Finding Maximal k-Edge-Connected Subgraphs from a Large Graph

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ABSTRACT

In this paper, we study how to find maximal k-edge-connected subgraphs from a large graph. k-edge-connected subgraphs can be used to capture closely related vertices, and finding such vertex clusters is interesting in many applications, e.g., social network analysis, bioinformatics, web link research. Compared with other explicit structures for modeling vertex clusters, such as quasi-clique, k-core, which only set the requirement on vertex degrees, k-edge-connected subgraph further requires high connectivity within a subgraph (a stronger requirement), and hence defines a more closely related vertex cluster.

To find maximal k-edge-connected subgraphs from a graph, a basic approach is to repeatedly apply minimum cut algorithm to the connected components of the input graph until all connected components are k-connected. However, the basic approach is very expensive if the input graph is large. To tackle the problem, we propose three major techniques: vertex reduction, edge reduction and cut pruning. These speed-up techniques are applied on top of the basic approach. We conduct extensive experiments and show that the speed-up techniques are very effective.

1. INTRODUCTION

Graphs are used to express the relationships of different objects for a wide range of applications. In social network analysis, individuals can be represented by vertices, and their friendship relations can be represented by edges. In bioinformatics, graphs can be used to model protein interactions and gene coexpressions. In web data management, web pages and their links can be considered as vertices and edges respectively. The common theme of these modelings is to represent an entity as a vertex (or node), and the relationship between two entities as an edge. As a result, many real-life problems can be transformed into mathematical problems on a graph, and then can be tackled with elegant solutions on the shelf.

In graph theory, connectivity is a fundamental subject. It has applications in a variety of traditional areas, such as network reliability analysis [8], VLSI chip design [14], transportation planning [3]. A k-edge-connected graph is a connected graph that cannot be disconnected by removing less than k edges, similarly, a k-vertex-connected graph is a connected graph that cannot be disconnected by removing less than k vertices. We only focus on edge connectivity in this paper, because k-vertex-connectivity can be reduced to k-edge-connectivity, so k-connected is short for k-edge-connected from now on.

On new types of data, finding k-connected subgraphs may be interesting as well. For example, in social network analysis, a k-connected subgraph could approximately model a community, here, k can be defined by a user to express how close the relationships are between members within a community. Different users may be interested in different k's. Efficiently discovering k-connected subgraphs helps users identify those closely related individuals, and such information could be useful for social behavior mining [2], viral marketing [4], etc. In computational biology, a k-connected subgraph could model a set of genes within the same functional module [26], here vertices represent the genes and edges represent coexpression relationships between the genes. A high-connected subgraph from a gene coexpression graph is likely to capture a functional gene cluster. Finding such subgraphs may assist biologists to analyze gene microarrays and develop reasonable conjectures before experiments. For a web-link graph, a high-connected subgraph may be a collection of web pages talking about a certain topic or discussing related topics. Such subgraphs may be useful for entity association mining from web pages or building a knowledge database based on web pages.

In a word, a k-connected subgraph captures a vertex cluster, where vertices within the cluster are closely related. There are some other defined structures playing a similar role, e.g., clique, quasi-clique (defined on vertices [30] or edges [1]), k-core [24], k-plex [23], etc. A clique defines a

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structure where every vertex is connected to the other vertices within the structure. Quasi-clique is a relaxed form of clique, and it asks a vertex to connect with other vertices no less than a predefined percentage, e.g., in a n-vertex γ-quasi-clique, each vertex is connected to at least $\lceil \gamma \cdot (n-1) \rceil$ other vertices. In a k-core, each vertex is connected to at least k other vertices. Similarly, in a n-vertex k-plex, each vertex is connected to at least $(n-k)$ vertices. After all, in presence of these existing explicit structures, why do we need to study k-connected subgraphs? We explain it in the next paragraph.

Firstly, cliques are too strong, because, in many real scenarios, it is unlikely that every entity would have a link to every other entity within the cluster. On the other hand, quasi-clique, k-core, k-plex are sort of weak in some situations. For example, in Fig. 1 (a), an 8-vertex graph is a 3/7-quasi-clique (defined on vertices), because each vertex is connected to at least three of the other vertices in the graph. Fig. 1 (b) is also a 3/7-quasi-clique. Comparing Fig 1 (a) and Fig 1 (b), they are both 3/7-quasi-cliques, having the same number of vertices and edges, and the same degree on each vertex. However, it is more appropriate to say: Fig. 1 (a) contains one vertex cluster while Fig. 1 (b) contains two vertex clusters. In Fig. 1 (c), the whole graph is a 5-core, because each vertex is connected to at least five other vertices. Its subgraph {A,B,C,D,E,F} (in a dashed rectangle) is also a 5-core. Comparing Fig. 1 (c) and its subgraph {A,B,C,D,E,F}, they are both 5-cores, but Fig. 1 (c) should be considered as two vertex clusters. k-plex is similar to k-core, and has a similar problem.

The above discussion reminds us that connectivity in a subgraph is not negligible. Unfortunately, most existing defined structures are based on node degrees, ignoring the connectivity within the defined subgraph. It is well-known that checking connectivity is more expensive than checking node degrees. As a result, an efficient approach to discover k-connected subgraph is highly sought after. In this paper, we aim to find all *maximal k-connected subgraphs*, that is, k-connected subgraphs not contained in other k-connected subgraphs (a formal definition will be given in Section 2), otherwise (if not maximal) we can find too many k-connected subgraphs.

To guarantee the resulting subgraphs are k-connected, cut-based processing steps are unavoidable. A basic approach is to repeatedly run a minimum cut algorithm on the connected components of the input graph, and decompose the connected components if a less-than-k cut can be found, until all connected components are k-connected. Such solution is acceptable on smaller graphs, but is very expensive on large graphs. To tackle the problem, we design a set of speed-up methods. On one hand, we try to reduce the size of graph so that any cut algorithm can run faster on smaller graphs. This includes vertex reduction and edge reduction. On the other hand, we introduce some pruning conditions with which we can tell directly whether a connected component is k-connected or not.

We summarize our contributions as follows:

- We show that k-connected subgraphs may be a better means to model node clusters, compared with some existing models, such as k-cores and quasi-cliques.

- We propose a basic minimum-cut-based approach to find maximal k-connected subgraphs. More importantly, we propose three speed-up methods, *node reduction*, *edge reduction* and *cut pruning* that can dramatically improve the performance of the basic approach. The correctness of the speed-up methods are proved theoretically.

- We conduct extensive experiments to test the algorithm performance when applying node reduction, edge reduction and cut pruning on top of the basic approach. The experiment results confirm that the speed-up methods are very effective.

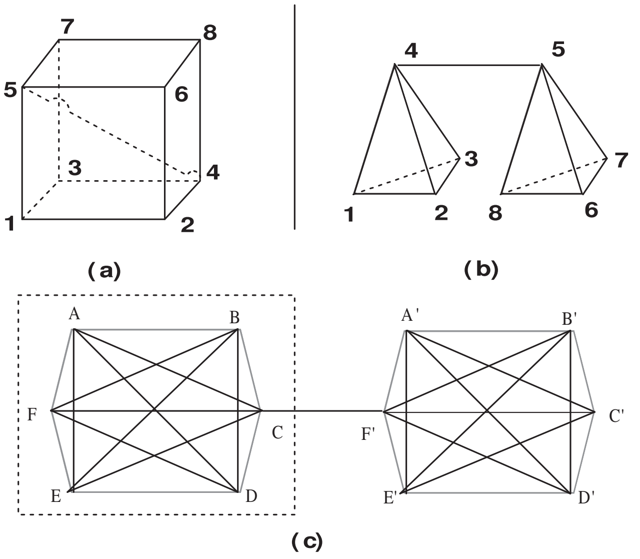


Figure 1: Compare k-connected subgraph with other structures

Here is a roadmap of this paper. In Section 2, we provide a formal definition of the problem, and introduce some necessary notations. In Section 4, we introduce vertex reduction and show its correctness. Edge reduction is introduced in Section 5. In Section 6, we introduce how to avoid cutting a connected component and how to cut a connected component into two halves earlier. Experiment results are shown in Section 7. Related works and conclusions are in Section 8 and Section 9 respectively.

2. PRELIMINARIES

We model networked data as a simple, unweighted, undirected graph $G = (V, E)$, where V is a set of vertices and E is a set of edges between vertices. Normally, V represents entities and E represents the relationships between entities. In real world, there may be different types of relationships between two entities, but in this paper, we do not distinguish the types of relationships. That means, as long as two entities are related, no matter how many types of relations there are, we consider the two entities are connected by a single edge.

A graph G is *k-connected* (short for k-edge-connected) if the removal of any up to $k-1$ edges does not make G disconnected, and there exists an edge set E_{cut} with $|E_{cut}| = k$ whose removal will make G disconnected. The edge set E_{cut} is called the cutset of a minimum cut. Note that G may have more than one minimum cut.

A graph $G_s = (V_s, E_s)$ is called an *induced subgraph* of a graph $G = (V, E)$, when $V_s \subseteq V$, $E_s \subseteq (V_s \times V_s) \cap E$, and for any two vertecies $x, y \in V_s$, edge $(x, y) \in E_s$ if and only if $(x, y) \in E$. Usually, an induced subgraph with vertex set V_s from G is denoted as $G[V_s]$.

A subgraph $G_s = (V_s, E_s)$ is a *maximal k-connected subgraph* of G, if there does not exist another k-connected subgraph $G'_s = (V'_s, E'_s)$, such that $V_s \subseteq V'_s$ and $E'_s \subseteq E_s$. Apparently, it implies that a maximal k-connected subgraph is an induced subgraph.

The problem we will study in this paper is, given a graph G and a user-specified integer k, how to find all maximal k-connected subgraphs from G efficiently.

3. BASIC APPROACH

In this section, we give a basic approach to find all maximal k-connected subgraphs. Important speed-up techniques will be introduced in Section 4, 5 and 6.

The idea of the basic approach is to repeatedly apply any minimum cut algorithm to the graph until each connected component is either a single vertex or a k-connected subgraph. Algorithm 1 describes the process. Throughout the process, R_0 stores the intermediate results for the graph decomposition, i.e. the produced connected components. If a produced connected component G_1 is at least k-connected, it will be added into the result set R (line 8); otherwise, G_1 will be decomposed into two pieces $\{G_2, G_3\}$ and added into R_0 for later inspection (line 5-6). Theorem 1 guarantees the correctness of the algorithm.

Algorithm 1 Basic Algorithm

Input: a graph G, connectivity threshold k;

Output: a set of maximal k-connected subgraphs R;

```
1: $R_0 := \{G\}$;
2: for each subgraph $G_1(V_1, E_1)$ ($|V_1| \neq 1$) in $R_0$ do
3:   find a minimum cut of $G_1$ (with cutset $E_{cut}$) using
     any minimum cut algorithm;
4:   if $|E_{cut}| < k$ then
5:     cut $G_1$ into $G_2$, $G_3$ by removing $E_{cut}$;
6:     $R_0 := R_0 \cup \{G_2, G_3\} - \{G_1\}$;
7:   else
8:     $R := R \cup \{G_1\}$;
9:   end if
10: end for
11: return $R$;
```

THEOREM 1. *Given a graph G and a connectivity threshold k, Algorithm 1 correctly finds all maximal k-connected subgraphs from G.*

PROOF. Obviously, all subgraphs in the result set R are k-connected. We need to show each of them is maximal as well. Suppose a graph $G_0 = (V_0, E_0) \in R$ is not maximal, then there must be a maximal k-connected subgraph $G_{max} = (V_{max}, E_{max})$ such that $V_0 \subset V_{max}$, and there must also exist a cut in a certain loop produced by the step 3, which separates a vertex (or some vertices) in V_{max} away from V_0. However, such a cut cannot exist, because $G[V_{max}]$ is supposed to be k-connected. As a result, G_0 should be maximal.

To show Algorithm 1 has found "all" maximal k-connected subgraphs: let (v_1, v_2) be an edge in a k-connected subgraph, since v_1 and v_2 are k-connected, (v_1, v_2) cannot be removed in the first loop (line 3-9) in Algorithm 1. Similarly, (v_1, v_2) cannot be removed in later loops. As a result, (v_1, v_2) will not be removed by Algorithm 1. This completes the proof of the "all" part. The theorem thus is correct. □

In Algorithm 1, the critical step is Step 3, i.e. performing a cut-based algorithm on a graph. In fact, it is likely that the cut-based algorithm cannot be avoided because k-connectivity needs to be guaranteed on the resulting subgraphs. As a result, if we can speed up the cut-step (Step 3), the k-connected subgraph discovery process can be accelerated. It is obvious that a fast minimum cut algorithm is preferred for Step 3. However, in this paper, most of the time, we constrain ourselves to a general minimum cut algorithm, because we aim to design a framework to accommodate any minimum cut algorithm, not a particular one. As such, if a novel minimum cut algorithm would be found, it could then be plugged into our framework without any modification. In case, users feel overwhelmed by the number of minimum cut algorithms to choose from, we suggest one minimum cut algorithm and explain the reason in Section 6. Given a general minimum cut algorithm, we briefly discuss some ideas to accelerate the cut algorithm on graph G_1, the details will be unfolded in Section 4, 5, 6:

- Reduce the size of G_1: The performance of most minimum cut algorithms are affected by the size of the graph, i.e. the number of vertices and the number of edges. Therefore, it is desirable if we can safely reduce the size of G_1 without affecting its connectivity, or exactly speaking without affecting the k-connectivity of those maximal k-connected subgraphs of G_1. Consequently, we can run a cut algorithm on a smaller graph but produce the same result. Vertex reduction and edge reduction will be introduced in Section 4 and Section 5 respectively.
- Avoid performing the cut algorithm: Some readers may have noticed that an unpromising connected component (with no k-connected subgraph inside) may be found earlier, no need to be cut into a few single vertices. For example, if a simple graph $G_1 = (V_1, E_1)$ has no more than k vertices ($|V_1| \leq k$), G_1 is at most $(k-1)$-connected (when G_1 is a clique), and cannot be k-connected. So G_1 can be disregarded earlier. Such speed-up tricks will be elaborated in Section 6.

4. VERTEX REDUCTION

In this section, we aim at reducing the number of vertices to speed up the basic minimum cut operation so that the whole discovery process can be accelerated. The idea is that if a subgraph G_s is k-connected we can *safely* contract the subgraph G_s into a new vertex v_{new} and the size of the original graph is reduced accordingly.

4.1 Contracting a k-connected subgraph

We will introduce the contraction process first and explain why this procedure is safe afterwards. Assume we have got a k-connected subgraph $G_s = (V_s, E_s)$, the contraction of G_s is as follows: (1) all vertices in V_s are replaced with a new vertex v_{new}; (2) all edges between vertices belonging to V_s

(not only those in E_s) will disappear, e.g., any edge (v_1, v_2) will be disregarded if $v_1, v_2 \in V_s$; (3) an edge between a vertex in V_s and a vertex in $V\backslash V_s$ will remain in the result graph but with one end-vertex modified, e.g., any edge (v_1, v_2) will become (v_{new}, v_2) if $v_1 \in V_s$ and $v_2 \in V\backslash V_s$. Note that the result graph may be a multiple graph, even though the original graph is simple, e.g., there are two edges (v_1, v_3), (v_2, v_3), let $V_s = \{v_1, v_2\}$ and $v_3 \in V\backslash V_s$, after the contraction, there will be two edges between v_{new} and v_3.

The following theorem guarantees that k-connectivity is consistent in the contracted graph and the original graph.

Theorem 2. *Given a graph $G = (V, E)$, let $G_s = (V_s, E_s)$ be a k-connected subgraph of G, let $G' = (V', E')$ be the graph produced from G by contracting G_s into a vertex v_{new}, for any vertex $v \in V$, we define $image(v) \in V'$ as: (1) $image(v) = v_{new}$, if $v \in V_s$; (2) $image(v) = v$, if $v \in V\backslash V_s$ (remains the same), then we have: for any vertices $v_1, v_2 \in V$, v_1, v_2 are k-connected in G, if and only if either $image(v_1) = image(v_2) = v_{new}$ or $image(v_1)$ and $image(v_2)$ are k-connected in G'.*

Proof. proof of "only if", given v_1, v_2 are k-connected in G:

Case (1): $v_1, v_2 \in V_s$, then obviously we have $image(v_1) = image(v_2) = v_{new}$.

Case (2): Without loss of generality, let $v_1 \in V_s$, $v_2 \in V\backslash V_s$, i.e. $image(v_1) = v_{new}$, $image(v_2) = v_2$, since v_1, v_2 are k-connected in G, there are k distinct paths between v_1 and v_2 in G, denoted as $\{p_1, ..., p_k\}$. Given one of these paths p_i, let v_{last} be the last vertex on p_i from v_1 to v_2 satisfying $image(v_{last}) = v_{new}$ and let $p'_i = p_{v_{last} \to v_2}$ denote the segment of p_i from v_{last} to v_2 in G, it is not difficult to see that p'_i is also a path from v_{new} to v_2 in G'. Given that p'_i is part of p_i, together with that $\{p_1, ..., p_k\}$ are distinct paths, we have $\{p'_1, ..., p'_k\}$ are distinct paths from v_{new} to v_2 in G'. As a result, v_{new}, v_2 (or equally $image(v_1)$, $image(v_2)$) are k-connected in G'.

Case (3): $v_1, v_2 \in V\backslash V_s$, i.e. $image(v_1) = v_1$, $image(v_2) = v_2$. Since v_1 and v_2 are k-connected in G, again there are k distinct paths between them in G, denoted as $\{p_1, ..., p_k\}$. Given one of these paths p_i, let v_{first} and v_{last} be the first and last vertices on p_i from v_1 to v_2 satisfying $image(v_{first}) = v_{new}$ and $image(v_{last}) = v_{new}$ (here, v_{first}, v_{last} may be the same vertex.), let $p_{v_1 \to v_{first}}$ be a segment of p_i from v_1 to v_{first} in G, $p_{v_{last} \to v_2}$ be a segment of p_i from v_{last} to v_2 in G, we can obtain $p'_i = p_{v_1 \to v_{first}} + p_{v_{last} \to v_2}$ by concatenating the two segments $p_{v_1 \to v_{first}}$ and $p_{v_{last} \to v_2}$ at v_{first} and v_{last}. It is not difficult to see that p'_i is a path from v_1 to v_2 in G'. Similarly, since $\{p_1, ..., p_k\}$ are distinct paths, we have $\{p'_1, ..., p'_k\}$ are distinct paths in G'. As a result, v_1, v_2 (or equally $image(v_1)$, $image(v_2)$) are k-connected in G'. Note that, for a path p_i, v_{first}, v_{last} may not always exist. If so, let $p'_i = p_i$ and the other parts of the proof remain the same.

Proof of "if", given either $image(v_1) = image(v_2) = v_{new}$ or $image(v_1)$ and $image(v_2)$ are k-connected in G':

Case (1): $image(v_1) = image(v_2) = v_{new}$, then obviously $v_1, v_2 \in V_s$. As G_s is a k-connected subgraph, v_1, v_2 is k-connected in G.

Case (2): Without loss of generality, let $v_1 \in V_s$, $v_2 \in V\backslash V_s$, i.e. $image(v_1) = v_{new}$, $image(v_2) = v_2$, since v_{new} and v_2 are k-connected in G', there are k distinct paths from v_{new} to v_2 in G'. Recall that v_{new} in G' actually represents multiple vertices in G, as a result, there are k distinct paths from subgraph G_s to v_2. We denote the starting vertices of these k distinct paths as $\{\bar{v}_1, ..., \bar{v}_k\}$, here $\{\bar{v}_1, ..., \bar{v}_k\} \subseteq V_s$ and $image(\bar{v}_1) = ... = image(\bar{v}_k) = v_{new}$. Note that $\bar{v}_i$, $\bar{v}_j$ $(i \in [1, k], j \in [1, k], i \neq j)$ may be the same vertex. Now we will prove v_1, v_2 is k-connected in G by contradiction. Suppose there exists a cut E_{cut} $(|E_{cut}| < k)$ of G separating v_1 and v_2 into two components $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ with $v_1 \in V_1$, $v_2 \in V_2$, (1) if $\{\bar{v}_1, ..., \bar{v}_k\} \subseteq V_1$, then $|E_{cut} < k|$ contradicts that there are k distinct paths from $\{\bar{v}_1, ..., \bar{v}_k\}$ to v_2; (2) otherwise, let $\bar{v}_i$ be a vertex separated into V_2 by E_{cut}, given G_s is k-connected and $v_1, \bar{v}_i \in V_s$, we have v_1 and $\bar{v}_i$ are k-connected, which also contradicts the existence of E_{cut} with $|E_{cut} < k|$. Therefore, v_1 and v_2 are k-connected in G.

Case (3): $v_1, v_2 \in V\backslash V_s$, i.e. $image(v_1) = v_1$, $image(v_2) = v_2$. Since v_1, v_2 are k-connected in G', there are k distinct paths from v_1 to v_2 in G', denoted as $\{p_1, ..., p_k\}$. If all these paths do not pass v_{new}, obviously, $\{p_1, ..., p_k\}$ are also distinct paths from v_1 to v_2 in G and the proof is over. Otherwise, suppose the first t $(t \in [1, k])$ paths pass vertex v_{new}, construct the following graph $G_t = (V_t = V_s \cup V_p, E_t = E_s \cup E_p)$ by combining G_s and paths $\{p_1, ..., p_t\}$, here V_p denotes the vertices on paths $\{p_1, ..., p_t\}$, E_p denotes the edges on paths $\{p_1, ..., p_t\}$. According to case (2), for any vertex $v \in V_s$, v_1, v are t-connected in G_t, similarly v_2, v are also t-connected in G_t. From Lemma 1, we have v_1, v_2 are t-connected in G_t, i.e. there are t distinct paths in G_t from v_1 to v_2. These t distinct paths, together with $\{p_{t+1}, ..., p_k\}$, form k distinct paths from v_1 to v_2 in G. Therefore, v_1, v_2 are k-connected in G. □

Lemma 1. *Given a graph $G = (V, E)$ and three vertices $\{v_a, v_b, v_c\} \subseteq V$, if v_a, v_b are k-connected and v_b, v_c are k-connected, then v_a, v_c are k-connected.*

Proof. We prove the lemma by contradiction. Suppose v_a, v_c are not k-connected, there must exist a cut E_{cut} separating G into two components $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ with $|E_{cut}| < k$. Without lose of generality, assume $v_a \in V_1$ and $v_c \in V_2$, since $V = V_1 \cup V_2$ and V_1, V_2 are disjoint, v_b must be in either V_1 or V_2. If $v_b \in V_1$, the existence of E_{cut} contradicts that v_b, v_c are k-connected, otherwise the existence of E_{cut} contradicts that v_b, v_a are k-connected. □

With Theorem 2, we can safely proceed to deal with the contracted graph, but now the problem is how to discover a few k-connected subgraphs in advance.

4.2 Finding a few k-connected subgraphs (not necessarily maximal)

We propose three methods to initially discover a few k-connected subgraphs. Intuitively, the more and the larger these discovered k-connected subgraphs are, the better the reduction effect can be achieved. However, it may take more time to discover more and larger k-connected subgraphs, and thus degrade the overall performance. It is unlikely to know a reasonable trade-off between the two aspects in advance. In our design, we put method efficiency at the first place, and the size of the initially discovered subgraphs at second due to the following reasons:

- It is difficult to find a maximal k-connected subgraph by expanding an existing k-connected subgraph[1]. We will give an example to show this in section 4.2.3. Consequently, it may not be worth the effort to find as many and as large k-connected subgraphs as possible, since finding these temporary subgraphs does not provide a shortcut to finding maximal k-connected subgraphs.

- To find k-connected subgraphs is only a subprocedure to reduce the number of vertices. It does not need to be perfect, but needs to be fast. So fast methods with reasonable quality are sufficient.

4.2.1 Using materialized views

If there are some precomputed maximal k'-connected subgraphs, as either materialized views or historical query results, we may use them as the bases to explore a few k-connected subgraphs.

- Case 1: If a maximal k'-connected subgraph G' has $k' \geq k$, obviously G' is also k-connected, but may not be maximal at k. If we have all maximal k'-connected subgraphs ($k' > k$), then we can safely contract these k'-connected subgraphs into a few supernodes (points) by Theorem 2. The size of the resulting graph is then significantly reduced in comparison with the original graph. To make the contraction more effective, we can first expand those materialized k'-connected subgraphs to obtain a set of larger k-connected subgraphs using the technique in Section 4.2.3, and then contract these k-connected subgraphs.

- Case 2: If a maximal k'-connected subgraph G' has $k' < k$, then G' may contain induced subgraphs which are k-connected. In such case, if $G' = (V', E')$ is not very large (e.g., $|V'| + |E'| \leq B$, where B is a predefined bound), we can find all maximal k-connected subgraphs from G' directly; otherwise, further vertex reduction and edge reduction can be performed on G'. Note that if we have got all maximal k'-connected subgraphs (when $k' < k$), we can start from these k'-connected subgraphs without resorting to the original graph, because a k-connected subgraph is also k'-connected and must be subsumed in one of those maximal k'-connected subgraphs (Lemma 2).

LEMMA 2. *For a given graph, its maximal k-connected subgraphs are disjoint, i.e. If $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ are two maximal k-connected subgraphs of the same graph G and $G_1 \neq G_2$, we have $V_1 \cap V_2 = \emptyset$.*

PROOF. We prove the lemma by contradiction. Suppose $V_1 \cap V_2 \neq \emptyset$ and let $v \in V_1 \cap V_2$, construct a new induced graph $G[V_1 \cup V_2]$, then for any two vertices $v_1, v_2 \in V_1 \cup V_2$, if $v_1, v_2 \in V_1$ or $v_1, v_2 \in V_2$, obviously v_1, v_2 are k-connected in $G[V_1 \cup V_2]$ since G_1, G_2 are k-connected and G_1, G_2 are subgraphs of $G[V_1 \cup V_2]$; otherwise, without loss of generality, let $v_1 \in V_1$, $v_2 \in V_2$, from $v_1, v \in V_1$, we have v_1, v are k-connected in $G[V_1 \cup V_2]$, similarly v_2, v are also k-connected in $G[V_1 \cup V_2]$. According to Lemma 1, v_1, v_2 are k-connected in $G[V_1 \cup V_2]$. As a result, $G[V_1 \cup V_2]$ is k-connected, but this contradicts that G_1, G_2 are maximal k-connected subgraphs. Consequently, we have $V_1 \cap V_2 = \emptyset$. □

In summary, as long as there is a precomputed maximal k'-connected subgraph G' (no matter k' is larger or smaller than k), we can use G' to help discover maximal k-connected subgraphs.

4.2.2 Using vertices with high degrees

The second method to find a few k-connected subgraphs is a heuristic method. It is inspired by the idea of work [7], which uses H*-graphs (comprised of vertices with higher degrees[2]) of an original graph to initially find some cliques, and then expands these cliques to find a portion of maximal cliques from the original graph.

Similarly, we can discover some initial k-connected subgraphs using vertices with high degrees. To be specific, we can load into memory the vertices with degrees above a certain level, e.g., $(1+f) \cdot k$ where $f > 0$, and find k-connected subgraphs using these "popular" vertices. The smaller f we choose, the more likely we can discover some k-connected subgraphs, but, at the same time, the more time we will spend on finding these initial k-connected subgraphs, because more nodes and edges need to be loaded into memory. In the implementation, given a memory pool to hold the vertices and edges, we can choose an f as small as possible on the condition that the memory pool does not overflow if we load all vertices with degree higher than $(1+f) \cdot k$.

In fact, the heuristic method introduced in this section is reciprocal to the method using materialized views. At the beginning, a system has no materialized views, so some initial k-connected subgraphs could be discovered from scratch using the method in this subsection. As the system runs on, more and more materialized views will be available, and the materialized view based method will play a more important role since it is usually more efficient than finding initial k-connected subgraphs from scratch.

4.2.3 Expanding existing k-connected subgraphs

The third method does not discover k-connected subgraphs from scratch. It takes existing k-connected subgraphs (possibly produced by the first and second methods) as input, and quickly expand the existing k-connected subgraphs in order to find larger ones. The expanding idea is: let a given k-connected subgraph be a core, let the core absorb *neighbor vertices* while keeping itself k-connected, stop the absorbing process when the core is not growing fast any more. Here, a *neighbor vertex* is a vertex not in the core, but is incident on an edge which has the other end in the core. Algorithm 2 illustrates the expanding process. The algorithm steps are self-explained. In step 4, the new G'_s is guaranteed to be k-connected by Lemma 3. In step 5, $\theta \in [0, 1)$ is a user-defined threshold. The larger θ is defined, the larger G'_s will be obtained and accordingly the more time the expanding process will take.

LEMMA 3. *Given a simple graph G, let $G_s = (V_s, E_s)$ be a k-connected subgraph of G, let V_n be a set of neighbor vertices of G_s in G, then induced subgraph $G[V_s \cup V_n]$ is k-connected if and only if $\forall v \in V_n, deg(v) \geq k$ in $G[V_s \cup V_n]$.*

PROOF. The "only if" part is obvious. We now prove the "if" part. Firstly, according to Theorem 2, we can safely

[1] There may be a brilliant method to achieve this, but, at the current stage, the problem is open.

[2] This is a rough idea, interested readers could refer to the paper for a more accurate definition of H*-graph.

Algorithm 2 Expanding a k-connected subgraph

```
Input: a $k$-connected subgraph $G_s = (V_s, E_s)$;
Output: a larger $k$-connected subgraph $G'_s$ ($G'_s$ may stay the same as $G_s$.);
$G'_s \leftarrow G_s$;
repeat
    let all neighbor vertices of subgraph $G'_s = (V'_s, E'_s)$ be $V_{neighbor}$, generate an induced subgraph $G[V'_s \cup V_{neighbor}]$ from the original graph;
    repeatedly remove vertices with degree less than $k$ from $G[V'_s \cup V_{neighbor}]$ and assign the result graph as the new $G'_s$ (to be used in the next loop), let the vertices removed in this step be $\Delta V_{neighbor}$;
until $\Delta V_{neighbor}/V_{neighbor} > \theta$
return $G'_s$;
```

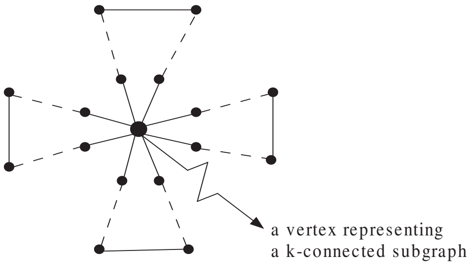


Figure 2: Expanding the graph to the end

contract G_s into a vertex v_{new} and then we can prove that for any $v \in V_n$, v, v_{new} are k-connected in the contracted graph $G'[V_s \cup V_n]$ of $G[V_s \cup V_n]$, given $deg(v) \geq k$, as follows:

Take any k edges incident on $v \in V_n$, assume t ($t \leq k$) edges of these k edges are between v and v_{new} and the other $k-t$ edges are incident on $\{v_1, ..., v_{k-t}\} \subseteq V_n$. Since G is a simple graph, we have vertices $\{v_1, ..., v_{k-t}\}$ are distinct. Therefore, there are $k-t$ one-hop disjoint paths from v to v_{new}, i.e. $\{v \to v_1 \to v_{new}, ..., v \to v_{k-t} \to v_{new}\}$. Together with t distinct edges between v and v_{new}, there are k distinct paths from v to v_{new}. As a result, v, v_{new} are k-connected in $G'[V_s \cup V_n]$.

Any two vertices $v_1, v_2 \in V_n$ are also k-connected in $G'[V_s \cup V_n]$ (Lemma 1), since v_1, v_{new} are k-connected and v_2, v_{new} are k-connected, and thus $G'[V_s \cup V_n]$ is k-connected. Finally, from the "if" part of Theorem 2, we have $G[V_s \cup V_n]$ is k-connected. □

The expanding process is heuristic. It looks one step forward and tries to recruit closely related vertices. Basically, it manages to find a relatively large k-connected subgraph with affordable time budget. Interestingly, looking several steps ahead may not be better, because a comparable efficient algorithm is not obvious. Furthermore, it is unlikely to find a maximal k-connected subgraph by straightforwardly expanding an existing k-connected subgraph. Fig. 2 shows an example. Starting from the initial vertex representing a contracted 2-connected subgraph, it is not until we see the whole graph that we can find the maximal 2-connected subgraph.

Note that k-subgraphs discovered using materialized views according to Case 2 (from k'-connected subgraphs, $k' < k$) do not need to be expanded any more, because they are already maximal.

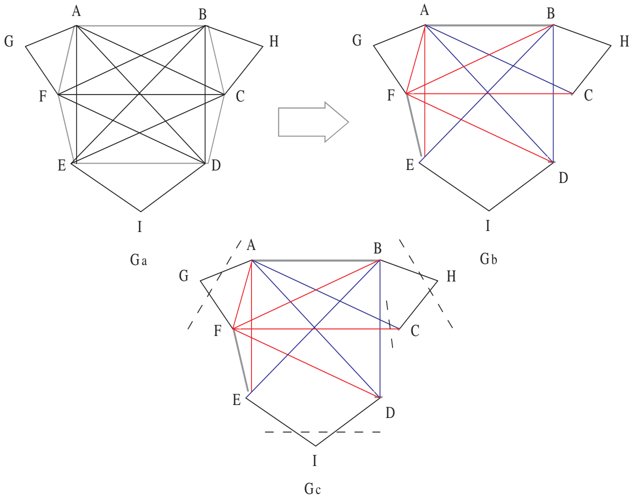


Figure 3: The example of graph reduction

5. EDGE REDUCTION

After contracting k-connected subgraphs into single vertices (in Section 4), the resulting connected components will usually be very dense. In order to efficiently cut the resulting connected components further, it is desirable to use a method not degraded by the large number of edges. In this section, we propose an iterative method based on edge reduction. We first introduce the idea, and then give theoretical foundations and algorithmic solutions for each step. To make the steps clear, we use a running example throughout the whole section.

5.1 Reduction Idea

The reduction idea is to remove vertices by inspecting a sparser subgraph. The reduction includes three steps:

1. Keep the vertex set unchanged and only remove the edges. A graph $G(V, E)$ will be reduced into $G'(V, E')$, where $E' \subseteq E$ and $|E'| \leq i(|V| - 1)$ (here $i \leq k$). We can guarantee that, if any two vertices in G are k-connected, then they are i-connected in G'.

2. With the above property, we know that all vertices in a k-connected subgraph $G_s(V_s, E_s)$ from G are pairwise i-connected in G'. Therefore, by discovering i-connected components from G', we can obtain a vertex superset V'_s of V_s, satisfying any two nodes in V'_s are i-connected in G'. We can guarantee that $V_s \subseteq V'_s$.

3. Using V'_s, we can get a smaller induced subgraph $G[V'_s]$ from G. Thereafter, we may either apply Algorithm 1 on $G[V'_s]$ directly or repeat the reduction process again on $G[V'_s]$ using another i' (here $i < i' \leq k$).

Readers are not required to fully understand the three steps at this stage, just need to know the rough picture. We will explain each step in detail in the following sections.

5.2 Theoretical Foundation for Step One

The initiative is to reduce the size of edges in G so that a minimum cut algorithm can be faster. For instance, in the extreme case, $|E|$ is in the size of $|V|^2$. After the reduction, $|E'|$ is in the size of $i|V|$ $(i < k)$, at a lower order. The complexity of a typical minimum cut algorithm, due to Stoer and Wagner [27], will be reduced from $O(|V|^3)$ to $O(|V|^2 \log |V|)$, recall the complexity of the algorithm is $O(|E||V| + |V|^2 \log |V|)$. It is obvious that this reduction is especially important when input graph G is dense. Now we come to two questions: (a) Is the edge reduction safe? (b) How to do the reduction?

For the question (a), Lemma 4 can guarantee that: if any two vertecies in G are k-connected, then they are i-connected in G'. For the question (b), $G'(V, E')$ can be constructed according to the description of G_i in Lemma 4. The set of spanning forests can be found in $O(|E| + |V|)$ time, due to Nagamochi and Ibaraki [15].

LEMMA 4. *For a graph $G = (V, E)$, simple or multiple, let $F_1 = (V, E_1)$ be a spanning forest in G and $F_i = (V, E_i)$ be a spanning forest in $G - E_1 \cup E_2 \cup \cdots \cup E_{i-1}$, for $i = 2, 3, \cdots, |E|$, where possibly $E_i = E_{i+1} = \cdots = E_{|E|} = \emptyset$ for some i. Then if an induced subgraph $G[V_s]$ $(V_s \subseteq V)$ of G is k-connected, then, for any $i <= k$, any two distinct vertices x, $y \in V_s$ is i-connected in $G_i = (V, E_1 \cup E_2 \cup \cdots \cup E_i)$.*

PROOF. For any two distinct vertices x, $y \in V_s$, considering that the local edge-connectivity between x, y in G is no less than the local edge-connectivity between x, y in $G[V_s]$, given $G[V_s]$ is k-connected, we have $\lambda(x, y; G) \geq k$. Here $\lambda(x, y; G)$ denotes the edge-connectivity between vertices x, y in G. According to lemma 2.1 in [16], $\lambda(x, y; G_i) \geq min\{\lambda(x, y; G), i\}$, combining with $\lambda(x, y; G) \geq k \geq i$, we have $\lambda(x, y; G_i) \geq i$. □

We use an example to illustrate the process. See Fig. 3, G_a is the original graph, let $k = 5$, so $G[\{A, B, C, D, E, F\}]$ is a maximal 5-connected subgraph. After reduction step one, G_b is the reduced graph using $i = 3$. G_b can be denoted as $G_3(V, E_1 \cup E_2 \cup E_3)$. Precisely speaking, E_1 is the outer circle except the edge CD, $E_2 = \{FA, FB, FC, FD, AE\}$, $E_3 = \{AC, AD, BD, BE\}$. E_1 is a spanning tree in G_a, E_2 is a spanning tree in $G_a - E_1$, E_3 is a spanning tree in $G_a - E_1 \cup E_2$, and E_1, E_2, E_3 are all spanning forests. Since G_a, $G_a - E_1$ and $G_a - E_1 \cup E_2$ are all connected, the spanning forests happen to be spanning trees. It is not difficult to verify that any two nodes from $\{A, B, C, D, E, F\}$ are 3-connected in G_b.

5.3 The Problem and Algorithmic Solutions for Step Two

As to the second step, after $G' = (V, E')$ is obtained from the first step, we want to find all i-connected components in G'. An i-connected component is a set of vertices, and any two vertices in the set are i-connected in G'. Suppose $G_s = (V_s, E_s)$ is a maximal k-connected subgraph in G, then all the vertices in V_s fall into a certain i-connected component in G', denoted as V'_s. Obviously, $V_s \subseteq V'_s$. The problem now is to find all these i-connected components from G'. In other words, i-connectivity on $G' = (V, E')$ is an equivalence relation on V, and we want to all non-singleton equivalence classes from V with respect to this equivalence relation.

A straightforward method is to find edge-connectivity for all vertex pairs in G', and then divide the vertices into groups. Lemma 1 guarantees the correctness. Naively, this process needs $\binom{n}{2}$ minimum s-t cut computation. Gomory and Hu [9] showed that $n - 1$ minimum s-t can do the job. Their algorithm computes a weighted cut-tree T from G', known as the Gomory-Hu tree, with the property that the edge connectivity between any two vertices s and t in G' exactly equals the weight of the lightest edge in the unique s-t path in T. Furthermore, the partition of the vertices produced by removing this edge from T produces a minimum s-t cut to the graph G'.

Among the candidate algorithms, one algorithm that is specially suitable to solve the problem is due to Hariharan et al. [11]. Their algorithm uses a graph and a user-specified k as input. The output of the algorithm is a tree T whose nodes represent k-connected components. To introduce more, the output is a weighted tree T whose nodes are vertex sets $V_1, V_2, \cdots, V_l$, a partition of V, with the property that the connectivity in G' between any two vertices $s \in V_i$ and $t \in V_j$, for $i \neq j$, is equal to the weight of the lightest edge on the path between V_i and V_j in T. Also, two vertices s and t belong to the same V_i for any i if and only if they are at least k-connected in G'. The complexity of the algorithm is $\tilde{O}(|E| + k^3|V|)$.

Return to the example in Fig. 3, on the reduced graph G_b, vertices A, B, C, D, E, F are pairwise 3-connected. They are in the same 3-connected component. Other vertices like G, H, I are singleton 3-connected components, and can be safely pruned. As a result, the only 3-connected equivalent class we can find from G_b is $\{A, B, C, D, E, F\}$.

5.4 An Example for Step Three

As to step three, apparently, if $V_s \subseteq V'_s$, then $G[V_s]$ is a subgraph of $G[V'_s]$. We are safe to deal with $G[V'_s]$, a smaller graph compared to G, since $G[V'_s]$ have filtered part of the vertices from the original graph G. In the example in Fig. 3, the maximal k-connected subgraph has vertex set V_s is $\{A, B, C, D, E, F\}$, and the corresponding superset V'_s in G_b is also $\{A, B, C, D, E, F\}$. $G[V'_s]$ and $G[V_s]$ happen to be the same. After we get $G[V'_s]$, we can either run Algorithm 1 to find the real results or repeat the reduction on $G[V'_s]$. In this example, $G[V'_s]$ will not be further reduced, if we repeat the reduction with $i' = 4$ or 5.

5.5 A Pitfall of Using Graph Reduction

In the second reduction step, suppose we have got G_i $(i \leq k)$ according to Lemma 4, some readers may ask whether we can perform Algorithm 1 on G_i to firstly obtain a set of induced i-connected subgraphs, and find induced subgraphs from G with the vertices in those i-connected subgraphs. For example, let $G'_s = (V'_s, E'_s)$ be an induced i-connected subgraph in G_i, is it safe to use $G[V'_s]$ as the input for further computation? In other words, given $G_s = (V_s, E_s)$ as a maximal k-connected subgraph in G, does it mean there must exist an induced i-connected subgraph $G'_s = (V'_s, E'_s)$ in G_i satisfying $V_s \subseteq V'_s$? Unfortunately, the answer is no. Refer to G_c in Fig. 3, an induced 3-connected subgraph of G_3 is $G[\{A, B, D, E, F\}]$. Here, vertex C is cut off from the graph, because after vertex H is cut off from the graph, C is no longer 3-connected to vertices $\{A, B, D, E, F\}$. Consequently, in reduction step two, finding i-connected components cannot be replaced with finding induced i-connected subgraphs.

6. CUT OPTIMIZATION

Given a connected component, there are several cases when we do not need to run a minimum cut algorithm on the connected component. We can tell whether the connected component is k-connected or not by inspecting its vertex degrees. This can dramatically improve the algorithm performance. We first list the cases and then explain the rationale afterwards. Let $G_1(V_1, E_1)$ be a connected component, $\Delta(G_1)$, $\delta(G_1)$ be the maximum and minimum vertex degree in G_1, v be a vertex in G_1.

1. When G_1 is simple and $|V_1| \leq k$, i.e. a connected component has no more than k vertices, the component does not have induced k-connected subgraphs, and hence can be disregarded.
2. When $\Delta(G_1) < k$, i.e. the maximum degree of the vertices in the connected component G_1 is less than k, it reflects that the component does not have induced k-connected subgraphs.
3. If a vertex v in G_1 has $deg(v) < k$, vertex v can be disregarded from the component. $G_1[V_1 - \{v\}]$ may still have induced k-connected subgraphs.
4. If $\delta(G_1) \geq k$, and $\delta(G_1) \geq \lfloor |V_1|/2 \rfloor$, then the connected component G_1 is k-connected. We do not need to apply the minimum cut algorithm on G_1.

We now explain the rationale behind those optimizations.

- For (1), if component G_1 is simple, in any induced subgraph $G_s(V_s, E_s)$ from G, separating a node v from G_s requires to remove at most $|V_s| - 1$ edges. (Here, $|V_s| - 1 \leq |V_1| - 1 < k$, because G_1 is simple and $|V_1| \leq k$.) In other words, there exists a cut set E_{cut} for G_s with $|E_{cut}| < k$. As a result, $\kappa(G_s) < k$. There is no induced k-connected subgraph in G_1.
- As to (2), in any induced subgraph $G_s(V_s, E_s)$, separating a node v from G_s requires to remove at most $\Delta(G_1)$ edges. Since $\Delta(G_1) < k$, we have $\kappa(G_s) < k$ for a similar reason as (1). Note that (2) also holds for multiple graphs, if G_1 is a simple graph, (1) is a special case of (2).
- For (3), the rationale is obvious, removing v is a special light-weighted cut.
- Finally, (4) is supported by Theorem 1 in [5]. We rephrase the theorem as Lemma 5 for easy reference. According to Lemma 5 and $\delta(G_1) \geq k$, we know G_1 is k-connected if the conditions in (4) are satisfied.

LEMMA 5. *Let $\delta(G)$ be minimum degree among all vertices in $G(V, E)$, if $\delta(G) \geq \lfloor |V|/2 \rfloor$, then $\kappa(G) = \delta(G)$.*

All the above four optimizations are designed to avoid performing the minimum cut algorithm in line 3 in Algorithm 1. Condition checks (e.g., checking $|V_1| \leq k$, $\Delta(G_1) < k$) and variable maintenance (e.g., updating $|V_1|$, $\Delta(G_1)$, $\delta(G_1)$, $deg(v)$) can be done together in $O(|V| + |E|)$ time.

A careful reader may have found that it is not a must to find a minimum cut in line 3 in Algorithm 1. *Any* cut E'_{cut} with $|E'_{cut}| < k$ can be used to cut G_1, and guarantees the correctness of the algorithm. So what is a desirable min-cut algorithm for our problem then? We suggest the minimum cut algorithm due to Stoer and Wagner [27], denoted by the SW algorithm, which provides an early-stop property, and is also reasonably efficient and easy to implement.

Algorithm 3 MinimumCut(G)

Description: find a minimum cut for graph G
Input: a graph $G(V, E)$;
Output: a min-cut edge set E_{cut};

```
1: initialize $E_{cut} = V$;
2: while $|V| > 1$ do
3:     $E'_{cut}$= MinimumCutPhase(G);
4:     if $|E_{cut}| > |E'_{cut}|$ then
5:         $E_{cut} = E'_{cut}$;
6:     end if
7: end while
8: return $E_{cut}$
```

Algorithm 4 MinimumCutPhase(G)

Description: find an s-t cut and merge two vertices
Input: a graph $G(V, E)$
Output: the edge set incident on v_{last};

```
1: randomly choose a vertex $v$, and let $A = \{v\}$;
2: while $A \neq V$ do
3:     add into $A$ the most tightly connected vertex from $V$;
4: end while
5: merging the last two vertices added into $A$;
6: return the edge set $E'_{cut}$ between the last added vertex
   $v_{last}$ and the rest vertices $V - \{v_{last}\}$;
```

We give the SW algorithm in Algorithm 3 and 4 and introduce it briefly. Algorithm 3 is the outer loop, runs $|V| - 1$ times, because after each loop, $|V|$ will be decreased by 1, for the reason that two vertices are merged into one after each loop (see line 5 in Algorithm 4). In each loop from line 2 to line 8 in Algorithm 3, it finds a new min-cut for the current graph (line 3), and compares with the current cut. A smaller cut will be recorded. At the end of this algorithm, the recorded cut will be the minimum cut. The key steps are in Algorithm 4. It first selects a seed vertex, and repeatedly take out other vertices from $|V|$ to join the seed vertex. In each round, the vertex having the highest connectivity with the seed set is selected and removed from $|V|$. In the end, the cut of the phase is the edge set E'_{cut} between the last added vertex v_{last} and the rest vertices $V - \{v_{last}\}$. The last two vertices added into the seed set will be merged before the current procedure return to the main loop.

The SW algorithm solves the minimum cut problem using $|V| - 1$ minimum s-t cut computations. A s-t cut is the minimum cut for graph G, which can separate vertex s, t into two different connected components. The global minimum cut is the lowest value among the $|V| - 1$ s-t cuts. Return to our problem, if any E_c among these $|V| - 1$ cuts satisfying $|E_c| < k$, we can stop finding other s-t cuts on G_1 and separate G_1 into two connected components safely using E_c. We refer this property as early-stop property. Furthermore, the SW algorithm has good theoretical complexity at $O(|E||V| + |V|^2 \log |V|)$. It is not a flow-based algorithm, and is easy to implement.

Algorithm 5 Combined Algorithm

```
Input: a graph $G$, connectivity threshold $k$;
Output: a set of maximal $k$-connected subgraphs $R$;
if there are maximal $k'$-connected subgraphs, $k' \in \{k_1, ..., k_n\}$ then
    let $\underline{k} = \max_{i \in [1,n]} \{k_i | k_i < k\}$ ;
    set the maximal $\underline{k}$-connected subgraphs as $R_0$, the initial set of connected components;
    let $\overline{k} = \min_{i \in [1,n]} \{k_i | k_i > k\}$ ;
    set the maximal $\overline{k}$-connected subgraphs as initially discovered $k$-connected subgraphs $V_0$;
else
    use the heuristic method in Section 4.2.2 to find a few $k$-connected subgraphs $V_0$;
end if
expand $V_0$ to find larger $k$-connected subgraphs according to Section 4.2.3;
perform vertex reduction to $R_0$ using $V_0$;
perform edge reduction to $R_0$;
for each component $G_1(V_1, E_1)$ ($|V_1| \neq 1$) in $R_0$ do
    if $G_1$ can be pruned without evaluating minimum cut then
        $R_0 := R_0 - \{G_1\}$;
    else
        if there exists any cut (not necessarily a minimum cut) with cutset $E_{cut}$ satisfying $|E_{cut}| < k$ then
            cut $G_1$ into $G_2$, $G_3$ by removing $E_{cut}$;
            $R_0 := R_0 \cup \{G_2, G_3\} - \{G_1\}$;
        else
            $R := R \cup \{G_1\}$;
        end if
    end if
end for
return $R$;
```

Finally, we give a combined algorithm in Algorithm 5 to incorporate all speed-up techniques in an overall framework. In Algorithm 5, we restrict ourselves to apply each reduction technique once. The order of the reduction techniques is carefully organized. However, we need to stress that Algorithm 5 is not the only acceptable solution. Each reduction technique may be applied multiple times and the order of some reduction techniques can be exchanged. For example, cut pruning check can be applied every time after a connected component is updated. We can also perform vertex reduction using available k-connected subgraphs first, and then expanding the resulting contracted vertices. Apparently, it is difficult to give an optimal algorithm that best organizes the speed-up techniques, because the effect of speed-up techniques is data-dependent. Nevertheless, Algorithm 5 is still valuable to provide a guideline on how to combine all the speed-up techniques in one framework.

7. EXPERIMENTS

In this section, we report the performance of the basic algorithm and the performance of applying different speed-up techniques on top of the basic algorithm. The results show that the speed-up techniques can improve the performance significantly. All experiments are done on a desktop with Intel(R) Core(TM) 2 Duo CPU E6550 at 2.33GHz and 3GB RAM. The operating system is Windows XP, and code is written in Java.

Table 1: Datasets

	vertices	edges	avg degree
Gnutella P2P network	6301	20777	3.30
Collaboration network	5242	28980	5.53
Epinions network	75879	508837	6.71

7.1 Datasets

We use three datasets to test the algorithms, Epinions social network (soc-Epinions1) [20], Arxiv GR-QC collaboration network (ca-GrQc) [13], and Gnutella peer-to-peer network (p2p-Gnutella08) [21]. Epinions social network is a who-trust-whom online social network of a general consumer review site (www.Epinions.com). Members of the site decide whether to trust each other. Arxiv GR-QC (General Relativity and Quantum Cosmology) collaboration network is from the e-print arXiv and it covers scientific collaborations between authors according to papers submitted to General Relativity and Quantum Cosmology category. Papers are from January 1993 to April 2003. Gnutella peer-to-peer network data is a snapshot of peer-to-peer file-sharing network in August 8, 2002. All the above datasets are in Stanford Large Network Dataset Collection[3]. We give the details of each dataset in terms of number of vertices, edges, and average degrees in Table 1.

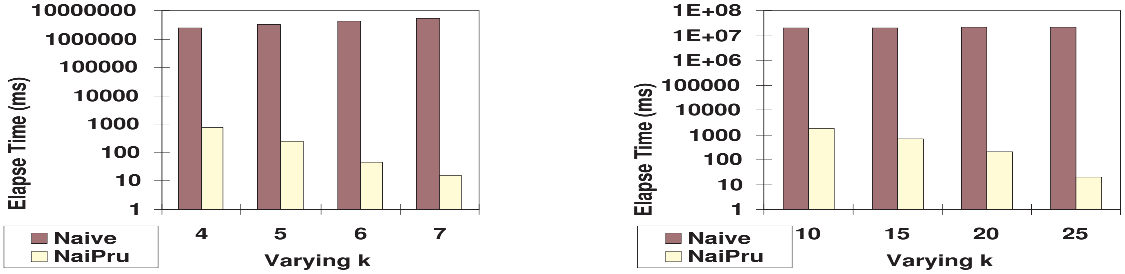
7.2 Effect of Cut Pruning

In this section, we report the effect of cut pruning (introduced in Section 6) solely without applying vertex reduction and edge reduction. Fig. 4 shows the result. We compare the basic approach (Naive) and the basic approach with cut pruning (NaiPru) on p2p network data and collaboration network data. On both datasets, the pure basic algorithm is rather slow, while after cut pruning, the performance is improved dramatically. When k becomes larger, the performance of NaiPru is improving as well. The reason is that when k is larger, more connected components can be pruned. In the following experiments, cut pruning is applied by default so that the baseline approach is not too slow. Cut pruning is orthogonal to vertex reduction and edge reduction.

7.3 Effect of Vertex Reduction

In this section, we report the effect of vertex reduction. We test the result on two relatively large datasets collaboration network and Epinions network. Four variant approaches using vertex reduction are tested and compared with the NaiPru approach. Table 2 gives the details of the approaches. Fig. 5 shows the experiment results. On collaboration network data, all four approaches have improved the performance significantly (note that the y-axis is in logarithmic scale, so looks not that impressive). Most of the time, expanding process can further improve the performance, especially when k is not large, because it is likely to find larger k-connected subgraphs using expansion. When k is large,

[3] http://snap.stanford.edu/data/



(a) P2P Network Data

(b) Collaboration Network Data

Figure 4: Cut Pruning

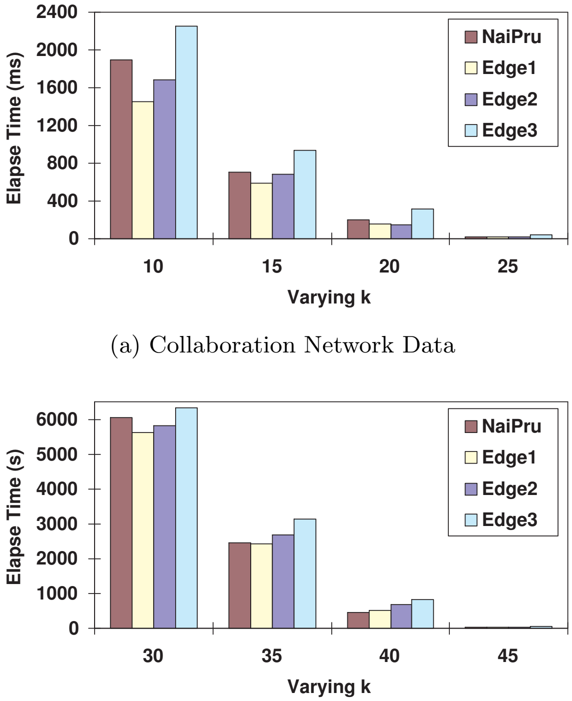
Table 2: The Meanings of the Approaches

HeuOly	use only the heuristic method in Section 4.2.2 to find a number of k-connected subgraphs, and then do vertex reduction using these subgraphs;
HeuExp	use the heuristic method in Section 4.2.2 and the expanding method in Section 4.2.3 to find a number of k-connected subgraphs, and then do vertex reduction
ViewOly	use only the materialized views (Section 4.2.1) to find a number of k-connected subgraphs, and then do vertex reduction using these views;
ViewExp	use the the materialized views (Section 4.2.1) together with the expanding method in Section 4.2.3

e.g., $k = 25$, the NaiPru approach is also acceptable, while the vertex reduction effect is not obvious. The reason is that the resulting connected components are already of a small size. On Epinions network data, the expanding process is always effective. The reason is that edges of Epinions network are not evenly distributed. There exists a large cluster, and thus it is very likely to find a larger k-connected subgraph by expanding a k'-connected subgraph ($k' > k$).

7.4 Effect of Edge Reduction

In this section, we test the effectiveness of edge reduction. As mentioned in Section 5, we can reduce edge iteratively. However, the iteration steps should not be too many, otherwise extra cost will overwhelm the inherent subgraph discovery cost. We compare three approaches with NaiPru, denoted as Edge1, Edge2 and Edge3. Edge1 preforms edge reduction once. Edge2 reduces the graph using a k' ($k' < k$) firstly, and then k. To generalize the case, we set $k' = k/2$. Similarly, Edge3 reduces the graph in three steps, $k/3$, $2k/3$ and then k. Unlike vertex reduction, we did not test $k = 6$ for the collaboration network, neither $k = 25$ for the Epinions network. We want to test the case when k is enough large so that approach Edge3 makes sense. From Fig. 6, we find that, on the network data, Edge1 is usually the best speed-up choice; when $k = 20$, Edge2 is slightly better than Edge1. The reason may be that the first step $k/3$ can effectively reduce the size of the graph. For all k's, Edge3 is the worst choice, even worse than ignoring edge reduction. This confirms that too much edge reduction is even more expensive. On the Epinions data, the result is similar. Edge1 is always better than the other approaches.



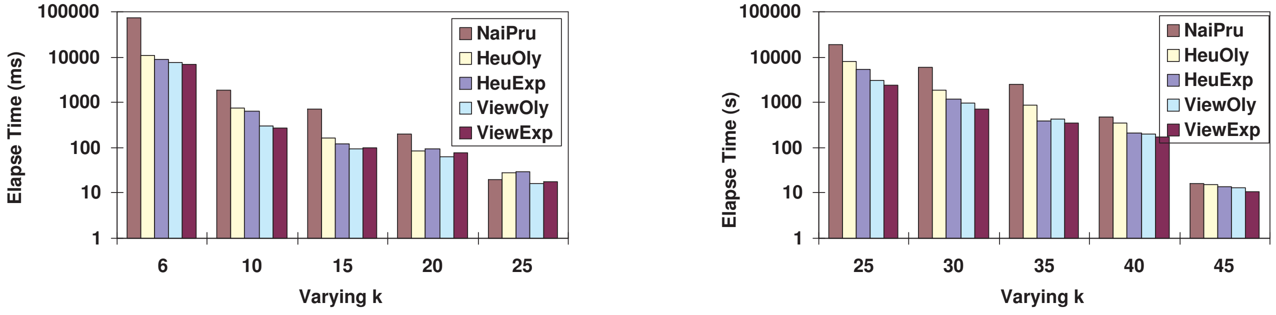
(a) Collaboration Network Data

(b) Epinions Social Network

Figure 6: Edge Reduction

7.5 Effect of All Speed-up Techniques

In this section, we report the effect of all the combined speed-up techniques. The BasicOpt approach in this section stands for an approach after applying both vertex reduction and edge reduction on top of the NaiPru method. As to vertex reduction, if there is no materialized views, HeuExp will be used to achieve the largest reduction probability; oth-

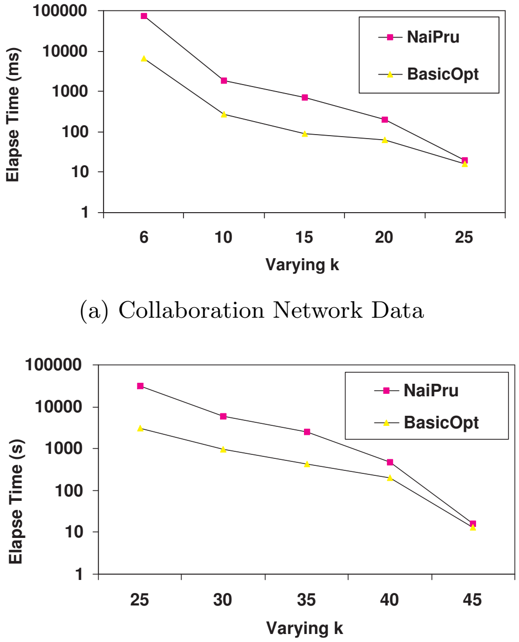


(a) Collaboration Network Data

(b) Epinions Social Network

Figure 5: Vertex Reduction

erwise ViewExp will be used to use materialized views to support vertex reduction. Edge reduction is iterated once in BasicOpt, because most of time, one edge-reduction iteration is the best, though sometimes not as good as twice. On both collaboration network data and Epinions social network data, the BasicOpt approach is up to 10 times faster than the NaiPru approach. Further combining with Fig. 4, BasicOpt is better than the Naive approach in order of magnitudes.



(a) Collaboration Network Data

(b) Epinions Social Network

Figure 7: The Combined Effect

8. RELATED WORK

Graph connectivity is a fundamental subject in graph theory. Graph connectivity is closely related to minimum cut, since the minimum cut gives the graph connectivity. A large number of have been done on the design of minimum-cut algorithm [10, 15, 27]. However, these works focus on the global connectivity, i.e. the connectivity of the whole graph, while, in this paper, we aim to find a subgraph with connectivity guarantee. The most similar work to ours is [11], where an algorithm is given to find all pairs of vertices, each of which has a connectivity no less than k, but again, the connectivity is defined on the global graph, not constrained on a local subgraph. In other works, Yan et al. [29] proposed to find frequent connected subgraphs from a large graph and connectivity is a constraint. Skygraph [18] proposed to find all maximal connected subgraphs from a given graph. It does not have the k-edge-connected requirement, and hence algorithms are in a progressive manner and cannot be adapted to our problem. Karypis and Kumar [12] developed a coarsening heuristic for a large graph. The aim is to reduce the input graph scale, similar to our graph reduction, but the techniques are different. Finally, in presence of many deterministic min-cut algorithms, Chekuri et al. [6] showed the algorithms in [10] and [15] are fast in practice, and may be good candidates to resort to.

On the other hand, works on extracting subgraphs from a given graph can be divided into two categories: explicit and implicit. In explicit works, such as, quasi-clique [30, 1], k-core [24], k-plex [23], a structure with certain property is predefined, and then the rest work is to design efficient algorithms to discover all the subgraphs with the structure requirement. In implicit works, some propose objective functions first, such as modulariy [17], normalized cut [25], and then partition the graph into a number of parts, here a good partition usually maximizes or minimizes the objective functions; Some works define neighbourhood distance, such as propinquity [31], structure closeness [28], and then group nearby nodes within a distance threshold around a given node to form a group; Some borrow the idea of Markov Clustering [22, 19] to repeat *random walk* for a few rounds until self-organized clusters turn up. Different from the implicit models, the maximal k-connected subgraphs we aim to find is explicitly defined.

9. CONCLUSIONS AND FUTURE WORK

In this paper, we have discussed how to find maximal k-edge-connected subgraphs from large graphs. We have proposed a basic cut-based approach and develop several important speed-up techniques: vertex reduction, edge reduction and optimizing the cut algorithm. We have conducted extensive experiments to test the performance of the speed-up techniques. Compared with the basic approach, experimental results show that using speed-up methods can dramatically improve the performance. Vertex reduction is very effective when there are suitable materialized views, because the size of the original graph can be reduced significantly. The effect of edge reduction is moderate, not remarkable, because after discovering the k-connected components, we still need to find maximal k-connected subgraphs from those k-connected components.The cut pruning is very effective, and it is also easy to implement. One direction for future work is to design external memory algorithms to find maximal k-connected subgraphs from massive graphs, because some real graphs are too large to fit into memory.

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