Theorem 1. Let V be a finite set, and let each element $e \in V$ have a non-negative weight $w_e \geq 0$. Define the set weight function

$$u(V) = \sum_{e \in V} w_e,$$

and a set function $F(V) = h\big(u(V)\big) : 2^V \to \mathbb{R}$. If $h(\cdot)$ is a monotone increasing and concave function, then F is a submodular function.

Proof. For sets $S \subseteq T \subseteq V$, let

$$u := u(S) = \sum_{e \in S} w_e, \quad u' := u(T) = \sum_{e \in T} w_e.$$

Then, for any element $x \in V \setminus T$,

$$F(S \cup \{x\}) - F(S) = h(u(S \cup \{x\})) - h(u(S)) = h(u + w_x) - h(u),$$

and similarly,

$$F(T \cup \{x\}) - F(T) = h(u' + w_x) - h(u').$$

Since $h(\cdot)$ is concave, by the definition of concavity, for $0 \leq u \leq u'$ and $v \geq 0$,

$$h(u + v) - h(u) \geq h(u' + v) - h(u').$$

Thus, the function $F(\cdot)$ satisfies

$$F(S \cup \{x\}) - F(S) \geq F(T \cup \{x\}) - F(T),$$

and is therefore submodular.