Structural Dynamics

Kun Wang

2019.10.05-2020.01.31

Chapter 1

Time-Domain Analysis of Continuous Systems

1.1 Basis of partial differential equations

常系数二阶偏微分方程

$$a\frac{\partial^2 f}{\partial x^2}+b\frac{\partial^2 f}{\partial x\partial y}+c\frac{\partial^2 f}{\partial y^2}=0 \tag{1.1}$$

若 $b^2-4ac>0$ 则为双曲线型 (hyperbolic) 方程;
若 $b^2-4ac=0$ 则为抛物线型方程;
若 $b^2-4ac<0$ 则为椭圆型方程。

连续体振动问题中的波动方程为

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{1.2}$$

其中 c 为波在物质中的传播速度。使用分离变量法可求得该微分方程的解。

令 $u(x,t)=X(x)T(t)$, 带入 (1.2) 得

$$c^2T(t)\frac{\partial^2 X}{\partial x^2}-X(x)\frac{\partial^2 T}{\partial t^2}=0$$

亦可写为

$$\frac{1}{X(x)}\frac{\partial^2 X}{\partial x^2}=\frac{1}{c^2T(t)}\frac{\partial^2 T}{\partial t^2}$$

由于等式两边变量不同，等式要成立，则等式两边必须均等于一常数，设为 $-\lambda$, 则可得

$$\frac{\partial^2 X}{\partial x^2}+\lambda X(x)=0 \tag{1.3}$$

$$\frac{\partial^2 T}{\partial t^2}+c^2\lambda T(x)=0 \tag{1.4}$$

表 1.1: general solution of 2-order ode

λ	$X(x)$
$\lambda < 0$	$X(x) = A_1 e^{\sqrt{-\lambda}x} + A_2 e^{\sqrt{\lambda}x}$
$\lambda = 0$	$X(x) = (A_1 + A_2 x)$
$\lambda > 0$	$X(x) = A_1 \sin\left(\sqrt{\lambda}x\right) + A_2 \cos\left(\sqrt{\lambda}x\right)$

这是两个常系数二阶常微分方程，根据常微分方程理论，式 (1.3) 的通解为 对于弦振动问题有约束条件 $u(0,t)=0, u(l,t)=0$。当 $\lambda \leq 0$ 时，方程 (1.2) 无非平凡解 (不恒等于零的解)。故 λ 只能大于 0，不妨设 $\lambda = (\omega/c)^2$。则对于弦振动问题，由方程 (1.3) 及边界条件得

$$\sin\left(\frac{\omega l}{c}\right) = 0 \tag{1.5}$$

可得自然频率

$$\omega_n = \frac{n\pi c}{l},\ n = 1, 2, \cdots \tag{1.6}$$

方程 (1.3) 的解为

$$X_n\left(x\right) = A_n \sin\frac{n\pi x}{l},\ n = 1, 2, \cdots \tag{1.7}$$

根据式 (1.4)(1.6) 可得

$$T_n\left(t\right) = B_n \sin\left(\frac{n\pi c}{l}t + \varphi_n\right),\ n = 1, 2, \cdots \tag{1.8}$$

波动方程的解为

$$u\left(x,t\right) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} \sin\left(\frac{n\pi c}{l}t + \varphi_n\right),\ n = 1, 2, \cdots \tag{1.9}$$

声波由多种单音振动组合而成，连续体的振动也是多种振动的合成，每种振动的波长为 $\frac{2l}{n}$, 振动周期为 $\frac{2l}{nc}$, 则波速即为 c。

1.2 一维弹性波动方程

1.2.1 弦的横向自由振动

assumptions: The transverse deflection is very small so that the length l of the string and the tension T are constant. The small deflection does not mean the small transverse motion but it meane the small deflection angle is small so $\frac{\partial^2 v}{\partial x^2}$ can be negelected.

假设弦的横向偏转很小，弦上取微元, 则其转角 θ 可近似为 $\theta = \sin\theta = \tan\theta$, 在横向方向上，根据 d'Alembert 原理可得

$$T\left(\theta + \frac{\partial\theta}{\partial x}\mathrm{d}x\right) - T\theta = \rho\mathrm{d}x\frac{\partial^2 v}{\partial t^2} \tag{1.10}$$

$\theta = \tan\theta = \frac{\partial v}{\partial x}$, 进一步得

$$\frac{\partial^2 v}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 v}{\partial t^2} \tag{1.11}$$

记 $c = \sqrt{T/\rho}$, 即得式

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} \tag{1.12}$$

弦两端加上固定约束，即 $v(0,t) = 0, v(l,t) = 0$。方程 (1.12) 是可使用分离变量法求解。根据 1.1节中理论，并设 $\lambda = \left(\frac{\omega}{c}\right)^2$(对于该问题 $\lambda \leq 0$ 无非平凡解)，可得

$$\begin{cases} X(x) = A\cos\frac{\omega}{c}x + B\sin\frac{\omega}{c}x \\ T(t) = C\cos\omega t + D\sin\omega t \end{cases} \tag{1.13}$$

将边界条件带入，得

$$\begin{cases} X(0)T(t) = AT(t) = 0 \\ X(l)T(t) = \left(A\cos\frac{\omega}{c}l + B\sin\frac{\omega}{c}l\right)T(t) = 0 \end{cases} \tag{1.14}$$

由于时间的任意性，要求得非平凡解 (non-trival solution)，必有

$$\sin\frac{\omega l}{c} = 0 \tag{1.15}$$

即

$$\frac{\omega l}{c} = n\pi \tag{1.16}$$

可得固有频率 (natrual frequency)

$$\omega_n = \frac{n\pi c}{l} \tag{1.17}$$

相应的模态振型 (mode shape) 为

$$\phi_n(x) = A_n\sin\frac{\omega_n}{c}x = A_n\sin\frac{n\pi x}{l} \tag{1.18}$$

弦振动的解为所有解的线性组合，即

$$\begin{aligned}v(x,t) &= \sum_{n=1}^{\infty}\phi_n(x)T_n(t) = \sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi c}{l}t + D\sin\frac{n\pi c}{l}t\right)\\ &= \sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{l}\sin\left(\frac{n\pi c}{l}t+\alpha_n\right)\end{aligned} \tag{1.19}$$

该解与琴弦振动发出声音由许多单音组合起来类似，每个频率为系统的固有频率，对应的振型为固有振型。

弦上点的振动会带动邻近点的振动，这种振动的传播即为波动。弦上每一点的振动频率均相同，所以波传播频率等于振动频率。弦内弹性波的波长为 $\lambda_n = \frac{2l}{n}$，波传播速度为 $\frac{\lambda_n\omega_n}{2\pi} = c$，故 c 为波传播速度。

1.2.2 弹性杆的轴向自由振动

取弹性杆内微元，截面处轴向内力为 P，其惯性力为 $\rho A\mathrm{d}x\frac{\partial^2 u}{\partial x^2}$, 根据 d'Alembert 原理可得

$$P + \frac{\partial P}{\partial x}\mathrm{d}x - P = \rho A\mathrm{d}x\frac{\partial^2 u}{\partial x^2} \tag{1.20}$$

根据应力应变关系 $\sigma_x = E\epsilon_x$，又 $P = \sigma_x A$, 有 $P = EA\epsilon_x = EA\frac{\partial u}{\partial x}$, 上式化为

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{E}\frac{\partial^2 u}{\partial x^2} \tag{1.21}$$

该式与式 (1.12) 相同，波在弹性杆中传播速度为 $c = \sqrt{E/\rho}$。式 (1.21) 可用分离变量法求解。

在弹性杆两端施加不同的约束，相同的偏微分方程可得到不同的结果。考虑如下三种边界条件

(1) 两端固定 $u(0,t) = 0, u(l,t) = 0$；

(2) 一端固定，一端自由，自由端无外力，即无应变，$u(0,t) = 0, \frac{\partial u}{\partial x}|(l,t) = 0$；

(3) 两端自由 $\frac{\partial u}{\partial x}|(0,t) = 0, \frac{\partial u}{\partial x}|(l,t) = 0$.

类似弦振动问题，微分方程的通解为

$$u(x,t) = \left(A\cos\frac{\omega}{c}x + B\sin\frac{\omega}{c}x\right)\sin(\omega t+\alpha) \tag{1.22}$$

将**第一种边界条件**带入得

固有频率

$$\omega_n = \frac{n\pi c}{l},$$

对应模态振型为

$$\phi_n(x) = A_n \sin \frac{n\pi x}{l},$$

方程解为

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \sin\left(\frac{n\pi c}{l} t + \alpha_n\right)$$

将**第二种边界条件**带入得

固有频率

$$\omega_n = \frac{(2n-1)\pi c}{2l},$$

对应模态振型为

$$\phi_n(x) = A_n \sin \frac{(2n-1)\pi x}{2l},$$

方程解为

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{(2n-1)\pi x}{2l} \sin\left(\frac{(2n-1)\pi c}{2l} t + \alpha_n\right)$$

将**第三种边界条件**带入得

固有频率

$$\omega_n = \frac{n\pi c}{l},$$

对应模态振型为

$$\phi_n(x) = A_n \cos \frac{n\pi x}{l},$$

方程解为

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos \frac{n\pi x}{l} \sin\left(\frac{n\pi c}{l} t + \alpha_n\right)$$

从以上结果看出，固有频率和固有振型和结构参数和边界条件有关，结构得振动响应不光和结构参数和边界条件有关，还和结构初始状态 (初值) 有关。

1.2.3 圆柱杆的自由扭转振动

假设圆柱杆截面没有翘曲，弹性杆截面扭矩为

$$T = GI_p \frac{\partial \theta}{\partial x} \tag{1.23}$$

其中 G 为剪切刚度，I_p 为截面扭转惯量。取微元，有

$$\left(T+\frac{\partial T}{\partial x}\mathrm{d}x\right)-T=\rho I_p\mathrm{d}x\frac{\partial^2\theta}{\partial t^2} \tag{1.24}$$

将式 (1.23) 带入式 (1.24) 中得

$$\frac{\partial^2\theta}{\partial x^2}=\frac{\rho}{G}\frac{\partial^2\theta}{\partial t^2} \tag{1.25}$$

则波动传播速度为 $c=\sqrt{G/\rho}$。式 (1.25) 与式 (1.21) 相同，其解也相同。

1.3 Euler-Bernoulli 梁的横向自由振动

Assumptions:

(1) The cross-section is infinitely rigid in its own plane.

(2) The cross-section of a beam remains plane after deformation.

(3) The cross-section remains normal to the deformed neutral axis of the beam.

Equilibrium equation of the deflection direciton

$$-Q-\frac{\partial Q}{\partial x}\mathrm{d}x+Q-\rho A(x)\mathrm{d}x\frac{\partial^2 v}{\partial t^2}=0 \tag{1.26}$$

Constitution law

$$M=EI(x)\frac{\partial^2 v}{\partial x^2} \tag{1.27}$$

Equilibrium equation of moment

$$Q\mathrm{d}x+M-\left(M+\frac{\partial M}{\partial x}\right)=0 \tag{1.28}$$

The following relationship between the shear force and the moment can be derived from Eq. (1.28)

$$Q=\frac{\partial M}{\partial x} \tag{1.29}$$

Substituting Eq. (1.27) and Eq. (1.29) into Eq. (1.26), the equilibrium equation of the differential element with respect to the deflection v becomes

$$EI\frac{\partial^4 v}{\partial x^4}+\rho A\frac{\partial^2 v}{\partial t^2}=0 \tag{1.30}$$

Eq. (1.30) is a separable linear fourth-order partial differential equation. The form of the solution would be

$$v(x,t) = X(x)\sin(\omega t + \alpha) \tag{1.31}$$

Substituting Eq.(1.31) into Eq.(1.30) can obtain the following ordinary differential equation

$$\frac{\mathrm{d}^4 X}{\mathrm{d}x^4} - \frac{\rho A \omega^2}{EI} X(x) = 0 \tag{1.32}$$

The characteristic equation of Eq. (1.32) is

$$\bar{\lambda}^4 - \frac{\rho A \omega^2}{EI} = 0 \tag{1.33}$$

Roots for the characteristic equation are $\pm\lambda, \pm\lambda i$, with

$$\lambda = \left(\frac{\rho A \omega^2}{EI}\right)^{\frac{1}{4}} \tag{1.34}$$

The generalized form of solutions of Eq. (1.32) is

$$X(x) = A_1 e^{\lambda x} + A_2 e^{-\lambda x} + A_3 e^{\lambda x i} + A_4 e^{-\lambda x i} \tag{1.35}$$

Based on the Euler formula, Eq. (1.35) can be expressed by the linear combination of the trigonometric and hyperbolic functioons as

$$X(x) = B_1 \sin\lambda x + B_2 \cos\lambda x + B_3 \sinh\lambda x + B_4 \cosh\lambda x \tag{1.36}$$

考虑三类边界条件

(1) 简支，位移和弯矩为 0，$v(x,t) = 0$, $\frac{\partial^2 v}{\partial x^2}|_{(x,t)} = 0$；

(2) 固定，位移和转角为 0，$v(x,t) = 0$, $\frac{\partial v}{\partial x}|_{(x,t)} = 0$；

(3) 自由，剪力和弯矩为 0，$\frac{\partial^3 v}{\partial x^3}|_{(x,t)} = 0$, $\frac{\partial^2 v}{\partial x^2}|_{(x,t)} = 0$.

不同的边界条件组合，研究如下四种情形梁的自由振动。

1.3.1 Simple supported beams

boundary conditions: $v(0,t) = v(l,t) = 0$, $\frac{\partial^2 v}{\partial x^2}|_{(0,t)} = \frac{\partial^2 v}{\partial x^2}|_{(l,t)} = 0$.

将边界条件带入式 (1.36) 得

$$\begin{cases} B_2 + B_4 = 0 \\ -\lambda^2 B_2 + \lambda^2 B_4 = 0 \\ B_1 \sin\lambda l + B_3 \sinh\lambda l = 0 \\ -\lambda^2 B_1 \sin\lambda l + \lambda^2 B_3 \sinh\lambda l = 0 \end{cases} \tag{1.37}$$

根据该方程组要想获得非平凡解，要求其系数矩阵行列式为 0，可得

$$\sin \lambda l = 0 \tag{1.38}$$

即要求

$$\lambda_n = \frac{n\pi}{l} \tag{1.39}$$

结合式 (1.34) 可得固有频率

$$\omega_n = \lambda_n^2 \sqrt{\frac{EI}{\rho A}} \tag{1.40}$$

解得 $B_2 = B_3 = B_4 = 0$，模态振型为

$$\phi_n(x) = A_n \sin \frac{n\pi x}{l} \tag{1.41}$$

1.3.2 Cantilever beams

boundary conditions: $v(0,t) = \frac{\partial v}{\partial x}|_{(0,t)} = 0$, $\frac{\partial^3 v}{\partial x^3}|_{(l,t)} = \frac{\partial^2 v}{\partial x^2}|_{(l,t)} = 0$.

将边界条件带入式 (1.36) 得

$$\begin{cases} B_2 + B_4 = 0 \\ \lambda B_1 + \lambda B_3 = 0 \\ -\lambda^2 B_1 \sin \lambda l - \lambda^2 B_2 \cos \lambda l + \lambda^2 B_3 \sinh \lambda l + \lambda^2 B_4 \cosh \lambda l = 0 \\ -\lambda^3 B_1 \cos \lambda l + \lambda^3 B_2 \sin \lambda l + \lambda^3 B_3 \cosh \lambda l + \lambda^3 B_4 \sinh \lambda l = 0 \end{cases} \tag{1.42}$$

系数矩阵行列式为 0，可得

$$\cos \lambda l \cosh \lambda l + 1 = 0 \tag{1.43}$$

该方程为超越方程，需借助于数值算法求得数值解，近似解为 $\lambda_1 l \approx 1.875$, $\lambda_2 l \approx 4.694$, $\lambda_3 l \approx 7.855$, $\lambda_n l \approx \left(n - \frac{1}{2}\right)$ when $n \geq 4$. 模态振型为

$$\phi_n(x) = A_n \left[\cosh \lambda_n x - \cos \lambda_n x - \frac{\cos \lambda_n l + \cosh \lambda_n l}{\sin \lambda_n l + \sinh \lambda_n l} \left(\sinh \lambda_n x - \sin \lambda_n x\right)\right] \tag{1.44}$$

1.3.3 Fixed-Fixed beams

boundary conditions: $v(0,t) = v(l,t) = 0$, $\frac{\partial v}{\partial x}|_{(0,t)} = \frac{\partial v}{\partial x}|_{(l,t)} = 0$.
将边界条件带入式 (1.36) 得

$$\begin{cases} B_2 + B_4 = 0 \\ \lambda B_1 + \lambda B_3 = 0 \\ B_1 \sin \lambda l + B_2 \cos \lambda l + B_3 \sinh \lambda l + B_4 \cosh \lambda l = 0 \\ B_1 \cos \lambda l - B_2 \sin \lambda l + B_3 \cosh \lambda l + B_4 \sinh \lambda l = 0 \end{cases} \tag{1.45}$$

系数矩阵行列式为 0，可得

$$\cos\lambda l\cosh\lambda l-1=0 \tag{1.46}$$

该方程为超越方程，需借助于数值算法求得数值解，近似解为 $\lambda_1 l\approx 4.730$, $\lambda_2 l\approx 7.853$, $\lambda_n l\approx \left(n+\frac{1}{2}\right)$ when $n\geq 3$. 模态振型为

$$\phi_n(x)=A_n\left[\cos\lambda_n x-\cosh\lambda_n x-\frac{\cos\lambda_n l-\cosh\lambda_n l}{\sin\lambda_n l-\sinh\lambda_n l}\left(\sinh\lambda_n x\sin\lambda_n x\right)\right] \tag{1.47}$$

1.3.4 Free-free beams

boundary conditions: $\frac{\partial^3 v}{\partial x^3}|_{(0,t)}=\frac{\partial^2 v}{\partial x^2}|_{(0,t)}=0$, $\frac{\partial^3 v}{\partial x^3}|_{(l,t)}=\frac{\partial^2 v}{\partial x^2}|_{(l,t)}=0$.

将边界条件带入式 (1.36) 得

$$\begin{cases}-\lambda^2B_2+\lambda^2B_4=0\\ -\lambda^3B_1+\lambda^3B_3=0\\ -\lambda^2B_1\sin\lambda l-\lambda^2B_2\cos\lambda l+\lambda^2B_3\sinh\lambda l+\lambda^2B_4\cosh\lambda l=0\\ -\lambda^3B_1\cos\lambda l+\lambda^3B_2\sin\lambda l+\lambda^3B_3\cosh\lambda l+\lambda^3B_4\sinh\lambda l=0\end{cases} \tag{1.48}$$

系数矩阵行列式为 0，可得

$$\cos\lambda l\cosh\lambda l-1=0 \tag{1.49}$$

该方程与式 (1.46) 相同，其解也就相同。模态振型为

$$\phi_n(x)=A_n\left[\cos\lambda_n x+\cosh\lambda_n x-\frac{\cos\lambda_n l-\cosh\lambda_n l}{\sin\lambda_n l-\sinh\lambda_n l}\left(\sin\lambda_n x+\sinh\lambda_n x\right)\right] \tag{1.50}$$

1.4 方形薄板的自由振动

1.5 模态振型的特性

1.5.1 Orthogonality

$$\int_0^l \rho A\phi_r\phi_s\mathrm{d}x=0,\ r\neq s. \tag{1.51}$$

1.5.2 Scaling

modal mass

$$M_r=\int_0^l \rho A\bar{\phi}_r^2\mathrm{d}x=1, \tag{1.52}$$

where $\bar{\phi}_r = c\phi$ is the normalized mode shape. c is the scaling factor and

$$c = \sqrt{\frac{1}{\int_0^l \rho A \phi_r^2 \mathrm{d}x}} \tag{1.53}$$

modal stiffness

$$K_r = \int_0^l EI \left(\frac{\mathrm{d}^2\phi_r}{\mathrm{d}x^2}\right)^2 \mathrm{d}x \tag{1.54}$$

$$\omega_n = \frac{K_r}{M_r} \tag{1.55}$$

1.5.3 Expasion theorm

在前几节已经得到张紧弦，Euler-Bernoulli 梁，方形薄板自由振动的模态振型。对于任意函数 $V(x)$，如果其满足如下条件

(1) 与一模态振型函数集满足相同的边界条件；

(2) $EI\left(\mathrm{d}^4V/\mathrm{d}x^4\right)$ 是连续函数。

则 $V(x)$ 可表示为收敛级数的形式。

$$V(x) = \sum_{r=1}^{\infty} q_r \phi_r \tag{1.56}$$

其中

$$q_r = \frac{\int_0^l \rho A V(x) \phi_r \mathrm{d}x}{\int_0^l \rho A \phi_r^2 \mathrm{d}x} \tag{1.57}$$

亦即模态坐标。

1.5.4 Rayleigh quotient

定义 Rayleigh quotient:

$$R(V) = \frac{\int_0^l EI \left(\frac{\mathrm{d}^2V}{\mathrm{d}x^2}\right)^2 \mathrm{d}x}{\int_0^l \rho A V^2 \mathrm{d}x} \tag{1.58}$$

根据模态振型的正交性，模态刚度的正交性，以及式 (1.55)(1.56)，可得

$$
\begin{aligned}
R(V) &= \frac{\sum_{r=1}^{\infty} q_r^2 \omega_r^2}{\sum_{r=1}^{\infty} q_r^2} \\
&= \omega_1^2 \frac{1 + \sum_{r=2}^{\infty} \left(\frac{q_r}{q_1}\right)^2 \left(\frac{\omega_r}{\omega_1}\right)^2}{1 + \sum_{r=2}^{\infty} \left(\frac{q_r}{q_1}\right)^2}
\end{aligned}
\tag{1.59}
$$

由于基频 ω_1 最小，$\omega_{n+1} > \omega_n$，故

$$
R(V) \geq \omega_1^2 \tag{1.60}
$$

式 (1.60) 表示 Rayleigh 商是基频的上界，当 $q_r = 0, r > 1$ 时取等号，此时 $V(x)$ 即为基频模态振型。如果可以选取适当的函数 $V(x)$ 使其与基频模态振型接近，那么就可以近似估计基频的值。

Chapter 2

Time-domain analysis of SDOF systems

Chapter 3

Time-domain analysis of MDOF systems

Chapter 4

Frequency-domain analysis