

WEIGHTED GEOMETRIC MEAN, MINIMUM MEDIATED SET, AND OPTIMAL SIMPLE SECOND-ORDER CONE REPRESENTATION*

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Abstract. We study optimal *simple second-order cone representations* (a particular subclass of second-order cone representations) for weighted geometric means, which turns out to be closely related to minimum mediated sets. Several lower bounds and upper bounds on the size of optimal simple second-order cone representations are proved. In the case of bivariate weighted geometric means (equivalently, one-dimensional mediated sets), we are able to prove the exact size of an optimal simple second-order cone representation and give an algorithm to compute one. In the general case, fast heuristic algorithms and traversal algorithms are proposed to compute an approximately optimal simple second-order cone representation. Finally, applications to polynomial optimization, matrix optimization, and quantum information are provided.

Key words. weighted geometric mean, minimum mediated set, second-order cone representation, polynomial optimization, semidefinite representation

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1. Introduction. This paper is concerned with the following problem:

(I) *Given a weighted geometric mean inequality $x_1^{\lambda_1} \cdots x_m^{\lambda_m} \geq x_{m+1}$ with weights $(\lambda_i)_{i=1}^m \in \mathbb{Q}_+^m$, $\sum_{i=1}^m \lambda_i = 1$, construct an equivalent representation using as few quadratic inequalities (i.e., $x_i x_j \geq x_k^2$) as possible.*

The study of problem (I) is motivated by the fact that any solution to problem (I) immediately gives rise to a second-order cone representation for the convex set defined by the weighted geometric mean inequality. One advantage of the second-order cone representation is that the related optimization problem can be solved with off-the-shelf second-order cone programming (SOCP) solvers [1], e.g., Mosek [2] and ECOS [6]. Naturally, to achieve the best efficiency, we wish to obtain such a second-order cone representation of as small size as possible.

Note that using the so-called tower-of-variables construction described in [4, Lecture 3.3], one is able to give an equivalent second-order cone representation for a weighted geometric mean inequality, whose size is, however, typically far from optimal. As far as the author knows, the problem of finding optimal second-order cone representations for general weighted geometric mean inequalities has not been investigated much in the literature, except for some special cases, which are mentioned below. Assume that in problem (I) the weights are written as $\lambda_i = \frac{s_i}{t}$, $s_i \in \mathbb{N}$ for $i = 1, \dots, m$ and $t \in \mathbb{N}$. In [19], Morenko et al. constructed an “economical” representation (without any proof concerning optimality) for a weighted geometric mean inequality in the case of $m = 3$ and $t = 2^l$ for some $l \in \mathbb{N}$ when studying p -norm cone programming. In [17], Kian, Berk, and Gürlér dealt with problem (I) in the case of

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$t = 2^l$ for some $l \in \mathbb{N}$ and proposed a heuristic algorithm for computing a second-order cone representation of small size.

On the other hand, there are plenty of applications of weighted geometric mean inequalities in optimization:

Second-order cone representations for other types of inequalities [4, Lecture 3.3.1].

There are other types of inequalities that can be expressed by weighted geometric mean inequalities, and so the second-order cone representations for weighted geometric mean inequalities can immediately lead to second-order cone representations for these inequalities with rational powers:

- $x_1^{\lambda_1} \cdots x_m^{\lambda_m} \geq x_{m+1} \iff x_1^{\lambda_1} \cdots x_m^{\lambda_m} y^{1-\sum_{i=1}^m \lambda_i} \geq x_{m+1}, y = 1 \ (\lambda_1, \dots, \lambda_m > 0, \sum_{i=1}^m \lambda_i < 1).$
- $x^\lambda \leq y \iff y^{\frac{1}{\lambda}} z^{1-\frac{1}{\lambda}} \geq x, z = 1 \ (\lambda > 1).$
- $x^\lambda \geq y \iff x^\lambda z^{1-\lambda} \geq y, z = 1 \ (0 < \lambda < 1).$
- $x^{-\lambda} \leq y \iff x^{\frac{\lambda}{1+\lambda}} y^{\frac{1}{1+\lambda}} \geq z, z = 1 \ (\lambda > 0).$
- $x_1^{-\lambda_1} \cdots x_m^{-\lambda_m} \leq y \ (\lambda_1, \dots, \lambda_m > 0) \iff x_1^{\frac{\lambda_1}{1+\sum_{i=1}^m \lambda_i}} \cdots x_m^{\frac{\lambda_m}{1+\sum_{i=1}^m \lambda_i}} y^{\frac{1}{1+\sum_{i=1}^m \lambda_i}} \geq z, z = 1.$

p-norm cone programming. Let $p \geq 1$ be a rational number. The p -norm of a vector $\mathbf{x} \in \mathbb{R}^n$ is $\|\mathbf{x}\|_p := \sum_{i=1}^n |x_i|^{\frac{1}{p}}$. The p -norm cone of dimension $n+1$ is defined by the set

$$\{(\mathbf{x}, z) \in \mathbb{R}^{n+1} \mid z \geq \|\mathbf{x}\|_p\}.$$

Note that the second-order cone is exactly the p -norm cone with $p = 2$. The p -norm cone admits second-order cone representations since the inequality $z \geq \|\mathbf{x}\|_p$ can be rewritten as $z \geq \sum_{i=1}^n |x_i|^p / z^{p-1}$, which is equivalent to [4, Lecture 3.3.1]

$$\exists (y_i)_{i=1}^n, (w_i)_{i=1}^n \in \mathbb{R}_+^n \text{ s.t. } \begin{cases} z^{1-\frac{1}{p}} y_i^{\frac{1}{p}} \geq w_i, |x_i| \leq w_i, i = 1, \dots, n, \\ \sum_{i=1}^n y_i = z. \end{cases}$$

Power cone programming [4, Lecture 3.3.1]. Given $\lambda_1, \dots, \lambda_m > 0$ with $\sum_{i=1}^m \lambda_i = 1$, the corresponding *power cone* is defined by the set

$$\{(\mathbf{x}, z) \in \mathbb{R}_+^n \times \mathbb{R} \mid x_1^{\lambda_1} \cdots x_m^{\lambda_m} \geq |z|\}.$$

The power cone admits second-order cone representations since

$$x_1^{\lambda_1} \cdots x_m^{\lambda_m} \geq |z| \iff \exists y \geq 0 \text{ s.t. } x_1^{\lambda_1} \cdots x_m^{\lambda_m} \geq y, |z| \leq y.$$

Matrix optimization. The second-order cone representation for weighted geometric mean inequalities allows one to give (approximate) semidefinite representations for many matrix functions, as revealed in [11, 12, 22]. These functions include the matrix power function [15], Lieb's function, the Tsallis entropy, the Tsallis relative entropy [11], and the matrix logarithm function [12], which have various applications in matrix optimization and quantum information.

A relevant problem to problem (I) is the following:

(II) *Given a set of points $\{\alpha_i\}_{i=1}^m \subseteq \mathbb{R}^{m-1}$ forming the vertices of a simplex and a point $\alpha_{m+1} = \sum_{i=1}^m \lambda_i \alpha_i$ with $(\lambda_i)_{i=1}^m \in \mathbb{Q}_+^m$ and $\sum_{i=1}^m \lambda_i = 1$, find as few points as possible (say, $\{\alpha_i\}_{i=m+2}^{m+n}$) such that every point in $\{\alpha_i\}_{i=m+1}^{m+n}$ is an average of two distinct points in $\{\alpha_i\}_{i=1}^{m+n}$.*

Problem (II) arises from the study of nonnegative circuit polynomials [18, 24]. Sums of nonnegative circuit polynomials (SONC) were proposed by Ilmanen and de Wolff as certificates of polynomial nonnegativity [16] and have been employed to solve sparse polynomial optimization problem in a “degree-free” manner [7, 8, 18]. The set of points $\{\alpha_i\}_{i=m+1}^{m+n}$ considered in problem (II) is called a (minimum) mediated set. Mediated sets over integers were initially introduced by Reznick in [21] to study agiforms and were extended to the rational case by Wang and Magron [24]. It was proved in [18] that a circuit polynomial is nonnegative if and only if it can be written as a sum of binomial squares supported on a mediated set, with the number of binomial squares equaling the number of points contained in the mediated set. In [5], the notion of mediated sets plays an important role in the study of the tropicalization of pseudomoment cones and discrete midpoint convexity. More recent research on mediated sets can be found in [14, 20].

Interestingly, it turns out that solutions to problem (I) are in a one-to-one correspondence to solutions to problem (II). As one would see later, the development of both theory and algorithms in the present paper deeply relies on this appealing connection.

We should point out that in the literature, there is a more general notion of second-order cone representations (also called second-order cone lifts) which are of the form $\pi(K^n \cap L)$ with K being the three-dimensional rotated second-order cone, L being an affine space, and π being a linear map [9, 10, 13]. However, in this paper, we would rather work on the more restrictive case (called *simple* second-order cone representations) considered in problem (I) where π specializes to a coordinate projection and L is an affine space that constrains certain coordinates to be equal due to the following reasons: (1) representations of this restrictive form can be lifted to semidefinite representations of matrix geometric means in a direct way (see Theorem 6.1 for a more formal statement about this); (2) representations of this restrictive form admit a one-to-one correspondence to mediated sets which have a nice geometric description; (3) we are able to propose efficient algorithms for computing simple second-order cone representations of (approximately) optimal size.

Our main contributions are summarized as follows:

- In section 3, we prove several lower bounds and upper bounds on the size of optimal simple second-order cone representations for weighted geometric mean inequalities (equivalently, the size of minimum mediated sets). In particular, the lower bounds are shown to be attainable by examples.
- In section 4, we prove the exact size of an optimal simple second-order cone representation for bivariate weighted geometric mean inequalities, which resolves a conjecture proposed in [18]. In addition, we provide a binary tree representation of one-dimensional “successive” minimum mediated sets.
- In section 5, we propose several heuristic algorithms for computing an approximately optimal simple second-order cone representation of a weighted geometric mean inequality (equivalently, an approximately minimum mediated set) and compare their practical performance. We also propose a brute-force algorithm for computing an exact optimal simple second-order cone representation of a weighted geometric mean inequality (equivalently, a minimum mediated set). Numerical experiments demonstrate that the heuristic algorithms can efficiently produce simple second-order cone representations of size being equal or close to the optimal one.
- In section 6, we provide applications of the proposed algorithms to polynomial optimization, matrix optimization, and quantum information and demonstrate their efficiency by numerical experiments.

2. Preliminaries. Let \mathbb{N} , \mathbb{R} , \mathbb{Q} be the set of nonnegative integers, real numbers, and rational numbers, respectively. Let \mathbb{N}^* , $\mathbb{R}_{\geq 0}$, \mathbb{R}_+ , \mathbb{Q}_+ be the set of positive integers, nonnegative real numbers, positive real numbers, and positive rational numbers, respectively. For $a \in \mathbb{R}$, $\lceil a \rceil$ stands for the least integer that is no less than a . For a tuple of integers $(s_1, \dots, s_m) \in \mathbb{N}^m$, we use \hat{s} to denote the sum of its entries, i.e., $\hat{s} := \sum_{i=1}^m s_i$. For a set A , we use $|A|$ to denote the cardinality of A . For a finite set $\mathcal{A} \subseteq \mathbb{N}^n$, we denote by $\text{conv}(\mathcal{A})$ the convex hull of \mathcal{A} and use $\text{conv}(\mathcal{A})^\circ$ to denote the relative interior of $\text{conv}(\mathcal{A})$. For integers s_1, s_2, \dots , we use the parentheses (s_1, s_2, \dots) to denote their greatest common divisor. Let \mathbf{S}^n , \mathbf{S}_+^n , \mathbf{S}_{++}^n be the set of symmetric matrices, positive semidefinite, and positive definite matrices of size n , respectively.

The n -dimensional *rotated second-order cone* is defined by the set

$$K_n := \left\{ (a_i)_{i=1}^n \in \mathbb{R}^n \left| 2a_1a_2 \geq \sum_{i=3}^n a_i^2, a_1 \geq 0, a_2 \geq 0 \right. \right\}.$$

In this paper, we are mostly interested in the three-dimensional rotated second-order cone¹ ($K := K_3$), which we simply refer to as the second-order cone.

3. Optimal simple second-order cone representations and minimum mediated sets. In this paper, a *weighted geometric mean inequality* refers to an inequality of form

$$x_1^{\lambda_1} \cdots x_m^{\lambda_m} \geq x_{m+1} \text{ with } (\lambda_i)_{i=1}^m \in \mathbb{R}_+^m \text{ and } \sum_{i=1}^m \lambda_i = 1,$$

where the variables x_1, \dots, x_{m+1} are assumed to be nonnegative. Throughout the paper, we assume that the weights $(\lambda_i)_{i=1}^m \in \mathbb{Q}_+^m$ are rational numbers. It is known that when the weights are rational, the weighted geometric mean inequality is second-order cone representable [4, Lecture 3.3]. As $(\lambda_i)_{i=1}^m$ are rational numbers by assumption, we can write $\lambda_i = \frac{s_i}{\hat{s}}$, $s_i \in \mathbb{N}$ for $i = 1, \dots, m$. In view of this, we also call $x_1^{s_1} \cdots x_m^{s_m} \geq x_{m+1}^{\hat{s}}$ with $(s_i)_{i=1}^m \in \mathbb{N}^m$, $(s_1, \dots, s_m) = 1$ a weighted geometric mean inequality.

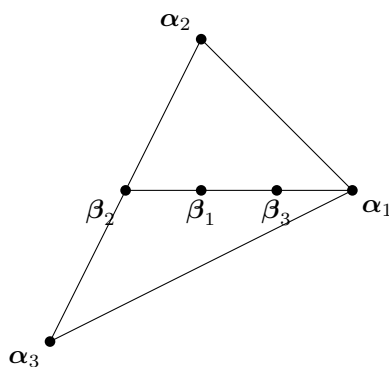
DEFINITION 3.1. A simple second-order cone representation for a weighted geometric mean inequality $x_1^{s_1} \cdots x_m^{s_m} \geq x_{m+1}^{\hat{s}}$ consists of a set of quadratic inequalities $x_{i_k} x_{j_k} \geq x_{m+k}^2$, $k = 1, \dots, n$, such that

$$(3.1) \quad x_1^{s_1} \cdots x_m^{s_m} \geq x_{m+1}^{\hat{s}} \iff \begin{cases} x_{i_1} x_{j_1} \geq x_{m+1}^2, \\ \vdots \\ x_{i_n} x_{j_n} \geq x_{m+n}^2, \\ i_k, j_k \in \{1, 2, \dots, m+n\}, \quad k = 1, \dots, n, \end{cases}$$

where x_{m+2}, \dots, x_{m+n} are $n-1$ auxiliary nonnegative variables. We call n the size of the second-order cone representation. A simple second-order cone representation of minimum size is called an *optimal simple second-order cone representation* whose size is denoted by $L(s_1, \dots, s_m)$.

Remark 3.2. The second-order cone representation in (3.1) is uniquely determined by the set of integer triples $\{(i_k, j_k, m+k)\}_{k=1}^n$, which is hereafter referred to as a *configuration*.

¹Actually, any n -dimensional second-order cone is representable using the three-dimensional second-order cone.

FIG. 3.1. $\{\beta_1, \beta_2, \beta_3\}$ forms an $\{\alpha_1, \alpha_2, \alpha_3\}$ -mediated set.

Remark 3.3. The value of $L(s_1, \dots, s_m)$ clearly does not depend on the ordering of s_1, \dots, s_m .

Example 3.4. The weighted geometric mean inequality $x_1^3 x_2^8 \geq x_3^{11}$ admits a simple second-order cone representation: $x_2 x_6 \geq x_3^2, x_1 x_3 \geq x_4^2, x_3 x_4 \geq x_5^2, x_4 x_5 \geq x_6^2$.

A set of points $\mathcal{A} = \{\alpha_1, \dots, \alpha_m\} \subseteq \mathbb{R}^{m-1}$ is called a *trellis* if they are affinely independent. Given a trellis \mathcal{A} , we say that a set of points \mathcal{B} is an \mathcal{A} -mediated set if every point $\beta \in \mathcal{B}$ is an average of two distinct points in $\mathcal{A} \cup \mathcal{B}$. One could immediately deduce that the points in an \mathcal{A} -mediated set belong to the convex hull of \mathcal{A} . An \mathcal{A} -mediated set containing a given point $\beta \in \text{conv}(\mathcal{A})^\circ$ is called an (\mathcal{A}, β) -mediated set and a *minimum* (\mathcal{A}, β) -mediated set is an (\mathcal{A}, β) -mediated set with the smallest cardinality. We sometimes omit the prefixes \mathcal{A}, β and simply say (minimum) mediated sets if there is no need to mention the specific \mathcal{A}, β . If $\beta \in \text{conv}(\mathcal{A})^\circ$, then there exists a unique tuple of scalars $\lambda_1, \dots, \lambda_m \in \mathbb{R}_+$ (the barycentric coordinates) such that $\sum_{i=1}^m \lambda_i = 1$ and $\beta = \sum_{i=1}^m \lambda_i \alpha_i$. It is easy to see that for an (\mathcal{A}, β) -mediated set to exist, the λ_i 's are necessarily rational numbers.

Example 3.5. Let $\mathcal{A} = \{\alpha_1 = (4, 2), \alpha_2 = (2, 4), \alpha_3 = (0, 0)\}$ and $\beta_1 = (2, 2), \beta_2 = (1, 2), \beta_3 = (3, 2)$. It is easy to check by hand that $\{\beta_1, \beta_2, \beta_3\}$ is an \mathcal{A} -mediated set; see Figure 3.1.

Assume that $\mathcal{A} = \{\alpha_1, \dots, \alpha_m\}$ is a trellis and $\alpha_{m+1} = \sum_{i=1}^m \frac{s_i}{s} \alpha_i$ with $(s_i)_{i=1}^m \in (\mathbb{N}^*)^m, (s_1, \dots, s_m) = 1$. If $\{\alpha_{m+1}, \dots, \alpha_{m+n}\}$ is an \mathcal{A} -mediated set, then by definition the following system of equations must be satisfied for appropriate subscripts i_k, j_k :

$$(3.2) \quad \begin{cases} \alpha_{i_1} + \alpha_{j_1} = 2\alpha_{m+1}, \\ \vdots \\ \alpha_{i_n} + \alpha_{j_n} = 2\alpha_{m+n}, \\ i_k, j_k \in \{1, 2, \dots, m+n\}, \quad k = 1, \dots, n. \end{cases}$$

Comparing (3.1) with (3.2), we see that there is a one-to-one correspondence between simple second-order cone representations for the weighted geometric mean inequality $x_1^{s_1} \cdots x_m^{s_m} \geq x_{m+1}^s$ and \mathcal{A} -mediated sets containing the point $\alpha_{m+1} = \sum_{i=1}^m \frac{s_i}{s} \alpha_i$:

simple second-order cone representations for $x_1^{s_1} \cdots x_m^{s_m} \geq x_{m+1}^{\hat{s}}$

$$\begin{array}{c} \updownarrow \\ \text{configurations } \{(i_k, j_k, m+k)\}_{k=1}^n \\ \updownarrow \\ \mathcal{A}\text{-mediated sets containing } \alpha_{m+1} = \sum_{i=1}^m \frac{s_i}{\hat{s}} \alpha_i. \end{array}$$

It follows that $L(s_1, \dots, s_m)$ also denotes the cardinality of a minimum $(\mathcal{A}, \alpha_{m+1})$ -mediated set.

We now prove a lower bound on the size of optimal simple second-order cone representations for a weighted geometric mean inequality in terms of the number of variables involved.

THEOREM 3.6. *Let $s_1, \dots, s_m \in \mathbb{N}^*$ be a tuple of integers with $(s_1, \dots, s_m) = 1$. Then*

$$(3.3) \quad L(s_1, \dots, s_m) \geq m - 1.$$

Proof. Suppose that $\mathcal{A} = \{\alpha_1, \dots, \alpha_m\}$ is a trellis and $\alpha_{m+1}, \dots, \alpha_{m+n}$ is a minimum $(\mathcal{A}, \alpha_{m+1})$ -mediated set such that $\sum_{i=1}^m s_i \alpha_i = \hat{s} \alpha_{m+1}$. Let us consider the system of equations (3.2). Note first that $\{1, \dots, m\} \subseteq \cup_{k=1}^n \{i_k, j_k\}$. Moreover, we must have $\{m+2, \dots, m+n\} \subseteq \cup_{k=1}^n \{i_k, j_k\}$ as if $k \notin \cup_{k=1}^n \{i_k, j_k\}$ for some $k \in \{m+2, \dots, m+n\}$, then we can delete α_k to obtain a smaller $(\mathcal{A}, \alpha_{m+1})$ -mediated set. From these facts, we get

$$2n \geq m + n - 1,$$

which yields the desired inequality. \square

The lower bound in Theorem 3.6 is attainable as the following example shows.

Example 3.7. The weighted geometric mean inequality $x_1 x_2 x_3 x_4 \geq x_5^4$ admits a simple second-order cone representation: $x_6 x_7 \geq x_5^2, x_1 x_2 \geq x_6^2, x_3 x_4 \geq x_7^2$.

In the language of mediated sets, Theorem 3.6 says that the cardinality of a minimum mediated set is bounded by its dimension (i.e., the dimension of its associated trellis) from below.

From the point of view of mediated sets, we are actually able to give another lower bound on $L(s_1, \dots, s_m)$ in terms of the sum \hat{s} . For the proof, we need the notion of M-matrices and two preliminary results.

DEFINITION 3.8. *Let $C = [c_{ij}]$ be a real matrix. Then C is called an M-matrix if it satisfies*

- (1) $c_{ij} \leq 0$ if $i \neq j$;
- (2) $C = tI - B$, where B is a matrix with nonnegative entries, I is an identity matrix, and t is no less than the spectral radius (the maximum of the moduli of the eigenvalues) of B .

A remarkable property of M-matrices is that the determinant of an M-matrix is bounded by the product of its diagonals from above.

LEMMA 3.9. *Let $C = [c_{ij}] \in \mathbb{R}^{n \times n}$ be an M-matrix. Then, it holds that*

$$(3.4) \quad \det(C) \leq \prod_{i=1}^n c_{ii}.$$

Proof. See, e.g., Corollary 4.1.2 of [3]. \square

The following lemma adapted from [21, Lemma 4.3] asserts that a particular class of matrices are M-matrices.

LEMMA 3.10. *Let $C = [c_{ij}]$ be a real matrix such that $c_{ii} = 2$ and $c_{ij} \in \{0, -1\}$ if $i \neq j$. Assume that each row of C has at most two -1 's and there is no principal submatrix of C in which each row has exactly two -1 's. Then C is a nonsingular M-matrix.*

Proof. Let us write $C = 2I - B$, where I is an identity matrix, and $B = [b_{ij}]$ is a matrix with entries being 0 or 1. We finish the proof by showing that the spectral radius of B is less than 2.

Let λ be any eigenvalue of B and $\alpha = (\alpha_i)_i$ a corresponding eigenvector such that $B\alpha = \lambda\alpha$. Let $\xi = \max_i \{|\alpha_i|\}$ and $I = \{i \mid |\alpha_i| = \xi\}$. For $i \in I$, let $T(i) = \{j \mid b_{ij} = 1\}$. Then for each $i \in I$, we have

$$|\lambda|\xi = |\lambda\alpha_i| = \left| \sum_j b_{ij}\alpha_j \right| = \left| \sum_{j \in T(i)} \alpha_j \right| \leq 2\xi.$$

So $|\lambda| \leq 2$ for $\xi > 0$. If $|\lambda| = 2$, then for any $i \in I$, we have $|T(i)| = 2$ and $|\alpha_j| = \xi$ for $j \in T(i)$ which implies $T(i) \subseteq I$. It follows that the principal submatrix of C indexed by I has exactly two -1 's in each row, which is impossible. Thus $|\lambda| < 2$ as desired. \square

Now we are ready to prove the promised result.

THEOREM 3.11. *Let $\mathcal{A} = \{\alpha_1, \dots, \alpha_m\} \subseteq \mathbb{R}^{m-1}$ be a trellis and $\beta = \sum_{i=1}^m \frac{s_i}{s} \alpha_i$ with $(s_i)_{i=1}^m \in (\mathbb{N}^*)^m$, $(s_1, \dots, s_m) = 1$. Then for any (\mathcal{A}, β) -mediated set \mathcal{B} , one has $|\mathcal{B}| \geq \lceil \log_2(\hat{s}) \rceil$.*

Proof. Assume that $\mathcal{B} = \{\beta_1, \dots, \beta_n\}$ with $\beta_1 = \beta$ is an (\mathcal{A}, β) -mediated set. By definition, for each β_i , we have that one of the following three equations holds for suitable subscripts $j, k \in \{1, \dots, m\}$:

$$(3.5a) \quad 2\beta_i = \alpha_j + \alpha_k,$$

$$(3.5b) \quad 2\beta_i - \beta_j = \alpha_k,$$

$$(3.5c) \quad 2\beta_i - \beta_j - \beta_k = \mathbf{0}.$$

Let us denote the coefficient matrix of (3.5) by C and let $B = [\beta_1, \dots, \beta_n]^\top$ with β_i 's being viewed as column vectors, so that we can rewrite (3.5) in matrix form as $CB = A$, where A is a matrix whose row vectors belong to the subspace generated by \mathcal{A} . It is not hard to see that C satisfies the conditions of Lemma 3.10; only the assumption that there is no principal submatrix with exactly two -1 's in each row is not obviously fulfilled. Suppose that C has a principal submatrix indexed by I with two -1 's in each row and let $\mathcal{B}' = \{\beta_i\}_{i \in I}$. Then by construction, every point β_i in \mathcal{B}' satisfies (3.5c) with $j, k \in I$ and so is an average of two other points in \mathcal{B}' , which is, however, impossible for a finite set. Hence C has no such principal submatrix. Applying Lemma 3.10, we deduce that C is an M-matrix. In addition by Lemma 3.9, $\det(C) \leq 2^n$. Solve $CB = A$ for β and we obtain $\beta = \frac{\sum_{i=1}^m r_i \alpha_i}{\det(C)}$ for some $r_i \in \mathbb{N}^*$, $i = 1, \dots, m$. It follows that $\hat{s} \leq \det(C) \leq 2^n$, which implies $|\mathcal{B}| = n \geq \lceil \log_2(\hat{s}) \rceil$. \square

From Theorem 3.11, we immediately obtain the following corollary.

COROLLARY 3.12. Let $s_1, \dots, s_m \in \mathbb{N}^*$ be a tuple of integers with $(s_1, \dots, s_m) = 1$. Then

$$(3.6) \quad L(s_1, \dots, s_m) \geq \lceil \log_2(\hat{s}) \rceil.$$

The lower bound in Corollary 3.12 is attainable as the following example shows.

Example 3.13. The weighted geometric mean inequality $x_1 x_2^2 x_3^3 \geq x_4^6$ admits a simple second-order cone representation: $x_3 x_5 \geq x_4^2, x_2 x_6 \geq x_5^2, x_1 x_5 \geq x_6^2$.

Remark 3.14. Combining Theorem 3.6 with Corollary 3.12, we get

$$(3.7) \quad L(s_1, \dots, s_m) \geq \max \{ \lceil \log_2(\hat{s}) \rceil, m-1 \}.$$

3.1. More general second-order cone representations. In this subsection, we consider more general second-order cone representations for a weighted geometric mean inequality and derive lower bounds on the size of such second-order cone representations. Let K be the three-dimensional rotated second-order cone. Given a convex cone C , we say that C admits a K^l -lift if $C = \pi(K^l \cap L)$ where L is an affine space and π is a linear map [10].

THEOREM 3.15. Let $s_1, \dots, s_m \in \mathbb{N}^*$ be a tuple of integers with $(s_1, \dots, s_m) = 1$. The size of any second-order cone representation for the weighted geometric mean inequality $x_1^{s_1} \cdots x_m^{s_m} \geq x_{m+1}^{\hat{s}}$ is bounded by $\frac{m}{2}$ from below.

Proof. Let us denote $S := \{(x_i)_{i=1}^{m+1} \in \mathbb{R}_{\geq 0}^{m+1} \mid x_1^{s_1} \cdots x_m^{s_m} \geq x_{m+1}^{\hat{s}}\}$. Note that S contains the nonnegative orthant $\mathbb{R}_{\geq 0}^{m+1}$ as a linear slice (with $x_{m+1} = 0$). Therefore if S admits a K^l -lift, then so does $\mathbb{R}_{\geq 0}^{m+1}$. Now since cone lifts induce order embeddings of face posets [10, section 5.1] and the longest chains of nonempty faces of $\mathbb{R}_{\geq 0}^m$ and K^l are respectively of length $m+1, 2l+1$, we deduce $m+1 \leq 2l+1$, equivalently, $l \geq \frac{m}{2}$ as desired. \square

4. A binary tree representation of successive minimum mediated sequences. In this section, we focus particularly on the case of one-dimensional mediated sets (equivalently, the case of bivariate weighted geometric mean inequalities). For the sake of conciseness, we use the terminology *mediated sequences* to refer to one-dimensional mediated sets. More concretely, given an integer $p > 0$, a set of integers $A \subseteq \mathbb{N}$ is a p -mediated sequence if every number in A is an average of two distinct numbers in $A \cup \{0, p\}$. A p -mediated sequence containing a given number q is called a (p, q) -mediated sequence. As being a mediated sequence is not changed by a scaling, there is no loss of generality in assuming $(p, q) = 1$. A *minimum* (p, q) -mediated sequence is a (p, q) -mediated sequence with the smallest cardinality. We sometimes omit the prefixes and simply say (minimum) mediated sequences if there is no need to mention the specific p, q .

Example 4.1. The set $A = \{2, 4, 5, 8\}$ is a minimum $(11, 2)$ -mediated sequence.

Given $p, q \in \mathbb{N}$ with $0 < q < p$, $(p, q) = 1$, there is an algorithm (Algorithm 4.1) for computing a minimum (p, q) -mediated sequence.

THEOREM 4.2. Algorithm 4.1 is correct.

Proof. Denote the initial values of $s_1, s_2, s_3, t_1, t_2, t_3$ respectively by $s_1^0, s_2^0, s_3^0, t_1^0, t_2^0, t_3^0$, and denote the values after the k th iteration of the loop respectively by $s_1^k, s_2^k, s_3^k, t_1^k, t_2^k, t_3^k$. We prove that

$$(4.1) \quad 2^{l-k} = s_1^k + s_2^k + s_3^k \text{ and } 2^{l-k}q = s_1^k t_1^k + s_2^k t_2^k + s_3^k t_3^k$$

Algorithm 4.1**Input:** Two integers p, q satisfying $0 < q < p$ and $(p, q) = 1$ **Output:** A minimum (p, q) -mediated sequence

```

1:  $l \leftarrow \lceil \log_2(p) \rceil$ ;
2:  $s_1 \leftarrow q, s_2 \leftarrow p - q, s_3 \leftarrow 2^l - p$ ;
3:  $t_1 \leftarrow p, t_2 \leftarrow 0, t_3 \leftarrow q$ ;
4: for  $k \leftarrow 1$  to  $l$  do
5:   Find  $1 \leq i \neq j \leq 3$  such that  $s_i \leq s_j$  are odd numbers;
6:    $q_k \leftarrow \frac{t_i + t_j}{2}, t_i \leftarrow q_k, r \leftarrow \{1, 2, 3\} \setminus \{i, j\}$ ;
7:    $s_j \leftarrow \frac{s_j - s_i}{2}, s_r \leftarrow \frac{s_r}{2}$ ;
8: end for
9: return  $\{q_k\}_{k=1}^l$ ;

```

hold true for $k = 0, 1, \dots, l$ by induction on k . By initialization, we clearly have $2^l = q + p - q + 2^l - p = s_1^0 + s_2^0 + s_3^0$ and $2^l q = q \cdot p + (p - q) \cdot 0 + (2^l - p) \cdot q = s_1^0 t_1^0 + s_2^0 t_2^0 + s_3^0 t_3^0$. Assume that $2^{l-k} = s_1^k + s_2^k + s_3^k$ and $2^{l-k} q = s_1^k t_1^k + s_2^k t_2^k + s_3^k t_3^k$ are true for some $k \geq 0$. Now consider the case of $k + 1$. We note first that at step 5 of Algorithm 4.1, such i, j indeed exist by construction. Then, $s_1^{k+1} + s_2^{k+1} + s_3^{k+1} = s_i^k + \frac{s_j^k - s_i^k}{2} + \frac{s_r^k}{2} = \frac{1}{2}(s_1^k + s_2^k + s_3^k) = 2^{l-k-1}$ and $s_1^{k+1} t_1^{k+1} + s_2^{k+1} t_2^{k+1} + s_3^{k+1} t_3^{k+1} = s_i^k \cdot \frac{t_i^k + t_j^k}{2} + \frac{s_j^k - s_i^k}{2} \cdot t_j^k + \frac{s_r^k}{2} \cdot t_r^k = \frac{1}{2}(s_1^k t_1^k + s_2^k t_2^k + s_3^k t_3^k) = 2^{l-k-1} q$. So we complete the induction.

Letting $k = l$ in (4.1), we obtain $1 = s_1^l + s_2^l + s_3^l$ and $q = s_1^l t_1^l + s_2^l t_2^l + s_3^l t_3^l$. Because s_1^l, s_2^l, s_3^l are nonnegative integers, the equality $1 = s_1^l + s_2^l + s_3^l$ implies $s_i^l = 1, s_j^l = s_r^l = 0$, and hence $q = t_i^l = q_l$. From this and by construction, we see that $\{q_k\}_{k=1}^l$ is indeed a (p, q) -mediated sequence. Moreover, by Theorem 3.11, $\{q_k\}_{k=1}^l$ is minimum. Thus we have proved the correctness of Algorithm 4.1. \square

Remark 4.3. Algorithm 4.1 can be readily adapted to produce an optimal simple second-order cone representation for the bivariate weighted geometric mean inequality $x_1^q x_2^{p-q} \geq x_3^p$ with $0 < q < p$.

Remark 4.4. The essence of Algorithm 4.1 has appeared in the proof of Proposition 5 of [19] in the context of second-order cone representations for trivariate weighted geometric mean inequalities $x_1^{s_1} x_2^{s_2} x_3^{s_3} \geq x_4^{2^l}$ with $s_1 + s_2 + s_3 = 2^l$.

Note that for integers $0 < q < p$, $L(q, p - q)$ denotes the cardinality of a minimum (p, q) -mediated sequence. By Theorem 4.2 we immediately obtain the exact value of $L(q, p - q)$, which resolves a conjecture concerning the value of $L(q, p - q)$ proposed in [18].

COROLLARY 4.5. *For integers $0 < q < p$ with $(p, q) = 1$, it holds that*

$$(4.2) \quad L(q, p - q) = \lceil \log_2(p) \rceil.$$

We notice that Algorithm 4.1 can be readily adapted to produce an optimal simple second-order cone representation for the inequality $x_1^{s_1} x_2^{s_2} x_3^{s_3} \geq x_4^{2^l}$ with $s_1 + s_2 + s_3 = 2^l$ and so obtain the following corollary.

COROLLARY 4.6. *For integers $s_1, s_2, s_3 \in \mathbb{N}^*$ with $(s_1, s_2, s_3) = 1$, if $s_1 + s_2 + s_3 = 2^l$ for some $l \in \mathbb{N}$, then*

$$(4.3) \quad L(s_1, s_2, s_3) = l.$$

The minimum mediated sequence produced by Algorithm 4.1 actually has a special structure which leads to the next definition.

DEFINITION 4.7. For integers $0 < q < p$ with $(p, q) = 1$, a minimum (p, q) -mediated sequence $A = \{q_1, \dots, q_l\}$ with $l = \lceil \log_2(p) \rceil$ is successive if it can be sorted in such a way that

$$(1) \quad q_i = \frac{q_{i-1} + t_i}{2} \text{ for } i = 1, \dots, l,$$

$$(2) \quad q_l = q,$$

where $t_i \in \{0, p, q, q_1, \dots, q_{i-2}\}$ for $i = 1, \dots, l$ and $q_0 \in \{p, q\}$.

Since the minimum mediated sequence output by Algorithm 4.1 is successive by construction, we know that successive minimum (p, q) -mediated sequences exist for any integers $0 < q < p$ with $(p, q) = 1$.

A successive minimum mediated sequence has a distinguished property, that is, it can be represented by a particular “binary tree.” To get a quick flavor of this fact, let us begin with an illustrative example. Let $p = 57, q = 11$, and $A = \{34, 17, 37, 27, 22, 11\}$. Noting

$$(4.4) \quad 34 = \frac{p+q}{2}, 17 = \frac{34}{2}, 37 = \frac{17+p}{2}, 27 = \frac{37+17}{2}, 22 = \frac{27+17}{2}, q = \frac{22}{2},$$

we see that A is a successive minimum (p, q) -mediated sequence. From (4.4), one can easily get the following iterated fraction representation of q :

$$(4.5) \quad q = \frac{\frac{\frac{\frac{p+q}{4} + p}{2} + \frac{p+q}{4}}{2} + \frac{p+q}{4}}{4}.$$

The mediated sequence A can be recovered from (4.5), which is visualized by the binary tree displayed in Figure 4.1.

Taking inspirations from the above example, we can construct a binary tree representation for any successive minimum (p, q) -mediated sequence $A = \{q_i\}_{i=1}^l$, where A is sorted according to Definition 4.7. For simplicity, we assume from now on that

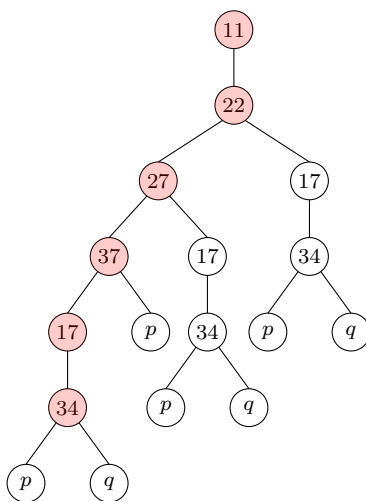


FIG. 4.1. The binary tree representation of a successive minimum (p, q) -mediated sequence with $p = 57, q = 11$.

p, q are both odd.² We shall describe the construction in an iterative manner. Let T_1 be the binary tree consisting of a root node labeled by q_1 along with two children labeled by p and q , respectively. For $2 \leq i \leq l$, we iteratively define

$$(4.6) \quad T_i = \begin{cases} \mathcal{C}(q_i, T_{i-1}) & \text{if } q_i = \frac{q_{i-1}}{2}, \\ \mathcal{C}(q_i, T_{i-1}, p) & \text{if } q_i = \frac{q_{i-1}+p}{2}, \\ \mathcal{C}(q_i, T_{i-1}, q) & \text{if } q_i = \frac{q_{i-1}+q}{2}, \\ \mathcal{C}(q_i, T_{i-1}, T_j) & \text{if } q_i = \frac{q_{i-1}+q_j}{2}, \end{cases}$$

where $\mathcal{C}(q_i, T_{i-1})$ denotes the binary tree obtained by adding a root node labeled by q_i connected to T_{i-1} (viewed as a left subtree); $\mathcal{C}(q_i, T_{i-1}, p)$ (resp., $\mathcal{C}(q_i, T_{i-1}, q)$) denotes the binary tree obtained by adding a root node labeled by q_i connected to the left to T_{i-1} and to the right to a leaf node labeled by p (resp., q); $\mathcal{C}(q_i, T_{i-1}, T_j)$ denotes the binary tree obtained by adding a root node labeled by q_i connected to the left to T_{i-1} and to the right to T_j . We then say that T_l is the *binary tree representation* of A . For a binary tree representation T , we can naturally define the height of any node such that the node at the bottom is of height 0. The height of T or its subtree is the height of the root node. We use $p(T)$ (resp., $q(T)$) to denote the set of heights of leaf nodes labeled by p (resp., q) of T .

THEOREM 4.8. *Suppose that T is the binary tree representation of a successive minimum (p, q) -mediated sequence A . Then the following hold:*

- (1) *The height of T is $\lceil \log_2(p) \rceil$.*
- (2) *Any leaf node of T is labeled by either p or q .*
- (3) *The root node of T is labeled by q .*
- (4) *The root node of the left subtree of T of height i is labeled by q_i .*
- (5) *Any right subtree of T either is a leaf node or coincides with some left subtree.*
- (6) *The label of any nonleaf node is the average of labels of its children or the half of the label of its child if there is only one child.*
- (7) *$\sum_{j \in p(T)} 2^j = q$ and $\sum_{j \in q(T)} 2^j = 2^{\lceil \log_2(p) \rceil} - p$.*

Proof. (1)–(6) are immediate from the construction.

Let $l = \lceil \log_2(p) \rceil$ and assume that $A = \{q_i\}_{i=1}^l$ is sorted according to Definition 4.7. For $i = 1, \dots, l$, let T_i be the left subtree of T with the root node labeled by q_i . We claim that

$$(4.7) \quad 2^i q_i = \left(\sum_{j \in p(T_i)} 2^j \right) p + \left(\sum_{j \in q(T_i)} 2^j \right) q$$

holds true for all i . Let us prove (4.7) by induction on i . For $i = 1$, we have $2q_1 = p + q$, which is exactly (4.7). Now assume that (4.7) is true for $i = k \geq 1$. Then if $q_{k+1} = \frac{q_k + q_t}{2}$, we have

$$\begin{aligned} 2^{k+1} q_{k+1} &= 2^k q_k + 2^k q_t \\ &= \left(\sum_{j \in p(T_k)} 2^j + 2^{k-t} \sum_{j \in p(T_t)} 2^j \right) p + \left(\sum_{j \in q(T_k)} 2^j + 2^{k-t} \sum_{j \in q(T_t)} 2^j \right) q \\ &= \left(\sum_{j \in p(T_{k+1})} 2^j \right) p + \left(\sum_{j \in q(T_{k+1})} 2^j \right) q. \end{aligned}$$

²The other cases can be easily converted to this case.

Similarly, we can prove (4.7) with $i = k + 1$ holds for the remaining cases (i.e., $q_{k+1} = \frac{q_k}{2}, \frac{q_k+p}{2}, \frac{q_k+q}{2}$). Hence (4.7) is true for $i = k + 1$ and we complete the induction. Letting $i = l$ in (4.7) gives $2^{\lceil \log_2(p) \rceil} q = (\sum_{j \in p(T)} 2^j) p + (\sum_{j \in q(T)} 2^j) q$, from which we deduce that $\sum_{j \in p(T)} 2^j = q$ and $\sum_{j \in q(T)} 2^j = 2^{\lceil \log_2(p) \rceil} - p$. \square

By virtue of Theorem 4.8, we are able to enumerate all successive minimum (p, q) -mediated sequences for given p, q via traversing the related binary tree representations. We emphasize that property (7) stated in Theorem 4.8 is crucial to reduce the search space of valid binary tree representations.

5. Algorithms. Now we study algorithms for computing simple second-order cone representations of weighted geometric mean inequalities. Unlike the bivariate case treated in the previous section, computing optimal simple second-order cone representations for general weighted geometric mean inequalities seems a notoriously difficult problem. Therefore, we are mostly interested in efficient heuristic algorithms that can produce approximately optimal simple second-order cone representations. We will propose two types of fast heuristic algorithms in subsection 5.1, and then extend them to certain “traversal-style” algorithms in subsection 5.2. For completeness and comparison, a brute-force algorithm for computing optimal simple second-order cone representations is provided in subsection 5.3. Finally, we evaluate all these algorithms via numerical experiments in subsection 5.4.

5.1. Fast heuristic algorithms. Our heuristic algorithms rely on a simple result which is stated in the following lemma.

LEMMA 5.1 (see [17, Lemma 1]). *Let $s_1, \dots, s_m \in \mathbb{N}^*$ be a tuple of integers. Then for a pair $i, j \in \{1, \dots, m\}$ and for any $\gamma \in \mathbb{N}$ with $0 < \gamma \leq \min\{s_i, s_j\}$, one has*

$$\prod_{k=1}^m x_k^{s_k} \geq x_{m+1}^{\hat{s}} \iff \exists y \geq 0 \text{ s.t. } x_i^{s_i-\gamma} x_j^{s_j-\gamma} y^{2\gamma} \prod_{\substack{k=1 \\ k \neq i, j}}^m x_k^{s_k} \geq x_{m+1}^{\hat{s}}, x_i x_j \geq y^2.$$

Suppose that we are given the inequality $\prod_{k=1}^m x_k^{s_k} \geq x_{m+1}^{\hat{s}}$ and let $l = \lceil \log_2(\hat{s}) \rceil$. Let us multiply $\prod_{k=1}^m x_k^{s_k} \geq x_{m+1}^{\hat{s}}$ by $x_{m+1}^{2^l - \hat{s}}$ to obtain $\prod_{k=1}^m x_k^{s_k} x_{m+1}^{2^l - \hat{s}} \geq x_{m+1}^{2^l}$ so that the exponent of x_{m+1} is a power of 2. Now we claim that if the exponent of x_{m+1} is a power of 2, then we are able to obtain a simple second-order cone representation for the weighted geometric mean inequality by iteratively applying Lemma 5.1 with appropriate i, j , and γ as detailed in Algorithm 5.1. Note that Algorithm 5.1 terminates when the tuple s_1, \dots, s_m reduces to two nonzero numbers, each being 2^{l-1} .

In the following, we propose two strategies to guide us in selecting such a pair i, j and the factor γ at step 4 of Algorithm 5.1. For an integer r , let $\Omega(r)$ be the set of exponents of 2 involved in the binary representation of r , and let $\Delta(r) := \min \Omega(r)$ be the minimal exponent of 2 involved in the binary representation of r . For example, with $r = 7$, one has $\Omega(r) = \{0, 1, 2\}$ and $\Delta(r) = 0$.

THEOREM 5.2 (cf. [17, Proposition 1]). *If we select the pair i, j satisfying $\Omega(s_i) \cap \Omega(s_j) \neq \emptyset$ and let $\gamma = \sum_{k \in \Omega(s_i) \cap \Omega(s_j)} 2^k$ at step 4, then Algorithm 5.1 terminates.*

Proof. As $s_1 + \dots + s_m = 2^l$ holds true at step 4, we can find a pair i, j such that $\Delta(s_i) = \Delta(s_j)$ and so $\Omega(s_i) \cap \Omega(s_j) \neq \emptyset$. Let us consider the quantity $h := \sum_{i=1}^m |\Omega(s_i)|$. After one iteration, s_i, s_j are replaced by $s_i - \gamma, s_j - \gamma$, and 2γ is added to the tuple. We then see that h decreases by $|\Omega(s_i) \cap \Omega(s_j)| \geq 1$ at each iteration. It

Algorithm 5.1**Input:** A tuple of positive integers s_1, \dots, s_m with $(s_1, \dots, s_m) = 1$ **Output:** A configuration $\{(i_k, j_k, t_k)\}_k$ that determines a simple second-order cone representation for $x_1^{s_1} \cdots x_m^{s_m} \geq x_{m+1}^{\hat{s}}$

```

1:  $l \leftarrow \lceil \log_2(\hat{s}) \rceil$ ,  $t \leftarrow m+1$ ,  $m \leftarrow m+1$ ,  $s_m \leftarrow 2^l - \hat{s}$ ;
2:  $\mathcal{S} \leftarrow \emptyset$ ;
3: repeat
4:   Select a pair  $i, j \in \{1, \dots, m\}$  and an integer  $\gamma$  satisfying  $0 < \gamma \leq \min\{s_i, s_j\}$ ;
5:    $m \leftarrow m+1$ ,  $s_i \leftarrow s_i - \gamma$ ,  $s_j \leftarrow s_j - \gamma$ ,  $s_m \leftarrow 2\gamma$ ;
6:    $\mathcal{S} \leftarrow \mathcal{S} \cup \{(i, j, m)\}$ ;
7: until  $\gamma = 2^{l-1}$ 
8: return  $\mathcal{S} \cup \{(i, j, t)\}$ ;

```

follows that the condition $\gamma = 2^{l-1}$ is satisfied in at most $\sum_{i=1}^m |\Omega(s_i)| - 2$ iterations. Therefore, Algorithm 5.1 terminates. \square

COROLLARY 5.3. Let $s_1, \dots, s_m \in \mathbb{N}^*$ be a tuple of integers with $(s_1, \dots, s_m) = 1$. Then

$$(5.1) \quad L(s_1, \dots, s_m) \leq \sum_{i=1}^m |\Omega(s_i)| - 1.$$

Proof. The proof of Theorem 5.2 immediately implies that Algorithm 5.1 yields a simple second-order cone representation for $x_1^{s_1} \cdots x_m^{s_m} \geq x_{m+1}^{\hat{s}}$ of size at most $\sum_{i=1}^m |\Omega(s_i)| - 1$, from which we get the desired inequality. \square

THEOREM 5.4. If we select the pair i, j satisfying

$$\Delta(s_i) = \Delta(s_j) = \min \{\Delta(s_k)\}_{k=1}^m,$$

and let $\gamma = \min\{s_i, s_j\}$ at step 4, then Algorithm 5.1 terminates.

Proof. As $s_1 + \cdots + s_m = 2^l$ holds true at step 4, we can find a pair i, j such that $\Delta(s_i) = \Delta(s_j) = \min \{\Delta(s_k)\}_{k=1}^m$ and so $\Delta(s_i - s_j) - \Delta(s_j) \geq 1$. Let us consider the quantity $h := \sum_{i=1}^m (l - \Delta(s_i))$ (set $\Delta(0) := l$). After one iteration, s_i, s_j are replaced by $\max\{s_i, s_j\} - \min\{s_i, s_j\}, 0$, and $2 \min\{s_i, s_j\}$ is added to the tuple. We then see that h decreases by $1 + \Delta(s_i - s_j) - \Delta(s_j) \geq 2$ at each iteration. It follows that the condition $\gamma = 2^{l-1}$ is satisfied in at most $\frac{1}{2} \sum_{i=1}^m (l - \Delta(s_i)) - 1$ iterations. Therefore, Algorithm 5.1 terminates. \square

COROLLARY 5.5. Let $s_1, \dots, s_m \in \mathbb{N}^*$ be a tuple of integers with $(s_1, \dots, s_m) = 1$ and let $l = \lceil \log_2(\hat{s}) \rceil$. Then

$$(5.2) \quad L(s_1, \dots, s_m) \leq \frac{1}{2} \sum_{i=1}^m (l - \Delta(s_i)).$$

Proof. The proof of Theorem 5.4 immediately implies that Algorithm 5.1 yields a simple second-order cone representation for $x_1^{s_1} \cdots x_m^{s_m} \geq x_{m+1}^{\hat{s}}$ of size at most $\frac{1}{2} \sum_{i=1}^m (l - \Delta(s_i))$. Thus we get the desired inequality. \square

Theorems 5.2 and 5.4 allow us to find an approximately optimal simple second-order cone representation for a weighted geometric mean inequality by implementing greedy strategies at each iteration of the loop in Algorithm 5.1:

- (1) selecting the pair i, j to maximize $|\Omega(s_i) \cap \Omega(s_j)|$ and letting $\gamma = \sum_{k \in \Omega(s_i) \cap \Omega(s_j)} 2^k$ (hereafter referred to as the greedy-common-one strategy), or alternatively,
- (2) selecting the pair i, j to maximize $\Delta(s_i - s_j)$ with $\Delta(s_i) = \Delta(s_j) = \min \{s_k\}_{k=1}^m$ and letting $\gamma = \min \{s_i, s_j\}$ (hereafter referred to as the greedy-power-two strategy).

Remark 5.6. Algorithm 5.1 that employs the greedy-common-one strategy has been used to compute simple second-order cone representations for weighted geometric mean inequalities $x_1^{s_1} \cdots x_m^{s_m} \geq x_{m+1}^{\hat{s}}$ with $\hat{s} = 2^l$ in [17].

The following lemmas whose proofs are straightforward allow us to further enhance the performance of Algorithm 5.1.

LEMMA 5.7. *Let $s_1, \dots, s_m \in \mathbb{N}^*$ be a tuple of integers. Assume $s_m = \max \{s_i\}_{i=1}^m$ and $l = \lceil \log_2(\hat{s}) \rceil$. Then,*

$$\prod_{k=1}^m x_k^{s_k} \geq x_{m+1}^{\hat{s}} \iff \exists y \geq 0 \text{ s.t. } \begin{cases} \prod_{k=1}^{m-1} x_k^{s_k} \geq y^{\frac{\hat{s}}{2}}, x_m y \geq x_{m+1}^2 & \text{if } s_m = \frac{\hat{s}}{2}; \\ \prod_{k=1}^{m-1} x_k^{s_k} x_{m+1}^{2s_m - \hat{s}} \geq y^{s_m}, x_m y \geq x_{m+1}^2 & \text{if } \frac{\hat{s}}{2} < s_m \leq 2^{l-1}; \\ \prod_{k=1}^{m-1} x_k^{s_k} x_m^{s_m - 2^{l-1}} x_{m+1}^{2^l - \hat{s}} \geq y^{2^{l-1}}, x_m y \geq x_{m+1}^2 & \text{if } s_m > 2^{l-1}, \hat{s} < 2^l; \\ \prod_{k=1}^{m-1} x_k^{s_k} x_m^{s_m - 2^{l-1}} \geq y^{2^{l-1}}, x_m y \geq x_{m+1}^2 & \text{if } s_m > 2^{l-1}, \hat{s} = 2^l. \end{cases}$$

LEMMA 5.8. *Let $s_1, \dots, s_m \in \mathbb{N}^*$ be a tuple of integers and assume $s_1 = s_2$. Then,*

$$\prod_{k=1}^m x_k^{s_k} \geq x_{m+1}^{\hat{s}} \iff \exists y \geq 0 \text{ s.t. } \prod_{k=3}^m x_k^{s_k} y^{2s_2} \geq x_{m+1}^{\hat{s}}, x_1 x_2 \geq y^2.$$

LEMMA 5.9. *Let $s_1, \dots, s_m \in \mathbb{N}^*$ be a tuple of integers. Assume $l = \lceil \log_2(\hat{s}) \rceil$ and s_1 is the unique odd number among s_1, \dots, s_m , which satisfies $s_1 \leq 2^l - \hat{s}$. Then,*

$$\prod_{k=1}^m x_k^{s_k} \geq x_{m+1}^{\hat{s}} \iff \exists y \geq 0 \text{ s.t. } \prod_{k=2}^m x_k^{\frac{s_k}{2}} y^{s_1} \geq x_{m+1}^{\frac{\hat{s}+s_1}{2}}, x_1 x_{m+1} \geq y^2.$$

By invoking either of Lemmas 5.7 and 5.9 if applicable, we get either a reduction of \hat{s} by a factor 2 or a decrease of m by 1 at the price of one quadratic inequality. We thereby embed Lemmas 5.7 and 5.9 into Algorithm 5.1 as detailed in Algorithm 5.2.

5.2. Traversal algorithms. Instead of considering only the “maximal” pair at each iteration (step 4 of Algorithm 5.1), we may take into account all pairs i, j such that $\Omega(s_i) \cap \Omega(s_j) \neq \emptyset$ (hereafter referred to as the common-one strategy) or $\Delta(s_i) = \Delta(s_j) = \min \{s_k\}_{k=1}^m$ (hereafter referred to as the power-two strategy). Such traversal algorithms enable us to obtain a simple second-order cone representation of possibly smaller size for a weighted geometric mean inequality by spending more time.

5.3. A brute-force algorithm. Besides the heuristics aiming to efficiently produce approximately optimal simple second-order cone representations, we also propose a brute-force algorithm to compute an exact optimal simple second-order cone representation for a weighted geometric mean inequality.

Let the configuration $\{(i_k, j_k, m+k)\}_{k=1}^n$ determine a simple second-order cone representation for $x_1^{s_1} \cdots x_m^{s_m} \geq x_{m+1}^{\hat{s}}$ (equivalently, an $(\mathcal{A}, \alpha_{m+1})$ -mediated set with

Algorithm 5.2 $\text{Heuristic}(s_1, \dots, s_m; t = m + 1)$

Input: A tuple of nonnegative integers s_1, \dots, s_m and a positive integer t equaling $m + 1$ by default

Output: A configuration $\{(i_k, j_k, t_k)\}_k$ that determines a simple second-order cone representation for $x_1^{s_1} \dots x_m^{s_m} \geq x_t^{\hat{s}}$

- 1: **if** $t = m + 1$ **then**
- 2: $m \leftarrow m + 1, s_m \leftarrow 0$;
- 3: **end if**
- 4: $l \leftarrow \lceil \log_2(\hat{s}) \rceil$;
- 5: **if** $\exists i, j \in \{1, \dots, m\}$ such that $s_i = s_j > 0$ **then**
- 6: **if** $s_i \neq 2^{l-1}$ **then**
- 7: $m \leftarrow m + 1, s_m \leftarrow 2s_i, s_i \leftarrow 0, s_j \leftarrow 0$;
- 8: **return** $\text{Heuristic}(s_1, \dots, s_m; t) \cup \{(i, j, m)\}$;
- 9: **else**
- 10: **return** $\{(i, j, t)\}$;
- 11: **end if**
- 12: **end if**
- 13: Find $k \in \{1, \dots, m\}$ such that $s_k = \max\{s_i\}_{i=1}^m$;
- 14: **if** $2s_k \geq \hat{s}$ **then**
- 15: **if** $2s_k = \hat{s}$ **then**
- 16: $s_k \leftarrow 0$;
- 17: **else if** $s_k \leq 2^{l-1}$ **then**
- 18: $s_t \leftarrow 2s_k - \hat{s}, s_k \leftarrow 0$;
- 19: **else if** $\hat{s} < 2^l$ **then**
- 20: $s_t \leftarrow 2^l - \hat{s}, s_k \leftarrow s_k - 2^{l-1}$;
- 21: **else**
- 22: $s_k \leftarrow s_k - 2^{l-1}$;
- 23: **end if**
- 24: **return** $\text{Heuristic}(s_1, \dots, s_m; m + 1) \cup \{(k, m + 1, t)\}$;
- 25: **end if**
- 26: **if** $\arg \min\{\Delta(s_i)\}_{i=1}^m = \{r\}$ and $s_r \leq 2^l - \hat{s}$ **then**
- 27: $m \leftarrow m + 1, s_m \leftarrow 2s_r, s_r \leftarrow 0$;
- 28: **return** $\text{Heuristic}(s_1, \dots, s_m; t) \cup \{(r, t, m)\}$;
- 29: **end if**
- 30: $s_t \leftarrow 2^l - \hat{s}$;
- 31: **if** $\exists i \in \{1, \dots, m\}, i \neq t$ such that $s_t = s_i > 0$ **then**
- 32: $m \leftarrow m + 1, s_m \leftarrow 2s_i, s_i \leftarrow 0, s_t \leftarrow 0$;
- 33: **return** $\text{Heuristic}(s_1, \dots, s_m; t) \cup \{(i, t, m)\}$;
- 34: **end if**
- 35: Select a pair $i, j \in \{1, \dots, m\}$ and an integer γ satisfying $0 < \gamma \leq \min\{s_i, s_j\}$;
- 36: $m \leftarrow m + 1, s_m \leftarrow 2\gamma, s_i \leftarrow s_i - \gamma, s_j \leftarrow s_j - \gamma$;
- 37: **return** $\text{Heuristic}(s_1, \dots, s_m; t) \cup \{(i, j, m)\}$;

$\mathcal{A} = \{\alpha_i\}_{i=1}^m$ being a trellis and $\alpha_{m+1} = \sum_{i=1}^m \frac{s_i}{\hat{s}} \alpha_i$ so that (3.2) holds. Denote the coefficient matrix of (3.2) by C and let $A = [\alpha_1, \dots, \alpha_{m+n}]^\top$ with α_i 's being viewed as column vectors, so that we can rewrite (3.2) in matrix form as $CA = 0$. By construction, the equality $\sum_{i=1}^m s_i \alpha_i = \hat{s} \alpha_{m+1}$ is implied by (3.2), which means that the vector $(s_1, \dots, s_m, -\hat{s}, 0, \dots, 0)$ belongs to the row space of C , i.e., there exist

Algorithm 5.3 Bruteforce(s_1, \dots, s_m)**Input:** A tuple of positive integers s_1, \dots, s_m with $(s_1, \dots, s_m) = 1$ **Output:** A configuration $\{(i_k, j_k, m+k)\}_{k=1}^n$ that determines an optimal simple second-order cone representation for $x_1^{s_1} \cdots x_m^{s_m} \geq x_{m+1}^{\hat{s}}$

- 1: $n \leftarrow \max\{\lceil \log_2(\hat{s}) \rceil, m-1\}$; #Remark 3.14
- 2: Enumerate all legitimate configurations $\{(i_k, j_k, m+k)\}_{k=1}^n$ which are denoted by $\mathcal{T}_{m,n}$;
- 3: **for** each \mathcal{S} in $\mathcal{T}_{m,n}$ **do**
- 4: **if** The system (5.3) is feasible **then**
- 5: **return** \mathcal{S} ;
- 6: **end if**
- 7: **end for**
- 8: $n \leftarrow n+1$;
- 9: **goto** Step 2;

$\gamma_1, \dots, \gamma_n \in \mathbb{Q}$ such that $(s_1, \dots, s_m, -\hat{s}, 0, \dots, 0) = \sum_{t=1}^n \gamma_t \mathbf{c}_t$ with $\{\mathbf{c}_t\}_{t=1}^n$ being the row vectors of C . It follows that the following linear system in variables $\gamma_1, \dots, \gamma_n$ admits a rational solution:

$$(5.3) \quad \begin{cases} \sum_{i_t=k} \gamma_t + \sum_{j_t=k} \gamma_t = s_k, & k=1, \dots, m, \\ \sum_{i_t=k} \gamma_t + \sum_{j_t=k} \gamma_t = 2\gamma_1 - \hat{s}, & k=m+1, \\ \sum_{i_t=k} \gamma_t + \sum_{j_t=k} \gamma_t = 2\gamma_{k-m}, & k=m+2, \dots, m+n. \end{cases}$$

Conversely, if the linear system (5.3) admits a rational solution for some configuration $\{(i_k, j_k, m+k)\}_{k=1}^n$, then this configuration gives rise to (3.2) and hence determines an $(\mathcal{A}, \alpha_{m+1})$ -mediated set with $\alpha_{m+1} = \sum_{i=1}^m \frac{s_i}{\hat{s}} \alpha_i$ as well as a simple second-order cone representation for $x_1^{s_1} \cdots x_m^{s_m} \geq x_{m+1}^{\hat{s}}$. Building upon these facts, we give the brute-force algorithm for computing optimal simple second-order cone representations in Algorithm 5.3.

The most expensive part of Algorithm 5.3 is step 2 as for given m, n , the number of legitimate configurations $\{(i_k, j_k, m+k)\}_{k=1}^n$ might be very large. In order to speed up the enumeration, we thereby impose some conditions on legitimate configurations which are stated in the following proposition.

PROPOSITION 5.10. *There is no loss of generality in assuming that any legitimate configuration $\{(i_k, j_k, m+k)\}_{k=1}^n$ satisfies the following conditions:*

- (1) $\{1, \dots, m\} \cup \{m+2, \dots, m+n\} \subseteq \cup_{k=1}^n \{i_k, j_k\}$;
- (2) $i_k < j_k$;
- (3) $i_k, j_k \neq m+k$;
- (4) $(i_{k_1}, j_{k_1}) \neq (i_{k_2}, j_{k_2})$ if $k_1 \neq k_2$;
- (5) $\{i_{k_1}, j_{k_1}, m+k_1\} \neq \{i_{k_2}, j_{k_2}, m+k_2\}$ if $k_1 \neq k_2$;
- (6) $t_k \geq m+k+1$ for $k=1, \dots, n-1$;
- (7) $i_1 \leq m, j_1 = m+2$ or $i_1 = m+2, j_1 = m+3$;
- (8) $i_k \leq t_{k-1}, j_k \leq t_{k-1}+1$ or $i_k = t_{k-1}+1, j_k = t_{k-1}+2$ for $k=2, \dots, n$,

where $t_k := \max\{j_1, \dots, j_k\}$ for $k=1, \dots, n-1$.

Proof. (1) follows from the proof of Theorem 3.6. (2)–(5) are immediate from the definition. (6) is due to the fact that we are seeking a configuration of minimum size and if $t_k < m+k+1$ for some $k \in \{1, \dots, n-1\}$, then $\{(i_l, j_l, m+l)\}_{l=1}^k$ is a

TABLE 5.1

The number of configurations that satisfy the conditions in Proposition 5.10.

(m, n)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
$ \mathcal{T}_{m,n} $	3	48	828	17178	419559	18	588	17016	514524

TABLE 5.2

Comparison of heuristic algorithms with $\hat{s} = 83$. For each m , the first column indicates the sum of sizes of simple second-order cone representations over partitions of length m , and the second column indicates the total running time in seconds. The symbol – means running time > 1 day.

Algorithm	$m = 3$		$m = 4$		$m = 5$		$m = 6$	
GreedyPowertwo	4567	0.001	37996	0.01	196262	0.11	697083	0.40
GreedyCommone	4695	0.025	39625	0.35	204927	2.36	728705	10.3
TraversalPowertwo	4561	0.007	37648	0.25	192654	9.82	681705	90.0
TraversalCommone	4660	0.311	38894	1074	–	–	–	–

configuration of smaller size. (7)–(8) are because we can arbitrarily label the auxiliary variables x_{m+2}, \dots, x_{m+n} . \square

Table 5.1 shows the cardinality of $\mathcal{T}_{m,n}$ derived from Proposition 5.10 for different (m, n) . We could see that $|\mathcal{T}_{m,n}|$ grows rapidly with m, n . However, for each (m, n) , the set $\mathcal{T}_{m,n}$ needs to be computed just once and then can be used forever.

5.4. Numerical experiments. The algorithms discussed above were implemented in the Julia package MiniSOC, which is available at <https://github.com/wangjie212/MiniSOC>. The numerical experiments were performed on a desktop computer with Windows 10 system, Intel Core i9-10900 CPU@2.80 GHz and 32G RAM.

5.4.1. Evaluating different heuristic algorithms. To test the performance of different heuristic algorithms, we run them on the partitions $s_1 + \dots + s_m$ of the integer $\hat{s} = 83$. The partitions are restricted to be of length $m = 3, 4, 5, 6$ and satisfy $(s_1, \dots, s_m) = 1$. We denote Algorithm 5.2 implementing the greedy-common-one strategy (resp., the greedy-power-two strategy) by **GreedyCommone** (resp., **GreedyPowertwo**); we denote the traversal algorithm implementing the common-one strategy (resp., the power-two strategy) by **TraversalCommone** (resp., **TraversalPowertwo**). The results are reported in Table 5.2. The data in Table 5.2 show that (1) the greedy algorithms are significantly faster than the traversal algorithms when $m \geq 5$; (2) the traversal algorithms may produce simple second-order cone representations of smaller size than the greedy algorithms; (3) **GreedyPowertwo** not only runs faster than **GreedyCommone** but also produces simple second-order cone representations of smaller size, and a similar statement also applies to **TraversalPowertwo** and **TraversalCommone**.

We mention that for $m = 3, 4, 5, 6$, the number of different partitions of 83 are 574, 4109, 18487, 58767, respectively, and the average sizes of second-order cone representations produced by the algorithm **GreedyPowertwo** are 8.0, 9.2, 10.6, 11.9, respectively.

To further compare the algorithm **GreedyPowertwo** with **GreedyCommone**, we generate random integers $s_1, \dots, s_m \in (0, 10^t)$ with $m = 10$. The related results are reported in Table 5.3, which confirms our observation that **GreedyPowertwo** not only runs faster than **GreedyCommone** (by a factor ~ 100) but also produces second-order cone representations of smaller size.

TABLE 5.3

Results of GreedyPowertwo and GreedyCommone on random instances. Each t corresponds to three different random trials. For each algorithm, the first column indicates the size of second-order cone representation, and the second column indicates the running time in seconds.

t	GreedyPowertwo		GreedyCommone	
10	91	0.0002	98	0.02
	90	0.0002	112	0.02
	91	0.0002	114	0.02
12	107	0.0002	135	0.03
	109	0.0003	119	0.03
	108	0.0003	121	0.04
14	133	0.0004	139	0.04
	122	0.0003	155	0.04
	128	0.0003	136	0.05
16	146	0.0004	179	0.07
	140	0.0004	152	0.06
	146	0.0004	195	0.07

5.4.2. Comparison with optimal simple second-order cone representations. We provide the sizes of optimal simple second-order cone representations for trivariate weighted geometric mean inequalities $x_1^{s_1} x_2^{s_2} x_3^{s_3} \geq x_4^{\hat{s}}$ with $\hat{s} \leq 15$ in Table 5.4, which are obtained with the brute-force algorithm (Algorithm 5.3).

According to Table 5.4, we emphasize that the four heuristic algorithms produce optimal simple second-order cone representations for all trivariate weighted geometric mean inequalities $x_1^{s_1} x_2^{s_2} x_3^{s_3} \geq x_4^{\hat{s}}$ with $\hat{s} \leq 15$ but 4 instances corresponding to $(s_1, s_2, s_3) = (5, 4, 3)$, $(7, 3, 2)$, $(6, 5, 3)$, $(11, 2, 1)$, respectively. For $(s_1, s_2, s_3) = (5, 4, 3)$, $(7, 3, 2)$, and $(11, 2, 1)$, the four heuristic algorithms yield simple second-order cone representations of size 5, while the optimal size is 4; for $(s_1, s_2, s_3) = (6, 5, 3)$, GreedyCommone and TraversalCommone yield simple second-order cone representations of size 6, and GreedyPowertwo and TraversalPowertwo yield simple second-order cone representations of size 5, while the optimal size is 4.

6. Applications. In this section, we give three applications of the proposed algorithm GreedyPowertwo in polynomial optimization, matrix optimization, and quantum information, respectively. All numerical experiments were performed on a desktop computer with Windows 10 system, Intel Core i9-10900 CPU@2.80 GHz and 32G RAM.

6.1. SONC optimization. Let $\mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \dots, x_n]$ be the ring of real n -variate polynomials. For $\boldsymbol{\alpha} = (\alpha_i)_i \in \mathbb{N}^n$, let $\mathbf{x}^{\boldsymbol{\alpha}} := x_1^{\alpha_1} \dots x_n^{\alpha_n}$. Suppose that $\mathcal{A} = \{\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_{n+1}\} \subseteq (2\mathbb{N})^n$ is a trellis. A polynomial $f = \sum_{i=1}^{n+1} c_i \mathbf{x}^{\boldsymbol{\alpha}_i} - d \mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$ is called a *circuit polynomial* if $c_i > 0$ for $i = 1, \dots, n+1$ and $\boldsymbol{\beta} \in \text{conv}(\mathcal{A})^\circ \cap \mathbb{N}^n$ [16]. The nonnegativity of a circuit polynomial f on \mathbb{R}^n can be easily verified by

$$(6.1) \quad f \geq 0 \iff \begin{cases} d < 0 \text{ or } \prod_{i=1}^{n+1} (c_i/\lambda_i)^{\lambda_i} \geq d \geq 0 & \text{if } \boldsymbol{\beta} \in (2\mathbb{N})^n, \\ \prod_{i=1}^{n+1} (c_i/\lambda_i)^{\lambda_i} \geq |d| & \text{if } \boldsymbol{\beta} \notin (2\mathbb{N})^n, \end{cases}$$

where $(\lambda_i)_{i=1}^{n+1} \in \mathbb{Q}_+^{n+1}$ are the barycentric coordinates of $\boldsymbol{\beta}$ with respect to \mathcal{A} satisfying $\boldsymbol{\beta} = \sum_{i=1}^{n+1} \lambda_i \boldsymbol{\alpha}_i$ and $\sum_{i=1}^{n+1} \lambda_i = 1$. Note that the conditions in (6.1) admit second-order cone representations as the inequalities $\prod_{i=1}^{n+1} (c_i/\lambda_i)^{\lambda_i} \geq d \geq 0$ are equivalent to

TABLE 5.4

Sizes of optimal simple second-order cone representations for trivariate weighted geometric mean inequalities $x_1^{s_1} x_2^{s_2} x_3^{s_3} \geq x_4^{\hat{s}}$ (labeled by (s_1, s_2, s_3)) with $\hat{s} \leq 15$. We exclude the cases $\hat{s} = 4, 8$ in view of Corollary 4.6.

$\hat{s} = 3$	(1, 1, 1)									
	3									
$\hat{s} = 5$	(2, 2, 1)	(3, 1, 1)								
	4	4								
$\hat{s} = 6$	(3, 2, 1)	(4, 1, 1)								
	3	3								
$\hat{s} = 7$	(3, 2, 2)	(3, 3, 1)	(4, 2, 1)	(5, 1, 1)						
	4	4	3	4						
$\hat{s} = 9$	(4, 3, 2)	(4, 4, 1)	(5, 2, 2)	(5, 3, 1)	(6, 2, 1)	(7, 1, 1)				
	4	5	5	5	4	5				
$\hat{s} = 10$	(4, 3, 3)	(5, 3, 2)	(5, 4, 1)	(6, 3, 1)	(7, 2, 1)	(8, 1, 1)				
	4	4	4	4	4	4				
$\hat{s} = 11$	(4, 4, 3)	(5, 3, 3)	(5, 4, 2)	(5, 5, 1)	(6, 3, 2)	(6, 4, 1)	(7, 2, 2)	(7, 3, 1)	(8, 2, 1)	(9, 1, 1)
	5	5	4	5	4	4	5	5	4	5
$\hat{s} = 12$	(5, 4, 3)	(5, 5, 2)	(6, 5, 1)	(7, 3, 2)	(7, 4, 1)	(8, 3, 1)	(9, 2, 1)	(10, 1, 1)		
	4	4	4	4	4	4	4	4		
$\hat{s} = 13$	(5, 4, 4)	(5, 5, 3)	(6, 4, 3)	(6, 5, 2)	(6, 6, 1)	(7, 3, 3)	(7, 4, 2)	(7, 5, 1)	(8, 3, 2)	(8, 4, 1)
	5	5	4	5	5	5	4	5	4	4
	(9, 2, 2)	(9, 3, 1)	(10, 2, 1)	(11, 1, 1)						
	5	5	5	5						
$\hat{s} = 14$	(5, 5, 4)	(6, 5, 3)	(7, 4, 3)	(7, 5, 2)	(7, 6, 1)	(8, 3, 3)	(8, 5, 1)	(9, 3, 2)	(9, 4, 1)	(10, 3, 1)
	4	4	4	4	4	4	4	5	4	5
	(11, 2, 1)	(12, 1, 1)								
	4	4								
$\hat{s} = 15$	(6, 5, 4)	(7, 4, 4)	(7, 5, 3)	(7, 6, 2)	(7, 7, 1)	(8, 4, 3)	(8, 5, 2)	(8, 6, 1)	(9, 4, 2)	(9, 5, 1)
	5	5	6	5	5	4	4	4	4	5
	(10, 3, 2)	(10, 4, 1)	(11, 2, 2)	(11, 3, 1)	(12, 2, 1)	(13, 1, 1)				
	5	4	5	5	4	5				

$$(6.2) \quad \exists y \geq 0 \text{ s.t. } \prod_{i=1}^{n+1} c_i^{\lambda_i} \geq y, d \prod_{i=1}^{n+1} \lambda_i^{\lambda_i} = y,$$

and the inequality $\prod_{i=1}^{n+1} (c_i/\lambda_i)^{\lambda_i} \geq |d|$ is equivalent to

$$(6.3) \quad \exists y \geq 0 \text{ s.t. } \prod_{i=1}^{n+1} c_i^{\lambda_i} \geq y, |d| \prod_{i=1}^{n+1} \lambda_i^{\lambda_i} \leq y.$$

One can certify the nonnegativity of a polynomial f by decomposing it into a SONC. Furthermore, one can provide a lower bound on the global minimum of f by solving the following SONC optimization problem:

$$(6.4) \quad \begin{cases} \sup & \gamma \\ \text{s.t.} & f - \gamma \text{ is a SONC.} \end{cases}$$

By virtue of (6.2) and (6.3), (6.4) can be further modeled as a second-order cone program; see [18] for more details.

We select 30 randomly generated polynomials from the database provided by Seidler and de Wolff in [23] and solve the related SONC optimization problem (6.4) with

TABLE 6.1

Results for the SONC optimization problem (6.4). n, d, t denote the number of variables, the degree, and the number of terms of the polynomial f , respectively; the column labeled “opt” indicates the optimum; the columns labeled “W” and “K-B-G” indicate the running time in seconds of the approaches with **GreedyPowertwo** and the algorithm from [17], respectively.

(n, d, t)	opt	W	K-B-G
(10, 40, 100)	5.4251	0.35	1.55
(10, 40, 200)	7.7161	0.78	3.26
(10, 40, 300)	38.662	1.15	4.73
(10, 50, 100)	0.1972	0.35	1.73
(10, 50, 200)	4.8642	0.91	4.00
(10, 50, 300)	7.0048	1.41	6.09
(10, 60, 100)	2.5232	0.43	2.57
(10, 60, 200)	26.017	1.04	4.92
(10, 60, 300)	23.417	1.76	8.27
(20, 40, 100)	2.8614	0.47	9.90
(20, 40, 200)	9.8565	1.21	24.0
(20, 40, 300)	1.9062	2.20	35.6
(20, 50, 100)	2.1435	0.53	12.9
(20, 50, 200)	8.1728	1.46	26.8
(20, 50, 300)	14.922	2.61	49.2
(20, 60, 100)	4.8874	0.59	16.1
(20, 60, 200)	3.0012	1.60	36.7
(20, 60, 300)	10.640	2.87	59.7
(30, 40, 100)	1.3083	0.86	45.2
(30, 40, 200)	3.3180	2.39	126
(30, 40, 300)	9.3044	4.07	198
(30, 50, 100)	3.3983	1.03	80.1
(30, 50, 200)	6.0753	2.79	186
(30, 50, 300)	1.9555	5.08	279
(30, 60, 100)	0.3946	1.17	83.6
(30, 60, 200)	5.8770	2.99	200
(30, 60, 300)	6.8931	4.97	348
(40, 50, 100)	3.3078	1.72	266
(40, 50, 200)	4.7519	5.50	779
(40, 50, 300)	2.9359	9.37	1142

the SOCP solver **Mosek 9.0**.³ For each instance, we use two algorithms to generate the required second-order cone representations: **GreedyPowertwo** and the one proposed in the paper [17]. The results are reported in Table 6.1. It is evident that the approach with **GreedyPowertwo** is more efficient than the one with the algorithm from [17], sometimes by orders of magnitude.

6.2. Matrix optimization.

6.2.1. Semidefinite representation for matrix geometric mean. For $A \in \mathbf{S}_{++}^n$, $B \in \mathbf{S}_{++}^n$, and $\lambda \in [0, 1]$, the λ -weighted geometric mean of A and B is defined by

$$G_\lambda(A, B) = A \#_\lambda B := A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^\lambda A^{\frac{1}{2}}.$$

It can be shown that a second-order cone representation drawn from a successive minimum mediated sequence for a scalar geometric mean can be lifted to a semidefinite

³The computation was performed with the Julia package **SONCSOCP**, which is available at <https://github.com/wangjie212/SONCSOCP>.

representation for the corresponding matrix geometric mean. That means Algorithm 4.1 can be readily adapted to produce a semidefinite representation for the matrix geometric mean G_λ when λ is a rational number.

THEOREM 6.1. *Let $\lambda = \frac{q}{p}$ with $p, q \in \mathbb{N}^*$, $0 < q < p$, and $(p, q) = 1$. Suppose that the scalar geometric mean inequality $x_1^{1-\lambda} x_2^\lambda \geq x_3$ admits a second-order cone representation drawn from a successive minimum mediated sequence: $x_a x_b \geq x_4^2$, $x_{i-1} x_{k_i} \geq x_i^2$, $i = 5, \dots, l$, where $a, b \in \{1, 2, 3\}$, $k_i \in \{1, 2, \dots, i-2\}$, $l = \lceil \log_2(p) \rceil + 3$, and $x_l = x_3$. Then the matrix geometric mean G_λ admits a semidefinite representation:*

$$\begin{aligned} & \{(X_1, X_2, T) \in \mathbf{S}_{++}^n \times \mathbf{S}_+^n \times \mathbf{S}^n \mid G_\lambda(X_1, X_2) \succeq T\} \\ &= \left\{ (X_1, X_2, T) \in \mathbf{S}_{++}^n \times \mathbf{S}_+^n \times \mathbf{S}^n \mid \exists \{X_i\}_{i=3}^l \subseteq \mathbf{S}^n \text{ s.t. } X_3 = X_l = T \text{ and} \right. \\ & \quad \left. \begin{bmatrix} X_a & X_4 \\ X_4 & X_b \end{bmatrix} \succeq 0, \begin{bmatrix} X_{i-1} & X_i \\ X_i & X_{k_i} \end{bmatrix} \succeq 0 \text{ for } i = 5, \dots, l \right\}. \end{aligned}$$

Proof. By adapting the proof of [22, Theorem 4.4], one can show that the matrix geometric mean G_λ satisfies the following extremal property:

$$\begin{aligned} G_\lambda(X_1, X_2) = \max_{\succeq} & \left\{ X \in \mathbf{S}_+^n \mid \exists \{X_i\}_{i=3}^l \subseteq \mathbf{S}^n \text{ s.t. } X_3 = X_l = X \text{ and} \right. \\ & \left. \begin{bmatrix} X_a & X_4 \\ X_4 & X_b \end{bmatrix} \succeq 0, \begin{bmatrix} X_{i-1} & X_i \\ X_i & X_{k_i} \end{bmatrix} \succeq 0 \text{ for } i = 5, \dots, l \right\}, \end{aligned}$$

where \max_{\succeq} takes the largest element with respect to the Löwner ordering (i.e., $A \succeq B$ if and only if $A - B \succeq 0$). The desired conclusion then easily follows. \square

COROLLARY 6.2. *Let $\lambda = \frac{q}{p}$ with $p, q \in \mathbb{N}^*$, $0 < q < p$, and $(p, q) = 1$. Then G_λ admits a semidefinite representation with $\lceil \log_2(p) \rceil$ linear matrix inequalities of size $2n \times 2n$ and one linear matrix inequality of size $n \times n$.*

Remark 6.3. As a comparison, the semidefinite representation for G_λ provided in [11, 22] needs as many as $2 \lceil \log_2(p) \rceil - 1$ linear matrix inequalities of size $2n \times 2n$ and one linear matrix inequality of size $n \times n$.

6.2.2. Semidefinite representation for the multivariate generalization of Lieb's function. Given $(\lambda_i)_{i=1}^m \in \mathbb{Q}_+^m$ with $\sum_{i=1}^m \lambda_i = 1$, the multivariate generalization of Lieb's function is defined by

$$(6.5) \quad (A_1, \dots, A_m) \in \mathbf{S}_+^{n_1} \times \dots \times \mathbf{S}_+^{n_m} \mapsto A_1^{\lambda_1} \otimes \dots \otimes A_m^{\lambda_m} \in \mathbf{S}_+^{n_1 \cdots n_m},$$

where \otimes denotes the Kronecker product of matrices. We are able to give a semidefinite representation for (6.5) by iteratively using

$$A_1^{\lambda_1} \otimes \dots \otimes A_m^{\lambda_m} \succeq T \iff \exists S \in \mathbf{S}_+^{n_1 n_2} \text{ s.t. } \begin{cases} A_1^{\frac{\lambda_1}{\lambda_1 + \lambda_2}} \otimes A_2^{\frac{\lambda_2}{\lambda_1 + \lambda_2}} \succeq S, \\ S^{\lambda_1 + \lambda_2} \otimes A_3^{\lambda_3} \otimes \dots \otimes A_m^{\lambda_m} \succeq T, \end{cases}$$

and applying Corollary 6.2 to

$$(6.6) \quad A^{1-\lambda} \otimes B^\lambda = (A \otimes I) \#_\lambda (I \otimes B),$$

where I denotes the identity matrix of appropriate size.

TABLE 6.2

Results for the trace optimization problem (6.5). For each n , “opt” means the optimum, and “time” means the running time in seconds.

$(\lambda_1, \lambda_2, \lambda_3)$	$n = 2$		$n = 3$	
	opt	time	opt	time
$(\frac{4}{9}, \frac{3}{9}, \frac{2}{9})$	4.656	0.03	10.46	2.38
$(\frac{8}{15}, \frac{4}{15}, \frac{3}{15})$	4.621	0.03	8.976	2.07
$(\frac{25}{44}, \frac{10}{44}, \frac{9}{44})$	5.054	0.05	9.474	3.13
$(\frac{37}{83}, \frac{24}{83}, \frac{22}{83})$	5.065	0.06	10.39	4.31
$(\frac{72}{169}, \frac{47}{169}, \frac{40}{169})$	4.579	0.07	10.03	5.57
$(\frac{159}{278}, \frac{64}{278}, \frac{55}{278})$	4.972	0.07	9.801	7.12
$(\frac{212}{453}, \frac{133}{453}, \frac{108}{453})$	4.039	0.07	9.188	5.92
$(\frac{391}{882}, \frac{280}{882}, \frac{211}{882})$	4.755	0.08	9.654	7.08

Let us consider the following trace optimization problem:

$$(6.7) \quad \begin{cases} \sup_{w_1, w_2, w_3} & \text{tr}((w_1 A_1 + A_2)^{\lambda_1} \otimes (w_2 A_3 + A_4)^{\lambda_2} \otimes (w_3 A_5 + A_6)^{\lambda_3}) \\ \text{s.t.} & w_1 + w_2 + w_3 = 1, \\ & w_1, w_2, w_3 \geq 0, \end{cases}$$

where $(\lambda_i)_{i=1}^3 \in \mathbb{Q}_+^3$ with $\sum_{i=1}^3 \lambda_i = 1$, and $A_1, \dots, A_6 \in \mathbf{S}_+^n$ are randomly generated positive definite matrices. By introducing a matrix variable T , this problem is equivalent to

$$(6.8) \quad \begin{cases} \sup_{w_1, w_2, w_3} & \text{tr}(T) \\ \text{s.t.} & (w_1 A_1 + A_2)^{\lambda_1} \otimes (w_2 A_3 + A_4)^{\lambda_2} \otimes (w_3 A_5 + A_6)^{\lambda_3} \succeq T, \\ & w_1 + w_2 + w_3 = 1, \\ & w_1, w_2, w_3 \geq 0. \end{cases}$$

We can further convert (6.8) into a semidefinite program (SDP) using the semidefinite representation for (6.5). In Table 6.2, we present the results of solving (6.7) with different $(\lambda_1, \lambda_2, \lambda_3)$ and $n = 2, 3$.⁴ Here **Mosek 9.0** serves as the SDP solver.

6.3. Quantum information. Building on simple second-order cone representations for bivariate weighted geometric means, Fawzi and Saunderson provided semidefinite representations for several matrix functions arising from quantum information [11]. We record their results below.

Suppose that $A \in \mathbf{S}_+^n$, $B \in \mathbf{S}_+^m$ are positive definite matrices and λ is a rational number in $(0, 1)$. The semidefinite representation for Lieb’s function $F_\lambda(A, B) := \text{tr}(K^\top A^{1-\lambda} K B^\lambda)$ ($K \in \mathbb{R}^{n \times m}$ are fixed) can be obtained via

$$(6.9) \quad \text{tr}(K^\top A^{1-\lambda} K B^\lambda) \geq t \iff \exists T \in \mathbf{S}_+^{nm} \text{ s.t. } \begin{cases} A^{1-\lambda} \otimes B^\lambda \succeq T, \\ \text{vec}(K)^\top T \text{vec}(K) \geq t, \end{cases}$$

⁴The script is available at <https://github.com/wangjie212/MiniSOC>.

where $\text{vec}(K)$ is a column vector of size nm obtained by concatenating the rows of K . The semidefinite representation for the Tsallis entropy $S_\lambda(A) := \frac{1}{\lambda} \text{tr}(A^{1-\lambda} - A)$ can be obtained via

$$(6.10) \quad \frac{1}{\lambda} \text{tr}(A^{1-\lambda} - A) \geq t \iff \exists T \in \mathbf{S}^n \text{ s.t. } \begin{cases} A \#_\lambda I \succeq T, \\ \frac{1}{\lambda} \text{tr}(T - A) \geq t, \end{cases}$$

where I is the identity matrix of appropriate size. Assuming $n = m$, the semidefinite representation for the Tsallis relative entropy $S_\lambda(A||B) := \frac{1}{\lambda} \text{tr}(A - A^{1-\lambda}B^\lambda)$ can be obtained via

$$(6.11) \quad \frac{1}{\lambda} \text{tr}(A - A^{1-\lambda}B^\lambda) \leq t \iff \exists s \in \mathbb{R} \text{ s.t. } \begin{cases} \text{tr}(A^{1-\lambda}B^\lambda) \geq s, \\ \frac{1}{\lambda}(\text{tr}(A) - s) \leq t. \end{cases}$$

Notice that $\text{tr}(A^{1-\lambda}B^\lambda)$ is Lieb's function $F_\lambda(A, B)$ with $K \in \mathbb{R}^{n \times n}$ being the identity matrix.

Remark 6.4. Because of Corollary 6.2, we can construct semidefinite representations of smaller size for Lieb's function, the Tsallis entropy, the Tsallis relative entropy than the semidefinite representations given in [11].

Now let us consider the following maximum entropy optimization problem (also tested in [11]):

$$(6.12) \quad \begin{cases} \sup_{w_i} & S_\lambda(\sum_{i=1}^m w_i A_i) \\ \text{s.t.} & \sum_{i=1}^m w_i = 1, \\ & w_i \geq 0, i = 1, \dots, m, \end{cases}$$

where $A_1, \dots, A_m \in \mathbf{S}_+^n$ are fixed positive semidefinite matrices of trace one. In Table 6.3 we present numerical results of solving (6.12) with $m = 10, n = 20, 40$, and different λ .⁵ The results were obtained with the SDP solver *Mosek 9.0*. For each instance, we use two algorithms to generate the required second-order cone representations: *GreedyPowertwo* and the one proposed in the paper [11]. As we can see from the table, the approach with *GreedyPowertwo* is more efficient than the one with the algorithm from [11] by a factor ~ 2 .

7. Conclusion and discussion. In this paper, we have studied the optimal size of simple second-order cone representations for a weighted geometric mean inequality and the minimum cardinality of mediated sets containing a given point. Fast heuristic algorithms have been proposed to compute approximately optimal simple second-order cone representations for weighted geometric mean inequalities. As (matrix) weighted geometric means widely appear in optimization, we believe that these results will lead to various applications and stimulate more research on this subject. We conclude the paper by listing some open problems for future research:

(1) Is there a polynomial time algorithm for computing an optimal simple second-order cone representation for a weighted geometric mean inequality? Or is this an NP-hard problem?

(2) We have proved $L(s_1, \dots, s_m) \geq m - 1$ in Theorem 3.6. On the other hand, we observed that for the tested examples the lower bound $m - 1$ is attained only if \hat{s} is even. Is this always true? Can we prove $L(s_1, \dots, s_m) \geq m$ given that \hat{s} is odd?

⁵The script is available at <https://github.com/wangjie212/MiniSOC>.

TABLE 6.3

Results for the maximum entropy optimization problem (6.12). The column labeled “opt” records the optimum; the columns labeled “W” and “F-S” record the running time in seconds of the approaches with GreedyPowerTwo and the algorithm from [11], respectively.

λ	$n = 20$			$n = 40$		
	opt	W	F-S	opt	W	F-S
$\frac{5}{27}$	3.9230	0.87	1.84	5.2130	21.5	52.7
$\frac{12}{41}$	4.7037	0.86	1.51	6.5351	20.9	41.6
$\frac{34}{63}$	7.3642	1.01	1.73	11.530	29.3	39.8
$\frac{21}{107}$	4.0125	1.42	2.77	5.3271	32.9	80.4
$\frac{79}{168}$	6.4613	1.21	2.25	9.7798	33.3	60.6
$\frac{43}{213}$	4.0429	1.73	3.12	5.3920	49.3	91.0
$\frac{135}{422}$	4.9303	1.77	3.33	6.9332	49.0	97.1
$\frac{341}{745}$	6.3203	1.65	3.63	9.4727	45.1	98.9

(3) We have provided two lower bounds on $L(s_1, \dots, s_m)$ in terms of either the dimension m or the degree \hat{s} . Is it possible to prove an improved lower bound on $L(s_1, \dots, s_m)$ combining m and \hat{s} ?

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REFERENCES

- [1] F. ALIZADEH AND D. GOLDFARB, *Second-order cone programming*, Math. Program., 95 (2003), pp. 3–51.
- [2] E. D. ANDERSEN AND K. D. ANDERSEN, *The Mosek interior point optimizer for linear programming: An implementation of the homogeneous algorithm*, in High Performance Optimization, Appl. Optim. 33, H. Frenk, K. Roos, T. Terlaky, and S. Zhang, eds., Springer, New York, 2000, pp. 197–232, https://doi.org/10.1007/978-1-4757-3216-0_8.
- [3] T. ANDO, *Inequalities for M-matrices*, Linear Multilinear Algebra, 8 (1980), pp. 291–316.
- [4] A. BEN-TAL AND A. NEMIROVSKI, *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*, SIAM, Philadelphia, 2001.
- [5] G. BLEKHERMAN, F. RINCÓN, R. SINN, C. VINZANT, AND J. YU, *Moments, Sums of Squares, and Tropicalization*, preprint, arXiv:2203.06291, 2022.
- [6] A. DOMAHIDI, E. CHU, AND S. BOYD, *ECOS: An SOCP solver for embedded systems*, in Proceedings of the 2013 European Control Conference, 2013, pp. 3071–3076, <https://doi.org/10.23919/ECC.2013.6669541>.
- [7] M. DRESSLER, S. İLİMAN, AND T. DE WOLFF, *An approach to constrained polynomial optimization via nonnegative circuit polynomials and geometric programming*, J. Symbolic Comput., 91 (2019), pp. 149–172.
- [8] M. DRESSLER, A. KURPISZ, AND T. DE WOLFF, *Optimization over the boolean hypercube via sums of nonnegative circuit polynomials*, Found. Comput. Math., 22 (2022), pp. 365–387.
- [9] H. FAWZI AND M. S. EL DIN, *A lower bound on the positive semidefinite rank of convex bodies*, SIAM J. Appl. Algebra Geom., 2 (2018), pp. 126–139.
- [10] H. FAWZI, J. GOUVEIA, P. A. PARRILO, J. SAUNDERSON, AND R. R. THOMAS, *Lifting for simplicity: Concise descriptions of convex sets*, SIAM Rev., 64 (2022), pp. 866–918.
- [11] H. FAWZI AND J. SAUNDERSON, *Lieb’s concavity theorem, matrix geometric means, and semidefinite optimization*, Linear Algebra Appl., 513 (2017), pp. 240–263.
- [12] H. FAWZI, J. SAUNDERSON, AND P. A. PARRILO, *Semidefinite approximations of the matrix logarithm*, Found. Comput. Math., 19 (2019), pp. 259–296.
- [13] J. GOUVEIA, P. A. PARRILO, AND R. THOMAS, *Lifts of convex sets and cone factorizations*, Math. Oper. Res., 38 (2013), pp. 248–264.

- [14] J. HARTZER, O. RÖHRIG, T. DE WOLFF, AND O. YÜRÜK, *Initial steps in the classification of maximal mediated sets*, J. Symbolic Comput., 109 (2022), pp. 404–425.
- [15] J. W. HELTON, J. NIE, AND J. S. SEMKO, *Free semidefinite representation of matrix power functions*, Linear Algebra Appl., 465 (2015), pp. 347–362.
- [16] S. ILIMAN AND T. DE WOLFF, *Amoebas, nonnegative polynomials and sums of squares supported on circuits*, Res. Math. Sci., 3 (2016), 9.
- [17] R. KIAN, E. BERK, AND Ü. GÜRLER, *Minimal conic quadratic reformulations and an optimization model*, Oper. Res. Lett., 47 (2019), pp. 489–493.
- [18] V. MAGRON AND J. WANG, *SONC optimization and exact nonnegativity certificates via second-order cone programming*, J. Symbolic Comput., 115 (2023), pp. 346–370.
- [19] Y. MORENKO, A. VINEL, Z. YU, AND P. KROKHMAL, *On p -norm linear discrimination*, European J. Oper. Res., 231 (2013), pp. 784–789.
- [20] V. POWERS AND B. REZNICK, *A note on mediated simplices*, J. Pure Appl. Algebra, 225 (2021), 106608.
- [21] B. REZNICK, *Forms derived from the arithmetic-geometric inequality*, Math. Ann., 283 (1989), pp. 431–464.
- [22] G. SAGNOL, *On the semidefinite representation of real functions applied to symmetric matrices*, Linear Algebra Appl., 439 (2013), pp. 2829–2843.
- [23] H. SEIDLER AND T. DE WOLFF, *An Experimental Comparison of SONC and SOS Certificates for Unconstrained Optimization*, preprint, arXiv:1808.08431, 2018.
- [24] J. WANG AND V. MAGRON, *A second order cone characterization for sums of nonnegative circuits*, in ISSAC’20: Proceedings of the 2020 ACM International Symposium on Symbolic and Algebraic Computation, 2020.