常微分方程作业

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2016年11月26日

二:考虑方程组 $X' = A(t)X$，其中$A(t)$是区间$[a,b]$上的连续$n \times n$矩阵,它的元为$a_{ij}(t)(i,j=1,2\cdots,n)$

(1)如果$x_1(t), x_2(t), \cdots x_n(t)$是方程的任意$n$个解，那么它们的朗斯基行列式$W[x_1(t), x_2(t), \cdots, x_n(t)] \equiv W(t)$满足一阶线性微分方程

$$W' = [a_{11}(t) + a_{22}(t) + \cdots + a_{nn}(t)]W$$

解：由题知$x_k'(t) = A(t)x_k(t) = \begin{pmatrix} a_{11}(t) & \cdots & a_{1k}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nk}(t) & \cdots & a_{nn}(t) \end{pmatrix} \begin{pmatrix} x_{1k}(t) \\ \vdots \\ x_{kk}(t) \\ \vdots \\ x_{kn}(t) \end{pmatrix}$

$W(t) = \begin{vmatrix} x_{11}(t) & \cdots & x_{1n}(t) \\ \vdots & \ddots & \vdots \\ x_{n1}(t) & \cdots & x_{nn}(t) \end{vmatrix} \Rightarrow$

$W'(t) = \sum_{k=1}^{n} \begin{vmatrix} x_{11}(t) & \cdots & x_{1k}(t) & \cdots & x_{1n}(t) \\ \vdots & & \vdots & & \vdots \\ x_{k1}'(t) & \cdots & x_{kk}'(t) & \cdots & x_{kn}'(t) \\ \vdots & & \vdots & & \vdots \\ x_{n1}(t) & \cdots & x_{nk}(t) & \cdots & x_{nn}(t) \end{vmatrix}$

$= \sum_{k=1}^{n} \begin{vmatrix} x_{11}(t) & \cdots & x_{1k}(t) \\ \vdots & & \vdots \\ a_{k1}(t)x_{11}(t) + \cdots + a_{kk}(t)x_{k1}(t) + \cdots + a_{kn}(t)x_{n1}(t) & \cdots & a_{k1}(t)x_{1k}(t) + \cdots + a_{kk}(t)x_{kk}(t) + \cdots + \\ \vdots & & \vdots \\ x_{n1}(t) & \cdots & x_{nk}(t) \end{vmatrix}$

$$= \sum_{k=1}^{n} \begin{vmatrix} x_{11}(t) & \cdots & x_{1k}(t) & \cdots & x_{1n}(t) \\ \vdots & & \vdots & & \vdots \\ a_{kk}(t)x_{k1}(t) & \cdots & a_{kk}(t)x_{kk}(t) & \cdots & a_{kk}(t)x_{kn}(t) \\ \vdots & & \vdots & & \vdots \\ x_{n1}(t) & \cdots & x_{nk}(t) & \cdots & x_{nn}(t) \end{vmatrix}$$
$= [a_{11}(t) + a_{22}(t) + \cdots + a_{nn}(t)]W$

(2)解上述一阶线性微分方程，证明以下公式

$$W(t) = W(t_0)e^{\int_{t_0}^{t}[a_{11}(s)+a_{22}(s)+\cdots+a_{nn}(s)]\mathrm{d}s}$$

解：$\frac{\mathrm{W}(t)}{\mathrm{d}t} = [a_{11}(s) + a_{22}(s) + \cdots + a_{nn}(s)]W(t) \Rightarrow$
$\ln W(t) = [a_{11}(s) + a_{22}(s) + \cdots + a_{nn}(s)] \Rightarrow$
$W(t) = Ce^{\int_{t_0}^{t}[a_{11}(s)+a_{22}(s)+\cdots+a_{nn}(s)]\mathrm{d}s}$
$\because W(t_0)$满足初值条件
$\therefore W(t) = W(t_0)e^{\int_{t_0}^{t}[a_{11}(s)+a_{22}(s)+\cdots+a_{nn}(s)]\mathrm{d}s}$

三：$A(t)$为在$[a,b]$上连续的$n \times n$实矩阵，$\Phi(t)$为方程$x' = A(t)x$的基解矩阵，而$x = \varphi(t)$为其一解，证明：

(1)对于方程$y' = -A^T(t)$的任意解$y = \psi(t)$必有$\psi^T\varphi(t) =$常数

解：$\psi^T\varphi(t)$对t求导
$\because (y')^T = -y^TA(t) \Longrightarrow (\psi'(t))^T = -\psi^T(t)A(t)$
$\therefore (\psi^T(t))'\varphi(t) + \psi^T\varphi'(t) = -\psi^T(t)A(t)\varphi(t) + \psi^T(t)A(t)\varphi(t) = 0$
$\therefore \psi^T\varphi(t) =$常数

(2)$\Psi(t)$为方程$y' = -A^T(t)$的基解矩阵的充要条件为存在非奇异的常数矩阵C使得$\Psi^T(t)\Phi(t) = C$

解：必要性：$\because y' = -A^T(t)$存在基解矩阵
$\therefore y' = -A^T(t)$存在n个线性无关的解构成矩阵$\Psi = [y_1, y_2, \cdots, y_n]$
由上题知对于方程$y' = -A^T(t)$的任意解$y = \psi(t)$必有$\psi^T\varphi(t) =$常数
$\therefore$必存在$x' = A(t)x$的基解矩阵$\Phi(t)$使得$\Psi^T(t)\Phi(t) = C$
$\because |C| = |\Phi(t)| \times |\Psi(t)|$且后两者均为基解矩阵
$\therefore C$非奇异
充分性：$\because \Psi^T(t)\Phi(t) = C, C$为非奇异矩阵，$\Phi(t)$为基解矩阵
$\therefore \Psi(t)$为非奇异矩阵
$\therefore \Phi = (\Psi^T)^{-1}C$
满足$\Phi' = A(t)\Phi \Longrightarrow [(\Psi^T)^{-1}]'C = A(t)(\Psi^T)^{-1}C \Longrightarrow (\Psi^{-1})' = \Psi^{-1}A^T(t)$
$\because (\Psi\Psi^{-1})' = \Psi'\Psi^{-1} + \Psi(\Psi^{-1})' = 0 \Longrightarrow \Psi'\Psi^{-1} = -\Psi(\Psi^{-1})'$
$\therefore \Psi(\Psi^{-1})' = A^T(t) \Longrightarrow -\Psi'\Psi^{-1} = A^T(t) \Longrightarrow \Psi' = -A^T(t)\Psi$
$\therefore \Psi$为基解矩阵