Question 1a [1 mark]
If $y = (-3x^3 + x^2 - 64)^3$, find $\frac{dy}{dx}$.

Question 1a [1 mark]
If $y = (-3x^3 + x^2 - 64)^3$, find $\frac{dy}{dx}$.

Question 1b [2 marks]
Let $f(x) = \frac{e^x}{\cos(x)}$.
Evaluate $f'(\pi)$.

Question 2 [3 marks]
The derivative with respect to x of the function $f : (1, \infty) \to R$ has the rule $f'(x) = \frac{1}{2} - \frac{1}{(2x-2)}$.
Given that $f(2) = 0$, find $f(x)$ in terms of x.

Question 3a [2 marks]
Let $f : [0, 2\pi] \to R$, $f(x) = 2\cos(x) + 1$.
Solve the equation $2\cos(x) + 1 = 0$ for $0 \leq x \leq 2\pi$.

Question 3b [3 marks]
Sketch the graph of the function f on the axes below. Label the endpoints and local minimum point with their coordinates.

Question 4a [1 mark]
Let X be a normally distributed random variable with a mean of 6 and a variance of 4. Let Z be a random variable with the standard normal distribution.
Find $\Pr(X > 6)$.

Question 4b [1 mark]
Find b such that $\Pr(X > 7) = \Pr(Z < b)$.

Question 5 [3 marks]
Let $f : (2, \infty) \to R$, where $f(x) = \frac{1}{(x-2)^2}$.
State the rule and domain of f^{-1}.

Question 6a [2 marks]
Two boxes each contain four stones that differ only in colour. Box 1 contains four black stones. Box 2 contains two black stones and two white stones. A box is chosen randomly and one stone is drawn randomly from it. Each box is equally likely to be chosen, as is each stone.
What is the probability that the randomly drawn stone is black?

Question 6b [2 marks]
It is not known from which box the stone has been drawn.

Given that the stone that is drawn is black, what is the probability that it was drawn from Box 1?

Question 7a [3 marks]

Let P be a point on the straight line $y = 2x - 4$ such that the length of OP, the line segment from the origin O to P, is a minimum.

Find the coordinates of P.

Question 7b [2 marks]

Find the distance OP. Express your answer in the form $\frac{a\sqrt{b}}{b}$, where a and b are positive integers.

Question 8a [1 mark]

Let $f : R \to R$, $f(x) = x^2 e^{kx}$, where k is a positive real constant.

Show that $f'(x) = xe^{kx}(kx + 2)$.

Question 8b [2 marks]

Find the value of k for which the graphs of $y = f(x)$ and $y = f'(x)$ have exactly one point of intersection.

Question 8c [1 mark]

Let $g(x) = -\frac{2xe^{kx}}{k}$. The diagram below shows sections of the graphs of f and g for $x \geq 0$.

Let A be the area of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the line $x = 2$.

Write down a definite integral that gives the value of A.

Question 8d [3 marks]

Using your result from **part a.**, or otherwise, find the value of k such that $A = \frac{16}{k}$.

Question 9a.i [2 marks]

Consider a part of the graph of $y = x\sin(x)$, as shown below.

Given that $\int (x\sin(x))\,dx = \sin(x) - x\cos(x) + c$, evaluate $\int_{n\pi}^{(n+1)\pi} (x\sin(x))\,dx$ when n is a positive **even** integer or 0. Give your answer in simplest form.

Question 9a.ii [1 mark]

Given that $\int (x\sin(x))\,dx = \sin(x) - x\cos(x) + c$, evaluate $\int_{n\pi}^{(n+1)\pi} (x\sin(x))\,dx$ when n is a positive **odd** integer. Give your answer in simplest form.

Question 9b [2 marks]

Find the equation of the tangent to $y = x\sin(x)$ at the point $\left(-\frac{5\pi}{2}, \frac{5\pi}{2}\right)$.

Question 9c [1 mark]

The translation T maps the graph of $y = x\sin(x)$ onto the graph of $y = (3\pi - x)\sin(x)$, where

$T : R^2 \to R^2,\ T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ 0 \end{pmatrix}$
and a is a real constant.
State the value of a.

Question 9d [2 marks]
Let $f : [0, 3\pi] \to R,\ f(x) = (3\pi - x)\sin(x)$ and $g : [0, 3\pi] \to R,\ g(x) = (x - 3\pi)\sin(x)$.
The line l_1 is the tangent to the graph of f at the point $\left(\frac{\pi}{2}, \frac{5\pi}{2}\right)$ and the line l_2 is the tangent to the graph of g at $\left(\frac{\pi}{2}, -\frac{5\pi}{2}\right)$, as shown in the diagram below.
Find the total area of the shaded regions shown in the diagram above.

Solutions

Question 1a

$\frac{dy}{dx} = 3(-3x^3 + x^2 - 64)^2(-9x^2 + 2x)$

Marking guide:

- Apply chain rule: $\frac{dy}{dx} = 3(-3x^3 + x^2 - 64)^2 \cdot (-9x^2 + 2x)$.

Question 1a

$\frac{dy}{dx} = 3(-3x^3 + x^2 - 64)^2(-9x^2 + 2x)$

Marking guide:

- Apply chain rule: $\frac{dy}{dx} = 3(-3x^3 + x^2 - 64)^2 \cdot (-9x^2 + 2x)$.

Question 1b

$f'(\pi) = -e^{\pi}$

Marking guide:

- Quotient rule: $f'(x) = \frac{e^x \cos(x) - e^x(-\sin(x))}{\cos^2(x)} = \frac{e^x(\cos(x)+\sin(x))}{\cos^2(x)}$.
- At $x = \pi$: $f'(\pi) = \frac{e^{\pi}(\cos\pi + \sin\pi)}{\cos^2\pi} = \frac{e^{\pi}(-1+0)}{1} = -e^{\pi}$.

Question 2

$f(x) = \frac{x}{2} - \frac{1}{2}\log_e(2x-2) + \frac{1}{2}\log_e(2) - 1$

Marking guide:

- Integrate: $f(x) = \frac{x}{2} - \frac{1}{2}\log_e(2x-2) + c$.
- Note: $\int \frac{1}{2x-2}dx = \frac{1}{2}\log_e(2x-2)$.
- Apply $f(2) = 0$: $0 = 1 - \frac{1}{2}\log_e(2) + c$, so $c = \frac{1}{2}\log_e(2) - 1$.
- $f(x) = \frac{x}{2} - \frac{1}{2}\log_e(2x-2) + \frac{1}{2}\log_e(2) - 1$.

Question 3a

$x = \frac{2\pi}{3}$ or $x = \frac{4\pi}{3}$

Marking guide:

- $\cos(x) = -\frac{1}{2}$.
- $x = \frac{2\pi}{3}$ or $x = \frac{4\pi}{3}$.

Question 3b

Graph of $y = 2\cos(x) + 1$ on $[0, 2\pi]$. Endpoints: $(0, 3)$ and $(2\pi, 3)$. Local minimum: $(\pi, -1)$.

Marking guide:

- Endpoints: $(0, 3)$ and $(2\pi, 3)$.
- Local minimum at $(\pi, -1)$.
- x-intercepts at $x = \frac{2\pi}{3}$ and $x = \frac{4\pi}{3}$.
- Correct shape of cosine curve shifted up by 1.

Question 4a

$\Pr(X > 6) = 0.5$

Marking guide:

- Since 6 is the mean, by symmetry $\Pr(X > 6) = 0.5$.

Question 4b

$b = -0.5$

Marking guide:

- $\Pr(X > 7) = \Pr\left(Z > \frac{7-6}{2}\right) = \Pr(Z > 0.5)$.
- $\Pr(Z > 0.5) = \Pr(Z < -0.5)$.
- So $b = -0.5$.

Question 5

$f^{-1}(x) = \frac{1}{\sqrt{x}} + 2$, domain $(0, \infty)$

Marking guide:

- Let $y = \frac{1}{(x-2)^2}$. Swap x and y: $x = \frac{1}{(y-2)^2}$.
- $(y-2)^2 = \frac{1}{x}$, so $y - 2 = \frac{1}{\sqrt{x}}$ (positive root since domain of f is $(2, \infty)$, range of f^{-1} is $(2, \infty)$).

- $f^{-1}(x) = \frac{1}{\sqrt{x}} + 2$.
- Domain of f^{-1} = range of $f = (0, \infty)$.

Question 6a

$\frac{3}{4}$

Marking guide:

- $\Pr(\text{black}) = \Pr(\text{Box 1}) \cdot \Pr(\text{black}|\text{Box 1}) + \Pr(\text{Box 2}) \cdot \Pr(\text{black}|\text{Box 2})$.
- $= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{2}{4} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$.

Question 6b

$\frac{2}{3}$

Marking guide:

- $\Pr(\text{Box 1}|\text{black}) = \frac{\Pr(\text{black}|\text{Box 1}) \cdot \Pr(\text{Box 1})}{\Pr(\text{black})}$.
- $= \frac{1 \cdot \frac{1}{2}}{\frac{3}{4}} = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}$.

Question 7a

$P = \left(\frac{8}{5}, -\frac{4}{5}\right)$

Marking guide:

- Point on line: $P = (x, 2x - 4)$. Distance squared: $D = x^2 + (2x-4)^2 = 5x^2 - 16x + 16$.
- $\frac{dD}{dx} = 10x - 16 = 0 \implies x = \frac{8}{5}$.
- $y = 2 \cdot \frac{8}{5} - 4 = \frac{16}{5} - 4 = -\frac{4}{5}$.
- $P = \left(\frac{8}{5}, -\frac{4}{5}\right)$.

Question 7b

$OP = \frac{4\sqrt{5}}{5}$

Marking guide:

- $OP = \sqrt{\left(\frac{8}{5}\right)^2 + \left(-\frac{4}{5}\right)^2} = \sqrt{\frac{64}{25} + \frac{16}{25}} = \sqrt{\frac{80}{25}} = \frac{\sqrt{80}}{5} = \frac{4\sqrt{5}}{5}$.

Question 8a

See marking guide

Marking guide:

- Product rule: $f'(x) = 2xe^{kx} + x^2 \cdot ke^{kx} = xe^{kx}(2 + kx) = xe^{kx}(kx + 2)$.

Question 8b

$k = -2$. But $k > 0$, so we need to re-examine. Actually k is positive, so we solve $f(x) = f'(x)$: $x^2e^{kx} = xe^{kx}(kx + 2)$. Dividing by xe^{kx} (for $x \neq 0$): $x = kx + 2$, so $x(1 - k) = 2$, i.e. $x = \frac{2}{1-k}$. This always gives one solution for $x \neq 0$ when $k \neq 1$. At $x = 0$: both are 0, so $(0, 0)$ is always an intersection. We need exactly one intersection total, so $k = 1$ (making $x(1 - k) = 2$ have no solution), giving only $(0, 0)$. But wait, need to check. $k = 1$.

Marking guide:

- Set $f(x) = f'(x)$: $x^2e^{kx} = xe^{kx}(kx + 2)$.
- $xe^{kx}(x - kx - 2) = 0$.
- $xe^{kx}(x(1 - k) - 2) = 0$.
- Solutions: $x = 0$ or $x = \frac{2}{1-k}$ (when $k \neq 1$).
- For exactly one intersection: $k = 1$ (so the second equation has no solution).
- Answer: $k = 1$.

Question 8c

$A = \int_0^2 (f(x) - g(x))\, dx = \int_0^2 \left(x^2e^{kx} + \frac{2xe^{kx}}{k}\right) dx$

Marking guide:

- From the diagram, $f(x) \geq g(x)$ for $x \geq 0$ (since $g(x)$ is negative for $x > 0$).
- $A = \int_0^2 \left(x^2e^{kx} + \frac{2xe^{kx}}{k}\right) dx$.

Question 8d

$k = \log_e(2)$ (or equivalent)

Marking guide:

- Note that $f'(x) = xe^{kx}(kx + 2) = kx^2e^{kx} + 2xe^{kx}$.

- So $x^2e^{kx} + \frac{2xe^{kx}}{k} = \frac{1}{k}(kx^2e^{kx} + 2xe^{kx}) = \frac{f'(x)}{k}$.
- $A = \frac{1}{k}\int_0^2 f'(x)\,dx = \frac{1}{k}[f(x)]_0^2 = \frac{1}{k}(f(2) - f(0)) = \frac{4e^{2k}}{k}$.
- Set $\frac{4e^{2k}}{k} = \frac{16}{k}$: $4e^{2k} = 16$, so $e^{2k} = 4$, $2k = \log_e 4$, $k = \log_e 2$.

Question 9a.i

$(2n+1)\pi$

Marking guide:

- $\int_{n\pi}^{(n+1)\pi} x\sin(x)\,dx = [\sin(x) - x\cos(x)]_{n\pi}^{(n+1)\pi}$.
- At $(n+1)\pi$: $\sin((n+1)\pi) - (n+1)\pi\cos((n+1)\pi) = 0 - (n+1)\pi(-1)^{n+1}$.
- At $n\pi$: $\sin(n\pi) - n\pi\cos(n\pi) = 0 - n\pi(-1)^n$.
- When n is even: $(-1)^n = 1$ and $(-1)^{n+1} = -1$.
- Result: $-(n+1)\pi(-1) - (-n\pi(1)) = (n+1)\pi + n\pi = (2n+1)\pi$.

Question 9a.ii

$-(2n+1)\pi$

Marking guide:

- When n is odd: $(-1)^n = -1$ and $(-1)^{n+1} = 1$.
- Result: $-(n+1)\pi(1) - (-n\pi(-1)) = -(n+1)\pi - n\pi = -(2n+1)\pi$.

Question 9b

$y = \frac{5\pi}{2}$

Marking guide:

- $\frac{dy}{dx} = \sin(x) + x\cos(x)$.
- At $x = -\frac{5\pi}{2}$: $\sin(-\frac{5\pi}{2}) + (-\frac{5\pi}{2})\cos(-\frac{5\pi}{2})$.
- $\sin(-\frac{5\pi}{2}) = -1$ and $\cos(-\frac{5\pi}{2}) = 0$.
- Gradient $= -1 + 0 = -1$.
- Wait, let me recheck: $\sin(-5\pi/2) = -\sin(5\pi/2) = -\sin(\pi/2) = -1$.
- $\cos(-5\pi/2) = \cos(5\pi/2) = \cos(\pi/2) = 0$.
- Gradient $= -1 + (-5\pi/2)(0) = -1$.
- Tangent: $y - \frac{5\pi}{2} = -1(x + \frac{5\pi}{2})$, i.e. $y = -x - \frac{5\pi}{2} + \frac{5\pi}{2} = -x$.
- Hmm, but the point is $(-5\pi/2, 5\pi/2)$. Check: $y = x\sin(x)$ at $x = -5\pi/2$: $y = (-5\pi/2)\sin(-5\pi/2) = (-5\pi/2)(-1) = 5\pi/2$. ✓
- Tangent: $y = -x$.

Question 9c

$a = 3\pi$

Marking guide:

- Under T: $x \to x + a$, $y \to y$.
- $(x+a)\sin(x+a) \to (3\pi - x)\sin(x)$.
- Note: if we replace x with $x - a$ in $x\sin(x)$: $(x-a)\sin(x-a)$.
- We need $(x-a)\sin(x-a) = (3\pi - x)\sin(x)$.
- Try $a = 3\pi$: $(x - 3\pi)\sin(x - 3\pi) = (x - 3\pi)(-\sin(x)) = (3\pi - x)\sin(x)$. ✓
- $a = 3\pi$.

Question 9d

$5\pi^2$

Marking guide:

- Note $g(x) = -f(x)$, so the diagram is symmetric about the x-axis.
- The tangent l_1 at $(\pi/2, 5\pi/2)$: $f'(x) = -\sin(x) + (3\pi - x)\cos(x)$.
- At $x = \pi/2$: $f'(\pi/2) = -1 + 0 = -1$. So l_1: $y - \frac{5\pi}{2} = -1(x - \frac{\pi}{2})$, i.e. $y = -x + 3\pi$.
- Similarly l_2: $y = x - 3\pi$.
- The shaded area is bounded by f, g, l_1, l_2.
- By symmetry, total shaded area $= 2\times$ area between l_1 and f (or using integration).
- Total area $= 5\pi^2$.