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Graph Reconstruction by Discrete Morse Theory

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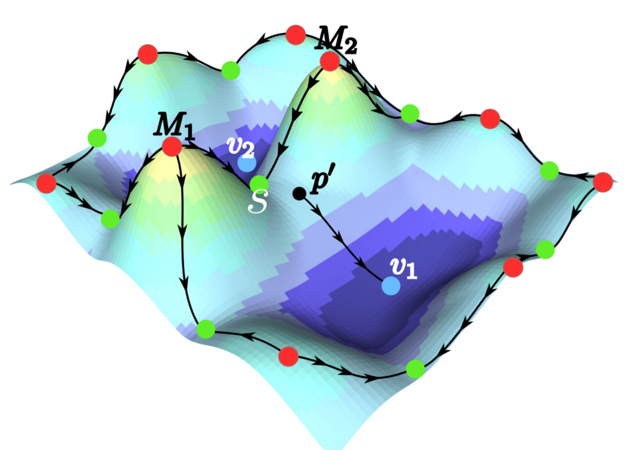
Problem: Recovering hidden structures from potentially noisy data.
Applications include reconstructing the road networks from GPS trajectories, inferring the filament structure of simulated density field of dark matters in universe.

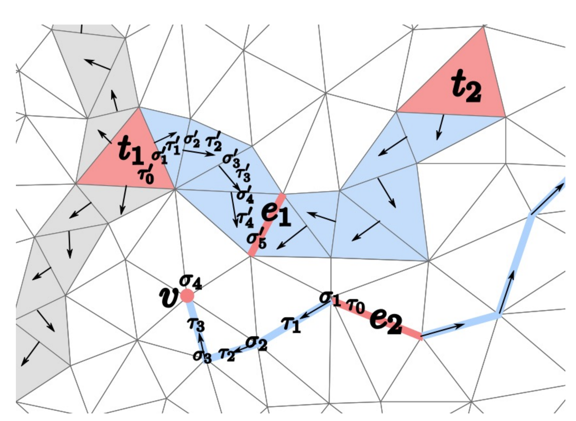
Our contributions:

1. **Simplify the existing algorithm.**
2. **Provide an editing strategy to add missing branches and loops for the 2D case.**
3. **Give the Geometric and topological reconstruction guarantees under a noise model**

Morse and Discrete Morse theory

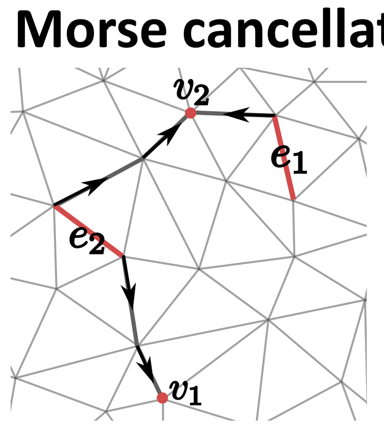


Integral line
1-stable manifold of saddles

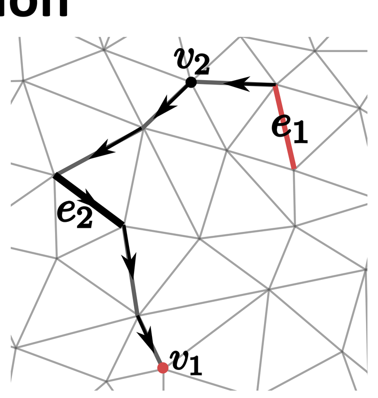


Vertex-edge gradient path:
$\tau_0 = e_2, \sigma_1, \tau_1, \sigma_2, \tau_2, \sigma_3, \tau_3, \sigma_4 = v$
Edge-triangle gradient path:
$\tau_0' = t_1, \sigma_1', \tau_1', \sigma_2', \tau_2', \sigma_3', \tau_3', \sigma_4', \tau_4', \sigma_5' = e_1$

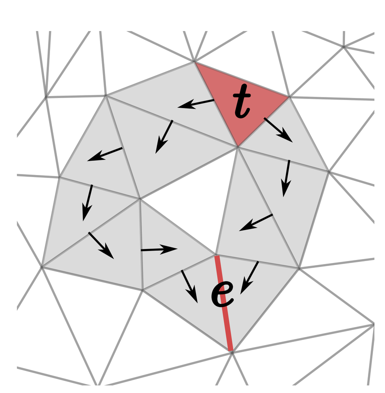
Morse cancellation



Before cancellation

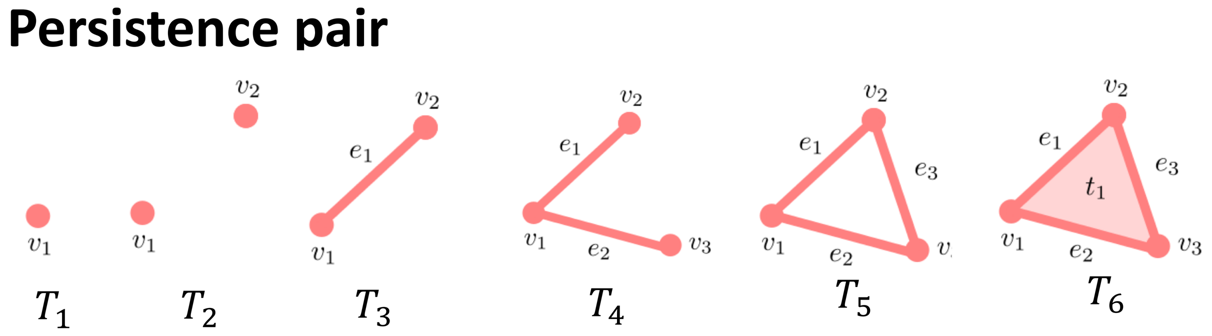


After cancellation



Not cancellable

Persistence pair



Persistent pair: (v_2, e_1), (v_3, e_2), (e_3, t_1)
$pers(v_2) = pers(e_1) = T_3 - T_2$
$pers(e_3) = pers(t_1) = T_6 - T_5$
Order of the simplices (Lower-star filtration):

1. Sort all vertices by their density values: $v_1 v_2 v_3 \ldots v_n$.
2. Edges and triangles come immediately after their adjacent vertices are in.

Existing algorithm

Algorithm 1: MorseRecon(K, ρ, δ)

Data: Triangulation K of Ω, density function $\rho : K \to \mathbb{R}$, threshold δ
Result: Reconstructed graph $\widehat{G}$
begin
  Compute persistence pairings $P(K)$ by the lower-star filtration of K w.r.t $g_\rho = -\rho$
  M = PerSimpVF($P(K), \delta$)
  $\widehat{G}$ = CollectOutputG(M)
  return $\widehat{G}$

Procedure PerSimpVF($P(K), \delta$)
  Set initial discrete gradient field M on K to be trivial
  Rank all persistence pairs in $P(K)$ in increasing order of their persistence
  for *each* $(\sigma, \tau) \in P(K)$ *with* $\mathrm{pers}(\sigma, \tau) \le \delta$ **do**
    If possible, perform discrete-Morse cancellation of (σ, τ) and update the discrete gradient vector field M
  return M

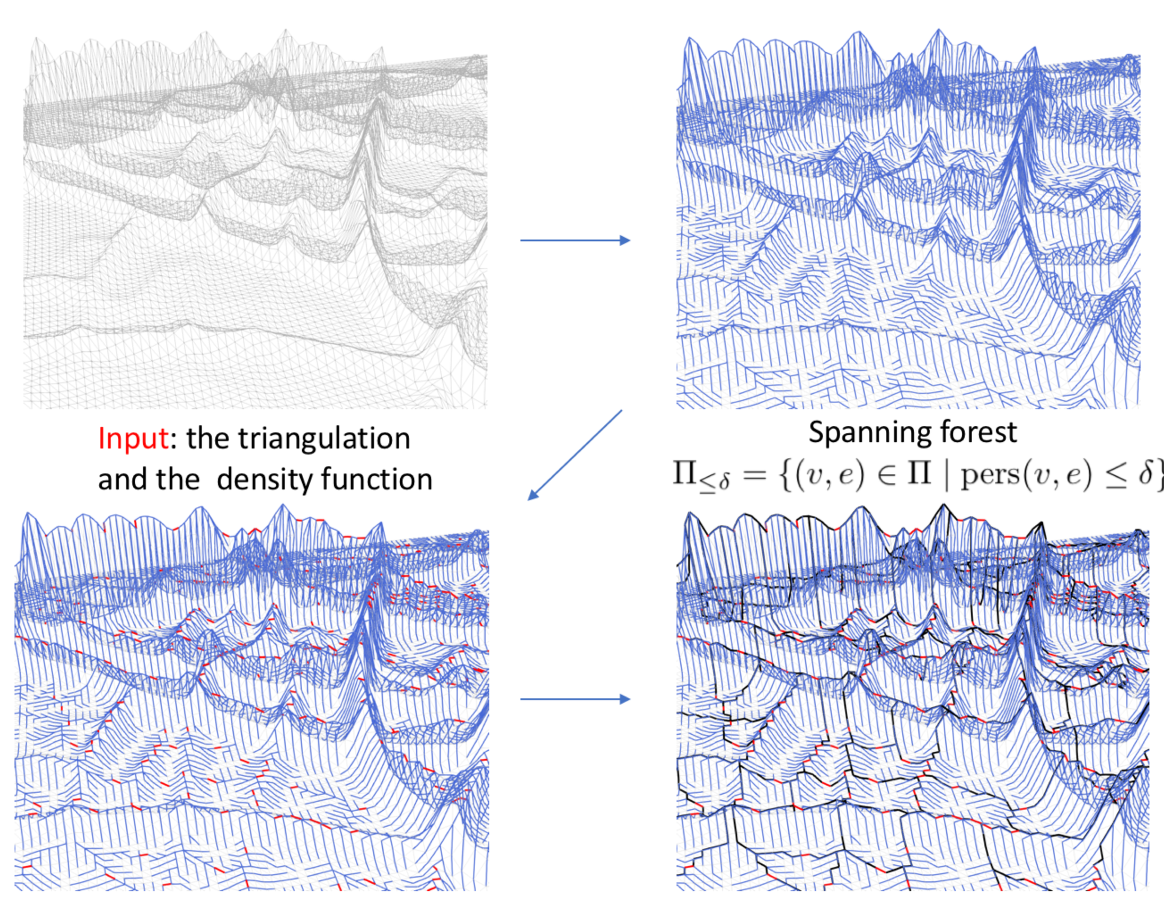
Procedure CollectOutputG(M)
  $\widehat{G} = \emptyset$
  for *each remaining critical edge e with* $\mathrm{pers}(e) > \delta$ **do**
    $\widehat{G} = \widehat{G} \bigcup \{$1-unstable manifold of $e\}$
  return $\widehat{G}$

Simplified algorithm

Algorithm 2: MorseReconSimp(K, ρ, δ)

Procedure PerSimpTree($P(K), \delta$) /* *This procedure replaces original* PerSimpVF() */
  $\Pi :=$ the set of vertex-edge persistence pairs from $P(K)$
  Set $\Pi_{\le\delta} \subseteq \Pi$ to be $\Pi_{\le\delta} = \{(v, e) \in \Pi \mid \mathrm{pers}(v, e) \le \delta\}$
  $\mathcal{T} := \bigcup_{(v,\sigma)\in\Pi_{\le\delta}} \{\sigma = \langle u_1, u_2 \rangle, u_1, u_2\}$
  return $\mathcal{T}$

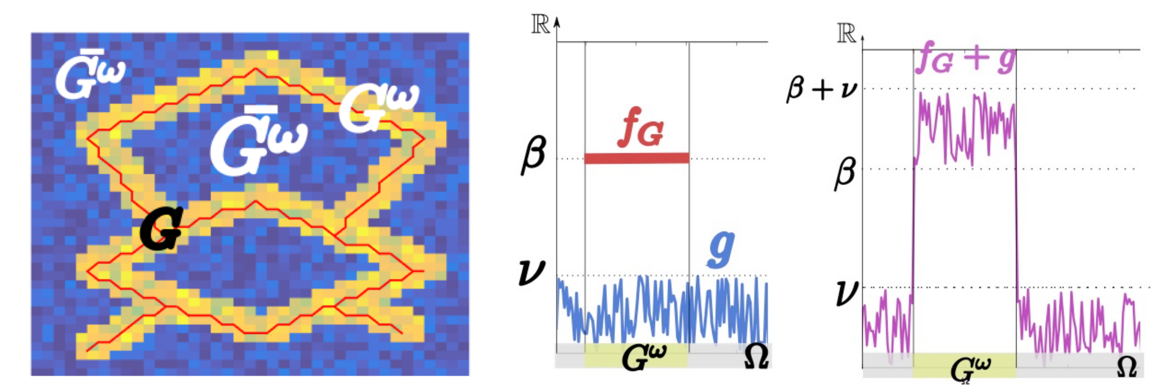
Procedure Treebased-OutputG($\mathcal{T}$) /* *This procedure replaces* CollectOutputG() */
  $\widehat{G} = \emptyset$
  for *each critical edge* $e = \langle u, v \rangle$ *with* $\mathrm{pers}(e) \ge \delta$ **do**
    Let $\pi(u)$ be the unique path from u to the sink of the tree T_i containing u
    Define $\pi(v)$ similarly; Set $\widehat{G} = \widehat{G} \cup \pi(u) \cup \pi(v) \cup \{e\}$
  return $\widehat{G}$



Input: the triangulation and the density function
Spanning forest $\Pi_{\le\delta} = \{(v, e) \in \Pi \mid \mathrm{pers}(v, e) \le \delta\}$
Red edges: $pers(e) > \delta$
Output: red and black edges

Illustration of the Simplified algorithm

Noise model



Definition A density function $\rho: \Omega \to \mathbb{R}$ is a $(\beta, \nu, \omega)-$*approximation* of a connected graph G is the following holds:
C-1 There is a ω-neighborhood G^ω of G such that G^ω deformation retracts to G.
C-2 $\rho(x) \in [\beta, \beta + \nu]$ for $x \in G^\omega$; and $\rho(x) \in [0, \nu]$ otherwise. Furthermore, $\beta > 2\nu$.

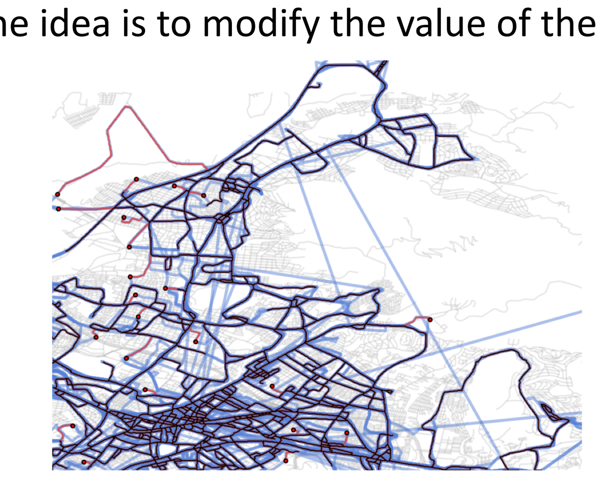
Theoretical guarantee

$\widehat{G}$ is geometrically close to G:
Theorem Under our noise model, the output graph satisfies $\widehat{G} \subseteq G^\omega$.

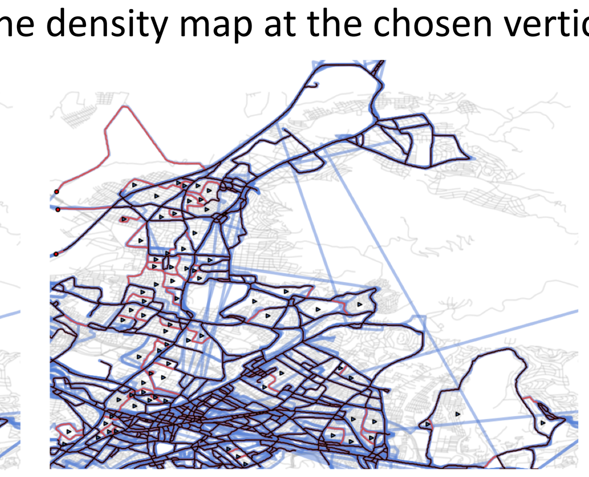
$\widehat{G}$ has the same topology with G:
Proposition Under our noise model, $\widehat{G}$ is homotopy equivalent to G.

Editing strategy

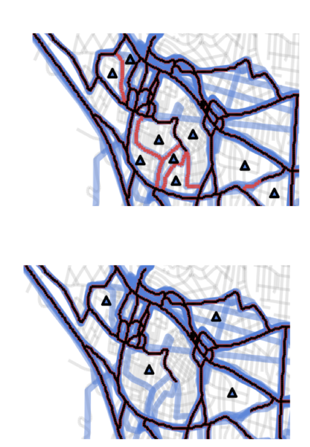
The idea is to modify the value of the density map at the chosen vertices.



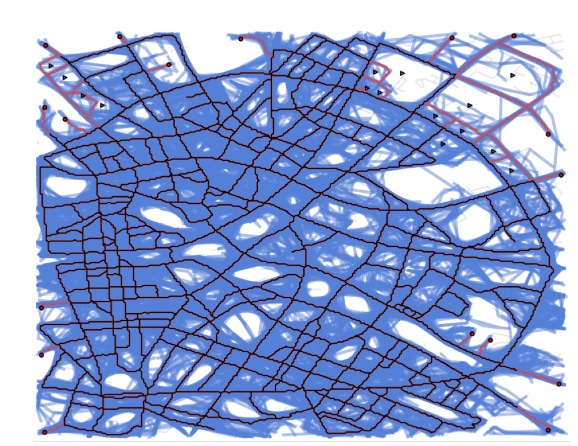
Add branches: Click at the end of missing branches, set to a large value.



Add loops: Click triangles at regions with missing roads that form loops, set to a small value.



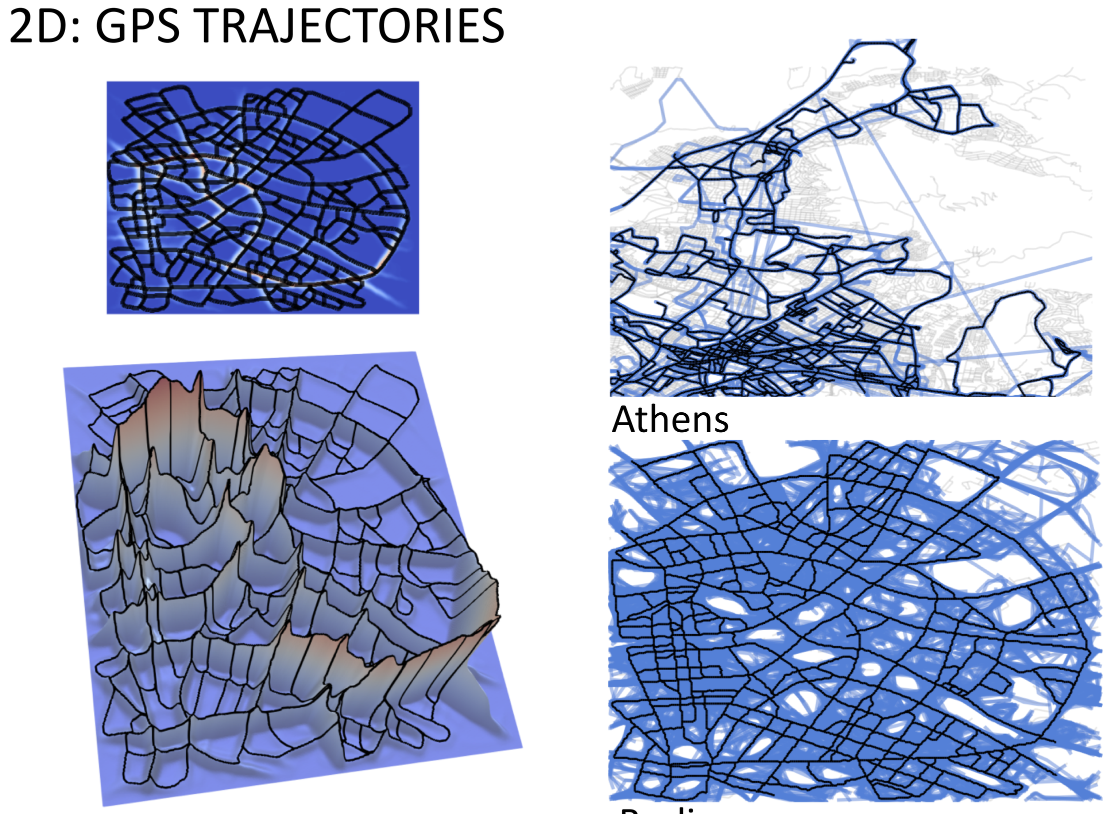
Adding extra triangles will capture more loops. One loop needs one triangle .



Blue: Input GPS trajectories Black: Original reconstruction Red: Newly added branches

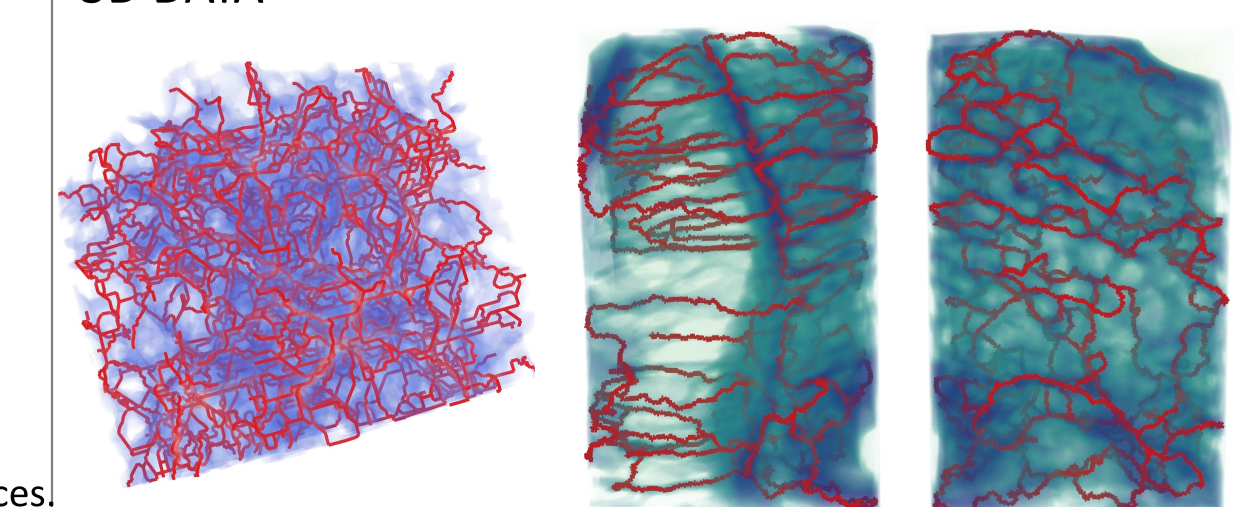
Experiments

2D: GPS TRAJECTORIES



Blue: Input GPS trajectories Black: Original reconstruction

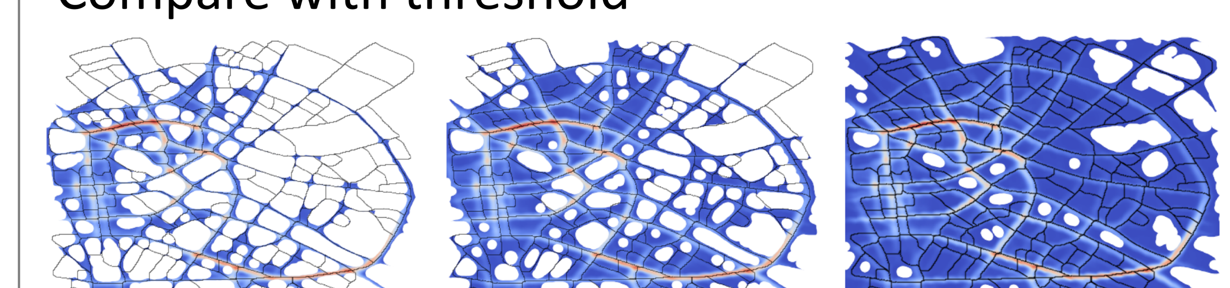
3D DATA



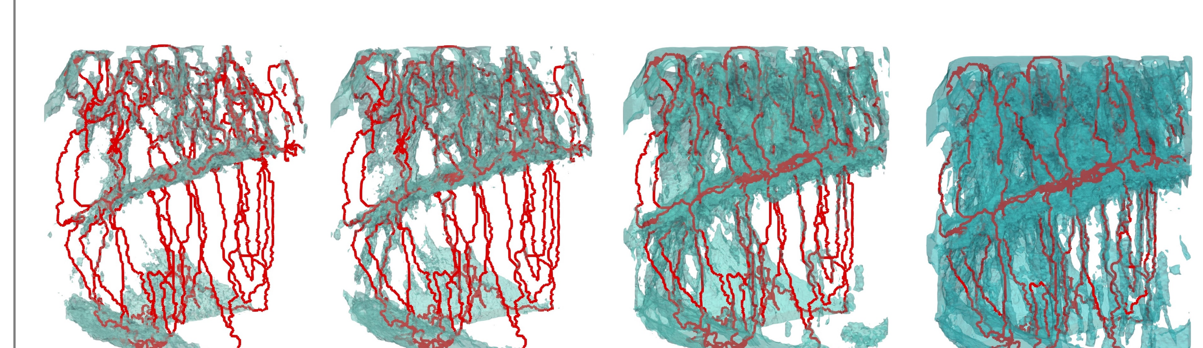
ENZO: Cosmological structure
Micro CT image of a bone

Red: Reconstruction result The volumes show the volume rendering of the input density functions.

Compare with threshold



Keep 50%
Keep 70%
Keep 90%



Left to right, decrease the threshold