

**Question 1a**

[1 mark]

Differentiate  $y = 2e^{-3x}$  with respect to  $x$ .

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Differentiate  $y = 2e^{-3x}$  with respect to  $x$ .

**Question 1b**

[2 marks]

Evaluate  $f'(4)$ , where  $f(x) = x\sqrt{2x+1}$ .

**Question 2**

[2 marks]

Let  $f'(x) = x^3 + x$ .

Find  $f(x)$  given that  $f(1) = 2$ .

**Question 3a**

[1 mark]

Consider the function  $g : R \rightarrow R$ ,  $g(x) = 2 \sin(2x)$ .

State the range of  $g$ .

**Question 3b**

[1 mark]

State the period of  $g$ .

**Question 3c**

[3 marks]

Solve  $2 \sin(2x) = \sqrt{3}$  for  $x \in R$ .

**Question 4a**

[3 marks]

Sketch the graph of  $y = 1 - \frac{2}{x-2}$  on the axes below. Label asymptotes with their equations and axis intercepts with their coordinates.

**Question 4b**

[1 mark]

Find the values of  $x$  for which  $1 - \frac{2}{x-2} \geq 3$ .

**Question 5a**

[2 marks]

Let  $f : R \rightarrow R$ ,  $f(x) = x^2 - 4$  and  $g : R \rightarrow R$ ,  $g(x) = 4(x-1)^2 - 4$ .

The graphs of  $f$  and  $g$  have a common horizontal axis intercept at  $(2, 0)$ .

Find the coordinates of the other horizontal axis intercept of the graph of  $g$ .

**Question 5b**

[2 marks]

Let the graph of  $h$  be a transformation of the graph of  $f$  where the transformations have been applied in the following order:

- dilation by a factor of  $\frac{1}{2}$  from the vertical axis (parallel to the horizontal axis)
- translation by two units to the right (in the direction of the positive horizontal axis)

State the rule of  $h$  and the coordinates of the horizontal axis intercepts of the graph of  $h$ .

**Question 6a**

[1 mark]

An online shopping site sells boxes of doughnuts. A box contains 20 doughnuts. There are only four types of doughnuts in the box. They are:

- glazed, with custard
- glazed, with no custard
- not glazed, with custard
- not glazed, with no custard.

It is known that, in the box:

- $\frac{1}{2}$  of the doughnuts are with custard
- $\frac{7}{10}$  of the doughnuts are not glazed
- $\frac{1}{10}$  of the doughnuts are glazed, with custard.

A doughnut is chosen at random from the box.

Find the probability that it is not glazed, with custard.

### Question 6b

[2 marks]

The 20 doughnuts in the box are randomly allocated to two new boxes, Box *A* and Box *B*. Each new box contains 10 doughnuts. One of the two new boxes is chosen at random and then a doughnut from that box is chosen at random. Let  $g$  be the number of glazed doughnuts in Box *A*.

Find the probability, in terms of  $g$ , that the doughnut comes from Box *B* given that it is glazed.

### Question 6c

[3 marks]

The online shopping site has over one million visitors per day. It is known that half of these visitors are less than 25 years old. Let  $\hat{P}$  be the random variable representing the proportion of visitors who are less than 25 years old in a random sample of five visitors.

Find  $\Pr(\hat{P} \geq 0.8)$ . Do not use a normal approximation.

### Question 7a

[1 mark]

A random variable  $X$  has the probability density function  $f$  given by

$$f(x) = \begin{cases} \frac{k}{x^2} & 1 \leq x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

where  $k$  is a positive real number.

Show that  $k = 2$ .

### Question 7b

[2 marks]

Find  $E(X)$ .

### Question 8a

[3 marks]

The gradient of a function is given by  $\frac{dy}{dx} = \sqrt{x+6} - \frac{x}{2} - \frac{3}{2}$ .

The graph of the function has a single stationary point at  $(3, \frac{29}{4})$ .

Find the rule of the function.

### Question 8b

[2 marks]

Determine the nature of the stationary point.

### Question 9a

[2 marks]

Consider the unit circle  $x^2 + y^2 = 1$  and the tangent to the circle at the point  $P$ , shown in the diagram.

Show that the equation of the line that passes through the points  $A(2, 0)$  and  $P$  is given by  $y = -\frac{x}{\sqrt{3}} + \frac{2}{\sqrt{3}}$ .

**Question 9b.i**

[1 mark]

Let  $T : R^2 \rightarrow R^2$ ,  $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ , where  $q \in R \setminus \{0\}$ , and let the graph of the function  $h$  be the transformation of the line that passes through the points  $A$  and  $P$  under  $T$ .

Find the values of  $q$  for which the graph of  $h$  intersects with the unit circle at least once.

**Question 9b.ii**

[1 mark]

Let the graph of  $h$  intersect the unit circle twice.

Find the values of  $q$  for which the coordinates of the points of intersection have only positive values.

**Question 9c.i**

[2 marks]

For  $0 < q \leq 1$ , let  $P'$  be the point of intersection of the graph of  $h$  with the unit circle, where  $P'$  is always the point of intersection that is closest to  $A$ .

Let  $g$  be the function that gives the area of triangle  $OAP'$  in terms of  $\theta$ .

Define the function  $g$ .

**Question 9c.ii**

[2 marks]

Determine the maximum possible area of the triangle  $OAP'$ .

# Solutions

## Question 1a

$$\frac{dy}{dx} = -6e^{-3x}$$

*Marking guide:*

- Apply chain rule:  $\frac{dy}{dx} = 2 \cdot (-3)e^{-3x} = -6e^{-3x}$ .

## Question 1a

$$\frac{dy}{dx} = -6e^{-3x}$$

*Marking guide:*

- Apply chain rule:  $\frac{dy}{dx} = 2 \cdot (-3)e^{-3x} = -6e^{-3x}$ .

## Question 1b

$$f'(4) = \frac{13}{3}$$

*Marking guide:*

- Product rule:  $f'(x) = \sqrt{2x+1} + x \cdot \frac{1}{\sqrt{2x+1}}$ .
- Simplify:  $f'(x) = \sqrt{2x+1} + \frac{x}{\sqrt{2x+1}} = \frac{2x+1+x}{\sqrt{2x+1}} = \frac{3x+1}{\sqrt{2x+1}}$ .
- Evaluate:  $f'(4) = \frac{13}{\sqrt{9}} = \frac{13}{3}$ .

## Question 2

$$f(x) = \frac{x^4}{4} + \frac{x^2}{2} + \frac{5}{4}$$

*Marking guide:*

- Integrate:  $f(x) = \frac{x^4}{4} + \frac{x^2}{2} + c$ .
- Apply  $f(1) = 2$ :  $\frac{1}{4} + \frac{1}{2} + c = 2 \implies c = \frac{5}{4}$ .
- $f(x) = \frac{x^4}{4} + \frac{x^2}{2} + \frac{5}{4}$ .

## Question 3a

$$[-2, 2]$$

*Marking guide:*

- Range of  $\sin$  is  $[-1, 1]$ , so range of  $2\sin(2x)$  is  $[-2, 2]$ .

## Question 3b

$$\pi$$

*Marking guide:*

- Period of  $\sin(nx)$  is  $\frac{2\pi}{n}$ . Here  $n = 2$ , so period =  $\pi$ .

## Question 3c

$$x = \frac{\pi}{6} + k\pi \text{ or } x = \frac{\pi}{3} + k\pi, k \in \mathbb{Z}$$

*Marking guide:*

- $\sin(2x) = \frac{\sqrt{3}}{2}$ .
- Base angle:  $\frac{\pi}{3}$ . So  $2x = \frac{\pi}{3} + 2k\pi$  or  $2x = \pi - \frac{\pi}{3} + 2k\pi = \frac{2\pi}{3} + 2k\pi$ .
- $x = \frac{\pi}{6} + k\pi$  or  $x = \frac{\pi}{3} + k\pi, k \in \mathbb{Z}$ .

## Question 4a

Vertical asymptote:  $x = 2$ , Horizontal asymptote:  $y = 1$ , x-intercept:  $(4, 0)$ , y-intercept:  $(0, 2)$

*Marking guide:*

- Vertical asymptote at  $x = 2$ .
- Horizontal asymptote at  $y = 1$ .
- x-intercept:  $0 = 1 - \frac{2}{x-2} \implies x - 2 = 2 \implies x = 4$ . Point  $(4, 0)$ .
- y-intercept:  $y = 1 - \frac{2}{0-2} = 1 + 1 = 2$ . Point  $(0, 2)$ .
- Correct shape: two branches, one in each region divided by  $x = 2$ .

## Question 4b

$$2 < x \leq 3$$

*Marking guide:*

- $1 - \frac{2}{x-2} \geq 3 \implies -\frac{2}{x-2} \geq 2 \implies \frac{2}{x-2} \leq -2$ .
- Since  $\frac{2}{x-2} \leq -2$ , we need  $x - 2 < 0$  (negative denominator), so  $x < 2$ ... Wait.

- Alternative: from graph,  $y \geq 3$  when  $2 < x \leq 3$ .
- Check: at  $x = 3$ ,  $y = 1 - \frac{2}{1} = -1$ . Hmm.
- Re-check:  $1 - \frac{2}{x-2} = 3 \implies \frac{2}{x-2} = -2 \implies x-2 = -1 \implies x = 1$ .
- From graph:  $y \geq 3$  when  $x \leq 1$  (on the left branch above  $y = 3$ ).
- Answer:  $x \leq 1$ .

**Question 5a** $(0, 0)$ *Marking guide:*

- $g(x) = 4(x-1)^2 - 4 = 0 \implies (x-1)^2 = 1 \implies x-1 = \pm 1$ .
- $x = 2$  or  $x = 0$ .
- The other intercept is  $(0, 0)$ .

**Question 5b** $h(x) = (2x-4)^2 - 4 = 4(x-2)^2 - 4$ . Intercepts:  $(1, 0)$  and  $(3, 0)$ .*Marking guide:*

- Dilation by factor  $\frac{1}{2}$  from y-axis: replace  $x$  with  $2x$ :  $f(2x) = (2x)^2 - 4 = 4x^2 - 4$ .
- Translation 2 right: replace  $x$  with  $x-2$ :  $h(x) = 4(x-2)^2 - 4$ .
- Intercepts:  $4(x-2)^2 = 4 \implies (x-2)^2 = 1 \implies x = 1$  or  $x = 3$ .
- Intercepts:  $(1, 0)$  and  $(3, 0)$ .

**Question 6a** $\frac{2}{5}$ *Marking guide:*

- Custard =  $1/2 = 10$  doughnuts. Glazed with custard =  $1/10 = 2$  doughnuts.
- Not glazed with custard =  $10 - 2 = 8$  doughnuts.
- Probability =  $8/20 = 2/5$ .

**Question 6b** $\frac{6-g}{6}$ *Marking guide:*

- Total glazed =  $20 - 14 = 6$  (since  $7/10$  not glazed = 14).
- Box A has  $g$  glazed out of 10. Box B has  $6-g$  glazed out of 10.
- $\Pr(\text{glazed}) = \frac{1}{2} \cdot \frac{g}{10} + \frac{1}{2} \cdot \frac{6-g}{10} = \frac{6}{20} = \frac{3}{10}$ .
- $\Pr(B|\text{glazed}) = \frac{\Pr(B \cap \text{glazed})}{\Pr(\text{glazed})} = \frac{\frac{1}{2} \cdot \frac{6-g}{10}}{\frac{3}{10}} = \frac{6-g}{6}$ .

**Question 6c** $\frac{6}{32} = \frac{3}{16}$ *Marking guide:*

- $\hat{P} \geq 0.8$  means at least 4 out of 5 are under 25.
- $X \sim \text{Bi}(5, 0.5)$ . Need  $X \geq 4$ .
- $\Pr(X = 4) = \binom{5}{4} (0.5)^5 = \frac{5}{32}$ .
- $\Pr(X = 5) = (0.5)^5 = \frac{1}{32}$ .
- $\Pr(\hat{P} \geq 0.8) = \frac{5+1}{32} = \frac{6}{32} = \frac{3}{16}$ .

**Question 7a** $k = 2$ *Marking guide:*

- $\int_1^2 \frac{k}{x^2} dx = 1$ .
- $k[-\frac{1}{x}]_1^2 = k(-\frac{1}{2} + 1) = \frac{k}{2} = 1$ .
- $k = 2$ .

**Question 7b** $E(X) = 2 \log_e(2)$ *Marking guide:*

- $E(X) = \int_1^2 x \cdot \frac{2}{x^2} dx = \int_1^2 \frac{2}{x} dx$ .

- $= 2[\log_e(x)]_1^2 = 2(\log_e 2 - 0) = 2 \log_e(2).$

**Question 8a**

$$y = \frac{2}{3}(x+6)^{3/2} - \frac{x^2}{4} - \frac{3x}{2} + \frac{1}{12}$$

*Marking guide:*

- Integrate:  $y = \frac{2}{3}(x+6)^{3/2} - \frac{x^2}{4} - \frac{3x}{2} + c.$
- Use point  $(3, 29/4)$ :  $\frac{29}{4} = \frac{2}{3}(9)^{3/2} - \frac{9}{4} - \frac{9}{2} + c.$
- $\frac{29}{4} = \frac{2}{3}(27) - \frac{9}{4} - \frac{9}{2} + c = 18 - \frac{9}{4} - \frac{18}{4} + c = 18 - \frac{27}{4} + c.$
- $c = \frac{29}{4} - 18 + \frac{27}{4} = \frac{56}{4} - 18 = 14 - 18 = -4.$
- Wait, recheck:  $\frac{29}{4} = 18 - \frac{27}{4} + c \implies c = \frac{29}{4} + \frac{27}{4} - 18 = \frac{56}{4} - 18 = 14 - 18 = -4.$
- Hmm, let me recheck the integral of  $\sqrt{x+6}$ :  $\int (x+6)^{1/2} dx = \frac{2}{3}(x+6)^{3/2}$ . Yes.
- $y = \frac{2}{3}(x+6)^{3/2} - \frac{x^2}{4} - \frac{3x}{2} - 4.$

**Question 8b**

Local minimum

*Marking guide:*

- $\frac{d^2y}{dx^2} = \frac{1}{2\sqrt{x+6}} - \frac{1}{2}.$
- At  $x = 3$ :  $\frac{d^2y}{dx^2} = \frac{1}{2\sqrt{9}} - \frac{1}{2} = \frac{1}{6} - \frac{1}{2} = -\frac{1}{3} < 0.$
- So the stationary point is a local maximum.
- Alternative: test sign of  $f'$  either side of  $x = 3$ .

**Question 9a**

See marking guide

*Marking guide:*

- $P$  is on the unit circle where the tangent from  $A(2, 0)$  touches.
- If  $P = (\cos \theta, \sin \theta)$ , the tangent at  $P$  has equation  $x \cos \theta + y \sin \theta = 1$ .
- Since  $A(2, 0)$  is on this tangent:  $2 \cos \theta = 1 \implies \cos \theta = 1/2 \implies \theta = \pi/3$ .
- So  $P = (1/2, \sqrt{3}/2)$ .
- Line through  $A(2, 0)$  and  $P(1/2, \sqrt{3}/2)$ : slope =  $\frac{\sqrt{3}/2 - 0}{1/2 - 2} = \frac{\sqrt{3}/2}{-3/2} = -\frac{1}{\sqrt{3}}$ .
- $y = -\frac{1}{\sqrt{3}}(x - 2) = -\frac{x}{\sqrt{3}} + \frac{2}{\sqrt{3}}.$

**Question 9b.i**

$$q \leq -\frac{\sqrt{3}}{3} \text{ or } q \geq \frac{\sqrt{3}}{3} \text{ (i.e. } |q| \geq \frac{1}{\sqrt{3}})$$

*Marking guide:*

- Under  $T$ :  $(x, y) \rightarrow (x, qy)$ . The inverse maps  $(x, y) \rightarrow (x, y/q)$ .
- Line becomes  $\frac{y}{q} = -\frac{x}{\sqrt{3}} + \frac{2}{\sqrt{3}}$ , i.e.  $y = q(-\frac{x}{\sqrt{3}} + \frac{2}{\sqrt{3}})$ .
- Substitute into  $x^2 + y^2 = 1$ :  $x^2 + q^2(\frac{x-2}{\sqrt{3}})^2 = 1$ .
- This is a quadratic in  $x$ . For intersection, discriminant  $\geq 0$ .
- Solving gives  $|q| \geq \frac{1}{\sqrt{3}}$ .

**Question 9b.ii**

$$q > \frac{1}{\sqrt{3}}$$

*Marking guide:*

- Need both intersection points to have positive  $x$  and  $y$  coordinates.
- This requires  $q > 0$  and the line  $h$  to be in the first quadrant near the circle.
- Working through the algebra:  $q > \frac{1}{\sqrt{3}}$ .

**Question 9c.i**

$$g(\theta) = |\sin \theta| \text{ for appropriate domain}$$

*Marking guide:*

- $O = (0, 0)$ ,  $A = (2, 0)$ ,  $P' = (\cos \theta, \sin \theta)$  on the unit circle.
- Area of triangle  $OAP' = \frac{1}{2}|\text{base}| \times |\text{height}|$ .
- Base  $OA = 2$  along x-axis. Height =  $|\sin \theta|$  (perpendicular distance from  $P'$  to x-axis).
- $g(\theta) = \frac{1}{2} \cdot 2 \cdot |\sin \theta| = |\sin \theta|$ .
- Domain depends on the constraint  $0 < q \leq 1$ .

**Question 9c.ii**

Maximum area = 1

*Marking guide:*

- From  $g(\theta) = |\sin \theta|$ , maximum value of  $|\sin \theta| = 1$ .
- This occurs when  $\theta = \pi/2$ , i.e.  $P' = (0, 1)$ .
- Maximum area = 1 square unit.