THE EGG GAME

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ABSTRACT. We present a game and proofs for an optimal solution.

1. THE GAME

You are presented with a building and some number of SUPERSTRONG eggs. These eggs can withstand drops from some height; you must figure out just what this height is. You are told that there is some floor, f_0, such that the eggs can survive the drop from f_0 but will break when dropped from floor $f_0 + 1$. f_0 may be the top of the building—that is, the eggs may not break at all. (Note that in each game all the eggs you are given have the same SUPERSTRENGTH, i.e., if one breaks when dropped from a certain height, they all will.)

You want to determine the value of f_0 as efficiently as possible, that is, by making the fewest number of droppings. Of course, you *must* guarantee that you find f_0 before you run out of eggs, so any strategy must take this into account.

As an example, take a 1 egg, 100 story game, which we call a (1,100) game. You have no choice but to start dropping the egg from the first floor, and proceeding in sequence until the egg breaks. This can take as many as 100 drops.

A more interesting example, is the (2,100) game—that is, a 2 egg, 100 story game. Before reading on, try to think of a good strategy to find the break floor. How well can you do?

Many people suggest a strategy seemingly inspired by binary search: do the first drop from the middle floor and cut your search space in half. Since you only have two eggs, it is easy to see that this can solve the (2,100) game in 51 drops. Two eggs is not enough to really execute a binary search. If you have 6 eggs you can do binary search and find the floor in 6 drops, but this is a much less interesting case. An improvement on this is to start by dropping off of every tenth floor. When the egg breaks you only have nine more drops to make. With this strategy it will take no more than 19 drops. This looks very good. Can you do better? If you use the method of equally spaced intervals then you cannot. However, the method of breaking the number of stories up into evenly spaced intervals turns out to be the wrong way to try to think of an optimal solution.

On next page is the optimal algorithm.

We present a 2-egg, 100-floor solution that takes at most 14 drops. To present it we say what floor you drop it off of if it DOES NOT break. We later see what you would do if it does break.

(1) First Drop: $d_1 = 1 + 13 = 14$, use 14th floor. If does not break, break-floor is in $[15, 100]$.
(2) Second Drop: $d_2 = 15 + 12 = 27$, use 27th floor. If does not break, break-floor is in $[28, 100]$.
(3) Third Drop: $d_3 = 28 + 11 = 39$, use 39th floor. If does not break, break-floor is in $[40, 100]$.
(4) Fourth Drop: $d_4 = 40 + 10 = 50$, use 50th floor. If does not break, break-floor is in $[51, 100]$.
(5) Fifth Drop: $d_5 = 51 + 9 = 60$, use 60th floor. If does not break, break-floor is in $[61, 100]$.
(6) Sixth Drop: $d_6 = 61 + 8 = 69$, use 69th floor. If does not break, break-floor is in $[70, 100]$.
(7) Seventh Drop: $d_7 = 70 + 7 = 77$, use 77th floor. If does not break, break-floor is in $[78, 100]$.
(8) Eighth Drop: $d_8 = 70 + 7 = 77$, use 77th floor. If does not break, break-floor is in $[78, 100]$.
(9) Ninth Drop: $d_9 = 78 + 6 = 84$, use 84th floor. If does not break, break-floor is in $[85, 100]$.
(10) Tenth Drop: $d_{10} = 85 + 5 = 90$, use 90th floor. If does not break, break-floor is in $[91, 100]$.
(11) Eleventh Drop: $d_{11} = 91 + 4 = 95$, use 95th floor. If does not break, break-floor is in $[96, 100]$.
(12) Twelveth Drop: $d_{12} = 96 + 3 = 99$, use 99th floor. If does not break, break-floor is in $[100, 100]$.
(13) Thirteenth Drop: Drop off $d_{13} = 100$th floor.

If the egg never breaks then clearly we use 13 drops, and we conclude that there is no break-floor. If the egg only breaks on the very last dropping then we still use just 13 breaks, and we conclude that the 100 is the break floor. But what if the egg does break?

(1) First Drop: If the egg breaks on the first drop then we know that the break floor is in $[1, 14]$. We also know that at 14 it does break. Hence we need only drop it off of floor $1, 2, \ldots, 13$. Thats 13 drops, in addition to the original drop, so 14 drops total.
(2) Second Drop: If the egg breaks on the second drop then we know that the break floor is in $[15, 27]$. We also know that at 27 it does break. Hence we need only drop it off of floor $15, 16, \ldots, 26$. Thats 12 drops, in addition to the original two drops,
(3) Third Drop: If the egg breaks on the third drop then we know that the break floor is in $[28, 39]$. We also know that at 39 it does break. Hence we need only drop it off of floor $28, 29, \ldots, 38$. Thats 11 drops, in addition to the original three drops, so 14 drops total.
(4) Fourth Drop and beyond: Fill it in yourself.

Note that this approach takes 14 drops in the worse case. This is better than the 19-drop approach.

2. Preliminaries

We will let $D(e, s)$ be the least number of drops it takes to guarantee finding the floor at which an egg breaks, given e eggs and an s-story building.

Let $FM(e, s)$ be a *First Move* in an optimal strategy for the $D(e, s)$ game.

Let t_i denote the ith triangle number ($t_i = \sum_{k=1}^{i} k$).

We use $\lg s$ to denote the logarithm base 2 of s.

Note that, in general, when $e \geq \lceil \lg s \rceil$ it is straightforward to find $D(e, s)$ in $\lfloor \lg s \rfloor + 1$ steps by (essentially) executing binary search. The number of eggs is not a constraint in this case. So the interesting cases are when $e < \lceil \lg s \rceil$.

Something else to keep in mind is that a $D(e, s)$ game is reduced one of two ways on a drop: the egg either breaks or it does not break. Suppose the move is to drop from floor j. If the egg breaks you are reduced to $D(e - 1, j - 1)$ since you are left with one less egg (for $e - 1$ total eggs) and you only have to check the $j - 1$ floors that are lower than j. That is, since the egg broke when dropped from the jth floor, you know it will break for all i, $j < i \leq s$; you no longer care about these floors. If the egg does not break you are reduced to $D(e, s - j)$ since you have lost no eggs and you are now only concerned with floors greater than j, of which there are $s - j$.

3. The one egg game

Claim 1. *For the one egg game, $D(1, s) = s$, with $FM(1, s) = 1$.*

This is very simple, but must be pointed out. If you have only one egg and s stories, you can execute linear search, dropping from floor $1, 2, 3, \ldots$ until the egg breaks or until you reach the sth floor. So s drops are sufficient to guarantee finding the break floor.

To show that s is also a lower bound on the number of drops needed to find the break floor, suppose you make fewer than s drops. Your first move can either be 1 or > 1. If the first move is > 1 it is clear that an adversary could always force a situation where the break floor cannot be determined at all: if the egg breaks on the first drop you cannot say whether the break floor is the floor you dropped from or a lower floor. So $FM(1, s) = 1$ is the more interesting case. In this case, since only $s - 1$ drops are made, an adversary can also force a situation where the break floor cannot be determined. As soon as a floor is skipped the adversary forces a break on the next drop. You do not know whether *that* is the break floor or the floor just skipped is the break floor. The situation where no floors are skipped, that is, the $s - 1$ drops are made from $1, 2, 3, \ldots, s - 1$ is also clear: you do not know whether the egg breaks from the sth floor or if it does not break at all (which is a possibility in how we have defined the game). So s drops is also a lower bound.

4. Bounds on the two egg game

Claim 2. *For the two egg game with $t_{i-1} < s \le t_i$, $D(2, s) = i$, with $FM(2, s) = i$.*

Proof of the upper bound. We prove the claim by strong induction on s, the number of stories.

The base case is to consider $t_1 = 1 < s \le t_2 = 3$ (i.e., $i = 2$).

Let $s = 2$. Determine $D(2, 2)$. For first drop from floor 2 there are two cases.

    i. Egg breaks, reduced to $D(1, 1)$ game. $D(1, 1) = 1$, resulting in 2 drops total.
    ii. Egg does not break, reduced to $D(2, 0)$ game, which is solved in 0 drops for a total of 1 drop.

So 2 drops suffices.

Let $s = 3$. Determine $D(2, 3)$. Taking $FM(2, 3) = 2$ there are again two cases.

    i. Egg breaks, reduced to $D(1, 1)$ game. $D(1, 1) = 1$, so total drops is 2.
    ii. Egg does not break, reduced to $D(2, 1)$ game. This takes 1 drop, for a total of 2 drops.

So 2 drops suffices here as well.

Now suppose for $s > 3$ (with $t_{i-1} < s \le t_i$) that $D(2, s) = i$ with $FM(2, s) = i$. We will show that $i + 1$ drops suffices for $t_i < s \le t_{i+1}$ (with $FM(2, s) = i + 1$).

Take s so that $t_i < s \le t_{i+1}$ and use $FM(2, s) = i + 1$ as the strategy. There are two cases to consider.

    i. If the egg breaks we are reduced to the $D(1, i)$ game, which can be solved in i drops. The total number of drops is $i + 1$.
    ii. If the egg does not break we are reduced to the $D(2, s - (i + 1))$ game. Let $s' = s - i - 1$ so we are in a $D(2, s')$ game, with this bound on s':

$$t_i < s \le t_{i+1}$$
$$t_i - (i + 1) < s - (i + 1) \le t_{i+1} - (i + 1)$$
$$t_{i-1} - 1 < s' \le t_i$$
$$t_{i-1} \le s' \le t_i \qquad \textit{Note the change in inequality}$$

By the inductive hypothesis, $D(2, s')$ is solvable in i drops (or $i - 1$ if $s' = t_{i-1}$). So $i + 1$ drops suffices.

In either case $e + 1$ drops suffices. $\square$

Lets compute a few of these numbers

$$t_1 = 1$$
$$t_2 = 1 + 2 = 3$$
$$t_3 = 1 + 2 + 3 = 6$$
$$t_4 = 1 + 2 + 3 + 4 = 10$$
$$t_5 = 1 + 2 + 3 + 4 + 5 = 15$$
$$t_6 = 1 + 2 + 3 + 4 + 5 + 6 = 21$$
$$t_7 = 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$$
$$t_8 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$$
$$t_9 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$$
$$t_{10} = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$$
$$t_{11} = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 = 66$$
$$t_{12} = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 = 78$$
$$t_{13} = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 = 91$$
$$t_{14} = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 = 105$$

Hence, if you have two eggs we have the following.

(1) Since $t_1 = 1$ and $t_2 = 3$, if $2 \le s \le 3$ then 2 drops suffice.
(2) Since $t_2 = 3$ and $t_3 = 6$, if $4 \le s \le 6$ then 3 drops suffice.
(3) Since $t_3 = 6$ and $t_4 = 10$, if $7 \le s \le 10$ then 4 drops suffice.
(4) Since $t_4 = 10$ and $t_5 = 15$, if $11 \le s \le 15$ then 5 drops suffice.
(5) Since $t_5 = 15$ and $t_6 = 21$, if $16 \le s \le 21$ then 6 drops suffice.
(6) Since $t_6 = 21$ and $t_7 = 28$, if $22 \le s \le 28$ then 7 drops suffice.
(7) Since $t_7 = 28$ and $t_8 = 36$, if $29 \le s \le 36$ then 8 drops suffice.
(8) Since $t_8 = 36$ and $t_9 = 45$, if $37 \le s \le 45$ then 9 drops suffice.
(9) Since $t_9 = 45$ and $t_{10} = 55$, if $46 \le s \le 55$ then 10 drops suffice.
(10) Since $t_{10} = 55$ and $t_{11} = 66$, if $56 \le s \le 66$ then 11 drops suffice.
(11) Since $t_{11} = 66$ and $t_{12} = 78$, if $67 \le s \le 78$ then 12 drops suffice.
(12) Since $t_{12} = 78$ and $t_{13} = 91$, if $79 \le s \le 91$ then 13 drops suffice.
(13) Since $t_{13}2 = 91$ and $t_{14} = 105$, if $92 \le s \le 105$ then 14 drops suffice.

If we have s stories is there some good approximation for the number of drops that are required? We need i such that

$1 + 2 + \cdots + (i - 1) < s \le 1 + 2 + \cdots + i$.

$\frac{i(i-1)}{2} < s \le \frac{(i+1)i}{2}$.

$\frac{i^2}{2} - \frac{i}{2} \le \frac{i^2}{2} + \frac{i}{2}$.

Note that the $\frac{i^2}{2}$ term is the important one. We approximate by ignoring the $\frac{i}{2}$ term. Hence we have that

$\frac{i^2}{2} = s$

$i^2 = 2s$

$i = \sqrt{2s}$.

SO if you have two eggs and s stories the number of drops is roughly $\sqrt{2s}$.

5. 3 EGGS

For three eggs, what we'd like is to follow the same basic strategy as for 2-eggs: regardless of when it breaks, we want the total number of drops to remain the same. But this means that if it breaks on drop $i + 1$, we want the subproblem with two eggs on the remaining unknown floors to take one less drop than the subproblem if it broke on drop i. So how do we make sure we decrement the number required for 2 eggs each time we increase?

Let's look at the pattern from before: 6 unknown floors (2 eggs) requires 3 drops, 10 unknown floors requires 4 drops, 15 unknown floors requires 5 drops, etc. Lets begin our 3 egg problem at floor 36. If it breaks, we require 8 more drops with our 2 remaining eggs to discover the floor. If it does not, then we add 28 floors and drop from floor 64. If it breaks we require only 7 drops with the remaining eggs (do you see why?). If not, we move up another 21 floors, then 15, then 10 etc. The total number of drops (regardless of when the first egg breaks) is 9.

So the final solution for 3 eggs (100 floors) is as follows:

Drop 1: floor 36
Drop 2: floor 36 + 28 = 64
Drop 3: floor 64 + 21 = 85
Drop 4: floor 85 + 15 = 100

When it breaks, we simply play the 2 egg game with the remaining unknown floors. For example, if it broke on floor 64, we'd have two eggs and 28 floors. We can do this in 7 drops. We'd simply start at 43 (36 + 7), then drop from 49 (42 + 6), 54, 58, 61, 63 and then 64.