1. The region for $\widetilde{K}_{m,5}$ with three steady states

The mass-action ODEs $\dot{x} = f(x)$ of $\widetilde{K}_{m,5}$ are given by ($r_4 > mr_5$):

$$(1)\qquad \begin{cases} f_1 &= -r_1x_1x_2 - r_5x_1 - r_6x_1 + r_1 + r_5 + r_6, \\ f_2 &= -r_1x_1x_2 - r_2x_2x_3 - r_7x_2 + r_1 + r_2 + r_7, \\ f_3 &= -r_2x_2x_3 - r_3x_3x_4 - r_8x_3 + r_2 + r_3 + r_8, \\ f_4 &= -r_3x_3x_4 - r_4x_4x_5 - r_9x_4 + r_3 + r_4 + r_9, \\ f_5 &= -r_4x_4x_4 + mr_5x_1 - r_{10}x_5 + r_4 - mr_5 + r_{10}. \end{cases}$$

Assume $r_6 = r_8 = r_9 = r_{10} = \varepsilon$. For ε sufficiently small and the other rate constants satisfy:

$$(2)\qquad \begin{cases} (m-1)r_1 &> r_7, \\ (m-1)r_1r_i &> (-1)^{i-1}m((r_1+r_5)r_7 - (m-1)r_1r_5), \quad i = 2,3,4, \end{cases}$$

the system (1) admits two steady states: $\delta_1 = (1, 1, \ldots, 1)$ and δ_2.

Let

$$(3)\qquad x = \varphi_1(y) : \begin{cases} x_i = y_i, i = 1,2,3, \\ x_4 = \frac{\varepsilon y_4}{y_5}, \\ x_5 = \frac{y_5}{\varepsilon}, \end{cases}$$

$$(4)\qquad x = \varphi_2(y) : \begin{cases} x_i = y_i, i = 1,2, \\ x_3 = \frac{\varepsilon y_3}{y_4}, \\ x_4 = \frac{y_4}{\varepsilon}, \\ x_5 = \frac{\varepsilon y_5}{y_4}, \end{cases}$$

and

$$(5)\qquad x = \varphi_3(y) : \begin{cases} x_1 = y_1, \\ x_2 = \frac{\varepsilon y_2}{y_3}, \\ x_3 = \frac{y_3}{\varepsilon}, \\ x_4 = \frac{\varepsilon y_4}{y_5}, \\ x_5 = \frac{y_5}{\varepsilon}. \end{cases}$$

Substitute $x = \varphi_i(y)$ ($i = 1,2,3$) into f with $y_1, \ldots, y_5$ viewed as new variables and the resulting polynomial system is denoted by $p^{(i)} := f|_{x=\varphi_i(y)}$ ($i = 1,2,3$). Then let $\varepsilon = 0$ and the resulting polynomial system is denoted by $g^{(i)} := p^{(i)}|_{\varepsilon=0}$ ($i = 1,2,3$).

Case 1: The system (1) admits exactly one steady state δ_3 of type $[0\infty]$ for rate constants:

$$(6)\qquad \begin{cases} (m-1)r_1 &> r_7, \\ r_3 &< r_1 + r_7, \\ (m-1)r_1r_3 &> m((r_1+r_5)r_7 - (m-1)r_1r_5). \end{cases}$$

Case 2: Eliminate y_2 from $p_1^{(2)}$ and $p_2^{(2)}$ and obtain

$$(7)\qquad h_2 := r_1r_5y_1^2 + (r_1r_2 - r_1r_5 + r_1r_7 + r_5r_7)y_1 - (r_1+r_5)r_7.$$

Let ξ_2 be the positive root of (7). Then we have

$$(8)\qquad \begin{cases} y_2 &= \frac{r_1+r_2+r_7}{r_1\xi_2+r_7}, \\ y_3 &= \frac{r_2+r_3}{r_3}, \\ y_4 &= mr_5(1-\xi_2)-r_2, \\ y_5 &= \frac{r_4-mr_5+mr_5\xi_2}{r_4}. \end{cases}$$

To ensure $y_2, y_3, y_4, y_5 > 0$, we need

$$(9)\qquad \xi_2 < 1 - \frac{r_2}{mr_5},$$

which is equivalent to $h_2(1-\frac{r_4}{mr_5}) > 0$. Note that

$$h_2(1-\frac{r_4}{mr_5}) = -\frac{r_2((m-1)r_1r_2 - m((m-1)r_1r_5-(r_1+r_5)r_7)}{m^2r_5}.$$

Hence $h_2(1-\frac{r_4}{mr_5}) > 0$ is equivalent to

$$(10)\qquad (m-1)r_1r_2 - m((m-1)r_1r_5-(r_1+r_5)r_7) < 0.$$

Hence the system (1) admits exactly one steady state δ_4 of type $[0\infty 0]$ for rate constants:

$$(11)\qquad \begin{cases} (m-1)r_1 &> r_7, \\ (m-1)r_1r_5 &> (r_1+r_5)r_7, \\ (m-1)r_1r_2 &< m((m-1)r_1r_5-(r_1+r_5)r_7). \end{cases}$$

Case 3: Eliminate y_1, y_2, y_3, y_5 from $p_i^{(3)}, i = 1, \ldots, 5$ and obtain

$$(12)\qquad h_3 := r_4(r_4-r_3)y_4^2 + (r_1r_3 - mr_1r_4 - 2r_4^2 + r_3r_7)y_4 + (mr_1+r_4)(r_3+r_4).$$

Let ξ_3 be the positive root of (12). Then we have

$$(13)\qquad \begin{cases} y_1 &= \frac{r_1+r_5}{r_5}, \\ y_2 &= \frac{r_1+r_2+r_7}{r_2}, \\ y_3 &= r_4(\xi_3-1)-r_1-r_7, \\ y_5 &= mr_1 - r_4(\xi_3-1). \end{cases}$$

To ensure $y_1, y_2, y_3, y_5 > 0$, we need

$$(14)\qquad 1 + \frac{r_1+r_7}{r_4} < \xi_3 < 1 + \frac{mr_1}{r_4}.$$

Note that

$$h_3(1+\frac{mr_1}{r_4}) = -\frac{r_3(mr_1+r_4)((m-1)r_1-r_7)}{r_4} < 0.$$

Hence we must have $h_3(1+\frac{r_1+r_7}{r_4}) > 0$, i.e.,

$$(15)\qquad h_3(1+\frac{r_1+r_7}{r_4}) = -((m-1)r_1-r_7)(r_1+r_7-r_3) > 0,$$

which is equivalent to $r_3 > r_1 + r_7$.

Hence the system (1) admits exactly one steady state δ_5 of type $[0\infty 0\infty]$ for rate constants:

$$(16)\qquad \begin{cases} (m-1)r_1 &> r_7, \\ r_3 &> r_1+r_7. \end{cases}$$

The steady states $\delta_i, i = 2,3,4,5$ can be divided into two classes:

Types	Steady states
$[], [0\infty 0]$	δ_2, δ_4
$[0\infty], [0\infty 0\infty]$	δ_3, δ_5

For specific rate constants, the system (1) admits at most one steady state from each class.

We summarize the open region for three steady states as follows:

$$\begin{cases} r_7 \ < \ (m-1)r_1, \\ r_5 \ < \ \frac{r_1 r_7}{(m-1)r_1 - r_7}, \\ r_4 \ > \ m r_5, \\ r_3 \ > \ \frac{m((r_1+r_5)r_7-(m-1)r_1r_5)}{(m-1)r_1}, \end{cases} \tag{17}$$

and

$$\begin{cases} r_7 \ < \ (m-1)r_1, \\ r_5 \ > \ \frac{r_1 r_7}{(m-1)r_1 - r_7}, \\ r_4 \ > \ m r_5. \end{cases} \tag{18}$$

2. The region for $\widetilde{K}_{m,7}$ with three steady states

Case 1: The system (1) admits two steady state $\delta_1 = (1,1,\ldots,1)$ and δ_2 of type [] for rate constants:

$$\begin{cases} (m-1)r_1 \ > \ r_9, \\ (m-1)r_1 r_i \ > \ (-1)^{i-1} m((r_1+r_7)r_9 - (m-1)r_1r_7), \quad i = 2,\ldots,6. \end{cases} \tag{19}$$

Case 2: The system (1) admits exactly one steady state δ_3 of type $[0\infty]$ for rate constants:

$$\begin{cases} (m-1)r_1 \ > \ r_9, \\ r_5 \ < \ r_3, \\ r_5 \ < \ r_1 + r_9, \\ (m-1)r_1r_5 \ > \ m((r_1+r_7)r_9 - (m-1)r_1r_7). \end{cases} \tag{20}$$

Case 3: The system (1) admits exactly one steady state δ_4 of type $[0\infty 0]$ for rate constants:

$$\begin{cases} (m-1)r_1 \ > \ r_9, \\ r_4 \ < \ r_2, \\ r_4 \ < \ m r_7, \\ (m-1)r_1r_4 \ < \ m((m-1)r_1r_7 - (r_1+r_7)r_9). \end{cases} \tag{21}$$

Case 4: The system (1) admits exactly one steady state δ_5 of type $[0\infty 0\infty]$ for rate constants:

$$\begin{cases} (m-1)r_1 \ > \ r_9, \\ r_3 \ < \ r_5, \\ r_3 \ < \ r_1 + r_9, \\ (m-1)r_1r_3 \ > \ m((r_1+r_7)r_9 - (m-1)r_1r_7). \end{cases} \tag{22}$$

Case 5: The system (1) admits exactly one steady state δ_6 of type $[0\infty0\infty0]$ for rate constants:

$$\begin{cases} (m-1)r_1 \ > \ r_9, \\ r_2 \ < \ r_4, \\ r_2 \ < \ mr_7, \\ (m-1)r_1r_2 \ < \ m((m-1)r_1r_7 - (r_1+r_7)r_9). \end{cases} \tag{23}$$

Case 6: The system (1) admits steady states δ_7 of type $[0\infty0\infty0\infty]$ for rate constants:

$$\begin{cases} (m-1)r_1 \ > \ r_9, \\ r_3 \ > \ r_1+r_9, \\ r_5 \ > \ r_1+r_9, \\ \cdots \end{cases} \tag{24}$$

We need to show in Case 6, the system (1) admits at most one steady state of type $[0\infty0\infty0\infty]$, i.e., the following system admits at most one steady state:

$$\begin{cases} r_1+r_7-r_7y_1, \\ r_1+r_2+r_9-r_2y_2, \\ r_2+r_3-r_2y_2-y_3-\frac{r_3y_3y_4}{y_5}, \\ r_3+r_4-r_4y_4-\frac{r_3y_3y_4}{y_5}, \\ r_4+r_5-r_4y_4-y_5-\frac{r_5y_5y_6}{y_7}, \\ r_5+r_6-r_6y_6-\frac{r_5y_5y_6}{y_7}, \\ r_6-mr_7+mr_7y_1-r_6y_6-y_7. \end{cases} \tag{25}$$

The steady states $\delta_i, i=2,\dots,7$ can be divided into two classes:

Types	Steady states
$[]$, $[0\infty0]$, $[0\infty0\infty0]$	$\delta_2, \delta_4, \delta_6$
$[0\infty]$, $[0\infty0\infty]$, $[0\infty0\infty0\infty]$	$\delta_3, \delta_5, \delta_7$

For specific rate constants, the system (1) admits at most one steady state from each class.