

Pattern Causality

1 Introduction

Pattern Causality(PC) is a novel causal inference algorithm and causal effect measurement tool based on chaos theory and systematics, PC extends the original series to a strange attractor, which obtains more information for causal inference. In the reconstructed phase space, we consider a kind of manifold as a source of information, using local projection and nearest neighbors to detect the causality, and measuring them by three types of causality, positive, negative and dark; Instead of two types of causality in traditional algorithms, PC provides the third causality type in causal inference.

2 Phase Space Reconstruction

The PC is started from phase space reconstruction, which is regarded as the state space in the whole process.

The attractor M_X can be expressed as

$$M_X(t) = \{X(t), \dots, X(t + (E - 1)\tau)\}, \quad (1)$$

where $1 \leq t \leq n$, with $X(\cdot)$ as a column vector and $M_X(t)$ as a vector space.

The distance matrix D_X can be calculated by $M_X(t)$ and defined as

$$D_X(t) = \{d(M_X(t), M_X(1)), \dots, d(M_X(t), M_X(n + (E - 1)\tau))\}, \quad (2)$$

where $d(M_X(t), M_X(\cdot))$ represents a column vector of distances between $M_X(t)$ and $M_X(\cdot)$, with appropriate dimensionality. In addition, the same procedure can be applied to Y_n to calculate M_Y and D_Y . After constructing the attractors $\{M_x, M_y\}$ and distance matrices $\{D_x, D_y\}$, a local projection analysis can be performed on the reconstructed state space to start prediction and causal inference.

3 Nearest Neighbors and Projection

In the reconstructed state space, we need to find the nearest neighbors in each time point t of M_X like.

$$\{t_1, t_2, \dots, t_{E+1}\} = NN_{E+1}(t) = \arg \min_{\{d_1, \dots, d_{E+1}\} \subset D_X(t)} \{d_1, \dots, d_{E+1}\}. \quad (3)$$

The set $\{t_1, t_2, \dots, t_{E+1}\}$ contains the nearest $E + 1$ time information in t , then we projected the time information in $\{t_1, t_2, \dots, t_{E+1}\}$ to M_Y and set h steps ahead to predict in $M_Y(t + h)$.

Now we need to define two other spaces and two mapping functions, the one is signature space,

$$S(t) = \mathcal{F}(M(t), \{t_i\}_{i=1}^{E+1}) = \{\Delta M(t_i, t_{i+1})\}_{i=1}^E. \quad (4)$$

The mapping function $\mathcal{F}(\cdot)$ is given by

$$\mathcal{F}_i(M, t_i, t_{i+1}) = \Delta M(t_i, t_{i+1}) = \frac{M(t_{i+1}) - M(t_i)}{M(t_i)}, \quad (5)$$

where M is obtained from Equation 1 like M_X and i is equal to E which is the embedded dimensions.

Now we have the signature space $S(t)$ and signature mapping function $\mathcal{F}(\cdot)$, then we define the pattern space and pattern mapping function.

$$P(t) = \mathcal{G}(S(t)), \quad (6)$$

Here, the space $S(t+h)$ is assigned to $P(t+h)$ via the pattern extraction function $\mathcal{G}(\cdot)$:

$$\mathcal{G}(S(t)) = \left\{ S(t) \xrightarrow{\mathbf{v}} P(t) \mid \mathbf{v} \in \mathbb{V} \right\}. \quad (7)$$

In this case, the size of $S(t)$ dictates the direction of the arrows, and \mathbb{V} is a set that includes all potential 3^{E-1} arrow directions. If $E = 2$, the pattern set will look like $\mathbb{V} = \{\nearrow, \longrightarrow, \searrow\}$, and if $E = 3$, the pattern set usually looks like $\mathbb{V} = \{\nearrow\nearrow, \nearrow\longrightarrow, \dots, \searrow\searrow\}$, $\mathbb{V} = \{\nearrow, \longrightarrow, \searrow\}$, where $P(t)$ is one of these combination patterns.

4 Local Prediction

As for the prediction step, we can use local weighted method to estimate the $t+h$ points in state space.

First of all, we can get the signature space in M_Y by

$$S_Y(t) = \mathcal{F}(M_Y(t), \{t_i\}_{i=1}^{E+1}) \quad (8)$$

In the local time information set $\{t_{1+h}, \dots, t_{E+1+h}\}$ in M_Y , we can find the distance between $M_Y(t_{1+h}, \dots, t_{E+1+h})$ and $M_Y(t)$ by.

$$d_y(t) = \{d(M_Y(t), M_Y(t_{1+h})), \dots, d(M_Y(t), M_Y(t_{E+1+h}))\}. \quad (9)$$

then we can obtain the weight by these distance

$$w(t) = \left\{ \frac{e^{d_y(t_1)}}{\sum e^{d_y(t)}}, \dots, \frac{e^{d_y(t_{E+1})}}{\sum e^{d_y(t)}} \right\}. \quad (10)$$

Finally, we construct the estimate $\hat{S}_Y(t+h)$ by

$$\hat{S}_Y(t+h) = w(t) \cdot S_Y(t) \quad (11)$$

Now the estimated $\hat{S}_Y(t+h)$ has been calculate by the time information of M_X and the information of M_Y , which considers both of their information.

5 Pattern Causality Matrix

Now it's time to find the real pattern by $\{M_X, M_Y\}$, as we define the signature space and pattern space before, we can get $\{P_X, P_Y, \hat{P}_Y\}$ by the following equations.

$$\begin{aligned} P_X(t) &= \mathcal{G}(S_X(t)) = \mathcal{G}(\mathcal{F}(M_X(t), \{t_i\}_{i=1}^{E+1})) \\ P_Y(t) &= \mathcal{G}(S_Y(t)) = \mathcal{G}(\mathcal{F}(M_Y(t), \{t_i\}_{i=1}^{E+1})) \\ \hat{P}_Y(t+h) &= \mathcal{G}(\hat{S}_Y(t+h)) \end{aligned} \quad (12)$$

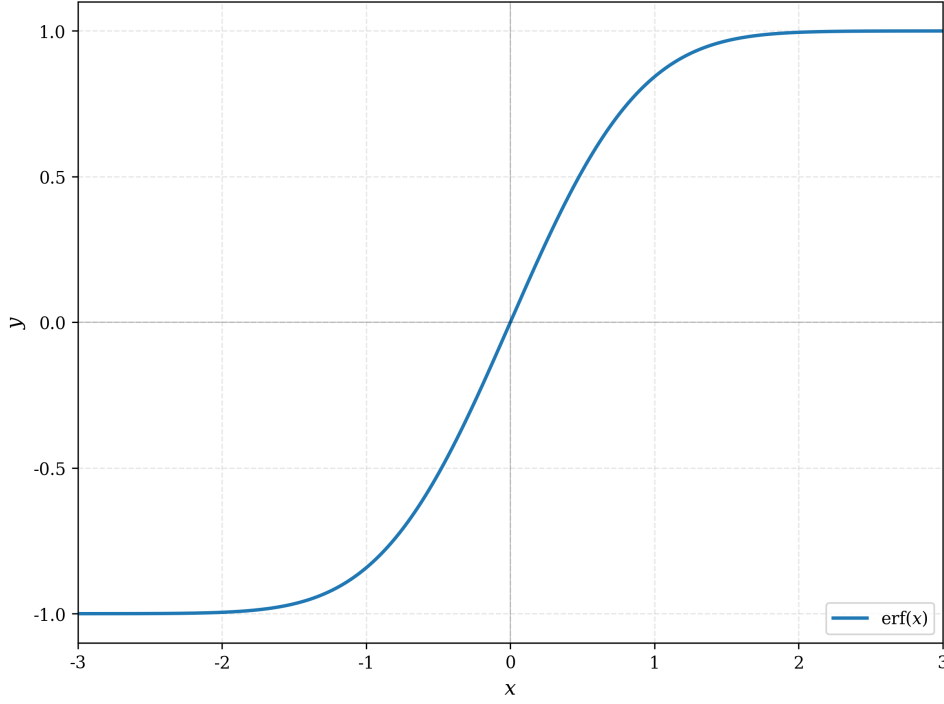


Figure 1: Gaussian erf function figure.

The $\hat{S}_Y(t+h)$ is obtained by Equation 11, then we match the P_Y and \hat{P}_Y , if and only $P_Y = \hat{P}_Y$, we define the causality exists, then we estimate the causal strength when the causality exists,

$$PC[P_x, P_y, t] = \frac{\|\hat{S}_Y(t+h)\|}{\|S_X(t)\|}, \quad (13)$$

where $\|\hat{S}_Y(t+h)\|$ is the predicted point \hat{S}_Y at time $t+h$, and $\|S_X(t)\|$ is the real signature space S_X at time t . It can be standardized using the Gaussian error function.

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \quad (14)$$

You can find the erf function by Figure 1, which means the value of causal strength will be limit in $[-1, 1]$, then we can construct the PC matrix at each time point t .

Lastly, we calculate the mean strength through all the time point t to get the overall causal strength of positive, negative and dark.

6 Conclusion

PC regards the chaos series as a dynamic system, doing causal inference in the view of systematics, the chaos theory has developed for more than 30 years, like Computer simulation of self-organization, Mandelbrot's non-integral geometry, KAM theory, as a manifold based method, it shows its own advantages in complex system, however, the clear definition of "information" in manifold or chaos is still waiting to be explored.

Algorithm 1 Pattern Causality Algorithm

Require: $\mathcal{D} = \{(\mathbf{x}_t, \mathbf{y}_t)\}_{t=1}^n$: series of observations of two random variables

Require: E : embedding dimension; τ : time delay; h : prediction horizon

Ensure: Causality percentages $P = \{P_{\text{positive}}, P_{\text{negative}}, P_{\text{dark}}\}$

1: **Step 1: Reconstruct State, Signature, and Pattern Spaces**

- 2: $M_s \leftarrow \{M_x, M_y\}$ ▷ Reconstruct state space by $\{\mathcal{D}, E, \tau\}$
 - 3: $S_s \leftarrow \{\mathcal{F}(M_x), \mathcal{F}(M_y)\}$ ▷ Calculate $S_s = \{S_x, S_y\}$
 - 4: $P_s \leftarrow \{\mathcal{G}(S_x), \mathcal{G}(S_y)\}$ ▷ Calculate $P_s = \{P_x, P_y\}$
 - 5: $D \leftarrow \{d(M_x), d(M_y)\}$ ▷ Calculate $D = \{D_x, D_y\}$
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7: **Step 2: Initialize Pattern Causality (PC) Matrix**

- 8: $m \leftarrow 3^{E-1}$ ▷ Initialize the number of all patterns from E
 - 9: $q \leftarrow t^*$ ▷ From the first causality point (FCP) to last causality point
 - 10: $PC \leftarrow \mathbf{0}_{m \times m \times q}$ ▷ Initialize three-dimensional zero matrix
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12: **Step 3: Main Causality Detection Loop**

- 13: **for** $i \leftarrow FCP$ **to** $n - (E - 1)\tau - h$ **do**
 - 14: $N_{x,i} \leftarrow \arg \min_{M_x} d(M_{x,i})$ ▷ Solve optimization problem
 - 15: $T \leftarrow T_{rec}(N_{x,i})$ ▷ Record the time of nearest neighbors
 - 16: $S_{y,T+h} \leftarrow S_y(T + h)$
 - 17: $W_{y,T+h} \leftarrow \frac{e^{D_{y,i}(T+h)}}{\sum e^{D_{y,i}(T+h)}}$ ▷ Estimate distance weight
 - 18: $\hat{S}_{y,i+h} \leftarrow S_{y,T+h} \odot W_{y,T+h}$ ▷ Estimate projection vector
 - 19: $\hat{P}_{y,i+h} \leftarrow \mathcal{G}(\hat{S}_{y,i+h})$ ▷ Estimate projection patterns
 - 20: **if** $\hat{P}_{y,i+h} = P_{y,i+h}$ **then**
 - 21: $c \leftarrow \text{erf}(\hat{S}_{y,i+h}, S_{x,i})$ (Equation 14) ▷ Estimate causal strength
 - 22: **else**
 - 23: $c \leftarrow 0$ ▷ Define no causality
 - 24: **end if**
 - 25: $PC^* \leftarrow c$ ▷ Put c into the corresponding position in Appendix ??
 - 26: **end for**
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28: **Step 4: Calculate Causality Percentages**

- 29: **if** $\mathbf{p}(PC^*) = \text{diag}(PC)$ **then** ▷ Find the position of PC^*
 - 30: $\theta^+ \leftarrow PC^*$ ▷ Estimate positive causal strength series
 - 31: **else if** $\mathbf{p}(PC^*) = \text{antidiag}(PC)$ **then**
 - 32: $\theta^- \leftarrow PC^*$ ▷ Estimate negative causal strength series
 - 33: **else**
 - 34: $\theta^* \leftarrow PC^*$ ▷ Estimate dark causal strength series
 - 35: **end if**
 - 36: $P_{\text{positive}}, P_{\text{negative}}, P_{\text{dark}} \leftarrow \left\{ \frac{\sum \theta^+}{\sum \theta^+ + \theta^- + \theta^*}, \frac{\sum \theta^-}{\sum \theta^+ + \theta^- + \theta^*}, \frac{\sum \theta^*}{\sum \theta^+ + \theta^- + \theta^*} \right\}$
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