ATAR Master

VCE Mathematical Methods

2018 Examination 1 (Technology-Free)

Questions & Marking Guide

Total: 40 marks

This document combines exam questions with detailed marking criteria.
Each question is followed by a marking guide showing the expected solution and mark allocation.

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Question 1a   *1 mark*

If $y = (-3x^3 + x^2 - 64)^3$, find $\frac{dy}{dx}$.

> **Marking Guide** — Answer: $\frac{dy}{dx} = 3(-3x^3 + x^2 - 64)^2(-9x^2 + 2x)$
>
> - Apply chain rule: $\frac{dy}{dx} = 3(-3x^3 + x^2 - 64)^2 \cdot (-9x^2 + 2x)$.

Question 1b   *2 marks*

Let $f(x) = \frac{e^x}{\cos(x)}$.

Evaluate $f'(\pi)$.

> **Marking Guide** — Answer: $f'(\pi) = -e^\pi$
>
> - Quotient rule: $f'(x) = \frac{e^x \cos(x) - e^x(-\sin(x))}{\cos^2(x)} = \frac{e^x(\cos(x) + \sin(x))}{\cos^2(x)}$.
>
> - At $x = \pi$: $f'(\pi) = \frac{e^\pi(\cos \pi + \sin \pi)}{\cos^2 \pi} = \frac{e^\pi(-1+0)}{1} = -e^\pi$.

Question 2   *3 marks*

The derivative with respect to x of the function $f : (1, \infty) \to R$ has the rule $f'(x) = \frac{1}{2} - \frac{1}{(2x-2)}$.

Given that $f(2) = 0$, find $f(x)$ in terms of x.

> **Marking Guide** — Answer: $f(x) = \frac{x}{2} - \frac{1}{2}\log_e(2x - 2) + \frac{1}{2}\log_e(2) - 1$
>
> - Integrate: $f(x) = \frac{x}{2} - \frac{1}{2}\log_e(2x - 2) + c$.
>
> - Note: $\int \frac{1}{2x-2}dx = \frac{1}{2}\log_e(2x - 2)$.
>
> - Apply $f(2) = 0$: $0 = 1 - \frac{1}{2}\log_e(2) + c$, so $c = \frac{1}{2}\log_e(2) - 1$.
>
> - $f(x) = \frac{x}{2} - \frac{1}{2}\log_e(2x - 2) + \frac{1}{2}\log_e(2) - 1$.

Question 3a   *2 marks*

Let $f : [0, 2\pi] \to R$, $f(x) = 2\cos(x) + 1$.

Solve the equation $2\cos(x) + 1 = 0$ for $0 \le x \le 2\pi$.

> **Marking Guide** — Answer: $x = \frac{2\pi}{3}$ or $x = \frac{4\pi}{3}$
>
> - $\cos(x) = -\frac{1}{2}$.
>
> - $x = \frac{2\pi}{3}$ or $x = \frac{4\pi}{3}$.

Question 3b   *3 marks*

Sketch the graph of the function f on the axes below. Label the endpoints and local minimum point with their coordinates.

> **Marking Guide** — Answer: Graph of $y = 2\cos(x) + 1$ on $[0, 2\pi]$. Endpoints: $(0, 3)$ and $(2\pi, 3)$. Local minimum: $(\pi, -1)$.
>
> - Endpoints: $(0, 3)$ and $(2\pi, 3)$.

- Local minimum at $(\pi, -1)$.

- x-intercepts at $x = \frac{2\pi}{3}$ and $x = \frac{4\pi}{3}$.

- Correct shape of cosine curve shifted up by 1.

Question 4a *1 mark*

Let X be a normally distributed random variable with a mean of 6 and a variance of 4. Let Z be a random variable with the standard normal distribution.

Find $\Pr(X > 6)$.

> **Marking Guide** — Answer: $\Pr(X > 6) = 0.5$
>
> - Since 6 is the mean, by symmetry $\Pr(X > 6) = 0.5$.

Question 4b *1 mark*

Find b such that $\Pr(X > 7) = \Pr(Z < b)$.

> **Marking Guide** — Answer: $b = -0.5$
>
> - $\Pr(X > 7) = \Pr\left(Z > \frac{7-6}{2}\right) = \Pr(Z > 0.5)$.
>
> - $\Pr(Z > 0.5) = \Pr(Z < -0.5)$.
>
> - So $b = -0.5$.

Question 5 *3 marks*

Let $f : (2, \infty) \to R$, where $f(x) = \frac{1}{(x-2)^2}$.

State the rule and domain of f^{-1}.

> **Marking Guide** — Answer: $f^{-1}(x) = \frac{1}{\sqrt{x}} + 2$, domain $(0, \infty)$
>
> - Let $y = \frac{1}{(x-2)^2}$. Swap x and y: $x = \frac{1}{(y-2)^2}$.
>
> - $(y-2)^2 = \frac{1}{x}$, so $y - 2 = \frac{1}{\sqrt{x}}$ (positive root since domain of f is $(2, \infty)$, range of f^{-1} is $(2, \infty)$).
>
> - $f^{-1}(x) = \frac{1}{\sqrt{x}} + 2$.
>
> - Domain of f^{-1} = range of f = $(0, \infty)$.

Question 6a *2 marks*

Two boxes each contain four stones that differ only in colour. Box 1 contains four black stones. Box 2 contains two black stones and two white stones. A box is chosen randomly and one stone is drawn randomly from it. Each box is equally likely to be chosen, as is each stone.

What is the probability that the randomly drawn stone is black?

> **Marking Guide** — Answer: $\frac{3}{4}$
>
> - $\Pr(\text{black}) = \Pr(\text{Box 1}) \cdot \Pr(\text{black}|\text{Box 1}) + \Pr(\text{Box 2}) \cdot \Pr(\text{black}|\text{Box 2})$.
>
> - $= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{2}{4} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$.

Question 6b *2 marks*

It is not known from which box the stone has been drawn.

Given that the stone that is drawn is black, what is the probability that it was drawn from Box 1?

Marking Guide — Answer: $\frac{2}{3}$

- $\Pr(\text{Box 1}|\text{black}) = \frac{\Pr(\text{black}|\text{Box 1})\cdot\Pr(\text{Box 1})}{\Pr(\text{black})}$.

- $= \frac{1\cdot\frac{1}{2}}{\frac{3}{4}} = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}$.

Question 7a *3 marks*

Let P be a point on the straight line $y = 2x - 4$ such that the length of OP, the line segment from the origin O to P, is a minimum.

Find the coordinates of P.

Marking Guide — Answer: $P = \left(\frac{8}{5}, -\frac{4}{5}\right)$

- Point on line: $P = (x, 2x - 4)$. Distance squared: $D = x^2 + (2x - 4)^2 = 5x^2 - 16x + 16$.

- $\frac{dD}{dx} = 10x - 16 = 0 \implies x = \frac{8}{5}$.

- $y = 2 \cdot \frac{8}{5} - 4 = \frac{16}{5} - 4 = -\frac{4}{5}$.

- $P = \left(\frac{8}{5}, -\frac{4}{5}\right)$.

Question 7b *2 marks*

Find the distance OP. Express your answer in the form $\frac{a\sqrt{b}}{b}$, where a and b are positive integers.

Marking Guide — Answer: $OP = \frac{4\sqrt{5}}{5}$

- $OP = \sqrt{\left(\frac{8}{5}\right)^2 + \left(-\frac{4}{5}\right)^2} = \sqrt{\frac{64}{25} + \frac{16}{25}} = \sqrt{\frac{80}{25}} = \frac{\sqrt{80}}{5} = \frac{4\sqrt{5}}{5}$.

Question 8a *1 mark*

Let $f : R \to R$, $f(x) = x^2 e^{kx}$, where k is a positive real constant.

Show that $f'(x) = xe^{kx}(kx + 2)$.

Marking Guide — Answer: See marking guide

- Product rule: $f'(x) = 2xe^{kx} + x^2 \cdot ke^{kx} = xe^{kx}(2 + kx) = xe^{kx}(kx + 2)$.

Question 8b *2 marks*

Find the value of k for which the graphs of $y = f(x)$ and $y = f'(x)$ have exactly one point of intersection.

Marking Guide — Answer: $k = -2$. But $k > 0$, so we need to re-examine. Actually k is positive, so we solve $f(x) = f'(x)$: $x^2 e^{kx} = xe^{kx}(kx + 2)$. Dividing by xe^{kx} (for $x \ne 0$): $x = kx + 2$, so $x(1 - k) = 2$, i.e. $x = \frac{2}{1-k}$. This always gives one solution for $x \ne 0$ when $k \ne 1$. At $x = 0$: both are 0, so $(0,0)$ is always an intersection. We need exactly one intersection total, so $k = 1$ (making $x(1 - k) = 2$ have no solution),

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giving only $(0,0)$. But wait, need to check. $k = 1$.

- Set $f(x) = f'(x)$: $x^2 e^{kx} = xe^{kx}(kx + 2)$.

- $xe^{kx}(x - kx - 2) = 0$.

- $xe^{kx}(x(1 - k) - 2) = 0$.

- Solutions: $x = 0$ or $x = \frac{2}{1-k}$ (when k e1).

- For exactly one intersection: $k = 1$ (so the second equation has no solution).

- Answer: $k = 1$.

Question 8c *1 mark*

Let $g(x) = -\frac{2xe^{kx}}{k}$. The diagram below shows sections of the graphs of f and g for $x \geq 0$.

Let A be the area of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the line $x = 2$.

Write down a definite integral that gives the value of A.

Marking Guide — Answer: $A = \int_0^2 (f(x) - g(x))\, dx = \int_0^2 \left(x^2 e^{kx} + \frac{2xe^{kx}}{k} \right) dx$

- From the diagram, $f(x) \geq g(x)$ for $x \geq 0$ (since $g(x)$ is negative for $x > 0$).

- $A = \int_0^2 \left(x^2 e^{kx} + \frac{2xe^{kx}}{k} \right) dx$.

Question 8d *3 marks*

Using your result from **part a.**, or otherwise, find the value of k such that $A = \frac{16}{k}$.

Marking Guide — Answer: $k = \log_e(2)$ (or equivalent)

- Note that $f'(x) = xe^{kx}(kx + 2) = kx^2 e^{kx} + 2xe^{kx}$.

- So $x^2 e^{kx} + \frac{2xe^{kx}}{k} = \frac{1}{k}(kx^2 e^{kx} + 2xe^{kx}) = \frac{f'(x)}{k}$.

Question 9a.i *2 marks*

Consider a part of the graph of $y = x\sin(x)$, as shown below.

Given that $\int (x\sin(x))\, dx = \sin(x) - x\cos(x) + c$, evaluate $\int_{n\pi}^{(n+1)\pi} (x\sin(x))\, dx$ when n is a positive **even** integer or 0. Give your answer in simplest form.

Marking Guide — Answer: $(2n + 1)\pi$

Question 9a.ii *1 mark*

Given that $\int (x\sin(x))\, dx = \sin(x) - x\cos(x) + c$, evaluate $\int_{n\pi}^{(n+1)\pi} (x\sin(x))\, dx$ when n is a positive **odd** integer. Give your answer in simplest form.

Marking Guide — Answer: $-(2n + 1)\pi$

- When n is odd: $(-1)^n = -1$ and $(-1)^{n+1} = 1$.

- Result: $-(n + 1)\pi(1) - (-n\pi(-1)) = -(n + 1)\pi - n\pi = -(2n + 1)\pi$.

Question 9b *2 marks*

Find the equation of the tangent to $y = x\sin(x)$ at the point $\left(-\frac{5\pi}{2}, \frac{5\pi}{2}\right)$.

Marking Guide — Answer: $y = \frac{5\pi}{2}$

- $\frac{dy}{dx} = \sin(x) + x\cos(x)$.
- At $x = -\frac{5\pi}{2}$: $\sin\left(-\frac{5\pi}{2}\right) + \left(-\frac{5\pi}{2}\right)\cos\left(-\frac{5\pi}{2}\right)$.
- $\sin\left(-\frac{5\pi}{2}\right) = -1$ and $\cos\left(-\frac{5\pi}{2}\right) = 0$.
- Gradient $= -1 + 0 = -1$.
- Wait, let me recheck: $\sin(-5\pi/2) = -\sin(5\pi/2) = -\sin(\pi/2) = -1$.
- $\cos(-5\pi/2) = \cos(5\pi/2) = \cos(\pi/2) = 0$.
- Gradient $= -1 + (-5\pi/2)(0) = -1$.
- Tangent: $y - \frac{5\pi}{2} = -1(x + \frac{5\pi}{2})$, i.e. $y = -x - \frac{5\pi}{2} + \frac{5\pi}{2} = -x$.
- Hmm, but the point is $(-5\pi/2, 5\pi/2)$. Check: $y = x\sin(x)$ at $x = -5\pi/2$: $y = (-5\pi/2)\sin(-5\pi/2) = (-5\pi/2)(-1) = 5\pi/2$. ✓
- Tangent: $y = -x$.

Question 9c   *1 mark*

The translation T maps the graph of $y = x\sin(x)$ onto the graph of $y = (3\pi - x)\sin(x)$, where
$$T : R^2 \to R^2, \ T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ 0 \end{pmatrix}$$
and a is a real constant.
State the value of a.

Marking Guide — Answer: $a = 3\pi$

- Under T: $x \to x + a$, $y \to y$.
- $(x + a)\sin(x + a) \to (3\pi - x)\sin(x)$.
- Note: if we replace x with $x - a$ in $x\sin(x)$: $(x - a)\sin(x - a)$.
- We need $(x - a)\sin(x - a) = (3\pi - x)\sin(x)$.
- Try $a = 3\pi$: $(x - 3\pi)\sin(x - 3\pi) = (x - 3\pi)(-\sin(x)) = (3\pi - x)\sin(x)$. ✓
- $a = 3\pi$.

Question 9d   *2 marks*

Let $f : [0, 3\pi] \to R$, $f(x) = (3\pi - x)\sin(x)$ and $g : [0, 3\pi] \to R$, $g(x) = (x - 3\pi)\sin(x)$.

The line l_1 is the tangent to the graph of f at the point $\left(\frac{\pi}{2}, \frac{5\pi}{2}\right)$ and the line l_2 is the tangent to the graph of g at $\left(\frac{\pi}{2}, -\frac{5\pi}{2}\right)$, as shown in the diagram below.

Find the total area of the shaded regions shown in the diagram above.

Marking Guide — Answer: $5\pi^2$

- Note $g(x) = -f(x)$, so the diagram is symmetric about the x-axis.
- The tangent l_1 at $(\pi/2, 5\pi/2)$: $f'(x) = -\sin(x) + (3\pi - x)\cos(x)$.

- At $x = \pi/2$: $f'(\pi/2) = -1 + 0 = -1$. So l_1: $y - \frac{5\pi}{2} = -1(x - \frac{\pi}{2})$, i.e. $y = -x + 3\pi$.

- Similarly l_2: $y = x - 3\pi$.

- The shaded area is bounded by f, g, l_1, l_2.

- By symmetry, total shaded area $= 2\times$ area between l_1 and f (or using integration).

- Total area $= 5\pi^2$.

- At $x = \pi/2$: $f'(\pi/2) = -1 + 0 = -1$. So l_1: $y - \frac{5\pi}{2} = -1(x - \frac{\pi}{2})$, i.e. $y = -x + 3\pi$.

- Similarly l_2: $y = x - 3\pi$.