Notes of J.E. Marsden's – *Foundations of Mechanics*

Regoon Wang, ChemE@UNSW
wang.regoon@gmail.com

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Contents

Part I

Preliminaries

Chapter 1

Differential Theory

1.1 Topology

Definition 1.1. A topological space is a set S together with a collect $\mathcal{O}$ of subsets called open sets s.t.

(T1) $\emptyset \in \mathcal{O}$ and $S \in \mathcal{O}$;

(T2) If $U_1, U_2 \in \mathcal{S}$, then $U_1 \cap U_2 \in \mathcal{O}$;

(T3) The union of any collection of open sets is open.

For such a topological space the **closed sets** are the elements of $\gamma = \{A \mid A^c \in \mathcal{O}\}$. An **open neighborhood of a point** u is a topological space S is an open set U s.t. $u \in U$. Similarly, for a subset A of S, U is an **open neighborhood** of A if U is open and $A \subset U$. If A is a subset of a topological space S, the **relative topology** on A is defined by $\mathcal{O}_A = \{U \cap A \mid U \in \mathcal{O}\}$.

Then a **basis** for the topology is a collection $\mathcal{B}$ of opensets s.t. every open set of S is a union of elements of $\mathcal{B}$. The topology is called **first coutable** if for each $u \in S$, there is a countable collection $\{U_n\}$ of neighborhoods of u s.t. for any neighborhood U of u, there is an n so $U_n \subset U$. The topology is called **second countable** if it has a countable basis.

Let $\{u_n\}$ be a sequence of points in S. The sequence is said to **coverge** if there is a point $u \in S$ s.t. for every neighborhood U of u, there is an N s.t. $n \geq N$ implies $u_n \in U$. We say that $\{u_n\}$ converges to u or u is a **limit point** of $\{u_n\}$.

Example 1.1. The standard topology of $\mathbb{R}$ is the unions of open intervals (a, b). Then $\mathbb{R}$ is second countable (hence first countable) with a basis

$$\left\{\left(r_n - \frac{1}{m}, r_n + \frac{1}{m}\right) \mid r_n \in \mathbb{Q}, m \in \mathbb{N}^+\right\}$$

.

Definition 1.2. Let S be a topological space and $A \subset S$. Then the **closure** of A, denoted $\overline{A}$ is the inersection of all colsed sets containing A. The **interior** of A, denoted $\mathring{A}$ is the union of all open sets contained in A. The **boundary** of A, denoted $\partial A := \overline{A} \cap \overline{A^c}$.

Thus, ∂A is closed, and $\partial A = \partial A^c$. Note that A is open iff $A = \mathring{A}$ and closed iff $A = \overline{A}$.

Propsition 1.1. *Let S be a topological space and $A \subset S$.*

(i) $u \in \overline{A}$ iff for every neighborhood U of u, $U \cap A \neq \emptyset$.

(ii) $u \in \mathring{A}$ iff there is a neighborhood U of u s.t. $U \subset A$.

(iii) $u \in \partial A$ iff for every neighborhood U of u, $U \cap A \neq \emptyset$ and $U \cap A^c \neq \emptyset$.

Definition 1.3. A point $u \in S$ is called **isolated** iff $\{u\}$ is open. A subset A of S is called **dense** in S iff $\overline{A} = S$ and is called **nowhere dense** iff $(\overline{A})^c$ is dense in S. Thus, A is nowhere dense iff $\mathring{\overline{A}} = \emptyset$.

Definition 1.4. A topological space S is called **Hausdorff** iff each two distinct points have disjoint neighborhoods. Similarly, S is called **normal** iff each two disjoint closed sets have disjoint neighborhoods.

Propsition 1.2. *(i) A space S is Hausdorff iff $\Delta_S = \{(u,u) \mid u \in S\}$ is closed in $S \times S$ is the product topology.*

(ii) A first countable space S is Hausdorff iff all sequences have at most one limit point.

Proof. If Δ_S is closed and u_1, u_2 are distinct, there is an open rectangle $U \times V$ containing (u_1, u_2) and $U \times V \subset \Delta_S^c$. Then in S, U and V are disjoint because if $\exists p \in U \cap V$, then $(p,p) \in U \times V$ it countered with closed set Δ_S. The converse is similar and we leave it as an exercise. □

Definition 1.5. Let $\mathbb{R}^+$ denote the nonnegative real numbers with a point $\{+\infty\}$ adjoined, and topology generated by the open intervals of the form (a,b). A **metric** on set M is a function $d : M \times M \to \mathbb{R}^+$ s.t.

(M1) $d(m_1, m_2) = 0$ iff $m_1 = m_2$;

(M2) $d(m_1, m_2) = d(m_2, m_1)$;

(M3) $d(m_1, m_3) \leq d(m_1, m_2) + d(m_2, m_3)$

For $\varepsilon > 0$, the ε **disk** about m is defined by $D_\varepsilon(m) = \{m' \in M \mid d(m', m) < \varepsilon$. The collection of subsets of M that are unions of such disks is the metric topology of the metric space (M,d). Two metrics on a set are called **equivalent** if they induce the same metric topology. $\{u_n\}$ is a **Cauchy sequence** iff for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ s.t. $n, m \geq N$ implies $d(u_n, u_m) < \varepsilon$. The space M is called **complete** if every Cauchy sequence converges. We define $d(u, A) = inf\{d(u,v) \mid v \in A\}$ and $d(u, \emptyset) = \infty$.

Propsition 1.3. *Every metric space is normal.*

Proof. Let A and B be closed, disjoint subsets of M, and let

$$U = \{u \in M \mid d(u,A) < d(u,B)\}$$
$$V = \{v \in M \mid d(v,A) > d(v,B)\}$$

It is verified that U and V are open, disjoint and $A \subset U, B \subset V$. □

Definition 1.6. If $\varphi : S \to T$ is **continuous** at $u \in S$ if $\forall V \ni \varphi(u), \exists U \ni u \Rightarrow \varphi(U) \subset V$. If $\forall V \subset T, \varphi^{-1}(V) = \{u \in S \mid \varphi(u) \in V$ is open in S, φ is **continuous**.

If $\varphi : S \to T$ is a **bijection**, φ and φ^{-1} are continuous, then φ is a **homeomorphism** and S and T are **homeomorphic**.

Propsition 1.4. *φ is continuous iff $\forall A \subset S \Rightarrow \varphi(\overline{A}) \subset \overline{\varphi(A)}$.*

Proof. If φ is continuous, then $\varphi^{-1}(\overline{\varphi(A)})$ is closed. But $A \subset \varphi^{-1}(\overline{\varphi(A)})$ and hence $\overline{A} \subset \varphi^{-1}(\overline{\varphi(A)})$ or $\varphi(\overline{A}) \subset \overline{\varphi(A)}$. Conversely, let $B \subset T$ be closed and $A = \varphi^{-1}(B)$. Then $\overline{(A)} \subset \varphi^{-1}(B) = A$, so A is closed. □

Propsition 1.5. *Let $\varphi : S \to T$ and S first countable set. Then φ is continuous iff $\forall u_n \to u \Rightarrow \varphi(u_n) \to \varphi(u)$.*

Propsition 1.6. *Let M and N be metric spaces with N complete. Then the collection $C(M,N)$ of all continuous maps $\varphi : M \to N$ forms a complete metric space with the metric $d^0(\varphi, \phi) = sup\{d(\varphi(u), \phi(u)) \mid u \in M\}$.*

Proof. It is readily verified that d^0 is a metric. Convergence of sequence $f_n \in C(M,N)$ to $f \in C(M,N)$ in the metric d^0 is the same as uniform convergence, that is, for all $\varepsilon > 0$ there is an N s.t. if $n \geq N$, $d(f_n(x), f(x)) \leq \varepsilon$ for all $x \in M$. If f_n is a Cauchy sequence in $C(M,N)$, then since $d(f_n(x), f_m(x)) \leq d^0(f_n, f_m)$. $f_n(x)$ is Cauchy for each point $x \in M$. Thus f_n converges pointwise, define a function $f(x)$. We must show that $f_n \to f$ uniformly and that f is continuous. First of all, given $\varepsilon > 0$, choose N s.t. $d^0(f_n, f_m) < \varepsilon/2$ if $n, m \geq N$. Then for any $x \in M$, pick $N_x \geq N$ s.t. $d(f_m(x), f(x)) < \varepsilon/2$ if $m \geq N_x$. Thus with $n \geq N$ and $m \geq N_x$, $d(f_n(x), f(x)) \leq d(f_n(x), f_m(x)) + d(f_m(x), f(x)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$ So $f_n \to f$ uniformly. □

Definition 1.7. S is called **compact** iff $\forall \cup_\alpha U_\alpha = S$ there is a finite subcovering. A subset $A \subset S$ is called **compact** iff A is compact in the relative topology. A space is called *locallycompact* iff each point has a neighborhood whose closure is compact.

Theorem 1.1 (Boizano-Weierstrass). *If S is a first countable space and is compact, then every sequence has a convergent subsequence.*

Proof. Suppose $\{u_n\}$ contains no convergent subsequences. Then we may assume all points are distinct. Each u_n has a neighborhood $\mathcal{O}_n$ that contains no other u_m. $\{u_n\}$ is closed, so that $\mathcal{O}_n$ together with $\{u_n\}^c$ forms an open covering of S, with no finit subcovering. □

In a metric space, every compact subset is closed and bounded(Heine Borel theorem).

Propsition 1.7. *Let S be a Hausdorff space. Then every compact subset of S is closed. Also, every compact Hausdorff space is normal.*

Proof. Let $u \in A^c$ and $v \in A$, where A is compact in S. There are disjoint neighborhoods of u and v and, since A is compact, there are disjoint neighborhood of u and A. Thus A^c is open. The second part is an exercise. □

Propsition 1.8. *Let S be a Hausdorff space that is locally homeomorphic to a locally compact Hausdorff space. Then S is locally compact.*

Proof. Let $U \subset S$ be homeomorphic to $\varphi(U) \subset T$. There is a neighborhood V of $\varphi(u)$ so $\overline{V} \subset \varphi(U)$ and $\overline{V}$ is compact. Then $\varphi^{-1}(\overline{V})$ is compact, and hence closed in S. $\varphi^{-1}(\overline{V}) \subset \overline{\varphi^{-1}(V)}$. Thus $\varphi^{-1}(V)$ has compact closure. $\square$

Definition 1.8. A covering $\{U_\alpha\}$ of S is called a **refinement** of a covering $\{V_i\}$ iff $\forall U_\alpha$ there is a V_i s.t. $U_\alpha \subset V_i$. A covering $\{U_\alpha\}$ of S is called **locally finite** iff each point $u \in S$ has a neighborhood U such that U intersects only a finite number of U_α. A space is called **paracompact** iff every open covering of S has a locally finite refinement of open sets, and S is Hausdorff.

Theorem 1.2. *Second countable, locally compact Hausdorff spaces are paracompact.*

Proof. S is the countable union of open sets U_n s.t. $\overline{U_n}$ is compact and $\overline{U_n} \subset U_{n+1}$. If W_α is a covering of S by open sets, and $K_n = \overline{U_n} - U_{n-1}$ then we can cover K_n by a finit number of open sets each of which is contained in some $W_\alpha \cap U_{n+1}$, and is disjoint from $\overline{U_{n-2}}$. The union of such collections yiels the desired refinement of $\{W_\alpha\}$. $\square$

Theorem 1.3. *Every paracompact space is normal.*

Proof. We first show that if A is closed and $u \in A^c$, there are disjoint neighborhoods of u and A. For each $v \in A$ let U_u, V_v be disjoint neighborhood of u and v. Let W_α be a locally fninit refinement of the covering V_v, A^c and $V = \cup W_\alpha$, the union over those α so $W_\alpha \cap A \neq \emptyset$. A neighborhood U_0 of u meets a finite number of W_α. Let U denote the intersection of U_0 and the corresponding U_u. Then V and U are the required neighborhoods. $\square$

Theorem 1.4. *If S is a Hausdorff space, the following are equivalent:*

(i) S is normal;

(ii) For any two closed nonempty disjoint set A, B ther is a continuous function $f : S \to [0,1]$ s.t. $f(A) = 0, f(B) = 1$.(Urysohn's Lemma)

(iii) For any closed set $A \subset S$ and continuous function $f : A \to [a,b]$, there is a continuous extension $\tilde{f} : S \to [a,b]$ of f (Tietze extension theorem)

Definition 1.9. The **support** of $f : s \to \mathbb{R}$ is $supp(f) = \overline{\{x \in S \mid f(x) \neq 0\}}$. A **partition of unity** on S is a familiy of continuous mappings $\{\varphi_i : S \to [0,1]\}$ s.t.

(i) $\{supp(\varphi_i)\}$ is locally finite.

(ii) $\sum_i \varphi_i(x) = 1$ for all x.

We say that a pratition of unity $\{\varphi_i\}$ is **subordinate** to a covering $\{A_\alpha\}$ of S if $supp(\varphi_i)$ is a refrinement of $\{A_\alpha\}$.

Theorem 1.5. *Let S be paracompact and $\{U_i\}$ be any open covering of S. Then ther is a partition of unity $\{\varphi_i\}$ subordinate to $\{U_i\}$.*

Definition 1.10. A topological space S is **connected** if $\emptyset$ and S are the only subsets of S that are both open and closed. A subset of S is connected iff it is connected in the relative topology. A **component** A of S is a nonempty connected subsect of S s.t. the only connected subset of S containing A is A; S is called **locally connected** iff each point x has an open neighborhood containing a connected neighborhood of x.

Propsition 1.9. *A space S is not connected iff either of the following holds.*

(i) There is a nonempty proper subset of S that is both open and closed.

(ii) S is the disjoint union of two nonempty open sets.

(iii) S is the disjoint union of two nonempty closed sets.

Propsition 1.10. *Let S be a connected space and $f : S \to \mathbb{R}$ be continuous. Then f assumes every value between any two values $f(u), f(v)$.*

Proof. Suppose $f(u) < a < f(v)$ and f doses not assume the value a. Then $U = \{u_0 \mid f(u_0) < a\}$ is both open and closed. $\square$

Propsition 1.11. *Let S be a topological space and $B \subset S$ be connected.*

(i) if $B \subset A \subset \overline{B}$, then A is connected;

(ii) if B_α are connected and $B_\alpha \cap B \neq \emptyset$, then $B \cup (U_\alpha B_\alpha)$ is connected.

Proof. If A is not connected, A is the disjoint union of $U_1 \cap A$ and $U_2 \cap A$ where U_1, U_2 are open in S. Then $U_1 \cap B \neq \emptyset, U_2 \cap B \neq \emptyset$, so B is not connected. $\square$

Definition 1.11. An **arc** φ in S is a continuous mapping $\varphi : I = [0, 1] \to S$. If $\varphi(0) = u, \varphi(1) = v$, we say φ joins u and v; S is called **arcwise connected** iff every two points in S can be joined by an arc in S. A space is called **locally arcwise connected** iff each point has an arcwise connected neighborhood.

Propsition 1.12. *Every arcwise connected space is connected. If a space is connected and locally arcwise connected, it is arcwise connected.*

Proof. If S is arcwise connected and not connected, write $S = U_1 \cup U_2$ where U_1, U_2 are nonempty, disjoint and open. Let $u_1 \in U_1, u_2 \in U_2$ and let φ be an arc joining u_1, u_2. Now $\varphi(I)$ is connected, and since $\varphi(I) \cap U_i \neq \emptyset$, $\varphi \cap U_1 \cap U_2 \neq \emptyset$. Hence $U_1 \cap U_2 \neq \emptyset$, a contradiction. Let $u \in S$and U denote all points that can be joined to u by an arc. An easy argument shows U and U^c are open and so $U = S$. $\square$

Definition 1.12. Let S be a metric space with metric d, and 2^S denote the set of all subsets of S. Define $\tilde{d}(A, B) = supd(a, B)|a \in A$. As this is not symmetric, we further define $\overline{d}(A, B) = sup\{\tilde{d}(A, B), \tilde{d}(B, A)\}$. If $A \neq \emptyset, B = \emptyset, \overline{d}(A, B) = \infty, \overline{d}(\emptyset, \emptyset) = 0$. We call it the **Hausdorff metric**.

Propsition 1.13. *Let S be a metric space and d the Hausdorff metric on 2^S. Then $f : S \to 2^S$ is continuous at $u_0 \in S$ iff for all $\varepsilon > 0$ there is a $\delta > 0$ s.t. $d(u, u_0) < \delta$ implies:*

(i) for all $a \in f(u)$, there is a $b \in f(u_0)$ s.t. $d(a,b) < \varepsilon$; that is

$$f(u) \subset \cup_{b \in f(u_0)} D_\varepsilon(b)$$

.

(ii) for all $b \in f(u_0)$, there is an $a \in f(u)$ s.t. $d(b,a) < \varepsilon$.

Definition 1.13. Let S be a set. An **equivalence relation** $\sim$ on S is a binary relation s.t. for all $u, v, w \in S$

(i) $u \sim u$;

(ii) $u \sim v$ iff $v \sim u$;

(iii) $u \sim v, v \sim w \Rightarrow u \sim w$.

The **equivalence class** containing u, denoted $[u]$ is defined by $[u] = \{v \in S \mid u \sim v\}$. The set of equivalence classes is denote $S/\sim$, and the mapping $\pi : S \to S/\sim; u \longmapsto [u]$ is called the **canonical projection**.

Definition 1.14. $\{U \subset S/\sim \mid \pi^{-1}(U) \text{is open in} S\}$ is called the **quotient topology**.

Example 1.2. Consider $\mathbb{R}^2$ and the relation $\sim$ defined by $(a_1, a_2) \sim (b_1, b_2)$ iff $a_1 - b_1, a_2 - b_2 \in \mathbb{Z}$. Then $T^2 = \mathbb{R}^2/\sim$ is called the **2-torus**. In addition to the quotient topology, it inherits a group structure in the usual way: $[(a_1, a_2)] + [(b_1, b_2)] = [(a_1, a_2) + (b_1, b_2)]$.

Example 1.3. The Klein bottle is obtained by reversing one of the orientations. Notice that K^2 is not "orientable" and does not inherit a group structure from $\mathbb{R}^2$.

Definition 1.15. Let Z be a topological space and $c : [0,1] \to Z$ a continuous map s.t. $c(0) = c(1) = p \in Z$. We call c a **loop** in Z based at p. The loop c is called **contractible** if there is a continuous map $H : [0,1] \times [0,1] \to Z$ s.t. $H(t,0) = c(t), H(t,1) = p, \forall t \in [0,1]$.

Definition 1.16. A space Z is called **simply connected** iff every loop in Z is contractible.

Definition 1.17. Let X be a topological space and $A \subset X$. Then A is called *residual* iff A is the intersection of a countable family of open dense subsets of X. A space X is called a **Baire space** iff every residual set is dense.

Lemma 1.1. *Let X be a locally Baire space; that is, each point $x \in X$ has a neighborhood U s.t. $\overline{U}$ is a Baire space. Then X is a Baire space.*

Proof. Let $A \subset X$ be residual; $A = \cap_1^\infty O_n$ where $\overline{O_n} = X$. Then $A \cap \overline{U} = \cap_1^\infty (O_n \cap \overline{U})$ Now $O_n \cap \overline{U}$ is dense in $\overline{U}$ for if $u \in \overline{U}, u \in O$ then $O \cap U \neq \emptyset, O \cap U \cap O_n \neq \emptyset$. Hence $\overline{U} \subset \overline{A}, \overline{A} = X$. $\square$

Theorem 1.6 (Baire category). *Every complete pseudometric space is a Baire space.*

Proof. Let $U \subset X$ be open and $A = \cap_1^\infty O_n$ be residual. We must show $U \cap A \neq \emptyset$. Now as $\overline{O_n} = X, U \cap O_n \neq \emptyset$ and so we can choose a disk of diameter less than one, say V_1, s.t. $\overline{V_1} \subset U \cap O_1$. Proceed inductively to obtain $\overline{V_n} \subset U \cap O_n \cap V_{n-1}$, where V_n has diameter $< 1/n$. Let $x_n \in \overline{V_n}$. Clearly $\{x_n\}$ is a Cauchy sequence, and by completeness has a convergence subsequence with limit point x. Then $x \in \cap_1^\infty \overline{V_n}, U \cap \cap_1^\infty O_n \neq \emptyset$. $\square$

Exercises

Exercise 1.1. *Let S and T be sets and $f : S \to T$. Show that f is a bijection iff there is a mapping $g : T \to S$ s.t. $f \circ g, g \circ f$ are identity mappings.*

Exercise 1.2. *Let X and Y be topological space with Y Hausdorff. Then show that, for any continuous maps $f, g : X \to Y, \{x \in X \mid f(x) = g(x)$ is closed. [Hint: Consider the mapping $x \longmapsto (f(x), g(x))$].*

Exercise 1.3. *Prove that in a Hausdorff space, single points are closed.*

Exercise 1.4. *Define a topological manifold as a space locally homeomorphic to $\mathbb{R}^n$. Find a topological manifold that is not Hausdorff and not locally compact. [Hint: Consier $\mathbb{R} \cup \{\pm\infty\}$].*

1.2 Finite-Dimensional Banach Sapce

Exercises

1.3 Local Differential Calculus

Exercises

1.4 Manifolds and Mappings

Exercises

1.5 Vector Bundles

Exercises

1.6 The Tangent Bundle

Exercises

1.7 Tensors

Exercises

Chapter 2

Calculus on Manifolds

2.1 Vector Fields as Dynamical Systems

Exercises

2.2 Vector Fields as Differential Operators

Exercises

2.3 Exterior Algebra

Exercises

2.4 Cartan's Calculus of Differential Forms

Exercises

2.5 Orientable Manifolds

Exercises

2.6 Integration on Manifolds

Exercises

2.7 Some Riemannian Geometry

Exercises

Part II

Analytical Dynamics

Chapter 3

Hamiltonian and Lagrangian Systems

3.1 Symplectic Algebra

Exercises

3.2 Symplectic Geometry

Exercises

3.3 Hamiltonian Vector Fields and Poisson Brackets

Exercises

3.4 Integral Invariants, Energy Surfaces, and Stability

Exercises

3.5 Lagrangian Systems

Exercises

3.6 The Legendre Transformation

Exercises

3.7 Mechanics on Riemannian Manifolds

Exercises

3.8 Variational Principles in Mechanics

Exercises

Chapter 4

Hamiltonian Systems with Symmetry

4.1 Lie Groups and Group Actions

Exercises

4.2 The Momentum Mapping

Exercises

4.3 Reduction of Phase Space with Symmetry

Exercises

4.4 Hamiltonian Systems on Lie Groups and the Rigid Body

Exercises

4.5 The Topology of Simple Mechanical Systems

Exercises

4.6 The Topology of the Rigid Body

Exercises

Chapter 5

Hamilton-Jacobi Theory and Mathematical Physics

5.1 Time-Dependent Systems

Exercises

5.2 Canonical Transformations and Hamilton-Jacobi Theory

Exercises

5.3 Lagrangian Submanifolds

Exercises

5.4 Quantization

Exercises

5.5 Introduction to Infinite-Dimensional Hamiltonian Systems

Exercises

5.6 Introduction to Nonlinear Oscillations

Part III

An Outline of Qualitative Dynamics

Chapter 6

Topological Dynamics

6.1 Limit and Minimal Sets

Exercises

6.2 Recurrence

Exercises

6.3 Stability

Exercises

Chapter 7

Differentiable Dynamics

7.1 Critical Elements

Exercises

7.2 Stable Manifolds

Exercises

7.3 Generic Properties

Exercises

7.4 Structural Stability

Exercises

7.5 Absolute Stability and Axiom A

Exercises

7.6 Bifurcations of Generic Arcs

Exercises

7.7 A Zoo of Stable Bifurcations

7.8 Experimental Dynamics

Chapter 8

Hamiltonian Dynamics

8.1 Critical Elements

8.2 Orbit Cylinders

Exercises

8.3 Stability of Orbits

Exercises

8.4 Generic Properties

Exercises

8.5 Structural Stability

8.6 A Zoo of stable Bifurcations

8.7 The General Pathology

8.8 Experimental Mechanics

Part IV

Celestial Mechanics

Chapter 9

The Two-Body Problem

9.1 Models for Two Bodies

Exercises

9.2 Elliptic Orbits and Kepler Elements

9.3 The Delaunay Variables

9.4 Lagrange Brackets of Kepler Elements

9.5 Whittaker's Method

9.6 Poincare Variables

Exercises

9.7 Summary of Models

Exercises

9.8 Topology of the Two-Body Problem

Chapter 10

The Three-Body Problem

10.1 Models for Three Bodies

Exercises

10.2 Critical Points in the Restricted Three-Body Problem

Exercises

10.3 Closed Orbits in the Restricted Three-Body Problem

Exercises

10.4 Topology of the Plannar n-Body Problem

Appendix

General Theory by Kolmogorov