Symplectic Geometry

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Abstract

This is a note while I reading Ana Cannas's *Lectures on Symplectic Geometry*. However, it's not just a copy. It contains my understanding, questions and solutions to homework. I believe it's a good way for me to self-study mathematics. Make it slow and carefully, learn it by doing it.

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1 Notations

In order to keep the text short, common used notations are introduced here.

- V be an m-dimensional vector space over $\mathbb{R}$.
- $\Omega : V \times V \to \mathbb{R}$ be a bilinear map.
- (V, Ω) is a **symplectic vector space**.

2 Symplectic Forms

Def. 2.1. The map Ω is **skew-symmetric** if $\Omega(u, v) = -\Omega(v, u), \forall u, v \in V$.

Thm. 2.1 (Standard Form for Skew-symmetric Bilinear Map). *$\exists$ a basis $u_1, \cdots, u_k, e_1, \cdots, e_n, f_1, \cdots, f_n$ of V s.t. $\forall i, j$ and $v \in V$*

$$\Omega(u_i, v) = \Omega(e_i, e_j) = \Omega(f_i, f_j) = 0, \quad \Omega(e_i, f_j) = \delta_{ij}$$

Remark.

1. The basis is not unique, though it is traditionally also called a "canonical" basis.
2. In matrix notation with respect to such basis, we have

$$\Omega(u, v) = \begin{bmatrix} - & u & - \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & -I_n & 0 \end{bmatrix} \begin{bmatrix} | \\ v \\ | \end{bmatrix}$$

and all normed linear independt enginvectors are basis.

Proof. This induction proof is a skew-symmetric version of the Gram-Schmidt process.
Let $U := \{u \in V \mid \Omega(u, v) = 0, \forall v \in V\}$. Choose a basis $u_1, \cdots, u_k$, and choose a complemetary space W,

$$V = U \oplus W$$

. Take any nonzero $e_1 \in W$. Then there is $f_1 \in W$ s.t. $\Omega(e_1, f_1) = 1$. Let

$$W_1 = span\{e_1, f_1\}$$
$$W_1^{\Omega} = \{w \in W \mid \Omega(w, v) = 0, \forall v \in W_1\}$$

Claim. $W_1 \cap W_1^{\Omega} = 0$.

Suppose that $v = ae_1 + bf_1 \in W_1 \cap W_1^{\Omega}$.

$$\left.\begin{aligned} 0 &= \Omega(v, e_1) = -b \\ 0 &= \Omega(v, f_1) = a \end{aligned}\right\} \Rightarrow v = 0$$

Claim. $W = W_1 \oplus W_1^{\Omega}$.

Suppose that $v \in W$ has $\Omega(v, e_1) = c, \Omega(v, f_1) = d$. Then

$$v = \underbrace{(-cf_1 + de_1)}_{\in W_1} + \underbrace{(v + cf_1 - de_1)}_{\in W_1^{\Omega}}$$

Go on with W_1^{Ω}: choose $e_2, f_2 \in W_1^{\Omega}$ s.t. $\Omega(e_2, f_2) = 1$, let $W_2 = span e_2, f_2$,etc. This process eventually stops because $\dim V < \infty$. We hence obtain

$$V = U \oplus W_1 \oplus \cdots \oplus W_n$$

. **Remark.**

1. $k := \dim U$ is an invariant of (V, Ω).

2. n is an invariant of (V, Ω); 2n is called the **rank** of Ω.

□

2.1 Skew-Symmetric Bilinear Maps

Def. 2.2. The map $\tilde{\Omega} : V \to V^*$ is the linear map defined by $\tilde{\Omega}(v)(u) = \Omega(v, u)$.

Def. 2.3. A skew-symetric bilinear map Ω is **symplectic** (or nondegenerate) if $\tilde{\Omega}$ is bijective, i.e., $U = 0$. The map Ω is then called a **linear symplectic structure** on V, and (V, Ω) is called a **symplectic vector space**.

Note. These are immediate properties of symplectic map:

1. Duality: the map $\Omega : V \overset{\simeq}{\to} V^*$ is a bijection.

2. $\dim U = 0, \dim V = 2n$.

3. (V, Ω) has a basis $e_1, \cdots, e_n, f_1, \cdots, f_n$ s.t.

$$\Omega(u, v) = \begin{bmatrix} - & u & - \end{bmatrix} \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} | \\ v \\ | \end{bmatrix}$$

Remark. Not all subspace W of a (V, Ω) look the same:

- W is **sysmplecitc** if $\Omega|_W$ is nondegenerate, for instance $W = span e_1, f_1$.
- W is **isotropic** if $\Omega|_W \equiv 0$, for instance $W = span e_1, e_1$.

Def. 2.4. A **symplectomorphism** φ between (V, Ω) and (V', Ω') is a linear isomorphism $\varphi : V \overset{\simeq}{\to} V'$ s.t. $\varphi^*\Omega' = \Omega, (\varphi^*\Omega')(u, v) = \Omega'(\varphi(u), \varphi(v))$. If a symplectomorphism exists, these two spaces are said to be **symplectomophic**.

Remark. Thm 2.1 shows that any symplectic space is symplectomorphic to $(\mathbb{R}^{2n}, \Omega_0)$.