Notes of J.E. Marsden's – ***Foundations of Mechanics***

Regoon Wang, ChemE@UNSW
wang.regoon@gmail.com

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Part I

Preliminaries

Chapter 1

Differential Theory

1.1 Topology

Definition 1.1. A topological space is a set S together with a collect $\mathcal{O}$ of subsets called open sets s.t.

(T1) $\emptyset \in \mathcal{O}$ and $S \in \mathcal{O}$;

(T2) If $U_1, U_2 \in \mathcal{S}$, then $U_1 \cap U_2 \in \mathcal{O}$;

(T3) The union of any collection of open sets is open.

For such a topological space the **closed sets** are the elements of $\gamma = \{A \mid A^c \in \mathcal{O}\}$. An **open neighborhood of a point** u is a topological space S is an open set U s.t. $u \in U$. Similarly, for a subset A of S, U is an **open neighborhood** of A if U is open and $A \subset U$. If A is a subset of a topological space S, the **relative topology** on A is defined by $\mathcal{O}_A = \{U \cap A \mid U \in \mathcal{O}\}$.

Then a **basis** for the topology is a collection $\mathcal{B}$ of opensets s.t. every open set of S is a union of elements of $\mathcal{B}$. The topology is called **first coutable** if for each $u \in S$, there is a countable collection $\{U_n\}$ of neighborhoods of u s.t. for any neighborhood U of u, there is an n so $U_n \subset U$. The topology is called **second countable** if it has a countable basis.

Let $\{u_n\}$ be a sequence of points in S. The sequence is said to **coverge** if there is a point $u \in S$ s.t. for every neighborhood U of u, there is an N s.t. $n \geq N$ implies $u_n \in U$. We say that $\{u_n\}$ converges to u or u is a **limit point** of $\{u_n\}$.

Example 1.1. The standard topology of $\mathbb{R}$ is the unions of open intervals (a, b). Then $\mathbb{R}$ is second countable (hence first countable) with a basis

$$\left\{ \left(r_n - \frac{1}{m}, r_n + \frac{1}{m} \right) \mid r_n \in \mathbb{Q}, m \in \mathbb{N}^+ \right\}$$

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Definition 1.2. Let S be a topological space and $A \subset S$. Then the **closure** of A, denoted $\overline{A}$ is the inersection of all colsed sets containing A. The **interior** of A, denoted $\mathring{A}$ is the union of all open sets contained in A. The **boundary** of A, denoted $\partial A := \overline{A} \cap \overline{A^c}$.

Thus, ∂A is closed, and $\partial A = \partial A^c$. Note that A is open iff $A = \mathring{A}$ and closed iff $A = \overline{A}$.

Propsition 1.1. *Let S be a topological space and $A \subset S$.*

(i) $u \in \overline{A}$ iff for every neighborhood U of u, $U \cap A \neq \emptyset$.

(ii) $u \in \mathring{A}$ iff there is a neighborhood U of u s.t. $U \subset A$.

(iii) $u \in \partial A$ iff for every neighborhood U of u, $U \cap A \neq \emptyset$ and $U \cap A^c \neq \emptyset$.

Definition 1.3. A point $u \in S$ is called **isolated** iff $\{u\}$ is open. A subset A of S is called **dense** in S iff $\overline{A} = S$ and is called **nowhere dense** iff $(\overline{A})^c$ is dense in S. Thus, A is nowhere dense iff $\mathring{\overline{A}} = \emptyset$.

Definition 1.4. A topological space S is called **Hausdorff** iff each two distinct points have disjoint neighborhoods. Similarly, S is called **normal** iff each two disjoint closed sets have disjoint neighborhoods.

Propsition 1.2. *(i) A space S is Hausdorff iff $\Delta_S = \{(u,u) \mid u \in S\}$ is closed in $S \times S$ is the product topology.*

(ii) A first countable space S is Hausdorff iff all sequences have at most one limit point.

Proof. If Δ_S is closed and u_1, u_2 are distinct, there is an open rectangle $U \times V$ containing (u_1, u_2) and $U \times V \subset \Delta_S^c$. Then in S, U and V are disjoint because if $\exists p \in U \cap V$, then $(p,p) \in U \times V$ it countered with closed set Δ_S. The converse is similar and we leave it as an exercise. $\square$

Definition 1.5. Let $\mathbb{R}^+$ denote the nonnegative real numbers with a point $\{+\infty\}$ adjoined, and topology generated by the open intervals of the form (a,b). A **metric** on set M is a function $d : M \times M \to \mathbb{R}^+$ s.t.

(M1) $d(m_1, m_2) = 0$ iff $m_1 = m_2$;

(M2) $d(m_1, m_2) = d(m_2, m_1)$;

(M3) $d(m_1, m_3) \leq d(m_1, m_2) + d(m_2, m_3)$

For $\varepsilon > 0$, the ε **disk** about m is defined by $D_\varepsilon(m) = \{m' \in M \mid d(m', m) < \varepsilon$. The collection of subsets of M that are unions of such disks is the metric topology of the metric space (M,d). Two metrics on a set are called **equivalent** if they induce the same metric topology. $\{u_n\}$ is a **Cauchy sequence** iff for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ s.t. $n, m \geq N$ implies $d(u_n, u_m) < \varepsilon$. The space M is called **complete** if every Cauchy sequence converges. We define $d(u, A) = inf\{d(u,v) \mid v \in A\}$ and $d(u, \emptyset) = \infty$.

Propsition 1.3. *Every metric space is normal.*

Proof. Let A and B be closed, disjoint subsets of M, and let

$$U = \{u \in M \mid d(u,A) < d(u,B)\}$$
$$V = \{v \in M \mid d(v,A) > d(v,B)\}$$

It is verified that U and V are open, disjoint and $A \subset U, B \subset V$. $\square$

Definition 1.6. If $\varphi : S \to T$ is **continuous** at $u \in S$ if $\forall V \ni \varphi(u), \exists U \ni u \Rightarrow \varphi(U) \subset V$. If $\forall V \subset T, \varphi^{-1}(V) = \{u \in S \mid \varphi(u) \in V$ is open in S, φ is **continuous**.

If $\varphi : S \to T$ is a **bijection**, φ and φ^{-1} are continuous, then φ is a **homeomorphism** and S and T are **homeomorphic**.

Propsition 1.4. *φ is continuous iff $\forall A \subset S \Rightarrow \varphi(\overline{A}) \subset \overline{\varphi(A)}$.*

Proof. If φ is continuous, then $\varphi^{-1}(\overline{\varphi(A)})$ is closed. But $A \subset \varphi^{-1}(\overline{\varphi(A)})$ and hence $\overline{A} \subset \varphi^{-1}(\overline{\varphi(A)})$ or $\varphi(\overline{A}) \subset \overline{\varphi(A)}$. Conversely, let $B \subset T$ be closed and $A = \varphi^{-1}(B)$. Then $\overline{(A)} \subset \varphi^{-1}(B) = A$, so A is closed. □

Propsition 1.5. *Let $\varphi : S \to T$ and S first countable set. Then φ is continuous iff $\forall u_n \to u \Rightarrow \varphi(u_n) \to \varphi(u)$.*

Propsition 1.6. *Let M and N be metric spaces with N complete. Then the collection $C(M, N)$ of all continuous maps $\varphi : M \to N$ forms a complete metric space with the metric $d^0(\varphi, \phi) = sup\{d(\varphi(u), \phi(u)) \mid u \in M\}$.*

Proof. It is readily verified that d^0 is a metric. Convergence of sequence $f_n \in C(M, N)$ to $f \in C(M, N)$ in the metric d^0 is the same as uniform convergence, that is, for all $\varepsilon > 0$ there is an N s.t. if $n \geq N$, $d(f_n(x), f(x)) \leq \varepsilon$ for all $x \in M$. If f_n is a Cauchy sequence in $C(M, N)$, then since $d(f_n(x), f_m(x)) \leq d^0(f_n, f_m)$. $f_n(x)$ is Cauchy for each point $x \in M$. Thus f_n converges pointwise, define a function $f(x)$. We must show that $f_n \to f$ uniformly and that f is continuous. First of all, given $\varepsilon > 0$, choose N s.t. $d^0(f_n, f_m) < \varepsilon/2$ if $n, m \geq N$. Then for any $x \in M$, pick $N_x \geq N$ s.t. $d(f_m(x), f(x)) < \varepsilon/2$ if $m \geq N_x$. Thus with $n \geq N$ and $m \geq N_x$, $d(f_n(x), f(x)) \leq d(f_n(x), f_m(x)) + d(f_m(x), f(x)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$ So $f_n \to f$ uniformly. □

Definition 1.7. S is called **compact** iff $\forall \cup_\alpha U_\alpha = S$ there is a finite subcovering. A subset $A \subset S$ is called **compact** iff A is compact in the relative topology. A space is called *locallycompact* iff each point has a neighborhood whose closure is compact.

Theorem 1.1 (**Boizano-Weierstrass**)**.** *If S is a first countable space and is compact, then every sequence has a convergent subsequence.*

Proof. Suppose $\{u_n\}$ contains no convergent subsequences. Then we may assume all points are distinct. Each u_n has a neighborhood $\mathcal{O}_n$ that contains no other u_m. $\{u_n\}$ is closed, so that $\mathcal{O}_n$ together with $\{u_n\}^c$ forms an open covering of S, with no finit subcovering. □

In a metric space, every compact subset is closed and bounded(Heine Borel theorem).

Propsition 1.7. *Let S be a Hausdorff space. Then every compact subset of S is closed. Also, every compact Hausdorff space is normal.*

Proof. Let $u \in A^c$ and $v \in A$, where A is compact in S. There are disjoint neighborhoods of u and v and, since A is compact, there are disjoint neighborhood of u and A. Thus A^c is open. The second part is an exercise. □

Propsition 1.8. *Let S be a Hausdorff space that is locally homeomorphic to a locally compact Hausdorff space. Then S is locally compact.*

Proof. Let $U \subset S$ be homeomorphic to $\varphi(U) \subset T$. There is a neighborhood V of $\varphi(u)$ so $\overline{V} \subset \varphi(U)$ and $\overline{V}$ is compact. Then $\varphi^{-1}(\overline{V})$ is compact, and hence closed in S. $\varphi^{-1}(\overline{V}) \subset \overline{\varphi^{-1}(V)}$. Thus $\varphi^{-1}(V)$ has compact closure. □

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