Statistics on Representation of Finite Groups

Contents

1 Definitions

Assume throughout that the base field is $\mathbb{C}$. The representations Reg_G, Ind_H^G and 1_G denote the regular representation on G, induced representation from H to G, and the trivial representation on G. Further we will use the same notation to denote the characters of the representation wherever it is not ambiguous.

We also denote $Cl(x)$ to be the conjugacy class of x.

Definition 1.1. *A finite group G is called* decomposable *if we can find at least one set of non-trivial proper subgroups $\{H_i\} = \{H_1, \cdots, H_k\}$ and positive rationals c_i, such that the following equivalence of characters hold:*

$$\mathrm{Reg}_G \equiv \sum_{i=1}^{k} c_i \,\mathrm{Ind}_{H_i}^G(1_{H_i}) \pmod{1_G}. \tag{1}$$

We refer to $\{H_i\}$ and c_i as a G-decomposition. c_0 will refer to the coefficient such that the following holds:

$$\mathrm{Reg}_G \equiv \sum_{i=1}^{k} c_i \,\mathrm{Ind}_{H_i}^G(1_{H_i}) - c_0 1_G.$$

Remark: We will show that if there exists positive reals c_i satisfying the above equation, then we can find a decomposition with positive rationals. Thus there is no loss in limiting the discussion to the rationals.

Definition 1.2. *A pair (G, H) of a finite group G and a subgroup H is called (G, H)* decomposable *if we can find at least one set of non-trivial proper subgroups $\{H_i\} = \{H_1, \cdots, H_k\}$ where H is a proper subgroup of each H_i, and positive rationals c_i, such that*

$$\mathrm{Ind}_H^G(1_H) \equiv \sum_{i=1}^{k} c_i \,\mathrm{Ind}_{H_i}^G(1_{H_i}) \pmod{1_G}. \tag{2}$$

Note that if H is the trivial subgroup we recover the previous definition. We refer to $\{H_i\}$ and c_i as a (G, H)-decomposition.

Definition 1.3. *A group G is said to have a* partition *if there exists a set of proper subgroups $\{H_i\}$, such that every non-trivial element of G is contained in exactly one H_i.*

Definition 1.4. *Let S be a subgroup of G. A S-*partition *is a set of proper subgroups $H_1, H_2, ..., H_n$, such that each H_i contains S, and every element not in S lies in exactly one H_i.*

2 G-Decomposable

2.1 Equivalence with partition of group

In the following, we show that the decomposability of a group is equivalent to the existence of a non-trivial partition. The groups satisfying the latter criteria has been completely classified in Baer (1961), Kegel (1961a) and Suzuki (1961).

Theorem 2.1. *G has a non-trivial partition if and only if G is decomposable.*

The strategy for the proof is to first describe a set S of subgroups that always form a partition unless this set contains only G. Then we will show that for some subgroup H in the decomposition of G, H must contain a subgroup in the set S. This immediately implies that S cannot just contain G, and thus forms a partition.

We will first construct one subgroup in this set, as per Young (1927). Let s be a non-trivial element of G.

- $G_{1,s}$ is defined as the group generated by s.
- $G_{2,s}$ is the group generated by the union of all elements in the groups in $G_{1,(s)}$, where

  $$G_{1,(s)} = \{G_{1,x} \mid s \in G_{1,x}\}$$

  is the set of all groups $G_{1,x}$ that contains s.
- Iterate this process for $G_{3,s}, G_{4,s}, ...$, where

  $$G_{n,(s)} = \{G_{n,x} \mid s \in G_{n,x}\}.$$

- Note that $G_{i,s} \leq G_{i+1,s}$, so there exists m such that:

  $$G_{m,s} = G_{m',s} \text{ for all } m' \geq m.$$

  Denote the terminating group as $G_s := G_{m,s}$.

Lemma 2.2. *Suppose G is decomposable and H is any subgroup of the decomposition. Then for all positive integers n and any non-trivial element $s \in H$, the following must be true:*

(i) $G_{n,s} \subseteq H$.

(ii) Suppose x is a non-trivial element in $G_{n,s}$. Then $\mathrm{Ind}_H^G 1_H(s') = \mathrm{Ind}_H^G 1_H(x') > 0$, *where s', x' denotes any element in the conjugacy classes of s, x respectively.*

Proof: First, it is not hard to check that for any $x, s \in G$ such that $x \in G_{n,s}$, we must have $yxy^{-1} \in G_{n,ysy^{-1}}$ for any $y \in G$. Define the set $F_{G,H}(x) = \{g \in G \text{ such that } gxg^{-1} \in H\}$. Let χ denote the sum of characters for the right hand side of (1).

We now perform a two step induction across all subgroups in the decomposition and non-trivial elements in those subgroups. For $n = 1$, (i) is clearly true.

Suppose both statements are true for some $n = i$, and yet (i) is not true for $n = i + 1$. Then (i) for $n = i$ implies that there exists $t \notin H$ such that $s \in G_{i,t}$. By induction we also have $\mathrm{Ind}_{H'}^{G}(t) = \mathrm{Ind}_{H'}^{G}(s) > 0$ for any subgroup H' in the decomposition containing any conjugate of t. Consider the set of conjugates of t that are in H. If it is empty, then it is clear that $0 = \mathrm{Ind}_{H}^{G}(t) < \mathrm{Ind}_{H}^{G}(s)$. This implies that $\chi(s) > \chi(t)$, but $\mathrm{Reg}(s) = \mathrm{Reg}(t)$ and $1_G(s) = 1_G(t)$, and so we have a contradiction. Thus there is at least one conjugate of t that is in H. Pick any such conjugate $t' := ptp^{-1}$, and note that $psp^{-1} \in G_{i,t'}$.

We claim that $F_{G,H}(t')$ is a proper subset of $F_{G,H}(psp^{-1})$. Suppose $g \in F_{G,H}(t')$. Then $gt'g^{-1} \in H$, so $G_{i,gt'g^{-1}} \subseteq H$ from induction, which implies that $g(psp^{-1})g^{-1} \in H$. Thus, $g \in F_{G,H}(psp^{-1})$ and our claim is true.

Finally note that $p^{-1} \in F_{G,H}(psp^{-1})$. Since by assumption $t \notin H$, we have $p^{-1} \notin F_{G,H}(t')$. Thus if we look at the following formula for the induced character,

$$\mathrm{Ind}_{H}^{G} 1_H(h) = \frac{|F_{G,H}(h)|}{|H|},$$

we note that $0 < \mathrm{Ind}_{H}^{G}(t) < \mathrm{Ind}_{H}^{G}(s)$ which is a contradiction as previously shown. Hence it must be true that if $t \in G_{i,s}$, then $t \in H$. This implies that $G_{i,(s)} \subseteq H$ and so $G_{i+1,s} \subseteq H$, proving (1) for $n = i + 1$.

Now we assume that (1) is true for $n = i + 1$ and (2) is true for $n = i$. To complete the induction it suffices to prove (2) for any x that is not contained in $G_{i,s}$ but contained in $G_{i+1,s}$.

For any $g \in F_{G,H}(s)$, we must have $G_{i,gsg^{-1}} \subseteq H$ by (1) when $n = i$, which implies that $gxg^{-1} \in H$. Hence $g \in F_{G,H}(x)$ and we have $\mathrm{Ind}_{H}^{G} 1_H(s) \leq \mathrm{Ind}_{H}^{G} 1_H(x)$. Since this inequality is true for any subgroup H, we immediately obtain that $\chi(x) \geq \chi(s)$. However G is decomposable, so equality must occur everywhere and (2) is true for $n = i + 1$. By induction, the lemma must be true for any n.

Theorem 2.3. *G has a non-trivial partition if any of the groups G_s is not the entire group G.*

Proof: This is precisely Theorem 2 of Young (1927).

Remark: Young (1927) also shows that this is the "minimal" partition of G, in the sense that for any other partition Π containing a subgroup H, H must be the union of some G_s.

Proof of Theorem 2.1: For the forward direction, notice that if G has a non-trivial partition, then it has a non-trivial primitive partition $\Pi = \{L_1, L_2, ..., L_k\}$ (a partition using $L(s)$).

Note further that the primitive partition Π is *normal*, that is if a group $L \in \Pi$, then $L^g \in \Pi$. With this condition, it is easy to check that if we choose $c_i = \frac{|L_i|}{k-1}$ for all $1 \leq i \leq k$, (2) is satisfied.

For the reverse direction, pick any subgroup H used to satisfy By Lemma 2.2, H contains a subgroup $L(s)$ for some $s \neq 1$, and this group cannot be G. Then by Theorem 2.3, G has a non-trivial partition.

Theorem 2.4. *A group G is decomposable if and only if one of the following is true.*

1. *G is a p-group with $H_p(G) \neq G$ and $|G| > p$.*
2. *G is a group of Hughes-Thompson type.*
3. *G is a Frobenius Group.*
4. *G is isomorphic to $PGL(2, p^n)$, where p is an odd prime and $n \in \mathbb{N}$. (Note that $PGL(2,3)$ is isomorphic to S_4)*
5. *G is isomorphic to $PSL(2, p^n)$, where p is a prime and $n \in \mathbb{N}$.*
6. *G is isomorphic to a Suzuki group $Sz(2^{2n+1})$, where $n \in \mathbb{N}$.*

Proof: From Zappa (2003) we verify that every group with a non-trivial partition must be one of the above 6 forms. By Theorem 2.1 we prove the theorem.

It is useful to look at some decomposable groups from the above list as it will become clear why these groups really work.

Example 1: Recall that a Frobenius group G is a semidirect product $G = N \rtimes H$, with the following properties: $gHg^{-1} \cap H = \{1\}$ or H. Every element (except the identity) belongs to either a conjugate of H, or they are in N. We can count that there are exactly $|N|$ in the conjugacy class of H.

$$\mathrm{Ind}_H^G(1_H)(x) = \frac{|G|}{|H|} \cdot \frac{|Cl(x) \cap H|}{|Cl(x)|}.$$

Pick the group H and N. N is normal, so all its elements take value $|H|$.

For every other element $x \neq 1$, suppose $Cl(x) \cap H$ has t elements. Then $Cl(x)$ must have $|N| \cdot t$ elements from the conjugation of H, so x will take value 1. Thus the following is a decomposition of the regular representation of G:

$$\mathrm{Ind}_N^G 1_N + |H| \cdot \mathrm{Ind}_H^G 1_H.$$

More explicitly, if we consider $G = S_3$, we can verify that $\langle(12)\rangle, \langle(123)\rangle$ with coefficients 2 and 1 respectively forms a decomposition. We can also find the partition of all Frobenius groups by taking N and all conjugates of H.

Example 2: We can show that the center of any non-trivial Hughes-Thompson group must be an elementary abelian group, such that $Z(G) \cong \underbrace{C_p \times ... \times C_p}_{n \geq 1}$, where p is the prime number such that $H_p(G) := \langle\{g \in G, \mathrm{ord}(g) \neq p\}\rangle \neq \{1\}$ or G (for a detailed discussion of these groups see Hughes and Thompson (1959)). Any element in $G\backslash H_p(G)$ must have order p and thus splits into cyclic subgroups of order p, where the conjugacy class of every non identity element in one such subgroup must have the same size. It is now straightforward to find appropriate coefficients for $H_p(G)$ and the cyclic subgroups to form a decomposition.

2.2 Rational coefficients

We prove the earlier remark in Definition 1.1 that it suffices to consider only positive rational coefficients.

Proposition 2.5. *If there exists positive reals* $\{c_i\}_{i=1}^k$ *satisfying*

$$\mathrm{Reg}_G \equiv \sum_{i=1}^{k} c_i \,\mathrm{Ind}_{H_i}^G(1_{H_i}) \qquad (\mathrm{mod}\ 1_G)$$

then we can replace $\{c_i\}_{i=1}^k$ *with positive rationals and the decomposition still holds.*

Proof: We express the above equation as $\mathrm{Reg}_G \equiv \sum_{i=1}^k c_i \,\mathrm{Ind}_{H_i}^G(1_{H_i}) + c_{k+1}1_G$, re-indexing $-c_0$ as c_{k+1} for convenience. Consider V to be the $\mathbb{Q}$ vector space generated by $\{c_i\}_{i=1}^{k+1}$ and $\mathbb{Q}$. Let $dim(V) = m$. We may establish a basis $\{v_j\}_{j=1}^m$ with $v_1 \in \mathbb{Q}$ and $v_j \in \mathbb{R}\backslash\mathbb{Q}$ for $j > 1$.

We may then express each c_i as a linear combination of elements in our basis with some rational coefficients a_{ij} with $1 \leq j \leq m$. For each $1 \leq i \leq k+1$:

$$c_i = a_{i1}v_1 + a_{i2}v_2 + ... + a_{im}v_m$$

Let $\vec{\mathbf{c}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{k+1} \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$, and $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & & & \\ \vdots & & \ddots & \\ a_{(k+1),1} & & & a_{(k+1),m} \end{bmatrix}$.

We can now easily see that $A\vec{\mathbf{v}} = \vec{\mathbf{c}}$. Furthermore, let $\vec{\mathbf{X}} = \begin{bmatrix} \mathrm{Ind}_{H_1}^G 1_{H_1} \\ \mathrm{Ind}_{H_2}^G 1_{H_2} \\ \vdots \\ \mathrm{Ind}_{H_k}^G 1_{H_k} \\ 1_G \end{bmatrix}$.

Now, we have $(A\vec{\mathbf{v}}) \cdot \vec{\mathbf{X}} = \vec{\mathbf{c}} \cdot \vec{\mathbf{X}} = c_1 \,\mathrm{Ind}_{H_1}^G 1_{H_1} + ... + c_k \,\mathrm{Ind}_{H_k}^G 1_{H_k} + c_{k+1}1_G = \mathrm{Reg}_G$ as expressed above.

Let A_j denote column j of A. Reg_G always takes rational values, so for $j > 1$, we have $(A_j v_j) \cdot \vec{\mathbf{X}} = 0$. By linearity, we see that this will hold for any choice of v_j, so for all $j > 1$, we may replace v_j with some positive $\alpha_j \in \mathbb{Q}$ and have our decomposition still hold. v_1 is already rational, so now we have expressed our coefficients $\{c_i\}_{i=1}^{k+1}$ as a linear combination of positive rational elements.

Thus, if there exists a character decomposition of a group G, then a character decomposition with rational coefficients of G can always be created. We establish the canonical form for the decomposition of a group to have rational coefficients.

2.3 Integer Coefficients

We also ask when does our groups admit a decomposition with positive integer coefficients.

Proposition 2.6. *If G is an elementary abelian group, then it has an integer decomposition*

Proof: Let $G = \underbrace{C_p \times ... \times C_p}_{n}$. We pick all the subgroups $H_1, ..., H_k$ of order p^{n-1} and let their coefficients in the decomposition be 1. There are exactly $\frac{p^n-1}{p-1}$ such subgroups. We provide a quick proof here:

We can view G as a n-dimensional vector space V in $\mathbb{F}_p$. Then we want to find the number of $(n-1)$-dimensional subspaces. For every such space W, there are

$$(p^{n-1} - 1)...(p^{n-1} - p^{n-2})$$

ways to choose the basis. On the other hand, there are

$$(p^n - 1)...(p^n - p^{n-2})$$

ways to choose $n-1$ linearly independent vectors in V. Thus the total number of distinct $(n-1)$-dimensional subspaces is

$$\frac{(p^n - 1)...(p^n - p^{n-2})}{(p^{n-1} - 1)...(p^{n-1} - p^{n-2})} = \frac{p^n - 1}{p - 1}.$$

This proves our claim on the number of such subgroups. Since each non-trivial element appears in $\frac{p^{n-1}-1}{p-1}$ of the subgroups, we can easily see that $\mathrm{Reg}_G = \sum_{i=1}^{k} \mathrm{Ind}_{H_i}^{G} 1_{H_i} - \frac{p^n-p}{p-1} \cdot 1_G$.

Theorem 2.7. *Let G be a decomposable group that is not a p-group. Then G admits a decomposition with positive integer coefficients, unless G is a group of Hughes-Thompson type with prime p, and the stabilizer of at least one element not in $H_p(G)$ is not isomorphic to C_p.*

Proof: We investigate the various types of decomposable groups.

Proposition 2.8. *If G is a Frobenius group, it has an integer decomposition.*

Proof: A Frobenius group $G = N \rtimes H$, satisfy the following properties: $gHg^{-1} \cap H = \{1\}$ or H. Every element (except the identity) belongs to either a conjugate of H, or they are in N. There are exactly $|N|$ elements in the conjugacy class of H. Since

$$\mathrm{Ind}_H^G(1_H)(x) = \frac{|G|}{|H|} \cdot \frac{|Cl(x) \cap H|}{|Cl(x)|}.$$

Pick the group H and N. Since N is normal, all its elements take value $|H|$ in the above character.

For every other element $x \neq 1$, suppose $Cl(x) \cap H$ has t elements. Then $Cl(x)$ must have $|N| \cdot t$ elements from the conjugation of H, so x will take value 1. Thus the following is a decomposition of the regular representation of G:

$$\mathrm{Ind}_N^G 1_N + |H| \cdot \mathrm{Ind}_H^G 1_H.$$

Proposition 2.9. *G has an integer decomposition if it satisfies one of the following:*

- *G is isomorphic to $PGL(2, p^n)$, where p is an odd prime and $n \in \mathbb{N}$.*
- *G is isomorphic to $PSL(2, p^n)$, where p is a prime and $n \in \mathbb{N}$.*
- *G is isomorphic to a Suzuki group $Sz(2^{2n+1})$, where $n \in \mathbb{N}$.*

Proof: We consider each case separately. In the following, c_H denotes the coefficient for H in the decomposition.

Case 1: Suzuki Group. Note that $Sz(2)$ is isomorphic to the Frobenius group F_5 and has integer decomposition as shown previously. Consider the Suzuki Group $Sz(q), q = 2^{2n+1} > 2$ of order $s = (q-1)q^2(q^2+1)$, with $r = 2^n$. From Theorem 4.1 in Farrokhi (2011) or Huppert and Blackburn (1982) we know that the following subgroups forms a normal partition of Suzuki Groups:

- $q^2 + 1$ Sylow 2-subgroups E of order q^2.
- $s/(4(q + 2r + 1))$ cyclic subgroups F of order $q + 2r + 1$.
- $s/(4(q - 2r + 1))$ cyclic subgroups G of order $q - 2r + 1$.
- $s/(2(q - 1))$ cyclic subgroups H of order $q - 1$.

In addition, each subgroup of the same order in the above categories are conjugate to each other. We let A, B, C, D denote the union of the elements excluding the identity e in each category above respectively. We introduce the Hall subgroup of $Sz(q)$ (type I) as defined in Theorem 3.1 of Héthelyi et al. (2015), which is a Frobenius group $K \rtimes L$, where K is of type E and L of type H. This group has order $q^2(q-1)$.

As such, the induced character for each subgroup type takes the following non-zero values:

- E: $(q^2 + 1)(q - 1)$ on e, $q - 1$ on A.

- F: $s/(q+2r+1)$ on e, 4 on B.
- G: $s/(q-2r+1)$ on e, 4 on C.
- H: $s/(q-1)$ on e, 2 on D.
- I: q^2+1 on e, 1 on A, 2 on D.

Finally, our integer coefficients for χ are

$$c_E = c_I = 1, c_F = c_G = q/4, c_H = q/2 - 1.$$

We can check that on A to D, χ takes value q, and on the identity χ is $s+q$, as desired.

Case 2: $PGL(2,q)$, for q odd. From Theorem 4.1 in Farrokhi (2011) and Huppert (1967) we obtain that $G = PGL(2,q)$ for $q \geq 5$ odd, has a partition Π by all the conjugates of subgroups H, K, L of order $q, q-1, q+1$ respectively. There are $q+1, q(q+1)/2, q(q-1)/2$ such conjugates respectively.

Now let the non identity elements of G be divided into sets $E(H), E(K), E(L)$, depending on which subgroup in the partition Π they belong in. The induced representations take the following values:

- H: $(q-1)(q+1)$ on id, $q-1$ on $E(H)$.
- K: $q(q+1)$ on id, 2 on $E(K)$.
- L: $q(q-1)$ on id, 2 on $E(L)$.

Thus picking coefficients $c_H = 1, c_K = c_L = (q-1)/2$ will suffice.

We are left with $PGL(2,3)$, which is isomorphic to S_4. We can take $H_1 = \langle(12)\rangle, H_2 = \langle(234)\rangle, H_3 = \langle(1234)\rangle, c_1 = c_2 = c_3 = 1$

Case 3: $PSL(2,q)$. Suppose $G = PSL(2,q)$ and $q \geq 4$. Define $d := gcd(2, q-1)$. We know that $|G| = q(q-1)(q+1)/d$. From Theorem 4.1 of Farrokhi (2011) and Huppert (1967), we have a partition Π by all the conjugates of subgroups H, K, L of order $q, (q-1)/d, (q+1)/d$. There are $q+1, q(q+1)/d, q(q-1)/d$ such conjugates respectively.

Now let the non identity elements of G be divided into sets $E(H), E(K), E(L)$, depending on which subgroup in the partition Π they belong in. In addition we introduce the Borel subgroup M of order $q(q-1)/d$, which is a Frobenius group $\simeq H \rtimes K$. Its existence is guaranteed by Theorem 6.51 in Gorenstein et al. (1998).

Thus we know that M can be partitioned by one conjugate of H and q conjugates of K. We can then deduce that the induced representations take the following values:

- H: $(q-1)(q+1)/d$ on id, $(q-1)/d$ on $E(H)$.
- L: $q(q-1)$ on id, d on $E(L)$.
- M: $(q+1)$ on id, 1 on $E(H)$, d on $E(K)$.

Thus picking coefficients $c_L = c_M = (q-1)/d, c_H = 1$, we obtain the desired integer decomposition. Finally we have $PSL(2,2) \simeq S_3$ and $PSL(2,3) \simeq A_4$, which are both Frobenius groups and covered by the earlier proposition.

Lemma 2.10. *If G is a group of Hughes-Thompson type with prime p and G is not a p-group, then G has an integer decomposition unless the stabalizer of at least one element not in $H_p(G)$ is not isomorphic to C_p.*

Proof: Firstly from Theorem 2 of Kegel (1961a), G must be decomposable and has a non-trivial center. We define $H_p(G)$ to be the subgroup of G generated by all the elements whose order is not p, as per Hughes and Thompson (1959). Furthermore we can write $G \simeq H_p(G) \rtimes \langle x \rangle$, where $\langle x \rangle \simeq C_p$. Every element not in $H_p(G)$ must have order p. These elements generate subgroups of order p, and we let X denote the set of such subgroups.

In the decomposition of Hughes-Thompson Group, one of the component must be $H_p(G)$. Without loss of generality $H_1 = H_p(G)$. As the only non-trivial subgroups that intersect trivially with $H_p(G)$ are from X, they must form the other subgroups in the decomposition.

Suppose that X is divided into conjugacy classes $X_1, X_2, ..., X_k$ of sizes $x_1, x_2, ..., x_k$. The centralizer for each subgroup in X contains the center, and thus $k \geq |Z(G)|$. Note further that $H_p(G)$ is a normal subgroup of G, thus $hxh^{-1}x^{-1} \in H_p(G)$ for any $h \in G$. As such, $|X_i| = |H_p(G)|/C_G(y_i)$ for any y_i in the union of elements of subgroups in X_i.

The decomposition must contain at least one subgroup P_i from each of X_i. We also note that $\mathrm{Ind}_{P_i}^G 1_{P_i}$ is $|H_p(G)|$ on the identity. Finally, if the coefficient for each subgroup is an integer, and thus at least 1, the following must be true:

$c_0 = c_1 \cdot p$ as any element in $H_p(G)$ that is not the identity is non-zero only on $Ind_{H_p(G)}^G 1_{H_p(G)}$. This implies that $\mathrm{Reg}_G = \sum_{i=1}^n c_i \cdot \mathrm{Ind}_{H_i}^G 1_{H_i} - c_0 \cdot 1_G$ is at least $k \cdot |H_p(G)|$ when evaluated at the identity. Hence $k = p$, $Z(G) \simeq C_p$ and $c_i = 1$ for all $i \geq 1$.

Thus a Hughes-Thompson Group is decomposable into integer coefficients, if and only if the center $Z(G) \simeq C_p$, and the stabalizer for every element not in $H_p(G)$ is precisely $Z(G)$.

Remark: Kegel (1961b) also shows that $H_p(G)$ is nilpotent, and hence the Fitting subgroup of G.

3 G-H Decomposable

It is natural to generalize our result to this problem. However, the corresponding generalization of a partition is no longer equivalent to the generalization of the decomposability of a group. An intuitive way to understand the difference is that in a (G, H) decomposition the elements conjugate to H have very different restrictions as opposed to elements in $G \backslash H$. However these restrictions do not apply when we restrict ourselves to $H = \{e\}$ which is equivalent to asking about G-decomposable.

3.1 Core-free

First we note the following lemma about conjugacy class:

Lemma 3.1. *Let N be a normal subgroup of G, and $x \in G$. Then $Cl(x)$ is equally distributed over the left cosets of N that contains at least one element of $Cl(x)$.*

Proof: If the conjugacy class is a subset of one of the left cosets we are done. Suppose otherwise that it intersects at least two cosets. Pick any two of them yN and N, and suppose $zN \cap Cl(x) = \{zn_1, ..., zn_k\}$. There exists some $g \in G$ such that $g(zn_1)g^{-1} = gzg^{-1}gn_1g^{-1} \in yN$. Since N is normal, we must have $gzg^{-1} = yn$ for some $n \in N$. Thus $gzn_ig^{-1} = yn(gn_ig^{-1}) \in yN$, which implies that $|yN \cap Cl(x)| \geq |zN \cap Cl(x)|$. Since the choice of y and z is arbitrary, we prove the lemma.

Definition 3.2. *For a group G, the* core *of the subgroup H is given by $\bigcap_{g \in G} gHg^{-1}$, the intersection of all conjugates of H.*

Theorem 3.3. *(G, H) is decomposable if and only if $(G/N, H/N)$ is decomposable, where N is the core of H.*

Proof: First we show that

$$\mathrm{Ind}_{H_i}^{G} 1_{H_i}(x) = \mathrm{Ind}_{H_i/N}^{G/N} 1_{H_i/N}(xN)$$

is true for any $x \in G$ and subgroup H_i that contains H (and thus H_i contains N).

Since the LHS and RHS are $\frac{|G|}{|H_i|} \cdot \frac{|Cl(x) \cap H_i|}{|Cl(x)|}$, $\frac{|G/N|}{|H_i/N|} \cdot \frac{|Cl(xN) \cap H_i/N|}{|Cl(xN)|}$ respectively, and the left cosets of N that intersect $Cl(x)$ non-trivially are precisely $Cl(xN)$, by the previous lemma we can see that the equation must be true.

Since H_i/N contains H/N, we then obtain that

$$\mathrm{Reg}_G = c_0 1_G + \sum_{i=1}^{k} c_i \,\mathrm{Ind}_{H_i}^{G} 1_{H_i}, c_i > 0 \text{ for } i \geq 1$$

if and only if

$$\mathrm{Reg}_{G/N} = \frac{c_0}{|N|} 1_{G/N} + \sum_{i=1}^{k} \frac{c_i}{|N|} \,\mathrm{Ind}_{H_i/N}^{G/N} 1_{H_i/N}, c_i > 0 \text{ for } i \geq 1$$

which implies the theorem.

From now on, we only consider pairs (G, H) where H is core-free, unless otherwised mentioned. Note that of a group is core-free, it cannot contain any non-trivial element from the center.

3.2 S-partition

We explore the relationship between the generalization of a partition of a group and (G, H) decomposability. Following Section 8 of Zappa (2003) or Zappa (1966), we cite the following definition and characterization of S-partition.

Theorem 3.4. *A group G has a S-partition by subgroups $\{H_i\}$ if and only if the following conditions are satisfied:*

1. *G is a Frobenius group with Frobenius complement S, and S is cyclic.*
2. *The Frobenius kernel K, is a p-group of exponent p and of class ≤ 2. It admits a non trivial partition, $\{K_1, ..., K_s\}$ such that the $S-$partition of G is given by $\{H_i\}$, where $H_i = K_i \rtimes S$.*

Remark: Note that, if G has a S-partition, the normalizer of each group K_i in G must contain S.

Next we show that, if a group G admits a S-partition, it is (G, H) decomposable with S being the subgroup H.

Theorem 3.5. *Any group G with a S-partition is (G, S) decomposable.*

Proof: We will prove this without using any condition on G as imposed by Theorem 3.4. Let $I(x) := |Cl(x) \cap S|$. Consider the following weighted induced trivial character of H_i, where the groups $H_1, .., H_s, s > 1$ are the groups that form the S-partition:

$$\frac{|H_i|}{|G|} \operatorname{Ind}_{H_i}^G 1_{H_i}(x) = \frac{|Cl(x) \cap H_i|}{|Cl(x)|}.$$

Then since every element not in S appears exactly once, we obtain

$$\sum_{i=1}^{s} \frac{|H_i|}{|G|} \operatorname{Ind}_{H_i}^G 1_{H_i}(x) = \frac{\sum_{i=1}^{s} |Cl(x) \cap H_i|}{|Cl(x)|} = \frac{s \cdot I(x) + |Cl(x)| - I(x)}{|Cl(x)|}$$

We now see that the desired decomposition is given by:

$$\operatorname{Ind}_S^G 1_S = \frac{|G|}{(s-1)|S|} \left(\sum_{i=1}^{s} \frac{|H_i|}{|G|} \operatorname{Ind}_{H_i}^G 1_{H_i}(x) - 1_G \right) = \sum_{i=1}^{s} \frac{|H_i|}{(s-1)|S|} \cdot \operatorname{Ind}_{H_i}^G 1_{H_i} - \frac{|G|}{(s-1)|S|} \cdot 1_G,$$

since we can check that the value at x for both sides is $\operatorname{Ind}_S^G 1_S(x) = \frac{|G|}{|S|} \cdot \frac{I(x)}{|Cl(x)|}$.

Example: The reverse is not true. (Order 40 example)

Example: For any prime $p \geq 3$, let $G \simeq C_p \rtimes D_p$ and $H = C_2$. (G,H) is decomposable.

3.3 Some observations on when G-H decomposable occurs

We begin by proving a few lemmas that will be needed later. Some of these results are well known, but we provide the proofs regardless.

Lemma 3.6. *Let C be any conjugacy class. Suppose there exists $b \notin C$ and $m > 1$ the smallest positive integer such that $b^m \in C$. Let the conjugacy class of b be D. Then for any element $b' \in D$, we must have $b'^m \in C$. Furthermore, $n = m$ is the smallest positive integer such that $b'^n \in C$.*

Hence we can denote D to be a generator for C, and m as the power of this set with respect to C.

Proof: Let $b' = gbg^{-1}$. Then $b'^m = gb^m g^{-1} \in C$. Now suppose $b'^n \in C$ and $n < m$. Then by the same argument, $b^n \in C$, a contradiction.

Lemma 3.7. *Suppose D is a generator of C with power m. For $c \in C$, let D_c be the set of elements $d \in D$, such that $d^m = c$. Then $|D_c|$ is a constant regardless of the choice of $c \in C$.*

Proof: Let $C = \{c_1, ..., c_k\}$. If $k = 1$ we are done. Suppose not, by Lemma 3.6, every element in D belongs in exactly one of the sets D_{c_i}, $1 \leq i \leq k$. We pick any set D_{c_s} and choose $d_s \in D_{c_s}$. Then there exists $g \in G$ and D_{c_t} where $t \neq s$, such that $gd_s g^{-1} \in D_{c_t}$.

Then $c_t = (gd_s g^{-1})^m = gd_s^m g^{-1} = gc_s g^{-1}$. Let $d'_s \in D_{c_s}$. Then $(gd'_s g^{-1})^m = gd'^m_s g^{-1} = gc_s g^{-1} = c_t$, which implies that $gD_{c_s} g^{-1} \in D_{c_t}$. Thus $D_{c_s} \subseteq D_{c_t}$. Since our choices of s and t were arbitrary, we must have $|D_{c_s}| = |D_{c_t}|$ for all $1 \leq s, t \leq k$.

Theorem 3.8. *Let (G, H) be decomposable where the subgroups in the decomposition are $H_1, \ldots, H_k$, and χ denotes the sum of characters in the RHS of (1). Furthermore let X denote the union of all the elements that are conjugate to some element of H. Then using the notations of Lemma 3.7, if $c \in H_i$ and $c \notin X$, we must have $D_c \subset H_i$.*

Proof: We claim that for any subgroup H_i $c \in C$ (c may be in X) and $d \in D$ we have

$$\mathrm{Ind}_{H_i}^G(1_{H_i})(c) \geq \mathrm{Ind}_{H_i}^G(1_{H_i})(d) = \frac{|G|}{|H_i|} \cdot \frac{|H_i \cap D|}{|D|},$$

and equality occurs if and only if whenever $c \in H_i$, we have $D_c \subset H_i$. Note that if $H_i \cap D_c \neq \emptyset$, then $c \in H_i$. If $H_i \cap C = \emptyset$ then the claim is clearly true.

Suppose otherwise, and let $H_i \cap C = \{c_1, ..., c_l\}$. Then $\mathrm{Ind}_{H_i}^G(1_{H_i})(c) = \frac{|G|}{|H_i|} \cdot \frac{l}{|C|}$ for all $c \in C$. Every element in $H_i \cap D$ must belong to one of $D_{c_i}, 1 \leq i \leq l$, thus $|H_i \cap D| \leq \sum_{i=1}^{l} |D_{c_i}| = l \cdot |D_c| = \frac{l|D|}{|C|}$, proving the claim.

Now if $c \in H_i, c \notin X$, we must have $\chi(c) = \chi(d)$ for (2) to hold true. This implies that equality must hold throughout for the inequality in the claim. This occurs if and only if $H_i \cap D = \bigcup_{j=1}^{l} D_{c_j}$. But c is one of the c_j, and so the theorem is proven.

Lemma 3.9. *If a subgroup N of G contains at least one element from every conjugacy class of G, then $N = G$.*

Proof: If N is a proper subgroup, consider all its conjugates. Note that the union must be G, since N contains elements in every conjugacy class. There are $|G|/|N|$ distinct conjugates, thus the total number of elements is at most $(|G|/|N|)(|N|-1)+1 < |G|$, a contradiction.

Lemma 3.10. *Let X be the set of elements whose conjugate is in H. If (G, H) is decomposable, then at least one of the H_i must contain an element not from X.*

Proof; H cannot contain an element from each conjugacy class, or by Lemma 3.9, $H = G$. Thus in the problem statement, there cannot be any copies of 1_G on both sides and we must have equality:

$$\operatorname{Ind}_H^G 1_H = \sum_{i=1}^{k} c_i \operatorname{Ind}_{H_i}^G 1_{H_i}. \tag{3}$$

Finally, note that each $H \subseteq H_i$. Thus for any non-identity element $x \in H$, we have $\frac{\operatorname{Ind}_{H_i}^G 1_{H_i}(x)}{\operatorname{Ind}_{H_i}^G 1_{H_i}(e)} \geq \frac{\operatorname{Ind}_H^G 1_H(x)}{\operatorname{Ind}_H^G 1_H(e)}$. Furthermore, this inequality is for at least one such x, which implies that (1) cannot be true, a contradiction.

With Theorem 3.8, we can find a criteria on the center of G if (G, H) is decomposable.

Lemma 3.11. *We use the notations as defined in Theorem 3.8. Suppose that the center of G, denoted by $Z(G)$, is non-trivial.*

- *If $Z(G)$ is not an elementary abelian group, then $P_H(G)$ is the group generated by H and all the elements not in X,*
- *If $Z(G)$ is an elementary abelian group, and there exists g such that $g^p \notin X$. Then $P_H(G)$ is the group generated by H and all the elements x such that $x^p \notin X$,*

If (G, H) is decomposable, then at least one of the subgroups H_i contains $P_H(G)$.

Proof: From Lemma 3.10 we can assume without loss of generality that H_1 contains an element outside X. Let this element be x. Note that for every element y outside X, $\chi(y)$ must be equal and non-zero.

Case 1: The center $Z(G)$ is not a p-group. First we show that this implies the center $Z(G) \subset H$ if it is not a p-group. Clearly we can pick $z, z' \in Z(G)$ whose orders p, q are greater than 1 and are coprime. Then consider $(xz)^p = x^p$ or $(xz')^q = x^q$. If x^p and $x^q \in X$, then $x \in X$, a contradiction. Thus at least one of them is not in X (and cannot be e). Without loss of generality let it be x^p. Then $xz \notin X$. Now note that xz generates x^p, and the order of xz and x^p are different, implying that they are in distinct conjugacy classes. Thus by Theorem 3.8 we have $xz \in H_1$ and so $z \in H_1$.

Now $Z(G) \cap X = \{e\}$, so by a similar argument as above, we show that $Z(G) \in H_1$. Lastly, note that if there exists $y \notin X$ such that $y \notin H_1$, we must have $\mathrm{Ind}_{H_i}^G 1_{H_i}(Z(G)) > \mathrm{Ind}_{H_1}^G 1_{H_1}(y)$. Using the same analysis as the beginning of Case 1, any group H_i containing a conjugate of y must contain $Z(G)$. This implies that $\mathrm{Ind}_{H_i}^G 1_{H_i}(Z(G)) \geq \mathrm{Ind}_{H_i}^G 1_{H_i}(y)$ for all i, a contradiction.

Case 2: $Z(G)$ is a p-group and $g^p \notin X$ for some $g \in G$. Without loss of generality suppose that g (or its conjugate) is in H_1. Using a similar analysis as Case 1, we obtain that $Z(G) \in H_1$, and any y such that $y^p \notin X$ is also in H_1.

Case 3: $Z(G)$ is a p-group that is not an elementary abelian group. Then $Z(G)$ must contain an element t of order p^2. Since $t^p \notin X$, the condition for Case 2 is satisfied.

Now consider any element $u \notin X$ such that $u^p \in X$. This immediately implies that the order of u is divisible by p. It is not possible for $(ut)^p \in X$, otherwise $t^p \in X$. Thus we must have $ut \in H_1$, which implies that $u \in H_1$.

Corollary 3.12. *If the center of a group G is non-trivial and not an elementary abelian group, then (G, H) is not decomposable for any core-free H.*

Proof: Suppose the decomposition exists. Then Lemma 3.10 and Lemma 3.11 implies that if the center of a group G is non-trivial, and is not a p-group, then $P_H(G) = G$ is contained in some subgroup H_i, a contradiction.

Example: An example of a group G whose center is an elementary abelian group can be seen from $G = C_2 \times F_7 \simeq \langle a^2 = b^7 = c^6 = 1 \,|\, ab = ba, ac = ca, cbc^{-1} = b^5\rangle$ of order 84. This group has a center $\langle a\rangle \simeq C_2$, and it is (G, H)-decomposable with $H = \langle c^2\rangle \simeq C_3$, as given by the following subgroups:

$H_1 = \langle c\rangle, H_2 = \langle ac\rangle, H_3 = \langle a, b, c^2\rangle$. The coefficients are $c_0 = -2, c_1 = 1, c_2 = 1, c_3 = 1$.

3.4 (G,H) decomposable groups

We move on to study when a (G, H) decomposition can occur.

Theorem 3.13. *Let $G = K \rtimes_\phi N$ and $H < N$, and let $H_c = \cap_{n\in N} H^n$. Suppose further that for all elements $n \in N \backslash H_c$, $\phi(n)$ has no non-trivial fixed points. Then (G, H) is decomposable.*

Proof: We show that it suffices to choose $K \rtimes H$ and N, with coefficients 1 and $|N|/|H|$.

For $h \in H$, let $S(h)$ denote the set of conjugates of h formed by K, and $T(h)$ denote the set of conjugates of h formed by N. We can thus calculate the induced characters as follows:

$$\mathrm{Ind}_N^G 1_N(x) = \begin{cases} \frac{|K|}{|S(h)|} & x \in S(h) \text{ for some } h \in H_c \\ 1 & x \in G\backslash(K \rtimes H_c) \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Ind}_{K\rtimes H}^{G} 1_{K\rtimes H}(x) = \begin{cases} \frac{|N|\cdot|T(h)\cap H|}{|H|\cdot|T(h)|} & x \in K \rtimes H \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Ind}_{H}^{G} 1_{H}(x) = \begin{cases} \frac{|G|}{|S(h)|\cdot|H|} & x \in S(h) \text{ for some } h \in H_c \\ \frac{|K|\cdot|T(h)\cap H|}{|H|\cdot|T(h)|} & x \in \cup_{g\in T(h)} S(g) \text{ for some } h \in H\backslash H_c \\ 0 & \text{otherwise} \end{cases}$$

Then we verify that

$$\mathrm{Ind}_{H}^{G} 1_{H} = \mathrm{Ind}_{K\rtimes H}^{G} 1_{K\rtimes H} + \frac{|N|}{|H|} \mathrm{Ind}_{N}^{G} 1_{N} - \frac{|N|}{|H|} 1_{G}.$$

Remarks: One class of groups that fall into this category is the Frobenius groups, which is the same as if ϕ has no non-trivial fixed points for all of N. Then Theorem 3.13 implies that for any proper subgroup H of the complement N, (G, H) will be decomposable.

Other concrete cases can be seen when $G \simeq A \rtimes B$, where $B \simeq C \rtimes K$ is Frobenius. Here N is replaced by $A \rtimes C$. More specifically for odd $n \geq 5$, we define F_n to be the Frobenius group with kernel or order n and complement C_{n-1}. Let $m \geq 3$ be an odd integer and $q = \frac{n-1}{2}$. If $G \simeq C_m \rtimes F_n$ and $H = C_q \subset F_n$, then (G, H) is decomposable.

Next we consider the following theorem, which allows us to build a (G, H)-decomposition from smaller ones. In the decompositions considered below, we do not require any of them to be core-free.

Theorem 3.14. *Assume that our group $G = K \rtimes N$ and $H = H_K \rtimes H_N$, where $H_K < K, H_N < N$. Then (G, H) is decomposable if the following are true:*

1. *(N, H_N) is decomposable, where the subgroups are $N_1, ..., N_n$ with coefficients $\alpha_1, ...\alpha_n$. We allow the decomposition where $N_1 = H_N$, $N_2 = N$, and $\alpha_1 = 1, \alpha_2 = |N|/|H_N|$.*
2. *For the subgroups N_i, $i \geq 2$, $(K \rtimes N_i, H)$ is decomposable, where the subgroups in the decomposition are $K \rtimes H_N, M_{i,1}, ...M_{i,n_i}$ with coefficients $1, \beta_{i,1}, ...\beta_{i,n_i}$.*

Proof: By considering the value at e in condition 1 and 2, we get that

$$\sum_{i=2}^{n} \frac{\alpha_i}{|N_i|} = \frac{1}{|H_N|}, \quad \beta_i := \sum_{j=1}^{n_i} \beta_{i,j} = \frac{|N_i|}{|H_N|}.$$

We claim that the decomposition is given precisely by the subgroups $K \rtimes N_1$ and $M_{i,j}$ for $2 \leq i \leq n, 1 \leq j \leq n_i$, with coefficents α_1 and $\frac{\alpha_i \cdot \beta_{i,j}}{\beta_i}$, and the coefficient for 1_G is $\frac{a_1 \cdot |N|}{|N_1|}$.

We write out the formulas that denote condition 1 and 2:

$$\operatorname{Ind}_{H_N}^{N} 1_{H_N} = \sum_{i=1}^{n} \alpha_i \operatorname{Ind}_{N_i}^{N} 1_{N_i} - \frac{\alpha_1 \cdot |N|}{|N_1|} 1_N \tag{4}$$

$$\operatorname{Ind}_{H}^{K\rtimes N_i} 1_H = \sum_{j=1}^{n_i} \beta_{i,j} \operatorname{Ind}_{M_{i,j}}^{K\rtimes N_i} 1_{M_{i,j}} + \operatorname{Ind}_{K\rtimes H_N}^{K\rtimes N_i} 1_{K\rtimes H_N} - \frac{|N_i|}{|H_N|} 1_{K\rtimes N_i} \text{ for all } i \geq 2. \tag{5}$$

Equation (5) can be further rewritten using the transitive property of an induced character to be

$$\operatorname{Ind}_{H}^{G} 1_H = \sum_{j=1}^{n_i} \beta_{i,j} \operatorname{Ind}_{M_{i,j}}^{G} 1_{M_{i,j}} + \operatorname{Ind}_{K\rtimes H_N}^{G} 1_{K\rtimes H_N} - \frac{|N_i|}{|H_N|} \operatorname{Ind}_{K\rtimes N_i}^{G} 1_{K\rtimes N_i} \text{ for all } i \geq 2. \tag{6}$$

Hence using the above equations we have

$$\begin{aligned}
&\sum_{i=2}^{n} \sum_{j=1}^{n_i} \frac{\alpha_i \cdot \beta_{i,j}}{\beta_i} \operatorname{Ind}_{M_{i,j}}^{G} 1_{M_{i,j}} + \alpha_1 \operatorname{Ind}_{K\rtimes N_1}^{G} 1_{K\rtimes N_1} - \frac{\alpha_1 \cdot |N|}{|N_1|} 1_G \\
&= \sum_{i=2}^{n} \frac{\alpha_i}{\beta_i} \left(\operatorname{Ind}_{H}^{G} 1_H - \operatorname{Ind}_{K\rtimes H_N}^{G} 1_{K\rtimes H_N} + \frac{|N_i|}{|H_N|} \operatorname{Ind}_{K\rtimes N_i}^{G} 1_{K\rtimes N_i} \right) + \alpha_1 \operatorname{Ind}_{K\rtimes N_1}^{G} 1_{K\rtimes N_1} - \frac{\alpha_1 \cdot |N|}{|N_1|} 1_G \\
&= \operatorname{Ind}_{H}^{G} 1_H - \operatorname{Ind}_{K\rtimes H_N}^{G} 1_{K\rtimes H_N} + \sum_{i=1}^{n} \alpha_i \operatorname{Ind}_{K\rtimes N_1}^{G} 1_{K\rtimes N_1} - \frac{\alpha_1 \cdot |N|}{|N_1|} 1_G.
\end{aligned}$$

It remains to show that $-\operatorname{Ind}_{K\rtimes H_N}^{G} 1_{K\rtimes H_N} + \sum_{i=1}^{n} \alpha_i \operatorname{Ind}_{K\rtimes N_1}^{G} 1_{K\rtimes N_1} - \frac{\alpha_1 \cdot |N|}{|N_1|} 1_G$ is 0 for any element. We observe that for any element kn, where $k \in K$ and $n \in N$, and $L < N$, $\operatorname{Ind}_{K\rtimes L}^{G} 1_{K\rtimes L}(kn) = \operatorname{Ind}_{L}^{N} 1_L(n)$. The justification for this is that for any $c \in N$, $\operatorname{Stab}_K(kn) = |\{x \in K, x(kn)x^{-1} = kn\}| = |\{x \in K^c, c^{-1}xc(kn)c^{-1}x^{-1}c = kn\}| = |\{x \in K, xc(kn)c^{-1}x^{-1} = c(kn)c^{-1}\}| = \operatorname{Stab}_K(c(kn)c^{-1})$. Hence $\frac{|Cl_L(n) \cap L|}{|L|} = \frac{|Cl_G(kn) \cap K\rtimes L|}{|K\rtimes L|}$, and we have the desired conclusion from equation (4).

It is useful to consider several examples illustrating the above theorem.

Example 1: Let $G = S_4 = C_2^2 \rtimes S_3 = \langle (12)(34), (13)(24) \rangle \rtimes \langle (123), (12) \rangle$ and $H = \langle (12), (34) \rangle \cong C_2$. Then H_n is trivial, and we pick $N_1 = \langle (123) \rangle, N_2 = \langle (12) \rangle, \alpha_1 = 1, \alpha_2 = 2$, which is a S_3 decomposition and satisfies condition 1. For the pair $(K \rtimes N_2, H) \cong (D_4, C_2)$, we then choose $M_{1,1} = \langle (12)(34), (12) \rangle \cong C_2^2$, $M_{1,2} = \langle (13)(24)(12) = (1324) \rangle \cong C_4$, $\beta_{1,1} = \beta_{1,2} = 1$ and together with $K \cong C_2^2$ we can easily verify this is a decomposition satisfying condition 2. Finally, we can readily verify that we have a (S_4, C_2) decomposition:

$$\operatorname{Ind}_{\langle (12),(34) \rangle}^{S_4} 1_{\langle (12),(34) \rangle} = \operatorname{Ind}_{\langle (12)(34),(12) \rangle}^{S_4} + \operatorname{Ind}_{\langle (1324) \rangle}^{S_4} + \operatorname{Ind}_{A_4}^{S_4} 1_{A_4} - 2 \cdot 1_G$$

Example 2: We assume the conditions in Theorem 3.13 where H_k is trivial, and we choose the trivial decomposition where $N_1 = H_N, N_2 = N$ with coefficients $\alpha_1 = 1, \alpha_2 = |N|/|H_N|$. Then condition 2 requires that (G, H) is decomposable. Although the theorem is vacuous for this case, this only occurs because we chose the trivial decomposition.

Example 3: We consider a generalization of Example 2. Let $G = B \rtimes_\psi C$ for some non-trivial action ψ. For the subgroup $H < C$, we will formulate a criteria for the existence of a (G, H)-decomposition. Note that H may not be core-free in this characterization. We denote H_c to be the core of H as a subgroup of C.

We can split the the elements of C into into the following sets: $W = \bigcup_{g \in C} H^g$, $X = \{x \in C \backslash W$ and $\psi(x)$ has no non-trivial fixed point$\}, Y = C \backslash \{W \cup X\}$. Note that $|\{b(cxc^{-1})b^{-1}, b \in B\}| = |\{bxb^{-1}, b \in B^c\}| = |\{bxb^{-1}, b \in B\}|$, which tells us that W, X, Y are all unions of conjugacy classes in C.

Suppose we can find a (C, H) decomposition (where H is not necessarily core-free in C) where the subgroups are $C_1, \ldots, C_k$, and coefficients $\alpha_1, \ldots, \alpha_k$ such that

- $W \backslash H_c \subset X$.
- C_1 contains Y and $C_2, \ldots, C_k$ is disjoint from Y.

Then we verify that $(B \rtimes C_i, H)$ has a decomposition with the subgroups $B \rtimes H, C_i$ with coefficients $1, \alpha_i$ using Theorem 3.13. Finally by Theorem 3.14 we can form a (G, H) decomposition using the subgroups $B \rtimes C_1, C_2, ..., C_k$.

Example 4: We now turn our attention towards discussion of (G, H) decompositions where $H \subset K$, the kernel of $G = K \rtimes N$. In these cases, for condition (1) of theorem 3.14, we require $(N, \{e\})$ to be decomposable, which is equivalent to N being G-decomposable (as in the previous section).

Take, as a motivating example, $G = C_5^2 \rtimes S_3 = K \rtimes N$. Let σ be the order 2 generator of S_3. Take $H = C_5 \subset C_5^2$ to be the order 5 subgroup in the kernel of G fixed by σ. $N = S_3$ is G-decomposable, and we take $N_1 = C_3$ and $N_2 = C_2$ with $\alpha_1 = 1$, $\alpha_2 = 2$. We claim $(K \rtimes N_2, H)$ is decomposable, where H is our chosen C_5 fixed by σ, $N_2 = C_2 = \langle \sigma \rangle$, and $K = C_5^2$. The calculations are as follows: $\mathrm{Ind}_H^{K \rtimes N_2} 1_H$ takes value 10 on elements of H and 0 elsewhere. $\mathrm{Ind}_K^{K \rtimes N_2} 1_K$ takes value 2 on elements of K and 0 elsewhere. $\mathrm{Ind}_{H \rtimes N_2}^{K \rtimes N_2} 1_{H \rtimes N_2}$ takes value 5 on elements of H, 1 on elements conjugate to σ, and 0 elsewhere. We then easily see that

$$\mathrm{Ind}_H^{K \rtimes N_2} 1_H = 2 \cdot \mathrm{Ind}_{H \rtimes C_2}^{K \rtimes N_2} 1_{H \rtimes N_2} + \mathrm{Ind}_K^{K \rtimes N_2} 1_K - 2 \cdot 1_{K \rtimes N_2}$$

which is our desired decomposition. Thus, from theorem 3.14, (G, H) is decomposable. Indeed, we observe our decomposition by:

$$\mathrm{Ind}_H^G 1_H = 2 \cdot \mathrm{Ind}_{H \rtimes C_2}^G 1_{H \rtimes C_2} + \mathrm{Ind}_K^G 1_K - 2 \cdot 1_G$$

Example 5: Here, we will generalize example 4 to prove the following: for prime p and odd q, let $G = C_p^2 \rtimes D_q = K \rtimes N$, and let $H = C_p$ be the order p subgroup of the kernel that is fixed by the order 2 elements of D_q. Then (G, H) is decomposable.

As before, D_q is Frobenius for odd q, thus N is decomposable with $N_1 = C_q$ and $N_2 = C_2$. $(K \rtimes N_2, H)$ is decomposable with subgroups K and $H \rtimes N_2$ using the same construction as in example 4, thus by theorem 3.14, (G,H) is decomposable. Indeed, we note the following decomposition predicted by theorem 3.14 is simple to calculate explicitly:

$$\mathrm{Ind}_H^G 1_H = 2 \cdot \mathrm{Ind}_{H \rtimes C_2}^G 1_{H \rtimes C_2} + \mathrm{Ind}_K^G 1_K - 2 \cdot 1_G$$

Remark: Finally, we note that in a computation for groups with order < 2000 (except for SmallGroup$(1944, 3870)$ and groups of order $512, 768, 1024, 1152, 1280, 1536, 1728, 1792, 1920$ - there are too many groups with those orders), it is verified that every (G,H) decomposable pair that is core-free falls under the category set out by either Theorem 3.5, Theorem 3.13, or Theorem 3.14, except for the group SmallGroup$(720, 765) = A_6.C_2$.

Discussion: We look at the exception $G =$SmallGroup$(720, 765) \simeq A_6.C_2$. In this case, there are two possible choices of H that are core-free, isomorphic to C_4 and D_4 respectively. The computation shows that

- For $H \simeq C_4$, we have $\mathrm{Ind}_H^G 1_H = \mathrm{Ind}_{H_1}^G 1_{H_1} + \mathrm{Ind}_{H_2}^G 1_{H_2} + \mathrm{Ind}_{H_3}^G 1_{H_3} - 2 \cdot 1_G$, where $H_1 \simeq A_6$, $H_2 \simeq C_8$, $H_3 \simeq Q_8$.
- For $H \simeq D_4$, we have $\mathrm{Ind}_H^G 1_H = \mathrm{Ind}_{H_1}^G 1_{H_1} + 2\,\mathrm{Ind}_{H_2}^G 1_{H_2} - 2 \cdot 1_G$, where $H_1 \simeq A_6$, $H_2 \simeq SD_{16}$.

************ Extra stuff:

Claim 1: For a group G, we can choose a representative σ of $\psi \in \mathrm{Out}(G)$, and form the extension $F = G.\langle\sigma\rangle$. If we choose another representative σ' of ψ and form $F' = G.\langle\sigma'\rangle$, then the conjugacy classes of $f = g\sigma^k$ and $f' = g'\sigma'^k$ $(k = 0, 1)$ have the same size. In other words, the conjugacy classes of F and F' are identical.

Proof: Note that σ, σ' have the same action on G upon conjugation by assumptions.

If $k = 0$, then the conclusion is immediate. If $k = 1$, we verify that for any $h \in G$, $hfh^{-1} = hg(h^{-1})^\sigma\sigma$ and $hf'h^{-1} = hg(h^{-1})^{\sigma'}\sigma' = hg(h^{-1})^\sigma\sigma'$. We also have $(h\sigma)f(h\sigma)^{-1} = hg^\sigma(h^{-1})^\sigma\sigma$ and $(h\sigma')f(h\sigma')^{-1} = hg^\sigma(h^{-1})^\sigma\sigma'$. Thus we can conclude that the two conjugacy classes have the same size.

Proposition: $PSL(2, p^2)$ for p odd, there exists an extension by $\langle\sigma\rangle \simeq C_2$ so that the resulting group is (G,H) decomposable.

Proof: Let ν be a generator of $\mathbb{F}_{p^2}^*$. We construct the group $G = PSL(2, p^2).\langle\sigma\rangle$, where $\sigma = \delta\phi$ as described in White (2013) and $\sigma^2 = \begin{pmatrix} \nu^{-1} & 0 \\ 0 & \nu \end{pmatrix}$. Hence $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\sigma = \begin{pmatrix} a & v^{-p-1}b \\ v^{p+1}c & d \end{pmatrix}$.

Next we characterize the conjugacy classes in $PSL(2, p^2)$. We can verify that the classes $1, c, d$ remains. The conjugacy classes a^m and $a^{\frac{p^2-1}{2}-m}$ merges, and similarly b^m merges with $b^{\frac{p^2+1}{2}-m}$, and so we have a^k for $k = 1, \ldots, \frac{p^2-1}{4}$ and b^l for $l = 1, \ldots, \frac{p^2-1}{4}$. We can also deduce their sizes from Freyre et al. (2010).

σ will merge some of the conjugacy classes in the $PSL(2, p^2)$ normal subgroup of G as follows:

- The conjugacy classes of the form b^m merge with those of the form b^s for $s \equiv \pm mp \pmod{p^2+1}$. Note that $p^2+1 \nmid m(p \pm 1)$, and so the number of classes is exactly $\frac{q-1}{8}$.
- The conjugacy classes of the form a^m, $1 \le m \le \frac{q-1}{4}$ merge with those of the form a^s for $s \equiv \pm mp \pmod{p^2-1}$, unless $\frac{p^2-1}{2} | m(p \pm 1)$. There are $\frac{p-1}{2}$ solutions to $\frac{p+1}{2}|m$ and $\frac{p+1}{2}$ solutions to $\frac{p-1}{2}|m$, for a total of $p-1$ solutions. Thus exactly $p-1$ conjugacy classes are intact, whereas the other conjugacy classes merge to form $\frac{p^2-1}{8} - \frac{p-1}{2}$ classes.
- A routine calculation shows that there are $\frac{p-1}{2}$ conjugacy classes represented by $\begin{pmatrix} \nu^k & 0 \\ 0 & \nu^{-k} \end{pmatrix} \sigma$ for $k = 0, \dots, \frac{p-3}{2}$ and $\frac{p+1}{2}$ conjugacy classes represented by $\begin{pmatrix} 0 & -\nu^k \\ \nu^{-k} & 0 \end{pmatrix} \sigma$ for $k = 1, \dots, \frac{p+1}{2}$. The stabilizers for them have order $2(p-1)$ and $2(p+1)$ respectively. Thus the total number of elements in these classes is $\frac{|G|}{2(p-1)} \cdot \frac{p-1}{2} + \frac{|G|}{2(p+1)} \cdot \frac{p+1}{2} = \frac{|G|}{2}$, and so they cover all the elements of the form $h\sigma$ for $h \in PSL(2, p^2)$.

The full table of conjugacy classes of G is as such ($a \sim b$ implies that we only consider one of a or b):

Class	Representative	Number of classes	Size
C_1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1	1
C_2	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	1	$p^4 - 1$
C_3	$a^k = \begin{pmatrix} \nu^k & 0 \\ 0 & \nu^{-k} \end{pmatrix}$, where $k \sim \pm kp \pmod{p^2-1}$, $\frac{p+1}{2} \nmid k$ and $\frac{p-1}{2} \nmid k$	$\frac{p^2-1}{8} - \frac{p-1}{2}$	$2p^2(p^2+1)$
C_4	$a^k = \begin{pmatrix} \nu^k & 0 \\ 0 & \nu^{-k} \end{pmatrix}$, where $k \neq \frac{p^2-1}{4}$, $\frac{p+1}{2}|k$ or $\frac{p-1}{2}|k$	$p-2$	$p^2(p^2+1)$
C_5	$\begin{pmatrix} \nu^{(p^2-1)/4} & 0 \\ 0 & \nu^{-(p^2-1)/4} \end{pmatrix}$	1	$\frac{p^2(p^2+1)}{2}$
C_6	$b^k = \begin{pmatrix} \gamma^k & 0 \\ 0 & \gamma^{-k} \end{pmatrix}$, where $k \sim \pm kp \pmod{p^2+1}$	$\frac{p^2-1}{8}$	$2p^2(p^2-1)$
C_7	$\begin{pmatrix} \nu^k & 0 \\ 0 & \nu^{-k} \end{pmatrix} \sigma$, $k = 0, \dots, \frac{p-3}{2}$	$\frac{p-1}{2}$	$\frac{(p+1)p^2(p^2+1)}{2}$
C_8	$\begin{pmatrix} 0 & -\nu^k \\ \nu^{-k} & 0 \end{pmatrix} \sigma$, $k = 1, \dots, \frac{p+1}{2}$	$\frac{p+1}{2}$	$\frac{(p-1)p^2(p^2+1)}{2}$

Desired subgroups H and subgroups in decomposition:

- $C_{(q-1)/2} \simeq \langle \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix} \rangle$:

$H_1 = PSL(2,p^2)$, $H_2 = \langle \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sigma \rangle$, $H_3 = \langle \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}, \sigma \rangle$, with coefficients 1 each.

- $D_{(q-1)/2} \simeq \langle \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rangle$:

  $H_1 = PSL(2,p^2)$, $H_2 = \langle \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sigma \rangle$, with coefficients 1 and 2 respectively.

It is straightforward to compute the values on each conjugacy class and verify that these constitute a valid decomposition.

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