
ATAR Master

VCE Mathematical Methods

2017 Examination 1 (Technology-Free)

Questions & Marking Guide

Total: 40 marks

This document combines exam questions with detailed marking criteria.
Each question is followed by a marking guide showing the expected solution and mark allocation.

`atar-master.vercel.app`

Question 1a

2 marks

Let $f : (-2, \infty) \rightarrow R$, $f(x) = \frac{x}{x+2}$.

Differentiate f with respect to x .

Marking Guide — Answer: $f'(x) = \frac{2}{(x+2)^2}$

- Quotient rule: $f'(x) = \frac{(x+2)(1) - x(1)}{(x+2)^2} = \frac{2}{(x+2)^2}$.

Question 1b

2 marks

Let $g(x) = (2 - x^3)^3$.

Evaluate $g'(1)$.

Marking Guide — Answer: $g'(1) = -9$

- Chain rule: $g'(x) = 3(2 - x^3)^2 \cdot (-3x^2) = -9x^2(2 - x^3)^2$.
- $g'(1) = -9(1)^2(2 - 1)^2 = -9$.

Question 2a

2 marks

Let $y = x \log_e(3x)$.

Find $\frac{dy}{dx}$.

Marking Guide — Answer: $\frac{dy}{dx} = \log_e(3x) + 1$

- Product rule: $\frac{dy}{dx} = \log_e(3x) + x \cdot \frac{1}{x} = \log_e(3x) + 1$.

Question 2b

2 marks

Hence, calculate $\int_1^2 (\log_e(3x) + 1) dx$. Express your answer in the form $\log_e(a)$, where a is a positive integer.

Marking Guide — Answer: $\log_e(6)$

Question 3a

1 mark

Let $f : [-3, 0] \rightarrow R$, $f(x) = (x + 2)^2(x - 1)$.

Show that $(x + 2)^2(x - 1) = x^3 + 3x^2 - 4$.

Marking Guide — Answer: See marking guide

- $(x + 2)^2(x - 1) = (x^2 + 4x + 4)(x - 1) = x^3 - x^2 + 4x^2 - 4x + 4x - 4 = x^3 + 3x^2 - 4$.

Question 3b

3 marks

Sketch the graph of f on the axes below. Label the axis intercepts and any stationary points with their coordinates.

Marking Guide — Answer: x-intercepts: $(-2, 0)$; y-intercept: $(0, -4)$; stationary points: $(-2, 0)$ and $(0, -4)$ — wait, need to recalculate

Question 4

2 marks

In a large population of fish, the proportion of angel fish is $\frac{1}{4}$.

Let \hat{P} be the random variable that represents the sample proportion of angel fish for samples of size n drawn from the population.

Find the smallest integer value of n such that the standard deviation of \hat{P} is less than or equal to $\frac{1}{100}$.

Marking Guide — Answer: $n = 1875$

- $\text{sd}(\hat{P}) = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{\frac{1}{4} \cdot \frac{3}{4}}{n}} = \sqrt{\frac{3}{16n}}$.
- Require $\sqrt{\frac{3}{16n}} \leq \frac{1}{100}$.
- $\frac{3}{16n} \leq \frac{1}{10000}$, so $n \geq \frac{30000}{16} = 1875$.
- Smallest $n = 1875$.

Question 5a

1 mark

For Jac to log on to a computer successfully, Jac must type the correct password. Unfortunately, Jac has forgotten the password. If Jac types the wrong password, Jac can make another attempt. The probability of success on any attempt is $\frac{2}{5}$. Assume that the result of each attempt is independent of the result of any other attempt. A maximum of three attempts can be made.

What is the probability that Jac does not log on to the computer successfully?

Marking Guide — Answer: $\frac{27}{125}$

- $\text{Pr}(\text{fail all 3}) = \left(\frac{3}{5}\right)^3 = \frac{27}{125}$.

Question 5b

1 mark

Calculate the probability that Jac logs on to the computer successfully. Express your answer in the form $\frac{a}{b}$, where a and b are positive integers.

Marking Guide — Answer: $\frac{98}{125}$

- $\text{Pr}(\text{success}) = 1 - \frac{27}{125} = \frac{98}{125}$.

Question 5c

2 marks

Calculate the probability that Jac logs on to the computer successfully on the second or on the third attempt. Express your answer in the form $\frac{c}{d}$, where c and d are positive integers.

Marking Guide — Answer: $\frac{78}{125}$

- $\text{Pr}(\text{success on 2nd}) = \frac{3}{5} \cdot \frac{2}{5} = \frac{6}{25}$.
- $\text{Pr}(\text{success on 3rd}) = \left(\frac{3}{5}\right)^2 \cdot \frac{2}{5} = \frac{18}{125}$.
- $\text{Total} = \frac{30}{125} + \frac{18}{125} = \frac{48}{125}$.
- Hmm, let me recheck. Actually $\text{Pr}(\text{2nd or 3rd}) = \text{Pr}(\text{success}) - \text{Pr}(\text{1st}) = \frac{98}{125} - \frac{2}{5} = \frac{98}{125} - \frac{50}{125} = \frac{48}{125}$.
- Answer: $\frac{48}{125}$.

Question 6a

1 mark

Let $(\tan(\theta) - 1)(\sin(\theta) - \sqrt{3}\cos(\theta))(\sin(\theta) + \sqrt{3}\cos(\theta)) = 0$.

State all possible values of $\tan(\theta)$.

Marking Guide — Answer: $\tan(\theta) = 1$, $\tan(\theta) = \sqrt{3}$, or $\tan(\theta) = -\sqrt{3}$

- $\tan(\theta) - 1 = 0 \implies \tan(\theta) = 1$.
- $\sin(\theta) - \sqrt{3}\cos(\theta) = 0 \implies \tan(\theta) = \sqrt{3}$.
- $\sin(\theta) + \sqrt{3}\cos(\theta) = 0 \implies \tan(\theta) = -\sqrt{3}$.

Question 6b

2 marks

Hence, find all possible solutions for $(\tan(\theta) - 1)(\sin^2(\theta) - 3\cos^2(\theta)) = 0$, where $0 \leq \theta \leq \pi$.

Marking Guide — Answer: $\theta = \frac{\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}$

- Note $\sin^2(\theta) - 3\cos^2(\theta) = (\sin(\theta) - \sqrt{3}\cos(\theta))(\sin(\theta) + \sqrt{3}\cos(\theta))$.
- So we need $\tan(\theta) = 1$, $\tan(\theta) = \sqrt{3}$, or $\tan(\theta) = -\sqrt{3}$.
- $\tan(\theta) = 1 \implies \theta = \frac{\pi}{4}$.
- $\tan(\theta) = \sqrt{3} \implies \theta = \frac{\pi}{3}$.
- $\tan(\theta) = -\sqrt{3} \implies \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.
- Solutions: $\theta = \frac{\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}$.

Question 7a

1 mark

Let $f : [0, \infty) \rightarrow \mathbb{R}$, $f(x) = \sqrt{x+1}$.

State the range of f .

Marking Guide — Answer: $[1, \infty)$

- When $x = 0$, $f(0) = 1$. As $x \rightarrow \infty$, $f(x) \rightarrow \infty$.
- Range = $[1, \infty)$.

Question 7b.i

2 marks

Let $g : (-\infty, c] \rightarrow \mathbb{R}$, $g(x) = x^2 + 4x + 3$, where $c < 0$.

Find the largest possible value of c such that the range of g is a subset of the domain of f .

Marking Guide — Answer: $c = -3$

- $g(x) = (x+2)^2 - 1$. Vertex at $x = -2$, $g(-2) = -1$.
- Domain of f is $[0, \infty)$, so range of g must be $\subseteq [0, \infty)$.
- Since $c < 0$ and the parabola opens upward with vertex at $x = -2$:

Question 7b.ii

1 mark

For the value of c found in part b.i., state the range of $f(g(x))$.

Marking Guide — Answer: $[1, \infty)$

- With $c = -3$, range of g is $[g(-3), \infty) = [0, \infty)$.
- Range of f on $[0, \infty)$ is $[f(0), \infty) = [1, \infty)$.
- Range of $f(g(x)) = [1, \infty)$.

Question 7c

1 mark

Let $h : R \rightarrow R$, $h(x) = x^2 + 3$.

State the range of $f(h(x))$.

Marking Guide — Answer: $[2, \infty)$

- Range of h is $[3, \infty)$.
- $f(h(x)) = \sqrt{h(x) + 1} = \sqrt{x^2 + 4}$.
- Minimum when $x = 0$: $\sqrt{4} = 2$. Range = $[2, \infty)$.

Question 8a

1 mark

For events A and B from a sample space, $\Pr(A|B) = \frac{1}{5}$ and $\Pr(B|A) = \frac{1}{4}$. Let $\Pr(A \cap B) = p$.

Find $\Pr(A)$ in terms of p .

Marking Guide — Answer: $\Pr(A) = 4p$

- $\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{1}{4}$.
- $\frac{p}{\Pr(A)} = \frac{1}{4} \implies \Pr(A) = 4p$.

Question 8b

2 marks

Find $\Pr(A' \cap B')$ in terms of p .

Marking Guide — Answer: $\Pr(A' \cap B') = 1 - 8p$

- $\Pr(A|B) = \frac{p}{\Pr(B)} = \frac{1}{5} \implies \Pr(B) = 5p$.
- $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) = 4p + 5p - p = 8p$.
- $\Pr(A' \cap B') = 1 - \Pr(A \cup B) = 1 - 8p$.

Question 8c

2 marks

Given that $\Pr(A \cup B) \leq \frac{1}{5}$, state the largest possible interval for p .

Marking Guide — Answer: $0 < p \leq \frac{1}{40}$

- $\Pr(A \cup B) = 8p \leq \frac{1}{5} \implies p \leq \frac{1}{40}$.
- Also $p > 0$ (since $\Pr(A|B)$ and $\Pr(B|A)$ are non-zero).
- Also need $\Pr(A) = 4p \leq 1$ and $\Pr(B) = 5p \leq 1$, but $p \leq \frac{1}{40}$ satisfies these.

- Interval: $0 < p \leq \frac{1}{40}$.

Question 9a

2 marks

The graph of $f : [0, 1] \rightarrow \mathbb{R}$, $f(x) = \sqrt{x}(1 - x)$ is shown below.

Calculate the area between the graph of f and the x -axis.

Marking Guide — Answer: $\frac{4}{15}$

- $\int_0^1 \sqrt{x}(1 - x) dx = \int_0^1 (x^{1/2} - x^{3/2}) dx.$

Question 9b

1 mark

For x in the interval $(0, 1)$, show that the gradient of the tangent to the graph of f is $\frac{1-3x}{2\sqrt{x}}$.

Marking Guide — Answer: See marking guide

- $f(x) = x^{1/2} - x^{3/2}.$
- $f'(x) = \frac{1}{2}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{1}{2\sqrt{x}} - \frac{3\sqrt{x}}{2} = \frac{1-3x}{2\sqrt{x}}.$

Question 9c

2 marks

The edges of the right-angled triangle ABC are the line segments AC and BC , which are tangent to the graph of f , and the line segment AB , which is part of the horizontal axis. Let θ be the angle that AC makes with the positive direction of the horizontal axis, where $45 \leq \theta < 90$.

Find the equation of the line through B and C in the form $y = mx + c$, for $\theta = 45$.

Marking Guide — Answer: $y = -x + 1$

- At $\theta = 45$, the tangent from A has gradient $\tan(45) = 1$.
- Set $f'(x) = 1$: $\frac{1-3x}{2\sqrt{x}} = 1 \implies 1 - 3x = 2\sqrt{x}.$
- Let $u = \sqrt{x}$: $1 - 3u^2 = 2u \implies 3u^2 + 2u - 1 = 0 \implies (3u - 1)(u + 1) = 0.$
- $u = \frac{1}{3}$, so $x = \frac{1}{9}$. Point on curve: $(\frac{1}{9}, f(\frac{1}{9})) = (\frac{1}{9}, \frac{1}{3} \cdot \frac{8}{9}) = (\frac{1}{9}, \frac{8}{27}).$
- Since ABC is right-angled and BC is also tangent, gradient of $BC = -1$ (perpendicular tangent).
- Set $f'(x) = -1$: $\frac{1-3x}{2\sqrt{x}} = -1 \implies 1 - 3x = -2\sqrt{x} \implies 3x - 2\sqrt{x} - 1 = 0.$
- Let $u = \sqrt{x}$: $3u^2 - 2u - 1 = 0 \implies (3u + 1)(u - 1) = 0$, so $u = 1$, $x = 1$.
- Point: $(1, 0)$. Line through $(1, 0)$ with gradient -1 : $y = -(x - 1) = -x + 1.$

Question 9d

4 marks

Find the coordinates of C when $\theta = 45$.

Marking Guide — Answer: $C = (\frac{11}{18}, \frac{7}{18})$

- Line AC passes through $(\frac{1}{9}, \frac{8}{27})$ with gradient 1: $y - \frac{8}{27} = 1(x - \frac{1}{9}).$
- $y = x - \frac{1}{9} + \frac{8}{27} = x + \frac{-3+8}{27} = x + \frac{5}{27}.$

- Line BC : $y = -x + 1$.
- Intersection: $x + \frac{5}{27} = -x + 1 \implies 2x = 1 - \frac{5}{27} = \frac{22}{27} \implies x = \frac{11}{27}$.
- $y = -\frac{11}{27} + 1 = \frac{16}{27}$.
- $C = \left(\frac{11}{27}, \frac{16}{27}\right)$.