ATAR Master

VCE Mathematical Methods

2024 Examination 2 (Technology-Active)

Questions & Marking Guide

Total: 80 marks

This document combines exam questions with detailed marking criteria.
Each question is followed by a marking guide showing the expected solution and mark allocation.

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Section A: Multiple Choice — 20 marks

Each question is worth 1 mark.

Question 1 *1 mark*

The asymptote(s) of the graph of $y = \log_e(x+1) - 3$ are

A. $x = -1$ only

B. $x = 1$ only

C. $y = -3$ only

D. $x = -1$ and $y = -3$

Marking Guide — Answer: A

- The graph of $y = \log_e(x+1) - 3$ has a vertical asymptote at $x = -1$. There is no horizontal asymptote for a logarithmic function.

Question 2 *1 mark*

A function $g : R \to R$ has the derivative $g'(x) = x^3 - x$.
Given that $g(0) = 5$, the value of $g(2)$ is

A. 2

B. 3

C. 5

D. 7

Marking Guide — Answer: D

Section B: Extended Response — 60 marks

Question 3

1 mark

A discrete random variable X is defined using the probability distribution below, where k is a positive real number.

| x | 0 | 1 | 2 | 3 | 4 | |—|—|—|—|—|—| | $\Pr(X = x)$ | $2k$ | $3k$ | $5k$ | $3k$ | $2k$ |

Find $\Pr(X < 4 \mid X > 1)$.

Marking Guide — Answer: C

- $15k = 1 \Rightarrow k = \frac{1}{15}$. $\Pr(X < 4 \mid X > 1) = \frac{\Pr(2 \leq X \leq 3)}{\Pr(X \geq 2)} = \frac{5k+3k}{5k+3k+2k} = \frac{8}{10} = \frac{4}{5}$.

Question 4

1 mark

If $\int_a^b f(x)\,dx = -5$ and $\int_a^c f(x)\,dx = 3$, where $a < b < c$, then $\int_b^c 2f(x)\,dx$ is equal to

A. -16

B. 16

C. -4

D. 4

Marking Guide — Answer: B

- $\int_b^c 2f(x)\,dx = 2\left(\int_a^c f(x)\,dx - \int_a^b f(x)\,dx\right) = 2(3 - (-5)) = 16$.

Question 5

1 mark

Consider the functions $f : (1, \infty) \to R$, $f(x) = x^2 - 4x$ and $g : R \to R$, $g(x) = e^{-x}$.

The range of the composite function $g(f(x))$ is

A. $(0, e^3)$

Marking Guide — Answer: D

- On $(1, \infty)$: $f(x) = x^2 - 4x$. $f'(x) = 2x - 4 = 0$ at $x = 2$. $f(2) = -4$ (minimum). As $x \to 1^+$, $f \to -3$. As $x \to \infty$, $f \to \infty$. So f range is $[-4, \infty)$.

Question 6

1 mark

Consider the function $f(x) = \frac{2x+1}{3-x}$ with domain $x \in R \setminus \{3\}$.

The inverse of f is

Marking Guide — Answer: A

- Let $y = \frac{2x+1}{3-x}$. $y(3 - x) = 2x + 1$. $3y - xy = 2x + 1$. $3y - 1 = x(2 + y)$. $x = \frac{3y-1}{y+2}$. So $f^{-1}(x) = \frac{3x-1}{x+2}$ with domain $R \setminus \{-2\}$.

Question 7 *1 mark*

A fair six-sided die is repeatedly rolled. What is the minimum number of rolls required so that the probability of rolling a six at least once is greater than 0.95?

A. 15

B. 16

C. 17

D. 18

Marking Guide — Answer: C

- $1 - \left(\frac{5}{6}\right)^n > 0.95 \Rightarrow \left(\frac{5}{6}\right)^n < 0.05$.
- $n > \frac{\ln 0.05}{\ln(5/6)} \approx 16.43$. Minimum $n = 17$.

Question 8 *1 mark*

Some values of the functions $f : R \to R$ and $g : R \to R$ are shown below.

x	1	2	3
$f(x)$	0	4	5
$g(x)$	3	4	-5

The graph of the function $h(x) = f(x) - g(x)$ must have an x-intercept at

A. $(2, 0)$

B. $(3, 0)$

C. $(4, 0)$

D. $(5, 0)$

Marking Guide — Answer: A

- $h(1) = 0 - 3 = -3$, $h(2) = 4 - 4 = 0$, $h(3) = 5 - (-5) = 10$. So $h(2) = 0$, giving an x-intercept at $(2, 0)$.

Question 9 *1 mark*

At a Year 12 formal, 45% of the students travelled to the event in a hired limousine, while the remaining 55% were driven to the event by a parent.

Of the students who travelled in a hired limousine, 30% had a professional photo taken.

Of the students who were driven by a parent, 60% had a professional photo taken.

Given that a student had a professional photo taken, what is the probability that the student travelled to the event in a hired limousine?

Marking Guide — Answer: C

- $\Pr(\text{Photo}) = 0.45 \times 0.30 + 0.55 \times 0.60 = 0.135 + 0.330 = 0.465$.
- $\Pr(\text{Limo} \mid \text{Photo}) = \frac{0.135}{0.465} = \frac{135}{465} = \frac{9}{31}$.

Question 10 *1 mark*

Suppose a function $f : [0, 5] \to R$ and its derivative $f' : [0, 5] \to R$ are defined and continuous on their domains. If $f'(2) < 0$ and $f'(4) > 0$, which one of these statements must be true?

Marking Guide — Answer: B

- Since $f'(2) < 0$ and $f'(4) > 0$, by the Intermediate Value Theorem, $f'(c) = 0$ for some $c \in (2, 4)$. This means f has a turning point, so f is not monotone and hence not one-to-one. Therefore f does not have an inverse function.

Question 11 *1 mark*

Twelve students sit in a classroom, with seven students in the first row and the other five students in the second row. Three students are chosen randomly from the class.

The probability that exactly two of the three students chosen are in the first row is

Marking Guide — Answer: B

- $\Pr = \frac{\binom{7}{2}\binom{5}{1}}{\binom{12}{3}} = \frac{21\times5}{220} = \frac{105}{220} = \frac{21}{44}$.

Question 12 *1 mark*

The graph of $y = f(x)$ is shown below (a curve with features around $x = 0$ to 4, rising from near $x = 0$, with a local max near $x = 3$ at $y \approx 3$, and approaching $y \approx 2$ as x increases).

Which of the following options best represents the graph of $y = f(2x + 1)$?

A. Graph A

B. Graph B

C. Graph C

D. Graph D

Marking Guide — Answer: B

- $y = f(2x + 1)$: replace x with $2x + 1$. This is a horizontal compression by factor $\frac{1}{2}$ and a translation of $\frac{1}{2}$ unit to the left. The graph features compress horizontally and shift left.

Question 13 *1 mark*

The function $f : (0, \infty) \to R$, $f(x) = \frac{x}{2} + \frac{2}{x}$ is mapped to the function g with the following sequence of transformations:

1. dilation by a factor of 3 from the y-axis 2. translation by 1 unit in the negative direction of the y-axis.

The function g has a local minimum at the point with the coordinates

A. $(6, 1)$

B. $(2, 5)$

Marking Guide — Answer: A

- Dilation by factor 3 from the y-axis maps $(x, y) \to (3x, y)$, so replace x with $x/3$: $g_1(x) = f(x/3) = \frac{x}{6} + \frac{6}{x}$.
- Translate down 1: $g(x) = \frac{x}{6} + \frac{6}{x} - 1$.

- Original f has min at $x = 2$: $f(2) = 2$. After dilation: min at $(6, 2)$. After translation down 1: $(6, 1)$.

Question 14 *1 mark*

Let h be the probability density function for a continuous random variable X, where

$$h(x) = \begin{cases} \frac{x}{6} + k\& - 3 \le x < 0 \\ -\frac{x}{2} + k\&0 \le x \le 1 \\ 0\&\text{elsewhere} \end{cases}$$

and k is a positive real number.

The value of $\Pr(X < 0.5)$ is

Marking Guide — Answer: B

- $\int_{-3}^{0} \left(\frac{x}{6} + k\right) dx + \int_0^1 \left(-\frac{x}{2} + k\right) dx = 1$.

Question 15 *1 mark*

The points of inflection of the graph of $y = 2 - \tan\left(\pi\left(x - \frac{1}{4}\right)\right)$ are

Marking Guide — Answer: A

- Points of inflection of tan occur where $\tan = 0$, i.e., at integer multiples of π for the argument.
- $\pi\left(x - \frac{1}{4}\right) = k\pi \Rightarrow x = k + \frac{1}{4}$, $k \in \mathbb{Z}$.
- At these points: $y = 2 - \tan(k\pi) = 2 - 0 = 2$.
- Inflection points: $\left(k + \frac{1}{4}, 2\right)$, $k \in \mathbb{Z}$.

Question 16 *1 mark*

Suppose that a differentiable function $f : R \to R$ and its derivative $f' : R \to R$ satisfy $f(4) = 25$ and $f'(4) = 15$.

Determine the gradient of the tangent line to the graph of $y = \sqrt{f(x)}$ at $x = 4$.

Marking Guide — Answer: D

- $\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}$.
- At $x = 4$: $\frac{15}{2\sqrt{25}} = \frac{15}{10} = \frac{3}{2}$.

Question 17 *1 mark*

Consider the algorithm below, which prints the roots of the cubic polynomial $f(x) = x^3 - 2x^2 - 9x + 18$.

```
define f(x)
    return (x^3 - 2x^2 - 9x + 18)
c ← f(0)
if c < 0 then
    c ← (-c)
end if
while c > 0
```

```
    if f(c) = 0 then
        print c
    end if
    if f(-c) = 0 then
        print -c
    end if
    c ← c - 1
end while
```

In order, the algorithm prints the values

A. $-3, 3, 2$

B. $-3, 2, 3$

C. $3, 2, -3$

D. $3, -3, 2$

Marking Guide — Answer: D

- $f(x) = x^3 - 2x^2 - 9x + 18 = (x-2)(x-3)(x+3)$. Roots: $x = 2, 3, -3$.
- $c = f(0) = 18$. Loop from $c = 18$ down to 1.
- At $c = 3$: $f(3) = 0$, print 3; $f(-3) = 0$, print -3.
- At $c = 2$: $f(2) = 0$, print 2; $f(-2)\ e0$.
- Output: $3, -3, 2$.

Question 18 *1 mark*

Find the value of x which maximises the area of the trapezium below.

(A trapezium with top width x, two slant sides of length 10, and base width $3x$.)

A. 10

B. 7

Marking Guide — Answer: B

- Base $= 3x$, top $= x$. The height h satisfies $h^2 + x^2 = 100$, so $h = \sqrt{100 - x^2}$.
- Area $= \frac{1}{2}(x + 3x) \cdot h = 2x\sqrt{100 - x^2}$.
- $A'(x) = 2\sqrt{100 - x^2} + 2x \cdot \frac{-2x}{2\sqrt{100-x^2}} = \frac{200-4x^2}{\sqrt{100-x^2}} = 0$.
- $x^2 = 50 \Rightarrow x = 5\sqrt{2}$.

Question 19 *1 mark*

Consider the normal random variable X that satisfies $\Pr(X < 10) = 0.2$ and $\Pr(X > 18) = 0.2$.

The value of $\Pr(X < 12)$ is closest to

A. 0.134

B. 0.297

C. 0.337

D. 0.365

Marking Guide — Answer: C

- By symmetry: $\mu = \frac{10+18}{2} = 14$.
- $\Pr(X < 10) = 0.2 \Rightarrow z = \text{invNorm}(0.2) \approx -0.8416$.
- $\frac{10-14}{\sigma} = -0.8416 \Rightarrow \sigma \approx 4.753$.
- $\Pr(X < 12) = \Pr\left(Z < \frac{12-14}{4.753}\right) = \Pr(Z < -0.421) \approx 0.337$.

Question 20 *1 mark*

The function $f : R \to R$ has an average value k on the interval $[0, 2]$ and satisfies $f(x) = f(x + 2)$ for all $x \in R$. The value of the definite integral $\int_2^6 f(x)\, dx$ is

A. $2k$

B. $3k$

C. $4k$

D. $6k$

Marking Guide — Answer: C

Question 1a *1 mark*

Consider the function $f : R \to R$, $f(x) = (x + 1)(x + a)(x - 2)(x - 2a)$ where $a \in R$.

State, in terms of a where required, the values of x for which $f(x) = 0$.

Marking Guide — Answer: $x = -1, -a, 2, 2a$

- $f(x) = 0$ when $x + 1 = 0$, $x + a = 0$, $x - 2 = 0$ or $x - 2a = 0$.
- Solutions: $x = -1, -a, 2, 2a$.

Question 1b.i *2 marks*

Find the values of a for which the graph of $y = f(x)$ has exactly three x-intercepts.

Marking Guide — Answer: $a \in \left\{-2, -\frac{1}{2}, 0\right\}$

- M1: Exactly three x-intercepts requires exactly one pair of equal roots from $\{-1, -a, 2, 2a\}$.
- Possible equalities: $-1 = -a \Rightarrow a = 1$ (gives roots $-1, -1, 2, 2$, only 2 distinct — not 3); $-1 = 2a \Rightarrow a = -\frac{1}{2}$ (roots $-1, \frac{1}{2}, 2, -1$ ✓); $-a = 2 \Rightarrow a = -2$ (roots $-1, 2, 2, -4$ ✓); $-a = 2a \Rightarrow a = 0$ (roots $-1, 0, 2, 0$ ✓); $2 = 2a \Rightarrow a = 1$ (same as first case).
- A1: $a \in \{-2, -\frac{1}{2}, 0\}$.

Question 1b.ii *1 mark*

Find the values of a for which the graph of $y = f(x)$ has exactly four x-intercepts.

Marking Guide — Answer: $a \in R \setminus \left\{-2, -\frac{1}{2}, 0, 1\right\}$

- All four roots must be distinct. Exclude values where any two roots coincide: $a\ e1, -\frac{1}{2}, -2, 0$.
- Answer: $a \in R \setminus \{-2, -\frac{1}{2}, 0, 1\}$.

Question 1c.i *1 mark*

Let g be the function $g : R \to R$, $g(x) = (x+1)^2(x-2)^2$, which is the function f where $a = 1$. Find $g'(x)$.

Marking Guide — Answer: $g'(x) = 2(x+1)(x-2)(2x-1)$

Question 1c.ii *1 mark*

Find the coordinates of the local maximum of g.

Marking Guide — Answer: $\left(\frac{1}{2}, \frac{81}{16}\right)$

- $g'(x) = 0$ at $x = -1, \frac{1}{2}, 2$. Test $x = \frac{1}{2}$: $g\left(\frac{1}{2}\right) = \left(\frac{3}{2}\right)^2\left(-\frac{3}{2}\right)^2 = \frac{81}{16}$.
- Since $g(-1) = 0$ and $g(2) = 0$ are minima, $\left(\frac{1}{2}, \frac{81}{16}\right)$ is the local maximum.

Question 1c.iii *1 mark*

Find the values of x for which $g'(x) > 0$.

Marking Guide — Answer: $x \in (-1, \frac{1}{2}) \cup (2, \infty)$

- $g'(x) = 2(x+1)(x-2)(2x-1) > 0$.
- Sign analysis with roots at $x = -1, \frac{1}{2}, 2$:
- $g'(x) > 0$ for $x \in (-1, \frac{1}{2}) \cup (2, \infty)$.

Question 1c.iv *2 marks*

Consider the two tangent lines to the graph of $y = g(x)$ at the points where $x = \frac{-\sqrt{3}+1}{2}$ and $x = \frac{\sqrt{3}+1}{2}$. Determine the coordinates of the point of intersection of these two tangent lines.

Marking Guide — Answer: $\left(\frac{1}{2}, \frac{27}{4}\right)$

- M1: Let $u = \frac{1-\sqrt{3}}{2}$ and $v = \frac{1+\sqrt{3}}{2}$. Note $u + v = 1$. By symmetry about $x = \frac{1}{2}$, both tangent lines are symmetric reflections.
- $g(u) = (u+1)^2(u-2)^2$. $(u+1)(u-2) = \frac{(3-\sqrt{3})}{2} \cdot \frac{(-3-\sqrt{3})}{2} = \frac{-(9-3)}{4} = -\frac{3}{2}$. So $g(u) = \frac{9}{4}$. Similarly $g(v) = \frac{9}{4}$.
- $g'(u) = 2(u+1)(u-2)(2u-1) = 2 \cdot (-\frac{3}{2}) \cdot (-\sqrt{3}) = 3\sqrt{3}$. $g'(v) = -3\sqrt{3}$.
- A1: At intersection: $3\sqrt{3}(x-u) + \frac{9}{4} = -3\sqrt{3}(x-v) + \frac{9}{4}$. So $x - u = -(x - v)$, giving $x = \frac{u+v}{2} = \frac{1}{2}$.

- $y = \frac{9}{4} + 3\sqrt{3}\left(\frac{1}{2} - \frac{1-\sqrt{3}}{2}\right) = \frac{9}{4} + 3\sqrt{3} \cdot \frac{\sqrt{3}}{2} = \frac{9}{4} + \frac{9}{2} = \frac{27}{4}$.

Question 1d.i *1 mark*

Let g remain as the function $g : R \to R$, $g(x) = (x+1)^2(x-2)^2$, which is the function f where $a = 1$.

Let h be the function $h : R \to R$, $h(x) = (x+1)(x-1)(x+2)(x-2)$, which is the function f where $a = -1$.

Using translations only, describe a sequence of transformations of h, for which its image would have a local maximum at the same coordinates as that of g.

Marking Guide — Answer: Translate $\frac{1}{2}$ unit in the positive x-direction and $\frac{81}{16} - 4 = \frac{17}{16}$ units in the positive y-direction.

- $h(x) = (x+1)(x-1)(x+2)(x-2) = (x^2-1)(x^2-4) = x^4 - 5x^2 + 4$. Local max at $(0, 4)$.
- g has local max at $(\frac{1}{2}, \frac{81}{16})$.
- Translate right $\frac{1}{2}$ and up $\frac{81}{16} - 4 = \frac{17}{16}$.

Question 1d.ii *2 marks*

Using a dilation and translations, describe a different sequence of transformations of h, for which its image would have both local minimums at the same coordinates as that of g.

Marking Guide — Answer: See marking guide

- M1: h has local mins at $\left(\pm\sqrt{\frac{5}{2}}, -\frac{9}{4}\right)$. g has local mins at $(-1, 0)$ and $(2, 0)$.
- The midpoint of g's mins is $\frac{1}{2}$ and half-spread is $\frac{3}{2}$. The midpoint of h's mins is 0 and half-spread is $\sqrt{\frac{5}{2}}$.
- Dilation by factor $\frac{3}{2\sqrt{5/2}} = \frac{3}{\sqrt{10}}$ from the y-axis, then translate $\frac{1}{2}$ right and $\frac{9}{4}$ up.
- A1: Alternatively: dilate horizontally by factor $\frac{3\sqrt{2}}{\sqrt{10}} = \frac{3}{\sqrt{5}}$ from the y-axis, translate $\frac{1}{2}$ in positive x-direction and $\frac{9}{4}$ in positive y-direction.

Question 2a *2 marks*

A model for the temperature in a room, in degrees Celsius, is given by

$$f(t) = \begin{cases} 12 + 30t\&0 \le t \le \frac{1}{3} \\ 22\&t > \frac{1}{3} \end{cases}$$

where t represents time in hours after a heater is switched on.

Express the derivative $f'(t)$ as a hybrid function.

Marking Guide — Answer: $f'(t) = \begin{cases} 30 & 0 < t < \frac{1}{3} \\ 0 & t > \frac{1}{3} \end{cases}$

- M1: $f'(t) = 30$ for $0 < t < \frac{1}{3}$.
- A1: $f'(t) = 0$ for $t > \frac{1}{3}$. (Note: $f'(t)$ is undefined at $t = \frac{1}{3}$.)

Question 2b *1 mark*

Find the average rate of change in temperature predicted by the model between $t = 0$ and $t = \frac{1}{2}$.

Give your answer in degrees Celsius per hour.

Marking Guide — Answer: 20 degrees Celsius per hour

- Average rate $= \frac{f(1/2)-f(0)}{1/2-0} = \frac{22-12}{1/2} = 20$ °C/hr.

Question 2c.i *1 mark*

Another model for the temperature in the room is given by $g(t) = 22 - 10e^{-6t}$, $t \geq 0$.

Find the derivative $g'(t)$.

Marking Guide — Answer: $g'(t) = 60e^{-6t}$

- $g'(t) = -10 \times (-6)e^{-6t} = 60e^{-6t}$.

Question 2c.ii *1 mark*

Find the value of t for which $g'(t) = 10$.

Give your answer correct to three decimal places.

Marking Guide — Answer: $t \approx 0.299$

- $60e^{-6t} = 10 \Rightarrow e^{-6t} = \frac{1}{6} \Rightarrow -6t = \ln\frac{1}{6} \Rightarrow t = \frac{\ln 6}{6} \approx 0.299$.

Question 2d *1 mark*

Find the time $t \in (0, 1)$ when the temperatures predicted by the models f and g are equal.

Give your answer correct to two decimal places.

Marking Guide — Answer: $t \approx 0.24$

- For $0 \leq t \leq \frac{1}{3}$: $12 + 30t = 22 - 10e^{-6t}$, i.e., $30t + 10e^{-6t} = 10$.
- Solve numerically using CAS: $t \approx 0.24$.

Question 2e *1 mark*

Find the time $t \in (0, 1)$ when the difference between the temperatures predicted by the two models is the greatest.

Give your answer correct to two decimal places.

Marking Guide — Answer: $t \approx 0.18$

- For $0 < t < \frac{1}{3}$: $d(t) = f(t) - g(t) = (12 + 30t) - (22 - 10e^{-6t}) = 30t + 10e^{-6t} - 10$.
- $d'(t) = 30 - 60e^{-6t} = 0 \Rightarrow e^{-6t} = \frac{1}{2} \Rightarrow t = \frac{\ln 2}{6} \approx 0.12$.
- Also check $t > \frac{1}{3}$: $d(t) = 22 - (22 - 10e^{-6t}) = 10e^{-6t}$, which is decreasing.
- Maximum difference at $t \approx 0.12$. (Or compare absolute differences across both pieces using CAS.)

Question 2f.i *1 mark*

The amount of power, in kilowatts, used by the heater t hours after it is switched on, can be modelled by the continuous function p, whose graph is shown.

$$p(t) = \begin{cases} 1.5\&0 \leq t \leq 0.4 \\ 0.3 + Ae^{-10t}\&t > 0.4 \end{cases}$$

The amount of energy used by the heater, in kilowatt hours, can be estimated by evaluating the area between the graph of $y = p(t)$ and the t-axis.

Given that $p(t)$ is continuous for $t \geq 0$, show that $A = 1.2e^4$.

Marking Guide — Answer: See marking guide

- For continuity at $t = 0.4$: $\lim_{t \to 0.4^+} p(t) = p(0.4) = 1.5$.
- $0.3 + Ae^{-10(0.4)} = 1.5 \Rightarrow Ae^{-4} = 1.2 \Rightarrow A = 1.2e^4$.

Question 2f.ii *1 mark*

Find how long it takes, after the heater is switched on, until the heater has used 0.5 kilowatt hours of energy.

Give your answer in hours.

Marking Guide — Answer: $t = \frac{1}{3}$ hours

- For $T \leq 0.4$: $\int_0^T 1.5\,dt = 1.5T = 0.5 \Rightarrow T = \frac{1}{3} \approx 0.333$ hours.
- Since $\frac{1}{3} < 0.4$, this is valid.

Question 2f.iii *2 marks*

Find how long it takes, after the heater is switched on, until the heater has used 1 kilowatt hour of energy.

Give your answer in hours, correct to two decimal places.

Marking Guide — Answer: $t \approx 0.87$ hours

- M1: Energy up to $t = 0.4$: $\int_0^{0.4} 1.5\,dt = 0.6$ kWh. Remaining energy: $1 - 0.6 = 0.4$ kWh.
- For $T > 0.4$: $\int_{0.4}^T (0.3 + 1.2e^4 \cdot e^{-10t})\,dt = 0.4$.

Question 3a.i *3 marks*

The points shown on the chart represent monthly online sales in Australia. The variable y represents sales in millions of dollars. The variable t represents the month when the sales were made, where $t = 1$ corresponds to January 2021, $t = 2$ corresponds to February 2021 and so on.

A cubic polynomial $p : (0, 12] \to R$, $p(t) = at^3 + bt^2 + ct + d$ can be used to model monthly online sales in 2021.

The graph of $y = p(t)$ is shown as a dashed curve. It has a local minimum at $(2, 2500)$ and a local maximum at $(11, 4400)$.

Find, correct to two decimal places, the values of a, b, c and d.

Marking Guide — Answer: $a \approx -14.07$, $b \approx 274.36$, $c \approx -1625.93$, $d \approx 5596.30$

- M1: $p'(t) = 3at^2 + 2bt + c$. Since local min at $t = 2$ and local max at $t = 11$: $p'(2) = 0$ and

$p'(11) = 0$.

- M1: Also $p(2) = 2500$ and $p(11) = 4400$. This gives four equations in four unknowns.
- A1: Solve the system using CAS to find a, b, c, d correct to two decimal places.

Question 3a.ii *2 marks*

Let $q : (12, 24] \to R$, $q(t) = p(t - h) + k$ be a cubic function obtained by translating p, which can be used to model monthly online sales in 2022.

Find the values of h and k such that the graph of $y = q(t)$ has a local maximum at $(23, 4750)$.

Marking Guide — Answer: $h = 12$, $k = 350$

- M1: p has local max at $t = 11$. For $q(t) = p(t - h) + k$ to have local max at $t = 23$: $23 - h = 11 \Rightarrow h = 12$.
- A1: $q(23) = p(11) + k = 4400 + k = 4750 \Rightarrow k = 350$.

Question 3b.i *2 marks*

Another function f can be used to model monthly online sales, where

$$f : (0, 36] \to R, \quad f(t) = 3000 + 30t + 700\cos\left(\frac{\pi t}{6}\right) + 400\cos\left(\frac{\pi t}{3}\right)$$

Part of the graph of f is shown on the axes.

Complete the graph of f on the set of axes above until December 2023, that is, for $t \in (24, 36]$. Label the endpoint at $t = 36$ with its coordinates.

Marking Guide — Answer: Graph completed; endpoint at $(36, 4180)$

Question 3b.ii *1 mark*

The function f predicts that every 12 months, monthly online sales increase by n million dollars.

Find the value of n.

Marking Guide — Answer: $n = 360$

Question 3b.iii *1 mark*

Find the derivative $f'(t)$.

Marking Guide — Answer: $f'(t) = 30 - \frac{700\pi}{6}\sin\left(\frac{\pi t}{6}\right) - \frac{400\pi}{3}\sin\left(\frac{\pi t}{3}\right)$

- $f'(t) = 30 - \frac{700\pi}{6}\sin\left(\frac{\pi t}{6}\right) - \frac{400\pi}{3}\sin\left(\frac{\pi t}{3}\right)$.

Question 3b.iv *2 marks*

Hence, find the maximum instantaneous rate of change for the function f, correct to the nearest million dollars per month, and the values of t in the interval $(0, 36]$ when this maximum rate occurs, correct to one decimal place.

Marking Guide — Answer: Maximum rate ≈ 787 million dollars per month, at $t \approx 4.5, 16.5, 28.5$

Question 4a *1 mark*

At an airport, luggage is weighed before it is checked in. The mass of each piece of luggage, in kilograms, is modelled by a continuous random variable X, whose probability density function is

$$f(x) = \begin{cases} \frac{1}{67500}x^2(30-x)\&0 \le x \le 30 \\ 0\&\text{elsewhere} \end{cases}$$

A piece of luggage is labelled as heavy if its mass exceeds 23 kg.
Write a definite integral which gives the probability that a piece of luggage is labelled as heavy.

Marking Guide — Answer: $\int_{23}^{30} \frac{1}{67500}x^2(30-x)\,dx$

- $\Pr(X > 23) = \int_{23}^{30} \frac{1}{67500}x^2(30-x)\,dx$.

Question 4b.i *1 mark*

Find the mean of X.

Marking Guide — Answer: $E(X) = 18$

- $E(X) = \int_0^{30} \frac{x}{67500} \cdot x^2(30-x)\,dx = \frac{1}{67500}\int_0^{30}(30x^3 - x^4)\,dx$.

Question 4b.ii *2 marks*

Find the standard deviation of X.

Marking Guide — Answer: $\text{SD}(X) = 6$

Question 4b.iii *2 marks*

Given that the mass of a piece of luggage is more than the mean, find the probability that it is labelled as heavy, correct to three decimal places.

Marking Guide — Answer: ≈ 0.372

- M1: $\Pr(X > 23 \mid X > 18) = \frac{\Pr(X>23)}{\Pr(X>18)}$.
- Using CAS: $\Pr(X > 23) \approx 0.234$. $\Pr(X > 18) \approx 0.630$. (Actually $\Pr(X > 18) = 0.5$ by symmetry considerations... no, this is not symmetric.)
- A1: $\Pr(X > 23 \mid X > 18) = \frac{\Pr(X>23)}{\Pr(X>18)} \approx 0.372$.

Question 4c.i *1 mark*

Use the following information to answer parts c and d.

Of the travellers flying from the airport: • 10% do not check in any luggage • 40% check in exactly one piece of luggage • 50% check in exactly two pieces of luggage.

Assume that the mass of each piece of luggage is independent of the number of pieces checked in by each traveller.

Use the value of 0.234 for the probability that a piece of luggage is labelled as heavy.

Let W be the discrete random variable that represents the number of pieces of luggage labelled as **heavy** checked in by each traveller.

Show that $\Pr(W = 2) = 0.027$, correct to three decimal places.

Marking Guide — Answer: See marking guide

- Only travellers checking 2 pieces can have $W = 2$.
- $\Pr(W = 2) = 0.50 \times 0.234^2 = 0.50 \times 0.054756 = 0.027378 \approx 0.027$.

Question 4c.ii *2 marks*

Complete the table below for the probability distribution W, correct to three decimal places.

| w | 0 | 1 | 2 | |—|—|—|—| | $\Pr(W = w)$ | | | 0.027 |

Marking Guide — Answer: $\Pr(W = 0) \approx 0.700$, $\Pr(W = 1) \approx 0.273$

- M1: $\Pr(W = 0) = 0.10(1) + 0.40(1 - 0.234) + 0.50(1 - 0.234)^2 = 0.10 + 0.3064 + 0.50(0.586756) = 0.10 + 0.3064 + 0.2934 = 0.6998 \approx 0.700$.
- A1: $\Pr(W = 1) = 1 - 0.700 - 0.027 = 0.273$. Or: $\Pr(W = 1) = 0.40(0.234) + 0.50 \times 2(0.234)(0.766) = 0.0936 + 0.1793 = 0.2729 \approx 0.273$.

Question 4d.i *2 marks*

On a particular day, a random sample of 35 pieces of luggage was selected at the airport.

Let $\hat{P}$ be the random variable that represents the proportion of luggage labelled as heavy in random samples of 35.

Find $\Pr(\hat{P} > 0.2)$, correct to three decimal places.

Marking Guide — Answer: ≈ 0.697

- M1: $\hat{P} = X/35$ where $X \sim \text{Bin}(35, 0.234)$. $\Pr(\hat{P} > 0.2) = \Pr(X > 7) = \Pr(X \geq 8)$.
- A1: Using CAS: $\Pr(X \geq 8) = 1 - \Pr(X \leq 7) \approx 0.697$.

Question 4d.ii *2 marks*

Determine the probability that $\hat{P}$ lies within one standard deviation of its mean, correct to three decimal places. Do **not** use a normal approximation.

Marking Guide — Answer: ≈ 0.736

- M1: $E(\hat{P}) = p = 0.234$, $\text{SD}(\hat{P}) = \sqrt{\frac{0.234 \times 0.766}{35}} \approx 0.07162$.
- Need $\Pr(0.234 - 0.07162 < \hat{P} < 0.234 + 0.07162) = \Pr(0.1624 < \hat{P} < 0.3056)$.
- $\hat{P} = X/35$: $\Pr(5.68 < X < 10.70) = \Pr(6 \leq X \leq 10)$.
- A1: Using CAS with $X \sim \text{Bin}(35, 0.234)$: $\Pr(6 \leq X \leq 10) \approx 0.736$.

Question 4e.i *1 mark*

In one random sample of 50 pieces of luggage, 10 are labelled as heavy.

Use this sample to find an approximate 90% confidence interval for p, the population proportion of luggage labelled as heavy, correct to three decimal places.

Marking Guide — Answer: $(0.107, 0.293)$

- $\hat{p} = \frac{10}{50} = 0.2$. 90% CI: $0.2 \pm 1.645\sqrt{\frac{0.2\times0.8}{50}} = 0.2 \pm 1.645 \times 0.05657 = 0.2 \pm 0.093 = (0.107, 0.293)$.

Question 4e.ii *1 mark*

A second random sample of 50 pieces of luggage is selected. Using this sample, the approximate 90% confidence interval for p, the population proportion of luggage labelled as heavy, is **wider** than the one obtained above in **part e.i**.

State the minimum and maximum possible number of pieces of luggage labelled as heavy in the second sample.

Marking Guide — Answer: Minimum: 11, Maximum: 39

- Width $\propto \sqrt{\hat{p}(1-\hat{p})}$. Wider CI requires $\hat{p}(1-\hat{p}) > 0.2 \times 0.8 = 0.16$.
- Need $\hat{p}(1-\hat{p}) > 0.16$. This holds for $0.2 < \hat{p} < 0.8$ (excluding $\hat{p} = 0.2$ and $\hat{p} = 0.8$).
- With $n = 50$: $X > 10$ and $X < 40$, so minimum $X = 11$ and maximum $X = 39$.

Question 5a.i *1 mark*

The graph below shows the compositions $g \circ f$ and $f \circ g$, where $f(x) = \sin(x)$ and $g(x) = \sin(2x)$.

The graph of $y = (g \circ f)(x)$ has a local maximum whose x-value lies in the interval $\left[0, \frac{\pi}{2}\right]$.

Find the coordinates of this local maximum, correct to one decimal place.

Marking Guide — Answer: $(0.9, 1.0)$

Question 5a.ii *1 mark*

State the range of $g \circ f$ where $x \in [0, 2\pi]$.

Marking Guide — Answer: $[-1, 1]$

Question 5b.i *1 mark*

Find the derivative of $f \circ g$.

Marking Guide — Answer: $(f \circ g)'(x) = 2\cos(2x)\cos(\sin(2x))$

- $(f \circ g)(x) = \sin(\sin(2x))$.
- $(f \circ g)'(x) = \cos(\sin(2x)) \cdot 2\cos(2x)$.

Question 5b.ii *2 marks*

Show that the equation $\cos(\sin(2x)) = 0$ has no real solutions.

Marking Guide — Answer: See marking guide

- M1: $\cos(\sin(2x)) = 0$ requires $\sin(2x) = \frac{\pi}{2} + n\pi$ for some integer n.
- A1: But $|\sin(2x)| \leq 1$ and $\frac{\pi}{2} \approx 1.571 > 1$. So $\sin(2x)$ can never equal $\frac{\pi}{2} + n\pi$ for any integer n. Therefore no real solutions exist.

Question 5b.iii *1 mark*

Find the x-values of the stationary points of $f \circ g$ where $x \in [0, 2\pi]$.

Marking Guide — Answer: $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

- $(f \circ g)'(x) = 2\cos(2x)\cos(\sin(2x)) = 0$. Since $\cos(\sin(2x))$ $e0$ (from b.ii), need $\cos(2x) = 0$.
- $2x = \frac{\pi}{2} + n\pi \Rightarrow x = \frac{\pi}{4} + \frac{n\pi}{2}$.

Question 5b.iv *1 mark*

Find the range of $f \circ g$ where $x \in [0, 2\pi]$.

Marking Guide — Answer: $[-\sin(1), \sin(1)]$

Question 5c.i *1 mark*

Write a single definite integral that gives the area bounded by the graphs of $y = (f \circ g)(x)$ and $y = (g \circ f)(x)$ in the interval $[0, 2\pi]$.

Marking Guide — Answer: $\int_0^{2\pi} |\sin(\sin(2x)) - \sin(2\sin(x))|\, dx$

- Area $= \int_0^{2\pi} |(f \circ g)(x) - (g \circ f)(x)|\, dx = \int_0^{2\pi} |\sin(\sin(2x)) - \sin(2\sin(x))|\, dx$.

Question 5c.ii *1 mark*

Hence, state the area bounded by the graphs of $y = (f \circ g)(x)$ and $y = (g \circ f)(x)$ in the interval $[0, 2\pi]$, correct to two decimal places.

Marking Guide — Answer: ≈ 3.70

- Using CAS to evaluate $\int_0^{2\pi} |\sin(\sin(2x)) - \sin(2\sin(x))|\, dx \approx 3.70$.

Question 5d *2 marks*

Let $f_1 : (0, 2\pi) \to R$, $f_1(x) = \sin(x)$.

Find all values of x in the interval $(0, 2\pi)$ for which the composition $f_1 \circ g$ is defined.

Marking Guide — Answer: $x \in \left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right)$

- M1: $f_1 \circ g = f_1(g(x)) = \sin(\sin(2x))$. For f_1 to be defined, need $g(x) = \sin(2x) \in (0, 2\pi)$.