Symplectic Geometry

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Abstract

This is a note while I reading Ana Cannas's *Lectures on Symplectic Geometry*. However, it's not just a copy. It contains my understanding, questions and solutions to homework. I believe it's a good way for me to self-study mathematics. Make it slow and carefully, learn it by doing it.

Contents

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1 Notations

In order to keep the text short, common used notations are introduced here.

- V be an m-dimensional vector space over $\mathbb{R}$.
- $\Omega : V \times V \to \mathbb{R}$ be a bilinear map.
- (V, Ω) is a **symplectic vector space**.

2 Symplectic Forms

Def. 2.1. The map Ω is **skew-symmetric** if $\Omega(u, v) = -\Omega(v, u), \forall u, v \in V$.

Thm. 2.1 (Standard Form for Skew-symmetric Bilinear Map). *$\exists$ a basis $u_1, \cdots, u_k, e_1, \cdots, e_n, f_1, \cdots, f_n$ of V s.t. $\forall i, j$ and $v \in V$*

$$\Omega(u_i, v) = \Omega(e_i, e_j) = \Omega(f_i, f_j) = 0, \quad \Omega(e_i, f_j) = \delta_{ij}$$

Remark.

1. The basis is not unique, though it is traditionally also called a "canonical" basis.

2. In matrix notation with respect to such basis, we have

$$\Omega(u,v) = \begin{bmatrix} - & u & - \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & -I_n & 0 \end{bmatrix} \begin{bmatrix} | \\ v \\ | \end{bmatrix}$$

and all normed linear independt enginvectors are basis.

Proof. This induction proof is a skew-symmetric version of the Gram-Schmidt process.
Let $U := \{u \in V \mid \Omega(u,v) = 0, \forall v \in V\}$. Choose a basis $u_1, \cdots, u_k$, and choose a complemetary space W,

$$V = U \oplus W$$

. Take any nonzero $e_1 \in W$. Then there is $f_1 \in W$ s.t. $\Omega(e_1, f_1) = 1$. Let

$$W_1 = span\{e_1, f_1\}$$
$$W_1^\Omega = \{w \in W \mid \Omega(w,v) = 0, \forall v \in W_1\}$$

Claim. $W_1 \cap W_1^\Omega = 0$.

Suppose that $v = ae_1 + bf_1 \in W_1 \cap W_1^\Omega$.

$$\left.\begin{aligned} 0 &= \Omega(v, e_1) = -b \\ 0 &= \Omega(v, f_1) = a \end{aligned}\right\} \Rightarrow v = 0$$

Claim. $W = W_1 \oplus W_1^\Omega$.

Suppose that $v \in W$ has $\Omega(v, e_1) = c, \Omega(v, f_1) = d$. Then

$$v = \underbrace{(-cf_1 + de_1)}_{\in W_1} + \underbrace{(v + cf_1 - de_1)}_{\in W_1^\Omega}$$

Go on with W_1^Ω: choose $e_2, f_2 \in W_1^\Omega$ s.t. $\Omega(e_2, f_2) = 1$, let $W_2 = span e_2, f_2$,etc. This process eventually stops because $\dim V < \infty$. We hence obtain

$$V = U \oplus W_1 \oplus \cdots \oplus W_n$$

. **Remark.**

1. $k := \dim U$ is an invariant of (V, Ω).

2. n is an invariant of (V, Ω); 2n is called the **rank** of Ω.

□

2.1 Skew-Symmetric Bilinear Maps

Def. 2.2. The map $\tilde{\Omega} : V \to V^*$ is the linear map defined by $\tilde{\Omega}(v)(u) = \Omega(v, u)$.

Def. 2.3. A skew-symetric bilinear map Ω is **symplectic** (or nondegenerate) if $\tilde{\Omega}$ is bijective, i.e., $U = 0$. The map Ω is then called a **linear symplectic structure** on V, and (V, Ω) is called a **symplectic vector space**.

Note. These are immediate properties of symplectic map:

1. Duality: the map $\Omega : V \overset{\simeq}{\rightarrow} V^*$ is a bijection.

2. $\dim U = 0, \dim V = 2n$.

3. (V, Ω) has a basis $e_1, \cdots, e_n, f_1, \cdots, f_n$ s.t.

$$\Omega(u,v) = \begin{bmatrix} - & u & - \end{bmatrix} \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} | \\ v \\ | \end{bmatrix}$$

Remark. Not all subspace W of a (V, Ω) look the same:

- W is **sysmplecitc** if $\Omega\mid_W$ is nondegenerate, for instance $W = span e_1, f_1$.
- W is **isotropic** if $\Omega\mid_W \equiv 0$, for instance $W = span e_1, e_1$.

2.2 Symplectic Vector Space

Def. 2.4. A **symplectomorphism** φ between (V, Ω) and (V', Ω') is a linear isomorphism $\varphi : V \overset{\backsimeq}{\rightarrow} V'$ s.t. $\varphi^*\Omega' = \Omega, (\varphi^*\Omega')(u,v) = \Omega'(\varphi(u), \varphi(v))$. If a symplectomorphism exists, these two spaces are said to be **symplectomophic**.

Remark. Thm 2.1 shows that any symplectic space is symplectomorphic to $(\mathbb{R}^{2n}, \Omega_0)$.

2.3 Symplectic Manifolds

Let ω be a de Rham 2-form on a manifold M, that is, for each $p \in M$, the map $\omega_p : T_pM \times T_pM \to \mathbb{R}$ is skew-symmetric bilinear on the tangent space to M at p, and ω_p varies smoothly in p. We say that ω is closed if it satisfies the differential equation $d\omega = 0$, where d is the de Rham differential.

Def. 2.5. The 2-form ω is symplectic if ω is closed and ω_p is symplectic for all $p \in M$.

If ω is symplectic, then $\dim T_mM = \dim M$ must be even.

Def. 2.6. A symplectic manifold is a pair (M, ω) where M is a manifold and ω is a symplectic form.

Example. Let $M = \mathbb{R}^{2n}$ with linear coordinates $x_1, \cdots, x_n, y_1, \cdots, y_n$. Then the form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$$

is symplectic, and the set

$$\left\{ \left(\frac{\partial}{\partial x_1}\right), \cdots, \left(\frac{\partial}{\partial x_n}\right), \left(\frac{\partial}{\partial y_1}\right), \cdots, \left(\frac{\partial}{\partial y_n}\right) \right\}$$

is a symplectic basis.

Example. Let $M = \mathbb{C}^n$ with linear coordinates $z_1, \cdots, z_n$. The form

$$\omega_0 = \frac{i}{2} \sum_{i=1}^{n} dz_k \wedge d\bar{z}_k$$

is symplectic. In fact $\mathbb{C}^n \equiv \mathbb{R}^{2n}$, $z_k = x_k + \mathbf{i}y_k$.

2.4 Symplectomorphisms

Def. 2.7. Let (M_1, ω_1) and (M_2, ω_2) be 2n-dim symplectic manifolds, and let $g : M_1 \to M2$ be a diffeomorphism. Then g is a **symplectomorphism** if $g^*\omega_2 = \omega_1$.

Thm. 2.2 (Darboux). *Let (M, ω) be 2n-dim sysmplectic manifold and point $p \in M$. Then there is a coordiante chart $(\mathcal{U}, x_1, \cdots, x_n, y_1, \cdots, y_n)$ centered at p s.t. the symplectic form is*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$

More precisely, any symplectic manifold (M^{2n}, ω) is locally symplectomorphic to $(\mathbb{R}^2, \omega_0)$.

2.5 Homework

Given a linear subspace Y of a symplectic vector space (V, Ω), its **symplectic orthogonal** is defined as

$$Y^{\Omega} := \{v \in V \mid \Omega(v, u) = 0, \forall u \in Y\}.$$

Exer 2.1. $\mathrm{dim}Y + \mathrm{dim}Y^{\Omega} = \mathrm{dim}V$.

Exer 2.2. $(Y^{\Omega})^{\Omega} = Y$.

Exer 2.3. *if* Y, W *are subspace, then* $Y \subseteq W \Leftrightarrow W^{\Omega} \subseteq Y^{\Omega}$.

Exer 2.4. Y *is symplectic* $\Leftrightarrow Y \cap Y^{\Omega} = \{0\} \Leftrightarrow V = Y \oplus Y^{\Omega}$.

Exer 2.5. Y *is isotropic* $\Rightarrow \mathrm{dim}Y \leq \frac{1}{2}\mathrm{dim}V$.

Exer 2.6. *An isotropic* Y *is called **Lagrangian** when* $\mathrm{dim}Y = \frac{1}{2}\mathrm{dim}V$. *Check that:* Y *is lagrangian* $\Leftrightarrow$ Y *is isotropic and coisotropic* $\Leftrightarrow Y = Y^{\Omega}$.

Exer 2.7. *We call* Y *coisotropic when* $Y^{\Omega} \Subset Y$. *Check that every codimension 1 subspace* Y *is coisotropic.*