Fourier series ---trigonometric

Liang Wang   2016 May

Recall that the mathematical series as follow expression

$$A_0 + \sum_{n=1}^{\infty} (A_n \, cos(nx) + B_n \, sin(nx))$$

We called it **Fourier series**.

Since this expression deals with convergence as n reach to infinite, we start by defining a similar expression when the sum is finite.

Definition. A **Fourier polynomial** is an expression of the form

$$F_n(x) = a_0 + (a_1 \, cos\, x + b_1 \, sin\, x) + \cdots + (a_n \, cos\, nx + b_n \, sin\, nx)$$

Which may rewritten as

$$F_n(x) = a_0 + \sum_{k=1}^{k=n} (a_k \, cos\, kx + b_k \, sin\, kx)$$

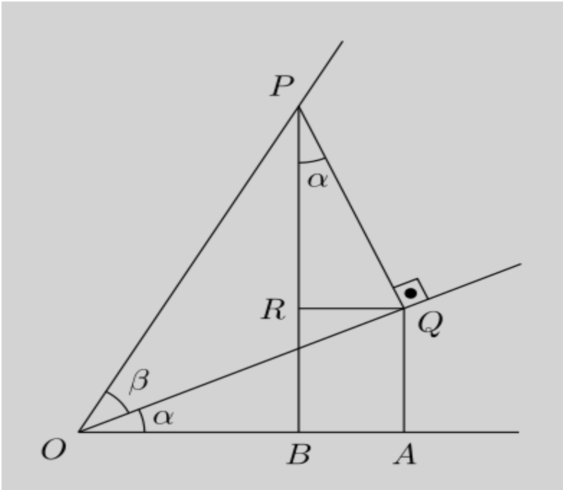
The constants a_0, a_i, b_i for $i = 1,2,3 \ldots n$ are called

the **coefficients** of $F_n(x)$.

The Fourier polynomials are 2π-periodic functions. Using the trigonometric identities

$$\begin{cases} sin(mx)\, cos(nx) = \frac{1}{2}[sin((m-n)x) + sin((m+n)x)] \\ cos(mx)\, cos(nx) = \frac{1}{2}[cos((m-n)x) + cos((m+n)x)] \\ sin(mx)\, sin(nx) = \frac{1}{2}[cos((m-n)x) - cos((m+n)x)] \end{cases}$$

Want to know why? See blue detail below, or please skip to black fonts directly.



Draw a horizontal line (the *x*-axis); mark an origin O. Draw a line from O at an angle α above the horizontal line and a second line at an angle β above that; the angle between the second line and the *x*-axis is $\alpha + \beta$.

Place P on the line defined by $\alpha + \beta$ at a unit distance from the origin.

Let PQ be a line perpendicular to line defined by angle α drawn from point Q on this line to point P. ∴ OQP is a right angle.

Let QA be a perpendicular from point A on the *x*-axis to Q and PB be a perpendicular from point B on the *x*-axis to P. ∴ OAQ and OBP are right angles.

Draw R on PB so that QR is parallel to the *x*-axis.

Now angle $RPQ = \alpha$ (because $OQA = 90^0 - \alpha$, making $RQO = \alpha, RQP = 90^0 - \alpha$, and finally $RPQ = \alpha$)

$$RPQ = 90^0 - RQP = 90^0 - (90^0 - RQO) = RQO = \alpha$$

$$OP = 1$$

$$PQ = sin\,\beta$$

$$OQ = cos\,\beta$$

$\frac{AQ}{OQ} = \sin\alpha$, so $AQ = sin\,a\,cos\,\beta$

$\frac{PR}{PQ} = \cos\alpha$, so $PR = \cos a \sin\beta$

$\sin(\alpha+\beta) = PB = PR + RB = PR + AQ = \cos a \sin\beta + \sin\alpha\cos\beta$

Similarly, we have follow identities

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

So we get

$$\sin(mx)\cos(nx) = \frac{1}{2}[\sin((m-n)x) + \sin((m+n)x)]$$
$$\cos(mx)\cos(nx) = \frac{1}{2}[\cos((m-n)x) + \cos((m+n)x)]$$
$$\sin(mx)\sin(nx) = \frac{1}{2}[\cos((m-n)x) - \cos((m+n)x)]$$

We can easily prove the integral formulas
(1)

for $n \geq 0$, we have

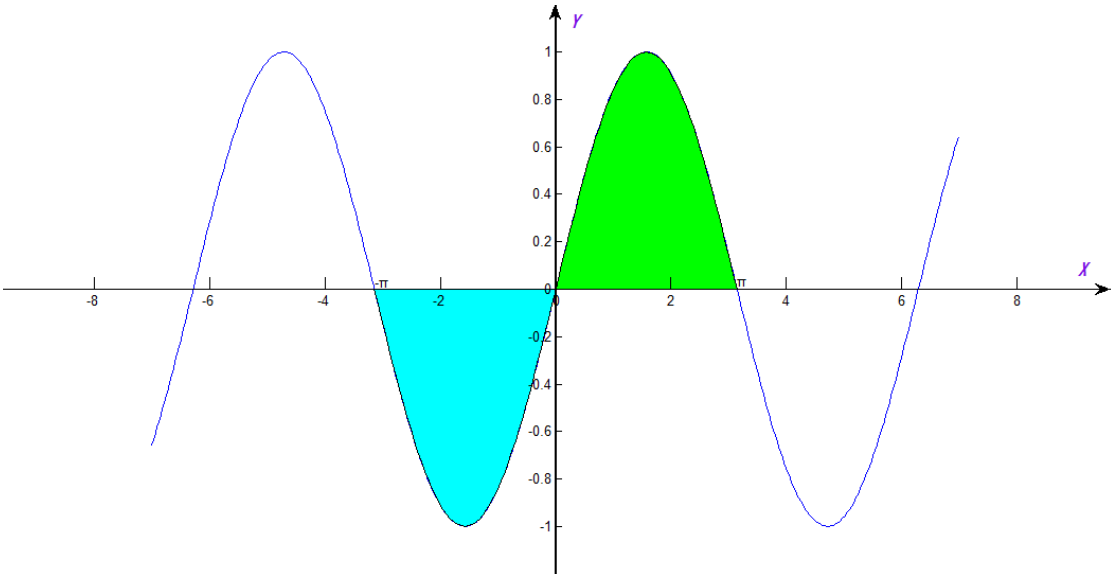
$$\int_{-\pi}^{\pi} \sin(nx)\, dx = 0$$

for $n \geq 1$ we have

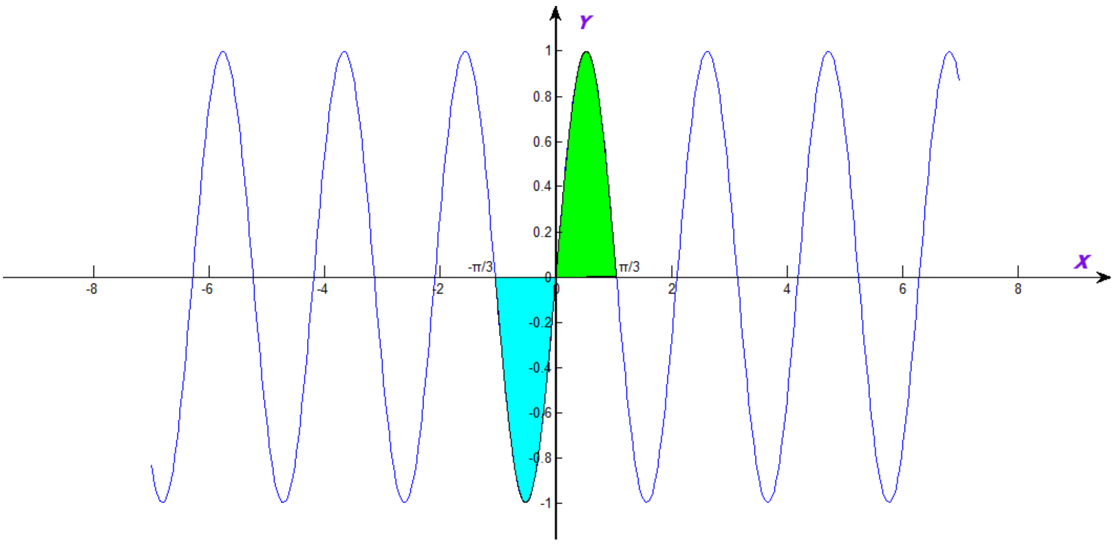
$$\int_{-\pi}^{\pi} \cos(nx)\, dx = 0$$

Want to know why? See blue detail below, or please skip to black fonts directly.

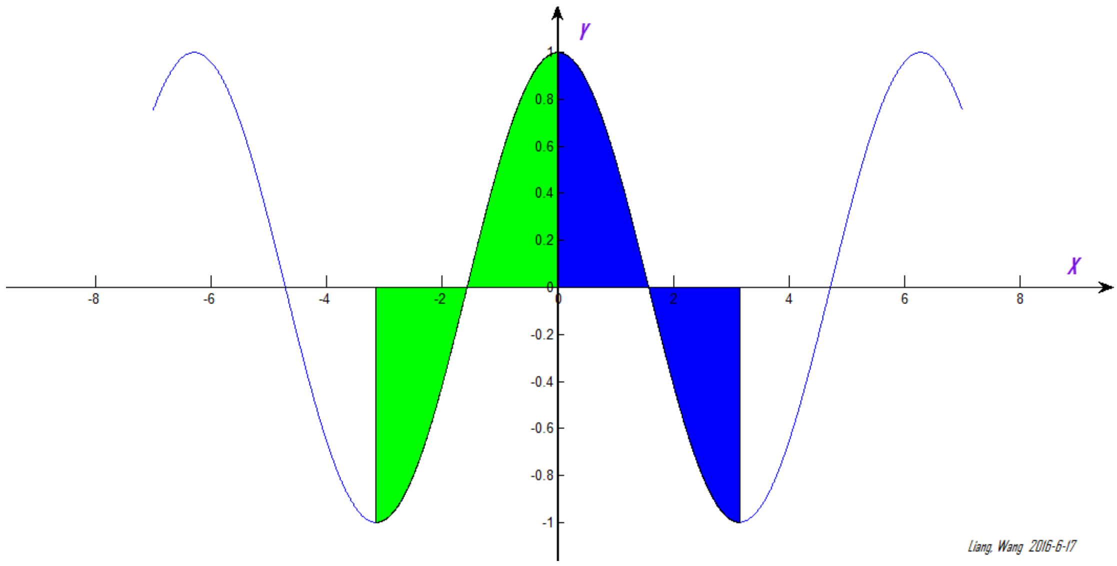
For any odd $f(x)$, we have $\int_{-T}^{T} f(x)dx = 0$, in symmetrical interval, obviously, seen below picture.



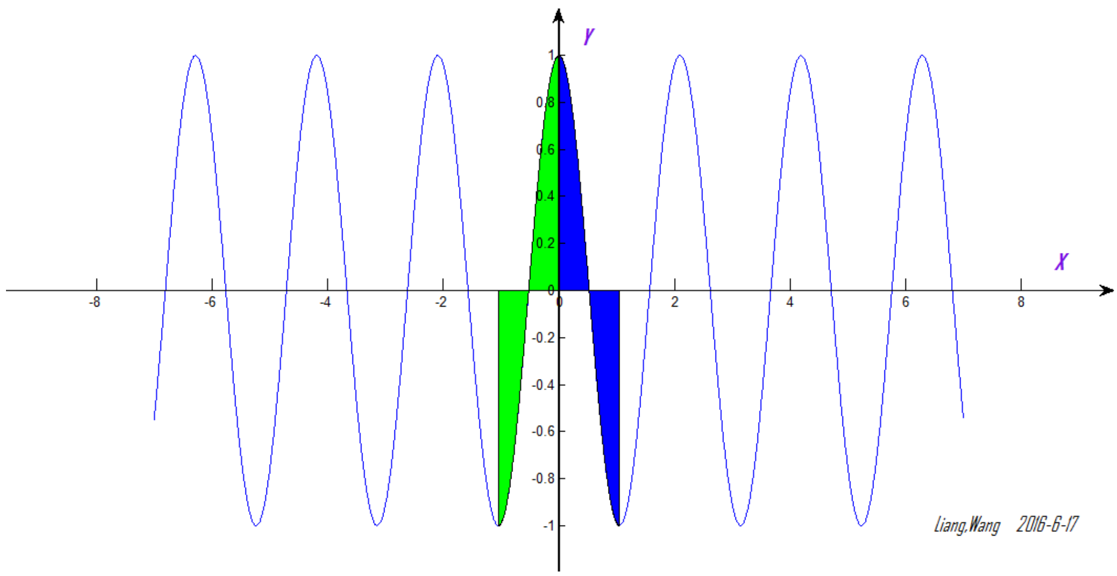
$sin(x)$



$sin(nx)$ ***n=3***



$cos(x)$



$cos(nx)$ ***n=3***

(2)

for m and n, we have

$$\int_{-\pi}^{\pi} sin(mx)\, cos(nx)\, dx = 0$$

Want to know why? See blue detail below, or please skip to black fonts directly.

As odd function* even function = odd function

(3)

for $n \neq m$, we have

$$\int_{-\pi}^{\pi} cos(mx)\, cos(nx)\, dx = 0\,, and\ \int_{-\pi}^{\pi} sin(mx)\, sin(nx)\, dx = 0$$

Want to know why? See blue detail below, or please skip to black fonts directly.

As

$$sin(mx)\, cos(nx) = \frac{1}{2}[sin((m-n)x) + sin((m+n)x)]$$
$$cos(mx)\, cos(nx) = \frac{1}{2}[cos((m-n)x) + cos((m+n)x)]$$
$$sin(mx)\, sin(nx) = \frac{1}{2}[cos((m-n)x) - cos((m+n)x)]$$

(4)

for $n \geq 1$, we have

$$\int_{-\pi}^{\pi} cos^2(nx) dx = \pi\,, \quad and \quad \int_{-\pi}^{\pi} sin^2(nx) dx = \pi\,.$$

Want to know why? See blue detail below, or please skip to black fonts directly.

As earlier formula, we can deduce

$$cos(nx)\ cos(nx) \ = \frac{1}{2}[cos(2nx) + 1]$$

$$\int_{-\pi}^{\pi} cos(nx)\ cos(nx)\ dx = \frac{1}{2}\int_{-\pi}^{\pi}[cos(2nx) + 1]dx = \pi$$

Same reason for

$$\int_{-\pi}^{\pi} sin(nx)\ sin(nx)\ dx = \frac{1}{2}\int_{-\pi}^{\pi}[1 - cos(2nx)]dx = \pi$$

Using the above formulas, we can easily deduce the following result:

Theorem. Let

$$F_n(x) = a_0 + \sum_{k=1}^{k=n} (a_k \cos kx + b_k \sin kx)$$

We have

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(x)\, dx,$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} F_n(x) \cos(kx)\, dx, \quad 1 \le k \le n$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} F_n(x) \sin(kx)\, dx, \quad 1 \le k \le n$$

Want to know why? See blue detail below, or please skip to black fonts directly.

For

$$F_n(x) = a_0 + \sum_{k=1}^{k=n} (a_k \cos kx + b_k \sin kx)$$

1) Do integral on two sides directly we get

$$\int_{-\pi}^{\pi} F_n(x)dx = \int_{-\pi}^{\pi} (a_0 + \sum_{k=1}^{k=n} (a_k \cos(kx) + b_k \sin(kx)))dx = 2\pi a_0$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(x)dx$$

2) Multiple $\cos(ix)$ on two sides first then do integral we get

$$\int_{-\pi}^{\pi} F_n(x)\, cos(ix)\, dx$$

$$= \int_{-\pi}^{\pi} (a_0\, cos(ix)$$

$$+ \sum_{k=1}^{k=n} (a_k\, cos(kx)\, cos(ix) + b_k\, sin(kx)\, cos(ix)))dx$$

$$= \int_{-\pi}^{\pi} a_i\, cos(ix)\, cos(ix) = \pi a_i$$

So

$$a_i = \frac{1}{\pi} \int_{-\pi}^{\pi} F_n(x)\, cos(ix)\, dx$$

3) Multiple $sin(ix)$ on two sides first then do integral we get

$$\int_{-\pi}^{\pi} F_n(x)\, sin(ix)\, dx$$

$$= \int_{-\pi}^{\pi} (a_0\, sin(ix)$$

$$+ \sum_{k=1}^{k=n} (a_k\, cos(kx)\, sin(ix) + b_k\, sin(kx)\, sin(ix)))dx$$

$$= \int_{-\pi}^{\pi} b_i\, sin(ix)\, sin(ix) = \pi b_i$$

So

$$b_i = \frac{1}{\pi} \int_{-\pi}^{\pi} F_n(x)\, sin(ix)\, dx$$

This theorem helps associate a Fourier series to any 2π-periodic function.

Definition. Let $f(x)$ be a 2π-periodic function which is integrable on $[-\pi, \pi]$. Set

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \quad n \geq 1$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx, \; n \geq 1$$

Want to know why? See blue detail below, or please skip to black fonts directly.

We just assume for any function *f(x)* which is 2π-periodic & integrable could be extended to one kind of series--- infinite trigonometric series, then check whether it is real benefit for engineer analysis. Actually it brings great benefit and being proved by much of experiments, gradually it becomes a very famous Engineer method. And why this assumption is true? That is a pure mathematics topic, we will try to discuss it on Fourier series ---Convergence part, like many other mathematics guess still not be proved, it is still open for a few kind of extreme functions, but in most cases it is convergence everywhere and value for our daily life case analysis, so just believe it, this assumption is **TRUE**☺

The trigonometric series

$$A_0 + \sum_{n=1}^{\infty}(A_n\cos(nx) + B_n\sin(nx))$$

is called the **Fourier series** associated to the function $f(x)$. We will use the notation

$$f(x) \sim A_0 + \sum_{n=1}^{K\to\infty}(A_n\cos(nx) + B_n\sin(nx))$$

$$f(x) = A_0 + \sum_{n=1}^{\infty} (A_n\, cos(nx) + B_n\, sin(nx))$$

Example. Find the Fourier series of the function

$$f(x) = x, \quad -\pi \leq\ x \leq \pi$$

Answer. Since $f(x)$ is odd, then $A_n = 0$, for $n \geq 0$. We turn our attention to the coefficients B_n. For any $n \geq 1$, we have

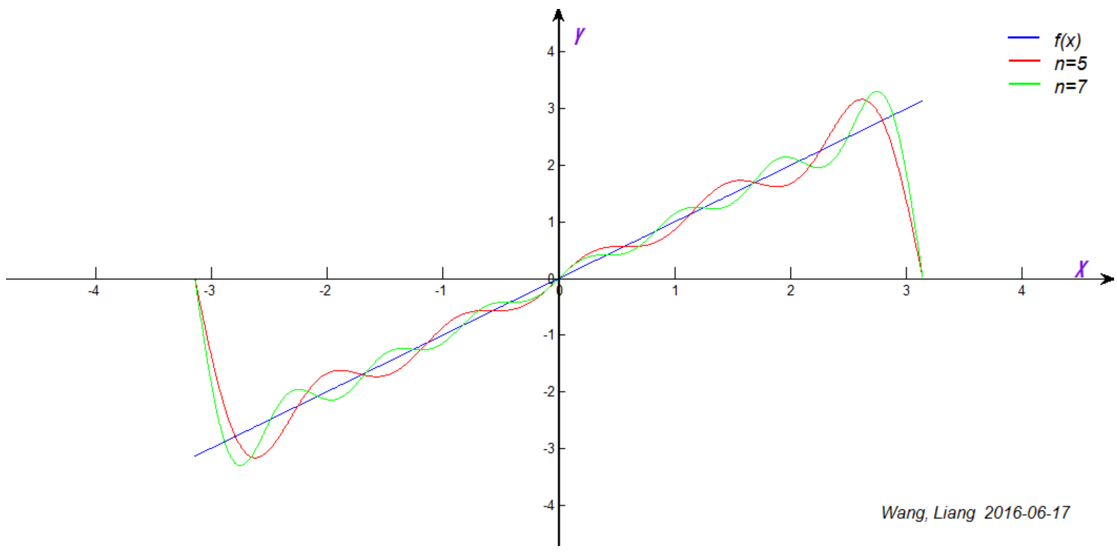
$$B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x\, sin(nx)\, dx = \frac{1}{\pi} \left[-\frac{x\, cos(nx)}{n} + \frac{sin(nx)}{n^2} \right]_{-\pi}^{\pi}$$

We deduce

$$B_n = -\frac{2}{n} cos(n\pi) = \frac{2}{n} (-1)^{n+1}$$

Hence

$$f(x) \sim 2 (sin\, x - \frac{sin(2x)}{2} + \frac{sin(3x)}{3} - \frac{sin(4x)}{4} \ldots)$$



Example. Find the Fourier series of the function

$$f(x) = \begin{cases} 0, & -\pi \leq x \leq 0 \\ \pi, & 0 \leq x \leq \pi \end{cases}$$

Answer. We have

$$a_0 = \frac{1}{2\pi}\left(\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} \pi\,dx\right) = \frac{\pi}{2}, \qquad a_n = \int_0^{\pi} \pi\,cos(nx)\,dx = 0, \qquad n \geq 1,$$

and

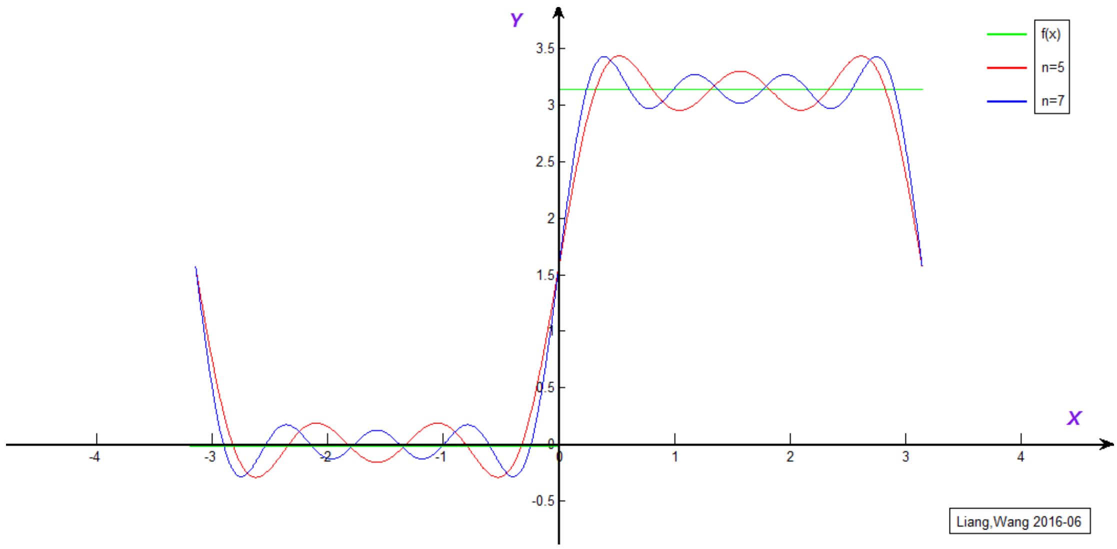
$$b_n = \int_0^{\pi} \pi\,sin(nx)dx = \frac{1}{n}(1 - cos(n\pi)) = \frac{1}{n}(1 - (-1)^n)$$

We obtain $b_{2n} = 0$ and

$$b_{2n+1} = \frac{2}{2n+1}$$

Therefore, the Fourier series of $f(x)$ is

$$f(x) \sim \frac{\pi}{2} + 2\left(sin(x) + \frac{sin(3x)}{3} + \frac{sin(5x)}{5} + \cdots\right).$$

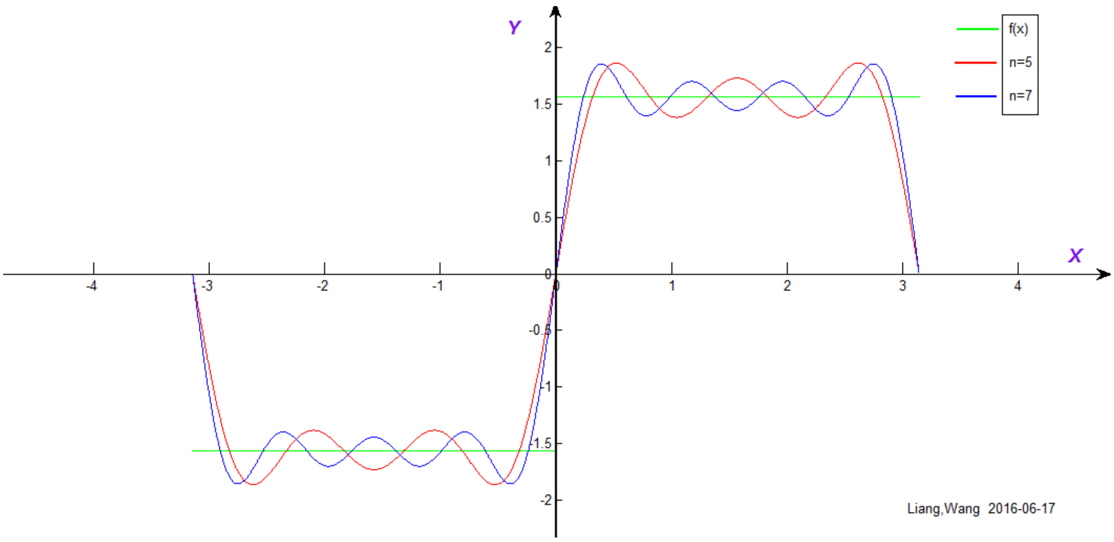


Example. Find the Fourier series of the function

$$f(x) = \begin{cases} -\frac{\pi}{2}, & -\pi \le x < 0 \\ \frac{\pi}{2}, & 0 \le x \le \pi \end{cases}$$

Answer. Since this function is the function of the example above minus the constant $\frac{\pi}{2}$. So Therefore, the Fourier series of *f*(*x*) is

$$f(x) \sim 2\left(sin(x) + \frac{sin(3x)}{3} + \frac{sin(5x)}{5} + \cdots\right)$$



Remark. We defined the Fourier series for functions which are 2π-periodic, one would wonder how to define a similar notion for functions which are L-periodic.

Assume that $f(x)$ is defined and integrable on the interval $[-L, L]$.

Let $t = \frac{\pi x}{L}$, so $t \in [-\pi, \pi]$ and

$$x = \frac{Lt}{\pi}$$

Set

$$f(x) = f\left(\frac{Lt}{\pi}\right) = F(t)$$

The function $F(t)$ is defined and integrable on $[-\pi, \pi]$. Consider the Fourier series of $F(t)$

$$F(t) = f\left(\frac{Lt}{\pi}\right) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt))$$

Using the substitution $t = \frac{\pi x}{L}$

, we obtain the following definition:

$$F(t) = f(x) = f\left(\frac{Lt}{\pi}\right) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \, cos\left(n\frac{\pi x}{L}\right) + b_n \, sin(n\frac{\pi x}{L}))$$

Definition. Let $f(x)$ be a function defined and integrable on $[-L,\ L]$. so

The Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \, cos\left(n\frac{\pi x}{L}\right) + b_n \, sin(n\frac{\pi x}{L}))$$

Where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{Lt}{\pi}\right) dt = \frac{1}{L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{Lt}{\pi}\right) cos(nt)\, dt = \frac{1}{L}\int_{-L}^{L} f(x)\, cos\left(n\frac{\pi x}{L}\right) dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{Lt}{\pi}\right) sin(nt)\, dt = \frac{1}{L}\int_{-L}^{L} f(x)\, sin\left(n\frac{\pi x}{L}\right) dx$$

for n ≥ 1.

Example. Find the Fourier series of

$$f(x) = \begin{cases} 0, & -2 \le x < 0 \\ x, & 0 \le x \le 2 \end{cases}$$

Answer. Since $L = 2$, we obtain

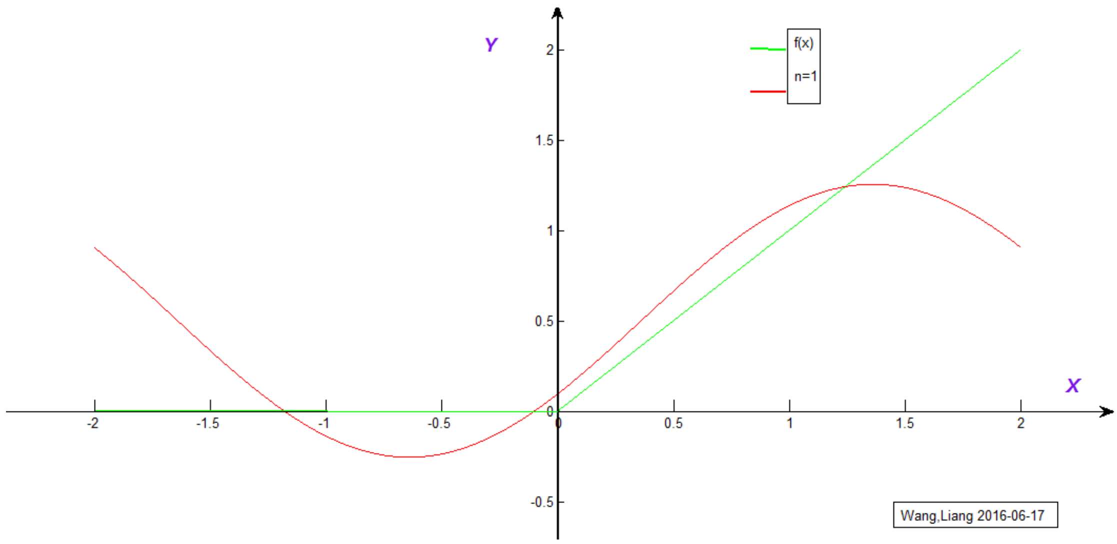
$$a_0 = \frac{1}{4}\int_0^2 x dx = \frac{1}{2}\ ,$$

$$a_n = \frac{1}{2}\int_0^2 x\,cos(n\frac{\pi x}{2})\,dx = \frac{1}{2}(\frac{2}{n\pi})^2(cos(n\pi)-1) = \frac{1}{2}\left(\frac{2}{n\pi}\right)^2((-1)^n-1),$$

$$b_n = \frac{1}{2}\int_0^2 x\,sin\left(n\frac{\pi x}{2}\right)dx = -\frac{2\,cos(n\pi)}{n\pi} = \frac{2}{n\pi}(-1)^{n+1}$$

for $n \geq 1$. Therefore, we have

$$f(x) \sim \frac{1}{2} + \sum_{n=1}^{\infty}\left[\frac{2}{n^2\pi^2}((-1)^n-1)\,cos\left(n\frac{\pi x}{2}\right) + \frac{2}{n\pi}(-1)^{n+1}\,sin(n\frac{\pi x}{2})\right]$$



Fourier series ---Exponential

In this Section we show how a Fourier series can be expressed more concisely if we introduce the imagery number $i = \sqrt{-1}$ and Euler formula:

$$e^{ix} = cos\,x + i\,sin\,x$$

So far we have discussed the trigonometric form of a Fourier series i.e. we have represented

Functions of period $[-L,\ L]$ in the terms of sinusoids, and possibly a constant term, using

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos\left(n\frac{\pi x}{L}\right) + b_n \sin(n\frac{\pi x}{L}))$$

If we use angular frequency

$$w_0 = \frac{2\pi}{T} = \frac{2\pi}{2L} = \frac{\pi}{L}$$

We could obtain more concise form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nw_0 x) + b_n \sin(nw_0 x))$$

We have seen that the Fourier coefficients are calculated using the following formula earlier:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(nw_0 x)dx \qquad\qquad n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(nw_0 x)dx \qquad\qquad n = 1, 2, \ldots$$

An alternative, more concise form, of a Fourier series is available using complex quantities.
As Euler formula

$$e^{ix} = \cos x + i \sin x$$

$$e^{-ix} = \cos x - i \sin x$$

We can deduce

$$cos\, x = \frac{1}{2}(e^{ix} + e^{-ix})$$

$$sin\, x = \frac{1}{2i}(e^{ix} - e^{-ix})$$

(You can imagine by vector add & subtract, it should be easily intuitive)

Using these results we can redraft an expression of the form

$$a_n\, cos(nw_0x) + b_n\, sin(nw_0x)$$

To

$$\frac{a_n}{2}\left(e^{inw_0x} + e^{-inw_0x}\right) + \frac{b_n}{2i}\left(e^{inw_0x} - e^{-inw_0x}\right)$$

$$\frac{(a_n - ib_n)}{2}e^{inw_0x} + \frac{(a_n + ib_n)}{2}e^{-inw_0x}$$

We Let

$$A_n = \frac{(a_n - ib_n)}{2}$$

And it has complex conjugate:

$$A_n^* = \frac{(a_n + ib_n)}{2}$$

Now we can clearly rewrite the trigonometric Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\, cos(nw_0x) + b_n\, sin(nw_0x))$$

As

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (A_n e^{inw_0 x} + A_n^* e^{-inw_0 x})$$

For $b_0 = 0$, we can combine $\frac{a_0}{2} = A_0 = \frac{(a_0 - i0)}{2}$

The Second part

$$\sum_{n=1}^{\infty} (A_n^* e^{-inw_0 x})$$

If we use symbol

$$B_{-n} = A_n^* \qquad (B_{-1}, B_{-2}, B_{-3} \dots means\ A_1^*, A_2^*, A_3^*\ )$$

We can write it

$$\sum_{n=1}^{\infty} \left(A_n^* e^{-inw_0 x}\right) = \sum_{n=1}^{\infty} (B_{-n} e^{-inw_0 x}) = \sum_{n=-1}^{-\infty} (B_n e^{inw_0 x})$$

If we define Series C_n $[-\infty, +\infty]$ as follow combine series:

$$\dots\ B_{-3},\ B_{-2}, \qquad B_{-1,}\ A_0, \qquad A_1, \qquad A_2, \qquad A_3 \dots$$

We could concise write Fourier Complex Series as follow

$$f(x) = \sum_{-\infty}^{\infty} (C_n e^{inw_0 x})$$

The complex Fourier coefficients C_n can be readily obtained as follows using its original definition for $a_n,\ b_n$.

Firstly

$$C_0 = \frac{a_0}{2} = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

For n = 1, 2, 3 . . . we have

$$C_n = A_n = \frac{(a_n - ib_n)}{2} = \frac{1}{2L}\int_{-L}^{L} f(x)[cos(nw_0x) - isin(nw_0x)]dx$$

$$C_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-inw_0x}dx$$

Meanwhile we have

$$C_n = B_{-n} = A_n^* = \frac{(a_n + ib_n)}{2} = \frac{1}{2L}\int_{-L}^{L} f(x)e^{inw_0x}dx$$

This last expression is equivalent to stating that for n = −1,−2,−3 . . .

$$C_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-inw_0x}dx$$

So we could say for n = 0,±1,±2,±3 . .

$$C_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-inw_0x}dx$$

Conclusion:

Fourier series in Complex Form could express as follow

A function $f(x)$ of period $[-L, L]$ has a complex Fourier series

$$f(x) = \sum_{-\infty}^{\infty} (C_n e^{inw_0 x})$$

Where

$$C_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-inw_0 x} dx$$

Fourier series ---Convergence

What a fantastic pure mathematics topic---TBD☺

References:

http://www.sosmath.com/fourier/fourier1/fourier1.html
https://en.wikipedia.org/wiki/Proofs_of_trigonometric_identities