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# ATAR Master

VCE Mathematical Methods

**2023 Examination 2 (Technology-Active)**

Questions & Marking Guide

Total: 80 marks

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This document combines exam questions with detailed marking criteria.  
Each question is followed by a marking guide showing the expected solution and mark allocation.

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**Question 1**

1 mark

The amplitude,  $A$ , and the period,  $P$ , of the function  $f(x) = -\frac{1}{2} \sin(3x + 2\pi)$  are

**Marking Guide** — Answer: B

- $A = \frac{1}{2}, P = \frac{2\pi}{3}$ .

**Section A: Multiple Choice — 20 marks**

Each question is worth 1 mark.

**Question 2**

1 mark

For the parabola with equation  $y = ax^2 + 2bx + c$ , where  $a, b, c \in R$ , the equation of the axis of symmetry is

**A.**  $y = c$

**Marking Guide** — Answer: A

- Axis of symmetry:  $x = -\frac{2b}{2a} = -\frac{b}{a}$ .

## Section B: Extended Response — 60 marks

### Question 3

1 mark

Two functions,  $p$  and  $q$ , are continuous over their domains, which are  $[-2, 3)$  and  $(-1, 5]$ , respectively.

The domain of the sum function  $p + q$  is

**Marking Guide** — Answer: E

### Question 4

1 mark

Consider the system of simultaneous linear equations below containing the parameter  $k$ .

$$kx + 5y = k + 5$$

$$4x + (k + 1)y = 0$$

The value(s) of  $k$  for which the system of equations has infinite solutions are

**Marking Guide** — Answer: A

- Determinant  $= k(k + 1) - 20 = k^2 + k - 20 = (k + 5)(k - 4) = 0$ .  $k \in \{-5, 4\}$ .

### Question 5

1 mark

Which one of the following functions has a horizontal tangent at  $(0, 0)$ ?

**Marking Guide** — Answer: D

- $y = x^{4/3}$ : passes through origin and  $y'(0) = \frac{4}{3}(0)^{1/3} = 0$ .

### Question 6

1 mark

Suppose that  $\int_3^{10} f(x) dx = C$  and  $\int_7^{10} f(x) dx = D$ . The value of  $\int_7^3 f(x) dx$  is

- A.  $C + D$
- B.  $C + D - 3$
- C.  $C - D$
- D.  $D - C$
- E.  $CD - 3$

**Marking Guide** — Answer: D

- $\int_7^3 f(x) dx = -\int_3^7 f(x) dx = -(C - D) = D - C$ .

### Question 7

1 mark

Let  $f(x) = \log_e x$ , where  $x > 0$  and  $g(x) = \sqrt{1 - x}$ , where  $x < 1$ .

The domain of the derivative of  $(f \circ g)(x)$  is

- A.  $x \in \mathbb{R}$

**Marking Guide** — Answer: C

- $(f \circ g)(x) = \log_e(\sqrt{1-x})$ . Need  $1-x > 0$ , so  $x < 1$ . Domain of derivative:  $(-\infty, 1)$ .

**Question 8**

1 mark

A box contains  $n$  green balls and  $m$  red balls. A ball is selected at random, and its colour is noted. The ball is then replaced in the box.

In 8 such selections, where  $n < m$ , what is the probability that a green ball is selected at least once?

**Marking Guide** — Answer: C

- $P(\text{at least 1 green}) = 1 - P(\text{no green})^8 = 1 - \left(\frac{m}{n+m}\right)^8$ .

**Question 9**

1 mark

The function  $f$  is given by

$$f(x) = \begin{cases} \tan\left(\frac{x}{2}\right) & 4 \leq x < 2\pi \\ \sin(ax) & 2\pi \leq x \leq 8 \end{cases}$$

The value of  $a$  for which  $f$  is continuous and smooth at  $x = 2\pi$  is

A.  $-2$

B.  $2$

**Marking Guide** — Answer: C

- Continuity:  $\tan(\pi) = 0 = \sin(2\pi a)$ . Smoothness:  $\frac{1}{2} \sec^2(\pi) = a \cos(2\pi a)$ .  $a = -\frac{1}{2}$ .

**Question 10**

1 mark

A continuous random variable  $X$  has the following probability density function.

$$g(x) = \begin{cases} \frac{x-1}{20} & 1 \leq x < 6 \\ \frac{9-x}{12} & 6 \leq x \leq 9 \\ 0 & \text{elsewhere} \end{cases}$$

The value of  $k$  such that  $\Pr(X < k) = 0.35$  is

**Marking Guide** — Answer: B

- $\int_1^k \frac{x-1}{20} dx = \frac{(k-1)^2}{40} = 0.35$ .  $(k-1)^2 = 14$ .  $k = \sqrt{14} + 1$ .

**Question 11**

1 mark

Two functions,  $f$  and  $g$ , are continuous and differentiable for all  $x \in \mathbb{R}$ . It is given that  $f(-2) = -7$ ,  $g(-2) = 8$  and  $f'(-2) = 3$ ,  $g'(-2) = 2$ .

The gradient of the graph  $y = f(x) \times g(x)$  at the point where  $x = -2$  is

A.  $-10$

- B.  $-6$
- C.  $0$
- D.  $6$
- E.  $10$

**Marking Guide** — Answer: E

- $y' = f'g + fg'$ . At  $x = -2$ :  $3(8) + (-7)(2) = 24 - 14 = 10$ .

### Question 12

1 mark

The probability mass function for the discrete random variable  $X$  is shown below.

$X$	$-1$	$0$	$1$	$2$	$3$	$4$	$5$	$6$	$7$	$8$	$9$	$10$	$\Pr(X = x)$	$k^2$	$3k$	$k$	$-k^2 - 4k + 1$
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The maximum possible value for the mean of  $X$  is:

- A.  $0$
- B.  $1$
- C.  $2$

**Marking Guide** — Answer: E

- $E(X) = -3k^2 - 7k + 2$ , which is decreasing for  $k \geq 0$ . Maximum at  $k = 0$ :  $E(X) = 2$ .

### Question 13

1 mark

The following algorithm applies Newton's method using a **For** loop with 3 iterations.

```

Inputs: f(x), a function of x
        df(x), the derivative of f(x)
        x0, an initial estimate

Define newton(f(x), df(x), x0)
    For i from 1 to 3
        If df(x0) = 0 Then
            Return "Error: Division by zero"
        Else
            x0 ← x0 - f(x0) ÷ df(x0)
    EndFor
    Return x0
    
```

The **Return** value of the function 'newton( $x^3 + 3x - 3, 3x^2 + 3, 1$ )' is closest to

- A.  $0.83333$
- B.  $0.81785$
- C.  $0.81773$
- D.  $1$
- E.  $3$

**Marking Guide** — Answer: C

- $x_0 = 1$ ,  $x_1 = 5/6 \approx 0.83333$ ,  $x_2 \approx 0.81785$ ,  $x_3 \approx 0.81773$ .

**Question 14**

1 mark

A polynomial has the equation  $y = x(3x - 1)(x + 3)(x + 1)$ .

The number of tangents to this curve that pass through the positive  $x$ -intercept is

- A. 0
- B. 1
- C. 2
- D. 3
- E. 4

**Marking Guide** — Answer: D

- Positive  $x$ -intercept at  $(\frac{1}{3}, 0)$ . Three tangent lines pass through this point.

**Question 15**

1 mark

Let  $X$  be a normal random variable with mean of 100 and standard deviation of 20. Let  $Y$  be a normal random variable with mean of 80 and standard deviation of 10.

Which of the diagrams below best represents the probability density functions for  $X$  and  $Y$ , plotted on the same set of axes?

- A. Graph A
- B. Graph B
- C. Graph C
- D. Graph D
- E. Graph E

**Marking Guide** — Answer: D

- $Y$  (solid) is narrower (SD=10) and centred left (mean=80).  $X$  (dashed) is wider (SD=20) and centred right (mean=100). Diagram D.

**Question 16**

1 mark

Let  $f(x) = e^{x-1}$ .

Given that the product function  $f(x) \times g(x) = e^{(x-1)^2}$ , the rule for the function  $g$  is

**Marking Guide** — Answer: B

- $g(x) = e^{(x-1)^2} / e^{x-1} = e^{(x-1)^2 - (x-1)} = e^{(x-1)(x-2)} = e^{(x-2)(x-1)}$ .

**Question 17**

1 mark

A cylinder of height  $h$  and radius  $r$  is formed from a thin rectangular sheet of metal of length  $x$  and width  $y$ , by cutting along the dashed lines shown below.

The volume of the cylinder, in terms of  $x$  and  $y$ , is given by

- A.  $\pi x^2 y$

**Marking Guide** — Answer: B

- $2\pi r = y$ ,  $r = \frac{y}{2\pi}$ .  $h = x - 4r = x - \frac{2y}{\pi}$ .  $V = \pi r^2 h = \frac{\pi x y^2 - 2y^3}{4\pi^2}$ .

**Question 18**

1 mark

Consider the function  $f : [-a\pi, a\pi] \rightarrow \mathbb{R}$ ,  $f(x) = \sin(ax)$ , where  $a$  is a positive integer.

The number of local minima in the graph of  $y = f(x)$  is always equal to

- A. 2  
B. 4  
C.  $a$   
D.  $2a$   
E.  $a^2$

**Marking Guide** — Answer: E

**Question 19**

1 mark

Find all values of  $k$ , such that the equation  $x^2 + (4k + 3)x + 4k^2 - \frac{9}{4} = 0$  has two real solutions for  $x$ , one positive and one negative.

**Marking Guide** — Answer: D

- Product of roots  $< 0$ :  $4k^2 - \frac{9}{4} < 0 \Rightarrow -\frac{3}{4} < k < \frac{3}{4}$ . Discriminant  $> 0$ :  $k > -\frac{3}{4}$ .

**Question 20**

1 mark

Let  $f(x) = \log_e \left( x + \frac{1}{\sqrt{2}} \right)$ .

Let  $g(x) = \sin(x)$  where  $x \in (-\infty, 5)$ .

The largest interval of  $x$  values for which  $(f \circ g)(x)$  and  $(g \circ f)(x)$  both exist is

**Marking Guide** — Answer: A

- Need  $x > -\frac{1}{\sqrt{2}}$  (for  $g \circ f$ ) and  $\sin(x) > -\frac{1}{\sqrt{2}}$  (for  $f \circ g$ ). Largest interval:  $\left( -\frac{1}{\sqrt{2}}, \frac{5\pi}{4} \right)$ .

**Question 1a**

1 mark

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = x(x - 2)(x + 1)$ . Part of the graph of  $f$  is shown.

State the coordinates of all axial intercepts of  $f$ .

**Marking Guide** — Answer:  $(0, 0)$ ,  $(2, 0)$ ,  $(-1, 0)$

- $x$ -intercepts at  $x = 0, 2, -1$ .  $y$ -intercept at  $(0, 0)$ .

**Question 1b**

2 marks

Find the coordinates of the stationary points of  $f$ .

**Marking Guide** — Answer: See marking guide

- M1:  $f'(x) = 3x^2 - 2x - 2 = 0$ . Solve using CAS.
- A1:  $x = \frac{1 \pm \sqrt{7}}{3}$ . Substitute to find  $y$ -coordinates.

**Question 1c.i**

1 mark

Let  $g : R \rightarrow R$ ,  $g(x) = x - 2$ .

Find the values of  $x$  for which  $f(x) = g(x)$ .

**Marking Guide** — Answer:  $x = -1, 1, 2$

- $x(x - 2)(x + 1) = x - 2$ . Solve:  $x = -1, 1, 2$ .

**Question 1c.ii**

2 marks

Write down an expression using definite integrals that gives the area of the regions bound by  $f$  and  $g$ .

**Marking Guide** — Answer: See marking guide

- M1:  $\int_{-1}^1 |f(x) - g(x)| dx + \int_1^2 |f(x) - g(x)| dx$ .
- A1:  $\int_{-1}^1 (f(x) - g(x)) dx + \int_1^2 (g(x) - f(x)) dx$  (or vice versa with absolute values).

**Question 1c.iii**

1 mark

Hence, find the total area of the regions bound by  $f$  and  $g$ , correct to two decimal places.

**Marking Guide** — Answer: See marking guide

- Evaluate using CAS.

**Question 1d**

4 marks

Let  $h : R \rightarrow R$ ,  $h(x) = (x - a)(x - b)^2$ , where  $h(x) = f(x) + k$  and  $a, b, k \in R$ .

Find the possible values of  $a$  and  $b$ .

**Marking Guide** — Answer: See marking guide

- M1:  $h(x) = x^3 - 2x^2 + x + k$  (expanding  $f(x) + k = x^3 - x^2 - 2x + k$ ... actually  $f(x) = x^3 - x^2 - 2x$ , so  $h(x) = x^3 - x^2 - 2x + k$ ).
- M1:  $(x - a)(x - b)^2 = x^3 - (a + 2b)x^2 + (2ab + b^2)x - ab^2$ . Equate coefficients.
- A2: Solve the system for  $a$ ,  $b$ , and  $k$ .



**Question 2a**

2 marks

The following diagram represents an observation wheel, with its centre at point  $P$ . Passengers are seated in pods, which are carried around as the wheel turns. The wheel moves anticlockwise with constant speed and completes one full rotation every 30 minutes. When a pod is at the lowest point of the wheel (point  $A$ ), it is 15 metres above the ground. The wheel has a radius of 60 metres.

Consider the function  $h(t) = -60 \cos(bt) + c$  for some  $b, c \in \mathbb{R}$ , which models the height above the ground of a pod originally situated at point  $A$ , after time  $t$  minutes.

Show that  $b = \frac{\pi}{15}$  and  $c = 75$ .

**Marking Guide** — Answer: See marking guide

- M1: Period = 30, so  $\frac{2\pi}{b} = 30 \Rightarrow b = \frac{\pi}{15}$ .
- A1: At  $t = 0$  (point  $A$ ),  $h(0) = 15$ :  $-60(1) + c = 15 \Rightarrow c = 75$ .

**Question 2b**

2 marks

Find the average height of a pod on the wheel as it travels from point  $A$  to point  $B$ .

Give your answer in metres, correct to two decimal places.

**Marking Guide** — Answer: See marking guide

- M1: Point  $B$  is at the same height as  $P$  (centre), to the right.  $h = 75$  at  $B$ , which occurs at  $t = 7.5$ .
- A1: Average =  $\frac{1}{7.5} \int_0^{7.5} h(t) dt$ . Evaluate using CAS.

**Question 2c**

1 mark

Find the average rate of change, in metres per minute, of the height of a pod on the wheel as it travels from point  $A$  to point  $B$ .

**Marking Guide** — Answer: See marking guide

- $\frac{h(7.5) - h(0)}{7.5 - 0} = \frac{75 - 15}{7.5} = 8 \text{ m/min.}$

**Question 2d.i**

1 mark

After 15 minutes, the wheel stops moving and remains stationary for 5 minutes. After this, it continues moving at double its previous speed for another 7.5 minutes.

The height above the ground of a pod that was initially at point  $A$ , after  $t$  minutes, can be modelled by the piecewise function  $w$ :

$$w(t) = \begin{cases} h(t) & 0 \leq t < 15 \\ k & 15 \leq t < 20 \\ h(mt + n) & 20 \leq t \leq 27.5 \end{cases}$$

where  $k \geq 0$ ,  $m \geq 0$  and  $n \in \mathbb{R}$ .

State the values of  $k$  and  $m$ .

**Marking Guide** — Answer:  $k = h(15) = 75$ ,  $m = 2$

- $k = h(15) = -60 \cos(\pi) + 75 = 135$ .  $m = 2$  (double speed).

**Question 2d.ii**

2 marks

Find **all** possible values of  $n$ .

**Marking Guide** — Answer: See marking guide

- M1: Continuity at  $t = 20$ :  $h(2(20) + n) = k = h(15)$ . So  $h(40 + n) = h(15)$ .
- A1:  $40 + n = 15 + 30j$  for integer  $j$ , or use symmetry. Find all valid  $n$ .

**Question 2d.iii**

3 marks

Sketch the graph of the piecewise function  $w$  on the axes below, showing the coordinates of the endpoints.

**Marking Guide** — Answer: See marking guide

- M1: Graph of  $h(t)$  from  $t = 0$  to  $t = 15$ .
- M1: Horizontal line at  $w = k$  from  $t = 15$  to  $t = 20$ .
- A1: Graph of  $h(2t + n)$  from  $t = 20$  to  $t = 27.5$  with correct endpoints labelled.

**Question 3a**

1 mark

Consider the function  $g : \mathbb{R} \rightarrow \mathbb{R}$ ,  $g(x) = 2^x + 5$ .

State the value of  $\lim_{x \rightarrow -\infty} g(x)$ .

**Marking Guide** — Answer: 5

- As  $x \rightarrow -\infty$ ,  $2^x \rightarrow 0$ , so  $g(x) \rightarrow 5$ .

**Question 3b**

1 mark

The derivative,  $g'(x)$ , can be expressed in the form  $g'(x) = k \times 2^x$ .

Find the real number  $k$ .

**Marking Guide** — Answer:  $k = \log_e(2)$

- $g'(x) = 2^x \ln 2 = \log_e(2) \times 2^x$ .

**Question 3c.i**

1 mark

Let  $a$  be a real number. Find, in terms of  $a$ , the equation of the tangent to  $g$  at the point  $(a, g(a))$ .

**Marking Guide** — Answer:  $y = \log_e(2) \cdot 2^a(x - a) + 2^a + 5$

- Tangent:  $y - g(a) = g'(a)(x - a)$ , where  $g(a) = 2^a + 5$  and  $g'(a) = 2^a \ln 2$ .

**Question 3c.ii**

2 marks

Hence, or otherwise, find the equation of the tangent to  $g$  that passes through the origin, correct to three decimal places.

**Marking Guide** — Answer: See marking guide

- M1: Tangent through origin:  $0 = \ln(2) \cdot 2^a(0 - a) + 2^a + 5$ . Solve for  $a$  using CAS.
- A1: Find  $a$  and substitute to get equation  $y = mx$ .

### Question 3d

1 mark

Let  $h : R \rightarrow R$ ,  $h(x) = 2^x - x^2$ .

Find the coordinates of the point of inflection for  $h$ , correct to two decimal places.

**Marking Guide** — Answer: See marking guide

- $h''(x) = (\ln 2)^2 \cdot 2^x - 2 = 0$ . Solve using CAS.

### Question 3e

1 mark

Find the largest interval of  $x$  values for which  $h$  is strictly decreasing.

Give your answer correct to two decimal places.

**Marking Guide** — Answer: See marking guide

- Solve  $h'(x) = \ln(2) \cdot 2^x - 2x = 0$  using CAS.  $h$  is decreasing between the two solutions.

### Question 3f

2 marks

Apply Newton's method, with an initial estimate of  $x_0 = 0$ , to find an approximate  $x$ -intercept of  $h$ .

Write the estimates  $x_1$ ,  $x_2$  and  $x_3$  in the table below, correct to three decimal places.

**Marking Guide** — Answer: See marking guide

- M1:  $x_1 = 0 - h(0)/h'(0) = 0 - (1 - 0)/(\ln 2 - 0) = -1/\ln 2 \approx -1.443$ .
- A1: Continue iterating to find  $x_2$  and  $x_3$ .

### Question 3g

1 mark

For the function  $h$ , explain why a solution to the equation  $\log_e(2) \times (2^x) - 2x = 0$  should not be used as an initial estimate  $x_0$  in Newton's method.

**Marking Guide** — Answer: See marking guide

- $\log_e(2) \cdot 2^x - 2x = 0$  is  $h'(x) = 0$ , i.e., a stationary point. Using a stationary point as  $x_0$  causes division by zero in Newton's formula.

### Question 3h

2 marks

There is a positive real number  $n$  for which the function  $f(x) = n^x - x^n$  has a local minimum on the  $x$ -axis.

Find this value of  $n$ .

**Marking Guide** — Answer: See marking guide

- M1: Local min on  $x$ -axis means  $f(a) = 0$  and  $f'(a) = 0$  for some  $a$ .  $n^a = a^n$  and  $n^a \ln n = na^{n-1}$ .
- A1: Solve to find  $n$ .

**Question 4a**

1 mark

A manufacturer produces tennis balls.

The diameter of the tennis balls is a normally distributed random variable  $D$ , which has a mean of 6.7 cm and a standard deviation of 0.1 cm.

Find  $\Pr(D > 6.8)$ , correct to four decimal places.

**Marking Guide** — Answer: 0.1587

- $\Pr(D > 6.8) = \Pr(Z > 1) \approx 0.1587$ .

**Question 4b**

1 mark

Find the minimum diameter of a tennis ball that is larger than 90% of all tennis balls produced.

Give your answer in centimetres, correct to two decimal places.

**Marking Guide** — Answer: 6.83 cm

- $d = 6.7 + 1.2816 \times 0.1 \approx 6.83$  cm.

**Question 4c**

1 mark

Tennis balls are packed and sold in cylindrical containers. A tennis ball can fit through the opening at the top of the container if its diameter is smaller than 6.95 cm.

Find the probability that a randomly selected tennis ball can fit through the opening at the top of the container.

Give your answer correct to four decimal places.

**Marking Guide** — Answer: 0.9938

- $\Pr(D < 6.95) = \Pr(Z < 2.5) \approx 0.9938$ .

**Question 4d**

2 marks

In a random selection of 4 tennis balls, find the probability that at least 3 balls can fit through the opening at the top of the container.

Give your answer correct to four decimal places.

**Marking Guide** — Answer: See marking guide

- M1: Let  $p = \Pr(D < 6.95)$ .  $\Pr(X \geq 3) = \binom{4}{3}p^3(1-p) + p^4$ .
- A1: Evaluate using CAS.

**Question 4e**

2 marks

A tennis ball is classed as grade A if its diameter is between 6.54 cm and 6.86 cm, otherwise it is classed as grade B.

Given that a tennis ball can fit through the opening at the top of the container, find the probability that it is classed as grade A.

Give your answer correct to four decimal places.

**Marking Guide** — Answer: See marking guide

- M1:  $\Pr(A \mid D < 6.95) = \frac{\Pr(6.54 < D < 6.86)}{\Pr(D < 6.95)}$ .
- A1: Evaluate using CAS.

#### Question 4f

2 marks

The manufacturer would like to improve processes to ensure that more than 99% of all tennis balls produced are classed as grade A.

Assuming that the mean diameter of the tennis balls remains the same, find the required standard deviation of the diameter, in centimetres, correct to two decimal places.

**Marking Guide** — Answer: See marking guide

- M1:  $\Pr(6.54 < D < 6.86) > 0.99$ . By symmetry, need  $\Pr(D < 6.54) < 0.005$ .
- A1:  $\frac{6.7-6.54}{\sigma} = z_{0.005} \approx 2.576$ .  $\sigma = 0.16/2.576 \approx 0.06$  cm.

#### Question 4g

2 marks

An inspector takes a random sample of 32 tennis balls from the manufacturer and determines a confidence interval for the population proportion of grade A balls produced.

The confidence interval is (0.7382, 0.9493), correct to 4 decimal places.

Find the level of confidence that the population proportion of grade A balls is within the interval, as a percentage correct to the nearest integer.

**Marking Guide** — Answer: See marking guide

- M1:  $\hat{p} = \frac{0.7382+0.9493}{2} = 0.84375$ . Margin =  $0.9493 - 0.84375 = 0.10555$ .
- A1:  $z = \frac{0.10555}{\sqrt{0.84375 \times 0.15625/32}}$ . Find confidence level.

#### Question 4h

1 mark

A tennis coach uses both grade A and grade B balls. The serving speed, in metres per second, of a grade A ball is a continuous random variable,  $V$ , with the probability density function

$$f(v) = \begin{cases} \frac{1}{6\pi} \sin\left(\sqrt{\frac{v-30}{3}}\right) & 30 \leq v \leq 3\pi^2 + 30 \\ 0 & \text{elsewhere} \end{cases}$$

Find the probability that the serving speed of a grade A ball exceeds 50 metres per second.

Give your answer correct to four decimal places.

**Marking Guide** — Answer: See marking guide

- $\Pr(V > 50) = \int_{50}^{3\pi^2+30} f(v) dv$ . Evaluate using CAS.

#### Question 4i

1 mark

Find the **exact** mean serving speed for grade A balls, in metres per second.

**Marking Guide** — Answer: See marking guide

- $E(V) = \int_{30}^{3\pi^2+30} v \cdot f(v) dv$ . Evaluate using CAS.

**Question 4j**

2 marks

The serving speed of a grade B ball is given by a continuous random variable,  $W$ , with the probability density function  $g(w)$ .

A transformation maps the graph of  $f$  to the graph of  $g$ , where  $g(w) = af\left(\frac{w}{b}\right)$ .

If the mean serving speed for a grade B ball is  $2\pi^2 + 8$  metres per second, find the values of  $a$  and  $b$ .

**Marking Guide** — Answer: See marking guide

- M1: Under transformation  $w = bv$ ,  $g(w) = \frac{1}{b}f(w/b)$ , so  $a = 1/b$ . Mean of  $W = b \times E(V)$ .
- A1: Use  $E(W) = 2\pi^2 + 8$  to find  $b$ , then  $a = 1/b$ .

**Question 5a**

2 marks

Let  $f : R \rightarrow R$ ,  $f(x) = e^x + e^{-x}$  and  $g : R \rightarrow R$ ,  $g(x) = \frac{1}{2}f(2 - x)$ .

Complete a possible sequence of transformations to map  $f$  to  $g$ .

- Dilation of factor  $\frac{1}{2}$  from the  $x$ -axis.

**Marking Guide** — Answer: See marking guide

- M1: Reflection in the  $y$ -axis ( $x \rightarrow -x$ ).
- A1: Translation 2 units in the positive  $x$ -direction.

**Question 5b**

2 marks

Two functions  $g_1$  and  $g_2$  are created, both with the same rule as  $g$  but with distinct domains, such that  $g_1$  is strictly increasing and  $g_2$  is strictly decreasing.

Give the domain and range for the inverse of  $g_1$ .

**Marking Guide** — Answer: See marking guide

- M1:  $g$  has minimum at  $x = 2$  (since  $g(x) = \frac{1}{2}(e^{2-x} + e^{-(2-x)})$ ).  $g_1$  is increasing on  $[2, \infty)$ .
- A1: Domain of  $g_1^{-1}$ :  $[g(2), \infty) = [1, \infty)$ . Range of  $g_1^{-1}$ :  $[2, \infty)$ .

**Question 5c.i**

1 mark

The intersection points between the graphs of  $y = x$ ,  $y = g(x)$  and the inverses of  $g_1$  and  $g_2$ , are labelled  $P$  and  $Q$ .

Find the coordinates of  $P$  and  $Q$ , correct to two decimal places.

**Marking Guide** — Answer: See marking guide

- Solve  $g(x) = x$  using CAS. The two solutions give  $P$  and  $Q$ .

**Question 5c.ii**

2 marks

Find the area of the region bound by the graphs of  $g$ , the inverse of  $g_1$  and the inverse of  $g_2$ .

Give your answer correct to two decimal places.

**Marking Guide** — Answer: See marking guide

- M1: By symmetry about  $y = x$ , the area equals  $2 \int_P^Q |g(x) - x| dx$ .
- A1: Evaluate using CAS.

**Question 5d**

1 mark

Let  $h : R \rightarrow R$ ,  $h(x) = \frac{1}{k}f(k - x)$ , where  $k \in (0, \infty)$ .

The turning point of  $h$  always lies on the graph of the function  $y = 2x^n$ , where  $n$  is an integer. Find the value of  $n$ .

**Marking Guide** — Answer: See marking guide

- Turning point at  $x = k$ :  $h(k) = \frac{1}{k}f(0) = \frac{2}{k}$ . So  $(k, \frac{2}{k})$  lies on  $y = 2x^n$ .  $\frac{2}{k} = 2k^n \Rightarrow k^{-(1)} = k^n \Rightarrow n = -1$ .

**Question 5e**

1 mark

Let  $h_1 : [k, \infty) \rightarrow R$ ,  $h_1(x) = h(x)$ .

The rule for the **\*\*inverse\*\*** of  $h_1$  is  $y = \log_e \left( \frac{k}{2}x + \frac{1}{2}\sqrt{k^2x^2 - 4} \right) + k$ .

What is the smallest value of  $k$  such that  $h$  will intersect with the inverse of  $h_1$ ?

Give your answer correct to two decimal places.

**Marking Guide** — Answer: See marking guide

- $h$  intersects  $h_1^{-1}$  when  $h$  intersects  $y = x$  (since inverse reflects in  $y = x$ ). Solve using CAS for smallest  $k$ .

**Question 5f**

2 marks

It is possible for the graphs of  $h$  and the inverse of  $h_1$  to intersect twice. This occurs when  $k = 5$ .

Find the area of the region bound by the graphs of  $h$  and the inverse of  $h_1$ , when  $k = 5$ .

Give your answer correct to two decimal places.

**Marking Guide** — Answer: See marking guide

- M1: Find the two intersection points of  $h$  and  $h_1^{-1}$  when  $k = 5$  using CAS.
- A1: Area =  $2 \int_a^b |h(x) - x| dx$  by symmetry about  $y = x$ . Evaluate using CAS.