ATAR Master

VCE Mathematical Methods

2022 Examination 1 (Technology-Free)

Questions & Marking Guide

Total: 40 marks

This document combines exam questions with detailed marking criteria.
Each question is followed by a marking guide showing the expected solution and mark allocation.

Question 1a *1 mark*

Let $y = 3xe^{2x}$. Find $\frac{dy}{dx}$.

> **Marking Guide** — Answer: $\frac{dy}{dx} = 3e^{2x} + 6xe^{2x} = 3e^{2x}(1 + 2x)$
>
> - Apply product rule: $u = 3x$, $v = e^{2x}$.
> - $u' = 3$, $v' = 2e^{2x}$.
> - $\frac{dy}{dx} = 3e^{2x} + 6xe^{2x}$ or equivalently $3e^{2x}(1 + 2x)$.

Question 1b *2 marks*

Find and simplify the rule of $f'(x)$, where $f : R \to R$, $f(x) = \frac{\cos(x)}{e^x}$.

> **Marking Guide** — Answer: $f'(x) = \frac{-e^x \sin(x) - e^x \cos(x)}{e^{2x}} = -\frac{\sin(x) + \cos(x)}{e^x}$
>
> - Apply quotient rule: $u = \cos(x)$, $v = e^x$.
> - $u' = -\sin(x)$, $v' = e^x$.
> - $f'(x) = \frac{-e^x \sin(x) - e^x \cos(x)}{e^{2x}}$.
> - Simplify: $f'(x) = -\frac{\sin(x) + \cos(x)}{e^x}$.

Question 2a *1 mark*

Let $g : \left(\frac{3}{2}, \infty\right) \to R$, $g(x) = \frac{3}{2x-3}$. Find the rule for an antiderivative of $g(x)$.

> **Marking Guide** — Answer: $\frac{3}{2} \log_e(2x - 3) + c$
>
> - Recognise $\int \frac{3}{2x-3} dx = \frac{3}{2} \log_e(2x - 3) + c$.

Question 2b *3 marks*

Evaluate $\int_0^1 f(x)\big(2f(x) - 3\big)\, dx$, where $\int_0^1 \big[f(x)\big]^2 dx = \frac{1}{5}$ and $\int_0^1 f(x)\, dx = \frac{1}{3}$.

> **Marking Guide** — Answer: $-\frac{3}{5}$

Question 3 *3 marks*

Consider the system of equations

$$kx - 5y = 4 + k$$
$$3x + (k + 8)y = -1$$

Determine the value of k for which the system of equations above has an infinite number of solutions.

> **Marking Guide** — Answer: $k = -3$
>
> - For infinite solutions, the ratios of coefficients must be equal: $\frac{k}{3} = \frac{-5}{k+8} = \frac{4+k}{-1}$.
> - From $\frac{k}{3} = \frac{-5}{k+8}$: $k(k + 8) = -15$, so $k^2 + 8k + 15 = 0$, $(k + 3)(k + 5) = 0$, giving $k = -3$ or $k = -5$.
> - Check $\frac{4+k}{-1}$: For $k = -3$: $\frac{1}{-1} = -1$ and $\frac{-3}{3} = -1$ ✓. For $k = -5$: $\frac{-1}{-1} = 1$ and $\frac{-5}{3}$ $eq1$ ✗.

- Therefore $k = -3$.

Question 4a *2 marks*

A card is drawn from a deck of red and blue cards. After verifying the colour, the card is replaced. This is performed four times. Each card has probability $\frac{1}{2}$ of being red and $\frac{1}{2}$ of being blue. Let X be the number of blue cards drawn.

Complete the table:

x	0	1	2	3	4
$\Pr(X = x)$	$\frac{1}{16}$		$\frac{6}{16}$		

Marking Guide — Answer: $\Pr(X = 1) = \frac{4}{16}$, $\Pr(X = 3) = \frac{4}{16}$, $\Pr(X = 4) = \frac{1}{16}$

- $X \sim \text{Bi}(4, \frac{1}{2})$.
- $\Pr(X = 1) = \binom{4}{1}(\frac{1}{2})^4 = \frac{4}{16}$.
- $\Pr(X = 3) = \binom{4}{3}(\frac{1}{2})^4 = \frac{4}{16}$.
- $\Pr(X = 4) = (\frac{1}{2})^4 = \frac{1}{16}$.

Question 4b *1 mark*

Given that the first card drawn is blue, find the probability that exactly two of the next three cards drawn will be red.

Marking Guide — Answer: $\frac{3}{8}$

- Given first card is blue, the remaining 3 draws are independent with $p = \frac{1}{2}$.
- $\Pr(\text{exactly 2 red out of 3}) = \binom{3}{2}(\frac{1}{2})^2(\frac{1}{2})^1 = \frac{3}{8}$.

Question 4c *2 marks*

The deck is changed so that the probability of a card being red is $\frac{2}{3}$ and the probability of a card being blue is $\frac{1}{3}$. Given that the first card drawn is blue, find the probability that exactly two of the next three cards drawn will be red.

Marking Guide — Answer: $\frac{4}{9}$

- Given first card is blue (already happened), remaining 3 draws: $p(\text{red}) = \frac{2}{3}$.
- $\Pr(\text{exactly 2 red out of 3}) = \binom{3}{2}(\frac{2}{3})^2(\frac{1}{3})^1 = 3 \times \frac{4}{9} \times \frac{1}{3} = \frac{12}{27} = \frac{4}{9}$.

Question 5a *2 marks*

Solve $10^{3x-13} = 100$ for x.

Marking Guide — Answer: $x = 5$

- Write $100 = 10^2$.
- $10^{3x-13} = 10^2$, so $3x - 13 = 2$.
- $3x = 15$, $x = 5$.

Question 5b *3 marks*

Find the maximal domain of f, where $f(x) = \log_e(x^2 - 2x - 3)$.

> **Marking Guide** — Answer: $(-\infty, -1) \cup (3, \infty)$
>
> - Require $x^2 - 2x - 3 > 0$.
> - Factorise: $(x - 3)(x + 1) > 0$.
> - Critical points: $x = -1$ and $x = 3$.
> - Solution: $x < -1$ or $x > 3$, i.e., $(-\infty, -1) \cup (3, \infty)$.

Question 6a *2 marks*

The graph of $y = f(x)$, where $f : [0, 2\pi] \to R$, $f(x) = 2\sin(2x) - 1$, is shown.
    On the axes, draw the graph of $y = g(x)$, where $g(x)$ is the reflection of $f(x)$ in the horizontal axis.

> **Marking Guide** — Answer: $g(x) = -f(x) = -2\sin(2x) + 1 = 1 - 2\sin(2x)$
>
> - Correct shape: reflection of $f(x)$ in the x-axis.
> - $g(x) = -(2\sin(2x) - 1) = 1 - 2\sin(2x)$.

Question 6b *3 marks*

Find all values of k such that $f(k) = 0$ and $k \in [0, 2\pi]$.

> **Marking Guide** — Answer: $k = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$
>
> - $2\sin(2k) - 1 = 0 \Rightarrow \sin(2k) = \frac{1}{2}$.

Question 6c.i *1 mark*

Let $h : D \to R$, $h(x) = 2\sin(2x) - 1$, where $h(x)$ has the same rule as $f(x)$ with a different domain. The graph of $y = h(x)$ is translated a units in the positive horizontal direction and b units in the positive vertical direction so that it is mapped onto the graph of $y = g(x)$, where $a, b \in (0, \infty)$.
    Find the value for b.

> **Marking Guide** — Answer: $b = 2$
>
> - $g(x) = 1 - 2\sin(2x)$ has vertical midline at $y = 1$; $h(x) = 2\sin(2x) - 1$ has midline at $y = -1$.
> - Vertical shift: $b = 1 - (-1) = 2$.

Question 6c.ii *1 mark*

Find the smallest positive value for a.

> **Marking Guide** — Answer: $a = \frac{\pi}{2}$
>
> - After vertical shift: $h(x) + 2 = 2\sin(2x) + 1$.
> - Need $2\sin(2(x - a)) + 1 = 1 - 2\sin(2x)$, so $\sin(2x - 2a) = -\sin(2x)$.
> - $\sin(2x - 2a) = \sin(2x + \pi)$, so $2a = \pi$ (smallest positive), $a = \frac{\pi}{2}$.

Question 6c.iii *1 mark*

Hence, or otherwise, state the domain D of $h(x)$.

> **Marking Guide** — Answer: $D = \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$

Question 7a.i *1 mark*

A tilemaker wants to make square tiles of size 20 cm $\times$ 20 cm. The front surface is painted with two colours meeting these conditions: - Condition 1: Each colour covers half the front surface. - Condition 2: Tiles can line up horizontally to form a continuous pattern.

For Type A, colours are divided using $f(x) = 4\sin\left(\frac{\pi x}{10}\right) + a$, where $a \in R$. Tile corners are at $(0,0)$, $(20,0)$, $(20,20)$, $(0,20)$.

Find the area of the front surface of each tile.

> **Marking Guide** — Answer: 400 cm²
>
> - Area $= 20 \times 20 = 400$ cm².

Question 7a.ii *1 mark*

Find the value of a so that a Type A tile meets Condition 1.

> **Marking Guide** — Answer: $a = 10$
>
> - Area below $f(x)$ from $x = 0$ to $x = 20$: $\int_0^{20} f(x)\,dx = \int_0^{20} 4\sin(\frac{\pi x}{10}) + a\,dx$.

Question 7b *3 marks*

Type B tiles are divided using $g(x) = -\frac{1}{100}x^3 + \frac{3}{10}x^2 - 2x + 10$.

Show that a Type B tile meets Condition 1.

> **Marking Guide** — Answer: $\int_0^{20} g(x)\,dx = 200$
>
> - $\int_0^{20} g(x)\,dx = \int_0^{20}\left(-\frac{x^3}{100} + \frac{3x^2}{10} - 2x + 10\right)dx$.

Question 7c *2 marks*

Determine the endpoints of $f(x)$ and $g(x)$ on each tile. Hence, use these values to confirm that Type A and Type B tiles can be placed in any order to produce a continuous pattern to meet Condition 2.

> **Marking Guide** — Answer: Both f and g have value 10 at $x = 0$ and $x = 20$.
>
> - $f(0) = 4\sin(0) + 10 = 10$ and $f(20) = 4\sin(2\pi) + 10 = 10$.
> - $g(0) = 0 + 0 - 0 + 10 = 10$ and $g(20) = -80 + 120 - 40 + 10 = 10$.
> - Both functions have the same value (10) at $x = 0$ and $x = 20$, so tiles connect continuously.

Question 8a *1 mark*

Part of the graph of $y = f(x)$ is shown. The rule $A(k) = k\sin(k)$ gives the area bounded by the graph of f, the horizontal axis and the line $x = k$.

State the value of $A\left(\frac{\pi}{3}\right)$.

> **Marking Guide** — Answer: $A\left(\frac{\pi}{3}\right) = \frac{\pi\sqrt{3}}{6}$

- $A\left(\frac{\pi}{3}\right) = \frac{\pi}{3} \cdot \sin\left(\frac{\pi}{3}\right) = \frac{\pi}{3} \cdot \frac{\sqrt{3}}{2} = \frac{\pi\sqrt{3}}{6}$.

Question 8b   *2 marks*
Evaluate $f\left(\frac{\pi}{3}\right)$.

Marking Guide — Answer: $f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} + \frac{\pi}{6}$

- Since $\int_0^k f(x)\,dx = A(k) = k\sin(k)$, by the Fundamental Theorem of Calculus: $f(k) = A'(k)$.
- $A'(k) = \sin(k) + k\cos(k)$.
- $f\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) + \frac{\pi}{3}\cos\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} + \frac{\pi}{3} \cdot \frac{1}{2} = \frac{\sqrt{3}}{2} + \frac{\pi}{6}$.

Question 8c   *2 marks*
Consider the average value of the function f over the interval $x \in [0, k]$, where $k \in [0, 2]$.
    Find the value of k that results in the maximum average value.

Marking Guide — Answer: $k = \frac{\pi}{2}$

- Average value $= \frac{1}{k}\int_0^k f(x)\,dx = \frac{A(k)}{k} = \frac{k\sin(k)}{k} = \sin(k)$.