Pregel Algorithms for Graph Connectivity Problems with Performance Guarantees

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ABSTRACT

Graphs in real life applications are often huge, such as the Web graph and various social networks. These massive graphs are often stored and processed in distributed sites. In this paper, we study graph algorithms that adopt Google's Pregel, an iterative vertex-centric framework for graph processing in the Cloud. We first identify a set of desirable properties of an efficient Pregel algorithm, such as linear space, communication and computation cost per iteration, and logarithmic number of iterations. We define such an algorithm as a *practical Pregel algorithm* (PPA). We then propose PPAs for computing *connected components* (CCs), *biconnected components* (BCCs) and *strongly connected components* (SCCs). The PPAs for computing BCCs and SCCs use the PPAs of many fundamental graph problems as building blocks, which are of interest by themselves. Extensive experiments over large real graphs verified the efficiency of our algorithms.

1. INTRODUCTION

With the popularity of online social networks, mobile communication networks and semantic web services, we have witnessed a growing interest in conducting efficient and effective analysis on massive real-world graphs. As a result, many distributed graph computing systems have emerged in recent years, which are deployed on a shared-nothing distributed computing infrastructure usually built on top of a cluster of low-cost commodity PCs. Pioneered by Google's Pregel [13], these systems adopt a vertex-centric computing paradigm, where programmers think naturally like a vertex when designing distributed graph algorithms. A Pregel-like system also takes care of fault recovery and scales to arbitrary cluster size without the need of changing the program code, both of which are indispensable properties for programs running in a cloud environment.

Pregel was shown to be more suitable for iterative graph computation than the popular MapReduce model [5, 13, 14]. However, existing work on Pregel algorithms [14] is rather ad hoc. It looks more like a demonstration of how the Pregel model can be adopted to solve graph problems, and lacks any analysis on the cost complexity.

In this paper, we study three fundamental graph connectivity problems and propose Pregel algorithms as solutions that have performance guarantees. The problems we study are *connected components* (**CC**s), *bi-connected components* (**BCC**s), and *strongly connected components* (**SCC**s). These problems have numerous real life applications and their solutions are essential building blocks for solving many other graph problems. For example, computing the BCCs of a telecommunications network can help detect the weaknesses in network design, while almost all reachability indices require SCC computation as a preprocessing step [3].

To avoid ad hoc algorithm design and ensure good performance, we define a class of Pregel algorithms that satisfy a set of rigid, but practical, constraints on various performance metrics: *(1)linear space usage, (2)linear computation cost per iteration*[1]*, (3)linear communication cost per iteration, and (4)at most logarithmic number of iterations*. We call such algorithms as **Practical Pregel Algorithm**s (**PPA**s). A similar but stricter set of constraints was proposed for the MapReduce model recently [19]. In contrast to our requirement of logarithmic number of iterations, their work demands a constant number of iterations, which is too restrictive for most graph problems. In fact, even list ranking (i.e., ranking vertices in a directed graph consisting of only one simple path) requires $O(\log n)$ time using $O(n)$ processors under the *shared-memory* PRAM model [21], where n is the number of vertices. This bound also applies to many other basic graph problems such as connected components and spanning tree [17].

It is challenging to design Pregel algorithms for problems such as BCCs and SCCs. Although there are simple sequential algorithms for computing BCCs and SCCs based on depth-first search (DFS), DFS is $\mathcal{P}$-Complete [16] and hence it cannot be applied to design parallel algorithms for computing BCCs and SCCs. In fact, as far as we know, only CC computation has been studied under the MapReduce framework [15], but none of the algorithms proposed in [15] achieve strictly linear space cost and $O(\log n)$ number of iterations for a general graph.

We apply the principle of PPA to develop Pregel algorithms that satisfy strict performance guarantees. In particular, to compute BCCs and SCCs, we develop a set of useful building blocks that are the PPAs of fundamental graph problems such as *breadth-first search*, *list ranking*, *spanning tree*, *Euler tour*, and *pre/post-order traversal*. As fundamental graph problems , their PPA solutions can also be applied to numerous other graph problems besides BCCs

[1] A Pregel algorithm proceeds in iterations.

and SCCs considered in this paper.

We evaluate the performance of our Pregel algorithms using a number of large real-world graphs with up to hundreds of millions of vertices and billions of edges. Our results verify that our algorithms are efficient for computing CCs, BCCs and SCCs in massive graphs.

The rest of this paper is organized as follows. Section 2 reviews Pregel and related work. We define PPA in Section 3. Sections 4-6 discuss algorithms for CCs, BCCs and SCCs. Then, we report the experimental results in Section 7 and conclude in Section 8.

2. PREGEL AND RELATED WORK

Pregel [13]. Pregel is designed based on the bulk synchronous parallel (BSP) model. It distributes vertices to different machines in a cluster, where each vertex v is associated with its adjacency list (i.e., the set of v's neighbors). A program in Pregel implements a user-defined *compute*() function and proceeds in iterations (called *supersteps*). In each superstep, the program calls *compute*() for each active vertex. The *compute*() function performs the user-specified task for a vertex v, such as processing v's incoming messages (sent in the previous superstep), sending messages to other vertices (to be received in the next superstep), and making v vote to halt. A halted vertex is reactivated if it receives a message in a subsequent superstep. The program terminates when all vertices vote to halt and there is no pending message for the next superstep.

Pregel numbers the supersteps so that a user may use the current superstep number when implementing the algorithm logic in the *compute*() function. As a result, a Pregel algorithm can perform different operations in different supersteps by branching on the current superstep number.

Pregel allows users to implement a *combine*() function, which specifies how to combine messages that are sent from a machine M_i to the same vertex v in a machine M_j. These messages are combined into a single message, which is then sent from M_i to v in M_j. Combiner is applied only when commutative and associative operations are to be applied to the messages. For example, in Pregel's PageRank algorithm [13], messages from machine M_i that are to be sent to the same target vertex in machine M_j can be combined into a single message that equals their sum, since the target vertex is only interested in the sum of the messages.

Pregel also supports aggregator, which is useful for global communication. Each vertex can provide a value to an aggregator in *compute*() in a superstep. The system aggregates those values and makes the aggregated result available to all vertices in the next superstep.

MapReduce [6]. MapReduce, and its open-source implementation Hadoop[1], is another distributed system popularly used for large scale graph processing [9, 10, 11, 15, 18]. However, many graph algorithms are intrinsically iterative, in which case a Pregel program is much more efficient than its MapReduce counterpart. This is because each MapReduce job can only perform one iteration of graph computation, and it requires to read the entire graph from a distributed file system (DFS) and write the processed graph back, which leads to excessive IO cost. Furthermore, a MapReduce job also needs to exchange the adjacency lists of vertices through the network, which results in heavy communication. In contrast, Pregel keeps vertices (along with their adjacency lists) in each local machine that processes their computation, and uses only message passing to exchange vertex states. The "think like a vertex" programming paradigm also makes it easier for programmers to design graph algorithms in Pregel than in MapReduce.

GraphLab [12] and PowerGraph [8]. GraphLab [12] is another vertex-centric parallel graph computing system but has a different design from Pregel. GraphLab supports both synchronous and asynchronous executions. However, the asynchronous mode does not support algorithms that require the superstep number to determine which operation to be performed at different supersteps, while the synchronous mode of GraphLab does not support combiner and aggregator, which limits both its performance and expressiveness. GraphLab 2.2, i.e., PowerGraph [8], partitions a graph by edges rather than by vertices in order to address imbalanced workload caused by high-degree vertices. However, a more complicated edge-centric Gather-Apply-Scatter (GAS) computing model should be used, and vertex values must be updated by a commutative and associative operation. This significantly reduces the expressiveness of the vertex-centric GraphLab model [12]. Both GraphLab and PowerGraph require more memory space than Pregel, as they need construct ghost copies for vertices in multiple machines.

PRAM. The PRAM model assumes that there are many processors and a shared memory. PRAM algorithms have been proposed for computing CCs [17], BCCs [20], and SCCs [1, 2, 7]. However, the PRAM model is not suitable for Cloud environments that are built on shared-nothing architectures. Furthermore, unlike Pregel and MapReduce, PRAM algorithms are not fault tolerant.

3. PRACTICAL PREGEL ALGORITHMS

We now define some frequently used notations and introduce the notion of practical Pregel algorithms.

Notations. Given a graph $G = (V, E)$, we denote the number of vertices $|V|$ by n, and the number of edges $|E|$ by m. We also denote the *diameter* of G by δ. For an undirected graph, we denote the set of *neighbors* of a vertex v by $\Gamma(v)$ and the *degree* of v by $d(v) = |\Gamma(v)|$. For a directed graph, we denote the set of *in-neighbors* and *out-neighbors* of v by $\Gamma_{in}(v)$ and Γ_{out}, and the *in-degree* and *out-degree* of v by $d_{in}(v) = |\Gamma_{in}(v)|$ and $d_{out}(v) = |\Gamma_{out}(v)|$, respectively.

Assume that all vertices in G are assigned a unique ID. For convenience of discussion, we simply use v to refer to the ID of vertex v in this paper, and thus the expression $u < v$ means that u's vertex ID is smaller than v's. We define the *color* of a (strongly) connected component in G to be the smallest vertex among all vertices in the component. The color of a vertex v, denoted by $color(v)$, is defined as the color of the component that contains v, and so, all vertices in G with the same color constitute a component.

A Pregel algorithm is called a **balanced practical Pregel algorithm** (**BPPA**) if it satisfies the following constraints:

1. *Linear space usage:* each vertex v uses $O(d(v))$ (or $O(d_{in}(v)+d_{out}(v))$) space of storage.
2. *Linear computation cost:* the time complexity of the $compute()$ function for each vertex v is $O(d(v))$ (or $O(d_{in}(v)+d_{out}(v))$).
3. *Linear communication cost:* at each superstep, the size of the messages sent/received by each vertex v is $O(d(v))$ (or $O(d_{in}(v) + d_{out}(v))$).
4. *At most logarithmic number of rounds:* the algorithm terminates after $O(\log n)$ supersteps.

Constraints 1-3 offers good load balancing and linear cost at each superstep, while Constraint 4 controls the total running time. As we shall see in later sections, some algorithms satisfying Constraints 1-3 require $O(\delta)$ rounds. Since most large real graphs have a small

[1] http://hadoop.apache.org

diameter δ, especially for social networks due to the small world phenomenon, we assume $O(\delta) = O(\log n)$ in this paper and consider algorithms requiring $O(\delta)$ rounds also satisfying Constraint 4.

For some problems the per-vertex requirements of BPPA can be too strict, and we can only achieve *overall linear space usage, computation and communication cost* (still in $O(\log n)$ rounds). We call a Pregel algorithm that satisfies these constraints simply as a **practical Pregel algorithm** (**PPA**).

Motivation. We define BPPA and PPA in order to characterize a set of Pregel algorithms that can run efficiently. Apart from the algorithms proposed in this paper, other Pregel algorithms, e.g., the four demo algorithms in the Pregel paper [13], also have these characteristics (we can show that they are BPPAs): PageRank (constant supersteps), single-source shortest paths ($O(\delta)$ supersteps), bipartite matching ($O(\log n)$ supersteps), and semi-clustering (constant supersteps). However, these existing Pregel algorithms are designed on a best-effort basis and there is no formal performance requirement to be met or design rule to be followed. For example, while the Pregel algorithm developed in [14] for diameter estimation is an $O(\delta)$-superstep BPPA, the Pregel algorithm for triangle counting and clustering coefficient computation in [14] is not a PPA. Specifically, in superstep 1 of the triangle counting algorithm, a vertex sends a message for each pair of neighbors, leading to a quadratic number of messages to be buffered and sent. With the concept of PPA/BPPA in mind, users of the triangle counting algorithm can then be aware of the scalability limitation when applying this algorithm. The requirements of PPAs/BPPAs also serve as a guideline for programmers/researchers who want to develop efficient Pregel algorithms, while they may also use any existing PPAs as a building block in their algorithms.

We now demonstrate how some fundamental graph problems can be solved by BPPAs. In Section 5, we will show how to use these BPPAs as building blocks to develop a more sophisticated PPA for computing BCCs.

3.1 List Ranking

Consider a linked list $\mathcal{L}$ with n objects, where each object v is associated with a value $val(v)$ and a link to its predecessor $pred(v)$. The object v at the head of $\mathcal{L}$ has $pred(v) = null$. For each object v in $\mathcal{L}$, let us define $sum(v)$ to be the sum of the values of all the objects from v following the predecessor link to the head. The *list ranking* problem computes $sum(v)$ for each object v. If $val(v) = 1$ for each v in $\mathcal{L}$, then $sum(v)$ is simply the rank of v in the list, i.e., the number of objects preceding v plus 1.

In list ranking, the objects in $\mathcal{L}$ are given in arbitrary order. We may regard $\mathcal{L}$ simply as a directed graph consisting of a single simple path. Albeit simple, list ranking is an important problem in parallel computing because it serves as a building block to many other parallel algorithms.

We now describe our BPPA for list ranking. Initially, each vertex v assigns $sum(v) = val(v)$. Then in each round, each vertex v does the following: If $pred(v) \neq null$, v sets $sum(v) = sum(v) + sum(pred(v))$ and $pred(v) = pred(pred(v))$; otherwise, v votes to halt. The if-branch is accomplished in three supersteps: (1)v sends a message to $u = pred(v)$ requesting for the values of $sum(u)$ and $pred(u)$; (2)u sends back the requested values to v; and (3)v updates $sum(v)$ and $pred(v)$. This process repeats until $pred(v) = null$ for every vertex v, at which point all vertices vote to halt and we have $sum(v)$ as desired.

Figure 1 illustrates how the algorithm works. Initially, objects v_1–v_5 form a linked list with $sum(v_i) = val(v_i) = 1$ and $pred(v_i) = v_{i-1}$. Let us now focus on v_5. In Round 1, we have $pred(v_5) = v_4$ and so we set $sum(v_5) = sum(v_5) + sum(v_4) = 1 + 1 = 2$ and $pred(v_5) = pred(v_4) = v_3$. One can verify the states of the other vertices similarly. In Round 2, we have $pred(v_5) = v_3$ and so we set $sum(v_5) = sum(v_5) + sum(v_3) = 2 + 2 = 4$ and $pred(v_5) = pred(v_3) = v_1$. In Round 3, we have $pred(v_5) = v_1$ and so we set $sum(v_5) = sum(v_5) + sum(v_1)$=4 + 1=5 and $pred(v_5) = pred(v_1) = null$. We can prove by induction that in Round i, we set $sum(v_j) = \sum_{k=j-2^i+1}^{j} val(v_k)$ and $pred(v_j) = v_{j-2^i}$. Furthermore, each object v_j sends at most one message to $v_{j-2^{i-1}}$ and receives at most one message from $v_{j+2^{i-1}}$. The algorithm is a BPPA because it terminates in $\log n$ rounds, and each object sends/receives at most one message per round.

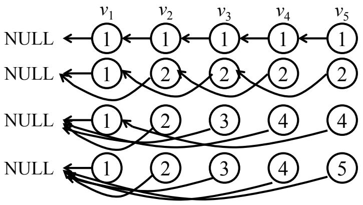


Figure 1: Illustration of BPPA for list ranking

3.2 Connected Components

The idea of the algorithm is to broadcast the smallest vertex (ID) seen so far by each vertex v, denoted by $min(v)$; when the process converges, $min(v)$=$color(v)$ for all v. A MapReduce algorithm, called Hash-Min, was also proposed to implement this idea recently [15]. Here, we propose a BPPA counterpart as follows.

In Superstep 1, each vertex v initializes $min(v)$ as the smallest vertex in the set $(\{v\} \cup \Gamma(v))$, sends $min(v)$ to all v's neighbors and votes to halt. In each subsequent superstep, a vertex v obtains the smallest vertex from the incoming messages, denoted by u. If $u < v$, v sets $min(v) = u$ and sends $min(v)$ it to all its neighbors. Finally, v votes to halt.

We prove that the algorithm is a BPPA as follows. For any CC, it takes at most δ supersteps for the ID of the smallest vertex to reach all the vertices in the CC, and in each superstep, each vertex v takes at most $O(d(v))$ time to compute $min(v)$ and sends/receives $O(d(v))$ messages each using $O(1)$ space.

Three other MapReduce algorithms were also proposed in [15] for computing CCs. However, they require that each vertex maintain a set whose size can be as large as the size of its CC, and that the whole set be sent to some vertices. Thus, they do not translate into efficient Pregel implementations due to the highly skewed communication and computation, and the excessive space cost.

3.3 BFS and Spanning Tree

We now present an $O(\delta)$-superstep BPPA that performs BFS and computes a spanning tree over an unweighted graph G from a source vertex s. We assume that the input graph G is connected; otherwise, we first compute $color(v)$ for each vertex v using the algorithm described in Section 3.2, and then pick the vertex s with $s = color(s)$ as the source for each CC. Since, multi-source BFS is done in parallel, the overall number of supersteps is still $O(\delta)$.

In the algorithm, each vertex v maintains two fields, the parent of v, denoted by $p(v)$; and the shortest-path distance (or BFS level) of v from s, denoted by $dist(v)$. Initially, only s is active with $p(s) = null$ and $dist(s) = 0$, and $dist(v) = \infty$ for all other v. In Superstep 1, s sends $\langle s, dist(s) \rangle$ to all its neighbors, and votes to halt. In each subsequent superstep, if a vertex v receives any message, it first checks whether v has been visited before (i.e.,

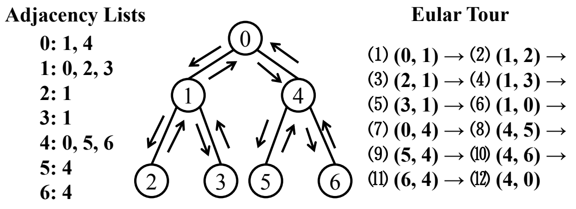


Figure 2: Euler tour

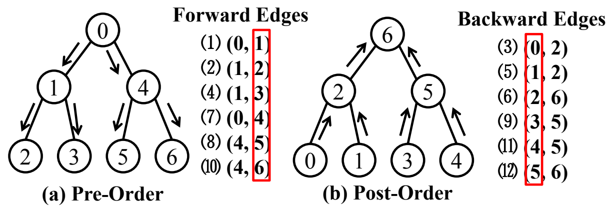


Figure 3: Pre-order & post-order

whether $dist(v) < \infty$): if not, it updates $dist(v) = dist(u) + 1$ and $p(v) = u$ with an arbitrary message $\langle u, dist(u)\rangle$ received, sends $\langle v, dist(v)\rangle$ to all v's neighbors. Finally, v votes to halt.

When the algorithm terminates, we obtain a tree edge $(p(v), v)$ from each vertex $v \neq s$, which constitute a spanning tree rooted at s. It is easy to see that the algorithm is an $O(\delta)$-superstep BPPA.

3.4 Euler Tour

A Euler tour is a representation of a tree which is useful in many parallel graph algorithms [20]. The tree is viewed as a directed graph, where each tree edge (u, v) is considered as two directed edges (u, v) and (v, u), and a Euler tour of the tree is simply a Eulerian circuit of the directed graph, i.e., a trail that visits every edge exactly once, and ends at the same vertex where it starts.

Assume that the neighbors of each vertex v are sorted according to their IDs, which is common for an adjacency list representation of a graph. For a vertex v, let $first(v)$ and $last(v)$ be the first and last neighbor of v in the sorted order; and for each neighbor u of v, if $u \neq last(v)$, let $next_v(u)$ be the neighbor of v next to u in the sorted adjacency list. We further define $next_v(last(v)) = first(v)$. Consider the example shown in Figure 2. For the adjacency list of vertex 4, we have $first(4) = 0$, $last(4) = 6$, $next_4(0) = 5$, $next_4(5) = 6$ and $next_4(6) = 0$.

If we translate $next_v(u) = w$ as specifying that the edge next to (u, v) is (v, w), we obtain a Euler tour of the tree. Referring to the example in Figure 2 again, where a Euler tour that starts and ends at vertex 0 is given. The next edge of $(2, 1)$ is $(1, 3)$, because $next_1(2) = 3$, while the next edge of $(6, 4)$ is $(4, 0)$ because $next_4(6) = 0$. In fact, starting from any vertex v and any neighbor u of v, $\langle(v, x = next_v(u))$, $(x, y = next_x(v))$, $(y, next_y(x))$, $\ldots$, $(u, v)\rangle$ defines a Euler tour.

We present a 2-superstep BPPA to construct the Euler tour as follows: In Superstep 1, each vertex v sends message $\langle u, next_v(u)\rangle$ to each neighbor u; in Supertep 2, each vertex u receives the message $\langle u, next_v(u)\rangle$ sent from each neighbor v, and stores $next_v(u)$ with v in u's adjacency list. When the algorithm finishes, for each vertex u and each neighbor v, the next edge of (u, v) is obtained as $(v, next_v(u))$.

The algorithm requires a constant number of supersteps, and in each superstep, each vertex v sends/receives $O(d(v))$ messages (each using $O(1)$ space). By implementing $next_v(.)$ as a hash table associated with v, we can obtain $next_v(u)$ in $O(1)$ expected time given u.

3.5 Pre-Order and Post-Order Traversal

Depth-first traversal of a tree generates pre-order or post-order numbers for the vertices in a tree, and the numbers are useful in many applications, such as deciding the ancestor-descendant relationship of two tree vertices, which we discuss in Section 3.6. Let $pre(v)$ and $post(v)$ be the pre-order and post-order number of each vertex v in the tree, respectively. We present a BPPA for traversing a tree starting from a source vertex s in pre-order/post-order below.

We first compute the Euler tour $\mathcal{P}$ of the tree starting from s, given by $\mathcal{P} = \langle(s, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k), (v_k, s)\rangle$. Next, we formulate a list ranking problem by treating each edge $e \in \mathcal{P}$ as a vertex and setting $val(e) = 1$. After obtaining $sum(e)$ for each $e \in \mathcal{P}$, we mark the edges in $\mathcal{P}$ as forward/backward edges using a two-superstep BPPA: in Superstep 1, each vertex $e = (u, v)$ sends $sum(e)$ to $e' = (v, u)$; in Superstep 2, each vertex $e' = (v, u)$ receives $sum(e)$ from $e = (u, v)$, sets e' itself as a forward edge if $sum(e') < sum(e)$, and a backward edge otherwise. In Figure 2, edge $(1, 2)$ is a forward edge because its rank (i.e., 2) is smaller than that of $(2, 1)$ (i.e., 3), while edge (4, 0) is a backward edge since its rank (i.e., 12) is larger than that of $(0, 4)$ (i.e., 7).

To compute $pre(v)$, we run a second round of list ranking by setting $val(e) = 1$ for each forward edge e in $\mathcal{P}$ and $val(e') = 0$ for each backward edge e'. Then, for each forward edge $e = (u, v)$, we get $pre(v) = sum(e)$ for vertex v. We set $pre(s) = 0$ for tree root s. For example, Figure 3(a) shows the forward edges (u, v) in the order in $\mathcal{P}$, where vertices are already labeled with pre-order numbers. Obviously, the rank of (u, v) gives $pre(v)$.

To compute $post(v)$, we run list ranking by setting $val(e) = 0$ for each forward edge e and $val(e') = 1$ for each backward edge e' in $\mathcal{P}$. Then, for each backward edge $e' = (v, u)$, we get $post(v) = sum(e')$ for vertex v. We set $post(s) = n - 1$ for tree root s, where n is the number of vertices in the tree. If n is not known, we can easily compute n using an aggregator in Pregel with each vertex providing a value of 1. (In the more general case where G is a forest, the aggregator counts the number of vertices for each tree/component). For example, Figure 3(b) shows the backward edges (v, u) in the order in $\mathcal{P}$, and vertices are relabeled with post-order numbers. Obviously, the rank of (v, u) gives $post(v)$.

The algorithm correctly computes $pre(v)$/$post(v)$ for all vertices v, because each vertex v in the tree (except root s) has exactly one parent u defined by the forward/backward edge (u, v)/(v, u). Finally, the proof for BPPA follows directly from the fact that both Euler tour and list ranking can be computed by BPPAs.

3.6 Ancestor-Descendant Query

Given a tree rooted at a source vertex s, we often need to know whether a vertex u is an ancestor of another vertex v. Let $pre(v)$ be the pre-order number of v and $nd(v)$ be the number of descendants of v in the tree. We show that if $pre(v)$ and $nd(v)$ is computed for each vertex v in the tree, then an ancestor-descendant query can be answered in $O(1)$ time. Given u and v, following the definition of pre-order numbering, we have: u is an ancestor of v iff $pre(u) \leq pre(v) < pre(u) + nd(u)$. For vertex 1 in Figure 3(a), we have $pre(1) = 1$ and $nd(1) = 3$, and therefore any vertex v with $1 \leq pre(v) < 1+3$ (i.e., vertices 1, 2 and 3) is a descendants of vertex 1.

We have discussed how to compute $pre(v)$ using a BPPA in Section 3.5. We now show that $nd(v)$ can be obtained in the same process: for each forward edge $e = (u, v)$, we set $nd(v) = sum(e') - sum(e) + 1$ where e' is the backward edge (v, u). For tree root s, we set $nd(s) = n$. For example, we compute $nd(1) = sum(1, 0) - sum(0, 1) + 1 = 3 - 1 + 1 = 3$ for vertex 1 in Figure 3(a).

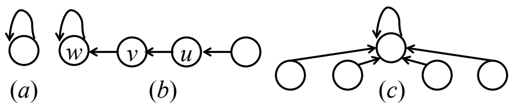
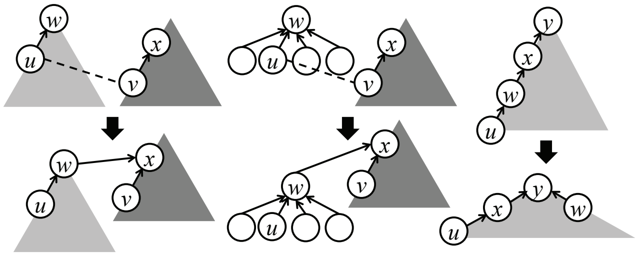


Figure 4: Forest structure of Shiloach-Vishkin's algorithm



(a) Tree Hooking (b) Star Hooking (c) Shortcutting

Figure 5: Tree hooking, star hooking, and shortcutting

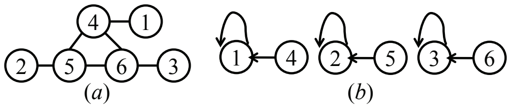


Figure 6: Illustration of the change to star hooking

4. CONNECTED COMPONENTS

In Section 3, we already proposed a BPPA for computing CCs, which requires $O(\delta)$ supersteps. However, the algorithm can be very slow for large-diameter graphs, such as spatial networks where $\delta \approx O(\sqrt{n})$. Now we present a $O(\log n)$-superstep PPA by adapting Shiloach-Vishkin's (S-V) algorithm for the PRAM model [17]. It is claimed in [15] that the requirement of concurrent writes makes the S-V algorithm difficult to be translated to MapReduce; we however show that a translation into a PPA is possible by altering a condition check in the S-V algorithm.

4.1 Algorithm Overview

In the S-V algorithm, each vertex u maintains a pointer $D[u]$, which is initialized as u, forming a self loop as shown Figure 4(a). Throughout the algorithm, vertices are organized by a forest such that all vertices in each tree in the forest belong to the same CC. The tree definition is relaxed a bit here to allow the tree root w to have a self-loop (see Figures 4(b) and 4(c)), i.e., $D[w] = w$; while $D[v]$ of any other vertex v in the tree points to v's parent.

The S-V algorithm proceeds in rounds, and in each round, the pointers are updated in three steps (illustrated in Figure 5): (1)*tree hooking*: for each edge (u, v), if u's parent $w = D[u]$ is a tree root, hook w as a child of v's parent $D[v]$ (i.e., merge the tree rooted at w into v's tree); (2)*star hooking*: for each edge (u, v), if u is in a star (see Figure 4(c) for an example of star), hook the star to v's tree as Step (1) does; (3)*shortcutting*: for each vertex v, move vertex v and its descendants closer to the tree root, by pointing v to the parent of v's parent, i.e., $D[D[v]]$. The algorithm ends when every vertex is in a star.

We perform tree hooking in Step (1) only if $D[v] < D[u]$, so that if u's tree is hooked to v's tree due to edge (u, v), then edge (v, u) will not hook v's tree to u's tree again.

The condition "$D[v] < D[u]$" is not required for star hooking, since if u's tree is hooked to v's tree, v's tree cannot be a star (e.g., in Figure 5(b), after hooking, u is two hops away from x). However, in the Pregel model, Step (2) cannot be processed. Consider the graph shown in Figure 6(a), and suppose that we obtain the three stars in Figure 6(b) right after Step (1). If the one-directional condition is not required, setting $D[D[u]]$ as $D[v]$ makes $D[1] = 2$, $D[2] = 3$ and $D[3] = 1$ through the edges $(4, 5)$, $(5, 6)$ and $(6, 4)$, thus forming a cycle and violating the tree formation required by the algorithm. Such problem does not exist in the PRAM model since the values of $D[u]$ and $D[v]$ are immediately updated after each write operation, while in Pregel the values are those received from the previous superstep. We fix this problem by requiring "$D[v] < D[u]$" for star hooking, without affecting the correctness.

Algorithm 1 The Shiloach-Vishkin Algorithm

```
1: par($u \in V$) {$D[u] \leftarrow u$}
2: par($(u, v) \in E$) {if($v < u$) $D[u] \leftarrow v$}
3: while($\exists u : star[u] = false$) {
4:    par($(u, v) \in E$)
5:       if($D[D[u]]$=$D[u]$ and $D[v]$<$D[u]$){ $D[D[u]] \leftarrow D[v]$}
6:    par($(u, v) \in E$)
7:       if($star[u]$ && $D[u] \neq D[v]$){ $D[D[u]] \leftarrow D[v]$}
8:    par($u \in V$) $D[u] \leftarrow D[D[u]]$
9: }
```

Algorithm 2 Computing $star[u]$

```
1: par($u \in V$) {
2:    $star[u] \leftarrow true$
3:    if($D[u] \neq D[D[u]]$){    $star[u] \leftarrow false$;
4:       $star[D[u]] \leftarrow false$; $star[D[D[u]]] \leftarrow false$}
5:    $star[u] \leftarrow star[D[u]]$
6: }
```

4.2 Algorithm Details

The S-V Algorithm. Algorithm 1 presents the original S-V algorithm for the PRAM model, where "par($o \in S$) {$op(o)$}" means that the operation $op(o)$ runs in parallel for all objects o in set S. Pointers $D[.]$ are updated by examining edge links in parallel. Initially, for each vertex u, $D[u]$ is set as a neighbor $v < u$ if such a v exists (Line 2). Then, in each round, the pointers are updated by tree hooking (Line 5), star hooking (Line 7) and shortcutting (Line 8).

Each vertex u is also associated with a flag $star[u]$ indicating whether it is in a star. The algorithm terminates when all trees become stars, and each star corresponds to a CC. The value of $star[u]$ is required in two places in the while-loop: Line 3 and Line 7. Algorithm 2 shows the PRAM algorithm for computing $star[u]$ for all $u \in V$ according to the current pointer setting, based on the fact that a vertex u with $D[u] \neq D[D[u]]$ invalidates a tree from being a star (see vertex u in Figure 4(b)).

We now illustrate how to translate the steps of Algorithm 1 into Pregel.

Tree Hooking in Pregel. We compute Lines 4–5 of Algorithm 1 in four supersteps: (1)each vertex u sends request to $w = D[u]$ for $D[w]$; (2)each vertex w responds to u by sending $D[w]$; and meanwhile, each vertex v sends $D[v]$ to all v's neighbors; (3)each vertex u obtains $D[w]$ and $D[v]$ from incoming messages and evaluates the if-condition, and then an arbitrary v that satisfies the condition (if it exists) is chosen and $D[v]$ is sent to w; (4)each vertex w that receives messages sets $D[w] \leftarrow D[v]$ using an arbitrary message $D[v]$.

Star Hooking in Pregel. We compute Lines 6–7 of Algorithm 1 similarly. However, the condition "$D[u] \neq D[v]$" in Line 7 should be changed to "$D[v] < D[u]$".

Computing $star[u]$ in Pregel. We compute Algorithm 2 in five supersteps: (1)each vertex u sets $star[u] \leftarrow true$, and sends request to $w = D[u]$ for $D[w]$; (2)each vertex w responds by sending back $D[w]$; (3)each vertex v checks whether $D[u] = D[w]$; if not, it sets $star[u] \leftarrow false$ and notifies w and $D[w]$; (4)each vertex w that gets notified sets $star[w] \leftarrow false$. To process Line 5, in Superstep (3), we also make each vertex u send request to $w = D[u]$ for $star[w]$; in Superstep (4), we also make w respond by sending $star[w]$ (note that if w get notified, it must set $star[w]$ to be false first). Finally, in Superstep (5), each vertex u gets $star[w]$ and uses it to update $star[u]$.

Analysis. The other steps can be translated to Pregel similarly. To check the condition in Line 3 of Algorithm 1, we use an aggregator that computes the AND of $star[u]$ for all vertices u. The algorithm terminates if its value equals $true$.

The correctness of this PPA can be justified as follows. Since we always require $v_a < v_b$ when setting $D[v_a] \leftarrow v_b$ during hooking, the pointer values monotonically decrease, and thus $D[v] = color(v)$ for any vertex v when the algorithm terminates.

Since each step requires a constant number of supersteps, each round of the S-V algorithm uses a constant number of supersteps (14 in our implementation). Following an analysis similar to [17], the algorithm computes CCs in $O(\log n)$ rounds, and therefore we obtain a PPA for computing CCs. However, the algorithm is not a BPPA since a vertex w may become the parent of more than $d(w)$ vertices and hence receives/sends more than $d(w)$ messages at a superstep, though the overall number of messages is always bounded by $O(n)$ at each superstep.

Extension for Computing Spanning Tree. In Section 3.3, we presented an $O(\delta)$-superstep BPPA for spanning tree construction based on BFS. We now illustrate how to modify the S-V algorithm to obtain an $O(\log n)$-superstep PPA that computes the spanning tree, which is faster for large-diameter graphs. The main idea is to mark an edge (u, v) as a tree-edge if a hooking operation is performed due to (u, v), which is straightforward for Line 2 of Algorithm 1. For Lines 4 and 6 that perform $D[D[u]] \leftarrow D[v]$, the last superstep is for $w = D[u]$ to pick an arbitrary message $D[v]$ sent by some v, and set $D[w] \leftarrow D[v]$. In this case, w needs to notify both u and v to mark their edge (u, v) as a tree-edge.

5. BI-CONNECTED COMPONENTS

A *bi-connected component* (**BCC**) of an undirected graph G is a maximal subgraph of G that remains connected after removing one arbitrary vertex. We illustrate the concept of BCC using the graph shown in Figure 7, where the dashed edges constitute one BCC, and the solid edges constitute another. Let R be the equivalence relation on the set of edges of G such that $e_1 R e_2$ iff $e_1 = e_2$ or e_1 and e_2 appear together in some simple cycle, then R defines the BCCs of G. For example, in Figure 7, edges $(4, 5)$ and $(5, 6)$ are in cycle $(4, 5, 6, 4)$, but there is no simple cycle containing both $(4, 5)$ and $(1, 2)$. A vertex is called an articulation point if it belongs to more than one BCC, such as vertex 1 in Figure 7. The removal of an articulation point disconnects the connected components containing it.

Obviously, if we construct a new graph G' whose vertices correspond to the edges of G, and an edge (e_1, e_2) exists in G' iff $e_1 R e_2$, then the CCs of G' correspond to the BCCs of G. However, the number of edges in G' can be superlinear to m. Tarjan-Vishkin's (T-V) PRAM algorithm [20] constructs a concise graph G^* containing a small subset of the edges in G', whose size is bounded by $O(m)$. They prove that it is sufficient to compute the CCs of G^* to obtain the BCCs of G. In the following, we first define G^*, and then propose a non-trivial PPA to compute G^*.

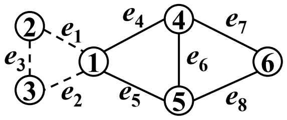
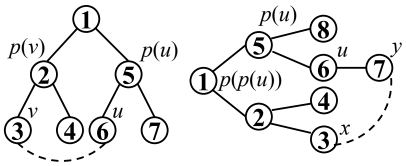


Figure 7: BCC illustration



(*a*) Case 1 & 2 (*b*) Case 3

Figure 8: Illustration of construction of G^*

Assume that $G = (V, E)$ is connected and T is a spanning tree of G. Also, assume that T is rooted and each vertex u in T is assigned a pre-order number $pre(u)$. Let $p(u)$ be the parent of a vertex u in T. We construct $G^* = (V^*, E^*)$ as follows. First, we set $V^* = E$. Then, we add an edge (e_1, e_2) to E^*, where $e_1, e_2 \in E$, iff e_1 and e_2 satisfy one of the following three cases:

- Case 1: $e_1 = (p(u), u)$ is a tree edge in T, and $e_2 = (v, u)$ is a non-tree edge (i.e., e_2 is not in T) with $pre(v) < pre(u)$.
- Case 2: $e_1 = (p(u), u)$ and $e_2 = (p(v), v)$ are two tree edges in T, u and v have no ascendant-descendant relationship in T, and $(u, v) \in E$.
- Case 3: $e_1 = (p(u), u)$ and $e_2 = (p(p(u)), p(u))$ are two tree edges in T, and $\exists(x, y) \in E$ s.t. x is a non-descendant of $p(u)$ in T and y is a descendant of u in T.

Figure 8 illustrates the three cases, where solid edges are tree edges in T, and dashed edges are non-tree edges in $(G - T)$. The vertices are labeled by their pre-order numbers. Case 1 is shown by $e_1 = (5, 6)$ and $e_2 = (3, 6)$ in Figure 8(a). Note that $e_1 R e_2$ as e_1 and e_2 are in a simple cycle $\langle 1, 2, 3, 6, 5, 1 \rangle$. Case 2 is also shown in Figure 8(a) by $e_1 = (5, 6)$ and $e_2 = (2, 3)$, and also e_1 and e_2 are in the same simple cycle. Case 3 is shown in Figure 8(b) by $e_1 = (5, 6)$ and $e_2 = (1, 5)$, where e_1 and e_2 are in the simple cycle $\langle 1, 2, 3, 7, 6, 5, 1 \rangle$.

Each non-tree edge (u, v) of G introduces at most one edge into G^* due to Case 1 (and Case 2), and each tree-edge $(p(u), u)$ introduces at most one edge due to Case 3. Therefore, $|E^*| = O(m)$.

PPA for BCC. The PPA for computing BCCs is a complicated Pregel algorithm. To clearly explain the algorithm, in our design we have separated the whole algorithm into a number of building blocks that are PPAs, most of which are already presented in Section 3 for clarity. Now we discuss how we integrate the building blocks, as well as a non-trivial algorithm to solve some remaining computations.

First, we need to compute the CCs of G^*, for which we only need to consider those vertices of G^* that correspond to the tree edges in T. This is because other vertices of G^* correspond to the non-tree edges e_2 of Case 1, and hence can be assigned to the CC of the corresponding e_1 later on. We already have a PPA to compute CCs from G^* in $O(\delta)$ or $O(\log n)$ supersteps (see Sections 3.2

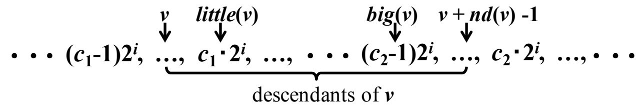

v
little(v)
big(v)
v + nd(v) -1
• • • (c1-1)2i, ..., c1·2i, ..., • • • (c2-1)2i, ..., c2·2i, ..., • • •
descendants of v

Figure 9: Illustration of computing $min(v)$

and 4). To design a PPA for computing BCCs, we still need a PPA for constructing G^* from G.

We first apply the BPPA in Section 3.3 or the PPA in Section 4 to compute a spanning tree T of G. Then, we use the BPPA in Section 3.5 to compute $pre(v)$ for each vertex v in T and the BPPA in Section 3.6 to compute $nd(v)$ for answering ancestor-descendant queries, after which both Case 1 and Case 2 can be checked in a constant number of supersteps by exchanging messages with neighbors. Case 3, however, is far more complicated to handle as vertices other than direct neighbors are involved.

We develop a PPA for handling Case 3 as follows. We first need to compute two more fields for each vertex v, which are defined recursively as follows:

- $min(v)$: the minimum of (1)$pre(v)$, (2)$min(u)$ for all of v's children u, and (3)$pre(w)$ for all non-tree edges (v, w).
- $max(v)$: the maximum of (1)$pre(v)$, (2)$max(u)$ for all of v's children u, and (3)$pre(w)$ for all non-tree edges (v, w).

Let $desc(v)$ be the descendants of v (including v itself) and $\Gamma_{desc}(v)$ be the set of vertices connected to any vertex in $desc(v)$ by a non-tree edge. Intuitively, $min(v)$ (or $max(v)$) is the smallest (or largest) pre-order number among $desc(v) \cup \Gamma_{desc}(v)$.

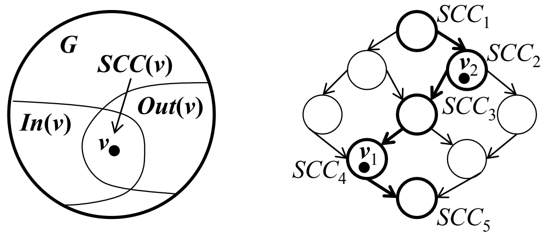
In Case 3, (x, y) exists iff $x \in \Gamma_{desc}(u) - desc(v)$. Since x is not a descendant of $p(u)$, either $pre(x) < pre(p(u))$ which implies $min(u) < pre(p(u))$, or $pre(x) \geq pre(p(u)) + nd(p(u))$ which implies $max(u) \geq pre(p(u)) + nd(p(u))$. To summarize, Case 3 holds for u iff $min(u) < pre(p(u))$ or $max(u) \geq pre(p(u)) + nd(p(u))$.

When $pre(v)$, $nd(v)$, $min(v)$ and $max(v)$ are available for each vertex v, all the three cases can be handled using $O(1)$ supersteps. We now show how to compute $min(v)$ for each v by a PPA in $O(\log n)$ supersteps (computing $max(v)$ is symmetric).

In the remainder of this section, we simply use v to denote $pre(v)$ for ease of presentation. We further define $local(v)$ to be the minimum among v and all the neighbors connected to v by a non-tree edge. Note that $min(v)$ is just the minimum of $local(u)$ among all of v's descendants u.

We compute $min(v)$ in $O(\log n)$ rounds. At the beginning of the i-th round, each vertex $v = c \cdot 2^i$ (c is a natural number) maintains a field $global(c \cdot 2^i) = \min\{local(u) : c \cdot 2^i \leq u < (c{+}1)2^i\}$. Then in the $(i{+}1)$-th round, for each vertex $v = c{\cdot}2^{i+1}$ we can simply update $global(v) = \min\{global(v), global(v + 2^i)\}$, i.e., merging the results from two consecutive segments of length 2^i. Here, each round can be done by a three-superstep PPA: (1) each v requests $global(v + 2^i)$ from $(v + 2^i)$; (2) $(v + 2^i)$ responds by sending $global(v{+}2^i)$ to v; (3) v receives $global(v{+}2^i)$ to update $global(v)$. Initially, $global(v) = local(v)$ for each v, and $local(v)$ can be computed similarly, by requesting $pre(u)$ from each neighbor u connected to v by a non-tree edge.

Given a vertex v, the descendants of v in T are $\{v, v{+}1, \ldots, v{+}nd(v) - 1\}$. At the beginning of the i-th round, we define $little(v)$ (and respectively, $big(v)$) to be the first (and respectively, the last) descendant that is a multiple of 2^i. Figure 9 illustrates the concept of $little(v)$ and $big(v)$.



(a) *In*(*v*), *Out*(*v*) & *SCC*(*v*)

(b) DAG Path

Figure 10: Concepts in SCC computation

We maintain the following invariant for each round:

$$\begin{aligned} min(v) \quad = \quad & \min\{local(u) : u \in [v, little(v)) \cup \\ & [big(v), v + nd(v) - 1]\}. \qquad (1) \end{aligned}$$

Obviously, the correct value of $min(v)$ is computed when $little(v) = big(v)$, which must happen for some value of i as i goes from 0 to $\lfloor \log_2 n \rfloor$. We perform the following operations for each v in the i-th round, which maintains the invariant given by Equation (1):

- If $little(v) < big(v)$ and $little(v)$ is not a multiple of 2^{i+1}, set $min(v) = \min\{min(v), global(little(v))\}$, and then set $little(v) = little(v) + 2^i$.
- If $little(v) < big(v)$ and $big(v)$ is not a multiple of 2^{i+1}, set $min(v) = \min\{min(v), global(big(v) - 2^i)\}$, and then set $big(v) = big(v) - 2^i$.

We do not update $little(v)$ (or $big(v)$) if it is a multiple of 2^{i+1}, so that in the $(i + 1)$-th round it is aligned with $c \cdot 2^{i+1}$. We update $little(v)$, $big(v)$, and $min(v)$ by Pregel operations similar to those for updating $global(v)$.

6. STRONGLY CONNECTED COMPONENTS

In this section, we present two novel Pregel algorithms that compute *strongly connected component*s (**SCC**s) from a directed graph $G = (V, E)$. Let $SCC(v)$ be the SCC that contains v, and let $Out(v)$ (and $In(v)$) be the set of vertices that can be reached from v (and respectively, that can reach v) in G. In the PRAM model, existing algorithms [7, 2, 1] were designed based on the observation that $SCC(v) = Out(v) \cap In(v)$ (see Figure 10(a)). Their approach computes $Out(v)$ and $In(v)$ by forward/backward BFS from source v that is randomly picked from G. This process then repeats on $G[Out(v) - SCC(v)]$, $G[In(v) - SCC(v)]$ and $G[V - (Out(v) \cup In(v))]$, where $G[X]$ denotes the subgraph of G induced by vertex set X. The correctness is guaranteed by the property that any remaining SCC must be in one of these subgraphs.

We also apply the relationship between $SCC(v)$, $Out(v)$ and $In(v)$ and design two Pregel algorithms based on label propagation. The first algorithm propagates the smallest vertex (ID) that every vertex has seen so far, while the second algorithm propagates multiple source vertices to speed up SCC computation. Before describing our SCC algorithms, we first present a BPPA for the graph decomposition operation that is employed in our algorithms.

Graph decomposition. Given a partition of V, denoted by V_1, V_2, …, V_ℓ, we decompose G into $G[V_1]$, $G[V_2]$, …, $G[V_\ell]$ in two supersteps (assume that each vertex v contains a label i indicating $v \in V_i$): (1)each vertex notifies all its in-neighbors and out-neighbors about its label i; (2)each vertex checks the incoming messages, removes the edges from/to the vertices whose label is different from its own label, and votes to halt.

6.1 Min-Label Algorithm

We first describe the Pregel operation for *min-label* propagation, where each vertex v maintains two labels $min_f(v)$ and $min_b(v)$.

Forward Min-Label Propagation. (1)Each vertex v initializes $min_f(v) = v$, propagates $min_f(v)$ to all v's out-neighbors, and votes to halt; (2)when a vertex v receives a set of messages from its in-neighbors $\Gamma_{in}(v)$, let min^* be the smallest message received, then if $min^* < min_f(v)$, v updates $min_f(v) = min^*$ and propagates $min_f(v)$ to all v's out-neighbors; v votes to halt at last. We repeat Step (2) until all vertices vote to halt.

Backward Min-Label Propagation. This operation is done after forward min-label propagation. The differences are that (1)initially, only vertices v satisfying $v = min_f(v)$ are active with $min_b(v) = v$, while for the other vertices u, $min_b(u) = \infty$; (2)each active vertex v propagates $min_b(v)$ towards all v's in-neighbors.

Both operations are BPPAs with $O(\delta)$ supersteps. After the forward and then backward min-label propagations, each vertex v obtains a label pair $(min_f(v), min_b(v))$. This labeling has the following property:

LEMMA 1. *Let $V_{(i,j)} = \{v \in V : (min_f(v), min_b(v)) = (i,j)\}$. Then, (i)any SCC is a subset of some $V_{(i,j)}$, and (ii)$V_{(i,i)}$ is a SCC with color i.*

PROOF. We first prove (i). Given a SCC and a vertex v in the SCC, suppose that v has a label pair (i,j), we show that for any other vertex u in the SCC, u also has the same label pair (i,j). We only prove $min_f(u) = i$, and the proof of $min_b(u) = j$ is symmetric. Let us denote $v_1 \to v_2$ iff v_1 can reach v_2. Since $i \to v \to u$, $min_f(u) > i$ is impossible. Also, since $min_f(u) \to u \to v$, $min_f(u) < i$ is impossible. Therefore, we have $min_f(u) = i$.

We now prove (ii). First, $\forall v \in V_{(i,i)}$, $v \to i$ and $i \to v$, and thus $V_{(i,i)} \subseteq SCC(i)$. Second, if $V_{(i,i)}$ exists, then $i \in V_{(i,i)}$, and by (i), we have $SCC(i) \subseteq V_{(i,i)}$. Therefore, $V_{(i,i)} = SCC(i)$. □

The **min-label algorithm** repeats the following operations: (1)forward min-label propagation; (2)backward min-label propagation; (3)an aggregator collects label pairs (i,j), and assigns a unique id $\mathcal{ID}$ to each $V_{(i,j)}$; then graph decomposition is performed to remove edges crossing different $G[V_{\mathcal{ID}}]$; finally, we mark each vertex v with label (i,i) to indicate that its SCC is found.

In each step, only unmarked vertices are active, and thus vertices do not participate in later rounds once its SCC is determined.

Each round of the algorithm refines the vertex partition of the previous round. Since all the three steps are BPPAs, each round of the min-label algorithm is a BPPA. The algorithm terminates once all vertices are marked.

The correctness of the algorithm follows directly from Lemma 1. We now analyze the number of rounds the min-label algorithm requires. Given a graph G, if we contract each SCC into a super-vertex, we obtain a DAG in which an edge directs from one super-vertex (representing a SCC, SCC_i) to another super-vertex (representing another SCC, SCC_j) iff there is an edge from some vertex in SCC_i to some vertex in SCC_j. Let $\mathcal{L}$ be the longest path length in the DAG, then we have the following bound.

THEOREM 1. *The min-label algorithm runs for at most $\mathcal{L}$ rounds.*

PROOF. We show the correctness of Theorem 1 by proving the following invariant: at the end of Round i, the length of any maximal path in the DAG is at most $\mathcal{L} - i$. We only need to show that in each round, a SCC is determined for any maximal path in the DAG. This is because, after the SCC is marked and the exterior edges are removed, (1) if the SCC is an endpoint of the DAG path (let us denote its length by ℓ), the path length becomes $\ell - 1$; (2) otherwise, the path is broken into two paths with length $< \ell - 1$. In either case, the new paths have length at most $\ell - 1$.

We now show that for an arbitrary maximal DAG path p, at least one SCC is found as $V_{(i,i)}$. First, let us assume i_1 is the smallest unmarked vertex in G, then $V_{(i_1,i_1)}$ is found as a SCC for any maximal path p_1 that contains $SCC(i_1)$. Now consider those maximal paths that does not contain $SCC(i_1)$, and let i_2 be the smallest vertex in the SCCs of these paths. Then $V_{(i_2,i_2)}$ is found as a SCC for any such path p_2 that contains $SCC(i_2)$. The reasoning continues for those maximal paths that do not contain $SCC(i_1)$ and $SCC(i_2)$ until all maximal paths are covered. □

The above bound is very loose, and often, more than one SCC is marked per DAG path in a round. We illustrate it using Figure 10(b), where there is a DAG path $P = \langle SCC_1, SCC_2, SCC_3, SCC_4, SCC_5 \rangle$, and v_1 is the smallest vertex in G. Obviously, for any vertex v in SCC_4–SCC_5, $min_f(v) = v_1$. Since v_1 is picked as a source for backward propagation, for any vertex v in SCC_1–SCC_4, $min_b(v) = v_1$. Thus, any vertex $v \in SCC_4$ has label pair (v_1, v_1) and is marked. Now consider the sub-path before SCC_4, i.e., $\langle SCC_1, SCC_2, SCC_3 \rangle$, and assume $v_2 \in SCC_2$ is the second smallest vertex in G, then a similar reasoning shows that SCC_2 is found as $V_{(v_2,v_2)}$. In this way, P is quickly broken into many subpaths, each can be processed in parallel, and hence in practice the number of rounds needed can be much less than $\mathcal{L}$.

In our implementation, we further perform two optimizations:

Optimization 1: Removing Trivial SCCs. If the in-degree or out-degree of a vertex v is 0, then v itself constitutes a trivial SCC and can be directly marked to avoid useless label propagation. We mark trivial SCCs before forward min-label propagation in each round.

We now describe a Pregel algorithm to mark all vertices with in-degree 0. Initially, each vertex v with in-degree 0 marks itself, sends itself to all out-neighbors and votes to halt. In subsequent supersteps, each vertex v removes its in-edges from the in-neighbors that appears in the incoming messages, and checks whether its in-degree is 0. If so, the vertex marks itself and sends itself to all out-neighbors. Finally, the vertex votes to halt. We also mark vertices with out-degree 0 in each superstep in a symmetric manner.

The algorithm takes only a small number of supersteps in practice (as shown by experiments in Section 7), since real world graphs (e.g., social networks) have a dense core, and the limited number of trivial SCCs only exist in the sparse boundary regions of the graphs. Furthermore, many vertices with zero in-degree/out-degree are marked as trivial SCCs in parallel in each superstep. On the other hand, removing trivial SCCs prevents them from participating in min-label propagation, which may otherwise degrade the algorithm effectiveness if some trivial SCC vertex has a small ID.

Optimization 2: Early Termination. We do not need to run the algorithm until all vertices are marked as a vertex of a SCC found. We also mark a vertex $v \in G[V_{\mathcal{ID}}]$ if $|V_{\mathcal{ID}}|$ is smaller than a threshold τ, so that all vertices in subgraph $G[V_{\mathcal{ID}}]$ remain inactive in later rounds. Here, $|V_{\mathcal{ID}}|$ is obtained using the aggregator of Step (3). We stop once all vertices are marked as a SCC/subgraph vertex. Then, we use one round of MapReduce to assign the subgraphs to different machines to compute the SCCs directly from each subgraph $G[V_{\mathcal{ID}}]$. Since each subgraph is small, its SCCs can be computed on a single machine using an efficient main memory algorithm, without inter-machine communication.

6.2 Multi-Label Algorithm

The multi-label algorithm aims to speed up SCC discovery and graph decomposition by propagating k source vertices in parallel, instead of just one randomly picked source.

In this algorithm, each vertex v maintains two label sets $Src_f(u)$ and $Src_b(u)$. The algorithm is similar to the min-label algorithm, except that the min-label propagation operation is replaced with the *k-label* propagation operation described below:

Forward k-Label Propagation. Suppose that the current vertex partition is V_1, V_2, …, V_ℓ. (1)In Superstep 1, an aggregator randomly selects k vertex samples from each subgraph $G[V_i]$. (2)In Superstep 2, each source u initializes $Src_f(u) = \{u\}$ and propagates label u to all its out-neighbors, while each non-source vertex v initializes $Src_f(v) = \emptyset$. Finally, the vertex votes to halt. (3)In subsequent supersteps, if a vertex v receives a label $u \notin Src_f(v)$ from an in-neighbor, it updates $Src_f(v) = Src_f(v) \cup \{u\}$ and propagates u to all its out-neighbors, before voting to halt.

The backward k-label propagation is symmetric. Unlike the min-label algorithm where backward propagation is done after forward propagation, in the multi-label algorithm, we perform both forward and backward propagation in parallel.

The k-label propagation operation is also a BPPA with $O(\delta)$ supersteps, and when it terminates, each vertex v obtains a label pair $(Src_f(v), Src_b(v))$. This labeling has the following property:

LEMMA 2. *Let $V_{(S_f,S_b)} = \{v \in V : (Src_f(v), Src_b(v)) = (S_f, S_b)\}$. Then, (i)any SCC is a subset of some $V_{(S_f,S_b)}$, and (ii)$V_{(S_f,S_b)}$ is a SCC if $S_f \cap S_b \neq \emptyset$.*

PROOF. We first prove (i). Given a SCC and a vertex v in the SCC, suppose that v has a label pair (S_f, S_b), we show that for any other u in the SCC, u also has the same label pair (S_f, S_b). We only prove $Src_f(u) = S_f$, and the proof of $Src_b(u) = S_b$ is symmetric. (1) For any vertex $s \in S_f$, we have $s \to v \to u$ and thus $s \in Src_f(u)$; therefore, $S_f \subseteq Src_f(u)$. (2) For any vertex $s \in Src_f(u)$, we have $s \to u \to v$ and thus $s \in Src_f(v) = S_f$; therefore, $Src_f(u) \subseteq S_f$. As a result, $Src_f(u) = S_f$.

We now prove (ii), Since $S_f \cap S_b \neq \emptyset$, $\exists u \in S_f \cap S_b$. First, $\forall v \in V_{(S_f,S_b)}$, $v \to u$ and $u \to v$, and thus $V_{(S_f,S_b)} \subseteq SCC(u)$. Second, if $V_{(S_f,S_b)}$ exists, then $u \in V_{(S_f,S_b)}$ and by (i), we have $SCC(u) \subseteq V_{(S_f,S_b)}$. Therefore, $V_{(S_f,S_b)} = SCC(u)$. □

We now analyze the number of rounds required. In any round, we have ℓ subgraphs and thus around ℓk source vertices. Since we do not know ℓ, We only give a very loose analysis assuming $\ell = 1$ (i.e., there are only k sources). Furthermore, we assume that vertices are only marked because they form a SCC, while in practice Optimization 2 of Section 6.1 is applied to also mark vertices of sufficiently small subgraphs.

Suppose that we can mark $(1-\theta)n$ vertices as SCC vertices in each round, where $0 < \theta < 1$. Then, after i rounds the graph has $\theta^i n$ vertices, and in $O(\log_{1/\theta} n)$ rounds the graph is sufficiently small to allow efficient single-machine SCC computation. We now study the relationship between θ and k.

Assume that there are c SCCs in G: SCC_1, $SCC_2, \ldots, SCC_c$. Let n_i be the number of vertices in SCC_i and $p_i = n_i/n$. We analyze how many vertices are marked in expectation after one round. Note that if x sampled source vertices belong to the same SCC, then we actually waste $x-1$ samples. Our goal is to show that such waste is limited.

We define a random variable X that refers to the number of vertices marked. We also define an indicator variable X_i for each SCC SCC_i as follows: $X_i = 1$ if at least one sample belongs to SCC_i, and $X_i = 0$ otherwise. Let s_j be the j-th sample. We have

$$\begin{aligned} E[X_i] &= Pr\{X_i = 1\} = 1 - \prod_{j=1}^{k} Pr\{s_j \notin SCC_i\} \\ &= 1 - \prod_{j=1}^{k}(1-p_i) = 1-(1-p_i)^k. \end{aligned}$$

Note that $X = \sum_{i=1}^{c} n_i \cdot X_i$. According to the linearity of expectation, we have

$$\begin{aligned} E[X] &= \sum_{i=1}^{c} n_i \cdot E[X_i] = \sum_{i=1}^{c}[n_i - n_i(1-p_i)^k] \\ &= n - \sum_{i=1}^{c} n_i(1-p_i)^k = n - n\sum_{i=1}^{c} p_i(1-p_i)^k. \end{aligned}$$

In other words, $\theta = \sum_{i=1}^{c} p_i(1-p_i)^k$. Since the number of vertices remaining unmarked is θn, we want θ to be as small as possible. In fact, if the size of SCCs are biased, θ is small. This is because if there is a very large SCC, it is likely that some of its vertices are sampled as source vertices, and hence many vertices will be marked as being a vertex of the SCC.

The worst case happens when all the SCCs are of equal size, i.e., $p_i = 1/c$ for all i, in which case $\theta = (1-1/c)^k$. Since $(1-1/c) < 1$, θ decreases with k, but the rate of decrement depends on c. For example, when $c = 1000$, to get $\theta = 0.9$ we need to set $k = 100$. However, we note that real world graphs rarely have all SCCs with similar sizes, and the analysis is very loose. In practise, k can be much smaller even for very large c.

We now present a theorem that formalizes the above discussion.

THEOREM 2. *If $p_i < \frac{2}{k+1}$ for all i, then $\theta \le (1-1/c)^k$. Otherwise, $\theta = \sum_{i=1}^{c} p_i(1-p_i)^k < 1-1/k$.*

PROOF. We first prove the case when there exists a SCC SCC_i with $p_i \ge \frac{2}{k+1} > \frac{1}{k}$. Since we sample k vertices in total, in expectation at least one vertex in SCC_i is sampled. As a result, in expectation at least $n_i > n/k$ vertices are marked, or equivalently, $\theta < 1-1/k$.

We now prove the case when $p_i < \frac{2}{k+1}$ for all i. Consider the following optimization problem:

$$\begin{aligned} &\text{maximize} \quad && \theta(p_1, \ldots, p_c) = \sum_{i=1}^{c} p_i(1-p_i)^k \\ &\text{subject to} \quad && \sum_{i=1}^{c} p_i = 1, \ (p_i > 0) \end{aligned}$$

Using the method of Lagrange multipliers, we obtain the following Lagrange function:

$$L(p_1, \ldots, p_c, \lambda) = \sum_{i=1}^{c} p_i(1-p_i)^k + \lambda(\sum_{i=1}^{c} p_i - 1)$$

The stationary points can be obtained by solving the following equations:

$$\begin{aligned} \frac{\partial L}{\partial p_i} &= [1-(k+1)p_i](1-p_i)^{k-1} + \lambda \triangleq 0 \\ \frac{\partial L}{\partial \lambda} &= \sum_{i=1}^{c} p_i - 1 \triangleq 0 \end{aligned}$$

Obviously, $p_i = 1/c$ and $\lambda = [(k+1)/c - 1](1 - 1/c)^{k-1}$ is a solution. To prove that θ is maximized at this point (i.e., $p_i = 1/c$ for all i), we need to show that θ is a concave function in the domain $\{(p_1, \ldots, p_c) \mid p_i > 0 \text{ for all } i \text{ and } \sum_{i=1}^{c} p_i = 1\}$. This is equivalent to showing that the hessian matrix of θ is negative definite.

We now compute the elements of the hessian matrix:

$$\begin{aligned} \frac{\partial^2 \theta}{\partial p_i \partial p_j} &= 0 \qquad\qquad (\text{when } i \neq j) \\ \frac{\partial^2 \theta}{\partial p_i^2} &= k[(k+1)p_i - 2](1 - p_i)^{k-2} \end{aligned}$$

Therefore, the hessian matrix is a diagonal matrix $diag(\frac{\partial^2 \theta}{\partial p_1^2}, \cdots, \frac{\partial^2 \theta}{\partial p_c^2})$. We now show that it is negative definite, which is based on the following property from linear algebra:

Lemma 3. *Matrix $M_{N \times N}$ is negative definite iff for all $r = 1, \cdots, N$, $(-1)^r \det({}_rM_r) > 0$, where ${}_sM_t$ is the submatrix composed of the first t rows and s columns of M.*

The proof is completed by observing that

$$\begin{aligned} & (-1)^r \det({}_rM_r) \\ = \; & (-1)^r \det(diag(\frac{\partial^2 \theta}{\partial p_1^2}, \cdots, \frac{\partial^2 \theta}{\partial p_r^2})) \\ = \; & (-1)^r \prod_{i=1}^{r} \frac{\partial^2 \theta}{\partial p_i^2} \\ = \; & k^r \cdot [\prod_{i=1}^{r} (2 - (k+1)p_i))] \cdot [\prod_{i=1}^{r} (1 - p_i)]^{k-2} \\ > \; & 0 \qquad\quad (\text{since } p_i < 2/(k+1) \text{ and } p_i < 1) \end{aligned}$$

□

Finally, we emphasize that Theorem 2 is very loose: when there exists a SCC SCC_i with p_i much greater than $\frac{2}{k+1}$, θ is much smaller than $1 - 1/k$.

7. EXPERIMENTAL EVALUATION

We evaluate the performance of our algorithms over large real-world graphs. We ran all experiments on a cluster of 16 machines, each with 24 processors (two Intel Xeon E5-2620 CPU) and 48GB RAM. One machine is used as the master that runs only one working process (or simply, worker), while the other 15 machines act as slaves each running 10 workers. The connectivity between any pair of nodes in the cluster is 1Gbps. All our algorithms were implemented in Pregel+[2], an open-source Pregel implementation built on top of Hadoop Distributed File System (HDFS), though any Pregel-like system can be used to implement our algorithms. All the source codes of the algorithms discussed in this paper can be found at `http://www.cse.cuhk.edu.hk/pregelplus/download.html`.

Datasets. We used five large real-world graph datasets, which are shown in Figure 11: (1)*BTC*[3]: a semantic graph converted from the Billion Triple Challenge 2009 RDF dataset [4]; (2)*LJ-UG*[4]: a bipartite network of LiveJournal users and their group memberships; (3)*USA*[5]: the USA road network; (4)*Twitter*[6]: Twitter who-follows-who network based on a snapshot taken in 2009; (5)*LJ-DG*[7]: a friendship network of the LiveJournal blogging community. The statistics of the graphs are given in Figure 11, where $|V|$ is the number of vertices and $|E|$ is the number of edges.

Data	Type	|*V*|	|*E*|
BTC	undirected	164,732,473	772,822,094
LJ-UG	undirected	10,690,276	224,614,770
USA	undirected	23,947,347	58,333,344
Twitter	directed	52,579,682	1,963,263,821
LJ-DG	directed	4,847,571	68,993,773

Figure 11: Datasets

7.1 Performance of CC & BCC Algorithms

In Section 3 we proposed BPPAs for a list of fundamental graph problems. Since they are used as building blocks in the PPA for computing BCCs, we also report their performance results as steps of the BCC computation. For the CC computation involved, we test both Hash-Min (see Section 3.2) and S-V (see Section 4). Before presenting the results, we first review the sequence of PPA tasks in the BCC computation: (1)*HashMin*: to compute $color(v)$ for all $v \in V$ using the BPPA of Section 3.2; (2)*BFS*: to compute a spanning forest of G using the BPPA of Section 3.3 with sources $\{s \in V | color(s) = s\}$; alternatively, we may obtain the spanning forest using the S-V algorithm of Section 4, denoted by *S-V*; (3)*EulerTour*: to construct Euler tours from the spanning forest using the BPPA of Section 3.4; (4)*ListRank1*: to break each Euler tour into a list and mark each edge as forward/backward using list ranking (see Section 3.5); (5)*ListRank2*: using the edge forward/backward marks to compute $pre(v)$ and $nd(v)$ for each $v \in V$ using list ranking (see Section 3.5); (6)*MinMax*: to compute $min(v)$ and $max(v)$ for each $v \in V$ using the PPA of Section 5; (7)*AuxGraph*: to construct G^* described in Section 5 using $pre(v)$, $nd(v)$, $min(v)$ and $max(v)$ information; (8)*HashMin2*: to compute the CCs of G^*, but only consider tree edges; alternatively, we may use the S-V algorithm for CC computation, denoted by *S-V2*; (9)*Case1Mark*: to decide the BCCs of the non-tree edges in G^* using Case 1 of Section 5.

We report the per-task performance of BCC computation over the two *small-diameter* graphs *BTC* and *LJ-UG* in Figure 12. Due to the small graph diameter, Hash-Min is much more efficient than S-V over these graphs. For example, Hash-Min ends in 18 supersteps over *LJ-UG* and takes only 11.85 seconds. In contrast, S-V takes 58 supersteps and 142.24 seconds. Thus, the results verify that it is more efficient to compute CCs using Hash-Min when graph diameter is small. We also give the total computational time of our BCC algorithm, and the results again show that using Hash-Min as a building block in the BCC computation achieves almost twice faster total time than using S-V.

Next, we report the per-task performance of BCC computation over the *large-diameter USA* road network in Figure 13. Due to the large graph diameter, Hash-Min is very time-consuming (1011.19 seconds) as it runs for 6262 supersteps. In contrast, S-V takes only 198 supersteps and only 368.20 seconds. This again verifies the difference between an $O(\delta)$-round PPA (i.e., Hash-Min) and an $O(\log n)$-round PPA (i.e., S-V). Similar behavior is also observed for CC computation over G^*, where HashMin2 takes 5437.72 seconds while S-V2 takes only 526.69 seconds. Overall, our BCC

[2] http://www.cse.cuhk.edu.hk/pregelplus

[3] http://km.aifb.kit.edu/projects/btc-2009/

[4] http://konect.uni-koblenz.de/networks/livejournal-groupmemberships

[5] http://www.dis.uniroma1.it/challenge9/download.shtml

[6] http://konect.uni-koblenz.de/networks/twitter_mpi

[7] http://snap.stanford.edu/data/soc-LiveJournal1.html

Task	BTC		LJ-UG	
	# of Steps	Comp. Time	# of Steps	Comp. Time
HashMin	30	32.24 s	18	11.85 s
BFS	31	20.56 s	19	8.61 s
S-V	86	449.97 s	58	142.24 s
EulerTour	3	14.26 s	3	2.26 s
ListRank1	49	544.71 s	53	97.98 s
ListRank2	49	541.86 s	53	98.59 s
MinMax	46	35.88 s	50	8.54 s
AuxGraph	4	58.05 s	4	21.94 s
HashMin2	34	42.91 s	11	21.04 s
S-V2	72	443.16 s	58	138.59 s
Case1Mark	4	35.62 s	4	20.83 s
Total Time (CC by HashMin)	1326.09 s		291.64 s	
Total Time (CC by S-V)	2123.51 s		530.97 s	

Figure 12: CC/BCC algorithm performance on BTC & LJ-UG

Task	USA	
	# of Steps	Comp. Time
HashMin	6262	1011.19 s
BFS	6263	964.11 s
S-V	198	368.20 s
EulerTour	3	3.04 s
ListRank1	55	203.05 s
ListRank2	55	197.78 s
MinMax	52	16.65 s
AuxGraph	4	12.29 s
HashMin2	7365	5437.72 s
S-V2	226	536.69 s
Case1Mark	4	2.90 s
Total Time (CC by HashMin)	7848.73 s	
Total Time (CC by S-V)	1340.60 s	

Figure 13: CC/BCC algorithm performance on USA

algorithm uses only 1340.60 seconds over *USA* when CC computation is done by S-V, while it takes more than 2 hours if Hash-Min is used instead. This demonstrates the advantage of our S-V algorithm for processing large-diameter graphs.

It might be argued that computing the CCs of a road network in Pregel is not important, as road networks are usually connected and not very large. However, some spatial networks are huge in size, such as the triangulated irregular network (TIN) that models terrain, where CC computation is useful when we want to compute the islands given a specific sea level. Also, CC computation is a critical building block in our PPA for computing BCCs, and finding BCCs of a spatial network is important for analyzing its weak connection points.

7.2 Performance of SCC Algorithms

We now report the performance of our min-label and multi-label algorithms for computing SCCs, over the two large directed graphs *Twitter* and *LJ-DG*.

7.2.1 *Performance of Min-Label Algorithm*

Before describing the results, we first review the sequence of tasks performed in each round of our min-label algorithm: (1)*Opt 1*: this task removes trivial SCCs as described in Optimization 1 of Section 6.1; (2)*MinLabel*: forward min-label propagation followed by backward min-label propagation; (3)*GDecom*: this task uses an aggregator to collect label pairs $(min_f(u), min_b(u))$ and assigns a new $\mathcal{ID}$ to each pair, sets the $\mathcal{ID}$ of each vertex u according to u's $(min_f(u), min_b(u))$, marks each vertex u with $min_f(u) = min_b(u)$ as being in a SCC, and performs graph decomposition using the algorithm described at the beginning of Section 6. Recall that we do not decompose a subgraph if its size (decided by number of vertices) is smaller than a user-defined threshold τ.

Round	Task	# of Steps	Comp. Time	Max Size
1	**Opt 1**	18	9.41 s	238,986
	MinLabel	15 + 14	75.80 s	
	GDecom	3	76.86 s	
2	**Opt 1**	5	0.81 s	22
	MinLabel	36 + 75	18.73 s	
	GDecom	3	1.74 s	
3	**Opt 1**	3	0.46 s	2
	MinLabel	6 + 5	2.02 s	
	GDecom	3	0.44 s	
4	**Opt 1**	1	0.21 s	0
	MinLabel	3 + 3	0.96 s	
	GDecom	3	0.50 s	
Total			187.94 s	

Figure 14: Min-label algorithm over Twitter ($\tau = 0$)

Round	Task	# of Steps	Comp. Time	Max Size
1	**Opt 1**	16	2.73 s	51,697
	MinLabel	14 + 16	14.91 s	
	GDecom	3	7.24 s	
2	**Opt 1**	4	0.34 s	81
	MinLabel	8 + 9	1.91 s	
	GDecom	3	0.72 s	
3	**Opt 1**	3	0.22 s	29
	MinLabel	6 + 7	1.08 s	
	GDecom	3	0.40 s	
4	**Opt 1**	1	0.14 s	12
	MinLabel	4 + 4	1.01 s	
	GDecom	3	0.32 s	
Total			31.02 s	

Figure 15: Min-label algorithm over LJ-DG ($\tau = 0$)

We first compute the SCCs of *Twitter*, where we only mark a vertex when its SCC is determined (i.e., $\tau = 0$). Figure 14 reports the number of supersteps required by each task, as well as the computational time. The last column "Max Size" shows the maximum $|V_{\mathcal{ID}}|$ among those subgraphs $G[V_{\mathcal{ID}}]$ that are not marked as a SCC after each round, and all SCCs are found when *MaxSize* becomes 0. As Figure 14 shows, the min-label algorithm takes only 4 rounds to compute all the SCCs over *Twitter*, which demonstrates that in practice the min-label algorithm requires much less than $\mathcal{L}$ rounds as established by Theorem 1. Besides, the min-label propagation operations take only a small number of supersteps due to the small graph diameter. For example, in Round 1, forward propagation takes only 15 supersteps, followed by a 14-superstep backward propagation. The total computational time is only 187.94 seconds for a graph with almost 2 billion edges, which is very efficient.

Note that after Round 2 in Figure 14, the largest unmarked subgraph has size merely 22. Therefore, another option is to distribute these small subgraphs to different machines for single-machine SCC computation using MapReduce. Thus, we also run our min-label algorithm over *Twitter* using $\tau = 50,000$, so that a subgraph is marked to avoid further decomposition once it contains less than 50,000 vertices. All vertices are marked after 2 rounds, and the performance is similar to those in Figure 14. We then run a MapReduce job to compute the SCCs of the marked subgraphs, which takes 199 seconds.

We remark that the min-label algorithm with $\tau = 0$ is not always able to find all the SCCs for an arbitrary graph. One example is the *LJ-DG* datasets, the performance of which is shown in Figure 15 (only the results of the first 4 rounds are shown). In fact, for the subsequent three rounds, "Max Size" decreases slowly as 11, 10

Round	Task	# of Steps	Comp. Time	Max Size
1	**Opt 1**	18	8.07 s	238,986
	MultiLabel	17	423.85 s	
	GDecom	3	102.98 s	
2	**Opt 1**	5	0.81 s	206,319
	MultiLabel	5	0.55 s	
	GDecom	3	0.41 s	
3	**Opt 1**	1	0.16 s	206,292
	MultiLabel	6	0.94 s	
	GDecom	3	0.38 s	
MapReduce			181 s	

Figure 16: Multi-label algorithm over Twitter ($\tau = 50,000$)

Round	Task	# of Steps	Comp. Time	Max Size
1	**Opt 1**	16	2.66 s	51,697
	MultiLabel	19	27.02 s	
	GDecom	3	6.30 s	
2	**Opt 1**	5	0.74 s	50,706
	MultiLabel	6	0.71 s	
	GDecom	3	0.25 s	
3	**Opt 1**	1	0.13 s	50,629
	MultiLabel	8	0.80 s	
	GDecom	3	0.36 s	
MapReduce			26 s	

Figure 17: Multi-label algorithm over LJ-DG ($\tau = 50,000$)

and 9. On the contrary, running min-label algorithm over *LJ-DG* using $\tau = 50,000$ takes only 2 rounds to mark all vertices. We then run a MapReduce job to compute the SCCs of the marked subgraphs, which takes 27 seconds.

7.2.2 Performance of Multi-Label Algorithm

We now report the performance of our multi-label algorithm. For the parallel forward and backward k-label propagation, we fix $k = 10$. We also set $\tau = 50,000$ and subgraphs with less than $50,000$ vertices are not further decomposed. The performance of the multi-label algorithm over *Twitter* and *LJ-DG* are shown in Figures 16 and 17, respectively. We can see that although Round 1 bounds the maximum unmarked subgraph size to a relatively small number, "Max Size" decreases slowly in the later Rounds. However, the subgraphs are small enough to be assigned to different machines for local SCC computation. We run MapReduce to compute the SCCs of all subgraphs after Round 3, which takes 181 seconds over *Twitter* and 26 seconds over *LJ-DG*.

Unlike the min-label algorithm for which we can afford to run until termination, the multi-label algorithm finds at most k SCCs in each round, and is only effective in earlier rounds when there are large SCCs. In fact, the multi-label algorithm almost always finds the largest SCC in the first round which usually contains the majority of the vertices in a graph, while the min-label algorithm may not find it in the first round. For example, let the largest SCC be SCC_{max}, then if there is a vertex $v \notin SCC_{max}$ whose ID is smaller than all vertices in SCC_{max}, and if v links to a vertex in SCC_{max}, then SCC_{max} cannot be found in Round 1 by the multi-label algorithm. Thus, in applications where only the largest SCC (also called the giant SCC) is needed, the multi-label algorithm will be a better choice; in applications where all SCCs are needed, running multi-label algorithm for Round 1 followed by min-label algorithm for the subsequent rounds can be a good choice.

8. CONCLUSIONS

We proposed efficient distributed algorithms for solving three fundamental graph connectivity problems, namely CC, BCC, and SCC. Specifically, we defined the notion of PPA to design Pregel algorithms that have guaranteed performance, i.e., requiring only linear space, communication and computation per iteration, and only $O(\log n)$ or $O(\delta)$ iterations of computation. Experiments on large real-world graphs verified that our algorithms have good performance in shared-nothing parallel computing platforms.

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