

Symplectic Geometry

Regoon Wang *

October 28, 2013

Abstract

This is a note while I reading Ana Cannas's *Lectures on Symplectic Geometry*. However, it's not just a copy. It contains my understanding, questions and solutions to homework. I believe it's a good way for me to self-study mathematics. Make it slow and carefully, learn it by doing it.

Contents

1	Notations	2
2	Symplectic Forms	2
2.1	Skew-Symmetric Bilinear Maps	3
2.2	Symplectic Vector Space	5
2.3	Symplectic Manifolds	5
2.4	Symplectomorphisms	5
2.5	Homework	5
3	Symplectic Form on the Cotangent Bundle	5
3.1	Cotangent Bundle	5
3.2	Tautological and Canonical Forms in Coordinates	5
3.3	Coordinate-Free Definitions	5
3.4	Naturality of the Tautological and Canonical Forms	5
3.5	Homework	5
4	Lagrangian Submanifolds	5
4.1	Submanifolds	5
4.2	Lagrangian Submanifolds of T^*X	5
4.3	Conormal Bundles	5
4.4	Application to Symplectomorphisms	5
4.5	Homework	5
5	Generating Functions	5
5.1	Constructing Symplectomorphisms	5
5.2	Method of Generating Functions	5
5.3	Application to Geodesic Flow	5
5.4	Homework	5
6	Recurrence	5
6.1	Periodic Points	5
6.2	Billiards	5
6.3	Poincaré Recurrence	5
6.4	Homework	5

*ChemE@UNSW, wang.regoon@gmail.com

7	Preparation for the Local Theory	5
7.1	Isotopies and Vector Fields	5
7.2	Tubular neighborhood Theorem	5
7.3	Homotopy Formula	5
7.4	Homework	5
8	Moser Theorems	5
8.1	Notions of Equivalence for Symplectic Structure	5
8.2	Moser Trick	5
8.3	Moser Local Theorem	5
8.4	Homework	5
9	Darboux-Moser-Weinstein Theory	5
9.1	Classical Darboux Theorem	5
9.2	Lagrangian Subspaces	5
9.3	Weinstein Lagrangian Neighborhood Theorem	5
9.4	Homework	5

1 Notations

In order to keep the text short, common used notations are introduced here.

- V be an m -dimensional vector space over \mathbb{R} .
- $\Omega : V \times V \rightarrow \mathbb{R}$ be a bilinear map.
- (V, Ω) is a **symplectic vector space**.

2 Symplectic Forms

Def. 2.1. The map Ω is **skew-symmetric** if $\Omega(u, v) = -\Omega(v, u), \forall u, v \in V$.

Thm. 2.1 (Standard Form for Skew-symmetric Bilinear Map). \exists a basis $u_1, \dots, u_k, e_1, \dots, e_n, f_1, \dots, f_n$ of V s.t. $\forall i, j$ and $v \in V$

$$\Omega(u_i, v) = \Omega(e_i, e_j) = \Omega(f_i, f_j) = 0, \quad \Omega(e_i, f_j) = \delta_{ij}$$

Remark.

1. The basis is not unique, though it is traditionally also called a "canonical" basis.
2. In matrix notation with respect to such basis, we have

$$\Omega(u, v) = \begin{bmatrix} - & u & - \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & -I_n & 0 \end{bmatrix} \begin{bmatrix} | \\ v \\ | \end{bmatrix}$$

and all normed linear independt enginvectors are basis.

Proof. This induction proof is a skew-symmetric version of the Gram-Schmidt process.

Let $U := \{u \in V \mid \Omega(u, v) = 0, \forall v \in V\}$. Choose a basis u_1, \dots, u_k , and choose a complemetary space W ,

$$V = U \oplus W$$

. Take any nonzero $e_1 \in W$. Then there is $f_1 \in W$ s.t. $\Omega(e_1, f_1) = 1$. Let

$$W_1 = \text{span}\{e_1, f_1\}$$

$$W_1^\Omega = \{w \in W \mid \Omega(w, v) = 0, \forall v \in W_1\}$$

Claim. $W_1 \cap W_1^\Omega = 0$.

Suppose that $v = ae_1 + bf_1 \in W_1 \cap W_1^\Omega$.

$$\left. \begin{aligned} 0 &= \Omega(v, e_1) = -b \\ 0 &= \Omega(v, f_1) = a \end{aligned} \right\} \Rightarrow v = 0$$

Claim. $W = W_1 \oplus W_1^\Omega$.

Suppose that $v \in W$ has $\Omega(v, e_1) = c, \Omega(v, f_1) = d$. Then

$$v = \underbrace{(-cf_1 + de_1)}_{\in W_1} + \underbrace{(v + cf_1 - de_1)}_{\in W_1^\Omega}$$

Go on with W_1^Ω : choose $e_2, f_2 \in W_1^\Omega$ s.t. $\Omega(e_2, f_2) = 1$, let $W_2 = \text{span}e_2, f_2$, etc. This process eventually stops because $\dim V < \infty$. We hence obtain

$$V = U \oplus W_1 \oplus \cdots \oplus W_n$$

Remark.

1. $k := \dim U$ is an invariant of (V, Ω) .
2. n is an invariant of (V, Ω) ; $2n$ is called the **rank** of Ω .

□

2.1 Skew-Symmetric Bilinear Maps

Def. 2.2. The map $\tilde{\Omega} : V \rightarrow V^*$ is the linear map defined by $\tilde{\Omega}(v)(u) = \Omega(v, u)$.

Def. 2.3. A skew-symmetric bilinear map Ω is **symplectic** (or nondegenerate) if $\tilde{\Omega}$ is bijective, i.e., $U = 0$. The map Ω is then called a **linear symplectic structure** on V , and (V, Ω) is called a **symplectic vector space**.

Note. These are immediate properties of symplectic map:

1. Duality: the map $\Omega : V \xrightarrow{\sim} V^*$ is a bijection.
2. $\dim U = 0, \dim V = 2n$.
3. (V, Ω) has a basis $e_1, \dots, e_n, f_1, \dots, f_n$ s.t.

$$\Omega(u, v) = \begin{bmatrix} - & u & - \end{bmatrix} \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} | \\ v \\ | \end{bmatrix}$$

Remark. Not all subspace W of a (V, Ω) look the same:

- W is **symplectic** if $\Omega|_W$ is nondegenerate, for instance $W = \text{span}e_1, f_1$.
- W is **isotropic** if $\Omega|_W \equiv 0$, for instance $W = \text{span}e_1, e_1$.

Def. 2.4. A **symplectomorphism** φ between (V, Ω) and (V', Ω') is a linear isomorphism $\varphi : V \xrightarrow{\sim} V'$ s.t. $\varphi^* \Omega' = \Omega, (\varphi^* \Omega')(u, v) = \Omega'(\varphi(u), \varphi(v))$. If a symplectomorphism exists, these two spaces are said to be **symplectomorphic**.

Remark. Thm 2.1 shows that any symplectic space is symplectomorphic to $(\mathbb{R}^{2n}, \Omega_0)$.

2.2	Symplectic Vector Space
2.3	Symplectic Manifolds
2.4	Symplectomorphisms
2.5	Homework
3	Symplectic Form on the Cotangent Bundle
3.1	Cotangent Bundle
3.2	Tautological and Canonical Forms in Coordinates
3.3	Coordinate-Free Definitions
3.4	Naturality of the Tautological and Canonical Forms
3.5	Homework
4	Lagrangian Submanifolds
4.1	Submanifolds
4.2	Lagrangian Submanifolds of T^*X
4.3	Conormal Bundles
4.4	Application to Symplectomorphisms
4.5	Homework
5	Generating Functions
5.1	Constructing Symplectomorphisms
5.2	Method of Generating Functions
5.3	Application to Geodesic Flow
5.4	Homework
6	Recurrence
6.1	Periodic Points
6.2	Billiards
6.3	Poincaré Recurrence
6.4	Homework
7	Preparation for the Local Theory
7.1	Isotopies and Vector Fields
7.2	Tubular neighborhood Theorem
7.3	Homotopy Formula
7.4	Homework
8	Moser Theorems
8.1	Notions of Equivalence for Symplectic Structure
8.2	Moser's Theorem