Notes of J.E. Marsden's – *Foundations of Mechanics*

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Part I

Preliminaries

Chapter 1

Differential Theory

1.1 Topology

Definition 1.1. A topological space is a set S together with a collect $\mathcal{O}$ of subsets called open sets s.t.

(T1) $\emptyset \in \mathcal{O}$ and $S \in \mathcal{O}$;

(T2) If $U_1, U_2 \in \mathcal{S}$, then $U_1 \cap U_2 \in \mathcal{O}$;

(T3) The union of any collection of open sets is open.

For such a topological space the **closed sets** are the elements of $\gamma = \{A \mid A^c \in \mathcal{O}\}$. An **open neighborhood of a point** u is a topological space S is an open set U s.t. $u \in U$. Similarly, for a subset A of S, U is an **open neighborhood** of A if U is open and $A \subset U$. If A is a subset of a topological space S, the **relative topology** on A is defined by $\mathcal{O}_A = \{U \cap A \mid U \in \mathcal{O}\}$.

Then a **basis** for the topology is a collection $\mathcal{B}$ of opensets s.t. every open set of S is a union of elements of $\mathcal{B}$. The topology is called **first coutable** if for each $u \in S$, there is a countable collection $\{U_n\}$ of neighborhoods of u s.t. for any neighborhood U of u, there is an n so $U_n \subset U$. The topology is called **second countable** if it has a countable basis.

Let $\{u_n\}$ be a sequence of points in S. The sequence is said to **coverge** if there is a point $u \in S$ s.t. for every neighborhood U of u, there is an N s.t. $n \geq N$ implies $u_n \in U$. We say that $\{u_n\}$ converges to u or u is a **limit point** of $\{u_n\}$.

Example 1.1. The standard topology of $\mathbb{R}$ is the unions of open intervals (a, b). Then $\mathbb{R}$ is second countable (hence first countable) with a basis

$$\left\{ \left(r_n - \frac{1}{m}, r_n + \frac{1}{m} \right) \mid r_n \in \mathbb{Q}, m \in \mathbb{N}^+ \right\}$$

.

Definition 1.2. Let S be a topological space and $A \subset S$. Then the **closure** of A, denoted $\overline{A}$ is the inersection of all colsed sets containing A. The **interior** of A, denoted $\mathring{A}$ is the union of all open sets contained in A. The **boundary** of A, denoted $\partial A := \overline{A} \cap \overline{A^c}$.

Thus, ∂A is closed, and $\partial A = \partial A^c$. Note that A is open iff $A = \mathring{A}$ and closed iff $A = \overline{A}$.

Propsition 1.1. *Let S be a topological space and $A \subset S$.*

(i) $u \in \overline{A}$ iff for every neighborhood U of u, $U \cap A \neq \emptyset$.

(ii) $u \in \mathring{A}$ iff there is a neighborhood U of u s.t. $U \subset A$.

(iii) $u \in \partial A$ iff for every neighborhood U of u, $U \cap A \neq \emptyset$ and $U \cap A^c \neq \emptyset$.

Definition 1.3. A point $u \in S$ is called **isolated** iff $\{u\}$ is open. A subset A of S is called **dense** in S iff $\overline{A} = S$ and is called **nowhere dense** iff $(\overline{A})^c$ is dense in S. Thus, A is nowhere dense iff $\mathring{\overline{A}} = \emptyset$.

Definition 1.4. A topological space S is called **Hausdorff** iff each two distinct points have disjoint neighborhoods. Similarly, S is called **normal** iff each two disjoint closed sets have disjoint neighborhoods.

Propsition 1.2. *(i) A space S is Hausdorff iff $\Delta_S = \{(u,u) \mid u \in S\}$ is closed in $S \times S$ is the product topology.*

(ii) A first countable space S is Hausdorff iff all sequences have at most one limit point.

Proof. If Δ_S is closed and u_1, u_2 are distinct, there is an open rectangle $U \times V$ containing (u_1, u_2) and $U \times V \subset \Delta_S^c$. Then in S, U and V are disjoint because if $\exists p \in U \cap V$, then $(p,p) \in U \times V$ it countered with closed set Δ_S. The converse is similar and we leave it as an exercise. □

Definition 1.5. Let $\mathbb{R}^+$ denote the nonnegative real numbers with a point $\{+\infty\}$ adjoined, and topology generated by the open intervals of the form (a,b). A **metric** on set M is a function $d : M \times M \to \mathbb{R}^+$ s.t.

(M1) $d(m_1, m_2) = 0$ iff $m_1 = m_2$;

(M2) $d(m_1, m_2) = d(m_2, m_1)$;

(M3) $d(m_1, m_3) \leq d(m_1, m_2) + d(m_2, m_3)$

For $\varepsilon > 0$, the ε **disk** about m is defined by $D_\varepsilon(m) = \{m' \in M \mid d(m', m) < \varepsilon$. The collection of subsets of M that are unions of such disks is the metric topology of the metric space (M,d). Two metrics on a set are called **equivalent** if they induce the same metric topology. $\{u_n\}$ is a **Cauchy sequence** iff for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ s.t. $n, m \geq N$ implies $d(u_n, u_m) < \varepsilon$. The space M is called **complete** if every Cauchy sequence converges. We define $d(u, A) = inf\{d(u,v) \mid v \in A\}$ and $d(u, \emptyset) = \infty$.

Propsition 1.3. *Every metric space is normal.*

Proof. Let A and B be closed, disjoint subsets of M, and let

$$U = \{u \in M \mid d(u,A) < d(u,B)\}$$
$$V = \{v \in M \mid d(v,A) > d(v,B)\}$$

It is verified that U and V are open, disjoint and $A \subset U, B \subset V$. □

Definition 1.6. If $\varphi : S \to T$ is **continuous** at $u \in S$ if $\forall V \ni \varphi(u), \exists U \ni u \Rightarrow \varphi(U) \subset V$. If $\forall V \subset T, \varphi^{-1}(V) = \{u \in S \mid \varphi(u) \in V$ is open in S, φ is **continuous**.

If $\varphi : S \to T$ is a **bijection**, φ and φ^{-1} are continuous, then φ is a **homeomorphism** and S and T are **homeomorphic**.

Propsition 1.4. *φ is continuous iff $\forall A \subset S \Rightarrow \varphi(\overline{A}) \subset \overline{\varphi(A)}$.*

Proof. If φ is continuous, then $\varphi^{-1}(\overline{\varphi(A)})$ is closed. But $A \subset \varphi^{-1}(\overline{\varphi(A)})$ and hence $\overline{A} \subset \varphi^{-1}(\overline{\varphi(A)})$ or $\varphi(\overline{A}) \subset \overline{\varphi(A)}$. Conversely, let $B \subset T$ be closed and $A = \varphi^{-1}(B)$. Then $\overline{(A)} \subset \varphi^{-1}(B) = A$, so A is closed. □

Propsition 1.5. *Let $\varphi : S \to T$ and S first countable set. Then φ is continuous iff $\forall u_n \to u \Rightarrow \varphi(u_n) \to \varphi(u)$.*

Propsition 1.6. *Let M and N be metric spaces with N complete. Then the collection $C(M, N)$ of all continuous maps $\varphi : M \to N$ forms a complete metric space with the metric $d^0(\varphi, \phi) = sup\{d(\varphi(u), \phi(u)) \mid u \in M\}$.*

Proof. It is readily verified that d^0 is a metric. Convergence of sequence $f_n \in C(M, N)$ to $f \in C(M, N)$ in the metric d^0 is the same as uniform convergence, that is, for all $\varepsilon > 0$ there is an N s.t. if $n \geq N$, $d(f_n(x), f(x)) \leq \varepsilon$ for all $x \in M$. If f_n is a Cauchy sequence in $C(M, N)$, then since $d(f_n(x), f_m(x)) \leq d^0(f_n, f_m)$. $f_n(x)$ is Cauchy for each point $x \in M$. Thus f_n converges pointwise, define a function $f(x)$. We must show that $f_n \to f$ uniformly and that f is continuous. First of all, given $\varepsilon > 0$, choose N s.t. $d^0(f_n, f_m) < \varepsilon/2$ if $n, m \geq N$. Then for any $x \in M$, pick $N_x \geq N$ s.t. $d(f_m(x), f(x)) < \varepsilon/2$ if $m \geq N_x$. Thus with $n \geq N$ and $m \geq N_x$, $d(f_n(x), f(x)) \leq d(f_n(x), f_m(x)) + d(f_m(x), f(x)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$ So $f_n \to f$ uniformly. □

Definition 1.7. S is called **compact** iff $\forall \cup_\alpha U_\alpha = S$ there is a finite subcovering. A subset $A \subset S$ is called **compact** iff A is compact in the relative topology. A space is called *locallycompact* iff each point has a neighborhood whose closure is compact.

Theorem 1.1 (**Boizano-Weierstrass**)**.** *If S is a first countable space and is compact, then every sequence has a convergent subsequence.*

Proof. Suppose $\{u_n\}$ contains no convergent subsequences. Then we may assume all points are distinct. Each u_n has a neighborhood $\mathcal{O}_n$ that contains no other u_m. $\{u_n\}$ is closed, so that $\mathcal{O}_n$ together with $\{u_n\}^c$ forms an open covering of S, with no finit subcovering. □

In a metric space, every compact subset is closed and bounded(Heine Borel theorem).

Propsition 1.7. *Let S be a Hausdorff space. Then every compact subset of S is closed. Also, every compact Hausdorff space is normal.*

Proof. Let $u \in A^c$ and $v \in A$, where A is compact in S. There are disjoint neighborhoods of u and v and, since A is compact, there are disjoint neighborhood of u and A. Thus A^c is open. The second part is an exercise. □

Propsition 1.8. *Let S be a Hausdorff space that is locally homeomorphic to a locally compact Hausdorff space. Then S is locally compact.*

Proof. Let $U \subset S$ be homeomorphic to $\varphi(U) \subset T$. There is a neighborhood V of $\varphi(u)$ so $\overline{V} \subset \varphi(U)$ and $\overline{V}$ is compact. Then $\varphi^{-1}(\overline{V})$ is compact, and hence closed in S. $\varphi^{-1}(\overline{V}) \subset \overline{\varphi^{-1}(V)}$. Thus $\varphi^{-1}(V)$ has compact closure. $\square$

Definition 1.8. A covering $\{U_\alpha\}$ of S is called a **refinement** of a covering $\{V_i\}$ iff $\forall U_\alpha$ there is a V_i s.t. $U_\alpha \subset V_i$. A covering $\{U_\alpha\}$ of S is called **locally finite** iff each point $u \in S$ has a neighborhood U such that U intersects only a finite number of U_α. A space is called **paracompact** iff every open covering of S has a locally finite refinement of open sets, and S is Hausdorff.

Theorem 1.2. *Second countable, locally compact Hausdorff spaces are paracompact.*

Proof. S is the countable union of open sets U_n s.t. $\overline{U_n}$ is compact and $\overline{U_n} \subset U_{n+1}$. If W_α is a covering of S by open sets, and $K_n = \overline{U_n} - U_{n-1}$ then we can cover K_n by a finit number of open sets each of which is contained in some $W_\alpha \cap U_{n+1}$, and is disjoint from $\overline{U_{n-2}}$. The union of such collections yiels the desired refinement of $\{W_\alpha\}$. $\square$

Theorem 1.3. *Every paracompact space is normal.*

Proof. We first show that if A is closed and $u \in A^c$, there are disjoint neighborhoods of u and A. For each $v \in A$ let U_u, V_v be disjoint neighborhood of u and v. Let W_α be a locally fninit refinement of the covering V_v, A^c and $V = \cup W_\alpha$, the union over those α so $W_\alpha \cap A \neq \emptyset$. A neighborhood U_0 of u meets a finite number of W_α. Let U denote the intersection of U_0 and the corresponding U_u. Then V and U are the required neighborhoods. $\square$

Theorem 1.4. *If S is a Hausdorff space, the following are equivalent:*

(i) *S is normal;*

(ii) *For any two closed nonempty disjoint set A, B ther is a continuous function $f : S \to [0,1]$ s.t. $f(A) = 0, f(B) = 1$.(Urysohn's Lemma)*

(iii) *For any closed set $A \subset S$ and continuous function $f : A \to [a,b]$, there is a continuous extension $\tilde{f} : S \to [a,b]$ of f (Tietze extension theorem)*

Definition 1.9. The **support** of $f : s \to \mathbb{R}$ is $supp(f) = \overline{\{x \in S \mid f(x) \neq 0\}}$. A **partition of unity** on S is a familiy of continuous mappings $\{\varphi_i : S \to [0,1]\}$ s.t.

(i) $\{supp(\varphi_i)\}$ is locally finite.

(ii) $\sum_i \varphi_i(x) = 1$ for all x.

We say that a pratition of unity $\{\varphi_i\}$ is **subordinate** to a covering $\{A_\alpha\}$ of S if $supp(\varphi_i)$ is a refrinement of $\{A_\alpha\}$.

Theorem 1.5. *Let S be paracompact and $\{U_i\}$ be any open covering of S. Then ther is a partition of unity $\{\varphi_i\}$ subordinate to $\{U_i\}$.*

Definition 1.10. A topological space S is **connected** if $\emptyset$ and S are the only subsets of S that are both open and closed. A subset of S is connected iff it is connected in the relative topology. A **component** A of S is a nonempty connected subsect of S s.t. the only connected subset of S containing A is A; S is called **locally connected** iff each point x has an open neighborhood containing a connected neighborhood of x.

Propsition 1.9. *A space S is not connected iff either of the following holds.*

(i) There is a nonempty proper subset of S that is both open and closed.

(ii) S is the disjoint union of two nonempty open sets.

(iii) S is the disjoint union of two nonempty closed sets.

Propsition 1.10. *Let S be a connected space and $f : S \to \mathbb{R}$ be continuous. Then f assumes every value between any two values $f(u), f(v)$.*

Proof. Suppose $f(u) < a < f(v)$ and f doses not assume the value a. Then $U = \{u_0 \mid f(u_0) < a\}$ is both open and closed. □

Propsition 1.11. *Let S be a topological space and $B \subset S$ be connected.*

(i) if $B \subset A \subset \overline{B}$, then A is connected;

(ii) if B_α are connected and $B_\alpha \cap B \neq \emptyset$, then $B \cup (U_\alpha B_\alpha)$ is connected.

Proof. If A is not connected, A is the disjoint union of $U_1 \cap A$ and $U_2 \cap A$ where U_1, U_2 are open in S. Then $U_1 \cap B \neq \emptyset, U_2 \cap B \neq \emptyset$, so B is not connected. □

Definition 1.11. An **arc** φ in S is a continuous mapping $\varphi : I = [0,1] \to S$. If $\varphi(0) = u, \varphi(1) = v$, we say φ joins u and v; S is called **arcwise connected** iff every two points in S can be joined by an arc in S. A space is called **locally arcwise connected** iff each point has an arcwise connected neighborhood.

Propsition 1.12. *Every arcwise connected space is connected. If a space is connected and locally arcwise connected, it is arcwise connected.*

Proof. If S is arcwise connected and not connected, write $S = U_1 \cup U_2$ where U_1, U_2 are nonempty, disjoint and open. Let $u_1 \in U_1, u_2 \in U_2$ and let φ be an arc joining u_1, u_2. Now $\varphi(I)$ is connected, and since $\varphi(I) \cap U_i \neq \emptyset$, $\varphi \cap U_1 \cap U_2 \neq \emptyset$. Hence $U_1 \cap U_2 \neq \emptyset$, a contradiction. Let $u \in S$and U denote all points that can be joined to u by an arc. An easy argument shows U and U^c are open and so $U = S$. □

Definition 1.12. Let S be a metric space with metric d, and 2^S denote the set of all subsets of S. Define $\tilde{d}(A,B) = supd(a,B)|a \in A$. As this is not symmetric, we further define $\overline{d}(A,B) = sup\{\tilde{d}(A,B), \tilde{d}(B,A)\}$. If $A \neq \emptyset, B = \emptyset, \overline{d}(A,B) = \infty, \overline{d}(\emptyset,\emptyset) = 0$. We call it the **Hausdorff metric**.

Propsition 1.13. *Let S be a metric space and d the Hausdorff metric on 2^S. Then $f : S \to 2^S$ is continuous at $u_0 \in S$ iff for all $\varepsilon > 0$ there is a $\delta > 0$ s.t. $d(u, u_0) < \delta$ implies:*

(i) *for all* $a \in f(u)$*, there is a* $b \in f(u_0)$ *s.t.* $d(a,b) < \varepsilon$*; that is*

$$f(u) \subset \cup_{b \in f(u_0)} D_\varepsilon(b)$$

.

(ii) *for all* $b \in f(u_0)$*, there is an* $a \in f(u)$ *s.t.* $d(b,a) < \varepsilon$*.*

Definition 1.13. Let S be a set. An **equivalence relation** $\sim$ on S is a binary relation s.t. for all $u, v, w \in S$

(i) $u \sim u$;

(ii) $u \sim v$ iff $v \sim u$;

(iii) $u \sim v, v \sim w \Rightarrow u \sim w$.

The **equivalence class** containing u, denoted $[u]$ is defined by $[u] = \{v \in S \mid u \sim v\}$. The set of equivalence classes is denote $S/\sim$, and the mapping $\pi : S \to S/\sim;\ u \longmapsto [u]$ is called the **canonical projection**.

Definition 1.14. $\{U \subset S/\sim \mid \pi^{-1}(U) \text{is open in} S\}$ is called the **quotient topology**.

Example 1.2. Consider $\mathbb{R}^2$ and the relation $\sim$ defined by $(a_1, a_2) \sim (b_1, b_2)$ iff $a_1 - b_1, a_2 - b_2 \in \mathbb{Z}$. Then $T^2 = \mathbb{R}^2/\sim$ is called the 2-**torus**. In addition to the quotient topology, it inherits a group structure in the usual way: $[(a_1, a_2)] + [(b_1, b_2)] = [(a_1, a_2) + (b_1, b_2)]$.

Example 1.3. The Klein bottle is obtained by reversing one of the orientations. Notice that K^2 is not "orientable" and does not inherit a group structure from $\mathbb{R}^2$.

Definition 1.15. Let Z be a topological space and $c : [0,1] \to Z$ a continuous map s.t. $c(0) = c(1) = p \in Z$. We call c a **loop** in Z based at p. The loop c is called **contractible** if there is a continuous map $H : [0,1] \times [0,1] \to Z$ s.t. $H(t,0) = c(t), H(t,1) = p, \forall t \in [0,1]$.

Definition 1.16. A space Z is called **simply connected** iff every loop in Z is contractible.

Definition 1.17. Let X be a topological space and $A \subset X$. Then A is called *residual* iff A is the intersection of a countable family of open dense subsets of X. A space X is called a **Baire space** iff every residual set is dense.

Lemma 1.1. *Let X be a locally Baire space; that is, each point* $x \in X$ *has a neighborhood* U *s.t.* $\overline{U}$ *is a Baire space. Then X is a Baire space.*

Proof. Let $A \subset X$ be residual; $A = \cap_1^\infty O_n$ where $\overline{O_n} = X$. Then $A \cap \overline{U} = \cap_1^\infty (O_n \cap \overline{U})$ Now $O_n \cap \overline{U}$ is dense in $\overline{U}$ for if $u \in \overline{U}, u \in O$ then $O \cap U \neq \emptyset, O \cap U \cap O_n \neq \emptyset$. Hence $\overline{U} \subset \overline{A}, \overline{A} = X$. □

Theorem 1.6 (**Baire category**). *Every complete pseudometric space is a Baire space.*

Proof. Let $U \subset X$ be open and $A = \cap_1^\infty O_n$ be residual. We must show $U \cap A \neq \emptyset$. Now as $\overline{O_n} = X, U \cap O_n \neq \emptyset$ and so we can choose a disk of diameter less than one, say V_1, s.t. $\overline{V_1} \subset U \cap O_1$. Proceed inductively to obtain $\overline{V_n} \subset U \cap O_n \cap V_{n-1}$, where V_n has diameter $< 1/n$. Let $x_n \in \overline{V_n}$. Clearly $\{x_n\}$ is a Cauchy sequence, and by completeness has a convergence subsequence with limit point x. Then $x \in \cap_1^\infty \overline{V_n}, U \cap \cap_1^\infty O_n \neq \emptyset$. □

Exercises

Exercise 1.1. *Let S and T be sets and $f : S \to T$. Show that f is a bijection iff there is a mapping $g : T \to S$ s.t. $f \circ g, g \circ f$ are identity mappings.*

Exercise 1.2. *Let X and Y be topological space with Y Hausdorff. Then show that, for any continuous maps $f, g : X \to Y, \{x \in X \mid f(x) = g(x)$ is closed. [Hint: Consider the mapping $x \longmapsto (f(x), g(x))$].*

Exercise 1.3. *Prove that in a Hausdorff space, single points are closed.*

Exercise 1.4. *Define a topological manifold as a space locally homeomorphic to $\mathbb{R}^n$. Find a topological manifold that is not Hausdorff and not locally compact. [Hint: Consier $\mathbb{R} \cup \{\pm\infty\}$].*

Exercise 1.5. *Show that the continuous image of a connected space is connected.*

1.2 Finite-Dimensional Banach Sapce

Definition 1.18. A **norm** on a vector space E is a mapping $\| \bullet \|: E \to \mathbb{R}$ s.t.

(N1) $\| \bullet \| \geq 0, \forall e \in E$ and $\| e \| = 0$ iff $e = 0$.

(N2) $\| \lambda e \| = |\lambda| \, \| e \|, \forall e \in E, \lambda \in \mathbb{R}$.

(N3) $\| e_1 + e_2 \| \leq \| e_1 \| + \| e_2 \|, \forall e_1, e_2 \in E$.

A normed space whose induced metric is complete is a **Banach space**.

Definition 1.19. Two norms on a vector space E are **equivalent** iff they induce the same topology on E.

Theorem 1.7. *Let E be a finite-dimensional real vector space. Then*

(i) there is a norm on E;

(ii) all norms on E are equivalent;

(iii) all norms on E are complete.

Theorem 1.8. *For finite-dimensional real vector spaces, linear and multilinear maps are continuous.*

Corollary 1.1. *Addition and scalar multiplication in a vector space are continuous maps from $E \times E \to E, \mathbb{R} \times E \to E$.*

Definition 1.20. Given E,F we let $L(E, F)$ denote the set of all linear maps from E into F together with the natural structure of finite dimensional real vector space. Similarly, $L^k(E, F)$ denote the space of multilinear maps from $E \times \cdots \times E$ into F, $L^k_s(E, F)$, the subspace of symmetric elements of $L^k(E, F)$ [that is, if π is any permutation of $\{1, 2 \cdots, k\}$, we have $f(e_1, \cdots, e_2) = f(e_{\pi(1)}, \cdots, e_{\pi(k)})$] and $L^k_a(E, F)$ the subspace of skew symmetric [That is if π is any permutation of $\{1, 2 \cdots, k\}$,we have $f(e_1, \cdots, e_2) = (sgn\pi) f(e_{\pi(1)}, \cdots, e_{\pi(k)})$, wher $sgn\pi = \pm 1$ according as π is an even or odd permutation].

Theorem 1.9. *There is a natural isomorphism $L(E, L^k(E, F)) \approx L^{k+1}(E, F)$.*

Exercises

Exercise 1.6. *Let $f \in L(E, F)$ so that f is continuous.*

(a) Show that there is a constant K s.t. $\| f(e) \| \leq K \| e \|$ for all $e \in E$. Define $\| f \|$ as the greatest lower bound of such K.

(b) Show that this is a norm on $L(E, F)$.

(c) Prove that $\| f \circ g \| \leq \| f \| \cdot \| g \|$.

Exercise 1.7. *Suppose $f \in L(E, F)$ and $dim(E) = dim(F)$. Then f is an isomorphism iff it is a monomorphism (one-to-one) and iff it is surjective (onto).*

Exercise 1.8. *Show that two norms $\| \bullet \|_1, \| \bullet \|_2$ are equivalent iff there is a constant M s.t. $M^{-1} \| e \|_1 \leq \| e \|_2 \leq M \| e \|_1$.*

Exercise 1.9. *Let E be the set of all C^1 functions $f : [0, 1] \to \mathbb{R}$ with the norm $\| f \| = sup_{x \in [0,1]} |f(x)| + sup_{x \in [0,1]} |f'(x)|$. Prove that E is a Banach space.*

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Exercises

9.8 Topology of the Two-Body Problem

Chapter 10

The Three-Body Problem

10.1 Models for Three Bodies

Exercises

10.2 Critical Points in the Restricted Three-Body Problem

Exercises

10.3 Closed Orbits in the Restricted Three-Body Problem

Exercises

10.4 Topology of the Plannar n-Body Problem

Appendix

General Theory by Kolmogorov