

Problems for Quantum Mechanics

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Chapter 1

Problems

1. Estimate De Broglies wavelength of the following object:

- A tennis ball at its world record speed (232 km/h),
- A gas molecule at room temperature,
- A electron in hydrogen atom.

For which one a quantum mechanical treatment is necessary ? Why ?

Chapter 2

Problems

1. Show that if the wavefunction $\Psi(\mathbf{r}) = c\tilde{\Psi}(\mathbf{r})$, where c is a constant and $\tilde{\Psi}(\mathbf{r})$ is real, then the corresponding current \mathbf{j} vanishes.
2. Consider a one-dimensional wavefunction $\Psi(x, t) = Ae^{ikx-i\omega t} + Be^{-ikx-i\omega t}$, compute the probability density ρ [Eq. (2.17)] and probability current \mathbf{j} [Eq. (2.20)] in the lecture. How do you understand your results ?
3. Consider the case where $V(\mathbf{r}) = V_R(\mathbf{r}) - iV_I$ where the imaginary part of the potential is constant. Is the Hamiltonian Hermitian ? Go through the derivation of the continuity equation and show that the total probability for finding the particle decrease exponentially as $e^{-2V_I t/\hbar}$.
4. Using the Schrödinger Equation $H\Psi = i\hbar\frac{\partial\Psi}{\partial t}$ with the Hamiltonian $H = -\frac{\hbar^2}{2\mu}\nabla^2 + V(\mathbf{r})$,

(a) Show that

$$\frac{d\mathbf{p}}{dt} = \mathbf{F},$$

where $\mathbf{p} = -i\hbar \int d\mathbf{r} \Psi^* \nabla \Psi$, and $\mathbf{F} = - \int d\mathbf{r} \Psi^* (\nabla V) \Psi$. What is the physical meaning of this equation ? Does that remind you something familiar from classical mechanics ?

(b) Show that

$$\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{S} = 0,$$

where $W = \frac{\hbar^2}{2\mu} \nabla \Psi^* \cdot \nabla \Psi + \Psi^* V \Psi$, and $\mathbf{S} = -\frac{\hbar^2}{2\mu} (\dot{\Psi}^* \nabla \Psi + \dot{\Psi} \nabla \Psi^*)$. What is the physical meaning of this equation, and the quantity W and \mathbf{S} respectively ?

5. Suppose $\Psi_1(\mathbf{r}, t)$ and $\Psi_2(\mathbf{r}, t)$ are two wave functions that satisfy the same Schrödinger equation. Show that their overlap $\int \Psi_1^* \Psi_2 d\mathbf{r}$ is time independent.
6. Show that the velocity field based on the solution of the Schrödinger equation satisfies $\nabla \times \mathbf{v} = 0$, i.e. the field is curl-less.
7. Consider a bound state solution $\Psi(\mathbf{r})$ of the Hamiltonian $H = -\frac{\hbar^2 \nabla^2}{2\mu} + V(\mathbf{r})$. Please write down the expression of the expected kinetic energy and potential energy. Show that the kinetic energy is always positive.

Chapter 3

Problems

1. A quantum particle of mass μ and energy E , arriving from left, strikes a potential wall at $x = 0$ given by

$$V(x) = \frac{\hbar^2}{\mu} \Omega \delta(x).$$

- (a) What is the dimensionality of the parameter Ω ?
 - (b) Please show the effect of the wall on the particle. Discuss the limiting case of $\Omega \rightarrow \infty$, what happens ?
2. A quantum particle of mass μ resides in the following potential

$$V(x) = -\frac{\hbar^2}{\mu} \Omega [\delta(x - a) + \delta(x + a)]. \quad (3.1)$$

- (a) Please solve the energy level and wavefunction of the quantum particle.
 - (b) What happens when $\Omega a \gg 1$, and $\Omega a \ll 1$ respectively ? What are the physical picture of these two cases ?
 - (c) In the case of $a = 0$, what is the kinetic energy and potential energy of the quantum particle ?
3. A quantum particle with energy $E = \frac{\hbar^2 k^2}{2\mu}$ hits the potential

$$V(x) = \begin{cases} 0, & x < 0, \\ V_0 & x > 0. \end{cases}$$

Please calculate the reflection and transmission coefficients.

4. A quantum particle resides in one dimensional space. Show that two bound states $\Psi_m(x)$ and $\Psi_n(x)$ with different eigenenergy $E_m \neq E_n$ are orthogonal, i.e.

$$\int_{-\infty}^{\infty} \Psi_n(x) \Psi_m(x) dx = 0.$$

5. Show that there is no degeneracy in one dimensional bound states. Moreover, show that the bound state wavefunction can always be chosen as to be real.
6. A particle of mass m in one dimension is bound to a fixed center by an attractive δ -function potential

$$V(x) = -\lambda\delta(x), \quad (\lambda > 0).$$

At time $t = 0$, the potential is suddenly switch off. Find the wavefunction for $t > 0$.

7. A particle is in the ground state of a box with hardwalls of length L . The box suddenly expands symmetrically to length λL , leaving the wavefunction undisturbed. What is the probability that the particle resides at the ground state of the new box ?
8. In the study of quark-antiquark bound system (known as *quarkonium*), it is relevant to consider a quantum particle with mass μ bound in the potential

$$V(x) = k|x|.$$

Please solve the energy spectrum of the particle. What is the parity of the ground state ? What is the ground state energy ? What is the excitation energy to the first excited state ? Hint: You may need to refresh your memory about Airy function and its zeros. Comment: This problem is also relevant if you bounce a quantum particle towards the floor under the action of gravity, c.f. Phys. Rev. D **67** 102002, (2003).

Chapter 4

Problems

1. Assuming H is a Hamiltonian which depends on parameter λ , and $|\Psi\rangle$ is its eigenstate, show that

$$\frac{\partial \langle \Psi | H | \Psi \rangle}{\partial \lambda} = \left\langle \Psi \left| \frac{\partial H}{\partial \lambda} \right| \Psi \right\rangle. \quad (4.1)$$

This is the so called *Hellmann-Feynman* Theorem which finds wide applications in quantum physics.

2. Assuming $H = \frac{\mathbf{p}^2}{2\mu} + V(\mathbf{r})$ and $|\Psi\rangle$ is its eigenstate, show that

$$\left\langle \Psi \left| \frac{\mathbf{p}^2}{2\mu} \right| \Psi \right\rangle = \frac{1}{2} \langle \Psi | \mathbf{r} \cdot \nabla V(\mathbf{r}) | \Psi \rangle. \quad (4.2)$$

This is the so called *Virial theorem*. Does this reminds you anything from classical mechanics ? If $V(\mathbf{r})$ is a δ -potential, what can you say about the energy relation from the Virial theorem ? Check this relation for the problem (3.1) you solved in Chapter 3.

3. Suppose the Hamiltonian commutes with the parity operator, i.e. $[H, P] = 0$, and $|\Psi\rangle$ is a nondegenerate eigenstate of H , show that $|\Psi\rangle$ is also a parity eigenstate.
4. Take your solution of (3.1) in the Chapter. 3 and compute $\Delta x \Delta p$. Check that it indeed satisfies the Heisenberg uncertainly principle.
5. Given operator A, B , and

$$\begin{aligned}
C_0 &= B, \\
C_1 &= [A, B] = AB - BA, \\
C_2 &= [A, C_1] = [A, [A, B]], \\
&\dots \\
C_n &= [A, C_{n-1}].
\end{aligned} \tag{4.3}$$

Show that $e^A B e^{-A} = C_0 + C_1 + \frac{1}{2!}C_2 + \frac{1}{3!}C_3 + \dots = \sum_{n=0}^{\infty} \frac{C_n}{n!}$. Apply the formula to the case $A = ip$, $B = x$.

6. Consider a quantum particle reside in a symmetric infinite deep potential of the width L , assuming a guess of ground wave-function $\Psi_{\text{guess}}(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4$.
 - (a) What can you say about the coefficients even without calculation ? (Hint: the ground state has even parity.)
 - (b) What is the expected energy of $\Psi_{\text{guess}}(x)$?
 - (c) Try to vary the coefficient to obtain the lowest possible energy. How does that compared to the exact solution ?
 - (d) Could you come up with a good guess of the first excited state ?
7. Show that every attractive potential ($x \rightarrow \pm\infty, V(x) \rightarrow 0$) in one dimension has at least one bound state. Hint: Evaluate the expected energy on a guessed wavefunction $\Psi_{\text{guess}}(x) \sim e^{-ax^2/2}$. What happens if $a \rightarrow 0$?

Chapter 5

Problems

1. The Hamiltonian of certain 2-level system is

$$H = E (|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|),$$

where $|1\rangle, |2\rangle$ are orthonormal basis and E is a number with the dimension of energy. What is the matrix representation of the Hamiltonian with respect to this basis ? Find its eigenvalues and eigenvectors as linear combination of $|1\rangle, |2\rangle$.

2. The coordinate space basis $|x\rangle$ and momentum space basis $|p\rangle$ satisfy $\langle x|x'\rangle = \delta(x - x')$, $\langle p|p'\rangle = \delta(p - p')$ and $\langle x|p\rangle = e^{ipx/\hbar}/\sqrt{2\pi\hbar}$, please write down the stationary Schrödinger equation for the wavefunction $\Psi(p) = \langle p|\Psi\rangle$ in the momentum space.
3. Solve the problems (3.1) of the Chapter 3 using Schrödinger equation in the momentum space. Hint1: Consider $a = 0$ if you think the full problem is too difficult. Hint2: Make sure you obtain exactly the same result as last time. Sometimes, solving the same problem multiple times using different methods and check consistency is a way to ensure correctness.
4. Consider Hamiltonian $H = \frac{p^2}{2\mu} + V(x)$, please write down the equation of motion for the operators x_H and p_H in the Heisenberg picture.
5. Prove the following sum rules in the energy representation

(a) Consider a quantum particle with mass μ , and $F(x)$ is a function of the coordinate,

$$\sum_n (E_n - E_k) |F_{nk}|^2 = \frac{\hbar^2}{2\mu} \left\langle k \left| \left| \frac{dF}{dx} \right|^2 \right| k \right\rangle \quad (5.1)$$

(b) H is the Hamiltonian, and F is an arbitrary Hermitian operator, show that

$$\sum_n (E_n - E_k)^2 |F_{nk}|^2 = -\langle k | [H, F]^2 | k \rangle. \quad (5.2)$$

Chapter 6

Problems

1. Find the expectation value of the potential energy in the n th state of the harmonic oscillator.
2. Evaluate the expected value $\langle n|x^4|n\rangle$ and $\langle n|x^3|n\rangle$.
3. Consider a particle subject to a one-dimensional potential of the following form

$$V(x) = \begin{cases} \frac{1}{2}\mu\omega^2 x^2, & x > 0, \\ \infty & x < 0. \end{cases}$$

What is the ground state energy ? What is the expectation value $\langle x^2 \rangle$ of the ground state ?

4. Consider a particle resides in a three dimensional harmonic potential $V(x, y, z) = \frac{\mu\omega^2}{2} (x^2 + y^2 + z^2)$. What is the energy spectrum of the system ? What is the degeneracy of each energy level ?
5. Suppose at time $t = 0$ the oscillator is in the coherent state $|\Psi(t = 0)\rangle = |\alpha\rangle$, please write down the wavefunction at time t , $|\Psi(t)\rangle$. Show that it is also a coherent state.
6. The magnetic length $\ell = \sqrt{\frac{\hbar c}{eB}}$, and the cyclotron frequency $\omega = \frac{eB}{\mu c}$. Try to estimate them for magnetic field strength $B = 1$ Tesla. What is the corresponding temperature scale of the cyclotron frequency ?
7. Let a and a^\dagger be the annihilation and creation operators for a harmonic oscillator, consider the Hamiltonian

$$H = \epsilon a^\dagger a + \eta (aa + a^\dagger a^\dagger),$$

where ϵ and η are real parameters. Compute the energy spectrum of the Hamiltonian. Hint: try to write the Hamiltonian using the operator $b = a \cosh \theta + a^\dagger \sinh \theta$ and $b^\dagger = a^\dagger \cosh \theta + a \sinh \theta$, where θ is a to be determined parameter. They are called Bogoliubov operator. You will encounter them in the study of superfluids and superconductors in future.

Chapter 7

Problems

1. Given a Hamiltonian $H = \frac{p^2}{2\mu} + V(x)$, does it conserve translation invariance in general ? Under which condition it will be translation invariant ?
2. Let $\Psi(x, t)$ be the wavefunction of a particle, show that $\Psi^*(x, -t)$ is the wavefunction of the particle with the momentum direction reversed.
3. For a given state $|x\rangle$, we have $\langle x'|x\rangle = \delta(x - x')$. Shows that $e^{-ipa}|x\rangle = |x + a\rangle$, where p is the momentum operator and a is a constant.
4. Let momentum state wave function for the state $\langle p|\Psi\rangle = \Psi(p)$. What is the momentum space wavefunction for the time-reversed state $\langle p|T|\Psi\rangle$?
5. Let Π be a parity operator, prove that if $[\Pi, H] = 0$, a system that starts out in a state of certain parity maintain its parity.
6. Show that magnetic field applied to a system in general breaks the time reversal invariance. How about an electric field ?

Chapter 8

Problems

1. Show that the expected values of L_x and L_y are both zero on the eigenstate of L_z .
2. Show that the Hamiltonian $H = \frac{p^2}{2\mu} + V$ commutes with all three components of the angular momentum \mathbf{L} , provided that the potential V only depends on r . Then show that $d\langle\mathbf{L}\rangle/dt = 0$ for any spherically symmetric potential.
3. A quantum particle is in the common eigenstate, Y_{20} , of the operators (\mathbf{L}^2, L_z) , please compute the possible observed values of L_x and their probabilities on the state.
4. Suppose that \mathbf{J} is the angular momentum operator, and $\mathbf{n}_1, \mathbf{n}_2$ commute with \mathbf{J} , show that

$$[\mathbf{J} \cdot \mathbf{n}_1, \mathbf{J} \cdot \mathbf{n}_2] = i\mathbf{J} \cdot (\mathbf{n}_1 \times \mathbf{n}_2).$$

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5. Given an angular momentum operator \mathbf{J} , $J_{\pm} = J_x \pm iJ_y$, show that

$$(a) \quad J_z^n J_{\pm} = J_{\pm} (J_z \pm 1)^n,$$

$$(b) \quad e^{i\lambda J_z} J_x e^{-i\lambda J_z} = J_x \cos \lambda - J_y \sin \lambda.$$

6. Show that $e^{i\lambda \mathbf{n} \cdot \boldsymbol{\sigma}} = \cos \lambda + i\mathbf{n} \cdot \boldsymbol{\sigma} \sin \lambda$, where λ is a constant and \mathbf{n} is a unit vector.
7. Given two $S = 1/2$ spins with the Hamiltonian $H = J\mathbf{S}_1 \cdot \mathbf{S}_2$, what is the ground state of the system for $J > 0$ and $J < 0$ respectively ?

Chapter 9

Problems

1. The radial part of a quantum particle in central potential is

$$\left[-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2} \right] u_{nl}(r) = E u_{nl}(r),$$

show that the bound state energy is larger for larger angular quantum number l , given the same radial quantum number n .

2. Show that for a bound state in a central potential, one has

$$\left\langle \frac{dV}{dr} \right\rangle - \frac{l(l+1)\hbar^2}{\mu} \left\langle \frac{1}{r^3} \right\rangle = \frac{2\pi\hbar^2}{\mu} |\Psi(0)|^2,$$

where $\Psi = R(r)Y_{lm}(\theta, \phi)$ is the eigenstate of (H, \mathbf{L}^2, L_z)

3. Try to relate the radial equation of the isotropic Harmonic oscillator $V(r) = \alpha r^2$ and the Coulomb potential $V(r) = \beta/r$. In this way, you can deduce the eigenenergy with the knowledge of one to another.
4. Show that the kinetic energy and potential energy of the electron in hydrogen atom satisfy $\langle T \rangle = -\langle V \rangle/2$. Based on that, compute the expected value $\langle \frac{1}{r} \rangle$.
5. Let μ be the mass of electron, and c be the speed of light.

(a) Convince yourself that $\mu c^2 \simeq 0.511 \times 10^6 \text{ eV}$, and $\alpha = \frac{1}{4\pi\epsilon_0} \frac{e^2}{\hbar c} \simeq \frac{1}{137.04}$ is a dimensionless number (fine-structure constant),

(b) Use these results to estimate the Bohr radius $a_0 = \frac{4\pi\epsilon_0\hbar^2}{\mu e^2}$ and Rydberg energy $\text{Ry} = \frac{\hbar^2}{2\mu a_0^2} = \frac{\mu}{2} c^2 \alpha^2 = \frac{\mu}{2} \left(\frac{e^2}{4\pi\epsilon_0\hbar^2} \right)^2$.

6. A single valence electron feel the following effective potential of the nuclei,

$$V(r) = \frac{-e^2}{r} - \lambda \frac{e^2 a_0}{r^2},$$

where a_0 is the Bohr radius, and $0 < \lambda \ll 1$. Please solve the energy spectrum and compare with the one of Hydrogen atom.