Homework 8, Math 3000

due on March 22, 2022

Before you start, please read the syllabus carefully.

1. Compute the determinant of the following matrices.

   (a)
   $$A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

   (b)
   $$A = \begin{pmatrix} 0 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}$$

   (c)
   $$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

   (d)
   $$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & b \\ 0 & 0 & a & 1 \end{pmatrix}$$

2. Recall the formula for determinant of $A \in M_{n\times n}(\mathbb{R})$

   $$\det(A) := \sum_P a_{i_1,1}a_{i_2,2}\cdots a_{i_n,n}\cdot \mathrm{sgn}(P),$$

   where the summation is over all permutations of $\{1, 2, \cdots, n\}$. For example, when $n = 3$, we have altogether 6 ways of permuting $\{1, 2, 3\}$, e.g. $(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$ or $(3, 2, 1)$.

   (a) For each permutation, determine its sign. (It is $+1$ if you need to do even times of swapping to go back to $(1, 2, 3)$, is -1 if you need to do odd times).

   (b) For $n = 4$, write down all possible permutations and determine their signs. How many permutations altogether?

(c) Use this formula to prove that for any matrix 4×4 matrix A in the form of

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where A_1 and A_2 are both 2×2, we have $\det(A) = \det(A_1)\det(A_2)$.

3. Prove that if $A \in M_{n\times n}(\mathbb{R})$ is in its row echelon form, then $\det(A) = \prod_i a_{ii}$.

4. Prove that if $B = C^{-1}AC$, then $\det(B - \lambda I) = \det(A - \lambda I)$.