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Stabilization and Asymptotic Path Tracking of a Rolling Disk

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Abstract

In this paper, the dynamics and control of a uniform disk (a thin wheel) rolling without slipping on a horizontal plane are considered. A model of the rolling disk is derived using the Lagrangian formulation, assuming that rolling, steering and leaning torques are available as control inputs. A dynamic extension is used to achieve a well defined vector relative degree. On the basis of the dynamic extension, a feedback control law is designed to stabilize the disk from falling over, while simultaneously allowing the disk to asymptotically track a ground reference trajectory. For this class of stabilization and tracking problems, the nonholonomic rolling without slipping constraint does not preclude the existence of smooth feedback that accomplishes the control objectives.

1. Introduction

In this paper, we consider the dynamics and control of a uniform disk (a thin wheel) rolling without slipping on a rough horizontal plane; the disk is not necessarily vertical. A model of the rolling disk is derived using the Lagrangian approach, assuming that rolling, steering, and leaning torques are available as control inputs. The model is nonlinear and nonholonomic. For outputs of interest, the system is found to have no well defined vector relative degree anywhere in the state space; yet a dynamic extension can be used to achieve a vector relative degree. On the basis of the dynamically extended system a feedback control law is designed to stabilize the disk from falling while simultaneously allowing it to asymptotically track a reference trajectory on the plane.

The motivation for this study is identified. A disk rolling without slipping is a well-known classical nonholonomic system and it has been studied as a simple example for motion planning and control strategies for nonholonomic systems [1], [2], assuming that the disk always remains vertical. The general rolling disk system, when the disk is not constrained to remain vertical, represents much richer dynamics and deserves attention. Since it is inherently nonlinear, nonholonomic and critically minimum phase, the general rolling disk system can serve well as a prototype for study of a class of nonlinear nonholonomic systems. With the recent development of tools for the control of nonholonomic systems, increasing attention has been paid to the control of wheeled mobile robots and vehicles [3], [4], [5]. These models assume that the wheel remains strictly vertical, but these models fail to represent the actual dynamics when the wheel is perturbed from the vertical position. Such instances arise for single-track vehicles, such as unicycles, bicycles and motorcycles. In [6], a model describing the dynamics of a bicycle is proposed. The assumption made is that the steering is decoupled from the vehicle dynamics. The model established in this paper is more general in the sense that it makes no assumption that the wheel (or disk) remains vertical. It should not be difficult to extend the model here to descriptions of various practical wheeled robots and vehicles.

This paper is organized as follows: In Section 2, the model of a disk rolling without slipping on a rough horizontal plane is established. In Section 3, control properties are investigated and a feedback control law is devised to stabilize the system and allow the disk to asymptotically track a ground reference trajectory. In Section 4, simulation results are presented.

2. Modeling of a Rolling Disk

Consider a disk rolling without slipping on a horizontal plane as shown in Figure 1. Let $XYZO$ be a fixed inertial reference frame with X axis and Y axis in the horizontal plane and Z axis vertical. Let p denote the point of contact on the disk, a denote the point of contact on the horizontal plane and $\overline{la}$ denote the contact line formed by the intersection of the horizontal plane with the plane of the disk. Let $\xi\eta\zeta c$ be a moving coordinate system with its origin at point c, the center of mass of the disk, the ξ axis is in the plane of the disk and parallel to the contact line $\overline{la}$, the η axis is in the plane of the disk but through the point of contact p and the center of mass c, and the ζ axis is normal to the plane of the disk. The configuration of the disk can be described by six generalized coordinates $(x_c, y_c, z_c, \varphi, \theta, \psi)$, where x_c, y_c, z_c represent the coordinates of the center of mass c with respect to the

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inertial system $XYZO$, φ is the steering angle measured from the X axis to the contact line $\overline{la}$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is the "tipping" angle measured from the Z axis to the η axis, and ψ is the rolling angle measured from the line $\overline{cp}$ to some fixed reference radius of the disk. Let $(\hat{e}_x, \hat{e}_y, \hat{e}_z)$

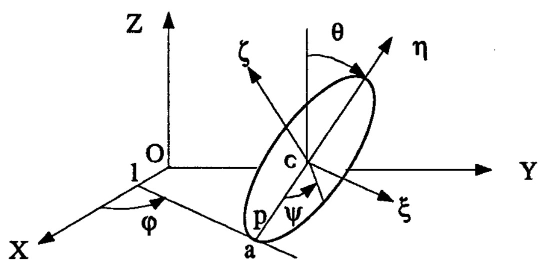

Figure 1: A Disk Rolling Without Slipping on a Horizontal Plane

and $(\hat{e}_\xi, \hat{e}_\eta, \hat{e}_\zeta)$ be unit basis vectors of the coordinate systems $XYZO$ and $\xi\eta\zeta c$ respectively. The transformation between the two coordinate systems is given by

$$\begin{bmatrix} \hat{e}_\xi \\ \hat{e}_\eta \\ \hat{e}_\zeta \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi\sin\theta & \cos\varphi\sin\theta & \cos\theta \\ \sin\varphi\cos\theta & -\cos\varphi\cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{bmatrix} \quad (1)$$

Let $\vec{V}_c$ denote the velocity of the center of mass. We have

$$\vec{V}_c = \dot{x}_c\hat{e}_x + \dot{y}_c\hat{e}_y + \dot{z}_c\hat{e}_z. \quad (2)$$

Let $\vec{\omega}$ denote the angular velocity vector of the disk with respect to the inertial frame $XYZO$, and it is given by

$$\vec{\omega} = -\dot{\theta}\hat{e}_\xi + \dot{\varphi}\cos\theta\hat{e}_\eta + (\dot{\psi} + \dot{\varphi}\sin\theta)\hat{e}_\zeta. \quad (3)$$

The constraints result from the requirement that the disk rolls without slipping on the horizontal plane, that is, the velocity of the point of contact on the disk is zero at any instant

$$\vec{V}_p = 0, \quad (4)$$

where $\vec{V}_p$ is the velocity of the point of contact p on the disk. Relative to the center of mass c, $\vec{V}_p$ can be represented as

$$\vec{V}_p = \vec{V}_c + \vec{\omega} \times \vec{r}_{cp}, \quad (5)$$

where $\vec{r}_{cp}$ is the position vector from the center of mass c to the point of contact p,

$$\vec{r}_{cp} = -R\hat{e}_\eta. \quad (6)$$

Substituting (1), (2) and (3) in (5) gives

$$\begin{aligned} \vec{V}_p = & [\dot{x}_c + R\dot{\theta}\sin\varphi\cos\theta + R(\dot{\psi} + \dot{\varphi}\sin\theta)\cos\varphi]\hat{e}_x \\ & +[\dot{y}_c - R\dot{\theta}\cos\varphi\cos\theta + R(\dot{\psi} + \dot{\varphi}\sin\theta)\sin\varphi]\hat{e}_y \\ & +(\dot{z}_c + R\dot{\theta}\sin\theta)\hat{e}_z. \end{aligned} \quad (7)$$

Equating (7) to zero yields two nonholonomic constraints and one holonomic constraint

$$\dot{x}_c + R\dot{\theta}\sin\varphi\cos\theta + R(\dot{\psi} + \dot{\varphi}\sin\theta)\cos\varphi = 0 \quad (8)$$
$$\dot{y}_c - R\dot{\theta}\cos\varphi\cos\theta + R(\dot{\psi} + \dot{\varphi}\sin\theta)\sin\varphi = 0 \quad (9)$$
$$\dot{z}_c + R\dot{\theta}\sin\theta = 0 \quad (10)$$

Equation (8) and (9) are not integrable and hence are nonholonomic; equation (10) can be integrated to obtain

$$z_c = R\cos\theta. \quad (11)$$

As the disk rolls without slipping, the locus of points of contact defines a smooth curve in the horizontal plane. Let (x_a, y_a) denote the coordinates of the point a on this locus that coincides with a point of contact p of the disk. In terms of generalized coordinates, x_a and y_a can be expressed as

$$x_a = x_c + R\sin\varphi\sin\theta, \quad (12)$$
$$y_a = y_c - R\cos\varphi\sin\theta. \quad (13)$$

Since the disk is uniform, the principal moments-of-inertia are $I_1 = \frac{1}{4}mR^2$, the moment of inertia of the wheel about the ξ axis and the η axis, and $I_2 = \frac{1}{2}mR^2$, the moment of inertia of the wheel about the ζ axis; m is the mass of the wheel and R is the radius of the wheel.

The kinetic energy of the system is

$$\begin{aligned} T = & \frac{1}{2}[I_1\dot{\theta}^2 + I_1\dot{\varphi}^2\cos^2\theta + I_2(\dot{\psi} + \dot{\varphi}\sin\theta)^2] \\ & +\frac{1}{2}m(\dot{x}_c^2 + \dot{y}_c^2 + \dot{z}_c^2) \end{aligned} \quad (14)$$

and by taking the horizontal plane as the reference, the potential energy is

$$V = mgz_c, \quad (15)$$

where g is the acceleration of gravity. Substituting the holonomic constraint (10) into (14) and (15), we can write the Lagrangian function for this system as

$$\begin{aligned} L = & T - V, \\ = & \frac{1}{2}[I_1\dot{\theta}^2 + I_1\dot{\varphi}^2\cos^2\theta + I_2(\dot{\psi} + \dot{\varphi}\sin\theta)^2] \\ & +\frac{1}{2}m(\dot{x}_c^2 + \dot{y}_c^2 + R^2\dot{\theta}^2\sin^2\theta) - mgR\cos\theta. \end{aligned} \quad (16)$$

Assume that there are control torques τ_ξ, τ_η, and τ_ζ about the axes ξ, η, and ζ respectively, where τ_ξ is called the leaning torque, τ_η is the steering torque, and τ_ζ is the rolling torque. Assume that the outputs to be controlled are the coordinates of the point of contact with the ground.

Using Lagrange multipliers, we arrive at the model of the system: the equations of motion are

$$M(q)\ddot{q} = F(q, \dot{q}) + A^T\lambda + Bu; \quad (17)$$

the nonholonomic constraints can be written as

$$A(q)\dot{q} = 0, \quad (18)$$

and the output equations are

$$y = h(q), \quad (19)$$

where

$$M(q) = \begin{bmatrix} m & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 \\ 0 & 0 & I_1\cos^2\theta + I_2\sin^2\theta & 0 & I_2\sin\theta \\ 0 & 0 & 0 & mR^2\sin^2\theta + I_1 & 0 \\ 0 & 0 & I_2\sin\theta & 0 & I_2 \end{bmatrix},$$

$U \xrightarrow{(21)} V \to \tilde{V} \to \hat{V}$

$$F(q,\dot q) = \begin{bmatrix} 0 \\ 0 \\ 2(I_1 - I_2)\dot\varphi\dot\theta\cos\theta\sin\theta - I_2\dot\theta\dot\psi\cos\theta \\ -mR^2\dot\theta^2\sin\theta\cos\theta - (I_1 - I_2)\dot\varphi^2\sin\theta\cos\theta + I_2\dot\varphi\dot\psi\cos\theta + mgR\sin\theta \\ -I_2\dot\varphi\dot\theta\cos\theta \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & 0 & R\cos\varphi\sin\theta & R\sin\varphi\cos\theta & R\cos\varphi \\ 0 & 1 & R\sin\varphi\sin\theta & -R\cos\varphi\cos\theta & R\sin\varphi \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$h(q) = \begin{bmatrix} x_c + R\sin\varphi\sin\theta \\ y_c - R\cos\varphi\sin\theta \end{bmatrix},$$

and the variables are

$$q = \begin{bmatrix} x_c \\ y_c \\ \varphi \\ \theta \\ \psi \end{bmatrix}, \lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}, u = \begin{bmatrix} \tau_\xi \\ \tau_\eta \\ \tau_\zeta \end{bmatrix}, y = \begin{bmatrix} x_a \\ y_a \end{bmatrix}.$$

This model is nonholonomic and can be used to characterize a rich set of controlled dynamics. The model of a rolling disk, assumed to remain vertical, is a special case corresponding to $\theta = 0$.

3. Control of the Rolling Disk

We investigate the control properties of the system. We show that the system does not have a well-defined vector relative degree anywhere in the state space, but a dynamic extension can be used to achieve a vector relative degree. On the basis of the augmented system, a feedback control law is designed that stabilizes the system and allows the disk to asymptotically track a ground reference trajectory.

We start by transforming the model into normal form. Following the procedure introduced in [1], we partition the matrix $A(q)$ as $A = [A_1 : A_2]$, where

设计控制器

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} R\cos\varphi\sin\theta & R\sin\varphi\cos\theta & R\cos\varphi \\ R\sin\varphi\sin\theta & -R\cos\varphi\cos\theta & R\sin\varphi \end{bmatrix}.$$

Let

5×3矩阵

$$C(q) = \begin{bmatrix} -A_1^{-1}A_2 \\ I_{3\times3} \end{bmatrix},$$

链式求导 $\dot q_2$

where $I_{3\times3}$ is a 3×3 identity matrix. Thus $C(q)$ spans the null space of A. Further, $\dot q$ can be expressed as $\dot q = C(q)\dot q_2$. Differentiating yields $\ddot q = C(q)\ddot q_2 + \dot C(q)\dot q_2$. Substituting into equation (17) and premultiplying the resulting equation by $C^T(q)$ gives

左乘 相减

目标方程

$$C^T(q)M(q)C(q)\ddot q_2 = C^T(q)[F(q, C(q)\dot q_2) - M(q)\dot C(q)\dot q_2 + Bu]. \quad (20)$$

It is easy to check that $C^T(q)M(q)C(q)$ is a 3×3 symmetric positive definite matrix function and $C^T(q)B(q)$ is a 3×3 invertible matrix. As a result, for any $u \in R^3$ there is a unique $\boxed{v \in R^3}$ such that

可逆

3×3矩阵 $v = \ddot q_2$

$$C^T(q)[F(q, C(q)\dot q_2) - M(q)\dot C(q)\dot q_2 + Bu] = C^T(q)M(q)C(q)v. \quad (21)$$

The relationship between u and v can be made more clear by writing out equation (21) as

分解

$$\begin{aligned} [I_1\cos^2\theta + (I_2 + mR^2)\sin^2\theta]v_1 + (I_2 + mR^2)\sin\theta v_3 \\ = \cos\theta u_2 + \sin\theta u_3 + 2(I_1 - I_2 - mR^2)\dot\varphi\dot\theta\cos\theta\sin\theta \\ -I_2\dot\theta\dot\psi\cos\theta, \end{aligned} \quad \text{一个式子}$$

$$\begin{aligned} (I_2 + mR^2)\sin\theta v_1 + (I_2 + mR^2)v_3 \\ = u_3 - (I_2 + 2mR^2)\dot\varphi\dot\theta\cos\theta, \end{aligned} \quad \text{一个式子}$$

由V求U，计算实际控制力矩.

$V_2 \to U_1$ (leaning torque)

$$\begin{aligned} (mR^2 + I_1)v_2 \\ = -u_1 + (I_2 + mR^2 - I_1)\dot\varphi^2\sin\theta\cos\theta \\ +(I_2 + mR^2)\dot\varphi\dot\psi\cos\theta + mgR\sin\theta. \end{aligned} \quad (22) \quad \text{一个式子}$$

$\{V_1, V_3\} \Rightarrow U_1\ U_3$

It is clear that v_2 is affected only by u_1 and not by u_2 and u_3 while v_1 and v_3 are affected only by u_2 and u_3 and not by u_1. Therefore the control u_1, the leaning torque, can be used to stabilize the disk to keep it vertical while the controls u_2 and u_3, i.e. the steering torque and the rolling torque can be used to achieve the tracking objective.

Using the control transformation given by (21), equation (20) becomes

$$\boxed{\ddot q_2 = v.} \quad (23)$$

Define

$$x_1 = \varphi, x_2 = \theta, x_3 = \psi, x_4 = x_c$$
$$x_5 = y_c, x_6 = \dot\varphi, x_7 = \dot\theta, x_8 = \dot\psi.$$

Then we obtain

$$\begin{aligned} \dot x &= f(x) + gv \\ y &= h(x) \end{aligned} \quad (24)$$

$v = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$

where

$$f(x) = \begin{bmatrix} x_6 \\ x_7 \\ x_8 \\ -R\sin x_2\cos x_1 x_6 - R\cos x_2\sin x_1 x_7 - R\cos x_1 x_8 \\ -R\sin x_2\sin x_1 x_6 + R\cos x_2\cos x_1 x_7 - R\sin x_1 x_8 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$g = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, h(x) = \begin{bmatrix} x_4 + R\sin x_2\sin x_1 \\ x_5 - R\sin x_2\cos x_1 \end{bmatrix}.$$

On the basis of this normal form we state and prove the following property.

Proposition 1 *The system with three control torques as described by (24) does not have a well defined vector relative degree anywhere in the state space. However the vector relative degree can be achieved via a dynamic extension when $x_8 \neq 0$.*

Proof:

Direct computation shows

$$L_g h_1 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, L_g h_2 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix},$$

$$L_f h_1 = -R\cos x_1 x_8, L_f h_2 = -R\sin x_1 x_8,$$

$$L_g L_f h_1 = \begin{bmatrix} 0 & 0 & -R\cos x_1 \end{bmatrix}, L_g L_f h_2 = \begin{bmatrix} 0 & 0 & -R\sin x_1 \end{bmatrix}$$

where h_i, $i = 1, 2$ are the rows of $h(x)$. Thus the decoupling matrix can be written as

$$A(x) = \begin{bmatrix} 0 & 0 & -R\cos x_1 \\ 0 & 0 & -R\sin x_1 \end{bmatrix}$$

矩阵为1,所以没有well defined vector

The rank of $A(x)$ is one everywhere in the state space. Therefore the system does not have a well defined vector relative degree anywhere in the state space.

To show that a dynamic extension can achieve a well defined relative degree, we differentiate the outputs three times to obtain

1. 为什么求3次导，并且怎么来的

$$y_1^{(3)} = R\cos x_1 x_6^2 x_8 + 2R\sin x_1 x_6 v_3 + R\sin x_1 x_8 v_1 - R\cos x_1 \dot{v}_3,$$

$$y_2^{(3)} = R\sin x_1 x_6^2 x_8 - 2R\cos x_1 x_6 v_3 + R\cos x_1 x_8 v_1 - R\sin x_1 \dot{v}_3.$$

多了 $\dot{v}_3$

由 $\tilde{v}_3$ 求 v_3

Let

$$x_9 = v_3, \tilde{v}_3 = \dot{v}_3, \tilde{v}_1 = v_1, \tilde{v}_2 = v_2.$$

$v_1 = \tilde{v}_1$, $v_2 = \tilde{v}_2 = \ddot{\theta}$, $\dot{v}_3 = \tilde{v}_3$, $v_3 = x_9$

In a matrix form, we obtain

3次导来求well defined vector

$$\begin{bmatrix} y_1^{(3)} \\ y_2^{(3)} \end{bmatrix} = \begin{bmatrix} b_1(x) \\ b_2(x) \end{bmatrix} + \tilde{A}(x) \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{v}_3 \end{bmatrix},$$

where

$$\begin{bmatrix} b_1(x) \\ b_2(x) \end{bmatrix} = \begin{bmatrix} R\cos x_1 x_6^2 x_8 + 2R\sin x_1 x_6 x_9 \\ R\sin x_1 x_6^2 x_8 - 2R\cos x_1 x_6 x_9 \end{bmatrix},$$

and the decoupling matrix is

$$\tilde{A}(x) = \begin{bmatrix} R\sin x_1 x_8 & 0 & -R\cos x_1 \\ -R\cos x_1 x_8 & 0 & -R\sin x_1 \end{bmatrix}.$$

秩为2,所以well defined

It is clear that the decoupling matrix $\tilde{A}(x)$ has rank 2, when $x_8 \neq 0$. Note that the second column corresponding to the input $\tilde{v}_2$ is zero. This indicates that the outputs are not affected by the leaning control torque. Only the steering and rolling torques affect the outputs.

With this dynamic extension, the system can be written as

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{f}(\tilde{x}) + \tilde{g}\tilde{v} \\ y &= h(\tilde{x}) \end{aligned} \quad (25)$$

实际上只有 $v \to \tilde{v}$ 发生变化, $g \to \tilde{g}$

核心:动态扩展 增加了一个状态 v_3

where

$$\tilde{f}(\tilde{x}) = \begin{bmatrix} x_6 \\ x_7 \\ x_8 \\ -R\sin x_2 \cos x_1 x_6 - R\cos x_2 \sin x_1 x_7 - R\cos x_1 x_8 \\ -R\sin x_2 \sin x_1 x_6 + R\cos x_2 \cos x_1 x_7 - R\sin x_1 x_8 \\ 0 \\ 0 \\ x_9 \\ 0 \end{bmatrix}$$

增加了一个状态

其它无变化

$\tilde{v} = \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{v}_3 \end{bmatrix} = \begin{bmatrix} \ddot{\theta} \end{bmatrix}$

$$\tilde{g} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad h(\tilde{x}) = \begin{bmatrix} x_4 + R\sin x_2 \sin x_1 \\ x_5 - R\sin x_2 \cos x_1 \end{bmatrix}$$

插入一行

This dynamically extended system has a vector relative degree $r_1 = 3$, $r_2 = 3$ when $x_8 \neq 0$. This is in contrast to claims we have made previously in [1] that such nonholonomic control systems cannot be smoothly stabilized and are not feedback linearizable. There is no contradiction, however, since the existence of a relative degree guarantees only that the input-output relation can be made linear. As we show later, the zero dynamics are characterized by the nonholonomic rolling dynamics of the disk.

Now we describe the asymptotic path tracking problem.

Asymptotic Path Tracking Problem 1
Given the system described by (17)-(19), design feedback controllers such that the disk is stabilized to remain vertical and the point of contact a on the ground tracks a ground reference trajectory

$$\vec{r}(t) = x_r(t)\hat{e}_x + y_r(t)\hat{e}_y$$

where $x_r(t)$, and $y_r(t)$ are continuously differentiable functions of time.

The starting point for solving the tracking problem is the dynamic extension (25). In order to design the controller, we consider the coordinate transformation

转换成对a点的状态,

怎么来的

$$z = \Phi(\tilde{x}) = \begin{bmatrix} x_4 + R\sin x_2 \sin x_1 \\ -R\cos x_1 x_8 \\ R\sin x_1 x_6 x_8 - R\cos x_1 x_9 \\ x_5 - R\sin x_2 \cos x_1 \\ -R\sin x_1 x_8 \\ -R\cos x_1 x_6 x_8 - R\sin x_1 x_9 \\ x_3 \\ x_7 \\ x_2 \end{bmatrix}. \quad (26)$$

$= y_1 = x_a$

$= y_2 = y_a$

$x_1 = \varphi$, $x_2 = \theta$, $x_3 = \psi$, $x_4 = x_c$, $x_5 = y_c$, $x_6 = \dot{\varphi}$, $x_7 = \dot{\theta}$, $x_8 = \dot{\psi}$, $x_9 = v_3$

$z_1 = x_a$, $z_2 = \dot{x}_a$, $z_3 =$, $z_4 = y_a$, $z_5 = \dot{y}_a$, $z_6 =$, $z_7 = \psi$, $z_8 = \dot{\theta}$, $z_9 = \theta$

It is easy to check that $det(\frac{\partial \Phi}{\partial \tilde{x}}) \neq 0$, when $x_8 \neq 0$. Thus $\Phi(\tilde{x})$ is a local diffeomorphism when $x_8 \neq 0$ and its inverse

2. z_3, z_6 怎么来的

逆

is given by

$$\tilde{x} = \Phi^{-1}(z) = \begin{bmatrix} \sin^{-1}\left(\frac{-z_5}{\sqrt{z_2^2+z_5^2}}\right) \\ z_9 \\ z_7 \\ z_1 - R\frac{z_5}{\sqrt{z_2^2+z_5^2}}\sin z_9 \\ z_4 + R\frac{z_2}{\sqrt{z_2^2+z_5^2}}\sin z_9 \\ \frac{z_2 z_6 - z_3 z_5}{z_2^2+z_5^2} \\ z_8 \\ -\frac{1}{R}\sqrt{z_2^2+z_5^2} \\ \frac{-z_2 z_3 - z_5 z_6}{R\sqrt{z_2^2+z_5^2}} \end{bmatrix}. \tag{27}$$

$-z_5$

In the transformed coordinate system, the system can be represented as

ode45求解微分方程组即可求出$\vec{z}$
但是!
需先求出$\hat{v}_1$, $\tilde{v}_2$, $\hat{v}_3$.

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= z_3, \\ \dot{z}_3 &= \hat{v}_1, \\ \dot{z}_4 &= z_5, \\ \dot{z}_5 &= z_6, \\ \dot{z}_6 &= \hat{v}_3, \\ \dot{z}_7 &= -\frac{1}{R}\sqrt{z_2^2+z_5^2}, \\ \dot{z}_8 &= \tilde{v}_2, \\ \dot{z}_9 &= z_8, \\ y_1 &= z_1, = x_c \\ y_2 &= z_4. = y_c \end{aligned} \tag{28}$$

设计z时,自然满足$\dot{z}=z$

$z_8=\dot{\theta}$, $\dot{z}_8=\ddot{\theta}=\tilde{v}_2$

where

$$\begin{bmatrix} \hat{v}_1 \\ \hat{v}_3 \end{bmatrix} = \begin{bmatrix} b_1(\tilde{x}) \\ b_2(\tilde{x}) \end{bmatrix} + \begin{bmatrix} a_{11}(\tilde{x}) & a_{12}(\tilde{x}) \\ a_{21}(\tilde{x}) & a_{22}(\tilde{x}) \end{bmatrix} \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_3 \end{bmatrix} \tag{29}$$

求$\tilde{v}$
求力矩u

which renders the system input-output noninteractive in the normal form, b_1 and b_2 are given previously, and

$\tilde{v} \to \hat{v}$

$$\begin{bmatrix} a_{11}(\tilde{x}) & a_{12}(\tilde{x}) \\ a_{21}(\tilde{x}) & a_{22}(\tilde{x}) \end{bmatrix} = \begin{bmatrix} R\sin x_1 x_8 & -R\cos x_1 \\ -R\cos x_1 x_8 & -R\sin x_1 \end{bmatrix}.$$

It is obvious that the output y_1 is affected only by $\hat{v}_1$ but not by $\hat{v}_3$ while the output y_2 is affected only by $\hat{v}_3$ but not by $\hat{v}_1$. Note also that the input $\tilde{v}_2$ does not affect either y_1 or y_2. But it does affect the internal dynamics. As a matter of fact, the subsystem associated with the input $\tilde{v}_2$

$$\begin{aligned} \dot{z}_8 &= \tilde{v}_2, \\ \dot{z}_9 &= z_8, \end{aligned} \tag{30}$$

describes how the leaning angle θ changes. To stabilize the disk to its vertical position, we can design $\tilde{v}_2$ as

$$\tilde{v}_2 = -K_1 z_9 - K_2 z_8, \tag{31}$$

where $K_1 > 0$, and $K_2 > 0$. This ensures that the disk is asymptotically stabilized to its vertical position when released at any arbitrary initial position.

Another feature of the system is that the internal dynamics are

内部动态

$$\dot{z}_7 = -\frac{1}{R}\sqrt{z_2^2+z_5^2} \tag{32}$$

and the zero dynamics are critically minimum phase. However, this does not adversely affect the solution to the asymptotic tracking problem since the implication is that the magnitude of the rolling angle ψ increases as time increases.

暗示

Suppose the ground reference trajectory is given by

$$\vec{r}(t) = x_r(t)\hat{e}_x + y_r(t)\hat{e}_y, \tag{33}$$

where $x_r(t)$, and $y_r(t)$ are given continuously differentiable functions of time. Let

$y_1 = x_a$
$y_2 = y_a$

$$\begin{aligned} e_1 &= y_1 - x_r(t), \\ e_2 &= y_2 - y_r(t). \end{aligned} \tag{34}$$

Consider a time varying change of variables

$$\begin{aligned} &\epsilon_1(t) = e_1, \epsilon_2(t) = \dot{e}_1, \epsilon_3(t) = \ddot{e}_1, \\ &\epsilon_4(t) = e_2, \epsilon_5(t) = \dot{e}_2, \epsilon_6(t) = \ddot{e}_2. \end{aligned} \tag{35}$$

Then the subsystem associated with the tracking objective is as follows

$$\begin{aligned} \dot{\epsilon}_1 &= \epsilon_2, \\ \dot{\epsilon}_2 &= \epsilon_3, \\ \dot{\epsilon}_3 &= \hat{v}_1 - x_r'''(t), \\ \dot{\epsilon}_4 &= \epsilon_5, \\ \dot{\epsilon}_5 &= \epsilon_6, \\ \dot{\epsilon}_6 &= \hat{v}_3 - y_r'''(t). \end{aligned} \tag{36}$$

Choose the controllers $\hat{v}_1$ and $\hat{v}_3$ to be

$$\begin{bmatrix} \hat{v}_1 \\ \hat{v}_3 \end{bmatrix} = \begin{bmatrix} x_r'''(t) \\ y_r'''(t) \end{bmatrix} + \begin{bmatrix} -c_{11}\epsilon_1 - c_{12}\epsilon_2 - c_{13}\epsilon_3 \\ -c_{21}\epsilon_3 - c_{22}\epsilon_5 - c_{23}\epsilon_6 \end{bmatrix} \tag{37}$$

so that

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c_{i1} & -c_{i2} & -c_{i3} \end{bmatrix}$$

$i = 1, 2$, are Hurwitz. This ensures that $y_1 \to x_r(t)$ and $y_2 \to y_r(t)$ as $t \to \infty$. Therefore the disk asymptotically approaches the reference trajectory

明确

$$\vec{r}(t) = x_r(t)\hat{e}_x + y_r(t)\hat{e}_y.$$

The explicit expression for the control torques τ_ξ, τ_η and τ_ζ can be easily obtained by combining equations (21), (29), (31) and (37).

4. Simulation Results

In the simulation, we use $m = 2kg$, $R = 0.2m$, $g = 9.8m/s^2$; $K_1 = 2$ and $K_2 = 1$ for the controller $\tilde{v}_2$; and $c_{i1} = 1$, $c_{i2} = 3$, and $c_{i3} = 2.5$, $i = 1, 2$, for controllers $\hat{v}_1$ and $\hat{v}_3$. The disk is initially released with its point of contact with the ground at $(0, -4)$; and $\varphi_0 = 90°$, $\theta_0 = 30°$, $\psi_0 = 0°$, $\dot{\varphi}_0 = 0$, $\dot{\theta}_0 = 0$, and $\dot{\psi}_0 = 0$. The ground reference trajectory is an ellipse

$$x_r(t) = 6\cos(\frac{\pi}{15}t), y_r(t) = 10\sin(\frac{\pi}{15}t).$$

The trajectory of the disk and the reference trajectory are shown in Figure 2. The control torques τ_ξ(dotted line), τ_η(broken line), τ_ζ(solid line) are shown in Figure 3, and the orientation angles φ, θ, and ψ of the disk are shown in Figures 4, 5, and 6, respectively. Figure 3 shows that approximately at time $t = 1$ second and $t = 3$ seconds, the rolling torque τ_ζ is greater than zero. Note that the rolling torque, when greater than zero, is braking. Figures 2 and 4 show that braking happens when the curvature of the output trajectory decreases. Also in Figure 3, the leaning torque τ_ξ is nonzero even when the disk has been stabilized to its vertical position, i.e. $\theta = 0$ (see Figure 5); a nonzero leaning torque is needed to maintain the disk at its vertical position.

5. Conclusions

In this paper, the dynamics and control of a disk (thin wheel) rolling without sliding on a horizontal plane have been studied. A model of the rolling disk is established which makes no assumptions that the disk remains vertical. The model is nonlinear and nonholonomic, and characterizes a rich set of controlled dynamics. The model is analyzed and found to have no vector relative degree; yet a well defined vector relative degree has been achieved using a dynamic extension. On the basis of the dynamic extension, a feedback control has been designed that stabilizes the disk to remain vertical while simultaneousely allowing it to asymptotically achieve a reference trajectory.

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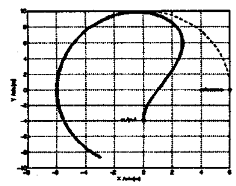

Figure 2: Asymptotic Path Tracking

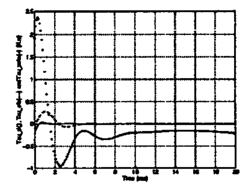

Figure 3: Control Torques τ_ξ, τ_η and τ_ζ

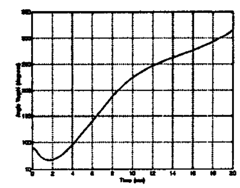

Figure 4: Steering Angle φ

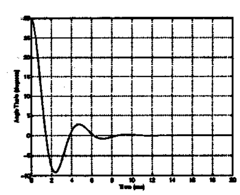

Figure 5: Leaning Angle θ

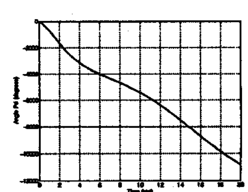

Figure 6: Rolling Angle ψ