A Mode-Aware Tucker Tensor Network for Learning Accurate Representation of High-Dimensional Incomplete and Unbalanced Tensor Supplementary File

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I. Introduction

This is the supplementary file for paper entitled *A Mode-Aware Tucker Tensor Network for Learning Accurate Representation of High-Dimensional Incomplete and Unbalanced Tensor*. Supplementary equations and experimental results are put into this file.

II. Supplementary proof of convergence

First, we review the formula for MSGD and rewrite it to separate the deterministic gradient part from the random noise part.

$$\begin{aligned} & v_{n+1} = \gamma v_n + \eta \nabla_{\omega_n} J\left(\omega_n\right) + \eta\left(\nabla_{\omega_n} J\left(\omega_n, \xi_n\right) - \nabla_{\omega_n} J\left(\omega_n\right)\right), \\ & \omega_{n+1} = \omega_n - v_{n+1}. \end{aligned} \tag{S1}$$

This transformation borrows the idea of random approximation. Then calculate the following Taylor expansion and do some processing to get:

$$J\left(\omega_{n+1}\right) - J\left(\omega_n\right) = -\nabla_{\omega} J\left(\omega_n\right)^T v_n + \frac{1}{2} v_n^T H_{\omega\omega}\left(\zeta_n\right) v_n. \tag{S2}$$

Using the above Taylor formula, we can make the following calculation:

$$\begin{aligned} & \nabla_{\omega_n} J\left(\omega_n\right)^T v_n \\ & = \left(\nabla_{\omega_{n-1}-v_{n-1}} J\left(\omega_{n-1} - v_{n-1}\right)\right)^T \left(\gamma v_{n-1} + \eta_n \nabla_{\omega_n} J\left(\omega_n, \xi_n\right)\right) \\ & = \gamma\left(\nabla_{\omega_{n-1}} J\left(\omega_{n-1}\right) - H_{\omega\omega}\left(\zeta_{n-1}\right) v_{n-1}\right)^T v_{n-1} + \eta_n \nabla_{\omega_n} J\left(\omega_n\right) \nabla_{\omega_n} J\left(\omega_n, \xi_n\right) \\ & = \gamma \nabla_{\omega_{n-1}} J\left(\omega_{n-1}\right)^T v_{n-1} - \gamma v_{n-1}^T H_{\omega\omega}\left(\zeta_{n-1}\right) v_{n-1} + \eta_n \nabla_{\omega_n} J\left(\omega_n\right) \nabla_{\omega_n} J\left(\omega_n, \xi_n\right). \end{aligned} \tag{S3}$$

Next, processing the above recursive formula, we get:

$$\nabla_{\omega_n} J\left(\omega_n\right)^T v_n = \gamma \nabla_{\omega_1} J\left(\omega_1\right)^T v_1 - \sum_{i=1}^{n-1} \gamma v_i^T H_{\omega\omega}\left(\zeta_i\right) v_i + \sum_{i=2}^{t} \gamma \eta_i \nabla_{\omega_i} J\left(\omega_i\right) \nabla_{\omega_i} J\left(\omega_i, \zeta_i\right). \tag{S4}$$

Then, bring (S4) back to (S2),

$$J\left(\omega_{n+1}\right) - J\left(\omega_n\right) = -\gamma \nabla_{\omega_1} J\left(\omega_1\right)^T v_1 - \sum_{i=2}^{t} \gamma \eta_i \nabla_{\omega_i} J\left(\omega_i\right) \nabla_{\omega_i} J\left(\omega_i, \xi_i\right) + \sum_{i=1}^{n-1} \gamma v_i^T H_{\omega\omega}\left(\zeta_i\right) v_i + \frac{1}{2} v_n^T H_{\omega\omega}\left(\zeta_n\right) v_n. \tag{S5}$$

Next, we will show that $J(\omega_{n+1})$ is almost certainly convergent. We do this by recursing on (S5) and dividing it into three parts:

$$\begin{aligned} J(\omega_{t+1}) = & \underbrace{J(\omega_1) - \frac{\gamma-\gamma^{t+1}}{1-\gamma}\nabla_{\omega_1} J(\omega_1)^T v_1 + \frac{1-\gamma^t}{1-\gamma}\eta_1 \nabla_{\omega_1} J(\omega_1) \nabla_{\omega_1} J(\omega_1, \xi_1)}_{A} \\ & \underbrace{-\sum_{n=1}^{t} \frac{1-\gamma^{t-n+1}}{1-\gamma}\eta_n \nabla_{\omega_n} J(\omega_n)^T \nabla_{\omega_n} J(\omega_n, \xi_n)}_{B} \\ & \underbrace{-\frac{1}{2}\sum_{n=1}^{t} v_n^T H_{\omega\omega}(\zeta_n) v_n + \sum_{n=1}^{t} \frac{1-\gamma^{t-n+1}}{1-\gamma} v_n^T H_{\omega\omega}(\zeta_n) v_n}_{C}. \end{aligned} \tag{S6}$$

Because a<1, a^n is convergent, which ensures that part A is converged. Then, Lemma 3 ensures that part C converges almost everywhere. For Part B, we have the following proof:

$$\begin{aligned} B = & \sum_{n=1}^{t} \frac{1-\gamma^{t-n+1}}{1-\gamma}\eta_n \nabla_{\omega_n} J(\omega_n)^T \nabla_{\omega_n} J(\omega_n, \xi_n) \\ = & \sum_{n=1}^{t} \frac{1-\gamma^{t-n+1}}{1-\gamma}\eta_n \left\|\nabla_{\omega_n} J(\omega_n)\right\|^2 \\ & + \sum_{n=1}^{t} \frac{1-\gamma^{t-n+1}}{1-\gamma}\eta_n \nabla_{\omega_n} J(\omega_n)^T \left(\nabla_{\omega_n} J(\omega_n, \xi_n) - \nabla_{\omega_n} J(\omega_n, \xi_n)\right). \end{aligned} \tag{S7}$$

By Lemma 4, we have:

$$\sum_{n=1}^{t} \frac{1-\gamma^{t-n+1}}{1-\gamma}\eta_n \left\|\nabla_{\omega_n} J(\omega_n)\right\|^2 < +\infty. \tag{S8}$$

According to Lemma 1 and Lemma 4, we can conclude

$$\sum_{n=1}^{t-1} \frac{1-\gamma^{t-n}}{1-\gamma}\eta_{n+1} \nabla_{\omega_n} J(\omega_n)^T \left(\nabla_{\omega_n} J(\omega_n, \xi_n) - \nabla_{\omega_n} J(\omega_n, \xi_n)\right). \tag{S9}$$

The above formula is convergent. So B is convergent, and $g(\omega_t+1)$ is also convergent. Substituting (S7) into (S6) we get:

$$J(\omega_{t+1}) = \zeta_t - \sum_{n=1}^{t} \eta_n \left\|\nabla_{\omega_n} J(\omega_n)\right\|^2, \tag{S10}$$

where $\{\zeta_t\}$ is defined as follows:

$$\begin{aligned} \zeta_t = & J(\omega_1) - \frac{\gamma-\gamma^{t+1}}{1-\gamma}\nabla_{\omega_1} J(\omega_1)^T v_1 + \frac{1-\gamma^t}{1-\gamma}\eta_1 \nabla_{\omega_1} J(\omega_1) \nabla_{\omega_1} J(\omega_1, \xi_1) \\ & + \sum_{n=1}^{t} \frac{1-\gamma^{t-n+1}}{1-\gamma}\eta_n \nabla_{\omega_n} J(\omega_n)^T \left(\nabla_{\omega_n} J(\omega_n, \xi_n) - \nabla_{\omega_n} J(\omega_n)\right) \\ & - \frac{1}{2}\sum_{n=1}^{t} v_n^T H_{\omega\omega}(\zeta_n) v_n + \sum_{n=1}^{t} \frac{1-\gamma^{t-n+1}}{1-\gamma} v_n^T H_{\omega\omega}(\zeta_n) v_n. \end{aligned} \tag{S11}$$

Since $\{\zeta_t\}$ almost certainly converges, Lemma 2 shows that,

$$\omega_t \to \omega^*. \tag{S12}$$

III. SUPPLEMENTARY TABLES

Here are some supplementary tables in the Experimental section.

TABLE S1

HYPER-PARAMETER SETTINGS OF M1-10 ON D1-8

Datasets	Hyper-parameter settings				
D1	M1: Adaptive	M2: $\eta=2^{-7}$, $\lambda=2^{-6}$	M3: $\eta=2^{-10}$, $\lambda=2^{-8}$	M4: $\eta=2^{-7}$, $\lambda=2^{-8}$	M5: $P=1.9$, $\lambda=2^{-6}$
	M6: $\lambda_1=2^{-10}$, $\lambda_2=2^{-9}$	M7: $\lambda_1=2^{-6}$, $\lambda_2=2^{-2}$	M8: $\lambda_1=2^{-3}$, $\lambda_2=2^{-4}$	M9: $\eta=2^{-9}$, $\lambda=2^{-10}$	M10: $\alpha=2^{-0}$, $\beta=2^{-8}$, $\lambda=2^{-7}$
D2	M1: Adaptive	M2: $\eta=2^{-6}$, $\lambda=2^{-6}$	M3: $\eta=2^{-10}$, $\lambda=2^{-10}$	M4: $\eta=2^{-8}$, $\lambda=2^{-6}$	M5: $P=1.9$, $\lambda=2^{-5}$
	M6: $\lambda_1=2^{-8}$, $\lambda_2=2^{-11}$	M7: $\lambda_1=2^{-3}$, $\lambda_2=2^{-2}$	M8: $\lambda_1=2^{-1}$, $\lambda_2=2^{-3}$	M9: $\eta=2^{-8}$, $\lambda=2^{-6}$	M10: $\alpha=2^{-0}$, $\beta=2^{-9}$, $\lambda=2^{-6}$
D3	M1: Adaptive	M2: $\eta=2^{-7}$, $\lambda=2^{-6}$	M3: $\eta=2^{-9}$, $\lambda=2^{-12}$	M4: $\eta=2^{-7}$, $\lambda=2^{-8}$	M5: $P=1.9$, $\lambda=2^{-5}$
	M6: $\lambda_1=2^{-7}$, $\lambda_2=2^{-10}$	M7: $\lambda_1=2^{-7}$, $\lambda_2=2^{-1}$	M8: $\lambda_1=2^{-9}$, $\lambda_2=2^{-3}$	M9: $\eta=2^{-8}$, $\lambda=2^{-2}$	M10: $\alpha=2^{-1}$, $\beta=2^{-7}$, $\lambda=2^{-6}$
D4	M1: Adaptive	M2: $\eta=2^{-7}$, $\lambda=2^{-6}$	M3: $\eta=2^{-9}$, $\lambda=2^{-12}$	M4: $\eta=2^{-8}$, $\lambda=2^{-5}$	M5: $P=1.9$, $\lambda=2^{-5}$
	M6: $\lambda_1=2^{-6}$, $\lambda_2=2^{-8}$	M7: $\lambda_1=2^{-5}$, $\lambda_2=2^{-2}$	M8: $\lambda_1=2^{-0}$, $\lambda_2=2^{-4}$	M9: $\eta=2^{-7}$, $\lambda=2^{-7}$	M10: $\alpha=2^{-0}$, $\beta=2^{-4}$, $\lambda=2^{-5}$
D5	M1: Adaptive	M2: $\eta=2^{-7}$, $\lambda=2^{-6}$	M3: $\eta=2^{-10}$, $\lambda=2^{-11}$	M4: $\eta=2^{-6}$, $\lambda=2^{-9}$	M5: $P=1.7$, $\lambda=2^{-6}$
	M6: $\lambda_1=2^{-7}$, $\lambda_2=2^{-9}$	M7: $\lambda_1=2^{-12}$, $\lambda_2=2^{-6}$	M8: $\lambda_1=2^{-2}$, $\lambda_2=2^{-4}$	M9: $\eta=2^{-9}$, $\lambda=2^{-11}$	M10: $\alpha=2^{-0}$, $\beta=2^{-3}$, $\lambda=2^{-8}$
D6	M1: Adaptive	M2: $\eta=2^{-7}$, $\lambda=2^{-7}$	M3: $\eta=2^{-9}$, $\lambda=2^{-12}$	M4: $\eta=2^{-5}$, $\lambda=2^{-10}$	M5: $P=1.9$, $\lambda=2^{-5}$
	M6: $\lambda_1=2^{-8}$, $\lambda_2=2^{-11}$	M7: $\lambda_1=2^{-9}$, $\lambda_2=2^{-6}$	M8: $\lambda_1=2^{-1}$, $\lambda_2=2^{-4}$	M9: $\eta=2^{-9}$, $\lambda=2^{-10}$	M10: $\alpha=2^{-2}$, $\beta=2^{-1}$, $\lambda=2^{-8}$
D7	M1: Adaptive	M2: $\eta=2^{-10}$, $\lambda=2^{-7}$	M3: $\eta=2^{-10}$, $\lambda=2^{-12}$	M4: $\eta=2^{-5}$, $\lambda=2^{-7}$	M5: $P=1.9$, $\lambda=2^{-5}$
	M6: $\lambda_1=2^{-9}$, $\lambda_2=2^{-12}$	M7: $\lambda_1=2^{-12}$, $\lambda_2=2^{-6}$	M8: $\lambda_1=2^{-0}$, $\lambda_2=2^{-5}$	M9: $\eta=2^{-9}$, $\lambda=2^{-9}$	M10: $\alpha=2^{-2}$, $\beta=2^{-2}$, $\lambda=2^{-8}$
D8	M1: Adaptive	M2: $\eta=2^{-7}$, $\lambda=2^{-7}$	M3: $\eta=2^{-8}$, $\lambda=2^{-12}$	M4: $\eta=2^{-5}$, $\lambda=2^{-9}$	M5: $P=1.9$, $\lambda=2^{-5}$
	M6: $\lambda_1=2^{-8}$, $\lambda_2=2^{-11}$	M7: $\lambda_1=2^{-5}$, $\lambda_2=2^{-6}$	M8: $\lambda_1=2^{-0}$, $\lambda_2=2^{-4}$	M9: $\eta=2^{-8}$, $\lambda=2^{-10}$	M10: $\alpha=2^{-2}$, $\beta=2^{-3}$, $\lambda=2^{-11}$

TABLE S2

TOTAL TIME ON D1-D8 FOR ALLTESTED MODELS (UNIT: MINUTES. NOTE THAT THE TIME INCLUDES THE TIME TO ADJUST THE HYPER-PARAMETERS.)

Models	D1	D2	D3	D4	D5	D6	D7	D8	Win/Loss
Time in RMSE↓									
CTTN-RL	$5.8_{\pm 3.3E-01}$	$\mathbf{3.3}_{\pm 6.2E-01}$	$\underline{2.0}_{\pm 2.4E-01}$	$\mathbf{0.7}_{\pm 6.1E-02}$	$28.0_{\pm 3.1E-00}$	$16.4_{\pm 7.2E-01}$	$8.1_{\pm 4.5E-01}$	$5.6_{\pm 1.0E-00}$	--
TW	$308.1_{\pm 1.3E+01}$	$71.7_{\pm 1.0E+00}$	$99.0_{\pm 2.5E+00}$	$32.0_{\pm 1.0E+00}$	$74.2_{\pm 1.1E+00}$	$74.4_{\pm 1.9E+01}$	$295.0_{\pm 2.6E+00}$	$22.3_{\pm 7.4E-01}$	8/0
Tucker	$306.7_{\pm 2.1E+01}$	$82.7_{\pm 3.7E-01}$	$23.7_{\pm 1.7E-01}$	$11.0_{\pm 2.1E-01}$	$28.8_{\pm 1.5E-00}$	$\mathbf{13.3}_{\pm 7.3E-02}$	$18.6_{\pm 1.7E-01}$	$\underline{1.8}_{\pm 5.4E-02}$	6/2
TR	$50.5_{\pm 4.5E-00}$	$80.6_{\pm 5.6E-00}$	$22.8_{\pm 5.6E-01}$	$20.5_{\pm 3.9E-01}$	$48.9_{\pm 7.8E-01}$	$38.1_{\pm 2.1E-00}$	$5.2_{\pm 1.8E-00}$	$11.1_{\pm 5.3E-01}$	7/1
GSNTD	$40.3_{\pm 5.2E-00}$	$43.0_{\pm 4.7E-00}$	$33.2_{\pm 3.2E-01}$	$11.5_{\pm 4.8E-01}$	$267.4_{\pm 6.3E-00}$	$125.2_{\pm 1.0E-01}$	$115.5_{\pm 5.6E-01}$	$86.1_{\pm 6.5E-00}$	8/0
SGCP	$\mathbf{2.2}_{\pm 6.8E-01}$	$\underline{5.9}_{\pm 6.4E-01}$	$\mathbf{1.3}_{\pm 4.8E-02}$	$\underline{0.8}_{\pm 7.2E-02}$	$\mathbf{3.4}_{\pm 2.7E-02}$	$17.6_{\pm 3.4E-00}$	$38.6_{\pm 2.2E-01}$	$10.5_{\pm 3.4E-01}$	5/3
BNTucF	$67.4_{\pm 1.4E+01}$	$56.2_{\pm 6.4E-00}$	$17.4_{\pm 6.8E-00}$	$20.3_{\pm 5.5E-00}$	$139.9_{\pm 1.1E-+01}$	$224.2_{\pm 6.6E-00}$	$123.9_{\pm 8.5E-00}$	$246.6_{\pm 2.1E-00}$	8/0
BCTL	$\underline{5.1}_{\pm 1.3E-00}$	$16.6_{\pm 2.6E-00}$	$10.9_{\pm 6.0E-02}$	$1.3_{\pm 1.9E-01}$	$55.1_{\pm 1.2E+01}$	$39.0_{\pm 6.4E-00}$	$5.2_{\pm 1.3E-01}$	$\mathbf{1.6}_{\pm 1.0E-01}$	5/3
TCA	$131.5_{\pm 2.8E+01}$	$45.5_{\pm 1.2E-00}$	$36.6_{\pm 9.3E-01}$	$9.7_{\pm 4.3E-01}$	$\underline{19.1}_{\pm 5.1E-01}$	$\underline{16.3}_{\pm 4.8E-01}$	$63.9_{\pm 1.6E-00}$	$8.8_{\pm 5.9E-01}$	6/2
DNL	$222.1_{\pm 8.8E-00}$	$202.2_{\pm 5.9E-00}$	$42.8_{\pm 4.3E-00}$	$1.9_{\pm 7.6E-02}$	$127.9_{\pm 3.1E-00}$	$109.7_{\pm 5.2E-01}$	$95.2_{\pm 1.6E-00}$	$32.0_{\pm 9.7E-01}$	8/0
Time in MAE↓									
CTTN-RL	$\underline{8.1}_{\pm 2.0E-01}$	$\mathbf{4.1}_{\pm 7.8E-01}$	$\underline{2.8}_{\pm 8.9E-02}$	$\mathbf{0.6}_{\pm 5.8E-02}$	$48.1_{\pm 1.2E-00}$	$26.7_{\pm 4.2E-00}$	$\mathbf{15.0}_{\pm 2.2E-00}$	$\underline{8.3}_{\pm 1.0E-00}$	--
TW	$515.8_{\pm 2.2E+01}$	$133.6_{\pm 4.6E+00}$	$190.6_{\pm 5.5E+00}$	$34.7_{\pm 1.5E+00}$	$252.7_{\pm 1.4E+00}$	$192.6_{\pm 3.5E+01}$	$294.7_{\pm 3.2E+00}$	$46.3_{\pm 1.4E+00}$	8/0
Tucker	$328.2_{\pm 2.0E+01}$	$106.2_{\pm 1.8E-00}$	$41.9_{\pm 4.2E-01}$	$13.2_{\pm 1.9E-01}$	$117.3_{\pm 2.9E-00}$	$63.5_{\pm 1.6E-00}$	$62.7_{\pm 1.0E-00}$	$\mathbf{4.0}_{\pm 4.4E-02}$	7/1
TR	$70.2_{\pm 6.4E-00}$	$110.8_{\pm 4.6E-00}$	$34.0_{\pm 1.2E-00}$	$18.4_{\pm 4.5E-01}$	$117.4_{\pm 2.6E-00}$	$94.1_{\pm 3.1E-00}$	$\underline{22.9}_{\pm 7.3E-01}$	$18.7_{\pm 6.4E-01}$	8/0
GSNTD	$40.0_{\pm 3.7E-00}$	$35.3_{\pm 1.9E-00}$	$27.5_{\pm 8.1E-01}$	$10.9_{\pm 4.8E-01}$	$197.8_{\pm 9.0E-00}$	$91.8_{\pm 4.7E-00}$	$82.1_{\pm 5.5E-01}$	$53.2_{\pm 3.6E-00}$	8/0
SGCP	$\mathbf{2.2}_{\pm 6.8E-01}$	$\underline{5.3}_{\pm 2.6E-02}$	$\mathbf{1.2}_{\pm 1.5E-01}$	$\underline{0.9}_{\pm 7.2E-02}$	$\mathbf{3.5}_{\pm 3.2E-02}$	$\underline{17.9}_{\pm 3.8E-00}$	$36.8_{\pm 7.4E-01}$	$10.6_{\pm 2.2E-01}$	4/4
BNTucF	$153.2_{\pm 3.6E-00}$	$71.2_{\pm 2.9E-00}$	$29.7_{\pm 1.1E-00}$	$20.3_{\pm 2.4E-00}$	$145.6_{\pm 1.0E+01}$	$205.9_{\pm 1.1E+01}$	$146.9_{\pm 1.0E+01}$	$235.0_{\pm 5.5E-00}$	8/0
BCTL	$46.8_{\pm 6.6E-00}$	$19.2_{\pm 1.1E-00}$	$11.8_{\pm 5.0E-02}$	$7.3_{\pm 2.7E-01}$	$55.8_{\pm 7.5E-00}$	$41.6_{\pm 3.1E-00}$	$40.8_{\pm 9.0E-00}$	$12.0_{\pm 2.7E-00}$	8/0
TCA	$131.0_{\pm 2.8E+01}$	$52.8_{\pm 2.6E-00}$	$50.0_{\pm 1.9E-00}$	$11.0_{\pm 3.8E-01}$	$\underline{19.1}_{\pm 5.1E-01}$	$\mathbf{16.3}_{\pm 4.8E-01}$	$64.7_{\pm 1.4E-00}$	$12.9_{\pm 7.1E-01}$	6/2
DNL	$389.5_{\pm 2.0E+01}$	$242.0_{\pm 2.7E-00}$	$66.7_{\pm 3.2E-00}$	$3.6_{\pm 3.0E-01}$	$127.6_{\pm 3.1E-00}$	$109.7_{\pm 5.2E-01}$	$95.2_{\pm 1.6E-00}$	$32.0_{\pm 9.7E-01}$	8/0
Time in R^2↓									
CTTN-RL	$5.8_{\pm 3.3E-01}$	$\mathbf{3.2}_{\pm 8.2E-01}$	$\underline{2.3}_{\pm 1.5E-01}$	$\mathbf{0.7}_{\pm 6.1E-02}$	$33.0_{\pm 3.1E-00}$	$16.6_{\pm 5.1E-01}$	$8.1_{\pm 3.6E-01}$	$5.6_{\pm 1.0E-00}$	--
TW	$366.2_{\pm 1.8E+01}$	$78.2_{\pm 1.1E+00}$	$123.1_{\pm 2.0E+00}$	$34.6_{\pm 1.2E+00}$	$78.7_{\pm 1.4E+00}$	$83.2_{\pm 2.3E+01}$	$294.7_{\pm 3.2E+00}$	$22.5_{\pm 8.1E-01}$	8/0
Tucker	$322.1_{\pm 2.3E+01}$	$101.6_{\pm 1.1E-00}$	$25.5_{\pm 2.9E-01}$	$11.6_{\pm 2.1E-01}$	$30.6_{\pm 1.3E-00}$	$\mathbf{13.9}_{\pm 3.1E-01}$	$20.3_{\pm 1.7E-01}$	$\underline{1.8}_{\pm 6.6E-02}$	5/3
TR	$53.0_{\pm 3.3E-00}$	$110.7_{\pm 6.2E-00}$	$24.5_{\pm 5.6E-01}$	$20.7_{\pm 3.0E-01}$	$65.8_{\pm 7.9E-01}$	$55.0_{\pm 2.1E-00}$	$\mathbf{5.1}_{\pm 2.8E-01}$	$18.5_{\pm 6.6E-01}$	7/1
GSNTD	$40.3_{\pm 1.2E-00}$	$42.1_{\pm 3.7E-00}$	$33.6_{\pm 6.6E-01}$	$12.0_{\pm 9.7E-01}$	$266.5_{\pm 4.7E-00}$	$124.7_{\pm 8.1E-00}$	$117.3_{\pm 3.1E-00}$	$84.4_{\pm 5.4E-00}$	8/0
SGCP	$\mathbf{2.2}_{\pm 6.8E-01}$	$\underline{5.9}_{\pm 6.4E-01}$	$\mathbf{1.3}_{\pm 4.8E-02}$	$\underline{0.8}_{\pm 7.2E-02}$	$\mathbf{3.4}_{\pm 4.4E-02}$	$17.6_{\pm 3.4E-00}$	$38.6_{\pm 1.8E-01}$	$10.4_{\pm 1.7E-01}$	5/3
BNTucF	$72.2_{\pm 2.4E+01}$	$73.3_{\pm 4.9E-00}$	$21.0_{\pm 9.3E-00}$	$23.8_{\pm 3.6E-00}$	$139.9_{\pm 1.1E+01}$	$229.8_{\pm 8.2E-00}$	$131.8_{\pm 7.2E-00}$	$249.1_{\pm 2.8E-00}$	8/0
BCTL	$\underline{5.1}_{\pm 1.3E-00}$	$16.6_{\pm 2.6E-00}$	$10.9_{\pm 6.0E-02}$	$1.3_{\pm 1.9E-01}$	$56.9_{\pm 1.0E+01}$	$39.0_{\pm 6.4E-00}$	$\underline{5.2}_{\pm 6.4E-02}$	$\mathbf{1.6}_{\pm 1.0E-01}$	5/3
TCA	$131.5_{\pm 2.8E+01}$	$46.8_{\pm 1.1E-00}$	$39.3_{\pm 3.4E-00}$	$9.9_{\pm 3.5E-01}$	$\underline{19.1}_{\pm 5.1E-01}$	$\underline{16.3}_{\pm 4.8E-01}$	$64.7_{\pm 1.4E-00}$	$9.3_{\pm 5.9E-01}$	6/2
DNL	$233.1_{\pm 4.3E-01}$	$203.7_{\pm 5.7E-00}$	$43.5_{\pm 4.7E-00}$	$1.9_{\pm 7.6E-02}$	$127.9_{\pm 3.1E-00}$	$109.7_{\pm 5.2E-01}$	$95.2_{\pm 1.6E-00}$	$32.0_{\pm 9.7E-01}$	8/0

TABLE S3

FRIEDMAN TEST RESULTS ON ACCURACY (RMSE, MAE, R^2) AND EFFICIENCY (TIME IN RMSE, TIME IN MAE, TIME IN R^2)

	CTTN-RL	TW	Tucker	TR	GSNTD	SGCP	BNTucF	BCTL	TCA	DNL
Accuracy	**1.00**	2.42	4.96	3.04	6.46	8.17	5.67	6.08	7.63	9.58
Efficiency	**2.25**	8.83	5.21	5.77	6.79	<u>2.38</u>	7.88	3.31	4.88	7.71