Question 1a [1 mark]
Let $y = 3xe^{2x}$. Find $\frac{dy}{dx}$.

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Question 1b [2 marks]
Find and simplify the rule of $f'(x)$, where $f : R \to R$, $f(x) = \frac{\cos(x)}{e^x}$.

Question 2a [1 mark]
Let $g : \left(\frac{3}{2}, \infty\right) \to R$, $g(x) = \frac{3}{2x-3}$. Find the rule for an antiderivative of $g(x)$.

Question 2b [3 marks]
Evaluate $\int_0^1 f(x)\big(2f(x) - 3\big)\,dx$, where $\int_0^1 \big[f(x)\big]^2\,dx = \frac{1}{5}$ and $\int_0^1 f(x)\,dx = \frac{1}{3}$.

Question 3 [3 marks]
Consider the system of equations

$$kx - 5y = 4 + k$$

$$3x + (k + 8)y = -1$$

Determine the value of k for which the system of equations above has an infinite number of solutions.

Question 4a [2 marks]
A card is drawn from a deck of red and blue cards. After verifying the colour, the card is replaced. This is performed four times. Each card has probability $\frac{1}{2}$ of being red and $\frac{1}{2}$ of being blue. Let X be the number of blue cards drawn.
Complete the table: | x | 0 | 1 | 2 | 3 | 4 | |—|—|—|—|—|—| | $\Pr(X = x)$ | $\frac{1}{16}$ | | $\frac{6}{16}$ | | |

Question 4b [1 mark]
Given that the first card drawn is blue, find the probability that exactly two of the next three cards drawn will be red.

Question 4c [2 marks]
The deck is changed so that the probability of a card being red is $\frac{2}{3}$ and the probability of a card being blue is $\frac{1}{3}$. Given that the first card drawn is blue, find the probability that exactly two of the next three cards drawn will be red.

Question 5a [2 marks]
Solve $10^{3x-13} = 100$ for x.

Question 5b [3 marks]
Find the maximal domain of f, where $f(x) = \log_e(x^2 - 2x - 3)$.

Question 6a [2 marks]

The graph of $y = f(x)$, where $f : [0, 2\pi] \to R$, $f(x) = 2\sin(2x) - 1$, is shown.

On the axes, draw the graph of $y = g(x)$, where $g(x)$ is the reflection of $f(x)$ in the horizontal axis.

Question 6b [3 marks]

Find all values of k such that $f(k) = 0$ and $k \in [0, 2\pi]$.

Question 6c.i [1 mark]

Let $h : D \to R$, $h(x) = 2\sin(2x) - 1$, where $h(x)$ has the same rule as $f(x)$ with a different domain. The graph of $y = h(x)$ is translated a units in the positive horizontal direction and b units in the positive vertical direction so that it is mapped onto the graph of $y = g(x)$, where $a, b \in (0, \infty)$.

Find the value for b.

Question 6c.ii [1 mark]

Find the smallest positive value for a.

Question 6c.iii [1 mark]

Hence, or otherwise, state the domain D of $h(x)$.

Question 7a.i [1 mark]

A tilemaker wants to make square tiles of size 20 cm × 20 cm. The front surface is painted with two colours meeting these conditions: - Condition 1: Each colour covers half the front surface. - Condition 2: Tiles can line up horizontally to form a continuous pattern.

For Type A, colours are divided using $f(x) = 4\sin\left(\frac{\pi x}{10}\right) + a$, where $a \in R$. Tile corners are at $(0, 0)$, $(20, 0)$, $(20, 20)$, $(0, 20)$.

Find the area of the front surface of each tile.

Question 7a.ii [1 mark]

Find the value of a so that a Type A tile meets Condition 1.

Question 7b [3 marks]

Type B tiles are divided using $g(x) = -\frac{1}{100}x^3 + \frac{3}{10}x^2 - 2x + 10$.

Show that a Type B tile meets Condition 1.

Question 7c [2 marks]

Determine the endpoints of $f(x)$ and $g(x)$ on each tile. Hence, use these values to confirm that Type A and Type B tiles can be placed in any order to produce a continuous pattern to meet Condition 2.

Question 8a [1 mark]

Part of the graph of $y = f(x)$ is shown. The rule $A(k) = k\sin(k)$ gives the area bounded by the graph of f, the horizontal axis and the line $x = k$.

State the value of $A\left(\frac{\pi}{3}\right)$.

Question 8b [2 marks]

Evaluate $f\left(\frac{\pi}{3}\right)$.

Question 8c [2 marks]

Consider the average value of the function f over the interval $x \in [0, k]$, where $k \in [0, 2]$.
Find the value of k that results in the maximum average value.

Solutions

Question 1a

$\frac{dy}{dx} = 3e^{2x} + 6xe^{2x} = 3e^{2x}(1 + 2x)$

Marking guide:

- Apply product rule: $u = 3x$, $v = e^{2x}$.
- $u' = 3$, $v' = 2e^{2x}$.
- $\frac{dy}{dx} = 3e^{2x} + 6xe^{2x}$ or equivalently $3e^{2x}(1 + 2x)$.

Question 1a

$\frac{dy}{dx} = 3e^{2x} + 6xe^{2x} = 3e^{2x}(1 + 2x)$

Marking guide:

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Question 1b

$f'(x) = \frac{-e^x \sin(x) - e^x \cos(x)}{e^{2x}} = -\frac{\sin(x)+\cos(x)}{e^x}$

Marking guide:

- Apply quotient rule: $u = \cos(x)$, $v = e^x$.
- $u' = -\sin(x)$, $v' = e^x$.
- $f'(x) = \frac{-e^x \sin(x) - e^x \cos(x)}{e^{2x}}$.
- Simplify: $f'(x) = -\frac{\sin(x)+\cos(x)}{e^x}$.

Question 2a

$\frac{3}{2}\log_e(2x - 3) + c$

Marking guide:

- Recognise $\int \frac{3}{2x-3} dx = \frac{3}{2}\log_e(2x - 3) + c$.

Question 2b

$-\frac{3}{5}$

Marking guide:

- Expand: $\int_0^1 f(x)(2f(x) - 3)\,dx = \int_0^1 2[f(x)]^2 - 3f(x)\,dx$.
- Split: $= 2\int_0^1 [f(x)]^2\,dx - 3\int_0^1 f(x)\,dx$.
- Substitute: $= 2 \times \frac{1}{5} - 3 \times \frac{1}{3} = \frac{2}{5} - 1 = -\frac{3}{5}$.

Question 3

$k = -3$

Marking guide:

- For infinite solutions, the ratios of coefficients must be equal: $\frac{k}{3} = \frac{-5}{k+8} = \frac{4+k}{-1}$.
- From $\frac{k}{3} = \frac{-5}{k+8}$: $k(k + 8) = -15$, so $k^2 + 8k + 15 = 0$, $(k + 3)(k + 5) = 0$, giving $k = -3$ or $k = -5$.
- Check $\frac{4+k}{-1}$: For $k = -3$: $\frac{1}{-1} = -1$ and $\frac{-3}{3} = -1$ ✓. For $k = -5$: $\frac{-1}{-1} = 1$ and $\frac{-5}{3} \neq 1$.
- Therefore $k = -3$.

Question 4a

$\Pr(X = 1) = \frac{4}{16}$, $\Pr(X = 3) = \frac{4}{16}$, $\Pr(X = 4) = \frac{1}{16}$

Marking guide:

- $X \sim \text{Bi}(4, \frac{1}{2})$.
- $\Pr(X = 1) = \binom{4}{1}(\frac{1}{2})^4 = \frac{4}{16}$.
- $\Pr(X = 3) = \binom{4}{3}(\frac{1}{2})^4 = \frac{4}{16}$.
- $\Pr(X = 4) = (\frac{1}{2})^4 = \frac{1}{16}$.

Question 4b

$\frac{3}{8}$

Marking guide:

- Given first card is blue, the remaining 3 draws are independent with $p = \frac{1}{2}$.
- Pr(exactly 2 red out of 3) $= \binom{3}{2}(\frac{1}{2})^2(\frac{1}{2})^1 = \frac{3}{8}$.

Question 4c

$\frac{4}{9}$

Marking guide:

- Given first card is blue (already happened), remaining 3 draws: $p(\text{red}) = \frac{2}{3}$.
- Pr(exactly 2 red out of 3) $= \binom{3}{2}(\frac{2}{3})^2(\frac{1}{3})^1 = 3 \times \frac{4}{9} \times \frac{1}{3} = \frac{12}{27} = \frac{4}{9}$.

Question 5a

$x = 5$

Marking guide:

- Write $100 = 10^2$.
- $10^{3x-13} = 10^2$, so $3x - 13 = 2$.
- $3x = 15$, $x = 5$.

Question 5b

$(-\infty, -1) \cup (3, \infty)$

Marking guide:

- Require $x^2 - 2x - 3 > 0$.
- Factorise: $(x-3)(x+1) > 0$.
- Critical points: $x = -1$ and $x = 3$.
- Solution: $x < -1$ or $x > 3$, i.e., $(-\infty, -1) \cup (3, \infty)$.

Question 6a

$g(x) = -f(x) = -2\sin(2x) + 1 = 1 - 2\sin(2x)$

Marking guide:

- Correct shape: reflection of $f(x)$ in the x-axis.
- $g(x) = -(2\sin(2x) - 1) = 1 - 2\sin(2x)$.
- Range of g: $[-1, 3]$, endpoints correct.

Question 6b

$k = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

Marking guide:

- $2\sin(2k) - 1 = 0 \Rightarrow \sin(2k) = \frac{1}{2}$.
- $2k = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$ (for $2k \in [0, 4\pi]$).
- $k = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$.

Question 6c.i

$b = 2$

Marking guide:

- $g(x) = 1 - 2\sin(2x)$ has vertical midline at $y = 1$; $h(x) = 2\sin(2x) - 1$ has midline at $y = -1$.
- Vertical shift: $b = 1 - (-1) = 2$.

Question 6c.ii

$a = \frac{\pi}{2}$

Marking guide:

- After vertical shift: $h(x) + 2 = 2\sin(2x) + 1$.
- Need $2\sin(2(x-a)) + 1 = 1 - 2\sin(2x)$, so $\sin(2x - 2a) = -\sin(2x)$.
- $\sin(2x - 2a) = \sin(2x + \pi)$, so $2a = \pi$ (smallest positive), $a = \frac{\pi}{2}$.

Question 6c.iii

$D = \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$

Marking guide:

- $g(x)$ is defined on $[0, 2\pi]$, and $h(x - \frac{\pi}{2}) + 2 = g(x)$.
- So h is evaluated at $x - \frac{\pi}{2}$ for $x \in [0, 2\pi]$.
- Domain of h: $[0 - \frac{\pi}{2}, 2\pi - \frac{\pi}{2}] = [-\frac{\pi}{2}, \frac{3\pi}{2}]$.

Question 7a.i
400 cm^2
Marking guide:
- Area $= 20 \times 20 = 400$ cm^2.

Question 7a.ii
$a = 10$
Marking guide:
- Area below $f(x)$ from $x = 0$ to $x = 20$: $\int_0^{20} f(x)\,dx = \int_0^{20} 4\sin(\frac{\pi x}{10}) + a\,dx$.
- $= \left[-\frac{40}{\pi}\cos(\frac{\pi x}{10}) + ax\right]_0^{20} = -\frac{40}{\pi}(\cos(2\pi) - \cos(0)) + 20a = 0 + 20a = 20a$.
- For Condition 1: $20a = 200$, so $a = 10$.

Question 7b
$\int_0^{20} g(x)\,dx = 200$
Marking guide:
- $\int_0^{20} g(x)\,dx = \int_0^{20} \left(-\frac{x^3}{100} + \frac{3x^2}{10} - 2x + 10\right) dx$.
- $= \left[-\frac{x^4}{400} + \frac{x^3}{10} - x^2 + 10x\right]_0^{20}$.
- $= -\frac{160000}{400} + \frac{8000}{10} - 400 + 200 = -400 + 800 - 400 + 200 = 200$.
- Since $200 = \frac{1}{2} \times 400$, Condition 1 is met.

Question 7c
Both f and g have value 10 at $x = 0$ and $x = 20$.
Marking guide:
- $f(0) = 4\sin(0) + 10 = 10$ and $f(20) = 4\sin(2\pi) + 10 = 10$.
- $g(0) = 0 + 0 - 0 + 10 = 10$ and $g(20) = -80 + 120 - 40 + 10 = 10$.
- Both functions have the same value (10) at $x = 0$ and $x = 20$, so tiles connect continuously.

Question 8a
$A\left(\frac{\pi}{3}\right) = \frac{\pi\sqrt{3}}{6}$
Marking guide:
- $A\left(\frac{\pi}{3}\right) = \frac{\pi}{3} \cdot \sin\left(\frac{\pi}{3}\right) = \frac{\pi}{3} \cdot \frac{\sqrt{3}}{2} = \frac{\pi\sqrt{3}}{6}$.

Question 8b
$f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} + \frac{\pi}{6}$
Marking guide:
- Since $\int_0^k f(x)\,dx = A(k) = k\sin(k)$, by the Fundamental Theorem of Calculus: $f(k) = A'(k)$.
- $A'(k) = \sin(k) + k\cos(k)$.
- $f\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) + \frac{\pi}{3}\cos\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} + \frac{\pi}{3} \cdot \frac{1}{2} = \frac{\sqrt{3}}{2} + \frac{\pi}{6}$.

Question 8c
$k = \frac{\pi}{2}$
Marking guide:
- Average value $= \frac{1}{k}\int_0^k f(x)\,dx = \frac{A(k)}{k} = \frac{k\sin(k)}{k} = \sin(k)$.
- Maximise $\sin(k)$ for $k \in [0, 2]$: maximum at $k = \frac{\pi}{2} \approx 1.571$.
- Since $\frac{\pi}{2} \in [0, 2]$, the maximum average value occurs at $k = \frac{\pi}{2}$.