Learning Notes of Apr 2019

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1 Lie Group and Lie Algebra

Algebraic set(Variety): set of solutions to an algebraic equation or system of algebraic equations. The set of zero point of a polynomial is 1 dimension less than the affine space.

Def 1.1 *Group: Set G with a group operation satisfying 4 axioms.*

1. *closure:*$\forall g_1, g_2 \in G, g_1 g_2 \in G$
2. *associativity:*$\forall g_1, g_2, g_3 \in G, (g_1 g_2) g_3 = g_1 (g_2 g_3)$
3. *identity:*$\exists e \in G, \forall g \in G, ge = eg = g$
4. *invertibility:*$\forall g \in G, \exists g^{-1} \in G, g g^{-1} = g^{-1} g = e$

Commutative group/Abel group: $\forall g_1, g_2 \in G, g_1 g_2 = g_2 g_1$.

Lie group: groups with continuty.

Examples:

1. $\mathbb{R}^n$
2. Complex numbers of unit modulus
3. Hamilton's quaternions: $q = a + bi + cj + dk$, $i^2 = j^2 = k^2 = -1,\ ijk = -1$, Conjugate of q is $\bar{q} = a - bi - cj - dk$, $G = \{q | \bar{q} q = 1\}$. group opration is quaternionic multiplication.
4. General linear group $GL(n, \mathbb{R})$, n means the invertible square matrice is of n dimensions. Group operation is matrix multiplication.
5. Special linear group $SL(n)$, $G = \{A | \det(A) = 1\}$. Its group manifold is of $n^2 - 1$ dimensions.
6. Orthogonal group $O(n)$: preserve the positive definite bilinear form.
7. Symplectic group $Sp(2n, \mathbb{R})$: preverse the bilineat anti-symmetric form.
8. Unitary group $U(n)$: preserve the Hermitian form. The transformation keep the complex scalars unchangable.

Def 1.2 *Homomorphism: differential map* $f : G \to H,\ \forall g_1, g_2 \in G, f(g_1 g_2) = f(g_1) f(g_2)$

矩阵指数函数

$$
\begin{aligned}
e^{At} &= I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots + \frac{(At)^n}{n!} + \dots \\
\frac{d}{dt}(e^{At}) &= A + A^2 t + \frac{A^3 t^2}{2!} + \cdots + \frac{A^n t^{n-1}}{(n-1)!} + \dots \\
&= A(I + At + \frac{(At)^2}{2!} + \cdots + \frac{(At)^{n-1}}{(n-1)!} + \dots) = Ae^{At}
\end{aligned}
$$

Orthogonal group: $\left\{M \in \mathbb{R}^{n\times n} | MM^T = I\right\}, det(M) = \pm 1$
Special Orthogonal group $SO(3) = \left\{\mathbf{R} \in \mathbb{R}^{3\times 3} | \mathbf{R}^T\mathbf{R} = \mathbf{I}, det(\mathbf{R}) = 1\right\}$
Special Euclidean group $SE(3) = \left\{\mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{x} \\ \mathbf{0}^T & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4} | \mathbf{R} \in SO(3), \mathbf{x} \in \mathbb{R}^3\right\}$
Lie Algebra

$$so(3) = \{\phi \in \mathbb{R}^3\}$$

$$se(3) = \{\xi \in \mathbb{R}^6\}$$

把物体的所有状态定义成一个集合。