

Symplectic Geometry

Regoon Wang *

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Abstract

This is a note while I reading Ana Cannas's *Lectures on Symplectic Geometry*. However, it's not just a copy. It contains my understanding, questions and solutions to homework. I believe it's a good way for me to self-study mathematics. Make it slow and carefully, learn it by doing it.

Contents

1	Notations	2
2	Symplectic Forms	2
2.1	Skew-Symmetric Bilinear Maps	3
2.2	Symplectic Vector Space	4
2.3	Symplectic Manifolds	4
2.4	Symplectomorphisms	4
2.5	Homework	5
3	Symplectic Form on the Cotangent Bundle	7
3.1	Cotangent Bundle	7
3.2	Tautological and Canonical Forms in Coordinates	7
3.3	Coordinate-Free Definitions	7
3.4	Naturality of the Tautological and Canonical Forms	7
3.5	Homework	7
4	Lagrangian Submanifolds	7
4.1	Submanifolds	7
4.2	Lagrangian Submanifolds of T^*X	7
4.3	Conormal Bundles	7
4.4	Application to Symplectomorphisms	7
4.5	Homework	7
5	Generating Functions	7
5.1	Constructing Symplectomorphisms	7
5.2	Method of Generating Functions	7
5.3	Application to Geodesic Flow	7
5.4	Homework	7
6	Recurrence	7
6.1	Periodic Points	7
6.2	Billiards	7
6.3	Poincaré Recurrence	7
6.4	Homework	7

*ChemE@UNSW, wang.regoon@gmail.com

7	Preparation for the Local Theory	7
7.1	Isotopies and Vector Fields	7
7.2	Tubular neighborhood Theorem	7
7.3	Homotopy Formula	7
7.4	Homework	7
8	Moser Theorems	7
8.1	Notions of Equivalence for Symplectic Structure	7
8.2	Moser Trick	7
8.3	Moser Local Theorem	7
8.4	Homework	7
9	Darboux-Moser-Weinstein Theory	7
9.1	Classical Darboux Theorem	7
9.2	Lagrangian Subspaces	7
9.3	Weinstein Lagrangian Neighborhood Theorem	7
9.4	Homework	7
10	Weinstein Tubular Neighborhood Theorem	7
10.1	Observation from linear algebra	7
10.2	Tubular Neighborhoods	7
10.3	Tangent space to the group of symplectomorphisms	7
10.4	Fixed points of symplectomorphisms	7
10.5	Homework	7
11	Contact Forms	7
11.1	Contact Structure	7
11.2	Examples	7
11.3	First Properties	7
11.4	Homework	7

1 Notations

In order to keep the text short, common used notations are introduced here.

- V be an m -dimensional vector space over \mathbb{R} .
- $\Omega : V \times V \rightarrow \mathbb{R}$ be a bilinear map.
- (V, Ω) is a **symplectic vector space**.

2 Symplectic Forms

Def. 2.1. The map Ω is **skew-symmetric** if $\Omega(u, v) = -\Omega(v, u), \forall u, v \in V$.

Thm. 2.1 (Standard Form for Skew-symmetric Bilinear Map). \exists a basis $u_1, \dots, u_k, e_1, \dots, e_n, f_1, \dots, f_n$ of V s.t. $\forall i, j$ and $v \in V$

$$\Omega(u_i, v) = \Omega(e_i, e_j) = \Omega(f_i, f_j) = 0, \quad \Omega(e_i, f_j) = \delta_{ij}$$

Remark.

1. The basis is not unique, though it is traditionally also called a "canonical" basis.

2. In matrix notation with respect to such basis, we have

$$\Omega(u, v) = \begin{bmatrix} - & u & - \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & -I_n & 0 \end{bmatrix} \begin{bmatrix} | \\ v \\ | \end{bmatrix}$$

and all normed linear independt enginvectors are basis.

Proof. This induction proof is a skew-symmetric version of the Gram-Schmidt process.

Let $U := \{u \in V \mid \Omega(u, v) = 0, \forall v \in V\}$. Choose a basis u_1, \dots, u_k , and choose a complemetary space W ,

$$V = U \oplus W$$

. Take any nonzero $e_1 \in W$. Then there is $f_1 \in W$ s.t. $\Omega(e_1, f_1) = 1$. Let

$$\begin{aligned} W_1 &= \text{span}\{e_1, f_1\} \\ W_1^\Omega &= \{w \in W \mid \Omega(w, v) = 0, \forall v \in W_1\} \end{aligned}$$

Claim. $W_1 \cap W_1^\Omega = 0$.

Suppose that $v = ae_1 + bf_1 \in W_1 \cap W_1^\Omega$.

$$\left. \begin{aligned} 0 &= \Omega(v, e_1) = -b \\ 0 &= \Omega(v, f_1) = a \end{aligned} \right\} \Rightarrow v = 0$$

Claim. $W = W_1 \oplus W_1^\Omega$.

Suppose that $v \in W$ has $\Omega(v, e_1) = c, \Omega(v, f_1) = d$. Then

$$v = \underbrace{(-cf_1 + de_1)}_{\in W_1} + \underbrace{(v + cf_1 - de_1)}_{\in W_1^\Omega}$$

Go on with W_1^Ω : choose $e_2, f_2 \in W_1^\Omega$ s.t. $\Omega(e_2, f_2) = 1$, let $W_2 = \text{span}e_2, f_2$, etc. This process eventually stops because $\dim V < \infty$. We hence obtain

$$V = U \oplus W_1 \oplus \dots \oplus W_n$$

. **Remark.**

1. $k := \dim U$ is an invariant of (V, Ω) .

2. n is an invariant of (V, Ω) ; $2n$ is called the **rank** of Ω .

□

2.1 Skew-Symmetric Bilinear Maps

Def. 2.2. The map $\tilde{\Omega} : V \rightarrow V^*$ is the linear map defined by $\tilde{\Omega}(v)(u) = \Omega(v, u)$.

Def. 2.3. A skew-symetric bilinear map Ω is **symplectic** (or nondegenerate) if $\tilde{\Omega}$ is bijective, i.e., $U = 0$. The map Ω is then called a **linear symplectic structure** on V , and (V, Ω) is called a **symplectic vector space**.

Note. These are immediate properties of symplectic map:

1. Duality: the map $\Omega : V \xrightarrow{\sim} V^*$ is a bijection.

2. $\dim U = 0, \dim V = 2n$.

3. (V, Ω) has a basis $e_1, \dots, e_n, f_1, \dots, f_n$ s.t.

$$\Omega(u, v) = \begin{bmatrix} - & u & - \end{bmatrix} \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} | \\ v \\ | \end{bmatrix}$$

Remark. Not all subspace W of a (V, Ω) look the same:

- W is **symplectic** if $\Omega|_W$ is nondegenerate, for instance $W = \text{span}\{e_1, f_1\}$.
- W is **isotropic** if $\Omega|_W \equiv 0$, for instance $W = \text{span}\{e_1, e_1\}$.

2.2 Symplectic Vector Space

Def. 2.4. A **symplectomorphism** φ between (V, Ω) and (V', Ω') is a linear isomorphism $\varphi : V \xrightarrow{\sim} V'$ s.t. $\varphi^* \Omega' = \Omega$, $(\varphi^* \Omega')(u, v) = \Omega'(\varphi(u), \varphi(v))$. If a symplectomorphism exists, these two spaces are said to be **symplectomorphic**.

Remark. Thm 2.1 shows that any symplectic space is symplectomorphic to $(\mathbb{R}^{2n}, \Omega_0)$.

2.3 Symplectic Manifolds

Let ω be a de Rham 2-form on a manifold M , that is, for each $p \in M$, the map $\omega_p : T_p M \times T_p M \rightarrow \mathbb{R}$ is skew-symmetric bilinear on the tangent space to M at p , and ω_p varies smoothly in p . We say that ω is closed if it satisfies the differential equation $d\omega = 0$, where d is the de Rham differential.

Def. 2.5. The 2-form ω is symplectic if ω is closed and ω_p is symplectic for all $p \in M$.

If ω is symplectic, then $\dim T_p M = \dim M$ must be even.

Def. 2.6. A symplectic manifold is a pair (M, ω) where M is a manifold and ω is a symplectic form.

Example. Let $M = \mathbb{R}^{2n}$ with linear coordinates $x_1, \dots, x_n, y_1, \dots, y_n$. Then the form

$$\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$$

is symplectic, and the set

$$\left\{ \left(\frac{\partial}{\partial x_1} \right), \dots, \left(\frac{\partial}{\partial x_n} \right), \left(\frac{\partial}{\partial y_1} \right), \dots, \left(\frac{\partial}{\partial y_n} \right) \right\}$$

is a symplectic basis.

Example. Let $M = \mathbb{C}^n$ with linear coordinates z_1, \dots, z_n . The form

$$\omega_0 = \frac{i}{2} \sum_{k=1}^n dz_k \wedge d\bar{z}_k$$

is symplectic. In fact $\mathbb{C}^n \equiv \mathbb{R}^{2n}$, $z_k = x_k + iy_k$.

2.4 Symplectomorphisms

Def. 2.7. Let (M_1, ω_1) and (M_2, ω_2) be $2n$ -dim symplectic manifolds, and let $g : M_1 \rightarrow M_2$ be a diffeomorphism. Then g is a **symplectomorphism** if $g^* \omega_2 = \omega_1$.

Thm. 2.2 (Darboux). Let (M, ω) be $2n$ -dim symplectic manifold and point $p \in M$. Then there is a coordinate chart $(\mathcal{U}, x_1, \dots, x_n, y_1, \dots, y_n)$ centered at p s.t. the symplectic form is

$$\omega = \sum_{i=1}^n dx_i \wedge dy_i$$

More precisely, any symplectic manifold (M^{2n}, ω) is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$.

2.5 Homework

Given a linear subspace Y of a symplectic vector space (V, Ω) , its **symplectic orthogonal** is defined as

$$Y^\Omega := \{v \in V \mid \Omega(v, u) = 0, \forall u \in Y\}.$$

Exer 2.1. $\dim Y + \dim Y^\Omega = \dim V$.

Exer 2.2. $(Y^\Omega)^\Omega = Y$.

Exer 2.3. if Y, W are subspace, then $Y \subseteq W \Leftrightarrow W^\Omega \subseteq Y^\Omega$.

Exer 2.4. Y is symplectic $\Leftrightarrow Y \cap Y^\Omega = \{0\} \Leftrightarrow V = Y \oplus Y^\Omega$.

Exer 2.5. Y is isotropic $\Rightarrow \dim Y \leq \frac{1}{2} \dim V$.

Exer 2.6. An isotropic Y is called **Lagrangian** when $\dim Y = \frac{1}{2} \dim V$. Check that: Y is lagrangian $\Leftrightarrow Y$ is isotropic and coisotropic $\Leftrightarrow Y = Y^\Omega$.

Exer 2.7. We call Y coisotropic when $Y^\Omega \subseteq Y$. Check that every codimension 1 subspace Y is coisotropic.

3 Symplectic Form on the Cotangent Bundle

3.1 Cotangent Bundle

3.2 Tautological and Canonical Forms in Coordinates

3.3 Coordinate-Free Definitions

3.4 Naturality of the Tautological and Canonical Forms

3.5 Homework

4 Lagrangian Submanifolds

4.1 Submanifolds

4.2 Lagrangian Submanifolds of T^*X

4.3 Conormal Bundles

4.4 Application to Symplectomorphisms

4.5 Homework

5 Generating Functions

5.1 Constructing Symplectomorphisms

5.2 Method of Generating Functions

5.3 Application to Geodesic Flow

5.4 Homework

6 Recurrence

6.1 Periodic Points

6.2 Billiards

6.3 Poincaré Recurrence

6.4 Homework

7 Preparation for the Local Theory

7.1 Isotopies and Vector Fields

7.2 Tubular neighborhood Theorem

7.3 Homotopy Formula

7.4 Homework

8 Moser Theorems

8.1 Notions of Equivalence for Symplectic Structure

8.2 Moser Trick

8.3 Moser Local Theorem

8.4 Homework

9 Darboux-Moser-Weinstein Theory