Homework 1

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1. (10') Simulate a sequence of uniform random numbers with R function `runif()`. Make 3-tuples with every 3 consecutive numbers from the sequence. Visualize the tuples in the cube $[0,1]^3$ and test the uniformity of these tuples with Pearson's χ^2 test.

2. (10') Prove that $F(X) \sim U(0,1)$ for a continuous random variable X with CDF $F(\cdot)$. And explain why this is not true if X follows a discrete distribution.

3. A fault-tolerant memory bank is built with 5 memory units. These units have independent random failure times, each with a distribution F. The memory bank is operable as long as 4 or more units are still working, so it fails when the second unit fails. Let $G(\cdot \mid \alpha, \beta)$ be the CDF of the $\mathrm{Beta}(\alpha, \beta)$.

   a) (5') Express the failure time of the memory bank as a function of a single variable $U \sim \boldsymbol{U}(0,1)$ via $F(\cdot)$ and $G(\cdot \mid \alpha, \beta)$ and/or their inverses.
   b) (5') Suppose that F is the exponential distribution with mean 500,000 (hours). Sample 10,000 memory bank lifetimes using the expression from part a). Estimate the mean memory bank lifetime and give a 99% confidence interval for it.
   c) (5') If we didn't use a redundant system, we might instead use 4 memory units and the bank would fail when the first of those failed. Find the expected lifetime of such a system without redundancy and give a 99% confidence interval. The redundant system takes 1.25 times as many memory units. Does it last at least 1.25 times as long on average?
   d) (5') Compare the speed of sampling by inversion with that of generating 5 exponential variables, sorting them and picking out the second smallest.

4. For acceptance-rejection sampling,

   a) (5') Show that the ratio $f(x)/g(x)$ of $N(0,1)$ to Cauchy densities is maximized at $x = \pm 1$.
   b) (5') Can we use acceptance-rejection with proposals from $N(0,1)$ to sample from Cauchy distribution? Explain why or why not.