LSTM

Many readers started learning LSTM from this blog post. But for those who want to dive deeper into its math derivations, this slide is a better choice.

The LSTM Unit

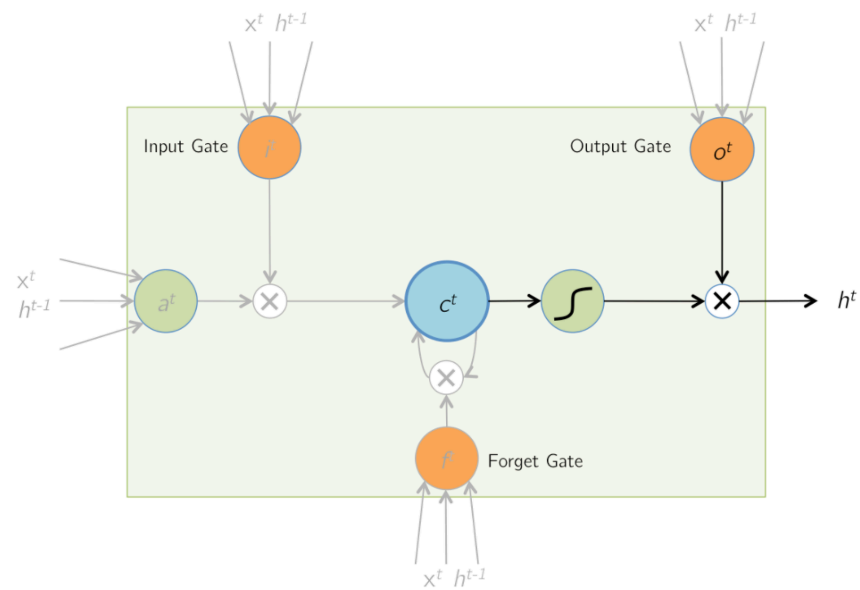

Figure 1:

Forward Pass

1. an input: $a^t = \tanh(\hat{a}^t) = \tanh(W_a x^t + U_a h^{t-1})$
2. the input gate: $i^t = \sigma(\hat{i}_t) = \sigma(W_i x^t + U_i h^{t-1})$
3. the forget gate: $f^t = \sigma(\hat{f}^t) = \sigma(W_f x^t + U_f h^{t-1})$
4. the memory cell: $c^t = a^t \odot i^t + c^{t-1} \odot f^t$
5. the output gate: $o^t = \sigma(\hat{o}^t) = \sigma(W_o x^t + U_o h^{t-1})$
6. the output: $h^t = \tanh(c^t) \odot o^t$

NOTE:

1. There is am implicit dataflow $c^{t+1} = c^t$ between LSTM units. This implies that c^t should accept gradients from *not only* h^t, *but also* h^{t+1}.

2. All gates uses the sigmoid function as their activations. This is because the gates' output must be in the range $[0, 1]$ so could they be used with $\odot$ as the gate.
3. The non-linearity of an LSTM cell comes from the tanh activations of a^t and h_t. Compared with sigmoid, whose gradient closes to zero when the input is very negative and stucks the SGD process, tanh doesn't have this problem.

Backward Pass

The backward pass updates W_o, U_o, W_f, U_f, W_i, and U_i.

Denoting the error of h^t by E, the derivation of the backpropagation algorithm tells that we will have $\frac{\partial E}{\partial h^t}$ form the cost layer.

$$\frac{\partial E}{\partial o^t} = \frac{\partial E}{\partial h^t} \cdot \frac{\partial h^t}{\partial o^t} = \frac{\partial E}{\partial h^t} \odot \tanh(c^t)$$

$$\frac{\partial E}{\partial c^t} = \frac{\partial E}{\partial h^t} \frac{\partial h^t}{\partial c^t} = \frac{\partial E}{\partial h^t} \odot o^t \odot \left[1 - \tanh^2(c^t)\right]$$

Due to the implicit dataflow, we introduce a varaible δc^t to accumulate gradients from h^T:

$$\delta c^t += \frac{\partial E}{\partial c^t}$$

Given δc^t, we can find

$$\begin{aligned}
\frac{\partial E}{\partial i^t} &= \delta c^t \odot a^t \\
\frac{\partial E}{\partial a^t} &= \delta c^t \odot i^t \\
\frac{\partial E}{\partial f^t} &= \delta c^t \odot c^{t-1} \\
\frac{\partial E}{\partial c^{t-1}} &= \delta c^t \odot f^t
\end{aligned}$$

NOTE: that *the last equation sets the initial value of δc^{t-1}, which will be updated later by $\frac{\partial E}{\partial c^{t-1}}$.*

$$\frac{\partial E}{\partial \hat{a}^t} = \frac{\partial E}{\partial a^t} \odot \left[1 - \tanh(\hat{a}^t)\right]$$

$$\frac{\partial E}{\partial \hat{i}^t} = \frac{\partial E}{\partial i^t} \odot \hat{a}^t \odot (1 - \hat{a}^t)$$

$$\frac{\partial E}{\partial \hat{f}^t} = \frac{\partial E}{\partial f^t} \odot \hat{f}^t \odot (1 - \hat{f}^t)$$

$$\frac{\partial E}{\partial \hat{o}^t} = \frac{\partial E}{\partial o^t} \odot \hat{o}^t \odot (1 - \hat{o}^t)$$

Because

$$\hat{I} = \Theta I$$

where

$$\hat{I}^t = [\hat{a}^t\ \hat{i}^t\ \hat{f}^t\ \hat{o}^t]^T$$
$$\Theta = [W\ U]$$
$$I^t = [x^t\ h^{t-1}]^T$$

we have

$$\delta\Theta^t = \frac{\partial E}{\partial \Theta} \mid_t = \frac{\partial E}{\partial \hat{I}^t} \cdot \frac{\partial \hat{I}^t}{\partial \Theta} \mid_t = \left[\frac{\partial E}{\partial \hat{a}^t}\ \frac{\partial E}{\partial \hat{i}^t}\ \frac{\partial E}{\partial \hat{f}^t}\ \frac{\partial E}{\partial \hat{o}^t} \right] \times I^t$$

According to the multivariate chain rule

$$\frac{\partial E}{\partial \Theta} = \sum_{t=1}^{T} \delta\Theta^t$$