln\int_Z |f|^p\,d\mu = \int_Z ln|f|\,d\mu.$$

Then

$$\begin{aligned}\lim_{p\to 0^+}\|f\|_p &= \lim_{p\to 0^+} exp(ln\|f\|_p) = exp(\lim_{p\to 0^+} ln\|f\|_p)\\ &= exp(\lim_{p\to 0^+}\frac{1}{p}ln\int_Z |f|^p\,d\mu) = exp(\int_Z ln|f|\,d\mu).\end{aligned}$$

□

11 Suppose $\mu(\Omega) = 1$ and suppose f and g are positive measurable functions on Ω such that $fg \geq 1$, prove that

$$\int_\Omega f\,d\mu, \quad \int_\Omega g\,d\mu \geq 1.$$

Proof: $fg \geq 1$ implies that $f^{\frac{1}{2}}g^{\frac{1}{2}} \geq 1$. Since

$$1 = \mu(\Omega) \leq \int_\Omega f^{\frac{1}{2}}g^{\frac{1}{2}}\,d\mu \leq (\int_\Omega f\,d\mu)^{\frac{1}{2}}\int_\Omega g\,d\mu)^{\frac{1}{2}},$$

then

$$1 \leq \int_\Omega f\,d\mu \int_\Omega g\,d\mu.$$

□

20 Suppose φ is a real function on $\mathbb{R}^1$ such that

$$\varphi(\int_0^1 f(x)\,dx) \leq \int_0^1 \varphi(f)\,dx$$

for every real bounded measurable f. Prove that ϕ is then convex.

Proof: $\forall x, y \in \mathbb{R}^1, \forall \alpha \in [0,1]$. Let $f(t) = x\chi_{[0,\alpha](t)} + y\chi_{[\alpha,1](t)}$, then $f(t)$ is bounded measurable in $\mathbb{R}^1$ and

$$\int_0^1 f(t)\,dt = x\alpha + (1-\alpha)y.$$

So

$$\begin{aligned}\varphi(\int_0^1 f(x)\,dx) &= \varphi(x\alpha + (1-\alpha)y) \leq \int_0^1 \varphi(f(t))\,dt = \int_{[0,\alpha]}\varphi(f(t))\,dt + \int_{[\alpha,1]}\varphi(f(t))\,dt\\ &= \int_{[0,\alpha]}\varphi(x)\,dt + \int_{[\alpha,1]}\varphi(y)\,dt = \varphi(x)m([0,\alpha]) + \varphi(y)m([\alpha,1])\\ &= \alpha\varphi(x) + (1-\alpha)\varphi(y).\end{aligned}$$

So φ is convex in $\mathbb{R}^1$. □

7 For some measures, the relation $r < s$ implies $L^r(\mu) \subset L^s(\mu)$, for others, the inclusion is resolved and there are some for which $L^r(\mu)$ does not contain $L^s(\mu)$ if $r \neq s$. Give examples of these situations and find condition on μ under which these situation will occur.

Proof: Example 1: Let X be any set, $m = 2^X$, μ be the counting measure, then if $p, q \in (0, \infty]$, $p < \varepsilon$, we have $L^p(\mu) \subset L^q(\mu)$. In fact, if $f \in L^p(\mu)$, let $F_n = \{x \in X : \ |f| > n\}$, then $E_1 \supset E_2 \supset \cdots E_n \supset E_{n+1} \cdots$. Since

$$n^p \mu(E_n) \leq \int |f|^p < +\infty,$$

which implies that $\mu(E_n) \to 0$. So

$$\mu(\cap_{n=1}^{+\infty} E_n) = \lim_{n\to\infty} \mu(E_n) = 0,$$

as μ is the counting measure, $\cap_{n=1}^{+\infty} E_n = \emptyset$.

Note that if $E \subset X$ with $\mu(E) > 0$, then $\mu(E) \geq 1$. So if $\forall k$, $\exists n_n > k$, then

$$\mu(E_{n_n}) = \lim_{n\to\infty} \mu(E_n) = \inf \mu(E_n) \geq 1,$$

a contradiction as $\mu(E_n) \to 0$. $\exists n_0$, if $n \geq n_0$, $\mu(E_n) < 1$.

$\mu(E_n) = 0$, $n \geq n_0$, $\Rightarrow \mu(E_n) = \emptyset$, $n \geq n_0$.

$f \leq n_0$, $f \in L^\infty(\mu)$, $\forall q > p$,

$$\int_Z |f|^q \, d\mu = \int_Z |f|^p |f|^{q-p} \, d\mu \leq \|f\|_\infty \int_Z |f|^p \, d\mu \leq n_0 \int_Z |f|^p \, d\mu < +\infty.$$

Example 2, Let $Z = [0, 1]$, m be the collection of Lebesgue measurable set on $[0, 1]$, μ be the Lebesgue measure restricted on $[0, 1]$, then by Hölder′s inequality, $L^p(\mu) \subset L^q(\mu)$ if $p > q$.

In fact, (Z, m, μ) be any measure space, $\mu(Z) < +\infty$, $p > q$ implies $L^p(\mu) \subset L^q(\mu)$.

Example 3, Let $Z = \mathbb{R}^1$, m be the collection of Lebesgue measurable sets on $\mathbb{R}^1$, μ be the Lebesgue measure on $Z = \mathbb{R}^1$, then if $r \neq s$, $L^r(\mu)$ does not contain $L^s(\mu)$.

$\forall\ 0 < r < s$, let

$$f(x) = \begin{cases} \frac{1}{x^{\frac{1}{s}}} & x \in (0, 1); \\ 0 & otherwise, \end{cases}$$

Then

$$\int_{(-\infty,+\infty)} |f|^r \, dx = \int_{[0,1]} \frac{1}{x^{\frac{r}{s}}} \, dx < +\infty.$$

So $f \in L^r(\mu)$. But

$$\int_{(-\infty,+\infty)} |f|^s \, dx = \int_{(0,1)} \frac{1}{x} \, dx = +\infty.$$

So $f \notin L^s(\mu)$.

Let

$$g(x) = \begin{cases} \frac{1}{x^{\frac{1}{r}}} & x \geq 1; \\ 0 & otherwise, \end{cases}$$

then if $s > r$,

$$\int_{(-\infty,+\infty)} |g|^s \, dx = \int_1^{+\infty} \frac{1}{x^{\frac{s}{r}}} \, dx < +\infty.$$

So $f \in L^s(\mu)$. But

$$\int_{(-\infty,+\infty)} |g|^r \, dx = \int_{(1,+\infty)} \frac{1}{x} \, dx = +\infty.$$

So $g \notin L^r(\mu)$. Thus $L^s(\mu) \neq L^r(\mu)$, $\forall r, s > 0$, $r \neq s$. □

Lemma 1, Let $p, q \in (1, \infty)$, the inclusion $L^p(\mu) \subset L^q(\mu)$ implies that the inclusion map i : $L^p(\mu) \to L^q(\mu)$ is continuous.

Proof: We will use Closed Graph Theorem.

Let X, Y be two Banach spaces, $T : D(T) \subset X \to Y$ is a linear operator, if T is closed, i.e. $x_n \in P(T)$, $x_n \to x$ in X, $Tx_n \to y$ in Y implies that $x \in D(T)$ and $Tx = y$.

If $L^p(\mu) \subset L^q(\mu)$, then the inclusion $i :\ L^p(\mu) \to L^q(\mu)$ is linear, i is also closed, $D(i) = L^p(\mu)$, if $f_n \to f$ in $L^p(\mu)$, $i(f_n) = f_n \to g$ in $L^q(\mu)$, we must have $f \in D(i) = L^p(\mu)$, $g = f$.

In fact, $f_n \to f$ in $L^p(\mu)$ implies a subsequence of f_n, f_{n_k} satisfies

$$f_{n_k} \to f \text{ a.e. in } X, \quad f_{n_k} \to g \text{ a.e. in } X.$$

By Riesz's theorem, $f = g$a.e., thus $f = g$ in $L^q(\mu)$.

By the Closed Graph Theorem, i is continuous, then $\exists\ k \geq 0$, such that $\|f\|_q \leq k\|f\|_p$, $\forall f \in L^p(\mu)$. □

Remark Lemma 1 also holds for p, $s \in (0, \infty]$ with the same proof, even though $L^p(\mu)$ is not a normal space for $0 < p < 1$.

$$d_i(f, g) = \|f - g\|_p^p \qquad \text{for } p \in (0, 1).$$

L^p is a metric space.

Assume $(\Omega, \tilde{A}, \mu)$ is a measure space. Let $\tilde{A}_0$ be two collection of all sets $A \in \tilde{A}$ with positive measure, then we have

Theorem 1. The following conditions on the measure space $(\Omega, \tilde{A}, \mu)$ are equivalent

(1) $L^p(\mu) \subset L^q(\mu)$ for some $p, q \in (0, \infty]$ with $p < q$;

(2) $\inf\limits_{E \in \tilde{A}_0} \mu(E) > 0$;

3 $L^p(\mu) \subset L^q(\mu)$ for all $p, q \in (0, \infty]$ with $p < q$;

Proof: (1) $\Rightarrow$ (2). Since $L^p(\mu) \subset L^q(\mu)$ implies $L^{pt}(\mu) \subset L^{qt}(\mu)$ for every $t \in (0, \infty)$, (since $\forall f \in L^{pt}(\mu)$, $|f|^t \in L^p(\Omega) \subset L^q(\Omega)$, then $\int_\Omega |f|^{qt} \, d\mu < +\infty$).

We can assume $p \geq 1$. Then $L^p(\mu)$ and $L^q(\mu)$ are normed spaces (in fact, Banach space) and by Lemma 1, there is a positive constant k such that $\|f\|_q \leq k\|f\|_p$ for every $f \in L^p(\mu)$. In particular, we have, for any $E \subset A$ with $0 < \mu(E) < +\infty$,

$$(\mu(E))^{\frac{1}{q}} \leq k(\mu(E))^{\frac{1}{p}}.$$

As $\chi_E \in L^p(\mu) \cap L^q(\mu)$ and hence $\mu(E) \geq k^{\frac{pq}{q-p}} > 0$, (2) is true.

(2) $\Rightarrow$ (3) Let $f \in L^p(\mu)$ and let $E(\mu) = \{|f| > n\}, n = 1, 2, \cdots$. Then

$$n^p \mu(E_n) \le \int_\Omega |f|^p \, d\mu < +\infty,$$

then $\mu(E_n) \to 0$ as $n \to \infty$. If $\forall\ k$, $\exists n_k \ge k$, s.t. $\mu(E_{n_k}) > 0$, then

$$0 = \lim_{k\to\infty} \mu(E_{n_k}) = \inf_k \mu(E_{n_k}) \ge \inf_{E\in A_0} \mu(E) > 0,$$

a contradiction. So $\exists n_0$, s.t. $n \ge n_0$, $\mu(E_n) = 0$. Thus $|f| \le n_0,\ \mu\ a.e.\ on\ \Omega$.

If $f \in L^p(\mu)$, then if $q > p$, we must have

$$\int_\Omega |f|^q \, d\mu = \int_\Omega |f|^q |f|^{q-p} \, d\mu \le n_0 \int_\Omega |f|^p \, d\mu < +\infty.$$

Thus $f \in L^q(\mu)$, so $f \in L^p(\mu) \subset L^q(\mu)$, $\forall 0 < p < q$.

(3) $\Rightarrow$ (2) is trivial. $\square$

Let $\tilde{A}_\infty$ be the collection of all sets $A \in \tilde{A}$ with finite measure. Then we have

Theorem 2. The following conditions on the measure space $(\Omega, \tilde{A}, \mu)$ are equivalent

(1) $L^p(\mu) \supset L^q(\mu)$ for some $p, q \in (0, \infty]$ with $p < q$;

(2) $\sup\limits_{E\in\tilde{A}_\infty} \mu(E) < \infty$;

3 $L^p(\mu) \supset L^q(\mu)$ for all $p, q \in (0, \infty]$ with $p < q$;

Proof: (1) $\Rightarrow$ (2). As in Theorem 1, we can assume $p \ge 1$. So by Lemma 2, there exists $k > 0$ such that $\|f\|_p \le k\|f\|_q$ for every $f \in L^q(\mu)$. It follows that for any $E \in \tilde{A}_\infty$, $\chi_E \in L^q(\mu) \subset L^p(\mu)$, $\mu(E) \le k^{\frac{pq}{q-p}}$, then

$$\sup_{E\in\tilde{A}_\infty} \mu(E) \le k^{\frac{pq}{q-p}} < +\infty.$$

(2) $\Rightarrow$ (3), Let $f \in L^q(\mu)$ and let

$$E_n = \{\frac{1}{n+1} \le |f| < \frac{1}{n},\ n = 1, 2, \cdots\}.$$

Then

$$\mu(E_n) \le (n+1)^q \int_\Omega |f|^q \, d\mu < \infty,\ n = 1, 2, \cdots\}.$$

As E_n are pairwise disjoint and $F_n = \bigcup\limits_{i=1}^{+\infty} E_i$ satisfies $\mu(F_n) = \sum_{i=1}^n \mu(E_i) < +\infty$. Then $\mu(F_n) \to \sum_{i=1}^{+\infty} \mu(E_i)$,

$$\sum_{i=1}^{+\infty} \mu(E_i) = \lim_{n\to\infty} \mu(F_n) = \sup_n \mu(F_n) \le \sup_{E\in\tilde{A}_\infty} \mu(E) < +\infty.$$

If $p < q$, then

$$\begin{aligned}
\int_\Omega |f|^p \, d\mu &= \int_{\{|f|\ge 1\}} |f|^p \, d\mu + \sum_{n=1}^{+\infty} \int_{E_n} |f|^p \, d\mu \\
&\le \int_\Omega |f|^q \, d\mu + \sum_{n=1}^{+\infty} \frac{1}{n^p} \mu(E_n) \le \int_\Omega |f|^q \, d\mu + \sum_{n=1}^{+\infty} \mu(E_n) < +\infty.
\end{aligned}$$

Since $f \in L^p$, then $L^q(\mu) \subset L^p(\mu)$. $\square$

Remark If $f \in L^\infty(\mu)$ in (2), then we can not get $\mu(E_n) \leq (n+1)^q \int_\Omega |f|^q \, d\mu < +\infty$ except that $\mu(\Omega) < +\infty$. If $\mu(\Omega) < +\infty$, then $L^q(\mu) \subset L^p(\mu)$ is obvious, then $(\Omega, \tilde{A}, \mu)$ satisfies $L^s(\mu)$ not include $L^r(\mu)$ when $r \neq s$, which is equivalent to

$$\sup_{E \in \tilde{A}_\infty} \mu(E) = +\infty \quad \text{and} \quad \sup_{E \in \tilde{A}_0} \mu(E) = 0.$$

For example: the Lebesgue measure in $\mathbb{R}^N$. $\square$

10. Suppose $f_n \in L^p(\mu)$ for $n = 1, 2, 3, \cdots$ and $\|f_n - f\|_p \to 0$ and $f_n \to g$ a.e. as $n \to +\infty$. What relation exists between f and g.

Solution: By page 68 3.12 Theorem(Riesz), $f_n \to f$ in $L^p(\mu)$, which implies that there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \to f$ a.e., so $f = g$ a.e.. $\square$

12 Suppose $\mu(\Omega) = 1$ and $h : \Omega \to [0, +\infty]$ is measurable. If $A = \int_\Omega h \, d\mu$, prove that

$$\sqrt{1 + A^2} \leq \int_\Omega \sqrt{1 + h^2} \, d\mu \leq 1 + A.$$

If μ is Lebesgue measure on $[0, 1]$, and if h is continuous $h = f'$, the above inequalities have a simple geometric interpretation. From this, conjecture (for general Ω), under what conditions on h, equality can hold in either of the above inequalities and prove your conjecture.

Proof: $f(x) = \sqrt{1 + x^2}$ satisfies

$$f'(x) = \frac{x}{\sqrt{1 + x^2}}, \quad f''(x) = \frac{1}{(1 + x^2)^{\frac{3}{2}}} > 0.$$

then f is convex on $(-\infty, +\infty)$.

$$\sqrt{1 + A^2} = f(\int_0^1 h \, d\mu) \leq \int_0^1 f \circ h \, d\mu = \int_0^1 \sqrt{1 + h^2} \, d\mu.$$

On the other hand,

$$\int_0^1 \sqrt{1 + h^2} \, d\mu \leq 1 + h, \qquad (= \mathit{hold} \Rightarrow h(x) = 0).$$

Then

$$\int_0^1 \sqrt{1 + h^2} \, d\mu \leq \int_0^1 (1 + h) \, d\mu = 1 + A.$$

So we have

$$\sqrt{1 + A^2} \leq \int_0^1 \sqrt{1 + h^2} \, d\mu \leq 1 + A.$$

If $h = f' \geq 0$, $f' \in C^0[0, 1]$, then

$$\int_0^1 \sqrt{1 + h^2} \, dt = \int_0^1 \sqrt{1 + (f')^2} \, dt = A, \quad t \in [0, 1].$$

Claim: $\int_0^1 \sqrt{1 + h^2} \, d\mu = 1 + A \Leftrightarrow h = 0 \; a.e. \; [\mu] \; on \; [0, 1]$.

since $1 + h - \sqrt{1 + h^2} \geq 0$, then $\int_0^1 \sqrt{1 + h^2} \, d\mu = 1 + A \Leftrightarrow \int_0^1 (1 + h - \sqrt{1 + h^2}) \, d\mu = 0$, then $1 + h - \sqrt{1 + h^2} = 0$ a.e. $[\mu]$ on $[0, 1]$. $\square$

Chapter 3 L^p SPACES

13, Under what conditions on f and g does equality hold in the conclusions of Theorems 3.8 and 3.9 ?

Proof:

for theorem 3.8

if $1 < p < \infty$,then $1 < q < \infty$. the equality holds in (1) of Theorem 3.8 if and only if that there are constants α ,β and $\alpha \times \beta \neq 0$ such that $\alpha|f|^p = \beta|g|^q$.

if $p = 1$,$q = \infty$ or $p = \infty$,$q = 1$ the equality holds in (1) of Theorem 3.8 if and only if that $|f(x)g(x)| = \|f\|_\infty |g(x)|$ a.e on X $\Longleftrightarrow |g(x)| = \|g\|_\infty$ for a.e $x \in E^c$ where $E = \{x, g(x) = 0\}$

for theorem 3.9

$g = \lambda f \ a.e \ for \ some \ \lambda \ when \ f \neq 0$ □

14, Suppose $1 < p < \infty$, $f \in L^p = L^p((0, \infty))$,relative to Lebesgue measure,and

$$F(x) = \frac{1}{x} \int_0^x f(t)dt \qquad (0 < x < \infty)$$

(a) Prove Hardy's inequality

$$\|F\|_p \leq \frac{p}{p-1} \|f\|_p$$

which shows that the mapping $f \to F$ carries L^pinto L^P

(b) Prove that equality holds only of $f = 0$ a.e.

(c) Prove that the constant $p/(p-1)$ cannot be replaced by a smaller one.

(d)If $f > 0 \ and \ f \in L^1$,prove that $F \overline{\in} L^1$.

suggestions:(a) We assume firstly that $f > 0 \ and \ f \in \mathcal{C}_c((0, \infty))$.Integration by parts gives

$$\int_0^\infty F^p(x)dx = -p \int_0^\infty F^{p-1}(x)xF^{'}(x)dx$$

Note that $xF^{'} = f - F$,and apply hölder's inequality to $\int F^{p-1}f$.Then derive the general case.(c) Take $f(x) = x^{\frac{-1}{p}}$ on$[1, A]$, $f(x) = 0$ elsewhere, for large a. See also Exercise 14,Chap.8.

Proof: for a:

Wa assume firstly that $f \geqslant 0, andf \in \mathcal{C}_c((0, \infty))$ Then:

$$F(x) = \frac{1}{x} \int_0^x f(t)dt \qquad (0 < x < \infty)$$

is differentiable on $(0, \infty), and \ from, xF(x) = \int_(0)^(x) f(t)dt$.We have $F(\infty) = \lim_{x \to \infty} = 0, F(0) = \lim_{x \to 0} = 0 \ xF^{'}(x) = f(x) - F^x, foranyx \in (0, \infty)$ Therefore:

$$\int_0^\infty F^p(x)dx = F^p(x)x\mid_0^\infty - p\int_0^\infty F^{p-1}(x)xF^{'}(x)dx$$
$$= -p\int_0^\infty F^{p-1}(x)xF^{'}(x)dx$$
$$= -p\int_0^\infty F^{p-1}(x)(f(x)-F(x))dx$$
$$= -p\int_0^\infty F^{p-1}(x)f(x)dx + p\int_0^\infty F^p(x)$$

$$\Longrightarrow (p-1)\int_0^\infty F^p(x)dx = p\int_0^\infty F^{p-1}(x)f(x)dx$$
$$\leqslant p(\int_0^\infty F^p(x)dx)^{\frac{p-1}{p}}(\int_0^\infty f^p(x)dx)^{\frac{1}{p}}$$

$$\Longrightarrow \|F\|_p \leqslant \frac{p}{p-1}\|f\|_p, \ \ for \ \ any \ \ f\in \mathcal{C}_c((0,\infty)), \ \ f\geqslant 0$$

Now if $f\in\mathcal{C}_c((0,\infty))$,we write $f=f^+-f^-$ then $f^\pm\geqslant 0,\ \ f^\pm\in\mathcal{C}_c((0,\infty))$,from the above result we have:

$$\|F^\pm\|_p \leqslant \frac{p}{p-1}\|f^\pm\|_p, \ \ where \ \ F^\pm = \frac{1}{x}\int_0^x f^\pm dt$$

Since:

$$F(x) = \frac{1}{x}\int_0^x f(t)dt = \frac{1}{x}\int_0^x (f^+-f^-)dt = F^+ - F^-$$

$$\Longrightarrow \|F\|_p \leqslant \|F^+\|_p + \|F^-\|_p \leqslant \frac{p}{p-1}(\|f^+\|_p + \|f^-\|_p) = \frac{p}{p-1}\|f\|_p$$

We notice that:$\|f\|_p = (\int(|f^+|+|f^-|)^p d\mu)^{\frac{1}{p}} = (\int|f^+|^p d\mu + \int|f^-|^p d\mu)^{\frac{1}{p}} = (\int|f|^p d\mu)^{\frac{1}{p}}$
$|f(x)|\in\mathcal{C}_c((0,\infty))\ \ |f(x)|\geqslant 0\ \ so:\ \ |F(x)|\leqslant\frac{1}{x}\int_0^x|f(t)|dt\ x\in(0,\infty)\ \|F\|_p\leqslant\|\frac{1}{x}\int_0^x|f(t)|dt\|_p\leqslant\frac{p}{p-1}\|f\|_p$
Given $f\in L^p((0,\infty))$ Since $\mathcal{C}_c((0,\infty))$ is dense in$L^p((0,\infty))(Th3.14\ P21)$,there is a sequence $\{f_n\}\subset\mathcal{C}_c((0,\infty))$ such that

$$\|f_n - f\|_p \longrightarrow 0 \ \ as \ n\longrightarrow\infty$$

Hence:$\|f_n\|_p\longrightarrow\|f\|_p$ by Minkousr inequality
If$f\in L^p((0,\infty))$,then $F(x)=\frac{1}{x}\int_0^x|f(x)|dt$ is well defined by $H\ddot{o}lder's\ Inequality$,so

$$F(x)\leqslant\frac{1}{x}\int_0^x|f(x)|dt\leqslant\frac{1}{x}(\int_0^x|f(x)|^p dt)^{\frac{1}{p}}(\int_0^x dt)^{\frac{1}{q}}\leqslant\frac{1}{x}(\int_0^\infty|f(x)|^p dt)^{\frac{1}{p}}x^{\frac{1}{q}} = x^{-\frac{1}{p}}\|f\|^{\frac{1}{p}}$$

Where $\frac{1}{p}+\frac{1}{q}=1$ also:

$$|F_n(x)-F(x)|\leqslant\frac{1}{x}\int_0^x|f_n(x)-f(x)|dt\leqslant\frac{1}{x}(\int_0^x|f_n(x)-f(x)|^p dt)^{\frac{1}{p}}x^{\frac{1}{q}} = x^{-\frac{1}{p}}\|F_n(x)-F(x)\|^p$$

Thus: $F_n(x) \longrightarrow F(x) \quad n \longrightarrow \infty \quad for\ any\ x \in (0,\infty)$
As $f_n \in \mathcal{C}_0((0,\infty))$ We have for n large enough:$\|F_n\|_p \leqslant \frac{p}{p-1}\|f_n\|_p \leqslant \frac{p}{p-1}(\|f\|_p + 1)$
Fatous lemma shows that:

$$\int_X |F|^p \leqslant \lim_{n\to\infty}\int_X |F_n|^p \leqslant \lim_{n\to\infty}(\frac{p}{p-1})^p\|f_n\|_p^p = (\frac{p}{p-1})^p\|f\|_p^p$$

Hence:$F \in L^p(\mu)$ and $\|F\|_p \leqslant \frac{p}{p-1}\|f\|_p \quad for\ any\ f \in L^p((0,\infty))$
For (b)
In (a),we actually proved that if $f \in \mathcal{C}_c((0,\infty)),\ f \geqslant 0$ then $\int_0^\infty |F|^p dx = p\int_0^\infty F^{p-1}f(x)dx$
If$f \in L^p((0,\infty))\ then\ |f| \in L^p((0,\infty))\ \exists g_n \in \mathcal{C}_c((0,\infty)),\ g_n \longrightarrow |f|\ in\ L^p((0,\infty))$ and from the process of the proof of $Lusinthorem$ and $Th3.13, Th3.14$,we have also arrange that $g_n \geqslant 0$ let$G_n(x) = \frac{1}{x}\int_0^x g_n(t)dt$ then $\|G_n\|_p \leqslant \frac{p}{p-1}\|g_n\|_p$

$$\|G_n - G_m\|_p \leqslant \frac{p}{p-1}\|g_n - g_m\|_p \longrightarrow 0 \quad as \quad n,m \longrightarrow \infty$$

We have :$G_n \longrightarrow G\ in\ L^p((0,\infty))$ and as $g_{n_k} \longrightarrow a.e$ we see that:$\|G\|_p \leqslant \frac{p}{p-1}\|f\|_p,\ G \in L^p$ and$\|G_n - G\|_p \leqslant \frac{p}{p-1}\|g_n - |f|\|_p \longrightarrow 0 \quad as \quad n \longrightarrow \infty$
As: $\int_0^\infty {G_n}^p dx = \frac{p}{p-1}\int_0^\infty G_n^{p-1}g_n(x)dx \quad (\because g_n \in \mathcal{C}_c((0,\infty)),\ g_n \geqslant 0)$Let $n \longrightarrow \infty$ we have:

$$\int_0^\infty G^p dx = \frac{p}{p-1}\int_0^\infty G^{p-1}|f|dx$$
$$\int_0^\infty \frac{1}{x}\int_0^x |f|dt^{\ p} dx = \frac{p}{p-1}\int_0^\infty \frac{1}{x}\int_0^x |f|dt^{\ p-1}|f|dx$$

If $f \in L^p$ we show that $\|F\|_p \leqslant \frac{p}{p-1}\|f\|_p$ Let $\tilde{F} = \frac{1}{x}\int_0 x|f(x)|dt$ then $\|F\|_p \leqslant \|\tilde{F}\|_p \leqslant \frac{p}{p-1}\||f|\|_p = \frac{p}{p-1}\|f\|_p = \|F\|_p \therefore \|\tilde{F}\|_p = \|F\|_p,\ \|\tilde{F}\|_p = \frac{p}{p-1}\|f\|_p$ So we may assume that $f \geqslant 0\ \ on\ (0,\infty)$.If we prove:$\|F\|_p = \frac{p}{p-1}\|f\|_p \Leftrightarrow f = 0\ a.e$ then the conclusion (b) is proved
Now we have :

$$\|F\|_p^p = \int_0^\infty F(x)^p dx = \frac{p}{p-1}\int_0^\infty F^{p-1}f(x)dx \leqslant \frac{p}{p-1}\|F\|_p^{p-1}\|f\|_p = \|F\|_p^p$$

So actually we have :

$$\int_0^\infty F^{p-1}f(x)dx = (\int_0^\infty F^p dx)^{\frac{p-1}{p}}(\int_0^\infty |f|^p dx)^{\frac{1}{p}}$$

Thus by the condition for the equality in $h\ddot{o}lder's\ inequality$ holds:
$(\int fg \leqslant \|f\|_p\|g\|_p \Longleftrightarrow \exists\alpha,\beta, not\ all\ zero\ such\ that\ \alpha(F^{p-1})^{\frac{p}{p-1}} = \beta f^p)$
$WLOG\ \exists\alpha,\beta,\alpha \neq 0$ then $F(x) = \frac{\beta}{\alpha}f(x) \doteq Cf(c), C \geqslant 0$ If$m\{x : f(x) \neq 0\} > 0$,Let $g(x) = \int_a^b f(t)dt$then as $f \in L^p(a,b),\ 0 < a < b < \infty,\ p \geqslant 1$ then $f \in L^1(a,b)$,g(x) is absolutely continuous on $[a,b]$, $g'(x) = f(x)\ a.e\ on\ (0,\infty)$
As $h(x) = \frac{1}{x}$ is differentiable for $x > 0$ and $h'(x) = -\frac{1}{x^2}$ is locally bounded $h(t)$ is absolutely continuous on any bounded interval $[a,b],\ (0 < a < b < \infty)$
$\therefore F(x)$ is absolutely continuous on $[a,b]$
Also,$J(x) = \ln x$ is differentiable for $x > 0$ and the $J'(x) = \frac{1}{x}$ is locally bounded on $[a,b],\ (a > 0, b > 0)$, so $J(x)$ is absolutely continuous on $[a,b]$. Hence it is differentiable on $[a,b],\ a.e$ and

$$\frac{d}{dx}\ln g(x) = \frac{f(x)}{\int_0^x f(t)dt} = \frac{C}{x},\ forx > a_0 \geqslant 0$$

and $g(x) > 0 \ if \ x > a_0$ $a_0 = inf\{x : g(t) > 0, if \ x > t\}$
$\therefore \frac{d}{dx}\ln g(x) = \frac{d}{dx}(C\ln x)) \ for \ a.e \ x > a_0$ for $b > a > a_0$ $\frac{d}{dx}(\ln g(x) - C\ln x) = 0, \ a.e \ on \ [a,b]$,$\ln g(x) - C\ln x$ is absolutely continuous on $[a,b]$, $(\infty > b > a > a_0)$ (see "*Real functions*" Theorem 3.8.6, Senlin Xu)Assume f absolutely continuous on$[a,b]$,then (1) $f^{'} exists \ a.e, \ and \ f^{'} \in L^1[a,b]$. (2)$f^{'} = 0 \ a.e \ on \ [a,b] \Rightarrow f(x) = constant \ for \ any \ x \in [a,b]$.
$\therefore \exists \ constant \ \tilde{C}_{a,b} \ s.t$

$$\ln g(x) - C\ln x = \tilde{C}_{a,b}$$

$\therefore \exists \ constant C_{a,b} > 0 \ s,t \ \ln g(x) = \ln C_{a,b}x^C \ x \in [a,b] \ \therefore g(x) = C_{a,b}x^C \ for \ any \ x \in [a,b]$
for any $a_n, b_n \ s.t \ a_n \to a_0 \ \ b_n \to \infty, \ mono \ as \ n \to \infty$ We have $(a_n b_n) \subset (a_{n+1}b_{n+1}) \ for \ any \ n \geqslant 1$ $C_{a_n,b_n} = C_{a_1,b_1} = constant = A$ which independent of n .So we have $g(x) = Ax^C \ for \ any \ x > a_0$.As g is absolutely continuous on $(0,\infty) g^{'} \ exists \ a.e \ on \ (0,\infty)$ and $g^{'}(x) = f(x) \ \therefore \ f(x) = ACx^{C-1} \ for \ any \ x > a_0$ But $f(x \in L^P(0,\infty) \subset L^P(a_0,\infty)$ if $C > 0$ then:

$$+\infty > \int_{a_0}^{\infty} |f|^p dx = \int_{a_0}^{\infty} (ACx^{C-1})^p dx = +\infty$$

So we must have $C = 0 \ ie \ f(x) = 0 \ for \ any \ (a_0, +\infty)$ also $g(a_0) = 0, g(x) = 0 \ for \ x < a_0 \ \therefore$ $f(x) = 0 \ a.e \ on(0,a_0) \ \therefore f(x) = 0 \ \ a.e \ on(0,\infty)$
This is contraster to $mx : f(x) > 0 > 0, \ \therefore \ m\{x : f(x) > 0\} = 0 \therefore f(x) = 0 \ a.e \ on \ (0,+\infty)$.
For (c)
Take

$$f(x) = \begin{cases} x^{-\frac{1}{x}} \ \ if \ \ x \in [1,A) \\ 0 \ \ oterwise, \end{cases}$$

then:

$$\int_0^{+\infty} \|f\|^p dx = \int_0^A \frac{1}{x} dx = \ln A$$

$$\frac{1}{x}\int_1^A f(t)dt = \frac{1}{x}\int_1^A x^{-\frac{1}{x}} dt = \frac{1}{x}\frac{p}{p-1}(A^{1-\frac{1}{p}} - 1)$$

If $1 \leqslant x < A$,then

$$\frac{1}{x}\int_1^x f(t)dt = \frac{1}{x}\frac{p}{p-1}(x^{1-\frac{1}{p}} - 1)$$

Then

$$F(x) = \frac{1}{x}\int_1^x f(t)dt = \begin{cases} 0 & if \ \ x \in (0,1) \\ \frac{1}{x}\frac{p}{p-1}(x^{1-\frac{1}{p}} - 1) & if \ \ x \in [1,A] \\ \frac{1}{x}\frac{p}{p-1}(A^{1-\frac{1}{p}} - 1) & if \ \ x \in (A,+\infty) \end{cases}$$

If there is a constant $C < \frac{p}{p-1}$ such that $\|F\|_p \leqslant \|f\|_p \quad for \ any \ \ f \in L^p$
then

$$\int_0^{+\infty} \|F(x)\|^p dx \leqslant C^p \int_0^{+\infty} \|f\|^p dx = C^p \ln A$$

for f given above

$$\int_0^{+\infty} \|F(x)\|^p dx = (\frac{p}{p-1})^p \int_0^A \frac{1}{x^p}(x^{1-\frac{1}{p}} - 1)^p dx + (\frac{p}{p-1})^p \int_A^{+\infty} \frac{1}{x^p} dx (A^{1-\frac{1}{p}} - 1)^p dx$$
$$= (\frac{p}{p-1})^p [\int_0^A \frac{1}{x}(1 - x^{\frac{1}{p}-1})^p dx + \frac{1}{p-1}(1 - A^{1-\frac{1-p}{p}})^p]$$

$$\therefore\ (\frac{p}{p-1})^p\int_0^A \frac{1}{x^p}(x^{1-\frac{1}{p}}-1)^p dx \leqslant \int_0^{+\infty}\|F(x)\|^p dx \leqslant C^p \ln A$$

$$\therefore\ (\frac{p}{p-1})^p \limsup_{A\to+\infty}\frac{\int_0^A \frac{1}{x^p}(x^{1-\frac{1}{p}}-1)^p dx}{lnA} \leqslant C^p$$

Hospital rule implies

$$(\frac{p}{p-1})^p \lim_{A\to+\infty}\frac{\frac{1}{A}(1-A^{\frac{1}{p}-1})^p}{\frac{1}{x}} \leqslant C^p$$

$$C < \frac{p}{p-1} \leqslant C$$

a contradiction so $\frac{p}{p-1}$ is the optimal constant

For (d)

Suppose $f > 0$ and $f \in L^1$, then $G = \int_0^{+\infty} f(t)dt < +\infty$ and there is an $N > 0$ such that if $x > N$,we have $\int_0^x f(t)dt > \frac{G}{2}$ thus

$$\int_0^{+\infty} F(t)dt \geqslant \int_N^{+\infty}\frac{1}{x}\int_0^x f(t)dt \geqslant \frac{G}{2}\int_N^{+\infty}\frac{1}{x}dx = +\infty$$

$$\therefore F \ \bar{\in}\ \ L^1.$$

□

15, Suppose $\{a_n\}$ is a sequence of positive numbers prove that

$$\sum_{N=1}^{\infty}(\frac{1}{N}\sum_{n=1}^{N}a_n)^p \leqslant (\frac{p}{p-1})^p\sum_{n=1}^{\infty}{a_n}^p \qquad (\circledast)$$

If1 $< p < \infty$ Hint: If $a_n \geqslant a_{n+1}$,the result can be made to follow Exercise 14 This special case implies the general one.

Proof: Let $f : (0,\infty) \longrightarrow (0,\infty)$ given as $f(x) = a_i$ of $i-1 < x \leqslant i$ then $f(x)$ is lebesgue measurable on $(0,\infty)$, $f \geqslant 0$

$$\int_0^{+\infty} f(x)^p dx = \sum_{i=1}^{+\infty}\int_{i-1}^{i} f(x)^p dx = \sum_{i=1}^{+\infty}\int_{i-1}^{i} {a_i}^p dx = \sum_{i=1}^{+\infty}{a_i}^p$$

If $\sum_0^\infty {a_i}^p = +\infty$ then $(\circledast)$ is clearly true.Suppose $\sum_{i=1}^\infty {a_i}^p \leqslant +\infty$ and $a_n \geqslant a_{n-1}$ $\quad (n = 1, 2, \cdots)$ then if $k < x \leqslant k+1$

$$\begin{aligned}
F(x) &= \frac{1}{x}\int_0^x f(t)dt = \frac{1}{x}[\int_0^k f(t)dt + \int_k^x f(t)dt] \\
&= \frac{1}{x}[\sum_0^k a_i + a_{k+1}(x-k)] \\
&= \frac{1}{x}[\sum_0^k a_i + a_{k+1} - a_{k+1}(k+1)] + a_{k+1} \\
&= \frac{1}{x}[\sum_0^{k+1} a_i - a_{k+1}(k+1)] + a_{k+1} \\
&\geqslant \frac{1}{k+1}[\sum_0^{k+1} a_i - a_{k+1}(k+1)] + a_{k+1} \\
&= \frac{1}{k+1}\sum_{i=1}^{k+1} a_i
\end{aligned}$$

By Hardy's inequality (see Exercise 14 above)

$$F \in L^p(0,+\infty) \ \ and \ \ \int_0^\infty F(x)^p dx \leqslant (\frac{p}{p-1})^p \int_0^\infty f(x)^p dx = (\frac{p}{p-1})^p \sum_{i=1}^{+\infty} {a_i}^p$$

$$\int_0^\infty F(x)^p dx = \sum_{N=1}^{+\infty} \int_{N-1}^N F(x)^p dx \geqslant \sum_{N=1}^{+\infty} (\frac{1}{N}\sum_{i=1}^N a_i)^p$$

$$\therefore \sum_{N=1}^{+\infty} (\frac{1}{N}\sum_{i=1}^N a_i)^p \leqslant (\frac{p}{p-1})^p \sum_{n=1}^{+\infty} {a_n}^p$$

For general $\{a_n\}_{n=1}^{+\infty}$ for any N set

$$f(x) = \frac{1}{x}\int_1^x f(t)dt = \begin{cases} a_i \ \ if \ \ i-1 < x \leqslant i \leqslant N \\ 0 \ \ if \ \ x \geqslant i > N \end{cases}$$

We renumber the finite numbers: $a_1, \cdots a_N$ such that $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_N$ $If x \leqslant k \leqslant N$ we have knows from above that

$$F(x)\frac{1}{x}\int_0^x f(t)dt \geqslant \frac{\sum_{i=1}^k a_i}{k}$$

$$\therefore (F(x))^p \geqslant \frac{\sum_{i=1}^k a_i}{k}^p \quad 1 \leqslant k \leqslant N$$

$$\sum_{k=1}^N \frac{\sum_{i=1}^k a_i}{k}^p \leqslant \sum_{k=1}^N \int_{k-1}^k F(x)^p dx \leqslant \int_0^N F(x)^p dx \leqslant \int_0^{+\infty} F(x)^p dx \leqslant (\frac{p}{p-1})^p \int_0^\infty f(t)^p dt$$

$$= (\frac{p}{p-1})^p \int_0^N f(t)^p dt = (\frac{p}{p-1})^p \sum_{i=1}^N \int_{i-1}^i f(t)^p dt = (\frac{p}{p-1})^p \sum_{i=1}^N {a_i}^p$$

$$\therefore \ \ for \ any \ N \ \ \sum_{k=1}^N \frac{\sum_{i=1}^k a_i}{k}^p \leqslant (\frac{p}{p-1})^p \sum_{i=1}^{+\infty} {a_i}^p$$

Letting $N \longrightarrow +\infty$ we have

$$\sum_{k=1}^{+\infty} \frac{\sum_{i=1}^k a_i}{k}^p \leqslant (\frac{p}{p-1})^p \sum_{i=1}^N {a_i}^p.$$

□

16, Prove Egoroff's theorem: If $\mu(X) \leqslant +\infty$,if$\{f_n\}$ is a sequence of complex measurable functions which converges pointwise at every point of X,and if $\epsilon > 0$,there is a measurable set $E \subset X$, with$\mu(X-E) < \epsilon$, such that $\{f_n\}$ converges uniformly on E.

(The conclusion is that by redefining the f_n on a set of arbitrarily small measure we can convert a pointwise convergent sequence to a uniformly convergent one;note the similarity with $Lusin's Theorem$).

Hint: Put

$$S(n,k) = \bigcap_{i,j>n} \{x : |f_i(x) - f_j(x)| < \frac{1}{k}\},$$

show that $\mu(S(n,k)) \to \mu(x)$ as $n \to \infty$,for each k,and hence that there is a suitably increasing sequence$\{n_k\}$ such that $E = \bigcap S(n_k, k)$ has the desired property.

Show that the theorem does not extend to $\sigma-$finite space.

Show that the theorem does extend, with essentially the same proof, to the situation in which the sequence $\{f_n\}$ is replaced by a family $\{f_t\}$ where t ranges over the positive reals;the assumptions are now that,for all $x \in X$,
(i)$\lim_{t\to\infty} f_t = f(x)$ and
(ii)$t \to f_t$ is continuous.
Proof: Assume $f_n \to f\ a.e\ [\mu]\ no\ X$. Define

$$C_{i,j} = \bigcup_{k=j}^{+\infty}\{x \in X : |f_k(x) - f(x)| > 2^{-i}\} \ \ (i, j = 1, 2, \cdots)$$

Then $C_{i,j+1} \subset C_{i,j}\ \ for\ all\ i,\ j$ and so since $\mu(X) < +\infty$,

$$\lim_{j\to\infty} \mu(X \bigcap C_{i,j}) = \mu(\bigcap_{j=1}^{+\infty}(X \bigcap C_{i,j})) = \mu(X \bigcap \bigcap_{j=1}^{+\infty} C_{i,j})$$

As $f_n \to f\ a.e\ [\mu]\ on\ X\ \ X \bigcap \bigcap_{j=1}^{+\infty} C_{i,j}$ is the set in which $f_n \nrightarrow f$,
so

$$\mu(X \bigcap \bigcap_{j=1}^{+\infty} C_{i,j}) = 0$$

Since all the points x at which $\lim_{k\to\infty} f_k(x) \neq f(x)$ form the set $D = \bigcup_{i=1}^{+\infty}\bigcap_{j=1}^{+\infty}\bigcup_{n=j}^{+\infty}\{x \in X : |f_k(x) - f(x)| \geqslant 2^{-i}\}$ ie $\mu(X \bigcap \bigcap_{j=1}^{+\infty} C_{i,j}) = 0, for\ any\ i$.So for any $i\ \exists\ N(i)$ s.t $\mu(X \bigcap C_{i,N(i)}) < \frac{\epsilon}{2^i}$
Let $E = X - \bigcup_{i=1}^{+\infty} C_{i,N(i)}$ then

$$\mu(X - E) \leqslant \mu(X \bigcap \bigcup_{i=1}^{+\infty} C_{i,N(i)}) \leqslant \sum_{i=1}^{+\infty} \mu(X \bigcap C_{i,N(i)}) = \sum_{i=1}^{+\infty} \frac{\varepsilon}{2^i} = \epsilon$$

For any $i\ \exists N(i)$ for any $x \in E$,$x \bar{\in} \bigcup_{j=1}^{+\infty} C_{j,N(j)} \Rightarrow x \bar{\in} C_{i,N(i)} \Rightarrow x \bar{\in} C_{i,k}\ (k > N(i))$

$$|f_k(x) - f(x)| < 2^{-i}$$

So

$$f_k \rightrightarrows f\ on\ E \quad \mu(X - E) < \epsilon$$

The condition $\mu(X) < +\infty$ is necessary Let $X = (-\infty, +\infty),\ \mu = m\ (lebesgue\ measure)$
$f_n(x) = 1\ \ x \in [n, n+1],\ otherwise\ 0$ then $\lim_{n\to+\infty} f_n(x) = 0,\ \ for\ any\ x \in X$
On the other hand Let $\delta = 1$, for any measurable set e (me < 1) for any $n\ \ m(n, n+1) = 1$,let $E_\delta = X - e$,then $E_\delta \bigcap [n, n+1] \neq \emptyset,\ \ n = 1, 2\cdots\ \therefore for\ any\ n\ \exists x_n \in E_\delta \bigcap [n, n+1],\ f_n(x) = 1$

$$|f_n(x) - f(x)| = 1 \quad \therefore f_n \not\rightrightarrows f\ \ on\ \ E_\delta$$

By the hint we may prove as follow
Since $f_n \to f\ on\ X$ for any k, $X = \bigcup_{n=1}^{+\infty} S(n,k)$,since f_n is measurable,so is $S(n,k)$ by its definition and $S(n,k) \subset S(n+1,k)$.It is clear that:$\mu(X) = \mu(\bigcup_{n=1}^{+\infty} S(n,k)) = \lim_{n\to+\infty} \mu(S(n,k))$ so $\mu(S(n,k)^c) = \mu(X - S(n,k)) = \mu(X) - \mu(S(n,k))\ \ (\because \mu(X) < +\infty)$

$$\lim_{n\to+\infty} \mu(S(n,k)^c) = 0\ \ (for\ any\ k)$$

$\therefore$ *for any* k $\exists n_k$ s.t $\mu(S(n_k,k)^c) < \frac{\epsilon}{2^k}$ Let $E = \bigcap_{k=1}^{+\infty} S(n_k,k)$ then we have

$$\mu(X-E) = \mu(X - \bigcap_{k=1}^{+\infty} S(n_k,k)) = \mu(\bigcup_{k=1}^{+\infty} S(n_k,k)^c) \leqslant \sum_{k=1}^{+\infty} \mu(S(n_k,k)^c) < \sum_{k=1}^{+\infty} \frac{\epsilon}{2^k} = \epsilon$$

By the definition of E ,For any k for any $x \in E$ $x \in S(n_k,k)$ so $|f_i(x) - f_j(x)| < \frac{1}{k}$,whenever $i,j > n_k$ which shows that $\{f_n\}$ converges uniformly on E

If $\{f_t\}$ $(t > 0)$ is a family of complex measurable functions and convergent as $t \to +\infty$ point wise on X then similar to what we did above for any k $X = \bigcup_{n=1}^{+\infty} \tilde{S}(n,k)$ where $\tilde{S}(n,k) = \bigcap_{s,t>n} \{x : |f_s(x) - f_t(x)| < \frac{1}{k}\}$.We claim that for any n,k $\tilde{S}(n,k)$ is measurable.In fact we show that $\tilde{S}(n,k) = \bigcap_{s,t>n;s,t\in Q} \{x : |f_s(x) - f_t(x)| < \frac{1}{k}\}$

Obviously

$$\tilde{S}(n,k) \subset \bigcap_{s,t>n;s,t\in Q} \{x : |f_s(x) - f_t(x)| < \frac{1}{k}\}$$

On the other hand, for $x \in \bigcap_{s,t\in Q} \{x : |f_s(x) - f_t(x)| < \frac{1}{k}\}$ for any $s_0, t_0 > n$ $(s_0, t_0 \in R^1) \exists s_l \in (s_0 + \delta, s_0)$ $and t_l \in (t_0 + \delta, t_0)$ $for \delta$ $small$ and $s_l, t_l \in Q$ So $|f_{s_l} - f_{t_s}| < \frac{1}{k}\}$ Letting $\sigma \to 0$ using the continuity of $t \to f_t(x)$ *for any* $x \in X$ We see that

$$|f_{s_0}(x) - f_{t_0}(x)| < \frac{1}{k}$$

$\therefore x \in \tilde{S}(n,k)$

Thus $\tilde{S}(n,k)$ is measurable.Clearly $\tilde{S}(n,k) \subset \tilde{S}(n{+}1,k) \therefore \lim_{n\to+\infty} \mu(\tilde{S}(n,k)) = \mu(\bigcup_{n=1}^{+\infty} \tilde{S}(n,k)) = \mu(X)$ $\mu(X) < \infty \Longrightarrow \lim_{n\to+\infty} \mu(\tilde{S}(n,k)^c) = 0$ .So for any k, $\exists n_k$, $\mu(\bigcup_{k=1}^{+\infty} \tilde{S}(n_k,k)^c) < \frac{\epsilon}{2^k}$

Let $E = \bigcap_{k=1}^{+\infty} \tilde{S}(n_k,k)$ then $\mu(X-E) = \mu(\bigcup_{k=1}^{+\infty} \tilde{S}(n_k,k)^c) < \sum_{k=1}^{+\infty} \frac{\epsilon}{2^k} = \epsilon$

For any k $\exists n_k$, *for any* $x \in E$ $x \in \tilde{S}(n_k,k) \subset \tilde{S}(m,k)$(for any $m \geqslant k$ and $m \in R^1$)$\Longrightarrow$ $|f_t(x) - f_s(x)| < \frac{1}{k}$ Letting $t \to +\infty$ using $\lim_{t\to+\infty} = f(x)$ we have $|f(x) - f_s(x)| < \frac{1}{k}$

$$\therefore f_t(x) \rightrightarrows f(x) \quad on \quad E.$$

$\square$

17, (a) If $0 < p < \infty$, put $\gamma_p = \max\{1, 2^{p-1}\}$, and show that

$$|\alpha - \beta|^p \leq \gamma_p(|\alpha|^p + |\beta|^p)$$

for arbitrary complex numbers α and β.

(b) Suppose μ is a positive measure on X, $0 < p < \infty$, $f \in L^p(\mu)$, $f_n \in L^p(\mu)$, $f_n(x) \to f(x)$ a.e, and $||f_n||_p \to ||f||_p$ as $n \to \infty$. Show that then $\lim ||f - f_n||_p = 0$, by completing the two proofs that are sketched below.

(i) By Egoroff's theorem, $X = A \cup B$ in such a way that $\int_A |f|^p < \varepsilon$, $\mu(B) < \infty$, and $f_n \to f$ uniformly on B. Fatou's lemma, applied to $\int_B |f_n|^p$, leads to

$$\limsup \int_A |f_n|^p d\mu \leq \varepsilon.$$

(ii) Put $h_n = \gamma_p(|f|^p + |f_n|^p) - |f - f_n|^p$, and use Fatou's lemma as in the proof of Theorem 1.34.

(c) Show that the conclusion of (b) is false if the hypothesis $||f_n||_p \to ||f||_p$ is omitted, even if $\mu(X) < \infty$.

Proof: (a) If $1 \le p < \infty$, the t^p is a convex function on $[0, +\infty)$, so

$$\begin{aligned} |\alpha - \beta|^p &\le (|\alpha| + |\beta|)^p = 2^p \left(\frac{|\alpha| + |\beta|}{2} \right)^p \\ &\le 2^p \frac{(|\alpha|^p + |\beta|^p)}{2} = 2^{p-1}(|\alpha|^p + |\beta|^p). \end{aligned}$$

If $0 < p < 1$, let $f(t) = 1 + t^p - (1+t)^p$ $(t \ge 0)$. We have $f(0) = 0$, $f'(t) = pt^{p-1} - p(1+t)^{p-1} = p\big(t^{p-1} - (1+t)^{p-1}\big) \ge 0$. Thus $f(t)$ is increasing on $[0, +\infty)$. Hence, $f(t) \ge f(0) = 0$ and

$$(1+t)^p \le 1 + t^p, \quad t \in [0, +\infty).$$

For any $\alpha, \beta \in \mathbb{C}$, if $\alpha = 0$, clearly $|\alpha - \beta|^p = |\alpha|^p + |\beta|^p$. Otherwise $\alpha \ne 0$, we have

$$\begin{aligned} |\alpha - \beta|^p &\le (|\alpha| + |\beta|)^p = |\alpha|^p \left(1 + \frac{|\beta|}{|\alpha|} \right)^p \\ &\le |\alpha|^p| \left(1 + \frac{|\beta|^p}{|\alpha|^p} \right) = |\alpha|^p + |\beta|^p. \end{aligned}$$

Consequently, for any $0 < p < \infty$,

$$|\alpha - \beta|^p \le \gamma_p |\alpha|^p + |\beta|^p,$$

where $\gamma_p = \max(1, 2^{p-1})$.

(b). (i) Since $f \in L^p(\mu)$, we have $|f|^p \in L^1(\mu)$ for $0 < p < \infty$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $E \in m$ with $\mu(E) < \delta$, we have

$$\int_E |f|^p d\mu < \varepsilon.$$

We claim if $f \in L^1(\mu)$, then for any $\varepsilon > 0$, there exists $A \in m$ with $X = A \cup B$ and $\mu(B) < +\infty$, such that $\int_A |f| d\mu < \varepsilon$.

Next we prove the claim. By Theorem 1.17, there exists a simple function series $\{s_n\}$ such that $s_n \to |f|$ $(n \to \infty)$ and

$$\lim_{n \to \infty} \int_X s_n d\mu = \int_X |f| d\mu.$$

Then for any $\varepsilon > 0$, there is $M > 0$, when $n \ge M$,

$$\int_X (|f| - s_n) d\mu < \varepsilon.$$

Let $A = \{x : s_n(x) = 0\}$. It is known that $s_n(x) = \sum\limits_{i=1}^{k_n} c_i \chi_{E_i}(x)$, where $E_i \in m$, $E_i \cap E_j = \emptyset$ $(i \ne j)$, and $c_i > 0$, $i = 1, \cdots, k_n$. Since $s_n \in L^1(\mu)$ and $c_i > 0$. Thus $\sum\limits_{i=1}^{k_n} c_i \mu(E_i) < +\infty$, and so $\mu(E_i) < +\infty$. Let $B = \bigcup\limits_{i=1}^{k_n} E_i$, we have $\mu(B) < +\infty$. Hence,

$$\begin{aligned} \int_A |f| d\mu &= \int_A |f| d\mu - \int_A s_n d\mu + \int_A s_n d\mu = \int_A (|f| - s_n) d\mu + \int_A s_n d\mu \\ &\le \int_X (|f| - s_n) d\mu + \int_A s_n d\mu < \varepsilon + \int_A s_n d\mu = \varepsilon + 0 = \varepsilon. \end{aligned}$$

Now we continue to prove (b). Following the claim, for any $\varepsilon > 0$, there exist $A \in m$ and $B \in m$ with $X = A \cup B$, $\mu(B) < +\infty$, such that $\int_A |f|^p d\mu < \varepsilon/8$. On the other hand, for $E \in m$, if there is $\delta > 0$ such that $\mu(E) < \delta$, then $\int_E |f|^p d\mu < \varepsilon/8$. Since $f_n \to f$ on B and $\mu(B) < +\infty$. Then by Egoroff Theorem, for the above $\delta > 0$, there exists $B_\delta \subset B$ such that $\mu(B - B_\delta) < \delta$ and $f_n \rightrightarrows f$ on B_δ. At the same time,

$$\int_{B-B_\delta} |f|^p d\mu < \varepsilon/4.$$

Now, we have $X = A \cup (B - B_\delta) \cup B_\delta$. In fact, we obtain that there exist $A, B \in m$, $A \cup B = X$, $\mu(B) < +\infty$, such that $\int_A |f|^p d\mu < \varepsilon/4$ and $f_n \rightrightarrows f$ on B. Hence, for any $\varepsilon > 0$, there exists a constant $N > 0$, when $n \geq N(B(\varepsilon), \varepsilon) = N(\varepsilon)$, we have $\int_B |f_n - f|^p d\mu < \varepsilon/4$.

Since $\lim\limits_{n\to+\infty} ||f_n||_p = ||f||_p$ and $\int_A |f_n|^p d\mu = \int_X |f_n|^p d\mu - \int_B |f_n|^p d\mu$. Fatou's Lemma applied to $\int_B |f_n|^p d\mu$ leads to

$$\limsup \int_A |f_n|^p d\mu = \int_X |f|^p d\mu - \liminf \int_B |f_n|^p d\mu \leq \int_X |f|^p d\mu - \int_B |f|^p d\mu = \int_A |f|^p d\mu < \varepsilon/4.$$

Hence,

$$\begin{aligned}
\int_X |f - f_n|^p d\mu &= \int_A |f - f_n|^p d\mu + \int_B |f - f_n|^p d\mu \\
&\leq \gamma_p \int_A (|f|^p + |f_n|^p) d\mu + \frac{\varepsilon}{2} \\
&\leq \gamma_p \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = C\varepsilon.
\end{aligned}$$

(ii) Let $h_n = \gamma_p(|f|^p + |f_n|^p) - |f - f_n|^p$. By Fatou's Lemma, we have

$$\int_X \liminf_{n\to\infty} h_n d\mu \leq \liminf_{n\to\infty} \int_X h_n d\mu.$$

So

$$\begin{aligned}
2\gamma_p \int_X |f|^p d\mu &= \int_X \gamma_p(|f|^p + |f|^p) d\mu \\
&= \int_X \liminf_{n\to\infty} \Big(\gamma_p(|f|^p + |f_n|^p) - |f - f_n|^p \Big) d\mu \\
&\leq \liminf_{n\to\infty} \int_X \Big(\gamma_p(|f|^p + |f_n|^p) - |f - f_n|^p \Big) d\mu \\
&= 2\gamma_p \int_X |f|^p d\mu - \limsup \int_X |f - f_n|^p d\mu.
\end{aligned}$$

We know $f \in L^p$, it follows that $\limsup \int_X |f - f_n|^p d\mu \leq 0$. Thus $\limsup \int_X |f - f_n|^p d\mu = 0$.

(c) Let $X = [0, +\infty)$, μ be the Lebesgue measure, and $f_k(x) = k\chi_{[0, \frac{1}{k^p}]}(x)$. Then for any $x > 0$, when k is sufficiently large, we have $f_k(x) = 0$, so $f_k(x) \to f(x) = 0$. On the other hand,

$$\left(\int_{[0,+\infty)} |f_k(x)|^p d\mu \right)^{1/p} = \left(\int_{[0,\frac{1}{k^p}]} k^p d\mu \right)^{1/p} = \left(k^p \mu([0, \frac{1}{k^p}]) \right)^{1/p} = (k^p \frac{1}{k^p})^{1/p} = 1.$$

Therefore,

$$\int_{[0,+\infty)} |f_k(x) - f(x)|^p d\mu = \int_X |f_k|^p d\mu = 1.$$

Hence, it follows from $||f_k||_p = 1 \nrightarrow 0 = ||f||_p$ that $f_k \nrightarrow f$ in $L^p(0, +\infty)$. $\square$

18, Let μ be a positive measure on X. A sequence $\{f_n\}$ of complex measurable functions on X is said to converge in measure to the measurable function f if to every $\varepsilon > 0$ there corresponds an N such that

$$\mu(\{x : |f_n(x) - f(x)| > \varepsilon\}) < \varepsilon$$

for all $n > N$. (This notion is of importance in probability theory.) Assume $\mu(X) < \infty$ and prove the following statements:

(a) If $f_n(x) \to f(x)$ a.e., then $f_n \to f$ in measure.

(b) If $f_n \in L^p(\mu)$ and $||f_n - f||_p \to 0$, then $f_n \to f$ in measure; here $1 \leq p \leq \infty$.

(c) If $f_n \to f$ in measure, then $\{f_n\}$ has a subsequence which converges to f a.e.

Investigate the converses of (a) and (b). What happens to (a), (b), and (c) if $\mu(X) = \infty$, for instance, if μ is Lebesgue measure on R^1?

Proof: (a) Since $f_n \to f$ a.e. (Noting that f_n, f are measurable) or by Egoroff Theorem, for any $\varepsilon > 0$, there exists a measurable subset $E \subset X$ with $\mu(E) < \varepsilon$, such that $f_n \rightrightarrows f$ on $X - E$. That is there exists $N(\varepsilon) > 0$, when $n \geq N$,

$$|f(x) - f_n(x)| \leq \varepsilon, \quad \forall\, x \in X - E.$$

Thus, $\{x : |f(x) - f_n(x)| \geq \varepsilon\} \subset E$ and $\mu(\{x : |f(x) - f_n(x)| \geq \varepsilon\}) \leq \mu(E) < \varepsilon$. So $f_n \to f$ in measure.

(b) If $1 \leq p < +\infty$. Assume $f_n \in L^p$ with $||f - f_n||_p \to 0$ $(n \to \infty)$. Then $f_n - f \in L^p$ and $f = f - f_n + f_n \in L^p$. On the other hand, for any $\varepsilon > 0$, there is a constant $N > 0$ such that

$$\int_X |f_n - f|^p d\mu < \varepsilon^{p+1}$$

for all $n \geq N$. Therefore,

$$\varepsilon^p \mu\{x : |f_n - f| > \varepsilon\} \leq \int_{\{x\colon\, |f_n - f| > \varepsilon\}} |f_n - f|^p d\mu \leq \int_X |f_n - f|^p d\mu < \varepsilon^{p+1},$$

i.e. $\mu\{x :\ |f_n - f| > \varepsilon\} < \varepsilon$. So $f_n \to f$ in measure.

If $p = +\infty$. It is well known that for any n, there exists a subset E_n of X with $\mu(E_n) = 0$, such that

$$||f_n - f||_\infty = \sup_{x \in X \setminus E_n} |f(x)|.$$

Let $E = \bigcup\limits_{n=1}^{+\infty} E_n$, we have $\mu(E) = 0$. Then for any $\varepsilon > 0$, there exists a constant $N > 0$ such that

$$\sup_{X \setminus E} |f_n(x) - f(x)| \leq \sup_{X \setminus E_n} |f_n(x) - f(x)| = ||f_n(x) - f(x)||_\infty < \varepsilon$$

for all $n \geq N$. Thus, when $n \geq N$, we see that $\{x :\ |f_n - f| > \varepsilon\} \subset E$ and $\mu\{x :\ |f_n - f| > \varepsilon\} \leq \mu(E) = 0$. So $f_n \to f$ in measure.

(c) We first assume $f_n \to f$ in measure. Then we can choose a sebsequence $\{f_{k_i}\}$, such that for any $k > 0$,

$$\sum_{i=1}^{\infty} \mu\{x \in X :\ |f_{k_i}(x) - f(x)| \geq \frac{1}{k}\} < +\infty.$$

In fact, it follows from $\mu\{x: \ |f_n(x)-f(x)| \geq \frac{1}{2}\} \to 0$ $(n \to \infty)$ that there is a subsequence $\{k_i\}$, here we may assume $k_1 < k_2 < k_3 < \cdots$, such that

$$\mu\{x: \ |f_{k_i}(x)-f(x)| \geq \frac{1}{2}\} \leq \frac{1}{2^i}.$$

Hence, for any $\varepsilon > 0$, there is a $i_0 > 0$ such that

$$\mu\{x \in X: \ |f_{k_i}(x)-f(x)| \geq \varepsilon\} \leq \mu\{x: \ |f_{k_i}(x)-f(x)| \geq \frac{1}{2}\} \leq \frac{1}{2^i}$$

for all $i \geq i_0$.

Now, note that for any $x_0 \in \{x: \ \limsup_{j\to\infty} |f_{k_j}(x)-f(x)| > \varepsilon\}$, we have

$$\varepsilon < \limsup_{j\to\infty} |f_{k_j}(x_0)-f(x_0)| = \inf_{i\geq 1} \sup_{j\geq i} |f_{k_j}(x_0)-f(x_0)|.$$

Then for any $i \geq 1$, we see $\sup_{j\geq i} |f_{k_j}(x_0)-f(x_0)| > \varepsilon$, and so there exists $j \geq i$ such that $|f_{k_j}(x_0)-f(x_0)| > \varepsilon$. Therefore,

$$\{x: \ \limsup_{j\to\infty} |f_{k_j}(x_0)-f(x_0)| > \varepsilon\} \subset \bigcap_{i=1}^{+\infty} \bigcup_{j=i}^{+\infty} \{x: \ |f_{k_j}(x)-f(x)| > \varepsilon\}.$$

Thus, for any $\varepsilon > 0$ and any $i \geq 1$,

$$\begin{aligned}
&\mu\{x: \ \limsup_{j\to\infty} |f_{k_j}(x)-f(x)| > \varepsilon\} \leq \mu\Big(\bigcap_{i=1}^{+\infty} \bigcup_{j=i}^{+\infty} \{x: \ |f_{k_j}(x)-f(x)| > \varepsilon\}\Big) \\
&\leq \sum_{j=i}^{+\infty} \mu\{x: \ |f_{k_j}(x)-f(x)| > \varepsilon\} \leq \sum_{j=i}^{+\infty} \frac{1}{2^i}.
\end{aligned}$$

Since $\sum\limits_{j=i}^{+\infty} \frac{1}{2^i} \to 0$ as $i \to +\infty$. Whence,

$$\mu\{x: \ \limsup_{j\to\infty} |f_{k_j}(x)-f(x)| > \varepsilon\} = 0.$$

It follows that $\limsup_{j\to\infty} |f_{k_j}(x)-f(x)| \leq \varepsilon$ a.e. on X. Finally, the arbitrary of ε yields $\limsup_{j\to\infty} |f_{k_j}(x)-f(x)| = 0$. Hence, $f_{k_j} \to f$ a.e. on X.

The converse of (a) is not true. For example, let $E = [0,1)$, define

$$f_1^{(1)}(x) = 1,$$

$$f_1^{(2)}(x) = \begin{cases} 1, & x \in [0, \frac{1}{2}), \\ 0, & x \in [\frac{1}{2}, 1). \end{cases}$$

$$f_1^{(2)}(x) = \begin{cases} 0, & x \in [0, \frac{1}{2}), \\ 1, & x \in [\frac{1}{2}, 1). \end{cases}$$

In generally, $[0,1]$ k, we define k functions on the k-th class by

$$f_i^{(k)}(x) = \begin{cases} 1, & x \in [\frac{i-1}{k}, \frac{i}{k}), \\ 0, & x \notin [\frac{i-1}{k}, \frac{i}{k}), \qquad (i = 1, 2, \cdots, k). \end{cases}$$

Now, we define a sequence as follows: $\varphi_1(x) = f_1^{(1)}(x)$, $\varphi_2(x) = f_1^{(2)}(x)$, $\varphi_3(x) = f_2^{(2)}(x)$, $\varphi_4(x) = f_1^{(3)}(x)$, $\varphi_5(x) = f_2^{(3)}(x)$, $\varphi_6(x) = f_3^{(3)}(x)$, $\cdots$. Then $\{\varphi_n(x)\}$ is a sequence of measurable functions which every taking finite value and defined a.e. in $(0,1)$. Let $\varphi(x) \equiv 0$, then for any $\sigma > 0$, if $\sigma > 1$, clearly,

$$E[x: \ |\varphi_n(x) - \varphi(x)| \geq \sigma] = 0,$$

and so

$$\lim_{n\to\infty} mE[x: \ |\varphi_n(x) - \varphi(x)| \geq \sigma] = 0.$$

If $\sigma \leq 1$ and $\varphi_n(x)$ is the k the i-th function, then

$$E[x: \ |\varphi_n(x) - \varphi(x)| \geq \sigma] = [\frac{i-1}{k}, \frac{1}{k}].$$

Thus,

$$mE[x: \ |\varphi_n(x) - \varphi(x)| \geq \sigma] = \frac{1}{k}.$$

We see that $n \to \infty$ implies $k \to \infty$, so

$$\lim_{n\to\infty} mE[x: \ |\varphi_n(x) - \varphi(x)| \geq \sigma] = 0.$$

Hence $\varphi_n(x) \to \varphi(x) \equiv 0$ in measure.

But for any $x_0 \in (0,1)$, the sequence $\{\varphi_n(x)\}$ have infinite functions which equiv to zero at x_0, and also have infinite functions satisfying $\varphi_n(x) \equiv 1$. Thus, $\{\varphi_n(x)\}$ does not converge to any number, although which has subsequences converge to 0 or 1.

The converse of (b) is also not true. That is, for $1 \leq p < \infty$, $f_n \to f$ in measure does not imply $||f_n - f||_p \to 0$. Take the example in (c) of exercise 16: let $X = [0,1]$ and $f_n(x) = n$ for $x \in [0, \frac{1}{n}]$, then

$$\int_X |f_n(x)|^p d\mu = \int_{[0,1]} |f_n(x)|^p d\mu = \int_0^1 |f_n(x)|^p dx = \int_0^{\frac{1}{n}} n^p dx = n^{p-1}.$$

So $f_n \in L^p(X)$. Now, for any $\sigma > 0$, we have

$$E[x: \ |f_n(x)| \geq \sigma] \subset [0, \frac{1}{n}],$$

and

$$mE[x: \ |f_n(x)| \geq \sigma] \leq m([0, \frac{1}{n}]) = \frac{1}{n} \to 0 \quad (\text{as } n \to \infty).$$

Then $f_n \to 0$ in measure. However, $||f_n||_p = n^{p-1}$ implies $||f_n||_p \to \infty$ as $n \to \infty$.

If $\mu(X) = +\infty$, then (a) may be not true. For example: let $X = [0, +\infty)$ and $f_n(x) = \chi_{[0,\ n]}(x)$. Thus, $f_n(x) \to 1$ a.e. in $[0, +\infty)$. But for any $0 < \sigma < 1$, we have $\{x: \ |f_n(x) - 1| > \sigma\} = [n, +\infty)$, and so $\mu\{x: \ |f_n(x) - 1| > \sigma\} = +\infty$. Hence $f_n \nrightarrow 1$ in measure.

The condition $\mu(X) < \infty$ is important for (b) and (c), Which can be found in their proofs. $\square$

21, Call a metric space Y a completion of a metric space X if X is dense in Y and Y is complete. In Sec. 3.15 reference was made to " the " completion of a metric space. State and prove a uniqueness theorem which justifies this terminology.

Solution: We have the following uniqueness result. If (Y, d) is a complete metric space, $(Z, \tilde{d})$ is also a complete metric space, $X \subset Y \cap Z$, X is dense in Y with respect to d and X is dense in Z with respect to $\tilde{d}$, and for any $x, y \in X$, $d(x, y) = \tilde{d}(x, y)$. Then $Y \cong Z$ and $d = \tilde{d}$ on $Y \times Y = Z \times Z$.

Proof: We first prove $Y \cong Z$. Since X is dense in Y, then for any $f \in Y$, there exists a sequence $\{f_n\} \subset X$ such that $d(f_n, f) \to 0$. Thus, $d(f_n, f_m) \to 0$ as $n, m \to +\infty$, and so $\tilde{d}(f_n, f_m) = d(f_n, f_m) \to 0$. Hence, $\{f_n\}$ is a cauchy sequence in $(Z, \tilde{d})$. The completeness of $(Z, \tilde{d})$ implies that there is a unique $g \in Z$, such that $\tilde{d}(f_n, g) \to 0$.

On the other hand, note that

$$d(f_n, f_m) = \tilde{d}(f_n, f_m).$$

Let $n \to \infty$, we have

$$d(f, f_m) = \tilde{d}(g, f_m).$$

Now, we will prove that $i : Y \to Z$ is well defined. If $f \in Y$, take $\{f_n\} \subset X$ with $f_n \to f$, and also $\{g_n\} \subset X$ with $g_n \to f$. Then there exist g and h, such that $f_n \to g$ in Z and $g_n \to h$ in Z. On the other hand,

$$\tilde{d}(g, h) \leq \tilde{d}(g, f_n) + \tilde{d}(f_n, g_n) + \tilde{d}(g_n, h) \leq \tilde{d}(g, f_n) + d(f_n, g_n) + \tilde{d}(g_n, h) \to 0.$$

Hence, $g = h$ and i is well defined. For any $f, h \in Y$, there exist $\{f_n\} \subset X$ and $\{h_n\} \subset X$, such that $f_n \to f$ in Y and $h_n \to h$ in Y. Then we have

$$\tilde{d}(i(h), i(f)) = \lim_{n\to\infty} \tilde{d}(h_n, f_n) = \lim_{n\to\infty} d(h_n, f_n) = d(h, f).$$

Therefore, i is an . □

22, Suppose X is a metric space in which every Cauchy sequence has a convergent subsequence. Dose it follow that X is complete? (See the proof of Theorem 3.11.)

Solution: Yes! In fact, if $\{f_n\}$ is a cauchy sequence, then by the assumption, there exists a subsequence $\{f_{n_i}\}$ and a $f \in X$, such that $d(f_{n_i}, f) \to 0$. Thus, for any $\varepsilon > 0$, there exists $N(\varepsilon) > 0$, when $n, m \geq N$,

$$d(f_n, f_m) < \varepsilon/2.$$

On the other hand, there is a $i_0 \geq N(\varepsilon)$, for $i \geq i_0$, also we have $n_i \geq i_0 \geq N$, such that

$$d(f_{n_i}, f) < \varepsilon/2.$$

So when $n \geq N(\varepsilon)$, take $i \geq N$, we have

$$d(f_n, f) \leq d(f_n, f_{n_i}) + d(f_{n_i}, f) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Then $\lim\limits_{n\to\infty} d(f_n, f) = 0$ and X is complete. □

23, Suppose μ is a positive measure on X, $\mu(X) < \infty$, $f \in L^\infty(\mu)$, $||f||_\infty > 0$, and

$$\alpha_n = \int_X |f|^n d\mu \qquad (n = 1, 2, 3, \cdots).$$

Prove that

$$\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = ||f||_\infty.$$

Proof: Assume $\mu(E) = 0$. We have

$$\begin{aligned}
\alpha_{n+1} &= \int_X |f|^{n+1} d\mu = \int_E |f|^{n+1} d\mu + \int_{X\setminus E} |f|^{n+1} d\mu \\
&= \int_{X\setminus E} |f|^{n+1} d\mu \le \sup_{x\in X\setminus E} |f(x)| \int_{X\setminus E} |f|^n d\mu \\
&= \sup_{x\in X\setminus E} |f(x)| \int_X |f|^n d\mu = \sup_{x\in X\setminus E} |f(x)| \alpha_n.
\end{aligned}$$

So

$$\alpha_{n+1} \le \inf_{E\subset m, \mu(E)=0} \sup_{x\in X\setminus E} |f(x)|\alpha_n = ||f||_\infty \alpha_n.$$

Since $||f||_\infty > 0$, there exists $E_0 \subset m$ with $\mu(E_0) > 0$ such that $\inf_{x\in E_0} |f(x)| > 0$. Thus, we see $\alpha_n > 0$. Hence,

$$\frac{\alpha_{n+1}}{\alpha_n} \le ||f||_\infty, \qquad \forall\ n \in \mathbb{N}.$$

On the other hand, by the definition, the set $E = \{x \in X : \ |f(x)| \ge ||f||_\infty - \varepsilon\}$ satisfies $\mu(E) > 0$. Otherwise,

$$||f||_\infty \le \sup_{x\in X\setminus E} |f(x)| \le ||f||_\infty - \varepsilon.$$

This is a contradiction. By Hölder inequality, we have

$$\alpha_n = \int_X |f|^n d\mu \le \left(\int_X |f|^{n\cdot\frac{n+1}{n}} d\mu\right)^{\frac{n}{n+1}} \left(\int_X 1 d\mu\right)^{\frac{1}{n+1}} = \mu(X)^{\frac{1}{n+1}} \alpha_{n+1}^{\frac{n}{n+1}}.$$

Thus,

$$\begin{aligned}
\frac{\alpha_{n+1}}{\alpha_n} &\ge \mu(X)^{-\frac{1}{n+1}} \alpha_{n+1}^{\frac{1}{n+1}} = \mu(X)^{-\frac{1}{n+1}} \left(\int_X |f|^{n+1} d\mu\right)^{\frac{1}{n+1}} \\
&\ge \mu(X)^{-\frac{1}{n+1}} \left(\int_E |f|^{n+1} d\mu\right)^{\frac{1}{n+1}} \ge \mu(X)^{-\frac{1}{n+1}} \mu(E)^{\frac{1}{n+1}} (||f||_\infty - \varepsilon).
\end{aligned}$$

Therefore,

$$\left(\frac{\mu(E)}{\mu(X)}\right)^{\frac{1}{n+1}} (||f||_\infty - \varepsilon) \le \frac{\alpha_{n+1}}{\alpha_n} \le ||f||_\infty, \qquad \forall\ n \in \mathbb{N}.$$

Let $n \to \infty$, we have

$$||f||_\infty - \varepsilon \le \liminf_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} \le \limsup_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} \le ||f||_\infty.$$

Since $\varepsilon > 0$ is arbitrary, so $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = ||f||_\infty$. $\square$

24, Suppose μ is a positive measure, $f \in L^p(\mu)$, $g \in L^p(\mu)$.

(a) If $0 < p < 1$, prove that

$$\int \big||f|^p - |g|^p\big| d\mu \leq \int |f-g|^p d\mu$$

that $\triangle(f,g) = \int |f-g|^p d\mu$ define a metric on $L^p(\mu)$, and that the resulting metric space is complete.

(b) If $1 \leq p < \infty$ and $||f||_p \leq R$, $||g||_p \leq R$, prove that

$$\int \big||f|^p - |g|^p\big| d\mu \leq 2pR^{p-1}||f-g||_p.$$

Hint: Prove first, for $x \geq 0$, $y \geq 0$, that

$$|x^p - y^p| \leq \begin{cases} |x-y|^p, & \text{if} \quad 0 < \mathrm{p} < 1, \\ p|x-y|(x^{p-1}+y^{p-1}), & \text{if} \quad 1 \leq \mathrm{p} < \infty. \end{cases}$$

Note that (a) and (b) establish the continuity of the mapping $f \to |f|^p$ that carries $L^p(\mu)$ into $L^1(\mu)$.

Proof: We first claim: for $x \geq 0$, $y \geq 0$, we have

$$|x^p - y^p| \leq \begin{cases} |x-y|^p, & \text{if} \quad 0 < \mathrm{p} < 1, \\ p|x-y|(x^{p-1}+y^{p-1}), & \text{if} \quad 1 \leq \mathrm{p} < \infty. \end{cases}$$

If $x = 0$ or $y = 0$, the result holds obviously. If $x \neq 0$ and $y \neq 0$, then we may assume $x \geq y > 0$, so we only need to show that

$$x^p - y^p \leq \begin{cases} (x-y)^p, & \text{if} \quad 0 < \mathrm{p} < 1, \\ p(x-y)(x^{p-1}+y^{p-1}), & \text{if} \quad 1 \leq \mathrm{p} < \infty. \end{cases}$$

If $0 < p < 1$, let $h(t) = t^p$, $0 < t \leq 1$. Then $t^p \geq t$ and $(1-t)^p \geq 1-t$. Thus

$$t^p + (1-t)^p \geq t + 1 - t = 1.$$

That is

$$0 \leq 1 - t^p \leq (1-t)^p, \qquad 0 < t \leq 1.$$

Let $t = y/x$, the $0 < t \leq 1$. Then $1 - (\frac{y}{x})^p \leq (1 - \frac{y}{x})^p$, which implies $x^p - y^p \leq (x-y)^p$.

If $1 \leq p < \infty$, let $h(t) = t^p - 1 - p(t-1)(t^{p-1}+1)$, $t \geq 1$. We have

$$h'(t) = pt^{p-1} - p(t^{p-1}+1) - p(t-1)(p-1)t^{p-2}$$

$$= -p((t-1)(p-1)t^{p-2}+1) \leq 0, \qquad \text{if } t \geq 1.$$

So $h(t) \leq h(1) = 0$ for any $t \geq 1$. Let $t = x/y$, we have $h(\frac{x}{y}) \leq 0$, i.e.

$$\left(\frac{x}{y}\right)^p - 1 \leq p\left(\frac{x}{y} - 1\right)\left(\left(\frac{x}{y}\right)^{p-1} + 1\right).$$

Thus, $x^p - y^p \leq p(x-y)(x^{p-1}+y^{p-1})$.

(a) If $0 < p < 1$, then by the above claim, it is easy to see that

$$\int \Big||f|^p - |g|^p\Big| d\mu \leq \int \Big||f| - |g|\Big|^p d\mu \leq \int |f-g|^p d\mu.$$

(i) For any $f, g \in L^p(\mu)$, we have $\triangle(f,g) = \int |f-g|^p d\mu < +\infty$, $\triangle(f,g) \geq 0$, and $\triangle(f,g) = 0$ if and only if $|f-g|^p = 0$ a.e. $[\mu]$, i.e. $f = g$ a.e. $[\mu]$.

(ii) $\triangle(f,g) = \int |f-g|^p d\mu = \int |g-f|^p d\mu = \triangle(g,f)$.

(iii) Since $\int ||f|^p - |g|^p| d\mu \leq \int |f-g|^p d\mu$. We have

$$\int \Big(|f|^p - |g|^p\Big) d\mu \leq \int \Big||f|^p - |g|^p\Big| d\mu \leq \int |f-g|^p d\mu.$$

So for any $f, g, h \in L^p(\mu)$,

$$\int \Big(|f-g|^p - |f-h|^p\Big) d\mu \leq \int |f-g-(f-h)|^p d\mu = \int |g-h|^p d\mu.$$

Then

$$\triangle(f,g) \leq \triangle(f,h) + \triangle(g,h).$$

Hence, $\triangle(\cdot,\cdot)$ defines a metric on $L^p(\mu)$.

If $0 < p < 1$, for any $f \in L^p(\mu)$, let $||f||_p = \int |f|^p d\mu$. Then for any $f, g \in L^p(\mu)$, we have

$$|f+g|^p \leq (|f| + |g|)^p \leq |f|^p + |g|^p.$$

So

$$||f+g||_p \leq ||f||_p + ||g||_p.$$

Now, if $\{f_n\}$ is a Cauchy sequence in $L^p(\mu)$ under the metric $\triangle(\cdot,\cdot)$. Then for any $\varepsilon > 0$, there exists N, when $n, m \geq N$, we have $\triangle(f_n, f_m) < \varepsilon$, i.e.

$$||f_n - f_m||_p < \varepsilon.$$

Similar to the proof of Theorem 3.11, there exists a subsequence $\{f_{n_i}\}$ such that

$$||f_{n_{i+1}} - f_{n_i}||_p < 2^{-i} \qquad (i = 1, 2, \cdots).$$

Put $g_k = \sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}|$ and $g = \sum_{i=1}^{\infty} |f_{n_{i+1}} - f_{n_i}|$. Then

$$||g_k||_p \leq \sum_{i=1}^{k} ||f_{n_{i+1}} - f_{n_i}||_p \leq \sum_{i=1}^{k} \frac{1}{2^i} \leq 1.$$

Fatous Lemma gives

$$\int g^p d\mu = \int \varliminf_{k\to\infty} |g_k|^p d\mu \leq \varliminf_{k\to\infty} \int |g_k|^p d\mu \leq 1.$$

Thus, $g \in L^p(\mu)$. In particular, $g(x) < \infty$ a.e. $[\mu]$ on X, so that the series

$$f_{n_1}(x) + \sum_{i=1}^{\infty} (f_{n_{i+1}}(x) - f_{n_i}(x))$$

converges absolutely for almost every $x \in X$.

Since $f_{n_k}(x) = f_{n_1}(x) + \sum_{i=1}^{k-1} (f_{n_{i+1}}(x) - f_{n_i}(x))$, we see that there exists a function $f(x)$ such that

$$f(x) = \lim_{i\to\infty} f_{n_i}(x) \qquad \text{a.e. on } X.$$

On the other hand, for every $m > N$, Fatous Lemma shows that

$$\int |f - f_m|^p d\mu \le \varliminf_{i\to\infty} \int |f_{n_i} - f_m|^p d\mu = \varliminf_{i\to\infty} ||f_{n_i} - f_m||_p \le \varepsilon.$$

So $\triangle(f_m, f) \to 0$ as $m \to \infty$, that is $f - f_m \in L^p(\mu)$. Thus, $f = f - f_m + f_m \in L^p(\mu)$. Hence, $(L^p(\mu), \triangle)$ is a complete metric space.

(b) If $1 \le p < \infty$, by the claim, we have

$$||f|^p - |g|^p| \le p||f| - |g||(|f|^{p-1} + |g|^{p-1}) \le p|f - g|(|f|^{p-1} + |g|^{p-1}).$$

If $||f||_p \le R$ and $||g||_p \le R$, Hölder inequality gives

$$\begin{aligned}
\int \Big| |f|^p - |g|^p \Big| d\mu &\le p \int |f - g|(|f|^{p-1} + |g|^{p-1}) d\mu \\
&= p\Big(\int |f - g|\ |f|^{p-1} d\mu + \int |f - g|\ |g|^{p-1} d\mu \Big) \\
&\le p\Big(||f - g||_p ||f||_p^{p-1} + ||f - g||_p ||g||_p^{p-1} \Big) \\
&= p||f - g||_p (||f||_p^{p-1} + ||g||_p^{p-1}) \le 2pR^{p-1} ||f - g||_p.
\end{aligned}$$

So $f \to |f|^p$ is a continuous map from $L^p(\mu)$ to $L^1(\mu)$. □

25, Suppose μ is a positive measure on X and $f : X \to (0, \infty)$ satisfies $\int_X f d\mu = 1$. Prove, for every $E \subset X$ with $0 < \mu(E) < \infty$, that

$$\int_E (\log f)\ d\mu \le \mu(E)\ \log \frac{1}{\mu(E)}$$

and, when $0 < p < 1$,

$$\int_E f^p\ d\mu \le \mu(E)^{1-p}.$$

Proof: First $\varphi(t) = \ln t$ satisfies $\varphi'(t) = \frac{1}{t}$ and $\varphi''(t) = -\frac{1}{t^2} < 0$. Then $-\varphi : (0, \infty) \to \mathbb{R}$ is convex. By Jensen's inequality, for any E with $\mu(E) < \infty$, we have

$$-\varphi\Big(\frac{1}{\mu(E)} \int_E f\ d\mu \Big) \le -\frac{1}{\mu(E)} \int_E \varphi(f)\ d\mu.$$

So

$$\frac{1}{\mu(E)} \int_E \varphi(f)\ d\mu \le \varphi\Big(\frac{1}{\mu(E)} \int_E f\ d\mu \Big).$$

Hence,

$$\frac{1}{\mu(E)} \int_E (\log f)\ d\mu \le \log\Big(\frac{1}{\mu(E)} \int_E f\ d\mu \Big) \le \log\Big(\frac{1}{\mu(E)} \int_X f\ d\mu \Big) = \log\Big(\frac{1}{\mu(E)} \Big).$$

That is

$$\int_E (\log f)\ d\mu \le \mu(E) \log\Big(\frac{1}{\mu(E)} \Big).$$

If $0 < p < 1$, $h(t) = t^p$ satisfies $h'(t) = pt^{p-1}$ and $h''(t) = p(p-1)t^{p-2} < 0$ $(t > 0)$. Thus, $h(t)$ is concave. So by Jensen's inequality,

$$\begin{aligned}
\int_E f^p\ d\mu &\le \mu(E)\Big(\frac{1}{\mu(E)} \int_E f\ d\mu \Big)^p \le \mu(E)\Big(\frac{1}{\mu(E)} \int_X f\ d\mu \Big)^p \\
&= \mu(E)\ \Big(\frac{1}{\mu(E)} \Big)^p = \mu(E)^{1-p}.
\end{aligned}$$

This completes the proof. □

Chapter 6 Complex Measures

1 If μ is a complex measure on a σ -algebra $\mathcal{M}$ and if $E \in \mathcal{M}$, define

$$\lambda(E) = \sup \sum |\mu(E_i)|,$$

the supremum being taken oven all finite partition $\{E_i\}$ of E. Does it follow that $\lambda = |\mu|$?

Solution: We prove that $\lambda = |\mu|$. For any $E \in \mathcal{M}$, if $\{E_i\}_{i=1}^{n}$ is a finite partition of E, then $E = \bigcup\limits_{i=1}^{n} F_i$, where

$$F_i = \begin{cases} E_i & \text{if } i \leq n; \\ \emptyset & \text{if } i > n. \end{cases}$$

Hence $\{F_i\}_{i=1}^{+\infty}$ is an infinite partition of E. So

$$\sum_{i=1}^{n} |\mu(E_i)| = \sum_{i=1}^{n} |\mu(F_i)| = \sum_{i=1}^{+\infty} |\mu(F_i)| \leq |\mu|(E),$$

as $\mu(\emptyset) = 0$.

On the other hand, if $\{E_i\}_{i=1}^{+\infty}$ is a partition of E, then

$$\mu(E) = \sum_{i=1}^{+\infty} \mu(E_i),$$

and the series is absolutely convergent. So $\sum\limits_{i=1}^{+\infty} |\mu(E_i)| < +\infty$. That is, for any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $\sum\limits_{i=n+1}^{+\infty} |\mu(E_i)| < \epsilon$. Let $F = \bigcup\limits_{i=n+1}^{+\infty} E_i$, then $F \in \mathcal{M}$ and $\{E_1, E_2, ..., E_n, F\}$ is a finite partition of E, $E = \bigcup\limits_{i=1}^{n} E_i \bigcup F$. So, as μ is a complex measure, we have

$$\begin{aligned} \sum_{i=1}^{+\infty} |\mu(E_i)| &= \sum_{i=1}^{n} |\mu(E_i)| + \sum_{i=n+1}^{+\infty} |\mu(E_i)| \\ &< \sum_{i=1}^{n} |\mu(E_i)| + \epsilon \leq \sum_{i=1}^{n} |\mu(E_i)| + |\mu(F)| + \epsilon \\ &\leq \lambda(E) + \epsilon. \end{aligned}$$

Hence $\sum\limits_{i=1}^{+\infty} |\mu(E_i)| \leq \lambda(E)$ and $|\mu|(E) \leq \lambda(E)$. Thus $|\mu| = \lambda$ is proved. □

2 Prove that the example given at the end of Sec.6.10 has the stated properties.

Solution: We shall prove that: if μ is the Lebesgue measure on $(0,1)$ and λ is the counting measure on the σ-algebra of all Lebesgue measurable set in $(0,1)$. Then λ has no Lebesgue decomposition relative to μ, and although $\mu \ll \lambda$ and μ is bounded, there is no $h \in L^1(\lambda)$ such that $d\mu = hd\lambda$.

Proof: Clearly, λ is not σ-finite. This is because that $\lambda(E) < +\infty \Leftrightarrow E$ is finite. So is $(0,1) = \bigcup\limits_{i=1}^{+\infty} E_i$ with $\lambda(E_i) < +\infty$ for each i, then $(0,1) = \bigcup\limits_{i=1}^{+\infty} E_i$ is countable, which is impossible.

Secondly, $\mu \ll \lambda$. In fact, $\lambda(E) = 0 \Rightarrow E = \emptyset$, $\mu(\emptyset) = 0$. Also, μ is bounded and $\mu(0,1) = 1$.

If there were a $h \in L^1(\lambda)$ such that

$$\mu(E) = \int_E h d\lambda \text{ for any } E \in \mathcal{M}.$$

Since $\mu(E) \geq 0$, we have $h \geq 0$ λ-a.e. on E. As $\lambda(F) = 0 \Leftrightarrow F = \emptyset$, we see that $h(x) \geq 0$ everywhere on $(0,1)$ (taking $E = (0,1)$).

For any $n \in \mathbb{N}$, let $E_n = \{x \in (0,1) : h(x) \geq \frac{1}{n}\}$. Then as $h \in L^1(\lambda)$, E_n is finite or empty. Let $F \triangleq \{x \in (0,1) : h(x) > 0\} = \bigcup_{n=1}^{+\infty} E_n$ is at most countable. Let $F = \{a_1, a_2, ..., a_n, ...\}$, $F \in \mathcal{M}$. So

$$0 = \mu(F) = \int_F h d\lambda = \sum_{i=1}^{+\infty} h(a_i)\lambda\{a_i\} = \sum_{i=1}^{+\infty} h(a_i).$$

But $h \geq 0 \Rightarrow h(a_i) = 0$, for each i. If $F = \emptyset$, then $h \equiv 0$. Thus

$$1 = \mu(0,1) = \int_F h d\lambda = 0.$$

This is a contradiction. $\square$

3 Prove that the vector space $M(X)$ of all complex regular Borel measures on a locally compact Hausdorff space X is a Banach space if $\|\mu\| = |\mu|(X)$. Hint: Compare Exercise 8, chapter 5. [That the difference of any two members of $M(X)$ is in $M(X)$ was used in the first paragraph of the proof of Theorem 6.19; supply a proof of this fact.]

Solution: It is clear that if $\mu, \nu \in M(X)$ and $\alpha, \beta \in \mathbb{C}$, then $\alpha\mu + \beta\nu$ is a complex measure (Borel measure). The main problem is to show that $\alpha\mu + \beta\nu$ is regular, i.e. $|\alpha\mu + \beta\nu|$ is a regular Borel measure. For that, we use the following lemma.

Lemma. (D.L.Cohn, Measure Theory P218, Proposition 7.3.3) Let X be a locally compact Hausdorff space, and let μ be a finite signed or complex measure on $(X, \mathcal{B}(X))$, then the following conditions are equivalent.

(a) μ is regular.

(b) Each of the positive measures appearing in the Jordan decomposition of μ is regular.

(c) μ is a linear combination of finite positive regular Borel measures.

Proof of the Lemma: Suppose (a) holds, and let μ^- be one of two measures appearing in the Jordan decomposition of μ, we claim that $\mu^- \leq |\mu|$. In fact, let $\mu = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4) = \mu_k + i\mu_I$, μ_i (i=1,2,3,4) are finite positive measures, then for any $E \in \mathcal{B}(X)$, we have

$$|\mu(E)| = (|\mu_k(E)|^2 + |\mu_I(E)|^2)^{\frac{1}{2}} \leq |\mu|(E).$$

So, $|\mu_k(E)| \leq |\mu|(E)$ and $|\mu_I(E)| \leq |\mu|(E)$. For any partition $E = \bigcup_{i=1}^{+\infty} E_i$, we get

$$\sum_{i=1}^{+\infty} |\mu_k(E_i)| \leq \sum_{i=1}^{+\infty} |\mu|(E_i) = |\mu|(E).$$

Hence, by the conclusion of Exercise 1 of this chapter we have

$$|\mu_k|(E) = \sup \sum_{i=1}^{+\infty} |\mu_k(E_i)| \leq |\mu|(E).$$

Similarly, we also get $|\mu_I|(E) \leq |\mu|(E)$. Since $\mu_k = \mu_k^+ - \mu_k^-$, $\mu_k^+ = \frac{1}{2}(|\mu_k| + \mu_k)$, $\mu_k^- = \frac{1}{2}(|\mu_k| - \mu_k)$, we have

$$0 \leq \mu_k^{\pm}(E) \leq |\mu_k|(E) \leq |\mu|(E).$$

Thus, $\mu^- \leq |\mu|$ is proved.

Recall that a finite signed or complex measure μ is regular if $|\mu|$ is a regular Borel measure. If $A \in \mathcal{B}(X)$, $\epsilon > 0$, and U is an open set that includes A and satisfies $|\mu|(U) < |\mu|(A) + \epsilon$, noting that $|\mu|$ is finite (see Theorem 6.4 of this chapter), then

$$\mu^-(U - A) \leq |\mu|(U - A) = |\mu|(U) - |\mu|(A) < \epsilon,$$

and so

$$\mu^-(U) = \mu^-(A) + \mu^-(U - A) < \mu^-(A) + \epsilon.$$

The outer regularity of μ^- follows.

And if, $A \in \mathcal{B}(X)$, $\epsilon > 0$, and $K \subset A$ (K compact) is such that $|\mu|(K) > |\mu|(A) - \epsilon$, i.e. $|\mu|(A - K) < \epsilon$, then $\mu^-(A - K) < \epsilon$, and so

$$\mu^-(A) = \mu^-(A - K) + \mu^-(K) < \epsilon + \mu^-(K).$$

The inner regularity of μ^- holds too.

Thus, μ^- is regular and $(a) \Rightarrow (b)$ is proved.

$(b) \Rightarrow (c)$ is trivial.

$(c) \Rightarrow (a)$ If $\mu_1, ..., \mu_n$ are finite positive regular Borel measures, $\mu = \sum\limits_{i=1}^{n} \alpha_i \mu_i$, $0 \neq \alpha_i \in \mathbb{C}$, then $|\mu| \leq \sum\limits_{i=1}^{n} |\alpha_i| \mu_i$. In fact, $\forall\ E \in \mathcal{B}(X)$, $|\mu(E)| \leq \sum\limits_{i=1}^{n} |\alpha_i| \mu_i(E)$, $\sum\limits_{i=1}^{n} |\alpha_i| \mu_i$ is clearly a positive measure. For any partition $E = \bigcup\limits_{i=1}^{+\infty} E_i$, we have

$$\sum_{i=1}^{+\infty} |\mu(E_i)| \leq \sum_{i=1}^{+\infty} \sum_{j=1}^{n} |\alpha_j| \mu_j(E_i) = \sum_{j=1}^{n} \sum_{i=1}^{+\infty} |\alpha_j| \mu_j(E_i) = \sum_{j=1}^{n} |\alpha_j| \mu_j(E).$$

So, we get $|\mu|(E) = \sup \sum\limits_{i=1}^{+\infty} |\mu(E_i)| \leq \sum\limits_{j=1}^{n} |\alpha_j| \mu_j(E)$.

Since μ_j are regular positive and finite, $\sum\limits_{j=1}^{n} |\alpha_j| \mu_j$ is also regular. In fact, for any $A \in \mathcal{B}(X)$, $\epsilon > 0$, there exists $U_j \supset A$ such that $\mu_j(U_j) < \mu_j(A) + \frac{\epsilon}{n|\alpha_j|}$. Let $U = \bigcap\limits_{i=1}^{+\infty} U_j$, then $\mu_j(U) \leq \mu_j(A) + \frac{\epsilon}{n|\alpha_j|}$ and

$$\sum_{j=1}^{n} |\alpha_j| \mu_j(U) < \sum_{j=1}^{n} |\alpha_j| (\mu_j(A) + \frac{\epsilon}{n|\alpha_j|}) = \sum_{j=1}^{n} |\alpha_j| \mu_j(A) + \epsilon.$$

The outer regularity of $\sum\limits_{j=1}^{n} |\alpha_j| \mu_j$ follows. The outer regularity of $\sum\limits_{j=1}^{n} |\alpha_j| \mu_j$ is proved similarly. Then as the proof of $(a) \Rightarrow (b)$, we can prove that $|\mu|$ is also regular. Thus $(c) \Rightarrow (a)$ holds. $\square$

So, if $\mu_1, \mu_2 \in \mathcal{M}(X)$, $\alpha, \beta \in \mathbb{C}$, μ_i (i=1,2) are regular. By the above lemma, we have $\mu_i = \sum\limits_{j=1}^{m_i} \alpha_j^i \mu_j^i$, μ_j^i are finite regular positive. Then $\alpha\mu_1 + \beta\mu_2$ is still a finite linear combination of several finite positive regular Borel measures. Thus $\alpha\mu_1 + \beta\mu_2$ is regular and $\alpha\mu_1 + \beta\mu_2 \in \mathcal{M}(X)$. $\mu \equiv 0 \in \mathcal{M}(X)$, which is the zero of $\mathcal{M}(X)$. So, $\mathcal{M}(X)$ is a linear space.

That $\|\mu\| = |\mu|(X)$ is a norm is proved easily.

$$\|\mu\| = 0 \Leftrightarrow |\mu|(X) \Leftrightarrow \mu = 0,$$

$$\|\lambda\mu\| = |\lambda\mu|(X) = \sup \sum |\lambda\mu(E_i)| = |\lambda| \sup \sum |\mu(E_i)| = |\lambda||\mu|(X) = \lambda\|\mu\|,$$

$$\begin{aligned}\|\mu + \nu\| &= |\mu + \nu|(X) = \sup \sum |(\mu + \nu)(E_i)| \leq \sup \sum (|\mu(E_i)| + |\nu(E_i)|) \\ &\leq \sup \sum |\mu(E_i)| + \sup \sum |\nu(E_i)| = |\mu|(X) + |\nu|(X) = \|\mu\| + \|\nu\|.\end{aligned}$$

So, $(\mathcal{M}(X), \|.\|)$ is a normed linear space.

Let $\{\mu_n\}$ be a Cauchy sequence in $(\mathcal{M}(X), \|.\|)$. For any $A \in \mathcal{B}(X)$, we have

$$|\mu_n(A) - \mu_m(A)| = |(\mu_n - \mu_m)(A)| \leq |\mu_n - \mu_m|(A) \leq |\mu_n - \mu_m|(X) = \|\mu_n - \mu_m\|.$$

So, $\{\mu_n(A)\}$ is a Cauchy sequence in $\mathbb{C}$. Hence, $\mu(A) := \lim\limits_{n\to+\infty} \mu_n(A)$ exists in $\mathbb{C}$.

We need to check that μ is a complex Borel regular measure on X. Notice that, $\forall\ \epsilon > 0$, there exists $N > 0$, if $n, m > N$, then $\|\mu_n - \mu_m\| < \epsilon$. So, $\forall\ A \in \mathcal{B}(X)$, $|\mu_n(A) - \mu_m(A)| \leq \|\mu_n - \mu_m\| < \epsilon$. Letting $n \to +\infty$, we get $|\mu(A) - \mu_m(A)| \leq \epsilon$ for all $m > N$ and $A \in \mathcal{B}(X)$. So, $\mu_n(A)$ uniformly converges to $\mu(A)$ for $A \in \mathcal{B}(X)$.

$\forall\ E \in \mathcal{B}(X)$, if $E = \bigcup\limits_{i=1}^{+\infty} E_i$, $E_i \in \mathcal{B}(X)$, $E_i \cap E_j = \emptyset$ if $i \neq j$, then

$$\mu(\bigcup_{i=1}^{+\infty} E_i) = \lim_{n\to+\infty} \mu_n(\bigcup_{i=1}^{+\infty} E_i) = \lim_{n\to+\infty} \sum_{i=1}^{+\infty} \mu_n(E_i) = \sum_{i=1}^{+\infty} \lim_{n\to+\infty} \mu_n(E_i) = \sum_{i=1}^{+\infty} \mu(E_i).$$

The last second equality holds since $\mu_n(A)$ uniformly converges to $\mu(A)$ for $A \in \mathcal{B}(X)$. So, μ is a complex measure (Borel measure)

To prove that $\|\mu - \mu_m\| \to 0$ as $m \to +\infty$, let $X = \bigcup\limits_{i=1}^{+\infty} E_i$ be a partition, then for $m > N$, we have

$$\begin{aligned}\sum_{i=1}^{+\infty} |(\mu - \mu_m)(E_i)| &= \sum_{i=1}^{+\infty} |\mu(E_i) - \mu_m(E_i)| = \sum_{i=1}^{+\infty} \lim_{n\to+\infty} |\mu_n(E_i) - \mu_m(E_i)| \\ &= \lim_{n\to+\infty} \sum_{i=1}^{+\infty} |\mu_n(E_i) - \mu_m(E_i)| \leq \limsup_{n\to+\infty} \sum_{i=1}^{+\infty} |\mu_n - \mu_m|(E_i) \\ &\leq \limsup_{n\to+\infty} |\mu_n - \mu_m|(X) = \limsup_{n\to+\infty} \|\mu_n - \mu_m\| \leq \epsilon.\end{aligned}$$

The third equality holds since $\mu_n(A)$ uniformly converges to $\mu(A)$ for $A \in \mathcal{B}(X)$. So, $\|\mu - \mu_m\| \leq \epsilon$ if $m > N$, which yields that $\|\mu - \mu_m\| \to 0$ as $m \to +\infty$.

Now we prove that $|\mu|$ is regular. $\forall\ A \in \mathcal{B}(X)$, $\forall\ \epsilon > 0$, there exists $m > 0$ such that $\|\mu - \mu_m\| < \frac{\epsilon}{2}$. Since μ_m is regular, there exists an open set $U \supset A$ such that $|\mu_m|(U - A) < \frac{\epsilon}{2}$. Therefore,

$$\begin{aligned}|\mu|(U - A) &= |\mu - \mu_m + \mu_m|(U - A) \leq |\mu - \mu_m|(U - A) + |\mu_m|(U - A) \\ &\leq |\mu - \mu_m|(X) + \frac{\epsilon}{2} = \|\mu - \mu_m\| + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

So, $|\mu|$ has outer regularity. The inner regularity is proved similarly. Thus, $\mu \in \mathcal{M}(X)$. $\square$

Remark: 1. The above proof shows that $\tilde{\mathcal{M}}(X)$ of all complex measures on a locally compact Hausdorff space X is a Banach space if $\|\mu\| = |\mu|(X)$. $(\mathcal{M}(X), \|.\|)$ of all complex regular Borel measures is a closed subspace of $\tilde{\mathcal{M}}(X)$, and $\bar{\mathcal{M}}(X)$ of all real signed measures is also a closed subspace of $\tilde{\mathcal{M}}(X)$.
2. Notice that Theorem 2.18 (see page 48). We use it to prove that $|\mu|$, the total variation of a complex measure μ, is regular. Specially for $X = \mathbb{R}^N$, any complex measure is regular since $|\mu|(X) < +\infty$. $\square$

11, Suppose μ is a positive measure on X, $\mu(X) < +\infty$, $f_n \in L^1(\mu)$ for $n = 1, 2, 3, ...$, $f_n(x) \to f(x)$ a.e., and there exists $p > 1$ and $C < \infty$ such that $\int_X |f_n|^p d\mu < C$ for all n, prove that

$$\lim_{n\to+\infty} \int_X |f - f_n| d\mu = 0.$$

Hint: $\{f_n\}$ is uniformly integrable.

Solution: Since $f_n(x) \to f(x)$ a.e. and $\int_X |f_n|^p d\mu < C$ for all n, then Fatou's lemma implies

$$\int_X |f|^p d\mu \le \liminf_{n\to+\infty} \int_X |f_n|^p d\mu \le C.$$

So, $f \in L^p(\mu)$ and $|f|^p \in L^1(\mu)$. By the absolute continuity of Lebesgue integrals, $\forall\ \epsilon > 0$, there exists $\delta > 0$ such that $\int_E |f| d\mu < \frac{\epsilon}{2}$ whenever $\mu(E) < \delta$. If $\mu(E) < (\frac{\epsilon}{2})^q / 2C^{\frac{1}{p}}$, then

$$\int_E |f - f_n| d\mu \le (\int_X |f - f_n|^p d\mu)^{\frac{1}{p}} (\mu(E))^{\frac{1}{q}} \le 2C^{\frac{1}{p}} (\mu(E))^{\frac{1}{q}} < \epsilon,$$

where $\frac{1}{q} + \frac{1}{p} = 1$. So, $(f - f_n)$ is uniformly integrable on X. Since $\mu(X) < +\infty$, then by Vitali Theorem we get that

$$\lim_{n\to+\infty} \int_X |f - f_n| d\mu = 0.$$

Chapter 6 Complex Measure

5, Suppose Z consists of two points a and b; define $\mu(\{a\}) = 1$, $\mu(\{b\}) = \mu(Z) = +\infty$, and $\mu(\phi) = 0$. It is true , for this μ, that $L^\infty(\mu)$ is the dual space of $L^1(\mu)$?
Proof: It is easy to see that $(Z, 2^Z, \mu)$ is a positive borel measure (if Z is discrete topological space) If $f \in L^1(\mu)$, then $+\infty > \int |f| d\mu = |f(a)|\mu(\{a\}) + |f(b)|\mu(\{b\}) = |f(a)| + |f(b)|(= \infty)$. $(\mu(\{a\}) = 1, \mu(\{b\}) = +\infty)$. So we must have $f(b) = 0$, and $|f(a)| < +\infty$. Let $i : L^1(\mu \to \mathbb{C})$ be defined as $i(f) = f(a)$. i is one to one. Then i is a linear map from $L^1(\mu)$ and in fact an nun singular one. Let $(L^1(\mu))'$ be the dual space of $L^1(\mu)$. Then L^1 introduces a non singular map $i^* : (L(\mu))' \to \mathbb{C}' = \mathbb{C}$. In fact $\forall\ \ell \in (L'(\mu))'$, and $x \in \mathbb{C}$, define $(i^*(\ell), x) = (\ell, (i^{-1}(x)))$, then $i^*(\ell)$ is in the dual space $\mathbb{C}'$ of $\mathbb{C}$.
Hence $\exists!\ \alpha_\ell \in \mathbb{C}$ (Riesz representation for Hilbert space), s.t. $(i^*(\ell), x) = \alpha_\ell \cdot x$, i.e. $(\ell, (i^{-1}(x))) = \alpha_\ell \cdot x$, or $(\ell, (f)) = \alpha_\ell \cdot f(a)$.

$\forall x \in \mathbb{C}, \exists!\ f \in L^1(\mu)$, s.t. $f(a) = x$.
So

$$\ell(f) = \alpha_\ell \cdot f(a) = \int_Z \alpha_\ell f(x) d\mu, \quad \forall f \in L^1(\mu)$$

, $\ell \leftrightarrow \alpha_\ell$ is one to one linear isometry. $\alpha_\ell \in L^\infty(\mu)$, so $(L^1(\mu))' = L^\infty(\mu)$.

$$\|\ell\| = |\alpha_\ell| = \|\alpha_\ell\|_\infty.$$

$$\sup_{\|f\|_1=1} \{(\ell, f)\} = \sup_{|f(a)|=1} |\alpha_\ell||f(a)| = |\alpha_\ell|.$$

□

6, Suppose $1 < p < \infty$ and prove that $L^q(\mu)$ is the dual space of $L^p(\mu)$ even if μ is not $\sigma-$finite (As usual, $1/p + 1/q = 1$)

Proof: Let F be a bounded linear functional on $L^p(\mu)$ with $1 < p < \infty$. We need to show that there is a unique element $g \in L^q$ such that $F(f) = \int fg d\mu$, and $\|F\| = \|g\|_{L^q}$.

If $E \subset Z$ is any measurable set of $\sigma-$finite measure , then $\forall\ f \in L^p(E, \mu)$ we could extent any $f \in L^p(E, \mu)$ by zero outside E to get a $\tilde{f} \in L^P(\mu)$. Since $F = L^p(\mu) \to \mathbb{C}$ is bounded linear map. Define $F_E = L^p(E, \mu) \to \mathbb{C}$, $F_E(f) = F(\tilde{f})$, then F_E is also a bounded linear operator on $L^p(E, \mu)$. In fact, F_E is clearly linear on $L^p(E, \mu)$. Because we may view $L^p(E, \mu)$ as a subspace of $L^p(\mu)$

$$\forall f \in L^p(E, \mu), \quad \tilde{f} = \begin{pmatrix} f & E \\ 0 & E^c \end{pmatrix}.$$

If $f_n \to f$ in $L^P(E, \mu)$, then the corresponding $\tilde{f_n}$, $\tilde{f}$ satisfies

$$\|\tilde{f_n} - \tilde{f}\|_{L^P(z)} = \|f_n - f\|_{L^p(E)} \to 0$$

$$\Rightarrow F_E(f_n - f) = F(\tilde{f_n} - \tilde{f}) \to 0$$

Since E is $\sigma-$finite, Theorem 6.16 show that there is a unique $g_E \in L^q(E, \mu)$ such that

$$F(\tilde{f}) = F_E(f) = \int_E g_E f d\mu = \int_Z \tilde{g}_E \cdot \tilde{f} d\mu.$$

In other words $\exists\ g_E \in L^q(\mu)$, s.t. $g_E = 0$ on E^c, and $F(f) = \int g_E f d\mu$ for any $f \in L^p(\mu)$ which vanish outside E.

The uniqueness of g_E implies that if $A \subset E$, then $g_A = g_E$ a.e. on A. In fact, Since $F'(f) = \int g_A f d\mu = \int_A g_A f d\mu$, $\forall f \in L^p(\mu)$, $f = 0$ on A^c, and $f \in L^q(\mu)$, $f = 0$ on A^c, thus $f \in L^q(\mu)$, $f = 0$ on E. So $Ff = \int g_E f d\mu = \int_A g_E f d\mu$, $\forall f \in L^q(\mu)$, $f = 0$ on A^c. By uniqueness, $g_A = g_E\ |_A$ in $L^q(A, \mu)$, So $g_A = g_E$ a.e. on A.

For each E of $\sigma-$finite measure, Set $\lambda(E) = \int |g_F|^q d\mu$. Then for $A \subset E$, we have $\lambda(A) \leq \lambda(E) \leq \|F\|^q$.

$$\begin{aligned}
&Th6.16 \Rightarrow \\
\|g_E\|_{L^q(E,\mu)} &= \|F_E\| = \sup_{f \in L^p(E,\mu)} |F_E(f)| \\
&= \sup_{\substack{f \in L^P(\mu) \\ f \neq 0, f=0 \text{ on } E^c}} \frac{|F_E(f)|}{\|f\|_p} = \sup_{\substack{\tilde{f} \in L^P(\mu) \\ \tilde{f} \neq 0, \tilde{f}=0 \text{ on } E^c}} \frac{|F(\tilde{f})|}{\|\tilde{f}\|_p} \\
&\leq \sup_{\substack{f \in L^P(\mu), \\ f \neq 0}} \frac{|F(f)|}{\|f\|_p} = \|F\|.
\end{aligned}$$

So $\lambda(E) = \|g_E\|_q^q \leq \|F\|^q$. Let $m = \sup\limits_{E\ \sigma-\text{finite}} \lambda(E)$.

Let $\{E_n\}$ be sequence of sets of $\sigma-$finite measure such that $\lambda(E_n)$ tends to m. Then $H = \cup_{n=1}^{+\infty} E_n$. We may assume that $E_1 \subset E_2 \subset \cdots E_n \cdots$. (Or denote $\hat{F}_n = \cup_{i=1}^{n} E_i$, then F_n is also $\sigma-$finite.)

H is a set of $\sigma-$finite measure and by the monotonicity of λ we have $m \geq \lambda(H) \geq \lambda(E_n) \to m$. And so $\lambda(H) = m$.

Let g be defined to be g_H, and H and O elsewhere. Then $g \in L^q$. If E is any set of $\sigma-$finite measure containing H, then $g_E = g_H$, a.e. on H. (It has been proved that $A \subset E \Rightarrow g_A = g_E$ a.e. on A.) But

$$\int |g_E|^q d\mu = \lambda(E) \leq \lambda(H) = m = \int |g_H|^q d\mu,$$
$$\int_H |g_H|^q + \int_{E-H} |g_E|^q = \int_H |g_E|^q d\mu + \int_{E-H} |g_E|^q d\mu \leq \int_H |g_H|^q d\mu.$$

So $\int_{E-H} |g_E|^q d\mu = 0$, then $g_E = 0$ a.e. on $E - H$. Thus $g_E = g$ a.e. on E.

If $f \in L^p$, then we see

$$N = \{x : f(x) \neq 0\} = \cup_{n=1}^{+\infty} \{x : |f(x)| > \frac{1}{n}\} \triangleq \cup_{n=1}^{+\infty} F_n.$$

$$n^p \mu(F_n) \leq \int_{F_n} |f|^p d\mu \leq \int_Z |f|^p d\mu,$$
$$\mu(F_n) < \frac{\int |f|^p d\mu}{n^p}.$$

So N is a set of $\sigma-$finite measure, and so is the set $E = N \cup H$, thus

$$F(f) = \int f g_E d\mu = \int f g d\mu, (\text{by the fact that} f = 0 \text{ on } E^c)$$

$$\alpha = \text{Sig} g = \begin{cases} 1, & g(t) > 0 \\ 0, & g(t) = 0 \\ -1, & g(t) < 0 \end{cases}, \quad \alpha g = |g|.$$

Let $F_n = \{x : |g(x)| \leq n\}$, $f = \chi_{E_n} |g|^{q-1} \alpha$.
Then $|f|^p = |g|^q$, on E_n, $f \in L^\infty(\mu)$.

$$\begin{aligned} \int_{E_n} |g|^q d\mu &= \int_Z f g d\mu = F(f) \leq \|F\| \left(\int (\chi_{E_n} |g|^{q-1})^p \right)^{\frac{1}{p}} \\ &= \|F\| \left(\int |g|^q \right)^{\frac{1}{p}} \end{aligned}$$

$$\int_Z \chi_{E_n} |g|^q d\mu \leq \|F\|^q.$$

monotone convergence theorem $\Rightarrow$

$$\int |g|^q d\mu = \lim_{n\to\infty} \int \chi_{E_n} |g|^q d\mu \leq \|F\|^q.$$

$$\Rightarrow \|g\|_q \le \|F\|,$$

$$F(f) = \int fgd\mu \le \|f\|_p\|g\|_q,$$
$$\Rightarrow \|F\| \le \|g\|_q,$$

thus $\|F\|_q = \|g\|_q$.

Uniqueness of g is trivial.

Note that $1 < p < \infty$, so that $\lambda(E) = \int |g_E|^q d\mu \le \|F\|^q$, and $1 < q < \infty$, $\lambda(E) \le \lambda(H)$. $(E \supset H) \Rightarrow g_E = 0$ a.e..

If $p = 1$, then $q = +\infty$, similarly it can be proved.

$$\tilde{\lambda}(E) = \|g_E\|_\infty \le \|F\|,$$

$\lambda(E) \le \lambda(H) \Rightarrow g_E = 0$ on $E - H$.

It follows that $\sup_{E-H} |g_E| \le \sup_H |g_H|$. □

4, Suppose $1 \le P \le \infty$, and q is the exponent conjugate to p. Suppose μ is a positive $\sigma-$finite measure and g ia a measurable function such that $fg \in L^1(\mu)$, for every $f \in L^P(\mu)$. Prove that then $g \in L^q(\mu)$.

proof:(i)$p = \infty$. We want to show that $g \in L^1(\mu)$. Let

$$f(x) = \begin{cases} \frac{\bar{g(x)}}{|g(x)|}, & \text{if} g(x \ne 0); \\ 0, & \text{if} g(t) = 0. \end{cases}$$

then $f \in L^\infty(\mu)$. and $\int_Z |g| d\mu = \int_Z fgd\mu < +\infty$, so $g \in L^1(\mu)$.

In fact, set $f_n(x) = \frac{\bar{g(x)}}{|g(x)| + \frac{1}{n}}$, then $f_n(t)$ is measurable, $f_n \to f(x)$ on Z.

(ii) $1 \le p < +\infty$. Since μ is a positive $\sigma-$finite measure. We may assume that $Z = \cup_{n=1}^{+\infty}$, $\mu(E_n) < +\infty$, and $E_n \subset E_{n+1}$ for all $n \in N$.

We claim that under the condition, $\mu\{x : |g(x)| = +\infty\} = c$, Otherwise,

$$0 < \mu(x; |g(x)| = +\infty) = \mu\left(\cup_{n=1}^{+\infty} E_n\{x : |g(x)| = +\infty\}\right) = \lim_{n \to +\infty} \mu(E_n\{x : |g(x)| = +\infty\}).$$

thus $\exists N$ s.t. $0 < \mu(E_N(x : |g(x)| = +\infty)) < \mu(E_N) < +\infty$.

Let $E = E_N x : |g(x)| = +\infty$, $h = \chi_E \cdot f$, where

$$f(x) = \begin{cases} \frac{\bar{g(x)}}{|g(x)|}, & g(x); \ne 0 \\ 0, & g(x) = 0. \end{cases}$$

then

$$\int_Z |h|^p \le \int_Z |\chi_E|^p |f|^p d\mu \le \int_Z \chi_E d\mu = \mu(E)$$
$$\mu(E_N\{x : |g(x)| = +\infty\}) < +\infty.$$

so $h \in L^1$.

$$\int_E |hg| d\mu = \int hgd\mu < +\infty.$$

But $\int_E |g| d\mu = \mu(E) \cdot \infty = +\infty$, so $\mu(E) > 0$. a contradiction.

Let $g_n(x) = \chi_{G_n(x)} g(x)$, where $G_n = E_n\{x : |g(x)| \le n\}$. then $|\{|g_n(x)|\}$ is nondecreasing in n for all $x \in Z$, $|g_n(x)| \to |g(x)|$ as $n \to +\infty$, and

$$\int |g_n|^q d\mu = \int_{G_n} |g|^q d\mu \le nm(G_n) \le nm(E_n) < +\infty$$

So $g_n \in L^q(\mu)$.

Define a linear functional on $L^p(\mu)$ as followings

$$\begin{aligned} T_n :\ L^p(\mu) &\quad \mapsto \mathbb{C} \\ f &\quad \mapsto T_n(f) \triangleq \int_Z \tilde{g}_n f d\mu, \quad F \in L^p(\mu). \end{aligned}$$

T_n is well defined and it is a bounded linear functional with $\|T_n\| \le \|\bar{g}_n\|_q = \|g_n\|_q$.
Actually we have $\|T_n\| = \|g_n\|_q$. Indeed, if $1 < p < \infty$ let $f = \|g_n\|_q^{-\frac{q}{p}} |q_n|^{q-2} g_n$, then

$$\int_Z |f|^p d\mu = \|g_n\|^{-q} \int_Z |g_n|^{(q-1)p} d\mu = \|g_n\|^{-q} \int |g_n|^q d\mu = 1.$$

as $(q-1)p = q(1-\frac{1}{q})p = q \cdot \frac{1}{p} p = q$

$$\begin{aligned} \|T_n\| &= \sup_{f \in L^p(\mu)} (T_n(f)) \ge T_n(f) = \|g_n\|_q^{-\frac{q}{p}} \int \bar{g}_n |g_n|^{q-2} g_n \\ &= \|g_n\|^{-\frac{q}{p}} \int_Z |g_n|^q d\mu = \|g_n\|_q^{-\frac{q}{p}+q} \\ &= \|g_n\|_q^{q(1-\frac{1}{p})} = \|g_n\|_q^{q\frac{1}{q}} = \|g_n\|_q \end{aligned}$$

So $\|T_n\| = \|g_n\|_q$.

If $p = 1$, if $g_n = 0$ a.e. $T_n(f) = 0 \Rightarrow \|T_n\| = 1 = \|g_n\|_\infty$, If $\|g_n\|_\infty > 0$. Let $\epsilon > 0$ be small enough s.t. $\|g_n\|_\infty - \epsilon > 0$, then

$$\mu(\{x : |g_n(x)| > \|g_n\|_\infty - \epsilon\}) > 0.$$

Otherwise $\|g_n\|_\infty \le \|g_n\|_\infty - \epsilon$, a contradiction.

As $|g(x)| = 0$ for $x \notin G_n$. We see that $A = \{x \in G_n, |g(x)| > \|g_n\|_\infty - \epsilon\}$ satisfies $\mu(A) > 0$. (Or $\mu(A) = 0 \Rightarrow |g_n| = 0$, $\|g\|_\infty = 0!$).
Let

$$f = \frac{1}{\mu(A)} \chi_A \frac{\bar{g}}{|g|} = \begin{cases} \frac{1}{\mu(A)} \frac{\bar{g}}{|g|}, & g \ne 0,\ x \in A; \\ 0, & g = 0. \end{cases}$$

Notice that $x \in A \Rightarrow |g_n| > 0 \Rightarrow |g| > 0$, $|g_n| = |g| > 0$. then

$$\int_Z |f| d\mu = \frac{1}{\mu(A)} \int_A \frac{|\bar{g}|}{|g|} d\mu = \frac{1}{\mu(A)} \int_A d\mu = 1.$$

and hence $f \in L^1(\mu)$, So

$$\|T_n\| \ge T_n(f) = \frac{1}{\mu(A)} \int_A |g| d\mu \ge (\|g_n\|_\infty - \epsilon) \frac{1}{\mu(A)} \int_A d\mu = \|g_n\|_\infty - \epsilon$$

$\epsilon \to 0 \Rightarrow \|T_n\| \ge \|g_n\|_\infty$, so $\|T_n\| = \|g_n\|_\infty$.
So $\|T_n\| = \|g_n\|_q$ for $1 \le p < \infty$.

Now, for any $f \in L^p(\mu)$, $|\bar{g}_n f| = |g_n||f| \uparrow |gf| \in L^1(\mu)$.
Lebesque's Dominated Convergence Theorem implies

$$\begin{aligned} |T_n(f) - \int_Z f\bar{g} d\mu| &\Leftarrow |\int_Z f(\bar{g}_n - \bar{g}) d\mu| \\ &\leq \int_Z |f|(|\bar{g}| - |\bar{g}_n|) d\mu \to 0 \quad \text{as} n \to +\infty \end{aligned}$$

Since $|f|(|\bar{g}| - |\bar{g}_n|) \leq |fg| + |f||g| = 2|f||g| \in L^1$, $|f||\bar{g}_n| = |f||g_n| \uparrow |gf| \Rightarrow |f|(|\bar{g}| - |\bar{g}_n| \to 0$, a.e. $[\mu]$.

So $\exists C(f) < +\infty$, s.t.

$$\sup_n |T_n(f)| \leq C(f) < +\infty.$$

By Banach-Steinlause Theorem (Uniform Boundedness Principle (P98) Th5-8) $\exists C(f) > 0$, s.t. $\|T_n\| \leq C$. So $\|g_n\|_q = \|T_n\| \leq C$, i.e. $\int_Z |g_n|^q d\mu \leq C$.
Fatou's lemma implies $\int_Z |g|^q \leq C$, i.e. $g \in L^q(\mu)$. $\square$

10, Let (X, M, μ) be a positive measure space. Call a set $\Phi \in L^1(\mu)$ uniformly integrable if to each $\epsilon > 0$ corresponds a $\delta > 0$ such that

$$|\int_E f \, d\mu| < \epsilon$$

whenever $f \in \Phi$ and $\mu(E) < \delta$,

(a) Prove that every finite subset of $L^1(\mu)$ is uniformly integrable,

(b) Prove the following convergence theorem of Vitali:

If (i) $\mu(X) < \infty$, (ii) $\{f_n\}$ is uniformly integrable, (iii) $f_n(x) \to f(x)$ a.e. as $n \to +\infty$ and (iv) $|f(x)| < \infty$ a.e., then $f \in L^1(\mu)$ and

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu| = 0.$$

Suggestion: Use Egoroff's theorem.

(c) Show that (b) fails if μ is Lebesgue measure on $(-\infty, +\infty)$, even if $\{\|f_n\|_1\}$ is assumed to be bounded. Hypothesis (i) can therefore not be omitted in (b).

(d) Show that hypothesis (iv) is redandant in (b) for some μ (for instance, for Lebesgue measure as a bounded interval but that there are finite measure for which the omission of (iv) would make (h) false.)

(e) Show that Vitali's theorem implies Lebesgue's dominate convergence theorem, for finite measure spaces. Construct an example in which Vitali's theorem applies although the hypotheses of Lebesgue's theorem do not hold.

(f) Construct a sequence $\{f_n\}$ say on $[0, 1]$, so that $f_n(t) \to 0$ for every n, $\int_E f_n \, d\mu \to 0$, but $\{f_n\}$ is not uniformly integrable (with respect to Lebesgue measure).

(g) However, the following converse of Vitali's theorem is true:

If $\mu(Z) < \infty$, $f_n \in L^1(\mu)$ and

$$\lim_{n\to\infty} \int_E f_n \, d\mu$$

exists for every $E \in M$, then f_n is uniformly integrable.

Prove this by completing the following outline.

Define $\rho(A, B) = \int |\chi_A - \chi_B| \, d\mu$. Then (M, ρ) is a complete metric space and $E \to \int_E f_n \, d\mu$ is continuous for each n. If $\epsilon > 0$, there exist E_0, δ, N (Exercise 13 Chap.5) so that

$$|\int_E (f_n - f_N) \, d\mu| < \epsilon \quad \text{if } \rho(E, E_0) < \delta, \ n > N. \tag{8}$$

If $\mu(A) < \delta$, (1) holds with $B = E_0 - A$ and $C = E_0 \cup A$ in place of E. This (1) holds with A in place of E and 2ϵ in place of ϵ. Now apply (a) to $\{f_1, \cdots f_N\}$: There exists $\delta' > 0$ such that

$$|\int_A f_n \, d\mu| < 3\epsilon \quad \text{if } \ \mu(A) < \delta, \quad n = 1, 2, 3, \cdots .$$

Solution: (a). Let $\{f_1, \cdots f_N\} \subset L^1(\mu)$. Then by Exercise 12 of Chap.1, (absolute continuity of Lebesgue integral), $\forall \epsilon > 0, \ \forall \ i, \ \exists \ \delta_i > 0$ such that $E \in M$, $\mu(E) < \delta_i \Rightarrow$

$$\int_E |f| \, d\mu| < \epsilon.$$

Let $\delta = min(\delta_1, \delta_2, \cdots, \delta_N)$, then $\delta > 0$ and if $E \in M$, $\mu(E) < \epsilon (\leq \delta_i)$, then $\forall \ i, \ 1 \leq i \leq N$,

$$|\int_E f_i \, d\mu| \leq \int_E |f_i| \, d\mu| < \epsilon.$$

So $\{f_1, \cdots f_N\}$ is uniformly integrable.

(b) First, let us claim that, if $\Phi \subset L^1(\mu)$ is uniformly integrable, then we have a strengthened form of uniformly integrable: $\forall \ \epsilon > 0, \exists \ \delta > 0$ and if $E \in M$, $\mu(E) < \delta$ implies

$$\int_E |f| \, d\mu| < \epsilon, \quad \forall \ f \in \Phi.$$

In fact, $\forall \ \epsilon > 0, \exists \ \delta > 0$ and if $E \in M$, $\mu(E) < \delta$, we have

$$\frac{\epsilon}{4} > |\int_E f \, d\mu| = |\int_E u \, d\mu + i \int_E v \, d\mu|, \qquad f = u + iv.$$

So

$$|\int_E u \, d\mu|^2 < \frac{\epsilon^2}{16}, \quad |\int_E v \, d\mu|^2 < \frac{\epsilon^2}{16}.$$

Let

$$A^+ = \{x \in E, u \geq 0\}, \qquad A^- = \{x \in E, u < 0\},$$
$$B^+ = \{x \in E, v \geq 0\}, \qquad B^- = \{x \in E, v < 0\}.$$

Then

$$\mu(A^\pm) \leq \mu(E) < \delta, \qquad \mu(B^\pm) \leq \mu(E) < \delta.$$

Since $u = u^+ - u^-$, $v = v^+ - v^-$, then

$$\begin{aligned}
\int_E |f| \, d\mu &\leq \int_E (|u| + |v|) \, d\mu = \int_E |u| \, d\mu + \int_E |v| \, d\mu \\
&= \int_{A^+} |u| \, d\mu + \int_{A^-} |u| \, d\mu + \int_{B^+} |v| \, d\mu + \int_{B^-} |v| \, d\mu \\
&= |\int_{A^+} u \, d\mu| + |\int_{A^-} u \, d\mu| + |\int_{B^+} v \, d\mu| + |\int_{B^-} v \, d\mu| \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon.
\end{aligned}$$

By condition (iii) and (iv), $\exists\ E \in M$, $\mu(E) = 0$ s.t.

$$f_n(x) \to f(x),\ \forall x \in E^c, \quad |f(x)| < +\infty,\ \forall x \in E^c.$$

Let

$$\tilde{f}_n(x) = \begin{cases} f_n(x) & x \in E^c; \\ 0 & x \in E. \end{cases} \qquad \tilde{f}(x) = \begin{cases} f(x) & x \in E^c; \\ 0 & x \in E. \end{cases}$$

Then

$$\tilde{f}_n(x) \to \tilde{f}(x),\ \forall x \in E, \quad |\tilde{f}(x)| < +\infty,\ \forall x \in E.$$

Since $\{f_n\} \subset L^1(\mu)$, $\{f_n\}$ are complex measurable, hence $\tilde{f}_n(x)$ are complex measurable, $\tilde{f}_n = \chi_{E^c} f_n(x)$. So page 15 Corollaries (a).

$\tilde{f}(x)$ is complex measurable. As $\tilde{f}(x) = f(x)$ on E^c and $det|E^c|$. $f(x)$ is measurable on E^c, $\tilde{f}(x) = f$ a.e. on X. We can change the value of f on E, so WL.OG. we may assume

$$f_n(x) \to f(x),\ \forall x \in Z, \quad |f(x)| < +\infty,\ \forall x \in Z,$$

and $f(x)$ is complex measurable.

We show next that $f \in L^1(\mu)$.

Since

$$f_n(x) \to f(x),\ \forall x \in Z, \quad |f(x)| < +\infty,\ \forall x \in Z,$$

Page 74, Exexcise 18 (a), Chapter 3 gives that $f_n \to f$ in measure. Since $\{f_n\}$ is uniformly integrable, $\forall\ i$, $\exists\ \delta_i$, $0 < \delta_i \le \frac{1}{2}i$, such that $A \subset Z$, $\mu(A) < \delta$, so that

$$\int_A |f_n(x)|\, d\mu < \frac{1}{2^{i+2}}, \qquad n \ge 1.$$

Let

$$E_n(i) = \{x \in Z : |f_n(x) - f(x)| \ge \frac{1}{2^i(\mu(z)+1)}\},$$

$$E_{m,n}(i) = \{x \in Z : |f_n(x) - f_m(x)| \ge \frac{1}{2^{i+1}(\mu(z)+1)}\}.$$

By $f_n \to f$ in measure, $\exists \delta_i$ such that $n \ge N_i$ and $m\ E_n(i) < \frac{\delta_i}{2}$. As

$$E_{m,n}(i) \subset \cup E_m(i),$$

so $m > n > N_i$ implies that $m\ E_{m,n}(i) < \delta_i$. So

$$\begin{aligned} \int_Z |f_m - f_n|\, d\mu &\le \int_{Z - E_{m,n}(i)} |f_m(x) - f_n(x)|\, d\mu + \int_{E_{m,n}(i)} |f_m(x)|\, d\mu + \int_{E_{m,n}(i)} |f_n(x)|\, d\mu \\ &< \frac{1}{2^{i+1}(\mu(z)+1)} \mu(E - E_{m,n}(i)) + \frac{1}{2^{i+2}} + \frac{1}{2^{i+2}} \\ &\le \frac{1}{2^{i+1}} + \frac{1}{2^{i+1}} = \frac{1}{2^i}. \end{aligned}$$

Then $\{f_n\}$ is a Candy sequence in $L^1(\mu)$. Since $L^1(\mu)$ is complete, there is a $\tilde{f} \in L^1(\mu)$ such that $\|f_n - \tilde{f}\| \to 0$. From Riesz Theorem we have $f_{n_i} \to \tilde{f}$ a.e. in Z. So by $f_n \to f$ a.e., we have $\tilde{f} = f$ a.e. in Z. Then $f \in L^1(\mu)$ and $\|f_n - f\|_1 \to 0$, i.e.

$$\lim_{n\to\infty} \int_Z |f_n - f|\, d\mu = 0.$$

We can also use Egoroff Theorem.

$\{f_n\}$ is uniformly integrable, then $\forall\ \epsilon > 0$, $\exists\ \delta > 0$, $\forall A \in m$ with $\mu(A) < \delta$ implies that $\int_A |f|\, d\mu < \epsilon$.

Fatou's Lemma and $f_n \to f$ pointwise on Z implies $\int_A |f|\, d\mu < \epsilon$.

For $\epsilon = 1$, $\exists\ \delta(1) = \delta_1 > 0$, $\forall A \in m$ with $\mu(A) < \delta_1$,

$$\int_A |f_n|\, d\mu < 1.$$

For the above $\delta_1 > 0$, Egoroff Theorem implies that $\exists E_1 = E_{\delta_1} \in m$, $\mu(E_{\delta_1}) < \delta_1$,

$$f_n(x) \rightrightarrows f(x) \text{ uniformly in } Z_{E_{\delta_1}}.$$

So $\exists\ N$ such that $|f_N(x) - f(x)| \le 1$, $\forall x \in Z - E_{\delta_1}$,

$$\begin{aligned}
\int_Z |f|\, d\mu &= \int_Z |f - f_N|\, d\mu + \int_Z |f_N|\, d\mu \\
&= \int_{Z-E_{\delta_1}} |f - f_N|\, d\mu + \int_{E_{\delta_1}} |f|\, d\mu + \int_{E_{\delta_1}} |f_N|\, d\mu + \int_Z |f_N|\, d\mu \\
&\le \mu(Z) + 1 + 1 + \int_Z |f_N|\, d\mu < +\infty.
\end{aligned}$$

Then $f \in L^1(\mu)$. And we have, $\forall\ \epsilon > 0$, $\exists\ \delta(\epsilon) > 0$, $\forall A \in m$ with $\mu(A) < \delta$ implies that

$$\int_A |f_n|\, d\mu < \frac{\epsilon}{4}, \qquad \int_A |f|\, d\mu < \frac{\epsilon}{4}.$$

By Egoroff Theorem, for ϵ and $\delta(\epsilon) > 0$ given above, $\exists E_\delta \in m$ such that $\mu(E_\delta) < \delta$ and

$$f_n(x) \rightrightarrows f(x) \ \text{ uniformly on } Z - E_\delta.$$

$\exists\ N = N(E_{\delta(\epsilon)}) = N(\epsilon) > 0$ such that

$$n > N \Rightarrow\ |f_n(x) - f(x)| < \frac{\epsilon}{4\mu(Z)}, \quad \forall\ x \in\ Z - E_\delta.$$

Then

$$\begin{aligned}
\int_Z |f - f_n|\, d\mu &\le \int_{Z-E_\delta} |f - f_n|\, d\mu + \int_{E_\delta} |f|\, d\mu + \int_{E_\delta} |f_n|\, d\mu \\
&< \frac{\epsilon}{4\mu(Z)}\mu(Z - E_\delta) + \frac{\epsilon}{4} + \frac{\epsilon}{4\mu} \\
&\le \frac{3\epsilon}{4\mu} < \epsilon.
\end{aligned}$$

Then $\lim\limits_{n\to\infty} \int_Z |f - f_n|\, d\mu = 0.$

(c) Let μ be the Lebesgue measure in $(-\infty, +\infty)$,

$$\tilde{f}_n(x) = \begin{cases} \frac{1}{n}, & 0 \le x < n; \\ 0, & n \le x < \infty \text{ or } x < 0. \end{cases}$$

Then $0 \le f_n \in L^1(\mu)$ and

$$\int_{-\infty}^{+\infty} f_n(x)\, d\mu = \int_0^{+\infty} f_n(x)\, d\mu + \int_0^n f_n(x)\, d\mu + \int_0^n \frac{1}{n}\, d\mu = 1.$$

So

$$\|f_n\|_1 = 1.$$

$\{f_n\}$ is uniformly integrable in $(-\infty, +\infty)$: $\forall\ \epsilon > 0$, let $\delta = \frac{\epsilon}{2}$, then if $A \subset (-\infty, +\infty)$ and A measurable $mA < \frac{\epsilon}{2}$, then

$$|\int_A f_n(x)\, d\mu| \le \int_A d\mu = mA < \epsilon.$$

Let $f = 0$, then $f_n \to f$ in measure and $f_n \to f$ pointwise in $(-\infty, +\infty)$. In fact, $f_n \to f$, $\forall\ x \in\ Z$ is clear and $\forall\ \sigma > 0$, $\exists\ N$, $n > N$ implies that $\frac{1}{n} < \sigma$, then $n > N$,

$$\{x : |f_n(x) - f(x)| \le \sigma\} = \emptyset.$$

Then

$$\mu\{x : |f_n(x) - f(x)| \le \sigma\} = 0 \text{ if } n > N.$$

So the condition of Vitali Th are satisfied except $\mu(-\infty, +\infty) = \infty$. However

$$1 = \int_{-\infty}^{+\infty} f_n(x)\, dx \not\to\ 0 = \int_{-\infty}^{+\infty} f(x)\, dx.$$

(d) First, we give an example which show that condition (iv) $|f(x)| < +\infty$ a.e. in necessary for Vitalli theorem.

Let Z be any set and $x_0 \in Z$, μ is the unit mass concentrated at x_0:

$$\mu(A) = \begin{cases} 1, & x_0 \in A; \\ 0, & x_0 \notin A. \end{cases}$$

$(Z, 2^Z, \mu)$ is then a measure space. μ is even a regular Borel measure if Z, τ is the discrete space.

$\forall\ n$, let

$$\tilde{f}_n(x) = \begin{cases} n, & x = x_0; \\ 0, & x \ne x_0. \end{cases} \qquad \tilde{f}(x) = \begin{cases} +\infty, & x = x; \\ 0, & x \ne x. \end{cases}$$

Then $f_n(x) \to f(x)$ pointwise on Z and $\mu(Z) = 1 < +\infty$, $\forall \epsilon > 0$, if $\delta < 1$, then $\forall A \in 2^Z$ with $\mu(A) < \delta$ implies that $x_0 \notin A$. So $\forall\ n$,

$$\int_A |f_n|\, d\mu = \int_A 0\, d\mu = 0 < \epsilon.$$

So $\{f_n\}$ is uniformly integrable. But

$$\int_Z |f|\,d\mu = \int_{x_0} |f|\,d\mu = (+\infty)\mu(x_0) = +\infty.$$

So $f \notin L^1(\mu)$ and

$$\int_Z |f_n - f|\,d\mu = \int_{x_0} |f_n - f|\,d\mu = \int_{x_0} [n - (+\infty)]\,d\mu = +\infty.$$

So the conclusion of Vitalli's Theorem fails. This is the condition $|f(x)| < +\infty$ a.e. no longer holds: $\mu(\{x_0\}) = 1 \neq 0$.

But for some measure μ, the condition (iv) $|f(x)| < +\infty$ a.e. is not necessary.

Claim: Let μ be a measure satisfying the following property: $\forall\, A \in\, m$ with $\mu(A) > 0$, $\forall\, \epsilon > 0$, $\exists\, B \subset A$, $B \in m$, $0 < \mu(B) \leq \epsilon$, then in Vitali's theorem the condition (iv) $|f(x)| < +\infty$ is not needed.

Proof: Assume (i) $\mu(Z) < +\infty$, (ii) $\{f_n\} \subset L^1(\mu)$ be uniformly integrable, (iii) $f_n(x) \to f(x)$ a.e. as $n \to \infty$. We show that $f \in L^1(\mu)$ and

$$\lim_{n\to\infty} \int_Z |f_n - f|\,d\mu = 0.$$

We need only show that $|f(x)| < +\infty$ a.e.. Let

$$E_\infty = \{x \in Z,\ |f(x)| = +\infty\}.$$

Suppose $\mu(E_\infty) > 0$, since $\{f_n\}$ is uniformly integrable, $\exists\, \delta > 0$ such that $A \in\, m$, $\mu(A) < \delta$ implies

$$\int_A |f_n|\,d\mu \leq 1, \quad \forall\, n \in N.$$

For the above given $\delta > 0$, $\exists A \in m$, $A \subset E_\infty$, $0 < \mu(A) < \delta$. So

$$\int_A |f_n|\,d\mu \leq 1, \quad \forall\, n \in N.$$

Fatou's Lemma and $f_n(x) \to f(x)$ a.e. gives that

$$\infty = (+\infty)\mu(A) = \int_A |f|\,d\mu \leq 1, \quad \text{since } \mu(A) > 0,$$

a contradiction. So $\mu(E_\infty) = 0$ and

$$|f(x)| < +\infty \quad \mu \text{ a.e. on Z}.$$

The claim is proved.

Remark: The Lebesgue measure in $\mathbb{R}^N$ satisfies the above property: $\forall E \subset m$, if $m(E) > c$, $f(t) = m(B(0,t) \cap E$, then $f(t)$ is continuous function in $\mathbb{R}^+$.

In fact, $\forall t, s \in [0, \infty)$, suppose $t > s$, then

$$\begin{aligned}
|f(t) - f(s)| &= |m(B(0,t) \cap E) - m(B(0,s) \cap E)| \\
&= m(B(0,t) \cap E) - m(B(0,s) \cap E) = m(B(0,t) \cap E - B(0,s) \cap E) \\
&= m((B(0,t) - B(0,s)) \cap E) \leq m((B(0,t) - B(0,s)) \\
&= w_n(t^n - s^n) \to 0.
\end{aligned}$$

Since $f(0) = 0$, so $\forall \epsilon > 0$, if $m(E) < \epsilon$, then $0 < m(E) < \epsilon$. Notice that $\mathbb{R}^N = \cup_{m=1}^{+\infty} B(0, m)$, then

$$\begin{aligned} m(E) &= m(E \cap \mathbb{R}^N) = m(E \cap \cup_{m=1}^{+\infty} B(0, m)) \\ &= m(\cup_{m=1}^{+\infty} E \cap B(0, m)) = \lim_{m \to \infty} m(B(0, m) \cap E) \\ &= \lim_{m \to \infty} f(m). \end{aligned}$$

If $0 < \epsilon < m(E)$, then when m is large enough, $0 < \epsilon < f(m)$, then $\exists r$ s.t. $f(r) = \epsilon$, that is $m(E \cap B(0, r)) = \epsilon$, but $E \cap B(0, r)) \subset E$, so Lebesgue measure in $\mathbb{R}^N$ satisfies $\mu(E) > 0$, then for $\forall\ \epsilon > 0$, $\exists A \subset E$, $0 < \mu(A) \leq \epsilon$. But unit mass concentrate at x_0, do not satisfy the property: $\mu\{\{x_0\}\} = 1$.

If $\epsilon < 1$, $\mu(A) \leq \epsilon$, $A \subset \{x_0\}$, so we have $A = \emptyset$, then $\mu(A) = 0$.

Dominated Convergence Theorem is the follows.

(e) If $\{f_n\} \subset L^1(\mu)$ satisfies (i) $f_n \to f$ a.e., (ii) $\exists g \in L^1(\mu)$, $|f_n| \leq g$ a.e., then $f_n \to f$ in $L^1(\mu)$.

Now with the additional condition $\mu(x) < +\infty$, we show that it can be deduced from Vitali's theorem.

In fact, by the absolute continuity of Lebesgue integral, Exercise 12 in Chap.1, from the fact that $g \in L^1(\mu)$, $\forall\ \epsilon > 0$, $\exists \delta > 0$ such that $E \in m$, $\mu(E) < \delta$ implies

$$\int_E |g|\, d\mu < \epsilon.$$

So by $|f_n| \leq g$ a.e., we have $\forall n \in N$,

$$\int_E |f_n|\, d\mu \leq \int_E |g|\, d\mu < \epsilon.$$

Hence $\{f_n\}$ is uniformly integral. Also

$$|f_n| \leq g \quad \text{a.e. and} \quad f_n \to f \text{ a.e.}$$

gives that

$$|f(x)| \leq g \quad \text{a.e.}$$

So $|f(x)| < +\infty$ a.e. as $g \in L^1(\mu)$ implies $|g(x)| < +\infty$ a.e.. Thus Vitali's Theorem is applied to get the conclusion.

Remark: We need $\mu(X) < +\infty$. In fact, we have Vitali's Theorem on unbounded domain:

Let (Z, m, μ) be a positive measure space. Let $\{f_n\} \subset L^1(\mu)$ satisfy

(i) $\{f_n\}$ is uniformly integrable on any $B \in m$ with $\mu(B) < +\infty$,

(ii) $f_n \to f$ a.e. on Z,

(iii) $|f(x)| < +\infty$ a.e. on any $B \in m$ with $\mu(B) < +\infty$,

(iv) $\{f_n\}$ is uniformly integrable at infinity in the sense that, $\forall\ \epsilon > 0$, $\exists\ A, B \in m$ such that $Z = A \cup B$ with $\mu(B) < +\infty$, $\int_A |f_n|\, d\mu < \epsilon$, $n = 1, 2, \cdots$.

Then $f \in L^1(\mu)$ and $\lim\limits_{n \to +\infty} \int_Z |f - f_n|\, d\mu = 0$.

Proof: $\forall\ \epsilon > 0$, by (iv), $\exists\ A_\epsilon, B_\epsilon \in m$, $Z = A_\epsilon \cup B_\epsilon$, $\mu(B_\epsilon) < +\infty$,

$$\int_A |f_n|\, d\mu < \frac{\epsilon}{4} \ \text{ for } \ \text{n} = 1, 2, \cdots.$$

(ii) and Fatou's Lemma implies that

$$\int_{A_\epsilon} |f|\, d\mu < \frac{\epsilon}{4}. \tag{9}$$

As $\mu(B_\epsilon) < +\infty$, $\{f_n\}$ satisfies all the conditions of Vitali's Theorem on B_ϵ, so $f \in L^1(B_\epsilon)$ and

$$\lim_{n\to\infty} \int_{B_\epsilon} |f - f_n|\, d\mu = 0.$$

So

$$\int_Z |f|\, d\mu \leq \int_{A_\epsilon} |f|\, d\mu + \int_{B_\epsilon} |f|\, d\mu < +\infty.$$

Thus $f \in L^1(\mu)$ and by (2), $\exists\ N(\epsilon) > 0$, $n > N$ such that

$$\int_{B_\epsilon} |f - f_n|\, d\mu < \frac{\epsilon}{2}.$$

So for $n > N$, we have

$$\begin{aligned} \int_Z |f - f_n|\, d\mu &\leq \int_{B_\epsilon} |f - f_n|\, d\mu + \int_{A_\epsilon} |f - f_n|\, d\mu \\ &< \frac{\epsilon}{2} + \int_{A_\epsilon} |f|\, d\mu + \int_{A_\epsilon} |f_n|\, d\mu \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon. \end{aligned}$$

So

$$\lim_{n\to\infty} \int_Z |f - f_n|\, d\mu = 0$$

is proved. $\square$.

Remark: If μ is Lebesgue measure in $\mathbb{R}^N$, then (iv) in the above is: $\forall\ \epsilon > 0$, $\exists\ N > 0$, when $n > N$,

$$\int_{B(0,N)} |f_n|\, d\mu < \epsilon.$$

The uniformly absolute continuity in infinity in Exercise 12, Chap 1, I had explain in addition that when $f \in L^1(\mu)$, $\forall\ \epsilon > 0$, $\exists\ N$, such that

$$\int_{B(0,N)^c} |f|\, d\mu < \epsilon.$$

So we know that the Dominated Convergence Theorem can be deduced from the above Vitali's Theorem on unbounded measurable space.

Of course, for some measure, the condition of $|f(x)| < +\infty$ a.e. can be removed.

The example of satisfies the condition of Vitali condition, but not satisfy the condition of Dominated Convergence Theorem.

Let $Z = (0, +\infty)$, μ is the Lebesgue measure in $\mathbb{R}^1$. $f_n(x) = \chi_{[n,n+1]}(x)\frac{1}{x}$, $f(x) = 0$, $f_n \to f$, $\forall\ x \in Z$. Then

$$\int_Z |f_n|\, d\mu = ln(n+1) - lnn = ln(1 + \frac{1}{n}) \to 0.$$

$$\int_Z |f_n - f|\, d\mu = \int_Z |f_n|\, d\mu \to 0.$$

Then $f - n \to f$. So we have $\{f_n\}$ uniformly integrable.

But $sup f_n(x) = \frac{1}{x}$, so $|f_n| = f_n$ have not dominated function $g \in L^1(\mu)$. This is because $|f_n| \leq g$ implies that $sup f_n \leq g$ and $sup f_n \in L^1$, but $\frac{1}{x} \in L^1(\mu)$.

In the above example, $\mu(Z) = +\infty$, next we create a example such that $\mu(Z) < +\infty$.

Let $Z = (0,1)$, μ is Lebesgue measure on L^1, $f_n(x) = \frac{1}{n}\chi_{[\frac{1}{n}],1}\frac{1}{x}$, $f(x) = 0$. $f_n(x) \to f(x)$, $\forall\ x \in (0,1)$. So

$$\int_{[0,1]} |f_n(x)|\, dx = \frac{1}{n}\int_{\frac{1}{n}}^{1} \frac{1}{x}\, dx = \frac{1}{n}(ln 1 - ln\frac{1}{n}) = \frac{lnn}{n} \to \int_{[0,1]} |f(x)|\, dx.$$

Thus $\{f_n\}$ is uniform integral on Z. $\square$

(f) $\forall\ n \geq 1$, let

$$\tilde{f_n}(x) = \begin{cases} n, & 0 < x < \frac{1}{n}; \\ 0, & x = 0 \text{ or } 1,\ \frac{1}{n} \leq x \leq 1 - \frac{1}{n}; \\ -n, & 1 - \frac{1}{n} < x < 1. \end{cases}$$

Then $f_n(x)$ is measurable w.r.τ. Lebesgue measure on $[0,1]$ and $f_n \in L^1(\mu)$, $f_n(x) \to 0$, $\forall\ x \in [0,1]$.

$$\begin{aligned} \int_{[0,1]} f_n(x)\, dx &= \int_{[0,\frac{1}{n}]} f_n(x)\, dx + \int_{[1-\frac{1}{n},1]} f_n(x)\, dx \\ &= \int_{[0,\frac{1}{n}]} n\, dx + \int_{[1-\frac{1}{n},1]} n\, dx = 1 - 1 = 0. \end{aligned}$$

But $\{f_n(x)\}$ is not uniformly integrable: $\forall\ n$, let $e_n = (0, \frac{1}{2n}) \subset [0,1]$, then $m(e_n) = \frac{1}{2n} \to 0$.

$$|\int_{e_n} f_n\, dx| = |\int_{(0,\frac{1}{2n})} n\, dx| = \frac{1}{2}.$$

$\square$

(e) We show that (m, ρ), where $\rho(A,B) = \int |\chi_A - \chi_B|\, d\mu$ is a complete metric space. $0 \leq \rho(A,B) = \rho(B,A)$ is trivial.

$$\rho(A,B) = 0 \ \Leftrightarrow\ \chi_A - \chi_B = 0 \ \ \mu\ a.e. \ \Leftrightarrow\ A = B \ \ \mu\ a.e..$$

$\forall\ A,, B, C \in m$,

$$\rho(A,B) = \int |\chi_A - \chi_B|\, d\mu \leq \int |\chi_A - \chi_C|\, d\mu + \int |\chi_C - \chi_B|\, d\mu = \rho(A,C) + \rho(C,B).$$

So (m,ρ) is a metric space. If $\{A_n\} \subset m$ is a Candy sequence, then $\{\chi_{A_n}\}$ is a Candy sequence in $L^1(\mu)$. Hence by the completeness of $L^1(\mu)$, there is a $f \in L^1(\mu)$ such that $\chi_{A_n} \to f$ in $L^1(\mu)$.

Also by Riesz theorem, there is a subsequence A_{n_k} such that $\chi_{A_{n_k}} \to f$ a.e. on Z. Since $\chi_{A_{n_k}}$ has only two possible value 1 and 0, so is f.

Let $A = \{x : f(x) = 1\}$, then $f = \chi_A$. We must have $\rho(A_n, A) \to 0$.

Let $\varphi_n : (m,\rho) \to c$ given by $\varphi_n(E) = \int_E f_n\, d\mu$ is a continuous map for each n. In fact, $f_n \in L^1(\mu)$. The absolute continuity of L^1 function, Exercise 12 Ch. implies that $\forall\ \epsilon > 0$, $\exists\ \delta_n > 0$, $\forall\ A \in\ m$, $\mu(A) < \delta$ implies that

$$\int_A |f_n|\, d\mu < \frac{\epsilon}{4}.$$

So $\forall\ E_0 \in\ m,\ E \in\ m$, if $\rho(E, E_0) < \delta$, then

$$\begin{aligned}
\delta &> \int |\chi_E - \chi_{E_0}|\, d\mu = \int_{E\backslash E_0} |\chi_E - \chi_{E_0}|\, d\mu + \int_{E_0\backslash E} |\chi_E - \chi_{E_0}|\, d\mu \\
&= \int_{E\backslash E_0} d\mu + \int_{E_0\backslash E} d\mu = \mu(E\backslash E_0) + \mu(E_0\backslash E).
\end{aligned}$$

Hence,

$$\begin{aligned}
\varphi_n(E) - \varphi_n(E_0) &= |\int_E f_n\, d\mu - \int_{E_0} f_n\, d\mu| = |\int_Z (\chi_E - \chi_{E_0}) f_n\, d\mu| \\
&\le \int_Z |\chi_E - \chi_{E_0}||f_n|\, d\mu = \int_{E\backslash E_0} |f_n|\, d\mu + \int_{E_0\backslash E} |f_n|\, d\mu \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

So $\varphi_n : (m, \rho) \to c$ is continuous, (m, ρ) is complete. $\forall\ E \in m$, $\lim\limits_{n\to\infty} \varphi_n(E) = \varphi(E)$ exists as a complex number, $\lim\limits_{n\to\infty} \int_E f_n\, d\mu$ exists by condition 1. Denote $\lim\limits_{n\to\infty} \varphi_n(E) = \varphi(E)$.

Page 113 Exercise 13, Let $\{f_n\}$ be a sequence of continuous complex functions on a (nonempty) complete metric space Z, such that $f(x) = \lim f_n(x)$ exists (as a complex number) for every $x \in Z$. Then

(a) $\exists$ open set $V \neq \emptyset$ and a number $M < +\infty$ such that for all $x \in\ V$ and for $n = 1, 2, \cdots$, $|f_n(x)| < M$,

(b) $\forall\ \epsilon > 0$, $\exists$ open set $V \neq \emptyset$ and an integer N such that $|f_m(x) - f_n(x)| \le \epsilon$ for $x \in V$ and $m, n \ge N$.

Using the (b) above, we see that $\forall\ \epsilon > 0$, $\exists$ open set $V \subset (m, \rho)$ and N, such that

$$\varphi_n(E) - \varphi_N(E) \le \frac{\epsilon}{2},\ \forall\ E \in V, \quad n \ge N.$$

So $\exists E_0 \in m$, $\delta > 0$ such that $\{E : \rho(E, E_0) < \delta\} \subset V$. So

$$\varphi_n(E) - \varphi_N(E) \le \frac{\epsilon}{3},\ \text{if } \rho(E, E_0) < \delta, \quad n \ge N.$$

Then

$$|\int_E (f_n - f_N)\, d\mu| \le \frac{\epsilon}{3},\ \text{if } \rho(E, E_0) < \delta, \quad n \ge N.$$

If $\mu(A) < \delta$, $B = E_0 - A$, $C = E_0 \cup A$, then

$$\rho(B, E_0) = \rho(E_0 - A, E_0) = \mu((E_0 - A) - E_0) \cup ((E_0 - A) - E_0) = \mu((E_0 - A) - E_0) \le \mu(A) < \delta.$$

$$\rho(C, E_0) = \rho(E_0 \cup A, E_0) = \mu((E_0 \cup A) - E_0) \cup ((E_0 \cup A) - E_0) = \mu(E_0 \cup A - E_0) \le \mu(A) < \delta.$$

Then

$$|\int_{E_0 - A} (f_n - f_N)\, d\mu| < \frac{\epsilon}{3}, \quad n > N.$$

$$|\int_{E_0\cup A}(f_n - f_N)\,d\mu| < \frac{\epsilon}{3}.$$

Since $E_0 = A \cup (E_0 - A)$, $A \cap (E_0 - A) = \emptyset$, then

$$\int_{E_0\cup A}(f_n - f_N)\,d\mu = \int_A (f_n - f_N)\,d\mu + \int_{E_0 - A}(f_n - f_N)\,d\mu.$$

$$\begin{aligned} |\int_A (f_n - f_N)\,d\mu| &= |\int_{E_0\cup A}(f_n - f_N)\,d\mu - \int_{E_0 - A}(f_n - f_N)\,d\mu| \\ &\leq |\int_{E_0\cup A}(f_n - f_N)\,d\mu| + |\int_{E_0 - A}(f_n - f_N)\,d\mu| < \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3}. \end{aligned}$$

Applying part (a) to $\{f_1, f_2, \cdots, f_N\}$, $\exists \delta_1 > 0$ such that

$$|\int_A f_n\,d\mu| < \frac{\epsilon}{3} \tag{10}$$

if $\mu(A) < \delta'$, where $n = 1, 2, \cdots, N$. Let $\delta_0 = min(\delta, \delta_1)$, then if $A \in m$ with$\mu(A) < \delta_0 \leq \delta'$ and if $1 \leq n \leq N$, (3) gives that

$$|\int_A f_n\,d\mu| < \frac{\epsilon}{3} < \epsilon.$$

If $n > N$, $\mu(A) \leq \delta_0 \leq \delta$, then

$$|\int_A (f_n - f)\,d\mu| < \epsilon.$$

So

$$\begin{aligned} |\int_A f_n\,d\mu| &\leq |\int_A (f_n - f_N)\,d\mu| + |\int_A (f_N)\,d\mu| \\ &\leq \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

So $\{f_n\}$ is uniformly integrable. □

Exercise 13, Chap 5

Proof: (a) Let $A_N = \{x : |f_n(x)| \leq N, n \geq 1\} = \cap_{n=1}^{+\infty}\{x : |f_n(x)| \leq N\}$.

Then since $\lim_{n\to\infty} f_n(x) = f(x)$ exists as complex number.

Since $Z = \bigcup_{N=1}^{+\infty} A_N$, $\{f_n\}$ are continuous, $E_{n,N} = \{x : |f_n(x)| \leq N\}$ is closed for each n, N. Hence $A_N = \cap_{n=1}^{+\infty} E_{n,N}$ is closed.

Since Z is complete, Baire's Theorem (page 97, Th 5.6) (Z is second category) If A_N is nowhere dense for each N, then Z is nowhere dense, a contradiction. So $\exists\ N$ and a open set $V \neq \emptyset$, such that $V \subset A_N$. So $\forall\ x \in V$, $|f_n(x)| \leq N$, (a) is proved.

(b) $\forall\ \epsilon > 0$. Let $A_N = \{x : |f_m(x) - f_n(x)| < \epsilon \text{ if } n, m \geq N\}$. Then $\bigcup_{N=1}^{+\infty} A_N = Z$, each A_N is closed. Then Z is complete implies that $\exists\ N$, $\exists$ open set $V \neq \emptyset$, $V \subset A_N$, then $\forall\ x \in N$, $|f_m(x) - f_n(x)| \leq \epsilon,\ n, m \geq N$. Thus $|f(x) - f_n(x)| \leq \epsilon,\ n \geq N$.

Another Baire theorem is :If Z is a complete space, $E = \cup_{n=1}^{+\infty} F_n$, F_n is closed. If each F_n have no interior point, then E have no interior point.

This can be deduced from Theorem 5.6. In fact, if F_n have no interior point, then F_n^c is dense in Z, otherwise $\exists\ \psi \neq\ V$ is open, such that $F_n^c \cap V = \emptyset$, so $V \subset\ (F_n^c)^c = F_n$, thus F_n has interior, contradiction.

$E^c = \cap_{n=1}^{+\infty} F_n^c$, from theorem 5.6, we have E^c is dense in Z. If E have interior, then $\exists$ open set $V \neq \psi$, $V \subset E$ such that $V \cap E^c = \emptyset$, this is contradict to E^c is dense in Z. □

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Proof: $\forall\ i, j \in N$, $i \leq j$,

$$\int_0^1 \varphi_i(t)\varphi_j(t)\,dt = \int_0^1 \varphi_0(2^i t)\varphi_0(2^j t)\,dt = \frac{1}{2^j}\int_0^{2j} \varphi_0(2^{i-j}s)\varphi_0(s)\,ds.$$

If $i = j$, then $\int_0^1 \varphi_i^2(t)\,dt = 1$,

If $i < j$, then

$$\frac{1}{2^j}\int_0^{2j} \varphi_0(2^{i-j}s)\varphi_0(s)\,ds = \frac{1}{2^j}\sum_{k=1}^{2^{j-1}}\int_{2(k-1)}^{2k} \varphi_0(2^{i-j}s)\varphi_0(s)\,ds.$$

However, $\forall\ s \in (2(k-1), 2k)$, $\varphi_0(2^{i-j}s)$ is constant for s. So

$$\int_{2(k-1)}^{2k} \varphi_0(2^{i-j}s)\varphi_0(s)\,ds = \pm\int_{2(k-1)}^{2k} \varphi_0(s)\,ds = 0.$$

This proves $\{\varphi_n\}$ is orthonormal on $[0, 1]$.

Since $\sum |c_n|^2 < \infty$, the series $\sum\limits_{n=1}^{\infty} c_n\varphi_n(t)$ converges to some function f in $L^2([0, 1])$.

Now for $a = j2^{-N}$, $b = (j+1)2^{-N}$, $a < t < b$ and $S_N = c_1\varphi_1 + c_2\varphi_2 + \cdots + c_N\varphi_N$, we claim that

$$S_N(t) = \frac{1}{b-a}\int_a^b S_N(t)\,dt = \frac{1}{b-a}\int_a^b S_n(t)\,dt \quad \text{for all } n > N.$$

Indeed, $\forall\ k \leq N$, $\varphi_k(t) = \varphi_0(2^k t)$ is constant on $[a, b]$, since $j2^{k-N} \leq 2^k t \leq (j+1)2^{k-N}$. And then $S_N(t)$ is constant on $[a, b]$,

$$S_N(t) = \frac{1}{b-a}\int_a^b S_N(t)\,dt.$$

On the other hand, $\forall\ k > N$,

$$\begin{aligned}\int_a^b \varphi_k(t)\,dt &= \int_a^b \varphi_0(2^k t)\,dt = \frac{1}{2^k}\int_{2^k a}^{2^k b} \varphi_0(s)\,ds \\ &= \frac{1}{2^k}\int_{j2^{k-N}}^{(j+1)2^{k-N}} \varphi_0(s)\,ds = 0.\end{aligned}$$

Since 2^{k-N} is a multiple of 2, the period of φ_0. So for all $n > N$, we have

$$S_N(t) = \frac{1}{b-a}\int_a^b S_N(t)\,dt = \frac{1}{b-a}\int_a^b S_n(t)\,dt.$$

Since $S_n(t) \to f(t)$ in L^2, we conclude that $\int_a^b S_n(t)\,dt \to \int_a^b f(t)\,dt$ by Hölder inequality. Therefore, $S_N(t) = \int_a^b f(t)\,dt$.

For any Lebesgue point x_0 of f (since $f \in L^2$ implies that $f \in L^1$), and for any $n \in N$, $\exists\, j_n \in N$ such that $x_0 \in [j_n 2^{-n}, (j_n+1)2^{-n}] = J_n$.

Therefore, $|J_n| \to 0$ as $n \to \infty$. By Theorem 7.10,

$$\lim_{n\to\infty} \frac{1}{m(J_n)} \int_{J_N} f(t)\, dt \to f(0)$$

as $n \to \infty$, i.e. $\sum_{n=1}^{\infty} c_n \varphi_n(x_0)$ converges. Hence the series $\sum_{n=1}^{\infty} c_n \varphi_n(t)$ converges for almost every $t \in [0,1]$, then for almost every $t \in \mathbb{R}$ due to the fact that $\varphi_n(t)$ have the same period 2, and $\varphi_n(t)$ are odd functions. $\square$

7, Suppose μ is a complex Boral measure on $[0, 2\pi)$ (Or on the unit circle T), and defined the Fourier coefficients of μ by

$$\hat{\mu}(n) = \int e^{-int} d\mu(t) \quad (n = 0, \pm 1, \pm 2, \ldots)$$

Assume that $\hat{\mu}(n) \to 0$ as $n \to +\infty$, and prove that then $\hat{\mu} \to 0$ as $n \to -\infty$.

Hint: the assumption also holds with $f d\mu$ in place of $d\mu$, f is any trigonometric polynomial, hence if f is continuous, hence if f is any bounded Boral function, hence if f is continuous, hence if f is any bounded Boral function, hence if $d\mu$ is replaced by $d\mu$!

$\hat{\mu}(n) = \int \cos ntd\mu(t) - i \int \sin ntd\mu(t)$, if μ is a real positive measure, then $\hat{\mu} \to 0 \Rightarrow n \Rightarrow +\infty \Rightarrow \int \cos ntd\mu(t) \to 0$, $\int \sin ntd\mu(t)$. So as $n \to +\infty$,

$$\int \cos nt = \int \cos(-nt) d\mu(t) \to 0$$

$$\sin ntd\mu(t) = -\sin(-nt)d\mu(t) \to 0$$

So we need to prove $\int e^{-int} d\mu(t) \to 0$ first.

Proof: For any trigonometric polynomial $f = \sum_{n=-N}^{N} C_k e^{ikt}$,

$$\int f e^{-int} d\mu(t) = \sum_{k=-N}^{N} C_k \int e^{-i(n-k)} d\mu(t) \to 0$$

as $n \to +\infty$.

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Here we use an estimate $|\int_Z f d\mu| \le \int_Z |f| d|\mu|$, $\forall f$ complex measure, which is deduced from Theorem 6.12. $d\mu = h d|\mu| (|h| = 1, \forall x \in Z)$.

$$|\int_Z f d\mu| = |\int_Z f h d|\mu|| \le \int_Z |f||h| d|\mu| = \int_Z |f| d|\mu|.$$

thus $\int e^{-int}f(t)d\mu(t) \to 0$, as $n \to +\infty$.

For any bounded Boral function f, then f is $|\mu|$ measurable and $|\mu|$ is a finite positive measure (Th 6.4). $\forall \epsilon > 0$, take $A = Z = [0, 2\pi)$ in Th 2.24 (Lusin Theorem). $\exists g \in C^0[0, 2\pi)$, s.t.

$$|\mu|\{x : f(t) \neq g(x)\} < \frac{\epsilon}{4\|f\|_\infty}.$$

(if $\|f\|_\infty = 0$, it is trivial that $\int e^{-int}f(t)d\mu(t) \to 0$, as $n \to +\infty$.)

and $\|g\|_\infty = \sup |g(x)| \leq \|f\|_\infty$. So for this fixed continuous g, by the result proved we have

$$\lim_{n\to\infty} \int e^{-int}g(t)d\mu(t) = 0.$$

So $\exists\ N$, $n > N$,

$$|\int e^{-int}f(t)d\mu(t)| < \frac{\epsilon}{2}.$$

So $n > N$,

$$\begin{aligned}
& |\int e^{-int}f(t)d\mu(t)| \\
\leq\ & |\int e^{\int}(f(t) - g(t))|d\mu(t)| + |\int e^{-int}g(t)d\mu(t)| \\
\leq\ & \int_{x:f\neq g} |f - g|d\mu(t) + \frac{\epsilon}{2} \\
\leq\ & 2\|f\|_\infty|\mu|\{x : f(x) \neq g(x)\} \\
\leq\ & \frac{2\|f\|_\infty\epsilon}{4\|f\|_\infty} \\
\leq\ & \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
\leq\ & \epsilon.
\end{aligned}$$

We see

$$\lim_{n\to+\infty} \int e^{-int}f(t)d\mu(t) = 0,$$

for any bounded Boral measurable function.

Theorem 6.12 says that there exists a Boral measurable function $h(x)$ with $h(x) = 1$ for all $x \in [0, 2\pi)$ s.t. $d\mu = h\mu$. So $\bar{h}d\mu = \bar{h}hd|\mu| = |h|^2d(\mu) = d|\mu|$.
Letting $f = \bar{h}$, then $f$is Bounded Boral function , So

$$\int e^{-int}\bar{h}d\mu(t) \to 0, \text{ as } n \to +\infty.$$

i.e.

$$\int e^{-int}d|\mu|(t) \to 0, \text{ as } \ n \to +\infty.$$

So as $n \to +\infty$, $\int \cos ntd|\mu|(t) \to 0$, $\sin ntd|\mu|(t) \to 0$.
So as $n \to -\infty$,

$$\begin{aligned}
&\int \cos ntd|\mu|(t) = \int \cos(-nt)d|\mu|(t) \to 0, \\
&-\int \sin ntd|\mu|(t) = +\int \sin(-nt)d|\mu|(t) \to 0, \\
&\int e^{-int}d|\mu|(t) \to 0 \ \text{ as } n \to -\infty
\end{aligned}$$

$$\int e^{-int} d|\mu|(t) \to 0 \text{ as } n \to -\infty$$

$$\text{i.e. } \int e^{-int} d|\mu|(t) \to 0 \text{ as } n \to +\infty.$$

The same argument shows that

$$\int e^{int} f d|\mu|(t) \to 0 \text{ as } n \to +\infty$$

Let $f = h$, then

$$\int e^{int} f d|\mu|(t) = \int e^{int} d|\mu|(t) \to 0,$$

as $n \to +\infty$,

$$\int e^{-int} d\mu(t) \to 0 \quad \text{as } n \to -\infty.$$

□

8, In the terminology of Exercise 7, find all μ such that $\hat{\mu}$ is periodic, with period k. [This means that $\hat{\mu}(n+k) = \hat{\mu}(n)$ for all integers n, of course, k is also assumed to be an integer].

Proof:

$$\begin{aligned} \hat{\mu}(n+k) - \hat{\mu}(n) &= \int e^{-int}(e^{-ikt} - 1) d\mu(t) \\ &\triangleq \int e^{-int} d\lambda(t) = \hat{\lambda}(n) \end{aligned}$$

where $d\lambda(t) = (e^{-int} - 1)d\mu(t)$ is a complex measure as well.

Thus $\hat{\mu}(n+k) = \hat{\mu}(n)$ iff $\hat{\lambda}(n) = 0$ for all n. Hence for any $m-$trionometric polynomial f, $\int f d\lambda(t) = 0$, Hence for continuous f, it is true.

By approximation $\forall$ bounded Boral function f, $\int f d\lambda(t) = 0$.

In particular, $\forall$ Boral E, we have $\int_E d\lambda(t) = 0$.

Thus $\lambda(E) = 0$, $\forall\ E \in \mathfrak{M}$, ($\sigma-$algebra of Boral sets)

Any partition, of E_i of E satisfies $\sum |\lambda(E_i)| = 0 \Rightarrow |\lambda|(E) = 0$, $\forall\ E \Rightarrow |\lambda| = 0$.

Theorem 6.12 $\Rightarrow d\mu(t) = hd|\mu|(t)$ for some Boral $|h| = 1$, $\forall x \in h$.

So $\forall$ Boral set E,

$$\int_E (e^{-int} - 1) d\mu(t) = 0.$$

i.e. $\int_E (e^{-int} - 1) hd|\mu|(t) = 0$, $\forall\ E \in \mathfrak{M}$.

We claim that μ is concentrated on $\{t : e^{-ikt} = 1\} \triangleq B$. Otherwise $\exists\ A$, $\mu(A) \neq 0$, $A \cap B = \phi$.

Then Theorem 1.39(h) P30, Note $|(e^{-iht} - 1)h| \leq 2$, $(e^{-iht} - 1)h \in L^\infty(|\mu|)$, $|u|(0, 2\pi) < +\infty \Rightarrow e^{-iht} - 1)h \in L^1$ implies that

$$(e^{-iht} - 1)h = 0, \quad |\mu| \ a.e. \ on \ E$$

for any $E \in \mathfrak{M}$ with $|\mu|(E) > 0$.

As $|h| = 1$, so $e^{-ht} = 1$ for any $E \in \mathfrak{M}$, with $|\mu|(E) > 0$. So if we let $A = \{t : e^{-1kt} = 1\}$, then $\forall A \in \mathfrak{M}$ with $A \cap B = \emptyset$, we must have $\mu(A) = 0$, otherwise $|\mu(A)| > 0 \Rightarrow |\mu|(A) > 0 \Rightarrow e^{-ikt} = 1$, $|\mu|$ a.e. on A.

So we get a contradiction.

Thus

$$\hat{u}(n+h) - \hat{u}(n), \quad \forall\, n \in Z.$$

$\Rightarrow$

μ is concentrated on $\{t : e^{-ikt} = 1\}$, we see that $\forall F \in \mathfrak{M}$

$$\begin{aligned} & \int (e^{-ikt} - 1) d\mu(t) = 0 \\ \Rightarrow \quad & \lambda(E) = 0, \ \forall\ E \in \mathfrak{M} \\ \Rightarrow \quad & \hat{u}(n+k) - \hat{u}(n) = 0. \end{aligned}$$

$$\{t : e^{ikt} = 1\} = \left\{ t = \frac{2m\pi}{k}, \quad m = 0, 1, 2, \cdots \right\}$$

So $\mu \perp \mathfrak{M}$. $\square$

9, Suppose that $\{g_n\}$ is a sequence of positive continuous functions on $I = [0, 1]$, that μ is a positive Boral measure on I, and that

(i) $\lim\limits_{n\to\infty} g_n(t) = 0$, a.e. $[m]$.

(ii)$\int_I g_n dm = 1$ for all n.

(iii) $\lim\limits_{n\to\infty} \int_I f g_n dm = \int f d\mu$ for every $f \in \mathbb{C}(I)$, Dose it follow that $\mu \perp m$?

Proof: The conclusion is false. For example: Let $\mu = m$,

$$g_k(x) = \frac{1}{k} \sum_{i=1}^{k} k^2 \chi_{(\frac{i}{k} - \frac{1}{k^2}, \frac{i}{k})}(x) \quad x \in [0, 1].$$

Then

(i) $\lim g_n(x) = 0$, m a.e. ($\forall x \in (0, 1)$, k large enough, $g_k(x)$)

(ii) $\int g_n dm = 1$, $\forall\ n$.

(iii) $\lim\limits_{n\to\infty} \int_I f g_n dm = \int f dm$, $\forall f \in C(I)$.

However $m \perp m$ is false! $\square$

?????

?????

Chapter 6 Abstract Integration

12 let $\mathcal{M}$ be the collection of all sets E in the unite interval $[0,1]$ such that either E or its complement is at most countable. Let μ be counting measure on this σ-algebra $\mathcal{M}$. If $g(x)=x$ for $0 \le x \le 1$, show that g is not m measurable, although the mapping

$$f \to \Sigma x f(x) = \int fgd\mu$$

makes sense for every $f \in L^1(\mu)$ and defines a bounded linear functional on $L^1(\mu)$. Thus $(L^1)^* \neq L^\infty$ in this situation.

Note: $\mathcal{M}$ here is different from that of Exercise 18 in Chapter 2.

Proof: First, $\mathcal{M}$ is indeed a σ-algebra. (i) $X \in \mathcal{M}$, as $\emptyset = X^c$ is at most countable; (ii)If $A \in \mathcal{M}$, then either A or A^c at most countable, so $((A)^c)^c$ or $((A)^c$ is at most countable, hence $((A))^c \in \mathcal{M}$; (iii) If $\{A_i\}_{i=1}^\infty \subset \mathcal{M}$ is a sequence sets, then for any i, either A_i or A_i^c is at most countable, if A_i is at most countable for all i, clearly $\bigcup_{i=1}^\infty A_i$ is at most countable, if $\exists\, i_0$ such that A_{i_0} is uncountable, then $A_{i_0}^c$ is at most countable because $A_{i_0} \in \mathcal{M}$, thus $(\bigcup_{i=1}^\infty A_i)^c = \bigcap_{i=1}^\infty A_i^c \subset A_{i_0}^c$, so $(\bigcup_{i=1}^\infty A_i)^c$ is at most countable, hence $(\bigcup_{i=1}^\infty A_i)^c \in \mathcal{M}$. So $\mathcal{M}$ is indeed a σ-algebra on $[0,1]$. That μ is a measure is trivial.

Next $g(x)=x$ is not $\mathcal{M}$-measurable. For any $r \in (0,1)$, $\{x : g(x) > r\}$ and $\{x : g(x) > r\}^c = \{x : g(x) \le r\}$ are both uncountable, so $\{x : g(x) > r\} = g^{-1}((r,+\infty))$is not in $\mathcal{M}$. But $(r,+\infty) \subset \mathbb{R}^1$ is open , this shows g is not $\mathcal{M}$-measurable. Although $|g(x)| = |x| \le 1$ for $x \in [0,1]$.

For any $f \in L^1(u)$, f must be $\mathcal{M}-$ measurable and $+\infty > \int_{[0,1]} |f| d\mu$, We claim that $\{x : |f(x)| > 0\} = \sqcup_{n=1}^{+\infty} \{x : |f(x)| > \frac{1}{n}\}$ is at most countable. In fact, for any n, $\{x : |f(x)| > \frac{1}{n}\}$ is finite. Otherwise, if $\{x : |f(x)| > \frac{1}{n}\}$ is infinite , then

$$+\infty > \int |f| d\mu \ge \int_{x:|f|>\frac{(}{1})n} |f| d\mu = \frac{1}{n}\mu\{x : |f| > \frac{1}{n}\} = \frac{1}{n}(+\infty) = +\infty,$$

a contradiction. So $\{x : |f(x)| > 0\}$ is at most countable.

Let $A = \{x_1, x_2, \cdots, x_m\}$ is at most finitely countable or $m=\infty$, then $f(x)=0$ if $x \in A^c$ and $f(x)g(x) = 0$ if $x \in A^c$. $\forall r \in \mathbb{R}^1$, $\{x : f(x)g(x)\, r\} \subset \{x : |f(x)| > 0\} \subset A$, so $\{x : f(x)g(x) > r\}$ is at most countable. Thus $f(x)g(x)$ is m-measurable, and

$$\int_{[0,1]} |f(x)g(x)| d\mu \le \int_{[0,1]} |f(x) d\mu < +\infty,$$

so $fg \in L^1(\mu)$ and

$$\int fgd\mu = \int_A fg\ d\mu = \sum_{i=1}^m f(x_i)g(x_i).$$

$f \longrightarrow \Lambda(f) \triangleq \int_X fgd(u)$ is clearly a linear functional on $L^1(\mu)$ and

$$|\Lambda(f)| \le \int |fg| d\mu \le \int |f| d\mu,$$

$$\|\Lambda\| = \sup_{f \in L^1(\mu), f \neq 0} \frac{|\Lambda(f)|}{\|f\|_1} \le 1.$$

Now we prove that $(L^1)^* \neq L^\infty$.

If there is a $h \in L^\infty(\mu)$ $s.t.$ $\Lambda(f) = \int fh\ d\mu$, $\forall f \in L^1(\mu)$. then $\int f(g-h)d\mu$, $\forall f \in L^1(\mu)$. $\forall x_0 \in [0,1]$, let

$$f_{x_0}(y) = \begin{cases} 1 & y = x_0; \\ 0 & y \neq x_0. \end{cases}$$

then clearly $f_{x_0} \in L^1(\mu)$ hence $\int f_x(g-h)\ d\mu = 0 \Longrightarrow 0 = f_{x_0}(x_0)(g(x_0) - h(x_0))\mu\{x_0\} = g(x_0) - h(x_0)$, so $g(x_0) = h(x_0)$, $h(x) = g(x) = x \in L^\infty$, this contradicts to the fact that $g \notin L^\infty(\mu)$. Since g is not $\mathcal{M}$-measurable.

Notice that for any positive measure space (X, m, μ), $\forall g \in L^\infty(\mu)$, $\Lambda g : L^1 \mapsto \mathcal{C}$ given by $\Lambda_g(f) = \int fgd\mu$ is a bounded linear functional and the map $L_\infty \mapsto (L^1)^*$ defined by $g \mapsto \Lambda g$ is an isometry if μ is σ-finite. Indeed,

$$\|\Lambda g\| = \sup_{f \in L^1(\mu), f \neq 0} \frac{|\Lambda_g f|}{\|f\|_1} = \| g \|_\infty$$

as Theorem 6.16 shows. (Note: Theorem 6.16 requires that μ is σ-finite, here although μ is the counting measure on $[0,1]$ and is NOT σ-finite, but we can still show that $g \mapsto \Lambda g$is an isometry.) In fact,$\| \Lambda g \| \leq \| g \|_\infty$ is trivial. And $\forall x \in [0,1]$, let

$$f(y) = \delta_x = \begin{cases} 1 & y = x; \\ 0 & y \neq x. \end{cases}$$

then $\| f \|_{L^1} = 1$, and

$$|\Lambda_g(f)| = |\int fgd\mu| = |g(x)| \leq \| \Lambda g \| .$$

So $\| g \|_\infty \leq \sup |g(x)| \leq \| g \|$, $\| \Lambda g \| = \| g \|_\infty$. Hence $g \mapsto \Lambda g$ is an isometry between L^∞ and a subspace of $(L^1)^*$. The above example with $g(x) \equiv x$, shows that the map $i:\ g \mapsto \Lambda g$ is not surjective . So $i(L^\infty) \subsetneq (L^1)^*$.

To prove that $(L^1)^* \neq (L^\infty)$, we need the following abstract result:

Proposition: If L and M are Banach spaces and $i:\ L \mapsto M$ is an linear norm preserving map with $\forall x \in L, \| i(x) \|_M = \| x \|$, then $i(L) = (M)$, i.e i is an isometry between L and M.

Proof: Since i is linear, $i(L)$is a linear subspaces of M. Also, since L is a Banach spaces and i is norm preserving, $i(L)$ is a closed subspaces of M. Indeed, if $y_n \in i(L)$, $y_n \to y$ in M, then $\exists x_n \in L$ such that $y_n = i(x_n)$, $\| y_n - y_m \|_M = \| x_n - x_m \| \to 0$, so $x_n \to x_0$ in L, $i(x_n) \to i(x_0) \Rightarrow i(x_0) = y$. (i is one to one), so $y \in i(L)$ and $i(L)$ is closed. On the other hand, i induces a bounded linear map $i^* : M^* \to L^*$, from the dual spaces M^* of M to the dual spaces L^* of L in a natural way : $\forall f \in M \forall x \in L$ define $i^*(f)(x) = f(i(x))$, then

$$\| i^*(f) \| = \sup x \in L \backslash \{0\} \frac{|f(i(x))|}{\|x\|_L} = \sup_{y \in i(L)} \frac{|f(y)|}{\|y\|_M} = \| f|_{i(L)} \| \leq \|f\|.$$

(NOTE: $i^* : M^* \to L^*$ is not necessarily isometry!)

—-TO BE FINISHED!

13 Let $L^\infty = L^\infty(m)$ where m is Lebesgue measure on $I = [0,1]$. Show that there is a bounded linear functional $\Lambda \neq 0$ on L^∞, that is O on $C(I)$ and that there is no $g \in L^1(m)$ that satisfies $\Lambda f = \int_I fgdm$ for every $f \in L^\infty$. Thus $(L^\infty)^* \neq L^1$.

Proof: $C(I)$ is a closed subspace of $L^\infty(m)$. We see that $\forall f \in L^\infty$ $C(I)$, $\exists \Lambda \in (L^\infty)$ s.t $\Lambda f \neq 0$ but $\Lambda g = 0$ for each $g \in C(I)$. We claim that for a fixed $\Lambda \in (L^\infty)^)$ with $\Lambda \neq 0$ and $\Lambda g = 0 (\forall g \in C(I))$, there is no $h \in (L^1)$ such that

$$\Lambda(f) = \int_I hf\ dm, \quad \forall f \in L^\infty(m).$$

In fact, if there is a $h \in L^1$ s.t. $\Lambda(f) = \int_I hf\ dm$, $\forall f \in L^\infty$, then $\int_I fhdm = 0$ for $f \in C(I)$.

Let $E = x \in I, h(x) 0$, then $m(E) \leq m(I) < +\infty$. Lusin Theorem (Theorem 2.24) implies that $\exists \{g_n\} \subset C(I)$ s.t. $\chi_E(x) = \lim_{n\to 0} g_n(x)$ a.e. and $|g_n(x)| \leq \sup \chi_E = 1$. So Lebesgue Dominated Convergence Theorem gives

$$\int_I \chi_E(x)h(x)dx = \lim_{n\to\infty} \int_I g_n(x)h(x)dx = 0,$$

so $\int_E h(x)dx = 0$, which implies that for each n,

$$\int_E hdx \geq \int_{x,h(x)\leq \frac{1}{n}} h(x)dx \leq \frac{1}{n} mx : h(x) > \frac{1}{n},$$

so

$$mx : h(x) > frac1n = 0 \Rightarrow m(E) \leq \sum_{n=1}^{+\infty} mx : h(x) > \frac{1}{n} = 0.$$

Similarly, $mx, h(x) < 0 = 0$, so $h = 0$ a.e in I. So $\Lambda \equiv 0$, a contradiction. So the imbedding $i : L^1 \to (L^{infty})^*$, $f \mapsto \Lambda_f$ given by $\Lambda_f(g) = \int fg\ dm$, $\forall \in L^\infty$ is NOT surjective. So $(L)^{+\infty^*} \neq (L^1)$. If there is an isometry $i : (L^\infty)^* \neq L^1$, So $L^{1^*} = L^{+\infty}$. Then since $L^1(m)$ is sparable $\Rightarrow (L)^\infty$ is too, a contradiction. $\square$

Chapter 7

1 Show that $|f(x)| \leq (Mf)(x)$ at every Lebesgue point of f if $f \in L^1(\mathbb{R}^n)$

Proof: If x is a Lebesgue point of f, then

$$\lim_{r\to 0}\frac{1}{m(B_r)}\int_{B_r}|f(y)-f(x)|dy = 0. \qquad (*)$$

So for any $r > 0$ small ,

$$\begin{aligned} |f(x)| &\leq |f(x) - \frac{1}{m(B_r)}\int_{B_r}|f(y)|dy| + |\frac{1}{m(B_r)}\int_{B_r}|f(y)|dy| \\ &\leq \frac{1}{m(B_r)}\int_{B_r}|f(y)-f(x)|dy + \frac{1}{m(B_r)}\int_{B_r}|f(y)|dy \\ &\leq \frac{1}{m(B_r)}\int_{B_r}|f(y)-f(x)|dy + Mf(x). \end{aligned}$$

Letting $r \to 0+$, we have $|f(x)| \leq Mf(x)$. (So by Th 7.6, a.e $x \in \mathbb{R}^k$ is a Lehesgue point, so $|f(x)| \leq Mf(x)$ a.e. on $\mathbb{R}^k$, so

$$\| f \|_\infty \leq \| Mf \|_\infty .$$

P173, Theorem 8.18 implies

$$\|Mf\|_p \leq C_p\|f\|_p.$$

) □

3 Suppose that E is a measurable set of real numbers with arbitrary small periods. Explicitly, this is means that there are positive numbers p_i, converging to 0 as $i \to \infty$ so that

$$E + P_i = E (i = 1, 2, 3, ...).$$

Prove that either or E or its complement has measure 0.

Hint: Pick $\alpha \in R^1$, put $F(x) = m(E\bigcap[\alpha, x])$ for $x > \alpha$, show that

$$F(x + p_i) - F(x - p_i) = F(y + p_i) - F(y - p_i)$$

if $\alpha + p_1 < x < y$. What does this imply about $F'(x)$ if $m(E) > 0$?

Solution: Claim: If E is Lebesgue measurable, t is a real number, and if $E + t = E$, then $(i)E - t = E$; $\forall m,\ k \in Z$, $E + kt = E = E - mt$,
$(ii)\forall\alpha \in R^1$,

$$\int_\alpha^{\alpha+t}\chi_E(x)dx = \int_0^t \chi_E(x)dx.$$

Proof of the Claim: (1) Since $E + t = E$, $\forall x \in E$, $x = x + t - t \in E - t$, hence $E \subset E - t$. On the other hand, $\forall x \in E - t$, $\exists y \in E = E + t$, s.t. $x = y - t$, and $\exists z \in E$ s.t. $y = z + t$. So $x = y - t = z + t - t = z \in E$, $E - t \subset E$, so $E = E - t$. Also trivially we have $E - kt = E = E - mt$, $\forall k,\ m,\ t$.

(2) $\forall \alpha \in \mathbb{R}^1$, $\exists\ k \in Z$ s.t. $(k-1)t < \alpha \leq kt$, E is lebesgue measurable hence $\chi_E \in L^1_{loc}(\mathbb{R})$,

$$\begin{aligned}
\int_\alpha^{\alpha+t} \chi_E(y)dy &= m(E\bigcap[\alpha,\alpha+t]) \\
&= m(E\bigcap[\alpha,\alpha+\varepsilon]) + m(E\bigcap[k+,\alpha+t]) \\
&= m(E\bigcap[\alpha,k+\varepsilon]) + m(E\bigcap[k+,\alpha+t]-t) \\
&= m(E\bigcap[\alpha,k+\varepsilon]) + m((E-t)\bigcap[(k-1)t,\alpha]) \\
&= m(E\bigcap[\alpha,k-t]) + m(E\bigcap[(k-1)t,\alpha\varepsilon]) \\
&= m(E\bigcap[(k-1)t,kt]) \\
&= m(E\bigcap[(k-1)t,kt]-(k-1)t) \\
&= m(E\bigcap[0,t]) = \int_0^t \chi_E(y)dy,
\end{aligned}$$

where the invariance of lebesgue measure under translation and $(A\bigcap B - t = (A-t)\bigcap(B-t))$ is used. So (ii) is true.

Now if $p_i > 0$ and such that $p_i \to 0$ and $E + P_i = E$, then $\forall x, y \in \mathbb{R}^1$

$$\begin{aligned}
\int_{[y-p_i,y+p_i]} \chi_E|z|dz &= \int_{[y-p_i,y]} \chi_E|z|dz + \int_{[y,y+p_i]} \chi_E|z|dz \\
&= \int_{[0,p_i]} \chi_E|z|dz + \int_{[0,p_i]} \chi_E|z|dz \\
&= 2\int_{[0,p_i]} \chi_E|z|dz \\
&= \int_{[x-p_i,x+p_i]} \chi_E|z|dz,
\end{aligned}$$

i.e.

$$\int_{]y-p_i,y+p_i[} \chi_E|z|dz = \int_{]x-p_i,x+p_i[} \chi_E|z|dz.$$

AS $\chi_E \in L^1_{loc}(\mathbb{R}^1)$, a easy variant of Theorem 7.7 shows that a.e $x \in \mathbb{R}^1$ are Lebesgue point of E, hence

$$\chi_E(x) = \lim_{r\to 0} \frac{1}{m(B_r)} \int_{B_r} |\chi_E(z)|dz,\ fora.e.x \in \mathbb{R}^1,$$

so

$$\chi_E(x) = \lim_{p_i\to 0} \frac{1}{2p_i} \int_{]p_i,p_i[} |\chi_E(z)|dz,\ a.e.x \in \mathbb{R}^1. \qquad (*)$$

Let $A \subset \mathbb{R}^1$ be sum that (*) holds for each $x \in A$ and $m(A^c) = 0$. We see that for every $x, y \in A$,

$$\begin{aligned}
\chi_E(x) &= \lim_{p_i\to 0} \frac{1}{2p_i} \int_{]y-p_i,y+p_i[} |\chi_E(z)|dz \\
&= \lim_{p_i\to 0} \frac{1}{2p_i} \int_{]y-p_i,y+p_i} |\chi(z)|dz \\
&= \chi_E(y),
\end{aligned}$$

i.e. $\forall x, y \in A, \chi_E(x) = \chi_E(y)$. So, $A \subset E$ or $A \subset E^C c$. If $A \subset E$, then $E^c \subset A^c$, $m(E^c) \leq m(A^c) = 0$, $m(E^c) = 0$; if $A \subset E^c$, then $E \subset A^c$, $m(E) = 0$. Th 7.7 says that $f \in L^1(\mathbb{R}^n) \Rightarrow$ a.e x is lebesgue point. In fact, this is true for $f \in L^1_{loc}(\mathbb{R}^n)$. $\forall m$, $\chi_B(0,m)f \in L^1(\mathbb{R}^n)$, so a.e. $x \in B_m$ is Lebesgue point of $\chi_B(0,m)f = f$. Thus, for almost all $x \in \cup_{m=1}^{m=+\infty} B_m = \mathbb{R}^k$ is Lebesgue point of f. $\square$

4 Call t a *period* of the function f on $\mathbb{R}^1$ if $f(x+t)=f(x)$ for all $x\in\mathbb{R}^1$. Suppose f is a real Lebesgue measurable function with periods s and t whose quotient s/t is irrational. Prove that there is a constant c that $f(x)=c$ a.e., but that f need not be constant.

Hint: Apply Exercise 3 to the sets $\{f>\lambda\}$.

Proof: Claim: Let $t>s$ be two positive numbers, $t/s\notin Q$, then there is a sequence $\{P_k\}_{k=1}^{\infty}$ with $P_k>0$, $P_k\to 0$ and $\forall\ k$, P_k is a linear combinations of $t,\ s$ with integer coefficient.

Proof of the Claim: Since $t>s>0$, and $t/s\notin Q$, i.e. $\forall\ l\in Z$, $t\neq l\cdot s$, so $\exists\ l\in Z_+$(the set of positive integers) s.t. $0<t-ls<s$. In fact, $A=\{l:\ l\in Z,\ 0<t-ls\}\neq\emptyset$ as $1\in A$. And $s>0$ implies that A is upper bounded. Let $l_0=\max_{l\in A}l$, then $0<t-l_0s$ and $t-(l_0+1)s<0$, i.e. $0<t-l_0s<s$.

Let $a_0=t,\ a_1=s,\ a_2=t-l_0s<a_1<a_0$. a_2 is a linear combination of a_0,a_1, hence of t,s with integer coefficients. Similarly, $\exists 0<a_3=a_1-m_0a_2<a_2<a_1<a_0,\ \ldots,a_k,\ \ldots$, (since $\frac{a_1}{a_2}=\frac{s}{t-l_0s}\notin Q$, otherwise $\frac{s}{t}\in Q$, a contradiction.) Then we get a sequence $\{a_k\}$, $a_k>0$, $\cdots<a_k<a_{k-1}<a_{k-2}\cdots<a_0$, and $\forall\ k$, a_k is a linear combination of a_{k-1},a_{k-2} hence of $s,\ t$ with integer coefficients. Since $a_n\downarrow$, $a_k>0$, $\lim_{n\to\infty}a_n$ exists as a nonnegative number. $0<a_n-a_{n-1}\to 0$, $P_n=a_n-a_{n-1}\to 0$.

Now if f has s,t as its period with $t>s>0$, $\frac{t}{s}\notin Q$ hence not in Z, then according to the Claim above, there exists a sequence $P_k>0$, which is a linear combination of s,t with integer coefficients, and $P_k\to 0$,

$$f(x+P_k)=f(x),\quad \forall x\in\mathbb{R}^1,$$

i.e. P_k are period of $f(x)$ because they are combination of periods with integer coefficients. Then $\forall\lambda\in\mathbb{R}^1$, $E_\lambda=\{x\in\mathbb{R}^k: f(x)>\lambda\}$ has period $P_k\to 0$ as well. Indeed,

$$\forall k,\ E_\lambda+P_k=\{x\in\mathbb{R}^k: f(x)>\lambda\}+P_k=E_\lambda.$$

($\because\ \ \forall x\in E_\lambda+P_k$, $\exists\ y\in\ E_\lambda$ s.t. $x=y+P_k$, $f(x)=f(y+P_k)=f(y)>\lambda\Rightarrow x\in\ E\lambda$, $\therefore\ E_\lambda+P_k\subset E_\lambda$. On the other hand, $\forall\ E_\lambda$, $x=x-P_k+P_k$ while $f(x-P_k)=f(x)>\lambda$ implies $x-P_k\in\ E_\lambda$, hence $x\in\ E_\lambda+P_k$, $\therefore\ E_\lambda\subset E_\lambda+P_k$.)

By Exercise 3 of Chapter 7, either $m(E_\lambda)=0$ or $m(E_\lambda^c)=0$. If $\forall\ \lambda\in\mathbb{R}^1$, $m(E_\lambda)=0$, then $\forall\ n\in N$, $m(E_{-n})=0$,so $f(x)\leq -n$ a.e., i.e.

$$\exists\ A_n,\ m(A_n^c)=0,\ f(x)\leq -n,\ \forall\ x\in A_n.$$

Let $A=\bigcap_{n=1}^{\infty}A_n$, then

$$m(A^c)=m(\bigcup_{n=1}^{\infty}A_n^c)\leq\Sigma_{n=1}^{\infty}m(A_n^c)=0.$$

And for all $x\in A$, $f(x)\leq-\infty$. So $f(x)=-\infty$ a.e.. This is impossible as $f(x)$ is real. (Note: $f(x)\in(-\infty,+\infty)$.)

So $\exists\lambda\in\mathbb{R}^1$, $m(E_\lambda)>0$ and hence $m(E_\lambda^c)=0$, thus

$$B=\{\lambda\in\mathbb{R}^1:\ m(E_\lambda)>0,m(E_\lambda^c)=0\}\neq\emptyset.$$

Let $\lambda_0 = \sup B$. If $\lambda = +\infty$, then $\exists\ \{\lambda_n\} \uparrow +\infty$, $m(E_{\lambda_n}) > 0$ and $m(E^c_{\lambda_n}) = 0$. So $\forall n$, $f(x) > \lambda_n$ a.e. $f(x) > \lim \lambda_n = +\infty \Rightarrow f(x) = +\infty$, which contradicts that f is real. If $\lambda < +\infty$, then

$$\exists\{\lambda_n\} \uparrow \lambda_0,\ m(E_{\lambda_n}) = 0,\ m(E^c_{\lambda_n}) = 0.$$

$\forall n$, $f(x) > \lambda_n$ a.e. in $\mathbb{R}^1$, so $f(x) > \lim \lambda_n = \lambda_0$ a.e. in $\mathbb{R}^1$. $\forall n$, $m(E_{\lambda_0+\frac{1}{n}}) = 0$, otherwise $m(E_{\lambda_0+\frac{1}{n}}) > 0$, $m(E^c_{\lambda_0+\frac{1}{n}}) = 0$, which implies $\lambda_0 + \frac{1}{n} \in B$, contradicts that $\sup B = \lambda_0$.

As $E_{\lambda_0} = \bigcup_{n=1}^{\infty} E_{\lambda_0+\frac{1}{n}}$, we have $m(E_\lambda) = 0$. Thus $f(x) \le \lambda_0$ a.e., hence $f(x) = \lambda_0$ a.e.

Note that if we assume $f : \mathbb{R}^1 \to [-\infty, +\infty]$ be measurable, then the condition is also true, $\exists\ c \in [-\infty, +\infty]$ s.t. $f(x) = c$ a.e.. Also f needs not be a constant.

Example: $\forall t > s > 0$, $\frac{t}{s} \notin Z$,

$$f(x) = \begin{cases} 1 & x \neq lt + ms,\ l, m \in Z; \\ 0 & x = lt + ms,\ l, m \in Z. \end{cases}$$

That $f(x + t) = f(x)$, $f(x + s) = f(x)$ is trivially true. But f is not a constant, although $B = \{x = lt + ms,\ l, m \in Z\}$ is countable, $m(B) = 0$, $f = 1$ a.e. in $\mathbb{R}^1$. □

5 If $A \subset \mathbb{R}^1$ and $B \subset \mathbb{R}^1$, define $A + B = \{a + b : a \in\ A,\ b \in\ B\}$. Suppose $m(A) > 0$, $m(B) > 0$. Prove that $A + B$ contains a segment, by completing the following outline.

There are points a_0 and b_0 where A and B have metric density 1. Choose a small $\delta > 0$. Put $c_0 = a_0 + b_0$. For each ϵ, positive or negative, define B_ϵ to be the set of all $c_0 + \epsilon - b$ for which $b \in B$ and $|b - b_0| < \delta$. Then $B_\epsilon \subset (a_0 + \epsilon - \delta, a_0 + \epsilon + \delta)$. If δ was well chosen and $|\epsilon|$ is sufficiently small, it follows that A intersects B_ϵ, so that $A + B \supset (c_0 + \epsilon_0, c_0 + \epsilon_0)$ for some $\epsilon_0 > 0$.

Let C be Cantor's "middle thirds" set and show that $C + C$ is an interval, although $m(C) = 0$. (See also Exercise 19, Chap.9.)

Solution: Since A and B are Lebesgue measurable, $\chi_A, \chi_B \in L^1_{loc}(\mathbb{R}^1)$ by the variant of TH. 7.6 (P139 or see P141, line (-6)), for almost very point the metric density of A (or B) is 1. So as $m(A) > 0$, $m(B) > 0$, there are $a_0 \in A$, $b_0 \in B$, such that

$$\lim_{r \to 0^+} \frac{m(A \cap B(a_0, r))}{m(B(a_0, r))} = \lim_{r \to 0^+} \frac{m(A \cap B(b_0, r))}{m(B(b_0, r))} = 1, \tag{1}$$

and a_0 is a Lebesgue point of χ_A and b_0 is a Lebesgue point of χ_B. Set $c_0 = a_0 + b_0$, $\forall \epsilon$ (positive or negative), define $B^\delta_\epsilon = \{c_0 + \epsilon - b : b \in B,\ |b - b_0| < \delta\}$, where $\delta > 0$ is to be determined. Now, for all $x = c_0 + \epsilon - b \in\ B^\delta_\epsilon$, we have

$$|x - (c_0 + \epsilon - b_0)| = |c_0 + \epsilon - b - (c_0 + \epsilon - b_0)| = |b - b_0| < \delta.$$

But $c_0 - b_0 = a_0$, so $|x - (a_0 + \epsilon)| < \delta$, thus $B^\delta_\epsilon \subset (a_0 + \epsilon - \delta, a_0 + \epsilon + \delta)$. For ϵ small enough, we choose $\delta = 2|\epsilon|$, then $(a_0 + \epsilon - \delta, a_0 + \epsilon + \delta) \subset (a_0 - 3|\epsilon|, a_0 + 3|\epsilon|) = B(a_0, 3|\epsilon|)$.

Notice that by (1), for $|\epsilon|$ small enough, $B_\epsilon \triangleq B^{2|\epsilon|}_\epsilon$ satisfies: $\exists\ \epsilon_0$ s.t. for all ϵ *s.t.* $|\epsilon| < \epsilon_0$,

$$m(B_\epsilon) = m(B_\epsilon - (c_0 + \epsilon)) = m(B \cap B(b_0, 2|\epsilon|)) > \frac{B(b_0, 2|\epsilon|}{2} = 2|\epsilon| > 0,$$

and $B_\epsilon \subset B(a_0, 3|\epsilon|$. Then we must have $A \cap B_\epsilon \neq \emptyset$ for all ϵ small. Otherwise, from (1) we know that there exists $0 < \tilde{\epsilon_0} < \epsilon_0$ s.t. for all ϵ *s.t.*$|\epsilon| < \tilde{\epsilon_0}$,

$$m(A \cap B(a_0, 3|\epsilon|)) > \frac{5}{6} m(B(a_0, 3|\epsilon|)) = 5|\epsilon|. \tag{2}$$

If $A \cap B_{\epsilon'} = \emptyset$ for some ϵ' with $0 < |\epsilon'| < \tilde{\epsilon_0}$, then (2) gives that

$$\begin{aligned} 5|\epsilon'| &= \frac{1}{2}m(B(a_0, 3|\epsilon'|)) < m(A \cap B(a_0, 3|\epsilon'|)) \\ &= m(A \cap B(a_0, 3|\epsilon'|)) - m(A \cap B_{\epsilon'}) \\ &= m(A \cap [B(a_0, 3|\epsilon'|) - B_{\epsilon'}]) \quad \because B_{\epsilon'} \subset B(a_0, 3|\epsilon'|) \\ &\leq m([B(a_0, 3|\epsilon'|) - B_{\epsilon'}]) = m(B(a_0, 3|\epsilon'|)) - m(B_{\epsilon'}) \\ &= 6|\epsilon'| - 2|\epsilon'| = 4|\epsilon'|, \end{aligned}$$

a contradiction.

So $\exists\ \tilde{\epsilon_0} > 0$ small enough s.t.

$$A \cap B_\epsilon \neq \emptyset \ \textit{for all}\ 0 < |\epsilon| < \tilde{\epsilon_0}.$$

If $\epsilon = 0$, then $\forall \delta > 0$, $B_\epsilon^d elta = \{c_0 - b, b \in B, |b - b_0| < \delta\}$, then $a_0 = c_0 - b_0 \in B_\epsilon^d elta$ for any $\delta < 0$. So if $\epsilon = 0$, let $\delta = \tilde{\epsilon_0}$, we still have

$$a_0 \in A \cap B_\epsilon \neq \emptyset, \ \textit{if}\ B_\epsilon = B_\epsilon^{\tilde{\epsilon_0}}.$$

So, $\exists\ \tilde{\epsilon_0} > 0,\ \textit{s.t.}\ A \cap B_\epsilon \neq \emptyset\ \textit{for all}\ 0 < |\epsilon| < \tilde{\epsilon_0}$. We claim that $(c_0 - \tilde{\epsilon_0}, c_0 + \tilde{\epsilon_0}) \subset A + B$. In fact, $\forall x \in (c_0 - \tilde{\epsilon_0}, c_0 + \tilde{\epsilon_0})$, let $\epsilon = x - c_0$,then $|\epsilon| = |x - c_0| < \tilde{\epsilon_0}$. For this ϵ, $\exists\ a \in A$, s.t. $a \in\ B_\epsilon$ i.e. $a = c_0 + \epsilon - b$ with $b \in B$ and $|b - b_0| < 2|\epsilon|$ if $|\epsilon| > 0$, and $|b - b_0| < \tilde{\epsilon_0}$ if $|\epsilon| = 0$. Therefore $a = c_0 + \epsilon - b = C_0 - (x - C_0) - b = x - b$, $\therefore$, $x = a + b$, $a \in A$, $b \in B$, $\therefore$ $x \in A + B$. Hence $(c_0 - \tilde{\epsilon_0}, c_0 + \tilde{\epsilon_0}) \subset A + B$, $A + B$ contains a segment.

Now let C be the Cantor's middle thirds set, $m(C) = 0$. We will show that $C + C = [0, 2]$ by showing that $\frac{1}{2}C + \frac{1}{2}C = [0, 1]$. Indeed, $\forall x \in C$, $x = 2\Sigma_{i=1}^{\infty} \frac{a_i}{3}$ where $a_i = 0\ \textit{or}\ 1$. If $x, y \in \frac{1}{2}C$, then

$$x = \Sigma_{i=1}^{\infty} \frac{a_i}{3^i}, \quad y = \Sigma_{i=1}^{\infty} \frac{b_i}{3^i},$$

for some $a_i, b_i \in \{0, 1\}$, $i = 1, 2, 3, \cdots$. Then

$$x + y = \Sigma_{i=1}^{\infty} \frac{a_i + b_i}{3^i}, \ \textit{where}\ a_i + b_i \in \{0, 1, 2\}.$$

So as $\Sigma_{i=1}^{\infty} \frac{2}{3^i} = 2 \cdot \frac{1/3}{1-1/3} = 1$, $x + y \in [0, 1]$.

On the other hand, $\forall a \in [0, 1]$, $a = \Sigma_{i=1}^{\infty} \frac{c_i}{3^i}$ with $c_i \in \{0, 1, 2\}$. Define

$$a_i = \begin{cases} 0 & \textit{if}\ c_i = 0,\ \textit{or}\ 1; \\ 1 & \textit{if}\ c_i = 2, \end{cases}$$

$$b_i = \begin{cases} 0 & \textit{if}\ c_i = 0; \\ 1 & \textit{if}\ c_i = 1,\ \textit{or}\ 2. \end{cases}$$

Then, $C_i = a_i + b_i$. Let $x = \Sigma_{i=1}^{\infty} \frac{a_i}{3^i}$, $y = \Sigma_{i=1}^{\infty} \frac{b_i}{3^i}$, then $x, y \in \frac{1}{2}C$, $a = x + y$, $[0, 1] \subset \frac{1}{2}C + \frac{1}{2}C$ Thus $[0, 1] = \frac{1}{2}C + \frac{1}{2}C$, $[0, 2] = C + C$.

NOTE: This example demonstrates that even if $m(A) = 0, m(B) = 0$, it is possible that $A + B$ contains an interval and thus $m(A + B) > 0$. □

6. suppose G is a subgroup of R^1,(relative to addition),G$\neq R^1$,and G is lebesgue measurable .Prove that then m(G)=0. (Hint use Exercise 5)

Proof: Since G is a subgroup of R^1 W.r.t. addition,then $\forall$ x , y $\in$ G. Also x $\in$ G, implies a x$\in$ G ,$\forall$ $a \in Z$,and 0$\in$ G. If on the contrary ,m(G)> 0,then by Exercise 5,$\exists$ segment $(a, b) \subset$ G+G$\subset$G. Hence $\exists$ c,d , c< d, [c,d]$\subset$ G. We show then we want have G=R^1 to get a contradiction to the condition G$\neq$ R^1. Then [0,d-c]$\subset$ G. and R^1=$\bigcup k[0, d-c] \subset G$. Giving that R^1=G. Indeed ,$\forall x \in [0, d-c], y = x + c \in [c, d] \subset G$,and $c \in G$,$\Longrightarrow$ -c $\in$G.$\therefore$ x=y+c-c$\in$G.$\therefore$ [0,d-c] $\subset$ G.$\Longrightarrow$ R^1=G.So we get a contradition .So m(G)=0. $\square$

8. Let V=(a,b) be a bounded segment in R_1. Choose segments $W_n \subset$ V in such a way that their union W is dense in V and the set K=V-W has m(k)>0.Choose continuous functions φ_n so that φ_n (x)=0 outside of W_n. $0 < \varphi(x) < 2^{-n}$ in W_n. Put $\varphi = \sum \varphi_n$ and define T(x)=$\int_a^x \varphi(t)dt$ $(a < x < b)$. Prove the following statements.

(a) T Satisfies the hypotheses of theorem 7.26, with Z=V.

(b) T' is continuous , T' (x) =0 on K, m(T(K))=0.

(c) If E is a nonmeasurable subset of K (See theorem 2.2) and A=T(E), Then χ_A is lebesgue measurable ,but $\chi_A T$ is not .

(d) The functions φ_n can be chosen that the resulting T is an infinitely differentiable homeomorphism of V onto some segment and (c) still holds.

Solution: Before we answer (a)-(c) ,we notice first that such W_n does exists according to the proof of EX7CH2. and W_n can even be choose such that $W_i \bigcap W_j = \Phi$,if i $\neq$ j , also $\varphi_n \geq 0$,$W_n = (\alpha_n, \beta_n)$. $\varphi_n(\beta_n) = \varphi_n(\alpha_n) = 0$.

(a) $0 < \varphi_n(x) < 2^{-n}$ on W_n. $\varphi_n = 0$ on W_n^c. $\Longrightarrow 0 \leq \varphi_n(x) \leq 2^{-n}$ on V. $\because \sum\limits_{n=1} 2^{-n} < +\infty$,$\therefore$ $\varphi = \Sigma\varphi_n$ is uniformly convergent .hence $\varphi(x)$ is continuous on V ,and $T(x) = \int_a^x \varphi(t)dt$ is differentiable on $a < x < b$, as (a,b) is a bounded interval ,and $\varphi(t)$ can be extended 0 outside V, $\varphi \in C^0[a, b]$ in fact!

In th7.26, if Z=V, then

(i) T: V$\longrightarrow$ R^1 is continuous

(ii) Z=V is clearly lebesgue measurable .T is differentiable in every point of Z=V, in fact $T'(x) = \varphi(x), \forall x \in V$.

(iii) $m(T(V - V)) = m(T(\Phi)) = m(\Phi) = 0$.

Also T is one-to-one ,If T(x)=T(y) if $y > x$. then

$$
\begin{aligned}
T(y) &= \int_a^y \varphi(t)dt \\
&= \int_a^x \varphi(t)dt + \int_x^y \varphi(t)dt \\
&= T(x) + \int_x^y \varphi(t)dt
\end{aligned}
$$

so $0 = \int_x^y \varphi(t)dt$ .But $y > x$ and W is dense in V $\Longrightarrow W \bigcap (x, y) \neq \Phi$, so $\exists x_0 \in W \bigcap (x, y)$,$\therefore$ $\exists n, x_0 \in W_n \bigcap (x, y), \varphi_n(x_0) > 0$ on $W_n \Longrightarrow \varphi(x_0) > 0, \varphi \in C^0(a, b)$. $\Longrightarrow \exists\delta > 0$, small enough S.T. $I_\delta = (x_0 - \delta, x_0 + \delta) \subset W_n \bigcap (x, b)$,$\varphi(Z) > 0$ on I_δ.

$\int_x^y \varphi(z)dz \geq \int_{I_\delta} \varphi(z)dz > 0$, a contradiction . So T(x)=T(y) $\Longrightarrow x = y$. and T is one-to-one . (a) is proved.

(b) T$'$=$\varphi(x)$ is continuous , $\varphi(x) = \sum\limits_n \varphi_n(x) = 0$. If $\forall n, \varphi_n(x)$, so if $x \in K = V - W$, $x \in W_n^c, \forall n$. $\varphi_n(x) = 0 \forall n$, so $T^{'}(x) = \varphi(x) = 0$ on K .

Since T is one-to-one.

$$T(K) = T(V - W) = T(V) - T(W)$$

Since each open interval $(\alpha, \beta) = \bigcup\limits_{n=1}^{+\infty} [\alpha + \frac{1}{n}, \beta - \frac{1}{n}]$, is σ-compact . T is continuous . Th2.10(P38), implies that $T(\alpha, \beta) = \bigcup\limits_{n=1}^{+\infty} T[\alpha + \frac{1}{n}, \beta - \frac{1}{n}]$, is σ-compact ,hence a F_σ set.

$W = \bigcup W_n, T(W) = \bigcup\limits_{n=1}^{+\infty} T(W_n), \Longrightarrow T(W)$ is F_σ set.

T(K)=T(V)-T(W) is a Borel set , T(K)$\in$M (Lebesgue measurable).

Hence f(x)=$\chi_{T(K)}(x)$ is a lebesgue measurable functions on V.

$fT(y) = \chi_{T(K)}(T(y)) = 1$, if $y \in K$,

$fT(y) = \chi_{T(K)}(T(y)) = 0$, if $y \in K^c$, This is because that T is one-to-one , $\forall y \in K, T(y) \in K, \chi_{T(K)}(T(y)) = 1$. $\forall y \in K^c$, if $T(y) \in T(K) \Longrightarrow \exists Z S.t. T(y) = T(Z)$ T is one-to-one, so y=z.This is a contradiction . So $y \in K^c, \Longrightarrow T(y) \overline{\exists} T(K)$ $\chi_{T(K)}(T(y)) = 0$. So $fT(y) = \chi_K(y)$, By Th7.26 since $K \subset V, T(K) \subset T(V)$,

$$\begin{aligned}
m(T(K)) &= \int\limits_{T(V)} \chi_{T(K)}(x) dm(x) \\
&= \int\limits_V \chi_{T(K)}(T(y)) \mid J_T \mid dm(y) \\
&= \int\limits_V \chi_K(y) \mid J_T(y) \mid dm(y) \\
&= \int\limits_K \chi_K(y) \mid J_T \mid dm(y) \\
&= \int\limits_K 0 dm(y) \\
&= 0
\end{aligned}$$

so (b) is true.

(c) If E is a nonmeasurable subset of K. (By Th2.22 and m(K) > 0, Such a E does exist!) and A=T(E), Then A$\subset$T(K),m(T(K))=0 By (b) A is lebesgue measurable and m(A)=0, SO χ_A is measurable . Since T is one-to-one.

$\chi_A(T(y)) = \chi_{T(E)}(T(y)) = 1, y \in E;$

$\chi_A(T(y)) = \chi_{T(E)}(T(y)) = 0, y \in E^c$

$\chi_A(T(y)) = \chi_E(y)$. is not a measurable function ,since E is not measurable .

(d) The proof of (c) tell the only fact that T is one-to-one, and the higher differentiability does not play any role . So the remaining problem is to construct T such that T is infinitely differentiable .

Let $W_n = (\alpha_n, \beta_n), W_i \bigcap W_j = \Phi$, if i$\neq$ j,

$$\varphi_n(x) = \begin{cases} 2^{-n}\exp(\frac{(\frac{\beta_n-\alpha_n}{2})^2}{(x-\frac{\alpha_n+\beta_n}{2})^2-(\frac{\beta_n-\alpha_n}{2})^2}), & x \in (\alpha_n, \beta_n); \\ 0, & x \in (\alpha_n, \beta_n)^c. \end{cases}$$

then $\varphi_n \in C^\infty(R^1), \varphi_n = 0$ on W_n^c. $\varphi(x) = \sum_{n=1}^{+\infty} \varphi_n(x)$, is infinitely differentiable . Because in W_n, $\varphi(x) = \varphi_n(x), x \in W_n, \varphi(x) = 0, x \in W^c$. and $\varphi_n \in C^\infty(R^1)$. □

2 For $\delta > 0$, let I(δ) be the segment $(-\delta, \delta) \subset R^1$. Given α and β, $0 \le \alpha \le \beta$. Construct a measurable set $E \subset R^1$ so that the upper and lower limits of $m(E \bigcap I(\delta))/2\delta$ are β and α, respectively ,as $\delta \longrightarrow 0$.

Solution: If $\alpha = \beta \in [0, 1]$, let $A = \bigcup_{n=1}^{+\infty}(\frac{\alpha}{n+1} + \frac{(1-\alpha)}{n}, \frac{1}{n})$.

If $\alpha = 0, \beta = 1$, then let

$$A = (\frac{1}{2!, \frac{1}{1!}}) \bigcup (\frac{1}{4!}, \frac{1}{3!}) \bigcup \cdots \bigcup (\frac{1}{(2n)!}, \frac{1}{(2n-1)!})$$

If $0 < \alpha < \beta < 1$, then let $\eta = \frac{\beta(1-\alpha)}{\alpha(1-\beta)} > 1, \theta = \frac{1-\beta}{1-\alpha} \in (0, 1)$, $A = \bigcup_{n=1}^{+\infty}(\theta\eta^{-n}, \eta^{-n})$, If $\alpha \in (0, 1), \beta = 1$, let $A = \bigcup_{n=1}^{+\infty}(\frac{\alpha}{n!} + \frac{(1-\alpha)}{(n+1)!}, \frac{1}{n!})$

If $\alpha = 0, \beta \in (0, 1)$, let $A = \bigcup_{n=1}^{+\infty}(\frac{1}{(n+1)!}, \frac{\alpha}{n!} + \frac{1-\alpha}{(n+1)!})$, Finally take $E = A \bigcup (-A)$, then E satisfies the requirement .

Notice that $A \bigcap (-A) = \Phi$, $A \bigcap (0, \delta) = (-\delta, 0) \bigcap (-A)$,So it enough to show that $\limsup_{\delta \longrightarrow 0} \frac{m(A \bigcap (0,\delta)}{\delta} = \beta$,

$\liminf_{\delta \longrightarrow 0} \frac{m(A \bigcap (0,\delta))}{\delta} = \alpha$.

(1).If $\alpha = \beta \in [0, 1]$,take $A = \bigcup_{n=1}^{+\infty}(\frac{\alpha}{n+1} + \frac{(1-\alpha)}{n}, \frac{1}{n})$,$\forall r \in (0, 1), \exists n, n+1 \in N$, S.t. $r \in [1/n + 1, 1/n[$.

So

$$\alpha \frac{n}{n+1} = \frac{\alpha \frac{1}{n+1}}{\frac{1}{n}} \le \frac{(A \bigcap [0, r[)}{r} \le \alpha \frac{\frac{1}{n}}{\frac{1}{n+1}} = \alpha((n+1)/n)$$

,Letting $r \longrightarrow 0^+$,then $n \longrightarrow +\infty$, so $\lim_{r \longrightarrow 0_+} \frac{A \bigcap [0,r]}{r} = \alpha$.

(2). If $\alpha = 0, \beta = 1$, take

$$A = (\frac{1}{2!}, \frac{1}{1!}) \bigcup (\frac{1}{4!}, \frac{1}{3!}) \bigcup \cdots \bigcup (\frac{1}{(2n)!}, \frac{1}{(2n-1)!})$$

,We have if $r_n = \frac{1}{(2n)!}$,then

$$0 \le \frac{A \bigcap [0, r_n[}{r_n} \le \frac{\frac{1}{(2n+1)!}}{(2n)!} = \frac{1}{2n+1} \longrightarrow 0$$

,as

$$n \longrightarrow +\infty, (\because A \bigcap [0, r_n) \subset [0, \frac{1}{(2n+1)!}])$$

If $r_n = \frac{1}{(2n-1)!}$,then

$$A \bigcap [0, r_n) \supset (\frac{1}{(2n)!}, \frac{1}{(2n-1)!})$$

$$1 \geq \frac{| A \bigcap [0, r_n[|}{r_n} \geq \frac{\frac{1}{(2n-1)!} - \frac{1}{(2n)!}}{\frac{1}{(2n-1)!}} = 1 - \frac{1}{2n} \longrightarrow 1$$

, as $n \longrightarrow +\infty$,

so $\limsup\limits_{r \longrightarrow 0^+} \frac{|A \bigcap [0,r[|}{r} = 1, \liminf\limits_{r \longrightarrow 0^+} \frac{|A \bigcap [0,r[|}{r} = 0.$

(3) If $0 < \alpha < \beta < 1$,take $A = \bigcup\limits_{n=1}^{+\infty} (\theta\eta^{-n}, \eta^{-n})$,where $\eta = \frac{\beta(1-\alpha)}{\alpha(1-\beta)} > 1, \theta = \frac{1-\beta}{1-\alpha} \in (0,1)$, Then as $\theta\eta > 1$,we have $\theta\eta^{-n} > \eta^{-n-1}$,for any $n \in N \bigcup 0$, Now $\forall r \in [0,1], \exists! n = n(r) \in N \bigcup 0$, S.t. $r \in [\eta^{-n-1}, \eta^{-n})$, Then

$$f(r) = \frac{|A \bigcap [0,r[|}{r} = \begin{cases} \frac{\sum\limits_{k=n+1}^{\infty} (1-\theta)\eta^{-k}}{r}, & r \in [\eta^{-n-1}, \theta\eta^{-n}]; \\ \frac{\sum\limits_{k=n+1}^{\infty} (1-\theta)\eta^{-k} + r - \theta\eta^{-n}}{r}, & r \in [\theta\eta^{-n}, \eta^{-n}]. \end{cases}$$

so at η^{-n}, f(r) attains its maximum

$$\frac{\sum\limits_{k=n}^{+\infty} (1-\theta)\eta^{-k}}{\eta^{-n}} = (1-\theta)\sum_{k=0}^{+\infty} \theta^{-k} = (1-\theta)\frac{1}{1-\frac{1}{\eta}} = (1-\theta)\frac{\eta}{\eta - 1} = \frac{(1-\theta)\eta}{\eta - 1} = \beta$$

in $[\theta\eta^{-n-1}, \theta\eta^{-n}]$, and at $\theta\eta^{-n}, f(x)$ take its minimum .

$$\begin{aligned} \frac{\sum\limits_{k=n+1}^{\infty} (1-\theta)\eta^{-k}}{\theta\eta^{-n}} &= \frac{1-\theta}{\theta} \sum_{n=1}^{+\infty} \eta^{-k} \\ &= \frac{1-\theta}{\theta} \frac{\eta^{-1}}{1 - \frac{1}{\eta}} \\ &= \frac{1-\theta}{\theta} \frac{1}{\eta - 1} \\ &= \frac{1 - \frac{1-\beta}{1-\alpha}}{\frac{1-\beta}{1-\alpha}} \frac{1}{\frac{\beta(1-\alpha)}{\alpha(1-\beta) - 1}} \\ &= \frac{\beta - \alpha}{1 - \beta} \frac{1}{\frac{\beta - \alpha}{\alpha(1-\beta)}} \\ &= \alpha \end{aligned}$$

, do not depend on n, we have $\alpha \leq f(r) \leq \beta, \forall r \in (0,1)$.

If we let $r_n = \eta^{-n}$,we have $f(r_n) = \beta$, and if we let $r_n = \theta\eta^{-n}$, we have $f(r_n) = \alpha$, so $\liminf\limits_{r \longrightarrow 0^+} f(r) = \alpha, \limsup\limits_{r \longrightarrow 0^+} f(r) = \beta$.

(4) If $\alpha \in (0,1), \beta = 1$, take $A = \bigcup\limits_{n=1}^{+\infty} (\frac{1}{(n+1)!}, \frac{\alpha}{n!} + \frac{1-\alpha}{(n+1)!}), \forall r \in (0,1), \exists! n = n(r) \in N$, S.t. $r \in [\frac{1}{(n+1)!}, \frac{1}{n!})$, similar to what we did in (3).

$$f(r) = \frac{|A \bigcap [0,r[|}{r} = \begin{cases} \frac{\sum\limits_{k=n+1}^{\infty} \alpha(\frac{1}{k!} - \frac{1}{(1+k)!}) + r - \frac{1}{(n+1)!}}{r}, & r \in [\frac{1}{(n+1)!}, \frac{\alpha}{n!} + \frac{1-\alpha}{(n+1)!}]; \\ \sum\limits_{k=n}^{\infty} \alpha(\frac{1}{k!} - \frac{1}{(1+k)!}), & r \in [\frac{\alpha}{n!} + \frac{1-\alpha}{(n+1)!}, \frac{1}{n!}[. \end{cases}$$

i.e. $f(r) = \begin{cases} \frac{\frac{\alpha}{(n+1)!} + r - \frac{1}{(n+1)!}}{r}, & r \in [\frac{1}{(n+1)!}, \frac{\alpha}{n!} + \frac{1-\alpha}{(n+1)!}]; \\ \frac{\frac{\alpha}{n!}}{r}, & r \in [\frac{\alpha}{n!} + \frac{1-\alpha}{(n+1)!}, \frac{1}{n!}[. \end{cases}$

for $g(r)=\frac{\frac{\alpha}{(n+1)!}+r-\frac{1}{(n+1)!}}{r}$,is nondecreasing in $[\frac{1}{(n+1)!},\frac{\alpha}{n!}+\frac{1-\alpha}{(n+1)!}]$. so it takes the maximum

$$\frac{\frac{\alpha}{(n+1)!}+\frac{\alpha}{n!}+\frac{1-\alpha}{(n+1)!}-\frac{1}{(n+1)!}}{\frac{\alpha}{n!}+\frac{1-\alpha}{(n+1)!}}=\frac{\frac{\alpha}{(n+1)!}+\alpha(\frac{1}{n!}-\frac{1}{(n+1)!})}{\frac{\alpha}{n!}+\frac{1-\alpha}{(n+1)!}}=\frac{\alpha}{\alpha+\frac{(1-\alpha)}{n+1}}$$

,And $h(r)=\frac{\frac{\alpha}{n!}}{r}$, take maximum at $\frac{\frac{\alpha}{n!}}{\frac{\alpha}{n!}+\frac{1-\alpha}{(n+1)!}}=\frac{\alpha}{\alpha+\frac{(1-\alpha)}{n+1}}$, so

$$\max_{[\frac{1}{(n+1)!},\frac{1}{n!}[} f(r)=\frac{\alpha}{\alpha+\frac{1-\alpha}{n+1}}=f(\frac{\alpha}{n!}+\frac{1-\alpha}{(n+1)!})$$

,similarly ,

$$\min_{[\frac{1}{(n+1)!},\frac{1}{n!}[} f(r)=\frac{\frac{\alpha}{(n+1)!}}{\frac{1}{(n+1)!}}=\alpha$$

, so let $r_n=\frac{\alpha}{n!}+\frac{1-\alpha}{(n+1)!}$,then $f(r_n)=\frac{\alpha}{\alpha+\frac{(1-\alpha)}{n+1}}\longrightarrow 1$, let $r_n=\frac{1}{n!}, f(r_n)\longrightarrow\alpha$, $\liminf\limits_{r\longrightarrow 0^+} f(r)=\alpha, \limsup\limits_{r\longrightarrow 0^+} f(r)=1$.

(5) If $\alpha=0,\beta\in(0,1)$,take $A=\bigcup\limits_{n=1}^{+\infty}(\frac{1-\beta}{n!}+\frac{\beta}{(n+1)!},\frac{1}{n!}), \forall r\in(0,1),\exists! n=n(r)\in N$,s.t.$r\in[\frac{1}{(n+1)!},\frac{1}{n!})$ □

11 Assume that $1<p<\infty$,f is absolutely continuous on [a,b] , $f^{'}\in L^p$, and $\alpha=1/\epsilon$, where ϵ is the conjugate exponent to p . Prove that $f\in$Lip α.

Solution: By the fact that f is A.C. on [a,b],Th7.20 giving that
$f(x)-f(y)=\int_y^x f^{'}(t)dt$, for any $a<y<x<b$.
Since $f^{'}\in$ L^p, Ho¨lder's inequality gives that

$$\begin{aligned}
\mid f(x)-f(y)\mid &\le \int_y^x \mid f^{'}(t)\mid dt\\
&\le (\int_y^x \mid f^{'}\mid^p dt)^{1/p}(\int_y^x dt)^{1/q}\\
&\le (\int_a^b \mid f^{'}\mid^p dt)^{1/p}\mid x-y\mid^{1/\epsilon}\\
&= \parallel f^{'}\parallel_{L^p[a,b]}\mid x-y\mid^{\alpha}
\end{aligned}$$

so $\sup\limits_{x,y\in(a,b),x\neq y}\mid f(x)-f(y)\mid/\mid x-y\mid^{\alpha}\le\parallel f^{'}\parallel_{L^p[a,b]}<+\infty$.
so $f\in$ Lipα. □

Chapter 7 Differentiation

11, Assume that $1 < p < \infty$, f absolutely continuous on $[a,b]$, $f' \in L^p$, and $\alpha = \frac{1}{q}$, where q is the exponent conjugate to p. Prove that $f \in \text{Lip } \alpha$.

Proof: By the fact that f is AC on $[a,b]$, Th7.20 gives that

$$f(x) - f(y) = \int_y^x f'(t)\ dt, \quad \text{for any } a \leq y \leq x \leq b.$$

Since $f' \in L^p$, Hölder's inequality gives that

$$\begin{aligned} |f(x) - f(y)| &\leq \int_y^x |f'(t)|\ dt \leq (\int_y^x |f'|^p\ dt)^{\frac{1}{p}} (\int_x^y\ dt)^{\frac{1}{\epsilon}} \\ &\leq (\int_a^b |f'|^p)^{\frac{1}{p}} |x - y|^{\frac{1}{\epsilon}} = \|f'\|_{L^p[a,b]} |x - y|^{\alpha} \end{aligned}$$

So

$$\sup_{\substack{x,y \in [a,b] \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}} \leq \|f'\|_{L^p[a,b]}$$

Thus

$$f \in \text{Lip } \alpha.$$

12, Suppose $\varphi : [a,b] \to \mathbb{R}^1$ is nondecreasing .

(a) Show that there is a left-continuous nondecreasing f on $[a,b]$ so that $\{f \neq \varphi\}$ is at most countable. [Left-continuous means: if $a < x \leq b$ and $\epsilon > 0$, then there is a $\delta > 0$ so that $|f(x) - f(x - t)| < \epsilon$ whenever $0 < t < \delta$.]

(b) Imitate the proof of Theorem 7.18 to show that there is a positive Borel measure μ on $[a,b]$ for which

$$f(x) - f(a) = \mu([a,x]) \quad (a \leq x \leq b).$$

(c) Deduce from (b) that $f'(x)$ exists a.e. $[m]$, that $f' \in L^1(m)$, and that

$$f(x) - f(a) = \int_a^x f'(t)\ dt + s(t) \quad (a \leq x \leq b)$$

where s is nondecreasing and $s'(x) = 0$ a.e. $[m]$.

(d) Show that $\mu \perp m$ if and only if $f'(x) = 0$ a.e. $[m]$, and that $\mu \ll m$ if and only if f is AC on $[a,b]$.

(e) Prove that $\varphi'(x) = f'(x)$ a.e. $[m]$.

Solution of (a): Since φ is nondecreasing, $\exists$ a at most countable set $A = \{x_i\}_{i=1}^{+\infty}$ s.t. for $\forall\ \ x \in [a,b] \setminus A$, $\varphi(x)$ is continuous at x and here we can define

$$f(x) = \begin{cases} \varphi(x) \quad \text{if} \quad x \in [a,b] \setminus A; \\ \varphi(x_i - 0) = \lim_{h \to 0+} \varphi(x_i - h) \quad \text{if} \quad x = x_i \in A. \end{cases}$$

Notice that $\varphi : [a,b] \to \mathbb{R}^1$ nondecreasing implies that $\varphi(x_i - 0)$ is a finite real number and f is a left-continuous.

Solution of (b): Let $k_i = f(x_i+0) - f(x_i-0) = f(x_i+0) - f(x_i)$. We know $0 < \sum_i k_i \le f(b) - f(a) < +\infty$, since $f : [a,b] \to \mathbb{R}^1, |f(b)| < +\infty$, and $|f(a)| < +\infty$. So $J(x) = \sum_i k_i \chi_{]x_i,b]}$ is uniformly convergent on $[a,b]$.

Claim 1: [by the left-continuity] f(x)-J(x) is continuous on $[a,b]$.
In fact, f is continuous at x_0 if $x_0 \in A = \{x_1, x_2, \cdots\}$, and taking $h \to 0$, we have

$$\lim_{h\to 0} J(x_0+h) = \sum_{i=1}^{+\infty} k_i \lim_{h\to 0} \chi_{]x_i,b]}(x_0+h).$$

$\forall$ fixed i, if $x_0 \in]x_i, b]$, $x_0 + h \in]x_i, b]$ for $|h| \ll 1$, So

$$\chi_{]x_i,b]}(x_0+h) = 1 = \chi_{]x_i,b]}(x_0),$$

thus

$$\lim_{h\to 0} \chi_{]x_i,b]}(x_0+h) = \chi_{]x_i,b]}(x_0).$$

If $x_0 \notin]x_i, b]$, then $x_0 < x_i$. So $x_0 + h < x_i$, for all $|h| \ll 1$. So $\chi_{]x_i,b]}(x_0+h) = 0 = \chi_{]x_i,b]}(x_0)$.

$$\lim_{h\to 0} \chi_{]x_i,b]}(x_0+h) = \chi_{]x_i,b]}(x_0)$$

By this we can deduce

$$\lim_{h\to 0} J(x_0+h) = J(x_0).$$

This show $f(x) - J(x)$ is continuous at x_0.

If $x_0 = x_{i_0} \in A$, and taking $h \to 0$, when $i \ne i_0$, we have

$$\chi_{]x_i,b]}(x_0+h) = \chi_{]x_i,b]}(x_0).$$

And $x_{i_0} + h \notin]x_{i_0}, b]$ if $h < 0$.

$$\begin{aligned}
\lim_{h\to 0^-} J(x_0+h) &= \sum_{i=1}^{+\infty} k_i \lim_{h\to 0^-} \chi_{]x_i,b]}(x_{i_0}+h) \\
&= \sum_{i=1}^{+\infty} k_i \chi_{]x_i,b]}(x_0) \\
&= J(x_0).
\end{aligned}$$

And $\lim_{h\to 0^-} f(x_{i_0}+h) = f(x_0)$, ($f$ is left-continuous.)
When $h \to 0^+$, if $i \ne i_0$, then

$$\chi_{]x_i,b]}(x_0+h) = \chi_{]x_i,b]}(x_0)$$

$$\chi_{]x_{i_0},b]}(x_0+h) = 1, \ \chi_{]x_{i_0},b]}(x_{i_0}) = 0$$

$$\begin{aligned}
&\lim_{h\to 0^+} [f(x_{i_0}+h) - J(x_{i_0}+h)] \\
= \ & f(x_{i_0}+0) - (\sum_{i\ne i_0} k_i \chi_{]x_i,b]}(x_0) + k_{i_0}) \\
= \ & f(x_{i_0}+0) - (f(x_{i_0}+0) - f(x_{i_0})) - \sum_{i=1}^{+\infty} k_i \chi_{]x_i,b]}(x_0) \\
= \ & f(x_{i_0}) - J(x_{i_0}).
\end{aligned}$$

$\therefore f(x) - J(x)$ is also right-continuous at x_{i_0}. The claim is proved.

Claim 2: $f(x) - J(x)$ is nondecreasing on $[a, b]$.

In fact $\forall x, y \in [a, b]$ with $x < y$.

$$\begin{aligned}
& J(y) - J(x) = \sum_{i=1}^{+\infty} k_i \chi_{]x_i,b]}(y) - \sum_{i=1}^{+\infty} k_i \chi_{]x_i,b]}(x) \\
= & \sum_{\{i: x_i < x\}} k_i \chi_{]x_i,b]}(y) + \sum_{\{i: x \le x_i < y\}} k_i \chi_{]x_i,b]}(y) - \sum_{\{i: x_i < x\}} k_i \chi_{]x_i,b]}(x) + 0 \\
= & \sum_{\{i: x \le x_i < y\}} k_i \\
\le & f(y) - f(x).
\end{aligned}$$

This is because that for any finite subset B 0f $\{i : x \le x_i < y\}$, without generality we set $B = \{i_1, \cdots, i_m\}$ and $x \le x_{i_k} < y, \ k = 1, \cdots, m$, where $x_{i_1} < x_{i_2} < \cdots < x_{i_m}$. Then it is clear that

$$\sum_{i \in B} k_i = \sum_{i \in B} (f(x_i + 0) - f(x_i)) \le f(y) - f(x)$$

Next, $\{i : x \le x_i < y\} = \{l_1, l_2, \cdots, l_m, \cdots\}$. Let $B_m = \{l_1, l_2, \cdots, l_m\}$, then $\lim\limits_{m \to +\infty} \sum\limits_{i \in R_m} k_i \le f(y) - f(x)$, and by monotone convergence theorem using to discrete measure space $\{2^N, \mu\}, \mu$ is the counting measure, then

$$\sum_{i \in B_m} k_i = \sum_{i=1}^{+\infty} \chi_{B_m}(i) k_i$$

$$\lim_{m \to \infty} \sum_{i \in B_m} k_i = \lim_{i=1} \sum_{i=1}^{+\infty} \chi_{B_m}(i) k_i = \sum_{\{i: x \le x_i < y\}} k_i$$

$$\therefore \sum_{\{i: x \le x_i < y\}} k_i \le f(y) - f(x).$$

$$\therefore J(y) - J(x) \le f(y) - f(x)$$

$$\Rightarrow f(x) - J(x) \le f(y) - J(y).$$

So $g(x) = x + f(x) - J(x)$ is strictly increasing and continuous on $[a, b]$, g^{-1} is also strictly increasing and continuous on $[g(a), g(b)]$. So for any Borel set $E \subset [a, b]$, $g(E) = (g^{-1})^{-1}(E)$ is a Borel set.

(In fact, $\forall$ function monotone on $[a, b]$, $\psi(x)$ is a Borel function. That is $\forall$ a open set $V \subset \mathbb{R}^1$, $\varphi^{-1}(V)$ is a Borel set.

Proof: Let $A = \{x_i\}_{i=1}^{+\infty}$ be distinct point set of ψ, at most countable.

$\forall$ open set v, $\psi^{-1}(V) = \big(\psi^{-1}(V) \cap ([a, b] \subset A)\big) \cup \big(\psi^{-1}(V) \cap A\big)$

$\psi \in C^0([a, b] \subset A)$, so $\psi^{-1} \cap ([a, b] \subset A)$ is a open subset of the subspace $[a, b] \subset A$. Then there exits an open set W in $\mathbb{R}^1$, such that

$$\psi^{-1}(V) \cap ([a, b] \subset A) = W \cap ([a, b] \subset A).$$

Where A is the union of countable close sets $\{x_i\}$, then is a F_σ set.

$\therefore \psi^{-1}(V) \cap A$ is a F_σ set. And since $W \cap ([a, b] \subset A)$ is a Borel set, $\psi^{-1}(V)$ is a Borel set.

Furthermore let $\mathcal{W} = \{E \subset \mathbb{R}^1, \psi^{-1}(E)$ is a Borel set$\}$, it is easy to prove $\mathcal{W}$ is a $\sigma-$ algebra, and contains all the open set in $\mathbb{R}^1$. So it contains all Borel sets. Therefore $\forall$ Borel set E, $\psi^{-1}(E)$ is a Borel set.

Now g^{-1} is monotone, $(g^{-1})^{-1} = g$ maps Borel sets to Borel sets. Define

$$\tilde{\mu}(E) = m(g(E)),$$

then $\tilde{\mu}$ is a positive finite Borel measure on $[a, b]$.

$$\mu(E) = \tilde{\mu}(E) - m(E) + \sum_{i=1}^{+\infty} k_i \delta_{x_i}(E)$$

where

$$\delta_{x_i}(E) = \begin{cases} 1 & \text{if } x_i \in E, \\ 0 & \text{if } x_i \notin E. \end{cases}$$

is the Unit-mass measure.

Claim 3: μ is a positive Borel measure.

Proof: μ is Complex but real Borel measure. For any open segment $(\alpha, \beta) \subset [a, b]$, since g is continuous and creasing. By Mean Value theorem,

$$m(g(\alpha, \beta)) = m(g(\alpha), g(\beta)) = g(\beta) - g(\alpha) = \beta - \alpha + f(\beta) - f(\alpha) - J(\beta) + J(\alpha).$$

$$\begin{aligned}
\tilde{\mu} &= m(g(\alpha, \beta)) - m(\alpha, \beta) + \sum k_i \delta_{x_i}(\alpha, \beta) \\
&= \beta - \alpha + f(\beta) - f(\alpha) - J(\beta) + J(\alpha) - (\beta - \alpha) + \sum_{\{i:\alpha < x_i < \beta\}} k_i. \\
&= f(\beta) - f(\alpha) - (J(\beta) - J(\alpha)) + \sum_{\{i:\alpha < x_i < \beta\}} k_i \\
&= f(\beta) - f(\alpha) - \sum_{\{i:\alpha < x_i \leq \beta\}} k_i \chi_{]x_i, b]}(\beta) + \sum_{\{i:\alpha < x_i < \beta\}} k_i \\
&= f(\beta) - f(\alpha) - \sum_{\{i:\alpha < x_i < \beta\}} k_i \chi_{]x_i, b]}(\beta) + \sum_{\{i:\alpha < x_i < \beta\}} k_i \\
&= f(\beta) - f(\alpha) - \sum_{\{i:\alpha < x_i < \beta\}} k_i + \sum_{\{i:\alpha < x_i < \beta\}} k_i \\
&= f(\beta) - f(\alpha) \geq 0.
\end{aligned}$$

So for any open set $V = \cup(\alpha_i, \beta_i)$,

$$\mu(V) = \sum_{i=1}^{+\infty} \mu(\alpha_i, \beta_i) \geq 0$$

and hence as μ is a real valued measure, it is a regular. So for any Borel set E, $\tilde{\mu}(E) = \inf\{\tilde{\mu}(V),\ E \subset V,\ V \text{is open}\} \geq 0$.(Th 2.18)

By Th 6.10 (Lebesgue-Radon-Nikodym), $\exists$ positive finite measure, μ_a, μ_s, such that

$$\mu_a \ll m,\ \mu_s \perp m,\ h \in L^1(m)([a, b])$$

and

$$\mu(E) = \int_E h\ dm + \mu_s(E)$$

for any Borel set $E \subset [a, b]$.

$$\begin{aligned}
\mu([a,x[) &= m(g([a,x[)) - m([a,x[) + \sum_{i=1}^{+\infty} k_i\delta_i([a,x[) \\
&= m([g(a),g(x)[) - m([a,x[) + \sum_{i=1}^{+\infty} k_i\delta_i([a,x[) \\
&= g(x) - g(a) - (x-a) + \sum_{\{i:a\le x_i<x\}}^{+\infty} k_i \\
&= x - a + f(x) - f(a) - \sum_{\{i:a\le x_i<x\}} k_i\chi_{]x_i,b]}(x) + \sum_{\{i:a\le x_i<x\}} k_i \\
&= f(x) - f(a) - \sum_{\{i:a\le x_i<x\}} k_i + \sum_{\{i:a\le x_i<x\}} k_i \\
&= f(x) - f(a)
\end{aligned}$$

$$f(x) - f(a) = \int_a^x h(t)\ dt + \mu_s([a,x[) \triangleq \int_a^x h(t)\ dt + s(x)$$

Since $h \in L^1(m, [a, b])$, a.e.$[m]$, $x \in [a, b]$. So x is a Lebesgue point of h. $F(x) = \int_a^x h\ dm$ is differentiable at Lebesgue point of h, and $F'(x) = h$ a.e.

By Th 7.13 and Th 7.14, $\lim\limits_{i\to\infty} \frac{\mu_s(E_i(x))}{m(E_i(x))} = 0$ a.e.$[m]$ (for any $\{E_i\}$ shrinks to x nicely.)

$$(D)\mu(x) = h(x)\text{a.e.}$$

in particular $\frac{\mu_s([x,x+h[)}{h} = \frac{s(x+h)-s(x)}{h} \to 0$ (or $\frac{\mu_s([x+h,x])}{h} \to 0$) for a.e.$[m]$ as $[x, x+h_i[$ (or $[x+h_i, x[$) shrinks nicely to x when $h_i \to 0^+$ (or $h_i \to 0^-$).
So $\frac{f(x+h)-f(x)}{h} = \frac{F(x+h)-F(x)}{h} + \frac{s(x+h)-s(x)}{h} \to h(x)(h \to 0)$ for a.e. $[m]$ $x \in [a, b]$.

$$\therefore f'(x) = h(x)\text{a.e.}[m] \text{ on } [a,b]$$

$$\therefore f(x) - f(a) = \int_a^x f'(t)\ dm + s(x)$$

(d) If $\mu \perp m$, then $h = 0$ a.e.$[m]$, i.e. $f'(x) = 0$ a.e.$[m]$ and vise versa. $h = 0 \Rightarrow \mu = \mu_s, \mu \perp m$. If $\mu \ll m \Rightarrow \mu_s = 0$.

$$\therefore f(x) - f(a) = \int_a^x f'(t)\ dt.$$

$f^1 \in L^1 \Rightarrow f$ is AC on $[a, b]$.

If f is AC on $[a, b]$ as f is nondecreasing. Th 7.18

$$\Rightarrow m(f(E)) = 0$$

for any Borel E with $m(E) = 0$.

If E is Borel and $m(E) = 0$, then $m(f(E)) = 0$.
$\forall\epsilon > 0, \exists$ open set V, s.t. $E \setminus [a, b] \subset V, m(v) < \frac{\epsilon}{2}$, $\exists U$ open, $f(E) \subset U, m(U) < \frac{\epsilon}{2}$.
$E \subset f^{-1}(U) \cap V \triangleq W$, W is open, $f(W) \subset U$.

$W = \bigcup_{i=1}^{+\infty}(\alpha_i, \beta_i), (\alpha_i, \beta_i)$ disjoint.

$$\begin{aligned}
\mu(E) &\le \mu(W) = \mu(\bigcup_i (\alpha_i, \beta_i) = \sum_{i=1}^{\infty} \mu(\alpha_i, \beta_i) \\
&\le \sum_{i=1}^{\infty} \mu([\alpha_i, \beta_i[) = \sum_{i=1}^{\infty} (f(\beta_i) - f(\alpha_i)) \\
&= \sum_{i=1}^{\infty} m(f\big((\alpha_i, \beta_i)\big)) = m(\bigcup_i (f\big((\alpha_i, \beta_i)\big) \\
&= m(f(\bigcup_i (\alpha_i, \beta_i))) = m(f(W)) \\
&\le m(U) < \frac{\epsilon}{2}
\end{aligned}$$

$$\therefore \mu(E) = 0, \mu \ll m.$$

Complementary to proof of (a).

1. f is nondecreasing.

$$\forall x, y \in [a, b], x < y, f(x) = \varphi(x - 0) = \lim_{h \to} \varphi(x - h) \le \lim_{h \to 0} \varphi(y - h) = \varphi(y - 0) = f(y)$$

2. f is left-continuous on $[a, b]$.

$$\forall x \in [a, b], \lim_{h \to} \varphi(x - h) = \varphi(x - 0)$$

$\therefore \forall \epsilon > 0, \exists \delta > 0$, when $0 \le s < \delta$

$$|\varphi(x - s) - \varphi(x - 0)| < \epsilon$$

So for $0 < h < \delta, 0 < k < \delta - h, 0 < h + k < \delta$,

$$\varphi(x - h - k) - \varphi(x - 0)| \le \epsilon.$$

i.e.

$$|f(x - h) - f(x)| \le \epsilon$$

$$\therefore f \text{ is left-continuous at } x.$$

3. Let $\{x_1, x_2, \cdots, \} = A$ be the set of all the points at which φ is not continuous then at x. $\forall \epsilon > 0$, if $x \notin A$, f is continuous at x, φ is continuous at x. $\Rightarrow \exists \delta > 0, |h| \ll \delta, \text{s.t} |\varphi(x + h) - \varphi(x) < \epsilon$

Fix h with $|h| < \delta$, if $|k| < \delta - |h|$ so that $|h - k| \le |h| + |k| \le \delta$, then $|\varphi(x + h - k) - \varphi(x)| < \epsilon$. Let $k \to 0$, $|\varphi(x + h-) - \varphi(x-)| \le \epsilon$. So $\varphi(x) = \varphi(x-)$.

$$\therefore |f(x + h) - f(x)| \le \epsilon.$$

$\therefore f$ is continuous at $x \in [a, b] \setminus A$.

$$f(a) = \varphi(a), f(b) = \varphi(b - 0).$$

By the same way, $k(x)$ is right-continuous on $[a,b]$, $k(x)$ is continuous on $[a,b]\setminus A$. $k(a)=\varphi(a+0), k(b)=\varphi(b)$.

(e) $\forall x\in[a,b]$

$$\begin{aligned}
\mu([a,x]) &= m(g([a,x]))-m([a,x])+\sum_{i=1}^{+\infty}k_i\delta_{x_i}([a,x])\\
&= g(x)-g(a)-(x-a)+\sum_{\{i:a\le x_i\le x\}}k_i\\
&= f(x)-f(a)+(x-a)-(x-a)-J(x)+\sum_{\{i:a\le x_i\le x\}}k_i\\
&= f(x)-f(a)-\sum_{\{i:x_i<x\}}k_i+\sum_{\{i:x_i\le x\}}k_i\\
&= f(x)-f(a)+\sum_{i=1}k_i\delta_{x_i}(x).
\end{aligned}$$

$$\delta_{x_i}(x)=\begin{cases}1 & x=x_i\\ 0 & x\ne x_i\end{cases}$$

If $x=x_i$ for some i, then

$$\begin{aligned}
m([a,x_i]) &= f(x_i)-f(a)+k_i\\
&= f(x_i)-f(a)+f(x_i+)-f(x_i-)\\
&= f(x_i+0)-f(a)\\
&= \varphi(x_i+0)-\varphi(a).
\end{aligned}$$

Where $f(x_i+)=\varphi(x_i+)$ because $\exists y_n\searrow x_i, \varphi(y_n)\searrow f(y_n)$.

If $x\ne x_i$, $\mu([a,x_i])=f(x)-f(a)=f(x+0)-f(a)$, So

$$\mu([a,x])=\varphi(x+)-f(a)=k(x)-f(a)=k(x)-\varphi(a)$$

$$k(x)-f(a)=\int_a^x h(t)\ dt+\mu_s([a,x])=\int_a^x h(t)\ dt+\tilde{s}(x).$$

By the same way $k'(x)=h(x)$ a.e. $\tilde{s}'(x)=0$ a.e. $[m]$.

Indeed $\frac{\tilde{s}(x+h)-\tilde{s}(x)}{h}=\frac{\mu_s(]x,x+h])}{h}=\frac{\mu_s(]x,x+h])}{m(]x,x+h])}$

$\forall h_i\to 0+,]x,x+h_i]$ shrinks nicely to x, and Th 7.13 $\Rightarrow D_{\mu_s}(x)=0$ a.e. $[m]$.

hence $\frac{\mu_s(]x,x+h_i])}{h_i}\to 0$ for any x with $D_{\mu_s}(x)=0$.

From the proof of Th 7.13

$$\frac{\mu_s(]x,x+h])}{h}\to 0 \text{ for a.e.}[m]\ x$$

$$\tilde{s}'(x)=0 \text{ a.e.}$$

$$\frac{d}{dx}\Big(\int_a^x h\ dt\Big)=h(x)\quad \text{a.e.}[m].$$

$$\exists E\subset[a,b], m([a,b]\setminus E)=0$$

$\forall x\in E, h\in\mathbb{R}^1, |h|\ll 1, k'(x)=h(x)=f'(x)$. Notice $\forall i, x\ne x_i$

$\frac{\varphi(x+h)-\varphi(x)}{h}=\frac{\varphi(x+h)-\varphi(x-)}{h}\ge\frac{\varphi(x+h-)-\varphi(x+)}{h}=\frac{k(x+h)-k(x)}{h}$

Similarly,
$\frac{\varphi(x+h)-\varphi(x)}{h}=\frac{\varphi(x+h)-\varphi(x+)}{h}\leq\frac{\varphi(x+h+)-\varphi(x+)}{h}=\frac{k(x+h)-k(x)}{h}$
$\Rightarrow\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}\leq\lim\limits_{h\to 0}\frac{\varphi(x+h)-\varphi(x)}{h}\leq\lim\limits_{h\to 0}\frac{k(x+h)-k(x)}{h}=k'(x).$
$f'(x)=\lim\limits_{h\to 0}\frac{\varphi(x+h)-\varphi(x)}{h}=f'(x)=k'(x)$
$\therefore\varphi'(x)$ exists almost everywhere in $[a,b]$, and $\varphi'(x)=f'(x)=k'(x)$ a.e. $[m]$.

So the decreasing function is a.e. differential.

$\varphi(x)-\varphi(a)\geq\varphi(x-0)-\varphi(a)=\int_0^x f'(t)\ dt+s(x)\geq\int_0^x f'(t)\ dt=\int_a^x\varphi(t)dt$
$\therefore\int_a^x\varphi'(t)\ dt\leq\varphi(x)-\varphi(a).$

13, Let BV be the class of all f on $[a,b]$ that have bounded variation on $[a,b]$, as defined after Theorem 7.19. Prove the following statements.
(a) Every monotonic bounded function on $[a,b]$ is in BV.
(b) If $f\in BV$ is real, there exist bounded monotonic functions f_1 and f_2 so that $f=f_1-f_2$.
Hint:Imitate the proof of Theorem 7.19.
(c) If $f\in BV$ is left-continuous then f_1 and f_2 can be chosen in (b) so as to be also left-continuous.
(d) If $f\in BV$ if left-continuous then there is a Borel measure μ on $[a,b]$ that satisfies

$$f(x)-f(a)=\mu([a,x))\ (a\leq x\leq b);$$

$\mu\ll m$ if and only if f is AC on $[a,b]$.
(e) Every $f\in BV$ is differentiable a.e.$[m]$, and $f'\in L^1(m)$.
Solution: (a) If $f:[a,b]\to\mathbb{R}^1$ is monotone (say nondecreasing). Then for any N and t_i with $\triangle: a=t_0<t_1<\cdots<t_N=x$
$V(\triangle)=\sum\limits_{i=1}^{N}|f(t_i)-f(t_{i-1})|=\sum\limits_{i=1}^{N}(f(t_i)-f(t_{i-1}))=f(t_N)-f(t_0)=f(b)-f(a)$
$V_a^b(f)=f(b)-f(a)<+\infty$
$\therefore f$ is in BV.

(b) If f is real and is in BV on $[a,b]$.

Th 7.19 write $F(x)=V_a^x(f)$
$f_1=\frac{V_a^x(f)+f(x)}{2}$ and $f_2=\frac{V_a^x(f)-f(x)}{2}$ are all nondecreasing,(Seeing the proof of Th 7.19.) and $f=f_1-f_2$, and f_1,f_2 are bounded.

(c) If f is real, BV and left-continuous.

We need only show that $V_a^x(f)$ is left-continuous, because f_1,f_2 are left continuous and linear combination of left-continuous functions is still left-continuous.
For $h>0$ small, $\forall x\in]a,b]$
$|V_a^x(f)-V_a^{x-h}(f)|=V_{x-h}^x(f)$ is non-increasing function of h. So it is enough to show that there is a sequence $\{h_n\}$ of positive numbers such that
$V_{x-h_n}^x(f)\to 0$

Because if $\inf\limits_{h>0}V_{x-h}^x(f)=0$, we are done.($\lim\limits_{h\to 0}V_{x-h}^x(f)=\inf\limits_{h>0}V_{x-h}^x(f)$).

Just suppose that $\exists\epsilon_0>0, V_{x_h}^x(f)\geq\epsilon$.
Since f is left-continuous at x, for $\epsilon>0$, and $\forall n\in N,\exists\delta_n>0$, s.t.

$$V_a^x(f)-\epsilon\leq\sum_{i=1}^{N}|f(t_i)-f(t_{i-1})|.$$

Proof: $\forall \epsilon < 0, \exists$ a division of $[a, x]$
$a = t_0 < t_1 < \cdots < t_N = x$, s.t.
$V_a^x(f) - \epsilon \le \sum_{i=1}^{N} |f(t_i) - f(t_{i-1})|$
$\exists \delta = \delta(x, \epsilon, t_{N-1}(\epsilon) > 0)$ s.t.
$t_{N-1} < x - h \le t_N = x$, and $|f(x) - f(x-h)| < \epsilon$,When $0 \le h < \delta$.
Thus

$$\begin{aligned} V_a^x(f) - \epsilon &\le \sum_{i=1}^{N-1} |f(t_i) - f(t_{i-1})| + |f(x-h) - f(t_{N-1})| + |f(t_N) - f(x-h)| \\ &\le V_a^{x-h}(f) + |f(x) - f(x-h)| \\ &\le V_a^x(f) + \epsilon \end{aligned}$$

for $0 \le h < \delta$.
So $\epsilon < V_a^x(f) - V_a^{x-h}(f) \le \epsilon$
$\lim_{h \to 0} V_a^{x-h}(f) = V_a^x(f)$
Hence $V_a^x(f)$ is left-continuous.
So $f_1 = \frac{V_a^x(f)+f}{2}, f_2 = \frac{V_a^x(f)-f}{2}$ are left-continuous and nondecreasing on $[a, b]$.

(d) If f is in BV and left-continuous on $[a, b]$ then by (c)
$f = f_1 - f_2 R$ and f_1, f_2 are left-continuous and non-decreasing on $[a, b]$.
12(b) implies that $\exists$ Borel positive measures μ_1, μ_2 bounded s.t.
$f_(x) - f(a) = \mu_1 9[a, x[), f_2(x) - f_2(a) = \mu_2([a, x[)$

Let $\mu = \mu_1 - \mu_2$, then since μ_i are bounded, μ is a Borel measure.
$\therefore f(x) - f(a) = (\mu_1 - \mu_2)([a, x[) = \mu([a, x[)$.

By Radon-Nkodym Theorem. Th 6.10
$\exists h_1, h_2 \in L^1(m, [a, b]), 0 \ll \mu_{1s}, \mu_{2s} \perp m$ s.t.
$d\mu_1 = h_1\ dm + \ d\mu_{1s},\ d\mu_2 = h_2\ dm + \ d\mu_{2s}$

By the uniqueness of the decomposition in Th 6.16.
$\mu(E) = \int_E (h_1 - h_2)\ dm + \mu_{1s}(E) - \mu_{2s}(E)$.
$\therefore\ d\mu_s = h\ dm + \mu_s, h = h_1 - h_2 \in L^1, \mu_s = \mu_{1s} - \mu_{2s}$.

If $\mu \ll m$, the $\mu_s = 0$.
$f(x) - f(a) = \int_a^x (h_1 - h_2)\ dm, f(x) = f(a) + \int_a^x (h_1 - h_2)\ dm = \int_a^x h\ dm + f(a)$.
$h \in L^1 \Rightarrow f(x)$ is AC on $[a, b]$.

If f is AC on $[a, b]$, and E is a Borel set, $m(E) = 0$. Then a variant of Th 7.18 (a)$\Rightarrow$ (b). It implies that $m(f(E)) = 0$.

Notice that $f_1 = \frac{V_a^x(f)+f(x)}{2}, f_2 = \frac{V_a^x(f)-f(x)}{2}$ Th 7.19 $\Rightarrow V_a^x(f), f_1, f_2$ are AC on $[a, b]$.
$\Rightarrow E(x) = V_a^x(f),\ f_1, f_2$ are AC on $[a, b]$ and nondecreasing.

So $\forall$ Borel set E, with $m(E) = 0$, from 12(d) $\mu_1 \ll m, \mu_2 \ll m$.
So $\mu = \mu_1 - \mu_2 \ll m$.

(e) $f \in BV \Rightarrow f = f_1 - f_2$
f_1, f_2 non-decreasing, 12(c)(e)$\Rightarrow f_1'(x), f_2'(x)$ exist a.e. $[m]$. and $f_1', f_2' \in L^1(m)$.
So $f' = f_1' - f_2'$ exists a.e. $[m]$ and $f' \in L^1(m)$.

In fact, from (d) we know, $f'(x) = h'(x)$. And we can prove $\int_a^b |f'(t)| \, dt \le V_a^b(f)$.(Seeing P181 Th 2.)

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Implies that $m(f(E))$=0.

Notice that

$$f_1 = \frac{V_a^x + f(x)}{2}, \quad f_2 = \frac{V_a^x - f(x)}{2},$$

Th7.19 $\Rightarrow$

$$V_a^x(f),\ f_1,\ f_2 \ are\ AC\ on\ [a,b]$$

$\Rightarrow$

$$F(x) = V_a^x(f),$$

hence f_1, f_2 are AC and nondecreasing on $[a,b]$.
So, $\forall$ Borel set E,with $m(E) = 0$, from $E_t12(d)$, $\mu_1 \le m$, $\mu_2 \le m$.
So

$$\mu = \mu_1 - \mu_2 \le m.$$

(e) $f \in BV \Rightarrow f = f_1 - f_2$, f_1, f_2 nondecreasing,
$E_t12(e) \Rightarrow f_1'(x), f_2'(x)$ exist a.e .(m),
and $f_1'(x), f_2'(x) \in L^1(m)$, so,

$$f'(x) = f_1'(x) - f_2'(x)$$

exists a.e [m] and $f' \in L^1(m)$.
In fact, from (d) we have, $f'(x) = h'(x)$.
Also

$$\int_a^b |f'(t)| dt \le V_a^b(f).$$

See Jian Zejian (P_{181} Th2). □

14. Show that the product of two absolutely continuous functions on [a,b] is absolutely continuous. Use this to derive a theorem about integration by parts.

Proof: If f, g are AC on $[a,b]$,then
$\forall \varepsilon > 0, \exists \delta_1 > 0, \delta_2 > 0, st$ $\forall$ open disjoint intervals $\{(a_i, b_i)\}_{i=1}^N$ with

$$\sum_{i=1}^N (\beta_i - \alpha_i) < \delta_1, \sum_{i=1}^N (\beta_i - \alpha_i) < \delta_2.$$

We have respectively,

$$\sum_{i=1}^N |f(\beta_i) - f(\alpha_i)| < \frac{\varepsilon}{2M},$$

$$\sum_{i=1}^N |g(\beta_i) - g(\alpha_i)| < \frac{\varepsilon}{2M},$$

where $M = max(max_{[a,b]}f(x), max_{[a,b]}g(x), 1)$.
So let
$$\delta = min(\delta_1, \delta_2) > 0, \sum_{i=1}^{N}(\beta_i - \alpha_i) < \delta.$$

$$\Longrightarrow \sum_{i=1}^{N} |f(\beta_i)g(\beta_i) - f(\alpha_i)g(\alpha_i)|$$
$$\leq \sum_{i=1}^{N}(|(f(\beta_i) - f(\alpha_i))g(\beta_i)| + |f(\alpha_i)(g(\beta_i) - g(\alpha_i))|)$$

$$\leq M(\sum_{i=1}^{N} |f(\beta_i) - f(\alpha_i)| + \sum_{i=1}^{N} |g(\beta_i) - g(\alpha_i)|)$$
$$\leq M(\frac{\varepsilon}{2M} + \frac{\varepsilon}{2M}) = \varepsilon.$$

So fg is AC on [a,b].
Since f, g, fg are AC on [a,b], $f', g', (fg)'$ exist a.e[m], and f', g', $(fg)' \in L^1[a, b]$.
Since $g, f \in C^0[a, b]$, gf', $fg' \in L^1[a, b]$. Let
$$A = \{x, f'(x), g'(x), (fg)'(x)\ exist\},$$

then
$$m([a, b] \bigcap A) = 0.$$
$\forall x \in A$,
$$(fg)'(x) = f(x)g'(x) + f'(x)g(x)$$

$$\int_{[a,y]}(fg)'dx = \int_{A\cap[a,y]}(fg)'dx = \int_{A\cap[a,y]} f(g)'dx + \int_{A\cap[a,y]}(f)'gdx.$$

$$= \int_{[a,y]}(f(x)g'(x) + f'(x)g(x))dx.$$

Th 7.20
$$\Rightarrow \forall y \in [a, b], \quad f(y)g(y) - f(a)g(a) = \int_a^y (fg)'dm$$
$$= \int_a^y (f'g + fg')dm = \int_a^y f'gdm + \int_a^y fg'dm.$$
So
$$\int_a^y fg'dm = -\int_a^y f'gdm + fg|_a^y.$$
The same as the classical formula. □

16. Suppose $E \subset [a,b], m(E) = 0$. Construct an absolutely continuous monotonic function f on $[a,b]$. So that $f'(x) = \infty$ at every $x \in E$.
Hint : $E \subset \cap V_n, V_n$ open, $m(V_n) < 2^{-n}$.
Consider the sum of the characteristic functions of these sets.

Solution: $m(E) = 0$, so $\forall n, \exists$ open set V_n,

$$E \subset V_n, m(V_n) < 2^{-n}.$$

let

$$h(x) = \sum_{n=1}^{+\infty} \chi_{V_n}(x),$$

then $h(x) \geq 0$, measurable,and (lebesgue ,Th1.27)

$$0 \leq \int_a^b h(x)dm = \sum_{n=1}^{+\infty} \int_a^b \chi_{V_n}(x)dm \leq \sum_{n=1}^{+\infty} m(V_n) \leq \sum_{n=1}^{+\infty} 2^{-n} < \infty.$$

So

$$h(x) \in L^1[a,b].$$

Let

$$f(x) = \int_a^x h(x)dt$$

Then f is nondecreasing , AC on $[a,b]$, by Th7.20 we see that

$$f' = h(x) \ a.e \ [m] \ on \ [a,b].$$

7.20 $f'(x) = h(x)$ $Th7.18, Th7.18$ $Th7.11,$ $Th7.11$ $h(x)$ $Lebesgue$ $f' = h(x)$. $Lebesgue$ $\Rightarrow |h(x)| < \infty$!.
$Th7.20$ $f'(x) = \infty = h(x), \forall x \in E$,()

$$\forall x \in E \subset \bigcap_{i=1}^{\infty} V_n, \forall n, x \in V_1, V_2 ... V_n.$$

so V_i open $\Rightarrow \exists \delta_i > 0$, st

$$[x, x+h], [x-h, h] \subset V_i, \ if \ 0 \leq |h| < \delta_i.$$

So,$\forall N, \exists \tilde{\delta}_N = \min\{\delta_1, \delta_2, \cdots, \delta_N\}$, $|h| < \tilde{\delta}_N$,

$$\Rightarrow [x, x+h], [x-h, x] \subset V_i, 1 \leq i \leq N, 0 < h < \tilde{\delta}_N,$$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{1}{h} \int_a^{x+h} f(x)dt - \frac{1}{h} \int_a^x f(x)dt.$$

$$= \frac{1}{h} \int_x^{x+h} f(x)dt \geq \frac{1}{h} \int_x^{x+h} \sum_{i=1}^{N} \chi_{V_i}(t)dt$$

$$= \frac{1}{h}N(x+h-x) = N.$$

so

$$\liminf_{h\to 0^+} \frac{f(x+h)-f(x)}{h} \geq N, \forall N.$$

so

$$\liminf_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = \infty.$$

$h < 0, |h| < \tilde{\delta}_N$,

$$\Rightarrow \frac{f(x+h)=f(x)}{h} = -\frac{f(x)-f(x+h)}{f(x)} = -\frac{1}{h}\int_{x-h}^{x} f(x)dt$$

$$\geq -\frac{1}{h}\int_{x+h}^{x}\sum_{i=1}^{N}\chi_{V_i}(x)dx = -\frac{1}{h}(x-(x+h))N = -\frac{1}{h}(-h)N = N.$$

So

$$\liminf_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = \infty,$$

so

$$\liminf_{h\to 0} \frac{f(x+h)-f(x)}{h} = \infty, \forall x \in E.$$

i.e

$$f'(x) = +\infty \quad \forall x \in E.$$

$m(E) = 0 \;\; f\ is\ AC \Rightarrow f'(x) \in L^1, |f'(x)| < +\infty\ a.e. m(E) > 0, f' = \infty$ □

17. Suppose μ_n is a sequence of positive Borel measures on R^k and

$$\mu(E) = \sum_{i=1}^{\infty} \mu_k(E),$$

Assume $\mu(R^k) < \infty$, show that μ is a Borel measure,what is the relation between the Lebesgue decompositions of the μ_n and μ? Prove that

$$D\mu(x)(x) = \sum_{n=1}^{\infty}(D\mu_n)(x)\ a.e[m].$$

Derive corresponding theorems for sequences $\{f_n\}$ of positive nondecreasing functions on R^1 and their sums $f = \sum f_n$.
Solution: 1. μ is a Borel measure, suppose $\{E_i\}_{i=1}^{+\infty}$ are Borel sets in R^k ,then since μ_n are positive Borel measures,

$$\mu_n(\bigcup E_i) = \sum_i \mu_n(E_i), \forall n,$$

so

$$\mu(\bigcup_{i=1}^{+\infty} E_i) = \sum_{n=1}^{+\infty}\mu_n(\bigcup_{i=1}^{+\infty} E_i) = \sum_{n=1}^{+\infty}\sum_{i=1}^{+\infty}(\mu_n(E_i)) = \sum_{i=1}^{+\infty}\sum_{n=1}^{+\infty}(\mu_n(E_i)) = \sum_{i=1}^{+\infty}\mu(E_i).$$

So, μ is a Borel measure . Also as $\mu(R^k) < \infty$,we see

$$\mu_n(R^k) \leq \mu(R^k) < \infty,$$

μ_n, μ are real-valued complex measures.
By $Th6.10(Lebesgue - Radon - Nikodgm)$,there are Lebesgue decomposition:

$$\mu = \mu_a + \mu_s, \mu_n = \mu_{na} + \mu_{ns}.$$

$\mu_a, \mu_s, \mu_{na}, \mu_{ns}$ are positive Borel measures,where $\mu_a \ll m, \mu_{na} \ll m, \mu_s \perp m, \mu_{ns} \perp m$. Since $\tilde{\mu}_n = \sum \mu_{na}, \tilde{\mu}_s = \sum \mu_{ns}$ are positive Borel measures and $\mu = \tilde{\mu}_a + \tilde{\mu}_s$. Since $\tilde{\mu} \ll m, \tilde{\mu_s} \perp m$. as $\mu_{na} \ll m, \mu_{ns} \perp m$ when $n = 1, 2, 3 \cdots$,
the uniquness of the Lebesgue decomposition show that

$$\mu_a = \tilde{\mu}_a = \sum_{n=1}^{+\infty} \mu_{na}, \mu_s = \tilde{\mu}_s = \sum_{n=1}^{+\infty} \mu_{ns}.$$

($\tilde{\mu} \ll m$ is trivial. $\tilde{\mu} \perp m$ $\mu_{ns} \perp m \Rightarrow \exists$ Borel set $n = 1, 2 \cdots, A_n \subset R^k$.) □

2. By

$$\mu_{ns}(E) = \mu_{ns}(E \bigcap A_n),\ m(E) = m(A_n \bigcap A_n^c),$$

$\Rightarrow$

$$m(A_n) = m(A_n \bigcap A_n)^c.$$

Let $A = \bigcup_{n=1}^{\infty} A_n$,
then

$$m(A) = 0.$$

$\forall$ Borel set $E \subset R^k$,

$$\mu_s(E) = \sum_{n=1}^{+\infty} \mu_{ns}(E) = \sum_{n=1}^{+\infty} \mu_{ns}(E \bigcap A_n)$$

$$= \sum_{n=1}^{+\infty} \mu_{ns}(E \bigcap A_n \bigcap A) = \sum_{n=1}^{+\infty} \mu_{ns}(E \bigcap A \bigcap A_n)$$

$$= \sum_{n=1}^{+\infty} \mu_{ns}(E \bigcap A) = \mu_s(E \bigcap A).$$

So

$$\mu_s \perp m.$$

To show that

$$(D\mu)(x) = \sum_{n=1}^{+\infty} (D\mu_n)(x)\ a.e\ [m],$$

we notice that $\mu(R^n) < \infty \Rightarrow \mu, \mu_n$ are positive,finite Borel measures.Th6.10

$$\Rightarrow \exists h \in L^1(m), h_n \in L^1(m), (n = 1, 2, \cdots)$$

s.t $\forall E$ (Borel sets in R^k),

$$\mu(E) = \int_E h dm, \mu_n(E) = \int_E h_n dm.$$

and Theorem7.8 gives that

$$D\mu_a = h \ a.e \ [m], D\mu_{na} = h_n \ a.e,$$

and Th7.14 gives that

$$D\mu(x) = D\mu_a = h \ a.e \ [m], D\mu_n(x) = D\mu_{na} = h_n \ a.e \ [m].$$

So we need only show that

$$h = \sum h_n.$$

Since $\mu_a(E) = \sum_{n=1}^{+\infty} \mu_{na}(E)$, we have by Lebesgue Th1.27 ($\mu \geq 0, \mu_n \geq 0, h, h_n \geq 0$ a.e [m]). For any Borel set E,

$$\int_E h dm = \sum_{n=1}^{+\infty} \int_E h_n dm = \int_E \sum h_n dm$$

i.e

$$\int_E (h - \sum_{n=1}^{+\infty} h_n) dm = 0. \ (*)$$

As $\mu_n(E) \geq \sum_{i=1}^{N} \mu_{na}(E), \forall N$. Fix N,

$$\int_E (h - \sum_{n=1}^{N} h_n) dx \geq 0, \forall E \ Borel.$$

$$\Rightarrow h - \sum_{n=1}^{N} h_n \geq 0 \ a.e \Rightarrow h \geq \sum_{n=1}^{N} h_n,$$

so

$$N \to +\infty \Rightarrow h \geq \sum_{n=1}^{N} h_n.$$

Thus

$$(*) \Rightarrow h - \sum_{n=1}^{+\infty} h_n = 0 \ a.e \ [m].$$

(: $f \in L^1(m)$ $\forall$ Borel set $E \subset R^k, \int_E f dm \geq 0$, then $f \geq 0$ a.e.)
Proof: $\forall$ Lebesgue E, $\exists F_\sigma$ F, $S \in M$, $m(S) = 0, E = S \bigcup F, S \bigcap F = \Phi$.
So

$$\int_E f dm = \int_S f dm + \int_F f dm \geq 0.$$

So $f \geq 0$ a.e. $\exists n, A_n = \{x; f(x) < -n\}$ and $m(A_n) > 0$
$\Rightarrow 0 \leq \int_{A_n} f \leq -n m(A_n) < 0$

Suppose now $\{f_n\}$ is a sequence of positive nondecreasing functions on R^1 and

$$f(x) = \sum_{n=1}^{+\infty} f_n(x).$$

We have the following results:
Claim :

If $\{f_n\}$ is a sequence of nondecreasing , real value functions on $R^1, f(x) = \sum f_n(x)$ converges for (finitely) each $x \in R^1$,then

$$f'(x) \ exists \ and \ f'(x) = \sum_n f_n'(x) \ a.e \ [m] \ on \ R^1.$$

($f_n \geq 0$!)

Proof: Clealy $f(x)$ is nondecreasing on R^1, and

$$|f(x)| < \infty, \forall x \in R^1, \forall -\infty < a < b < \infty.$$

$\forall n$, let

$$\tilde{f}_n = f_n(x-), if \ x \in]a,b], \quad \tilde{f}_n(a) = f_n(a).$$
$$\tilde{\tilde{f}}_n = f_n(x+), if \ x \in [a,b[, \quad \tilde{\tilde{f}}_n(b) = f_n(b).$$

Then $\tilde{f}_n$ is left-continuous on $[a,b]$ and nondecreasing, $\tilde{\tilde{f}}_n$ is right-continuous and nondecreasing, and let $x_1^n, \cdots, x_n^n, \cdots$ be the points in $[a,b]$ at which f_n is not continuous ,then

$$f_n = \tilde{f}_n = \tilde{\tilde{f}}_n \ if \ x \in [a,b] \backslash A_n.$$

when $A_n = \{x_i^n\}_i$. Note $\quad \tilde{f}_n \leq f_n \leq \tilde{\tilde{f}}_n$.

Let $\tilde{F}(x) := \sum_{n=1}^{+\infty} \tilde{f}_n$, then $\tilde{F}(x)$ is well difined because $\sum_{n=1}^{+\infty} \tilde{f}_n(x)$ converges which could be seen by $Cauchy'$s principle:

$$\sum_{i=n}^{n+k} f_i(a) = \sum_{i=n}^{n+k} \tilde{f}_i(a) \leq \sum_{i=n}^{n+k} \tilde{f}_i(x) \leq \sum_{i=n}^{n+k} f_i(x) \leq \sum_{i=n}^{n+k} f_i(b).$$

So in fact, $\sum_{n=1}^{n+k} \tilde{f}_n(x)$ converges uniformly on $[a,b]$, to $\tilde{F}(x)$ because

$$\sum_{i=1}^{+\infty} f_i(a), \quad \sum_{i=1}^{+\infty} f_i(b) \ converges,$$

hence $\tilde{F}(x)$ is even left-continuous on $[a,b]$.

By Et12,we konw that $\forall n$, $\exists$ positive Borel measure $\mu_n \ on$ $[a,b]$ s.t

$$\tilde{f}_n(x) - \tilde{f}_n(a) = \mu_n([a,x[) \quad (a \leq x \leq b)$$

and

$$f_n'(x) = \tilde{\tilde{f}}_n'(x) = \tilde{f}_n'(x) = D\mu_n(x) \ a.e \ [m].$$

Let

$$\mu = \sum \mu_n,$$

then $\mu \geq 0$ is a positive Borel measure on $\mu([a,b]) = \sum \mu_n[a,b]($ Et12 $)$

$$= \sum(\tilde{\tilde{f_n}}(b) - f_n(a)) \leq \sum_{n\to\infty}(f_n(b) - f_n(a))(<\infty) = F(b) - F(a).$$

So μ is finite, according to what we proved,

$$D\mu(x) = \sum D\mu_n(x)a.e \quad [m].$$

with obvious small varicut:

$$(\tilde{(\mu)}(E) = \mu(E\bigcap[a,b], \tilde{\mu}(R^1) = \mu([a,b]) < \infty,$$

$$\tilde{\mu_n}(E) = \mu_n(E\bigcap[a,b]), \forall E \ Borel \ set \ in \ R^1))$$

So

$$\exists E \ s.t \bigcup_{n=1}^{+\infty} A_n \subset E, a \in E, b \in E, m(E) = 0.$$

$\forall x \in [a,b]\backslash E, D\mu(x)$ exists.(maybe $+\infty$)
and

$$D\mu(x) = \sum_{n=1}^{+\infty} D\mu_n(x), \quad f_n(x) = \tilde{f}_n(x) = \tilde{\tilde{f_n}}(x), \quad \forall x \in ([a,b]\backslash E).$$

$$D\mu_n(x) = f_n'(x), \quad F'(x) \ exists.(by \ Ex12, F \nearrow, F'(x) \ exists \ a.e \ [m].)$$

Now take any $x \in [a,b]\backslash E, h > 0$.

$$\frac{F(x+h) - F(h)}{h} = \sum_{n=1}^{+\infty} \frac{f_n(x+h) - f_n(x)}{h}$$

$$\Rightarrow \sum_{n=1}^{+\infty} \frac{\tilde{f}_n(x+h) - f_n(x)}{h} = \sum_{n=1}^{+\infty} \frac{\tilde{f}_n(x+h) - \tilde{f}_n(x)}{h}$$

$$= \frac{1}{h}\sum_{n=1}^{+\infty} \mu_n([x,x+h[) = \frac{1}{h}\mu[x,x+h[$$

$$\longrightarrow \ D\mu(x)_{\ (as \ h\to 0, \ by \ Th7.14)} = \sum_{n=1}^{+\infty} D\mu_n(x) = \sum_{n=1}^{+\infty} f_n'(x).$$

Similarly, notice by $Ex12(e)^{'}s$ proof,

$$\mu[a,x] = \sum_{n=1}^{+\infty} \mu_n[a,x] = \sum_{n=1}^{+\infty}(\tilde{\tilde{f_n}}(x) - f_n(a)).$$

$$\frac{\mu([a,x+b])-\mu([a,x])}{h}=\frac{\mu(]x,x+h])}{h}$$

$$=\sum_{n=1}^{+\infty}\frac{\tilde{\tilde{f_n}}(x+h)-\tilde{\tilde{f_n}}(x)}{h}=\sum_{n=1}^{+\infty}\frac{\mu_n(]x,x+h])}{h}.$$

By 7.14

$$\frac{\mu(]x,x+h])}{h}\to D\mu(x)$$

So

$$\frac{F(x+h)-F(x)}{h}=\sum_{n=1}^{+\infty}\frac{f_n(x+h)}{f_n(x)}$$

$$\leq\sum_{n=1}^{+\infty}\frac{\tilde{\tilde{f_n}}(x+h)-f_n(x)}{h}=\sum_{n=1}^{+\infty}\frac{\tilde{\tilde{f_n}}(x+h)-\tilde{\tilde{f_n}}(x)}{h}$$

$$=\frac{\mu(]x,x+h])}{h}\to D\mu(x)=\sum_n D\mu_n(x)=\sum_{n=1}^{+\infty}f_n'(x).$$

So

$$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=\sum f_n'(x),\ \forall x\in[a,b]\backslash E.$$

$$\Rightarrow F'(x)=\sum f_n'(x)\quad a.e\ on\ [a,b].$$

Hence

$$F'(x)=\sum f_n'(x)\ \ a.e\ on\ R^1.$$

□

21.If f is a real function on $[0,1]$ and

$$\gamma(t)=t+if(t)$$

the length of the graph of f is, by definition,the total variation of γ *on* $[0,1]$. Show that this length is finite if and only if $f\in BV$.(see Exercrise 13.)Show that it is equal to

$$\int_0^1\sqrt{1+[f'(t)]^2}$$

if f is absolutely continuous.

Solution: For any partition $0=t_0<t_1<\cdots<t_n=1$

$$\sum_{i=1}^n|f(t_i)-f(t_{i-1})|\leq\sum_{i=1}^n\sqrt{(t_i-t_{i-1})^2+(f(t_i)-f(t_{i-1}))^2}$$

$$\leq\sum_{i=1}^n(|t_i-t_{i-1}|^2+|f(t_i)-f(t_{i-1})|)=\sum_{i=1}^n(t_i-t_{i-1})^2+\sum_{i=1}^n|f(t_i)-f(t_{i-1})|$$

$$=1+\sum_{i=1}^n|f(t_i)-f(t_{i-1})|$$

(Note $\sqrt{a^2+b^2} \le |a|+|b|$.)

Thus the length of the graph of f (which is the total variation of $\gamma(t)$.)satisfies

$$V_0^1(f) \le V_0^1(\gamma) \le 1 + V_0^1(f).$$

So it is finite $\Longleftrightarrow f \in BV$.

Let $l(x) = V_0^x(\gamma)$,then $l(x)$ is nondecreasing on $[0,1]$.

So by Exercise 12, () $l'(x)$ exists a.e [m], and

$$\int l'(x)dx \le l(1) - l(0) = l(1) = V_0^1(\gamma)$$

on the other hand,$\forall 0 \le x \le y \le 1$

$$\frac{l(y)-l(x)}{y-x} \ge \sqrt{1+(\frac{f(y)-f(x)}{y-x})^2}$$

because

$$l(y)-l(x) = V_0^y(\gamma) - V_0^x(\gamma) = (_{P148(5)})V_x^y(\gamma)$$

$$\ge \sqrt{(y-x)^2+(f(y)-f(x))^2} = \sqrt{|\gamma(y)-\gamma(x)|}.$$

([x,y] [x,y])

as f AC $f'(x)$ a.e [m] [0,1] $f' \in L^1$.

So $l'(x) \ge \sqrt{1+(f'(x))^2}$ a.e [m] on [0,1].

So

$$\int_0^1 \sqrt{1+(f'(t))^2}dt \le \int_0^1 l'(t)dt \le l(1)-l(0) = l(1).$$

Notice $|\sqrt{1+\alpha^2} - \sqrt{1+\beta^2}| \le |\alpha - \beta|$. $\forall \alpha, \beta \in R^1$.

$$(because \;\; h(x) = \sqrt{1+x^2}, h'(x) = \frac{x}{\sqrt{1+x^2}}, |h'(x) \le 1|. \;\; h(\alpha) - h(\beta) = h'(\xi)(\alpha - \beta))$$

$\forall \varepsilon > 0, \exists$ Partition $\Delta : 0 = t_0 < t_1 < \cdots < t_n = 1$

s.t

$$V_f(\Delta) := \sum_{i=1}^n \sqrt{(t_i - t_{i-1})^2 + (f(t_{i+1}) - f(t_i))^2} \ge l(1) - \varepsilon$$

If in addition $f' \in C^0[0,1]$, then the mean value theorem gives that

$$V_f(\Delta) = \sum_{i=1}^n (t_i - t_{i-1})\sqrt{1+(\frac{f(t_i)-f(t_{i-1})}{t_i - t_{i-1}})^2}$$

$$= \sum_{i=1}^n (t_i - t_{i-1})\sqrt{1+|f'(\xi_i)|^2} \;\; (t_i \le \xi_{i-1} \le \xi_i).$$

$$= I(\sqrt{1+|f'|^2}, \Delta)$$

(on[0,1])

$$\leq \int_o^1 \sqrt{1+|f'|^2}$$

As $V_f(\Delta) \leq V_f(\Delta')$ if Δ' is a refinement division of Δ in the sense that $\Delta = \{x_i\}, \Delta' = \{y_i\},\ \Delta \subset \Delta'()$ $\sqrt{1+|f'|^2 \in C^0[a,b]}$ is Riemann integrable ,
so

$$\forall \varepsilon > 0, \exists \delta > 0, \Delta, \Delta', if\ max_{\{1\leq i\leq n\}}\delta_i < \delta\ where\ \ \Delta' \supset \Delta$$

$$\Delta' :\ 0 = t'_0 < t'_1 < \cdots < t'_m, \delta_i = (t'_i - t'_{i-1})$$

Then

$$I(\sqrt{1+|f'|^2}, \Delta) < \int_0^1 \sqrt{1+|f'(t)|^2}dt + \varepsilon$$

So for any

$$l(1) - \varepsilon \leq V_\Delta(f) \leq V_{\Delta'}(f) \leq \int_a^b \sqrt{1+|f'(t)|^2 dt + \varepsilon}$$

So

$$l(1) \leq \int_a^b \sqrt{1+|f'(t)|^2}dt$$

If now we only have $f' \in L^1[a,b]$, then $\forall \varepsilon > 0$, CH3, Th3.14
$\Rightarrow, \exists g \in C^0[a,b]$ s.t

$$\int_0^1 |g - f'|dm < \varepsilon.$$

Let

$$G(x) = \int_0^x g(t)dt,$$

then $G(x)$ is in $C^1[0,1],\ AC$ and $G'(x) = g(x).(\forall t)$

$$\forall\ Partition\ \Delta : 0 = t_0 < t_1 < \cdots < t_n = 1$$

$|V_f(\Delta) - V_G(\Delta)|$

$$= |\sum_{i=1}^n \sqrt{(t_i - t_{i-1})^2 + (f(t_i - f(t_{i-1})))^2} - \sum_{i=1}^n \sqrt{(t_i - t_{i-1})^2 + (G(t_i) - G(t_{i-1}))^2}|$$

$$= \sum_{i=1}^n (t_i - t_{i-1})|\sqrt{1 + (\frac{f(t_i) - f(t_{i-1})}{t_i - t_{i-1}})^2} - \sqrt{1 + (\frac{G(t_i) - G(t_{i-1})}{t_i - t_{i-1}})^2}|$$

$$\leq \sum_{i=1}^n (t_i - t_{i-1})|\frac{f(t_i) - f_(t_{i-1})}{t_i - t_{i-1}} - \frac{G(t_i) - G_(t_{i-1})}{t_i - t_{i-1}}|$$

$$= \sum | \int_{t_{i-1}}^{t_i} f'(s)ds - \int_{t_{i-1}}^{t_i} g(s)ds|$$

$$= \sum_{i=1}^n \int_{t_{i-1}}^{t_i} |f'(s) - g(s)|ds = \int_0^1 |f'(s) - g(s)|ds < \varepsilon$$

$\forall \varepsilon > 0, \exists \Delta : 0 = t_0 < t_1 < \cdots < t_n = 1$ s.t

$$l(1) - \varepsilon \leq V_f(\Delta) \leq V_G(\Delta) + \varepsilon$$

$$\leq V_G(\Delta') + \varepsilon \ (\Delta \subset \Delta'), \qquad (max_{(0\leq i\leq m)} \ll 1)$$

$$= \int_0^1 \sqrt{1+|G'(t)|^2dt} + \varepsilon = \int_0^1 \sqrt{1+|g(t)|^2dt} + \varepsilon$$

$$< \int_0^1 \sqrt{1+|f'(t)|^2dt} + 2\varepsilon$$

$$(Because \ |\int_0^1 \sqrt{1+(f'(t))^2} - \int_0^1 \sqrt{1+(g(t))^2}| < \int_0^1 |f'-g|dt < \varepsilon)$$

So

$$l(1) = \int_0^1 \sqrt{1+|f'(t)|^2dt} \quad is\ true.$$

□

23. The definition of Lebesgue points, as made in Sec.7.6, applies to indiviual interable functions, not to the equivalence classes discussed in Sec.3.10. However, if $F \in L^1(R^k)$ is one of these equivalence classes, one may call a point $x \in R^k$ a *Lebesgue point* of F if there is a complex number, let us call it $(SF)(x)$, such that

$$\lim_{r\to 0} \frac{1}{m(B_r)} \int_{B(x,r)} |f - (SF)(x)|dm = 0$$

for one (hence for every) $f \in F$.
Define$(SF)(x)$ to be 0 at those points $x \in R^k$ that are not Lebesgue points of F.
Prove the following statement:If $f \in F$, and x is a Lebesgue point of f, then a also a Lebesgue point of F, and $f(x) = (SF)(x)$. Hence $SF \in F$.
Thus S "selects" a member of F that has a *maximal* set of Lebesgue points.

Proof: Let $F \in L^1(R^k)$ be an equivalence classes. If x is a Lebesgue Points of F, then $(SF)(x)$ is well defined, ie if $\alpha, \beta \in C$ s.t

$$\lim_{r\to 0} \frac{1}{m(B_r)} \int_{B(x,r)} |f - \alpha|dm = \lim_{r\to 0} \frac{1}{m(B_r)} \int_{B(x,r)} |g - \beta|dm$$

for any members $f, g \in F$, then $\alpha = \beta$.
In fact,

$$|\alpha - \beta| = \frac{1}{m(B_r)} \int_{B(x,r)} |\alpha - \beta|dm$$

$$\leq \frac{1}{m(B_r)} \int_{B(x,r)} (|\alpha - f| + |f - g| + |\beta - g|)dm$$

$$= \frac{1}{m(B_r)} \int_{B(x,r)} |\alpha - f|dm + \frac{1}{m(B_r)} \int_{B(x,r)} |g - \beta|dm$$

$$\to 0. \quad (Because\ f = g \quad a.e)$$

So

$$|\alpha - \beta| = 0 \ \Rightarrow \alpha = \beta.$$

So

$$(SF)(x)\ is\ well\ defined.$$

If x is a Lebesgue Point of $f \in F$, then

$$\lim_{r \to 0} \frac{1}{m(B_r)} \int_{B(x,r)} |f - f(x)| dm = 0$$

So $f(x) = (SF)(x)$ by the well-definedness of $(SF)(x)$.
As a.e $x \in R^k$ is a Lebesgue Point of f,
we see

$$f(x) = (SF)(x) \quad a.e \ on \ R^k.$$

So $SF \in F$, and SF is a member of F that has a *maximal* set of Lebesgue Points. □

22. (a) Assume that both f and its maximal function Mf are in $L^1(\mathbb{R}^k)$. Prove that then $f(x) = 0$ a.e.[m]. *Hint:* To every other $f \in L^1(\mathbb{R}^k)$ corresponds a constant $c = c(f) > 0$ such that

$$(Mf)(x) \geq c|x|^{-k}$$

whenever $|x|$ is sufficiently large.
(b) Define $f(x) = x^{-1}(logx)^{-2}$ if $0 < x < \frac{1}{2}$, $f(x) = 0$ on the rest of $\mathbb{R}^1$. Then $f \in L^1(\mathbb{R}^1)$. Show that

$$(Mf)(x) \leq |2x \ log(2x)|^{-1} \quad (0 < x < 1/4)$$

so that $\int_0^1 (Mf)(x) \ dx = \infty$.
Solution: (a) Suppose $f, \ Mf \in L^1(\mathbb{R}^1)$. We claim that if $f \neq 0$ a.e.[m]. is not true, then $\exists \ c(f) > 0$ s.t. $(Mf)(x) \geq c|x|^{-k}$ for $|x|$ is sufficiently large. Indeed, if the claim is not true, then $\forall n \in N$, $\exists x_n \in \mathbb{R}^k$ with $|x_n| > n$ s.t.

$$Mf(x_n) < \frac{1}{n^{k+1}} |x|^{-k}. \tag{$*$}$$

This means that for all $r > 0$, we have

$$\int_{B(x_n,r)} |f| dm < \frac{c_k r^k}{n^{k+1}} |x|^{-k},$$

where $c_k r^k = m(B_r)$. Take $r_n = n|x_n|$, then

$$\int_{B(x_n,n|x_n|)} |f| \ dm < \frac{c_n n^k |x_n| k |x_n| - k}{n^{k+1}} = \frac{c_k}{n} \tag{$**$}$$

Notice that for all $x \in \mathbb{R}^k$, $\exists N(x) \in \mathbb{N}$, s.t. $N(x) > 0$, $n \geq N(x)$ implies that $|x| < n < |x_n|$, hence

$$|x - x_n| \leq |x| + |x_n| < n + |x_n| < |x_n| + |x_n| = 2|x_n| < n|x_n|,$$

i.e. $x \in B(x_n, n|x_n|)$. Thus $\forall \ x \in \mathbb{R}^k$, $\chi_B(x_n, n|x_n|)(x) \to 1$ as $n \to +\infty$. The Dominant Convergence Theorem and the fact that $f \in L^(\mathbb{R}^k)$ imply that

$$\int_{B(x_n,n|x_n|)} |f| \ dm = \int_{\mathbb{R}^k} \chi_B(x_n, n|x_n|)(x)|f(x)| \ dm \to \int_{|R^k} |f| \ dm.$$

So (**) gives that

$$\int_{|R^k} |f| \ dm \leq \lim_{n \to +\infty} \frac{c_k}{n} = 0,$$

and $f=0$ a.e., a contradiction. So the claim is true. Thus if $f \neq 0 a.e.$ does not hold, there is a $c(f)>0$ s.t. $\exists\ k=k(t)>0$,

$$(Mf)(x) \geq c|x|^{-k}, \quad for |x| \geq k,$$

$$\int_{\mathbb{R}^k} (Mf)(x)\ dx \geq c \int_{\mathbb{R}^k} |x|-k\ dx = +\infty.$$

This contradicts to that $Mf \in L^1$. So $f(x)=0$ a.e.[m] on $\mathbb{R}^k$.

(b) $\forall\ x \in (0,1/4)$, let $r=x$, then $x+r \leq 2x < 1/2$,

$$\begin{aligned}
Q_r f &\triangleq \frac{1}{m(B_r)} \int_{x-r}^{x+r} |f|\ dm = \frac{1}{2x} \int_0^{2x} \frac{1}{t(\ln t)^2}\ dt \\
&= \frac{1}{2x} \lim_{n\to+\infty} \int_{1/n}^{2x} \frac{1}{t(\ln t)^2}\ dt \qquad (Monotone Convergence Theorem) \\
&= \frac{1}{2x} \lim_{n\to+\infty} (R) \int_{1/n}^{2x} \frac{1}{t(\ln t)^2}\ dt \qquad (\frac{1}{t(\ln t)^2}\ is\ continuous\ on (1/n, 2x]) \\
&= \frac{1}{2x} \lim_{n\to\infty} (-\frac{1}{\ln t}) \mid_{1/n}^{2x} = \frac{1}{2x}(-\frac{1}{\ln 2x}) = |\frac{1}{2x \ln 2x}|
\end{aligned}$$

So $Mf(x) = \sup_{0<r<infty} Q_r f \geq |\frac{1}{2x\ln 2x}|$. Next,

$$\begin{aligned}
\int_0^{1/4} \frac{1}{|2x \ln 2x|}\ dx &= -\int_0^{1/4} \frac{1}{2x \ln 2x}\ dx \\
&= -\lim_{n\to+\infty} (-(R) \int_{1/n}^{1/4} \frac{1}{2x\ln 2x}\ dx = -\lim_{n\to+\infty} \frac{1}{2} \ln|\ln 2x| \mid_{1/n} 1/4 \\
&= -\lim_{n\to+\infty} [\frac{1}{2} \ln|\ln 1/2| - \frac{1}{2} \ln|\ln 2/n|] = +\infty,
\end{aligned}$$

$$\int_0^1 Mf(x)dx \geq \int_0^{1/4} Mf(x)dx \geq +\infty.$$

This shows that $Mf(x) \in L^1(\mathbb{R}^k)$ does not hold.

Additionally, (a) can be proved more simply:

$$\begin{aligned}
Mf(x) &= \sup_{0<r<\infty} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f| dm \geq \frac{1}{m(B(x,2|x|))} \int_{B(x,2|x|)} |f| dm \\
&= \frac{1}{2^k c_k |x|^k} \int_{B(x,2|x|)} |f| dm.
\end{aligned}$$

By the absolutely continuity of $f \in L^1$, $\forall\ \epsilon > 0$, $\exists\ R_0$ s.t. $\int_{B(0,R_0)} |f|\ dx < \epsilon$, so,

$$0 \leq \int_{\mathbb{R}^k} |f|\ dx \leq \int_{B(0,R_0)} |f|\ dx + \epsilon.$$

And we could R_0 larger here. If $\|f\|_{L^1} \neq 0$, then let $\epsilon = \frac{1}{2}\|f\|_{L^1}$, then take $R_0 >> 1$, we have, when $|x| > R_0$, we have $B(0,R_0) \subset B(|x|, 2|x|)$, and

$$\frac{1}{2}\|f\|_{L^1} = \int_{\mathbb{R}^k} |f|\ dm - \frac{1}{2} \int_{\mathbb{R}^k} |f|\ dm < \int_{B(0,R_0)} |f|\ dm \leq \int_{B(x,2|x|)} |f|\ dx.$$

So $Mf(x) \geq \frac{\frac{1}{2}\|f\|_{L^1}}{c_k 2^k |x|^k}$ if $|x| > R_0$. □

19. Suppose f is continuous on R^1, $f(x) > 0$ if $0 < x < 1$, $f(x) = 0$ otherwise. Define

$$h_\epsilon(x) = sup\{n^c f(nx) : n = 1, 2, 3, \dots\}.$$

Prove that
(a) h_c is in $L^1(\mathbb{R}^1)$ if $0 < c < 1$,
(b) h_1 is weak L^1 but not in $L^1(\mathbb{R}^1)$,
(c) h_c is not in weak L^1 if $c > 1$.
Proof: (a) $f \in C^0(\mathbb{R}^1)$ and $f = 0$ outside $(0, 1)$, $f > 0$ in $(0, 1)$, implies that $0 < M = \max_{\mathbb{R}^1} |f| = \max_{[0,1]} f < +\infty$. For $x < 0$ or $x > 1$, then $\forall n \in \mathbb{N}$, $nx < 0$ or $nx \geq x \geq 1$, hence, $f(nx) = 0$, $h_c(x) = 0$. For $x \in (0, 1)$, $f(nx) \neq 0$ if and only if $0 < nx < 1$ i.e. $0 < n < 1/x$, so

$$h_c(x) = sup_n n^c f(nx) = sup_{n<1/x, n\in\mathbb{N}} n^c f(nx) \leq (\frac{1}{x})^c M.$$

And if $0 < c < 1$, letting $F(x) = (\frac{1}{x})^c M,\ for\, 0 < x < 1; = 0, for\ x < 0\ or\ x \geq 1$, we have $F \in L^1(\mathbb{R}^1)$. And $f \in C^0(\mathbb{R}^1)$ implies $n^c f(nx) \in C^0$ and hence is measurable, from which we know h_c is measurable. This together with the fact that $h_c \geq 0$ and $h_c \leq F$ lead to that $h_c \in L^1(\mathbb{R}^1)$.

(b) Similar to (a) we see that if $c = 1$,

$$h_c(t) \leq \begin{cases} \frac{1}{x} M, & 0 < x < 1; \\ 0, & x < 0\ or\ x \geq 1. \end{cases}$$

So for any $\lambda > 0$, $h_1 x > \lambda$ implies $x \in (0, 1)$ and $\frac{1}{x} M\ 1$. So $\{x :\ h_1 x > \lambda\} \subset \{x :\ 0 < x < \frac{M}{\lambda}\}$. Thus $\lambda\ m\{x :\ h_1 x > \lambda\} \leq \lambda\ m\{x :\ 0 < x < \frac{M}{\lambda}\} = M < +\infty$, so $h_1 \in weak - L^1$.

As there exists $[a, b]$ on which $f(x) \geq \frac{M}{2}$, we see that for any $x \in (0, b - a)$, $\frac{b-a}{x} > 1$. Take $N = [\frac{a}{x} + 1$, then $\frac{a}{x} < N \leq \frac{a}{x} + 1 < \frac{b}{x}$, i.e. $Nx \in (a, b)$. So $f(Nx) \geq \frac{M}{2}$. Therefore, $h_1(x) \geq N f(Nx) \geq N \cdot \frac{M}{2} > \frac{a}{x} \cdot \frac{M}{2} = \frac{aM}{2}\frac{1}{x}$, so h_1 is not in L^1.

(c) If $c > 1$, let $[a, b] \subset (0, 1)$ be the set on which $f(x) > M$. For all $n \in N$, if $x \in [\frac{a}{n}, \frac{b}{n}]$, then $a \leq nx \leq b$, and $n^c f(nx) > n^c \frac{M}{2}$, so $h_c(x) > n^c \frac{M}{2}$. Thus $[\frac{a}{n}, \frac{b}{n}] \subset \{x :\ h_c(x) > n^c \frac{M}{2}\}$, and $n^c \frac{M}{2}\ m\{x :\ h_c(x) > n^c \frac{M}{2}\} \geq n^c \frac{M}{2}\ m[\frac{a}{n}, \frac{b}{n}] = n^c \frac{M}{2} \frac{b-a}{n} \to +\infty$ since $c > 1$. So h_c is not in weak L^1 if $c > 1$. □

18 Let $\phi_0(t) = 1$ on $[0, 1)$, $\phi_0(t) = -1$ on $[1, 2)$, extend ϕ_0 to $\mathbb{R}^1$ so as to have period 2, and define $\phi_n(t) = \phi_0(2^n t)$, $n = 1, 2, 3, \dots$. Assume that $\Sigma|c_n|2 < \infty$ and prove that the series

$$\Sigma_{n=1}^\infty c_n \phi_n(t) \tag{*}$$

converges then for almost every t.
Probabilistic interpretation: The series $\Sigma(\pm c_n)$ converges with probability 1.
Suggestion: ϕ_n is orthonormal on $[0, 1]$, hence (*) is the Fourier series of some $f \in L^2$. If $a = j \cdot 2^{-N}$, $b = (j + 1) \cdot 2^{-N}$, $a < t < b$, and $s_N = c_1\phi_1 + \cdots + c_N\phi_N$, then, for $n > N$,

$$s_N(t) = \frac{1}{b-a}\int_a^b s_N\ dm = \frac{1}{b-a}\int_a^b s_n\ dm,$$

and the integral converges to $\int_a^b f\ dm$ as $n \to \infty$. Show that (*) converges to $f(t)$ at almost every Lebesgue point of f.

Proof: We have

$$\int_0^1 \phi_n^2(t)\ dt = \int_0^1 \phi_0^2(2^n t)\ dt = \frac{1}{2^n}\int_0^{2^n} \phi_0^2(x)\ dx = \frac{1}{2^n}\int_0^{2^n} dx = 1,$$

where Theorem 7.26 is used in the second equality. If $m > n$, then

$$\int_0^1 \phi_n(x)\phi_m(x)\ dx = \int_0^1 \phi_0(2^n x)\phi_0(2^m x) = \frac{1}{2^n}\int_0^{2^n} \phi_0(y)\phi_0(2^{m-n}y)dy.$$

Since ϕ_0 has period 2, $f(y) = \phi_0(y)\phi_0(2^{m-n}y)$ has period 2. Notice that the integration of a function over any interval of length T is the same if $0 < T$ is a period of the function, and that the integration of $\phi_n(t)$ over any interval of period length is 0, we have

$$\int_0^{2^n} \phi_0(y)\phi_0(2^{m-n}y)dy = \frac{2^{n-1}}{2^n}\int_0^2 \phi_0(y)\phi_0(2^{m-n}y)dy = \frac{1}{2}[(\int_0^1 - \int_1^2)\phi_0(2^{m-n}y\ dy)] = \frac{1}{2}[0{-}0] = 0.$$

Thus $\phi_n(t)$ is orthonormal on $[0,1]$. By Riesz-Fisher Theorem, and $\Sigma|c_n|2 < \infty$, there is an $f \in L^2[0,1]$ such that $c_n = \int_0^1 f\phi_n\ dm$, $\Sigma_{n=1}^{\infty} c_n\phi_n(t)$ is the Fourier series of f, and $s_n(t) = \Sigma_{n=1}^{n} c_i\phi_i(t) \to f$ in L^2. If $a = j\cdot 2^{-N}$, $b = (j+1)\cdot 2^{-N}$, $a < t < b$, and $s_N = c_1\phi_1 + \cdots + c_N\phi_N$, then for $n > N$

$$\int_a^b c_n\phi_n(t)\ dt = c_n\int_a^b \phi_0(2^n t)\ dt = \frac{c_n}{2^n}\int_{a2^n}^{b2^n} \phi_0(t)dt.$$

Since $b2^n - a2^n = (j+1)\cdot 2^{-N+n} - j\cdot 2^{-N+n} = 2^{-N+n} = 22^{-N+n-1}$, we have

$$\int_a^b c_n\phi_n(t)\ dt = \frac{c_n}{2^n}2^{-N+n-1}\int_0^2 \phi_0(t)dt = 0.$$

So $\frac{1}{b-a}\int_a^b s_n\ dm = \frac{1}{b-a}\int_a^b s_N\ dm$. For $1 \le i \le N$, $a < t < b$, we have $j\cdot 2^{-N+i} < 2^i t < (j+1)\cdot 2^{-N+i}$, and since $(j+1)\cdot 2^{-N+i} - j\cdot 2^{-N+i} = 2^{-N+i} < 1$ and it lies in half a period, we know that $\phi_0 \equiv 1 or -1$ over $[j\cdot 2^{-N+i}, (j+1)\cdot 2^{-N+i}]$ and thus $\phi_i \equiv 1 or - 1$ over $[a,b]$, which implies that

$$\frac{1}{b-a}\int_a^b \phi_i(t)dt = \phi_i \equiv 1 or - 1,\ \forall 1 \le i \le N.$$

So, we have $s_N(t) = \frac{1}{b-a}\int_a^b s_N\ dm = \frac{1}{b-a}\int_a^b s_n\ dm$ for all $n > N$. Note that $s_n \to f$ in $L^2[0,1]$,so $f \in L^1[0,1]$ and since $[a,b] \subset [0,1]$, we have

$$|\int_a^b (s_n - f)\ dm| \le (\int_a^b |s_n - f|^2)^{\frac{1}{2}}(b-a)^{\frac{1}{2}} \le \|s_n - f\|_{L^2[0,1]} \to 0.$$

That is

$$\int_a^b s_n(t)\ dt \to \int_a^b f(t)\ dt,$$

and thus, $s_N(t) = \frac{1}{b-a}\int_a^b f(t)\ dt$.

For any Lebesgue point x_0 of f in $[0,1]$, and for any $n \in \mathbb{N}$, $\exists j_n \in \mathbb{N}$ s.t. $x_0 \in [j_n 2^{-n}, j_{n+1}2^{-n}] = J_n$, J_n shrinks to x_0 nicely, $J_n \subset B(x_0, 2^{-n})$, $m(J_n) = 2^{-n} \ge \frac{1}{2}m(B(x_0, 2^{-n})) = 2^{-n}$, $|J_n| \to 0$. So Theorem 7.10 (use the form that $f \in L^1_{loc}(\mathbb{R}^1)$ only) implies

$$\lim_{n\to+\infty}\frac{1}{m(J_n)}\int_{J_n} f(t)\ dt = f(x_0),$$

but $\forall n$, $\exists j_n \in N$, $a_n = j_n 2^{-n}$, $b_n = (j_n+1)2^{-n}$, $s_n(x_0) = \frac{1}{b_n - a_n}\int_{a_n}^{b_n} f(t)\ dt \to f(x_0)$, i.e. $\Sigma_{n=1}^{\infty} c_n\phi_n(x_0)$ converges to $f(x_0)$. Thus $\Sigma_{n=1}^{\infty} c_n\phi_n(t) \to f(t)$ a.e. in $[0,1]$.

Notice that $\forall\ n > 1$, $\phi_n(t+1) = \phi_0(2^n)(t+1) = \phi_0(2^n t) = \phi_n(t)$, so ϕ_n is 1-period. Hence $f(t)$ and $\Sigma_{n=1}^{\infty} c_n\phi_n(t)$ is also 1-period. So $\Sigma_{n=1}^{\infty} c_n\phi_n(t) \to f(t)$ a.e. in $\mathbb{R}^1$. □

20, (a) For any set $E \subset \mathbb{R}^2$ the boundary ∂E of E is by definition, the closure of E minus the interior of E. Show that E is Lebesgue measurable whenever $m(\partial E) = 0$.
(b) Suppose that E is the union of a (*possibly uncountable*) collection of *closed* discs in $\mathbb{R}^2$ whose radii are least 1. Use (a) to show that E is Lebesgue measurable.
(c) Show that the conclusion of (b) is true when the radii are unrestricted.
(d) Show that some unions of closed discs of radius 1 are unrestricted.
(e) Can discs be replaced by triangles, rectangles, arbitrary polygons, etc., in all this? What is the relevant geometric property?

Solution: (a) $\partial E = \overline{E} - E^o$. If $m(\partial E) = 0$, then since $E^o \subset E \subset \overline{E}$, we have $m(\overline{E} - E^o) = m(\partial E) = 0$. And $\overline{E}$, E^o are Borel sets, and m is complete, so E is lebesgue measurable.

(b) Suppose $E = \cup_{x\in\Lambda}\overline{B(x, r_x)}$, $r_x \geq 0$. $\forall n$, let $E_n = E \cap B(0, n)$. Let $\mathcal{F}_n = \{\overline{B(z, r)},\ r > 0, \overline{B(z, r)} \subset E_n\}$. Then $\forall\ x_0 \in E_n$, $\exists\ y \in \Lambda$ s.t. $x_0 \in \overline{B(y, r_y)} \subset E_n$, $r_y > 0$. And if $x_0 \in B(y, r_y)$, then clearly, $\overline{B(x_0, r)} \subset \overline{B(y, r_y)} \cap B(0, n)$ for $0 < r << 1$ and $\inf\{diam B | x_0 \in B,\ B \in \mathcal{F}_n\} = 0$. If $x_0 \in \partial B(y, r_y) \cap B(0, n)$, there is a sequence $\{y_m\}$ s.t. $y_m \in B(y, r_y)$ and $|x_0 - y_m| \to 0$, $x_0 \in \overline{B(y_m, |x_0 - y_m|)} \subset \overline{B(y, r_y)} \cap B(0, n)$. So $x_0 \in \cup_{B\in\mathcal{F}_n} B$ and $\inf\{diam B | x_0 \in B,\ B \in \mathcal{F}_n\} = 0$. (In the above argument we have used that $E = \cup_{x\in\Lambda}\overline{B(x, r_x)}$ and $B(x, r_x)$ is a ball.)

So, $E_n \subset \cup_{B\in\mathcal{F}_n} B$, and $\forall\ x_0 \in E_n$,

$$\inf\{diam B | x_0 \in B,\ B \in \mathcal{F}_n\} = 0.$$

By Vitali's Covering Theorem's Corollary (Evans, Measure Theory and Fine Properties of Functions, P28), there exists a countable family of G of disjoint balls in $\mathcal{F}_n$ such that for each finite $\{B_1, \cdots, B_m\} \subset \mathcal{F}_n$, we have

$$E_n - \cup_{k=1}^m B_k \subset \bigcup_{B \in G - \{B_1, \cdots, B_m\}} \hat{B},$$

where $\hat{B}$ is the ball with the center of B and radius five times of that of B. Denote $G = \{B_1, \cdots, B_m, \cdots\}$, then for any m,

$$E_n - \cup_{k=1}^m B_k \subset \bigcup_{j=m+1}^{+\infty} \hat{B}j.$$

As $B_i \subset B(0, n)$ and they are disjoint,

$$\Sigma_{i=1}^{+\infty} m(B_i) = m(\bigcup_{i=1}^{+\infty} B_i) \leq m(B(0, n)) < +\infty.$$

So $\forall \epsilon > 0$, $\exists m$ s.t. $\Sigma_{j=m+1}^{+\infty} m(B_j) < \frac{\epsilon}{5^k}$,

$$m(E_n - \bigcup_{k=1}^m B_k) \leq \Sigma_{j=m+1}^{+\infty} m(\hat{B}) = 5^k \Sigma_{j=m+1}^{+\infty} m(B) = \epsilon.$$

And for any m, there exists K_m, s.t.

$$E_n - \bigcup_{j=1}^{+\infty} B_j \subset E_n - \bigcup_{j=1}^{K_m} B_j \subset \bigcup_{j=K_m+1}^{+\infty} \hat{B}_j \triangleq G_m$$

and $m(G_m) < \frac{1}{m}$. Then $E_n - \bigcup_{j=1}^{+\infty} B_j \subset \bigcap_{m=1}^{+\infty} G_m$. And since $m(\bigcap_{m=1}^{+\infty} G_m) < m(G_m) < \frac{1}{m}$, so $m(\bigcap_{m=1}^{+\infty} G_m) = 0$. Thus $m(E_n - \bigcup_{j=1}^{+\infty} B_j) = 0$ and $E_n - \bigcup_{j=1}^{+\infty} B_j \in \mathcal{M}$. Then

$$E_n = (E_n - \bigcup_{j=1}^{+\infty} B_j)\bigcup(\bigcup_{j=1}^{+\infty} B_j)$$

(since $\bigcup_{j=1}^{+\infty} B_j \subset E_n$) is measurable, and thus $E = \bigcup_{j=n}^{+\infty} E_n$ is measurable.

The above argument holds if the ball is replaced by other family of sets which satisfy interior ball condition.

In the following we prove it in another way. First we state a fact:

Claim: In $\mathbb{R}^2$, if $p_0 \in \partial B(x,r)$, $r > 0$, then $\forall 0 < s < r$, $m(B(p_0,s) \cap B(x,r)) \geq \frac{1}{4} m(B(p_0,s))$.

Now we begin to prove: in $\mathbb{R}^2$, if $E = \bigcup_{x\in\Lambda} \overline{B(x,r_x)}$, $r_x \geq 0$, $\forall x \in \Lambda$, $(\Lambda \in \mathbb{R}^2,\ E \in \mathbb{R}^2)$, then E is Lebesgue measurable. Let

$$E = \bigcup_{n=1}^{+\infty} E_n, \quad where\ \ E_n = \bigcup_{\{x\in\Lambda: r_x > 1/n\}} \overline{B(x,r_x)}.$$

Then the problem becomes: if $E = \bigcup_{x\in\Lambda} \overline{B(x,r_x)}$, and there exists $c > 0$ s.t. $r_x \geq c$ for all $x \in A$, then E is Lebesgue measurable. By (a) it suffices to show that $m(\partial E) = 0$.

Suppose $m(\partial E) > 0$, then since E^o (the interior of E which is an open set) is Lebeague measurable,, and its character function $\chi_{E^o} \in L^1_{loc}(\mathbb{R}^2)$. By Theorem 7.10, almost all the point in $\mathbb{R}^2$ is Lebeague point of χ_{E^o}. As $m(\partial E) > 0$, there exists $x_0 \in \partial E$ s.t. x_0 is a Lebesgue point of χ_{E^o}. For any $n \in \mathbb{N}$, $\exists y_n \in E \cap B(x_0, 1/n)$, so $E_n \triangleq B(y_n, 1/2n) \subset B(0,1/n)$ and $m(E_n(x_0)) = \frac{\pi}{4}(\frac{1}{n})^2 = \frac{1}{4} m(B(x_0,1/n))$. Thus $\{E_n(x_0)\}$ shrinks nicely to x_0. since x_0 is not in E^o, then Theorem 7.10 implies

$$0 = \chi_{E^o}(x_0) = \lim_{n\to+\infty} \frac{1}{m(E_n(x_0))} \int_{E_n(x_0)} \chi_{E^o}\ dm. \tag{*}$$

Since $y_n \in E = \bigcup_{x\in\Lambda} \overline{B(x,r_x)}$, there exists $z_n \in \Lambda$ s.t. $y_n \in \overline{B(z_n, r_{z_n})}$. If $y_n \in \partial B(z_n, r_{z_n})$, the above Claim means that for n large, $1/2n < c < r_{z_n}$,

$$\int_{E_n(x_0)} \chi_{E^o} dm = m(E_n(x_0) \cap E^o) \geq m(E_n(x_0) \cap B(z_n, r_{z_n})) \geq \frac{1}{4} m(E_n(x_0)).$$

If $y_n \in B(z_n, r_{z_n})$, then it is easy to see that when n is large, the above inequality also holds. Thus there holds

$$\frac{1}{m(E_n(x_0))} \int_{E_n(x_0)} \chi_{E^o} dm \geq \frac{1}{4}, \quad \forall n,$$

which contradicts with (*). Thus $m(\partial E) = 0$ and (b)(c) holds.

(b)(c) could be generalized to $\mathbb{R}^k$. In this case the Claim should be replaced by

Claim: If $p \in \partial B(x,r)$, $r > 0$, then $\forall 0 < s < r$, $m(B(p_0,s) \cap B(x,r)) \geq c_k m(B(p_0,s))$.

Where the constant $c_k > 0$ needs more detailed calculation. And the result of (b)(c) could be generalized to:

If $F = \bigcup_{\tau\in I} A_\tau$ and (1) $\forall i$, $diam A_i > c > 0$; (2) $\exists \rho(c) > 0$ s.t. $\forall P \in \partial A_i$, if $n >> 1$, $m(B(P,1/n) \cap A_i^o) \geq \rho(c) m(B(P,1/n))$ (Which implies that A_i is not an isolated point).

The results hold if discs replaced by triangles, rectangles, arbitrary polygons, etc. The relevant geometric property is convex. As for example, the for polygons the Claim holds for some constant determined by the minimum angles. We can also prove that: in $\mathbb{R}^2$, if $z_0 \in \partial B(x,r)$, then

$$\lim_{\epsilon \to 0^+} \frac{m(B(z_0,\epsilon) \cap B(x,r))}{m(B(z_0,\epsilon))} = \frac{1}{2}.$$

$\square$

Chapter 8 Integration on Product Spaces

1. Find a monotone class $\mathfrak{M}$ in R^1 which is not a $\sigma - algebra$ even though $R^1 \in \mathfrak{M}$ and $R^1 - A \in \mathfrak{M}$ for every $A \in \mathfrak{M}$.

Solution: Let $\mathfrak{M} = \{(-\infty,+\infty) = R^1, \emptyset(-\infty a], (-\infty,b), [c,+\infty), (d,+\infty),$ for every $a,b,c,d \in R^1\}$.Then $\mathfrak{M}$ is not a $\sigma - algebra$,because $(-\infty,1) \in \mathfrak{M}$, $(2,+\infty) \in \mathfrak{M}$,but $(-\infty,1) \bigcup (2,+\infty) \notin \mathfrak{M}$.

Cleary,$R^1 \in \mathfrak{M}$,and if $A \in \mathfrak{M}$,we have$R^1 - A \in \mathfrak{M}$.It is easy to see also that $\mathfrak{M}$ is a monotone class on R^1.In fact ,if $A_1 \subset A_2 \subset \cdots$,and if $A_1 = (-\infty,a_1]$, or $(-\infty,a_1)$,then $A_i (i \geq 2)$ must not be the form $[a,+\infty)$,or$(-\infty,+\infty)$ for $a \in R^1$,and $\bigcup\limits_{i=1}^{\infty} A_i = (-\infty,a]$ or $(-\infty,a)$, or $(-\infty,\infty)$ for some $a \in R^1$,so $\bigcup\limits_{i=1}^{\infty} A_i \in \mathfrak{M}$.

Similary,if $A_1 \subset A_2 \subset \cdots$,and if $A_1 = [a_1,+\infty)$,or $(a_1,+\infty)$,then $A_i (i \geq 2)$ must not be the form $(-\infty,b]$ or $(-\infty,b)$ for $b \in R_1$. $\bigcup\limits_{i=1}^{\infty} A_i = (a,+\infty)$,or $[a,+\infty)$,or $(-\infty,+\infty)$ for some $a \in R^1$,so $\bigcup\limits_{i=1}^{\infty} A_i \in \mathfrak{M}$.

In the same way,we see that if $A_1 \supset A_2 \supset \cdots$,and if $A_1 = (-\infty,a_1)$,or $(-\infty,a_1]$,then $\bigcap\limits_{i=1}^{\infty} A_i = \emptyset$, or $(-\infty,b)$ or $(-\infty,b]$.And if $A_1 = (a,+\infty)$, or $[a,+\infty)$,then $\bigcap\limits_{i=1}^{\infty} A_i = \emptyset$, or $(b,+\infty)$,or $[b,+\infty)$.

Then $\bigcap\limits_{i=1}^{\infty} A_i \in \mathfrak{M}$,so $\mathfrak{M}$ is monotone class. $\square$

2. Suppose f is a Lebesgue measurable nonnegative real function on R^1 and $A(f)$ is the ordinate set of f.This is the set of all points $(x,y) \in R^2$ for which $0 < y < f(x)$.

(a)Is it true that $A(f)$ is Lebesgue measurable,in the two-dimensional sense?

(b)If the answer to (a)is affirmative,is the integral of f over R^1 equal to the measure of $A(f)$?

(c)Is the graph of a measurable set of R^2?

(d)If the answer to (c)is Affirmative,is the measure of the graph equal to zero?

Solution: (a)We prove a more general result:

If $E \subset R^k$ is a Lebesgue measurable set,$f : E \to R^1$ is a nonnegative Lebesgue measurable function,then

$$A(f,E) = \{(x,y) \in R^{k+1}, x- \in E, 0 < y < f(x)\}$$

is a Lebesgue measurable set in $R^k \times R^1 = R^{k+1}$.

Pf:Let $\{r_1, r_2, r_3, \cdots\}$ be the set of all positive rational numbers.$G_m = E_m \times (0, r_m)$ where $E_m = \{x \in R^k, f(x) > r_m\}$ which is a Lebesgue measurable set since f is Lebesgue measurable.We prove in the following that $\bigcup\limits_{m=1}^{\infty} G_m = A(f, E)$, and then as G_m is a measurable rectangle hence measurable in R^{k+1},so $A(f, E)$ is Lebesgue measurable in $R^k \times R^1 = R^{k+1}$.If $(x, y) \in G_m$,then $x \in E_m$ and $0 < y < r_m$,$\therefore 0 < y < r_m < f(x)$,so $(x, y) \in A(f, E)$.

On the other hand,if $(x, y) \in A(f, E)$,then$0 < y < f(x)$.By the fact that $\bar{Q}_+ = R^1_+, \exists r_m > 0, r_m \in Q$, st $0 < y < r_m < f(x)$, so $x \in E_m$ and $y \in (0, r_m), (x, y) \in E_m \times (0, r_m) = G_m$,thus $A(f, E) \subset \bigcup\limits_{m=1}^{\infty} G_m$.So (1) is true.This provs that $A(f, E)$ is Lebesgue measurable.

(This conclusion is suitable to measurable space $(X, \mathfrak{M}, \mu)$,

$f : X \to R^1_+$ is $\mu-$measurable,

$$A(f, X) = \{(x, y) \in X \times R^1, 0 < y < f(x)\},$$

we can prove that $A(f, X)$ is $\mu \times m_1-$measurable in the same way.)

(b)Still we assume even more generally that $E \subset R^k$ is Lebesgue measurable and $f : E \to R$ is Lebesgue measurable then we have that $A(f, E)$ is Lebesgue measurable in $R^k \times R^1 = R^{k+1}$. as we have that

Claim:$m_{k+1}(A(f, E)) = \int_E f dm_k$

Pf:(i) If $f(x) = \chi_A(x)$, where A is Lebesgue measurable set and $A \subset E$,Then $\int_E f dm_k = \int_E \chi_A dm_k = m_k(A)$.

on the other hand,$A(\chi_A, E) = \{(x, y), 0 < y < \chi_A(x)\} = \{(x, y) \mid x \in A, 0 < y < 1\} = A \times (0, 1), m_k \times m_1(A(\chi_A, E)) = m_k \times m_1(A \times (0, 1)) = m_k(A)m_1((0, 1)) = m_k(A) = \int_E \chi_A dm_k$.

(ii)If $c \geq 0, f(x) = c\chi_A(x)$,where A is a Lebesgue measurable set in E,

then similarly,

if $c > 0$,$m_k \times m_1(A(f, E)) = m_k \times m_1(A \times (0, c)) = m_k(A)m_1((0, c)) = cm_k(A) = \int_E c\chi_A dm_k$,

if $c = 0$, then $A(f, E) = \emptyset, m_k \times m_1(A(f, E)) = 0 = \int_E c\chi_A dm_k$

(iii)If $E = \bigcup\limits_{i=1}^{m} E_i, E_i$ are Lebesgue measurable and $E_i \bigcap E_j = \emptyset$ for $i \neq j$,then $A(f, \bigcup\limits_{i=1}^{m} E_i) = \bigcup\limits_{i=1}^{m} A(f, E_i)$ and $A(f, E_i)$ are disjoint Lebesgue set.This is trival.

(iv)If $f = \sum\limits_{i=1}^{m} c_i\chi_{E_i}(x), c_i \geq 0$, and $\bigcup\limits_{i=1}^{m} E_i = E, E_i$ are mutually disjoint Lebesgue measurable sets.

Then by(iii),$A(f, E) = A(f, \bigcup\limits_{i=1}^{m} E_i) = \bigcup\limits_{i=1}^{m} A(f, E_i)$,

$\therefore\ m_{k+1}(A(f, E)) = m_{k+1}(\bigcup\limits_{i=1}^{m} A(f, E_i)) = \sum\limits_{i=1}^{m} m_{k+1}(A(f, E_i)) = \sum\limits_{i=1}^{m} m_{k+1}(E_i \times (0, c_i)) = \sum\limits_{i=1}^{m} c_i m_k(E_i) = \int_E f dm_k$.

so for simple nonnegative function s,$m_{k+1}(A(s, E)) = \int_E s dm_k$.

(v)If f is nonnegative Lebesgue measurable function,then there is a sequence of nonnegative functions $\{s_n\}$ on $E, s_n \nearrow f(x)$ for each $x \in E$,so clearly $A(s_n, E) \subset A(s_{n+1}, E)$.

We claim that: $\bigcup\limits_{n=1}^{\infty} A(s_n, E) = A(f, E)$ (2)

In fact,if $(x, y) \in A(s_n, E)$ clearly $(x, y) \in A(f, E)$ as $0 < g < s_n(x) < f(x)$,on the other hand if $(x, y) \in A(f, E)$,then $0 < y < f(x)$,from the fact that $\lim\limits_{n\to\infty} s_n(x) = f(x), (\forall x \in E)$,we see that

$\exists n, 0 < y < s_n(x) \le f(x), \therefore (x,y) \in A(s_n, E)$, so (2) is true.

By the result proved in (iv) and monotone convergence theorem $m(A(f,E)) = m(\bigcup\limits_{n=1}^{\infty} A(s_n,E)) = \lim\limits_{n\to\infty} m(A(s_n,E)) = \lim\limits_{n\to\infty} \int_E s_n dm_k = \int_E f dm_k$.

(c)$f : E \subset R^k \to R^1_+$($E$ is measurable and f is measurable)

Let

$$F(f,E) = \{(x,y) \in E \times R^1, f(x) < y\},$$

let $\{r_1, r_2, r_3, \cdots, r_m, \cdots\}$ be the set of all positive rational numbers.

Then $F_m = \{x \in E, f(x) < r\}$ is measurable,and $F_m \times (r_m, +\infty)$ is a measurable rectangle in $R^k \times R^1$.

$F(f,E) = \bigcup F_m \times (r_m, +\infty)$ is clearly true,so $F(f,E)$ is also Lebesgue measurable in $R^k \times R^1$.

Let

$$G(f,E) = \{(x,y) \in E \times R^1, f(x) = y\},$$

then in fact we can prove that

$$A^\alpha(f,E) \triangleq \{(x,y) \in E \times R^1, 0 \le y < f(x)\}$$

is also measurable in R^{k+1},as $A^\alpha(f,E) = \bigcup\limits_{m=1}^{\infty} F_m \times [0, r_m)$ is easily proved by a similar method as what we did in (a),thus $E \times [0,\infty) = A^\alpha(f,E) \bigcup G(f,E) \bigcup F(f,E)$ and $A^\alpha(f,E), G(f,E), F(f,E)$ are disjoint.

$\therefore\ G(f,E) = E \times [0,+\infty) - A^\alpha(f,E) \bigcup F(f,E)$is Lebesgue measurable in R^{k+1} as $E \times [0,+\infty), A^\alpha(f,E), F(f,E)$ are.

(d)claim 1:If $f \in C([a,b], R^1_+)$,then

$$G(f,[a,b]) = \{(x,y) \in [a,b] \times R^1, y = f(x)\}$$

satisfies $m(G(f,[a,b])) = 0$,here $[a,b]$ is a bounded closed interval.

Pf:$\forall \epsilon > 0$, as $f \in C^0([a,b], R^1)$, f is uniformly continuous on $[a,b]$, so $\exists \delta > 0$, st $x^{'}, x^{''} \in [a,b]$ and

$$|x^{'} - x^{''}| < \delta \Rightarrow |f(x^{'}) - f(x^{''})| < \frac{\epsilon}{16(b-a)}.$$

Take $N = N(\epsilon)$, st $n \ge N \Rightarrow \frac{4(b-a)}{n} < \delta$. Divide $[a,b]$ into n equal with the points $a = x_0 < x_1 < \cdots < x_n = b$ then $x_i - x_{i-1} = \frac{b-a}{n}, i = 1,2,\cdots,n$.

For $i = 0,1,2,\cdots,n$, let

$$\begin{aligned} I_i^n \ &\triangleq\ \{(x,y) \in R^2, x_i - \frac{2(b-a)}{n} < x < x_i + \frac{2(b-a)}{n}, \\ &\qquad f(x_i) - \frac{\epsilon}{16(b-a)} < y < f(x_i) + \frac{\epsilon}{16(b-a)}\}, \end{aligned}$$

then I_i^n is a Lebesgue measurable rectangle in R^2.If we let

$$A_i = \{(x,y), x_i \le x \le x_{i+1}, y = f(x), i = 0,1,2,\cdots\}$$

,then $A_i \subset I_i^n$, indeed if $(x, f(x)) \in A_i$ then $x_i \le x \le x_{i+1}, |x - x_i| \le |x_{i+1} - x_i| = \frac{b-a}{n} < \frac{2(b-a)}{n} < \delta$, so $|f(x) - f(x_i)| < \frac{\epsilon}{16(b-a)}$, i.e. $(x, f(x)) \in I_i^n$. Notice that $G(f[a,b]) \subset \bigcup\limits_{i=0}^{n} A_i \subset \bigcup\limits_{i=0}^{n} I_i^n$ and by(c),$G(f[a,b])$ is measurable, so

$$m(G(f[a,b])) \le \sum_{i=0}^{n} m_i(I_i^n) = \sum_{i=0}^{n} \frac{4(b-a)}{n} \frac{\epsilon}{8(b-a)} = \sum_{i=0}^{n} \frac{\epsilon}{2n} = \frac{(n+1)\epsilon}{2n} \le \epsilon,$$

since $\epsilon > 0$ is arbitrary,we see that $m(G(f[a,b])) = 0$.

(In fact,if we have no demand of $f \ge 0$ upward,we can also prove $m(G(f[a,b])) = 0$)

We use outer measure to prove: $\forall m$, let $G_m = \bigcup_{i=0}^{n} I_i^n$, st $m(\bigcup_{i=0}^{n} I_i^n) \le \frac{1}{m}$, $B = \bigcap_{m=0}^{\infty} G_m$, then B is measurable and $m_2(B) \le \frac{1}{m}(\forall m)$,$\therefore$ $m_2(B) = 0, G(f[a,b]) \subset B$,$\therefore$ $G(f[a,b]) m_2-$ measurable and $m_2(G(f[a,b])) = 0$.

claim 2:If $f \in C^0(R^1, R^1), G(f, R^1) = \{(x,y), x \in R^1, f(x) = y\}$,then $m(G(f, R^1)) = 0$.

In fact $G(F, R^1) \subset \bigcup_{k=-\infty}^{+\infty} G(f, [k, k+1])$ and $f \in C^0(R^1, R^1) \Rightarrow f \in C^0([k, k+1], R^1)$, so claim $2 \Rightarrow m(G(f, [k, k+1])) = 0$, hence $m(G(f, R^1)) = 0$.

Now,we are ready to prove that

claim 3:If $E \subset R^1$ is a Lebesgue measurable set, and $f : E \to R^1_+$ is a Lebesgue measurable function,then $m(G(f, E)) = 0$ where $G(f, E) = \{(x, y), x \in E, f(x) = y\}$.

To prove this ,first assume that $m(E) < \infty$, and $|f| < M < +\infty$ and let

$$\widetilde{f}(x) = \begin{cases} f(x), & x \in E; \\ 0, & x \notin E. \end{cases}$$

By Lusin's theorem $(Th2.24, P_{55})$, $\forall \epsilon > 0, \exists g_\epsilon \in C_c(R^1)$ st $m(\{x \in R^1, \widetilde{f}(x) \ne g_\epsilon(x)\}) < \epsilon, \therefore$ $m(\{x \in E, f(x) \ne g_\epsilon(x)\}) < \epsilon$. Let $E_\epsilon = \{x \in E, f(x) \ne g_\epsilon(x)\}\}$ and $G_\epsilon = E_\epsilon \times [-M, M]$, then $G(f, E) = G(f, E_\epsilon^c) \bigcup G(f, E_\epsilon) = G(g_\epsilon, E_\epsilon^c) \bigcup G_\epsilon$, so by claim 2 $m_2(G(g_\epsilon, R^1)) = 0$, since $g_\epsilon \in C_c(R^1)$,thus $m(G(f, E)) \le m_2(G(g_\epsilon, R)) + m_2(G_\epsilon) = 0 + m_1(E_\epsilon) m([-M, M]) = 2M\epsilon, \epsilon \to 0$ implies that $m(G(f, E)) = 0$.

Next assume $m(E) < 0$,but $0 \le f$ is arbitrary measurable function on E.$\forall n$, let

$$[f]_n = \begin{cases} f(x), & \text{if } f(x) \le M; \\ M, & \text{if } f(x) > M. \end{cases}$$

then $[f]_n$ is a bounded measurable function and $m(G([f]_n, E)) = 0$. But $f : E \to R^1 \Rightarrow G(f, E) \subset \bigcup_{n=1}^{+\infty} G([f]_n, E)$($\because$ $\forall (x, y)$,when $f(x) = y$, then $\exists n$, st $f(x) < n, [f]_n(x) = f(x) = y, \therefore (x, y) \in G([f]_n, E)$).So $m(G(f, E)) = 0$ .Finally if $m(E) = +\infty$, then $E_n = E \bigcap [-n, n]$ is bounded and $m(G(f, E_n)) = 0$, so from $G(f, E) \subset \bigcup_{n=1}^{\infty} G(f, E_n)$,we see that $m(G(f, E)) = 0$. This complete the proof. □

7. Suppose $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{T}, \lambda)$ are σ-finite measurable spaces,and suppose ψ is a measure on $\mathscr{S} \times \mathscr{T}$ such that $\psi(A \times B) = \mu(A) \times lambda(B)$ whenever $A \in \mathscr{S}$ and $B \in \mathscr{T}$.Prove that then $\psi(E) = (\mu \times \lambda)(E)$ for every $E \in \mathscr{S} \times \mathscr{T}$.

Solution: claim 1:ψ and $\mu \times \lambda$ are σ-finite.

Pf:since $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{T}, \lambda)$ are σ-finite,there are $\{X_n\}_{n=1}^{\infty} \subset \mathscr{S}, \{Y_m\}_{m=1}^{\infty} \subset \mathscr{T}$ st $X = \bigcup_{n=1}^{\infty} X_n, Y = \bigcup_{m=1}^{\infty} Y_m$ and $\mu(X_n) < \infty, \lambda(Y_m) < \infty$ for $m, n = 1, 2, \cdots$. By the structure of $\mu \times \lambda$ and ψ, we see that

$$\psi(X_n \times Y_m) = (\mu \times \lambda)(X_n \times Y_m) = \mu(X_n)\lambda(Y_m) < \infty.$$

Let

$$\Omega = \{E \subset \mathscr{S} \times \mathscr{T}, \psi(E) = (\mu \times \lambda)(E)\},$$

then if $A_i \in \Omega, A_i \bigcap A_j = \emptyset$ for $i \neq j$, then $\bigcup\limits_{i=1}^{\infty} A_i \subset \Omega$ as well.

Also if $A_i \in \Omega, A_i \nearrow$, then $\bigcup\limits_{i=1}^{\infty} A_i \subset \Omega$. Let

$$\mathfrak{M} = \{E \in \mathscr{S} \times \mathscr{T}, \forall n, m, E \bigcap (X_n \times Y_m) \subset \Omega\},$$

then

(i) $\mathfrak{M}$ is a monotone class;

(ii)$\mathscr{E} \subset \mathfrak{M}$.

In fact,if $\{A_i\}_{i=1}^{m} \subset \mathfrak{M}, A_i \nearrow$, then $\forall n, m, A_i \bigcap (X_n \times Y_m) \in \Omega, A_i \in \mathscr{S} \times \mathscr{T}$, and

$$A_i \bigcap (X_n \times Y_m) \nearrow, A_i \bigcap (X_n \times Y_m) \in \mathscr{S} \times \mathscr{T},$$

$$\begin{aligned}
\psi(A_i \bigcap (X_n \times Y_m)) &= (\mu \times \lambda)(A_i \bigcap (X_n \times Y_m)) \\
&= \psi((\bigcup_{i=1}^{\infty} A_i) \bigcap (X_n \times Y_m)) \\
&= \psi(\bigcup_{i=1}^{\infty} (A_i \bigcap (X_n \times Y_m))) \\
&= \lim_{n \to \infty} \psi(A_i \bigcap (X_n \times Y_m) \\
&= \lim_{n \to \infty} (\mu \times \lambda)(A_i \bigcap (X_n \times Y_m) \\
&= (\mu \times \lambda)(\bigcup_{i=1}^{\infty} (A_i \bigcap (X_n \times Y_m))) \\
&= (\mu \times \lambda)((\bigcup_{i=1}^{\infty} A_i) \bigcap (X_n \times Y_m))
\end{aligned}$$

and $(\bigcup\limits_{i=1}^{\infty} A_i) \bigcap (X_n \times Y_m) = \bigcup\limits_{i=1}^{\infty} (A_i \bigcap (X_n \times Y_m)) \in \mathscr{S} \times \mathscr{T}, \therefore \bigcup\limits_{i=1}^{\infty} A_i \in \mathfrak{M}$.

If $\{A_i\}_{i=1}^{m} \subset \mathfrak{M}, A_i \searrow$, then $\forall n, m, A_i \bigcap (X_n \times Y_m) \in \Omega, A_i \in \mathscr{S} \times \mathscr{T}$, and

$$A_i \bigcap (X_n \times Y_m) \searrow, A_i \bigcap (X_n \times Y_m) \in \mathscr{S} \times \mathscr{T},$$

$$\begin{aligned}
\psi(A_i\bigcap(X_n\times Y_m)) &= (\mu\times\lambda)(A_i\bigcap(X_n\times Y_m)) \\
&= \psi((\bigcap_{i=1}^{\infty}A_i)\bigcap(X_n\times Y_m)) \\
&= \psi(\bigcap_{i=1}^{\infty}(A_i\bigcap(X_n\times Y_m))) \\
&= \lim_{n\to\infty}\psi(A_i\bigcap(X_n\times Y_m) \\
&= \lim_{n\to\infty}(\mu\times\lambda)(A_i\bigcap(X_n\times Y_m) \\
&= (\mu\times\lambda)(\bigcap_{i=1}^{\infty}(A_i\bigcap(X_n\times Y_m))) \\
&= (\mu\times\lambda)((\bigcap_{i=1}^{\infty}A_i)\bigcap(X_n\times Y_m))
\end{aligned}$$

and

$$(\bigcup_{i=1}^{\infty}A_i)\bigcap(X_n\times Y_m)=\bigcup_{i=1}^{\infty}(A_i\bigcap(X_n\times Y_m))\in\mathscr{S}\times\mathscr{T},$$

$\therefore \bigcup\limits_{i=1}^{\infty}A_i\in\mathfrak{M}$. $\therefore \mathfrak{M}$ is a monotone class.

If $R=R_1\bigcup R_2\bigcup\cdots\bigcup R_m\in\mathscr{E}, R_i$ are measurable rectangle then $R_i\in\Omega$ and $R_i\bigcap(X_n\times Y_m)$ are measurable rectangle too. so $R_i\bigcap(X_n\times Y_m)\in\Omega, \therefore R_i\in\mathfrak{M}$. It is easy to see that $R\in\mathfrak{M}$. So $\mathfrak{M}$ is a monotone class containing $\mathscr{E}$,so $\mathscr{S}\times\mathscr{T}\subset\mathfrak{M}\subset\mathscr{S}\times\mathscr{T}$ implies $\mathfrak{M}=\mathscr{S}\times\mathscr{T}$.

Thus $\forall E\in\mathfrak{M},\forall n,m, E\bigcap(X_n\times Y_m)\in\Omega, \psi(E\bigcap(X_n\times Y_m))=(\mu\times\lambda)(E\bigcap(X_n\times Y_m))$, so

$$\begin{aligned}
\psi(E) &= \sum_{m,n=1}^{\infty}\psi(E\bigcap(X_n\times Y_m)) \\
&= \sum_{m,n=1}^{\infty}(\mu\times\lambda)(E\bigcap(X_n\times Y_m)) \\
&= (\mu\times\lambda)(E),
\end{aligned}$$

$\therefore \forall E\in\mathscr{S}\times\mathscr{T}, \psi(E)=(\mu\times\lambda)(E)$, so $\psi=\mu\times\lambda$ on $\mathscr{S}\times\mathscr{T}$. □

11. Let $\mathscr{B}_k$ be the -algebra of Borel sets in R^k.Prove that $\mathscr{B}_{m+n}=\mathscr{B}_m\times\mathscr{B}_n$.This is relevant in Theorem 8.14.

Proof: As a topological space,R^{m+n} is second countable,and the sets of all rational open intervals in R^{m+n} forms a countable base of the topology of R^{m+n},so any open set in R^{m+n} is a countable union of open intervals.As any open interval $\alpha=\{(x_1,\cdots,x_{m+n})\in R^{m+n},\alpha_i\le x_i\le\beta_i, i=1,2,\cdots,m+n\}$ is a Borel measurable rectangle in $R^m\times R^n$, and by definition ,$\mathscr{B}_m\times\mathscr{B}_n$ is the smallest σ-algebra containing all the borel measurable rectangle in $R^m\times R^n=R^{m+n}$. So $\mathscr{B}_m\times\mathscr{B}_n$ contains all open sets in R^{m+n}, thus contains all Borel sets in R^{m+n}, So $\mathscr{B}_{m+n}\subset\mathscr{B}_m\times\mathscr{B}_n$.

On the other hand,by Th8.3, $\mathscr{B}_m\times\mathscr{B}_n$ is the smallest monotone class containing all elementary sets ,let $R=\bigcup\limits_{i=1}^{\infty}R_i$, ($R_i$ are disjoint and R_i are measurable rectangles, i.e. $R_i=A_i\times B_i$ where $A_i\in\mathscr{B}_m, B_i\in\mathscr{B}_n$)

It is enough to prove $\mathscr{B}_{m+n}$ is a monotone class and contains all the elementary sets.Since $\mathscr{B}_{m+n}$ is a $\sigma-algebra$, it is a monotone class.Since all elementary set R is a union of disjoint measurable rectangle,it is enough to prove that

claim:If $A \in \mathscr{B}_m, B \in \mathscr{B}_n$, then $A \times B \in \mathscr{B}_{m+n}$.

Pf:We use the method of the proof of Th8.3.

First,for any open set $A \subset X$, let $\Omega_A = \{B \subset R^n | A \times B \in \mathscr{B}_{m+n}\}$, it is clear that if B is an open set in R^n,then $A \times B$ is an open set in R^{m+n}, hence $A \times B \in \mathscr{B}_{m+n}$. (This can be known from the conclusion of topological product.Let $(X, \mathscr{T})$ and $(Y, \mathscr{S})$ be topological spaces,then the topology bases of $\mathscr{T} \times \mathscr{S}$ are composed of all elements like $A \times B(A \in \mathscr{T}, B \in \mathscr{S})$)

We show that Ω_A is a $\sigma - algebra$.

(i)X is open, so $X \in \Omega_A$.

(ii)If $B \in \Omega_A$, then

$$A \times B^c = A \times (R^n - B) = A \times R^n - A \times B \in \mathscr{B}_{m+n},$$

because $\mathscr{B}_{m+n}$ is a $\sigma - algebra$ and $A \times R^n \in \mathscr{B}_{m+n}, A \times B \in \mathscr{B}_{m+n}$. $\therefore B^c \in \Omega_A$.

(iii)If $B_i \in \Omega_A, i = 1, 2, \cdots$, then $A \times B_i \in \mathscr{B}_{m+n}, A \times (\bigcup\limits_{i=1}^{\infty} B_i) = \bigcup\limits_{i=1}^{\infty} (A \times B_i) \in \mathscr{B}_{m+n}$, because $\mathscr{B}_{m+n}$ is a $\sigma - algebra$. Thus for any open set $A \subset X$, $\Omega_A$is a $\sigma - algebra$ containing all open sets in R^n, so $\mathscr{B}_n \subset \Omega_A$, i.e. $A \times B \in \mathscr{B}_{m+n}$ for any Borel set $B \subset R^n$.

Now for any $B \in \mathscr{B}_n$,let $Q_B = \{A \subset R^m, A \times B \in \mathscr{B}_{m+n}\}$,then Q_B contains all open sets in R^m and it is a $\sigma - algebra$ which could be proved similarly as above.So $\mathscr{B}_m \subset \mathscr{B}_{m+n}$.

Thus $\forall$ Borel set $A \in \mathscr{B}_m, B \in \mathscr{B}_n, A \times B \in \mathscr{B}_{m+n}$.

So the claim is proved.

Then $\mathscr{B}_m \times \mathscr{B}_n \subset \mathscr{B}_{m+n}$, and so $\mathscr{B}_m \times \mathscr{B}_n = \mathscr{B}_{m+n}$ is proved. □

12. Use Fubuni's theorem and the relation

$$\frac{1}{x} = \int_0^{\infty} e^{-xt} dt (x > 0)$$

to prove that

$$\lim_{A \to \infty} \int_0^A \frac{\sin x}{x} dx = \frac{\pi}{2}$$

(from the mathematical analysis we know that $\int_0^{\infty} \frac{\sin x}{x} dx$,conditional convergence, we can use Dirichlet judgement to show that $|\int_0^A \sin x dx| \leq 2$, and $\frac{1}{x} \searrow 0$).

Proof: It is easy to see that

$$\frac{d}{dx}[\frac{e^{-x\lambda}}{1+\lambda^2}(-\cos x - \lambda \sin x)] = e^{-x\lambda} \sin x.$$

(Method of undetermined coefficient: let

$$\frac{d}{dx}[e^{-x\lambda}(a \cos x + b \sin x)] = e^{-x\lambda} \sin x,$$

then

$$e^{-x\lambda}[(b - \lambda a) \cos x + (-a - b\lambda) \sin x] = e^{-x\lambda} \sin x,$$

$$\therefore \begin{cases} b - \lambda a = 0, \\ a + b\lambda = -1, \end{cases} \qquad \therefore \begin{cases} a = \frac{-1}{1+\lambda^2}, \\ b = \frac{-\lambda}{1+\lambda^2}, \end{cases}$$

)

Clearly $f(x,t) = \sin x e^{-xt}$ is continuous in R^2,and hence measurable in R^2 (Lebesgue measurable). $\forall A > 0, f(x) = \int_0^{+\infty} |f(x,t)|dt = \int_0^{+\infty} |\sin x| e^{-xt} dt = \frac{\sin x}{x}$,

$$\int_0^A \frac{|\sin x|}{x} dx \le C(A) < \infty (\because \frac{\sin x}{x} \to 1 (x \to 0)).$$

By Fubuni's theorem $f \in L^1(R^2)$, and

$$\int_0^A (\int_0^{+\infty} f(x,t)dt)dx = \int_0^{+\infty} (\int_0^A f(x,t)dx)dt$$

$$\begin{aligned}
\int_0^A \frac{\sin x}{x} dx &= \int_0^A \sin x \int_0^\infty e^{-xt} dt dx \\
&= \int_0^\infty (\int_0^A e^{-xt} \sin x dx) dt \\
&= \int_0^\infty [-\frac{e^{-xt}}{1+t^2} (\cos x + t \sin x)]|_{x=0}^{x=A} dt \\
&= \int_0^\infty [\frac{1}{1+t^2} - \frac{e^{-xt}}{1+t^2} (\cos A + t \sin A)] dt \\
&= \int_0^\infty \frac{1}{1+t^2} dt - \int_0^\infty \frac{e^{-xt}}{1+t^2} (\cos A + t \sin A)] dt
\end{aligned}$$

Notice that for $t \ge 0, A \ge 1$,

$$\begin{aligned}
|\frac{e^{-At}(\cos A + t \sin A)}{1+t^2}| \le \frac{e^{-At}(1+t)}{1+t^2} &= \frac{e^{-At}}{1+t^2} + e^{-At} \frac{t}{1+t^2} \\
&\le e^{-t} + te^{-t} = (1+t)e^{-t} \in L^1((0,\infty)),
\end{aligned}$$

so by the dominated convergence theorem,

$$\begin{aligned}
\lim_{n\to\infty} \int_0^{+\infty} \frac{e^{-A_n t}(\cos A_n + t \sin A_n)}{1+t^2} dt &= \int_0^{+\infty} \lim_{n\to\infty} \frac{e^{-A_n t}(\cos A_n + t \sin A_n)}{1+t^2} dt \\
&= \int_0^{+\infty} 0 dt \\
&= 0,
\end{aligned}$$

for any sequence $A_n \to +\infty$, thus

$$\lim_{A\to\infty} \int_0^{+\infty} \frac{e^{-At}(\cos A + t \sin A)}{1+t^2} dt = 0,$$

$$\lim_{A\to\infty} \int_0^A \frac{\sin x}{x} dx = \int_0^\infty \frac{1}{1+t^2} dt = \frac{\pi}{2}.$$

□

14. Complete the following proof of Hardy inequality (chap3,exercise 14)

Suppose $f \geq 0$ on $(0, \infty), f \in L^p, 1 < p < \infty$, and

$$F(x) = \frac{1}{x}\int_0^x f(t)dt$$

, write $xF(x) = \int_0^x f(t)t^\alpha t^{-\alpha}dt$, where $0 < \alpha < \frac{1}{q}$, use Holder's inequality to get an upper bound for $F(x)^p$, and integrate to obtain

$$\int_0^\infty F(x)^p dx \leq (1-\alpha q)^{1-p}(\alpha p)^{-1}\int_0^\infty f(t)^p dt.$$

Show that the best choice of α yields

$$\int_0^\infty F(x)^p dx \leq (\frac{p}{p-1})^p \int_0^\infty f(t)^p dt.$$

Solution: Since

$$xF(x) = \int_0^x f(t)dt = \int_0^x f(t)t^\alpha t^{-\alpha}dt \leq (\int_0^x (f(t)t^\alpha)^p)^{\frac{1}{p}}(\int_0^x (t^{-\alpha})^q)^{\frac{1}{q}},$$

$$\therefore x^p F(x)^p \leq \int_0^x f(t)^p t^{\alpha p}dt(\frac{1}{1-\alpha q}x^{1-\alpha q})^{\frac{p}{q}},$$

so

$$F(x)^p \leq (\frac{1}{1-\alpha q})^{p-1}x^{-1-\alpha q}\int_0^x f(t)^p t^{\alpha p}dt$$

$$(\because \frac{1}{p}+\frac{1}{q}=1, \therefore q=\frac{p}{p-1}, \frac{p}{q}=p-1, p(\frac{1}{q}-1) = -\frac{p}{p} = -1).$$

Since $x^{-1-\alpha p}f(t)^p t^{\alpha p}$ is nonnegative and measurable in $(0,\infty)\times(0,\infty)$. (because it is continuous). By Fubini's theorem

$$\begin{aligned}
\int_0^\infty F(x)^p dx &\leq (\frac{1}{1-\alpha q})^{p-1}\int_0^\infty x^{-1-\alpha p}(\int_0^x f(t)^p t^\alpha dt)dx \\
&= (1-\alpha q)^{1-p}\int_0^\infty x^{-1-\alpha p}(\int_0^\infty \chi_{[0,x]}(t)f(t)^p t^{\alpha p}dt)dx \\
&= (1-\alpha q)^{1-p}\int_0^\infty (\int_0^\infty \chi_{[0,x]}(t)f(t)^p t^{\alpha p}x^{-1-\alpha p}dx)dt \\
&= (1-\alpha q)^{1-p}\int_0^\infty dt\int_t^\infty f(t)^p t^{\alpha p}x^{-1-\alpha p}dx \\
&= (1-\alpha q)^{1-p}\int_0^\infty dt[-\frac{1}{\alpha p}x^{-\alpha p}f(t)^p t^{\alpha p}] \\
&= (1-\alpha q)^{1-p}\int_0^\infty \frac{f(t)^p}{\alpha p}dt \\
&= \frac{(1-\alpha q)^{1-p}}{\alpha p}\int_0^\infty f(t)^p dt
\end{aligned}$$

It is easy to see that

$$g(\alpha) = (1-\alpha q)^{1-p}(\alpha p)^{-1}$$

has a minimum in $(0, \frac{1}{q})$ at $\alpha = \frac{1}{pq}$,

$$\begin{aligned}
(g(\alpha)^{'} &= \frac{-q(1-\alpha q)^{-p}(1-p)}{\alpha p} - \frac{(1-\alpha q)^{1-p}}{\alpha^2 p} \\
&= -\frac{1}{\alpha^2 p}[\alpha q(1-\alpha q)^{-p}(1-p) - (1-\alpha q)^{1-p}],
\end{aligned}$$

$$\begin{aligned} \therefore g(\alpha)' = 0 \Leftrightarrow 0 &= \alpha q(1-\alpha q)^{-p}(1-p) - (1-\alpha q)^{1-p} \\ &= -(1-\alpha q)^{-p}(\alpha q(1-p) + 1 - \alpha q), \end{aligned}$$

since $0 < \alpha < \frac{1}{2}$,

$$g(\alpha)' = 0 \Leftrightarrow \alpha q(1-p) + 1 - \alpha q = 0, \therefore \alpha q(1-p-1) = 1, \therefore \alpha = \frac{1}{pq},)$$

and $g(\alpha)' < 0$, if $\alpha < \frac{1}{pq}$, $g(\alpha)' > 0$, if $\alpha > \frac{1}{pq}$, then

$$\begin{aligned} \min g(\alpha) &= g(\frac{1}{pq}) \\ &= (1 - \frac{1}{pq}q)^{1-p}(\frac{1}{pq}p)^{-1} \\ &= (1 - \frac{1}{p})^{1-p}(\frac{1}{q})^{-1} \\ &= \frac{p}{p-1}(\frac{p-1}{p})^{1-p} \\ &= \frac{p}{p-1}(\frac{p}{p-1})^{p-1} \\ &= (\frac{p}{p-1})^p, \end{aligned}$$

so $\int_0^\infty F(x)^p dx \leq (\frac{p}{p-1})^p \int_0^\infty f(t)^p dt$. □

6. (Polar coordinates in $\mathbb{R}^k$) Let S_{k-1} be the unit sphere in $\mathbb{R}^k$, i.e. the set of all $u \in \mathbb{R}^k$ whose distance from the origin 0 is 1. Show that every $x \in \mathbb{R}^k$, except for $x = 0$, has a unique representation of the form $x = ru$, where r is a positive real number and $u \in S_{k-1}$. Thus $\mathbb{R}^k - \{0\}$ may be regarded as the cartesian product $(0, \infty) \times S_{k-1}$.

Let m_k be Lebesgue measure on $\mathbb{R}^k$, and define a measure σ_{k-1} on S_{k-1} as follows: If $A \subset S_{k-1}$ and A is a Borel set, let $\tilde{A}$ be the set of all points ru, where $0 < r < 1$ and $u \in A$, and define

$$\sigma_{k-1}(A) = k \cdot m_k(\tilde{A}).$$

Prove that the formula

$$\int_{\mathbb{R}^k} f\ dm = \int_0^\infty r^{k-1} dr \int_{S_{k-1}} f(ru) d\sigma_{k-1}(u)$$

is valid for every nonnegative Borel function f on $\mathbb{R}^k$. Check that this coincides with familiar results when $k = 2$ and when $k = 3$.

Suggestion: If $0 < r_1 < r_2$ and if A is an open subset of S_{k-1}, let E be the set of all ru with $r_1 < r < r_2$, $u \in A$, and verify that the formula holds for the characteristic function of E. Pass from these to characteristic functions of Borel sets in $\mathbb{R}^k$.

Solution: If $x \in \mathbb{R}^k - \{0\}$, then $x = |x|\frac{x}{|x|}$, i.e. $x = ru$ with $r = |x| > 0$, $u = \frac{x}{|x|} \in S_{k-1}$. This representation is unique: If $x = r_1 u_1 = r_2 u_2$, where $r_1, r_2 > 0$, $u_1, u_2 \in S_{k-1}$, then $|x| = r_1|u_1| = r_2|u_2|$, $r_1 = r_2$, hence $u_1 = \frac{x}{r_1} = \frac{x}{r_2} = u_2$. Thus the map $\mathbb{R}^k - \{0\} \to (0, \infty) \times S_{k-1} : x \mapsto (|x|, \frac{x}{|x|})$ is one-to-one and onto, continuous.Also the map P: $\mathbb{R}^k - \{0\} \to S_{k-1} : x \mapsto \frac{x}{|x|}$ is continuous, where they are viewed as subspaces of $\mathbb{R}^k$ with induced topology from $\mathbb{R}^k$. And thus P is a Borel function.

Now $\forall\ A \subset S_{k-1}$ and A is a Borel set (w.r.t. S_{k-1} topology), then $\tilde{A} = P^{-1}(A) \cap B(0,1) = P^{-1} \cap (B(0,1) - \{0\})$, and $\tilde{A}$ is a Borel set in $\mathbb{R}^k - \{0\}$ (hence a Borel set in $\mathbb{R}^k$, as $\mathbb{R}^k - \{0\}$ is open in $\mathbb{R}^k$) because $P^{-1}(A)$ and $B(0,1)$ are! And $P^{-1}(A) \cap B(0,1) = \{x | x = ru,\ 0 < r < 1,\ u \in A\}$. If we let

$$\sigma_{k-1}(A) = km_k(\tilde{A}) = km_k(P^{-1}(A) \cap B(0,1)),$$

then $\sigma_{k-1} \geq 0$ and for any Borel sets $\{A_i\} \subset S_{k-1}$, $A_i \cap A_j =$ if $i \neq j$, then

$$\begin{aligned}
\sigma_{k-1}(\bigcup_{i=1}^{+\infty} A_i) &= km_k(P_{-1}(\bigcup_{i=1}^{+\infty})A_i \bigcap B(0,1)) \\
&= km_k(\bigcup_{i=1}^{+\infty} P^{-1}(A_i) \bigcap B(0,1)) \\
&= \Sigma_{i=1}^{+\infty} m_k(P^{-1}(A_i) \bigcap B(0,1)) = \Sigma_{i=1}^{+\infty} \sigma_{k-1}(A_i)
\end{aligned}$$

have we noticed that $(P^{-1}(A_i) \cap B(0,1)) \cap (P^{-1}(A_j) \cap B(0,1)) = \emptyset$ if $i \neq j$, since $A_i \cap A_j =$ if $i \neq j$. Then σ_{k-1} is a Borel measure on S_{k-1}. We also have the following observation: For any $r > 0$ and $A \subset S_{k-1}$ (A is a Borel set),

$$m_k(P^{-1}(A) \cap B(0,r)) = r^k m_k(P^{-1}(A) \cap B(0,1)) = \frac{r^k}{k} \sigma_{k-1}(A).$$

This is because the fact that

$$P^{-1}(A) \bigcap B(0,r) = r(P^{-1}(A) \cap B(0,1)).$$

If $A \subset S_{k-1}$ is a Borel set, $0 < r_1 < r_2$ and $E = P^{-1}(A) \cap B(0,r_2) - P^{-1}(A) \cap \overline{B(0,r_1)}$, then

$$\begin{aligned}
m_k(E) &= \int_{\mathbb{R}^k} \chi_E \ dm_k = m(P^{-1}(A) \cap B(0,r_2)) - m(P^{-1}(A) \cap \overline{B(0,r_1)}) \\
&= m_k(P^{-1}(A) \cap B(0,r_2)) - m_k(P^{-1}(A) \cap B(0,r_1)).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_0^{+\infty} r^{k-1} dr \int_{S_{k-1}} \chi_E(ru) \ d\sigma_{k-1}(u) \\
= &\int_{r_1}^{r_2} r^{k-1} \ dr \int_{A \cap S_{k-1}} d\sigma_{k-1}(u) = \int_{r_1}^{r_2} r^{k-1} \ dr \sigma_{k-1}(A) \\
= &\int_{r_1}^{r_2} r^{k-1} \ dr \ km_k(P^{-1}(A) \cap B(0,1)) \\
= &(r_2^k - r_1^k) m_k(P^{-1}(A) \cap B(0,1)) \\
= &m_k(P^{-1}(A) \cap B(0,r_2)) - m_k(P^{-1}(A) \cap B(0,r_1)) = m_k(E).
\end{aligned}$$

View it in another way: $\phi : \mathbb{R}^k - \{0\} \to (0,+\infty) \times S_{k-1}$ given by $\phi(x) = (|x|, \frac{x}{|x|})$, is a continuous homeomorphism. Here $\mathbb{R}^k - \{0\}$, $(0,+\infty) \times S_{k-1}$ are viewed as subspace of $\mathbb{R}^k$, with the nature topology as subspaces. So $\phi^{-1}(r,u) = ru$, $\forall (r,u) \in (0,+\infty) \times S_{k-1}$. Let $\mathcal{B}_{\mathbb{R}^k - \{0\}}$ and $\mathcal{B}_{(0,+\infty) \times S_{k-1}}$ be the collections of all Borel sets in $\mathbb{R}^k - \{0\}$ and $(0,+\infty) \times S_{k-1}$ respectively. Then since ϕ is a homeomorphism, $\phi(\mathcal{B}_{\mathbb{R}^k - \{0\}}) = \mathcal{B}_{(0,+\infty) \times S_{k-1}}$, $\phi^{-1}(\mathcal{B}_{(0,+\infty) \times S_{k-1}}) = \mathcal{B}_{\mathbb{R}^k - \{0\}}$.

Now using the idea in proving Exercise 11 ($\mathcal{B}_{m+n} = \mathcal{B}_m \times \mathcal{B}_n$), we can show that $\mathcal{B}_{(0,+\infty)\times S_{k-1}} = \mathcal{B}_{(0,+\infty)} \times \mathcal{B}_{S_{k-1}}$. Define now two measures on $\mathcal{B}_{(0,+\infty)} \times \mathcal{B}_{S_{k-1}} = \mathcal{B}_{(0,+\infty)\times S_{k-1}}$. $\forall E \subset \mathcal{B}_{(0,+\infty)\times S_{k-1}} =$, let $\tilde{m_k}(E) = m_k(\phi^{-1}(E))$ and let $\rho \times \sigma_{k-1}$ be the product measure induced from $\sigma_{k-1}(A) = k m_k(P^{-1}(A) \cap B(0,1))$, $\rho(E) = \int_E r^{k-1} dr$ (which is a measure). Then we actually showed that if $[a,b] \times A \subset \mathcal{B}_{(0,+\infty)} \times \mathcal{B}_{S_{k-1}}$ then $\tilde{m_k}([a,b] \times A) = \rho \times \sigma_{k-1}([a,b] \times A)$, since

$$\begin{aligned}
& \tilde{m_k}([a,b] \times A) = m_k(\phi^{-1}([a,b] \times A)) = m_k([a,b] \times A) \\
= \;& m_k(P^{-1}(A) \cap (\overline{B(0,b)} - \overline{B(0,a)}) = (b^k - a^k) m_k(P^{-1}(A) \cap \overline{B(0,1)}) \\
= \;& \frac{b^k - a^k}{k} \sigma_{k-1}(A) = \int_{[a,b]} r^{k-1} dr \sigma_{k-1}(A) \\
= \;& \rho([a,b]) \times \sigma^{k-1}(A) = \rho \times \sigma_{k-1}([a,b] \times A).
\end{aligned}$$

Let $\mu(N) = \tilde{m_k}(N \times A)$, $\nu(N) = (\rho \times \sigma_{k-1})(N \times A)$. For any $N \in \mathcal{B}_{(0,+\infty)}$, then μ, ν are Borel measures on $\mathbb{R}^+$, $\mu([a,b]) = \nu([a,b])$ for every interval in $\mathbb{R}^+$. Then it is easy to see that $\mu(N) = \nu(N)$ for any $N \in \mathcal{B}_{(0,+\infty)}$. So $\forall N \times A \in \mathcal{B}_{(0,+\infty)} \times \mathcal{B}_{S_{k-1}}$,

$$\tilde{m_k}(N \times A) = \rho \times \sigma_{k-1}(N \times A).$$

Since $\tilde{m_k}$, $\rho \times \sigma_{k-1}$ are clearly σ-finite, so Exercise 7 shows that $\tilde{m_k} = \rho \times \sigma_{k-1}$ on $\mathcal{B}_{(0,+\infty)} \times \mathcal{B}_{S_{k-1}} = \mathcal{B}_{(0,+\infty)\times S_{k-1}}$. Now for all $E \in \mathcal{B}_{\mathbb{R}^k - \{0\}}$,

$$\begin{aligned}
& m_k(E) = m_k(\phi^{-1}(\phi(E))) = m_k(\tilde{\phi}(E)) \\
= \;& (\rho \times \sigma_{k-1})(\phi(E)) = \int_0^{+\infty} r^{k-1} dr \int_{S_{k-1}} \chi_{\phi(E)}(r,u) d\sigma_{k-1}(u) \\
= \;& \int_0^{+\infty} r^{k-1} dr \int_{S_{k-1}} \chi_E(ru) d\sigma_{k-1}(u),
\end{aligned}$$

since

$$\chi_{\phi(E)}(r,u) = \begin{cases} 1 & \textit{iff } (r,u) \in \phi(E); \\ 0, & \textit{otherwise,} \end{cases}$$

$\chi_{\phi(E)}(r,u) = \chi_E(ru)$. If $f = \chi_E$, $E \in B_{\mathbb{R}^k - \{0\}}$, then

$$\int_E f \; dm_k = \int_0^{+\infty} r^{k-1} dr \int_{S_{k-1}} f(ru) \; d\sigma_{k-1}(u). \tag{$*$}$$

For all $E \in \mathcal{B}_{\mathbb{R}^k}$, $E - \{0\} \in \mathcal{B}_{\mathbb{R}^k - \{0\}}$,

$$\begin{aligned}
& m_k(E) = m_k(E - \{0\}) = \int_0^{+\infty} r^{k-1} dr \int_{S_{k-1}} \chi_{E-\{0\}}(ru) d\sigma_{k-1}(u) \\
= \;& \int_0^{+\infty} r^{k-1} dr \int_{S_{k-1}} (\chi_{E-\{0\}} - \chi_{\{0\}})(ru) d\sigma_{k-1}(u) \\
= \;& \int_0^{+\infty} r^{k-1} dr \int_{S_{k-1}} \chi_E(ru) d\sigma_{k-1}(u).
\end{aligned}$$

Thus (*) holds for χ_E. Then for all nonnegative Borel function f, by simple Borel function approximation and Monotone Convergence Theorem, we can get the result. □

8 (a) Suppose f is a real function on $\mathbb{R}^2$ such that each section f_x is Borel measurable and each section f^y is continuous. Prove that f is Borel measurable on $\mathbb{R}^2$. Note the contrast between this and Example 8.9(c).

(b) Suppose g is a real function on $\mathbb{R}^k$ which is continuous in each of the k variables separately. More explicitly, for every choice of $x_2, \ldots, x_k$, the mapping $x_k \to g(x_1, x_2, \ldots, x_k)$ is continuous, etc. Prove that g is a Borel function.

Hint: If $(i-1)/n = a_(i-1) \le x \le a_i = i/n$, put

$$f_n(x,y) = \frac{a_i - x}{a_i - a_{i-1}} f(a_{i-1}, y) + \frac{x - a_{i-1}}{a_i - a_{i-1}} f(a_i, y).$$

Proof: For all n, for all $(x,y) \in \mathbb{R}^2$, let

$$f_n(x,y) = \frac{a_i - x}{a_i - a_{i-1}} f(a_{i-1}, y) + \frac{x - a_{i-1}}{a_i - a_{i-1}} f(a_i, y),$$

if $(i-1)/n = a_(i-1) \le x \le a_i = i/n$. For any fixed n, $\forall i$, let

$$g_i(x,y) = \frac{a_i - x}{a_i - a_{i-1}} f(a_{i-1}, y) + \frac{x - a_{i-1}}{a_i - a_{i-1}} f(a_i, y).$$

Then

$$f_n(x,y) = \Sigma_{i=-\infty} {+\infty} \chi_{[a_{i-1}, a_i)}(x) g_i(x,y).$$

We want to show that $\chi_{[a_{i-1}, a_i)}(x)$ and $g_i(x,y)$ are Borel function as functions of variable $(x,y) \in \mathbb{R}^2$. $\forall \alpha \in \mathbb{R}^1$,

$$\{(x,y) : \chi_{[a_{i-1}, a_i)}(x) > \alpha\} = \begin{cases} [a_{i-1}, a_i) \times \mathbb{R}^1, & \textit{if } 0 \le \alpha < 1; \\ \emptyset, \quad , \textit{ if } \alpha \ge 1; \\ \mathbb{R}^1 \times \mathbb{R}^1, \textit{ if } \alpha < 0. \end{cases}$$

So $\chi_{[a_{i-1}, a_i)}(x)$ is a Borel function, because $[a_{i-1}, a_i) \times \mathbb{R}^1$, $\emptyset$, $\mathbb{R}^1 \times \mathbb{R}^1$ are Borel sets in $\mathbb{R}^2$. (Note in Exercise 11: $\mathcal{B}_{m+n} = \mathcal{B}_m \times \mathcal{B}_n$).

Claim: If $g(x)$ is Borel function for one variable, then it is a Borel function as a function of two variable (x,y)'s function.

Proof: $\forall \alpha \in \mathbb{R}^1$, $\{(x,y) : g(x) > \alpha\} = g^{-1}((\infty, \alpha)) \times \mathbb{R}^1$. Since $g^{-1}((\infty, \alpha))$ and $\mathbb{R}^1$ are Borel sets, so is $\{(x,y) : g(x) > \alpha\}$.

From this Claim, we see that $fraca_i - xa_i - a_{i-1} f(a_{i-1}, y)$, $\frac{x - a_{i-1}}{a_i - a_{i-1}} f(a_i, y)$ are Borel functions. Hence $g_i(x,y)$ is a Borel function. Hence $f_n(x,y)$ is also a Borel function.

Now we show that $f_n(x,y) \to f(x,y)$ for $(x,y) \in \mathbb{R}^2$. $\forall (x,y) \in \mathbb{R}^2$, $\forall n$, $\exists i_n$ s.t. $a_{i_n - 1} = \frac{i_n - 1}{n} < x \frac{i_n}{n} = a_{i_n}$, $0 \le a_{i_n} - x < a_{i_n} - a_{i_n - 1} = 1/n \to 0$ and $0 \le x - a_{i_n - 1} < a_{i_n} - a_{i_n - 1} = 1/n \to 0$, so $a_{i_n} \to x$ and $a_{i_n - 1} \to x$. $\forall \epsilon > 0$, since $f(a_{i_n}, y) \to f(x,y)$ and $f(a_{i_n - 1}, y) \to f(x,y)$, there exists $N = N(\epsilon, x) > 0$ s.t. if $n \ge N$, $|f(a_{i_n}, y) - f(x,y)| < \frac{\epsilon}{2}$, $|f(a_{i_n - 1}, y) - f(x,y)| < \frac{\epsilon}{2}$. Thus for $n \ge N$,

$$\begin{aligned} |f_n(x,y) - f(x,y)| &\le \frac{a_{i_n} - x}{a_{i_n} - a_{i_n - 1}} |f(a_{i_n}, y) - f(x,y)| + \frac{x - a_{i_n - 1}}{a_{i_n} - a_{i_n - 1}} |f(a_{i_n - 1}, y) - f(x,y)| \\ &\le \frac{a_{i_n} - x}{a_{i_n} - a_{i_n - 1}} \frac{\epsilon}{2} + \frac{x - a_{i_n - 1}}{a_{i_n} - a_{i_n - 1}} \frac{\epsilon}{2} \\ &= \frac{\epsilon}{2}. \end{aligned}$$

Thus $f_n(x,y) \to f(x,y)$. So $f(x,y)$ is a Borel function.

Obviously, this result could be generalized to several variable: If $f : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$, $\forall y \in \mathbb{R}^k$, f^y is continuous in x, and $\forall x \in \mathbb{R}^k$, $f(x,y)$ is Borel in y, then f is Borel. The proof is exactly the same.

(b) Induction on k. For $k = 1$, it is true. Suppose if for $k = n$, the result is true, i.e., if $g(x_1, \dots, x_n)$ is continuous in x_i separately, then g is Borel in $\mathbb{R}^k$. Now if $k = n+1$, $g(x_1, \dots, x_n, x_{n+1})$ is continuous in x_i separately, then $\forall x_{n+1} \in \mathbb{R}^1$, $f_{x_{n+1}}(x_1, \dots, x_n) \triangleq g(x_1, \dots, x_n, x_{n+1})$ is continuous separately in $x_1, \dots, x_n$. By the induction assumption, $g(x_1, \dots, x_n, x_{n+1})$ is Borel in $x_1, \dots, x_n$ and continuous in x_{n+1}. Thus g is Borel in $x_1, \dots, x_n, x_{n+1}$ as well. So the result is true. $\square$

4 Suppose $1 \le p \le \infty$, $f \in L^1(\mathbb{R}^1)$, and $g \in L^p(\mathbb{R}^1)$.

(a) Imitate the proof of Theorem 8.14 to show that the integral defining $(f * g)(x)$ exists for almost all x, that $f * g \in L^p(\mathbb{R}^1)$ and that

$$\|f * g\|_p \le \|f\|_1 \|g\|_p.$$

(b) Show that equality can hold in (a) if $p = 1$ and if $p = \infty$, and find the conditions under which this happens.

(c)Assume $1 < p < \infty$, and equality holds in (a). Show that then either $f = 0$ a.e. or $g = 0$ a.e.

(d) Assume $1 \le p \le \infty$, $\epsilon > 0$, and show that there exist $f \in L^1(\mathbb{R}^1)$ and $g \in L^p(\mathbb{R}^1)$ such that

$$\|f * g\|_p > (1-\epsilon)\|f\|_1 \|g\|_p.$$

Solution: (a) If $p = 1$, Theorem 8.14 gives the result. If $p = \infty$, then

$$h(x) = \int_{-\infty}^{+\infty} f(x-t)g(t)dt, \tag{$*$}$$

$$\begin{aligned} |h(x)| &\le \int_{-\infty}^{+\infty} |f(x-t)||g(t)|dt \le \|g\|_\infty \int_{-\infty}^{+\infty} |f(x-t)|dt \\ &= \|g\|_\infty \int_{-\infty}^{+\infty} |f(t)|dt = \|g\|_\infty \|f\|_1. \end{aligned}$$

Now assume $1 < p < \infty$, then if $h(x)$ is defined by (*), then we know from Theorem 8.14 that we may assume without lost of generality that f and g are Borel measurable and then so is $f(x-t)g(t)$ in $\mathbb{R}^2$. If $h(x)$ is measurable, then

$$\begin{aligned} \int_{-\infty}^{+\infty} |h(x)|^p \, dx &= \int_{-\infty}^{+\infty} |\int_{-\infty}^{+\infty} f(x-t)g(t)dt|^p \, dx \\ \le & \int_{-\infty}^{+\infty} |\int_{-\infty}^{+\infty} |f(x-t)|^{\frac{1}{q}} |f(x-t)|^{1-\frac{1}{q}} |g(t)|dt|^p \, dx \\ \le & \int_{-\infty}^{+\infty} (\int_{-\infty}^{+\infty} |f(x-t)|dt)^{\frac{p}{q}} \int_{-\infty}^{+\infty} |f(x-t)||g(t)|^p dt| \, dx \\ = & \|f\|_1^{\frac{p}{q}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f(x-t)||g(t)|^p dt| \, dx \\ \le & \|f\|_1^{\frac{p}{q}} \|f\|_1 \|g\|_p^p = \|f\|_1^P \|g\|_p^p. \end{aligned}$$

Then

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

What is remained, which is the key point, is to show that when $1 < p \leq \infty$, $h(x)$ is measurable. If $f \geq 0$, $g \geq 0$, then by Fubini's Theorem and the fact that $f(x-t)g(t)$ is measurable (in fact, we may assume that f, g are Borel measurable by Lemma 1, P169, or Theorem 8.12.), $h(x) = \int_{-\infty}^{+\infty} f(x-t)g(t)dt$ is Lebesgue measurable. Hence $h \in L^p$ and $\|h\|_p \leq \|f\|_1 \|g\|_1$. For general, f, g, $f = f^+ - f^-$, $g = g^+ - g^-$, $h_1(x) \triangleq \int_{-\infty}^{+\infty} f^+(x-t)g^+(t)dt$, $h_2(x) \triangleq \int_{-\infty}^{+\infty} f^-(x-t)g^+(t)dt$, $h_3(x) \triangleq \int_{-\infty}^{+\infty} f^+(x-t)g^-(t)dt$, $h_4(x) \triangleq \int_{-\infty}^{+\infty} f^-(x-t)g^-(t)dt$ are in $L^p(\mathbb{R}^1)$ by the above argument. Hence $h_1(x) - h_2(x) - h_3(x) + h_4(x) \in L^p(\mathbb{R}^1)$ as $L^p(\mathbb{R}^1)$ is a linear space. So $h_i(x) < +\infty$ a.e. in $\mathbb{R}^1$, and $h(x) = \int_{-\infty}^{+\infty} f^+(x-t)g^+(t)dt = h_1(x) - h_2(x) - h_3(x) + h_4(x)$ is finite for a.e. $x \in \mathbb{R}^1$ and hence $h(x) \in L^p(\mathbb{R}^1)$, so h is measurable.

(b) If $p = 1$ and $f \geq 0,\ g \geq 0$, then $h(x) = \int_{-\infty}^{+\infty} f(x-t)g(t)dt \geq 0$ and Fubini's Theorem lead to

$$\begin{aligned}
\|f * g\|_1 &= \int_{-\infty}^{+\infty} h(x)dx \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x-t)g(t)dt \\
&= \int_{-\infty}^{+\infty} g(x)dt \int_{-\infty}^{+\infty} f(x-t))dt = \|g\|_1 \|f\|_1.
\end{aligned}$$

If $p = \infty$, $f \geq 0$, $g = const \geq 0$, then $\|f * g\|_\infty = \|h\|_\infty = |\int_{-\infty}^{+\infty} f(x-t)g\ dt| = \|f\|_1 \|g\|_\infty$.

(c) If $a < p < \infty$, $f \in L^1(\mathbb{R}^1)$, $g \in L^p(\mathbb{R}^1)$, $\|f * g\|_p = \|f\|_1 \|g\|_p$, then WLOG we may assume that $f, g \geq 0$. In fact, it is clear that $\|f * g\|_p \leq \||f| * |g|\|_p \leq \|f\|_1 \|g\|_p = \|f * g\|_p \Rightarrow \||f| * |g|\|_p = \||f|\|_1 \||g|\|_p$. Notice in the proof of the inequality, the "¡" is caused by using the *Hölder* Inequality: if $f, g \geq 0$ for a.e. $x \in \mathbb{R}^1$,

$$\begin{aligned}
\int_{-\infty}^{+\infty} f(x-t)g(t)dt &\leq \int f(x-t)^{\frac{1}{q}} f(x-t)^{1-\frac{1}{q}} g(t)dt \\
&\leq (\int f(x-t)dt)(\int f(x-t)^{p(1-\frac{1}{q})} g(t)^p dt)^{\frac{1}{p}}.
\end{aligned}$$

So if $\|f * g\|_p = \||f|\|_1 \|g\|_p$, then for a.e. $x \in \mathbb{R}^1$ the equality of the above inequality holds. By the caution for the equality in *Hölder*'s inequality, $existsP \subset \mathbb{R}^1$, $m(P) = 0$, $\forall x \in P$, $\exists \alpha(x),\ \beta(x)$ (not all zero) s.t. $\alpha(x)f(x-t) = \beta(x)f(x-t)g(t)$ for a.e. $t \in \mathbb{R}^1$. If $x_0 \in P$ is such that $\beta(x_0) = 0$, then $\alpha(x_0) \neq 0$ and $f(x-t) = 0$ a.e. t. So $\exists N \subset \mathbb{R}^1$ s.t. $m(N^c) = 0$ and $\forall t \in N,\ f(x-t) = 0$. So $\forall y \in x - N = \{x - t : t \in N\}$, $f(y) = 0$ and $m(x-N) = m(N) = 0$ (the translation invariance of m, $m(TA) = |det(T)|m(A)$). Thus $f = 0$ *a.e.*.

The following to be completed: If $x_0 \in P$ is such that $\alpha(x_0) = 0$, then $\beta(x_0) \neq 0$ and hence $f(x-t)g(t) = 0$ for a.e. $t \in \mathbb{R}^1$. So $\exists N(x_0) \subset \mathbb{R}^1$, $m(N(x_0)^c) = 0$, $\forall t \in N(x_0)$, $f(x_0 - t)g(t) = 0$. If $f = 0$ *a.e.* if not true, then $\exists M \subset \mathbb{R}^1$, $m(M) > 0$, $f(t) > 0$. So, $0 < f(x-t)$ and $g(t) = 0$ if $t \in x - M$. If $\exists t_0 \in (x-M)^c$ s.t. $g(t_0) > 0$, then————————-

UNKNOWN PART:

Since $E_0 \cup A = A \cup (E_0 - A)$, $\int_{E_0 \cup A}(f_n - f_N)d\mu = (\int_A + \int_{E_0 - A})(f_n - f_N)d\mu$,

$$
\begin{aligned}
|\int_A (f_n - f_N)d\mu| &= |\int_{E_0 \cup A}(f_n - f_N)d\mu - \int_{E_0 - A}(f_n - f_N)d\mu| \\
&\leq |\int_{E_0 \cup A}(f_n - f_N)d\mu| + |\int_{E_0 - A}(f_n - f_N)d\mu| < \infty
\end{aligned}
$$

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