Homework 1, Math 401

due on January 13, 2020

Before you start, please read the syllabus carefully.

1. Given $(G, \cdot)$ a group. Prove the following:
   (a) There exists a unique element $e \in G$ such that $e \cdot a = a \cdot e = a$ for any element $a \in G$. (*)We call this element *identity*
   (b) For any element $a \in G$, there exists a unique $b \in G$ such that $a \cdot b = b \cdot a = e$. (*)We call this element *inverse* of a, and denote it by a^{-1}
   (c) If $a \cdot c = b \cdot c$, then $a = b$ (cancellation law)
   (d) $(a^{-1})^{-1} = a$
   (e) $(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$
2. Given $(R, +, \cdot)$ a ring. We will denote the identity with respect to "+" by 0, and the identity respect to "$\cdot$" by 1, the inverse of $a \in R$ with respect to "+" by $-a$. Prove the following:
   (a) $0 \cdot a = a \cdot 0 = 0$
   (b) $-a = (-1) \cdot a$
   (c) $-(-a) = a$
3. Prove the following:
   (a) $(\mathbb{Z}, +, \times)$ forms a ring.
   (b) $(\mathbb{R}, +, \times)$ forms a ring.
   (c) $(\mathbb{Q}, +, \times)$ forms a ring.
   (d) $(\mathbb{C}, +, \times)$ forms a ring.

   Here $+$ and $\times$ are interpreted as the usual addition and multiplication for each set. $\mathbb{Z}$ means integers, $\mathbb{R}$ means real numbers, $\mathbb{Q}$ means rational numbers, $\mathbb{C}$ means complex numbers.
4. Prove that $(M_{n\times n}(\mathbb{R}), +, \times)$ forms a ring. Here $M_{n\times n}(\mathbb{R})$ is the set of all n-by-n matrices over real numbers $\mathbb{R}$. Here $+$ and $\times$ are interpreted as the usual addition and multiplication for matrices.
5. Determine whether:
   (a) $(\mathbb{Z}\backslash\{0\}, \times)$ forms a group?
   (b) $(\mathbb{R}\backslash\{0\}, \times)$ forms a group?

(c) $(\mathbb{Q}\backslash\{0\}, \times)$ forms a group?

(d) $(\mathbb{C}\backslash\{0\}, \times)$ forms a group?

If yes, give a proof; if no, explain why.

6. Determine whether:

   (a) $(M_{n\times n}(\mathbb{R})\backslash\{0\}, \times)$ forms a group?

   (b) $(M_{n\times n}(\mathbb{Z})\backslash\{0\}, \times)$ forms a group?

   Here 0 means zero matrix. If yes, give a proof; if no, explain why.
   (**Bonus**): If no, come up with a set of matrices that forms a group with respect to usual multiplication for matrices.

7. Given $(R, +, \cdot)$ a ring. Do we have cancellation law for $\cdot$, i.e, if $a \cdot c = b \cdot c$, do we have $a = b$? If yes, give a proof; if no, explain why.

8. Given $(R, +, \cdot)$ a ring. Show that if there exists an inverse of 0 with respect to "$\cdot$", i.e., if there exists $x \in R$ such that $x \cdot 0 = 0 \cdot x = 1$, then R contains only one element.
   (*)We say that such a ring is *trivial*. This exercise shows why we do not have $\frac{a}{0}$.

   Answer:
   Since $x \cdot 0 = 0 \cdot x = 0$, we have $0 = 1$. For all element $y \in R$, we have $y = y \cdot 1 = y \cdot 0 = 0$. Therefore there is only one element in R.

   Remark:
   In order to define $\frac{a}{0}$, we need to first find a multiplicative inverse 0^{-1} of 0, and then define $\frac{a}{0} := a \cdot 0^{-1}$. Question 8 determines that we should not be able to find any multiplicative inverse of 0 at all, unless in a very extreme situation. Thus, there is no way to talk about $\frac{a}{0}$.