多项式优化入门

王杰
中国科学院数学与系统科学研究院

北京师范大学, 2024 年秋季

课程内容

多项式优化问题

$$f_{\min} := \begin{cases} \inf\limits_{\mathbf{x}\in\mathbb{R}^n} & f(\mathbf{x}) \\ \text{s.t.} & g_j(\mathbf{x}) \geq 0, \quad j \in [m] \end{cases}$$

变量稀疏性

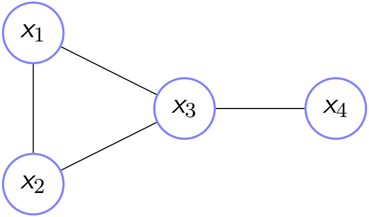
- 变量（correlative）稀疏性（Waki et al. 2006）：变量之间的关联
- 变量稀疏型（csp）图 $G^{\mathrm{csp}}(V, E)$：
  - $V := \{x_1, \ldots, x_n\}$
  - $\{x_i, x_j\} \in E \iff x_i, x_j$ 出现在 f 的同一项或同一个约束多项式 g_k 中

变量稀疏性

- 变量（correlative）稀疏性（Waki et al. 2006）：变量之间的关联
- 变量稀疏型（csp）图 $G^{\mathrm{csp}}(V, E)$：
  - $V := \{x_1, \ldots, x_n\}$
  - $\{x_i, x_j\} \in E \Longleftrightarrow x_i, x_j$ 出现在 f 的同一项或同一个约束多项式 g_k 中

例子

$f = x_1^4 + x_1x_2^2 + x_2x_3 + x_3^2x_4^2$, $g_1 = 1 - x_1^2 - x_2^2 - x_3^2$, $g_2 = 1 - x_3x_4$



两个极大团：$\{x_1, x_2, x_3\}$ **和** $\{x_3, x_4\}$

稀疏矩方阵与局部化矩阵

- $G^{\mathrm{csp}}(V, E)$ 的极大团集： $I_1, \ldots, I_p \subseteq [n]$
  - $f = f_1 + \cdots + f_p,\ f_k \in \mathbb{R}[\mathbf{x}, I_k],\ k \in [p]$
- 约束指标集 $J_1, \ldots, J_p \subseteq [m]$: $\cup_{k=1}^{p} J_k = [m]$
  - $\forall k \in [p],\ \forall j \in J_k,\ g_j \in \mathbb{R}[\mathbf{x}, I_k]$
- $I_k \longmapsto \mathbf{M}_r(\mathbf{y}, I_k),\ \mathbf{M}_{r-d_j}(g_j\mathbf{y}, I_k),\ j \in J_k$

稀疏矩方阵与局部化矩阵

- $G^{\mathrm{csp}}(V, E)$ 的极大团集： $I_1, \ldots, I_p \subseteq [n]$
  - $f = f_1 + \cdots + f_p$, $f_k \in \mathbb{R}[\mathbf{x}, I_k]$, $k \in [p]$
- 约束指标集 $J_1, \ldots, J_p \subseteq [m]$: $\cup_{k=1}^{p} J_k = [m]$
  - $\forall k \in [p]$, $\forall j \in J_k$, $g_j \in \mathbb{R}[\mathbf{x}, I_k]$
- $I_k \longmapsto \mathbf{M}_r(\mathbf{y}, I_k)$, $\mathbf{M}_{r-d_j}(g_j\mathbf{y}, I_k)$, $j \in J_k$

稀疏矩方阵与局部化矩阵

- $G^{\text{csp}}(V,E)$ 的极大团集： $I_1,\ldots,I_p\subseteq[n]$
  - $f=f_1+\cdots+f_p,\ f_k\in\mathbb{R}[\mathbf{x},I_k],\ k\in[p]$
- 约束指标集 $J_1,\ldots,J_p\subseteq[m]$: $\cup_{k=1}^{p}J_k=[m]$
  - $\forall k\in[p],\ \forall j\in J_k,\ g_j\in\mathbb{R}[\mathbf{x},I_k]$
- $I_k\longmapsto\mathbf{M}_r(\mathbf{y},I_k),\ \mathbf{M}_{r-d_j}(g_j\mathbf{y},I_k),\ j\in J_k$

当 $r = r_{\min}$

- $J' := \{j \in [m] \mid d_j = r_{\min}\}$
- $\{x_i, x_j\} \in E \Longleftrightarrow x_i, x_j$ 出现在 f 或 $g_k, k \in J'$ 的同一项或同一个约束多项式 $g_k, k \in [m] \setminus J'$ 中
  - $f = f_1 + \cdots + f_p$, $f_k \in \mathbb{R}[\mathbf{x}, I_k]$, $k \in [p]$
  - $g_j = g_j^1 + \cdots + g_j^p$, $g_j^k \in \mathbb{R}[\mathbf{x}, I_k]$, $j \in J'$
  - $J_1, \ldots, J_p \subseteq [m]$: $\cup_{k=1}^p J_k = [m] \setminus J'$
  - $\forall k \in [p]$, $\forall j \in J_k$, $g_j \in \mathbb{R}[\mathbf{x}, I_k]$

当 $r = r_{\min}$

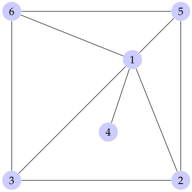
- $J' := \{j \in [m] \mid d_j = r_{\min}\}$
- $\{x_i, x_j\} \in E \iff x_i, x_j$ 出现在 f 或 $g_k, k \in J'$ 的同一项或同一个约束多项式 $g_k, k \in [m] \setminus J'$ 中
  - $f = f_1 + \cdots + f_p,\ f_k \in \mathbb{R}[\mathbf{x}, I_k],\ k \in [p]$
  - $g_j = g_j^1 + \cdots + g_j^p,\ g_j^k \in \mathbb{R}[\mathbf{x}, I_k],\ j \in J'$
  - $J_1, \ldots, J_p \subseteq [m]$: $\cup_{k=1}^p J_k = [m] \setminus J'$
  - $\forall k \in [p], \forall j \in J_k, g_j \in \mathbb{R}[\mathbf{x}, I_k]$

当 $r = r_{\min}$

- $J' := \{j \in [m] \mid d_j = r_{\min}\}$
- $\{x_i, x_j\} \in E \iff x_i, x_j$ 出现在 f 或 $g_k, k \in J'$ 的同一项或同一个约束多项式 $g_k, k \in [m] \setminus J'$ 中
  - $f = f_1 + \cdots + f_p,\ f_k \in \mathbb{R}[\mathbf{x}, I_k],\ k \in [p]$
  - $g_j = g_j^1 + \cdots + g_j^p,\ g_j^k \in \mathbb{R}[\mathbf{x}, I_k],\ j \in J'$
  - $J_1, \ldots, J_p \subseteq [m]$: $\cup_{k=1}^p J_k = [m] \setminus J'$
  - $\forall k \in [p],\ \forall j \in J_k,\ g_j \in \mathbb{R}[\mathbf{x}, I_k]$

例子

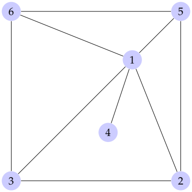
- $f = x_2x_5 + x_3x_6 - x_2x_3 - x_5x_6 + x_1(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6)$
- $g_i = (6.36 - x_i)(x_i - 4), i = 1, \ldots, 6,\ g_7 = x_2^2 + x_6^2 - 1$



- $r = r_{\min} = 1,\ J' = [7]$
- $I_1 = \{1,4\},\ I_2 = \{1,2,5\},\ I_3 = \{1,5,6\},\ I_4 = \{1,3,6\},\ I_5 = \{1,2,3\}$
- $J_k = \emptyset,\ k \in [5]$

例子

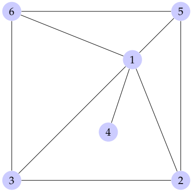
- $f = x_2x_5 + x_3x_6 - x_2x_3 - x_5x_6 + x_1(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6)$
- $g_i = (6.36 - x_i)(x_i - 4),\ i = 1, \ldots, 6,\ g_7 = x_2^2 + x_6^2 - 1$



- $r = r_{\min} = 1,\ J' = [7]$
- $I_1 = \{1, 4\},\ I_2 = \{1, 2, 5\},\ I_3 = \{1, 5, 6\},\ I_4 = \{1, 3, 6\},\ I_5 = \{1, 2, 3\}$
- $J_k = \emptyset,\ k \in [5]$

例子

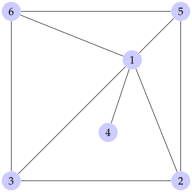
- $f = x_2x_5 + x_3x_6 - x_2x_3 - x_5x_6 + x_1(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6)$
- $g_i = (6.36 - x_i)(x_i - 4), i = 1, \ldots, 6,\ g_7 = x_2^2 + x_6^2 - 1$



- $r \geq 2,\ J' = \emptyset$
- $I_1 = \{1, 4\},\ I_2 = \{1, 2, 3, 6\},\ I_3 = \{1, 2, 5, 6\}$
- $J_1 = \{1, 4\},\ J_2 = \{1, 2, 3, 6, 7\},\ J_3 = \{1, 2, 5, 6, 7\}$

例子

- $f = x_2x_5 + x_3x_6 - x_2x_3 - x_5x_6 + x_1(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6)$
- $g_i = (6.36 - x_i)(x_i - 4), i = 1, \ldots, 6,\ g_7 = x_2^2 + x_6^2 - 1$



➤ $r \geq 2,\ J' = \emptyset$

➤ $I_1 = \{1, 4\},\ I_2 = \{1, 2, 3, 6\},\ I_3 = \{1, 2, 5, 6\}$

➤ $J_1 = \{1, 4\},\ J_2 = \{1, 2, 3, 6, 7\},\ J_3 = \{1, 2, 5, 6, 7\}$

稀疏矩松弛

$$\lambda_r^{\mathrm{cs}} := \begin{cases} \inf\limits_{\mathbf{y}} & L_{\mathbf{y}}(f) \\ \text{s.t.} & \mathbf{M}_r(\mathbf{y}, I_k) \succeq 0, \quad k \in [p] \\ & \mathbf{M}_{r-d_j}(g_j\mathbf{y}, I_k) \succeq 0, \quad j \in J_k, k \in [p] \\ & L_{\mathbf{y}}(g_j) \geq 0, \quad j \in J' \\ & y_{\mathbf{0}} = 1 \end{cases}$$

- $\lambda_r^{\mathrm{cs}} \leq \lambda_r$
- 对于 QCQP, $\lambda_1^{\mathrm{cs}} = \lambda_1$

稀疏矩松弛

$$\lambda_r^{\mathrm{cs}} := \begin{cases} \inf\limits_{\mathbf{y}} & L_{\mathbf{y}}(f) \\ \mathrm{s.t.} & \mathbf{M}_r(\mathbf{y}, I_k) \succeq 0, \quad k \in [p] \\ & \mathbf{M}_{r-d_j}(g_j\mathbf{y}, I_k) \succeq 0, \quad j \in J_k, k \in [p] \\ & L_{\mathbf{y}}(g_j) \geq 0, \quad j \in J' \\ & y_{\mathbf{0}} = 1 \end{cases}$$

- $\lambda_r^{\mathrm{cs}} \leq \lambda_r$
- 对于 QCQP, $\lambda_1^{\mathrm{cs}} = \lambda_1$

稀疏矩松弛

$$\lambda_r^{\mathrm{cs}} := \begin{cases} \inf\limits_{\mathbf{y}} & L_{\mathbf{y}}(f) \\ \text{s.t.} & \mathbf{M}_r(\mathbf{y}, I_k) \succeq 0, \quad k \in [p] \\ & \mathbf{M}_{r-d_j}(g_j\mathbf{y}, I_k) \succeq 0, \quad j \in J_k, k \in [p] \\ & L_{\mathbf{y}}(g_j) \geq 0, \quad j \in J' \\ & y_{\mathbf{0}} = 1 \end{cases}$$

- $\lambda_r^{\mathrm{cs}} \leq \lambda_r$
- 对于 QCQP, $\lambda_1^{\mathrm{cs}} = \lambda_1$

稀疏 SOS 松弛

$$
\begin{cases}
\sup\limits_{\lambda,\sigma_{k,j},\tau_j} & \lambda \\
\text{s.t.} & f-\lambda=\sum_{k=1}^{p}\left(\sigma_{k,0}+\sum_{j\in J_k}\sigma_{k,j}g_j\right)+\sum_{j\in J'}\tau_j g_j \\
 & \sigma_{k,0},\sigma_{k,j}\in\Sigma[\mathbf{x},I_k],\quad j\in J_k, k\in[p] \\
 & \deg(\sigma_{k,0}),\deg(\sigma_{k,j}g_j)\le 2r,\quad j\in J_k, k\in[p] \\
 & \tau_j\ge 0,\quad j\in J'
\end{cases}
$$

复杂度比较

	稠密	稀疏
矩阵阶数	$\binom{n+r}{r}$	$\binom{\max\{\lvert I_k\rvert\}_{k=1}^p+r}{r}$
等式约束个数	$\binom{n+2r}{2r}$	$\sim\sum_{k=1}^p\binom{\lvert I_k\rvert+2r}{2r}$

Running intersection property (RIP)

- $I_1, \ldots, I_p \subseteq [n]$ 满足 RIP：对任意 $k \in [p-1]$，存在 $i \leq k$ 使得

$$\left(I_{k+1} \cap \bigcup_{j \leq k} I_j \right) \subseteq I_i$$

定理

一个连通图是弦图当且仅当它的极大团集经过合适地排序满足 *RIP*.

稀疏 Putinar's Positivstellensatz

- $S := \{\mathbf{x} \in \mathbb{R}^n \mid g_1(\mathbf{x}) \geq 0, \ldots, g_m(\mathbf{x}) \geq 0\}$

定理 (Lasserre, 2006)

- $I_1, \ldots, I_p \subseteq [n]$ 满足 *RIP*;
- $\forall k \in [p], \mathcal{Q}(\{g_j\}_{j \in J_k})$ 满足阿基米德条件.

如果 $f \in \mathbb{R}[\mathbf{x}]$ 在 S 上是严格正的，则 f 存在稀疏 *SOS* 表示：

$$f = \sum_{k=1}^{p} \left(\sigma_{k,0} + \sum_{j \in J_k} \sigma_{k,j} g_j \right),$$

其中 $\sigma_{k,0}, \sigma_{k,j} \in \Sigma[\mathbf{x}, I_k]$, $j \in J_k$, $k \in [p]$.

渐近收敛性

- $I_1, \ldots, I_p \subseteq [n]$ 满足 RIP
- $\forall k \in [p]$, $\mathcal{Q}(\{g_j\}_{j \in J_k})$ 满足阿基米德条件

➤ $\lambda_r^{\mathrm{cs}} \nearrow f_{\min}$, $r \to \infty$

最优性与最优解

定理

- 令 $a_k := \max_{j\in J_k}\{d_j\}$, $k \in [p]$. 若稀疏矩松弛的最优解 $\mathbf{y}$ 满足

$$\operatorname{rank} \mathbf{M}_r(\mathbf{y}, I_k) = \operatorname{rank} \mathbf{M}_{r-a_k}(\mathbf{y}, I_k), \quad \forall k \in [p],$$

 且对所有 (j,k) 使得 $I_j \cap I_k \neq \emptyset$, 有 $\operatorname{rank} \mathbf{M}_r(\mathbf{y}, I_j \cap I_k) = 1$,

 则 $\lambda_r^{\mathrm{cs}} = f_{\min}$.

- 对每个 $k \in [p]$, 令 Δ_k 是从 $\mathbf{M}_r(\mathbf{y}, I_k)$ 提取的点集. 则任一 $\mathbf{x} \in \mathbb{R}^n$ 满足 $(x_i)_{i\in I_k} \in \Delta_k$ 是多项式优化问题的最优解.

变量-项联合稀疏松弛

- CS-TSSOS 松弛分层:

1. 利用变量稀疏性得到变量块 $I_1, \ldots, I_p$
2. 对每个变量块利用项稀疏性得到项稀疏型图
3. 建立变量-项联合稀疏松弛

变量-项联合稀疏松弛

- CS-TSSOS 松弛分层：
  1. 利用变量稀疏性得到变量块 $I_1, \ldots, I_p$
  2. 对每个变量块利用项稀疏性得到项稀疏型图
  3. 建立变量-项联合稀疏松弛

变量-项联合稀疏松弛

- CS-TSSOS 松弛分层：
  1. 利用变量稀疏性得到变量块 $I_1, \ldots, I_p$
  2. 对每个变量块利用项稀疏性得到项稀疏型图
  3. 建立变量-项联合稀疏松弛

稀疏 Putinar's Positivstellensatz

定理

- $I_1, \ldots, I_p \subseteq [n]$ 满足 *RIP;*
- $\forall k \in [p]$, $\mathcal{Q}(\{g_j\}_{j \in J_k})$ 满足阿基米德条件.

令 R 是 $f(\mathbf{x}), g_1(\mathbf{x}), \ldots, g_m(\mathbf{x})$ 的所有符号对称性给出的二元矩阵. 如果 $f \in \mathbb{R}[\mathbf{x}]$ 在 S 上是严格正的, 则 f 存在稀疏 *SOS* 表示:

$$f = \sum_{k=1}^{p} \left(\sigma_{k,0} + \sum_{j \in J_k} \sigma_{k,j} g_j \right),$$

其中 $\sigma_{k,j} \in \Sigma[\mathbf{x}, I_k]$ 满足 $R^\intercal \boldsymbol{\alpha} \equiv 0 \bmod 2$, $\forall \mathbf{x}^{\boldsymbol{\alpha}} \in \operatorname{supp}(\sigma_{k,j})$.

例子

- The Broyden banded function:

$$f_{\mathrm{Bb}}(\mathbf{x}) = \sum_{i=1}^{n} \left(x_i \left(2 + 5x_i^2\right) + 1 - \sum_{j \in J_i} (1 + x_j)\, x_j \right)^2,$$

where $J_i = \{j \mid j \neq i, \max(1, i-5) \leq j \leq \min(n, i+1)\}$.

例子: The Broyden banded function

n	CS			TS			CS+TS		
	mb	opt	time	mb	opt	time	mb	opt	time
20	120	0	21.7	33	0	4.39	19	0	2.24
40	120	0	44.6	52	0	231	19	0	6.95
60	120	0	81.8	-	-	-	19	0	13.0
80	120	0	116	-	-	-	19	0	19.6
100	120	0	151	-	-	-	19	0	27.0
200	120	-	-	-	-	-	19	0	72.9
300	120	-	-	-	-	-	19	0	132
400	120	-	-	-	-	-	19	0	220
500	120	-	-	-	-	-	19	0	313

例子

- The generalized Rosenbrock function:

$$f_{\mathrm{gR}}(\mathbf{x}) = 1 + \sum_{i=2}^{n} \left(100(x_i - x_{i-1}^2)^2 + (1 - x_i)^2\right)$$

- The Broyden tridiagonal function:

$$f_{\mathrm{Bt}}(\mathbf{x}) = ((3 - 2x_1)x_1 - 2x_2 + 1)^2 + \sum_{i=2}^{n-1} ((3 - 2x_i)x_i - x_{i-1} - 2x_{i+1} + 1)^2$$

$$+ ((3 - 2x_n)x_n - x_{n-1} + 1)^2$$

- The chained Wood function:

$$f_{\mathrm{cW}}(\mathbf{x}) = 1 + \sum_{i \in J} (100(x_{i+1} - x_i^2)^2 + (1 - x_i)^2 + 90(x_{i+3} - x_{i+2}^2)^2$$

$$+ (1 - x_{i+2})^2 + 10(x_{i+1} + x_{i+3} - 2)^2 + 0.1(x_{i+1} - x_{i+3})^2),$$

where $J = \{1, 3, 5, \ldots, n-3\}$ and $4|n$.

例子: The generalized Rosenbrock function

n	CS			TS			CS+TS		
	mb	opt	time	mb	opt	time	mb	opt	time
40	231	38.051	126	41	38.049	0.61	21	38.049	0.23
60	231	57.849	232	61	57.845	3.31	21	57.845	0.32
80	231	77.647	306	81	77.641	11.7	21	77.641	0.41
100	231	97.445	377	101	97.436	31.3	21	97.436	0.54
200	231	-	-	201	196.41	1327	21	196.41	1.27
300	231	-	-	-	-	-	21	295.39	2.26
400	231	-	-	-	-	-	21	394.37	3.36
500	231	-	-	-	-	-	21	493.35	4.65
1000	231	-	-	-	-	-	21	988.24	15.8

例子: The Broyden tridiagonal function

n	CS			TS			CS+TS		
	mb	opt	time	mb	opt	time	mb	opt	time
40	231	31.234	168	43	31.234	1.95	23	31.234	0.64
60	231	47.434	273	63	47.434	8.33	23	47.434	1.14
80	231	63.634	413	83	63.634	33.9	23	63.634	1.50
100	231	79.834	519	103	79.834	104	23	79.834	1.96
200	231	-	-	-	-	-	23	160.83	4.88
300	231	-	-	-	-	-	23	241.83	8.67
400	231	-	-	-	-	-	23	322.83	13.3
500	231	-	-	-	-	-	23	403.83	19.9
1000	231	-	-	-	-	-	23	808.83	57.5

例子: The chained Wood function

n	CS			TS			CS+TS		
	mb	opt	time	mb	opt	time	mb	opt	time
40	231	574.51	164	41	574.51	0.81	21	574.51	0.26
60	231	878.26	254	61	878.26	3.61	21	878.26	0.40
80	231	1182.0	393	81	1182.0	15.3	21	1182.0	0.57
100	231	1485.8	505	101	1485.8	43.2	21	1485.8	0.73
200	231	-	-	201	3004.5	1238	21	3004.5	1.91
300	231	-	-	-	-	-	21	4523.6	3.39
400	231	-	-	-	-	-	21	6042.0	5.72
500	231	-	-	-	-	-	21	7560.7	7.88
1000	231	-	-	-	-	-	21	15155	23.0

最优电力流问题（AC-OPF）

$$
\begin{cases}
\inf\limits_{V_i, S_k^g \in \mathbb{C}} & \sum\limits_{k \in G} \left(\mathbf{c}_{2k} \Re(S_k^g)^2 + \mathbf{c}_{1k} \Re(S_k^g) + \mathbf{c}_{0k} \right) \\
\text{s.t.} & \angle V_r = 0, \\
& \mathbf{S}_k^{gl} \le S_k^g \le \mathbf{S}_k^{gu}, \quad \forall k \in G, \\
& \boldsymbol{v}_i^l \le |V_i| \le \boldsymbol{v}_i^u, \quad \forall i \in N, \\
& \sum_{k \in G_i} S_k^g - \mathbf{S}_i^d - \mathbf{Y}_i^s |V_i|^2 = \sum_{(i,j) \in E_i \cup E_i^R} S_{ij}, \quad \forall i \in N, \\
& S_{ij} = (\mathbf{Y}_{ij}^* - \mathbf{i} \frac{\mathbf{b}_{ij}^c}{2}) \frac{|V_i|^2}{|\mathbf{T}_{ij}|^2} - \mathbf{Y}_{ij}^* \frac{V_i V_j^*}{\mathbf{T}_{ij}}, \quad \forall (i,j) \in E, \\
& S_{ji} = (\mathbf{Y}_{ij}^* - \mathbf{i} \frac{\mathbf{b}_{ij}^c}{2}) |V_j|^2 - \mathbf{Y}_{ij}^* \frac{V_i^* V_j}{\mathbf{T}_{ij}^*}, \quad \forall (i,j) \in E, \\
& |S_{ij}| \le \mathbf{s}_{ij}^u, \quad \forall (i,j) \in E \cup E^R, \\
& \boldsymbol{\theta}_{ij}^{\Delta l} \le \angle(V_i V_j^*) \le \boldsymbol{\theta}_{ij}^{\Delta u}, \quad \forall (i,j) \in E.
\end{cases}
$$

最优电力流问题（AC-OPF）

n	m+l	CS ($r=2$)				CS+TS ($r=2, s=1$)			
		mb	opt	time	gap	mb	opt	time	gap
20	55	28	1.7543e4	0.56	0.05%	22	1.7543e4	0.30	0.05%
72	297	45	4.9927e3	4.43	0.07%	22	4.9920e3	2.69	0.08%
114	315	120	7.6943e4	94.9	0.00%	39	7.6942e4	14.8	0.00%
344	1325	253	–	–	–	73	1.0470e5	169	0.50%
348	1809	253	–	–	–	34	1.2096e5	201	0.03%
766	3322	153	3.3072e6	585	0.68%	44	3.3042e6	33.9	0.77%
1112	4613	496	–	–	–	31	7.2396e4	410	0.25%
4356	18257	378	–	–	–	27	1.3953e6	934	0.51%
6698	29283	1326	–	–	–	76	5.9858e5	1886	0.47%

对称性

- **置换对称性**: $(x_1, \ldots, x_n) \to (x_{\tau(1)}, \ldots, x_{\tau(n)})$
- **平移对称性**: $(x_1, \ldots, x_n) \to (x_{1+i}, \ldots, x_{n+i})$, $x_{n+i} = x_i$
- **符号对称性**: $(x_{i_1}, \ldots, x_{i_k}) \to (-x_{i_1}, \ldots, -x_{i_k})$

对称性

1. 确定多项式优化问题的对称群
2. 计算对称群的不可约表示
3. 计算每个同型分支的一组基
4. 建立块对角矩-平方和松弛

例子

- $f = 1 + \sum_{i=1}^{4} x_i^2 + x_1x_2x_3 + x_2x_3x_4 + x_1x_3x_4 + x_1x_2x_4$
- 对称群：4 阶循环群
- 次数 ≤ 2 的不变多项式：$b_0 = 1, b_1 = \frac{1}{4}\sum_{i=1}^{4} x_i, b_2 = \frac{1}{4}\sum_{i=1}^{4} x_i^2, b_3 = \frac{1}{4}\sum_{i=1}^{4} x_ix_{i+1}, b_4 = \frac{1}{2}(x_1x_3 + x_2x_4)$
- 令 $y_i = L(b_i), i = 0, \ldots, 4$

例子

- 对称基下的矩阵:

$$
\mathbf{M}_2^G(\mathbf{y}) = \begin{bmatrix} 1 & y_1 & y_1 & y_1 & y_1 \\ y_1 & y_2 & y_3 & y_4 & y_3 \\ y_1 & y_3 & y_2 & y_3 & y_4 \\ y_1 & y_4 & y_3 & y_2 & y_3 \\ y_1 & y_3 & y_4 & y_3 & y_2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2y_1 & 0 & 0 & 0 \\ 2y_1 & y_2 + 2y_3 + y_4 & 0 & 0 & 0 \\ 0 & 0 & y_2 - y_4 & 0 & 0 \\ 0 & 0 & 0 & y_2 - 2y_3 + y_4 & 0 \\ 0 & 0 & 0 & 0 & y_2 - y_4 \end{bmatrix}
$$

下次课

- 扩展与应用

参考文献

- J. Wang and V. Magron, **Sparse Polynomial Optimization: Theory and Practice**, World Scientific Publishing, 2023.
- H. Waki, S. Kim, M. Kojima, and M. Muramatsu, **Sums of squares and semidefinite program relaxations for polynomial optimization problems with structured sparsity**, SIAM Journal on Optimization, 2006.
- Jean B. Lasserre, **Convergent SDP-relaxations in polynomial optimization with sparsity**, SIAM Journal on optimization, 2006.
- K. Gatermann and P. A. Parrilo, **Symmetry groups, semidefinite programs, and sums of squares**, Journal of Pure and Applied Algebra, 2004.
- C. Riener, T. Theobald, L.J. Andrén, and J. B. Lasserre, **Exploiting symmetries in SDP-relaxations for polynomial optimization**, Mathematics of Operations Research, 2013.

更多信息见个人主页

https://wangjie212.github.io/jiewang