Mortar method for planar frictionless contact problems with small deformation

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1 Introduction

In the case of non-matching meshes, the gap can not be evaluated by node-to-node. To deal with the non-matching meshes, the node-to-segment (NTS) method [1] and segment-to-segment (STS) method have been developed. A typical STS method is the so-called mortar contact method [2, 1]. The two methods are all developed for the discretization of the integral of the weak form of the contact contributions

$$\delta\Pi_c = \int_{\gamma_c} \delta\lambda_N g_N + \lambda_N \delta g_N da. \tag{1}$$

In this report, the NTS method and the mortar method will be introduced to discretize the contact interfaces. The contact constraints are implemented by the Lagrange multiplier method. For simplicity, the bilinar planar FEM elements are used and the contact interfaces are also interpolated linearly.

2 Kinematics

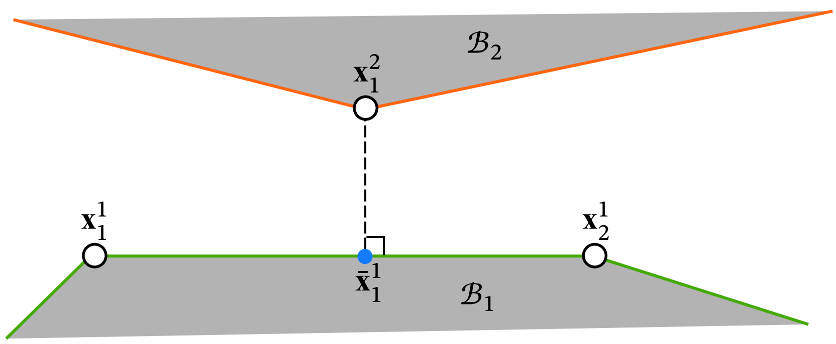


Figure 1: Kinematics of contact between discretized bodies

An illustration of the contact between two discretized bodies is shown in Figure 1. The superscripts in the notations index the bodies. The boundaries are obtained from the discretized elements. For bilinear elements, the position of points at the edge defined by the node $\mathbf{x}_1^1$ and $\mathbf{x}_2^1$ is interpolated as

$$\mathbf{x}^1 = N_1(\xi)\mathbf{x}_1^1 + N_2(\xi)\mathbf{x}_2^1, \tag{2}$$

where

$$N_1(\xi) = \frac{1}{2}(1-\xi), \quad N_1(\xi) = \frac{1}{2}(1+\xi) \tag{3}$$

Similarly, the interpolation of the displacement has the same form.

$$\mathbf{u}^1 = N_1(\xi)\mathbf{u}_1^1 + N_2(\xi)\mathbf{u}_2^1 \tag{4}$$

3 Node-to-segment method

The NTS method define the contact boundary of one body as the master side and the opposite as the slave side. For an example shown in Figure 1, the boundary of body $\mathcal{B}_1$ is regarded as the master side and the boundary of the opposite body is called the slave side. Assuming the node $\mathbf{x}_1^2$ is in contact with the master side, the gap function is defined as

$$g_{N1} = \left[\mathbf{u}_1^2 - N_1(\bar{\xi})\mathbf{u}_1^1 - N_2(\bar{\xi})\mathbf{u}_2^1\right]^T \mathbf{n}^1(\bar{\xi}), \tag{5}$$

where $\bar{\xi}$ is obtained by the closest projection map

$$\left[\mathbf{x}_1^2 - N_1(\xi)\mathbf{x}_1^1 - N_2(\xi)\mathbf{x}_2^1\right] \times \mathbf{n}^1(\xi) = \mathbf{0} \tag{6}$$

The weak form of the contact contribution is integrated along the slave side and the Lagrange multipliers are defined at slave nodes. The contact force is defined as $\lambda_{Ni}a_i$, where a_i denotes the area of the contact element. The first integral in Eq.(1) is approximated as

$$\int_{\gamma_c} \delta\lambda_N g_N da \approx \sum_{e=1}^{n_A} \delta\lambda_{Ne} g_{Ne} a_e \tag{7}$$

For one contact element,

$$\begin{aligned} \delta\lambda_{Ne} g_{Ne} &= \delta\lambda_{Ne} \left[\mathbf{u}_e^2 - N_1(\bar{\xi}_e)\mathbf{u}_{e1}^1 - N_2(\bar{\xi}_e)\mathbf{u}_{e2}^1\right]^T a_e \mathbf{n}^1(\bar{\xi}_e) \\ &= \delta\lambda_{Ne} \left[\begin{array}{ccc} -N_1(\bar{\xi}_e)\mathbf{n}^{1T}(\bar{\xi}_e) & -N_2(\bar{\xi}_e)\mathbf{n}^{1T}(\bar{\xi}_e) & \mathbf{n}^{1T}(\bar{\xi}_e) \end{array}\right] a_e \mathbf{u}_e \end{aligned} \tag{8}$$

where $\mathbf{u}_e^T = \left[\begin{array}{ccc} \mathbf{u}_{e1}^{1T} & \mathbf{u}_{e2}^{1T} & \mathbf{u}_e^{2T} \end{array}\right]$. One can see that the contact reactions along the edge are directly approximated by the reactions at the adjacent node.

4 Mortar method

The mortar method, originally used in parallel computations, was introduced by McDevitt and Laursen [2] to solve contact problems. Unlike the NTS method which use the contact reactions at nodes to approximate the contact reactions along the interfaces, the main idea of the mortar method is to interpolate the contact reactions by reactions at nodes along the non-mortar side and then integrate the contact contibutions along the non-mortar side.

4.1 Contact segment

In the mortar method, contact interfaces are seperated into small segments which are the so-called contact segments. A two dimensional contact segment is illustrated in Figure 2. In these notations, the overhead bar means the closest projection point. $\bar{\mathbf{x}}_2^1$ is the closest projection of point $\mathbf{x}_2^2$ on the mortar side. Similarly, $\bar{\mathbf{x}}_2^2$ is the closest projection of point $\mathbf{x}_2^1$ on the non-mortar side. The contact segment is defined by the quadrilateral with the four nodes $\bar{\mathbf{x}}_2^1$, $\mathbf{x}_2^1$, $\bar{\mathbf{x}}_2^2$, $\mathbf{x}_2^2$. Different segment definition has been introduced in [3].

The coordinate of the closest projection point of node $\mathbf{x}_2^2$, denoted by ξ_1 shown in Figure 3, can be obtained by solving Eq. (6). Similarly, the coordinate of the closest projection of node $\mathbf{x}_2^1$ onto the non-mortar side is denoted by ζ_1.

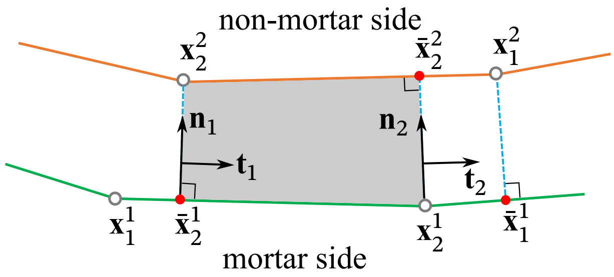


Figure 2: The contact segment

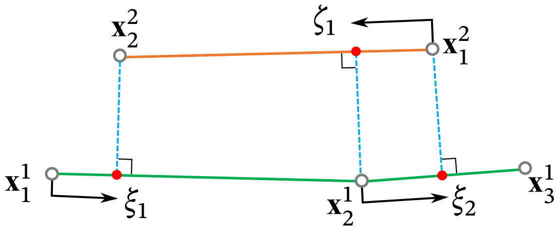


Figure 3: The quadrature of the mortar element

4.2 Integrals of mortar elements

In each pair of contact surfaces, one side is choosen as the mortar side and another as the non-mortar side [4], which is similar to the master/slave side in the NTS method. The surface of body $\mathcal{B}_1$ is regarded as the mortar side in this study. The gap function is defined by the displacement field of the contact surfaces. It can be written as

$$g_N = \left[\mathbf{u}^2(\zeta) - \mathbf{u}^1(\bar{\xi})\right]^T \mathbf{n}^1(\bar{\xi}), \tag{9}$$

where ζ is the coordinate that denote the non-mortar side and the mortar side is denoted by the surface coordinate ξ. $\bar{\xi}$ means the coordinate is obtained from the non-mortar side coordinate ζ by the closest projection map. The Lagrange multipliers are defined on the non-mortar side. On a segment defined by the nodes $\mathbf{x}_1^2$ and $\mathbf{x}_2^2$, the Lagrange multiplier is interpolated as

$$\lambda_N = N_1(\zeta)\lambda_{N1} + N_2(\zeta)\lambda_{N2}. \tag{10}$$

The integral of the contact contribution is implemented along the non-mortar side. The first integral in Eq.(1) can be written as

$$\int_{\gamma_c^{nm}} \delta\lambda_N g_N da = \sum_{l=1}^{n_s} \int_{\gamma_{sl}^{nm}} \delta\lambda_N g_N da, \tag{11}$$

where n_s is the number of the contact segment. According to Eq.(4), Eq.(9) and Eq.(10), the integral within a contact segment shown in Figure 3 is derived as

$$\begin{aligned}
\int_{\gamma_c^{nm}} \delta\lambda_N g_N da &= \sum_{i=1}^{2} \delta\lambda_{Ni} \int_{\gamma_c^{nm}} N_i(\zeta) g_N da \\
&= \sum_{i=1}^{2} \delta\lambda_{Ni} \int_{\gamma_c^{nm}} N_i(\zeta) \left[\sum_{j=1}^{2} N_j(\zeta)\mathbf{u}_j^2 - \sum_{k=1}^{2} N_k(\bar{\xi})\mathbf{u}_k^1\right]^T \mathbf{n}^1 da \\
&= \sum_{i=1}^{2}\sum_{j=1}^{2} \delta\lambda_{Ni} \int_{\gamma_c^{nm}} N_i(\zeta)N_j(\zeta) da (\mathbf{n}^1)^T \mathbf{u}_j^2 \\
&\quad - \sum_{i=1}^{2}\sum_{k=1}^{2} \delta\lambda_{Ni} \int_{\gamma_c^{nm}} N_i(\zeta)N_k(\bar{\xi}) da (\mathbf{n}^1)^T \mathbf{u}_k^1
\end{aligned} \tag{12}$$

In the matrix notation, Eq.(12) is written as the following form

$$\int_{\gamma_c^{nm}} \delta\lambda_N g_N da = \begin{bmatrix}\delta\lambda_{N1} & \delta\lambda_{N2}\end{bmatrix}\left(\mathbf{D}\begin{bmatrix}\mathbf{u}_1^2\\ \mathbf{u}_2^2\end{bmatrix} - \mathbf{M}\begin{bmatrix}\mathbf{u}_1^1\\ \mathbf{u}_2^1\end{bmatrix}\right), \tag{13}$$

where

$$\mathbf{D} = \tilde{\mathbf{D}}\begin{bmatrix}\left(\mathbf{n}^1\right)^T & \mathbf{0}_{1\times 2}\\ \mathbf{0}_{1\times 2} & \left(\mathbf{n}^1\right)^T\end{bmatrix}, \quad \mathbf{M} = \tilde{\mathbf{M}}\begin{bmatrix}\left(\mathbf{n}^1\right)^T & \mathbf{0}_{1\times 2}\\ \mathbf{0}_{1\times 2} & \left(\mathbf{n}^1\right)^T\end{bmatrix}, \tag{14}$$

where $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{M}}$ are all 2×2 matrix. Their elements are

$$\tilde{\mathbf{D}}_{ij} = \frac{L}{2}\int_{\zeta_1}^{1} N_i(\zeta)N_j(\zeta)d\zeta, \quad \tilde{\mathbf{M}}_{ij} = \frac{L}{2}\int_{\zeta_1}^{1} N_i(\zeta)N_j(\bar{\xi})d\zeta \ , \tag{15}$$

where L is the length of the segment defined by $\mathbf{x}_1^2$ and $\mathbf{x}_2^2$. The integrals in Eq.(15) can be implemented by Gauss quadrature.

5 Results

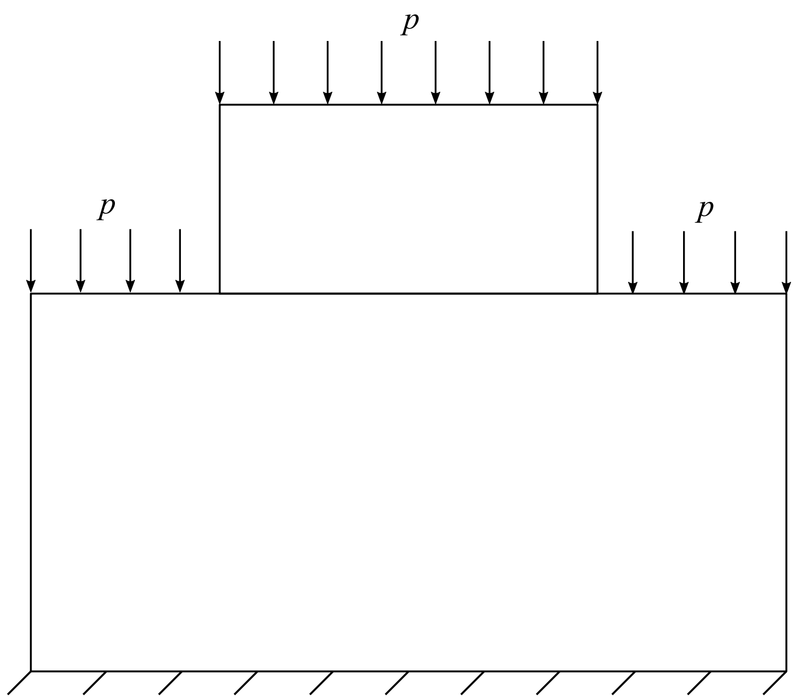
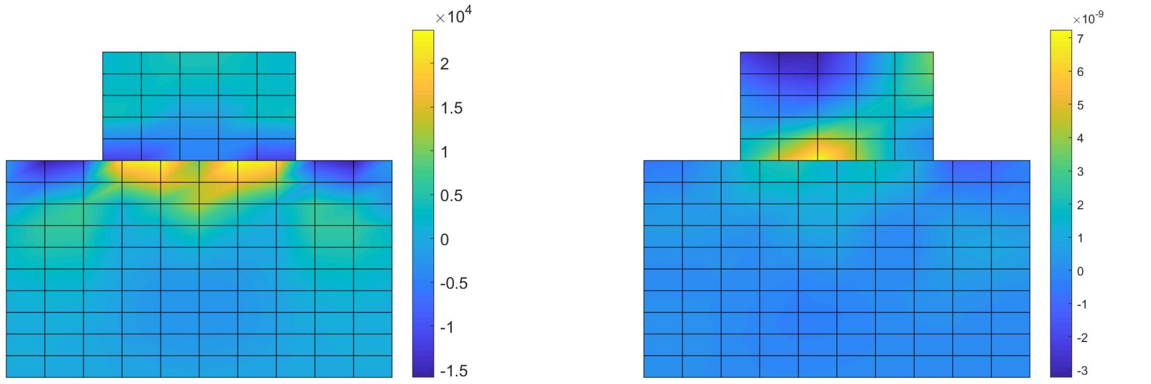


Figure 4: The patch test model

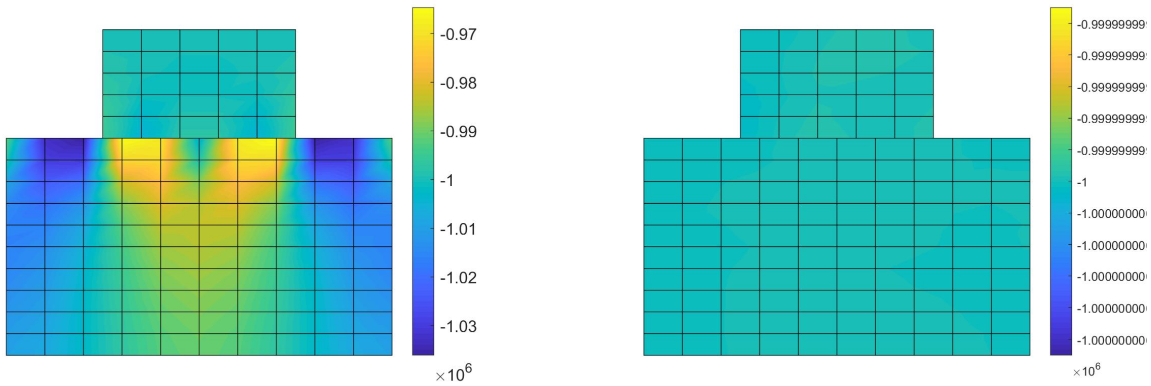
A model of the patch test is shown in Figure 4. The lower big rectangle plate has a width and height of 1 m × 0.5 m and is meshed with a 10 × 10 grid. The upper one is 0.5 m × 0.25 m in width and height and meshed with a 5 × 5 grid. The pressure load is $p = 1$ MPa and the bottem edge of the lower rectangle plate is fixed. The results form the NTS method and the mortar method are listed in Figure 5-7.

It can be seen from Figure 5-7 that the mortar method passed the patch test and conveyed the stress continuously. The NTS method didn't pass the patch test. It can be concluded that using the node contact reaction to replace the contact reactions in the adjacent area has larger error than interpolating the contact reactions along the surface by contact reactions at nodes.



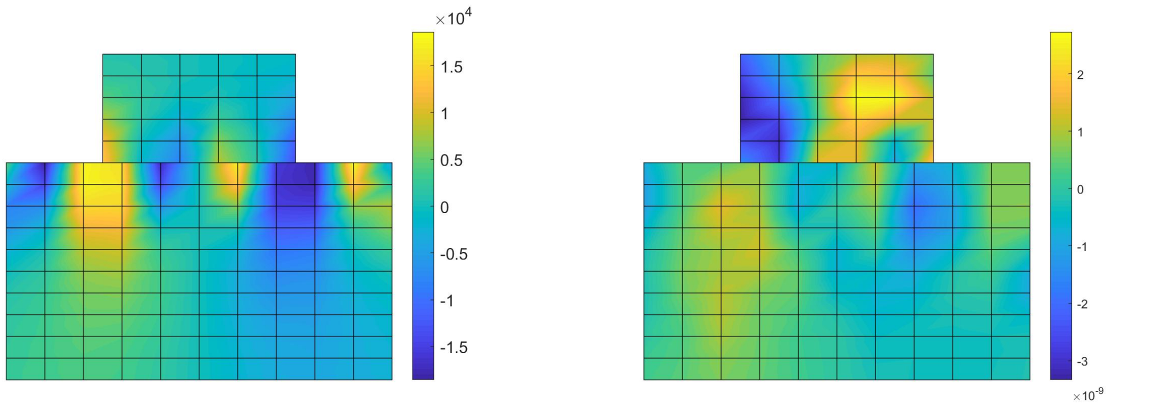
(a) By NTS method (b) By mortar method

Figure 5: σ_x



(a) By NTS method (b) By mortar method

Figure 6: σ_y



(a) By NTS method (b) By mortar method

Figure 7: σ_z

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