NONNEGATIVE POLYNOMIALS AND CIRCUIT POLYNOMIALS

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Abstract. Recently, S. Iliman and T. Wolff introduced the concept of sums of nonnegative circuit polynomials (SONC) as a new certificate of nonnegativity of polynomials. For a polynomial with a simplex Newton polytope satisfying certain conditions, it is nonnegative if and only if it is a sum of nonnegative circuit polynomials. In this paper, we generalize this conclusion to polynomials with general Newton polytopes. Moreover, we prove that every SONC polynomial decomposes into a sum of nonnegative circuit polynomials with the same support, which reveals the advantage of SONC decompositions for certifying nonnegativity of sparse polynomials compared with the classical SOS decompositions. By virtue of this fact, we can decide $f \in$ SONC through relative entropy programming more efficiently.

1. Introduction

A real polynomial $f \in \mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \ldots, x_n]$ is called a *nonnegative polynomial* if its evaluation on every real point is nonnegative. All of nonnegative polynomials form a convex cone, denoted by PSD. Certifying nonnegativity of polynomials is a central problem of real algebraic geometry and also has applications in many fields such as polynomial optimization, control, engineering, probability, statistics and physics. The classical method for certifying nonnegativity of polynomials is using sums of squares (SOS) programming which can be effectively solved by semidefinite programming (SDP) ([12, 13]). However, the size of the corresponding semidefinite program problems for SOS decompositions grows rapidly as the size of polynomials increases ([11]). Therefore, to deal with huge polynomials, sparsity must be exploited.

The key idea of SOS decompositions is representing a nonnegative polynomial as a sum of a certain class of nonnegative polynomials whose nonnegativity is easy to check. Recently in [7], Iliman and Wolff introduced the concept of sums of nonnegative circuit polynomials as a substitute of sums of squares of polynomials to represent nonnegative polynomials. A polynomial f is called a *circuit polynomial* if it is of the form

$$f(\mathbf{x}) = \sum_{i=1}^{m} c_i \mathbf{x}^{\boldsymbol{\alpha}_i} - d\mathbf{x}^{\boldsymbol{\beta}}, \tag{1.1}$$

where the Newton polytope $\Delta = \mathrm{New}(f)$ is a lattice simplex with the vertex set $\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m\}$, $\boldsymbol{\beta}$ an interior point of Δ and $c_i > 0$ for $i = 1, \ldots, m$. For every

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circuit polynomial f, we associate it with the *circuit number* defined as $\Theta_f := \prod_{i=1}^m (c_i/\lambda_i)^{\lambda_i}$, where the λ_i's are uniquely given by the convex combination $\boldsymbol{\beta} = \sum_{i=1}^m \lambda_i \boldsymbol{\alpha}_i$ with $\lambda_i > 0$ and $\sum_{i=1}^m \lambda_i = 1$. The nonnegativity of circuit polynomials is easy to check. Actually the circuit polynomial f is nonnegative if and only if $\boldsymbol{\alpha}_i \in (2\mathbb{N})^n$ for all i, and $-\Theta_f \le d \le \Theta_f$ if $\boldsymbol{\beta} \notin (2\mathbb{N})^n$ or $d \le \Theta_f$ if $\boldsymbol{\beta} \in (2\mathbb{N})^n$.

If a polynomial f can be written as a sum of nonnegative circuit polynomials, then f is obviously nonnegative. All of polynomials which can be written as sums of nonnegative circuit polynomials also form a convex cone, denoted by SONC. Based on these SONC decompositions for nonnegativity certificates, new approaches were proposed for both unconstrained polynomial optimization problems and constrained polynomial optimization problems, which were proved to be significantly more efficient than the classic semidefinite programming method in many cases ([4, 5, 6, 8, 15]).

However, the SONC cone is not understood well. It is natural to ask which types of nonnegative polynomials admit SONC decompositions and how big the gap between the PSD cone and the SONC cone is. In [7], it was proved that if the Newton polytope New(f) of f is a simplex and there exists a point such that all terms of f except for those corresponding to vertices have the negative sign on this point, then $f \in$ PSD if and only if $f \in$ SONC (Theorem 2.5). In this paper, we generalize this conclusion to polynomials with general Newton polytopes.

Moveover, we clarify an important fact that every SONC polynomial decomposes into a sum of nonnegative circuit polynomials with the same support. In other words, SONC decompositions for nonnegative polynomials exactly maintain sparsity of polynomials. It is dramatically unlike the case of SOS decompositions for nonnegative polynomials, in which case many extra support monomials are needed in general. This reveals the advantage of SONC decompositions for certifying nonnegativity of sparse polynomials compared with the classical SOS decompositions.

Finally, from the perspective of computation, we put the problem of deciding $f \in$ SONC down to the feasibility of a relative entropy program problem. Since relative entropy programming is convex, it can be solved very efficiently ([1, 10]).

The rest of this paper organized as follows. In Section 2, we introduce some notions on nonnegative polynomials and recall some results on circuit polynomials. After that we consider which types of nonnegative polynomials with general Newton polytopes admit SONC decompositions. In Section 3, we deal with the case of nonnegative polynomials with one negative term. In Section 4, we deal with the case of nonnegative polynomials with multiple negative terms. In Section 5, we prove that every SONC polynomial decomposes into a sum of nonnegative circuit polynomials with the same support. By virtue of this fact, the problem of deciding $f \in$ SONC is solved via relative entropy programming more efficiently in Section 6.

2. Preliminaries

2.1. **Nonnegative Polynomials.** Let $\mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \dots, x_n]$ be the ring of real n-variate polynomial, $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$, and $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$. Let $\mathbb{R}_+$ be the set of positive real numbers and $\mathbb{R}_{\ge 0}$ the set of nonnegative real numbers. For a finite set $A \subset \mathbb{N}^n$, we denote by cone(A) the conic hull of A, by conv(A) the convex hull of A, and by $V(A)$ the vertices of the convex hull of A. Also we denote by $V(P)$ the vertex set of a polytope P. We consider polynomials $f \in \mathbb{R}[\mathbf{x}]$ supported on $A \subset \mathbb{N}^n$, i.e. f is of the form $f(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \in A} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$ with $c_{\boldsymbol{\alpha}} \in \mathbb{R}, \mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The support of f is $\mathrm{supp}(f) := \{\boldsymbol{\alpha} \in \mathscr{A} \mid c_{\boldsymbol{\alpha}} \neq 0\}$ and the Newton polytope is defined as $\mathrm{New}(f) = \mathrm{conv}(\mathrm{supp}(f))$. For a polytope P, we use P° to denote the interior of P. For $m \in \mathbb{N}$, let $[m] := \{1, \ldots, m\}$.

A polynomial $f \in \mathbb{R}[\mathbf{x}]$ which is nonnegative over $\mathbb{R}^n$ is called a *nonnegative polynomial*. The class of nonnegative polynomials is denoted by PSD, which is a closed convex cone.

A nonnegative polynomial must satisfy the following necessary conditions.

Proposition 2.1. *(*[14, Theorem 3.6]*) Let $\mathscr{A} \subset \mathbb{N}^n$ and $f = \sum_{\boldsymbol{\alpha} \in \mathscr{A}} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} \in \mathbb{R}[\mathbf{x}]$ with* $\mathrm{supp}(f) = \mathscr{A}$*. Then f is nonnegative only if the followings hold:*

(1) $V(\mathscr{A}) \subset (2\mathbb{N})^n$;

(2) If $\boldsymbol{\alpha} \in V(\mathscr{A})$, then the corresponding coefficient $c_{\boldsymbol{\alpha}}$ is positive.

For the remainder of this paper, we assume that the monomial factor of f is 1, that is, if $f = \mathbf{x}^{\boldsymbol{\alpha}'}(\sum c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}})$ such that $\sum c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} \in \mathbb{R}[\mathbf{x}]$ and $\boldsymbol{\alpha}' \in \mathbb{N}^n$, then $\mathbf{x}^{\boldsymbol{\alpha}'} = 1$.

2.2. **Nonnegative Polynomials Supported on Circuits.** A subset $\mathscr{A} \subseteq (2\mathbb{N})^n$ is called a *trellis* if $\mathscr{A}$ comprises the vertices of a simplex ([14]).

Definition 2.2. *Let $\mathscr{A}$ be a trellis and $f \in \mathbb{R}[\mathbf{x}]$. Then f is called a* circuit polynomial *if it is of the form*

$$f(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \in \mathscr{A}} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - d\mathbf{x}^{\boldsymbol{\beta}}, \tag{2.1}$$

with $c_{\boldsymbol{\alpha}} > 0$ and $\boldsymbol{\beta} \in \mathrm{conv}(\mathscr{A})^{\circ}$. Assume

$$\boldsymbol{\beta} = \sum_{\boldsymbol{\alpha} \in \mathscr{A}} \lambda_{\boldsymbol{\alpha}} \boldsymbol{\alpha} \text{ with } \lambda_{\boldsymbol{\alpha}} > 0 \text{ and } \sum_{\boldsymbol{\alpha} \in \mathscr{A}} \lambda_{\boldsymbol{\alpha}} = 1. \tag{2.2}$$

For every circuit polynomial f, we define the corresponding circuit number *as $\Theta_f := \prod_{\boldsymbol{\alpha} \in \mathscr{A}} (c_{\boldsymbol{\alpha}}/\lambda_{\boldsymbol{\alpha}})^{\lambda_{\boldsymbol{\alpha}}}$.*

The nonnegativity of a circuit polynomial f is decided by its circuit number alone.

Theorem 2.3. *(*[7, Theorem 3.8]*) Let $f = \sum_{\boldsymbol{\alpha} \in \mathscr{A}} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - d\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$ be a circuit polynomial and Θ_f its circuit number. Then f is nonnegative if and only if $\boldsymbol{\beta} \notin (2\mathbb{N})^n$ and $|d| \leq \Theta_f$, or $\boldsymbol{\beta} \in (2\mathbb{N})^n$ and $d \leq \Theta_f$.*

Proposition 2.4. *(*[7, Proposition 3.4 and Corollary 3.9]*) Let $f = \sum_{i=0}^{n} c_i \mathbf{x}^{\boldsymbol{\alpha}_i} - \Theta_f \mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$ be a circuit polynomial, Θ_f the circuit number and $\boldsymbol{\beta} = \sum_{i=0}^{n} \lambda_i \boldsymbol{\alpha}_i$ with $\lambda_i > 0$ and $\sum_{i=0}^{n} \lambda_i = 1$. Then f has exactly one zero $\mathbf{x}_*$ in $\mathbb{R}_+^n$ which satisfies:*

$$\frac{c_0 \mathbf{x}_*^{\boldsymbol{\alpha}_0}}{\lambda_0} = \cdots = \frac{c_n \mathbf{x}_*^{\boldsymbol{\alpha}_n}}{\lambda_n} = \Theta_f \mathbf{x}_*^{\boldsymbol{\beta}}. \tag{2.3}$$

Moreover, if $\mathbf{x}$ is any zero of f, then $|\mathbf{x}| = \mathbf{x}_*$, which means $|x_i| = (x_*)_i$ for $i = 1, \ldots, n$.

In analogy with writing a nonnegative polynomial as a sum of squares of polynomials, writing a nonnegative polynomial as a sum of nonnegative circuit polynomials is a certificate of its nonnegativity. We denote by SONC both the class of polynomials which can be written as sums of nonnegative circuit polynomials and the property of a polynomial to be in this class.

If the Newton polytope of f is a simplex, then under certain conditions f is nonnegative if and only if $f \in$ SONC.

Theorem 2.5. *([7, Corollary 7.5]) Let* $f = \sum_{i=0}^{n} c_i \mathbf{x}^{\boldsymbol{\alpha}_i} - \sum_{j=1}^{l} d_j \mathbf{x}^{\boldsymbol{\beta}_j} \in \mathbb{R}[\mathbf{x}]$ *with* $\boldsymbol{\alpha}_i \in (2\mathbb{N})^n, c_i > 0, i = 0, \ldots, n$ *such that* $\mathrm{New}(f)$ *is a simplex and* $\boldsymbol{\beta}_j \in \mathrm{New}(f)^\circ \cap \mathbb{N}^n$ *for* $j = 1, \ldots, l$*. If there exists a point* $\boldsymbol{v} = (v_j) \in (\mathbb{R}^*)^n$ *such that* $d_j \boldsymbol{v}^{\boldsymbol{\beta}_j} > 0$ *for all* j*, then* f *is nonnegative if and only if* $f \in \mathrm{SONC}$*.*

3. Nonnegative Polynomials with One Negative Term

Now we consider which types of nonnegative polynomials with general Newton polytopes admit SONC decompositions. In this section, we deal with the case of nonnegative polynomials with one negative term.

Let $f_d = \sum_{i=1}^{m} c_i \mathbf{x}^{\boldsymbol{\alpha}_i} - d\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$, $\boldsymbol{\alpha}_i \in (2\mathbb{N})^n, c_i > 0, i = 1, \ldots, m$, $\boldsymbol{\beta} \in \mathrm{New}(f_d)^\circ \cap \mathbb{N}^n$. Without loss of generality, assume $\dim(\mathrm{New}(f_d)) = n$ and $m > n+1$. It is easy to see that the set $\{d \in \mathbb{R} \mid f_d \text{ is nonnegative}\}$ is nonempty and has upper bounds. So the supremum exists. Let

$$d^* \triangleq \sup\{d \in \mathbb{R} \mid f_d \text{ is nonnegative}\}. \tag{3.1}$$

Theorem 3.1. *([9, Theorem 1.5]) Consider the following system of polynomial equations*

$$\sum_{i=1}^{m} c_i \boldsymbol{\alpha}_i \mathbf{x}^{\boldsymbol{\alpha}_i} = \mathbf{b}, \tag{3.2}$$

where $\boldsymbol{\alpha}_i \in \mathbb{R}^n, c_i > 0, i = 1, \ldots, m$*. Moreover, assume* $\dim(\mathrm{conv}(\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m\})) = n$*. Then for any* $\mathbf{b} \in \mathrm{cone}(\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m\})^\circ$*, (3.2) has exactly one zero in* $\mathbb{R}_+^n$*. (*$\mathrm{cone}(\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m\})$ *denotes the cone generated by* $\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m\}$*.)*

Lemma 3.2. *Assume* $\dim(\mathrm{conv}(\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m\})) = n$*. The following system of polynomial equations on variables* $(\mathbf{x}, d)$

$$\begin{cases} \sum_{i=1}^{m} c_i \mathbf{x}^{\boldsymbol{\alpha}_i} - d\mathbf{x}^{\boldsymbol{\beta}} = 0 \\ \sum_{i=1}^{m} c_i \boldsymbol{\alpha}_i \mathbf{x}^{\boldsymbol{\alpha}_i} - d\boldsymbol{\beta}\mathbf{x}^{\boldsymbol{\beta}} = \mathbf{0} \end{cases}, \tag{3.3}$$

where $\boldsymbol{\alpha}_i \in \mathbb{R}^n, c_i > 0, i = 1, \ldots, m$*,* $\boldsymbol{\beta} \in \mathrm{conv}(\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m\})^\circ$*, has exactly one zero in* $\mathbb{R}_+^{n+1}$*.*

Proof. Eliminate d from (3.3) and we obtain

$$\sum_{i=1}^{m} c_i (\boldsymbol{\alpha}_i - \boldsymbol{\beta}) \mathbf{x}^{\boldsymbol{\alpha}_i} = \mathbf{0}. \tag{3.4}$$

Divide (3.4) by $\mathbf{x}^{\boldsymbol{\beta}}$, and we have

$$\sum_{i=1}^{m} c_i (\boldsymbol{\alpha}_i - \boldsymbol{\beta}) \mathbf{x}^{\boldsymbol{\alpha}_i - \boldsymbol{\beta}} = \mathbf{0}. \tag{3.5}$$

Since $\boldsymbol{\beta} \in \mathrm{conv}(\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m\})^\circ$, we have $\mathbf{0} \in \mathrm{cone}(\{\boldsymbol{\alpha}_1 - \boldsymbol{\beta}, \ldots, \boldsymbol{\alpha}_m - \boldsymbol{\beta}\})^\circ$. Thus by Theorem 3.1, (3.5) and hence (3.4) have exactly one zero in $\mathbb{R}_+^n$, say $\mathbf{x}_*$. Substitute $\mathbf{x}_*$ into the first equation of (3.3), and we obtain $d = \sum_{i=1}^{m} c_i \mathbf{x}_*^{\boldsymbol{\alpha}_i - \boldsymbol{\beta}} > 0$. □

Theorem 3.3. *Let* $f_d = \sum_{i=1}^{m} c_i \mathbf{x}^{\boldsymbol{\alpha}_i} - d\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$*,* $\boldsymbol{\alpha}_i \in (2\mathbb{N})^n, c_i > 0, i = 1, \ldots, m$*,* $\boldsymbol{\beta} \in \mathrm{New}(f_d)^\circ \cap \mathbb{N}^n$*,* $\dim(\mathrm{New}(f_d)) = n$*, and* d^* *be defined as (3.1). Then* f_d *is nonnegative if and only if* $\boldsymbol{\beta} \notin (2\mathbb{N})^n$ *and* $|d| \le d^*$*, or* $\boldsymbol{\beta} \in (2\mathbb{N})^n$ *and* $d \le d^*$*. Moreover,* f_{d^*} *has exactly one zero in* $\mathbb{R}_+^n$*.*

Proof. First, if $\boldsymbol{\beta} \in (2\mathbb{N})^n$ and $d \leq 0$, then obviously f_d is nonnegative since it is a sum of monomial squares. If $\boldsymbol{\beta} \notin (2\mathbb{N})^n$ and $d \leq 0$, then f_d is nonnegative if and only if f_{-d} is nonnegative. So without loss of generality, we can always assume $d > 0$. Since the only negative term in f_d is $-d\mathbf{x}^{\boldsymbol{\beta}}$, f_d is nonnegative over $\mathbb{R}^n$ if and only if f_d is nonnegative over $\mathbb{R}^n_+$. Therefore, by the definition of d^*, f_d is nonnegative for $d \leq d^*$. The zeros of f_{d^*} are also the minimums of f_{d^*}. So they satisfy the system of equations $f_{d^*}(\mathbf{x}) = \nabla(f_{d^*}(\mathbf{x})) = 0$ which is equivalent to

$$\begin{cases} \sum_{i=1}^m c_i \mathbf{x}^{\boldsymbol{\alpha}_i} - d^* \mathbf{x}^{\boldsymbol{\beta}} = 0 \\ \sum_{i=1}^m c_i \boldsymbol{\alpha}_i \mathbf{x}^{\boldsymbol{\alpha}_i} - d^* \boldsymbol{\beta} \mathbf{x}^{\boldsymbol{\beta}} = \mathbf{0} \end{cases}. \tag{3.6}$$

By Lemma 3.2, (3.6) has exactly one zero in $\mathbb{R}^n_+$, and so does f_{d^*}. □

We need the following theorem from discrete geometry.

Theorem 3.4 (Helly, [3])**.** *Let $X_1, \ldots, X_r$ be a finite collection of convex subsets of $\mathbb{R}^n$ with $r > n$. If the intersection of every $n+1$ of these sets is nonempty, then the whole collection has a nonempty intersection.*

Corollary 3.5. *Let $X_1, \ldots, X_r$ be a finite collection of convex subsets of $\mathbb{R}^n$ with $r > n+1$. If the intersection of every $r-1$ of these sets is nonempty, then the whole collection has a nonempty intersection.*

Proof. Since $r > n+1$, the condition that the intersection of every $r-1$ of these sets is nonempty implies that the intersection of every $n+1$ of these sets is nonempty. So the corollary is immediate from Theorem 3.4. □

Lemma 3.6. *Let $A = (a_{ij}) \in \mathbb{R}^{m\times r}$, $\mathbf{b} = (b_j) \in \mathbb{R}^r$ and $\mathbf{z} = (z_1, \ldots, z_r)^T$ a set of variables. For each j, let A_j be the submatrix by deleting all of the i-th rows with $a_{ij} \neq 0$ and the j-th column from A. Assume that $A\mathbf{z} = \mathbf{b}$ has a solution,* $\operatorname{rank}(A) > 1$ *and* $\operatorname{rank}(A_j) = \operatorname{rank}(A) - 1$ *for all j. Then $A\mathbf{z} = \mathbf{b}$ has a nonnegative solution if and only if $A_j \bar{\mathbf{z}}_j = \bar{\mathbf{b}}_j$ has a nonnegative solution for $j = 1, \ldots, r$, where $\bar{\mathbf{z}}_j = \mathbf{z} \backslash z_j$, $\bar{\mathbf{b}}_j = \mathbf{b} \backslash b_j$.*

Proof. Let $t = \operatorname{rank}(A) > 1$. Then the system of linear equations $A\mathbf{z} = \mathbf{b}$ has $r - t$ free variables. Without loss of generality, let the $r - t$ free variables be $\{z_1, \ldots, z_{r-t}\}$. We can figure out $\{z_{r-t+1}, \ldots, z_r\}$ from $A\mathbf{z} = \mathbf{b}$ and assume $z_j = f_{j-r+t}(z_1, \ldots, z_{r-t})$ for $j = r-t+1, \ldots, r$. Then $A\mathbf{z} = \mathbf{b}$ has a nonnegative solution if and only if

$$\{(z_1, \ldots, z_{r-t}) \mid z_1 \geq 0, \ldots, z_{r-t} \geq 0, z_{r-t+1} = f_1 \geq 0, \ldots, z_r = f_t \geq 0\} \tag{3.7}$$

is nonempty. Since both $z_j \geq 0, 1 \leq j \leq r-t$ and $f_j \geq 0, 1 \leq j \leq t$ define convex subsets of $\mathbb{R}^{r-t}$, then by Corollary 3.5, (3.7) is nonempty if and only if

$$\begin{aligned} \{(z_1, \ldots, z_{r-t}) \mid & z_1 \geq 0, \ldots, z_{j-1} \geq 0, z_{j+1} \geq 0, \ldots, z_{r-t} \geq 0, \\ & z_{r-t+1} = f_1 \geq 0, \ldots, z_r = f_t \geq 0\} \end{aligned} \tag{3.8}$$

is nonempty for $j = 1, \ldots, r-t$ and

$$\begin{aligned} \{(z_1, \ldots, z_{r-t}) \mid & z_1 \geq 0, \ldots, z_{r-t} \geq 0, z_{r-t+1} = f_1 \geq 0, \ldots, z_{j-1+r-t} = f_{j-1} \geq 0, \\ & z_{j+1+r-t} = f_{j+1} \geq 0, \ldots, z_r = f_t \geq 0\} \end{aligned} \tag{3.9}$$

is nonempty for $j = 1, \ldots, t$.

For $j \in [r-t]$, (3.8) is nonempty if and only if $A\mathbf{z} = \mathbf{b}$ has a solution with $\bar{\mathbf{z}}_j \in \mathbb{R}_{\geq 0}^{r-1}$ and $z_j \in \mathbb{R}$, which is equivalent to the condition that $A_j\bar{\mathbf{z}}_j = \bar{\mathbf{b}}_j$ has a nonnegative solution since $\operatorname{rank}(A_j) = \operatorname{rank}(A) - 1$. For $j \in [t]$, (3.9) is nonempty if and only if $\{f_1 \geq 0, \ldots, f_{j-1} \geq 0, f_{j+1} \geq 0, \ldots, f_t \geq 0\}$ has a nonnegative solution, which is also equivalent to the condition that $A_{j+r-t}\bar{\mathbf{z}}_{j+r-t} = \bar{\mathbf{b}}_{j+r-t}$ has a nonnegative solution since $\operatorname{rank}(A_{j+r-t}) = \operatorname{rank}(A) - 1$. Put all above together and we have that $A\mathbf{z} = \mathbf{b}$ has a nonnegative solution if and only if $A_j\bar{\mathbf{z}}_j = \bar{\mathbf{b}}_j$ has a nonnegative solution for $j = 1, \ldots, r$ as desired. □

We know that the system of linear equations $A\mathbf{z} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ belongs to the image of A. For the later use, we give a more concrete description for the condition that $A\mathbf{z} = \mathbf{b}$ has a solution here.

Lemma 3.7. *Let $A = (a_{ij}) \in \mathbb{R}^{m\times r}$, $\mathbf{b} = (b_j) \in \mathbb{R}^r$, and $\mathbf{z} = (z_1, \ldots, z_r)^T$, $\mathbf{y} = (y_1, \ldots, y_r)^T$ be sets of variables. Let $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ be the set of row vectors of A and let $I = (\mathbf{a}_1\mathbf{z} - y_1, \ldots, \mathbf{a}_m\mathbf{z} - y_m) \cap \mathbb{R}[y_1, \ldots, y_m]$. Assume that $\{\mathbf{c}_1\mathbf{y}, \ldots, \mathbf{c}_l\mathbf{y}\}$ is a set of generators of I and let C be the matrix whose row vectors are $\{\mathbf{c}_1, \ldots, \mathbf{c}_l\}$. Then $\operatorname{rank}(C) = m - \operatorname{rank}(A)$ and $A\mathbf{z} = \mathbf{b}$ has a solution if and only if $C\mathbf{b} = \mathbf{0}$.*

Proof. Observe that $\{\mathbf{c}_1, \ldots, \mathbf{c}_l\}$ generates the linear space of all linear relationships among $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$. In other words, $\{\mathbf{c}_1^T, \ldots, \mathbf{c}_l^T\}$ generates the kernel space of A^T. Thus $\operatorname{rank}(C) = \operatorname{rank}(\ker(A^T)) = m - \operatorname{rank}(A)$.

If $C\mathbf{b} = \mathbf{0}$, i.e. $\mathbf{b}$ is a zero of the elimination ideal I, then by the Extension Theorem in p.125 of [2], we can extend $\mathbf{b}$ to a zero of the ideal $(\mathbf{a}_1\mathbf{z} - y_1, \ldots, \mathbf{a}_m\mathbf{z} - y_m)$. So $A\mathbf{z} = \mathbf{b}$ has a solution. The converse is obvious. □

Lemma 3.8. *Let $f_d = \sum_{i=1}^m c_i\mathbf{x}^{\boldsymbol{\alpha}_i} - d\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$, $\boldsymbol{\alpha}_i \in (2\mathbb{N})^n, c_i > 0, i = 1, \ldots, m$, $\boldsymbol{\beta} \in \operatorname{New}(f_d)^\circ \cap \mathbb{N}^n$, $\dim(\operatorname{New}(f_d)) = n$, and d^* be defined as (3.1). Then $f_{d^*} \in$ SONC.*

Proof. By Theorem 3.3, f_{d^*} has exactly one zero in $\mathbb{R}_+^n$, which is denoted by $\mathbf{x}_*$. Let

$$\{\Delta_1, \ldots, \Delta_r\} := \{\Delta \mid \Delta \text{ is a simplex }, \boldsymbol{\beta} \in \Delta^\circ, V(\Delta) \subseteq \{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m\}\}$$

and $I_k := \{i \in [m] \mid \boldsymbol{\alpha}_i \in V(\Delta_k)\}$ for $k = 1, \ldots, r$. Firstly, we assume $\dim(\Delta_k) = n$ for all k. Hence $|I_k| = n + 1$ for all k. For each Δ_k, since $\boldsymbol{\beta} \in \Delta_k^\circ$, we have $\boldsymbol{\beta} = \sum_{i\in I_k} \lambda_{ik}\boldsymbol{\alpha}_i$, where $\sum_{i\in I_k} \lambda_{ik} = 1, \lambda_{ik} > 0, i \in I_k$. Let us consider the following system of linear equations on variables c_{ik} and s_k:

$$\begin{cases} \frac{c_{ik}\mathbf{x}_*^{\boldsymbol{\alpha}_i}}{\lambda_{ik}} = s_k, & \text{for } i \in I_k, k = 1, \ldots, r \\ \sum_{i\in I_k} c_{ik} = c_i, & \text{for } i = 1, \ldots, m \end{cases}. \tag{3.10}$$

Eliminate c_{ik} from (3.10) and we obtain:

$$\sum_{i\in I_k} \lambda_{ik}s_k = c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}, \quad \text{for } i = 1, \ldots, m. \tag{3.11}$$

Claim: The linear system (3.11) on variables $\{s_1, \ldots, s_r\}$ has a nonnegative solution.

Denote the coefficient matrix of (3.11) by A. Add up all of the equations of (3.11) and we obtain:

$$\sum_{k=1}^r s_k = \sum_{i=1}^m \sum_{i\in I_k} \lambda_{ik}s_k = \sum_{i=1}^m c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}. \tag{3.12}$$

Multiply the i-th equation of (3.11) by $\boldsymbol{\alpha}_i$ and then add up all of them. We obtain:

$$\boldsymbol{\beta}\sum_{k=1}^{r} s_k = \sum_{i=1}^{m}\sum_{i\in I_k}\lambda_{ik}\boldsymbol{\alpha}_i s_k = \sum_{i=1}^{m} c_i\boldsymbol{\alpha}_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}. \tag{3.13}$$

(3.13) $-$ (3.12) $\times\,\boldsymbol{\beta}$ gives

$$\sum_{i=1}^{m} c_i(\boldsymbol{\alpha}_i-\boldsymbol{\beta})\mathbf{x}_*^{\boldsymbol{\alpha}_i} = \mathbf{0}. \tag{3.14}$$

By (3.6) in the proof of Theorem 3.3, $\{c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}\}_{i=1}^m$ satisfies (3.14). Thus by Lemma 3.7, (3.11) has a solution. Moreover, since $\boldsymbol{\beta}\in\mathrm{New}(f_d)^\circ$ and $\dim(\mathrm{New}(f_d))=n$, then $\mathrm{rank}(\{\boldsymbol{\alpha}_i-\boldsymbol{\beta}\}_{i=1}^m)=n$. So $\mathrm{rank}(A)=m-n$.

For each j, denote the coefficient matrix of

$$\{\sum_{i\in I_k}\lambda_{ik}s_k = c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}\mid i\notin I_j\} \tag{3.15}$$

by A_j. For every $i\notin I_j$, since $\boldsymbol{\beta}\in\Delta_j^\circ$, there exists a facet F of Δ_j such that $\boldsymbol{\beta}\in\mathrm{conv}(V(F)\cup\{\boldsymbol{\alpha}_i\})^\circ$. Assume $\mathrm{conv}(V(F)\cup\{\boldsymbol{\alpha}_i\})=\Delta_{p_i}$. For every $k\notin\{j\}\cup\bigcup_{i\notin I_j}\{p_i\}$, let $s_k=0$ in (3.15) and we obtain:

$$\{\lambda_{ip_i}s_{p_i} = c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}\mid i\notin I_j\}. \tag{3.16}$$

Thus $\mathrm{rank}(A_j)=m-|I_j|=m-(n+1)=\mathrm{rank}(A)-1$. Therefore by Lemma 3.6, to prove the claim, we only need to show that the linear system (3.15) on variables $\{s_1,\dots,s_r\}\backslash\{s_j\}$ has a nonnegative solution for $j=1,\dots,r$.

Given $j\in[r]$, from (3.16) we have $s_{p_i}=c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}/\lambda_{ip_i}$ for $i\notin I_j$ and hence

$$\begin{cases} s_k=0, & \text{for } k\notin\{j\}\cup\bigcup_{i\notin I_j}\{p_i\}\\ s_{p_i}=c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}/\lambda_{ip_i}, & \text{for } i\notin I_j\end{cases} \tag{3.17}$$

is a nonnegative solution for (3.15). So the claim is proved.

Assume that $\{s_1^*,\dots,s_r^*\}$ is a nonnegative solution for the system of equations (3.11). Substitute $\{s_1^*,\dots,s_r^*\}$ into the system of equations (3.10), and we have $c_{ik}=\lambda_{ik}s_k^*/\mathbf{x}_*^{\boldsymbol{\alpha}_i}$ for $i\in I_k,k=1,\dots,r$. Let $d_k=s_k^*/\mathbf{x}_*^{\boldsymbol{\beta}}$ and $f_k=\sum_{i\in I_k}c_{ik}\mathbf{x}^{\boldsymbol{\alpha}_i}-d_k\mathbf{x}^{\boldsymbol{\beta}}$ for $k=1,\dots,r$. Then by (3.10) and by Proposition 2.4, d_k is the circuit number of f_k and f_k is a nonnegative circuit polynomial for all k. By (3.10), $\sum_{k=1}^r d_k\mathbf{x}_*^{\boldsymbol{\beta}}=\sum_{k=1}^r\sum_{i\in I_k}c_{ik}\mathbf{x}_*^{\boldsymbol{\alpha}_i}=\sum_{i=1}^m c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}=d^*\mathbf{x}_*^{\boldsymbol{\beta}}$. So we have $\sum_{k=1}^r d_k=d^*$. It follows $f_{d^*}=\sum_{k=1}^r f_k$ as desired.

For the case that $\dim(\Delta_k)=n$ does not hold for all k, note that all results above remain valid for $\boldsymbol{\beta}\in\mathbb{R}^n$. We give $\boldsymbol{\beta}$ a small perturbation, say $\boldsymbol{\delta}$, such that $\dim(\Delta_k)=n$ holds for all k. Then the new linear system (3.11) for $\boldsymbol{\beta}+\boldsymbol{\delta}$ has a nonnegative solution. Let $\boldsymbol{\delta}\to\mathbf{0}$, we obtain that (3.11) also has a nonnegative solution for $\boldsymbol{\beta}$. Thus the theorem remains true in this case. □

Theorem 3.9. *Let* $f_d=\sum_{i=1}^m c_i\mathbf{x}^{\boldsymbol{\alpha}_i}-d\mathbf{x}^{\boldsymbol{\beta}}\in\mathbb{R}[\mathbf{x}]$, $\boldsymbol{\alpha}_i\in(2\mathbb{N})^n, c_i>0, i=1,\dots,m$, $\boldsymbol{\beta}\in\mathrm{New}(f_d)^\circ\cap\mathbb{N}^n$, $\dim(\mathrm{New}(f_d))=n$. *Then* f_d *is nonnegative if and only if* $f_d\in\mathrm{SONC}$.

Proof. The sufficiency is obvious. Assume that f_d is nonnegative. If $\boldsymbol{\beta}\in(2\mathbb{N})^n$ and $d<0$, or $d=0$, then f_d is a sum of monomial squares and obviously $f_d\in\mathrm{SONC}$. If $\boldsymbol{\beta}\notin(2\mathbb{N})^n$ and $d<0$, through a variable transformation $x_j\mapsto -x_j$ for some odd number β_j, we can always assume $d>0$. Let d^* be defined as (3.1). By

Lemma 3.8, $f_{d^*} \in \text{SONC}$. Suppose $f_{d^*} = \sum_{k=1}^{r}(\sum_{i\in I_k} c_{ik}\mathbf{x}^{\boldsymbol{\alpha}_i} - d_k\mathbf{x}^{\boldsymbol{\beta}})$, where $\sum_{i\in I_k} c_{ik}\mathbf{x}^{\boldsymbol{\alpha}_i} - d_k\mathbf{x}^{\boldsymbol{\beta}}$ is a circuit polynomials with d_k the corresponding circuit number for all k. Since f_d is nonnegative, then $d \le d^*$ by Theorem 3.3. We have $f_d = \sum_{k=1}^{r}(\sum_{i\in I_k} c_{ik}\mathbf{x}^{\boldsymbol{\alpha}_i} - \frac{d}{d^*}d_k\mathbf{x}^{\boldsymbol{\beta}})$, where $\sum_{i\in I_k} c_{ik}\mathbf{x}^{\boldsymbol{\alpha}_i} - \frac{d}{d^*}d_k\mathbf{x}^{\boldsymbol{\beta}}$ is a nonnegative circuit polynomial for all k by Theorem 2.3. Thus $f_d \in \text{SONC}$. □

4. Nonnegative Polynomials with Multiple Negative Terms

In this section, we deal with the case of nonnegative polynomials with multiple negative terms. Let Δ be a polytope of dimension d. For a vertex $\boldsymbol{\alpha}$ of Δ, if $\boldsymbol{\alpha}$ is the intersection of precisely d edges, then we say Δ is *simple* at $\boldsymbol{\alpha}$.

Theorem 4.1. *Let $f = \sum_{i=1}^{m} c_i\mathbf{x}^{\boldsymbol{\alpha}_i} - \sum_{j=1}^{l} d_j\mathbf{x}^{\boldsymbol{\beta}_j} \in \mathbb{R}[\mathbf{x}]$, $\boldsymbol{\alpha}_i \in (2\mathbb{N})^n, c_i > 0, i = 1,\dots,m$, $\boldsymbol{\beta}_j \in \text{New}(f)^\circ \cap \mathbb{N}^n, j = 1,\dots,l$. Assume that $\text{New}(f)$ is simple at some vertex, all of the $\boldsymbol{\beta}_j$'s lie in the same side of every hyperplane determined by points among $\{\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_m\}$ and there exists a point $\boldsymbol{v} = (v_k) \in (\mathbb{R}^*)^n$ such that $d_j\boldsymbol{v}^{\boldsymbol{\beta}_j} > 0$ for all j. Then f is nonnegative if and only if $f \in \text{SONC}$.*

Proof. Without loss of generality, assume $\dim(\text{New}(f)) = n$ and $m > n+1$. The sufficiency is obvious. Suppose f is nonnegative. After a variable transformation $x_k \mapsto -x_k$ for all k with $v_k < 0$, we can assume all $d_j > 0$. Let

$$d_l^* \triangleq \sup\{d_l \in \mathbb{R} \mid f = \sum_{i=1}^{m} c_i\mathbf{x}^{\boldsymbol{\alpha}_i} - \sum_{j=1}^{l-1} d_j\mathbf{x}^{\boldsymbol{\beta}_j} - d_l\mathbf{x}^{\boldsymbol{\beta}_l} \text{ is nonnegative}\}. \tag{4.1}$$

Note that d_l^* is well-defined since the set in (4.1) is nonempty and has upper bounds. Let $f^* = \sum_{i=1}^{m} c_i\mathbf{x}^{\boldsymbol{\alpha}_i} - \sum_{j=1}^{l-1} d_j\mathbf{x}^{\boldsymbol{\beta}_j} - d_l^*\mathbf{x}^{\boldsymbol{\beta}_l}$. Then $f^* = 0$ has a zero in $\mathbb{R}_+^n$ ([16, Lemma 4.6]), which is denoted by $\mathbf{x}_*$. The condition that all of the $\boldsymbol{\beta}_j$'s lie in the same side of every hyperplane determined by points among $\{\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_m\}$ implies if a simplex Δ whose vertices come from $\{\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_m\}$ contains some $\boldsymbol{\beta}_j$, then $\dim(\Delta) = n$ and it contains all of $\{\boldsymbol{\beta}_1,\dots,\boldsymbol{\beta}_l\}$. Let

$$\{\Delta_1,\dots,\Delta_r\} := \{\Delta \mid \Delta \text{ is a simplex }, \boldsymbol{\beta}_j \in \Delta^\circ, j \in [l], V(\Delta) \subseteq \{\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_m\}\}$$

and $I_k := \{i \in [m] \mid \boldsymbol{\alpha}_i \in V(\Delta_k)\}$ for $k = 1,\dots,r$. Then $\dim(\Delta_k) = n$ for all k. For every $\boldsymbol{\beta}_j$ and every Δ_k, since $\boldsymbol{\beta}_j \in \Delta_k^\circ$, we have $\boldsymbol{\beta}_j = \sum_{i\in I_k}\lambda_{ijk}\boldsymbol{\alpha}_i$, where $\sum_{i\in I_k}\lambda_{ijk} = 1, \lambda_{ijk} > 0, i \in I_k$. Let us consider the following system of linear equations on variables c_{ijk}, d_{jk} and s_{jk}:

$$\begin{cases} \frac{c_{ijk}\mathbf{x}_*^{\boldsymbol{\alpha}_i}}{\lambda_{ijk}} = d_{jk}\mathbf{x}_*^{\boldsymbol{\beta}_j} = s_{jk}, & \text{for } i \in I_k, k = 1,\dots,r, j = 1,\dots,l \\ \sum_{k=1}^{r} d_{jk} = d_j, & \text{for } j = 1,\dots,l-1 \\ \sum_{k=1}^{r} d_{lk} = d_l^*, & \\ \sum_{j=1}^{l}\sum_{i\in I_k} c_{ijk} = c_i, & \text{for } i = 1,\dots,m \end{cases}. \tag{4.2}$$

Eliminate c_{ijk} and d_{jk} from (4.2) and we obtain:

$$\begin{cases} \sum_{j=1}^{l}\sum_{i\in I_k}\lambda_{ijk}s_{jk} = c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}, & \text{for } i = 1,\dots,m \\ \sum_{k=1}^{r} s_{jk} = d_j\mathbf{x}_*^{\boldsymbol{\beta}_j}, & \text{for } j = 1,\dots,l-1 \\ \sum_{k=1}^{r} s_{lk} = d_l^*\mathbf{x}_*^{\boldsymbol{\beta}_l}, & \end{cases}. \tag{4.3}$$

Claim: The linear system (4.3) on variables s_{jk} has a nonnegative solution.

Denote the coefficient matrix of (4.3) by A. Add up all of the equations of the first part of (4.3), and we obtain:

$$\sum_{j=1}^{l}\sum_{k=1}^{r} s_{jk} = \sum_{i=1}^{m}\sum_{j=1}^{l}\sum_{i\in I_k} \lambda_{ijk}s_{jk} = \sum_{i=1}^{m} c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}. \tag{4.4}$$

Multiply the i-th equation of the first part of (4.3) by $\boldsymbol{\alpha}_i$ and then add up all of them. We obtain:

$$\sum_{j=1}^{l}\boldsymbol{\beta}_j\sum_{k=1}^{r} s_{jk} = \sum_{i=1}^{m}\sum_{j=1}^{l}\sum_{i\in I_k} \lambda_{ijk}\boldsymbol{\alpha}_i s_{jk} = \sum_{i=1}^{m} c_i\boldsymbol{\alpha}_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}. \tag{4.5}$$

Substitute the second and the third part of (4.3) into (4.4) and (4.5), and we obtain:

$$\begin{cases}\sum_{j=1}^{l-1} d_j\mathbf{x}_*^{\boldsymbol{\beta}_j} + d_l^*\mathbf{x}_*^{\boldsymbol{\beta}_l} = \sum_{i=1}^{m} c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}\\ \sum_{j=1}^{l-1} d_j\boldsymbol{\beta}_j\mathbf{x}_*^{\boldsymbol{\beta}_j} + d_l^*\boldsymbol{\beta}_l\mathbf{x}_*^{\boldsymbol{\beta}_l} = \sum_{i=1}^{m} c_i\boldsymbol{\alpha}_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}\end{cases}. \tag{4.6}$$

The zero $\mathbf{x}_*$ of f^* is also the minimizer of f^*. So it satisfies $f^*(\mathbf{x}_*) = \nabla(f^*(\mathbf{x}_*)) = 0$ which is equivalent to (4.6). Thus by Lemma 3.7, (4.3) has a solution. Moreover, since $\dim(\Delta_1) = n$, the volume of Δ_1, which equals $\frac{1}{n!}|\det(\{\begin{pmatrix}1\\ \boldsymbol{\alpha}_i\end{pmatrix}\}_{i\in I_1})|$, is nonzero. So $\mathrm{rank}(\{\begin{pmatrix}1\\ \boldsymbol{\alpha}_1\end{pmatrix},\ldots,\begin{pmatrix}1\\ \boldsymbol{\alpha}_m\end{pmatrix},\begin{pmatrix}-1\\ -\boldsymbol{\beta}_1\end{pmatrix},\ldots,\begin{pmatrix}-1\\ -\boldsymbol{\beta}_l\end{pmatrix}\}) = n+1$ and hence by Lemma 3.7, $\mathrm{rank}(A) = m + l - (n+1) > 1$.

For every $u \in [l]$ and every $v \in [r]$, denote the coefficient matrix of

$$\begin{cases}\sum_{i\in I_k} \lambda_{ijk}s_{jk} = c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}, & \text{for } i \notin I_v\\ \sum_{k=1}^{r} s_{jk} = d_j\mathbf{x}_*^{\boldsymbol{\beta}_j}, & \text{for } j \neq u, l\\ \sum_{k=1}^{r} s_{lk} = d_l^*\mathbf{x}_*^{\boldsymbol{\beta}_l}, & \end{cases} \tag{4.7}$$

by A_{uv}. For every $i \notin I_v$, since $\boldsymbol{\beta}_u \in \Delta_v^\circ$, there exists a facet F of Δ_v such that $\boldsymbol{\beta}_u \in \mathrm{conv}(V(F)\cup\{\boldsymbol{\alpha}_i\})^\circ$. Assume $\mathrm{conv}(V(F)\cup\{\boldsymbol{\alpha}_i\}) = \Delta_{p_i}$. For $j = u, k \notin \cup_{i\notin I_v}\{p_i\}$ or $j \neq u, k \neq v$, let $s_{jk} = 0$ in (4.7), and we obtain:

$$\begin{cases}\lambda_{iup_i}s_{up_i} = c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}, & \text{for } i \notin I_v\\ s_{jv} = d_j\mathbf{x}_*^{\boldsymbol{\beta}_j}, & \text{for } j \neq u, l\\ s_{lv} = d_l^*\mathbf{x}_*^{\boldsymbol{\beta}_l}, & \end{cases}. \tag{4.8}$$

Thus $\mathrm{rank}(A_{uv}) = m - |I_v| + l - 1 = m - (n+1) + l - 1 = \mathrm{rank}(A) - 1$. Therefore by Lemma 3.6, to prove the claim, we only need to show that the linear system (4.7) on variables $\{s_{jk}\}_{j,k}\backslash\{s_{uv}\}$ has a nonnegative solution for all $u \in [l]$ and all $v \in [r]$.

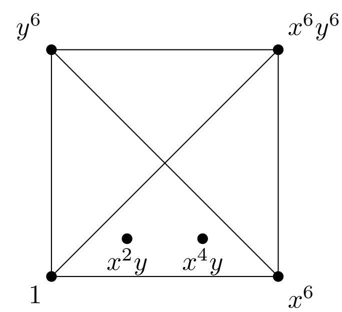
Given $v \in [r]$, from (4.8) we have $s_{up_i} = c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}/\lambda_{iup_i}$ for $i \notin I_v$, $s_{jv} = d_j\mathbf{x}_*^{\boldsymbol{\beta}_j}$ for $j \neq u, l$, and $s_{lv} = d_l^*\mathbf{x}_*^{\boldsymbol{\beta}_l}$. Hence

$$\begin{cases}s_{jk} = 0, & \text{for } j = u, k \notin \cup_{i\notin I_v}\{p_i\} \text{ or } j \neq u, k \neq v\\ s_{up_i} = c_i\mathbf{x}_*^{\boldsymbol{\alpha}_i}/\lambda_{iup_i}, & \text{for } i \notin I_v\\ s_{jv} = d_j\mathbf{x}_*^{\boldsymbol{\beta}_j}, & \text{for } j \neq u, l\\ s_{lv} = d_l^*\mathbf{x}_*^{\boldsymbol{\beta}_l}, & \end{cases} \tag{4.9}$$

is a nonnegative solution for (4.7). So the claim is proved.

Assume that $\{s_{jk}^*\}_{j,k}$ is a nonnegative solution for the system of equations (4.3). Substitute $\{s_{jk}^*\}_{j,k}$ into the system of equations (4.2), and we have $c_{ijk} = \lambda_{ijk}s_{jk}^*/\mathbf{x}_*^{\boldsymbol{\alpha}_i}$ for $i \in I_k, k = 1, \ldots, r, j = 1, \ldots, l$. Let $f_{jk} = \sum_{i\in I_k} c_{ijk}\mathbf{x}^{\boldsymbol{\alpha}_i} - d_{jk}\mathbf{x}^{\boldsymbol{\beta}_j}$ for $k = 1, \ldots, r, j = 1, \ldots, l$. Then by (4.2) and by Proposition 2.4, d_{jk} is the circuit number of f_{jk} and f_{jk} is a nonnegative circuit polynomial for all j, k. By (4.2), we have $f = \sum_{j=1}^{l-1}\sum_{k=1}^{r} f_{jk} + \sum_{k=1}^{r}(\sum_{i\in I_k} c_{ilk}\mathbf{x}^{\boldsymbol{\alpha}_i} - \frac{d_l}{d_l^*}d_{lk}\mathbf{x}^{\boldsymbol{\beta}_l})$. Since $d_l \le d_l^*$, $\sum_{i\in I_k} c_{ilk}\mathbf{x}^{\boldsymbol{\alpha}_i} - \frac{d_l}{d_l^*}d_{lk}\mathbf{x}^{\boldsymbol{\beta}_l}$ is a nonnegative circuit polynomial for all k by Theorem 2.3. Thus $f \in \mathrm{SONC}$. $\square$

Example 4.2. *Let* $d^* = \sup\{d \in \mathbb{R}_+ \mid 1+x^6+y^6+x^6y^6-x^2y-dx^4y$ *is nonnegative*$\}$ *and* $f = 1+x^6+y^6+x^6y^6-x^2y-d^*x^4y$. $(2,1) = \frac{1}{6}(6,6)+\frac{1}{6}(6,0)+\frac{2}{3}(0,0) = \frac{1}{3}(6,0)+\frac{1}{6}(0,6)+\frac{1}{2}(0,0)$, $(4,1) = \frac{1}{6}(6,6)+\frac{1}{2}(6,0)+\frac{1}{3}(0,0) = \frac{2}{3}(6,0)+\frac{1}{6}(0,6)+\frac{1}{6}(0,0)$.



The system $f = \nabla(f) = 0$ *has exactly one zero* $(x_* \approx 1.04521, y_* \approx 0.764724, d^* \approx 2.11373)$ *in* $\mathbb{R}_+^n$. *The following linear system*

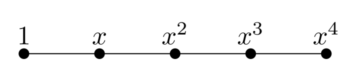
$$\begin{cases} 1 = \frac{2}{3}s_{11} + \frac{1}{2}s_{12} + \frac{1}{3}s_{21} + \frac{1}{6}s_{22} \\ x_*^6 = \frac{1}{6}s_{11} + \frac{1}{3}s_{12} + \frac{1}{2}s_{21} + \frac{2}{3}s_{22} \\ y_*^6 = \frac{1}{6}s_{12} + \frac{1}{6}s_{22} \\ x_*^6y_*^6 = \frac{1}{6}s_{11} + \frac{1}{6}s_{21} \\ x_*^2y_* = s_{11} + s_{12} \\ d^*x_*^4y_* = s_{21} + s_{22} \end{cases} \tag{4.10}$$

on variables $\{s_{11}, s_{12}, s_{21}, s_{22}\}$ *has a nonnegative solution* $(s_{11} \approx 0.835429, s_{12} = 0, s_{21} \approx 0.729142, s_{22} = 1.2)$. *So from the proof of Theorem 4.1, we obtain* $f \approx (0.556953+0.106793x^6+0.533967x^6y^6-x^2y)+(0.243047+0.27962x^6+0.466033x^6y^6-0.798909x^4y)+(0.2+0.613587x^6+y^6-1.31482x^4y)$ *as a sum of nonnegative circuit polynomials. Therefore,* $f \in \mathrm{SONC}$.

The condition that all of the $\boldsymbol{\beta}_j$'s lie in the same side of every hyperplane determined by points among $\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m\}$ in Theorem 4.1 cannot be dropped. We give an example to illustrate this.

Example 4.3. *Let* $f = 1 + 4x^2 + x^4 - 3x - 3x^3$. *Then* $f \in \mathrm{PSD}$, *but* $f \notin \mathrm{SONC}$.

Proof. The minimum of f is 0 with the only minimizer $x_* = 1$. We have $1 = \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 2 = \frac{3}{4}\cdot 0 + \frac{1}{4}\cdot 4$, $3 = \frac{1}{2}\cdot 2 + \frac{1}{2}\cdot 4 = \frac{1}{4}\cdot 0 + \frac{3}{4}\cdot 4$.



From the proof of Theorem 4.1, we have that if $f \in \text{SONC}$, then the following linear system

$$\begin{cases} 1 = \frac{1}{2}s_1 + \frac{3}{4}s_3 + \frac{1}{4}s_4 \\ 4x_*^2 = \frac{1}{2}s_1 + \frac{1}{2}s_3 \\ x_*^4 = \frac{1}{4}s_2 + \frac{1}{2}s_3 + \frac{3}{4}s_4 \\ 3x_* = s_1 + s_2 \\ 3x_*^3 = s_3 + s_4 \end{cases} \tag{4.11}$$

on variables $\{s_1, s_2, s_3, s_4\}$ should have a nonnegative solution. However, (4.11) has no nonnegative solutions. So $f \notin \text{SONC}$. □

Corollary 4.4. *Let $f = \sum_{i=1}^m c_i \mathbf{x}^{\boldsymbol{\alpha}_i} - \sum_{j=1}^l d_j \mathbf{x}^{\boldsymbol{\beta}_j} \in \mathbb{R}[\mathbf{x}]$, $\boldsymbol{\alpha}_i \in (2\mathbb{N})^n, c_i > 0, i = 1,\ldots,m$, $\boldsymbol{\beta}_j \in \text{New}(f)^\circ \cap \mathbb{N}^n, d_j > 0, j = 1,\ldots,l$, $\dim(\text{New}(f)) = n$. Assume that f is nonnegative and has a zero, $\text{New}(f)$ is simple at some vertex, and all of the $\boldsymbol{\beta}_j$'s lie in the same side of every hyperplane determined by points among $\{\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_m\}$. Then f has exactly one zero in $\mathbb{R}_+^n$.*

Proof. By Theorem 4.1, $f \in \text{SONC}$. Suppose $f = \sum_{k=1}^r f_k$, where all f_k are nonnegative circuit polynomials. Let $\mathbf{x}$ be a zero of f. Then $f_k(\mathbf{x}) = 0$ for all k. By Proposition 2.4, $f_k(|\mathbf{x}|) = 0$ and $|\mathbf{x}|$ is the only zero of f_k in $\mathbb{R}_+^n$ for all k. Hence $|\mathbf{x}|$ is the only zero of f in $\mathbb{R}_+^n$. □

5. Supports of Sums of Nonnegative Circuit Polynomials

For $f \in \mathbb{R}[\mathbf{x}]$, let $\Lambda(f) := \{\boldsymbol{\alpha} \in \text{supp}(f) \mid \boldsymbol{\alpha} \in (2\mathbb{N})^n \text{ and } c_{\boldsymbol{\alpha}} > 0\}$ and $\Gamma(f) := \text{supp}(f)\backslash\Lambda(f)$. Then we can write $f = \sum_{\boldsymbol{\alpha}\in\Lambda(f)} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\Gamma(f)} d_{\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\beta}}$ with $c_{\boldsymbol{\alpha}} > 0$. Assume $\Gamma(f) \subset \text{New}(f)^\circ$. For every $\boldsymbol{\beta} \in \Gamma(f)$, let

$$\Delta(\boldsymbol{\beta}) := \{\Delta \mid \Delta \text{ is a simplex}, \boldsymbol{\beta} \in \Delta^\circ, V(\Delta) \subseteq \Lambda(f)\}. \tag{5.1}$$

If we can write f as $f = \sum_{\boldsymbol{\beta}\in\Gamma(f)} \sum_{\Delta\in\Delta(\boldsymbol{\beta})} (\sum_{\boldsymbol{\alpha}\in V(\Delta)} c_{\boldsymbol{\beta}\Delta\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - d_{\boldsymbol{\beta}\Delta} \mathbf{x}^{\boldsymbol{\beta}})$ such that every $\sum_{\boldsymbol{\alpha}\in V(\Delta)} c_{\boldsymbol{\beta}\Delta\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - d_{\boldsymbol{\beta}\Delta} \mathbf{x}^{\boldsymbol{\beta}}$ is a nonnegative circuit polynomial, then we say that f is a *sum of nonnegative circuit polynomials with the same support.*

In Theorem 3.9 and Theorem 4.1, we see that nonnegative polynomials satisfying certain conditions decompose into sums of nonnegative circuit polynomials with the same support. Actually, every SONC polynomial decomposes into a sum of nonnegative circuit polynomials with the same support. The proof needs a connection between nonnegative circuit polynomials and sums of squares of binomials (SOSB).

5.1. Nonnegative Circuit Polynomials and Sums of Squares of Binomials.

We call a lattice point is *even* if it is in $(2\mathbb{N})^n$. For a subset $M \subseteq \mathbb{N}^n$, define $\overline{A}(M) := \{\frac{1}{2}(\boldsymbol{u}+\boldsymbol{v}) \mid \boldsymbol{u} \neq \boldsymbol{v}, \boldsymbol{u}, \boldsymbol{v} \in M \cap (2\mathbb{N})^n\}$ as the set of averages of distinct even points in M. For a trellis $\mathscr{A}$, we sat that M is an *$\mathscr{A}$-mediated set* if $\mathscr{A} \subseteq M \subseteq \overline{A}(M) \cup \mathscr{A}$.

Theorem 5.1. *Let $f = \sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - d\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}], d \neq 0$ be a nonnegative circuit polynomial with $\boldsymbol{\beta} \in \text{New}(f)^\circ$. If $\boldsymbol{\beta}$ belongs to an $\mathscr{A}$-mediated set M, then f is a sum of squares of the form $(a_{\boldsymbol{u}} \mathbf{x}^{\boldsymbol{u}} - b_{\boldsymbol{v}} \mathbf{x}^{\boldsymbol{v}})^2$, where $2\boldsymbol{u}, 2\boldsymbol{v} \in M$.*

Proof. For the proof, please refer to Theorem 5.2 in [7] which exploits Theorem 4.4 in [14]. □

Inspired by Theorem 5.1, we are interested in the problem of deciding if there exists an $\mathscr{A}$-mediated set containing a given lattice point and computing one if there exists. However, there are no effective algorithms to do such thing as far as I know. On the other hand, for a trellis $\mathscr{A}$, there is a maximal $\mathscr{A}$-mediated set $\mathscr{A}^*$ satisfying $A(\mathscr{A}) \subseteq \mathscr{A}^* \subseteq \mathrm{conv}(\mathscr{A}) \cap \mathbb{N}^n$ which contains every $\mathscr{A}$-mediated set. Following [14], a trellis $\mathscr{A}$ is called an *H-trellis* if $\mathscr{A}^* = \mathrm{conv}(\mathscr{A}) \cap \mathbb{N}^n$. A sufficient condition for H-trellises is given in [7] which has the following useful corollary.

Proposition 5.2. *(*[7, Corollary 5.12]*) Let $\mathscr{A} \subseteq \mathbb{N}^n$ be a trellis. Then $k\mathscr{A}$ is an H-trellis for $k \geq n$.*

From Proposition 5.2 together with Theorem 5.1, we know that every n-variate nonnegative circuit polynomial supported on $k\mathscr{A}$ and a lattice point in the interior of $\mathrm{conv}(k\mathscr{A})$ is a sum of squares of binomials for a trellis $\mathscr{A}$ and $k \geq n$.

Lemma 5.3. *Suppose that $f(x_1, \ldots, x_n)$ is a sum of nonnegative circuit polynomials. Then $f(x_1^k, \ldots, x_n^k)$ is a sum of squares of binomials for $k \geq n$.*

Proof. Assume $f = \sum f_i$, where f_i's are nonnegative circuit polynomials. For $k \geq n$, since every $f_i(x_1^k, \ldots, x_n^k)$ is a sum of squares of binomials, so is $f(x_1^k, \ldots, x_n^k)$. □

5.2. **Supports of Sums of Nonnegative Circuit Polynomials.** Now we can prove: every SONC polynomial decomposes into a sum of nonnegative circuit polynomials with the same support. The proof will take use of the SOSB decompositions for SONC polynomials, so we apply the map $x_i \mapsto x_i^k$ to $f \in \mathbb{R}[\mathbf{x}]$.

Lemma 5.4. *Let $f(x_1, \ldots, x_n) \in \mathbb{R}[\mathbf{x}]$. Then $f(x_1, \ldots, x_n)$ is a sum of nonnegative circuit polynomials with the same support if and only if $f(x_1^k, \ldots, x_n^k)$ is a sum of nonnegative circuit polynomials with the same support for an odd number k.*

Proof. It is immediate from the fact that a polynomial $g(x_1, \ldots, x_n)$ is a nonnegative circuit polynomial if and only if $g(x_1^k, \ldots, x_n^k)$ is a nonnegative circuit polynomial for an odd number k. □

If a polynomial $g \in \mathbb{R}[\mathbf{x}]$ has the form $\sum_{\boldsymbol{\alpha} \in \Lambda(g)} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - d\mathbf{x}^{\boldsymbol{\beta}}$, where $\boldsymbol{\beta} \in (2\mathbb{N})^n$ and $d > 0$, or $\boldsymbol{\beta} \notin (2\mathbb{N})^n$, then we call g a *banana polynomial*. By Theorem 3.9, a nonnegative banana polynomial is a sum of nonnegative circuit polynomials with the same support. Moreover, if a polynomial f can be written as a sum of nonnegative banana polynomials, then $f \in$ SONC. For a polynomial f, if we can write f as $f = \sum_{\boldsymbol{\beta} \in \Gamma(f)} (\sum_{\boldsymbol{\alpha} \in \Lambda(f)} c_{\boldsymbol{\beta\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - d_{\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\beta}})$ such that every $\sum_{\boldsymbol{\alpha} \in \Lambda(f)} c_{\boldsymbol{\beta\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - d_{\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\beta}}$ is a nonnegative banana polynomial, then we say that f is a *sum of nonnegative banana polynomials with the same support.*

Theorem 5.5. *Let $f = \sum_{\boldsymbol{\alpha} \in \Lambda(f)} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta} \in \Gamma(f)} d_{\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$. If $f \in$ SONC, then f is a sum of nonnegative circuit polynomials with the same support.*

Proof. By Lemma 5.4, we only need to show that $f(x_1^{2n+1}, \ldots, x_n^{2n+1})$ is a sum of nonnegative circuit polynomials with the same support. Since by Theorem 3.9, a nonnegative banana polynomial is a sum of nonnegative circuit polynomials with the same support, we finish the proof by showing that $f(x_1^{2n+1}, \ldots, x_n^{2n+1})$ is a sum of nonnegative banana polynomials with the same support.

For simplicity, let $h = f(x_1^{2n+1}, \ldots, x_n^{2n+1})$. By Theorem 5.3, we can assume $h = \sum_{i=1}^m (a_i \mathbf{x}^{\boldsymbol{u}_i} - b_i \mathbf{x}^{\boldsymbol{v}_i})^2$. Let us do induction on m. When $m = 1$, $h = (a_1 \mathbf{x}^{\boldsymbol{u}_1} - b_1 \mathbf{x}^{\boldsymbol{v}_1})^2 = a_1^2 \mathbf{x}^{2\boldsymbol{u}_1} + b_1^2 \mathbf{x}^{2\boldsymbol{v}_1} - 2a_1 b_1 \mathbf{x}^{\boldsymbol{u}_1 + \boldsymbol{v}_1}$ and the conclusion is obvious. Now assume that the conclusion is correct for $m-1$. Without loss of generality, assume $\boldsymbol{u}_m + \boldsymbol{v}_m \in \Gamma(h)$. Let $h' = \sum_{i=1}^{m-1}(a_i\mathbf{x}^{\boldsymbol{u}_i} - b_i\mathbf{x}^{\boldsymbol{v}_i})^2 = \sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\Gamma(h')} d'_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}$. By the induction hypothesis, we can write $h' = \sum_{\boldsymbol{\beta}\in\Gamma(h')}(\sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{\boldsymbol{\beta\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d'_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}})$ as a sum of nonnegative banana polynomials with the same support. Then

$$h = \sum_{\boldsymbol{\beta}\in\Gamma(h')}(\sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{\boldsymbol{\beta\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d'_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}) + (a_m\mathbf{x}^{\boldsymbol{u}_m} - b_m\mathbf{x}^{\boldsymbol{v}_m})^2. \tag{5.2}$$

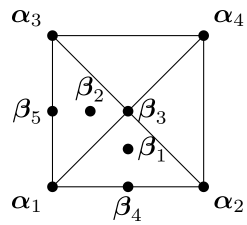
From $h = h' + (a_m\mathbf{x}^{\boldsymbol{u}_m} - b_m\mathbf{x}^{\boldsymbol{v}_m})^2$, it follows that $\mathrm{supp}(h)$ and $\mathrm{supp}(h')$ differ among three elements: $2\boldsymbol{u}_m, 2\boldsymbol{v}_m, \boldsymbol{u}_m + \boldsymbol{v}_m$. We obtain the expression of $h = \sum_{\boldsymbol{\beta}\in\Gamma(h)}(\sum_{\boldsymbol{\alpha}\in\Lambda(h)} c_{\boldsymbol{\beta\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}})$ as a sum of nonnegative banana polynomials with the same support from (5.2) by adjusting the terms involving $2\boldsymbol{u}_m, 2\boldsymbol{v}_m, \boldsymbol{u}_m + \boldsymbol{v}_m$ in (5.2).

First let us consider the terms involving $2\boldsymbol{u}_m$. If $2\boldsymbol{u}_m \notin \Gamma(h')$, then we have nothing to do. If $2\boldsymbol{u}_m \in \Gamma(h')$ and $2\boldsymbol{u}_m \in \Gamma(h)$, then we must have $d'_{2\boldsymbol{u}_m} > a_m^2$. By the equality $\sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{2\boldsymbol{u}_m\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d'_{2\boldsymbol{u}_m}\mathbf{x}^{2\boldsymbol{u}_m} + a_m^2\mathbf{x}^{2\boldsymbol{u}_m} + b_m^2\mathbf{x}^{2\boldsymbol{v}_m} - 2a_mb_m\mathbf{x}^{\boldsymbol{u}_m+\boldsymbol{v}_m} = (1 - \frac{a_m^2}{d'_{2\boldsymbol{u}_m}})(\sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{2\boldsymbol{u}_m\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d'_{2\boldsymbol{u}_m}\mathbf{x}^{2\boldsymbol{u}_m}) + \sum_{\boldsymbol{\alpha}\in\Lambda(h')} \frac{c'_{2\boldsymbol{u}_m\boldsymbol{\alpha}}a_m^2}{d'_{2\boldsymbol{u}_m}}\mathbf{x}^{\boldsymbol{\alpha}} + b_m^2\mathbf{x}^{2\boldsymbol{v}_m} - 2a_mb_m\mathbf{x}^{\boldsymbol{u}_m+\boldsymbol{v}_m}$, we obtain the expression of $h = \sum_{\boldsymbol{\beta}\in\Gamma(h')\backslash\{2\boldsymbol{u}_m\}}(\sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{\boldsymbol{\beta\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d'_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}) + (1 - \frac{a_m^2}{d'_{2\boldsymbol{u}_m}})(\sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{2\boldsymbol{u}_m\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d'_{2\boldsymbol{u}_m}\mathbf{x}^{2\boldsymbol{u}_m}) + \sum_{\boldsymbol{\alpha}\in\Lambda(h')} \frac{c'_{2\boldsymbol{u}_m\boldsymbol{\alpha}}a_m^2}{d'_{2\boldsymbol{u}_m}}\mathbf{x}^{\boldsymbol{\alpha}} + b_m^2\mathbf{x}^{2\boldsymbol{v}_m} - 2a_mb_m\mathbf{x}^{\boldsymbol{u}_m+\boldsymbol{v}_m}$ which is still a sum of nonnegative banana polynomials. If $2\boldsymbol{u}_m \in \Gamma(h')$ and $2\boldsymbol{u}_m \in \Lambda(h)$, then we must have $a_m^2 > d'_{2\boldsymbol{u}_m}$ and we can write h as $h = \sum_{\boldsymbol{\beta}\in\Gamma(h')\backslash\{2\boldsymbol{u}_m\}}(\sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{\boldsymbol{\beta\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d'_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}) + \sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{2\boldsymbol{u}_m\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} + (a_m^2 - d'_{2\boldsymbol{u}_m})\mathbf{x}^{2\boldsymbol{u}_m} + b_m^2\mathbf{x}^{2\boldsymbol{v}_m} - 2a_mb_m\mathbf{x}^{\boldsymbol{u}_m+\boldsymbol{v}_m}$ which is still a sum of nonnegative banana polynomials. If $2\boldsymbol{u}_m \in \Gamma(h')$ and $2\boldsymbol{u}_m \notin \mathrm{supp}(h)$, then the terms $-d'_{2\boldsymbol{u}_m}\mathbf{x}^{2\boldsymbol{u}_m}$ and $a_m^2\mathbf{x}^{2\boldsymbol{u}_m}$ must be cancelled. Hence we obtain the expression of h as $h = \sum_{\boldsymbol{\beta}\in\Gamma(h')\backslash\{2\boldsymbol{u}_m\}}(\sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{\boldsymbol{\beta\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d'_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}) + \sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{2\boldsymbol{u}_m\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} + b_m^2\mathbf{x}^{2\boldsymbol{v}_m} - 2a_mb_m\mathbf{x}^{\boldsymbol{u}_m+\boldsymbol{v}_m}$ which is still a sum of nonnegative banana polynomials.

Continue adjusting the terms involving $2\boldsymbol{v}_m$ and $\boldsymbol{u}_m + \boldsymbol{v}_m$ in the expression of h in a similar way. Eventually we can write h as a sum of nonnegative banana polynomials with the same support as desired. $\square$

Remark 5.6. *Theorem 5.5 essentially says that the SONC decompositions for nonnegative polynomials exactly maintain sparsity of polynomials. It is dramatically unlike the case of SOS decompositions for nonnegative polynomials, in which case many extra support monomials are needed in general.*

Example 5.7. *Let* $f = 50x^4y^4 + x^4 + 3y^4 + 800 - 300xy^2 - 180x^2y$ *which is nonnegative. Let* $\mathscr{A} = \{\boldsymbol{\alpha}_1 = (0,0), \boldsymbol{\alpha}_2 = (4,0), \boldsymbol{\alpha}_3 = (0,4), \boldsymbol{\alpha}_4 = (4,4)\}$ *and* $\boldsymbol{\beta}_1 = (2,1), \boldsymbol{\beta}_2 = (1,2)$. *There are two simplexes contain* $\boldsymbol{\beta}_1$*:* Δ_1 *with vertices* $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3\}$ *and* Δ_2 *with vertices* $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_4\}$. *There are two simplexes contain* $\boldsymbol{\beta}_2$*:* Δ_1 *and* Δ_3 *with vertices* $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4\}$. *If* $f \in \mathrm{SONC}$, *then by Theorem 5.5,* f *is a sum of nonnegative circuit polynomials supported on* $\Delta_1, \Delta_2, \Delta_1, \Delta_3$ *respectively. Let* $\boldsymbol{\beta}_3 = (2,2), \boldsymbol{\beta}_4 = (2,0), \boldsymbol{\beta}_5 = (0,2)$. *A* $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3\}$*-mediated set containing* $\boldsymbol{\beta}_1$ *is* $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\beta}_1, \boldsymbol{\beta}_3, \boldsymbol{\beta}_4\}$. *A* $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_4\}$*-mediated set containing* $\boldsymbol{\beta}_1$ *is* $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_4, \boldsymbol{\beta}_1, \boldsymbol{\beta}_3, \boldsymbol{\beta}_4\}$. *A* $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\,\boldsymbol{\alpha}_3\}$*-mediated set containing* $\boldsymbol{\beta}_2$ *is* $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\beta}_1, \boldsymbol{\beta}_3, \boldsymbol{\beta}_5\}$. *A* $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4\}$*-mediated set containing* $\boldsymbol{\beta}_2$ *is* $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4, \boldsymbol{\beta}_1, \boldsymbol{\beta}_3, \boldsymbol{\beta}_5\}$. *So by Theorem 5.1,* f *is a sum of squares of binomials. However, in fact* f *is even not a sum of squares. Thus* $f \notin \mathrm{SONC}$.



6. Computation via Relative Entropy Programming

As shown in [4], testing whether a polynomial belongs to SONC can be converted to a relative entropy programming (REP) problem, which is convex and can be solved efficiently via interior point methods ([1]). By virtue of Theorem 5.5, the supports of nonnegative circuit polynomials appearing in the SONC decomposition are contained in the support of the given polynomial. Thus we can compute SONC decompositions for nonnegative polynomials via relative entropy programming more efficiently.

Theorem 6.1. *([4, Theorem 3.2]) Let $f = \sum_{i=1}^m c_i\mathbf{x}^{\boldsymbol{\alpha}_i} - d\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$ be a circuit polynomial, which is not a sum of monomial squares. Assume $\boldsymbol{\beta} = \sum_{i=1}^m \lambda_i\mathbf{x}^{\boldsymbol{\alpha}_i}$, where $\sum_{i=1}^m \lambda_i = 1, \lambda_i > 0, i = 1,\dots,m$. Then f is nonnegative if and only if the following REP on variables ν_i and δ_i is feasible:*

$$\begin{cases} \textit{minimize} & 1 \\ \nu_i = d\lambda_i, & \textit{for } i = 1,\dots,m \\ \nu_i \log(\nu_i/c_i) \le \delta_i, & \textit{for } i = 1,\dots,m \\ \sum_{i=1}^m \delta_i \le 0, & \end{cases}. \tag{6.1}$$

We make the following assumption for the rest of this section.

Assumption: Let $f = \sum_{i=1}^m c_i\mathbf{x}^{\boldsymbol{\alpha}_i} - \sum_{j=1}^l d_j\mathbf{x}^{\boldsymbol{\beta}_j} \in \mathbb{R}[\mathbf{x}]$ with $\Lambda(f) = \{\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_m\}$ and $\Gamma(f) = \{\boldsymbol{\beta}_1,\dots,\boldsymbol{\beta}_l\}$. For every $\boldsymbol{\beta}_j$, let

$$\{\Delta_{j1},\dots,\Delta_{js_j}\} := \{\Delta \mid \Delta \text{ is a simplex }, \boldsymbol{\beta}_j \in \Delta^\circ, V(\Delta) \subseteq \Lambda(f)\}$$

and $I_{jk} := \{i \in [m] \mid \boldsymbol{\alpha}_i \in V(\Delta_{jk})\}$ for $k = 1,\dots,s_j$ and $j = 1,\dots,l$. For every $\boldsymbol{\beta}_j$ and every Δ_{jk}, since $\boldsymbol{\beta}_j \in \Delta_{jk}^\circ$, we can write $\boldsymbol{\beta}_j = \sum_{i\in I_{jk}} \lambda_{ijk}\boldsymbol{\alpha}_i$, where $\sum_{i\in I_{jk}} \lambda_{ijk} = 1, \lambda_{ijk} > 0, i \in I_{jk}$.

Theorem 6.2. *Let $f = \sum_{i=1}^m c_i\mathbf{x}^{\boldsymbol{\alpha}_i} - \sum_{j=1}^l d_j\mathbf{x}^{\boldsymbol{\beta}_j} \in \mathbb{R}[\mathbf{x}]$ with $\Lambda(f) = \{\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_m\}$, $\Gamma(f) = \{\boldsymbol{\beta}_1,\dots,\boldsymbol{\beta}_l\}$ and $V(\mathrm{New}(f)) \subseteq \Lambda(f)$, which is not a sum of monomial squares. Then $f \in$ SONC if and only if the following REP on variables d_{jk}, ν_{ijk}, c_{ijk} and δ_{ijk} is feasible:*

$$\begin{cases} \textit{minimize} & 1 \\ \nu_{ijk} = d_{jk}\lambda_{ijk}, & \textit{for } i \in I_{jk}, k = 1,\dots,s_j, j = 1,\dots,l \\ \nu_{ijk} \log(\nu_{ijk}/c_{ijk}) \le \delta_{ijk}, & \textit{for } i \in I_{jk}, k = 1,\dots,s_j, j = 1,\dots,l \\ \sum_{i\in I_{jk}} \delta_{ijk} \le 0, & \textit{for } k = 1\dots,s_j, j = 1,\dots,l \\ \sum_{j=1}^l \sum_{i\in I_{jk}} c_{ijk} = c_i, & \textit{for } i = 1,\dots,m \\ \sum_{k=1}^{s_j} d_{jk} = d_j, & \textit{for } j = 1,\dots,l \end{cases}. \tag{6.2}$$

Proof. Suppose $f_{jk} = \sum_{i\in I_{jk}} c_{ijk}\mathbf{x}^{\boldsymbol{\alpha}_i} - d_{jk}\mathbf{x}^{\boldsymbol{\beta}_j}$ is a nonnegative circuit polynomial for $k = 1,\dots,s_j, j = 1,\dots,l$ and $f = \sum_{j=1}^{l}\sum_{k=1}^{r} f_{jk}$. Then by Theorem 6.1, $(d_{jk})_{j,k}$, $(\nu_{ijk})_{i,j,k} = (d_{jk}\lambda_{ijk})_{i,j,k}$, $(c_{ijk})_{i,j,k}$ and $(\delta_{ijk})_{i,j,k} = (\nu_{ijk}\log(\nu_{ijk}/c_{ijk}))_{i,j,k}$ is a feasible solution of (6.2).

Conversely, suppose that $(d_{jk})_{j,k}$, $(\nu_{ijk})_{i,j,k}$, $(c_{ijk})_{i,j,k}$ and $(\delta_{ijk})_{i,j,k}$ is a feasible solution of (6.2). Let $f_{jk} = \sum_{i\in I_{jk}} c_{ijk}\mathbf{x}^{\boldsymbol{\alpha}_i} - d_{jk}\mathbf{x}^{\boldsymbol{\beta}_j}$ for $k = 1,\dots,s_j, j = 1,\dots,l$. Then by Theorem 6.1, f_{jk} is a nonnegative circuit polynomial for all j,k. Moreover, by the last two equality conditions in (6.2), we have $f = \sum_{j=1}^{l}\sum_{k=1}^{r} f_{jk}$. Thus, $f \in$ SONC. $\square$

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