Notes of J.E. Marsden's – *Foundations of Mechanics*

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Part I

Preliminaries

Chapter 1

Differential Theory

1.1 Topology

Definition 1.1. A topological space is a set S together with a collect $\mathcal{O}$ of subsets called open sets s.t.

(T1) $\emptyset \in \mathcal{O}$ and $S \in \mathcal{O}$;

(T2) If $U_1, U_2 \in \mathcal{S}$, then $U_1 \cap U_2 \in \mathcal{O}$;

(T3) The union of any collection of open sets is open.

For such a topological space the **closed sets** are the elements of $\gamma = \{A \mid A^c \in \mathcal{O}\}$. An **open neighborhood of a point** u is a topological space S is an open set U s.t. $u \in U$. Similarly, for a subset A of S, U is an **open neighborhood** of A if U is open and $A \subset U$. If A is a subset of a topological space S, the **relative topology** on A is defined by $\mathcal{O}_A = \{U \cap A \mid U \in \mathcal{O}\}$.

Then a **basis** for the topology is a collection $\mathcal{B}$ of opensets s.t. every open set of S is a union of elements of $\mathcal{B}$. The topology is called **first coutable** if for each $u \in S$, there is a countable collection $\{U_n\}$ of neighborhoods of u s.t. for any neighborhood U of u, there is an n so $U_n \subset U$. The topology is called **second countable** if it has a countable basis.

Let $\{u_n\}$ be a sequence of points in S. The sequence is said to **coverge** if there is a point $u \in S$ s.t. for every neighborhood U of u, there is an N s.t. $n \geq N$ implies $u_n \in U$. We say that $\{u_n\}$ converges to u or u is a **limit point** of $\{u_n\}$.

Example 1.1. The standard topology of $\mathbb{R}$ is the unions of open intervals (a, b). Then $\mathbb{R}$ is second countable (hence first countable) with a basis

$$\left\{ \left(r_n - \frac{1}{m}, r_n + \frac{1}{m} \right) \mid r_n \in \mathbb{Q}, m \in \mathbb{N}^+ \right\}$$

.

Definition 1.2. Let S be a topological space and $A \subset S$. Then the **closure** of A, denoted $\overline{A}$ is the inersection of all colsed sets containing A. The **interior** of A, denoted $\mathring{A}$ is the union of all open sets contained in A. The **boundary** of A, denoted $\partial A := \overline{A} \cap \overline{A^c}$.

Thus, ∂A is closed, and $\partial A = \partial A^c$. Note that A is open iff $A = \mathring{A}$ and closed iff $A = \overline{A}$.

Propsition 1.1. *Let S be a topological space and $A \subset S$.*

(i) $u \in \overline{A}$ iff for every neighborhood U of u, $U \cap A \neq \emptyset$.

(ii) $u \in \mathring{A}$ iff there is a neighborhood U of u s.t. $U \subset A$.

(iii) $u \in \partial A$ iff for every neighborhood U of u, $U \cap A \neq \emptyset$ and $U \cap A^c \neq \emptyset$.

Definition 1.3. A point $u \in S$ is called **isolated** iff $\{u\}$ is open. A subset A of S is called **dense** in S iff $\overline{A} = S$ and is called **nowhere dense** iff $(\overline{A})^c$ is dense in S. Thus, A is nowhere dense iff $\mathring{\overline{A}} = \emptyset$.

Definition 1.4. A topological space S is called **Hausdorff** iff each two distinct points have disjoint neighborhoods. Similarly, S is called **normal** iff each two disjoint closed sets have disjoint neighborhoods.

Propsition 1.2. *(i) A space S is Hausdorff iff $\Delta_S = \{(u,u) \mid u \in S\}$ is closed in $S \times S$ is the product topology.*

(ii) A first countable space S is Hausdorff iff all sequences have at most one limit point.

Proof. If Δ_S is closed and u_1, u_2 are distinct, there is an open rectangle $U \times V$ containing (u_1, u_2) and $U \times V \subset \Delta_S^c$. Then in S, U and V are disjoint because if $\exists p \in U \cap V$, then $(p,p) \in U \times V$ it countered with closed set Δ_S. The converse is similar and we leave it as an exercise. □

Definition 1.5. Let $\mathbb{R}^+$ denote the nonnegative real numbers with a point $\{+\infty\}$ adjoined, and topology generated by the open intervals of the form (a,b). A **metric** on set M is a function $d : M \times M \to \mathbb{R}^+$ s.t.

(M1) $d(m_1, m_2) = 0$ iff $m_1 = m_2$;

(M2) $d(m_1, m_2) = d(m_2, m_1)$;

(M3) $d(m_1, m_3) \leq d(m_1, m_2) + d(m_2, m_3)$

For $\varepsilon > 0$, the ε **disk** about m is defined by $D_\varepsilon(m) = \{m' \in M \mid d(m', m) < \varepsilon$. The collection of subsets of M that are unions of such disks is the metric topology of the metric space (M,d). Two metrics on a set are called **equivalent** if they induce the same metric topology. $\{u_n\}$ is a **Cauchy sequence** iff for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ s.t. $n, m \geq N$ implies $d(u_n, u_m) < \varepsilon$. The space M is called **complete** if every Cauchy sequence converges. We define $d(u, A) = inf\{d(u,v) \mid v \in A\}$ and $d(u, \emptyset) = \infty$.

Propsition 1.3. *Every metric space is normal.*

Proof. Let A and B be closed, disjoint subsets of M, and let

$$U = \{u \in M \mid d(u,A) < d(u,B)\}$$
$$V = \{v \in M \mid d(v,A) > d(v,B)\}$$

It is verified that U and V are open, disjoint and $A \subset U, B \subset V$. □

Definition 1.6. If $\varphi : S \to T$ is **continuous** at $u \in S$ if $\forall V \ni \varphi(u), \exists U \ni u \Rightarrow \varphi(U) \subset V$. If $\forall V \subset T, \varphi^{-1}(V) = \{u \in S \mid \varphi(u) \in V$ is open in S, φ is **continuous**.

If $\varphi : S \to T$ is a **bijection**, φ and φ^{-1} are continuous, then φ is a **homeomorphism** and S and T are **homeomorphic**.

Propsition 1.4. *φ is continuous iff $\forall A \subset S \Rightarrow \varphi(\overline{A}) \subset \overline{\varphi(A)}$.*

Proof. If φ is continuous, then $\varphi^{-1}(\overline{\varphi(A)})$ is closed. But $A \subset \varphi^{-1}(\overline{\varphi(A)})$ and hence $\overline{A} \subset \varphi^{-1}(\overline{\varphi(A)})$ or $\varphi(\overline{A}) \subset \overline{\varphi(A)}$. Conversely, let $B \subset T$ be closed and $A = \varphi^{-1}(B)$. Then $\overline{(A)} \subset \varphi^{-1}(B) = A$, so A is closed. □

Propsition 1.5. *Let $\varphi : S \to T$ and S first countable set. Then φ is continuous iff $\forall u_n \to u \Rightarrow \varphi(u_n) \to \varphi(u)$.*

Propsition 1.6. *Let M and N be metric spaces with N complete. Then the collection $C(M, N)$ of all continuous maps $\varphi : M \to N$ forms a complete metric space with the metric $d^0(\varphi, \phi) = sup\{d(\varphi(u), \phi(u)) \mid u \in M\}$.*

Proof. It is readily verified that d^0 is a metric. Convergence of sequence $f_n \in C(M, N)$ to $f \in C(M, N)$ in the metric d^0 is the same as uniform convergence, that is, for all $\varepsilon > 0$ there is an N s.t. if $n \geq N$, $d(f_n(x), f(x)) \leq \varepsilon$ for all $x \in M$. If f_n is a Cauchy sequence in $C(M, N)$, then since $d(f_n(x), f_m(x)) \leq d^0(f_n, f_m)$. $f_n(x)$ is Cauchy for each point $x \in M$. Thus f_n converges pointwise, define a function $f(x)$. We must show that $f_n \to f$ uniformly and that f is continuous. First of all, given $\varepsilon > 0$, choose N s.t. $d^0(f_n, f_m) < \varepsilon/2$ if $n, m \geq N$. Then for any $x \in M$, pick $N_x \geq N$ s.t. $d(f_m(x), f(x)) < \varepsilon/2$ if $m \geq N_x$. Thus with $n \geq N$ and $m \geq N_x$, $d(f_n(x), f(x)) \leq d(f_n(x), f_m(x)) + d(f_m(x), f(x)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$ So $f_n \to f$ uniformly. □

Definition 1.7. S is called **compact** iff $\forall \cup_\alpha U_\alpha = S$ there is a finite subcovering. A subset $A \subset S$ is called **compact** iff A is compact in the relative topology. A space is called *locallycompact* iff each point has a neighborhood whose closure is compact.

Theorem 1.1 (**Boizano-Weierstrass**)**.** *If S is a first countable space and is compact, then every sequence has a convergent subsequence.*

Proof. Suppose $\{u_n\}$ contains no convergent subsequences. Then we may assume all points are distinct. Each u_n has a neighborhood $\mathcal{O}_n$ that contains no other u_m. $\{u_n\}$ is closed, so that $\mathcal{O}_n$ together with $\{u_n\}^c$ forms an open covering of S, with no finit subcovering. □

In a metric space, every compact subset is closed and bounded(Heine Borel theorem).

Propsition 1.7. *Let S be a Hausdorff space. Then every compact subset of S is closed. Also, every compact Hausdorff space is normal.*

Proof. Let $u \in A^c$ and $v \in A$, where A is compact in S. There are disjoint neighborhoods of u and v and, since A is compact, there are disjoint neighborhood of u and A. Thus A^c is open. The second part is an exercise. □

Propsition 1.8. *Let S be a Hausdorff space that is locally homeomorphic to a locally compact Hausdorff space. Then S is locally compact.*

Proof. Let $U \subset S$ be homeomorphic to $\varphi(U) \subset T$. There is a neighborhood V of $\varphi(u)$ so $\overline{V} \subset \varphi(U)$ and $\overline{V}$ is compact. Then $\varphi^{-1}(\overline{V})$ is compact, and hence closed in S. $\varphi^{-1}(\overline{V}) \subset \overline{\varphi^{-1}(V)}$. Thus $\varphi^{-1}(V)$ has compact closure. $\square$

Definition 1.8. A covering $\{U_\alpha\}$ of S is called a **refinement** of a covering $\{V_i\}$ iff $\forall U_\alpha$ there is a V_i s.t. $U_\alpha \subset V_i$. A covering $\{U_\alpha\}$ of S is called **locally finite** iff each point $u \in S$ has a neighborhood U such that U intersects only a finite number of U_α. A space is called **paracompact** iff every open covering of S has a locally finite refinement of open sets, and S is Hausdorff.

Theorem 1.2. *Second countable, locally compact Hausdorff spaces are paracompact.*

Proof. S is the countable union of open sets U_n s.t. $\overline{U_n}$ is compact and $\overline{U_n} \subset U_{n+1}$. If W_α is a covering of S by open sets, and $K_n = \overline{U_n} - U_{n-1}$ then we can cover K_n by a finit number of open sets each of which is contained in some $W_\alpha \cap U_{n+1}$, and is disjoint from $\overline{U_{n-2}}$. The union of such collections yiels the desired refinement of $\{W_\alpha\}$. $\square$

Theorem 1.3. *Every paracompact space is normal.*

Proof. We first show that if A is closed and $u \in A^c$, there are disjoint neighborhoods of u and A. For each $v \in A$ let U_u, V_v be disjoint neighborhood of u and v. Let W_α be a locally fninit refinement of the covering V_v, A^c and $V = \cup W_\alpha$, the union over those α so $W_\alpha \cap A \neq \emptyset$. A neighborhood U_0 of u meets a finite number of W_α. Let U denote the intersection of U_0 and the corresponding U_u. Then V and U are the required neighborhoods. $\square$

Theorem 1.4. *If S is a Hausdorff space, the following are equivalent:*

(i) *S is normal;*

(ii) *For any two closed nonempty disjoint set A, B ther is a continuous function $f : S \to [0, 1]$ s.t. $f(A) = 0, f(B) = 1$.(Urysohn's Lemma)*

(iii) *For any closed set $A \subset S$ and continuous function $f : A \to [a, b]$, there is a continuous extension $\tilde{f} : S \to [a, b]$ of f (Tietze extension theorem)*

Definition 1.9. The **support** of $f : s \to \mathbb{R}$ is $supp(f) = \overline{\{x \in S \mid f(x) \neq 0\}}$. A **partition of unity** on S is a familiy of continuous mappings $\{\varphi_i : S \to [0, 1]\}$ s.t.

(i) $\{supp(\varphi_i)\}$ is locally finite.

(ii) $\sum_i \varphi_i(x) = 1$ for all x.

We say that a pratition of unity $\{\varphi_i\}$ is **subordinate** to a covering $\{A_\alpha\}$ of S if $supp(\varphi_i)$ is a refrinement of $\{A_\alpha\}$.

Theorem 1.5. *Let S be paracompact and $\{U_i\}$ be any open covering of S. Then ther is a partition of unity $\{\varphi_i\}$ subordinate to $\{U_i\}$.*

Definition 1.10. A topological space S is **connected** if $\emptyset$ and S are the only subsets of S that are both open and closed. A subset of S is connected iff it is connected in the relative topology. A **component** A of S is a nonempty connected subsect of S s.t. the only connected subset of S containing A is A; S is called **locally connected** iff each point x has an open neighborhood containing a connected neighborhood of x.

Propsition 1.9. *A space S is not connected iff either of the following holds.*

(i) There is a nonempty proper subset of S that is both open and closed.

(ii) S is the disjoint union of two nonempty open sets.

(iii) S is the disjoint union of two nonempty closed sets.

Propsition 1.10. *Let S be a connected space and $f : S \to \mathbb{R}$ be continuous. Then f assumes every value between any two values $f(u), f(v)$.*

Proof. Suppose $f(u) < a < f(v)$ and f doses not assume the value a. Then $U = \{u_0 \mid f(u_0) < a\}$ is both open and closed. □

Propsition 1.11. *Let S be a topological space and $B \subset S$ be connected.*

(i) if $B \subset A \subset \overline{B}$, then A is connected;

(ii) if B_α are connected and $B_\alpha \cap B \neq \emptyset$, then $B \cup (U_\alpha B_\alpha)$ is connected.

Proof. If A is not connected, A is the disjoint union of $U_1 \cap A$ and $U_2 \cap A$ where U_1, U_2 are open in S. Then $U_1 \cap B \neq \emptyset, U_2 \cap B \neq \emptyset$, so B is not connected. □

Definition 1.11. An **arc** φ in S is a continuous mapping $\varphi : I = [0, 1] \to S$. If $\varphi(0) = u, \varphi(1) = v$, we say φ joins u and v; S is called **arcwise connected** iff every two points in S can be joined by an arc in S. A space is called **locally arcwise connected** iff each point has an arcwise connected neighborhood.

Propsition 1.12. *Every arcwise connected space is connected. If a space is connected and locally arcwise connected, it is arcwise connected.*

Proof. If S is arcwise connected and not connected, write $S = U_1 \cup U_2$ where U_1, U_2 are nonempty, disjoint and open. Let $u_1 \in U_1, u_2 \in U_2$ and let φ be an arc joining u_1, u_2. Now $\varphi(I)$ is connected, and since $\varphi(I) \cap U_i \neq \emptyset$, $\varphi \cap U_1 \cap U_2 \neq \emptyset$. Hence $U_1 \cap U_2 \neq \emptyset$, a contradiction. Let $u \in S$and U denote all points that can be joined to u by an arc. An easy argument shows U and U^c are open and so $U = S$. □

Definition 1.12. Let S be a metric space with metric d, and 2^S denote the set of all subsets of S. Define $\tilde{d}(A, B) = supd(a, B)|a \in A$. As this is not symmetric, we further define $\overline{d}(A, B) = sup\{\tilde{d}(A, B), \tilde{d}(B, A)\}$. If $A \neq \emptyset, B = \emptyset, \overline{d}(A, B) = \infty, \overline{d}(\emptyset, \emptyset) = 0$. We call it the **Hausdorff metric**.

Propsition 1.13. *Let S be a metric space and d the Hausdorff metric on 2^S. Then $f : S \to 2^S$ is continuous at $u_0 \in S$ iff for all $\varepsilon > 0$ there is a $\delta > 0$ s.t. $d(u, u_0) < \delta$ implies:*

(i) *for all* $a \in f(u)$*, there is a* $b \in f(u_0)$ *s.t.* $d(a,b) < \varepsilon$*; that is*

$$f(u) \subset \bigcup_{b \in f(u_0)} D_\varepsilon(b)$$

.

(ii) *for all* $b \in f(u_0)$*, there is an* $a \in f(u)$ *s.t.* $d(b,a) < \varepsilon$*.*

Definition 1.13. Let S be a set. An **equivalence relation** $\sim$ on S is a binary relation s.t. for all $u, v, w \in S$

(i) $u \sim u$;

(ii) $u \sim v$ iff $v \sim u$;

(iii) $u \sim v, v \sim w \Rightarrow u \sim w$.

The **equivalence class** containing u, denoted $[u]$ is defined by $[u] = \{v \in S \mid u \sim v\}$. The set of equivalence classes is denote $S/\sim$, and the mapping $\pi : S \to S/\sim; u \longmapsto [u]$ is called the **canonical projection**.

Definition 1.14. $\{U \subset S/\sim \mid \pi^{-1}(U) \text{is open in} S\}$ is called the **quotient topology**.

Example 1.2. Consider $\mathbb{R}^2$ and the relation $\sim$ defined by $(a_1, a_2) \sim (b_1, b_2)$ iff $a_1 - b_1, a_2 - b_2 \in \mathbb{Z}$. Then $T^2 = \mathbb{R}^2/\sim$ is called the 2-**torus**. In addition to the quotient topology, it inherits a group structure in the usual way: $[(a_1, a_2)] + [(b_1, b_2)] = [(a_1, a_2) + (b_1, b_2)]$.

Example 1.3. The Klein bottle is obtained by reversing one of the orientations. Notice that K^2 is not "orientable" and does not inherit a group structure from $\mathbb{R}^2$.

Definition 1.15. Let Z be a topological space and $c : [0,1] \to Z$ a continuous map s.t. $c(0) = c(1) = p \in Z$. We call c a **loop** in Z based at p. The loop c is called **contractible** if there is a continuous map $H : [0,1] \times [0,1] \to Z$ s.t. $H(t,0) = c(t), H(t,1) = p, \forall t \in [0,1]$.

Definition 1.16. A space Z is called **simply connected** iff every loop in Z is contractible.

Definition 1.17. Let X be a topological space and $A \subset X$. Then A is called *residual* iff A is the intersection of a countable family of open dense subsets of X. A space X is called a **Baire space** iff every residual set is dense.

Lemma 1.1. *Let X be a locally Baire space; that is, each point* $x \in X$ *has a neighborhood* U *s.t.* $\overline{U}$ *is a Baire space. Then X is a Baire space.*

Proof. Let $A \subset X$ be residual; $A = \cap_1^\infty O_n$ where $\overline{O_n} = X$. Then $A \cap \overline{U} = \cap_1^\infty (O_n \cap \overline{U})$ Now $O_n \cap \overline{U}$ is dense in $\overline{U}$ for if $u \in \overline{U}, u \in O$ then $O \cap U \neq \emptyset, O \cap U \cap O_n \neq \emptyset$. Hence $\overline{U} \subset \overline{A}, \overline{A} = X$. □

Theorem 1.6 (**Baire category**)**.** *Every complete pseudometric space is a Baire space.*

Proof. Let $U \subset X$ be open and $A = \cap_1^\infty O_n$ be residual. We must show $U \cap A \neq \emptyset$. Now as $\overline{O_n} = X, U \cap O_n \neq \emptyset$ and so we can choose a disk of diameter less than one, say V_1, s.t. $\overline{V_1} \subset U \cap O_1$. Proceed inductively to obtain $\overline{V_n} \subset U \cap O_n \cap V_{n-1}$, where V_n has diameter $< 1/n$. Let $x_n \in \overline{V_n}$. Clearly $\{x_n\}$ is a Cauchy sequence, and by completeness has a convergence subsequence with limit point x. Then $x \in \cap_1^\infty \overline{V_n}, U \cap \cap_1^\infty O_n \neq \emptyset$. □

Exercises

Exercise 1.1. *Let S and T be sets and $f : S \to T$. Show that f is a bijection iff there is a mapping $g : T \to S$ s.t. $f \circ g, g \circ f$ are identity mappings.*

Exercise 1.2. *Let X and Y be topological space with Y Hausdorff. Then show that, for any continuous maps $f, g : X \to Y, \{x \in X \mid f(x) = g(x)$ is closed. [Hint: Consider the mapping $x \longmapsto (f(x), g(x))$].*

Exercise 1.3. *Prove that in a Hausdorff space, single points are closed.*

Exercise 1.4. *Define a topological manifold as a space locally homeomorphic to $\mathbb{R}^n$. Find a topological manifold that is not Hausdorff and not locally compact. [Hint: Consier $\mathbb{R} \cup \{\pm\infty\}$].*

Exercise 1.5. *Show that the continuous image of a connected space is connected.*

1.2 Finite-Dimensional Banach Sapce

Definition 1.18. A **norm** on a vector space E is a mapping $\| \bullet \|: E \to \mathbb{R}$ s.t.

(N1) $\| \bullet \| \geq 0, \forall e \in E$ and $\| e \| = 0$ iff $e = 0$.

(N2) $\| \lambda e \| = |\lambda| \| e \|, \forall e \in E, \lambda \in \mathbb{R}$.

(N3) $\| e_1 + e_2 \| \leq \| e_1 \| + \| e_2 \|, \forall e_1, e_2 \in E$.

A normed space whose induced metric is complete is a **Banach space**.

Definition 1.19. Two norms on a vector space E are **equivalent** iff they induce the same topology on E.

Theorem 1.7. *Let E be a finite-dimensional real vector space. Then*

(i) there is a norm on E;

(ii) all norms on E are equivalent;

(iii) all norms on E are complete.

Theorem 1.8. *For finite-dimensional real vector spaces, linear and multilinear maps are continuous.*

Corollary 1.1. *Addition and scalar multiplication in a vector space are continuous maps from $E \times E \to E, \mathbb{R} \times E \to E$.*

Definition 1.20. Given E,F we let $L(E, F)$ denote the set of all linear maps from E into F together with the natural structure of finite dimensional real vector space. Similarly, $L^k(E, F)$ denote the space of multilinear maps from $E \times \cdots \times E$ into F, $L^k_s(E, F)$, the subspace of symmetric elements of $L^k(E, F)$ [that is, if π is any permutation of $\{1, 2 \cdots, k\}$, we have $f(e_1, \cdots, e_2) = f(e_{\pi(1)}, \cdots, e_{\pi(k)})$] and $L^k_a(E, F)$ the subspace of skew symmetric [That is if π is any permutation of $\{1, 2 \cdots, k\}$,we have $f(e_1, \cdots, e_2) = (sgn\pi) f(e_{\pi(1)}, \cdots, e_{\pi(k)})$, wher $sgn\pi = \pm 1$ according as π is an even or odd permutation].

Theorem 1.9. *There is a natural isomorphism $L(E, L^k(E, F)) \approx L^{k+1}(E, F)$.*

Exercises

Exercise 1.6. *Let $f \in L(E, F)$ so that f is continuous.*

(a) Show that there is a constant K s.t. $\| f(e) \| \leq K \| e \|$ for all $e \in E$. Define $\| f \|$ as the greatest lower bound of such K.

(b) Show that this is a norm on $L(E, F)$.

(c) Prove that $\| f \circ g \| \leq \| f \| \cdot \| g \|$.

Exercise 1.7. *Suppose $f \in L(E, F)$ and $dim(E) = dim(F)$. Then f is an isomorphism iff it is a monomorphism (one-to-one) and iff it is surjective (onto).*

Exercise 1.8. *Show that two norms $\| \bullet \|_1, \| \bullet \|_2$ are equivalent iff there is a constant M s.t. $M^{-1} \| e \|_1 \leq \| e \|_2 \leq M \| e \|_1$.*

Exercise 1.9. *Let E be the set of all C^1 functions $f : [0, 1] \to \mathbb{R}$ with the norm $\| f \| = sup_{x \in [0,1]} |f(x)| + sup_{x \in [0,1]} |f'(x)|$. Prove that E is a Banach space.*

1.3 Local Differential Calculus

Definition 1.21. Let E,F be two vector spaces with maps $f, g : U \subset E \to F$. We say f and g are **tangent** at $u_0 \in U$ iff

$$\lim_{u \to u_0} \frac{\| f(u) - g(u) \|}{\| u - u_0 \|} = 0$$

Theorem 1.10. *For $f : U \subset E \to F$ there is at most one $L \in L(E, F)$ so that the map $g_L : U \subset E \to F$ given by $g_L(u) = f(u_0) + L(u - u_0)$ is tangent to f at u_0.*

Definition 1.22. If there is such an L we say f is **differentiable** at u_0, and define the **derivative** of f at u_0 to be $Df(u_0) = L$. If f is differentiable at each $u \in U$, the map $\mathrm{D}f : U \to L(E, F); u \mapsto \mathrm{D}f(u)$ is the **derivative** of f. Moreover, if $\mathrm{D}f$ is a continuous map we say f is of class C^1.

Definition 1.23. Suppose $f : U \subset E \to F$ is of class C^1. Define the **tangent** of f to be the map: $Tf : U \times E \to F \times F$ given by $Tf(u, e) = (f(u), Df(u) \cdot e$.

From a geometrical point of view, T is more natural than D. If we take (u, e) as a vector with base point u, then $(f(u), Df(u) \cdot e)$ is the image vector with its base point. Another reason for this is its behavior under composition, so T is a covariant functor.

Theorem 1.11.

$$T(g \circ f) = T(g) \circ T(f)$$
$$T^r(g \circ f) = T^r(g) \circ T^r(f)$$

For $f : E \to F, c : I \to U$. So $Df(u) \cdot e = \frac{\mathrm{d}}{\mathrm{d}t}\{f(u+te)\} \mid_{t=0}$. $Df(u)$ is represented by the usual Jacobian matrix. If we apply the fundamental theorm of calculus to $t \mapsto f(tx+(1-t)y)$ and $\| Df(tx+(1-t)y) \| \leq M$, we obtain the mean value inequality:$\| f(x) - f(y) \| \leq M \| x - y \|$.

Definition 1.24. Let $U_1 \subset E_1, U_2 \subset E_2$ be open and suppose $f : U_1 \times U_2 \to F$. Then the **partial derivative** of f with respect to E_1 denoted $D_1 f$ is defined by $D_1 f(u_1, u_2) : E_1 \to F : e_1 \mapsto D_1 f(u_1, u_2) \cdot e_1 = Df(u_1, u_2) \cdot (e_1, 0)$. Thus $Df = D_1 f + D_2 f$.

Theorem 1.12 (**Inverse Mapping**). *Let $f : U \subset E \to F$ be of class C^r and suppose $Df(x_0)$ is a linear isomorphism. Then f is a C^r diffeomorphism of some neighborhood of x_0 onto some neighborhood of $f(x_0)$.*

Lemma 1.2. *Let M be a complete metric space, Let $F : M \to M$ and assume there is a constant $0 \leq \lambda < 1$ s.t. $\forall x, y \in M, d(F(x), F(y)) \leq \lambda d(x, y)$. Then F has a unique fixed point $x_0 \in M, F(x_0) = x_0$.*

Proof. Pick $x_1 \in M$ and define $x_{n+1} = F(x_n)$. Thus $d(x_{n+1}, x_n) \leq \lambda^{n-1} d(F(x_1), x_1)$ and $d(x_{n+k}, x_n) \leq \left(\sum_{j=n-1}^{n+k-1} \lambda^j\right) d(F(x_1), x_1)$. Thus x_n is a Cauchy sequence. Since F is obviously uniformly continuous, then $x_0 = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n) = F(x_0)$. ☐

Theorem 1.13 (**Implicit Function**). *Let $U \subset E, V \subset F$ and $f : U \times V \to G$ be C^r. For some x_0, y_0 assume $D_2 f(x_0, y_0) : F \to G$ is an isomorphism. Then there are neighborhoods $x_0 \in U_0, f(x_0, y_0) \in W_0$ and a unique C^r map $g : U_0 \times W_0 \to V$ s.t. $\forall (x, w) \in U_0 \times W_0, f(x, g(x, w)) = w$.*

Proof. Consider the map $\Phi : (x, y) \mapsto (x, f(x, y))$, then

$$D\Phi(x_0, y_0) \cdot (x_1, y_1) = \begin{pmatrix} I & 0 \\ D_1 f(x_0, y_0) & D_2 f(x_0, y_0) \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

which is easily seen to be an isomorphism. Thus Φ has a unique C^r local inverse $\Phi^{-1} : (x, w) \mapsto (x, g(x, w))$. Then g so defined is the desired map. ☐

Exercises

Exercise 1.10. *Show that*

$$T^2 f : (U \times E) \times (E \times E) \to (F \times F) \times F \times F$$

$$(u, e_1, e_2, e_3) \mapsto (f(u), Df(u) \cdot e_1, Df(u) \cdot e_2, D^2 f(u) \cdot (e_1, e_2) + Df(u) \cdot e_3$$

Exercise 1.11. *Develop a formula for $D^r(f \circ g), D^r(fg)$.*

1.4 Manifolds and Mappings

Definition 1.25. A **local chart** on S is a bijection φ from a subsut U of S to an open subset of some (fimite-dimemtional, real) vector space F, denoted as (U, φ). An **atlas** on S is a family $\mathcal{A}$ of charts $\{(U_i, \varphi_i)\}$ s.t.

(1) $S = \cup U_i$;

(2) Any two charts are compatable in the sense that the overlap maps between members of $\mathcal{A}$ are C^∞ diffeomorphisms.

Two atlases $\mathcal{A}_1, \mathcal{A}_2$ are **equivalent** iff $\mathcal{A}_1 \cup \mathcal{A}_2$ is an atlas. A **differentiable structure** $\mathcal{S}$ on S is an equivalence class of atlases on S. The union of the atlases $\mathcal{A}_\} = \cup \mathcal{A}$ is the **maximal atlas**. A **differentiable manifold** M is a pair $(S, \mathcal{S})$. A manifold will always mean a Hausdorff, second countable, differentiable manifold.

Definition 1.26. Let $(S_1, \mathcal{S}_1), (S_2, \mathcal{S}_2)$ be two manifolds. The **product manifold** is $(S_1 \times S_2, \mathcal{S}_1 \times \mathcal{S}_2)$.

Definition 1.27. A **submanifold** of a manifold M is a subset $B \subset M$ with the property that for each $b \in B$ there is admissible chart (U, φ) in M with $b \in U$ which has the **submanifold property**, $\varphi : U \to E \times F$, and $\varphi(U \cap B) = \varphi(U) \cap (E \times \{0\})$. And its differentiable structure generated by the atlas is $\{(U \cap B, \varphi \mid U \cap B)\}$.

Definition 1.28. Suppose we have $f : M \to N$ and charts $(U, \varphi), (V, \phi)$. So the **local representative** of f, $f_{\varphi\phi} = \phi \circ f \circ \varphi^{-1}$.

Definition 1.29. A map $f : M \to N$ is called a diffeomorphism if f is of class C^r, is a bijection, and f^{-1} is of class C^r.

Exercises

Exercise 1.12. *Prove that S^1 is a submanifold of $\mathbb{R}^2$.*

1.5 Vector Bundles

Definition 1.30. Let E and F be vector spaces with U an open subset of E. We call the product $U \times F$ a **local vector bundle**. We call U the **base space**, which can be identified with $U \times \{0\}$, the zero section. For $u \in U$, $\{u\} \times F$ is called the **fiber** over u, which we can endow with the vector space structure of F. The map $\pi : U \times F \to U$ given by $\pi(u, f) = u$ is called the **projection** of $U \times F$. Thus, the fiber over u is $\pi^{-1}(u)$.

Definition 1.31. Let S be a set. A **local bundle chart** of S is a pair (U, φ) where $U \subset S$ and $\varphi : U \to U' \times F'$ is a bijection. A **vector bundle atlas** on S is a family $\mathcal{B} = \{(U_i, \varphi_i)\}$ statisfying:

(1) it covers S;

(2) for any two local bundle charts $(U_i, \varphi_i), (U_j, \varphi_j)$ with $U_i \cap U_j \neq \emptyset$ and the overlap map $\phi_{ji} = \varphi_j \circ \varphi_i^{-1} \mid \varphi_i(U_i \cap U_j)$ is a local vector bundle isomorphism.

A **vector bundle** E is a pair $(S, \mathcal{V})$, where $\mathcal{V}$ is a vector bundle structure on set S. We define the **zero section** by $E_0 = \{e \in E \mid \text{there exists}(U, \varphi), e = \varphi^{-1}(u', 0)$.

1.6 The Tangent Bundle

Let $\tau_U : TU \to U$ be the projections. We identify U with zero section $U \times \{0\}$. Then the diagram

$$\begin{tikzcd} TU \arrow[r, "Tf"] \arrow[d, "\tau_U"'] & TV \arrow[d, "\tau_V"] \\ U \arrow[r, "f"'] & V \end{tikzcd}$$

is commutative, that is, $f \circ \tau_U = \tau_V \circ Tf$. We will now extend the tangent functior T from this local context to the category of differentiable manifolds and mappings.

Definition 1.32. Let M be a manifold and $m \in M$. A curve at m is a C^1 map $c : I \to M$ with $0 \in I, c(0) = m$. Let c_1, c_2 be cureves at m and (U, φ) an admissible chart with $m \in U$. Then we say c_1, c_2 are **tangent** at m with respect φ iff $\varphi \circ c_1, \varphi \circ c_2$ are tangent at 0.

Propsition 1.14. *Suppose (U_β, φ_β) are admissble charts. Then c_1, c_2 are tangent at m with respect to φ_1 iff they are tangent at m with respect to φ_2.*

Proof. Let $D(\varphi_1 \circ c_1)(0) = D(\varphi_1 \circ c_2)(0)$ and $u_1 = U_2$. Then we have $\varphi_2 \circ c_i = (\varphi_2 \circ \varphi_1^{-1}) \circ (\varphi_1 \circ c_i)$. Then it follows that $D(\varphi_2 \circ c_1)(0) = D(\varphi_2 \circ c_2)(0)$. ☐

From the Proposition, we can say that c_1, c_2 are tangent at m for any local chart φ. An **equivalence class** of such curves is denoted $[c]_m$.

Definition 1.33. The **tangent space** of M at m is the set of equivalence calsses of curves at m $T_m(M) = \{[c]_m \mid c \text{is a curve at m}\}$. For a subset $A \subset M$, let $TM \mid A = m \underset{\cup}{\in} AT_m(M)$. We call $TM = TM \mid M$ the tangent bundle of M. The mapping $\tau_M : TM \to M$ defined by $\tau_M([c]_m) = m$, is the **tangent bundle projection**.

Propsition 1.15. *There is a unique $e \in E$ s.t. the curve $c_{u,e}$ defined by $c_{u,e}(t) = u + te$ is tangent to c at u.*

Proof. $Dc(0)$ is the unique linear map in $L(\mathbb{R}, E)$ s.t. the curve $g : \mathbb{R} \to E$ given by $g(t) = u + Dc(0) * t$ is tangent to c at t=0. If $e = Dc(0) \cdot 1$, then $g = c_{u,e}$. ☐

Propsition 1.16. *Suppose c_1, c_2 are tangent at m. Let $f : M \to N$ be of class C^1. Then $f \circ c_1, f \circ c_2$ are tangent at $f(m) \in N$.*

Proof. $(\phi \circ f \circ c_1)'(0) = (\phi \circ f' \circ \varphi^{-1})(\varphi \circ c_1)'(0) = (\phi \circ f' \circ \varphi^{-1})(\varphi \circ c_2)'(0) = (\phi \circ f \circ c_2)'(0)$ ☐

Definition 1.34. We define $Tf : TM \to TN$ by $Tf([c]_m) = [f \circ c]_{f(m)}$. We call Tf the **tangent** of f.

Theorem 1.14. *(1) Suppose $M\vec{f}N\vec{g}K$. Then $f \circ f : M \to K$ is of class C^1 and $T(g \circ f) = Tg \circ Tf$.*

(2) If $h : M \to M$ is the identity map, then Th is the identity map.

(3) If f is a diffeomorphism, then Tf is a bijection and $(Tf)^{-1} = T(f^-1)$.

Proof. Let $(U, \varphi), (V, \phi), (W, \rho)$ be charts of M,N,K. Then for the local representatives

$$\begin{aligned}(g \circ f)_{\varphi\rho} &= \rho \circ g \circ f \circ \varphi^{-1} \\ &= \rho \circ g \circ \phi^{-1} \circ \phi \circ f \circ \varphi^{-1} \\ &= g_{\phi\rho} \circ f_{\varphi\phi}\end{aligned}$$

(1) Thus, by the C^1 composite mapping theorem,

$$\begin{aligned}&T(g \circ f)[c]_m = [g \circ f \circ c]_{g \circ f(m)} \\ &Tg \circ Tf[c]_m = Tg([f \circ c]_{f(m)}) = [g \circ f \circ c]_{g \circ f(m)}\end{aligned}$$

(2) it is obvious. (3) f, f^{-1} are diffeomorphisms with $f \circ f^{-1}$ the identity on N, while $f^{-1} \circ f$ is the identity on M. By (1,2), $Tf \circ Tf^{-1}$ is the identity on TN, while $Tf^{-1} \circ Tf$ is the identity on TM. □

Theorem 1.15. *Let M be a manifold and $\mathcal{A}$ an atlas of admissible charts. Then $T\mathcal{A} = \{(TU, T\varphi) \mid (U, \varphi) \in \mathcal{A}\}$ is a vector bundle atlas of TM called a **natural atlas**.*

Proof. Since union of U covers M, the union of TU covers TM. Suppose we have $TU_i \cap TU_j \neq \emptyset$. Then $U_i \cap U_j \neq \emptyset$ and we can verify $T\varphi_i \circ (T\varphi_j)^{-1} = T(\varphi_i \circ \varphi_j^{-1})$ is a local vector bundle isomorphism. □

Theorem 1.16. *Suppose f and g are C^r mappings. Then $g \circ f$ is of C^r and $T^r(g \circ f) = T^r g \circ T^r f$.*

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General Theory by Kolmogorov