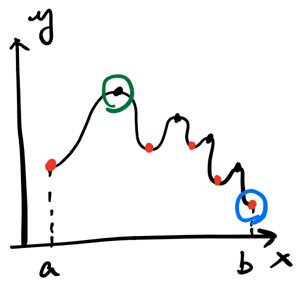
Max / Min Problem

Recall in calculus, given $f(x)$, what is maximal/minimal value of $f(x)$?

- Local



To locate critical pts

- $f'(x)=0$

or

$f'(x)$ not defined

or

x at boundary

To tell max/min for x with $f'(x)=0$

- $f''(x)>0$ local min

  $f''(x)<0$ local max.

- Global.

To locate candidates:

- $f'(x)=0$

or

$f'(x)$ not defined

or

x at boundary.

To determine global max/min.

- Compute $f(x)$ for all candidates

Idea: Taylor Expansion of a smooth function

$$f(x) \approx f(x_0) + f'(x_0)\cdot(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2$$

For more variable case. given $f(\vec{x})$

$$f(\vec{x}) \approx f(\vec{x}_0) + \nabla f(\vec{x}_0)\cdot(\vec{x}-\vec{x}_0) + \frac{1}{2}\sum_{i,j}\frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x}_0)\,(x_i - x_{0,i})\cdot(x_j - x_{0,j})$$

eg. 2 variables.

$$\frac{1}{2}\left[f_{xx}(\vec{x}_0)\cdot x^2 + 2 f_{xy}(\vec{x}_0)\cdot xy + f_{yy}(\vec{x}_0)\cdot y^2\right]$$

- Local

To locate critical pts:

- $\nabla f(x)=0$

or

∇f is defined

or

$\vec{x}$ at boundary.

To tell max/min/saddle

- $A = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$

- Global

To locate candidates

- $\nabla f = 0$

or

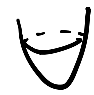
∇f not defined

or

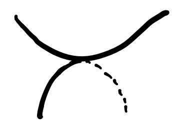
$\vec{x}$ at boundary

To determine max/min.

- Compute values for every candidates.

$\det(A) > 0 \quad f_{xx} > 0$ local min 

$\det(A) > 0 \quad f_{xx} < 0$ local max 

$\det(A) < 0$ saddle pts 

Lagrange Multiplier: Optimation with Constraints.

Goal: $f(\vec{x})$

Constraints: $g(\vec{x}) = 0$

To locate candidates

- $\nabla g = 0$

or

- $\nabla f = \lambda \cdot \nabla g$ for some λ.

$\Bigg\}$ along with $g(\vec{x}) = 0$

(*) For more constraints

- $\nabla g = 0$ or $\nabla h = 0$ or $\nabla g = k \nabla h$
- $\nabla f = \lambda_1 \nabla g + \lambda_2 \nabla h$

To determine max/min.

- Compute the value for all candidates.

Example: Minimize $f(x,y) = xy$ subject to the constraint

$$g(x,y) = x^2 + \frac{1}{4}y^2 - 1 = 0$$

Ans. To look for candidates.

① $$\begin{cases} \nabla g(\vec{x}) = 0 \\ g(\vec{x}) = 0 \end{cases} \Rightarrow \begin{cases} \begin{pmatrix} 2x \\ \frac{1}{2}y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ x^2 + \frac{1}{4}y^2 = 1 \end{cases} \Rightarrow \text{no solution.}$$

②. $$\begin{cases} \lambda \nabla g(\vec{x}) = \nabla f(\vec{x}) \\ g(\vec{x}) = 0 \end{cases} \Rightarrow \begin{cases} \begin{pmatrix} y \\ x \end{pmatrix} = \lambda \cdot \begin{pmatrix} 2x \\ \frac{1}{2}y \end{pmatrix} \\ x^2 + \frac{1}{4}y^2 = 1 \end{cases}$$

$$\begin{cases} y = \lambda \cdot 2x \\ x = \lambda \cdot \frac{1}{2} y \end{cases} \Rightarrow \quad x = \lambda \cdot \frac{1}{2} \cdot \lambda \cdot 2x \quad \Rightarrow \quad x \cdot (\lambda^2 - 1) = 0$$

$$\Rightarrow \quad \underset{\text{impossible.}}{\underset{\Downarrow}{x = 0}} \quad \text{or} \quad \lambda = \pm 1$$

If $\lambda = 1$. $\quad x^2 + \frac{1}{4} \cdot (2x)^2 = 1 \Rightarrow x = \pm \frac{\sqrt{2}}{2}$.

$$y = 2x = \pm \sqrt{2}$$

If $\lambda = -1$ $\quad x^2 + \frac{1}{4}(-2x)^2 = 1 \Rightarrow x = \pm \frac{\sqrt{2}}{2}$

$$y = -2x = \mp \sqrt{2}.$$

So all critical pts are $\left(\frac{\sqrt{2}}{2}, \sqrt{2}\right) \to f = 1$

$\left(-\frac{\sqrt{2}}{2}, \sqrt{2}\right)$, $\left(\frac{\sqrt{2}}{2}, -\sqrt{2}\right)$ $\left(-\frac{\sqrt{2}}{2}, -\sqrt{2}\right)$.

$f = -1$ $\quad f = -1$ $\quad f = 1$

Compare the value at all points.

the minimal value is obtained at

$\left(\frac{\sqrt{2}}{2}, -\sqrt{2}\right)$ and $\left(-\frac{\sqrt{2}}{2}, \sqrt{2}\right)$, and is -1.

Example: $\quad f(x, y) = y^2 - 18x^2 + x^4$

1) What are the local min/max ?

$$\nabla f = \begin{pmatrix} -36x + 4x^3 \\ 2y \end{pmatrix} = \vec{0} \Rightarrow \begin{cases} 4x^3 - 36x = 4x \cdot (x^2 - 9) = 0 \\ y = 0 \end{cases} \Rightarrow$$

$x = 0, \pm 3$ and $y = 0$. So critical pts are

$(0, 0) \quad (-3, 0) \quad (3, 0)$

$$A = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} -36 + 12x^2 & 0 \\ 0 & 2 \end{pmatrix}$$

At $(0,0)$ $\quad \det(A) = -72 < 0 \quad$ saddle pts.

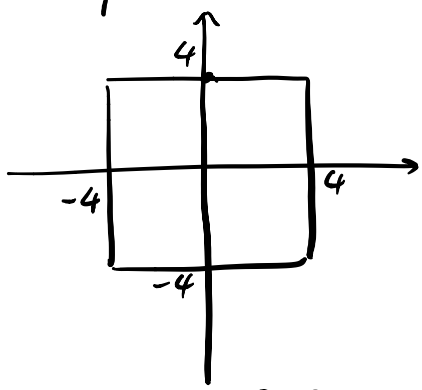
At $(3,0)$ $\quad \det(A) > 0 \quad f_{xx} = 12\times 9 - 36 = 72 > 0 \quad$ local min.

At $(-3,0)$ $\quad$ local min

2) $D = [-4,4]\times[-4,4]$. What is the global max/min in D?

To locate candidates $(0,0)$ $\quad (\pm 3, 0)$

points on the boundary.



$f(0,0) = 0 \qquad f(\pm 3, 0) = 0 - 18\times 9 + 81 = 81 - 162 = -81$

$f(\pm 4, y) = y^2 - 18\times 4^2 + 4^4 = y^2 + 4^2\cdot(4^2 - 18) = y^2 - 32$

at most $\quad 16 - 32 = -16$ at $|y| = 4$

at least $\quad 0 - 32 = -32$ at $|y| = 0$

$f(x, \pm 4) = 16 - 18x^2 + x^4 = \underbrace{(x^2-9)^2 - 81}_{} + 16$

$= (x^2 - 9)^2 - 65$

at most $\quad 9^2 - 65 = 16$ at $|x| = 0$

at least $\quad 7^2 - 65 = -16$ at $|x| = 4$

Global max is 16 at $x=0$ $\quad y = \pm 4$

min is -81 at $x=0$ $\quad y=0$.