Question 1a [1 mark]

Let $y = x^2 \cos(x)$. Find $\frac{dy}{dx}$.

Question 1b [2 marks]

Let $f(x) = 6\sqrt{x+1} + 5$. Find the gradient of the tangent to $y = f(x)$ at $x = 8$.

Question 2 [2 marks]

Let $g(x)$ be a function defined for $x > -\frac{3}{2}$ so that $g'(x) = \frac{1}{2x+3}$ and $g(1) = 0$. Find $g(x)$.

Question 3a [1 mark]

Let $f : [0, 2\pi] \to R$, $f(x) = 2\cos(2x) + 1$.

State the range of f.

Question 3b [3 marks]

Let $f : [0, 2\pi] \to R$, $f(x) = 2\cos(2x) + 1$.

Solve $f(x) = 0$ for x.

Question 3c [2 marks]

Let $f : [0, 2\pi] \to R$, $f(x) = 2\cos(2x) + 1$.

Sketch the graph of $y = f(x)$ for $x \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ on the axes provided. Label the endpoints with their coordinates.

Question 4a [2 marks]

The probability distribution for the discrete random variable X is given in the table below, where k is a positive real number.

x	0	1	2	3
$\Pr(X = x)$	$\frac{4}{k}$	$\frac{2k}{75}$	$\frac{k}{75}$	$\frac{2}{k}$

Show that $k = 10$ or $k = 15$.

Question 4b.i [1 mark]

Let $k = 15$.

Find $\Pr(X > 1)$.

Question 4b.ii [1 mark]

Let $k = 15$.

Find $E(X)$.

Question 5a [2 marks]

Solve $e^{2x} - 8e^x + 7 = 0$ for x.

Question 5b [2 marks]

Let $g(x) = e^{2x} - 8e^x + 7$, where $x \in R$. The function $g(x)$ has exactly one stationary point, a local minimum.

Find the largest value of a such that when g is restricted to the domain $(-\infty, a]$, it has an inverse function.

Question 6a [1 mark]

Consider the binomial random variable $X \sim \text{Bi}(6, \frac{1}{4})$.

Find $\text{var}(X)$.

Question 6b [2 marks]

Consider the binomial random variable $X \sim \text{Bi}(6, \frac{1}{4})$.

Determine $\Pr(X \geq 5)$. Give your answer in the form $\frac{a}{2^b}$, where $a, b \in Z$.

Question 7a [1 mark]

Let $f : R \to R, f(x) = x^3 - x^2 - 16x - 20$.

Verify that $x = 5$ is a solution of $f(x) = 0$.

Question 7b [2 marks]

Express $f(x)$ in the form $(x + d)^2(x - 5)$, where $d \in R$.

Question 7c [1 mark]

Consider the graph of $y = f(x)$, as shown.

Complete the coordinate pairs of all axial intercepts of $y = f(x)$.

Question 7d.i [1 mark]

Let $g : R \to R, g(x) = x + 2$.

State the coordinates of the stationary point of inflection for the graph of $y = f(x)g(x)$.

Question 7d.ii [1 mark]

Write down the values of x for which $f(x)g(x) \geq 0$.

Question 8a [3 marks]

Consider $f(x) = \begin{cases} \frac{3}{8}(4 - 3x) & 0 \leq x \leq \frac{4}{3} \\ 0 & \text{otherwise} \end{cases}$.

The continuous random variable X has probability density function $f(x)$.

Find k such that $\Pr(X > k) = \frac{9}{16}$.

Question 8b [2 marks]

The function $h(x)$ is a transformation of $f(x)$ such that $h(x) = mf(x) + n$, where m and n are real numbers.

Find $\int_0^{4/3} h(x)\, dx$ in terms of m and n.

Question 9a [3 marks]

Consider the functions $f : R \setminus \{1\} \to R, f(x) = \frac{w^2}{(x-1)^2}$ and $g : R \to R, g(x) = (x - w)^2$, where $w \in R$.

If $w = -3$, find the four solutions to $f(x) = g(x)$.

Question 9b.i [2 marks]

Consider the case where $w > 0$.

Find, in terms of w, the coordinates of the minimum point of the graph of $y = (x-1)(x-w)$.

Question 9b.ii [2 marks]

Hence, or otherwise, find the positive values of w for which $f(x) = g(x)$ has exactly three solutions.

Solutions

Question 1a

$2x\cos(x) - x^2\sin(x)$

Marking guide:

- Apply product rule: $u = x^2, v = \cos(x)$.
- $\frac{dy}{dx} = 2x\cos(x) - x^2\sin(x)$.
- A – brackets not required around argument, either order, clearly only 2 terms.

Question 1b

1

Marking guide:

- Write $f(x) = 6(x+1)^{1/2} + 5$.
- M – attempt at derivative: $f'(x) = \frac{3}{\sqrt{x+1}}$.
- A – $f'(8) = \frac{3}{\sqrt{9}} = \frac{3}{3} = 1$.

Question 2

$g(x) = \frac{1}{2}\log_e(2x+3) - \frac{1}{2}\log_e(5)$

Marking guide:

- M – integrate to get $g(x) = \frac{1}{2}\log_e(2x+3) + c$, must have $+c$.
- A – apply $g(1) = 0$: $c = -\frac{1}{2}\log_e(5)$. Any correct form, base e or ln.

Question 3a

$[-1, 3]$

Marking guide:

- Range: $[-1, 3]$ or equivalent, between -1 and 3 inclusive.
- A – not $y =$ or $f =$, just the interval.

Question 3b

$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

Marking guide:

- $2\cos(2x) + 1 = 0 \implies \cos(2x) = -\frac{1}{2}$.
- M – see base angle of $\frac{\pi}{3}$.
- M – see 2 correct results for $2x$.
- A – 4 correct values only: $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$.

Question 3c

Graph from $(\frac{\pi}{2}, -1)$ to $(\frac{3\pi}{2}, -1)$ with correct shape

Marking guide:

- M – accurately represents correct shape with symmetry and correct range, may extend beyond domain.
- A – correct endpoints labelled: $(\frac{\pi}{2}, -1)$ and $(\frac{3\pi}{2}, -1)$, graph restricted to domain.

Question 4a

$k = 10$ or $k = 15$

Marking guide:

- Sum of probabilities $= 1$: $\frac{4}{k} + \frac{2k}{75} + \frac{k}{75} + \frac{2}{k} = 1$.
- $\frac{6}{k} + \frac{3k}{75} = 1 \implies \frac{6}{k} + \frac{k}{25} = 1$.
- M – multiply through: $150 + k^2 = 25k$.
- A – $k^2 - 25k + 150 = 0 \implies (k-10)(k-15) = 0$.

Question 4b.i

$\frac{1}{3}$

Marking guide:
- With $k = 15$: $\Pr(X = 2) = \frac{15}{75} = \frac{1}{5}$, $\Pr(X = 3) = \frac{2}{15}$.
- $\Pr(X > 1) = \frac{1}{5} + \frac{2}{15} = \frac{3+2}{15} = \frac{5}{15} = \frac{1}{3}$.

Question 4b.ii

$\frac{6}{5}$ or 1.2

Marking guide:
- Probs: $\frac{4}{15}, \frac{6}{15}, \frac{3}{15}, \frac{2}{15}$.
- $E(X) = 0 \cdot \frac{4}{15} + 1 \cdot \frac{6}{15} + 2 \cdot \frac{3}{15} + 3 \cdot \frac{2}{15} = \frac{6+6+6}{15} = \frac{18}{15} = \frac{6}{5}$.

Question 5a

$x = 0$ or $x = \log_e(7)$

Marking guide:
- Let $u = e^x$: $u^2 - 8u + 7 = 0$.
- M $- (u - 1)(u - 7) = 0$, so $e^x = 1$ or $e^x = 7$.
- A $- x = 0$ or $x = \log_e(7)$.

Question 5b

$a = \log_e(4)$

Marking guide:
- Find stationary point: $g'(x) = 2e^{2x} - 8e^x = 0$.
- M $- 2e^x(e^x - 4) = 0 \implies e^x = 4$.
- A $- x = \log_e(4)$. Function is one-to-one on $(-\infty, \log_e(4)]$.

Question 6a

$\frac{9}{8}$

Marking guide:
- $\text{var}(X) = np(1 - p) = 6 \times \frac{1}{4} \times \frac{3}{4} = \frac{18}{16} = \frac{9}{8}$.

Question 6b

$\frac{19}{2^{12}}$

Marking guide:
- $\Pr(X = 5) = \binom{6}{5}(\frac{1}{4})^5(\frac{3}{4})^1 = \frac{18}{4^6}$.
- $\Pr(X = 6) = (\frac{1}{4})^6 = \frac{1}{4^6}$.
- M – correct binomial expressions.
- A – $\Pr(X \geq 5) = \frac{19}{4^6} = \frac{19}{2^{12}}$.

Question 7a

$f(5) = 125 - 25 - 80 - 20 = 0$

Marking guide:
- Substitute: $f(5) = 125 - 25 - 80 - 20 = 0$ ✓.

Question 7b

$f(x) = (x + 2)^2(x - 5)$, so $d = 2$

Marking guide:
- Divide $f(x)$ by $(x - 5)$: $f(x) = (x - 5)(x^2 + 4x + 4)$.
- M – polynomial division or factor theorem.
- A – $(x - 5)(x + 2)^2$, so $d = 2$.

Question 7c

$(-2, 0), (5, 0), (0, -20)$

Marking guide:
- x-intercepts at $x = -2$ (touch) and $x = 5$ (cross).
- y-intercept: $f(0) = -20$.
- All three coordinate pairs: $(-2, 0), (5, 0), (0, -20)$.

Question 7d.i

$(-2, 0)$

Marking guide:
- $f(x)g(x) = (x+2)^2(x-5)(x+2) = (x+2)^3(x-5)$.
- Stationary point of inflection at $x = -2$ (triple root).
- Coordinates: $(-2, 0)$.

Question 7d.ii

$x = -2$ or $x \geq 5$

Marking guide:
- $y = (x+2)^3(x-5)$. Quartic, positive leading coefficient.
- Zero at $x = -2$ (inflection, touches) and $x = 5$ (crosses).
- $y \geq 0$ when $x = -2$ or $x \geq 5$.

Question 8a

$k = \frac{1}{3}$

Marking guide:
- $\Pr(X > k) = \int_k^{4/3} \frac{3}{8}(4 - 3x)\,dx = \frac{9}{16}$.
- Note: $f(x)$ is linear from $\frac{3}{2}$ at $x = 0$ to 0 at $x = \frac{4}{3}$. The region from k to $\frac{4}{3}$ forms a triangle.
- Area $= \frac{1}{2}(\frac{4}{3} - k) \cdot f(k) = \frac{1}{2}(\frac{4}{3} - k) \cdot \frac{3}{8}(4 - 3k) = \frac{9}{16}(\frac{4}{3} - k)^2$.
- Solve $\frac{9}{16}(\frac{4}{3} - k)^2 = \frac{9}{16}$, so $(\frac{4}{3} - k)^2 = 1$.
- $\frac{4}{3} - k = \pm 1$, giving $k = \frac{1}{3}$ or $k = \frac{7}{3}$.
- Since $0 \leq k \leq \frac{4}{3}$, $k = \frac{1}{3}$.

Question 8b

$m + \frac{4n}{3}$

Marking guide:
- $\int_0^{4/3} h(x)\,dx = \int_0^{4/3}[mf(x) + n]\,dx$.
- $= m\int_0^{4/3} f(x)\,dx + n\int_0^{4/3} 1\,dx$.
- Since f is a PDF, $\int_0^{4/3} f(x)\,dx = 1$.
- M – recognise $\int f = 1$ and set up correctly.
- A – $= m + \frac{4n}{3}$.

Question 9a

$x = -1 - \sqrt{7},\ -1 + \sqrt{7},\ -2,\ 0$

Marking guide:
- $\frac{9}{(x-1)^2} = (x+3)^2 \implies \left(\frac{3}{x-1}\right)^2 = (x+3)^2$.
- Take square root: $(x-1)(x+3) = \pm 3$.
- Case 1: $x^2 + 2x - 3 = 3 \implies x^2 + 2x - 6 = 0 \implies x = -1 \pm \sqrt{7}$.
- Case 2: $x^2 + 2x - 3 = -3 \implies x^2 + 2x = 0 \implies x(x+2) = 0 \implies x = 0, -2$.
- M – setting up squared equation correctly.
- M – solving one case.
- A – all four solutions.

Question 9b.i

$\left(\frac{w+1}{2}, -\frac{(w-1)^2}{4}\right)$

Marking guide:
- Axis of symmetry: $x = \frac{1+w}{2}$.
- M – find x-coordinate of vertex.
- $y = (\frac{w+1}{2} - 1)(\frac{w+1}{2} - w) = \frac{w-1}{2} \cdot \frac{1-w}{2} = -\frac{(w-1)^2}{4}$.
- A – coordinates $\left(\frac{w+1}{2}, -\frac{(w-1)^2}{4}\right)$.

Question 9b.ii

$w = 3 - 2\sqrt{2}$ or $w = 3 + 2\sqrt{2}$

Marking guide:

- $f(x) = g(x) \implies (x-1)(x-w) = \pm w$.
- The parabola $y = (x-1)(x-w)$ intersects $y = w$ (2 solutions always for $w > 0$) and $y = -w$.
- For exactly 3 total solutions, $y = -w$ must be tangent to the parabola (1 solution).
- Set vertex value equal to $-w$: $-\frac{(w-1)^2}{4} = -w$.
- $(w-1)^2 = 4w \implies w^2 - 6w + 1 = 0$.
- A $- w = 3 \pm 2\sqrt{2}$. Both positive.