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Upper bounds on the average eccentricity

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ABSTRACT

Sharp upper bounds on the average eccentricity of a connected graph of given order in terms of its independence number, chromatic number, domination number or connected domination number are given. Our results settle two conjectures of the computer program AutoGraphiX (Aouchiche et al., 2005).

1. Introduction

Let $G = (V, E)$ be a connected graph of order n. The *distance* $d_G(u, v)$ between vertices u and v in G is the length of a shortest u–v path in G. The *eccentricity* $\mathrm{ec}_G(v)$ of a vertex v of G is the distance from v to a vertex furthest away from v. The *average eccentricity* $\mathrm{avec}(G)$ of G is defined as $\mathrm{avec}(G) = \frac{1}{n}\sum_{x\in V} \mathrm{ec}_G(x)$.

In this paper we give sharp upper bounds on the average eccentricity for graphs of given order and independence number, domination number, connected domination number or chromatic number. We prove, as corollaries to our results, two conjectures produced by the computer program AutoGraphiX.

The average eccentricity was introduced by Buckley and Harary [5] as the eccentric mean, and further studied in [10], where bounds in terms of order, radius, diameter and minimum degree were given, and for more recent results see [13]. The average eccentricity has also been the subject of a number of conjectures produced by the computer program AutoGraphiX. For information on the program AutoGraphiX see [2, 4, 6]; and for a list of its conjectures see [1, 11].

The average eccentricity is related to the average distance, defined as the average of the distances between all pairs of vertices. Many extremal graphs that maximise the average distance for a given parameter set also maximise the average eccentricity, and the results in this paper are a case in point. The average distance is also related to some new graph invariants studied in chemical graph theory, specifically to the eccentric connectivity index (see for example [10, 13] and references therein) and the eccentric distance sum (see for example [18]).

All graphs in this paper are finite and connected. Our notation follows [5]. In particular we denote the order, independence number, chromatic number, domination number and connected domination number of a graph G by n, α, χ, γ and γ_c. We denote the sum $\sum_{x\in V} \mathrm{ec}_G(x)$ by $\zeta(G)$, so $\mathrm{avec}(G) = \frac{\zeta(G)}{n}$. In most statements and proofs we use $\zeta(G)$, rather than $\mathrm{avec}(G)$,

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in order to avoid unpleasant fractions. If x and y are vertices of a connected graph, then $P(x, y)$ denotes a shortest path between them.

2. Preliminary results

We begin with a few preparatory results which will be needed for the proofs of our main results.

The first proposition derives from the fact that the path maximises average eccentricity amongst all connected graphs of a given order, which was proved in [10]. The second proposition, a sharp upper bound on the average eccentricity of a tree of given order and diameter, is a refinement of the first. The third result is a classical result on eccentric sequences by Lesniak [14].

Proposition 2.1 *([10]). Let G be a connected graph of order n. Then*

$$\zeta(G) \le \left\lfloor \frac{3}{4}n^2 - \frac{1}{2}n \right\rfloor,$$

with equality if and only if G is a path. □

Lemma 2.1. *Let T be a tree and let u, v be two vertices at distance $\mathrm{diam}(T)$. Then*

$$\mathrm{ec}(x) = \max\{d(x, u), d(x, v)\}$$

for all $x \in V(T)$.

Proof. Suppose not. Then there exist vertices x and y with $d(x, y) > \max\{d(x, u), d(x, v)\}$.

Case 1: $P(u, v)$ and $P(x, y)$ have at most one vertex in common.

Let w and z be vertices on $P(u, v)$ and $P(x, y)$, respectively, at minimum distance, so $w = z$ if and only if the two paths have a vertex in common. By our assumption we have $d(z, u), d(z, v) < d(z, y)$, and so $d(w, y) > d(w, u), d(w, v)$. Since a shortest (u, y)-path contains w, we obtain the contradiction $d(u, y) = d(u, w) + d(w, y) > d(u, w) + d(w, v) = \mathrm{diam}(T)$.

Case 2: $P(u, v)$ and $P(x, y)$ share more than one vertex.

Then the two paths have a path segment $P(w, z)$ in common. We may assume that w is closer to u on $P(u, v)$ and z is closer to v. As above, $d(x, y) > d(x, v)$ implies $d(z, y) > d(z, v)$, and so we obtain the contradiction $d(u, y) = d(u, z) + d(z, y) > d(u, z) + d(z, v) = d(u, v) = \mathrm{diam}(T)$. □

Proposition 2.2. *Let T be a tree of order n and diameter d. Then*

$$\zeta(T) \le \left\lfloor dn - \frac{1}{4}d^2 + \frac{1}{4} \right\rfloor.$$

Proof. Let $P = v_0, v_1, \ldots, v_d$ be a diametral path. By Lemma 2.1 we have $\mathrm{ec}(v_i) = d - i$ for $i \le \frac{1}{2}d$ and $\mathrm{ec}(v_i) = i$ for $i > \frac{1}{2}d$. Summation yields $\sum_{x \in V(P)} \mathrm{ec}(x) \le \lfloor \frac{3}{4}(d+1)^2 - \frac{1}{2}(d+1) \rfloor$. Hence,

$$\begin{aligned}
\zeta(T) &= \sum_{x \in V(P)} \mathrm{ec}(x) + \sum_{x \in V(T) - V(P)} \mathrm{ec}(x) \\
&\le \left\lfloor \frac{3}{4}(d+1)^2 - \frac{1}{2}(d+1) \right\rfloor + (n - d - 1)d \\
&= \left\lfloor dn - \frac{1}{4}d^2 + \frac{1}{4} \right\rfloor,
\end{aligned}$$

as desired. □

Theorem 2.1 *([14]). Let G be a connected graph of order n. Then for every integer k with $\mathrm{rad}(G) < k < \mathrm{diam}(G)$ there exist at least two vertices in G of eccentricity k.* □

Lemma 2.2. *Let G be a connected graph of order n, and let T be a connected dominating subgraph of G with vertex set S. Then*

$$\zeta(G) \le \zeta(T) + |S| + \Big(|V(G)| - |S|\Big)\Big(\mathrm{diam}(T) + 2\Big).$$

Proof. Since every vertex of G not contained in S is adjacent to a vertex in S, we have $\mathrm{ec}_G(x) \le \mathrm{ec}_T(x) + 1$ for all $x \in S$. Moreover, since every two vertices of G are connected by a path whose internal vertices are in S, we have $\mathrm{ec}_G(x) \le$

$\mathrm{diam}(T) + 2$ for all $x \in V(G) - S$. It follows that

$$\begin{aligned}\zeta(G) &= \sum_{x \in S} \mathrm{ec}_G(x) + \sum_{x \in V(G)-S} \mathrm{ec}_G(x)\\ &\le \sum_{x\in S}[\mathrm{ec}_T(x) + 1] + \sum_{x \in V(G)-S}[\mathrm{diam}(T) + 2]\\ &= \zeta(T) + |S| + (|V(G)| - |S|)[\mathrm{diam}(T) + 2]\end{aligned}$$

as desired. □

3. The main results

Our first main result relates the average eccentricity to the connected domination number.

Corollary 3.1. *Let G be a connected graph of order n and connected domination number γ_c. Then*

$$\zeta(G) \le \left\lfloor n(\gamma_c + 1) - \frac{1}{4}\gamma_c^2 - \frac{1}{2}\gamma_c \right\rfloor,$$

and this bound is sharp.

Proof. Let S be a minimum connected dominating set of G, and let T be the subgraph of G induced by S. By Lemma 2.2 we have

$$\zeta(G) \le \zeta(T) + |S| + (|V(G)| - |S|)[\mathrm{diam}(T) + 2]. \tag{1}$$

Now T has γ_c vertices. Hence $\mathrm{diam}(T) \le \gamma_c - 1$ and, by Proposition 2.2,

$$\zeta(T) \le \left\lfloor \frac{3}{4}\gamma_c^2 - \frac{1}{2}\gamma_c \right\rfloor.$$

Substituting this into (1) yields the bound of the theorem.

To see that the bound is sharp, consider for given positive integers n and γ_c with $\gamma_c \le n - 2$ the tree obtained from a path P_{γ_c+1} by attaching $n - 1 - \gamma_c$ end vertices to one end of the path. It is easy to verify that $\gamma_c(G) = \gamma_c$ and $\zeta(G) = \left\lfloor n(\gamma_c + 1) - \frac{1}{4}\gamma_c^2 - \frac{1}{2}\gamma_c \right\rfloor$. □

Our next result is a bound on the average eccentricity in terms of order and independence number. This result mirrors results by Chung [7], who proved that the average distance is bounded from above by the independence number, and by Dankelmann [8], who showed that the maximum value of the average distance among connected graphs of given order n and independence number α is attained by graphs consisting of two cliques whose cardinalities differ by at most 1 joined by a path, provided $\alpha \le \frac{n+1}{2}$. (For further results on this topic see [3, 15].) It was conjectured by the computer program AutoGraphiX (Conjecture A.479 in [1] and in the list of conjectures [11]) that this also holds true for graphs maximising the average eccentricity, and our results prove this conjecture. We treat the two cases $\alpha \le \frac{n}{2}$ and $\alpha > \frac{n}{2}$ separately.

Theorem 3.1. *Let G be a connected graph of order n and independence number α, where $\alpha \le \frac{n}{2}$. Then*

$$\zeta(G) \le (2\alpha - 1)n - \alpha^2 + \alpha,$$

and this bound is sharp.

Proof. Construct a sequence $A_1 \subset A_2 \subset A_3 \subset \cdots$ of independent sets, and a sequence $T_1 \le T_2 \le T_3 \le \cdots$ of subtrees of G such that $A_i \subseteq V(T_i)$ for $i = 1, 2, \ldots$ using the following procedure. Choose an arbitrary vertex v of G, and let $A_1 = \{v\}$. Let $T_1 \le G$ be the tree on one vertex, v. Now assume that A_{i-1}, and tree T_{i-1} have been found. If there exists a vertex x in G with $d(x, A_{i-1}) = 2$, let $A_i = A_{i-1} \cup \{x\}$. To obtain T_i, let xya, $a \in A_{i-1}$, be an x–A_{i-1} shortest path in G and define $T_i = xya \cup T_{i-1}$. Eventually one of the sets, A_k say, is maximally independent and so every vertex not in A_k is adjacent to some vertex in A_k. Thus T_k is a dominating tree of G. Since T_k has at most $2\alpha - 1$ vertices, we have the following:

$$\gamma_c(G) \le 2\alpha - 1. \tag{2}$$

Let T be a dominating tree of G of order $\gamma_c(G)$ and, among those, of minimum diameter. We consider three cases.

Case 1: $\gamma_c(G) \le 2\alpha - 2$.

By Lemma 2.2 and the fact that $\mathrm{diam}(T) \le \gamma_c(G) - 1$, we have

$$\zeta(G) \le \zeta(T) + \gamma_c(G) + (n - \gamma_c(G))[\gamma_c(G) + 1] \le \zeta(T) + (\gamma_c(G) + 1)n - \gamma_c(G)^2.$$

Applying Proposition 2.2 to bound $\zeta(T)$, we obtain after simplification

$$\zeta(G) \le (\gamma_c + 1)n - \frac{1}{4}\gamma_c^2 - \frac{1}{2}\gamma_c.$$

The right hand side of the last inequality is easily seen to be increasing in γ_c. Substituting $2\alpha - 2$ for γ_c yields

$$\zeta(G) \le (2\alpha - 1)n - (\alpha - 1)^2 - (\alpha - 1) = (2\alpha - 1)n - \alpha^2 + \alpha,$$

as desired.

Case 2: $\gamma_c(G) = 2\alpha - 1$ and T is not a path.

Let T have diameter d. Since T has order $2\alpha - 1$ and T is not a path, we have $d \le 2\alpha - 3$. By Lemma 2.2 and applying Proposition 2.2, we obtain

$$\begin{aligned}\zeta(G) &\le \zeta(T) + (2\alpha - 1) + \Big(n - (2\alpha - 1)\Big)\Big(d + 2\Big)\\ &\le d(2\alpha - 1) - \frac{1}{4}d^2 + \frac{1}{4} + (2\alpha - 1) + \Big(n - (2\alpha - 1)\Big)\Big(d + 2\Big)\\ &= -\frac{1}{4}d^2 + dn + 2n - 2\alpha + \frac{5}{4}.\end{aligned}$$

The function $f(d) := -\frac{1}{4}d^2 + dn + 2n - 2\alpha + \frac{5}{4}$ is increasing in d. Subject to $d \le 2\alpha - 3$, f is maximised for $d = 2\alpha - 3$ to give

$$\zeta(G) \le f(2\alpha - 3) = (2\alpha - 1)n - \alpha^2 + \alpha - 1 < (2\alpha - 1)n - \alpha^2 + \alpha,$$

as desired.

Case 3: $\gamma_c(G) = 2\alpha - 1$ and T is a path.

Let $T = v_1, v_2, \ldots, v_{2\alpha-1}$. We first show that the bound in the theorem holds if T is not an induced path of G. If $v_1v_{2\alpha-1} \in E(G)$, then the cycle $C = v_1, v_2, \ldots, v_{2\alpha-1}, v_1$ has diameter at most $\alpha - 1$ and dominates G. It follows, by Lemma 2.2, that

$$\begin{aligned}\zeta(G) &\le \zeta(C) + |V(C)| + \Big(|V(G)| - |V(C)|\Big)\Big(\mathrm{diam}(C) + 2\Big)\\ &\le (2\alpha - 1)(\alpha - 1) + (2\alpha - 1) + (n - 2\alpha + 1)(\alpha - 1 + 2)\\ &= (\alpha + 1)n - 2\alpha + 1 < (2\alpha - 1)n - \alpha^2 + \alpha,\end{aligned}$$

as desired. Now assume that for some $i, j \in \{1, 2, \ldots, 2\alpha - 1\}$, where $j - i \ge 2$ and $v_iv_j \ne v_1v_{2\alpha-1}$, we have $v_iv_j \in E(G)$. Then the tree T' obtained from T by removing the edge v_iv_{i+1} or v_jv_{j-1} and adding the edge v_iv_j is a dominating tree of G of order $2\alpha - 1$ that is not a path. By Lemma 2.2,

$$\zeta(G) \le \zeta(T') + (2\alpha - 1) + \Big(n - (2\alpha - 1)\Big)\Big(\mathrm{diam}(T') + 2\Big).$$

As in Case 2, the right hand side of the above expression is at most $(2\alpha - 1)n - \alpha^2 + \alpha$, as desired.

We now assume that $T = v_1, v_2, \ldots, v_{2\alpha-1}$ is an induced path of G. Since T is a minimal dominating tree, the trees $T - v_1$ and $T - v_{2\alpha-1}$ are not dominating trees. Hence there exist private neighbours u_1 of v_1 and $u_{2\alpha-1}$ of $v_{2\alpha-1}$ in T, i.e., $N_G(u_i) \cap V(T) = \{v_i\}$ for $i = 1, 2\alpha - 1$.

We claim that u_1 and $u_{2\alpha-1}$ are adjacent in G. Suppose not. Then since T is an induced path of G, $P = u_1, v_1, v_2, \ldots, v_{2\alpha-1}, u_{2\alpha-1}$ is also an induced path of G, and so $\{u_1, v_2, v_4, \ldots, v_{2\alpha-2}, u_{2\alpha-1}\}$ is an independent set of order $\alpha + 1$, a contradiction. Hence $u_1u_{2\alpha-1} \in E(G)$.

It follows that $V(T) \cup \{u_1, u_{2\alpha-1}\}$ induces a dominating cycle C of length $2\alpha + 1$. Since $\mathrm{diam}(C) = \alpha$, we conclude by Lemma 2.2 that

$$\begin{aligned}\zeta(G) &\le \zeta(C) + |V(C)| + \Big(|V(G)| - |V(C)|\Big)\Big(\mathrm{diam}(C) + 2\Big)\\ &\le (2\alpha + 1)\alpha + (2\alpha + 1) + (n - 2\alpha - 1)(\alpha + 2)\\ &= (\alpha + 2)n - 2\alpha - 1.\end{aligned}$$

If $\alpha \ge 3$, then it is easy to verify that the last right hand side is not more than $(2\alpha - 1)n - \alpha^2 + \alpha$, and so the theorem follows in that case.

It remains to show the theorem for the case $\alpha = 2$. Let G be a connected graph of order n and independence number 2. Then $\mathrm{diam}(G) \le 3$. If $\mathrm{diam}(G) \le 2$, then $\zeta(G) \le 2n$ and so the theorem holds. Hence we may assume that G has two vertices at distance 3. Let v_0, v_1, v_2, v_3 be a shortest path between these vertices. Since no three vertices of G are independent, the sets $N[v_0]$ and $N[v_3]$ form a partition of $V(G)$, and each of the two sets induces a complete graph. The edge v_1v_2 joins the two cliques, and so $\mathrm{ec}(v_1) = \mathrm{ec}(v_2) = 2$. Since the eccentricity of each remaining vertex is at most 3, we have

$$\zeta(G) \le 2 + 2 + 3(n-2) = 3n - 2,$$

as desired.

Finally, to see that the bound is tight, consider the graph $G_{n,\alpha}$ obtained by taking a path $P_{2\alpha-2}$, of order $2\alpha - 2$, with end vertices u and v, and joining u to every vertex of a complete graph K_a, and joining v to every vertex of a complete graph K_c, where $a + c = n - 2\alpha + 2$, $a, c \ge 1$. □

The question of which graph G of given order n attains the maximum value of $\frac{\mathrm{avec}(G)}{\alpha(G)}$ is considered in Conjecture A479 in [1]. Let the independence number of such a graph be α_n, where α_n depends on n. Then it follows from Theorem 3.1 that for such a graph $\frac{\mathrm{avec}(G)}{\alpha(G)} \le 2 - \frac{1}{\alpha_n} - \frac{\alpha_n}{n} + \frac{1}{n}$. Equality holds for exactly the graphs of order n and independence number α_n that maximise $\zeta(G)$. By Theorem 3.1 the graphs obtained from a path $P_{2\alpha_n-2}$ by appending a clique each to both ends of the path maximise $\frac{\mathrm{avec}(G)}{\alpha(G)}$. Hence Conjecture A479, which says that a graph obtained from a path by appending cliques of equal or almost equal cardinality maximises $\frac{\mathrm{avec}(G)}{\alpha(G)}$, follows. A simple maximisation of the term $\frac{\mathrm{avec}(G)}{\alpha(G)} \le 2 - \frac{1}{\alpha_n} - \frac{\alpha_n}{n} + \frac{1}{n}$ shows that $\alpha_n = \lfloor\sqrt{n}\rfloor$ or $\alpha_n = \lceil\sqrt{n}\rceil$, and hence the path has order $2\lfloor\sqrt{n}\rfloor - 2$ or $2\lceil\sqrt{n}\rceil - 2$.

Theorem 3.2. *Let G be a connected graph of order n and independence number α, where $\alpha > \frac{n}{2}$. Then*

$$\zeta(G) \le n^2 - \alpha^2,$$

and this bound is sharp.

Proof. Since the bound is decreasing in α, it is sufficient to show that the above bound holds for all connected graphs of order n with independence number at least α.

Let G be a connected graph of order n and independence number at least α and, among those, a graph of maximum average eccentricity. We may assume that G is a tree, since removing edges of G decreases neither the independence number nor the average eccentricity.

We show that $\mathrm{rad}(G) \le n - \alpha$. Since G is a tree it suffices to show that $\mathrm{diam}(G) \le 2(n-\alpha)$. Consider a shortest path P between two diametral vertices u and v. If S is a maximum independent set of G, then at most $\frac{|V(P)|+1}{2}$ vertices of P are in S, and so $|V(P) - S| \ge \frac{|V(P)|-1}{2}$, implying $n - \alpha \ge \frac{d(u,v)}{2} = \frac{\mathrm{diam}(G)}{2}$, as desired.

Consider the eccentric sequence $e_1, e_2, \ldots, e_n$ of G. It follows from the above and Theorem 2.1 that $e_1 \le n - \alpha$, $e_n \le 2(n-\alpha)$, and that every integer k with $e_1 < k < e_n$ appears at least twice in the eccentric sequence. Subject to these conditions, $\sum_{i=1}^{n} e_i$ is maximised for the sequence $(n-\alpha), (n-\alpha+1)^{(2)}, (n-\alpha+2)^{(2)}, (n-\alpha+3)^{(2)}, \ldots, (2n-2\alpha-1)^{(2)}, (2n-2\alpha)^{(2\alpha-n+1)}$. Hence

$$\begin{aligned}\zeta(G) \le\ & (n-\alpha) + 2(n-\alpha+1) + 2(n-\alpha+2) + 2(n-\alpha+3) \\ & + \cdots + 2(2n-2\alpha-1) + (2\alpha-n+1)(2n-2\alpha) \\ =\ & n^2 - \alpha^2,\end{aligned}$$

as desired.

To see that the above bound is sharp, consider the tree T obtained from a path of order $2n - 2\alpha$ by appending $2\alpha - n$ end vertices to one end of the path. It is easy to verify that T has n vertices and independence number α and that $\zeta(T) = n^2 - \alpha^2$. □

We now consider bounds on the average eccentricity in terms of order and chromatic number. We show that a graph obtained from the disjoint union of a clique and a path by joining one end of the path to a vertex of the clique has the maximum average eccentricity among all connected graphs of given order and chromatic number. This was conjectured by the computer program AutoGraphiX (see Conjecture A492 in [1]). This result mirrors a result for the average distance by Tomescu and Melter [17] who showed that this graph also maximises the average distance among all graphs of given order and chromatic number.

Theorem 3.3. *Let G be a connected graph of order n and chromatic number χ, where $2 \le \chi \le n - 1$. Then*

$$\zeta(G) \le \left\lfloor \frac{3}{4}n^2 - \frac{1}{4}\chi^2 - \frac{1}{2}\chi n + \frac{1}{2}n + \frac{1}{2}\chi \right\rfloor.$$

Equality in the above bound is attained for the graph obtained from the union of a complete graph of order χ and a path of order $n - \chi$ by adding an edge joining an end vertex of the path to a vertex of the complete graph.

Proof. If G is an odd cycle, then $\chi = 3$, $\zeta(G) = \frac{n(n-1)}{2}$, and it is easy to see that the theorem holds in this case. So we may assume that G is neither an odd cycle, nor complete.

Since G has chromatic number χ, it follows by Brooks' Theorem that $\Delta(G) \geq \chi$. Hence G has a vertex of degree at least χ and thus a spanning tree T of maximum degree at least χ, and so $\mathrm{diam}(T) \leq n+1-\chi$. It follows by Proposition 2.2 that

$$\zeta(T) \leq \left\lfloor \frac{3}{4}n^2 - \frac{1}{4}\chi^2 - \frac{1}{2}\chi n + \frac{1}{2}n + \frac{1}{2}\chi \right\rfloor,$$

as desired.

The second statement can be verified by calculating $\zeta(G)$ for the described graph. □

Finally we determine the maximum value of the average eccentricity of graphs of given order and domination number. Our results mirror results on the average distance obtained by Dankelmann [9] (see also [16]), who showed that the graphs maximising the average distance among all graphs of given order and domination number are graphs obtained from a path by appending end vertices to both ends of the path (for $\gamma \leq \frac{n}{3}$) and graphs obtained from a path by appending one end vertex each to some of its vertices. For the proof we need the well-known observations that every connected graph has a spanning tree with the same domination number, and that the diameter of a graph G does not exceed $3\gamma(G) - 1$ (see for example [12]). We treat the cases $\gamma \leq \frac{n}{3}$ and $\gamma \geq \frac{n}{3}$ separately.

Theorem 3.4. *Let G be a connected graph of order n and domination number $\gamma \leq \frac{n}{3}$. Then*

$$\zeta(G) \leq \left\lfloor 3n\gamma - n - \frac{9}{4}\gamma^2 + \frac{3}{2}\gamma \right\rfloor,$$

and this bound is sharp.

Proof. Since G has a spanning tree with the same domination number we may assume that G is a tree. Let G have diameter d. Applying Proposition 2.2, we have

$$\zeta(G) \leq \left\lfloor dn - \frac{1}{4}d^2 + \frac{1}{4} \right\rfloor.$$

Now $d \leq 3\gamma - 1$. Subject to this condition, the function $f(d) := dn - \frac{1}{4}d^2 + \frac{1}{4}$ is maximised for $d = 3\gamma - 1$ to give

$$\zeta(G) \leq \left\lfloor 3n\gamma - n - \frac{9}{4}\gamma^2 + \frac{3}{2}\gamma \right\rfloor,$$

as desired.

Equality in the above bound is attained for the graph obtained by taking a path $P_{3\gamma} = v_1, v_2, \ldots, v_{3\gamma}$, of order 3γ, and attaching $n - 3\gamma$ end vertices to v_2. □

For the case $\gamma \geq \frac{n}{3}$ we need the following lemma which is almost identical to a lemma of Rautenbach [16].

Lemma 3.1. *Let T be a tree, and v a vertex of T with three distinct neighbours v_0, v_1, v_2. Let the component of $T - v$ containing v_0 be non-trivial and let u be a vertex in this component adjacent to an end vertex of T. If $T_i = T - vv_i + uv_i$ for $i = 1, 2$, then*

$$\gamma(T_1) \geq \gamma(T) \quad \text{or} \quad \gamma(T_2) \geq \gamma(T).$$

Proof. Suppose to the contrary that $\gamma(T_1), \gamma(T_2) < \gamma(T)$. Since T_1 is a supergraph of $T - vv_1$, and since $\gamma(T - vv_1) \geq \gamma(T) - 1$, we have $\gamma(T_1) = \gamma(T) - 1$, and similarly $\gamma(T_2) = \gamma(T) - 1$. Let D, D_1, and D_2 be minimum dominating sets of T, T_1, and T_2, respectively. Since u is adjacent to an end vertex, we may assume that u is in D, D_1 and D_2. Also $v \notin D_1$ since otherwise D_1 would be a dominating set of T of cardinality $|D| - 1$. Similarly $v \notin D_2$.

For $i = 1, 2$ let V_i be the vertex set of the component of $T - vv_i$ containing v_i. For $i = 1, 2$ let $A_i = V_1 \cap D_i$, $B_i = V_2 \cap D_i$, and $C_i = D_i \cap (V(T) - (V_1 \cup V_2))$. Clearly, $B_1 \cup C_1$ dominates the component of $T_1 - uv_1 = T - vv_1$ containing v, and A_2 dominates the component of $T_2 - vv_1$ containing v_1, which is the component of $T - vv_1$ containing v_1. Hence $A_2 \cup B_1 \cup C_1$ is a dominating set of T. Similarly, $A_1 \cup B_2 \cup C_2$ is a dominating set of T. Hence we obtain the contradiction

$$2\gamma(T) \leq |A_2 \cup B_1 \cup C_1| + |A_1 \cup B_2 \cup C_2| \leq |A_1| + |B_1| + |C_1| + |A_2| + |B_2| + |C_2| = \gamma(T_1) + \gamma(T_2),$$

and the lemma follows. □

Theorem 3.5. *Let G be a connected graph of order n and domination number γ, where $\gamma \geq \frac{n}{3}$. Then*

$$\zeta(G) \leq \zeta(T_{n,\gamma}),$$

where $T_{n,\gamma}$ is the tree of order n obtained from a path of order $2n - 3\gamma$ by appending an end vertex to each of the first $\lfloor \frac{3\gamma - n}{2} \rfloor$ vertices and each of the last $\lceil \frac{3\gamma - n}{2} \rceil$ vertices.

Proof. It is easy to verify that $\zeta(T_{n,\gamma})$ is decreasing for fixed n. Hence it suffices to show that $\zeta(G) \leq \zeta(T_{n,\gamma})$ for every connected graph of order n and with $\gamma(G) \geq \gamma$.

Let G be a graph of order n and domination number at least γ for which $\zeta(G)$ is maximum. Since G has a spanning tree with the same domination number we may assume that G is a tree. Let d be the diameter of G. Let T_0 be the tree obtained from G by removing all end vertices, so $\mathrm{diam}(T_0) = d - 2$, let x, y be two diametral vertices of T_0 and let P be a path joining them.

Claim 1. *Every vertex of T_0 is adjacent to at most one end vertex of G.*

Suppose to the contrary that a vertex v of T_0 is adjacent to two end vertices v_1, v_2 of G. Then the graph $G - vv_2 + v_1v_2$ has domination number at least γ, but since the eccentricity of v_2 is greater in the new graph, we have $\zeta(G - vv_1 + v_1v_2) > \zeta(G)$, a contradiction to the maximality of $\zeta(G)$.

Claim 2. *Every vertex of degree at least 3 is on P.*

Suppose to the contrary that there exists a vertex v not on P of degree at least 3. Let v_0 be the neighbour of v on the path from v to P, and let v_1 and v_2 be two other neighbours of v. By the construction of T_0, vertex x is adjacent to an end vertex of G. Let T_i be the graph $G - vv_i + xv_i$ for $i = 1, 2$. By Lemma 3.1 we have $\gamma(T_i) \geq \gamma(G)$ for some $i \in \{1, 2\}$, and without loss of generality $\gamma(T_1) \geq \gamma(G)$. Let V_1 be the vertex set of the component of $G - vv_1$ containing v_1. Now $\mathrm{ec}_G(x) = d - 1$ and so $\mathrm{ec}_{T_1}(w) \geq d$ for all $w \in V_1$, with equality only for $w = v_1$. So $\mathrm{ec}_{T_1}(w) > \mathrm{ec}_G(w)$ for all $w \in V_1 - \{v_1\}$, while $\mathrm{ec}_{T_1}(w) \geq \mathrm{ec}_G(w)$ for all $w \in V(G) - (V_1 - \{v_1\})$, and we have $\zeta(T_1) > \zeta(G)$, a contradiction to G having maximum average eccentricity. Hence Claim 1 follows.

Claim 3. *T_0 is a path.*

Suppose to the contrary that T_0 is not a path. Then P contains a vertex v of degree at least 3 in T_0. Let v be such a vertex closest to y on P. Let v_1 and v_2 be the neighbours of v on P closest to x and y, respectively, and let v_0 be a third neighbour of v in T_0. Let u be an end vertex of T_0 in the component of $G - vv_0$ containing v_0. Then u is adjacent to an end vertex u' of G. Let T_i be the tree $G - vv_i + v_iu$ for $i = 1, 2$. By Lemma 3.1 the tree T_i satisfies $\gamma(T_i) \geq \gamma(G)$ for some $i \in \{1, 2\}$.

We now show that $\zeta(T_i) > \zeta(G)$. Denote the end vertices of G adjacent to x and y by x' and y', respectively. Let A be the set of vertices of the (x', y')-path, and let B be the set of vertices on the (v_0, u)-path. Denote the set of end vertices of G adjacent to some vertex in B and distinct from u' by B'. Let $D = V(G) - (A \cup B \cup B')$.

The vertices of A induce a diametral path in G, say, $w_0, w_1, \ldots, w_d$. Hence it follows from Lemma 2.1 that

$$\sum_{w \in A} \mathrm{ec}_G(w) = \sum_{i=1}^{d} \mathrm{ec}_G(w_i) = \sum_{i=0}^{d} \max(i, d - i) \leq \frac{3}{4}(d + 1)^2 - \frac{1}{2}(d + 1). \tag{3}$$

Since $\mathrm{ec}_G(u') \leq \mathrm{diam}(G) = d$, it follows that the vertices on the path from u' to P have eccentricities at most $d, d-1, d-2, \ldots$ and so

$$\sum_{w \in B} \mathrm{ec}_G(w) \leq \sum_{i=1}^{|B|} (d - i) = \frac{d(d - 1)}{2} - \frac{(d - |B| - 1)(d - |B|)}{2}. \tag{4}$$

Now the graph $T_i[A \cup B]$ induces a path of order $d + 1 + |B|$, and hence

$$\sum_{w \in A \cup B} \mathrm{ec}_{T_i}(w) \geq \zeta(P_{d+1+|B|}) \geq \frac{3}{4}(d + 1 + |B|)^2 - \frac{1}{2}(d + 1 + |B|) - \frac{1}{4}. \tag{5}$$

By Claim 2 the vertices of $B - \{u\}$ all have degree 2 in T_0, so the component of $T_0 - vv_0$ containing v_0 consists just of a path with vertex set $B \cup \{u'\}$. By Claim 1 each of the vertices in B is adjacent to at most one end vertex in G, so $|B'| \leq |B|$. For each $w \in B'$, the distance between w and any other vertex in G and T_1 differ by at most $|B|$. Hence

$$\sum_{w \in B'} \mathrm{ec}_{T_1}(w) - \sum_{w \in B'} \mathrm{ec}_G(w) \geq -|B'| \cdot |B| \geq -|B|^2. \tag{6}$$

For each vertex $w \in V - (A \cup B \cup B')$ we have $\mathrm{ec}_G(w) - \mathrm{ec}_{T_i}(w) \leq 0$, and so

$$\sum_{w \in V - (A \cup B \cup B')} \mathrm{ec}_{T_i}(w) - \sum_{w \in V - (A \cup B \cup B')} \mathrm{ec}_G(w) \geq 0. \tag{7}$$

Adding inequalities (5)–(7) and subtracting (3), (4) we obtain that

$$\zeta(T_i) - \zeta(G) \geq \frac{1}{4}|B|^2 + \frac{1}{2}|B|d + \frac{3}{2}|B| - \frac{1}{4} > 0,$$

a contradiction which concludes the proof of Claim 3.

Claim 4. $G = T_{n,\gamma}$.

It follows from Claims 1 and 3 that G consists of a path P of length $d-2$ with additional end vertices, such that each vertex on P is adjacent to at most one end vertex, and each end of P is adjacent to an end vertex. We assume d is odd; the proof for even d is similar. Label the vertices of P as $v_1, v_2, \ldots, v_{(d-1)/2}, w_{(d-1)/2}, w_{(d-2)/2}, \ldots, w_2, w_1$. We claim that there exist integers j_1, j_2 such that v_i is adjacent to an end vertex if $i \leq j_1$ and not adjacent to an end vertex if $i > j_1$, and w_i is adjacent to an end vertex if $i \leq j_2$ and not adjacent to an end vertex if $i > j_2$. Suppose not. Then there exist integers a and b such that, say, v_a and v_b are adjacent to an end vertex, and $v_{a+1}, v_{a+2}, \ldots, v_{b-1}$ are not. It is easy to verify that moving the end vertex adjacent to v_b to v_{a+1} increases its eccentricity from $d+1-b$ to $d-a$, and so $\zeta(G)$ is increased while $\gamma(G)$ remains unchanged, contradicting the maximality of $\zeta(G)$. Hence there exists j_1, and similarly j_2, as defined above. A similar argument shows that $|j_1 - j_2| \leq 1$. It is easy to verify that the graph G satisfies $n = d + 1 + j_1 + j_2$ and $\gamma = j_1 + j_2 + \lceil \frac{d+1-j_1-j_2-2}{3} \rceil$ and hence $G = T_{n,\gamma}$. □

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