Structural Dynamics

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Chapter 1

Time-Domain Analysis of Continuous Systems

1.1 Basis of partial differential equations

常系数二阶偏微分方程

$$a\frac{\partial^2 f}{\partial x^2}+b\frac{\partial^2 f}{\partial x\partial y}+c\frac{\partial^2 f}{\partial y^2}=0 \tag{1.1}$$

若 $b^2-4ac>0$ 则为双曲线型 (hyperbolic) 方程;
若 $b^2-4ac=0$ 则为抛物线型方程;
若 $b^2-4ac<0$ 则为椭圆型方程。

连续体振动问题中的波动方程为

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{1.2}$$

其中 c 为波在物质中的传播速度。使用分离变量法可求得该微分方程的解。

令 $u\left(x,t\right)=X\left(x\right)T\left(t\right)$, 带入 (1.2) 得

$$c^2T(t)\frac{\partial^2 X}{\partial x^2}-X(x)\frac{\partial^2 T}{\partial t^2}=0$$

亦可写为

$$\frac{1}{X(x)}\frac{\partial^2 X}{\partial x^2}=\frac{1}{c^2T(t)}\frac{\partial^2 T}{\partial t^2}$$

由于等式两边变量不同，等式要成立，则等式两边必须均等于一常数，设为 $-\lambda$, 则可得

$$\frac{\partial^2 X}{\partial x^2}+\lambda X\left(x\right)=0 \tag{1.3}$$

$$\frac{\partial^2 T}{\partial t^2}+c^2\lambda T\left(x\right)=0 \tag{1.4}$$

表 1.1: general solution of 2-order ode

λ	$X(x)$
$\lambda < 0$	$X(x) = A_1 e^{\sqrt{-\lambda}x} + A_2 e^{\sqrt{\lambda}x}$
$\lambda = 0$	$X(x) = (A_1 + A_2 x)$
$\lambda > 0$	$X(x) = A_1 \sin\left(\sqrt{\lambda}x\right) + A_2 \cos\left(\sqrt{\lambda}x\right)$

这是两个常系数二阶常微分方程，根据常微分方程理论，式 (1.3) 的通解为 对于弦振动问题有约束条件 $u(0,t)=0, u(l,t)=0$。当 $\lambda \le 0$ 时，方程 (1.2) 无非平凡解 (不恒等于零的解)。故 λ 只能大于 0，不妨设 $\lambda = (\omega/c)^2$。则对于弦振动问题，由方程 (1.3) 及边界条件得

$$\sin\left(\frac{\omega l}{c}\right) = 0 \tag{1.5}$$

可得自然频率

$$\omega_n = \frac{n\pi c}{l},\ n = 1, 2, \cdots \tag{1.6}$$

方程 (1.3) 的解为

$$X_n(x) = A_n \sin\frac{n\pi x}{l},\ n = 1, 2, \cdots \tag{1.7}$$

根据式 (1.4)(1.6) 可得

$$T_n(t) = B_n \sin\left(\frac{n\pi c}{l}t + \varphi_n\right),\ n = 1, 2, \cdots \tag{1.8}$$

波动方程的解为

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} \sin\left(\frac{n\pi c}{l}t + \varphi_n\right),\ n = 1, 2, \cdots \tag{1.9}$$

声波由多种单音振动组合而成，连续体的振动也是多种振动的合成，每种振动的波长为 $\frac{2l}{n}$, 振动周期为 $\frac{2l}{nc}$, 则波速即为 c。

1.2 一维弹性波动方程

1.2.1 弦的横向自由振动

assumptions: The transverse deflection is very small so that the length l of the string and the tension T are constant. The small deflection does not mean the small transverse motion but it meane the small deflection angle is small so $\frac{\partial^2 v}{\partial x^2}$ can be negelected.

假设弦的横向偏转很小，弦上取微元, 则其转角 θ 可近似为 $\theta = \sin\theta = \tan\theta$, 在横向方向上，根据 d'Alembert 原理可得

$$T\left(\theta + \frac{\partial\theta}{\partial x}\mathrm{d}x\right) - T\theta = \rho\mathrm{d}x\frac{\partial^2 v}{\partial t^2} \tag{1.10}$$

$\theta = \tan\theta = \frac{\partial v}{\partial x}$, 进一步得

$$\frac{\partial^2 v}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 v}{\partial t^2} \tag{1.11}$$

记 $c = \sqrt{T/\rho}$, 即得式

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} \tag{1.12}$$

弦两端加上固定约束，即 $v(0,t) = 0, v(l,t) = 0$。方程 (1.12) 是可使用分离变量法求解。根据 1.1节中理论，并设 $\lambda = \left(\frac{\omega}{c}\right)^2$(对于该问题 $\lambda \leq 0$ 无非平凡解)，可得

$$\begin{cases} X(x) = A\cos\frac{\omega}{c}x + B\sin\frac{\omega}{c}x \\ T(t) = C\cos\omega t + D\sin\omega t \end{cases} \tag{1.13}$$

将边界条件带入，得

$$\begin{cases} X(0)T(t) = AT(t) = 0 \\ X(l)T(t) = \left(A\cos\frac{\omega}{c}l + B\sin\frac{\omega}{c}l\right)T(t) = 0 \end{cases} \tag{1.14}$$

由于时间的任意性，要求得非平凡解 (non-trival solution)，必有

$$\sin\frac{\omega l}{c} = 0 \tag{1.15}$$

即

$$\frac{\omega l}{c} = n\pi \tag{1.16}$$

可得固有频率 (natrual frequency)

$$\omega_n = \frac{n\pi c}{l} \tag{1.17}$$

相应的模态振型 (mode shape) 为

$$\phi_n(x) = A_n\sin\frac{\omega_n}{c}x = A_n\sin\frac{n\pi x}{l} \tag{1.18}$$

弦振动的解为所有解的线性组合，即

$$
\begin{aligned}
v(x,t) &= \sum_{n=1}^{\infty} \phi_n(x) T_n(t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \left(C \cos \frac{n\pi c}{l} t + D \sin \frac{n\pi c}{l} t \right) \\
&= \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \sin \left(\frac{n\pi c}{l} t + \alpha_n \right)
\end{aligned}
\tag{1.19}
$$

该解与琴弦振动发出声音由许多单音组合起来类似，每个频率为系统的固有频率，对应的振型为固有振型。

弦上点的振动会带动邻近点的振动，这种振动的传播即为波动。弦上每一点的振动频率均相同，所以波传播频率等于振动频率。弦内弹性波的波长为 $\lambda_n = \frac{2l}{n}$，波传播速度为 $\frac{\lambda_n \omega_n}{2\pi} = c$，故 c 为波传播速度。

1.2.2 弹性杆的轴向自由振动

取弹性杆内微元，截面处轴向内力为 P，其惯性力为 $\rho A \mathrm{d}x \frac{\partial^2 u}{\partial x^2}$, 根据 d'Alembert 原理可得

$$
P + \frac{\partial P}{\partial x} \mathrm{d}x - P = \rho A \mathrm{d}x \frac{\partial^2 u}{\partial x^2} \tag{1.20}
$$

根据应力应变关系 $\sigma_x = E\epsilon_x$，又 $P = \sigma_x A$, 有 $P = EA\epsilon_x = EA\frac{\partial u}{\partial x}$, 上式化为

$$
\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{E} \frac{\partial^2 u}{\partial x^2} \tag{1.21}
$$

该式与式 (1.12) 相同，波在弹性杆中传播速度为 $c = \sqrt{E/\rho}$。式 (1.21) 可用分离变量法求解。

在弹性杆两端施加不同的约束，相同的偏微分方程可得到不同的结果。考虑如下三种边界条件

(1) 两端固定 $u(0,t) = 0, u(l,t) = 0$；

(2) 一端固定，一端自由，自由端无外力，即无应变，$u(0,t) = 0, \frac{\partial u}{\partial x}|(l,t) = 0$；

(3) 两端自由 $\frac{\partial u}{\partial x}|(0,t) = 0, \frac{\partial u}{\partial x}|(l,t) = 0$.

类似弦振动问题，微分方程的通解为

$$
u(x,t) = \left(A \cos \frac{\omega}{c} x + B \sin \frac{\omega}{c} x \right) \sin(\omega t + \alpha) \tag{1.22}
$$

将**第一种边界条件**带入得

固有频率

$$
\omega_n = \frac{n\pi c}{l},
$$

对应模态振型为

$$\phi_n(x) = A_n \sin \frac{n\pi x}{l},$$

方程解为

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \sin\left(\frac{n\pi c}{l} t + \alpha_n\right)$$

将**第二种边界条件**带入得

固有频率

$$\omega_n = \frac{(2n-1)\pi c}{2l},$$

对应模态振型为

$$\phi_n(x) = A_n \sin \frac{(2n-1)\pi x}{2l},$$

方程解为

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{(2n-1)\pi x}{2l} \sin\left(\frac{(2n-1)\pi c}{2l} t + \alpha_n\right)$$

将**第三种边界条件**带入得

固有频率

$$\omega_n = \frac{n\pi c}{l},$$

对应模态振型为

$$\phi_n(x) = A_n \cos \frac{n\pi x}{l},$$

方程解为

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos \frac{n\pi x}{l} \sin\left(\frac{n\pi c}{l} t + \alpha_n\right)$$

从以上结果看出，固有频率和固有振型和结构参数和边界条件有关，结构得振动响应不光和结构参数和边界条件有关，还和结构初始状态 (初值) 有关。

1.2.3 圆柱杆的自由扭转振动

假设圆柱杆截面没有翘曲，弹性杆截面扭矩为

$$T = GI_p \frac{\partial \theta}{\partial x} \tag{1.23}$$

其中 G 为剪切刚度，I_p 为截面扭转惯量。取微元，有

$$\left(T+\frac{\partial T}{\partial x}\mathrm{d}x\right)-T=\rho I_p\mathrm{d}x\frac{\partial^2\theta}{\partial t^2} \tag{1.24}$$

将式 (1.23) 带入式 (1.24) 中得

$$\frac{\partial^2\theta}{\partial x^2}=\frac{\rho}{G}\frac{\partial^2\theta}{\partial t^2} \tag{1.25}$$

则波动传播速度为 $c=\sqrt{G/\rho}$。式 (1.25) 与式 (1.21) 相同，其解也相同。

1.3 Euler-Bernoulli 梁的横向自由振动

Assumptions:

(1) The cross-section is infinitely rigid in its own plane.

(2) The cross-section of a beam remains plane after deformation.

(3) The cross-section remains normal to the deformed neutral axis of the beam.

Equilibrium equation of the deflection direciton

$$-Q-\frac{\partial Q}{\partial x}\mathrm{d}x+Q-\rho A(x)\mathrm{d}x\frac{\partial^2 v}{\partial t^2}=0 \tag{1.26}$$

Constitution law

$$M=EI(x)\frac{\partial^2 v}{\partial x^2} \tag{1.27}$$

Equilibrium equation of moment

$$Q\mathrm{d}x+M-\left(M+\frac{\partial M}{\partial x}\right)=0 \tag{1.28}$$

The following relationship between the shear force and the moment can be derived from Eq. (1.28)

$$Q=\frac{\partial M}{\partial x} \tag{1.29}$$

Substituting Eq. (1.27) and Eq. (1.29) into Eq. (1.26), the equilibrium equation of the differential element with respect to the deflection v becomes

$$EI\frac{\partial^4 v}{\partial x^4}+\rho A\frac{\partial^2 v}{\partial t^2}=0 \tag{1.30}$$

Eq. (1.30) is a separable linear fourth-order partial differential equation. The form of the solution would be

$$v(x,t) = X(x)\sin(\omega t + \alpha) \tag{1.31}$$

Substituting Eq.(1.31) into Eq.(1.30) can obtain the following ordinary differential equation

$$\frac{\mathrm{d}^4 X}{\mathrm{d}x^4} - \frac{\rho A \omega^2}{EI} X(x) = 0 \tag{1.32}$$

The characteristic equation of Eq. (1.32) is

$$\bar{\lambda}^4 - \frac{\rho A \omega^2}{EI} = 0 \tag{1.33}$$

Roots for the characteristic equation are $\pm\lambda, \pm\lambda i$, with

$$\lambda = \left(\frac{\rho A \omega^2}{EI}\right)^{\frac{1}{4}} \tag{1.34}$$

The generalized form of solutions of Eq. (1.32) is

$$X(x) = A_1 e^{\lambda x} + A_2 e^{-\lambda x} + A_3 e^{\lambda x i} + A_4 e^{-\lambda x i} \tag{1.35}$$

Based on the Euler formula, Eq. (1.35) can be expressed by the linear combination of the trigonometric and hyperbolic functioons as

$$X(x) = B_1 \sin\lambda x + B_2 \cos\lambda x + B_3 \sinh\lambda x + B_4 \cosh\lambda x \tag{1.36}$$

考虑三类边界条件

(1) 简支，位移和弯矩为 0，$v(x,t) = 0, \frac{\partial^2 v}{\partial x^2}|_{(x,t)} = 0$；

(2) 固定，位移和转角为 0，$v(x,t) = 0, \frac{\partial v}{\partial x}|_{(x,t)} = 0$；

(3) 自由，剪力和弯矩为 0，$\frac{\partial^3 v}{\partial x^3}|_{(x,t)} = 0, \frac{\partial^2 v}{\partial x^2}|_{(x,t)} = 0$.

不同的边界条件组合，研究如下四种情形梁的自由振动。

1.3.1 Simple supported beams

boundary conditions: $v(0,t) = v(l,t) = 0, \frac{\partial^2 v}{\partial x^2}|_{(0,t)} = \frac{\partial^2 v}{\partial x^2}|_{(l,t)} = 0$.

将边界条件带入式 (1.36) 得

$$\begin{cases} B_2 + B_4 = 0 \\ -\lambda^2 B_2 + \lambda^2 B_4 = 0 \\ B_1 \sin\lambda l + B_3 \sinh\lambda l = 0 \\ -\lambda^2 B_1 \sin\lambda l + \lambda^2 B_3 \sinh\lambda l = 0 \end{cases} \tag{1.37}$$

根据该方程组要想获得非平凡解，要求其系数矩阵行列式为 0，可得

$$\sin \lambda l = 0 \tag{1.38}$$

即要求

$$\lambda_n = \frac{n\pi}{l} \tag{1.39}$$

结合式 (1.34) 可得固有频率

$$\omega_n = \lambda_n^2 \sqrt{\frac{EI}{\rho A}} \tag{1.40}$$

解得 $B_2 = B_3 = B_4 = 0$，模态振型为

$$\phi_n(x) = A_n \sin \frac{n\pi x}{l} \tag{1.41}$$

1.3.2 Cantilever beams

boundary conditions: $v(0,t) = \frac{\partial v}{\partial x}|_{(0,t)} = 0$, $\frac{\partial^3 v}{\partial x^3}|_{(l,t)} = \frac{\partial^2 v}{\partial x^2}|_{(l,t)} = 0$.

将边界条件带入式 (1.36) 得

$$\begin{cases} B_2 + B_4 = 0 \\ \lambda B_1 + \lambda B_3 = 0 \\ -\lambda^2 B_1 \sin \lambda l - \lambda^2 B_2 \cos \lambda l + \lambda^2 B_3 \sinh \lambda l + \lambda^2 B_4 \cosh \lambda l = 0 \\ -\lambda^3 B_1 \cos \lambda l + \lambda^3 B_2 \sin \lambda l + \lambda^3 B_3 \cosh \lambda l + \lambda^3 B_4 \sinh \lambda l = 0 \end{cases} \tag{1.42}$$

系数矩阵行列式为 0，可得

$$\cos \lambda l \cosh \lambda l + 1 = 0 \tag{1.43}$$

该方程为超越方程，需借助于数值算法求得数值解，近似解为 $\lambda_1 l \approx 1.875$, $\lambda_2 l \approx 4.694$, $\lambda_3 l \approx 7.855$, $\lambda_n l \approx \left(n - \frac{1}{2}\right)$ when $n \geq 4$. 模态振型为

$$\phi_n(x) = A_n \left[\cosh \lambda_n x - \cos \lambda_n x - \frac{\cos \lambda_n l + \cosh \lambda_n l}{\sin \lambda_n l + \sinh \lambda_n l} \left(\sinh \lambda_n x - \sin \lambda_n x\right)\right] \tag{1.44}$$

1.3.3 Fixed-Fixed beams

boundary conditions: $v(0,t) = v(l,t) = 0$, $\frac{\partial v}{\partial x}|_{(0,t)} = \frac{\partial v}{\partial x}|_{(l,t)} = 0$.
将边界条件带入式 (1.36) 得

$$\begin{cases} B_2 + B_4 = 0 \\ \lambda B_1 + \lambda B_3 = 0 \\ B_1 \sin \lambda l + B_2 \cos \lambda l + B_3 \sinh \lambda l + B_4 \cosh \lambda l = 0 \\ B_1 \cos \lambda l - B_2 \sin \lambda l + B_3 \cosh \lambda l + B_4 \sinh \lambda l = 0 \end{cases} \tag{1.45}$$

系数矩阵行列式为 0，可得

$$\cos\lambda l\cosh\lambda l-1=0 \tag{1.46}$$

该方程为超越方程，需借助于数值算法求得数值解，近似解为 $\lambda_1 l\approx 4.730$, $\lambda_2 l\approx 7.853$, $\lambda_n l\approx \left(n+\frac{1}{2}\right)$ when $n\geq 3$. 模态振型为

$$\phi_n(x)=A_n\left[\cos\lambda_n x-\cosh\lambda_n x-\frac{\cos\lambda_n l-\cosh\lambda_n l}{\sin\lambda_n l-\sinh\lambda_n l}\left(\sinh\lambda_n x\sin\lambda_n x\right)\right] \tag{1.47}$$

1.3.4 Free-free beams

boundary conditions: $\frac{\partial^3 v}{\partial x^3}|_{(0,t)}=\frac{\partial^2 v}{\partial x^2}|_{(0,t)}=0$, $\frac{\partial^3 v}{\partial x^3}|_{(l,t)}=\frac{\partial^2 v}{\partial x^2}|_{(l,t)}=0$.

将边界条件带入式 (1.36) 得

$$\begin{cases}-\lambda^2B_2+\lambda^2B_4=0\\-\lambda^3B_1+\lambda^3B_3=0\\-\lambda^2B_1\sin\lambda l-\lambda^2B_2\cos\lambda l+\lambda^2B_3\sinh\lambda l+\lambda^2B_4\cosh\lambda l=0\\-\lambda^3B_1\cos\lambda l+\lambda^3B_2\sin\lambda l+\lambda^3B_3\cosh\lambda l+\lambda^3B_4\sinh\lambda l=0\end{cases} \tag{1.48}$$

系数矩阵行列式为 0，可得

$$\cos\lambda l\cosh\lambda l-1=0 \tag{1.49}$$

该方程与式 (1.46) 相同，其解也就相同。模态振型为

$$\phi_n(x)=A_n\left[\cos\lambda_n x+\cosh\lambda_n x-\frac{\cos\lambda_n l-\cosh\lambda_n l}{\sin\lambda_n l-\sinh\lambda_n l}\left(\sin\lambda_n x+\sinh\lambda_n x\right)\right] \tag{1.50}$$

1.4 方形薄板的自由振动

Kinematic equations

Kirchhoff Assumptions:

(1) The normal material line to the neutral plane is infinitely rigid along its own, which means that the length of the material line is constant.

(2) The normal material line to the neutral plane remains a straight line after deformation.

(3) The straight normal material line remains normal to the deformed neutral plane.

根据假设 (1), 垂直于中性面的应变为 0，即 $\epsilon_z=\frac{\partial w}{\partial z}=0$. 所以位移 w 与物质坐标 z 无关。即

$$w(x,y,z)=\bar{w}(x,y,z) \tag{1.51}$$

根据假设 2，面内位移可表示为

$$\begin{aligned}u(x,y,z)&=\bar{u}(x,y)+z\theta_2(x,y)\\v(x,y,z)&=\bar{v}(x,y)-z\theta_1(x,y)\end{aligned} \tag{1.52}$$

其中 θ_1, θ_2 分别为中性面法线绕 x, y 轴的偏转角。根据假设 (3)，则有

$$\theta_1(x,y) = \frac{\partial w}{\partial y},\ \theta_2 = -\frac{\partial w}{\partial x}. \tag{1.53}$$

几何方程

$$\begin{aligned}
\epsilon_x &= \frac{\partial u}{\partial x} = \frac{\partial \bar{u}}{\partial x} - z\frac{\partial^2 \bar{w}}{\partial x^2} \\
\epsilon_y &= \frac{\partial v}{\partial y} = \frac{\partial \bar{v}}{\partial y} - z\frac{\partial^2 \bar{w}}{\partial y^2} \\
\gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\partial \bar{u}}{\partial z} + \theta_2 + \frac{\partial w}{\partial x} = 0 \\
\gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \frac{\partial \bar{v}}{\partial z} - \theta_1 + \frac{\partial w}{\partial y} = 0 \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} - 2z\frac{\partial^2 \bar{w}}{\partial x \partial y}
\end{aligned} \tag{1.54}$$

纯弯曲时可忽略面内应变，即 $\partial\bar{u}/\partial x = 0, \partial\bar{u}/\partial y = 0, \partial\bar{v}/\partial x = 0, \partial\bar{v}/\partial y = 0$。则有

$$\epsilon_x = -z\frac{\partial^2 \bar{w}}{\partial x^2},\ \epsilon_y = -z\frac{\partial^2 \bar{w}}{\partial y^2},\ \gamma_{xy} = -2z\frac{\partial^2 \bar{w}}{\partial x \partial y} \tag{1.55}$$

本构方程

$$\begin{aligned}
\sigma_x &= -\frac{Ez}{1-\mu^2}\left(\frac{\partial^2 \bar{w}}{\partial x^2} + \mu\frac{\partial^2 \bar{w}}{\partial y^2}\right) \\
\sigma_y &= -\frac{Ez}{1-\mu^2}\left(\frac{\partial^2 \bar{w}}{\partial y^2} + \mu\frac{\partial^2 \bar{w}}{\partial x^2}\right) \\
\tau_{xy} &= -\frac{Ez}{1-\mu^2}\frac{\partial^2 \bar{w}}{\partial x \partial y}
\end{aligned} \tag{1.56}$$

取微元，有

$$\begin{aligned}
M_x &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x z \mathrm{d}z = -D\left(\frac{\partial^2 \bar{w}}{\partial x^2} + \mu\frac{\partial^2 \bar{w}}{\partial y^2}\right) \\
M_y &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y z \mathrm{d}z = -D\left(\frac{\partial^2 \bar{w}}{\partial y^2} + \mu\frac{\partial^2 \bar{w}}{\partial x^2}\right) \\
M_{xy} &= -(1-\mu)D\frac{\partial^2 \bar{w}}{\partial x \partial y}
\end{aligned} \tag{1.57}$$

其中

$$D = \frac{Eh^3}{12(1-\mu^2)}$$

. 对微元列平衡方程

$$\begin{aligned} \sum F_z &= 0 \\ \sum M_x &= 0 \\ \sum M_y &= 0 \end{aligned} \tag{1.58}$$

可得

$$\begin{aligned} \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} &= 0 \\ \frac{\partial M_{xy}}{\partial y} + \frac{\partial M_x}{\partial x} &= 0 \\ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \rho h \frac{\partial^2 \bar{w}}{\partial t^2} &= 0 \end{aligned} \tag{1.59}$$

将式 (1.57) 带入上式中得

$$D\left(\frac{\partial^4 \bar{w}}{\partial x^4} + 2\frac{\partial^4 \bar{w}}{\partial x^2 \partial y^2} + \frac{\partial^4 \bar{w}}{\partial y^4}\right) + \rho h \frac{\partial^2 \bar{w}}{\partial t^2} = 0 \tag{1.60}$$

该方程也可用分离变量法求解。
边界条件

(1) 简支边。位移和弯矩为 0，即 $w = 0, M_n = 0$。

(2) 固定边。位移和转角为 0，即 $w = 0, \partial w/\partial n = 0$。

(3) 自由边。剪力和弯矩为 0，即 $M_n = 0, Q_n - \frac{\partial M_{ns}}{\partial s} = 0$。

方程 (1.60) 的通解形式为

$$\bar{w}(x, y, t) = \bar{W}(x, y)e^{\omega t i} \tag{1.61}$$

$$\begin{aligned} \bar{W}(x, y) = &A_1 \sin\alpha x \sin\beta y + A_2 \sin\alpha x \cos\beta y + A_3 \cos\alpha x \sin\beta y + A_4 \cos\alpha x \cos\beta y + \\ &A_5 \sinh\alpha x \sinh\beta y + A_6 \sinh\alpha x \cosh\beta y + \\ &A_7 \cosh\alpha x \sinh\beta y + A_8 \cosh\alpha x \cosh\beta y \end{aligned} \tag{1.62}$$

根据不同的边界条件可求得特解。

对于四边简支的矩形薄板，根据边界条件可得其模态振型为

$$\bar{W}(x, y) = A_1 \sin\alpha x \sin\beta y \tag{1.63}$$

且要满足

$$\sin\alpha x a = 0, \sin\beta b = 0 \tag{1.64}$$

其中 a, b 为矩阵板的边长。则有

$$\alpha = \frac{n\pi}{a},\ \beta = \frac{m\pi}{b} \tag{1.65}$$

其固有频率为

$$\omega_{nm} = \pi^2\left[\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2\right]\sqrt{\frac{D}{\rho h}} \tag{1.66}$$

其模态振型为

$$\phi_{nm} = A_{nm}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b} \tag{1.67}$$

1.5 模态振型的特性

1.5.1 Orthogonality

$$\int_0^l \rho A\phi_r\phi_s \mathrm{d}x = 0,\ r \neq s. \tag{1.68}$$

1.5.2 Scaling

modal mass

$$M_r = \int_0^l \rho A\bar{\phi}_r^2 \mathrm{d}x = 1, \tag{1.69}$$

where $\bar{\phi}_r = c\phi$ is the normalized mode shape. c is the scaling factor and

$$c = \sqrt{\frac{1}{\int_0^l \rho A\phi_r^2 \mathrm{d}x}} \tag{1.70}$$

modal stiffness

$$K_r = \int_0^l EI\left(\frac{\mathrm{d}^2\phi_r}{\mathrm{d}x^2}\right)^2 \mathrm{d}x \tag{1.71}$$

$$\omega_n = \frac{K_r}{M_r} \tag{1.72}$$

1.5.3 Expasion theorm

在前几节已经得到张紧弦，Euler-Bernoulli 梁，方形薄板自由振动的模态振型。对于任意函数 $V(x)$，如果其满足如下条件

(1) 与一模态振型函数集满足相同的边界条件；

(2) $EI\left(\mathrm{d}^4 V/\mathrm{d}x^4\right)$ 是连续函数。

则 $V(x)$ 可表示为收敛级数的形式。

$$V(x)=\sum_{r=1}^{\infty} q_r \phi_r \tag{1.73}$$

其中

$$q_r=\frac{\int_0^l \rho A V(x)\phi_r \mathrm{d}x}{\int_0^l \rho A \phi_r^2 \mathrm{d}x} \tag{1.74}$$

亦即模态坐标。

1.5.4 Rayleigh quotient

定义 Rayleigh quotient:

$$R(V)=\frac{\int_0^l EI\left(\frac{\mathrm{d}^2 V}{\mathrm{d}x^2}\right)^2 \mathrm{d}x}{\int_0^l \rho A V^2 \mathrm{d}x} \tag{1.75}$$

根据模态振型的正交性，模态刚度的正交性，以及式 (1.72)(1.73)，可得

$$\begin{aligned} R(V)&=\frac{\sum_{r=1}^{\infty} q_r^2\omega_r^2}{\sum_{r=1}^{\infty} q_r^2}\\ &=\omega_1^2\frac{1+\sum_{r=2}^{\infty}\left(\frac{q_r}{q_1}\right)^2\left(\frac{\omega_r}{\omega_1}\right)^2}{1+\sum_{r=2}^{\infty}\left(\frac{q_r}{q_1}\right)^2} \end{aligned} \tag{1.76}$$

由于基频 ω_1 最小，$\omega_{n+1}>\omega_n$，故

$$R(V)\geq\omega_1^2 \tag{1.77}$$

式 (1.77) 表示 Rayleigh 商是基频的上界，当 $q_r=0, r>1$ 时取等号，此时 $V(x)$ 即为基频模态振型。如果可以选取适当的函数 $V(x)$ 使其与基频模态振型接近，那么就可以近似估计基频的值。

Chapter 2

Time-domain analysis of SDOF systems

2.1 Mathematic modelling of lumped-parameter systems

2.2 Free vibration of SDOF systems

2.3 Dynamic behaviour of undamped SDOF systems under harmonic excitation

2.4 Viscous-damped SDOF systems under Harmonic excitation

2.5 Expansion to periodic excitation via Fourier series

Chapter 3

Time-domain analysis of MDOF systems

Chapter 4

Frequency-domain analysis