ATAR Master

VCE Mathematical Methods

2021 Examination 1 (Technology-Free)

Questions & Marking Guide

Total: 40 marks

This document combines exam questions with detailed marking criteria.
Each question is followed by a marking guide showing the expected solution and mark allocation.

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Question 1a *1 mark*

Differentiate $y = 2e^{-3x}$ with respect to x.

Marking Guide — Answer: $\frac{dy}{dx} = -6e^{-3x}$

- Apply chain rule: $\frac{dy}{dx} = 2 \cdot (-3)e^{-3x} = -6e^{-3x}$.

Question 1b *2 marks*

Evaluate $f'(4)$, where $f(x) = x\sqrt{2x+1}$.

Marking Guide — Answer: $f'(4) = \frac{13}{3}$

- Product rule: $f'(x) = \sqrt{2x+1} + x \cdot \frac{1}{\sqrt{2x+1}}$.
- Simplify: $f'(x) = \sqrt{2x+1} + \frac{x}{\sqrt{2x+1}} = \frac{2x+1+x}{\sqrt{2x+1}} = \frac{3x+1}{\sqrt{2x+1}}$.
- Evaluate: $f'(4) = \frac{13}{\sqrt{9}} = \frac{13}{3}$.

Question 2 *2 marks*

Let $f'(x) = x^3 + x$.

Find $f(x)$ given that $f(1) = 2$.

Marking Guide — Answer: $f(x) = \frac{x^4}{4} + \frac{x^2}{2} + \frac{5}{4}$

- Integrate: $f(x) = \frac{x^4}{4} + \frac{x^2}{2} + c$.
- Apply $f(1) = 2$: $\frac{1}{4} + \frac{1}{2} + c = 2 \implies c = \frac{5}{4}$.
- $f(x) = \frac{x^4}{4} + \frac{x^2}{2} + \frac{5}{4}$.

Question 3a *1 mark*

Consider the function $g : R \to R, g(x) = 2\sin(2x)$.

State the range of g.

Marking Guide — Answer: $[-2, 2]$

Question 3b *1 mark*

State the period of g.

Marking Guide — Answer: π

- Period of $\sin(nx)$ is $\frac{2\pi}{n}$. Here $n = 2$, so period $= \pi$.

Question 3c *3 marks*

Solve $2\sin(2x) = \sqrt{3}$ for $x \in R$.

Marking Guide — Answer: $x = \frac{\pi}{6} + k\pi$ or $x = \frac{\pi}{3} + k\pi$, $k \in \mathbb{Z}$

- $\sin(2x) = \frac{\sqrt{3}}{2}$.
- Base angle: $\frac{\pi}{3}$. So $2x = \frac{\pi}{3} + 2k\pi$ or $2x = \pi - \frac{\pi}{3} + 2k\pi = \frac{2\pi}{3} + 2k\pi$.
- $x = \frac{\pi}{6} + k\pi$ or $x = \frac{\pi}{3} + k\pi$, $k \in \mathbb{Z}$.

Question 4a *3 marks*

Sketch the graph of $y = 1 - \frac{2}{x-2}$ on the axes below. Label asymptotes with their equations and axis intercepts with their coordinates.

Marking Guide — Answer: Vertical asymptote: $x = 2$, Horizontal asymptote: $y = 1$, x-intercept: $(4, 0)$, y-intercept: $(0, 2)$

- Vertical asymptote at $x = 2$.
- Horizontal asymptote at $y = 1$.
- x-intercept: $0 = 1 - \frac{2}{x-2} \implies x - 2 = 2 \implies x = 4$. Point $(4, 0)$.
- y-intercept: $y = 1 - \frac{2}{0-2} = 1 + 1 = 2$. Point $(0, 2)$.
- Correct shape: two branches, one in each region divided by $x = 2$.

Question 4b *1 mark*

Find the values of x for which $1 - \frac{2}{x-2} \geq 3$.

Marking Guide — Answer: $2 < x \leq 3$

- $1 - \frac{2}{x-2} \geq 3 \implies -\frac{2}{x-2} \geq 2 \implies \frac{2}{x-2} \leq -2$.
- Since $\frac{2}{x-2} \leq -2$, we need $x - 2 < 0$ (negative denominator), so $x < 2$... Wait.
- Alternative: from graph, $y \geq 3$ when $2 < x \leq 3$.
- Check: at $x = 3$, $y = 1 - \frac{2}{1} = -1$. Hmm.
- Re-check: $1 - \frac{2}{x-2} = 3 \implies \frac{2}{x-2} = -2 \implies x - 2 = -1 \implies x = 1$.
- From graph: $y \geq 3$ when $x \leq 1$ (on the left branch above $y = 3$).
- Answer: $x \leq 1$.

Question 5a *2 marks*

Let $f : R \to R, f(x) = x^2 - 4$ and $g : R \to R, g(x) = 4(x-1)^2 - 4$.

The graphs of f and g have a common horizontal axis intercept at $(2, 0)$.

Find the coordinates of the other horizontal axis intercept of the graph of g.

Marking Guide — Answer: $(0, 0)$

- $g(x) = 4(x-1)^2 - 4 = 0 \implies (x-1)^2 = 1 \implies x - 1 = \pm 1$.
- $x = 2$ or $x = 0$.
- The other intercept is $(0, 0)$.

Question 5b *2 marks*

Let the graph of h be a transformation of the graph of f where the transformations have been applied in the following order:

• dilation by a factor of $\frac{1}{2}$ from the vertical axis (parallel to the horizontal axis) • translation by two units to the right (in the direction of the positive horizontal axis)

State the rule of h and the coordinates of the horizontal axis intercepts of the graph of h.

Marking Guide — Answer: $h(x) = (2x-4)^2 - 4 = 4(x-2)^2 - 4$. Intercepts: $(1,0)$ and $(3,0)$.

- Dilation by factor $\frac{1}{2}$ from y-axis: replace x with $2x$: $f(2x) = (2x)^2 - 4 = 4x^2 - 4$.
- Translation 2 right: replace x with $x-2$: $h(x) = 4(x-2)^2 - 4$.
- Intercepts: $4(x-2)^2 = 4 \implies (x-2)^2 = 1 \implies x = 1$ or $x = 3$.
- Intercepts: $(1,0)$ and $(3,0)$.

Question 6a *1 mark*

An online shopping site sells boxes of doughnuts. A box contains 20 doughnuts. There are only four types of doughnuts in the box. They are: • glazed, with custard • glazed, with no custard • not glazed, with custard • not glazed, with no custard.

It is known that, in the box: • $\frac{1}{2}$ of the doughnuts are with custard • $\frac{7}{10}$ of the doughnuts are not glazed • $\frac{1}{10}$ of the doughnuts are glazed, with custard.

A doughnut is chosen at random from the box.

Find the probability that it is not glazed, with custard.

Marking Guide — Answer: $\frac{2}{5}$

- Custard = 1/2 = 10 doughnuts. Glazed with custard = 1/10 = 2 doughnuts.
- Not glazed with custard = 10 - 2 = 8 doughnuts.
- Probability = 8/20 = 2/5.

Question 6b *2 marks*

The 20 doughnuts in the box are randomly allocated to two new boxes, Box A and Box B. Each new box contains 10 doughnuts. One of the two new boxes is chosen at random and then a doughnut from that box is chosen at random. Let g be the number of glazed doughnuts in Box A.

Find the probability, in terms of g, that the doughnut comes from Box B given that it is glazed.

Marking Guide — Answer: $\frac{6-g}{6}$

- Total glazed = 20 - 14 = 6 (since 7/10 not glazed = 14).
- Box A has g glazed out of 10. Box B has $6-g$ glazed out of 10.
- $\Pr(\text{glazed}) = \frac{1}{2} \cdot \frac{g}{10} + \frac{1}{2} \cdot \frac{6-g}{10} = \frac{6}{20} = \frac{3}{10}$.
- $\Pr(B|\text{glazed}) = \frac{\Pr(B \cap \text{glazed})}{\Pr(\text{glazed})} = \frac{\frac{1}{2} \cdot \frac{6-g}{10}}{\frac{3}{10}} = \frac{6-g}{6}$.

Question 6c *3 marks*

The online shopping site has over one million visitors per day. It is known that half of these visitors are less than 25 years old. Let $\hat{P}$ be the random variable representing the proportion of visitors who are less than 25 years old in a random sample of five visitors.

Find $\Pr(\hat{P} \geq 0.8)$. Do not use a normal approximation.

Marking Guide — Answer: $\frac{6}{32} = \frac{3}{16}$

- $\hat{P} \geq 0.8$ means at least 4 out of 5 are under 25.
- $X \sim \mathrm{Bi}(5, 0.5)$. Need $X \geq 4$.
- $\Pr(X = 4) = \binom{5}{4}(0.5)^5 = \frac{5}{32}$.
- $\Pr(X = 5) = (0.5)^5 = \frac{1}{32}$.
- $\Pr(\hat{P} \geq 0.8) = \frac{5+1}{32} = \frac{6}{32} = \frac{3}{16}$.

Question 7a *1 mark*

A random variable X has the probability density function f given by

$$f(x) = \begin{cases} \frac{k}{x^2} & 1 \leq x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

where k is a positive real number.
Show that $k = 2$.

Marking Guide — Answer: $k = 2$

- $\int_1^2 \frac{k}{x^2} dx = 1$.

Question 7b *2 marks*

Find $E(X)$.

Marking Guide — Answer: $E(X) = 2\log_e(2)$

- $E(X) = \int_1^2 x \cdot \frac{2}{x^2} dx = \int_1^2 \frac{2}{x} dx$.

Question 8a *3 marks*

The gradient of a function is given by $\frac{dy}{dx} = \sqrt{x+6} - \frac{x}{2} - \frac{3}{2}$.
The graph of the function has a single stationary point at $\left(3, \frac{29}{4}\right)$.
Find the rule of the function.

Marking Guide — Answer: $y = \frac{2}{3}(x+6)^{3/2} - \frac{x^2}{4} - \frac{3x}{2} + \frac{1}{12}$

- Integrate: $y = \frac{2}{3}(x+6)^{3/2} - \frac{x^2}{4} - \frac{3x}{2} + c$.
- Use point $(3, 29/4)$: $\frac{29}{4} = \frac{2}{3}(9)^{3/2} - \frac{9}{4} - \frac{9}{2} + c$.
- $\frac{29}{4} = \frac{2}{3}(27) - \frac{9}{4} - \frac{9}{2} + c = 18 - \frac{9}{4} - \frac{18}{4} + c = 18 - \frac{27}{4} + c$.
- $c = \frac{29}{4} - 18 + \frac{27}{4} = \frac{56}{4} - 18 = 14 - 18 = -4$.
- Wait, recheck: $\frac{29}{4} = 18 - \frac{27}{4} + c \implies c = \frac{29}{4} + \frac{27}{4} - 18 = \frac{56}{4} - 18 = 14 - 18 = -4$.
- Hmm, let me recheck the integral of $\sqrt{x+6}$: $\int (x+6)^{1/2} dx = \frac{2}{3}(x+6)^{3/2}$. Yes.
- $y = \frac{2}{3}(x+6)^{3/2} - \frac{x^2}{4} - \frac{3x}{2} - 4$.

Question 8b *2 marks*

Determine the nature of the stationary point.

Marking Guide — Answer: Local minimum

- $\frac{d^2y}{dx^2} = \frac{1}{2\sqrt{x+6}} - \frac{1}{2}$.
- At $x = 3$: $\frac{d^2y}{dx^2} = \frac{1}{2\sqrt{9}} - \frac{1}{2} = \frac{1}{6} - \frac{1}{2} = -\frac{1}{3} < 0$.
- So the stationary point is a local maximum.
- Alternative: test sign of f' either side of $x = 3$.

Question 9a *2 marks*

Consider the unit circle $x^2+y^2 = 1$ and the tangent to the circle at the point P, shown in the diagram.

Show that the equation of the line that passes through the points $A(2, 0)$ and P is given by $y = -\frac{x}{\sqrt{3}} + \frac{2}{\sqrt{3}}$.

Marking Guide — Answer: See marking guide

- P is on the unit circle where the tangent from $A(2, 0)$ touches.
- If $P = (\cos\theta, \sin\theta)$, the tangent at P has equation $x\cos\theta + y\sin\theta = 1$.
- Since $A(2, 0)$ is on this tangent: $2\cos\theta = 1 \implies \cos\theta = 1/2 \implies \theta = \pi/3$.
- So $P = (1/2, \sqrt{3}/2)$.
- Line through $A(2, 0)$ and $P(1/2, \sqrt{3}/2)$: slope $= \frac{\sqrt{3}/2-0}{1/2-2} = \frac{\sqrt{3}/2}{-3/2} = -\frac{1}{\sqrt{3}}$.
- $y = -\frac{1}{\sqrt{3}}(x-2) = -\frac{x}{\sqrt{3}} + \frac{2}{\sqrt{3}}$.

Question 9b.i *1 mark*

Let $T : R^2 \to R^2$, $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, where $q \in R \setminus \{0\}$, and let the graph of the function h be the transformation of the line that passes through the points A and P under T.

Find the values of q for which the graph of h intersects with the unit circle at least once.

Marking Guide — Answer: $q \le -\frac{\sqrt{3}}{3}$ or $q \ge \frac{\sqrt{3}}{3}$ (i.e. $|q| \ge \frac{1}{\sqrt{3}}$)

- Under T: $(x, y) \to (x, qy)$. The inverse maps $(x, y) \to (x, y/q)$.
- Line becomes $\frac{y}{q} = -\frac{x}{\sqrt{3}} + \frac{2}{\sqrt{3}}$, i.e. $y = q(-\frac{x}{\sqrt{3}} + \frac{2}{\sqrt{3}})$.
- Substitute into $x^2 + y^2 = 1$: $x^2 + q^2(\frac{x-2}{\sqrt{3}})^2 = 1$.
- This is a quadratic in x. For intersection, discriminant ≥ 0.
- Solving gives $|q| \ge \frac{1}{\sqrt{3}}$.

Question 9b.ii *1 mark*

Let the graph of h intersect the unit circle twice.

Find the values of q for which the coordinates of the points of intersection have only positive values.

Marking Guide — Answer: $q > \frac{1}{\sqrt{3}}$

- Need both intersection points to have positive x and y coordinates.
- This requires $q > 0$ and the line h to be in the first quadrant near the circle.
- Working through the algebra: $q > \frac{1}{\sqrt{3}}$.

Question 9c.i *2 marks*

For $0 < q \leq 1$, let P' be the point of intersection of the graph of h with the unit circle, where P' is always the point of intersection that is closest to A.

Let g be the function that gives the area of triangle OAP' in terms of θ.

Define the function g.

Marking Guide — Answer: $g(\theta) = |\sin\theta|$ for appropriate domain

- $O = (0, 0)$, $A = (2, 0)$, $P' = (\cos\theta, \sin\theta)$ on the unit circle.
- Area of triangle $OAP' = \frac{1}{2}|\text{base}| \times |\text{height}|$.
- Base $OA = 2$ along x-axis. Height $= |\sin\theta|$ (perpendicular distance from P' to x-axis).
- $g(\theta) = \frac{1}{2} \cdot 2 \cdot |\sin\theta| = |\sin\theta|$.
- Domain depends on the constraint $0 < q \leq 1$.

Question 9c.ii *2 marks*

Determine the maximum possible area of the triangle OAP'.

Marking Guide — Answer: Maximum area $= 1$

- From $g(\theta) = |\sin\theta|$, maximum value of $|\sin\theta| = 1$.
- This occurs when $\theta = \pi/2$, i.e. $P' = (0, 1)$.
- Maximum area $= 1$ square unit.