Literature Review on Nonnegative Dynamics, Compartmental Modeling, Vector Lyapunov Function and Vector Dissipativity Theory

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Abstract

Nonnegative dynamical system models are derived from mass and energy balance considerations that involves dynamic states whose values are nonnegative. These model are widespread in biological and ecological sciences and play a key role in the understanding of these processes.An unified framework(linear & nonlinear) involving compartmetal modeling, stability analysis and vector dissipativity theory was developed in [1, 2, 3, 4]. This work contains systematic review on these subjects and some rough ideas about their applications in chemical process control.

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1 Introduction

2 Nonnegative dynamics

Defination 2.1. Let $A \in \mathbb{R}^{m\times n}$. Then A is **nonnegative** (resp., **positive**) if $A_{ij} \geqslant 0$ (resp., $A_{ij} > 0$) for all $i = 1, \cdots, m$ and $j = 1, \cdots, n$.

Defination 2.2. Let $T > 0$. A real function $u : [0, T] \to \mathbb{R}^m$ is a nonegative (resp., positive) function if $u(t) \geqslant\geqslant 0$ (resp., $u(t) \gg 0$), which means $u_i(t) \geqslant 0$ (resp., $u_i(t) > 0$) for $i = 1, \cdots, m$.

Defination 2.3. Let $A \in \mathbb{R}^{n\times n}$. A is a **Z-matrix** if $A_{ij} \leqslant 0, i, j = 1, \cdots, n, i \neq j$. A is an **M-matrix** (resp., nonsingular M-matrix) if A is a Z-matrix and all the principal minors of A are nonnegative (resp., positive). A is **essentially nonnegative** if -A is a Z-matrix, which means $A_{ij} \geqslant 0, \forall i, j = 1, \cdots, n, i \neq j$.

Lemma 2.1. *Assume A is a Z-matrix. Then the following statement are equivalent:*

(i) *A is an M-matrix.*

(ii) *$\exists \alpha > 0, B \geqslant\geqslant 0$ s.t. $\alpha > \rho(B)$ and $A = \alpha I - B$.*

(iii) *$Re\lambda \geq 0$, $\lambda \in \mathrm{spec}(A)$.*

(iv) *If $\lambda \in \mathrm{spec}(A)$, then either $\lambda = 0$ or $\lambda > 0$.*

Furthermore, in the case where A is a nonsigular Z-matrix, then the following statements are equivlent:

(v) *A is a nonsingular M-matrix.*

(vi) $det(A) \neq 0$ *and* $A^{-1} \geqslant\!\!\geqslant 0$.

(vii) $y \in \mathbb{R}^n, y \geqslant\!\!\geqslant 0$, *then* $\exists x \in \mathbb{R}^n, x \geqslant\!\!\geqslant 0$ *s.t.* $Ax = y$.

(viii) $\exists x \in \mathbb{R}^n, x \geqslant\!\!\geqslant 0$, *s.t.* $Ax \gg 0$.

(ix) $\exists x \in \mathbb{R}^n, x \gg 0$, *s.t.* $Ax \gg 0$.

Consider the linear dynamical system of the form

$$\dot{x} = Ax, \qquad x(0) = x_0, t \geqslant 0 \tag{1}$$

Lemma 2.2. *Let $A \in \mathbb{R}^{n\times n}$. A is essentially nonnegative iff e^{At} is nonnegative for all $t \geqslant 0$. Furthermore, if A is essentially nonnegative and $x_0 \geqslant\!\!\geqslant 0$, then $x(t) \geqslant\!\!\geqslant 0$ and Sys.1 is called* ***linear nonnegative dynamical system****.*

Defination 2.4. The quilibrium solution $x(t) = x_e$ of Sys.1 is

- **Lyapunov stable** if, $\forall \epsilon > 0, \exists \delta = \delta(\epsilon) > 0$ s.t. if $x_0 \in \mathcal{B}_\delta(x_e) \cap \tilde{\mathbb{R}}^n_+$, then $x(t) \in \mathcal{B}_\delta(x_e) \cap \tilde{\mathbb{R}}^n_+, t \geqslant 0$.
- **semistable** if it is Lyapunove stable and $\exists \delta > 0$ s.t. if $x_0 \in \mathcal{B}_\delta(x_e) \cap \tilde{\mathbb{R}}^n_+$, then $\lim_{t\to\infty} x(t)$ exists and converges to a Lyapunov stable equilibrium point.
- **asymptotically stable** if it is Lyapunov stable and $\exists \delta > 0$ s.t. if $x_0 \in \mathcal{B}_\delta(x_e) \cap \tilde{\mathbb{R}}^n_+$, then $\lim_{t\to\infty} x(t) = x_e$.
- **globally asmptotically stable** if it is asymptotically stable respect to all $x_0 \in \mathbb{R}^n$.

Theorem 2.1. *Let $A \in \mathbb{R}^{n\times n}$ be essentially nonnegative. If $\exists p, r \in \mathbb{R}^n$ s.t. $p \gg 0, r \geqslant\!\!\geqslant 0$ statisfy*

$$0 = A^T p + r \tag{2}$$

then the following properties hold:

(i) -A *is an M-matrix.*

(ii) *if* $\lambda \in \operatorname{spec}(A)$, *then either* $\lambda = 0$ *or* $\lambda > 0$.

(iii) $ind(A) \leqslant 1$, *so* A *has generalized group inverse* $A^\#$.

(iv) A *is semistable and* $\lim_{t\to\infty} e^{At} = I - AA^\# \geqslant\!\!\geqslant 0$.

(v) $\mathcal{R}(A) = \mathcal{N}(I - AA^\#)$, $\mathcal{N}(A) = \mathcal{R}(I - AA^\#)$.

(vi) $\int_0^t e^{Atau} \mathrm{d}\tau = A^\#(e^{At} - I) + (I - AA^\#)t, t \geqslant 0$.

(vii) A *is nonsingular iff* -A *is a nonsingular M-matrix.*

(viii) *if* A *is nonsingular, then* A *is asymptotical stable and* $A^{-1} \leqslant\!\!\leqslant 0$.

3 Compartmental modeling

4 Vector Lyapunove function

5 Vector dissipativity theory

6 Application in Chemical Process Control

References

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