CS-TSSOS: CORRELATIVE AND TERM SPARSITY FOR LARGE-SCALE POLYNOMIAL OPTIMIZATION

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ABSTRACT. This work proposes a new moment-SOS hierarchy, called *CS-TSSOS*, for solving large-scale sparse polynomial optimization problems. Its novelty is to exploit simultaneously *correlative sparsity* and *term sparsity* by combining advantages of two existing frameworks for sparse polynomial optimization. The former is due to Waki et al. [WKKM06] while the latter was initially proposed by Wang et al. [WLX19] and later exploited in the TSSOS hierarchy [WML19, WML20]. In doing so we obtain CS-TSSOS – a two-level hierarchy of semidefinite programming relaxations with (i), the crucial property to involve quasi block-diagonal matrices and (ii), the guarantee of convergence to the global optimum. We demonstrate its efficiency on several large-scale instances of the celebrated Max-Cut problem and the important industrial optimal power flow problem, involving up to several thousands of variables and ten thousands of constraints.

1. Introduction

This paper is concerned with solving large-scale polynomial optimization problems. As is often the case, the polynomials in the problem description involve only a few monomials of low degree and the ultimate goal is to exploit this crucial feature to provide semidefinite relaxations that are computationally much cheaper than those of the standard SOS-based hierarchy [Las01] or their sparse version [Las06, WKKM06] based on correlative sparsity.

Throughout the paper, we consider large-scale instances of the following polynomial optimization problem (POP):

(Q):
$$\rho^* = \inf_{\mathbf{x}} \{ f(\mathbf{x}) : \mathbf{x} \in \mathbf{K} \},$$

where the objective function f is assumed to be a polynomial in n variables $\mathbf{x} = (x_1, \dots, x_n)$ and the set of constraints $\mathbf{K} \subseteq \mathbb{R}^n$ is assumed to be defined by a finite conjunction of m polynomial inequalities, namely

$$\mathbf{K} := {\mathbf{x} \in \mathbb{R}^n : q_1(\mathbf{x}) > 0, \dots, q_m(\mathbf{x}) > 0},$$

for some polynomials g_1, \ldots, g_m in \mathbf{x} . Here "large-scale" means that the magnitude of the number of variables n and the number of inequalities m can be both proportional to several thousands. A nowadays well-established scheme to handle (Q) is the moment-SOS hierarchy [Las01], where SOS is the abbreviation of sum of squares. The moment-SOS hierarchy provides a sequence of semidefinite programming (SDP) relaxations, whose optimal values are non-decreasing lower bounds of

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the value ρ^* of (Q). Under mild assumption slightly stronger than compactness, the sequence generically converges in finite steps [Nie14]. SDP solvers [WSV12] address a specific class of convex optimization problems, with linear cost and linear matrix inequalities. With a priory fixed precision, an SDP can be solved in polynomial time with respect to its input size. Modern SDP solvers via the interior-point method (e.g. Mosek [AA00]) can solve an SDP problem involving matrices of moderate size (say, $\leq 5,000$) and equality constraints of moderate number (say, $\leq 20,000$) in reasonable time on a standard laptop [Toh18]. The SDP relaxations arising from the moment-SOS hierarchy typically involve matrices of size $\binom{n+d}{d}$ and equality constraints of number $\binom{n+2d}{2d}$, where d is the relaxation order. For problems with $n \simeq 200$, it is thus possible to compute the first-order SDP relaxation of a quadratically constrained quadratic problem (QCQP), as one can take d=1, yielding $\binom{n+d}{d} \simeq 200$ and $\binom{n+2d}{2d} \simeq 20,000$ (in this case, this relaxation is also known as Shor relaxation [Sho87]). But the quality of the resulting approximation is often not satisfactory and it is required to go beyond the first-order relaxation. For the second-order relaxation, the matrix size $\binom{n+2}{2} \simeq 20,000$ and the number of equality constraints $\binom{n+4}{4} \simeq 7 \times 10^7$ of the resulting program is out of reach for modern solvers. Therefore, in view of the current state of SDP solvers, the dense moment-SOS hierarchy does not scale well enough. For instance for solving the second-order relaxation (d=2) one is limited to problems of small size, typically with $\binom{n+4}{4} \leq 20,000$ (hence with $n \leq 24$) on a standard laptop. One possible remedy is to rely on alternative weaker positivity certificates, such as the hierarchy of linear programs (LP) based on Krivine-Stengle's certificates [Kri64, Ste74, LTY17] or the one based on (scaled) diagonal sums of squares [AM14] to approximate from below the solution of (Q). Even though modern LP solvers can handle much larger problems, by comparison with SDP solvers, the resulting relaxations have been shown to provide less accurate bounds, in particular for combinatorial problems [Lau03], do not have the finite converge property for continuous problems, and not even finite convergence for convex QCQP problems [Las15, Section 9.3].

Related work for unconstrained POPs. A first option is to exploit term sparsity for sparse unconstrained SOS problems, i.e. when $\mathbf{K} = \mathbb{R}^n$, f involves few terms (monomials) and admits an SOS decomposition. The algorithm consists of automatically reducing the size of the corresponding SDP matrix by eliminating the monomial terms which never appear among the support of the SOS decomposition [R⁺78]. Other classes of positivity certificates have been recently developed with a specific focus on sparse unconstrained problems. Instead of trying to decompose a positive polynomial as an SOS, one can try to decompose it as a sum of nonnegative circuits (SONC), by solving a geometric program [IDW16] or a second-order cone program [Ave19, WM19], or alternatively as a sum of arithmeticgeometric-mean-exponentials (SAGE) [CS16] by solving a relative entropy program. Despite their efficiency of certain sub-classes of unconstrained POPs, these methods share the common drawback to not provide systematic guarantees of convergence for constrained problems. Since several large-scale real-world problems, such as optimal power flow instances, involve equality or inequality constraints, it is mandatory to design more suitable efficient relaxation schemes.

Related work on correlative sparsity. In order to reduce the computational burden associated to the dense moment-SOS hierarchy while keeping its nice convergence properties, one possibility is to take into account the sparsity pattern satisfied by the variables of the POP [Las06, WKKM06]. The resulting algorithm has been implemented in the SparsePOP solver [WKK⁺08] and can handle unconstrained sparse problems with up to thousand variables. Many applications of interest have been successfully handled thanks to this framework, for instance certified roundoff error bounds in computer arithmetics [MCD17, Mag18] with up to several hundred variables and constraints, optimal powerflow problems [JM18] with up to several thousands of variables and constraints. More recent extensions have been developed for volume computation of sparse semialgebraic sets [TWLH19], approximating regions of attraction of sparse polynomial systems [TCHL19], noncommutative POPs [KMP19], Lipschitz constant estimation of deep networks [LRC20, CLMP20] and for sparse positive definite functions [MML20]. In these applications both polynomial cost functions and polynomial constraints have a specific correlative sparsity pattern. The resulting sparse moment-SOS hierarchy is obtained by building quasi block-diagonal SDP matrices with respect to some subsets (or cliques) $I_1, \ldots, I_p \subseteq \{1, \ldots, n\}$ of the input variables. When the maximal cardinal of these cliques is reasonably small, one can expect to handle problems with larger number of input variables. For instance, the maximal cardinal is less than 10 for some unconstrained problems in [WKKM06] or roundoff error problems in [MCD17], and less than 20 for the optimal power flow problems handled in [JM18]. Even though correlative sparsity has been successfully used for several interesting applications, there is still room for improvement. There are POPs admitting correlative sparsity patterns with variable subsets of too large cardinality, yielding untractable SDP, typically with more than 20 variables, or which do not fully satisfy the pattern required in [Las06, WKKM06].

Related work on term sparsity. To overcome these issues, one can exploit term sparsity as described in [WLX19, WML19, WML20]. The TSSOS hierarchy from [WLX19, WML19] as well as the complementary *Chordal-TSSOS* from [WML20] offer some alternative to problems where correlative sparsity cannot be successfully exploited. In both TSSOS and Chordal-TSSOS frameworks a so-called term sparsity pattern (tsp) graph is associated with the POP. The nodes of this tsp graph are monomials (from a vector of monomials) needed to construct SOS relaxations of the POP. Two nodes are connected via an edge when the product of the corresponding monomials appears in the supports of polynomials involved in the POP or is a monomial of even degree. Note that this graph differs from the *correlative sparsity* pattern (csp) graph used in [WKKM06], where the nodes are the input variables and the edges connect two nodes when the corresponding variables appear in the same term of the objective function or in the same constraint. A two-step iterative algorithm takes as input the tsp graph and enlarges it to exploit the term sparsity in (Q). Each iteration consists of two steps: (i) a support-extension procedure and (ii) either a block-closure operation on adjacency matrices in the case of TSSOS [WML19] or an (approximately) minimal chordal extension in the case of Chordal-TSSOS [WML20]. In doing so one obtains a two-level moment-SOS hierarchy with quasi block-diagonal SDP matrices. If the block sizes are small then the resulting SDP relaxations become tractable as their computational cost is reduced significantly. Another interesting feature of TSSOS is that the adjacency matrix of the graph obtained at the end of the iterative algorithm automatically induces a partition of the monomials (from the monomials vector), which can be interpreted in terms of sign-symmetries of the initial POP. Sign-symmetries were previously investigated in [Lof09]. One can also rely on symmetry exploitation as in [RTAL13] but this requires quite strong assumptions on the input data, such as invariance of each polynomial f, g_1, \ldots, g_m under the action of a finite group. TSSOS and Chordal-TSSOS allow one to solve POPs with several hundreds of variables, which do not fulfill such correlative sparsity or symmetry patterns.

A natural idea already mentioned in [WML20] is to simultaneously benefit from correlative and term sparsity patterns to solve large-scale POPs. This is the spirit of our contribution. Also in the same vein the work in [MZSP19] combines the (S)DSOS framework [AM14] with the TSSOS hierarchy [WML19] but does not provide systematic convergence guarantees.

Contribution. Our main contribution is as follows:

- For large-scale POPs with a correlative sparsity pattern, we first apply the usual sparse polynomial optimization framework [Las06, WKKM06] to get a coarse decomposition in terms of cliques of the variables. Next we apply a term sparsity strategy (either TSSOS or Chordal-TSSOS) to each subsystem (which involves only one clique of variables). While the overall strategy is quite clear and simple, its implementation is not trivial and needs some care. Indeed for its coherency one needs to take extra care of the monomials which involve variables that belong to intersections of variable cliques (those obtained after applying correlative sparsity). The resulting combination of correlative sparsity (CS for short) and term sparsity produces what we call the CS-TSSOS hierarchy a two-level hierarchy of SDP relaxations with quasi block-diagonal matrices which yields a converging sequence of certified approximations for POPs. In contrast with other techniques exploiting sparsity, we obtain tighter bounds, while overcoming even deeper scalability issues.
- Our algorithmic development of the CS-TSSOS hierarchy is fully implemented in the TSSOS tool. The most recent version of TSSOS has been released within the Julia [BEKS17] programming language and relies on Mosek [ApS17] for solving the SDP relaxations. Our tool is freely available online and documented. Accuracy and performance are evaluated on several large-scale benchmarks coming from the continuous and combinatorial optimization literature. In particular, numerical experiments demonstrate that the CS-TSSOS hierarchy can handle Max-Cut and optimal power flow problem instances with up to thousands of variables and ten thousands of constraints.

The rest of the paper is organized as follows: in Section 2, we provide preliminary background on SOS polynomials, the moment-SOS hierarchy, correlative sparsity and the TSSOS and Chordal-TSSOS hierarchies exploiting term sparsity. In Section 3, we explain how to combine correlative and term sparsities to obtain the resulting CS-TSSOS hierarchy. Its convergence is analyzed in Section 4. Eventually, we provide numerical experiments for large-scale POP instances in Section 5. Discussions and conclusions are made in Section 6.

¹https://github.com/wangjie212/TSSOS

2. Notation and Preliminaries

2.1. Notation and SOS polynomials. Let $\mathbf{x} = (x_1, \dots, x_n)$ be a tuple of variables and $\mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \dots, x_n]$ be the ring of real n-variate polynomials. For $d \in \mathbb{N}$, the set of polynomials of degree no more than 2d is denoted by $\mathbb{R}_{2d}[\mathbf{x}]$. A polynomial $f \in \mathbb{R}[\mathbf{x}]$ can be written as $f(\mathbf{x}) = \sum_{\alpha \in \mathscr{A}} f_{\alpha} \mathbf{x}^{\alpha}$ with $\mathscr{A} \subseteq \mathbb{N}^n$ and $f_{\alpha} \in \mathbb{R}, \mathbf{x}^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The support of f is defined by $\sup(f) = \{\alpha \in \mathscr{A} \mid f_{\alpha} \neq 0\}$. We use $|\cdot|$ to denote the cardinality of a set. For a finite set $\mathscr{A} \subseteq \mathbb{N}^n$, let $\mathbf{x}^{\mathscr{A}}$ be the $|\mathscr{A}|$ -dimensional column vector consisting of elements $\mathbf{x}^{\alpha}, \alpha \in \mathscr{A}$ (fix any ordering on \mathbb{N}^n). For a positive integer r, the set of $r \times r$ symmetric matrices is denoted by \mathbf{S}^r and the set of $r \times r$ positive semidefinite (PSD) matrices is denoted by \mathbf{S}^r . Let us denote by $\langle A, B \rangle \in \mathbb{R}$ the trace inner-product, defined by $\langle A, B \rangle = \operatorname{Tr}(A^T B)$. For $d \in \mathbb{N}$, let $\mathbb{N}_d^n := \{\alpha = (\alpha_i) \in \mathbb{N}^n \mid \sum_{i=1}^n \alpha_i \leq d\}$.

Given a polynomial $f(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$, if there exist polynomials $f_1(\mathbf{x}), \ldots, f_t(\mathbf{x})$ such that $f(\mathbf{x}) = \sum_{i=1}^t f_i(\mathbf{x})^2$, then we call $f(\mathbf{x})$ a sum of squares (SOS) polynomial. The set of SOS polynomials is denoted by $\Sigma[\mathbf{x}]$. Assume that $f \in \Sigma_{2d}[\mathbf{x}] := \Sigma[\mathbf{x}] \cap \mathbb{R}_{2d}[\mathbf{x}]$ and $\mathbf{x}^{\mathbb{N}_d^n}$ is the standard monomial basis. Then the SOS condition for f is equivalent to the existence of a PSD matrix Q, which is called a Gram matrix $[\mathbb{R}^+78]$, such that $f = (\mathbf{x}^{\mathbb{N}_d^n})^T Q \mathbf{x}^{\mathbb{N}_d^n}$. For convenience, we abuse notation in the sequel and denote by \mathbb{N}_d^n instead of $\mathbf{x}^{\mathbb{N}_d^n}$ the standard monomial basis and use the exponent α to represent a monomial \mathbf{x}^{α} .

2.2. **Moment-SOS** hierarchy for POPs. Consider the constrained polynomial optimization problem:

(2.1)
$$(Q): \quad \rho^* := \inf_{\mathbf{x}} \{ f(\mathbf{x}) : \mathbf{x} \in \mathbf{K} \} ,$$

where $f(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$ is a polynomial and $\mathbf{K} \subseteq \mathbb{R}^n$ is the basic semialgebraic set

(2.2)
$$\mathbf{K} = \{ \mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \ge 0, j = 1, \dots, m \},$$

for some polynomials $g_j(\mathbf{x}) \in \mathbb{R}[\mathbf{x}], j = 1, \dots, m$.

With $\mathbf{y} = (y_{\alpha})$ being a sequence indexed by the standard monomial basis \mathbb{N}^n of $\mathbb{R}[\mathbf{x}]$, let $L_{\mathbf{y}} : \mathbb{R}[\mathbf{x}] \to \mathbb{R}$ be the linear functional

$$f = \sum_{\alpha} f_{\alpha} \mathbf{x}^{\alpha} \mapsto L_{\mathbf{y}}(f) = \sum_{\alpha} f_{\alpha} y_{\alpha}.$$

With the standard monomial basis \mathbb{N}_d^n , the moment matrix $M_d(\mathbf{y})$ associated with \mathbf{y} is the matrix with rows and columns indexed by \mathbb{N}_d^n such that

$$M(\mathbf{y})_{\beta\gamma} := L_{\mathbf{y}}(\mathbf{x}^{\beta}\mathbf{x}^{\gamma}) = y_{\beta+\gamma}, \quad \forall \beta, \gamma \in \mathbb{N}_d^n.$$

Suppose $g = \sum_{\alpha} g_{\alpha} \mathbf{x}^{\alpha} \in \mathbb{R}[\mathbf{x}]$ and let $\mathbf{y} = (y_{\alpha})$ be given. For a positive integer d, the *localizing* matrix $M_d(g\mathbf{y})$ associated with g and \mathbf{y} is the matrix with rows and columns indexed by \mathbb{N}_d^n such that

$$M_d(g\,\mathbf{y})_{\boldsymbol{\beta}\boldsymbol{\gamma}} := L_{\mathbf{y}}(g\,\mathbf{x}^{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\gamma}}) = \sum_{\boldsymbol{\alpha}} g_{\boldsymbol{\alpha}} y_{\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{\gamma}}, \quad \forall \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{N}_d^n.$$

Let $d_j := \lceil \deg(g_j)/2 \rceil, j = 1, ..., m$ and let $\hat{d} \ge \max\{\lceil \deg(f)/2 \rceil, d_1, ..., d_m\}$ be a positive integer. Then the Lasserre hierarchy indexed by \hat{d} of primal moment

SDP relaxations of (Q) is defined by ([Las01]):

(2.3)
$$(\mathbf{Q}_{\hat{d}}): \begin{cases} \inf & L_{\mathbf{y}}(f) \\ \text{s.t.} & M_{\hat{d}}(\mathbf{y}) \succeq 0, \\ & M_{\hat{d}-d_{j}}(g_{j}\mathbf{y}) \succeq 0, \quad j = 1, \dots, m, \\ & y_{0} = 1. \end{cases}$$

We call \hat{d} the relaxation order.

Set $g_0 := 1$ and $d_0 := 0$. For each j, writing $M_{\hat{d}-d_j}(g_j\mathbf{y}) = \sum_{\alpha} D_{\alpha}^j y_{\alpha}$ for appropriate (0,1)-binary matrices $\{D_{\alpha}^j\}$, we can write the dual of (2.3) as

(2.4)
$$(Q_{\hat{d}})^* : \begin{cases} \sup & \rho \\ \text{s.t.} & \sum_{j=0}^{m} \langle Q_j, D_{\boldsymbol{\alpha}}^j \rangle + \rho \delta_{\boldsymbol{0}\boldsymbol{\alpha}} = f_{\boldsymbol{\alpha}}, \quad \forall \boldsymbol{\alpha} \in \mathbb{N}_{2\hat{d}}^n, \\ Q_j \succeq 0, \quad j = 0, \dots, m, \end{cases}$$

where $\delta_{0\alpha}$ is the usual Kronecker symbol.

2.3. Chordal graphs and sparse matrices. We briefly recall some basic notions from graph theory. An (undirected) graph G(V, E) or simply G consists of a set of nodes V and a set of edges $E \subseteq \{\{v_i, v_j\} \mid (v_i, v_j) \in V \times V\}$. Note that we admit self-loops (i.e. edges that connect the same node) in the edge set E. If G is a graph, we will use V(G) and E(G) to indicate the set of nodes of G and the set of edges of G, respectively. For two graphs G, H, we say that G is a subgraph of H if $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$, denoted by $G \subseteq H$. For a graph G(V, E), a cycle of length E is a set of nodes $\{v_1, v_2, \ldots, v_k\} \subseteq V$ with $\{v_k, v_1\} \in E$ and $\{v_i, v_{i+1}\} \in E$, for $i = 1, \ldots, k-1$. A chord in a cycle $\{v_1, v_2, \ldots, v_k\}$ is an edge $\{v_i, v_j\}$ that joins two nonconsecutive nodes in the cycle. A clique $C \subseteq V$ of G is a subset of nodes where $\{v_i, v_j\} \in E$ for any $v_i, v_j \in C$. If a clique C is not a subset of any other clique, then it is called a maximal clique.

A graph is called a *chordal graph* if all its cycles of length at least four have a chord. Note that any non-chordal graph G(V,E) can always be extended to a chordal graph $\overline{G}(V,\overline{E})$ by adding appropriate edges to E, which is called a *chordal extension* of G(V,E). The chordal extension of G is usually not unique. We use \overline{G} to indicate any specific chordal extension of G. For graphs $G\subseteq H$, we assume that $\overline{G}\subseteq \overline{H}$ always holds in this paper. It is known that maximal cliques of a chordal graph can be enumerated efficiently in linear time in the number of nodes and edges of the graph. See e.g. [BP93, FG65, Gol04] for the details.

Suppose G(V, E) is a graph with the node set $V \subseteq \mathbb{N}^n$. We define the *support* of G by

$$\operatorname{supp}(G) := \{ \beta + \gamma \mid \{ \beta, \gamma \} \in E \}.$$

Given a graph G(V, E), a symmetric matrix Q with row and column indices labeled by V is said to have sparsity pattern G if $Q_{\beta\gamma} = Q_{\gamma\beta} = 0$ whenever $\{\beta, \gamma\} \notin E$. Let \mathbf{S}_G be the set of symmetric matrices with sparsity pattern G. The PSD matrices with sparsity pattern G form a convex cone

(2.5)
$$\mathbf{S}_{+}^{|V|} \cap \mathbf{S}_{G} = \{ Q \in \mathbf{S}_{G} \mid Q \succeq 0 \}.$$

Remark 2.1. For a graph G, among all chordal extensions of G, there is a maximal one \overline{G} : making every connected component of G to be a complete subgraph. Accordingly, the matrix with sparsity pattern \overline{G} is block diagonal (up to permutation). We hereafter refer to this chordal extension as the maximal chordal extension. In this paper, we only consider chordal extensions that are subgraphs of the maximal chordal extension.

Given a maximal clique C of G(V, E), we define a matrix $P_C \in \mathbb{R}^{|C| \times |V|}$ as

(2.6)
$$(P_C)_{i\beta} = \begin{cases} 1, & \text{if } C(i) = \beta, \\ 0, & \text{otherwise.} \end{cases}$$

where C(i) denotes the *i*-th node in C, sorted with respect to an ordering compatible with V. Note that $Q_C = P_C Q P_C^T \in \mathbf{S}^{|C|}$ extracts a principal submatrix Q_C defined by the indices in the clique C from a symmetry matrix Q, and $Q = P_C^T Q_C P_C$ inflates a $|C| \times |C|$ matrix Q_C into a sparse $|V| \times |V|$ matrix Q.

When the sparsity pattern graph G is chordal, the cone $\mathbf{S}_{+}^{|V|} \cap \mathbf{S}_{G}$ can be decomposed as a sum of simple convex cones, as stated in the following theorem.

Theorem 2.2 ([VA⁺15], Theorem 9.2). Let G(V, E) be a chordal graph and assume that C_1, \ldots, C_t are all of the maximal cliques of G(V, E). Then a matrix $Q \in \mathbf{S}_+^{|V|} \cap \mathbf{S}_G$ if and only if there exist $Q_k \in \mathbf{S}_+^{|C_k|}$ for $k = 1, \ldots, t$ such that $Q = \sum_{k=1}^{t} P_{C_k}^T Q_k P_{C_k}$.

Given a graph G(V, E), let Π_G be the projection from $\mathbf{S}^{|V|}$ to the subspace \mathbf{S}_G , i.e., for $Q \in \mathbf{S}^{|V|}$,

(2.7)
$$\Pi_G(Q)_{\beta\gamma} = \begin{cases} Q_{\beta\gamma}, & \text{if } \{\beta,\gamma\} \in E, \\ 0, & \text{otherwise.} \end{cases}$$

We denote by $\Pi_G(\mathbf{S}_+^{|V|})$ the set of matrices in \mathbf{S}_G that have a PSD completion, i.e.,

(2.8)
$$\Pi_G(\mathbf{S}_+^{|V|}) = \{ \Pi_G(Q) \mid Q \in \mathbf{S}_+^{|V|} \}.$$

One can check that the PSD completable cone $\Pi_G(\mathbf{S}_+^{|V|})$ and the PSD cone $\mathbf{S}_+^{|V|} \cap \mathbf{S}_G$ form a pair of dual cones in \mathbf{S}_G . Moreover, for a chordal graph G, the decomposition result for the cone $\mathbf{S}_+^{|V|} \cap \mathbf{S}_G$ in Theorem 2.2 leads to the following characterization of the PSD completable cone $\Pi_G(\mathbf{S}_+^{|V|})$.

Theorem 2.3 ([VA⁺15], Theorem 10.1). Let G(V, E) be a chordal graph and assume that C_1, \ldots, C_t are all of the maximal cliques of G(V, E). Then a matrix $Q \in \Pi_G(\mathbf{S}_+^{|V|})$ if and only if $Q_k = P_{C_k}QP_{C_k}^T \succeq 0$ for $k = 1, \ldots, t$.

For more details about sparse matrices and chordal graphs, the reader may refer to $[VA^+15]$.

2.4. Correlative sparsity. To exploit correlative sparsity in the moment-SOS hierarchy for POPs, one proceeds in two steps: 1) partition the set of variables into cliques according to the links between variables emerging in the input polynomial system, and 2) construct a quasi block moment-SOS hierarchy with respect to the former partition of variables [WKKM06].

More concretely, we define the correlative sparsity pattern (csp) graph associated to POP (2.1) to be the graph G^{csp} with nodes $V = [n] := \{1, 2, \dots, n\}$ and edges E satisfying $\{i, j\} \in E$ if one of followings holds:

- (i) there exists $\alpha \in \text{supp}(f)$ s.t. $\alpha_i > 0, \alpha_i > 0$;
- (ii) there exists k, with $1 \le k \le m$, s.t. $x_i, x_j \in \text{var}(g_k)$, where $\text{var}(g_k)$ is the set of variables involved in g_k .

Let $\overline{G}^{\text{csp}}$ be a chordal extension of G^{csp} and $I_l, l = 1..., p$ be the maximal cliques of $\overline{G}^{\text{csp}}$ with cardinal denoted by n_l . Let $\mathbb{R}[\mathbf{x}(I_l)]$ denote the ring of polynomials in the n_l variables $\mathbf{x}(I_l) = \{x_i \mid i \in I_l\}$. We then partition the constraints g_1, \dots, g_m into groups $\{g_j \mid j \in J_l\}, l = 1, \dots, p$ which satisfy:

- (i) $J_1, \ldots, J_p \subseteq [m] := \{1, 2, \ldots, m\}$ are pairwise disjoint and $\bigcup_{l=1}^p J_l = [m]$; (ii) for any $j \in J_l$, $\operatorname{var}(g_j) \subseteq I_l$, $l = 1, \ldots, p$.

Next, with $l \in \{1, ..., p\}$ fixed, d a positive integer and $g \in \mathbb{R}[\mathbf{x}(I_l)]$, let $M_d(\mathbf{y}, I_l)$ (resp. $M_d(q\mathbf{y}, I_l)$) be the moment (resp. localizing) submatrix obtained from $M_d(\mathbf{y})$ (resp. $M_d(g\mathbf{y})$) by retaining only those rows (and columns) $\boldsymbol{\beta} = (\beta_i) \in \mathbb{N}_d^n$ of $M_d(\mathbf{y})$ (resp. $M_d(g\mathbf{y})$) with supp $(\boldsymbol{\beta}) \subseteq I_l$, where supp $(\boldsymbol{\beta}) = \{i \mid \beta_i \neq 0\}$.

Then with $\hat{d} \geq d := \max\{\lceil \deg(f)/2 \rceil, d_1, \ldots, d_m\}$, the moment-SOS hierarchy based on correlative sparsity for (2.1) is defined as:

(2.9)
$$(Q_{\hat{d}}^{cs}): \begin{cases} \inf & L_{\mathbf{y}}(f) \\ \text{s.t.} & M_{\hat{d}}(\mathbf{y}, I_{l}) \succeq 0, \quad l = 1, \dots, p, \\ & M_{\hat{d} - d_{j}}(g_{j}\mathbf{y}, I_{l}) \succeq 0, \quad j \in J_{l}, l = 1, \dots, p, \\ & y_{\mathbf{0}} = 1, \end{cases}$$

with optimal value denoted by $\rho_{\hat{d}}$. In the following, we refer to $(Q_{\hat{d}}^{cs})$ (2.9) as the CSSOS hierarchy for the POP (2.1).

Remark 2.4. As shown in [Las06] under some compactness assumption, the sequence $(\rho_{\hat{d}})_{\hat{d}}$ converges to the global optimum ρ^* of the original POP (2.1).

2.5. **Term sparsity.** In contrast to the correlative sparsity pattern which focuses on links between variables, the term sparsity pattern focuses on links between monomials (or terms). To exploit term sparsity in the moment-SOS hierarchy one also proceeds in two steps: 1) partition each involved monomial basis into blocks according to the links between monomials emerging in the input polynomial system, and 2) construct a quasi block moment-SOS hierarchy with respect to the former partition of monomial base [WLX19, WML19].

More concretely, let $\mathscr{A} = \operatorname{supp}(f) \cup \bigcup_{j=1}^m \operatorname{supp}(g_j)$ and $\mathbb{N}^n_{\hat{d}-d_j}$ be the standard monomial basis for j = 0, ..., m. We define the term sparsity pattern (tsp) graph with relaxation order \hat{d} associated to POP (2.1) or \mathscr{A} , to be the graph $G_{\hat{d}}^{\text{tsp}}$ with nodes $V = \mathbb{N}^n_{\hat{d}-d_i}$ and edges

(2.10)
$$E := \{ \{ \boldsymbol{\beta}, \boldsymbol{\gamma} \} \mid \boldsymbol{\beta} + \boldsymbol{\gamma} \in \mathscr{A} \cup (2\mathbb{N}_{\hat{d}-d_j}^n) \}.$$

Assume that $G^{(0)}_{\hat{d},0}=G^{\mathrm{tsp}}_{\hat{d}}$ and $G^{(0)}_{\hat{d},j}, j=1,\ldots,m$ are empty graphs. Then we recursively define a sequence of graphs $(G^{(k)}_{\hat{d},j}(V_{\hat{d},j},E^{(k)}_{\hat{d},j}))_{k\geq 1}$ with $V_{\hat{d},j}=\mathbb{N}^n_{\hat{d}-d_j}$ for $j=0,\ldots,m$ by

(2.11)
$$G_{\hat{d},j}^{(k)} := \overline{F_{\hat{d},j}^{(k)}}, \quad j = 0, \dots, m,$$

where $F_{\hat{d},j}^{(k)}$ is the graph with $V(F_{\hat{d},j}^{(k)}) = \mathbb{N}_{\hat{d}-d_j}^n$ and

$$(2.12) E(F_{\hat{d},j}^{(k)}) = \{\{\boldsymbol{\beta}, \boldsymbol{\gamma}\} \mid (\operatorname{supp}(g_j) + \boldsymbol{\beta} + \boldsymbol{\gamma}) \cap (\cup_{i=0}^m \operatorname{supp}(G_{\hat{d},i}^{(k-1)})) \neq \emptyset\},$$
for each $i = 0$ m

for each $j=0,\ldots,m$. Let $r_j:=\binom{n+\hat{d}-d_j}{\hat{d}-d_j}, j=0,\ldots,m$. Then with $\hat{d}\geq d$ and $k\geq 1$, the moment-SOS hierarchy based on term sparsity for (2.1) is defined as:

(2.13)
$$(\mathbf{Q}_{\hat{d},k}^{\text{ts}}) : \begin{cases} \inf & L_{\mathbf{y}}(f) \\ \text{s.t.} & M_{\hat{d}}(\mathbf{y}) \in \Pi_{G_{\hat{d},0}^{(k)}}(\mathbf{S}_{+}^{r_{0}}), \\ & M_{\hat{d}-d_{j}}(g_{j}\mathbf{y}) \in \Pi_{G_{\hat{d},j}^{(k)}}(\mathbf{S}_{+}^{r_{j}}), \quad j = 1, \dots, m, \\ y_{0} = 1. \end{cases}$$

We call k the *sparse order* and in the remaining of the paper, the *TSSOS* hierarchy for POP (2.1) refers to the hierarchy $(Q_{\hat{J}k}^{ts})$.

Remark 2.5. In $(Q_{\hat{d},k}^{ts})$, one has the freedom to choose a specific chordal extension for any involved graph $G_{\hat{d},j}^{(k)}$. As shown in [WLX19], if one chooses the maximal chordal extension then with \hat{d} fixed, the resulting sequence of optimal values of the TSSOS hierarchy (as k increases) converges in finitely many steps to the optimal value of the corresponding dense moment-SOS relaxation for POP (2.1).

3. The CS-TSSOS Hierarchy

For large-scale POPs, it is natural to ask whether one can combine correlative sparsity and term sparsity to further exploit both sparsity features possessed by the problem. As we shall see in the following sections, the answer is affirmative.

3.1. The CS-TSSOS Hierarchy for general POPs. Consider POP $(2.1)^2$. A first natural idea to combine correlative sparsity and term sparsity for (2.1) would be to apply the TSSOS hierarchy for each clique *separately*, once the cliques of variables have been obtained from the csp graph of (2.1). However, with this naive approach convergence may be lost and below we describe the extra care needed to avoid this annoying consequence.

Let G^{csp} be the csp graph associated to (2.1), $\overline{G}^{\text{csp}}$ a chordal extension of G^{csp} and $I_l, l = 1, \ldots, p$ be the maximal cliques of $\overline{G}^{\text{csp}}$ with cardinal denoted by n_l . As in Sec. 2.4, the set of variables \mathbf{x} is partitioned into $\mathbf{x}(I_1), \mathbf{x}(I_2), \ldots, \mathbf{x}(I_p)$. Let J_1, \ldots, J_p be defined as in Sec. 2.4.

Now we apply the term sparsity pattern to each subsystem involving variables $\mathbf{x}(I_i), l = 1, ..., p$ respectively as follows. Let

(3.1)
$$\mathscr{A} := \operatorname{supp}(f) \cup \bigcup_{j=1}^{m} \operatorname{supp}(g_j) \text{ and } \mathscr{A}_l := \{ \boldsymbol{\alpha} \in \mathscr{A} \mid \operatorname{supp}(\boldsymbol{\alpha}) \subseteq I_l \}$$

for $l=1,\ldots,p$. As before, $d:=\max\{\lceil \deg(f)/2\rceil,d_1,\ldots,d_m\},\ d_0:=0$ and $g_0:=1$. Fix a relaxation order $\hat{d}\geq d$ and let $\mathbb{N}^{n_l}_{\hat{d}-d_j}$ be the standard monomial basis for $j\in\{0\}\cup J_l, l=1\ldots,p$. Let $G^{\mathrm{tsp}}_{\hat{d},l}$ be the tsp graph with nodes $\mathbb{N}^{n_l}_{\hat{d}-d_j}$ associated

² Though we only include inequality constraints in the definition of \mathbf{K} (2.2) for the sake of simplicity, equality constraints can be treated in a similar way.

to \mathscr{A}_l defined as in Sec. 2.5. Note that we embed $\mathbb{N}^{n_l}_{\hat{d}-d_j}$ into $\mathbb{N}^n_{\hat{d}-d_j}$ via the map $\boldsymbol{\alpha}=(\alpha_i)\in\mathbb{N}^{n_l}_{\hat{d}-d_j}\mapsto \boldsymbol{\alpha}'=(\alpha_i')\in\mathbb{N}^n_{\hat{d}-d_j}$ which satisfies

$$\alpha_i' = \begin{cases} \alpha_i, & \text{if } i \in I_l, \\ 0, & \text{otherwise.} \end{cases}$$

Assume that $G_{\hat{d},l,0}^{(0)}=G_{\hat{d},l}^{\mathrm{tsp}}$ and $G_{\hat{d},l,j}^{(0)}, j\in J_l, l=1,\ldots,p$ are empty graphs. Letting

(3.2)
$$\mathscr{C}_{\hat{d}}^{(k-1)} := \bigcup_{l=1}^{p} \bigcup_{j \in \{0\} \cup J_{l}} (\operatorname{supp}(g_{j}) + \operatorname{supp}(G_{\hat{d},l,j}^{(k-1)})), \quad k \geq 1,$$

we recursively define a sequence of graphs $(G_{\hat{d},l,j}^{(k)}(V_{\hat{d},l,j},E_{\hat{d},l,j}^{(k)}))_{k\geq 1}$ with $V_{\hat{d},l,j}=\mathbb{N}_{\hat{d}-d_j}^{n_l}$ for $j\in\{0\}\cup J_l, l=1,\ldots,p$ by

(3.3)
$$G_{\hat{d},l,j}^{(k)} := \overline{F_{\hat{d},l,j}^{(k)}},$$

where $F_{\hat{d},l,j}^{(k)}$ is the graph with $V(F_{\hat{d},l,j}^{(k)}) = \mathbb{N}_{\hat{d}-d_i}^{n_l}$ and

(3.4)
$$E(F_{\hat{d},l,j}^{(k)}) = \{ \{ \boldsymbol{\beta}, \boldsymbol{\gamma} \} \mid (\operatorname{supp}(g_j) + \boldsymbol{\beta} + \boldsymbol{\gamma}) \cap \mathscr{C}_{\hat{d}}^{(k-1)} \neq \emptyset \}.$$

Example 3.1. Let $f = 1 + x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_3$ and consider the unconstrained POP: $\min\{f(\mathbf{x}): \mathbf{x} \in \mathbb{R}^n\}$. The variables is then partitioned into two cliques: $\{x_1, x_2\}$ and $\{x_2, x_3\}$. The tsp graphs for these two cliques are illustrated in Figure 3.1 (the left (resp. right) graph corresponds to the first (resp. second) clique). If we apply the TSSOS hierarchy (using the maximal chordal extension in (3.3)) separately for each clique, then the graph sequences $(G_{1,l}^{(k)})_{k\geq 1}, l=1,2$ (the subscript j is omitted since there are no constraints) stabilize at k=1. However, the added (dashed) edge in the right graph corresponds to the monomial x_2 , which only involves the variable x_2 belonging to the first clique. Hence we need to add the edge connecting 1 and x_2 to the left graph in order to get the guarantee of convergence as we shall see in Sec. 4.1. Consequently, the graph sequences $(G_{1,l}^{(k)})_{k\geq 1}, l=1,2$ stabilize at k=2.

FIGURE 1. The tsp graphs of Example 3.1



Each node has a self-loop which is not displayed for simplicity. The dashed edge is added after the maximal chordal extension.

Let $r_{l,j} := \binom{n_l + \hat{d} - d_j}{\hat{d} - d_j}$ for all l, j. Then for each $k \ge 1$, the moment-SOS hierarchy based on combined correlative and term sparsities for (2.1) is defined as:

(3.5)
$$(\mathbf{Q}_{\hat{d},k}^{\text{cs-ts}}) : \begin{cases} \inf & L_{\mathbf{y}}(f) \\ \text{s.t.} & M_{\hat{d}}(\mathbf{y}, I_{l}) \in \Pi_{G_{\hat{d},l,0}^{(k)}}(\mathbf{S}_{+}^{r_{l,0}}), \quad l = 1, \dots, p, \\ & M_{\hat{d}-d_{j}}(g_{j}\mathbf{y}, I_{l}) \in \Pi_{G_{\hat{d},l,j}^{(k)}}(\mathbf{S}_{+}^{r_{l,j}}), \quad j \in J_{l}, l = 1, \dots, p, \\ & y_{\mathbf{0}} = 1, \end{cases}$$

with optimal value denoted by $\rho_{\hat{d}}^{(k)}$.

Proposition 3.2. For any $\hat{d} \geq d$, the sequence $(\rho_{\hat{d}}^{(k)})_{k\geq 1}$ is monotone non-decreasing and $\rho_{\hat{d}}^{(k)} \leq \rho_{\hat{d}}$ for all k.

Proof. By construction, we have $G_{\hat{d},l,j}^{(k)} \subseteq G_{\hat{d},l,j}^{(k+1)}$ for all \hat{d},l,j and all k. It follows that each maximal clique of $G_{\hat{d},l,j}^{(k)}$ is a subset of some maximal clique of $G_{\hat{d},l,j}^{(k+1)}$. Hence by Theorem 2.3, $(Q_{\hat{d},k}^{\text{cs-ts}})$ is a relaxation of $(Q_{\hat{d},k+1}^{\text{cs-ts}})$ and is clearly also a relaxation of $(Q_{\hat{d}}^{\text{cs}})$. Therefore, $(\rho_{\hat{d}}^{(k)})_{k\geq 1}$ is monotone non-decreasing and $\rho_{\hat{d}}^{(k)} \leq \rho_{\hat{d}}$ for all k.

Proposition 3.3. For any $k \geq 1$, the sequence $(\rho_{\hat{d}}^{(k)})_{\hat{d} > d}$ is monotone non-decreasing.

Proof. The conclusion follows if we can show that $G_{\hat{d},l,j}^{(k)}\subseteq G_{\hat{d}+1,l,j}^{(k)}$ for all l,j,\hat{d},k since by Theorem 2.3 this implies that $(Q_{\hat{d},k}^{\text{cs-ts}})$ is a relaxation of $(Q_{\hat{d}+1,k}^{\text{cs-ts}})$. Let us prove $G_{\hat{d},l,j}^{(k)}\subseteq G_{\hat{d}+1,l,j}^{(k)}$ by induction on k. For k=1, from (2.10), we have $G_{\hat{d},l,0}^{(0)}=G_{\hat{d},l,0}^{\text{tsp}}\subseteq G_{\hat{d}+1,l,0}^{(0)}=G_{\hat{d}+1,l,0}^{\text{tsp}}$ which together with (3.4) implies that $F_{\hat{d},l,j}^{(1)}\subseteq F_{\hat{d}+1,l,j}^{(1)}$ for $j\in\{0\}\cup J_l, l=1,\ldots,p$. It follows that $G_{\hat{d},l,j}^{(1)}=\overline{F_{\hat{d},l,j}^{(1)}}\subseteq G_{\hat{d}+1,l,j}^{(1)}=\overline{F_{\hat{d}+1,l,j}^{(1)}}$. Now assume that $G_{\hat{d},l,j}^{(k)}\subseteq G_{\hat{d}+1,l,j}^{(k)}$, $j\in\{0\}\cup J_l, l=1,\ldots,p$, holds for some $k\geq 1$. Then from (3.4) and by the induction hypothesis, we have $F_{\hat{d},l,j}^{(k+1)}\subseteq F_{\hat{d}+1,l,j}^{(k+1)}$ for $j\in\{0\}\cup J_l, l=1,\ldots,p$. Thus $G_{\hat{d},l,j}^{(k+1)}=\overline{F_{\hat{d},l,j}^{(k+1)}}\subseteq G_{\hat{d}+1,l,j}^{(k+1)}=\overline{F_{\hat{d}+1,l,j}^{(k+1)}}$ which completes the induction.

From Proposition 3.2 and Proposition 3.3, we deduce the following two-level hierarchy of lower bounds for the optimum of (Q) (2.1):

The array of lower bounds (3.6) (and its associated SDP relaxations (3.5)) is what we call the CS-TSSOS moment-SOS hierarchy (in short CS-TSSOS hierarchy) associated to (Q).

Example 3.4. Let $f = 1 + \sum_{i=1}^{6} x_i^4 + x_1x_2x_3 + x_3x_4x_5 + x_3x_4x_6 + x_3x_5x_6 + x_4x_5x_6$, and consider the unconstrained POP: $\min\{f(\mathbf{x}): \mathbf{x} \in \mathbb{R}^n\}$. Apply the CSTSSOS hierarchy (using the maximal chordal extension in (3.3)) to this problem. First, using the csp graph (see Figure 2), partition variables into the two cliques $\{x_1, x_2, x_3\}$ and $\{x_3, x_4, x_5, x_6\}$. Figure 3 and Figure 4 illustrate the tsp graphs for the first clique and the second clique respectively. For the first clique one obtains four blocks of SDP matrices with respective sizes 4, 2, 2, 2. For the second clique one obtains two blocks of SDP matrices with respective sizes 5, 10. Thus the original size 28 of the SDP matrix has been reduced to a maximal size of 10.

If one applies the TSSOS hierarchy (using the maximal chordal extension in (3.3)) directly to the problem (i.e., without partitioning variables), then the tsp graph is illustrated in Figure 5. One obtains five blocks of SDP matrices with respective size 7, 2, 2, 2, 10. Compared to the CS-TSSOS case, the two blocks of SDP matrices with respective sizes 4, 5 are replaced by a single block SDP matrix with size 7.

FIGURE 2. The csp graph of Example 3.4

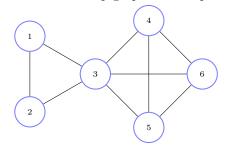
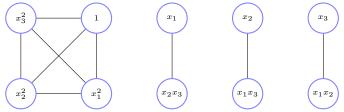


FIGURE 3. The tsp graph for the first clique of Example 3.4



Each node has a self-loop which is not displayed for simplicity.

The CS-TSSOS hierarchy entails a trade-off. One has the freedom to choose a specific chordal extension for any graph involved in (3.5). This choice affects the resulting size of (submatrix) blocks and the quality of optimal values of the corresponding CS-TSSOS hierarchy. Intuitively, chordal extensions with less edges should lead to (submatrix) blocks of smaller size and optimal values of (possibly)

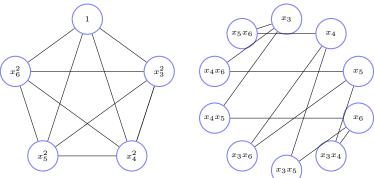
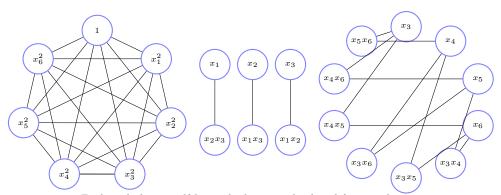


FIGURE 4. The tsp graph for the second clique of Example 3.4

Each node has a self-loop which is not displayed for simplicity.

FIGURE 5. The tsp graph without partitioning variables of Example 3.4



Each node has a self-loop which is not displayed for simplicity.

lower quality while chordal extensions with more edges should lead to (submatrix) blocks with larger size and optimal values of (possibly) higher quality.

For any l,j, write $M_{\hat{d}-d_j}(g_j\mathbf{y},I_l)=\sum_{\boldsymbol{\alpha}}D_{\boldsymbol{\alpha}}^{l,j}y_{\boldsymbol{\alpha}}$ for appropriate (0,1)-binary matrices $\{D_{\boldsymbol{\alpha}}^{l,j}\}$. Then for each $k\geq 1$, the dual of $(\mathbf{Q}_{\hat{d},k}^{\text{cs-ts}})$ reads:

$$(3.7) \quad (\mathbf{Q}_{\hat{d},k}^{\text{cs-ts}})^* : \begin{cases} \sup & \rho \\ \text{s.t.} & \sum_{l=1}^p \sum_{j \in \{0\} \cup J_l} \langle Q_{l,j}, D_{\boldsymbol{\alpha}}^{l,j} \rangle + \rho \delta_{\mathbf{0}\boldsymbol{\alpha}} = f_{\boldsymbol{\alpha}}, \quad \forall \boldsymbol{\alpha} \in \mathscr{C}_{\hat{d}}^{(k)}, \\ Q_{l,j} \in \mathbf{S}_{+}^{r_{l,j}} \cap \mathbf{S}_{G_{\hat{d},l,j}^{(k)}}, & j \in \{0\} \cup J_l, l = 1, \dots, p, \end{cases}$$

where $\mathscr{C}_{\hat{d}}^{(k)}$ is defined as in (3.2).

Proposition 3.5. Let $f \in \mathbb{R}[\mathbf{x}]$ and \mathbf{K} be as in (2.2). Assume that K has a nonempty interior. Then there is no duality gap between $(Q_{\hat{d},k}^{cs-ts})$ and $(Q_{\hat{d},k}^{cs-ts})^*$ for any $\hat{d} \geq d$ and $k \geq 1$.

Proof. By the duality theory of convex programming, this easily follows from Theorem 3.6 of [Las06] and Theorem 2.3.

Note that the number of equality constraints in (3.7) is equal to the cardinal of $\mathscr{C}_{\hat{j}}^{(k)}$. We next give a description of the elements in $\mathscr{C}_{\hat{j}}^{(k)}$ in terms of sign-symmetries.

3.2. Sign-symmetries.

Definition 3.6. Given a finite set $\mathscr{A} \subseteq \mathbb{N}^n$, the sign-symmetries of \mathscr{A} are defined by all vectors $\mathbf{r} \in \mathbb{Z}_2^n := \{0,1\}^n$ such that $\mathbf{r}^T \boldsymbol{\alpha} \equiv 0 \pmod{2}$ for all $\boldsymbol{\alpha} \in \mathscr{A}$.

For any $\alpha \in \mathbb{N}^n$, we define $(\alpha)_2 := (\alpha_1 \pmod{2}, \dots, \alpha_n \pmod{2}) \in \mathbb{Z}_2^n$. We use the same notation for any subset $\mathscr{A} \subseteq \mathbb{N}^n$, i.e. $(\mathscr{A})_2 = \{\alpha_2 \mid \alpha \in \mathscr{A}\} \subseteq \mathbb{Z}_2^n$. For a subset $S \subseteq \mathbb{Z}_2^n$, the *orthogonal complement space* of S in \mathbb{Z}_2^n , denoted by S^{\perp} , is the set $\{\alpha \in \mathbb{Z}_2^n \mid \alpha^T \mathbf{s} \equiv 0 \pmod{2}, \forall \mathbf{s} \in S\}$.

Remark 3.7. By definition, the set of sign-symmetries of $\mathscr A$ is just the orthogonal complement space $(\mathscr A)_2^{\perp}$ in $\mathbb Z_2^n$. Hence the sign-symmetries of $\mathscr A$ can be essentially represented by a basis of the subspace $(\mathscr A)_2^{\perp}$ in $\mathbb Z_2^n$.

For a subset $S \subseteq \mathbb{Z}_2^n$, we say that S is closed under addition modulo 2 if $\mathbf{s}_1, \mathbf{s}_2 \in S$ implies $(\mathbf{s}_1 + \mathbf{s}_2)_2 \in S$. The minimal set containing S with elements which are closed under addition modulo 2 is denoted by \overline{S} . Note that $\overline{S} \subseteq \mathbb{Z}_2^n$ is just the subspace spanned by S in \mathbb{Z}_2^n . This is because the subspace spanned by S is $\{(\sum_i \mathbf{s}_i)_2 \mid \mathbf{s}_i \in S\}$ which is closed under addition modulo 2.

Lemma 3.8. Let $S \subseteq \mathbb{Z}_2^n$. Then $(S^{\perp})^{\perp} = \overline{S}$.

Proof. It is immediate from the definitions.

Lemma 3.9. Suppose G is a graph with $V \subseteq \mathbb{N}^n$. Then $(\operatorname{supp}(\overline{G}))_2 \subseteq \overline{(\operatorname{supp}(G))_2}$.

Proof. By definition, for any $\{\beta, \gamma\} \in E(\overline{G})$, we need to show $(\beta+\gamma)_2 \in \overline{(\operatorname{supp}(G))_2}$. Since in the process of chordal extensions, edges are added only if two nodes belong to the same connected component, there is a path connecting β and γ in $G: \{\beta, v_1, \ldots, v_r, \gamma\}$ with $\{\beta, v_1\}, \{v_r, \gamma\} \in E(G)$ and $\{v_i, v_{i+1}\} \in E(G), i = 1, \ldots, r-1$. From $(\beta+v_1)_2, (v_1+v_2)_2 \in (\operatorname{supp}(G))_2$, we deduce that $(\beta+v_2)_2 \in \overline{(\operatorname{supp}(G))_2}$ because $\overline{(\operatorname{supp}(G))_2}$ is closed under addition modulo 2. Likewise, we can prove $\underline{(\beta+v_i)_2} \in \overline{(\operatorname{supp}(G))_2}$ for $i=3,\ldots,r$ with $v_{r+1}:=\gamma$. Hence $(\beta+\gamma)_2 \in \overline{(\operatorname{supp}(G))_2}$ as desired.

Proposition 3.10. Let \mathscr{A} be defined as in (3.1), $\mathscr{C}^{(k)}_{\hat{d}}$ be defined as in (3.2) and assume that the sign-symmetries of \mathscr{A} are represented by the columns of a binary matrix, denoted by R. Then for any $k \geq 1$ and any $\alpha \in \mathscr{C}^{(k)}_{\hat{d}}$, one has $R^T \alpha \equiv 0 \pmod{2}$. In other words, $(\mathscr{C}^{(k)}_{\hat{d}})_2 \subseteq R^{\perp}$.

Proof. By Lemma 3.8, we only need to prove $(\mathscr{C}_{\hat{d}}^{(k)})_2 \subseteq \overline{(\mathscr{A})_2}$. Let us do induction on $k \geq 0$. For k = 0, by (3.2), $\mathscr{C}_{\hat{d}}^{(0)} = \bigcup_{l=1}^p \operatorname{supp}(G_{\hat{d},l,0}^{(0)}) = \bigcup_{l=1}^p \operatorname{supp}(G_{\hat{d},l}^{\operatorname{tsp}}) \subseteq \bigcup_{l=1}^p (\mathscr{A}_l \cup \mathbb{N}_{\hat{d}}^{n_l}) = \mathscr{A} \cup (2\mathbb{N})^n$. Hence $(\mathscr{C}_{\hat{d}}^{(0)})_2 \subseteq \overline{\mathscr{A}_2}$. Now assume that $(\mathscr{C}_{\hat{d}}^{(k)})_2 \subseteq \overline{\mathscr{A}_2}$ holds for some $k \geq 0$. By (3.4), for any l, j and any $\{\beta, \gamma\} \in E(F_{\hat{d},l,j}^{(k+1)})$, we have $(\operatorname{supp}(g_j) + \beta + \gamma) \cap \mathscr{C}_{\hat{d}}^{(k)} \neq \emptyset$, i.e., there exists $\alpha \in \operatorname{supp}(g_j)$ such that $\alpha + \beta + \beta = \emptyset$

 $\gamma \in \mathscr{C}^{(k)}_{\hat{d}}$, which implies $(\alpha + \beta + \gamma)_2 \in (\mathscr{C}^{(k)}_{\hat{d}})_2$. Hence by the induction hypothesis, $(\alpha + \beta + \gamma)_2 \in \overline{\mathscr{A}}_2$. Since $\overline{\mathscr{A}}_2$ is closed under addition modulo 2 and $(\alpha)_2 \in \mathscr{A}_2$, we have $(\beta + \gamma)_2 \in \overline{\mathscr{A}}_2$. It follows that $(\sup(F^{(k+1)}_{\hat{d},l,j}))_2 \subseteq \overline{\mathscr{A}}_2$. Because $G^{(k+1)}_{\hat{d},l,j} = \overline{F^{(k+1)}_{\hat{d},l,j}}$, by Lemma 3.9, we have $(\sup(G^{(k+1)}_{\hat{d},l,j}))_2 \subseteq \overline{(\sup(F^{(k+1)}_{\hat{d},l,j}))_2} \subseteq \overline{\mathscr{A}}_2$. From this, (3.2) and the fact that $\overline{\mathscr{A}}_2$ is closed under addition modulo 2, we then deduce that $(\mathscr{C}^{(k+1)}_{\hat{d}})_2 \subseteq \overline{\mathscr{A}}_2$ which completes the induction. \square

4. Convergence analysis

4.1. **Global convergence.** We next prove that if for any graph involved in (3.5), the chordal extension is chosen to be maximal, then for fixed relaxation order \hat{d} the sequence of optimal values $(\rho_{\hat{d}}^{(k)})_{k\geq 1}$ of the CS-TSSOS hierarchy converges to the optimal value $\rho_{\hat{d}}$ of the corresponding CSSOS hierarchy (2.9). In turn, as \hat{d} increases, the latter sequence converges to the global optimum ρ^* of the original POP (2.1) (after adding some redundant quadratic constraints) as shown in [Las06].

Obviously, the sequence of graphs $(G_{\hat{d},l,j}^{(k)}(V_{\hat{d},l,j},E_{\hat{d},l,j}^{(k)}))_{k\geq 1}$ stabilizes for all $j\in\{0\}\cup J_l, l=1,\ldots,p$ after finitely many steps. We denote the resulting stabilized graphs by $G_{\hat{d},l,j}^{(*)}, j\in\{0\}\cup J_l, l=1,\ldots,p$ and the corresponding SDP (3.5) by $(Q_{\hat{d},*}^{\text{cs-ts}})$.

Theorem 4.1. Assume that the chordal extension in (3.3) is the maximal chordal extension. Then for any $\hat{d} \geq d$, the sequence $(\rho_{\hat{d}}^{(k)})_{k\geq 1}$ converges to $\rho_{\hat{d}}$ in finitely many steps.

Proof. Let $\mathbf{y} = (y_{\alpha})$ be an arbitrary feasible solution of $(\mathbf{Q}^{\text{cs-ts}}_{\hat{d},*})$ and $\rho^*_{\hat{d}}$ be the optimal value of $(\mathbf{Q}^{\text{cs-ts}}_{\hat{d},*})$. Note that $\{y_{\alpha} \mid \alpha \in \cup_{l=1}^p \cup_{j \in \{0\} \cup J_l} (\text{supp}(g_j) + \text{supp}(G^{(*)}_{\hat{d},l,j}))\}$ is the set of variables effectively appearing in $(\mathbf{Q}^{\text{cs}}_{\hat{d},*})$. Let \mathscr{R} be the set of variables effectively appearing in $(\mathbf{Q}^{\text{cs}}_{\hat{d}})$ (2.9). We then define a vector $\overline{\mathbf{y}} = (\overline{y}_{\alpha})_{\alpha \in \mathscr{R}}$ as follows:

$$\overline{y}_{\alpha} = \begin{cases} y_{\alpha}, & \text{if } \alpha \in \bigcup_{l=1}^{p} \bigcup_{j \in \{0\} \cup J_{l}} \left(\text{supp}(g_{j}) + \text{supp}(G_{\hat{d},l,j}^{(*)}) \right), \\ 0, & \text{otherwise.} \end{cases}$$

By construction and by the fact that $M_{\hat{d}-d_j}(g_j\mathbf{y},I_l)$ is block diagonal (up to permutation), we have that $M_{\hat{d}-d_j}(g_j\mathbf{y},I_l)\in\Pi_{G_{l,j,\hat{d}}^{(*)}}(\mathbf{S}_+^{r_{l,j}})$ implies $M_{\hat{d}-d_j}(g_j\overline{\mathbf{y}},I_l)\succeq 0$ for $j\in\{0\}\cup J_l, l=1,\ldots,p$. Therefore $\overline{\mathbf{y}}$ is a feasible solution of $(\mathbf{Q}_{\hat{d}}^{\mathrm{cs}})$ and hence $L_{\mathbf{y}}(f)=L_{\overline{\mathbf{y}}}(f)\geq\rho_{\hat{d}}$. Hence $\rho_{\hat{d}}^*\geq\rho_{\hat{d}}$ since \mathbf{y} is an arbitrary feasible solution of $(\mathbf{Q}_{\hat{d},*}^{\mathrm{cs-ts}})$. By Proposition 3.2, we already have $\rho_{\hat{d}}^*\leq\rho_{\hat{d}}$. Therefore, $\rho_{\hat{d}}^*=\rho_{\hat{d}}$.

To guarantee the global optimality, consider the following compactness assumption on the feasible set.

Assumption 1. Let **K** be as in (2.2). There exists an M > 0 such that $||\mathbf{x}||_{\infty} < M$ for all $\mathbf{x} \in \mathbf{K}$.

Because of Assumption 1, one has $||\mathbf{x}(I_k)||^2 \le n_k M^2$, k = 1, ..., p. Therefore, we can add the p redundant quadratic constraints

(4.1)
$$q_{m+k}(\mathbf{x}) := n_k M^2 - ||\mathbf{x}(I_k)||^2 > 0, \quad k = 1, \dots, p$$

in the definition (2.2) of **K** and set m' = m + p, so that **K** is now defined by

(4.2)
$$\mathbf{K} := \{ \mathbf{x} \in \mathbb{R}^n \mid g_j \ge 0, \quad j = 1, \dots, m' \}.$$

Note that $g_{m+k} \in \mathbb{R}[\mathbf{x}(I_k)]$ for $k = 1, \dots, p$.

Then by Theorem 3.6 in [Las06], the sequence $(\rho_{\hat{d}})_{\hat{d} \geq d}$ converges to the globally optimal value ρ^* of Q (2.1).

4.2. A sparse representation theorem. Proceeding along Theorem 4.1, we next provide a *sparse representation* theorem for a polynomial positive on a compact basic semialgebraic set.

Theorem 4.2 (sparse representation). Let $f \in \mathbb{R}[\mathbf{x}]$ and \mathbf{K} be as in (4.2) with the additional quadratic constraints (4.1). Let I_l , J_l be defined as in Sec. 3.1 and $\mathscr{A} = \operatorname{supp}(f) \cup \bigcup_{j=1}^m \operatorname{supp}(g_j)$. Assume that the sign-symmetries of \mathscr{A} are represented by the columns of the binary matrix R. If f is positive on \mathbf{K} , then

(4.3)
$$f = \sum_{l=1}^{p} (\sigma_{l,0} + \sum_{j \in J_l} \sigma_{l,j} g_j),$$

for some polynomials $\sigma_{l,j} \in \Sigma[\mathbf{x}(I_l)], j \in \{0\} \cup J_l, l = 1, ..., p$, satisfying $R^T \boldsymbol{\alpha} \equiv 0$ (mod 2) for any $\boldsymbol{\alpha} \in \operatorname{supp}(\sigma_{l,j})$, i.e., $\operatorname{supp}(\sigma_{l,j})_2 \subseteq R^{\perp}$.

That is, (4.3) provides a certificate of positivity of f on K.

Proof. By Corollary 3.9 of [Las06] (or Theorem 5 of [GNS07]), there exist polynomials $\sigma'_{l,j} \in \Sigma[\mathbf{x}(I_l)], j \in \{0\} \cup J_l, l = 1, \ldots, p$ such that

(4.4)
$$f = \sum_{l=1}^{p} (\sigma'_{l,0} + \sum_{j \in J_l} \sigma'_{l,j} g_j).$$

Let $d = \max\{\lceil \deg(\sigma'_{l,j}g_j)/2 \rceil : j \in \{0\} \cup J_l, l = 1, \ldots, p\}$. Let $Q'_{l,j}$ be a Gram matrix associated to $\sigma'_{l,j}$ and indexed by the monomial basis $\mathbb{N}^{n_l}_{d-d_j}$. Then for all l, j, we define $Q_{l,j} \in \mathbf{S}^{r_{l,j}}$ with $r_{l,j} = \binom{n_l + \hat{d} - d_j}{\hat{d} - d_j}$ (indexed by $\mathbb{N}^{n_l}_{d-d_j}$) by

$$[Q_{l,j}]_{\boldsymbol{\beta}\boldsymbol{\gamma}} := \begin{cases} [Q'_{l,j}]_{\boldsymbol{\beta}\boldsymbol{\gamma}}, & \text{if } R^T(\boldsymbol{\beta} + \boldsymbol{\gamma}) \equiv 0 \text{ (mod 2)}, \\ 0, & \text{otherwise,} \end{cases}$$

and let $\sigma_{l,j} = (\mathbf{x}^{\mathbb{N}_{d-d_j}^{n_l}})^T Q_{l,j} \mathbf{x}^{\mathbb{N}_{d-d_j}^{n_l}}$. One can easily verify that $Q_{l,j}$ is block diagonal up to permutation (also see [WML19]) and each block is a principal submatrix of $Q'_{l,j}$. Then the positive semidefiniteness of $Q'_{l,j}$ implies that $Q_{l,j}$ is also positive semidefinite. Thus $\sigma_{l,j} \in \Sigma[\mathbf{x}(I_l)]$.

By construction, substituting $\sigma'_{l,j}$ by $\sigma_{l,j}$ in (4.4) boils down to removing the terms with exponent α that do not satisfy $R^T \alpha \equiv 0 \pmod{2}$ from the right hand side of (4.4). Since any $\alpha \in \text{supp}(f)$ satisfies $R^T \alpha \equiv 0 \pmod{2}$, this does not change the match of coefficients on both sides of the equality. Thus we obtain

$$f = \sum_{l=1}^{p} (\sigma_{l,0} + \sum_{j \in J_l} \sigma_{l,j} g_j),$$

with the desired property.

4.3. Extracting a solution. In the case of dense moment-SOS relaxations, there is a standard procedure described in [HL05] to extract globally optimal solutions when the so-called flatness condition is satisfied and this procedure is also generalized to the correlative sparse setting in [Las06, § 3.3]. However, in our combined sparsity setting, the corresponding procedure cannot be directly applied because the moment matrix associated to each clique does not involve enough moment variables. In order to extract a solution, we may add an order-one (dense) moment matrix for each clique in (3.5):

(4.5)
$$(Q_{\hat{d},k}^{\text{cs-ts}})'$$
:
$$\begin{cases} \inf & L_{\mathbf{y}}(f) \\ \text{s.t.} & M_{\hat{d}}(\mathbf{y}, I_{l}) \in \Pi_{G_{\hat{d},l,0}^{(k)}}(\mathbf{S}_{+}^{r_{l,0}}), \quad l = 1, \dots, p, \\ M_{1}(\mathbf{y}, I_{l}) \succeq 0, \quad l = 1, \dots, p, \\ M_{\hat{d}-d_{j}}(g_{j}\mathbf{y}, I_{l}) \in \Pi_{G_{\hat{d},l,j}^{(k)}}(\mathbf{S}_{+}^{r_{l,j}}), \quad j \in J_{l}, l = 1, \dots, p, \\ y_{0} = 1. \end{cases}$$

Let \mathbf{y}^* be an optimal solution of $(\mathbf{Q}_{\hat{d},k}^{\text{cs-ts}})'$. Typically, $M_1(\mathbf{y}^*, I_l)$ (after identifying sufficiently small entries with zero) is a block diagonal matrix (up to permutation). If for all l, every block of $M_1(\mathbf{y}^*, I_l)$) (see [Las06, Theorem 3.7]) has rank one, then a globally optimal solution \mathbf{x}^* to Q (2.1) can be extracted. At the same time, the global optimality is certified. Otherwise, the relaxation might be not exact or yield multiple global solutions. In the latter case, adding a small perturbation to the objective function, as in [WKKM06], may yield a unique global solution.

Remark 4.3. Note that $(Q_{\hat{d},k}^{cs-ts})'$ is a stronger relaxation of (Q) than $(Q_{\hat{d},k}^{cs-ts})$. Therefore, even if the globally optimal value is not achieved, $(Q_{\hat{d},k}^{cs-ts})'$ still provides a better lower bound for (Q) than $(Q_{\hat{d},k}^{cs-ts})$. If (Q) is a QCQP, then $(Q_{\hat{d},k}^{cs-ts})'$ is always stronger than Shor relaxation for (Q).

5. Applications and numerical experiments

In this section, we conduct numerical experiments for the proposed CS-TSSOS hierarchies (3.5)-(3.7) and apply them to two important classes of POPs: Max-Cut problems and AC optimal power flow (AC-OPF) problems. The tool TSSOS which executes the CS-TSSOS hierarchy is implemented in Julia, and constructs instances of the dual SDP problems (3.7), then relies on Mosek [ApS17] to solve them. TSSOS is available on the website:

https://github.com/wangjie212/TSSOS.

All numerical examples were computed on an Intel Core i5-8265U@1.60GHz CPU with 8GB RAM memory and the WINDOWS 10 system. The timings includes the time required to model SDP and the time spent to solve it.

The notation that we use are listed in Table 5.

We consider two types of chordal extensions: the maximal chordal extension and the approximately minimal chordal extension, aiming to minimize the number of edges added. There are different ways to generate an approximately minimal chordal extension. In TSSOS, one has two choices: the one based on minimum degree heuristic which relies on the Matlab function "amd" and the one based on greedy fill-in heuristic used in the Julia package SumsOfSquares [WLC+19]. For correlative sparsity patterns, the approximately minimal chordal extension is

var	the number of variables
cons	the number of constraints
mc	the maximal size of maximal cliques
mb	the maximal size of blocks of SDP matrices
opt	the optimal value
time	the running time in seconds
gap	the optimality gap
CE	the method to generate a chordal extension in (3.3)
min	the approximately minimal chordal extension
max	the maximal chordal extension
-	out of memory

Table 1. The notation

always used. For term sparsity patterns in (3.3), we use either the maximal chordal extension or the approximately minimal chordal extension. The most appropriate choice depends on the problem.

5.1. **Benchmarks for unconstrained POPs.** The Broyden banded function is defined as

$$f_{\text{Bb}}(\mathbf{x}) = \sum_{i=1}^{n} (x_i(2+5x_i^2) + 1 - \sum_{j \in J_i} (1+x_j)x_j)^2,$$

where $J_i = \{j \mid j \neq i, \max(1, i - 5) \leq j \leq \min(n, i + 1)\}.$

The task is to minimize the Broyden banded function over \mathbb{R}^n which is formulated as an unconstrained POP. We execute the CSSOS hierarchy $(Q_{\hat{d}}^{cs})$ (2.9) $(\hat{d}=3)$ and the CS-TSSOS hierarchy $(\hat{d}=3)$ with sparse order k=1, where the approximately minimal chordal extension is used in (3.3). For these problems, CS-TSSOS reduces the maximal size of SDP matrices from 120 to 22. The results are displayed in Table 2.

We observe from Table 2 that when $n \leq 100$, the CS-TSSOS hierarchy is nearly five times faster than the CSSOS hierarchy; when $n \geq 120$, the CSSOS hierarchy is running out of memory while the CS-TSSOS hierarchy can easily handle up to 400 variables.

5.2. Benchmarks for constrained POPs.

• The generalized Rosenbrock function

$$f_{\rm gR}(\mathbf{x}) = 1 + \sum_{i=1}^{n} (100(x_i - x_{i-1}^2)^2 + (1 - x_i)^2).$$

• The Broyden tridiagonal function

$$f_{\text{Bt}}(\mathbf{x}) = ((3 - 2x_1)x_1 - 2x_2 + 1)^2 + \sum_{i=1}^{n-1} ((3 - 2x_i)x_i - x_{i-1} - 2x_{i+1} + 1)^2 + ((3 - 2x_n)x_n - x_{n-1} + 1)^2.$$

var	$_{ m mc}$	CS	SSOS (3	nd)	CS-TSSOS (3)					
	mc	mb	opt	time	mb	opt	time			
20	7	120	3.4e-6	20.3	22	2.6e-7	3.87			
40	7	120	1.9e-9	49.9	22	1.2e-7	11.2			
60	7	120	1.7e-8	86.3	22	1.6e-7	17.8			
80	7	120	1.6e-6	128	22	1.5e-7	27.3			
100	7	120	2.4e-7	165	22	1.3e-6	34.8			
120	7	120	-	-	22	7.1e-9	48.0			
140	7	120	-	-	22	5.4e-7	58.5			
160	7	120	-	-	22	1.2e-6	70.1			
180	7	120	-	-	22	1.6e-7	87.2			
200	7	120	-	-	22	4.0e-7	109			
250	7	120	-	-	22	6.1e-7	135			
300	7	120	-	-	22	1.9e-7	186			
400	7	120	-	-	22	4.2e-6	288			

Table 2. The result for Broyden banded functions

• The chained Wood function

$$f_{cW}(\mathbf{x}) = 1 + \sum_{i \in J} (100(x_{i+1} - x_i^2)^2 + (1 - x_i)^2 + 90(x_{i+3} - x_{i+2}^2)^2 + (1 - x_{i+2})^2 + 10(x_{i+1} + x_{i+3} - 2)^2 + 0.1(x_{i+1} - x_{i+3})^2),$$

where $J = \{1, 3, 5, \dots, n-3\}$ and 4|n.

With the generalized Rosenbrock (resp. Broyden tridiagonal or chained Wood) function as the objective function, we consider the following constrained POP:

(5.1)
$$\begin{cases} \inf & f_{gR} \quad (\text{resp. } f_{Bt} \text{ or } f_{cW}) \\ \text{s.t.} & 1 - \left(\sum_{i=20j-19}^{20j} x_i^2\right) \ge 0, \quad j = 1, 2, \dots, n/20, \end{cases}$$

where 20|n. The generalized Rosenbrock function, the Broyden tridiagonal function and the chained Wood function involve cliques with 2 or 3 variables, which can be efficiently handled by the CSSOS hierarchy; see [WKKM06]. For them, the CSTSSOS hierarchy gives the same results with the CSSOS hierarchy. Hence we add the sphere constraints to increase the clique size and to show the difference.

We execute the second-order CSSOS hierarchy and the second-order CS-TSSOS hierarchy of sparse order k=1 where the approximately minimal chordal extension is used in (3.3). For these problems, CS-TSSOS reduces the maximal size of SDP matrices from 231 to 21 or 24. The results are displayed in Table 3, Table 4 and Table 5.

As for unconstrained problems, Table 3 indicates that when $n \leq 80$, the CS-TSSOS hierarchy is over 50 times faster than the CSSOS hierarchy; when $n \geq 100$, the CSSOS hierarchy runs out of memory while the CS-TSSOS hierarchy can tackle up to thousand variables. The relative gap between the value of the CSSOS hierarchy and the value of the CS-TSSOS hierarchy is less than 3e-5.

Similarly, from Table 4 (resp. Table 5), we see that when $n \le 80$, the CS-TSSOS hierarchy is nearly 100 (resp. 200) times faster than the CSSOS hierarchy; when $n \ge 100$, the CSSOS hierarchy runs out of memory while CS-TSSOS scales to

thousand variables. The relative gap between the values of CSSOS and CS-TSSOS is less than 4e-4 (resp. 1e-6).

Table 3. The result for the generalized Rosenbrock function

var	mc	C	CSSOS (2n	id)	CS-TSSOS (2nd)			
vai	IIIC	$^{\mathrm{mb}}$	opt	time	mb	opt	time	
40	20	231	38.0513	199	21	38.0508	3.24	
60	20	231	57.8493	324	21	57.8481	5.96	
80	20	231	77.6471	421	21	77.6454	8.24	
100	20	231	-	-	21	97.4425	9.36	
120	20	231	-	-	21	117.240	11.1	
140	20	231	-	-	21	137.037	14.5	
160	20	231	-	-	21	156.835	16.5	
180	20	231	-	-	21	176.632	19.0	
200	20	231	-	-	21	196.429	23.2	
300	20	231	-	-	21	295.416	40.1	
400	20	231	=	_	21	394.403	61.9	
500	20	231	=	_	21	493.389	81.3	
1000	20	231	-	-	21	988.320	245	

Table 4. The result for the Broyden tridiagonal function

var	mc	C	SSOS (2n	.d)	CS-TSSOS (2nd)			
	IIIC	$^{\mathrm{mb}}$	opt	time	$^{\mathrm{mb}}$	opt	time	
40	20	231	31.2332	143	24	31.2332	1.52	
60	20	231	47.4335	256	24	47.4248	2.35	
80	20	231	63.6335	358	24	63.6115	3.32	
100	20	231	-	-	24	79.7967	4.34	
120	20	231	-	-	24	95.9814	5.62	
140	20	231	-	-	24	112.166	5.96	
160	20	231	-	-	24	128.351	6.96	
180	20	231	-	-	24	144.535	8.50	
200	20	231	-	-	24	160.720	10.6	
300	20	231	-	-	24	241.643	18.4	
400	20	231	-	-	24	322.566	27.7	
500	20	231	-	-	24	403.488	40.1	
1000	20	231	-	-	24	808.103	87.6	

5.3. Max-Cut problems. The Max-Cut problem is one of the basic combinatorial optimization problems, which is known to be NP-hard. Let G(V, E) be an undirected graph with $V = \{1, \ldots, n\}$ and with edge weights w_{ij} for $\{i, j\} \in E$. Then the Max-Cut problem for G can be formulated as a QCQP on binary variables:

(5.2)
$$\begin{cases} \inf & \frac{1}{2} \sum_{\{i,j\} \in E} w_{ij} (1 - x_i x_j) \\ \text{s.t.} & 1 - x_i^2 = 1, \quad i = 1, \dots, n. \end{cases}$$

var	mc	C	SSOS (2n	id)	CS-TSSOS (2nd)			
	IIIC	$^{\mathrm{mb}}$	opt	time	mb	opt	time	
40	20	231	574.512	151	21	574.511	0.76	
60	20	231	878.260	237	21	878.259	1.00	
80	20	231	1182.01	351	21	1182.01	1.38	
100	20	231	-	_	21	1485.76	1.86	
120	20	231	-	_	21	1789.51	2.21	
140	20	231	-	_	21	2093.26	2.75	
160	20	231	-	_	21	2397.01	3.11	
180	20	231	-	_	21	2700.75	3.30	
200	20	231	-	_	21	3004.50	3.78	
300	20	231	-	-	21	4523.25	6.52	
400	20	231	=	_	21	6042.00	11.0	
500	20	231	=	_	21	7560.74	14.4	
1000	20	231	_	_	21	15154.5	39.3	

Table 5. The result for the chained Wood function

We construct random instances of Max-Cut problems with a block-band sparsity pattern, illustrated in Figure 6, which consists of l blocks of size b and two bands of width h. Here we select b=25 and h=5. For a given l, we generate a random sparse binary matrix $A \in \mathbf{S}^{lb+h}$ according to the block-arrow sparsity pattern: the entries out of the blue area take zero; the entries in the block area take one with probability 0.16; the entries in the band area take one with probability $2/\sqrt{l}$. Then we construct the graph G with adjacency matrix A. For each edge $\{i,j\} \in E(G)$, the weight w_{ij} randomly takes the value 1 or -1. Doing so, we build nine Max-Cut instances with l=20,40,60,80,100,120,140,160,180, respectively³. The largest number of nodes is 4505.

 $l \text{ blocks} \cdot$

FIGURE 6. The block-band sparsity pattern

l: the number of blocks; b: the size of blocks; h: the width of bands.

For each instance, we execute the sparse first-order moment-SOS relaxation (Shor relaxation) and the second-order CS-TSSOS hierarchy of sparse order k = 1 where

³The instances can be downloaded at https://wangjie212.github.io/jiewang/code.html.

the maximal chordal extension is used in (3.3). For these problems, the second-order CSSOS hierarchy cannot be executed due to the memory limit. The result is displayed in Table 6. One can see that for each instance, CS-TSSOS significantly improves the bound obtained by Shor relaxation. In addition, one could obtain more accurate bounds by increasing the sparse order k. Note that here we rely on a naive SDP formulation of Max-Cut problems. There are more technical ones in the literature, e.g., [GPP+12, RRW10], that could be combined with CS-TSSOS to achieve better performance.

instance	nodes	edges	$^{ m mc}$	Shor	CS-	ΓSSOS	S (2nd)	
ilistance	nodes	cuges	IIIC	opt	mb	opt	$_{ m time}$	
g20	505	2045	15	570	55	537	85.7	
g40	1005	3441	15	1032	46	992	70.7	
g60	1505	4874	15	1439	46	1387	152	
g80	2005	6035	15	1899	51	1838	157	
g100	2505	7320	16	2398	53	2328	222	
g120	3005	8431	15	2731	39	2655	275	
g140	3505	9658	15	3115	43	3027	357	
g160	4005	10677	15	3670	37	3589	407	
g180	4505	12081	15	4054	44	3953	631	

Table 6. The result for Max-Cut problems

In this table, only the integer part of optimal values are kept.

5.4. **AC-OPF** problems. The AC optimal power flow (AC-OPF) is a central problem in power systems. It can be formulated as the following POP:

problem in power systems. It can be formulated as the following POP:
$$\begin{cases} \inf_{V_{i},S_{k}^{g},S_{ij}} & \sum_{k\in G} (\mathbf{c}_{2k}(\Re(S_{k}^{g}))^{2} + \mathbf{c}_{1k}\Re(S_{k}^{g}) + \mathbf{c}_{0k}) \\ \text{s.t.} & \angle V_{r} = 0, \\ \mathbf{S}_{k}^{gl} \leq S_{k}^{g} \leq \mathbf{S}_{k}^{gu}, & \forall k \in G, \\ v_{i}^{l} \leq |V_{i}| \leq v_{i}^{u}, & \forall i \in N, \\ \sum_{k\in G_{i}} S_{k}^{g} - \mathbf{S}_{i}^{d} - \mathbf{Y}_{i}^{s}|V_{i}|^{2} = \sum_{(i,j)\in E_{i}\cup E_{i}^{R}} S_{ij}, & \forall i \in N, \\ S_{ij} = (\mathbf{Y}_{ij}^{*} - \mathbf{i} \frac{\mathbf{b}_{ij}^{c}}{2}) \frac{|V_{i}|^{2}}{|\mathbf{T}_{ij}|^{2}} - \mathbf{Y}_{ij}^{*} \frac{V_{i}^{*}V_{j}^{*}}{\mathbf{T}_{ij}^{*}}, & \forall (i,j) \in E, \\ S_{ji} = (\mathbf{Y}_{ij}^{*} - \mathbf{i} \frac{\mathbf{b}_{ij}^{c}}{2})|V_{j}|^{2} - \mathbf{Y}_{ij}^{*} \frac{V_{i}^{*}V_{j}^{*}}{\mathbf{T}_{ij}^{*}}, & \forall (i,j) \in E, \\ |S_{ij}| \leq \mathbf{s}_{ij}^{u}, & \forall (i,j) \in E \cup E^{R}, \\ \theta_{ij}^{\Delta l} \leq \angle(V_{i}V_{j}^{*}) \leq \theta_{ij}^{\Delta u}, & \forall (i,j) \in E, \end{cases}$$
where V_{i} is the voltage, S_{k}^{g} is the power generation, S_{ij} is the power flow

where V_i is the voltage, S_k^g is the power generation, S_{ij} is the power flow (all are complex variables; $\Re(\cdot)$ and $\angle \cdot$ stand for the real part and the angle of a complex number, respectively) and all symbols in boldface are constants. For a full description on AC-OPF problems, the reader may refer to [BBC⁺19]. By introducing real variables for both real and imaginary part of each complex variable, we can convert an AC-OPF problem to a POP involving only real variables.

To tackle an AC-OPF problem, we first compute a locally optimal solution with a local solver and then rely on an SDP relaxation to certify the global optimality. Suppose that the optimal value reported by the local solver is AC and the optimal

value of the SDP relaxation is opt. The *optimality gap* between the locally optimal solution and the SDP relaxation is defined by

$$gap := \frac{AC - opt}{AC}.$$

If the optimality gap is less than 1.00%, then we accept the locally optimal solution as globally optimal. For many AC-OPF problems, the first-order moment-SOS relaxation (Shor relaxation) is already able to certify the global optimality (with an optimality gap less than 1.00%). We focus on more challenging AC-OPF problems, for which the gap is greater than 1.00%. We select such benchmarks from the AC-OPF library pglib [BBC+19]. Since we shall go to the second-order moment-SOS relaxation, we can replace the variables S_{ij} and S_{ji} by their right-hand side values in (5.3) and then convert the resulting problem to a real POP. The data for these selected AC-OPF benchmarks are displayed in Table 7, where the AC values are from pglib.

We execute (sparse) Shor relaxation, the second-order CSSOS hierarchy and the second-order CS-TSSOS hierarchy of sparse order k=1. The result for these AC-OPF benchmarks is displayed in Table 8. For each instance, the CS-TSSOS hierarchy succeeds to reduce the optimality gap to less than 1.00%. Again, one can still reduce the optimality gap further by increasing the sparse order k. We also observe that even if the bound obtained by CSSOS should be theoretically better than the one obtained by CS-TSSOS, CS-TSSOS practically provides slightly more accurate bounds than CSSOS for the tested instances (when CSSOS can be executed), due to numerical uncertainties arising when solving the SDP relaxations related to CSSOS.

Table 7. The data for AC-OPF problems

					Shor		
case name	var	cons	mc	AC	opt	gap	
3_lmbd_api	12	28	6	1.1242e4	1.0417e4	7.34%	
5 _ pjm	20	55	6	1.7552e4	1.6634e4	5.22%	
24_ieee_rts_api	114	315	10	1.3495e5	1.3216e5	2.06%	
24_ieee_rts_sad	114	315	14	7.6943e4	7.3592e4	4.36%	
30_as_api	72	297	8	4.9962e3	4.9256e3	1.41%	
73_ieee_rts_api	344	971	16	4.2273e5	4.1041e5	2.91%	
73_ieee_rts_sad	344	971	16	2.2775e5	2.2148e5	2.75%	
118_ieee_api	344	1325	21	2.4205e5	2.1504e5	11.16%	
118_ieee_sad	344	1325	21	1.0522e5	1.0181e5	3.24%	
162_ieee_dtc	348	1809	21	1.0808e5	1.0616e5	1.78%	
162_ieee_dtc_api	348	1809	21	1.2100e5	1.1928e5	1.42%	
240_pserc	766	3322	16	3.3297e6	3.2818e6	1.44%	
500_tamu_api	1112	4613	20	4.2776e4	4.2286e4	1.14%	
500_tamu	1112	4613	30	7.2578e4	7.1034e4	2.12%	
1888_rte	4356	18257	26	1.4025e6	1.3748e6	1.97%	

ango namo		CSSOS	(2nd)		CS-TSSOS (2nd)				
case name	mb	opt	time	gap	mb	opt	time	gap	CE
3_lmbd_api	28	1.1242e4	0.21	0.00%	22	1.1242e4	0.09	0.00%	max
5_pjm	28	1.7437e4	0.46	0.66%	22	1.7543e4	0.30	0.05%	max
24_ieee_rts_api	66	1.3339e5	4.75	1.16%	31	1.3396e5	2.01	0.73%	max
24_ieee_rts_sad	120	7.5108e4	98.3	2.38%	39	7.6942e4	14.8	0.00%	max
30_as_api	45	4.9485e3	3.40	0.95%	22	4.9833e3	2.66	0.26%	max
73_ieee_rts_api	153	4.1523e5	502	1.77%	44	4.1942e5	72.8	0.78%	max
73_ieee_rts_sad	153	2.2383e5	445	1.72%	44	2.2755e5	79.1	0.09%	max
118_ieee_api	253	_	_	_	31	2.4180e5	82.7	0.11%	min
118_ieee_sad	253	_	_	_	73	1.0470e5	169	0.50%	max
162_ieee_dtc	253	_	_	_	34	1.0802e5	278	0.05%	min
162_ieee_dtc_api	253	_	_	_	34	1.2096e5	201	0.03%	min
240_pserc	153	3.2883e6	300	1.24%	44	3.3042e6	33.9	0.77%	max
500_tamu_api	231	4.2321e4	893	1.06%	43	4.2412e4	50.3	0.85%	max
500_tamu	496		_	_	31	7.2396e4	410	0.25%	min
1888 rte	378	_	_		27	1.3953e6	934	0.51%	min

Table 8. The results for AC-OPF problems

6. Discussion and conclusions

This paper introduces the CS-TSSOS hierarchy, a sparse variant of the moment-SOS hierarchy, which can be used to solve large-scale real-world nonlinear optimization problems, assuming that the input data are sparse polynomials. In addition to its theoretical convergence guarantees, CS-TSSOS allows one to make a trade-off between the quality of optimal values and the computational efficiency by controlling the types of chordal extensions and the sparse order k.

By fully exploiting sparsity, CS-TSSOS allows one to go beyond Shor relaxation and solve the second-order moment-SOS relaxation associated with large-scale POPs to obtain more accurate bounds. Indeed CS-TSSOS can handle second-order relaxations of POP instances with thousands of variables and constraints on a standard laptop in tens of minutes. Such instances include the optimal power flow (OPF) problem, an important challenge in the management of electricity networks. In particular, next step is to perform advanced numerical experiments on HPC cluster, for OPF instances with larger number of buses [EDA19].

This work suggests additional investigation tracks for further research:

- 1) Recall that chordal extension plays an important role for both correlative and term sparsity patterns. It turns out that the size of the resulting maximal cliques is crucial for the overall computational efficiency of CS-TSSOS. So far, we have only considered maximal chordal extensions (for convergence guarantee) and approximately minimal chordal extensions (whose goal is to minimize the number of additional edges). However, as [SAL+19] suggests, it would be worth investigating chordal extensions that minimize the size of maximal cliques.
- 2) The CS-TSSOS strategy could be adapted to other applications involving sparse polynomial problems, including deep learning [CLMP20] or noncommutative optimization problems [KMP19] arising in quantum information.

3) At last but not least, a challenging research issue is to establish serious computationally cheaper alternatives to interior-point methods for solving SDP relaxations of POPs. The recent work [YTF⁺19] which reports spectacular results for standard SDPs (and Max-Cut problems in particular) is a positive sign in this direction.

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References

- [AA00] Erling D Andersen and Knud D Andersen. The mosek interior point optimizer for linear programming: an implementation of the homogeneous algorithm. In *High per*formance optimization, pages 197–232. Springer, 2000.
- [AM14] Amir Ali Ahmadi and Anirudha Majumdar. Dsos and sdsos optimization: Lp and socp-based alternatives to sum of squares optimization. In 2014 48th annual conference on information sciences and systems (CISS), pages 1–5. IEEE, 2014.
- [ApS17] MOSEK ApS. The MOSEK optimization toolbox. Version 8.1., 2017.
- [Ave19] Gennadiy Averkov. Optimal size of linear matrix inequalities in semidefinite approaches to polynomial optimization. SIAM Journal on Applied Algebra and Geometry, 3(1):128–151, 2019.
- [BBC⁺19] Sogol Babaeinejadsarookolaee, Adam Birchfield, Richard D Christie, Carleton Coffrin, Christopher DeMarco, Ruisheng Diao, Michael Ferris, Stephane Fliscounakis, Scott Greene, Renke Huang, et al. The power grid library for benchmarking ac optimal power flow algorithms. arXiv preprint arXiv:1908.02788, 2019.
- [BEKS17] Jeff Bezanson, Alan Edelman, Stefan Karpinski, and Viral B Shah. Julia: A fresh approach to numerical computing. SIAM review, 59(1):65–98, 2017.
- [BP93] Jean RS Blair and Barry Peyton. An introduction to chordal graphs and clique trees. In Graph theory and sparse matrix computation, pages 1–29. Springer, 1993.
- [CLMP20] Tong Chen, Jean-Bernard Lasserre, Victor Magron, and Edouard Pauwels. Polynomial optimization for bounding lipschitz constants of deep networks. arXiv preprint arXiv:2002.03657, 2020.
- [CS16] Venkat Chandrasekaran and Parikshit Shah. Relative entropy relaxations for signomial optimization. SIAM Journal on Optimization, 26(2):1147–1173, 2016.
- [EDA19] Anders Eltved, Joachim Dahl, and Martin S Andersen. On the robustness and scalability of semidefinite relaxation for optimal power flow problems. *Optimization and Engineering*, pages 1–18, 2019.
- [FG65] Delbert Fulkerson and Oliver Gross. Incidence matrices and interval graphs. *Pacific journal of mathematics*, 15(3):835–855, 1965.
- [GNS07] David Grimm, Tim Netzer, and Markus Schweighofer. A note on the representation of positive polynomials with structured sparsity. Archiv der Mathematik, 89(5):399–403, 2007.
- [Gol04] Martin Charles Golumbic. Algorithmic graph theory and perfect graphs. Elsevier, 2004
- [GPP+12] Luigi Grippo, Laura Palagi, Mauro Piacentini, Veronica Piccialli, and Giovanni Rinaldi. Speedp: an algorithm to compute sdp bounds for very large max-cut instances. Mathematical programming, 136(2):353-373, 2012.

- [HL05] Didier Henrion and Jean-Bernard Lasserre. Detecting global optimality and extracting solutions in gloptipoly. In *Positive polynomials in control*, pages 293–310. Springer, 2005.
- [IDW16] Sadik Iliman and Timo De Wolff. Amoebas, nonnegative polynomials and sums of squares supported on circuits. Research in the Mathematical Sciences, 3(1):9, 2016.
- [JM18] Cédric Josz and Daniel K Molzahn. Lasserre hierarchy for large scale polynomial optimization in real and complex variables. SIAM Journal on Optimization, 28(2):1017–1048, 2018.
- [KMP19] Igor Klep, Victor Magron, and Janez Povh. Sparse noncommutative polynomial optimization. arXiv preprint arXiv:1909.00569, 2019.
- [Kri64] Jean-Louis Krivine. Anneaux préordonnés. Journal d'analyse mathématique, 12(1):307–326, 1964.
- [Las01] J.-B. Lasserre. Global Optimization with Polynomials and the Problem of Moments. SIAM Journal on Optimization, 11(3):796–817, 2001.
- [Las06] J.-B. Lasserre. Convergent sdp-relaxations in polynomial optimization with sparsity. SIAM Journal on Optimization, 17(3):822-843, 2006.
- [Las15] J. B. Lasserre. An Introduction to Polynomial and Semi-Algbraic Optimization. Cambridge University Press, Cambridge, UK, 2015.
- [Lau03] Monique Laurent. A comparison of the sherali-adams, lovász-schrijver, and lasserre relaxations for 0–1 programming. *Mathematics of Operations Research*, 28(3):470–496, 2003.
- [Lof09] Johan Lofberg. Pre-and post-processing sum-of-squares programs in practice. *IEEE transactions on automatic control*, 54(5):1007–1011, 2009.
- [LRC20] Fabian Latorre, Paul Rolland, and Volkan Cevher. Lipschitz constant estimation of neural networks via sparse polynomial optimization. arXiv preprint arXiv:2004.08688, 2020.
- [LTY17] Jean B Lasserre, Kim-Chuan Toh, and Shouguang Yang. A bounded degree sos hierarchy for polynomial optimization. EURO Journal on Computational Optimization, 5(1-2):87–117, 2017.
- [Mag18] Victor Magron. Interval enclosures of upper bounds of roundoff errors using semidefinite programming. ACM Transactions on Mathematical Software (TOMS), 44(4):1–18, 2018.
- [MCD17] V. Magron, G. Constantinides, and A. Donaldson. Certified Roundoff Error Bounds Using Semidefinite Programming. ACM Trans. Math. Softw., 43(4):1–34, 2017.
- [MML20] Ngoc Hoang Anh Mai, Victor Magron, and J-B Lasserre. A sparse version of reznick's positivstellensatz. arXiv preprint arXiv:2002.05101, 2020.
- [MZSP19] Jared Miller, Yang Zheng, Mario Sznaier, and Antonis Papachristodoulou. Decomposed structured subsets for semidefinite and sum-of-squares optimization. arXiv preprint arXiv:1911.12859, 2019.
- [Nie14] J. Nie. Optimality conditions and finite convergence of Lasserre's hierarchy. Mathematical Programming, 146(1):97–121, Aug 2014.
- [R⁺78] Bruce Reznick et al. Extremal psd forms with few terms. *Duke mathematical journal*, 45(2):363–374, 1978.
- [RRW10] Franz Rendl, Giovanni Rinaldi, and Angelika Wiegele. Solving max-cut to optimality by intersecting semidefinite and polyhedral relaxations. *Mathematical Programming*, 121(2):307, 2010.
- [RTAL13] Cordian Riener, Thorsten Theobald, Lina Jansson Andrén, and Jean B Lasserre. Exploiting symmetries in sdp-relaxations for polynomial optimization. Mathematics of Operations Research, 38(1):122–141, 2013.
- [SAL+19] Julie Sliwak, Miguel Anjos, Lucas Létocart, Jean Maeght, and Emiliano Traversi. Improving clique decompositions of semidefinite relaxations for optimal power flow problems. arXiv preprint arXiv:1912.09232, 2019.
- [Sho87] Naum Z Shor. Quadratic optimization problems. Soviet Journal of Computer and Systems Sciences, 25:1–11, 1987.
- [Ste74] Gilbert Stengle. A nullstellensatz and a positivstellensatz in semialgebraic geometry. Mathematische Annalen, 207(2):87–97, 1974.

- [TCHL19] Matteo Tacchi, Carmen Cardozo, Didier Henrion, and Jean Lasserre. Approximating regions of attraction of a sparse polynomial differential system. arXiv preprint arXiv:1911.09500, 2019.
- [Toh18] K.-C Toh. Some numerical issues in the development of sdp algorithms. *Informs OS Today*, 8(2):7–20, 2018.
- [TWLH19] M. Tacchi, T. Weisser, J.-B. Lasserre, and D. Henrion. Exploiting Sparsity for Semi-Algebraic Set Volume Computation. preprint arXiv:1902.02976, 2019.
- [VA⁺15] Lieven Vandenberghe, Martin S Andersen, et al. Chordal graphs and semidefinite optimization. Foundations and Trends® in Optimization, 1(4):241–433, 2015.
- [WKK+08] Hayato Waki, Sunyoung Kim, Masakazu Kojima, Masakazu Muramatsu, and Hiroshi Sugimoto. Algorithm 883: Sparsepop—a sparse semidefinite programming relaxation of polynomial optimization problems. ACM Transactions on Mathematical Software (TOMS), 35(2):1–13, 2008.
- [WKKM06] H. Waki, S. Kim, M. Kojima, and M. Muramatsu. Sums of Squares and Semidefinite Programming Relaxations for Polynomial Optimization Problems with Structured Sparsity. SIAM Journal on Optimization, 17(1):218–242, 2006.
- [WLC+19] Tillmann Weisser, Benoît Legat, Chris Coey, Lea Kapelevich, and Juan Pablo Vielma. Polynomial and moment optimization in julia and jump. In JuliaCon, 2019.
- [WLX19] Jie Wang, Haokun Li, and Bican Xia. A new sparse sos decomposition algorithm based on term sparsity. In Proceedings of the 2019 on International Symposium on Symbolic and Algebraic Computation, pages 347–354, 2019.
- [WM19] J. Wang and V. Magron. A second order cone characterization for sums of nonnegative circuits. arXiv preprint arXiv:1906.06179, 2019.
- [WML19] Jie Wang, Victor Magron, and Jean-Bernard Lasserre. Tssos: A moment-sos hierarchy that exploits term sparsity. arXiv preprint arXiv:1912.08899, 2019.
- [WML20] Jie Wang, Victor Magron, and Jean-Bernard Lasserre. Chordal-tssos: a moment-sos hierarchy that exploits term sparsity with chordal extension. $arXiv\ preprint\ arXiv:2003.03210,\ 2020.$
- [WSV12] Henry Wolkowicz, Romesh Saigal, and Lieven Vandenberghe. *Handbook of semidefinite programming: theory, algorithms, and applications*, volume 27. Springer Science & Business Media, 2012.
- [YTF⁺19] Alp Yurtsever, Joel A Tropp, Olivier Fercoq, Madeleine Udell, and Volkan Cevher. Scalable semidefinite programming. arXiv preprint arXiv:1912.02949, 2019.