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Graph classes with structured neighborhoods and algorithmic applications

Rémy Belmonte, Martin Vatshelle *

Department of Informatics, University of Bergen, P.O. Box 7803, N-5020 Bergen, Norway

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ABSTRACT

Given a graph in any of the following graph classes: trapezoid graphs, circular permutation graphs, convex graphs, Dilworth k graphs, k-polygon graphs, circular arc graphs and complements of k-degenerate graphs, we show how to compute decompositions with the number of d-neighborhoods bounded by a polynomial of the input size. Combined with results of Bui-Xuan, Telle and Vatshelle (2013) [1] this leads to polynomial time algorithms for a large class of locally checkable vertex subset and vertex partitioning problems on all of these graph classes. The boolean-width of a graph is related to the number of 1-neighborhoods and our results imply that any of these graph classes have boolean-width $O(\log n)$.

1. Introduction

When solving graph problems by divide and conquer, we need to recursively divide the input graph G. A natural way to do this is to recursively partition the vertices of the graph into two parts. The resulting decomposition of G can be stored as a full binary tree whose leaves are in bijection with the n vertices of G, called *decomposition tree*. In a companion paper [1], Bui-Xuan, Telle and Vatshelle define for a given cut an equivalence relation on vertex subsets, namely d-neighborhood equivalence, for every fixed integer d. Further, they define nec_d as the maximum number of equivalence classes of the d-neighborhood equivalence relation over the cuts defined by a given decomposition tree. In this paper we give polynomial time algorithms for computing decomposition trees with nec_d polynomial in n for any graph belonging to one of the following classes: circular k-trapezoid graphs, k-polygon graphs, Dilworth k graphs, complements of k-degenerate graphs and convex graphs (see Group II of Fig. 1). In [1], it is shown that given a graph G and a decomposition tree of G the large class of locally checkable vertex subset and vertex partitioning problems (LC-VSP problems as defined in [2]) can be solved in time polynomial in n and nec_d. Combined with the results in this paper we get polynomial time algorithms solving any LC-VSP problem on any graph class in Group I or II of Fig. 1.

In a seminal paper by Courcelle, Makowski and Rotics [3], it was shown that every problem expressible in MSO_1 logic can be solved in linear time on graphs of bounded clique-width, i.e Group I of Fig. 1. However, since e.g. the Maximum Clique problem is NP-hard on complements of planar graphs [4] (a subclass of co-5-degenerate graphs), we cannot expect to obtain such a strong result on all the graph classes in Group II of Fig. 1. Instead our results imply polynomial time algorithms for the LC-VSP problems, a subclass of the MSO_1 problems, via their relation to d-neighborhoods. This includes many problems related to independence, domination and homomorphism. Many of the LC-VSP problems have been studied separately on many of the graph classes in Group II of Fig. 1; see [5–15]. A previous result closely related to ours is by Kratochvíl, Manuel and Miller in [16] who showed that a subset of LC-VSP problems is solvable in polynomial time on interval graphs.

* Corresponding author. Tel.: +47 55 58 42 00; fax: +47 55 58 41 99.
E-mail addresses: remy.belmonte@uib.no (R. Belmonte), martinv@ii.uib.no, vatshelle@ii.uib.no (M. Vatshelle).

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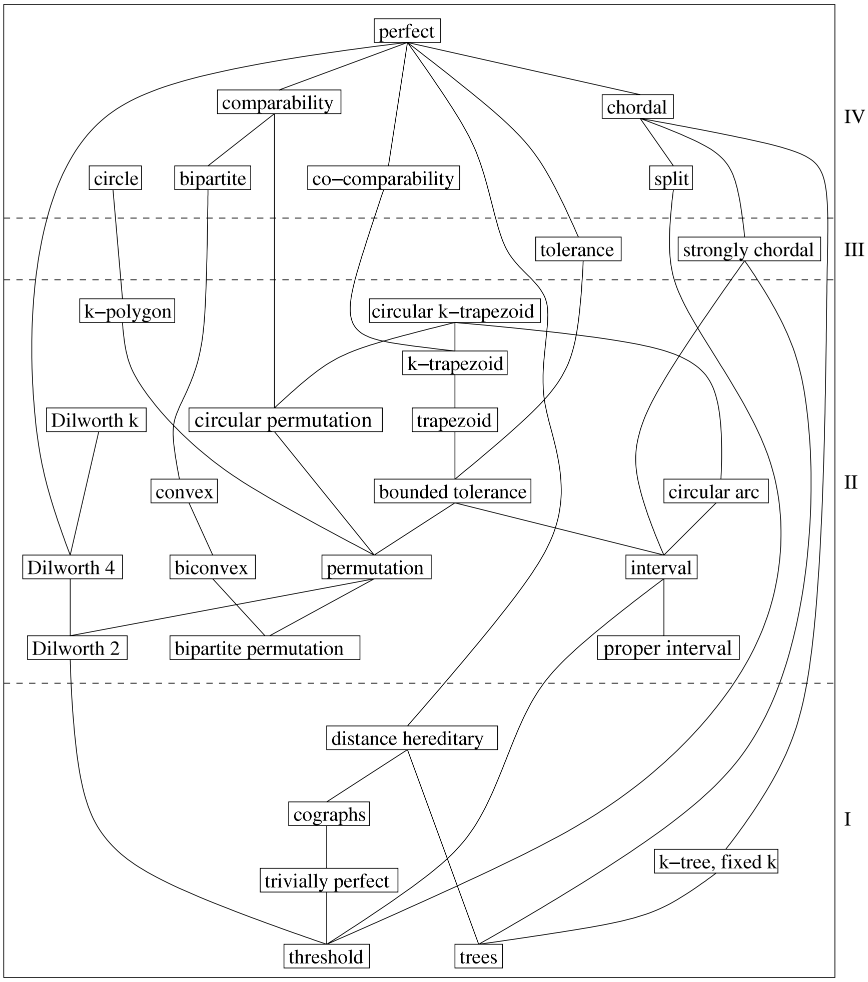

Fig. 1. Inclusion diagram of some well-known graph classes. (I) Graph classes where clique-width and boolean-width are bounded by a constant. (II) Graph classes having decomposition trees with nec_d polynomial in n, boolean-width $O(\log n)$ and all LC-VSP problems solvable in polynomial time (for k-trapezoid graphs assume an intersection model is given). (III) It is unknown whether these classes have boolean-width $O(\log n)$. (IV) Either boolean-width is not $O(\log n)$ or it is NP-hard to compute such decompositions.

The notion of d-neighborhood generalizes the classical notion of neighborhood, i.e. two subsets of a set A of vertices are 1-neighborhood equivalent with respect to the cut $(A, \overline{A})$ if they have the same neighborhood in $\overline{A}$. Boolean-width of a graph G is a parameter introduced by Bui-Xuan, Telle and Vatshelle [17], defined as $\log_2(nec_1)$ of an optimal decomposition of G. In particular, they gave FPT algorithms for solving Maximum Weight Independent Set and Minimum Weight Dominating Set in $2^{O(k)} \cdot poly(n)$ time, assuming a decomposition tree of boolean-width k is given. As a corollary of the results in this paper, we get that all the classes in Group II of Fig. 1 have boolean-width $O(\log n)$ and thus the two above algorithms are polynomial on these graph classes. To our knowledge, this is the first time an FPT algorithm is used to give polynomial time algorithms on a graph class where the parameter value is not bounded by a constant.

In the simple case of interval graphs and permutation graphs we show how to construct decompositions such that every cut defined by the decomposition has nested neighborhoods, i.e. the neighborhoods across the cut are totally ordered by inclusion. Bipartite graphs with nested neighborhoods are called chain graphs, and Yannakakis [18] showed that a bipartite graph is a chain graph if and only if the size of a maximum induced matching is 1. Twin free chain graphs, called Hsu graphs, are used in Hsu's generalized join [19]. We generalize the idea of a cut with nested neighborhoods by considering cuts inducing bipartite graphs with a constant size maximum induced matching. We use the fact that the size of a maximum induced matching across a cut bounds the number of d-neighborhood equivalence classes, to give polynomial upper bounds on the value of nec_d for all of the graph classes in Group II of Fig. 1.

In Section 2, we start by introducing standard graph theoretic notions and terminology. We define d-neighborhoods and relate these to induced matchings. In Section 3, we show that for interval graphs, circular arc graphs, permutation graphs, circular permutation graphs, trapezoid graphs, convex graphs, k-polygon graphs, Dilworth k graphs and complement of k-degenerate graphs we can in polynomial time compute decompositions where all cuts have the number of d-neighborhoods bounded by a polynomial of n. For k-trapezoid graphs and circular k-trapezoid graphs similar results are proven, but we must assume that an intersection model is given along with the input graph. In Section 4, we show that for all the graph classes in Group II of Fig. 1, except possibly Dilworth k graphs (for $k \geq 2$), our upper bounds are essentially tight. We do so by showing that these classes have rank-width $\Omega(\sqrt{n})$, which implies that none of these classes can have decompositions with nec_d at most $n^{o(1)}$. Moreover we show that for the graph classes in Group *IV* of Fig. 1 we cannot find decompositions with nec_d bounded by a polynomial of n, unless $P = NP$. Finally in Section 5 we conclude and give some open problems.

2. Framework

All graphs considered in this paper are undirected, finite and simple. The *neighborhood* of a vertex u, denoted by $N(u)$, is the set of vertices u such that the edge (u, v) is in E and for a subset of vertices X we denote the union of the neighborhoods of vertices in X by $N(X) = \bigcup_{x \in X} N(x)$. Given a set $A \subseteq V$, we denote by $\overline{A}$ the *complement* of A in V, i.e. $V \setminus A$. We call a bipartition $\left(A, \overline{A}\right)$ of V a *cut* of G. We denote by $G[X]$ the subgraph of G induced by X and $G[X, Y]$ the bipartite subgraph of G induced by those edges with one endpoint in X and the other in Y.

We now define formally the notion of decomposition tree. The choice of a decomposition tree greatly influences the running time of any algorithm using the decomposition tree. In order to choose the best decomposition tree, we evaluate a decomposition tree by using a cut function. The following formalism is referred to as *branch decomposition* of a cut function and is standard in graph and matroid theory, see e.g. [20–22].

Definition 1 (*Decomposition Tree*)**.** A decomposition tree of a graph G is a pair (T, δ) where T is a full binary tree (i.e. T rooted with every non-leaf having two children) and δ a bijection between the leaf set of T and the vertex set of G. For a node w of T let the subset of $V(G)$ in bijection δ with the leaves of the subtree of T rooted at w be denoted by V_w. We say the decomposition defines the cut $\left(V_w, \overline{V_w}\right)$.

Definition 2 (*Rank-width*)**.** Let G be a graph and (T, δ) a decomposition of G. The rank-width of (T, δ), denoted by $rw(T, \delta)$, is defined as the maximum over all $w \in V(T)$ of the rank of the adjacency matrix of the bipartite graph $G[V_w, \overline{V_w}]$. The rank-width of G, denoted by $rw(G)$, is defined as the minimum $rw(T, \delta)$ over all decompositions (T, δ) of G.

Caterpillar decompositions are decompositions where the underlying tree is a path with one leaf added as neighbor of each internal node of the path. Many of our proofs will construct caterpillar decompositions. To describe a caterpillar decomposition of a graph G, we only give an ordering $v_1, \ldots, v_n$ of the vertices of G. To construct the caterpillar decomposition (T, δ) from an ordering, first construct a caterpillar T from a path $u_1, \ldots, u_n$ of length $|V|$. Then let δ be a mapping of v_1 to u_1, v_n to u_n and for all $i \in \{2, \ldots, n-1\}$, of v_i to a new leaf attached to u_i. Finally, subdivide any edge and root the decomposition at the newly added vertex.

One of the essential notions in [17] and [1] is that of a representative. For a cut $\left(A, \overline{A}\right)$ in a graph G a representative of a subset $X \subseteq A$ is a subset R having the same neighbors in $\overline{A}$ as X. Maximum induced matchings bound the maximal size of representatives needed to represent any neighborhood across a cut.

Definition 3 (*Maximum Induced Matching Over a Cut*)**.** Let G be a graph, then we say a subset $M \subseteq V(G)$ of vertices is an induced matching in G if $G[M]$ is a disjoint union of edges. For $A \subseteq V(G)$ we define *mim* (A) as the maximum number of edges in an induced matching in $G[A, \overline{A}]$.

To see the relation to representative, note that no two subsets of $M \cap A$ has the same neighborhood in $\overline{A}$, hence all $2^{|M \cap A|}$ subsets have different neighborhoods in $\overline{A}$, and for the converse relation see Lemma 1. In the companion paper [1] the notion of neighborhoods is generalized to d-neighborhoods, we now formally define the notion of d-neighborhood equivalence.

Definition 4 (*d-Neighborhood Equivalence*)**.** Let d be a non-negative integer, G be a graph and $A \subseteq V(G)$. Two vertex subsets $X \subseteq A$ and $Y \subseteq A$ are d-neighbor equivalent with respect to A, denoted by $X \equiv_A^d Y$, if:

$$\forall v \in \overline{A} : \min(d, |X \cap N(v)|) = \min(d, |Y \cap N(v)|).$$

Let $nec(\equiv_A^d)$ be the number of equivalence classes of $\equiv_A^d$. For (T, δ) a decomposition tree of G define $nec_d(T, \delta)$ as the maximum $nec(\equiv_{V_w}^d)$ and $nec(\equiv_{\overline{V_w}}^d)$ over all nodes w in $V(T)$. We refer to $nec(\equiv_A^d)$ as the number of d-neighborhoods of the cut $\left(A, \overline{A}\right)$ and to $nec_d(T, \delta)$ as the number of d-neighborhoods of (T, δ).

Note that $nec(\equiv_A^1) = nec(\equiv_A)$ since $\min(1, |X \cap N(v)|)$ is 1 if and only if $v \in N(X)$ and 0 otherwise. Hence for two vertex subsets $X, Y \subseteq A$ we have $X \equiv_A^1 Y$ if and only if $\forall v \in \overline{A} : v \in N(X) \Leftrightarrow v \in N(Y))$ which means $N(X) \cap \overline{A} = N(Y) \cap \overline{A}$.

Definition 5 (*Boolean-width*)**.** Let G be a graph and (T, δ) a decomposition of G. The boolean-width of (T, δ), denoted by $boolw(T, \delta)$, is defined as the maximum over all $w \in V(T)$ of $\log_2\left(nec\left(\equiv_{V_w}^1\right)\right)$. The boolean-width of G, denoted by $boolw(G)$, is defined as the minimum $boolw(T, \delta)$ over all decompositions (T, δ) of G.

As a corollary of our results, we give logarithmic bounds on the boolean-width of the graph classes in Group II of Fig. 1. Moreover, in Section 4 we use boolean-width to provide lower bounds on the value of nec_d on these graph classes.

In Section 3 we will show how to compute decompositions where the number of d-neighborhoods is polynomial in the size of the graph. To do this we will use the following connection between the number of d-neighborhoods and maximum induced matchings.

Lemma 1. *Let G be a graph and $A \subseteq V(G)$. $mim(A) \leq k$ if and only if for every $S \subseteq A$, there is $R \subseteq S$ such that $N(R) \cap \overline{A} = N(S) \cap \overline{A}$ and $|R| \leq k$.*

Proof. We will show the converse namely, $mim(A) > k$ if and only if there exist $S \subseteq A$ such that for all $R \subseteq S$ either $N(R) \cap \overline{A} \neq N(S) \cap \overline{A}$ or $|R| > k$.

First we prove $mim(A) > k$ implies there exist $S \subseteq A$ such that for all $R \subseteq S$ either $N(R) \cap \overline{A} \neq N(S) \cap \overline{A}$ or $|R| > k$. Let M be a maximum induced matching over the cut $(A, \overline{A})$ and $S = M \cap A$, then clearly $mim(A) = |S| > k$. If $R = S$ we have $|R| > k$, for every proper subset R there is a vertex in $S \setminus R$, this vertex is part of an edge in M hence the other endpoint of that edge is in $N(S) \cap \overline{A}$ but not in $N(R) \cap \overline{A}$. Hence $N(R) \cap \overline{A} \neq N(S) \cap \overline{A}$.

For the second implication let S be an inclusion minimal set such that for all $R \subseteq S$ either $N(R) \cap \overline{A} \neq N(S) \cap \overline{A}$ or $|R| > k$. Clearly we must have $|S| > k$, otherwise the statement would not hold for $R = S$. Assume there is a vertex $v \in S$ such that $N(S) \cap \overline{A} = N(S \setminus v) \cap \overline{A}$, then the statement would be true for $S \setminus v$ contradicting minimality of S. For every $v \in S$ we have $N(S) \cap \overline{A} \neq N(S \setminus v) \cap \overline{A}$. We build an induced matching M as follows: For every $v \in S$, let u be any vertex in $\left(N(S) \cap \overline{A}\right) \setminus \left(N(S \setminus v) \cap \overline{A}\right)$ and add the edge (v, u) to M. Since $mim(A) \geq |M| = |S| > k$ the theorem follows. □

Lemma 2. *Let G be a graph and $A \subseteq V(G)$. Then $nec\left(\equiv_A^d\right) \leq n^{d \cdot mim(A)}$.*

Proof. First we prove the following:

Claim. For every subset $S \subseteq A$, there exist $R \subseteq S$ such that $R \equiv_A^d S$ and $|R| \leq mim(A) \cdot d$.

Proof of the claim. This proof is by induction on d and similar to that of [1, Lemma 5]. For $d = 1$, this follows from Lemma 1 by using $k = mim(A)$. Now, assume the induction hypothesis true up to $d-1$, then we show it true for d. Let $S' \subseteq S$ be an inclusion minimal set such that $N(S') \cap \overline{A} = N(S) \cap \overline{A}$ i.e. $S' \equiv_A^1 S$. Hence from Lemma 1 we have that $|S'| \leq mim(A)$. By induction hypothesis there exists $R' \subseteq (S \setminus S')$ such that $R' \equiv_A^{d-1} (S \setminus S')$ and $|R'| \leq mim(A) \cdot (d-1)$. Thus it is enough to show $R = R' \cup S' \equiv_A^d S$. We partition the nodes of $\overline{A}$ into (P, Q) such that $\forall v \in P$, we have $|N(v) \cap (S \setminus S')| = |N(v) \cap R'| < d-1$ and $\forall v \in Q$, we have $|N(v) \cap (S \setminus S')| \geq d-1$ and $|N(v) \cap R'| \geq d-1$. For every vertex $v \in P$, since $S' \cap R' = \emptyset$ and $S' \subseteq S$, we know $|N(v) \cap S| = |N(v) \cap (S \setminus S')| + |N(v) \cap S'| = |N(v) \cap R'| + |N(v) \cap S'| = |N(v) \cap R|$. We have $N(S) = N(S')$ and since $d > 1$ we have $Q \subseteq N(S')$. For every vertex $v \in Q$, since $|N(v) \cap (S \setminus S')| \geq d-1$ we get $|N(v) \cap S| \geq d$ and since $|N(v) \cap R'| \geq d-1$ we get $|N(v) \cap R| \geq d$. Since (P, Q) is a partition we get $R \equiv_A^d X$ and $|R| \leq mim(A) \cdot d$, thus by induction the claim holds.

To bound the number of equivalence classes $nec\left(\equiv_A^d\right)$ we know from the claim that we only need to find the equivalence classes among the subsets of A of size at most $d \cdot mim(A)$. A trivial bound on number of subsets of A with size $d \cdot mim(A)$ gives us: $nec\left(\equiv_A^d\right) \leq |A|^{d \cdot mim(A)} \leq n^{d \cdot mim(A)}$. □

3. Finding good decompositions on restricted graph classes

In this section we will show how to compute decomposition trees with $nec_d(T, \delta)$ bounded by a polynomial of n if the graph belongs to a certain graph-class. In almost all proofs, we construct a caterpillar decomposition by giving an ordering of the vertices of the graph, and then argue using Lemma 1 that for each cut of the decomposition the size of a maximum induced matching is bounded, and thus we can apply Lemma 2.

For all graph classes considered in this paper, except for the complements of k-degenerate graphs and Dilworth k graphs, we use definitions via a geometrical intersection model.

Definition 6 (*Intersection Model*)**.** Let $\mathcal{F}$ be a family of nonempty sets. For a graph G, we say $\mathcal{F}$ is an intersection model of G if there exists a bijection φ from $\mathcal{F}$ to $V(G)$ such that two vertices $u, v \in V(G)$ are adjacent if and only if $\varphi(u)$ and $\varphi(v)$ intersect.

The sets in the intersection model usually consists of geometrical objects such as lines, circles or polygons.

3.1. Interval graphs

An interval $I = \langle i, j \rangle$ is represented by an ordered pair of real numbers with $i < j$ and represent the set of real numbers $\{x : i < x < j\}$. Let $I_1 = (a, b)$ and $I_2 = (c, d)$ be two intervals, then I_1 intersects I_2 if and only if $a < d$ and $c < b$.

Definition 7 (*Interval Graph*)**.** A graph is an interval graph if it has an intersection model consisting of intervals.

Lemma 3. *Given an interval graph G and any positive integer d, we can, in polynomial time, compute a decomposition tree (T, δ) of G having $nec_d(T, \delta) \leq n^d$.*

Proof. Any interval graph has an intersection model where no interval starts or ends at the same point and we can find such an intersection model in linear time [23]. We build a caterpillar decomposition by sorting the vertices in increasing order by

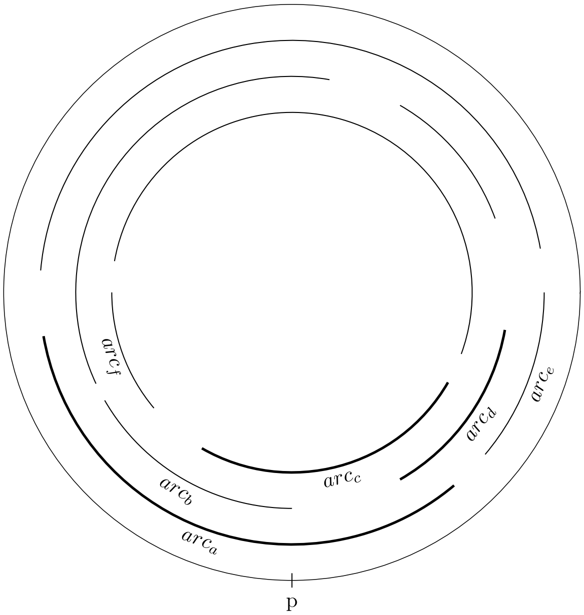

Fig. 2. An intersection model of a circular-arc graph with 10 vertices. Let $A = \{a, b, c, d, e, f\}$. Note that A forms a cut of the decomposition since no other arcs are closer to p. Let $S = \{a, c, d\}$. Note the arcs of S are drawn in bold. We then get $arc_l = arc_a$ and $arc_r = arc_d$, thus $S' = \{a, d\} \equiv_A S$.

the left/smallest endpoint of their corresponding intervals. Let us now consider any cut $\left(A, \overline{A}\right)$ defined by the decomposition. We want to bound $mim\,(A)$ by applying Lemma 1. Thus we will show that for every set $S \subseteq A$, there is a set R such that $N\,(R) \cap \overline{A} = N\,(S) \cap \overline{A}$ and $|R| \leq 1$. Since we sorted the intervals all left endpoints of intervals corresponding to vertices of A are to the left of all left endpoints of intervals corresponding to vertices of $\overline{A}$. Let σ be the total ordering of the vertices of A sorted by their right endpoint, two vertices $u \in A, u' \in \overline{A}$ are neighbors if and only if the right endpoint of u is to the right of the left endpoint of u'. Hence, for every pair of vertices $u, v \in A$ if $\sigma(u) \leq \sigma(v)$ then $N\,(u) \cap \overline{A} \subseteq N\,(v) \cap \overline{A}$. For every set $S \subseteq A$, let R contain the unique vertex of S whose interval has the rightmost right endpoint. We then have $N\,(R) \cap \overline{A} = N\,(S) \cap \overline{A}$ and $|R| \leq 1$. Therefore, by Lemma 1, $mim\,(A) \leq 1$ for all cuts. Then for any d, by Lemma 2 we have $nec_d\,(T, \delta) \leq n^d$. $\square$

3.2. Circular-arc graphs

Circular-arc graphs is a natural generalization of interval graphs.

Definition 8 (*Circular arc Graph*)**.** A graph is a circular arc graph if it has an intersection model consisting of arcs of a circle.

Lemma 4. *Given a circular-arc graph G and any positive integer d, we can, in polynomial time, compute a decomposition tree (T, δ) of G having $nec_d\,(T, \delta) \leq n^{2d}$.*

Proof. We can compute the circular-arc intersection model of G in linear time [24]. Fix an arbitrary point p on the circle. We define the distance of an arc from p as follows: The distance is the minimum distance between p and any point of the arc. For any vertex u, we denote by arc_u the arc corresponding to u.

Build a caterpillar decomposition by totally ordering the vertices in order of increasing distance of their associated arc from p, breaking ties arbitrarily. Note that this decomposition can be computed in polynomial time. We now consider any cut $\left(A, \overline{A}\right)$ of this decomposition. By construction, for every $x \in A$, $y \in \overline{A}$, the distance of arc_x from p is less than or equal to the distance of arc_y from p.

Now, we prove that for any set $S \subseteq A$, there exists a subset $S' \subseteq S$ such that $|S'| \leq 2$ and $N\,(S) \cap \overline{A} = N\left(S'\right) \cap \overline{A}$. Let arc_l be the arc extending the furthest from p in clockwise direction and arc_r the arc extending the furthest from p in counter-clockwise direction and $S' = \{l, r\}$. In Fig. 2 we get $arc_l = arc_a$ and $arc_r = arc_e$. Assume for contradiction that there exist $v \in \overline{A}$ such that $v \in N(S) \setminus N(S')$. Since arc_v does not intersect arc_l or arc_r and the distance of arc_v is larger than both arc_l and arc_r. The arc corresponding to the vertex in S overlapping with arc_v must extend further than either arc_l or arc_r leading to a contradiction.

For every set $S \subseteq A$, let R contain the vertices l and r as defined above. We then have $N\,(R) \cap \overline{A} = N\,(S) \cap \overline{A}$ and $|R| \leq 2$. Therefore, by Lemma 1, $mim\,(A) \leq 2$ for all cuts. Then for any d, by Lemma 2 we have $nec_d\,(T, \delta) \leq n^{2d}$. $\square$

3.3. Permutation graphs

Definition 9 (*Permutation Graph*). Let L and U be two distinct infinite parallel lines. A graph is a permutation graph if it has an intersection model consisting of straight line-segments with one endpoint on L and one on U.

Lemma 5. *Given a permutation graph G and any positive integer d, we can, in polynomial time, compute a decomposition tree (T, δ) of G having $nec_d(T, \delta) \leq n^d$.*

Proof. We compute the permutation model of G in linear time [25]. We build a caterpillar decomposition by sorting the vertices by the upper endpoint of their corresponding line. Let us now consider a cut $(A, \overline{A})$ of the decomposition. Let σ be the total ordering of the vertices of A sorted by their lower endpoint, hence $\forall u, v \in A$, $\sigma(u) \leq \sigma(v)$ iff the lower endpoint of u is to the left of the lower endpoint of v. Since all upper endpoints of lines corresponding to vertices of A are to the left of all upper endpoints of lines corresponding to vertices of $\overline{A}$, two vertices $u \in A$, $u' \in \overline{A}$ are neighbors if and only if the lower endpoint of u is to the right of the lower endpoint of u'.

Hence for any set $S \subseteq A$ there exists $x \in S$ such that $N(S) \cap \overline{A} = N(x) \cap \overline{A}$, namely the vertex of S with the rightmost lower endpoint, by Lemma 1, $mim(A) \leq 1$ for all cuts. Then for any d, by Lemma 2 we have $nec_d(T, \delta) \leq n^d$. □

3.4. Circular permutation graphs

Definition 10 (*Circular Permutation Graph*). Let L and U be two parallel different circles on a cylinder. A graph is a circular permutation graph if it has an intersection model consisting of straight line-segments with one endpoint on L and one on U.

Lemma 6. *Given a circular permutation graph G and any positive integer d, we can, in polynomial time, compute a decomposition tree (T, δ) of G having $nec_d(T, \delta) \leq n^{2d}$.*

Proof. We compute the circular permutation model of G in linear time [26]. Let s_v be the line corresponding to the vertex v. We build a caterpillar decomposition using an ordering obtained by sorting the vertices by the endpoint on L of their corresponding lines in clockwise order starting from any point p. Let us now consider a cut $(A, \overline{A})$ of the decomposition. For any $S \subseteq A$ we show that we can find $S' \subseteq S$ such that $N(S) \cap \overline{A} = N(S') \cap \overline{A}$ and $|S'| \leq 2$. Let l (resp. r) be the line corresponding to the vertex $v \in S$ that extend the furthest from p in clockwise (resp. counter-clockwise) direction.

Let $S' = \{l, r\}$ and assume for contradiction that there exist $v \in \overline{A}$ such that $v \in N(S) \setminus N(S')$. Since s_v does not intersect s_l nor s_r and that the distance from p to the point of s_v on L is greater in clockwise (resp. counter-clockwise) direction than the point of s_l (resp. s_r) on L we have that the distance from p to the point of s_v on U is greater in clockwise (resp. counter-clockwise) direction than the point of s_l (resp. s_r) on U.

For every set $S \subseteq A$, let R contain the vertices l and r as defined above. We then have $N(R) \cap \overline{A} = N(S) \cap \overline{A}$ and $|R| \leq 2$. Therefore, by Lemma 1, $mim(A) \leq 2$ for all cuts. Then for any d, by Lemma 2 we have $nec_d(T, \delta) \leq n^{2d}$. □

3.5. Trapezoid graphs

Let L and U be two infinite parallel lines. Let A and B be two non-crossing straight line-segments with one endpoint on L and the other on U. The finite area defined by the four lines L, U, A, B is called a trapezoid between L and U. Trapezoid graphs is a generalization of permutation graphs, i.e. using $A = B$ and a generalization of interval graphs, i.e. using $L = U$.

Definition 11 (*Trapezoid Graph*). A graph is a trapezoid graph if it has an intersection model consisting of trapezoids between two parallel lines.

Lemma 7. *Given a trapezoid graph G and any positive integer d, we can, in polynomial time, compute a decomposition tree (T, δ) of G having $nec_d(T, \delta) \leq n^{2d}$.*

Proof. We compute the trapezoid intersection model of G in $O(n^2)$ time [27]. We build a caterpillar decomposition by sorting the vertices by the upper right corner of their corresponding trapezoid from left to right. Let us now consider a cut $(A, \overline{A})$ of the decomposition. We show that for any $S \subseteq A$, we can find a set $S' \subseteq S$ with $N(S) \cap \overline{A} = N(S') \cap \overline{A}$ and $|S'| \leq 2$: For the upper line (resp. lower), we take the trapezoid u (resp. l) with the rightmost upper (resp. lower) right corner, we then set $S' = \{u, l\}$. Let us assume for contradiction that $\exists x \in \overline{A} : x \in N(S) \setminus N(S')$. The trapezoid of x must intersect some trapezoid of S on the upper or lower line. If it does not intersect u or l, then the whole trapezoid of x is to the right of u and l. By construction of the decomposition, x would have been in A, thus $N(S) \cap \overline{A} = N(S') \cap \overline{A}$. For every set $S \subseteq A$, let R contain the vertices u and l as defined above. We then have $N(R) \cap \overline{A} = N(S) \cap \overline{A}$ and $|R| \leq 2$. Therefore, by Lemma 1, $mim(A) \leq 2$ for all cuts. Then for any d, by Lemma 2 we have $nec_d(T, \delta) \leq n^{2d}$. □

3.6. k-trapezoid graphs

k-trapezoid graphs are a natural generalization of trapezoid graphs and interval graphs in the sense that the 2-trapezoid graphs are exactly the trapezoid graphs and 1-trapezoid graphs are exactly interval graphs.

Definition 12 (*k-trapezoid Graphs*)**.** Let $L_1, \ldots, L_k$ be k parallel lines. In order to build a k-trapezoid, first choose two points s_i and e_i on each line such that $s_i < e_i$. Then, make two non-intersecting paths s and e by joining s_i to s_{i+1} and e_i to e_{i+1} respectively by straight lines for each $i \in \{1, \ldots, k-1\}$. A k-trapezoid is the polygon defined by s, e and the lines going from s_1 to e_1 and s_k to e_k in clockwise direction. A k-trapezoid graph is the intersection graph of k-trapezoids.

Note that k-trapezoid graphs are equivalent to comparability graphs of partial orders of interval dimension k [28]. Moreover, Yannakakis showed in [29] that deciding if a partial order of height 1 has dimension at most 3 is *NP*-complete. Therefore, recognizing k-trapezoid graphs for any fixed $k \geq 3$ is NP-complete. Additionally, by [30] the smallest integer k such that a given graph G is a k-trapezoid graph cannot be approximated within a factor better than $\sqrt{n}$.

Lemma 8. *Given a k-trapezoid graph G together with a k-trapezoid model of G and any positive integer d, we can, in polynomial time, compute a decomposition tree (T, δ) of G having $nec_d\,(T, \delta) \leq n^{kd}$.*

Proof. We build a caterpillar decomposition by sorting the vertices by the rightmost corner of their corresponding k-trapezoid. Let us now consider a cut $\left(A, \overline{A}\right)$ of the decomposition. We show that for any $S \subseteq A$, we can find a set $S' \subseteq S$ with $N\,(S) \cap \overline{A} = N\left(S'\right) \cap \overline{A}$ and $|S'| \leq k$: For each line i, we take the k-trapezoid r_i, corresponding to a vertex of S, which contains the rightmost point on L_i. We set S' as the set of all r_i. Let us assume for contradiction that $\exists v \in \overline{A} : v \in N(S) \setminus N(S')$, let t_v be the trapezoid corresponding to v. Since v is in $\overline{A}$ there must exist some x such that t_v contains a point to the right of r_x on L_x. Also there must exist some y, such that t_v intersects some trapezoid of S on L_y hence t_v contains a point to the left of the rightmost point of r_y on L_y. Since both t_v and r_y are continuous they have to intersect at some point between L_x and L_y. Thus $N\,(S) \cap \overline{A} = N\left(S'\right) \cap \overline{A}$ hence $mim\,(A) \leq k$ then by Lemma 2 the lemma holds.

For every set $S \subseteq A$, let R contain all the vertices r_i as defined above. We then have $N\,(R) \cap \overline{A} = N\,(S) \cap \overline{A}$ and $|R| \leq k$. Therefore, by Lemma 1, $mim\,(A) \leq k$ for all cuts. Then for any d, by Lemma 2 we have $nec_d\,(T, \delta) \leq n^{kd}$. □

It is an interesting open problem whether one, given a k-trapezoid graph can build a decomposition tree having $nec_d\,(T, \delta) = n^{O(k)}$.

3.7. Circular k-trapezoid graphs

Circular k-trapezoid graphs form a natural extension of k-trapezoid graphs introduced in [31], see [32].

Definition 13 (*Circular k-trapezoid Graph*)**.** Let $C_1, \ldots, C_k$ be k circles on the surface of a cylinder, all orthogonal to its axis. In order to build a circular k-trapezoid, first choose two points s_i and e_i on each line. Then, make two non-intersecting paths s and e by joining s_i to s_{i+1} and e_i to e_{i+1} respectively by straight lines for each $i \in \{1, \ldots, k-1\}$. A circular k-trapezoid is the polygon defined by s, e and the arcs going from s_1 to e_1 and s_k to e_k in clockwise direction. A circular k-trapezoid graph is the intersection graph of circular k-trapezoids.

Lemma 9. *Given a circular k-trapezoid graph G and a circular k-trapezoid model and any positive integer d, we can, in polynomial time, compute a decomposition tree (T, δ) of G having $nec_d\,(T, \delta) \leq n^{2kd}$.*

Proof. Let p be an arbitrary point on C_1. We define the distance of a k-trapezoid from p as the minimum distance between p and any point of the arc of the k-trapezoid on C_1. For any vertex u, we denote by t_u the k-trapezoid corresponding to u and $arc_{u,i}$ the arc of C_i contained in t_u. Build a caterpillar decomposition by adding the vertices in order of increasing distance of their associated k-trapezoid from p, breaking ties arbitrarily. We now consider any cut $\left(A, \overline{A}\right)$ of this decomposition.

By construction, for every $x \in A$, $y \in \overline{A}$, the distance of t_x from p is less than or equal to the distance of t_y from p. Now, we prove that for any set $S \subseteq A$, there exists a subset $S' \subseteq S$ such that $|S'| \leq 2 \cdot k$ and $S \equiv_A S'$. Let $r(S, i)$(resp. $l(S, i)$) be the vertex $v \in S$ such that t_v contains the extremal point of L_i in clockwise (respectively counter-clockwise) direction. Let $S' = \bigcup_{i \leq k}\{r(S, i), l(S, i)\}$, assume for contradiction that there $\exists v \in \overline{A} : v \in N(S) \setminus N(S')$. By construction of the decomposition $arc_{v,1}$ must contain a point more extreme than the points on $arc_{l(S,1),1}$ and $arc_{r(S,1),1}$. Since $v \in N(S)$ there must exist a j such that, without loss of generality, $arc_{r(S,j),j}$ contains a more extreme point than the least extreme point of $arc_{v,j}$, but then $t_{r(S,j)}$ contains both a point less extreme and more extreme than t_v, hence they must intersect.

For every set $S \subseteq A$, let R contain all the vertices $r(S, i)$ and $l(S, i)$ as defined above. We then have $N\,(R) \cap \overline{A} = N\,(S) \cap \overline{A}$ and $|R| \leq 2k$. Therefore, by Lemma 1, $mim\,(A) \leq 2k$ for all cuts. Then for any d, by Lemma 2 we have $nec_d\,(T, \delta) \leq n^{2kd}$. □

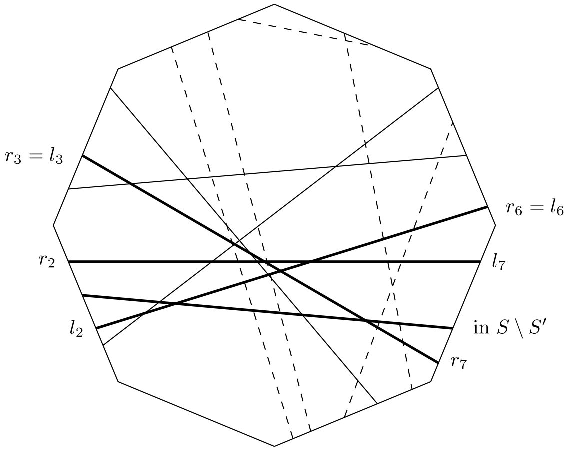

Fig. 3. A 8-polygon graph with 12 vertices, the vertices in S are represented by bold lines, those in $A \setminus S$ by thin lines and those in $\overline{A}$ by dashed lines.

3.8. Convex graphs

Definition 14 (*Convex Graph*)**.** A graph $G = (V, E)$ is convex if G is bipartite with color classes X and Y and an ordering $x_1, \ldots, x_{|X|}$ of X such that for every vertex $u \in Y$ and $x_i, x_j \in N(u)$, we have for every vertex $x_t \in X$ that if $i < t < j$ then $x_t \in N(u)$, i.e. the vertices in $N[u]$ are consecutive in the ordering of X.

Lemma 10. *Given a convex graph G and any positive integer d, we can, in polynomial time, compute a decomposition tree (T, δ) of G having $nec_d\ (T, \delta) \leq n^d$.*

Proof. Since G is convex we can in polynomial time find a bipartition (X, Y) of V and σ_X an ordering of X such that for every vertex $u \in Y$ and $x, y \in N(u)$ we have for every vertex $z \in X$ that if $\sigma_X(x) < \sigma_X(z) < \sigma_X(y)$ then $z \in N(u)$ [23]. We construct a total ordering σ of V from σ_X by keeping the ordering of vertices in X and for each vertex $v \in Y$ we insert v immediately after the last element (according to σ_X) of $N(v)$. We construct a caterpillar decomposition from the order σ.

Let us now consider a cut $\left(A, \overline{A}\right)$ of the decomposition and $S \subseteq A$. Note that by construction of σ, we have $\forall v \in Y \cap A, N(v) \cap \overline{A} = \emptyset$, hence $S \equiv_A S \cap X$. Let $v_1, v_2, \ldots, v_t$ be the ordering of the vertices of $X \cap S$ induced by σ and $R = \{v_t\}$. We show that $R \equiv_A S \cap X$. Assume for contradiction that there is a vertex $y \in (N(S \cap X) \setminus N(v_t)) \cap \overline{A}$. We know that $y \in Y$ and appear later in σ than v_t and hence by definition of σ we must have $x \in N\ (y) \cap \overline{A}$ such that x appears later in σ than v_t, then by definition of σ_X, either $v_t \in N(y)$ or $N(y) \cap S = \emptyset$ leading to a contradiction.

We then have $N\ (R) \cap \overline{A} = N\ (S) \cap \overline{A}$ and $|R| \leq 1$. Therefore, by Lemma 1, $mim\ (A) \leq 1$, hence for any d, we have by Lemma 2 that $nec_d\ (T, \delta) \leq n^d$. $\square$

3.9. k-polygon graphs

Definition 15 (*k-polygon Graph*)**.** A k-polygon graph is the intersection graph of chords (straight lines between two points on distinct sides) of a convex k sided polygon.

Lemma 11. *Given a k-polygon graph G and any positive integer d, we can, in polynomial time, compute a decomposition tree (T, δ) of G having $nec_d\ (T, \delta) \leq n^{2kd}$.*

Proof. We compute the k-polygon intersection model of G in $O(4^k \cdot n^2)$ time [33]. Let p be an arbitrary corner of the k-polygon. We measure the distance of a point from p as the distance around the edge of the k-polygon in clockwise direction. We define the distance of a chord from p as the minimum distance of any endpoint of the chord from p. We build a caterpillar decomposition of G by ordering the vertices of G by increasing distance from p of their corresponding chords.

Consider any cut $\overline{A}$ of the decomposition. We prove that for any set $S \subseteq A$, there exists a subset $S' \subseteq S$ such that $|S'| \leq 2k$ and $S' \equiv_A S$. We denote by t the maximum distance from p to a chord of any vertex in A. We can observe that, by construction of the decomposition, for every vertex u in A, if both endpoints of the chord corresponding to u are at distance at most t from p, then $N(u) \cap \overline{A} = \emptyset$. Now, we associate each side of the k-polygon with an index $i \in \{1, \ldots, k\}$ ordered in clockwise direction starting from p. For each side we define $S_i \subseteq S$ as the set of vertices of which line has an endpoint on side i. Each vertex of G belongs to exactly 2 such sets. We also define for each side i such that $S_i \neq \emptyset$, the point l_i on side i as the endpoint of a chord corresponding to a vertex in S_i closest to p, likewise r_i is the endpoint of a chord on side i corresponding to a vertex in S_i furthest from p (see Fig. 3).

Let $S' = \bigcup_{i \leq k}\{l_i, r_i\}$, we claim that $S' \equiv_A S$. Let us assume for contradiction that there exists a vertex $x \in (N(S) \setminus N(S')) \cap \overline{A}$. Let c_x be the chord corresponding to x, p_a and p_b be the endpoints of c_x such that $t < p_a < p_b$.

Let $y \in S$ be a neighbor of x and c_y the chord corresponding to y. Let p_y be the endpoint of c_y such that $t < p_y$ and j the index of the side containing p_y, then we know since y is a neighbor of x that $p_a < p_y < p_b$. We also know by definition that $l_j < p_y < r_j$, recall that both l_j and r_j are endpoints of some cord with the other endpoint at distance at most t from p. Since no chord can have both endpoints on the same side we have either $a < j$ or $j < b$, if $a < j$ then $p_a < l_j < p_y < p_b$ hence c_x intersects the chord ending at l_j, likewise we can argue if $j < b$ leading to a contradiction.

For every set $S \subseteq A$, let R contain all the vertices l_i and r_i as defined above. We then have $N(R) \cap \overline{A} = N(S) \cap \overline{A}$ and $|R| \leq 2k$. Therefore, by Lemma 1, $mim(A) \leq 2k$ for all cuts. Then for any d, by Lemma 2 we have $nec_d(T, \delta) \leq n^{2kd}$. □

3.10. Dilworth k graphs

Definition 16 (*Dilworth k Graph*). Two vertices x and y are said to be comparable if either $N(y) \subseteq N[x]$ or $N(x) \subseteq N[y]$. The Dilworth number of a graph is the largest number of pairwise incomparable vertices of the graph. A graph is a Dilworth k graph if it has Dilworth number k.

Dilworth k graphs can be recognized in $O(k^2 \cdot n^2)$ time [34].

Lemma 12. *Given a Dilworth k graph G and any positive integer d, we can, in polynomial time, compute a decomposition tree (T, δ) of G having $nec_d(T, \delta) \leq n^{kd}$.*

Proof. Let us consider any cut $(A, \overline{A})$ of G and $S \subseteq A$. We want to prove that there exists S' such that $|S'| \leq k$ and $N(S') \cap \overline{A} = N(S) \cap \overline{A}$. Let S' be an inclusion minimal subset of S such that $N(S') \cap \overline{A} = N(S) \cap \overline{A}$. If S' contains two vertices x and y such that $N(y) \subseteq N[x]$, then $S' \setminus \{y\}$ contradicts the minimality of S'. Since V cannot contain more than k pairwise incomparable vertices, $|S'| \leq k$. Therefore $mim(A) \leq k$ and by applying Lemma 2 the lemma follows. □

3.11. Complements of k degenerate graphs

Definition 17 (*k Degenerate Graph*). A graph $G = (V, E)$ is k-degenerate if there exists an elimination ordering $v_1, \ldots, v_n$ of the vertices of G such that $\forall i \in \{1, \ldots, n\}$, $|\{v_j : j > i \text{ and } v_j \in N(v_i)\}| \leq k$.

Lemma 13. *Given a graph G that is the complement of a k-degenerate graph, and any positive integer d, we can, in polynomial time, compute a decomposition tree (T, δ) of G having $nec_d(T, \delta) \leq n^d \cdot 2^{kd}$.*

Proof. We build a caterpillar decomposition of G using the elimination ordering induced by the k-degeneracy of $\overline{G}$. We consider a cut $(A, \overline{A})$ of the decomposition.

Note first that since $\overline{G}$ is k-degenerate, every vertex of A has at most k neighbors in $\overline{A}$. Therefore, in G every vertex of A has at least $|\overline{A}| - k$ neighbors in $\overline{A}$. Moreover every $S \subseteq A$ with size d sees at least $|\overline{A}| - kd$ vertices d times. There is at most n^d ways to choose S. For each such choice there is at most kd missing vertices, hence at most 2^{kd} non-equivalent subsets containing S. □

3.12. Boolean-width

We say that a class of graphs $\mathcal{C}$ has boolean-width $O(f(n))$ if for every G in $\mathcal{C}$ we have $boolw(G) \in O(f(n))$. We now restate the results in this section in terms of boolean-width, using the fact that, for any graph G and any decomposition (T, δ) of G, we have $boolw(G) \leq boolw(T, \delta) = \log_2 nec_1(T, \delta) \leq \log_2 nec_d(T, \delta)$. Combining this fact with the results in this section we get the following.

Corollary 14. *The following graph classes all have boolean-width $O(\log n)$: interval graphs, circular arc graphs, permutation graphs, circular permutation graphs, trapezoid graphs, k-trapezoid graphs, circular k-trapezoid graphs, convex graphs, k-polygon graphs, Dilworth k graphs and complements of k-degenerate graphs.*

4. Lower bounds

In this section we provide lower bounds for $nec_d(T, \delta)$ of optimal decompositions for various classes of graphs. We do so by providing lower bounds for the boolean-width of these classes. Note that a lower bound on $boolw(T, \delta)$ also implies the same lower bound for $nec_1(T, \delta)$, which in turn gives the same lower bound for $nec_d(T, \delta)$ for $d \geq 2$. However, the converse of this statement is not true for values of $d \geq 2$. We show that the upper bounds we gave in Section 4 are tight in two senses. We say that a class of graphs $\mathcal{C}$ has boolean-width $\Omega(f(n))$ if for any integers k and n there exists a graph $G \in \mathcal{C}$ with $|V(G) \geq n|$ having boolean-width larger than $k \times f(|V(G)|)$. Firstly, we show that all graph classes (except possibly Dilworth k graphs) in Group II of Fig. 1 have boolean-width $\Omega(\log n)$. Secondly, we show that for all graph classes in Group IV of Fig. 1, it is highly unlikely that they have boolean-width $O(\log n)$. We use the following result on the relation between boolean-width and some other width parameters.

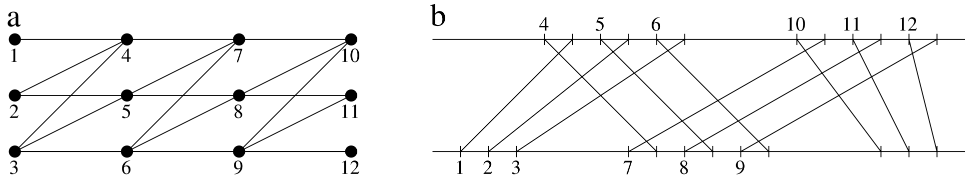

Fig. 4. (4 × 3) Hsu-stable chain (a) and its permutation representation (b).

Theorem 15 (*Bui-Xuan, Telle, Vatshelle [17,1]*). *For any graph G and any decomposition* (T, δ) *of G, it holds that* $\log rw(T, \delta) - 1 \le \log cw(T, \delta) - 1 \le boolw(T, \delta) \le \log nec_d(T, \delta)$*, where* $boolw(T, \delta)$, $rw(T, \delta)$ *and* $cw(T, \delta)$ *denote respectively the boolean-width, rank-width and clique-width of* (T, δ).

Hence if a graph class has rank-width or clique-width $\Omega(n^c)$ for some constant $c > 0$, then this graph class also has boolean-width $\Omega(\log n)$. We now give a lower bound for the rank-width of proper interval, bipartite permutation graphs and complement of grids, which implies the desired $\Omega(\log n)$ lower bound for boolean-width.

A *Hsu-graph* is a bipartite graph $H = (V, E)$ with $V = \{v_1, v_2, \dots v_p\}, \{u_1, u_2, \dots u_p\}$ and $v_i, u_j \in E(H)$ if and only if $i \le j$. A *Hsu-join-chain* of length q and width p is a graph which can be constructed as follows: Let $\mathcal{F} = G_1, G_2, \dots, G_q$ be a family of graphs, all on at least p vertices. For $j \in \{1, n\}$, let $S_j \subseteq V(G_j)$ such that $|S_j| = p$ and σ_j an ordering of S_j. Then, for every $1 \le i \le q-1$, let $G[S_i, S_{i+1}]$ be isomorphic to a Hsu-graph where we identify $\sigma_i(r)$ with v_r and $\sigma_{i+1}(r)$ with u_r.

Lemma 16. *If G is a Hsu-join-chain of length q and width p where* $q > 3p$ *then* $rw(G) \ge p/3$.

Proof. Let $\mathcal{F} = G_1, G_2, \dots, G_q$ be the family of graphs used to construct G. Without loss of generality, we assume $|V(G_i)| = p$ for $1 \le i \le q$ since deleting a vertex from G_i can only decrease the rank-width. To show that G has high rank-width, note that every decomposition tree of G contains a $(\frac{1}{3}, \frac{2}{3})$-balanced cut, let $(A, \overline{A})$ be such a balanced cut. We show $(A, \overline{A})$ has *cut–rank* at least $p/3$. Assume for contradiction that *cut–rank*$(A) < p/3$. We distinguish two cases:

Case 1: At least p graphs G_i in $\mathcal{F}$ contains vertices from both A and $\overline{A}$. For every i we know that $G[V(G_i), V(G_{i+1})]$ is connected and hence contains an edge (u, v) with $u \in A$ and $v \in \overline{A}$. There is at least $p/3$ such edges forming an induced matching contradicting that *cut-rank*$(A) < p/3$.

Case 2: At most $p-1$ graphs G_i in $\mathcal{F}$ contains vertices from both A and $\overline{A}$, hence at least $2p+1$ graphs in $\mathcal{F}$ contains vertices all from the same side of the cut $(A, \overline{A})$. Since the cut is balanced there must be at least one graph with all its vertices in A and one graph with all its vertices in $\overline{A}$. Let G_i and $G_{i'}$ be two such graphs, without loss of generality assume $i < i'$ and $V(G_i) \subseteq A$. Then for each $r \in \{1, 2, \dots, p\}$ there must be a j such that $i \le j \le i'$ and $\sigma_j(r) \in A$ and $\sigma_{j+1}(r) \in \overline{A}$, let S be a set containing all such pairs of vertices and $H = G[S]$ the subgraph induced by these vertices. The graph H is formed by a collection of Hsu-graphs whose size sums up to at least $2p$, there is an induced subgraph of H which is a collection of vertex disjoint Hsu-graphs whose size sums up to at least $2p/3$, namely partitioning the edges into 3 by j mod 3 and picking the largest partition. This graph has rank-width at least $p/3$ leading to a contradiction. □

We now describe two distinct families of Hsu-join-chains, one being a subclass of bipartite permutation graphs and the other a subclass of proper interval.

Corollary 17. *Bipartite permutation graphs have rank-width* $\Omega(\sqrt{n})$ *and boolean-width* $\Theta(\log n)$.

Proof. Let a *Hsu-stable-chain* of length q and width p be the Hsu-join-chain of length q and width p where for every i, G_i is a stable set of size p. Clearly, Hsu-stable-chains are bipartite permutation graphs (see Fig. 4). If $q \ge 3p$ then by Lemma 16 and Theorem 15 they have rank-width $\Theta(p)$ and boolean-width $\Theta(\log p)$. Let $p = \frac{\sqrt{n}}{2}$ and $q = 2\sqrt{n}$, then the lower bounds follow, the upper bound on boolean-width follows from Lemma 5. □

Recall that proper interval graphs are interval graphs admitting an interval model where all the intervals have same length.

Corollary 18. *Proper interval graphs have rank-width* $\Omega(\sqrt{n})$ *and boolean-width* $\Theta(\log n)$.

Proof. Let a *Hsu-clique-chain* of length q and width p be the Hsu-join-chain of length q and width p where for every i, G_i is a clique of size p. Clearly, Hsu-clique-chains are proper interval graphs (see Fig. 5), and by Lemma 16 and Theorem 15 they have rank-width $\Theta(p)$ and boolean-width $\Theta(\log n)$. □

Moreover, note that Jelínek showed in [35] that $\sqrt{n} \times \sqrt{n}$ grids have rank-width exactly $\sqrt{n} - 1$. Since the rank-width of a graph differs by at most one from the rank-width of its complement, then complement of grids have rank-width at least $\sqrt{n} - 2$. Since grids are 2-degenerate, then complement of k-degenerate graphs have rank-width $\Omega(\sqrt{n})$.

Finally, we can summarize these lower bounds as follows:

Lemma 19. *All graph classes in Group II of Fig. 1 (except possibly Dilworth k graphs), have boolean-width* $\Omega(\log n)$.

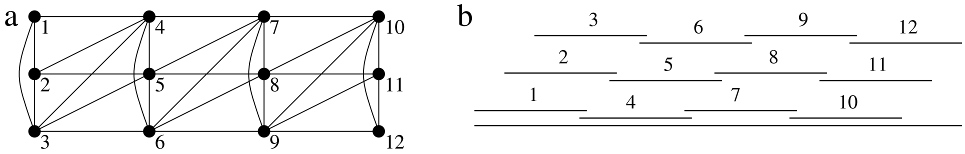

Fig. 5. (4×3) Hsu-clique chain (a) and its proper interval representation (b).

Another interesting question to ask is whether there exist more graph classes having logarithmic boolean-width. The usual way to answer this question is by either showing how to construct a decomposition of small width, or by showing an infinite family of graphs of large width. For some graph classes it is possible to provide such examples of graphs having non-logarithmic boolean-width, like for the $q \times q$ grid. However, for other classes of graphs, we do not know any example of infinite family of graphs having non-logarithmic boolean-width. We are nonetheless able to provide some lower bounds:

Lemma 20. *For all the classes in Group IV of Fig. 1, either they do not have boolean-width $O(\log n)$, or such a decomposition cannot be computed in polynomial time unless $P = NP$.*

Proof. Note first that for all the classes of graphs in Group IV of Fig. 1, Minimum Weight Dominating Set is NP-complete (see [36,37] and [38]). Moreover, Minimum Weight Dominating Set can be solved in time $O(2^{3 \cdot boolw} \cdot poly(n))$ [17]. Assume now that there exists a class C in Group IV of Fig. 1 having boolean-width $O(\log n)$ and where such decompositions can be computed in polynomial time. Then Minimum Weight Dominating Set can be computed in time $O(2^{O(log n)} \cdot poly(n))$ which is polynomial in n. Hence if a class of graphs on which Minimum Weight Dominating Set is NP-complete has boolean-width $O(\log n)$, then computing such decompositions is NP-hard. □

Note that this holds not only for Minimum Weight Dominating Set, but as long as there exists a problem which can be solved in $O(2^{O(boolw} \cdot poly(n))$ time. Finally, we can get stronger lower bounds by working under a stronger hypothesis. The *E*xponential *T*ime *H*ypothesis (ETH) states that there does not exist an algorithm for solving 3-Sat running in time $2^{o(n)}$. We can reformulate Lemma 20 as follows:

Lemma 21. *For all the classes in Group IV of Fig. 1, either they do not have boolean-width $O(n^{o(1)})$, or such a decomposition cannot be computed in time $2^{n^{o(1)}}$, unless ETH fails.*

Proof. Assume for contradiction that there exists a class of graphs $\mathcal{C}$ in Group IV of Fig. 1 for which a decomposition of boolean-width $n^{o(1)}$ can be computed in time $2^{n^{o(1)}}$. Recall that Minimum Weight Dominating Set is NP-complete on all the classes in Group IV of Fig. 1. Hence, there is a polynomial time reduction from k-SAT to Minimum Weight Dominating Set on $\mathcal{C}$ such that from any instance I of k-SAT, a graph $G = (V, E)$ belonging to $\mathcal{C}$ can be built such that $n \leq |I|^c$, for some constant $c > 0$ and solving Minimum Weight Dominating Set on G implies a solution to k-SAT on I. Recall that Minimum Weight Dominating Set can be solved in $2^{3 \cdot boolw(G)} \cdot poly(n)$. Finally, since we assumed we could compute a decomposition of boolean-width $n^{o(1)}$ in time $2^{n^{o(1)}}$, the instance I can be solved in $2^{3 \cdot n^{o(1)}} \cdot poly(n)$, which is equivalent to $2^{3 \cdot |I|^{o(c)}} \cdot poly(|I|^c)$. This would imply that we could solve the instance I in time $2^{o(|I|)}$. Hence the Lemma follows. □

This means for instance that if split graphs have boolean-width polylogarithmic in n, then it is NP-hard to compute a decomposition of split graphs having boolean-width within a factor $\log(n)$ of the optimum.

5. Conclusion

We have shown that all graph classes in Group II of Fig. 1 admit a decomposition where nec_d is bounded by a polynomial of n, and we can compute such decompositions in polynomial time if the intersection model is given. This answers an open question from [17]. The following theorem is the main motivation for our results.

Theorem 22 (*Main Theorem of [1]*)**.** *Let G be a graph given along with a decomposition tree (T, δ). For every LC-VSP problem Π, there are constants d and q such that Π can be solved in time $O(n^4 \cdot q \cdot nec_d(T, \delta)^{3q})$.*

Combined with the results in Section 3 we get the following theorem:

Theorem 23. *Let $\mathcal{C}$ be one of the following graph classes: Dilworth k graphs, convex graphs, trapezoid graphs, circular permutation graphs, circular arc graphs or circular k-trapezoid graphs. Then, every LC-VSP problem can be solved in polynomial time on $\mathcal{C}$.*

For the particular case of complement of k-degenerate, we gave a bound for $nec_d(T, \delta)$ of the form $2^{d \cdot k} \cdot n^d$, which implies the following:

Theorem 24. *Let G be the complement of a k-degenerate graph, given along with a decomposition tree (T, δ). Every LC-VSP problem can be solved in time $2^{O(k)} \cdot poly(n)$.*

This means that every LC-VSP problem can be solved in FPT time on a graph G when parameterized by the degeneracy of the complement of G, with single exponential dependence in the parameter. Finally, we leave open the question of whether the classes in Group III of Fig. 1 have logarithmic boolean-width.

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