Pattern Causality

1 Introduction

Pattern Causality(PC) is a novel causal inference algorithm and causal effect measurement tool based on chaos theory and systematics, PC extends the original series to a strange attractor, which obtains more information for causal inference. In the reconstructed phase space, we consider a kind of manifold as a source of information, using local projection and nearest neighbors to detect the causality, and measuring them by three types of causality, positive, negative and dark; Instead of two types of causality in traditional algorithms, PC provides the third causality type in causal inference.

2 Phase Space Reconstruction

The PC is started from phase space reconstruction, which is regarded as the state space in the whole process.

The attractor M_x can be expressed as

$$M_X(t) = \{X(t), \ldots, X(t + (E-1)\tau)\}, \tag{1}$$

where $1 \leq t \leq n$, with $X(\cdot)$ as a column vector and $M_X(t)$ as a vector space.

The distance matrix D_X can be calculated by $M_X(t)$ and defined as

$$D_X(t) = \{d(M_X(t), M_X(1)), \ldots, d(M_X(t), M_X(n + (E-1)\tau))\}, \tag{2}$$

where $d(M_X(t), M_X(\cdot))$ represents a column vector of distances between $M_X(t)$ and $M_X(\cdot)$, with appropriate dimensionality. In addition, the same procedure can be applied to Y_n to calculate M_Y and D_Y. After constructing the attractors $\{M_x, M_y\}$ and distance matrices $\{D_x, D_y\}$, a local projection analysis can be performed on the reconstructed state space to start prediction and causal inference.

3 Nearest Neighbors and Projection

In the reconstructed state space, we need to find the nearest neighbors in each time point t of M_X like.

$$\{t_1, t_2, \ldots, t_{E+1}\} = NN_{E+1}(t) = \underset{\{d_1, \ldots, d_{E+1}\} \subset D_X(t)}{\arg\min} \{d_1, \ldots, d_{E+1}\}. \tag{3}$$

The set $\{t_1, t_2, \ldots, t_{E+1}\}$ contains the nearest $E+1$ time information in t, then we projected the time information in $\{t_1, t_2, \ldots, t_{E+1}\}$ to M_Y and set h steps ahead to predict in $M_Y(t+h)$.

Now we need to define two other spaces and two mapping functions, the one is signature space,

$$S(t) = \mathcal{F}\left(M(t), \{t_i\}_{i=1}^{E+1}\right) = \{\Delta M(t_i, t_{i+1})\}_{i=1}^{E}. \tag{4}$$

The mapping function $\mathcal{F}(\cdot)$ is given by

$$\mathcal{F}_i(M, t_i, t_{i+1}) = \Delta M(t_i, t_{i+1}) = \frac{M(t_{i+1}) - M(t_i)}{M(t_i)}, \tag{5}$$

where M is obtained from Equation 1 like M_X and i is equal to E which is the embedded dimensions. The corresponding signature space $S(t)$ is given by

Now we have the signature space $S(t)$ and signature mapping function $\mathcal{F}(\cdot)$, then we define the pattern space and pattern mapping function.

$$P(t+h) = \mathcal{G}(S(t+h)), \tag{6}$$

Here, the space $S(t+h)$ is assigned to $P(t+h)$ via the pattern extraction function $\mathcal{G}(\cdot)$:

$$\mathcal{G}(S(t+h)) = \left\{ S(t+h) \xrightarrow{\mathbf{v}} P(t+h) \mid \mathbf{v} \in \mathbb{V} \right\}. \tag{7}$$

In this case, the size of $S(t+h)$ dictates the direction of the arrows, and $\mathbb{V}$ is a set that includes all potential 3^{E-1} arrow directions. If $E = 2$, the pattern set will look like $\mathbb{V} = \{\nearrow, \longrightarrow, \searrow\}$, and if $E = 3$, the pattern set usually looks like $\mathbb{V} = \{\nearrow\nearrow, \nearrow\longrightarrow, \ldots, \searrow\searrow\}$, $\mathbb{V} = \{\nearrow, \longrightarrow, \searrow\}$, where $P(t+h)$ is one of these combination patterns.

4 Local Prediction

As for the prediction step, we can use local weighted method to estimate the $t+h$ points in state space.

First of all, we can get the signature space in M_Y by

$$S_Y(t) = \mathcal{F}\left(M_Y(t), \{t_i\}_{i=1}^{E+1}\right) \tag{8}$$

In the local time information set $\{t_{1+h}, \ldots, t_{E+1+h}\}$ in M_Y, we can find the distance between $M_Y(t_{1+h}, \ldots, t_{E+1+h})$ and $M_Y(t)$ by.

$$d_y(t) = \{d(M_Y(t), M_Y(t_{1+h})), \ldots, d(M_Y(t), M_Y(t_{E+1+h}))\}. \tag{9}$$

then we can obtain the weight by these distance

$$w(t) = \left\{ \frac{e^{d_y(t_1)}}{\sum e^{d_y(t)}}, \ldots, \frac{e^{d_y(t_{E+1})}}{\sum e^{d_y(t)}} \right\}. \tag{10}$$

Finally, we construct the estimate $\hat{S}_Y(t+h)$ by

$$\hat{S}_Y(t+h) = w(t) \cdot S_Y(t) \tag{11}$$

Now the estimated $\hat{S}_Y(t+h)$ has been calculate by the time information of M_X and the information of M_Y, which considers both of their information.

5 Pattern Causality Matrix

Now it's time to find the real pattern by $\{M_X, M_Y\}$, as we define the signature space and pattern space before, we can get $\{P_X, P_Y, \hat{P}_Y\}$ by the following equations.

$$\begin{aligned} P_X(t) &= \mathcal{G}(S_X(t)) = \mathcal{G}(\mathcal{F}\left(M_X(t), \{t_i\}_{i=1}^{E+1}\right)) \\ P_Y(t) &= \mathcal{G}(S_Y(t)) = \mathcal{G}(\mathcal{F}\left(M_Y(t), \{t_i\}_{i=1}^{E+1}\right)) \\ \hat{P}_Y(t+h) &= \mathcal{G}(\hat{S}_Y(t+h)) \end{aligned} \tag{12}$$

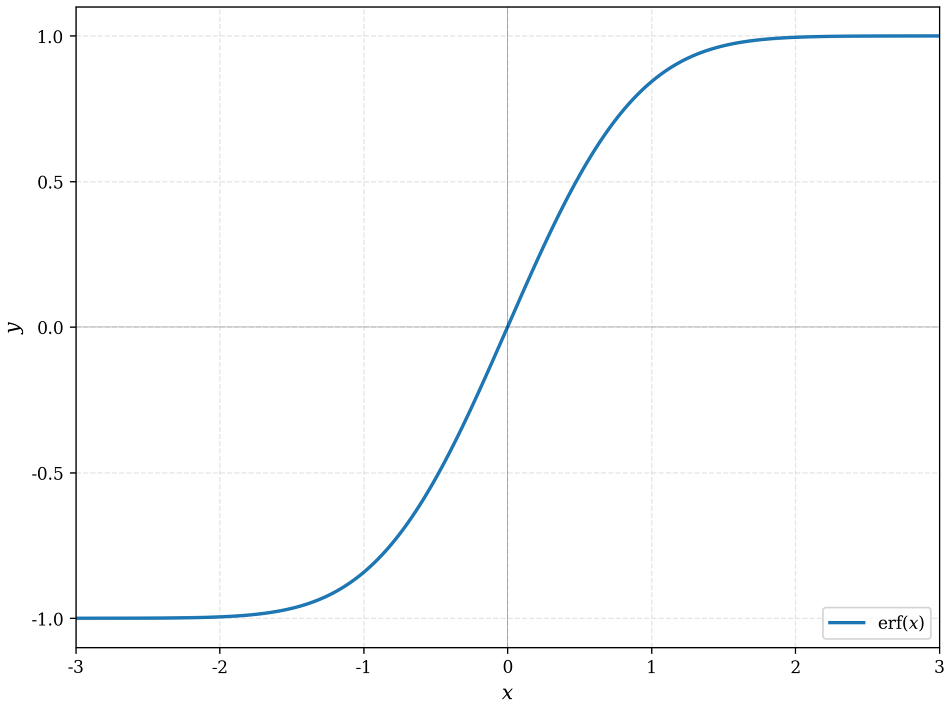

Figure 1: Gaussian erf function figure.

The $\hat{S}_Y(t+h)$ is obtained by Equation 11, then we match the P_Y and $\hat{P}_Y$, if and only $P_Y = \hat{P}_Y$, we define the causality exists, then we estimate the causal strength when the causality exists,

$$PC[P_x, P_y, t] = \frac{\left\| \hat{S}_Y(t+h) \right\|}{\| S_X(t) \|}, \tag{13}$$

where $\left\| \hat{S}_Y(t+h) \right\|$ is the predicted point $\hat{S}_Y$ at time $t+h$, and $\|S_X(t)\|$ is the real signature space S_X at time t. It can be standardized using the Gaussian error function.

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \tag{14}$$

You can find the erf function by Figure 1, which means the value of causal strength will be limit in [-1, 1], then we can construct the PC matrix at each time point t.

Lastly, we calculate the mean strength through all the time point t to get the overall causal strength of positive, negative and dark.

6 Conclusion

PC regards the chaos series as a dynamic system, doing causal inference in the view of systematics, the chaos theory has developed for more than 30 years, like Computer simulation of self-organization, Mandelbrot's non-integral geometry, KAM theory, as a manifold based method, it shows its own advantages in complex system, however, the clear definition of "information" in manifold or chaos is still waiting to be explored.

Algorithm 1 Pattern Causality Algorithm

Require: $\boldsymbol{\mathcal{D}} = \{(\mathbf{x}_t, \mathbf{y}_t)\}_{t=1}^{n}$: series of observations of two random variables
Require: E: embedding dimension; τ: time delay; h: prediction horizon
Ensure: Causality percentages $P = \{P_{\text{positive}}, P_{\text{negative}}, P_{\text{dark}}\}$

Step 1: Reconstruct State, Signature, and Pattern Spaces
$M_s \leftarrow \{M_x, M_y\}$ ▷ Reconstruct state space by $\{\boldsymbol{\mathcal{D}}, E, \tau\}$
$S_s \leftarrow \{\mathcal{F}(M_x), \mathcal{F}(M_y)\}$ ▷ Calculate $S_s = \{S_x, S_y\}$
$P_s \leftarrow \{\mathcal{G}(S_x), \mathcal{G}(S_y)\}$ ▷ Calculate $P_s = \{P_x, P_y\}$
$D \leftarrow \{d(M_x), d(M_y)\}$ ▷ Calculate $D = \{D_x, D_y\}$

Step 2: Initialize Pattern Causality (PC) Matrix
$m \leftarrow 3^{E-1}$ ▷ Initialize the number of all patterns from E
$q \leftarrow t^*$ ▷ From the first causality point(FCP) to last causality point
$PC \leftarrow \mathbf{0}_{m\times m\times q}$ ▷ Initialize three-dimensional zero matrix

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Step 3: Main Causality Detection Loop
for $i \leftarrow FCP$ **to** $n-(E-1)\tau-h$ **do**
  $N_{x,i} \leftarrow \arg\min_{M_x} d(M_{x,i})$ ▷ Solve optimization problem
  $T \leftarrow T_{rec}(N_{x,i})$ ▷ Record the time of nearest neighbors
  $S_{y,T+h} \leftarrow S_y(T+h)$
  $W_{y,T+h} \leftarrow \frac{e^{D_{y,i}(T+h)}}{\sum e^{D_{y,i}(T+h)}}$ ▷ Estimate distance weight
  $\hat{S}_{y,i+h} \leftarrow S_{y,T+h} \odot W_{y,T+h}$ ▷ Estimate projection vector
  $\hat{P}_{y,i+h} \leftarrow \mathcal{G}(\hat{S}_{y,i+h})$ ▷ Estimate projection patterns
  if $\hat{P}_{y,i+h} = P_{y,i+h}$ **then**
    $c \leftarrow \text{erf}(\hat{S}_{y,i+h}, S_{x,i})$(Equation 14) ▷ Estimate causal strength
  else
    $c \leftarrow 0$ ▷ Define no causality
  end if
  $PC^* \leftarrow c$ ▷ Put c into the corresponding position in Appendix **??**
end for

Step 4: Calculate Causality Percentages
if $\mathbf{p}(PC^*) = \text{diag}(PC)$ **then** ▷ Find the position of PC^*
  $\boldsymbol{\theta}^+ \leftarrow PC^*$ ▷ Estimate positive causal strength series
else if $\mathbf{p}(PC^*) = \text{antidiag}(PC)$ **then**
  $\boldsymbol{\theta}^- \leftarrow PC^*$ ▷ Estimate negative causal strength series
else
  $\boldsymbol{\theta}^* \leftarrow PC^*$ ▷ Estimate dark causal strength series
end if
$P_{\text{positive}}, P_{\text{negative}}, P_{\text{dark}} \leftarrow \{\frac{\sum \boldsymbol{\theta}^+}{\sum \boldsymbol{\theta}^+ + \boldsymbol{\theta}^- + \boldsymbol{\theta}^*}, \frac{\sum \boldsymbol{\theta}^-}{\sum \boldsymbol{\theta}^+ + \boldsymbol{\theta}^- + \boldsymbol{\theta}^*}, \frac{\sum \boldsymbol{\theta}^*}{\sum \boldsymbol{\theta}^+ + \boldsymbol{\theta}^- + \boldsymbol{\theta}^*}\}$