08/24/21. Week 2.

Recall linear equation is

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

b, a_i constant
x_i variable.

system of linear equations

b_i, a_{ij} constant.
$x_1 \cdots x_n$ variables

$$\begin{cases} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2 \\ \vdots \qquad \vdots \qquad \qquad \vdots \qquad \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = b_m \end{cases}$$

solution set is all $\{(x_1, \cdots, x_n)$ s.t. it satisfies all equations in the system $\}$.

Q: How to solve the system?

Ex:

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 & ① \\ -2x_1 - x_2 + 0 \cdot x_3 = -1 & ② \\ x_1 - x_2 + 2x_3 = 2 & ③ \end{cases}$$

3 variable
3 equations.
$m=3 \quad n=3$

"Gaussian Elimination".

$② + ① \times 2$:
$$\begin{aligned} &-2x_1 - x_2 + 0 \cdot x_3 &&= -1 \\ &+ (x_1 + 2x_2 + x_3) \cdot \times 2 &&+ 2 \times 2 \end{aligned}$$

simplify: $\quad 3x_2 + 2x_3 = 3 \quad ②'$

$③ - ①$: $\quad 0 \cdot x_1 - 3x_2 + x_3 = 0 \quad ③'$

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 & ① \\ 3x_2 + 2x_3 = 3 & ②' \\ -3x_2 + x_3 = 0 & ③' \end{cases}$$

$③' + ②'$: $\quad 0 x_2 + 3x_3 = 3 \quad ③''$

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 & ① \\ 3x_2 + 2x_3 = 3 & ②' \\ 3x_3 = 3 & ③'' \end{cases} \longleftarrow \text{"upper triangular form"}$$

Using ③'': $x_3 = 1$

then ②': $3x_2 + 2 \cdot 1 = 3 \Rightarrow x_2 = \frac{1}{3}$

then ①: $x_1 + \frac{2}{3} + 1 = 2 \Rightarrow x_1 = \frac{1}{3}$

So the solution for this system of equations is

$$\begin{cases} x_1 = \frac{1}{3} \\ x_2 = \frac{1}{3} \\ x_3 = 1 \end{cases} \quad \text{or} \quad \left(\frac{1}{3}, \frac{1}{3}, 1\right)$$

<u>Matrix Notation</u>

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad \qquad \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

<u>coefficient matrix</u> is

$$m \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$
$$n$$

<u>augmented matrix</u> is

$$m\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

$n+1$

Ex:

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 & ① \\ -2x_1 - x_2 + 0\cdot x_3 = -1 & ② \\ x_1 - x_2 + 2x_3 = 2 & ③ \end{cases}$$

c.m. $\begin{pmatrix} 1 & 2 & 1 \\ -2 & -1 & 0 \\ 1 & -1 & 2 \end{pmatrix}$ a.m. $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ -2 & -1 & 0 & -1 \\ 1 & -1 & 2 & 2 \end{array}\right)$

②' = ② + ① × 2

③' = ③ − ①

Replace row 2 by row 2 + row1 × 2;
Replace row 3 by row 3 − row 1.

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 & ① \\ 3x_2 + 2x_3 = 3 & ②' \\ -3x_2 + x_3 = 0 & ③' \end{cases}$$

c. m. $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & -3 & 1 \end{pmatrix}$ a.m. $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 3 & 2 & 3 \\ 0 & -3 & 1 & 0 \end{array}\right)$

Row Operations (on augmented matrix):

1) Replace row i by row i + row j × number. ; ($i \neq j$)

2) Switch row i with row j. ($i \neq j$)

3) Multiply row i by number. (non-zero)

Replace row 3 by row 3 + row 2.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 3 & 2 & 3 \\ 0 & 0 & 3 & 3 \end{array}\right) \leftarrow \text{echelon form.}$$

Def: A rectangular matrix is in <u>echelon form</u> if

1. All non-zero rows are above any rows of all zeros.

2. Each leading entry of a row is in a column to the strict right of leading entry of the row above it.

eg. ① $\begin{pmatrix} \times & \times & \times \\ 0 & 0 & \times \\ 0 & 0 & 0 \end{pmatrix}$ ② $\begin{pmatrix} \times & \times & \times \\ 0 & \times & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ⑤ ✗ $\begin{pmatrix} \times & \times & \times \\ 0 & \times & 0 \\ 0 & \times & \times \end{pmatrix}$

③ ✗ $\begin{pmatrix} \times & \times & \times \\ 0 & 0 & 0 \\ 0 & 0 & \times \end{pmatrix}$ ④ $\begin{pmatrix} 0 & \times & \times & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Ex. $\begin{cases} x_1 + 2x_2 - x_3 = 2 & ① \\ 2x_1 + 4x_2 - 3x_3 = 1 & ② \\ 3x_1 + 6x_2 - x_3 = -2 & ③ \end{cases}$

$$\begin{pmatrix} 1 & 2 & -1 & 2 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -1 & -2 \end{pmatrix} \xrightarrow[\text{row 3} - (\text{row 1}) \times 3]{\text{row 2} - (\text{row 1}) \times 2} \begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 2 & -8 \end{pmatrix}$$

$$\xrightarrow{\text{row 3} + \text{row 2} \times 2} \begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & \boxed{-14} \end{pmatrix}$$

pivot

$$\begin{cases} x_1 + 2x_2 - x_3 = 2 \\ \qquad\qquad - x_3 = -3 \\ 0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 = -14 \end{cases}$$ $\leftarrow$ impossible.

So there is no solution.

<u>Conclusion</u>: A system of linear equations has no solutions (called inconsistent) if and only if there is a pivot in the last column of the echelon form of a.m.

Ex: $$\begin{cases} x_2 + 2x_3 + x_4 = 0 & ① \\ x_1 - x_2 + x_4 = 1 & ② \\ 2x_1 - x_2 + 2x_3 + 3x_4 = 2 & ③ \end{cases}$$

a.m. $$\left(\begin{array}{cccc|c} 0 & 1 & 2 & 1 & 0 \\ 1 & -1 & 0 & 1 & 1 \\ 2 & -1 & 2 & 3 & 2 \end{array}\right)$$

② $\leftrightarrow$ ①

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 2 & -1 & 2 & 3 & 2 \end{array}\right)$$

③ $\rightsquigarrow$ ③ $-$ ① $\times 2$

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & -3 & 2 & 1 & 0 \end{array}\right)$$

③ $\rightsquigarrow$ ③ $+$ ② $\times 3$

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 8 & 4 & 0 \end{array}\right)$$

$$\begin{cases} x_1 - x_2 \qquad\qquad + x_4 = 1 \\ \qquad x_2 + 2x_3 + x_4 = 0 \\ \qquad\qquad 8x_3 + 4x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 1 + x_2 - x_4 = 1 - x_4 \\ x_2 = -2 \cdot (-\frac{1}{2}) x_4 - x_4 = 0 \\ x_3 = -\frac{1}{2} x_4 \\ x_4 \text{ is free} \end{cases}$$