

Question 1a

[1 mark]

Let $y = e^x \cos(3x)$. Find $\frac{dy}{dx}$.

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Let $y = e^x \cos(3x)$. Find $\frac{dy}{dx}$.

Question 1b

[2 marks]

Let $f(x) = \log_e(x^3 - 3x + 2)$. Find $f'(3)$.

Question 2

[3 marks]

Consider the simultaneous linear equations: $3kx - 2y = k+4$ ($k-4)x + ky = -k$ Determine the value of k for which the system of equations has no real solution.

Question 3a

[3 marks]

Let $g : R \setminus \{-3\} \rightarrow R$, $g(x) = \frac{1}{(x+3)^2} - 2$. On the axes below, sketch the graph of $y = g(x)$ labelling all asymptotes with their equations and axis intercepts with their coordinates.

Question 3b

[2 marks]

Determine the area of the region bounded by the line $x = -2$, the x-axis, the y-axis and the graph of $y = g(x)$.

Question 4a

[1 mark]

Let X be a binomial random variable where $X \sim Bi(4, \frac{9}{10})$. Find the standard deviation of X .

Question 4b

[2 marks]

Find $\Pr(X < 2)$.

Question 5a

[1 mark]

The function $h : [0, \infty) \rightarrow R$, $h(t) = \frac{3000}{t+1}$ models the population of a town after t years. Use the model to predict the population after four years.

Question 5b

[2 marks]

A new function, h_1 , models a population where $h_1(0) = h(0)$ but h_1 decreases at half the rate of h at any point in time. State a sequence of two transformations that maps h to this new model h_1 .

Question 5c

[3 marks]

In the town, 100 people were randomly selected and surveyed, with 60 people indicating that they were unhappy with the roads. i. Determine an approximate 95% confidence interval for the proportion of people in the town who are unhappy with the roads. Use $z = 2$ for this confidence interval. (2 marks) ii. A new sample of n people results in the same sample proportion. Find the smallest value of n to achieve a standard deviation of $\frac{\sqrt{2}}{50}$ for the sample proportion. (1 mark)

Question 6

[4 marks]

Solve $2 \log_3(x - 4) + \log_3(x) = 2$ for x .

Question 7a

[3 marks]

Use the trapezium rule with a step size of $\frac{\pi}{3}$ to determine an approximation of the total area between the graph of $y = x \sin(x)$ and the x-axis over the interval $x \in [0, \pi]$.

Question 7b

[3 marks]

i. Find $f'(x)$ where $f(x) = x \sin(x)$. ii. Determine the range of $f'(x)$ over the interval $[\frac{\pi}{2}, \frac{2\pi}{3}]$. iii. Hence, verify that $f(x)$ has a stationary point for $x \in [\frac{\pi}{2}, \frac{2\pi}{3}]$.

Question 7c

[3 marks]

On the set of axes below, sketch the graph of $y = f'(x)$ on the domain $[-\pi, \pi]$, labelling the endpoints with their coordinates.

Question 8a

[1 mark]

Let $g : R \rightarrow R$, $g(x) = \sqrt[3]{x - k} + m$, where $k \in R \setminus \{0\}$ and $m \in R$. Let the point P be the y-intercept of the graph of $y = g(x)$. Find the coordinates of P , in terms of k and m .

Question 8b

[2 marks]

Find the gradient of g at P , in terms of k .

Question 8c

[1 mark]

Given that the graph of $y = g(x)$ passes through the origin, express k in terms of m .

Question 8d

[3 marks]

Let the point Q be a point different from the point P , such that the gradient of g at points P and Q are equal. Given that the graph of $y = g(x)$ passes through the origin, find the coordinates of Q in terms of m .

Solutions

Question 1a

$$e^x(\cos(3x) - 3\sin(3x))$$

Marking guide:

- Apply product rule: $u = e^x, v = \cos(3x)$
- Derivatives: $u' = e^x, v' = -3\sin(3x)$
- Combine: $e^x \cos(3x) - 3e^x \sin(3x)$

Question 1a

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Question 1b

$\frac{6}{5}$

Marking guide:

- Apply chain rule: $f'(x) = \frac{1}{x^3 - 3x + 2} \times (3x^2 - 3)$
- Substitute $x = 3$: $\frac{3(9)-3}{27-9+2} = \frac{24}{20}$
- Simplify to $6/5$

Question 2

$$k = \frac{4}{3}$$

Marking guide:

- Matrix determinant method or gradient comparison.
- Determinant $\Delta = (3k)(k) - (-2)(k - 4) = 3k^2 + 2k - 8$.
- Set $\Delta = 0$: $(3k - 4)(k + 2) = 0 \implies k = 4/3, k = -2$.
- Check $k = -2$: Eqs become $-6x - 2y = 2$ and $-6x - 2y = 2$. Coincident lines (infinite solutions).
- Check $k = 4/3$: Slopes are equal, but intercepts differ (parallel lines). No solution.

Question 3a

Sketch showing asymptotes $x=-3$, $y=-2$ and intercepts

Marking guide:

- Vertical asymptote $x = -3$. Horizontal asymptote $y = -2$.
- y-intercept: $x = 0, y = 1/9 - 2 = -17/9$. Point $(0, -17/9)$.
- x-intercepts: $\frac{1}{(x+3)^2} = 2 \implies (x+3)^2 = 1/2$.
- $x = -3 \pm \frac{1}{\sqrt{2}}$. Points $(-3 - \frac{\sqrt{2}}{2}, 0)$ and $(-3 + \frac{\sqrt{2}}{2}, 0)$.

Question 3b

$\frac{10}{3}$

Marking guide:

- Region is below x-axis from $x = -2$ to $x = 0$.
- Area = $\left| \int_{-2}^0 ((x+3)^{-2} - 2) dx \right|$ or $\int_{-2}^0 (2 - (x+3)^{-2}) dx$.
- Antiderivative of $(x+3)^{-2}$ is $-(x+3)^{-1}$.
- $\int_{-2}^0 g(x) dx = [-(x+3)^{-1} - 2x]_{-2}^0 = (-1/3 - 0) - (-1/1 - 2(-2)) = -1/3 - (-1 + 4) = -1/3 - 3 = -10/3$.
- Area is magnitude: $10/3$.

Question 4a

0.6

Marking guide:

- $\text{Var}(X) = np(1-p) = 4 \times 0.9 \times 0.1 = 0.36$.
- $\text{SD}(X) = \sqrt{0.36} = 0.6$.

Question 4b

0.0037

Marking guide:

- $\Pr(X < 2) = \Pr(X = 0) + \Pr(X = 1).$
- $\Pr(X = 0) = (0.1)^4 = 0.0001.$
- $\Pr(X = 1) = \binom{4}{1}(0.9)^1(0.1)^3 = 4(0.9)(0.001) = 0.0036.$
- Total = $0.0001 + 0.0036 = 0.0037.$

Question 5a

600

Marking guide:

- Substitute $t = 4$: $h(4) = \frac{3000}{4+1} = \frac{3000}{5} = 600.$

Question 5b

Dilation by factor 1/2 from t-axis (or x-axis), followed by Translation of 1500 units up.

Marking guide:

- Rate is derivative. $h'_1(t) = 0.5h'(t)$. Integrating gives $h_1(t) = 0.5h(t) + c.$
- Initial condition: $h_1(0) = 0.5h(0) + c \implies 3000 = 1500 + c \implies c = 1500.$
- Model is $h_1(t) = 0.5h(t) + 1500.$
- Transformation 1: Dilation factor 0.5 from t-axis.
- Transformation 2: Translation +1500 units in y-direction.

Question 5c

- i. $(0.5, 0.7)$ ii. $n = 300$

Marking guide:

- i. $\hat{p} = 60/100 = 0.6$. Formula: $\hat{p} \pm z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$
- Margin = $2\sqrt{\frac{0.6 \times 0.4}{100}} = 2\sqrt{0.0024}.$
- Wait, $\frac{0.24}{100} = 0.0024$. $\sqrt{0.0024} = \frac{\sqrt{24}}{100} = \frac{2\sqrt{6}}{100} \approx 0.049.$
- Using approx: $2 \times 0.05 = 0.1$. Interval $0.6 \pm 0.1 = (0.5, 0.7).$
- ii. $SD(\hat{p}) = \sqrt{\frac{0.24}{n}} = \frac{\sqrt{2}}{50}.$
- Square both sides: $\frac{0.24}{n} = \frac{2}{2500} = \frac{1}{1250}.$
- $n = 0.24 \times 1250 = 300.$

Question 6

$$x = \frac{7+\sqrt{13}}{2}$$

Marking guide:

- Combine logs: $\log_3((x-4)^2x) = 2.$
- Exponential form: $x(x^2 - 8x + 16) = 3^2 = 9.$
- Cubic: $x^3 - 8x^2 + 16x - 9 = 0.$
- Use Factor Theorem: $P(1) = 1 - 8 + 16 - 9 = 0$. So $(x-1)$ is a factor.
- Division: $(x-1)(x^2 - 7x + 9) = 0.$
- Roots: $x = 1$ or $x = \frac{7 \pm \sqrt{49-36}}{2} = \frac{7 \pm \sqrt{13}}{2}.$
- Domain constraint: $x-4 > 0 \implies x > 4.$
- Reject $x = 1$ and $x = \frac{7-\sqrt{13}}{2} \approx 1.7.$
- Only solution: $x = \frac{7+\sqrt{13}}{2}$ (approx 5.3).

Question 7a

$$\frac{\pi^2 \sqrt{3}}{6}$$

Marking guide:

- Step $h = \pi/3$. Points: $x_0 = 0, x_1 = \pi/3, x_2 = 2\pi/3, x_3 = \pi.$
- $y_0 = 0, y_3 = 0.$
- $y_1 = (\pi/3) \sin(\pi/3) = \frac{\pi}{3} \frac{\sqrt{3}}{2} = \frac{\pi\sqrt{3}}{6}.$
- $y_2 = (2\pi/3) \sin(2\pi/3) = \frac{2\pi}{3} \frac{\sqrt{3}}{2} = \frac{\pi\sqrt{3}}{3}.$

- Area $\approx \frac{h}{2}(y_0 + 2(y_1 + y_2) + y_3) = \frac{\pi}{6}(0 + 2(\frac{\pi\sqrt{3}}{6} + \frac{2\pi\sqrt{3}}{6}) + 0).$
- $= \frac{\pi}{6}(2 \times \frac{3\pi\sqrt{3}}{6}) = \frac{\pi}{6}(\pi\sqrt{3}) = \frac{\pi^2\sqrt{3}}{6}.$

Question 7b

- i. $\sin(x) + x \cos(x)$ ii. $[\frac{\sqrt{3}}{2} - \frac{\pi}{3}, 1]$

Marking guide:

- i. Product rule: $f'(x) = 1 \cdot \sin(x) + x \cos(x).$
- ii. f' is continuous. $f'(\pi/2) = 1 + 0 = 1.$
- $f'(2\pi/3) = \frac{\sqrt{3}}{2} + \frac{2\pi}{3}(-\frac{1}{2}) = \frac{\sqrt{3}}{2} - \frac{\pi}{3}.$
- Check monotonicity: $f''(x) = 2 \cos(x) - x \sin(x).$ In Q2 both terms negative, so f' is decreasing.
- Range is $[f'(2\pi/3), f'(\pi/2)].$
- iii. $f'(\pi/2) = 1 > 0$ and $f'(2\pi/3) \approx 0.86 - 1.05 < 0.$
- Since signs change and function is continuous (IVT), there is a root (stationary point) in the interval.

Question 7cGraph of sine-like wave with endpoints $(-\pi, \pi)$ and $(\pi, -\pi)$ *Marking guide:*

- Endpoints: $f'(-\pi) = 0 + (-\pi)(-1) = \pi.$ Point $(-\pi, \pi).$
- $f'(\pi) = 0 + \pi(-1) = -\pi.$ Point $(\pi, -\pi).$
- Passes through origin $(0, 0).$
- Odd function symmetry.

Question 8a $(0, m - k^{1/3})$ *Marking guide:*

- Set $x = 0.$ $y = \sqrt[3]{-k} + m = -k^{1/3} + m.$

Question 8b $\frac{1}{3k^{2/3}}$ *Marking guide:*

- $g(x) = (x - k)^{1/3} + m.$
- $g'(x) = \frac{1}{3}(x - k)^{-2/3}.$
- $g'(0) = \frac{1}{3}(-k)^{-2/3} = \frac{1}{3}((-1)^2 k^2)^{-1/3} = \frac{1}{3k^{2/3}}.$

Question 8c $k = m^3$ *Marking guide:*

- Passes through $(0, 0) \implies P$ is origin.
- y-coord of P is 0: $m - k^{1/3} = 0 \implies m = k^{1/3}.$
- Cube both sides: $k = m^3.$

Question 8d $(2m^3, 2m)$ *Marking guide:*

- Set $g'(x) = g'(0).$ $\frac{1}{3}(x - k)^{-2/3} = \frac{1}{3}(-k)^{-2/3}.$
- $(x - k)^2 = (-k)^2 = k^2.$
- $x - k = k$ or $x - k = -k.$
- $x = 2k$ or $x = 0.$ Since $Q \neq P$ (where $x = 0), x_Q = 2k.$
- Substitute $k = m^3:$ $x_Q = 2m^3.$
- Find y: $g(2k) = \sqrt[3]{2k - k} + m = k^{1/3} + m.$
- Since $k^{1/3} = m,$ $y_Q = m + m = 2m.$