常微分方程作业

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题十：求方程通解$x'-8x=e^{2t}$

解：考查其特征方程$\lambda^2-8=0 \Longrightarrow \lambda_1=2\lambda_2=\sqrt{3}i-2\lambda_3=-\sqrt{3}i-2$
$\because f(t)=e^{2t}$
$\therefore$方程有特解$x=Ate^{2t}$代入原式得$A(12e^{2t}+8te^{2t})-8Ate^{2t}=e^{2t}$
$\therefore A=\frac{1}{12}$
$\therefore$通解为$x=c_1e^{2t}+c_2e^{(\sqrt{3}i-2)t}+c_3e^{(-\sqrt{3}i-2)t}+\frac{te^{2t}}{12}$

题十二:已知方程组

$$y=\begin{cases}\frac{\mathrm{d}x_1}{\mathrm{d}t}=x_1\cos^2 t-x_2(1-\sin t\cos t)\\ \frac{\mathrm{d}x_2}{\mathrm{d}t}=x_1(1+\cos t\sin t)+x_2\sin^2 t\end{cases}$$

有解$\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}-\sin t\\ \cos t\end{pmatrix}$求其通解

解：$\begin{pmatrix}x_1\\x_2\end{pmatrix}'=\begin{pmatrix}\cos^2 t & -1+sint\cos t\\ 1+\sin t\cos t & \sin^2 t\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$ 由$P217$题四

题一：A为$n\times n$矩阵，证明对任意正整数k都有$(\exp\ A)^k=\exp\ kA$

证明：$k=0$时，$(\exp\ A)^0=I, \exp\ 0=I,$命题成立
k为正整数时，$(\exp\ A)^k=\underbrace{(\exp\ A)\times(\exp\ A)\times\cdots\times(\exp\ A)}=\exp\ kA$
k为负整数时, $(\exp\ A)^k=[(\exp\ A)^{(}-1)]^{-k}$
由以上结论得
$(\exp\ A)^k=\exp\,(-k)(-A)=\exp\ kA$

题五：求方程组$x'=Ax$的基解矩阵，并求满足初值条件$\varphi_0=\eta$的解$\varphi(t)$

$$A=\begin{pmatrix}1&0&3\\8&1&-1\\5&1&-1\end{pmatrix},\eta=\begin{pmatrix}0\\-2\\-7\end{pmatrix}$$

解：$\lambda I-A=\begin{pmatrix}\lambda-1&0&-3\\-8&\lambda&1\\-5&-1&\lambda+1\end{pmatrix}$
$\therefore f(\lambda)=\lambda^3-\lambda^2-15\lambda-9=(\lambda+3)(\lambda-2+\sqrt{7})(\lambda-2-\sqrt{7})$

$\therefore \lambda_1 = -3, \lambda_2 = 2+\sqrt{7}, \lambda_3 = 2-\sqrt{7}$

算得矩阵属于$-3, 2+\sqrt{7}, 2-\sqrt{7}$的三个特征向量为

$$\begin{pmatrix} -3 \\ 7 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 4\sqrt{7}-5 \\ \sqrt{7}+1 \end{pmatrix}, \begin{pmatrix} 3 \\ -4\sqrt{7}+5 \\ 1-\sqrt{7} \end{pmatrix}$$

$\therefore$ 基解矩阵 $\Phi(t) = \begin{pmatrix} -3e^{e^{-3t}} & 3e^{(2+\sqrt{7})t} & 3e^{(2-\sqrt{7})t} \\ 7e^{-3t} & (4\sqrt{7}-5)e^{(2+\sqrt{7})t} & (-4\sqrt{7}+5)e^{(2-\sqrt{7})t} \\ 4e^{-3t} & (\sqrt{7}+1)e^{(2+\sqrt{7})t} & (1-\sqrt{7})e^{(2-\sqrt{7})t} \end{pmatrix}$

$\therefore \varphi(t) = \Phi(t)\Phi(0)^{-1}\eta + \Phi(t)\int_0^t \Phi^{-1}(s) \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \mathrm{d}s$

$= \frac{1}{4\sqrt{7}} \begin{pmatrix} \frac{52\sqrt{7}}{3}e^{-3t} + \frac{-4-26\sqrt{7}}{3}e^{(2+\sqrt{7})t} + \frac{-4-26\sqrt{7}}{3}e^{(2-\sqrt{7})t} \\ \frac{-364\sqrt{7}}{9}e^{-3t} + \frac{-748+146\sqrt{7}}{9}e^{(2+\sqrt{7})t} + \frac{748+146\sqrt{7}}{9}e^{(2-\sqrt{7})t} \\ \frac{-208\sqrt{7}}{9}e^{-3t} + \frac{-178-22\sqrt{7}}{9}e^{(2+\sqrt{7})t} + \frac{178-22\sqrt{7}}{9}e^{(2-\sqrt{7})t} \end{pmatrix}$