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# ATAR Master

VCE Mathematical Methods

**2022 Examination 2 (Technology-Active)**

Questions & Marking Guide

Total: 80 marks

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This document combines exam questions with detailed marking criteria.  
Each question is followed by a marking guide showing the expected solution and mark allocation.

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## Section A: Multiple Choice — 20 marks

Each question is worth 1 mark.

### Question 1

1 mark

The period of the function  $f(x) = 3 \cos(2x + \pi)$  is

- A.  $2\pi$
- B.  $\pi$
- C. 2
- D. 3

**Marking Guide** — Answer: B

- Period =  $\frac{2\pi}{2} = \pi$ .

### Question 2

1 mark

The graph of  $y = \frac{1}{(x+3)^2} + 4$  has a horizontal asymptote with the equation

- A.  $y = 4$
- B.  $y = 3$
- C.  $y = 0$
- D.  $x = -2$
- E.  $x = -3$

**Marking Guide** — Answer: A

- As  $x \rightarrow \pm\infty$ ,  $\frac{1}{(x+3)^2} \rightarrow 0$ , so  $y \rightarrow 4$ .

### Question 3

1 mark

The gradient of the graph of  $y = e^{3x}$  at the point where the graph crosses the vertical axis is equal to

- A. 0
- B. 1
- C.  $e$
- D. 3

**Marking Guide** — Answer: E

- $\frac{dy}{dx} = 3e^{3x}$ . At  $x = 0$ :  $\frac{dy}{dx} = 3e^0 = 3$ .

## Section B: Extended Response — 60 marks

**Question 4***1 mark*

Which one of the following functions is not continuous over the interval  $x \in [0, 5]$ ?

**Marking Guide** — Answer: D**Question 5***1 mark*

The largest value of  $a$  such that the function  $f : (-\infty, a] \rightarrow R$ ,  $f(x) = x^2 + 3x - 10$ , where  $f$  is one-to-one, is

- A. -12.25
- B. -5
- C. -1.5
- D. 0
- E. 2

**Marking Guide** — Answer: C**Question 6***1 mark*

Which of the pairs of functions below are \*\*not\*\* inverse functions?

**Marking Guide** — Answer: C

- C:  $f(x) = x^2$  for  $x < 0$  and  $g(x) = \sqrt{x}$  for  $x > 0$ . The inverse of  $f(x) = x^2, x < 0$  is  $g(x) = -\sqrt{x}$ , not  $\sqrt{x}$ .

**Question 7***1 mark*

The graph of  $y = f(x)$  is shown (a curve with a local minimum to the left of the y-axis and a local maximum to the right, with an inflection point near the origin). The graph of  $y = f'(x)$ , the first derivative of  $f(x)$  with respect to  $x$ , could be

- A. Graph A (positive hump then dip, zeros at two points)
- B. Graph B (tall positive hump, one zero)
- C. Graph C (negative dip then positive, two zeros)
- D. Graph D (vertical asymptote shape)
- E. Graph E (starts positive, crosses to negative, single smooth curve)

**Marking Guide** — Answer: E

- The original function has a local min (left) and local max (right), so  $f'(x) = 0$  at two points, positive between them. Graph E shows a positive hump crossing zero at both turning points, consistent with a cubic-like derivative.

**Question 8***1 mark*

If  $\int_0^b f(x) dx = 10$  and  $\int_0^a f(x) dx = -4$ , where  $0 < a < b$ , then  $\int_a^b f(x) dx$  is equal to

- A. -6
- B. -4
- C. 0
- D. 10
- E. 14

**Marking Guide** — Answer: E

- $\int_a^b f(x) dx = \int_0^b f(x) dx - \int_0^a f(x) dx = 10 - (-4) = 14.$

**Question 9***1 mark*

Let  $f : [0, \infty) \rightarrow R$ ,  $f(x) = \sqrt{2x + 1}$ .

The shortest distance,  $d$ , from the origin to the point  $(x, y)$  on the graph of  $f$  is given by

- A.  $d = x^2 + 2x + 1$
- B.  $d = x + 1$
- C.  $d = 2x + 1$

**Marking Guide** — Answer: D

- $d = \sqrt{x^2 + y^2} = \sqrt{x^2 + 2x + 1} = \sqrt{(x + 1)^2} = x + 1$  (since  $x \geq 0$ ).

**Question 10***1 mark*

An organisation randomly surveyed 1000 Australian adults and found that 55% of those surveyed were happy with their level of physical activity.

An approximate 95% confidence interval for the percentage of Australian adults who were happy with their level of physical activity is closest to

- A. (4.1, 6.9)
- B. (50.9, 59.1)
- C. (52.4, 57.6)
- D. (51.9, 58.1)
- E. (45.2, 64.8)

**Marking Guide** — Answer: D

- $\hat{p} = 0.55$ ,  $n = 1000$ .  $E = 1.96 \sqrt{\frac{0.55 \times 0.45}{1000}} \approx 0.0308$ .

- CI:  $(0.55 - 0.031, 0.55 + 0.031) \approx (0.519, 0.581)$ , as percentage (51.9, 58.1).

**Question 11**

1 mark

If  $\frac{d}{dx}(x \cdot \sin(x)) = \sin(x) + x \cdot \cos(x)$ , then  $\frac{1}{k} \int x \cos(x) dx$  is equal to

- A.  $k(x \sin(x) - \int \sin(x) dx) + c$

**Marking Guide** — Answer: C

- From the product rule:  $x \cos(x) = \frac{d}{dx}(x \sin(x)) - \sin(x)$ .
- $\int x \cos(x) dx = x \sin(x) - \int \sin(x) dx + c$ .
- So  $\frac{1}{k} \int x \cos(x) dx = \frac{1}{k} (x \sin(x) - \int \sin(x) dx) + c$ .

**Question 12**

1 mark

A bag contains three red pens and  $x$  black pens. Two pens are randomly drawn from the bag without replacement.

The probability of drawing a pen of each colour is equal to

**Marking Guide** — Answer: A

- Total pens:  $3 + x$ .  $P = \frac{\binom{3}{1}\binom{x}{1}}{\binom{3+x}{2}} = \frac{3x}{\frac{(3+x)(2+x)}{2}} = \frac{6x}{(2+x)(3+x)}$ .

**Question 13**

1 mark

The function  $f(x) = \log_e \left( \frac{x+a}{x-a} \right)$ , where  $a$  is a positive real constant, has the maximal domain

**Marking Guide** — Answer: C

**Question 14**

1 mark

A continuous random variable,  $X$ , has a probability density function given by

$$f(x) = \begin{cases} \frac{2}{9}xe^{-\frac{1}{9}x^2} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

The expected value of  $X$ , correct to three decimal places, is

- A. 1.000  
 B. 2.659  
 C. 3.730  
 D. 6.341  
 E. 9.000

**Marking Guide** — Answer: B

- $E(X) = \int_0^\infty x \cdot \frac{2}{9}xe^{-\frac{1}{9}x^2} dx = \int_0^\infty \frac{2}{9}x^2e^{-\frac{1}{9}x^2} dx \approx 2.659.$

**Question 15***1 mark*

The maximal domain of the function with rule  $f(x) = \sqrt{x^2 - 2x - 3}$  is given by

- $(-\infty, \infty)$
- $(-\infty, -3) \cup (1, \infty)$
- $(-1, 3)$

**Marking Guide** — Answer: E

**Question 16***1 mark*

The function  $f(x) = \frac{1}{3}x^3 + mx^2 + nx + p$ , for  $m, n, p \in R$ , has turning points at  $x = -3$  and  $x = 1$  and passes through the point  $(3, 4)$ .

The values of  $m$ ,  $n$  and  $p$  respectively are

- $m = 1, n = -3, p = -5$
- $m = -1, n = -3, p = 13$

**Marking Guide** — Answer: B

- $f'(x) = x^2 + 2mx + n$ . Turning points at  $x = -3$  and  $x = 1$ :  $f'(x) = (x+3)(x-1) = x^2 + 2x - 3$ .
- So  $2m = 2 \Rightarrow m = 1$  and  $n = -3$ .
- $f(3) = 9 + 9 - 9 + p = 4 \Rightarrow p = -5$ .

**Question 17***1 mark*

A function  $g$  is continuous on the domain  $x \in [a, b]$  and has the following properties:

- The average rate of change of  $g$  between  $x = a$  and  $x = b$  is positive.
- The instantaneous rate of change of  $g$  at  $x = \frac{a+b}{2}$  is negative.

Therefore, on the interval  $x \in [a, b]$ , the function must be

- many-to-one.
- one-to-many.
- one-to-one.
- strictly decreasing.
- strictly increasing.

**Marking Guide** — Answer: A

- $g(b) > g(a)$  (positive average rate), but  $g'(\frac{a+b}{2}) < 0$  (decreasing at midpoint). The function increases overall but decreases somewhere in between, so it must be many-to-one.

**Question 18***1 mark*

If  $X$  is a binomial random variable where  $n = 20$ ,  $p = 0.88$  and  $\Pr(X \geq 16 | X \geq a) = 0.9175$ , correct to four decimal places, then  $a$  is equal to

- A. 11
- B. 12
- C. 13
- D. 14
- E. 15

**Marking Guide** — Answer: B

- $\Pr(X \geq 16 | X \geq a) = \frac{\Pr(X \geq 16)}{\Pr(X \geq a)} = 0.9175$  (since  $a < 16$ ).
- Use CAS to find  $\Pr(X \geq 16)$  and solve for  $a$ .  $a = 12$ .

**Question 19***1 mark*

A box is formed from a rectangular sheet of cardboard, which has a width of  $a$  units and a length of  $b$  units, by first cutting out squares of side length  $x$  units from each corner and then folding upwards to form a container with an open top.

The maximum volume of the box occurs when  $x$  is equal to

**Marking Guide** — Answer: D

- $V = x(a - 2x)(b - 2x)$ .  $V' = 12x^2 - 4(a + b)x + ab = 0$ .
- $x = \frac{4(a+b) \pm \sqrt{16(a+b)^2 - 48ab}}{24} = \frac{a+b \pm \sqrt{a^2 - ab + b^2}}{6}$ .
- Maximum at the smaller root:  $x = \frac{a+b - \sqrt{a^2 - ab + b^2}}{6}$ .

**Question 20***1 mark*

A soccer player kicks a ball with an angle of elevation of  $\theta$ , where  $\theta$  is a normally distributed random variable with a mean of 42 and a standard deviation of 8.

The horizontal distance that the ball travels before landing is given by the function  $d = 50 \sin(2\theta)$ .

The probability that the ball travels more than 40 m horizontally before landing is closest to

- A. 0.969
- B. 0.937
- C. 0.226
- D. 0.149
- E. 0.027

**Marking Guide** — Answer: A

- Solve  $50 \sin(2\theta) > 40$ , i.e.  $\sin(2\theta) > 0.8$ .
- $2\theta > \sin^{-1}(0.8) \approx 53.13$  or  $2\theta < 180 - 53.13 = 126.87$ .

- So  $26.57 < \theta < 63.43$ .
- $\Pr(26.57 < \theta < 63.43)$  where  $\theta \sim N(42, 8^2)$ .
- Using CAS:  $\approx 0.969$ .

**Question 1a***1 mark*

The diagram shows part of the graph of  $y = f(x)$ , where  $f(x) = \frac{x^2}{12}$ .

State the equation of the axis of symmetry of the graph of  $f$ .

**Marking Guide** — Answer:  $x = 0$

- The parabola  $f(x) = \frac{x^2}{12}$  is symmetric about the y-axis. Axis of symmetry:  $x = 0$ .

**Question 1b***1 mark*

State the derivative of  $f$  with respect to  $x$ .

**Marking Guide** — Answer:  $f'(x) = \frac{x}{6}$

- $f'(x) = \frac{2x}{12} = \frac{x}{6}$ .

**Question 1c***2 marks*

The tangent to  $f$  at point  $M$  has gradient  $-2$ .

Find the equation of the tangent to  $f$  at point  $M$ .

**Marking Guide** — Answer:  $y = -2x - 12$

- M1: Find x-coordinate:  $f'(x) = -2 \Rightarrow \frac{x}{6} = -2 \Rightarrow x = -12$ .
- A1:  $f(-12) = \frac{144}{12} = 12$ . Tangent:  $y - 12 = -2(x + 12) \Rightarrow y = -2x - 12$ .

**Question 1d.i***1 mark*

Find the equation of the line perpendicular to the tangent passing through point  $M$ .

**Marking Guide** — Answer:  $y = \frac{1}{2}x + 18$

- Perpendicular gradient:  $\frac{1}{2}$ . Through  $M(-12, 12)$ :  $y - 12 = \frac{1}{2}(x + 12) \Rightarrow y = \frac{1}{2}x + 18$ .

**Question 1d.ii***2 marks*

The line perpendicular to the tangent at point  $M$  also cuts  $f$  at point  $N$ .

Find the area enclosed by this line and the curve  $y = f(x)$ .

**Marking Guide** — Answer:  $\frac{10976}{9} \approx 1219.6$

- M1: Find intersection:  $\frac{x^2}{12} = \frac{1}{2}x + 18 \Rightarrow x^2 - 6x - 216 = 0 \Rightarrow (x + 12)(x - 18) = 0$ .  $N$  at  $x = 18$ .
- A1: Area =  $\int_{-12}^{18} \left( \frac{1}{2}x + 18 - \frac{x^2}{12} \right) dx$ .

**Question 1e***4 marks*

Another parabola is defined by the rule  $g(x) = \frac{x^2}{4a^2}$ , where  $a > 0$ .

A tangent to  $g$  and the line perpendicular to the tangent at  $x = -b$ , where  $b > 0$ , are shown.

Find the value of  $b$ , in terms of  $a$ , such that the shaded area is a minimum.

**Marking Guide** — Answer:  $b = a\sqrt[3]{2}$

- M1:  $g'(x) = \frac{x}{2a^2}$ . At  $x = -b$ : gradient  $= -\frac{b}{2a^2}$ , perpendicular gradient  $= \frac{2a^2}{b}$ .
- M1: Find intersection of perpendicular line with  $g(x)$ , set up area integral.
- M1: Express area as a function of  $b$  and differentiate.

**Question 2a.i***1 mark*

On a remote island, there are only two species of animals: foxes and rabbits. The populations increase and decrease in a periodic pattern.

The population of rabbits can be modelled by the rule  $r(t) = 1700 \sin\left(\frac{\pi t}{80}\right) + 2500$ .

One point of minimum fox population, (20, 700), and one point of maximum fox population, (100, 2500), are shown on the graph.

State the initial population of rabbits.

**Marking Guide** — Answer: 2500

- $r(0) = 1700 \sin(0) + 2500 = 2500$ .

**Question 2a.ii***1 mark*

State the minimum and maximum population of rabbits.

**Marking Guide** — Answer: Min: 800, Max: 4200

- $\text{Min} = 2500 - 1700 = 800$ .  $\text{Max} = 2500 + 1700 = 4200$ .

**Question 2a.iii***1 mark*

State the number of weeks between maximum populations of rabbits.

**Marking Guide** — Answer: 160

- Period  $= \frac{2\pi}{\pi/80} = 160$  weeks.

**Question 2b***2 marks*

The population of foxes can be modelled by the rule  $f(t) = a \sin\left(\frac{\pi}{60}(t - b)\right) + 1600$ .

Show that  $a = 900$  and  $b = 80$ .

**Marking Guide** — Answer: See marking guide

- M1: From graph: min fox pop = 700, max = 2500. Mean  $= \frac{700+2500}{2} = 1600 \checkmark$ . Amplitude  $a = 2500 - 1600 = 900$ .
- A1: Period same as rabbits = 160 weeks, so  $\frac{2\pi}{\pi/60} = 120\dots$  Actually from graph, period  $= 2 \times (100 - 20) = 160$ . Min at  $t = 20$ :  $\sin\left(\frac{\pi}{60}(20 - b)\right) = -1 \Rightarrow \frac{\pi(20-b)}{60} = -\frac{\pi}{2} \Rightarrow 20 - b = -30\dots$

Alternatively, max at  $t = 100$ , min at  $t = 20$ .  $b = 80$ .

**Question 2c***1 mark*

Find the maximum combined population of foxes and rabbits. Give your answer correct to the nearest whole number.

**Marking Guide** — Answer: 5339

- Using CAS, maximise  $r(t) + f(t)$ . Maximum combined population  $\approx 5339$ .

**Question 2d***1 mark*

What is the number of weeks between the periods when the combined population of foxes and rabbits is a maximum?

**Marking Guide** — Answer: 160

- Both populations have the same period of 160 weeks, so the combined population also has period 160 weeks.

**Question 2e***4 marks*

The population of foxes is better modelled by the transformation of  $y = \sin(t)$  under  $Q$  given by

$$Q : \begin{pmatrix} t \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{90}{\pi} & 0 \\ 0 & 900 \end{pmatrix} \begin{pmatrix} t \\ y \end{pmatrix} + \begin{pmatrix} 60 \\ 1600 \end{pmatrix}$$

Find the average population during the first 300 weeks for the combined population of foxes and rabbits, where the population of foxes is modelled by the transformation of  $y = \sin(t)$  under the transformation  $Q$ . Give your answer correct to the nearest whole number.

**Marking Guide** — Answer: 4100 (to nearest whole number)

- M1: Under transformation  $Q$ :  $t_{new} = \frac{90}{\pi}t + 60$ ,  $y_{new} = 900y + 1600$ .
- So fox model becomes  $f(t) = 900 \sin\left(\frac{\pi(t-60)}{90}\right) + 1600$ .
- M1: Combined population =  $r(t) + f(t) = 1700 \sin\left(\frac{\pi t}{80}\right) + 2500 + 900 \sin\left(\frac{\pi(t-60)}{90}\right) + 1600$ .
- M1: Average =  $\frac{1}{300} \int_0^{300} (r(t) + f(t)) dt$ .
- A1: Evaluate using CAS  $\approx 4100$ .

**Question 2f***2 marks*

Over a longer period of time, it is found that the increase and decrease in the population of rabbits gets smaller and smaller.

The population of rabbits over a longer period of time can be modelled by the rule

$$s(t) = 1700e^{-0.003t} \sin\left(\frac{\pi t}{80}\right) + 2500, \quad \text{for all } t \geq 0$$

Find the average rate of change between the first two times when the population of rabbits is at a maximum. Give your answer correct to one decimal place.

**Marking Guide** — Answer: -3.0

- M1: Find the first two maximum points of  $s(t)$  using CAS (e.g.,  $t_1 \approx 40$ ,  $t_2 \approx 200$ , or solve  $s'(t) = 0$ ).
- A1: Average rate of change =  $\frac{s(t_2) - s(t_1)}{t_2 - t_1} \approx -3.0$ .

### Question 2g

*2 marks*

Find the time, where  $t > 40$ , in weeks, when the rate of change of the rabbit population is at its greatest positive value. Give your answer correct to the nearest whole number.

**Marking Guide** — Answer:  $t \approx 156$  weeks

- M1: Find  $s'(t)$  and maximise it for  $t > 40$  using CAS.
- A1:  $t \approx 156$  weeks.

### Question 2h

*1 mark*

Over time, the rabbit population approaches a particular value.

State this value.

**Marking Guide** — Answer: 2500

- As  $t \rightarrow \infty$ ,  $e^{-0.003t} \rightarrow 0$ , so  $s(t) \rightarrow 2500$ .

### Question 3a.i

*1 mark*

Mika is flipping a coin. The unbiased coin has a probability of  $\frac{1}{2}$  of landing on heads and  $\frac{1}{2}$  of landing on tails.

Let  $X$  be the binomial random variable representing the number of times that the coin lands on heads.

Mika flips the coin five times.

Find  $\Pr(X = 5)$ .

**Marking Guide** — Answer:  $\frac{1}{32}$

- $\Pr(X = 5) = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$ .

### Question 3a.ii

*1 mark*

Find  $\Pr(X \geq 2)$ .

**Marking Guide** — Answer:  $\frac{13}{16}$

- $\Pr(X \geq 2) = 1 - \Pr(X = 0) - \Pr(X = 1) = 1 - \frac{1}{32} - \frac{5}{32} = \frac{26}{32} = \frac{13}{16}$ .

### Question 3a.iii

*2 marks*

Find  $\Pr(X \geq 2 | X < 5)$ , correct to three decimal places.

**Marking Guide** — Answer: 0.806

- M1:  $\Pr(X \geq 2 | X < 5) = \frac{\Pr(2 \leq X < 5)}{\Pr(X < 5)} = \frac{\Pr(X \geq 2) - \Pr(X = 5)}{1 - \Pr(X = 5)}.$
- A1:  $= \frac{\frac{13}{16} - \frac{1}{32}}{1 - \frac{1}{32}} = \frac{\frac{25}{32}}{\frac{31}{32}} = \frac{25}{31} \approx 0.806.$

**Question 3a.iv***2 marks*

Find the expected value and the standard deviation for  $X$ .

**Marking Guide** — Answer:  $E(X) = \frac{5}{2}$ ,  $\text{SD}(X) = \frac{\sqrt{5}}{2}$

- M1:  $E(X) = np = 5 \times \frac{1}{2} = \frac{5}{2}.$
- A1:  $\text{Var}(X) = np(1-p) = \frac{5}{4}$ .  $\text{SD}(X) = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}.$

**Question 3b.i***1 mark*

The height reached by each of Mika's coin flips is given by a continuous random variable,  $H$ , with the probability density function

$$f(h) = \begin{cases} ah^2 + bh + c & 1.5 \leq h \leq 3 \\ 0 & \text{elsewhere} \end{cases}$$

where  $h$  is the vertical height reached by the coin flip, in metres, between the coin and the floor, and  $a$ ,  $b$  and  $c$  are real constants.

State the value of the definite integral  $\int_{1.5}^3 f(h) dh$ .

**Marking Guide** — Answer: 1

- Total area under a PDF is 1.

**Question 3b.ii***3 marks*

Given that  $\Pr(H \leq 2) = 0.35$  and  $\Pr(H \geq 2.5) = 0.25$ , find the values of  $a$ ,  $b$  and  $c$ .

**Marking Guide** — Answer:  $a = \frac{16}{15}$ ,  $b = -\frac{16}{3}$ ,  $c = \frac{32}{5}$

- M1: Set up three equations:  $\int_{1.5}^3 (ah^2 + bh + c) dh = 1$ ,  $\int_{1.5}^2 (ah^2 + bh + c) dh = 0.35$ ,  $\int_{2.5}^3 (ah^2 + bh + c) dh = 0.25$ .
- M1: Solve the system of three equations in three unknowns.
- A1:  $a = \frac{16}{15}$ ,  $b = -\frac{16}{3}$ ,  $c = \frac{32}{5}$ .

**Question 3b.iii***1 mark*

The ceiling of Mika's room is 3 m above the floor. The minimum distance between the coin and the ceiling is a continuous random variable,  $D$ , with probability density function  $g$ .

The function  $g$  is a transformation of the function  $f$  given by  $g(d) = f(rd + s)$ , where  $d$  is the minimum distance between the coin and the ceiling, and  $r$  and  $s$  are real constants.

Find the values of  $r$  and  $s$ .

**Marking Guide** — Answer:  $r = -1$ ,  $s = 3$

- If ceiling is 3 m, then  $d = 3 - h$ , so  $h = 3 - d = -d + 3$ . Hence  $g(d) = f(-d + 3) = f(-1 \cdot d + 3)$ , so  $r = -1$ ,  $s = 3$ .

**Question 3c.i**

1 mark

Mika's sister Bella also has a coin. On each flip, Bella's coin has a probability of  $p$  of landing on heads and  $(1 - p)$  of landing on tails, where  $p$  is a constant value between 0 and 1.

Bella flips her coin 25 times in order to estimate  $p$ .

Let  $\hat{P}$  be the random variable representing the proportion of times that Bella's coin lands on heads in her sample.

Is the random variable  $\hat{P}$  discrete or continuous? Justify your answer.

**Marking Guide** — Answer: Discrete

- Discrete, because  $\hat{P} = \frac{X}{25}$  where  $X$  can only take integer values 0, 1, 2, ..., 25. So  $\hat{P}$  can only take a countable number of values.

**Question 3c.ii**

1 mark

If  $\hat{p} = 0.4$ , find an approximate 95% confidence interval for  $p$ , correct to three decimal places.

**Marking Guide** — Answer: (0.208, 0.592)

- $CI = 0.4 \pm 1.96 \sqrt{\frac{0.4 \times 0.6}{25}} = 0.4 \pm 1.96 \times 0.09798 = 0.4 \pm 0.192 = (0.208, 0.592)$ .

**Question 3c.iii**

1 mark

Bella knows that she can decrease the width of a 95% confidence interval by using a larger sample of coin flips.

If  $\hat{p} = 0.4$ , how many coin flips would be required to halve the width of the confidence interval found in part c.ii.?

**Marking Guide** — Answer: 100

- To halve the width, need to multiply  $n$  by 4 (since width  $\propto \frac{1}{\sqrt{n}}$ ).  $4 \times 25 = 100$ .

**Question 4a**

1 mark

Consider the function  $f$ , where  $f : (-\frac{1}{2}, \frac{1}{2}) \rightarrow R$ ,  $f(x) = \log_e \left( \frac{\frac{1}{2}+x}{\frac{1}{2}-x} \right)$ .

State the range of  $f(x)$ .

**Marking Guide** — Answer:  $R$

- As  $x \rightarrow \frac{1}{2}^-$ ,  $f(x) \rightarrow +\infty$ . As  $x \rightarrow -\frac{1}{2}^+$ ,  $f(x) \rightarrow -\infty$ . Range is  $R$ .

**Question 4b.i**

2 marks

Find  $f'(0)$ .

**Marking Guide** — Answer:  $f'(0) = 4$

- M1:  $f(x) = \log_e \left( \frac{1}{2} + x \right) - \log_e \left( \frac{1}{2} - x \right)$ .

- A1:  $f'(x) = \frac{1}{\frac{1}{2}+x} + \frac{1}{\frac{1}{2}-x}$ . At  $x = 0$ :  $f'(0) = 2 + 2 = 4$ .

**Question 4b.ii***1 mark*

State the maximal domain over which  $f$  is strictly increasing.

**Marking Guide** — Answer:  $(-\frac{1}{2}, \frac{1}{2})$

- $f'(x) > 0$  for all  $x \in (-\frac{1}{2}, \frac{1}{2})$  (the entire domain). So  $f$  is strictly increasing on its whole domain.

**Question 4c***1 mark*

Show that  $f(x) + f(-x) = 0$ .

**Marking Guide** — Answer: See marking guide

- A1:  $f(-x) = \log_e \left( \frac{\frac{1}{2}-x}{\frac{1}{2}+x} \right) = -\log_e \left( \frac{\frac{1}{2}+x}{\frac{1}{2}-x} \right) = -f(x)$ . Therefore  $f(x) + f(-x) = 0$ .

**Question 4d***3 marks*

Find the domain and the rule of  $f^{-1}$ , the inverse of  $f$ .

**Marking Guide** — Answer:  $f^{-1} : R \rightarrow R$ ,  $f^{-1}(x) = \frac{e^x - 1}{2(e^x + 1)}$

- M1: Let  $y = \log_e \left( \frac{\frac{1}{2}+x}{\frac{1}{2}-x} \right)$ . Swap  $x$  and  $y$ :  $x = \log_e \left( \frac{\frac{1}{2}+y}{\frac{1}{2}-y} \right)$ .
- M1:  $e^x = \frac{\frac{1}{2}+y}{\frac{1}{2}-y}$ . Solve for  $y$ :  $e^x(\frac{1}{2} - y) = \frac{1}{2} + y$ ,  $\frac{e^x}{2} - ye^x = \frac{1}{2} + y$ ,  $y(1 + e^x) = \frac{e^x - 1}{2}$ .
- A1:  $f^{-1}(x) = \frac{e^x - 1}{2(e^x + 1)}$ . Domain of  $f^{-1}$  is  $R$ .

**Question 4e.i***1 mark*

Let  $h$  be the function  $h : (-\frac{1}{2}, \frac{1}{2}) \rightarrow R$ ,  $h(x) = k \log_e \left( \frac{\frac{1}{2}+x}{\frac{1}{2}-x} \right)$ , where  $k \in R$  and  $k > 0$ .

The inverse function of  $h$  is defined by  $h^{-1} : R \rightarrow R$ ,  $h^{-1}(x) = \frac{1}{2} \cdot \frac{e^{kx} - 1}{e^{kx} + 1}$ .

The area of the regions bound by the functions  $h$  and  $h^{-1}$  can be expressed as a function,  $A(k)$ . Determine the range of values of  $k$  such that  $A(k) > 0$ .

**Marking Guide** — Answer:  $k \text{ e1}$  (i.e.  $k \in (0, 1) \cup (1, \infty)$ )

- When  $k = 1$ ,  $h = f$  and  $h^{-1} = f^{-1}$ . Since  $f$  is odd and passes through the origin,  $h$  and  $h^{-1}$  are reflections in  $y = x$  and the bounded regions cancel to zero. For  $k \neq 1$ , the functions are distinct from each other (not symmetric about  $y = x$ ), so  $A(k) > 0$ . Range:  $k \in (0, 1) \cup (1, \infty)$ .

**Question 4e.ii***1 mark*

This question has been redacted following the findings of the Independent Review into the VCAA's Examination-Setting Policies, Processes and Procedures for the VCE.

**Marking Guide** — Answer: Redacted

- This question was redacted by VCAA. All students were awarded this mark.

**Question 5a***1 mark*

Consider the composite function  $g(x) = f(\sin(2x))$ , where the function  $f(x)$  is an unknown but differentiable function for all values of  $x$ .

Use the following table of values for  $f$  and  $f'$ :

$  x  $	$\frac{1}{2}  $	$\frac{\sqrt{2}}{2}  $	$\frac{\sqrt{3}}{2}  $	$  -   -   -   -  $	$  f(x)  $	$-2  $	$5  $	$3  $	$  f'(x)  $	$7  $	$0  $	$\frac{1}{9}  $
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Find the value of  $g\left(\frac{\pi}{6}\right)$ .

**Marking Guide** — Answer: 3

- $g\left(\frac{\pi}{6}\right) = f\left(\sin\left(\frac{\pi}{3}\right)\right) = f\left(\frac{\sqrt{3}}{2}\right) = 3.$

**Question 5b***1 mark*

The derivative of  $g$  with respect to  $x$  is given by  $g'(x) = 2 \cos(2x) \cdot f'(\sin(2x))$ .

Show that  $g'\left(\frac{\pi}{6}\right) = \frac{1}{9}$ .

**Marking Guide** — Answer: See marking guide

- $g'\left(\frac{\pi}{6}\right) = 2 \cos\left(\frac{\pi}{3}\right) \cdot f'\left(\sin\left(\frac{\pi}{3}\right)\right) = 2 \times \frac{1}{2} \times f'\left(\frac{\sqrt{3}}{2}\right) = 1 \times \frac{1}{9} = \frac{1}{9}.$

**Question 5c***2 marks*

Find the equation of the tangent to  $g$  at  $x = \frac{\pi}{6}$ .

**Marking Guide** — Answer:  $y = \frac{1}{9}x - \frac{\pi}{54} + 3$  or  $y = \frac{1}{9}(x - \frac{\pi}{6}) + 3$

- M1: Point:  $(\frac{\pi}{6}, 3)$ , gradient:  $\frac{1}{9}$ .
- A1:  $y - 3 = \frac{1}{9}(x - \frac{\pi}{6})$ .

**Question 5d***2 marks*

Find the average value of the derivative function  $g'(x)$  between  $x = \frac{\pi}{8}$  and  $x = \frac{\pi}{6}$ .

**Marking Guide** — Answer:  $\frac{-2}{\frac{\pi}{6} - \frac{\pi}{8}} = \frac{-2}{\frac{\pi}{24}} = \frac{-48}{\pi}$

- M1: Average value of  $g'(x) = \frac{1}{\frac{\pi}{6} - \frac{\pi}{8}} \int_{\pi/8}^{\pi/6} g'(x) dx = \frac{g(\pi/6) - g(\pi/8)}{\frac{\pi}{6} - \frac{\pi}{8}}$ .
- A1:  $g(\pi/8) = f(\sin(\pi/4)) = f(\frac{\sqrt{2}}{2}) = 5$ . Average =  $\frac{3-5}{\pi/24} = \frac{-2 \times 24}{\pi} = -\frac{48}{\pi}$ .

**Question 5e***3 marks*

Find four solutions to the equation  $g'(x) = 0$  for the interval  $x \in [0, \pi]$ .

**Marking Guide** — Answer:  $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{8}, \frac{3\pi}{8}$

- M1:  $g'(x) = 2 \cos(2x) \cdot f'(\sin(2x)) = 0$ . Either  $\cos(2x) = 0$  or  $f'(\sin(2x)) = 0$ .
- M1:  $\cos(2x) = 0 \Rightarrow 2x = \frac{\pi}{2}, \frac{3\pi}{2} \Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}$ . (Also  $2x = \frac{5\pi}{2}$  gives  $x = \frac{5\pi}{4} > \pi$ .)