Question 1a [2 marks]

Let $f : (-2, \infty) \to R,\ f(x) = \frac{x}{x+2}$.
Differentiate f with respect to x.

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Let $f : (-2, \infty) \to R,\ f(x) = \frac{x}{x+2}$.
Differentiate f with respect to x.

Question 1b [2 marks]

Let $g(x) = (2 - x^3)^3$.
Evaluate $g'(1)$.

Question 2a [2 marks]

Let $y = x \log_e(3x)$.
Find $\frac{dy}{dx}$.

Question 2b [2 marks]

Hence, calculate $\int_1^2 (\log_e(3x) + 1)\, dx$. Express your answer in the form $\log_e(a)$, where a is a positive integer.

Question 3a [1 mark]

Let $f : [-3, 0] \to R,\ f(x) = (x+2)^2(x-1)$.
Show that $(x+2)^2(x-1) = x^3 + 3x^2 - 4$.

Question 3b [3 marks]

Sketch the graph of f on the axes below. Label the axis intercepts and any stationary points with their coordinates.

Question 4 [2 marks]

In a large population of fish, the proportion of angel fish is $\frac{1}{4}$.

Let $\hat{P}$ be the random variable that represents the sample proportion of angel fish for samples of size n drawn from the population.

Find the smallest integer value of n such that the standard deviation of $\hat{P}$ is less than or equal to $\frac{1}{100}$.

Question 5a [1 mark]

For Jac to log on to a computer successfully, Jac must type the correct password. Unfortunately, Jac has forgotten the password. If Jac types the wrong password, Jac can make another attempt. The probability of success on any attempt is $\frac{2}{5}$. Assume that the result of each attempt is independent of the result of any other attempt. A maximum of three attempts can be made.

What is the probability that Jac does not log on to the computer successfully?

Question 5b [1 mark]

Calculate the probability that Jac logs on to the computer successfully. Express your answer in the form $\frac{a}{b}$, where a and b are positive integers.

Question 5c [2 marks]

Calculate the probability that Jac logs on to the computer successfully on the second or on the third attempt. Express your answer in the form $\frac{c}{d}$, where c and d are positive integers.

Question 6a [1 mark]

Let $(\tan(\theta) - 1)(\sin(\theta) - \sqrt{3}\cos(\theta))(\sin(\theta) + \sqrt{3}\cos(\theta)) = 0$.
State all possible values of $\tan(\theta)$.

Question 6b [2 marks]

Hence, find all possible solutions for $(\tan(\theta) - 1)(\sin^2(\theta) - 3\cos^2(\theta)) = 0$, where $0 \leq \theta \leq \pi$.

Question 7a [1 mark]

Let $f : [0, \infty) \to R$, $f(x) = \sqrt{x+1}$.
State the range of f.

Question 7b.i [2 marks]

Let $g : (-\infty, c] \to R$, $g(x) = x^2 + 4x + 3$, where $c < 0$.
Find the largest possible value of c such that the range of g is a subset of the domain of f.

Question 7b.ii [1 mark]

For the value of c found in part b.i., state the range of $f(g(x))$.

Question 7c [1 mark]

Let $h : R \to R$, $h(x) = x^2 + 3$.
State the range of $f(h(x))$.

Question 8a [1 mark]

For events A and B from a sample space, $\Pr(A|B) = \frac{1}{5}$ and $\Pr(B|A) = \frac{1}{4}$. Let $\Pr(A \cap B) = p$.
Find $\Pr(A)$ in terms of p.

Question 8b [2 marks]

Find $\Pr(A' \cap B')$ in terms of p.

Question 8c [2 marks]

Given that $\Pr(A \cup B) \leq \frac{1}{5}$, state the largest possible interval for p.

Question 9a [2 marks]

The graph of $f : [0, 1] \to R$, $f(x) = \sqrt{x}(1 - x)$ is shown below.
Calculate the area between the graph of f and the x-axis.

Question 9b [1 mark]

For x in the interval $(0, 1)$, show that the gradient of the tangent to the graph of f is $\frac{1-3x}{2\sqrt{x}}$.

Question 9c [2 marks]

The edges of the right-angled triangle ABC are the line segments AC and BC, which are tangent to the graph of f, and the line segment AB, which is part of the horizontal axis. Let θ be the angle that AC makes with the positive direction of the horizontal axis, where 45ř $\leq \theta <$ 90ř.

Find the equation of the line through B and C in the form $y = mx + c$, for $\theta =$ 45ř.

Question 9d [4 marks]

Find the coordinates of C when $\theta =$ 45ř.

Solutions

Question 1a

$f'(x) = \frac{2}{(x+2)^2}$

Marking guide:

- Quotient rule: $f'(x) = \frac{(x+2)(1)-x(1)}{(x+2)^2} = \frac{2}{(x+2)^2}$.

Question 1a

$f'(x) = \frac{2}{(x+2)^2}$

Marking guide:

- Quotient rule: $f'(x) = \frac{(x+2)(1)-x(1)}{(x+2)^2} = \frac{2}{(x+2)^2}$.

Question 1b

$g'(1) = -9$

Marking guide:

- Chain rule: $g'(x) = 3(2-x^3)^2 \cdot (-3x^2) = -9x^2(2-x^3)^2$.
- $g'(1) = -9(1)^2(2-1)^2 = -9$.

Question 2a

$\frac{dy}{dx} = \log_e(3x) + 1$

Marking guide:

- Product rule: $\frac{dy}{dx} = \log_e(3x) + x \cdot \frac{1}{x} = \log_e(3x) + 1$.

Question 2b

$\log_e(6)$

Marking guide:

- Since $\frac{d}{dx}[x\log_e(3x)] = \log_e(3x) + 1$,
- $\int_1^2 (\log_e(3x) + 1)\,dx = [x\log_e(3x)]_1^2 = 2\log_e(6) - \log_e(3) = \log_e(36) - \log_e(3) = \log_e(12)$.
- Wait, recheck: $2\log_e(6) - 1 \cdot \log_e(3) = \log_e(36) - \log_e(3) = \log_e(12)$.
- Answer: $\log_e(12)$.

Question 3a

See marking guide

Marking guide:

- $(x+2)^2(x-1) = (x^2+4x+4)(x-1) = x^3 - x^2 + 4x^2 - 4x + 4x - 4 = x^3 + 3x^2 - 4$.

Question 3b

x-intercepts: $(-2, 0)$; y-intercept: $(0, -4)$; stationary points: $(-2, 0)$ and $(0, -4)$ — wait, need to recalculate

Marking guide:

- x-intercepts: $(x+2)^2(x-1) = 0$ gives $x = -2$ (touch) and $x = 1$. But domain is $[-3, 0]$, so $x = 1$ is excluded.
- Only x-intercept in domain: $(-2, 0)$.
- y-intercept: $f(0) = (2)^2(-1) = -4$, so $(0, -4)$.
- $f'(x) = 3x^2 + 6x = 3x(x+2) = 0$ gives $x = 0$ and $x = -2$.
- Stationary points: $(-2, 0)$ (local max) and $(0, -4)$ (endpoint/local min).
- Endpoints: $f(-3) = (-1)^2(-4) = -4$.
- Graph: starts at $(-3, -4)$, rises to $(-2, 0)$, then falls to $(0, -4)$.

Question 4

$n = 1875$

Marking guide:

- $\text{sd}(\hat{P}) = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{\frac{1}{4}\cdot\frac{3}{4}}{n}} = \sqrt{\frac{3}{16n}}$.
- Require $\sqrt{\frac{3}{16n}} \leq \frac{1}{100}$.
- $\frac{3}{16n} \leq \frac{1}{10000}$, so $n \geq \frac{30000}{16} = 1875$.
- Smallest $n = 1875$.

Question 5a
$\frac{27}{125}$
Marking guide:
- $\text{Pr(fail all 3)} = \left(\frac{3}{5}\right)^3 = \frac{27}{125}$.

Question 5b
$\frac{98}{125}$
Marking guide:
- $\text{Pr(success)} = 1 - \frac{27}{125} = \frac{98}{125}$.

Question 5c
$\frac{78}{125}$
Marking guide:
- $\text{Pr(success on 2nd)} = \frac{3}{5} \cdot \frac{2}{5} = \frac{6}{25}$.
- $\text{Pr(success on 3rd)} = \left(\frac{3}{5}\right)^2 \cdot \frac{2}{5} = \frac{18}{125}$.
- $\text{Total} = \frac{30}{125} + \frac{18}{125} = \frac{48}{125}$.
- Hmm, let me recheck. Actually Pr(2nd or 3rd) = Pr(success) - Pr(1st) = $\frac{98}{125} - \frac{2}{5} = \frac{98}{125} - \frac{50}{125} = \frac{48}{125}$.
- Answer: $\frac{48}{125}$.

Question 6a
$\tan(\theta) = 1$, $\tan(\theta) = \sqrt{3}$, or $\tan(\theta) = -\sqrt{3}$
Marking guide:
- $\tan(\theta) - 1 = 0 \implies \tan(\theta) = 1$.
- $\sin(\theta) - \sqrt{3}\cos(\theta) = 0 \implies \tan(\theta) = \sqrt{3}$.
- $\sin(\theta) + \sqrt{3}\cos(\theta) = 0 \implies \tan(\theta) = -\sqrt{3}$.

Question 6b
$\theta = \frac{\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}$
Marking guide:
- Note $\sin^2(\theta) - 3\cos^2(\theta) = (\sin(\theta) - \sqrt{3}\cos(\theta))(\sin(\theta) + \sqrt{3}\cos(\theta))$.
- So we need $\tan(\theta) = 1$, $\tan(\theta) = \sqrt{3}$, or $\tan(\theta) = -\sqrt{3}$.
- $\tan(\theta) = 1 \implies \theta = \frac{\pi}{4}$.
- $\tan(\theta) = \sqrt{3} \implies \theta = \frac{\pi}{3}$.
- $\tan(\theta) = -\sqrt{3} \implies \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.
- Solutions: $\theta = \frac{\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}$.

Question 7a
$[1, \infty)$
Marking guide:
- When $x = 0$, $f(0) = 1$. As $x \to \infty$, $f(x) \to \infty$.
- Range $= [1, \infty)$.

Question 7b.i
$c = -3$
Marking guide:
- $g(x) = (x+2)^2 - 1$. Vertex at $x = -2$, $g(-2) = -1$.
- Domain of f is $[0, \infty)$, so range of g must be $\subseteq [0, \infty)$.
- Since $c < 0$ and the parabola opens upward with vertex at $x = -2$:
- If $c \leq -2$, then g is decreasing on $(-\infty, c]$, minimum is $g(c)$.
- Wait: for $c < 0$, the function on $(-\infty, c]$ has range $[g(c), \infty)$ if $c \leq -2$, or range $[-1, \infty)$ if $-2 < c < 0$... no.
- Actually if $c \leq -2$: g is decreasing on the domain, so range is $[g(c), \infty)$. Need $g(c) \geq 0$. $g(c) = c^2 + 4c + 3 = (c+1)(c+3) \geq 0$. Since $c \leq -2 < -1$, we need $c \leq -3$. Largest is $c = -3$.

- If $-2 < c < 0$: minimum of g is $g(-2) = -1 < 0$, so range includes negative values. Not valid.
- Answer: $c = -3$.

Question 7b.ii

$[1, \infty)$

Marking guide:

- With $c = -3$, range of g is $[g(-3), \infty) = [0, \infty)$.
- Range of f on $[0, \infty)$ is $[f(0), \infty) = [1, \infty)$.
- Range of $f(g(x)) = [1, \infty)$.

Question 7c

$[2, \infty)$

Marking guide:

- Range of h is $[3, \infty)$.
- $f(h(x)) = \sqrt{h(x) + 1} = \sqrt{x^2 + 4}$.
- Minimum when $x = 0$: $\sqrt{4} = 2$. Range $= [2, \infty)$.

Question 8a

$\Pr(A) = 4p$

Marking guide:

- $\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{1}{4}$.
- $\frac{p}{\Pr(A)} = \frac{1}{4} \implies \Pr(A) = 4p$.

Question 8b

$\Pr(A' \cap B') = 1 - 8p$

Marking guide:

- $\Pr(A|B) = \frac{p}{\Pr(B)} = \frac{1}{5} \implies \Pr(B) = 5p$.
- $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) = 4p + 5p - p = 8p$.
- $\Pr(A' \cap B') = 1 - \Pr(A \cup B) = 1 - 8p$.

Question 8c

$0 < p \leq \frac{1}{40}$

Marking guide:

- $\Pr(A \cup B) = 8p \leq \frac{1}{5} \implies p \leq \frac{1}{40}$.
- Also $p > 0$ (since $\Pr(A|B)$ and $\Pr(B|A)$ are non-zero).
- Also need $\Pr(A) = 4p \leq 1$ and $\Pr(B) = 5p \leq 1$, but $p \leq \frac{1}{40}$ satisfies these.
- Interval: $0 < p \leq \frac{1}{40}$.

Question 9a

$\frac{4}{15}$

Marking guide:

- $\int_0^1 \sqrt{x}(1 - x)\,dx = \int_0^1 (x^{1/2} - x^{3/2})\,dx$.
- $= \left[\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^1 = \frac{2}{3} - \frac{2}{5} = \frac{4}{15}$.

Question 9b

See marking guide

Marking guide:

- $f(x) = x^{1/2} - x^{3/2}$.
- $f'(x) = \frac{1}{2}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{1}{2\sqrt{x}} - \frac{3\sqrt{x}}{2} = \frac{1-3x}{2\sqrt{x}}$.

Question 9c

$y = -x + 1$

Marking guide:

- At $\theta = 45ř$, the tangent from A has gradient $\tan(45ř) = 1$.
- Set $f'(x) = 1$: $\frac{1-3x}{2\sqrt{x}} = 1 \implies 1 - 3x = 2\sqrt{x}$.
- Let $u = \sqrt{x}$: $1 - 3u^2 = 2u \implies 3u^2 + 2u - 1 = 0 \implies (3u - 1)(u + 1) = 0$.

- $u = \frac{1}{3}$, so $x = \frac{1}{9}$. Point on curve: $(\frac{1}{9}, f(\frac{1}{9})) = (\frac{1}{9}, \frac{1}{3} \cdot \frac{8}{9}) = (\frac{1}{9}, \frac{8}{27})$.
- Since ABC is right-angled and BC is also tangent, gradient of $BC = -1$ (perpendicular tangent).
- Set $f'(x) = -1$: $\frac{1-3x}{2\sqrt{x}} = -1 \implies 1 - 3x = -2\sqrt{x} \implies 3x - 2\sqrt{x} - 1 = 0$.
- Let $u = \sqrt{x}$: $3u^2 - 2u - 1 = 0 \implies (3u+1)(u-1) = 0$, so $u = 1$, $x = 1$.
- Point: $(1, 0)$. Line through $(1, 0)$ with gradient -1: $y = -(x-1) = -x + 1$.

Question 9d

$C = \left(\frac{11}{18}, \frac{7}{18}\right)$

Marking guide:

- Line AC passes through $(\frac{1}{9}, \frac{8}{27})$ with gradient 1: $y - \frac{8}{27} = 1(x - \frac{1}{9})$.
- $y = x - \frac{1}{9} + \frac{8}{27} = x + \frac{-3+8}{27} = x + \frac{5}{27}$.
- Line BC: $y = -x + 1$.
- Intersection: $x + \frac{5}{27} = -x + 1 \implies 2x = 1 - \frac{5}{27} = \frac{22}{27} \implies x = \frac{11}{27}$.
- $y = -\frac{11}{27} + 1 = \frac{16}{27}$.
- $C = \left(\frac{11}{27}, \frac{16}{27}\right)$.