

# Derivatives of Common Functions

## The Power Function

The power function has the form

$$f(x) = x^n$$

According to the definition of derivatives, we have

$$\frac{d}{dx}x^n = \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h}$$

Use the Binomial Theorem, we have:

$$(x+h)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} h^k$$

Because the first term in the above summation is  $x^n$ , so we have:

$$\frac{(x+h)^n - x^n}{h} = \frac{\sum_{k=1}^n \binom{n}{k} x^{n-k} h^k}{h}$$

Please be aware of the change of the range of  $k$  from 0 to 1 in the summation.

Then, let us cancel  $k$ , the denominator, and get:

$$\frac{\sum_{k=1}^n \binom{n}{k} x^{n-k} h^k}{h} = \sum_{k=1}^n \binom{n}{k} x^{n-k} h^{k-1}$$

Please be aware of the change of the superscription of  $h$  from  $k$  to  $k-1$ .

This change inspires us to substitute  $k-1$  by  $k'$ , which ranges from 0 to  $n-1$ :

$$\sum_{k=1}^n \binom{n}{k} x^{n-k} h^{k-1} = \sum_{k'=0}^{n-1} \binom{n}{k'} x^{n-k} h^{k'}$$

When  $h \rightarrow 0$ , most terms in above summation becomes 0, except for the first one

$$\binom{n}{1} x^{n-1} h^0 = nx^{n-1}$$

So we have

$$\frac{d}{dx}x^n = nx^{n-1}$$

## The Exponential Function

The exponential function has the form

$$f(x) = a^x$$

I didn't derive its derivative from the definition; instead, I used a property that the derivative of the *natural exponential function* is itself, or

$$\frac{d}{dx}e^x = e^x$$

We will derive this property later. For now, we just use it to derive the derivative of the general exponential function

$$\frac{d}{dx}a^x = \frac{d}{dx}(e^{\ln(a)})^x = \frac{d}{dx}e^{x \ln(a)}$$

Here we need to use the chain rule by assuming that  $g(x) = x \ln(a)$ , so we have

$$\frac{d}{dx}e^{x \ln(a)} = \frac{d}{dg}e^g \frac{d}{dx}x \ln(a) = e^{x \ln(a)} \ln(a) = a^x \ln(a)$$

## The Natural Exponential Function

By the definition of derivatives, we have

$$\frac{d}{dx}e^x = \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \rightarrow 0} e^x \frac{(e^h - 1)}{h}$$

By the definition of  $e$ , or the Euler's number, we have

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

Please be aware that  $n \rightarrow \infty$  implies that  $1/n \rightarrow 0$ . By introducing  $h = 1/n$ , we get another form of the definition of  $e$ :

$$e = \lim_{h \rightarrow 0} (1 + h)^{\frac{1}{h}}$$

Substitue this definition into the above derivative

$$\frac{d}{dx} e^x = \lim_{h \rightarrow 0} e^x \frac{((1 + h)^{\frac{1}{h}})^h - 1}{h} = \lim_{h \rightarrow 0} e^x$$

## The Logarithm Function

The natural logarithm function has the form

$$f(x) = \ln(x)$$

Given that

$$e^{\ln(x)} = x$$

The derivatives of both sides should be the same

$$\frac{d}{dx} e^{\ln(x)} = \frac{d}{dx} x = 1$$

Using the chain rule for the left side

$$e^{\ln(x)} \frac{d}{dx} \ln(x) = 1$$

So

$$\frac{d}{dx} \ln(x) = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$$

We can extends to logarithm with arbitrary base

$$f(x) = \log_a(x)$$

1. Using the quotient rule:

$$\frac{d}{dx} \log_a(x) = \frac{d}{dx} \frac{\ln(x)}{\ln(a)} = \frac{\ln'(x) \ln(a) - \ln(x) \ln'(a)}{\ln^2(a)}$$

Because  $\ln'(a) = 0$ , so

$$\frac{d}{dx} \log_a(x) = \frac{1}{x \ln(a)}$$

2. Using the equality trick

$$\frac{d}{dx} a^{\log_a(x)} = \frac{d}{dx} x = 1$$

and by the chain rule

$$\frac{d}{dx} a^{\log_a(x)} = a^{\log_a(x)} \ln(a) \frac{d}{dx} \log_a(x) = x \ln(a) \frac{d}{dx} \log_a(x) = 1$$

so we have

$$\frac{d}{dx} \log_a(x) = \frac{1}{x \ln(a)}$$

## Hyperbolic Functions

The tanh function is defined as

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

where

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

and

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

It is easy to verify that

$$\frac{d}{dx} \sinh(x) = \cosh(x)$$

and

$$\frac{d}{dx} \cosh(x) = \sinh(x)$$

so according to the quotient rule, we have

$$\frac{d}{dx} \tanh(x) = \frac{\frac{d}{dx} \sinh(x) \cosh(x) - \sinh(x) \frac{d}{dx} \cosh(x)}{\cosh^2(x)} = 1 - \tanh^2(x)$$