ON SUPPORTS OF SUMS OF NONNEGATIVE CIRCUIT POLYNOMIALS

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Abstract. In this paper, we prove that every SONC polynomial decomposes into a sum of nonnegative circuit polynomials with the same support, which reveals the advantage of SONC decompositions for certifying nonnegativity of sparse polynomials compared with the classical SOS decompositions. By virtue of this fact, we can decide $f \in \mathrm{SONC}$ through relative entropy programming more efficiently.

1. Introduction

A real polynomial $f \in \mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \ldots, x_n]$ is called a *nonnegative polynomial* if its evaluation on every real point is nonnegative. Certifying nonnegativity of polynomials is a central problem of real algebraic geometry and also has applications in many fields such as polynomial optimization, control, engineering, probability, statistics and physics. The classical method for certifying nonnegativity of polynomials is using sums of squares (SOS) programming which can be effectively solved by semidefinite programming (SDP) ([7, 8]). However, the size of the corresponding semidefinite program problems for SOS decompositions grows rapidly as the size of polynomials increases. Therefore, to deal with huge polynomials, sparsity must be exploited.

Recently in [5], Iliman and Wolff introduced the concept of sums of nonnegative circuit polynomials (SONC) as a substitute of sums of squares of polynomials to represent nonnegative polynomials. A polynomial f is called a *circuit polynomial* if it is of the form

$$f(\mathbf{x}) = \sum_{i=1}^{m} c_i \mathbf{x}^{\boldsymbol{\alpha}_i} - d\mathbf{x}^{\boldsymbol{\beta}}, \tag{1.1}$$

where the Newton polytope $\Delta = \mathrm{New}(f)$ is a lattice simplex with the vertex set $\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m\}$, $\boldsymbol{\beta}$ an interior point of Δ and $c_i > 0$ for $i = 1, \ldots, m$. For every circuit polynomial f, we associate it with the *circuit number* defined as $\Theta_f := \prod_{i=1}^{m} (c_i/\lambda_i)^{\lambda_i}$, where the λ_i's are uniquely given by the convex combination $\boldsymbol{\beta} = \sum_{i=1}^{m} \lambda_i \boldsymbol{\alpha}_i$ with $\lambda_i > 0$ and $\sum_{i=1}^{m} \lambda_i = 1$. The nonnegativity of circuit polynomials is easy to check. Actually the circuit polynomial f is nonnegative if and only if $\boldsymbol{\alpha}_i \in (2\mathbb{N})^n$ for all i, and $-\Theta_f \le d \le \Theta_f$ if $\boldsymbol{\beta} \notin (2\mathbb{N})^n$ or $d \le \Theta_f$ if $\boldsymbol{\beta} \in (2\mathbb{N})^n$.

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If a polynomial f can be written as a sum of nonnegative circuit polynomials, then f is obviously nonnegative. Based on these SONC decompositions for certificates of nonnegativity, new approaches were proposed for both unconstrained polynomial optimization problems and constrained polynomial optimization problems, which were proved to be significantly more efficient than the SOS programming method in many cases ([2, 3, 4, 6, 10]).

In my previous paper [11], it was proved that certain kinds of nonnegative polynomials decompose into sums of nonnegative circuit polynomials with the same support. In this paper, we clarify an important fact that every SONC polynomial decomposes into a sum of nonnegative circuit polynomials with the same support. In other words, SONC decompositions for nonnegative polynomials exactly maintain sparsity of polynomials. It is dramatically unlike the case of SOS decompositions for nonnegative polynomials, in which case many extra support monomials are needed in general. This reveals the advantage of SONC decompositions for certifying nonnegativity of sparse polynomials compared with the classical SOS decompositions. By virtue of this fact, we can decide $f \in \text{SONC}$ via relative entropy programming more efficiently.

2. Preliminaries

2.1. **Nonnegative Polynomials.** Let $\mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \ldots, x_n]$ be the ring of real n-variate polynomial, and $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$. For a finite set $\mathscr{A} \subset \mathbb{N}^n$, we denote by $\text{conv}(\mathscr{A})$ the convex hull of $\mathscr{A}$, and by $V(\mathscr{A})$ the vertices of the convex hull of $\mathscr{A}$. Also we denote by $V(P)$ the vertex set of a polytope P. We consider polynomials $f \in \mathbb{R}[\mathbf{x}]$ supported on $\mathscr{A} \subset \mathbb{N}^n$, i.e. f is of the form $f(\mathbf{x}) = \sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}}$ with $c_{\boldsymbol{\alpha}} \in \mathbb{R}, \mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$. The support of f is $\text{supp}(f) := \{\boldsymbol{\alpha} \in \mathscr{A} \mid c_{\boldsymbol{\alpha}} \neq 0\}$ and the Newton polytope is defined as $\text{New}(f) = \text{conv}(\text{supp}(f))$. For a polytope P, we use P° to denote the interior of P.

A polynomial $f \in \mathbb{R}[\mathbf{x}]$ which is nonnegative over $\mathbb{R}^n$ is called a *nonnegative polynomial*. A nonnegative polynomial must satisfy the following necessary conditions.

Proposition 2.1. *(*[9, Theorem 3.6]*) Let $\mathscr{A} \subset \mathbb{N}^n$ and $f = \sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} \in \mathbb{R}[\mathbf{x}]$ with $\text{supp}(f) = \mathscr{A}$. Then f is nonnegative only if the followings hold:*

(1) $V(\mathscr{A}) \subset (2\mathbb{N})^n$;
(2) If $\boldsymbol{\alpha} \in V(\mathscr{A})$, then the corresponding coefficient $c_{\boldsymbol{\alpha}}$ is positive.

For $f \in \mathbb{R}[\mathbf{x}]$, let $\Lambda(f) := \{\boldsymbol{\alpha} \in \text{supp}(f) \mid \boldsymbol{\alpha} \in (2\mathbb{N})^n \text{ and } c_{\boldsymbol{\alpha}} > 0\}$ and $\Gamma(f) := \text{supp}(f)\backslash\Lambda(f)$. Then we can write $f = \sum_{\boldsymbol{\alpha}\in\Lambda(f)} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\Gamma(f)} d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}$ with $c_{\boldsymbol{\alpha}} > 0$. The necessary conditions in Proposition 2.1 restate as $V(\text{New}(f)) \subseteq \Lambda(f)$.

2.2. **Sums of Nonnegative Circuit Polynomials.** A subset $\mathscr{A} \subseteq (2\mathbb{N})^n$ is called a *trellis* if $\mathscr{A}$ comprises the vertices of a simplex ([9]).

Definition 2.2. *Let $\mathscr{A}$ be a trellis and $f \in \mathbb{R}[\mathbf{x}]$. Then f is called a* circuit polynomial *if it is of the form*

$$f(\mathbf{x}) = \sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d\mathbf{x}^{\boldsymbol{\beta}}, \tag{2.1}$$

with $c_{\boldsymbol{\alpha}} > 0$ *and* $\boldsymbol{\beta} \in \operatorname{conv}(\mathscr{A})^\circ$. *Assume*

$$\boldsymbol{\beta} = \sum_{\boldsymbol{\alpha}\in\mathscr{A}} \lambda_{\boldsymbol{\alpha}}\boldsymbol{\alpha} \text{ with } \lambda_{\boldsymbol{\alpha}} > 0 \text{ and } \sum_{\boldsymbol{\alpha}\in\mathscr{A}} \lambda_{\boldsymbol{\alpha}} = 1. \tag{2.2}$$

For every circuit polynomial f, we define the corresponding circuit number *as* $\Theta_f := \prod_{\boldsymbol{\alpha}\in\mathscr{A}}(c_{\boldsymbol{\alpha}}/\lambda_{\boldsymbol{\alpha}})^{\lambda_{\boldsymbol{\alpha}}}$.

The nonnegativity of a circuit polynomial f is decided by its circuit number alone.

Theorem 2.3. *([5, Theorem 3.8]) Let $f = \sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$ be a circuit polynomial and Θ_f its circuit number. Then f is nonnegative if and only if $\boldsymbol{\beta} \notin (2\mathbb{N})^n$ and $|d| \le \Theta_f$, or $\boldsymbol{\beta} \in (2\mathbb{N})^n$ and $d \le \Theta_f$.*

In analogy with writing a nonnegative polynomial as a sum of squares of polynomials, writing a nonnegative polynomial as a sum of nonnegative circuit polynomials is a certificate of its nonnegativity. We denote by SONC both the class of polynomials which can be written as sums of nonnegative circuit polynomials and the property of a polynomial to be in this class.

Suppose $f = \sum_{\boldsymbol{\alpha}\in\Lambda(f)} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\Gamma(f)} d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$ with $\Gamma(f) \subset \operatorname{New}(f)^\circ$. For every $\boldsymbol{\beta} \in \Gamma(f)$, let

$$\Delta(\boldsymbol{\beta}) := \{\Delta \mid \Delta \text{ is a simplex, } \boldsymbol{\beta} \in \Delta^\circ, V(\Delta) \subseteq \Lambda(f)\}. \tag{2.3}$$

If we can write f as $f = \sum_{\boldsymbol{\beta}\in\Gamma(f)}\sum_{\Delta\in\Delta(\boldsymbol{\beta})}(\sum_{\boldsymbol{\alpha}\in V(\Delta)} c_{\boldsymbol{\beta}\Delta\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d_{\boldsymbol{\beta}\Delta}\mathbf{x}^{\boldsymbol{\beta}})$ such that every $\sum_{\boldsymbol{\alpha}\in V(\Delta)} c_{\boldsymbol{\beta}\Delta\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d_{\boldsymbol{\beta}\Delta}\mathbf{x}^{\boldsymbol{\beta}}$ is a nonnegative circuit polynomial, then we say that f is a *sum of nonnegative circuit polynomials with the same support.*

In [11], it was proved that if a nonnegative polynomial has at most one negative term, then it decomposes into a sum of nonnegative circuit polynomials with the same support.

Theorem 2.4. *([11, Theorem 3.9]) Let $f = \sum_{\boldsymbol{\alpha}\in\Lambda(f)} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$ with $\boldsymbol{\beta} \in \operatorname{New}(f)^\circ$. If f is nonnegative, then f is a sum of nonnegative circuit polynomials with the same support.*

3. Nonnegative Circuit Polynomials and Sums of Squares of Binomials

In this section, we give a connection between nonnegative circuit polynomials and sums of squares of binomials (SOSB).

We call a lattice point is *even* if it is in $(2\mathbb{N})^n$. For a subset $M \subseteq \mathbb{N}^n$, define $\overline{A}(M) := \{\frac{1}{2}(\boldsymbol{u}+\boldsymbol{v}) \mid \boldsymbol{u} \neq \boldsymbol{v}, \boldsymbol{u}, \boldsymbol{v} \in M \cap (2\mathbb{N})^n\}$ as the set of averages of distinct even points in M. For a trellis $\mathscr{A}$, we sat that M is an $\mathscr{A}$*-mediated set* if $\mathscr{A} \subseteq M \subseteq \overline{A}(M) \cup \mathscr{A}$.

Theorem 3.1. *Let $f = \sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$ be a nonnegative circuit polynomial with $\boldsymbol{\beta} \in \operatorname{conv}(\mathscr{A})^\circ, d \neq 0$. If $\boldsymbol{\beta}$ belongs to an $\mathscr{A}$-mediated set M, then f is a sum of squares of the form $(a_{\boldsymbol{u}}\mathbf{x}^{\boldsymbol{u}} - b_{\boldsymbol{v}}\mathbf{x}^{\boldsymbol{v}})^2$, where $2\boldsymbol{u}, 2\boldsymbol{v} \in M$.*

Proof. For the proof, please refer to Theorem 5.2 in [5] which exploits Theorem 4.4 in [9]. □

Inspired by Theorem 3.1, we are interested in the problem of deciding if there exists an $\mathscr{A}$-mediated set containing a given lattice point and computing one if there exists. However, there are no effective algorithms to do such thing as far as I know. On the other hand, for a trellis $\mathscr{A}$, there is a maximal $\mathscr{A}$-mediated set $\mathscr{A}^*$ satisfying $A(\mathscr{A}) \subseteq \mathscr{A}^* \subseteq \mathrm{conv}(\mathscr{A}) \cap \mathbb{N}^n$ which contains every $\mathscr{A}$-mediated set. Following [9], a trellis $\mathscr{A}$ is called an *H-trellis* if $\mathscr{A}^* = \mathrm{conv}(\mathscr{A}) \cap \mathbb{N}^n$. A sufficient condition for H-trellises is given in [5] which has the following useful corollary.

Proposition 3.2. *([5, Corollary 5.12]) Let $\mathscr{A} \subseteq \mathbb{N}^n$ be a trellis. Then $k\mathscr{A}$ is an H-trellis for $k \geq n$.*

From Proposition 3.2 together with Theorem 3.1, we know that every n-variate nonnegative circuit polynomial supported on $k\mathscr{A}$ and a lattice point in the interior of $\mathrm{conv}(k\mathscr{A})$ is a sum of squares of binomials for a trellis $\mathscr{A}$ and $k \geq n$.

Lemma 3.3. *Suppose that $f(x_1, \ldots, x_n)$ is a sum of nonnegative circuit polynomials. Then $f(x_1^k, \ldots, x_n^k)$ is a sum of squares of binomials for $k \geq n$.*

Proof. Assume $f = \sum f_i$, where f_i's are nonnegative circuit polynomials. For $k \geq n$, since every $f_i(x_1^k, \ldots, x_n^k)$ is a sum of squares of binomials, so is $f(x_1^k, \ldots, x_n^k)$. □

4. Supports of Sums of Nonnegative Circuit Polynomials

In this section, we prove the main result of this paper: every SONC polynomial decomposes into a sum of nonnegative circuit polynomials with the same support. The proof will take use of the SOSB decompositions for SONC polynomials, so we apply the map $x_i \mapsto x_i^k$ to $f \in \mathbb{R}[\mathbf{x}]$.

Lemma 4.1. *Let $f(x_1, \ldots, x_n) \in \mathbb{R}[\mathbf{x}]$. Then $f(x_1, \ldots, x_n)$ is a sum of nonnegative circuit polynomials with the same support if and only if $f(x_1^k, \ldots, x_n^k)$ is a sum of nonnegative circuit polynomials with the same support for an odd number k.*

Proof. It is immediate from the fact that a polynomial $g(x_1, \ldots, x_n)$ is a nonnegative circuit polynomial if and only if $g(x_1^k, \ldots, x_n^k)$ is a nonnegative circuit polynomial for an odd number k. □

If a polynomial $g \in \mathbb{R}[\mathbf{x}]$ has at most one negative term, i.e. g has the form $\sum_{\boldsymbol{\alpha} \in \Lambda(g)} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - d\mathbf{x}^{\boldsymbol{\beta}}$, where $\boldsymbol{\beta} \in (2\mathbb{N})^n, d > 0$ or $\boldsymbol{\beta} \notin (2\mathbb{N})^n$, then we call g a *banana polynomial*. By Theorem 2.4, a nonnegative banana polynomial is a sum of nonnegative circuit polynomials with the same support. Moreover, if a polynomial f can be written as a sum of nonnegative banana polynomials, then $f \in$ SONC. For a polynomial f, if we can write f as $f = \sum_{\boldsymbol{\beta} \in \Gamma(f)} (\sum_{\boldsymbol{\alpha} \in \Lambda(f)} c_{\boldsymbol{\beta}\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - d_{\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\beta}})$ such that every $\sum_{\boldsymbol{\alpha} \in \Lambda(f)} c_{\boldsymbol{\beta}\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - d_{\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\beta}}$ is a nonnegative banana polynomial, then we say that f is a *sum of nonnegative banana polynomials with the same support*.

Theorem 4.2. *Let $f = \sum_{\boldsymbol{\alpha} \in \Lambda(f)} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta} \in \Gamma(f)} d_{\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$. If $f \in$ SONC, then f is a sum of nonnegative circuit polynomials with the same support.*

Proof. By Lemma 4.1, we only need to show that $f(x_1^{2n+1}, \ldots, x_n^{2n+1})$ is a sum of nonnegative circuit polynomials with the same support. Since by Theorem 2.4, a nonnegative banana polynomial is a sum of nonnegative circuit polynomials with the same support, we finish the proof by showing that $f(x_1^{2n+1}, \ldots, x_n^{2n+1})$ is a sum of nonnegative banana polynomials with the same support.

For simplicity, let $h = f(x_1^{2n+1}, \ldots, x_n^{2n+1})$. By Theorem 3.3, we can assume $h = \sum_{i=1}^{m}(a_i\mathbf{x}^{\boldsymbol{u}_i} - b_i\mathbf{x}^{\boldsymbol{v}_i})^2$. Let us do induction on m. When $m = 1$, $h = (a_1\mathbf{x}^{\boldsymbol{u}_1} - b_1\mathbf{x}^{\boldsymbol{v}_1})^2 = a_1^2\mathbf{x}^{2\boldsymbol{u}_1} + b_1^2\mathbf{x}^{2\boldsymbol{v}_1} - 2a_1b_1\mathbf{x}^{\boldsymbol{u}_1+\boldsymbol{v}_1}$ and the conclusion is obvious. Now assume that the conclusion is correct for $m-1$. Without loss of generality, assume $\boldsymbol{u}_m + \boldsymbol{v}_m \in \Gamma(h)$. Let $h' = \sum_{i=1}^{m-1}(a_i\mathbf{x}^{\boldsymbol{u}_i} - b_i\mathbf{x}^{\boldsymbol{v}_i})^2 = \sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\Gamma(h')} d'_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}$. By the induction hypothesis, we can write $h' = \sum_{\boldsymbol{\beta}\in\Gamma(h')}(\sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{\boldsymbol{\beta}\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d'_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}})$ as a sum of nonnegative banana polynomials with the same support. Then

$$h = \sum_{\boldsymbol{\beta}\in\Gamma(h')}(\sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{\boldsymbol{\beta}\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d'_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}) + (a_m\mathbf{x}^{\boldsymbol{u}_m} - b_m\mathbf{x}^{\boldsymbol{v}_m})^2. \tag{4.1}$$

From $h = h' + (a_m\mathbf{x}^{\boldsymbol{u}_m} - b_m\mathbf{x}^{\boldsymbol{v}_m})^2$, it follows that $\mathrm{supp}(h)$ and $\mathrm{supp}(h')$ differ among three elements: $2\boldsymbol{u}_m, 2\boldsymbol{v}_m, \boldsymbol{u}_m + \boldsymbol{v}_m$. We obtain the expression of h as a sum of nonnegative banana polynomials with the same support from (4.1) by adjusting the terms involving $2\boldsymbol{u}_m, 2\boldsymbol{v}_m, \boldsymbol{u}_m + \boldsymbol{v}_m$ in (4.1).

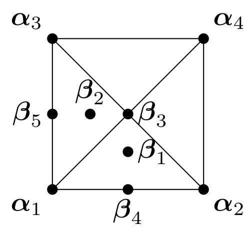
First let us consider the terms involving $2\boldsymbol{u}_m$. If $2\boldsymbol{u}_m \in \Gamma(h)$, then we must have $2\boldsymbol{u}_m \in \Gamma(h')$ and $d'_{2\boldsymbol{u}_m} > a_m^2$. By the equality $\sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{2\boldsymbol{u}_m\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d'_{2\boldsymbol{u}_m}\mathbf{x}^{2\boldsymbol{u}_m} + a_m^2\mathbf{x}^{2\boldsymbol{u}_m} + b_m^2\mathbf{x}^{2\boldsymbol{v}_m} - 2a_mb_m\mathbf{x}^{\boldsymbol{u}_m+\boldsymbol{v}_m} = (1 - \frac{a_m^2}{d'_{2\boldsymbol{u}_m}})(\sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{2\boldsymbol{u}_m\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d'_{2\boldsymbol{u}_m}\mathbf{x}^{2\boldsymbol{u}_m}) + \sum_{\boldsymbol{\alpha}\in\Lambda(h')} \frac{c'_{2\boldsymbol{u}_m\boldsymbol{\alpha}}a_m^2}{d'_{2\boldsymbol{u}_m}}\mathbf{x}^{\boldsymbol{\alpha}} + b_m^2\mathbf{x}^{2\boldsymbol{v}_m} - 2a_mb_m\mathbf{x}^{\boldsymbol{u}_m+\boldsymbol{v}_m}$, we obtain the expression of h as $h = \sum_{\boldsymbol{\beta}\in\Gamma(h')\setminus\{2\boldsymbol{u}_m\}}(\sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{\boldsymbol{\beta}\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d'_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}) + (1 - \frac{a_m^2}{d'_{2\boldsymbol{u}_m}})(\sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{2\boldsymbol{u}_m\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d'_{2\boldsymbol{u}_m}\mathbf{x}^{2\boldsymbol{u}_m}) + \sum_{\boldsymbol{\alpha}\in\Lambda(h')} \frac{c'_{2\boldsymbol{u}_m\boldsymbol{\alpha}}a_m^2}{d'_{2\boldsymbol{u}_m}}\mathbf{x}^{\boldsymbol{\alpha}} + b_m^2\mathbf{x}^{2\boldsymbol{v}_m} - 2a_mb_m\mathbf{x}^{\boldsymbol{u}_m+\boldsymbol{v}_m}$ which is a sum of nonnegative banana polynomials. If $2\boldsymbol{u}_m \in \Lambda(h)$ and $2\boldsymbol{u}_m \in \Gamma(h')$, then we must have $a_m^2 > d'_{2\boldsymbol{u}_m}$ and we can write h as $h = \sum_{\boldsymbol{\beta}\in\Gamma(h')\setminus\{2\boldsymbol{u}_m\}}(\sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{\boldsymbol{\beta}\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d'_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}) + \sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{2\boldsymbol{u}_m\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} + (a_m^2 - d'_{2\boldsymbol{u}_m})\mathbf{x}^{2\boldsymbol{u}_m} + b_m^2\mathbf{x}^{2\boldsymbol{v}_m} - 2a_mb_m\mathbf{x}^{\boldsymbol{u}_m+\boldsymbol{v}_m}$ which is a sum of nonnegative banana polynomials. If $2\boldsymbol{u}_m \notin \mathrm{supp}(h)$, then $2\boldsymbol{u}_m \in \Gamma(h')$ and the terms $-d'_{2\boldsymbol{u}_m}\mathbf{x}^{2\boldsymbol{u}_m}$ and $a_m^2\mathbf{x}^{2\boldsymbol{u}_m}$ must be cancelled. Hence we obtain the expression of h as $h = \sum_{\boldsymbol{\beta}\in\Gamma(h')\setminus\{2\boldsymbol{u}_m\}}(\sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{\boldsymbol{\beta}\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d'_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}) + \sum_{\boldsymbol{\alpha}\in\Lambda(h')} c'_{2\boldsymbol{u}_m\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} + b_m^2\mathbf{x}^{2\boldsymbol{v}_m} - 2a_mb_m\mathbf{x}^{\boldsymbol{u}_m+\boldsymbol{v}_m}$ which is a sum of nonnegative banana polynomials.

Continue adjusting the terms involving $2\boldsymbol{v}_m$ and $\boldsymbol{u}_m + \boldsymbol{v}_m$ in the expression of h in a similar way. Eventually we can write h as a sum of nonnegative banana polynomials with the same support as desired. $\square$

Remark 4.3. *Theorem 4.2 essentially says that the SONC decompositions for nonnegative polynomials exactly maintain sparsity of polynomials. It is dramatically unlike the case of SOS decompositions for nonnegative polynomials, in which case many extra support monomials are needed in general.*

As an application of Theorem 4.2, we give a nonnegative polynomial which is not a SONC polynomial.

Example 4.4. *Let* $f = 50x^4y^4 + x^4 + 3y^4 + 800 - 300xy^2 - 180x^2y$ *which is nonnegative. Let* $\mathscr{A} = \{\boldsymbol{\alpha}_1 = (0,0), \boldsymbol{\alpha}_2 = (4,0), \boldsymbol{\alpha}_3 = (0,4), \boldsymbol{\alpha}_4 = (4,4)\}$ *and* $\boldsymbol{\beta}_1 = (2,1), \boldsymbol{\beta}_2 = (1,2)$. *There are two simplexes contain* $\boldsymbol{\beta}_1$: Δ_1 *with vertices* $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3\}$ *and* Δ_2 *with vertices* $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_4\}$. *There are two simplexes contain* $\boldsymbol{\beta}_2$: Δ_1 *and* Δ_3 *with vertices* $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4\}$. *If* $f \in \mathrm{SONC}$, *then by Theorem 4.2,* f *is a sum of nonnegative circuit polynomials supported on* $\Delta_1, \Delta_2, \Delta_1, \Delta_3$ *respectively. Let* $\boldsymbol{\beta}_3 = (2,2), \boldsymbol{\beta}_4 = (2,0), \boldsymbol{\beta}_5 = (0,2)$. *A* $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3\}$*-mediated* *set containing* $\boldsymbol{\beta}_1$ *is* $\{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3,\boldsymbol{\beta}_1,\boldsymbol{\beta}_3,\boldsymbol{\beta}_4\}$*. A* $\{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_4\}$*-mediated set containing* $\boldsymbol{\beta}_1$ *is* $\{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_4,\boldsymbol{\beta}_1,\boldsymbol{\beta}_3,\boldsymbol{\beta}_4\}$*. A* $\{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2\,\boldsymbol{\alpha}_3\}$*-mediated set containing* $\boldsymbol{\beta}_2$ *is* $\{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3,\boldsymbol{\beta}_1,\boldsymbol{\beta}_3,\boldsymbol{\beta}_5\}$*. A* $\{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_3,\boldsymbol{\alpha}_4\}$*-mediated set containing* $\boldsymbol{\beta}_2$ *is* $\{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_3,\boldsymbol{\alpha}_4,\boldsymbol{\beta}_1,\boldsymbol{\beta}_3,\boldsymbol{\beta}_5\}$*. So by Theorem 3.1, f is a sum of squares of binomials. However, in fact f is even not a sum of squares. Thus* $f\notin\mathrm{SONC}$.



5. Computation via Relative Entropy Programming

As shown in [2], testing whether a polynomial belongs to SONC can be converted to a relative entropy programming (REP) problem, which is convex and can be solved efficiently via interior point methods ([1]). By Theorem 4.2, the supports of nonnegative circuit polynomials appearing in the SONC decomposition are contained in the support of the given polynomial. Thus we can compute SONC decompositions for nonnegative polynomials via relative entropy programming more efficiently ([11]).

Theorem 5.1. ([2, Theorem 3.2]) *Let* $f=\sum_{i=1}^m c_i\mathbf{x}^{\boldsymbol{\alpha}_i}-d\mathbf{x}^{\boldsymbol{\beta}}\in\mathbb{R}[\mathbf{x}]$ *be a circuit polynomial, which is not a sum of monomial squares. Assume* $\boldsymbol{\beta}=\sum_{i=1}^m\lambda_i\mathbf{x}^{\boldsymbol{\alpha}_i}$*, where* $\sum_{i=1}^m\lambda_i=1,\lambda_i>0,i=1,\ldots,m$*. Then f is nonnegative if and only if the following REP on variables ν_i and δ_i is feasible:*

$$\tag{5.1}\begin{cases}\textit{minimize} & 1\\ \nu_i=d\lambda_i, & \textit{for } i=1,\ldots,m\\ \nu_i\log(\nu_i/c_i)\le\delta_i, & \textit{for } i=1,\ldots,m\\ \sum_{i=1}^m\delta_i\le 0, & \end{cases}.$$

We make the following assumption for the rest of this section.
Assumption: Let $f=\sum_{i=1}^m c_i\mathbf{x}^{\boldsymbol{\alpha}_i}-\sum_{j=1}^l d_j\mathbf{x}^{\boldsymbol{\beta}_j}\in\mathbb{R}[\mathbf{x}]$ with $\Lambda(f)=\{\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_m\}$ and $\Gamma(f)=\{\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_l\}$. For every $\boldsymbol{\beta}_j$, let

$$\{\Delta_{j1},\ldots,\Delta_{js_j}\}:=\{\Delta\mid\Delta\text{ is a simplex },\boldsymbol{\beta}_j\in\Delta^\circ,V(\Delta)\subseteq\Lambda(f)\}$$

and $I_{jk}:=\{i\in[m]\mid\boldsymbol{\alpha}_i\in V(\Delta_{jk})\}$ for $k=1,\ldots,s_j$ and $j=1,\ldots,l$. For every $\boldsymbol{\beta}_j$ and every Δ_{jk}, since $\boldsymbol{\beta}_j\in\Delta_{jk}^\circ$, we can write $\boldsymbol{\beta}_j=\sum_{i\in I_{jk}}\lambda_{ijk}\boldsymbol{\alpha}_i$, where $\sum_{i\in I_{jk}}\lambda_{ijk}=1,\lambda_{ijk}>0,i\in I_{jk}$.

Theorem 5.2. *Let* $f=\sum_{i=1}^m c_i\mathbf{x}^{\boldsymbol{\alpha}_i}-\sum_{j=1}^l d_j\mathbf{x}^{\boldsymbol{\beta}_j}\in\mathbb{R}[\mathbf{x}]$ *with* $\Lambda(f)=\{\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_m\}$, $\Gamma(f)=\{\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_l\}$ *and* $V(\mathrm{New}(f))\subseteq\Lambda(f)$*, which is not a sum of monomial squares. Then* $f\in\mathrm{SONC}$ *if and only if the following REP on variables* d_{jk}, ν_{ijk},

c_{ijk} and δ_{ijk} is feasible:

$$
(5.2)\qquad \begin{cases} \textit{minimize} \quad 1 \\ \nu_{ijk} = d_{jk}\lambda_{ijk}, & \textit{for } i \in I_{jk}, k = 1,\ldots,s_j, j = 1,\ldots,l \\ \nu_{ijk}\log(\nu_{ijk}/c_{ijk}) \leq \delta_{ijk}, & \textit{for } i \in I_{jk}, k = 1,\ldots,s_j, j = 1,\ldots,l \\ \sum_{i\in I_{jk}} \delta_{ijk} \leq 0, & \textit{for } k = 1\ldots,s_j, j = 1,\ldots,l \\ \sum_{j=1}^{l}\sum_{i\in I_{jk}} c_{ijk} = c_i, & \textit{for } i = 1,\ldots,m \\ \sum_{k=1}^{s_j} d_{jk} = d_j, & \textit{for } j = 1,\ldots,l \end{cases}.
$$

Proof. Suppose $f_{jk} = \sum_{i\in I_{jk}} c_{ijk}\mathbf{x}^{\boldsymbol{\alpha}_i} - d_{jk}\mathbf{x}^{\boldsymbol{\beta}_j}$ is a nonnegative circuit polynomial for $k = 1,\ldots,s_j, j = 1,\ldots,l$ and $f = \sum_{j=1}^{l}\sum_{k=1}^{r} f_{jk}$. Then by Theorem 5.1, $(d_{jk})_{j,k}$, $(\nu_{ijk})_{i,j,k} = (d_{jk}\lambda_{ijk})_{i,j,k}$, $(c_{ijk})_{i,j,k}$ and $(\delta_{ijk})_{i,j,k} = (\nu_{ijk}\log(\nu_{ijk}/c_{ijk}))_{i,j,k}$ is a feasible solution of (5.2).

Conversely, suppose that $(d_{jk})_{j,k}$, $(\nu_{ijk})_{i,j,k}$, $(c_{ijk})_{i,j,k}$ and $(\delta_{ijk})_{i,j,k}$ is a feasible solution of (5.2). Let $f_{jk} = \sum_{i\in I_{jk}} c_{ijk}\mathbf{x}^{\boldsymbol{\alpha}_i} - d_{jk}\mathbf{x}^{\boldsymbol{\beta}_j}$ for $k = 1,\ldots,s_j, j = 1,\ldots,l$. Then by Theorem 5.1, f_{jk} is a nonnegative circuit polynomial for all j,k. Moreover, by the last two equality conditions in (5.2), we have $f = \sum_{j=1}^{l}\sum_{k=1}^{r} f_{jk}$. Thus, $f \in$ SONC. □

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