ATAR Master

VCE Mathematical Methods

2024 Examination 1 (Technology-Free)

Questions & Marking Guide

Total: 40 marks

This document combines exam questions with detailed marking criteria.
Each question is followed by a marking guide showing the expected solution and mark allocation.

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Question 1a *1 mark*

Let $y = e^x \cos(3x)$. Find $\frac{dy}{dx}$.

Marking Guide — Answer: $e^x(\cos(3x) - 3\sin(3x))$

- Apply product rule: $u = e^x, v = \cos(3x)$
- Derivatives: $u' = e^x, v' = -3\sin(3x)$
- Combine: $e^x \cos(3x) - 3e^x \sin(3x)$

Question 1b *2 marks*

Let $f(x) = \log_e(x^3 - 3x + 2)$. Find $f'(3)$.

Marking Guide — Answer: $\frac{6}{5}$

- Apply chain rule: $f'(x) = \frac{1}{x^3-3x+2} \times (3x^2 - 3)$
- Substitute $x = 3$: $\frac{3(9)-3}{27-9+2} = \frac{24}{20}$
- Simplify to $6/5$

Question 2 *3 marks*

Consider the simultaneous linear equations: $3kx - 2y = k + 4$ $(k-4)x + ky = -k$ Determine the value of k for which the system of equations has no real solution.

Marking Guide — Answer: $k = \frac{4}{3}$

- Matrix determinant method or gradient comparison.
- Determinant $\Delta = (3k)(k) - (-2)(k-4) = 3k^2 + 2k - 8$.
- Set $\Delta = 0$: $(3k-4)(k+2) = 0 \implies k = 4/3, k = -2$.
- Check $k = -2$: Eqs become $-6x - 2y = 2$ and $-6x - 2y = 2$. Coincident lines (infinite solutions).
- Check $k = 4/3$: Slopes are equal, but intercepts differ (parallel lines). No solution.

Question 3a *3 marks*

Let $g : R \setminus \{-3\} \to R, g(x) = \frac{1}{(x+3)^2} - 2$. On the axes below, sketch the graph of $y = g(x)$ labelling all asymptotes with their equations and axis intercepts with their coordinates.

Marking Guide — Answer: Sketch showing asymptotes x=-3, y=-2 and intercepts

- Vertical asymptote $x = -3$. Horizontal asymptote $y = -2$.
- y-intercept: $x = 0, y = 1/9 - 2 = -17/9$. Point $(0, -17/9)$.
- x-intercepts: $\frac{1}{(x+3)^2} = 2 \implies (x+3)^2 = 1/2$.
- $x = -3 \pm \frac{1}{\sqrt{2}}$. Points $(-3 - \frac{\sqrt{2}}{2}, 0)$ and $(-3 + \frac{\sqrt{2}}{2}, 0)$.

Question 3b *2 marks*

Determine the area of the region bounded by the line $x = -2$, the x-axis, the y-axis and the graph of $y = g(x)$.

Marking Guide — Answer: $\frac{10}{3}$

- Region is below x-axis from $x = -2$ to $x = 0$.
- Area $= \left|\int_{-2}^{0}((x+3)^{-2} - 2)dx\right|$ or $\int_{-2}^{0}(2 - (x+3)^{-2})dx$.
- Antiderivative of $(x+3)^{-2}$ is $-(x+3)^{-1}$.

Question 4a
1 mark

Let X be a binomial random variable where $X \sim \text{Bi}(4, \frac{9}{10})$. Find the standard deviation of X.

Marking Guide — Answer: 0.6

- $\text{Var}(X) = np(1-p) = 4 \times 0.9 \times 0.1 = 0.36$.
- $\text{SD}(X) = \sqrt{0.36} = 0.6$.

Question 4b
2 marks

Find $\Pr(X < 2)$.

Marking Guide — Answer: 0.0037

- $\Pr(X < 2) = \Pr(X = 0) + \Pr(X = 1)$.
- $\Pr(X = 0) = (0.1)^4 = 0.0001$.
- $\Pr(X = 1) = \binom{4}{1}(0.9)^1(0.1)^3 = 4(0.9)(0.001) = 0.0036$.
- Total $= 0.0001 + 0.0036 = 0.0037$.

Question 5a
1 mark

The function $h : [0, \infty) \to R, h(t) = \frac{3000}{t+1}$ models the population of a town after t years. Use the model to predict the population after four years.

Marking Guide — Answer: 600

- Substitute $t = 4$: $h(4) = \frac{3000}{4+1} = \frac{3000}{5} = 600$.

Question 5b
2 marks

A new function, h_1, models a population where $h_1(0) = h(0)$ but h_1 decreases at half the rate of h at any point in time. State a sequence of two transformations that maps h to this new model h_1.

Marking Guide — Answer: Dilation by factor 1/2 from t-axis (or x-axis), followed by Translation of 1500 units up.

- Rate is derivative. $h_1'(t) = 0.5h'(t)$. Integrating gives $h_1(t) = 0.5h(t) + c$.
- Initial condition: $h_1(0) = 0.5h(0) + c \implies 3000 = 1500 + c \implies c = 1500$.

- Model is $h_1(t) = 0.5h(t) + 1500$.
- Transformation 1: Dilation factor 0.5 from t-axis.
- Transformation 2: Translation +1500 units in y-direction.

Question 5c *3 marks*

In the town, 100 people were randomly selected and surveyed, with 60 people indicating that they were unhappy with the roads. i. Determine an approximate 95% confidence interval for the proportion of people in the town who are unhappy with the roads. Use $z = 2$ for this confidence interval. (2 marks) ii. A new sample of n people results in the same sample proportion. Find the smallest value of n to achieve a standard deviation of $\frac{\sqrt{2}}{50}$ for the sample proportion. (1 mark)

Marking Guide — Answer: i. $(0.5, 0.7)$ ii. $n = 300$

- i. $\hat{p} = 60/100 = 0.6$. Formula: $\hat{p} \pm z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$.
- Margin $= 2\sqrt{\frac{0.6\times0.4}{100}} = 2\sqrt{0.0024}$.
- Wait, $\frac{0.24}{100} = 0.0024$. $\sqrt{0.0024} = \frac{\sqrt{24}}{100} = \frac{2\sqrt{6}}{100} \approx 0.049$.
- Using approx: $2 \times 0.05 = 0.1$. Interval $0.6 \pm 0.1 = (0.5, 0.7)$.
- ii. $SD(\hat{p}) = \sqrt{\frac{0.24}{n}} = \frac{\sqrt{2}}{50}$.
- Square both sides: $\frac{0.24}{n} = \frac{2}{2500} = \frac{1}{1250}$.
- $n = 0.24 \times 1250 = 300$.

Question 6 *4 marks*

Solve $2\log_3(x-4) + \log_3(x) = 2$ for x.

Marking Guide — Answer: $x = \frac{7+\sqrt{13}}{2}$

- Combine logs: $\log_3((x-4)^2 x) = 2$.
- Exponential form: $x(x^2 - 8x + 16) = 3^2 = 9$.
- Cubic: $x^3 - 8x^2 + 16x - 9 = 0$.
- Use Factor Theorem: $P(1) = 1 - 8 + 16 - 9 = 0$. So $(x-1)$ is a factor.
- Division: $(x-1)(x^2 - 7x + 9) = 0$.
- Roots: $x = 1$ or $x = \frac{7\pm\sqrt{49-36}}{2} = \frac{7\pm\sqrt{13}}{2}$.
- Domain constraint: $x - 4 > 0 \implies x > 4$.
- Reject $x = 1$ and $x = \frac{7-\sqrt{13}}{2} \approx 1.7$.
- Only solution: $x = \frac{7+\sqrt{13}}{2}$ (approx 5.3).

Question 7a *3 marks*

Use the trapezium rule with a step size of $\frac{\pi}{3}$ to determine an approximation of the total area between the graph of $y = x\sin(x)$ and the x-axis over the interval $x \in [0, \pi]$.

Marking Guide — Answer: $\frac{\pi^2\sqrt{3}}{6}$

- Step $h = \pi/3$. Points: $x_0 = 0, x_1 = \pi/3, x_2 = 2\pi/3, x_3 = \pi$.
- $y_0 = 0$. $y_3 = 0$.
- $y_1 = (\pi/3)\sin(\pi/3) = \frac{\pi}{3}\frac{\sqrt{3}}{2} = \frac{\pi\sqrt{3}}{6}$.
- $y_2 = (2\pi/3)\sin(2\pi/3) = \frac{2\pi}{3}\frac{\sqrt{3}}{2} = \frac{\pi\sqrt{3}}{3}$.
- Area $\approx \frac{h}{2}(y_0 + 2(y_1 + y_2) + y_3) = \frac{\pi}{6}(0 + 2(\frac{\pi\sqrt{3}}{6} + \frac{2\pi\sqrt{3}}{6}) + 0)$.
- $= \frac{\pi}{6}(2 \times \frac{3\pi\sqrt{3}}{6}) = \frac{\pi}{6}(\pi\sqrt{3}) = \frac{\pi^2\sqrt{3}}{6}$.

Question 7b

3 marks

i. Find $f'(x)$ where $f(x) = x\sin(x)$. ii. Determine the range of $f'(x)$ over the interval $[\frac{\pi}{2}, \frac{2\pi}{3}]$. iii. Hence, verify that $f(x)$ has a stationary point for $x \in [\frac{\pi}{2}, \frac{2\pi}{3}]$.

Marking Guide — Answer: i. $\sin(x) + x\cos(x)$ ii. $[\frac{\sqrt{3}}{2} - \frac{\pi}{3}, 1]$

- i. Product rule: $f'(x) = 1 \cdot \sin(x) + x\cos(x)$.
- ii. f' is continuous. $f'(\pi/2) = 1 + 0 = 1$.
- $f'(2\pi/3) = \frac{\sqrt{3}}{2} + \frac{2\pi}{3}(-\frac{1}{2}) = \frac{\sqrt{3}}{2} - \frac{\pi}{3}$.
- Check monotonicity: $f''(x) = 2\cos(x) - x\sin(x)$. In Q2 both terms negative, so f' is decreasing.

Question 7c

3 marks

On the set of axes below, sketch the graph of $y = f'(x)$ on the domain $[-\pi, \pi]$, labelling the endpoints with their coordinates.

Marking Guide — Answer: Graph of sine-like wave with endpoints $(-\pi, \pi)$ and $(\pi, -\pi)$

- Endpoints: $f'(-\pi) = 0 + (-\pi)(-1) = \pi$. Point $(-\pi, \pi)$.
- $f'(\pi) = 0 + \pi(-1) = -\pi$. Point $(\pi, -\pi)$.
- Passes through origin $(0, 0)$.
- Odd function symmetry.

Question 8a

1 mark

Let $g : R \to R, g(x) = \sqrt[3]{x - k} + m$, where $k \in R \setminus \{0\}$ and $m \in R$. Let the point P be the y-intercept of the graph of $y = g(x)$. Find the coordinates of P, in terms of k and m.

Marking Guide — Answer: $(0, m - k^{1/3})$

Question 8b

2 marks

Find the gradient of g at P, in terms of k.

Marking Guide — Answer: $\frac{1}{3k^{2/3}}$

- $g(x) = (x-k)^{1/3} + m$.
- $g'(x) = \frac{1}{3}(x-k)^{-2/3}$.
- $g'(0) = \frac{1}{3}(-k)^{-2/3} = \frac{1}{3}((-1)^2k^2)^{-1/3} = \frac{1}{3k^{2/3}}$.

Question 8c *1 mark*

Given that the graph of $y = g(x)$ passes through the origin, express k in terms of m.

Marking Guide — Answer: $k = m^3$

- Passes through $(0, 0) \implies P$ is origin.
- y-coord of P is 0: $m - k^{1/3} = 0 \implies m = k^{1/3}$.
- Cube both sides: $k = m^3$.

Question 8d *3 marks*

Let the point Q be a point different from the point P, such that the gradient of g at points P and Q are equal. Given that the graph of $y = g(x)$ passes through the origin, find the coordinates of Q in terms of m.

Marking Guide — Answer: $(2m^3, 2m)$

- Set $g'(x) = g'(0)$. $\frac{1}{3}(x-k)^{-2/3} = \frac{1}{3}(-k)^{-2/3}$.
- $(x-k)^2 = (-k)^2 = k^2$.
- $x - k = k$ or $x - k = -k$.
- $x = 2k$ or $x = 0$. Since $Q\ eP$ (where $x = 0$), $x_Q = 2k$.
- Substitute $k = m^3$: $x_Q = 2m^3$.