Approximate Closest Community Search in Networks

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ABSTRACT

Recently, there has been significant interest in the study of the community search problem in social and information networks: given one or more query nodes, find densely connected communities containing the query nodes. However, most existing studies do not address the "free rider" issue, that is, nodes far away from query nodes and irrelevant to them are included in the detected community. Some state-of-the-art models have attempted to address this issue, but not only are their formulated problems NP-hard, they do not admit any approximations without restrictive assumptions, which may not always hold in practice.

In this paper, given an undirected graph G and a set of query nodes Q, we study community search using the k-truss based community model. We formulate our problem of finding a *closest truss community* (CTC), as finding a connected k-truss subgraph with the largest k that contains Q, and has the minimum diameter among such subgraphs. We prove this problem is NP-hard. Furthermore, it is NP-hard to approximate the problem within a factor $(2-\varepsilon)$, for any $\varepsilon > 0$. However, we develop a greedy algorithmic framework, which first finds a CTC containing Q, and then iteratively removes the furthest nodes from Q, from the graph. The method achieves 2-approximation to the optimal solution. To further improve the efficiency, we make use of a compact truss index and develop efficient algorithms for k-truss identification and maintenance as nodes get eliminated. In addition, using bulk deletion optimization and local exploration strategies, we propose two more efficient algorithms. One of them trades some approximation quality for efficiency while the other is a very efficient heuristic. Extensive experiments on 6 real-world networks show the effectiveness and efficiency of our community model and search algorithms.

1. INTRODUCTION

Community structures naturally exist in many real-world networks such as social, biological, collaboration, and communication networks. The task of community detection is to identify all communities in a network, which is a fundamental and well-studied problem in the literature. Recently, several papers have studied a related but different problem called *community search*, which is to find the community containing a given set of query nodes. The need for community search naturally arises in many real application scenarios, where one is motivated by the discovery of the communities in which given query nodes participate. Since the communities defined by different nodes in a network may be quite different, community search with query nodes opens up the prospects of user-centered and personalized search, with the potential of the answers being more meaningful to a user [13]. As just one example, in a social network, the community formed by a person's high school classmates can be significantly different from the community formed by her family members which in turn can be quite different from the one formed by her colleagues [17].

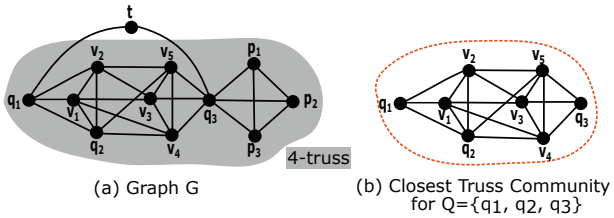


Figure 1: Closest truss community example

Various community models have been proposed based on different dense subgraph structures such as k-core [23, 10, 16] , k-truss [13], quasi-clique [9], weighted densest subgraph [28], to name a few major examples. Of these, the k-truss as a definition of cohesive subgraph of a graph G, requires that each edge be contained in at least $(k-2)$ triangles within this subgraph. Consider the graph G in Figure 1, in the subgraph in the whole grey region (i.e., excluding the node t), each edge is contained in two triangles. Thus, it is a 4-truss. It is well known that most of real-world social networks are triangle-based, which always have high local clustering coefficient. Triangles are known as the fundamental building blocks of networks [25]. In a social network, a triangle indicates two friends have a common friend, which shows a strong and stable relationship among three friends. Intuitively, the more common friends two people have, the stronger their relationship. In a k-truss, each pair of friends is "endorsed" by at least $(k-2)$ common friends. Thus, a k-truss with a large value of k signifies strong inner-connections between members of the subgraph. Huang et al. [13] proposed a community model based on the notion of k-truss as follows. Given *one* query node q and a parameter k, a k-truss community containing q is a maximal k-truss containing q, in which each edge is "triangle connected" with other edges. Triangle connectivity is strictly stronger than connectivity. The k-truss community model works well to find all overlapping communities containing a query node q. It is natural to search for communities containing a *set* of query nodes in real applications, and the above community model, extended for multiple query nodes, has

the following limitations. Due to the strict requirement of triangle connectivity constraint, the model may fail to discover any community for query nodes. For example, for the graph of Figure 1(a), and query nodes $Q = \{v_4, q_3, p_1\}$ the above k-truss community model cannot find a qualified community for any k, since the edges (v_4, q_3) and (q_3, p_1) are not triangle connected in any k-truss. A detailed comparison of various community search models and techniques can be found in the Section 7.

In this paper, we study the problem of *close community search*, i.e., given a set of query nodes, find a dense connected subgraph that contains the query nodes, in which nodes are close to each other. As a qualifying cohesive structure, we use the notion of k-truss for modeling a densely connected community, which inherits several good structural properties, such as k-edge connectivity, bounded diameter and hierarchical structure. In addition, to ensure every node included in the community is tightly related to query nodes and other nodes, we use graph diameter to measure the closeness of all nodes in the community. Thus, based on k-truss and graph diameter, we propose a novel community model as **closest truss community (CTC)**, which requires that the all query nodes are connected in this community, the graph structure is a k-truss with the largest trussness k. In general, several such candidate communities may exist. Some of them may suffer from the so-called "free rider effect" formally defined and studied in [28]. While we discuss this in detail in Section 3.2, we illustrate it with an example here. In Figure 1(a), for the query nodes $\{q_1, q_2, q_3\}$, the subgraph shaded grey is a 4-truss containing the query nodes. It includes the nodes p_1, p_2, p_3 which are intuitively not relevant to the query nodes. Specifically, p_2 is far away from q_1. This 4-truss is said to suffer from the free rider effect. On the other hand, the subgraph without the nodes $\{p_1, p_2, p_3\}$ is also a 4-truss, it has the smallest diameter among all 4-trusses containing the query nodes, and does not suffer from the free rider effect. Motivated by this, we define a closest truss community as a connected k-truss with the largest k containing the query nodes and having the smallest diameter. We show that such a definition avoids the free rider effect. A connected k-truss with the largest k containing given query nodes can be found in polynomial time. However, we show that finding such a k-truss with the minimum diameter is NP-hard and it hard to approximate within a factor better than 2. Here, the approximation is w.r.t. the minimum diameter. On the other hand, we develop a greedy strategy for finding a CTC that delivers a 2-approximation to the optimal solution, thus essentially matching the lower bound. In order to make our algorithm scalable to large real networks, we propose two techniques. One of them is based on bulk deletion of nodes far away from query nodes. The second is a heuristic exploration of the local neighborhood of a Steiner tree containing the query nodes. The challenge here is that a naive application of Steiner trees may yield a k-truss with a low value of k, which is undesirable. We address this challenge by developing a new notion of distances based on edge trussness. Specifically, we make the following contributions in this paper.

- We propose a novel community search model called *closest truss community* (CTC) and motivate the problem of finding CTC containing given query nodes (Section 2).
- We analyze the structural and computational properties of CTC and show that it avoids the free rider effect, is NP-hard to compute exactly or to approximate within a factor of $(2 - \varepsilon)$, for any $\varepsilon > 0$ (Section 3).
- We develop a greedy 2-approximation algorithm for finding a CTC given a set of query nodes. The algorithm is based on finding, in linear time, a connected k-truss with maximum k containing the query nodes, using a simple truss index. Then successively nodes far away from the query nodes are eliminated (Section 4).
- We further speed up CTC search in two ways: (1) we make use of a clever bulk deletion strategy and (2) find a Steiner tree of the query nodes and expand it into a k-truss by exploring the local neighborhood of the Steiner tree. The first of these slightly degrades the approximation factor while the second is a heuristic (Section 5).
- We extensively experiment with the various algorithms on 6 real networks. Our results show that our closest truss community model can efficiently and effectively discover the queried communities on real-world networks with ground-truth communities. (Section 6).

A detailed comparison of related work appears in Section 7. We summarize the paper in Section 8 and discuss future work.

Table 1: Frequently Used Notations

Notation	Description
$G = (V(G), E(G))$	An undirected and connected simple graph G
$n; m$	The number of vertices/edges in G
$N(v)$	The set of neighbors of v
$\sup_H(e)$	The *support* of edge e in H
$\tau(H)$	Trussness of graph H
$\tau(e)$	Trussness of edge e
$\tau(v)$	Trussness of vertex v
$\bar{\tau}(S)$	The maximum trussness of connected graphs containing S
$\mathsf{diam}(H)$	The diameter of graph H
$\mathsf{dist}_H(v, u)$	The shortest distance between v and u in H
$\mathsf{dist}_H(R, Q)$	$\mathsf{dist}_H(R, Q) = \max_{v \in R, u \in Q} \mathsf{dist}_H(v, u)$

2. PROBLEM DEFINITION

We consider an undirected, unweighted simple graph $G = (V(G), E(G))$ with $n = |V(G)|$ vertices and $m = |E(G)|$ edges. We denote the set of neighbors of a vertex v by $N(v)$, i.e., $N(v) = \{u \in V : (v, u) \in E\}$, and the degree of v by $d(v) = |N(v)|$. We use $d_{max} = \max_{v \in V} d(v)$ to denote the maximum vertex degree in G. W.l.o.g we assume in this paper that the graph G we consider is connected. Note that this implies that $m \geq n - 1$. Table 1 summarizes the frequently used notations in the paper.

A *triangle* in G is a cycle of length 3. Let $u, v, w \in V$ be the three vertices on the cycle, then we denote this triangle by $\triangle_{uvw}$. The *support* of an edge $e(u, v) \in E$ in G, denoted $sup_G(e)$, is defined as $|\{\triangle_{uvw} : w \in V\}|$. When the context is obvious, we drop the subscript and denote the support as $sup(e)$. Based on the definition of k-truss [7, 25], we define a connected k-truss as follows.

DEFINITION 1 (CONNECTED K-TRUSS). *Given a graph G and an integer k, a connected k-truss is a connected subgraph $H \subseteq G$, such that $\forall e \in E(H)$, $sup_H(e) \geq (k - 2)$.*

Intuitively, a connected k-truss is a connected subgraph such that each edge (u, v) in the subgraph is "endorsed" by $k - 2$ common neighbors of u and v [7]. In a connected k-truss graph, each node has degree at least $k - 1$ and a connected k-truss is also a $(k - 1)$-core [2]. Next, we define the *trussness* of a subgraph, an edge, and a vertex as follows.

DEFINITION 2 (TRUSSNESS). *The trussness of a subgraph $H \subseteq G$ is the minimum support of an edge in H plus 2, i.e., $\tau(H) = 2 + \min_{e \in E(H)}\{sup_H(e)\}$. The trussness of an edge $e \in E(G)$ is $\tau(e) = \max_{H \subseteq G \wedge e \in E(H)}\{\tau(H)\}$. The trussness of a vertex $v \in V(G)$ is $\tau(v) = \max_{H \subseteq G \wedge v \in V(H)} \{\tau(H)\}$.*

Consider the graph G in Figure 1(a). Edge $e(q_2, v_2)$ is contained in three triangles $\triangle_{q_2v_2q_1}$, $\triangle_{q_2v_2v_1}$ and $\triangle_{q_2v_2v_5}$, thus its support is $sup_G(e(q_2, v_2)) = 3$. Suppose H is the triangle $\triangle_{q_2v_2q_1}$, then the trussness of the subgraph H is $\tau(H) = 2+\min_{e\in H} sup_H(e) = 3$, since each edge is contained in one triangle in H. The trussness of the edge $e(q_2, v_2)$ is 4, because in the induced subgraph on vertices $\{q_1, q_2, v_1, v_2\}$, each edge is contained in two triangles in the subgraph and any subgraph H containing $e(q_2, v_2)$ has $\tau(H) \leq 4$, i.e., $\tau(e(q_2, v_2)) = \max_{H\subseteq G\wedge e\in E(H)} \{\tau(H)\} = 4$. Note that the trussness of an edge e of a graph G could be less than $sup_G(e)+2$, e.g., $\tau(e(q_2, v_2)) = 4 < 5 = sup(e(q_2, v_2)) + 2$. Moreover, the vertex trussness of q_2 is also 4, i.e. $\tau(q_2) = 4$.

For a set of vertices $S \subseteq V(G)$, we use $\bar{\tau}(S)$ to denote the maximum trussness of a *conncted* subgraph H containing S, i.e., $\bar{\tau}(S) = \max_{S\subseteq H\subseteq G\wedge H \text{is connected}} \{\tau(H)\}$. Notice that by definition, for $S = \emptyset$, $\bar{\tau}(\emptyset)$ is the maximum trussness of any edge in G. In Figure 1(a), the whole subgraph in the grey region is a 4-truss. There exists no 5-truss in G, and $\bar{\tau}(\emptyset) = 4$. We will make use of $\bar{\tau}(\emptyset)$ in Section 5.

For two nodes $u, v \in G$, we denote by $\mathsf{dist}_G(u, v)$ the length of the shortest path between u and v in G, where $\mathsf{dist}_G(u, v) = +\infty$ if u and v are not connected.

DEFINITION 3 (GRAPH DIAMETER). *The diameter of a graph G is defined as the maximum length of a shortest path in G, i.e.,* $\mathsf{diam}(G) = \max_{u,v\in G}\{\mathsf{dist}_G(u, v)\}$.

For the graph H in Figure 1(b), the shortest path between q_1 and q_3 is $\langle(q_1, v_1), (v_1, v_3), (v_3, q_3)\rangle$, and $\mathsf{dist}_H(q_1, q_3) = 3$. This is also the longest shortest path in H, so $\mathsf{diam}(H) = 3$.

On the basis of the definitions of k-truss and graph diameter, we define the *closest truss community* on a graph G as follows.

DEFINITION 4 (CLOSEST TRUSS COMMUNITY). *Given a graph G and a set of query nodes Q, G' is a closest truss community (CTC), if G' satisfies the following two conditions:*

(1) Connected k-Truss. *G' is a connected k-truss containing Q with the largest k, i.e., $Q \subseteq G' \subseteq G$ and $\forall e \in E(G')$, $sup(e) \geq k - 2$;*

(2) Smallest Diameter. *G' is a subgraph of smallest diameter satisfying condition (1). That is, $\nexists G'' \subseteq G$, such that $diam(G'') < diam(G')$, and G'' satisfies condition (1).*

Condition (1) requires that the closest community containing the query nodes Q be densely connected. In addition, Condition (2) makes sure that each node is as close as possible to every query node in the community.

EXAMPLE 1. Definition 4 firstly considers the connected k-truss of G containing query nodes with the largest trussness, and then among of all such subgraphs, the smallest diameter one is regarded as the closest truss community. Consider the graph G in Figure 1(a), and $Q = \{q_1, q_2, q_3\}$, the subgraph in the region shaded grey is a 4-truss containing Q, and is the subgraph containing Q with the largest trussness. Then, we can discover that the closest truss community for Q that has smallest diameter 3, is shown in Figure 1(b). In addition, we can see that in Figure 1(a), even though the nodes p_1, p_2, p_3 are also in the 4-truss and are strongly connected with q_3, they are far away from the other two query nodes q_1 and q_2. By Condition (2) of Definition 4, we can see that the 4-truss graph in Figure 1(a), is not considered the closest truss community, because its diameter of 4 is larger than the diameter of the community in Figure 1(b). Hence, we do not consider that these nodes belong to the closest truss community for Q. We will see in Section 3.2 that the definition of CTC above avoids the so-called "free rider effect".

In contrast, consider the opposite situation of Definition 4, where we first minimize the diameter among connected subgraphs of G containing Q and look for the k-truss subgraph with the largest k among those. Firstly, the circle of $\{(q_1, t), (t, q_3), (q_3, v_4), (v_4, q_2), (q_2, q_1)\}$, which is the connected subgraph containing Q with the smallest diameter 2. Thus, we can find that this circle is the k-truss subgraph with the largest k containing itself, which is only a 2-truss with loosely connected structure. □

The problem of **closest truss community (CTC) search** studied in this paper is defined as follows.

PROBLEM 1 (CTC-Problem). *Given a graph $G(V, E)$ and a set of query vertices $Q = \{v_1, ..., v_r\} \subseteq V$, find a closest truss community containing Q.*

3. PROBLEM ANALYSIS

3.1 Structural Properties

Since our closest truss community model is based on the concept of k-truss, the communities caputure good structral properties of k-truss, such as *k-edge-connected* and *hierarchical structure*. In addition, since CTC is required to have minimum diameter, it also has *bounded diameter*. As a result, CTC avoids the "free rider effect" [23, 28] (see Section 3.2).

Small diameter, k-edge-connected, hierarchical structure. First, the diameter of a connected k-truss with n vertices is no more than $\lfloor\frac{2n-2}{k}\rfloor$ [7]. The diameter of a community is considered as an important feature of a good community [11]. Moreover, a k-truss community is $(k-1)$-edge-connected [7], as it remains connected whenever fewer than $k-1$ edges are removed [12]. In addition, k-truss based community has *hierarchical structure* that represents the cores of a community at different levels of granularity [13], that is, k-truss is always contained in the $(k-1)$-truss for any $k \geq 3$.

Largest k. We have a trivial upper bound on the maximum possible trussness of a connected k-truss containing the query nodes.

LEMMA 1. *For a connected k-truss H satisfying definition of CTC for Q, we have $k \leq \min\{\tau(q_1), ..., \tau(q_r)\}$ holds.*

PROOF. First, we have $Q \subseteq H$. For each node $q \in Q$, q cannot be contained in a k-truss in G, whenever $k > \tau(q)$. Thus, the fact that H is a k-truss subgraph containing Q implies that $k \leq \min\{\tau(q_1), ..., \tau(q_r)\}$. □

3.2 Free Rider Effect

In previous work on community detection, researchers [23, 28] have identified an undesirable phenomenon called "free rider effect". Intuitively, if a definition of community admits irrelevant subgraphs in the detected community, we refer to such irrelevant subgraphs as free riders. As an example, suppose we define a closest truss community simply as a k-truss with the largest trussness that is connected and contains the query nodes. In Figure 1(a), for the query nodes $Q = \{q_1, q_2, q_3\}$, the entire subgraph shaded in grey (i.e., excluding node t) is a connected 4-truss containing Q and has the maximum truss among such subgraphs. However, the subgraph induced by the nodes $\{p_1, p_2, p_3\}$ is far away from q_1. Removing these nodes leads to the subgraph in Figure 1(b) which still is a connected 4-truss where each node is closer to the query nodes. Thus, $\{p_1, p_2, p_3\}$ are free riders, which makes no contributions to graph trussness. Following [23, 28], we define the free rider effect as follows. Typically, a community definition is based on a goodness metric $f(H)$ for a subgraph H and subgraphs with

minimum[1] value $f(H)$ are defined as communities. E.g., for our CTC problem, diameter is the goodness: the smaller the diameter of H, the better it is as a community. The definition of free rider effect is based on this goodness metric.

DEFINITION 5 (FRE). *Let H be a solution to a community definition based on a goodness metric $f(.)$. We say that the definition suffers from free rider effect, provided whenever there is an optimal solution H^* to the community detection problem, then $f(H \cup H^*) \leq f(H)$, i.e., the union of the subgraphs H and H^* is no worse a solution than H.*

We next show that our definition of CTC avoids the problem of free rider effect.

PROPOSITION 1. *For any graph G and query nodes $Q \subset V(G)$, there is a solution H to the CTC search problem such that for all optimal solutions H^*, either $H \cup H^*$ is not disconnected or has a strictly larger diameter than H.*

PROOF. Let $\mathcal{C}(G, Q)$ denote the set of optimal solutions to the CTC search problem on graph G and query nodes Q. $\mathcal{C}(G, Q)$ is partially ordered w.r.t. graph containment order $\subseteq$. Let H be any maximal element of $\mathcal{C}(G, Q)$, let H^* be any optimal solution in $\mathcal{C}(G, Q)$ and consider $H \cup H^*$. Suppose $H \cup H^*$ is a connected k-truss with maximum trussness containing Q, and $\mathsf{diam}(H \cup H^*) \leq \mathsf{diam}(H)$. This contradicts the maximality of H. □

3.3 Hardness and Approximation

Hardness. In the following, we show the CTC-Problem as NP-hard. Thereto, we define the decision version of the CTC-Problem.

PROBLEM 2 (CTCk-Problem). *Given a graph $G(V, E)$, a set of query nodes $Q = \{v_1, ..., v_r\} \subseteq V$ and parameters k and d, test whether G contains a connected k-truss subgraph with diameter at most d, that contains Q.*

THEOREM 1. *The* CTCk-Problem *is **NP-hard**.*

PROOF. We reduce the well-known NP-hard problem of Maximum Clique (decision version) to CTCk-Problem. Given a graph $G(V, E)$ and number k, the Maximum Clique Decision problem is to check whether G contains a clique of size k. From this, construct an instance of CTCk-Problem, consisting of graph G, parameters k and $d = 1$, and the empty set of query nodes $Q = \emptyset$. We show that the instance of the Maximum Clique Decision problem is a YES-instance iff the corresponding instance of CTCk-Problem is a YES-instance. Clearly, any clique with at least k nodes is a connected k-truss with diameter 1. On the other hand, given a solution H for CTCk-Problem, H must contain at least k nodes since H is a k-truss, and $diam(H) = d = 1$, which implies H is a clique. □

The hardness of CTC-Problem follows from this. The next natural question is whether CTC-Problem can be approximated.

Approximation. For $\alpha \geq 1$, we say that an algorithm achieves an α-approximation to the closest truss community (CTC) search problem if it outputs a connected k-truss subgraph $H \subseteq G$ such that $Q \subseteq H$, $\tau(H) = \tau(H^*)$ and $\mathsf{diam}(H) \leq \alpha \cdot \mathsf{diam}(H^*)$, where H^* is the optimal CTC. That is, H^* is a connected k-truss with the largest k s.t. $Q \subseteq H^*$, and $diam(H^*)$ is the minimum among all such CTCs containing Q. Notice that the trussness of the output subgraph H matches that of the optimal solution H^* and that the approximation is only w.r.t. the diameter: the diameter of H is required to be no more than $\alpha \cdot diam(H^*)$.

[1]We use minimum w.l.o.g.

Non-Approximability. We next prove that CTC-Problem cannot be approximated within a factor better than 2. We establish this result through a reduction, again from the Maximum Clique Decision problem to the problem of approximating CTC-Problem, given k. In the next section, we develop a 2-approximation algorithm for CTC-Problem, thus essentially matching this lower bound. Notice that the CTC-Problem with given parameter k is essentially the CTCk-Problem.

THEOREM 2. *Unless* P = NP, *for any $\varepsilon > 0$, the* CTC-Problem *with given parameter k cannot be approximated in polynomial time within a factor $(2 - \varepsilon)$ of the optimal.*

PROOF. Suppose there exists a polynomial time algorithm $\mathbb{A}$ for the CTC-Problem with a given k that provides a solution H with an approximation factor $(2 - \varepsilon)$ of the optimal solution H^*. Set the query nodes $Q = \emptyset$. By our assumption, we have $Q \subseteq H$, $\tau(H) = \tau(H^*) = k$ and $\mathsf{diam}(H) \leq (2 - \varepsilon) \cdot \mathsf{diam}(H^*)$. Next, we use this approximation solution to exactly solve the Maximum Clique Decision problem as follows. Since the latter cannot be done in polynomial time unless P = NP, the theorem follows.

Run algorithm $\mathbb{A}$ on a given instance G of the Maximum Clique Decision problem, with parameter k and query nodes $Q = \emptyset$. We claim that G contains a clique of size k iff $\mathbb{A}$ outputs a solution H with $\tau(H) = k$ and $diam(H) = 1$. To see this, suppose $diam(H) = 1$, then the optimal solution H^* has $\mathsf{diam}(H^*) \leq \mathsf{diam}(H) = 1$, and H^* is a connected k-truss, which shows H^* is a clique of size k in G. On the other hand, suppose $diam(H) \geq 2$. Then we have $2 \cdot \mathsf{diam}(H^*) > (2-\varepsilon) \cdot \mathsf{diam}(H^*) \geq \mathsf{diam}(H) \geq 2$. Since diameter is an integer, we deduce that $diam(H^*) \geq 2$. In this case, G cannot possibly contain a clique of size k, for if it did, that clique would be the optimal solution to the CTC-Problem on G, with parameter k, whose diameter is 1, which contradicts the optimality of H^*. Thus, using algorithm $\mathbb{A}$, we can distinguish between the YES and NO instances of the Maximum Clique Decision problem. This was to be shown. □

4. ALGORITHMS

In this section, we present a greedy algorithm called Basic for the CTC search problem. Then, we show that this algorithm achieves a 2-approximation to the optimal result. Finally, we discuss procedures for an efficient implementation of the algorithm and analyze its time and space complexity.

4.1 Basic Algorithmic Framework

Here is an overview of our algorithm Basic. First, given a graph G and query nodes Q, we find a maximal connected k-truss, say G_0, containing Q and having the largest trussness. In general G_0 can have a large diameter so we iteratively remove nodes far away from the query nodes, while maintaining the trussness of the remainder graph at k.

Query distance. Before describing the algorithm in detail, we first define query distance, based on the shortest distance from query nodes.

DEFINITION 6 (QUERY DISTANCE). *Given a graph G and a set of query nodes $Q \subset V$, for each vertex $v \in G$, the vertex query distance of v is the maximum length of a shortest path from v to a query node $q \in Q$, i.e., $\mathsf{dist}_G(v, Q) = \max_{q \in Q} \mathsf{dist}_G(v, q)$. For a subgraph $H \subseteq G$, the graph query distance of H is defined as $\mathsf{dist}_G(H, Q) = \max_{u \in H} \mathsf{dist}_G(u, Q) = \max_{u \in H, q \in Q} \mathsf{dist}_G(u, q)$.*

EXAMPLE 2. For the graph G in Figure 1(a) and $Q = \{q_2, q_3\}$, the vertex query distance $\mathsf{dist}_G(v_2, Q) = \max_{q \in Q} \{dist_G(v_2, q)\} = 2$, since $\mathsf{dist}_G(v_2, q_3) = 2$ and $\mathsf{dist}_G(v_2, q_2) = 1$. The graph query distance $\mathsf{dist}_G(G, Q) = \max_{u \in G} \mathsf{dist}_G(u, Q) = \mathsf{dist}(p_1, q_2) = 3$. The diameter of G is $\mathsf{diam}(G) = 4$. □

Since the distance function satisfies the triangular inequality, i.e., for all nodes u, v, w, $\mathsf{dist}_G(u, v) \leq \mathsf{dist}_G(u, w) + \mathsf{dist}_G(w, v)$, we can express the lower and upper bounds on the graph diameter in terms of the query distances as follows.

LEMMA 2. *For a graph $G(V, E)$ and a set of nodes $Q \subseteq G$, we have $\mathsf{dist}_G(G, Q) \leq \mathsf{diam}(G) \leq 2\mathsf{dist}_G(G, Q)$.*

PROOF. First, the diameter $\mathsf{diam}(G) = \max_{v,u \in G} \mathsf{dist}_G(v, u)$, which is clearly no less than than $\mathsf{dist}_G(G, Q) = \max_{v \in G, q \in Q} \mathsf{dist}_G(v, q)$ for $Q \subseteq G$. Thus, $\mathsf{dist}_G(G, Q) \leq \mathsf{diam}(G)$.

Second, suppose that the longest shortest path in G is between v and u. Then $\forall q \in Q$, then we have $\mathsf{diam}(G) = \mathsf{dist}(v, u) \leq \mathsf{dist}(v, q) + \mathsf{dist}(q, u) \leq 2\mathsf{dist}_G(G, Q)$. The lemma follows. □

Algorithm. Algorithm 1 outlines a framework for finding a closest truss community based on a greedy strategy. For query nodes Q, we first find a maximal connected k-truss G_0 that contains Q, s.t. $k = \tau(G_0)$ is the largest (line 1). Then, we set $l = 0$. For all $u \in G_l$ and $q \in Q$, we compute the shortest distance between u and q (line 4), and obtain the vertex query distance $\mathsf{dist}_{G_l}(u, Q)$. Among all vertices, we pick up a vertex u^* with the maximum $\mathsf{dist}_{G_l}(u^*, Q)$, which is also the graph query distance $\mathsf{dist}_{G_l}(G_l, Q)$ (lines 5-6). Next, we remove the vertex u^* and its incident edges from G_l, and delete any nodes and edges needed to restore the k-truss property of G_l (lines 7-8). We assign the updated graph as a new G_l. Then, we repeat the above steps until G_l does not have a connected subgraph containing Q (lines 3-9). Finally, we terminate by outputting graph R as the closest truss community, where R is any graph $G' \in \{G_0, ..., G_{l-1}\}$ with the smallest graph query distance $\mathsf{dist}_{G'}(G', Q)$ (line 10). Note that each intermediate graph $G' \in \{G_0, ..., G_{l-1}\}$ is a k-truss with the maximum trussness as required.

EXAMPLE 3. *Continuing with the above example, we apply Algorithm 1 on G in Figure 1 for $Q = \{q_1, q_2, q_3\}$. First, we obtain the the 4-truss subgraph G_0 shaded in grey, using a procedure we will shortly explain. Then, we compute all shortest distances, and get the maximum vertex query distance as $\mathsf{dist}_{G_0}(p_1, Q) = 4$, and $u^* = p_1$. We delete the node p_1 and its incident edges from G_0; we also delete p_2 and p_3, in order to restore the 4-truss property. The resulting subgraph is G_1. Any further deletion of a node in the next iteration of the while loop will induce a series of deletions in line 8, eventually making the graph disconnected or containing just a part of query nodes. As a result, the output graph R, shown in Figure 1(b), is just G_1. Also $\mathsf{dist}_R(R, Q) = 3$, and R happens to be the exact CTC with diameter 3, which is optimal.*

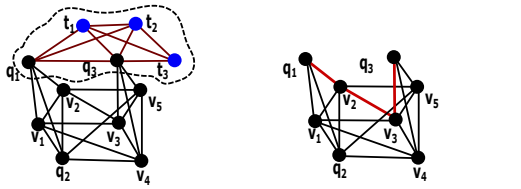
4.2 Approximation Analysis

Algorithm 1 can achieve 2-approxiamtion to the optimal solution, that is, the obtained connected k-truss community R satisfies $Q \subseteq R$, $\tau(R) = \tau(H^*)$ and $\mathsf{diam}(R) \leq 2\mathsf{diam}(H^*)$, for any optimal solution H^*. Since any graph in $\{G_0, ..., G_{l-1}\}$ is a conneted k-truss with the largest k containing Q by Algorithm 1, and $R \in \{G_0, ..., G_{l-1}\}$, we have $Q \subseteq R$, and $\tau(R) = \tau(H^*)$. In the following, we will prove that $\mathsf{diam}(R) \leq 2\mathsf{diam}(H^*)$. We start with a few key results. For graphs G_1, G_2, we write $G_1 \subseteq G_2$ to mean $V(G_1) \subseteq V(G_2)$ and $E(G_1) \subseteq E(G_2)$.

Algorithm 1 Basic (G, Q)

Input: A graph $G = (V, E)$, a set of query nodes $Q = \{q_1, ..., q_r\}$.
Output: A connected k-truss R with a small diameter.

1: Find a maximal connected k-truss containing Q with the largest k as G_0 //`see Algorithm 2`.
2: $l \leftarrow 0$;
3: **while** $\mathsf{connect}_{G_l}(Q) =$ **true do**
4: Compute $\mathsf{dist}_{G_l}(q, u), \forall q \in Q$ and $\forall u \in G_l$;
5: $u^* \leftarrow \arg\max_{u \in G_l} \mathsf{dist}_{G_l}(u, Q)$;
6: $\mathsf{dist}_{G_l}(G_l, Q) \leftarrow \mathsf{dist}_{G_l}(u^*, Q)$;
7: Delete u^* and its incident edges from G_l;
8: Maintain k-truss property of G_l //`see Algorithm 3`;
9: $G_{l+1} \leftarrow G_l$; $l \leftarrow l + 1$;
10: $R \leftarrow \arg\min_{G' \in \{G_0, ..., G_{l-1}\}} \mathsf{dist}_{G'}(G', Q)$;



(a) G_0 of 4-truss (b) G_1 after deleting t_3 by Alg.1

Figure 2: Closest truss community example

FACT 1. *Given two graphs G_1 and G_2 with $G_1 \subseteq G_2$, for $u, v \in V(G_1)$, $\mathsf{dist}_{G_2}(u, v) \leq \mathsf{dist}_{G_1}(u, v)$ holds. Moreover, if $Q \subseteq V(G_1)$, then $\mathsf{dist}_{G_2}(G_1, Q) \leq \mathsf{dist}_{G_1}(G_1, Q)$ also holds.*

PROOF. Trivially follows from the fact that G_2 preserves paths between nodes in G_1. □

Recall that in Algorithm 1, in each iteration i, a node u^* with maximum $\mathsf{dist}(u^*, Q)$ is deleted from G_i, but $\mathsf{dist}_{G_i}(G_i, Q)$ is *not* monotone nonincreasing during the process, hence $\mathsf{dist}_{G_{l-1}}(G_{l-1}, Q)$ is not necessarily the minimum. Note that in Algorithm 1, G_l is not the last feasible graph (i.e., connected k-truss containing Q), but G_{l-1} is. The observation is shown in the following lemma.

LEMMA 3. *In Algorithm 1, it is possible that for some $0 \leq i < j < l$, we have $G_j \subset G_i$, and $\mathsf{dist}_{G_i}(G_i, Q) < \mathsf{dist}_{G_j}(G_j, Q)$ hold.*

PROOF. It is easily to be realized, because for a vertex $v \in G$, $\mathsf{dist}_G(v, Q)$ is non-decreasing monotone w.r.t. subgraphs of G. More precisely, for $v \in G_i \cap G_j$, $\mathsf{dist}_{G_i}(v, Q) \leq \mathsf{dist}_{G_j}(v, Q)$ holds. □

EXAMPLE 4. To illustrate the lemma, suppose the graph in Figure 2(a) is G_0, a connected 4-truss containing the query nodes $Q = \{q_1\}$ in some initial graph G (not shown) and suppose the maximum trussness of such a subgraph is 4. One of furthest nodes from Q in G_0 is t_3, which has query distance $\mathsf{dist}_{G_0}(t_3, Q) = 2$. After deleting the node t_3 from G_0, we remove the all incident edges of nodes t_1, t_2 and t_3, since the 4-truss subgraph induced by $\{q_1, q_3, t_1, t_2, t_3\}$ in the dashed region does not exist any more in G_1 in Figure 2. Thus, we have the largest query distance as $\mathsf{dist}_{G_1}(G_1, Q) = \mathsf{dist}_{G_1}(q_3, Q) = 3$, which is larger than $\mathsf{dist}_{G_0}(G_0, Q) = 2$. □

We have an important observation that if an intermediate graph G_i obtained by Algorithm 1 contains an optimal solution H^*, i.e., $H^* \subset G_i$and $\mathsf{dist}_{G_i}(G_i, Q) > \mathsf{dist}_{G_i}(H^*, Q)$, then algorithm will not terminate at G_{i+1}.

LEMMA 4. *In Algorithm 1, for any intermediate graph G_i, we have $H^* \subseteq G_i$, and $\mathsf{dist}_{G_i}(G_i, Q) > \mathsf{dist}_{G_i}(H^*, Q)$, then G_{i+1} is a connected k-truss containing Q and $H^* \subseteq G_{i+1}$.*

PROOF. Suppose $H^* \subseteq G_i$ and $\mathsf{dist}_{G_i}(G_i, Q) > \mathsf{dist}_{G_i}(H^*, Q)$. Then there exists a node $u \in G_i \backslash H^*$ s.t. $\mathsf{dist}_{G_i}(u, Q) = \mathsf{dist}_{G_i}(G_i, Q) > \mathsf{dist}_{G_i}(H^*, Q)$. Clearly, $u \notin Q$. In the next iteration, Algorithm 1 will delete u from G_i (Step 7), and perform Step 8. The graph resulting from restoring the k-truss property is G_{i+1}. Since H^* is a connected k-truss containing Q, the restoration step (line 8) must find a subgraph G_{i+1} s.t. $H^* \subseteq G_{i+1}$, and G_{i+1} is a connected k-truss containing Q. Thus, the algorithm will not terminate in iteration $(i+1)$. □

We are ready to establish the main result of this section.

LEMMA 5. $diam(R) \leq 2diam(H^*)$.

PROOF. First, suppose that the following statement holds:

$$\textbf{(}A_1\textbf{)} \quad \mathsf{dist}_R(R, Q) \leq \mathsf{dist}_{H^*}(H^*, Q).$$

Then we get $diam(R) \leq 2\mathsf{dist}_R(R, Q) \leq 2\mathsf{dist}_{H^*}(H^*, Q) \leq 2diam(H^*)$ by Lemma 2. The lemma follows from this. Now, we prove **(**A_1**)**. The following cases arise for G_{l-1}, which is the last feasible graph obtained by Algorithm 1.

Case (a): $H^* \subseteq G_{l-1}$. We have $\mathsf{dist}_{G_{l-1}}(G_{l-1}, Q) \leq \mathsf{dist}_{G_{l-1}}(H^*, Q)$; for otherwise, if $\mathsf{dist}_{G_{l-1}}(G_{l-1}, Q) > \mathsf{dist}_{G_{l-1}}(H^*, Q)$, we can deduce from Lemma 4 that G_{l-1} is not the last feasible graph obtained by Algorithm 1, a contradiction. Thus, according to Step 10 in Algorithm 1 and $\mathsf{dist}_{G_{l-1}}(G_{l-1}, Q) \leq \mathsf{dist}_{G_{l-1}}(H^*, Q)$, we have $\mathsf{dist}_R(R, Q) \leq \mathsf{dist}_{G_{l-1}}(G_{l-1}, Q) \leq \mathsf{dist}_{G_{l-1}}(H^*, Q) \leq \mathsf{dist}_{H^*}(H^*, Q)$.

Case (b): $H^* \nsubseteq G_{l-1}$. There exists a vertex $v \in H^*$ deleted from one of the subgraphs $\{G_0, ..., G_{l-2}\}$. Suppose the first deleted vertex $v^* \in H^*$ is in graph G_i, where $0 \leq i \leq l-2$, then v^* must be deleted in Step 7, but not in Step 8. This is because each vertex/edge of H^* satisfies the condition of k-truss, and will not be removed before any vertex is removed from G_i. Then, we have $\mathsf{dist}_{G_i}(G_i, Q) = \mathsf{dist}_{G_i}(v^*, Q) = \mathsf{dist}_{G_i}(H^*, Q)$, and $\mathsf{dist}_{G_i}(G_i, Q) \geq \mathsf{dist}_R(R, Q)$ by Step 10. As a result, $\mathsf{dist}_R(R, Q) \leq \mathsf{dist}_{G_i}(H^*, Q) \leq \mathsf{dist}_{H^*}(H^*, Q)$. □

Based on the preceding lemmas, we have:

THEOREM 3. *Algorithm 1 provides a 2-approximation to the* CTC-Problem.

4.3 K-truss Indentification and Maintenance

A key step in Algorithm 1 is finding G_0, a maximal connected k-truss containing Q and with the largest k (Step 1). In this section, we make use of an index structure from [13] for performing this step efficiently. The index is based on organizing edges in order of their trussness.

Index Construction. We first apply a truss decomposition algorithm such as [25] and compute the trussness of each edge of graph G. We omit the details of this algorithm due to space limitation.

Based on the obtained edge trussness, we construct our truss index as follows. For each vertex $v \in V$, we sort its neighbors $N(v)$ in descending order of the edge trussness $\tau(e(v, u))$, for $u \in N(v)$. For each distinct trussness value $k \geq 2$, we mark the position of the first vertex u in the sorted adjacency list where $\tau(e(u, v)) = k$. This supports efficient retrieval of v's incident edges with a certain trussness value. The vertex trussness of v is also kept as $\tau(v) = \max\{\tau(v, u) | u \in N(v)\}$, which is the trussness of first edge in the sorted adjacency list. Moreover, we also build a hashtable to keep all the edges and their trussness values. This is identical to the simple truss index of [13] and we refer to it as the truss index.

Algorithm 2 Find$G_0(G, Q)$

Input: A graph $G = (V, E)$, a set of query nodes $Q = \{q_1, ..., q_r\}$.
Output: A connected k-truss G_0 containing Q with the largest k.

```
1: k ← min{τ(q_1), ..., τ(q_r)} //see Lemma 1;
2: V(G_0) ← ∅; S_k = Q;
3: while connect_{G_0}(Q) = false do
4:     for v ∈ S_k do
5:         if v ∈ V(G_0) then
6:             k_max ← k + 1; V(G_0) ← V(G_0) ∪ {v};
7:         else
8:             k_max ← +∞;
9:         for (v, u) ∈ G with k ≤ τ(v, u) < k_max do
10:            G_0 ← G_0 ∪ {(v, u)};
11:            if u ∉ S_k then S_k ← S_k ∪ {u};
12:        l ← max{τ(v, u) | u ∉ G_0};
13:        S_l ← S_l ∪ {v};
14:    k ← k − 1;
15: Compute the edge support sup(v, u) in G_0, for all (v, u) ∈ G_0;
```

Algorithm 3 K-trussMaintenance (G, V_d)

Input: A graph $G = (V, E)$, a set of nodes to be removed as V_d.
Output: A k-truss graph.

```
1: S ← ∅;
2: for v ∈ V_d and (v, u) ∈ G do
3:     S ← S ∪ (v, u);
4: for (v, u) ∈ S do
5:     for w ∈ N(v) ∩ N(u) do
6:         sup(v, w) ← sup(v, w) − 1; sup(u, w) ← sup(u, w) − 1;
7:         if sup(v, w) < k − 2 and (v, w) ∉ S then S ← S ∪ (v, w);
8:         if sup(u, w) < k − 2 and (u, w) ∉ S then S ← S ∪ (u, w);
9:     Remove (v, u) from G;
10: Remove isolated vertices from G;
```

In the following, we will show that this truss index is sufficient to design an algorithm for finding the maximal connected k-truss containing given query nodes Q in time $O(m')$, where $m' = |E(G_0)|$. This time complexity is essentially *optimal*. We remark the complexity of this k-truss index construction below.

REMARK 1. *The construction of this truss index takes $O(\rho \cdot m)$ time and $O(m)$ space, where ρ is the arboricity of graph G, i.e., the minimum number of spanning forests needed to cover all edges of G. Notice that $\rho \leq \min\{d_{max}, \sqrt{m}\}$ [6].*

Find G_0. Based on the constructed index, we present Algorithm 2 to find G_0, the maximal connected k-truss containing Q with largest trussness.

We initialize G_0 to the empty graph and expand it level by level, from highest to lowest, where "level" corresponds to trussness. The initial level is computed as the $k = \min\{\tau(q_1), ..., \tau(q_r)\}$ (line 1). This is motivated by the fact that, by Lemma 1, for any $k' > k$, no connected k'-truss can contain Q. We use S_k to denote the set of nodes to be visited within level k. We start with $S_k = Q$ (line 2). For a given k, we process each node $v \in S_k$, and visit its neighbors in a BFS manner, and then insert those of its incident edges (v, u), with $k \leq \tau(v, u) \leq k_{max}$ into G_0. The truss index facilitates efficient access of v's neighbors based on their trussness. We use k_{max} to keep track of the maximum possible trussness of unvisited edges. Whenever the current node $v \in V(G_0)$ is a node newly added to G_0, we set $k_{max} = +\infty$; otherwise, we set $k_{max} = k+1$ (line 5-8). We only need to check the edges at the levels between k and k_{max} (line 4-11). If the neighbor u is not in S_k, we add u into S_k (line 11). After checking all edges incident to v, we add v to S_l,

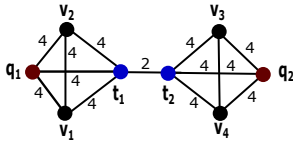


Figure 3: An example graph G of finding G_0

where $l = \max\{\tau(v,u) \mid u \in N(v), \tau(v,u) < k\}$ (line 12-13). Notice that l is the next highest level for which a connected l-truss contains the node v. After traversing all vertices in S_k, the algorithm checks whether Q is connected in G_0. If yes, the algorithm terminates, and G_0 is returned; otherwise, we decrease the present level k by 1 line 14), and repeat above steps (lines 4-14). After obtaining G_0, we compute all edge supports by counting triangles in G_0, which is used for the k-truss maintenance (line 15). This can be done efficiently again using the truss index.

The following example illustrates the algorithm.

EXAMPLE 5. Consider the graph G in Figure 3 with $Q = \{q_1, q_2\}$. The trussness of each edge is displayed, e.g., $\tau(q_1, v_1) = 4$. Now, we apply Algorithm 2 on G to find G_0 containing Q. We can verify that $\tau(q_1) = \tau(q_2) = 4$ so we start with level $k = 4$ and set $S_4 = \{q_1, q_2\}$. Then, we process the node $q_1 \in S_4$, and insert all its incident edges into G_0, for the trussness of each edge is 4. Meanwhile, all its neighbors are inserted into S_4. We repeat above process for each node in S_4. Note that for nodes t_1, t_2, $\tau(t_1, t_2) = 2$, so we insert t_1, t_2 into S_2 (lines 11-12 of Algorithm 2). Then, at level $k = 4$, we get the 4-truss as the whole graph in Figure 3 minus the edge (t_1, t_2), for $\tau(t_1, t_2) = 2$. Since the current G_0 is not connected, we decrease the truss level k to 3, and find that $S_3 = \emptyset$. Then, we decrease k to 2, and find that $S_2 = \{t_1, t_2\}$. So we expand from the edge incident to t_1, and insert the edge (t_1, t_2) into G_0, and find that the resulting graph contains Q and is connected. In this example, G_0 happens to coincide with G. □

REMARK 2. *Based on the truss index, for each vertex v, in line 9 of Algorithm 2, each edge (v,u) can be accessed constant time using the sorted adjacent list of v, and in line 12, we can compute l in constant time. Algorithm 2 takes time $O(m')$ where $m' = |E(G_0)|$.*

Computing Query Distance. For a vertex v, to compute the query distance $\mathsf{dist}_{G_i}(v, Q)$, we need to perform $|Q|$ BFS traversals on graph G_i. Specifically, for each query node $q \in Q$, with one BFS traversal starting from q in G_i, we can obtain the shortest distance $\mathsf{dist}_{G_i}(v, q)$ for each node $v \in G_i$. Then, $\mathsf{dist}_{G_i}(v, Q)$ is the maximum of all shortest distances $\mathsf{dist}_{G_i}(v, q)$, for $q \in Q$.

K-truss Maintenance. Algorithm 3 describes the procedure for maintaining G as a k-truss after the deletion of nodes V_d from G. In Algorithm 1, $V_d = \{u^*\}$ (see line 8).[2] Algorithm 3 firstly pushes all edges incident to nodes V_d into set S (lines 1-3). Then, for each edge $(u,v) \in S$, the algorithm checks every traingle $\triangle_{uvw}$ where $w \in N(u) \cap N(v)$, and decreases the support of edges (u,w) and (v,w) by 1; For any edge $e \notin S$, with resulting support $\sup(e) < k-2$, e is added to S. After traversing all triangles containing (u,v), the edge (u,v) is deleted from G. This process continues until S becomes empty (lines 4-9), and then the algorithm removes all isolated vertices form G (line 10).

4.4 Complexity analysis

In the implementation of Algorithm 1, we do not need to keep all immediate graphs, but just record the removal of vertices/edges at each iteration. Let G_0 be the maximal connected k-truss found in line 1 of Algorithm 1. Let $n' = |V(G_0)|$ and $m' = |E(G_0)|$, and let d'_{max} be the maximum degree of a vertex in G_0.

At each iteration i of Algorithm 1, we delete at least one node and its incident edges from G_i. Clearly, the number of removed edges is no less than $k-1$, thus the total number of iterations is $t \leq \min\{n'-k, m'/(k-1)\}$, i.e., t is $O(\min\{n', m'/k\})$. We have:

THEOREM 4. *Algorithm 1 takes $O((|Q|t+\rho)m')$ time and $O(m')$ space, where $t \in O(\min\{n', m'/k\})$, and ρ is the arboricity of graph G_0. Furthermore, we have $\rho \leq \min\{d'_{max}, \sqrt{m'}\}$.*

PROOF. First, finding the k-truss G_0, listing all triangles of G_0 and creating a series of k-truss graphs $\{G_0, ..., G_{l-1}\}$ takes $O(\rho \cdot m')$ time in all, where ρ is the arboricity of graph G_0.

Second, in each iteration, the algorithm needs to compute the shortest distances by a BFS traversal strating from each query node $q \in Q$, which takes $O(|Q|m')$ time. Since the algorithm runs in t iterations, the total time cost is $O(t|Q|m')$. Thus, the overall time complexity of Algorithm 1 is $O((|Q|t+\rho)m')$.

Next, we analyze the space complexity. For graphs $\{G_0, ..., G_l\}$, we only record the sequence of removed edges from G_0 for attaching a corresponding label to a graph G_i at each iteration i, which takes $O(m')$ space in all. For each vertex $v \in G_i$, we only keep $\mathsf{dist}(v, Q)$ instead of all query distances $\mathsf{dist}(v, q)$ for $q \in Q$, which takes $O(n)$ space. Hence, the space complexity of Algorithm 1 is $O(m'+n)$, which is $O(m')$, as G_0 is connected. □

5. FAST SEARCH ALGORITHMS

In this section, we focus on improving the efficiency of CTC search in two ways. First, we develop a new greedy strategy to speed up the pruning process process in Section 5.1, by deleting at least k nodes in batch, to achieve quick termination while sacrificing some approximation ratio. Second, we also propose a heuristic strategy to quickly find the closest truss community in the local neighborhood of query nodes.

5.1 Bulk Deletion Optimization

In this subsection, we propose a new algorithm called BulkDelete following the framework of Algorithm 1, which is based on deletion of a set of nodes in batch when maintaining a k-truss. The algorithm is described in detail in Algorithm 4, which can terminate quicker than Algorithm 1. It is based on the following two observations.

First, in Algorithm 1, if a graph G_i has query distance $\mathsf{dist}_{G_i}(G_i, Q) = d$, only one vertex u^* with $\mathsf{dist}_{G_i}(u^*, Q) = d$ is removed from G_i. Instead, we can delete *all* nodes u with $\mathsf{dist}_{G_i}(u, Q) = d$, from G_i, in one shot. The reason is that $\mathsf{dist}_{G_i}(u, Q)$ is monotone non-decreasing with decreasing graphs, i.e., $dist_{G_j}(u, Q) \geq \mathsf{dist}_{G_i}(u, Q) = d$, for $j > i$. Thus, removing a set of vertices $L = \{u^* | \mathsf{dist}_{G_i}(u^*, Q) \geq d, u^* \in G_i\}$ in each iteration i will improve the efficiency. This improvement indeed works in real applications. However, in theory, it is possible that $|L| = 1$ in every iteration.

Our second observation, shown in the next lemma, is that a vertex u^* with $\mathsf{dist}_{G_i}(u^*, Q) = d$ has at least $k-1$ neighbors v with $\mathsf{dist}_{G_i}(v, Q) = d-1$. If we remove $L = \{u | \mathsf{dist}_{G_i}(u, Q) \geq d-1, u \in G_i\}$ at each iteration, then the resulting number of iterations is $O(n'/k)$, where $n' = |V(G_0)|$.

LEMMA 6. *Algorithm 4 terminates in $O(n'/k)$ iterations.*

PROOF. In Algorithm 4, at each iteration i, the graph G_i has at least one node u^* with $\mathsf{dist}_{G_l}(u^*, Q) = d$, which belongs to

[2] In Section 5, we will discuss deleting a set of nodes V_d in batch.

Algorithm 4 BulkDelete (G, Q)

Input: A graph $G = (V, E)$, a set of query nodes $Q = \{q_1, ..., q_r\}$.
Output: A connected k-truss R with a small diameter.

1: Find$G0$ (G, Q) //see Algorithm 2;
2: $d \leftarrow +\infty$; $l \leftarrow 0$;
3: **while** $\mathsf{connect}_{G_l}(Q) =$ **true do**
4: Compute $\mathsf{dist}_{G_l}(q, u), \forall q \in Q$ and $\forall u \in G_l$;
5: $\mathsf{dist}_{G_l}(G_l, Q) \leftarrow \max_{u^* \in G_l} \mathsf{dist}_{G_l}(u^*, Q)$;
6: **if** $\mathsf{dist}_{G_l}(G_l, Q) < d$ **then**
7: $d \leftarrow \mathsf{dist}_{G_l}(G_l, Q)$;
8: $L = \{u^* | \mathsf{dist}_{G_l}(u^*, Q) \geq d - 1, u^* \in G_l\}$;
9: Maintain k-truss property of G_l //see Algorithm 3;
10: $G_{l+1} \leftarrow G_l$; $l \leftarrow l + 1$;
11: $R \leftarrow \arg\min_{G' \in \{G_0, ..., G_{l-1}\}} \mathsf{dist}_{G'}(G', Q)$;

L, and will be deleted in this iteration (lines 4-10). Since G_i is a connected k-truss and $u^* \in G_i$, u^* has at least $k - 1$ neighbors, i.e., $|N_{G_i}(u^*)| \geq k - 1$. Moreover, $\forall v \in N_{G_i}(u^*)$, we have $\mathsf{dist}_{G_i}(v, Q) \geq d - 1$: otherwise, if $\exists v \in N_{G_i}(u^*)$ with $\mathsf{dist}_{G_i}(v, Q) < d - 1$, we can obtain $\mathsf{dist}_{G_i}(u^*, Q) < d$, a contradiction. As a result, we have $u^* \in L$ and $N(u) \subset L$, and $|L| \geq k$. Thus, at least k nodes are deleted at each iteration, and the algorithm terminates in $O(n'/k')$ iterations. □

Thus, the number of iterations is improved from $O(\min\{n', m'/k\})$ to $O(n'/k)$ (see Theorem 4). We just proved:

THEOREM 5. *Algorithm 4 takes $O((|Q|t' + \rho')m')$ time using $O(m')$ space, where $t' \in O(n'/k)$, and $\rho' \leq \min\{d'_{max}, \sqrt{m'}\}$.*

The approximation quality of Algorithm 4 is characterized below.

THEOREM 6. *Algorithm 4 is a $(2 + \varepsilon)$-approximation solution of* CTC-Problem, *where $\varepsilon = 2/diam(H^*)$.*

PROOF. Similar with Lemma 5, to prove this theorem, we only need to ensure $\mathsf{dist}_R(R, Q) \leq \mathsf{dist}_{H^*}(H^*, Q)+1$. Because $diam(R) \leq 2\mathsf{dist}_R(R, Q) \leq 2\mathsf{dist}_{H^*}(H^*, Q) + 2 \leq 2(diam(H^*) + 1)$ by Lemma 2, then approximation ratio is $2+\varepsilon$, where $\varepsilon = 2/diam(H^*)$. The detailed proof is similar with Lemma 5, which is omitted here, due to space limitation. □

EXAMPLE 6. Continuing with the previous example, we apply Algorithm 4 on Figure 1(a) to find the closest truss community for $Q = \{q_1, q_2, q_3\}$. In G_0, we compute $d = \max_{u \in G_0} \mathsf{dist}_{G_0}(u, Q) = 4$, and $L = \{q_1, q_3, p_1, p_2, p_3\}$, as each node $u \in L$ has query distance $\mathsf{dist}_{G_0}(u, Q) = 3 \geq d - 1$. After removing L from G_0, the remaining graph does not contain Q, and the algorithm terminates. Thus, Algorithm 4 reports the entire 4-truss G_0 as the answer, which has diameter 4, compared to the answer of Figure 1(b) reported by Algorithm 1, which has diameter 3. □

5.2 Local Exploration

In this subsection, we develop a heuristic strategy to quickly find the closest truss community by local exploration. The key idea is as follows. We first form a Steiner tree to connect all query nodes, and then expand it to a graph G'_0 by involving the local neighborhood of the query nodes. From this new graph G'_0, we find a connnected k-truss with the highest k containing Q, and then iteratively remove the furthest nodes from this k-truss using the BulkDelete algorithm discussed earlier.

Algorithm 5 Local-CTC (G, Q)

Input: A graph $G = (V, E)$, a set of query nodes $Q = \{q_1, ..., q_r\}$, a node size threshold η.
Output: A connected k-truss R with a small diameter.

1: Compute a Steiner Tree T containing Q using truss distance fuctions;
2: $k_t \leftarrow \min_{e \in T} \tau(e)$;
3: Expand T to a graph $G_t = \{e \in G | \tau(e) \geq k_t\}$, s.t. $T \subseteq G_t$ and $|G_t| \leq \eta$;
4: Extract the maximal connected k-truss H_t containing Q from G_t, where $k \leq k_t$ is the maximum possible trussness;
5: Apply BulkDelete algorithm on H_t to identify closest community.

Connect query nodes with a Steiner tree. As explained above, the Steiner tree found is used as a seed for expanding into a k-truss. It is well-known that finding a minimal weight Steiner tree is NP-hard but it admits a 2-approximation [14, 18]. However, a naive application of these algorithms may produce a result with a small trussness. To see this, consider the graph G and the query $Q = \{q_1, q_2, q_3\}$ in Figure 1(a). Suppose all edges are uniformly weighted. Then it is obvious that the tree $T_1 = \{(q_2, q_1), (q_1, t), (t, q_3)\}$ with total weight 3 is an optimal (i.e., minimum weight) Steiner tree for Q. However, the smallest trussness of the edges in T_1 is 2, which suggests growing T_1 into a larger graph will yield a low trussness. By contrast, the Steiner tree $T_2\{(q_1, q_2), (q_2, v_4), (v_4, q_3)\}$ has the total weight 3 and all its edges have the trussness at least 4, indicating it could be expanded into a more dense graph. To help discriminate between such Steiner trees, we define edge weights as follows. Recall the definition of $\bar{\tau}(S)$ from Section 2.

DEFINITION 7 (TRUSS DISTANCE). *Given a path P between nodes u, v in G, we define the truss distance of u and v as $\hat{\mathsf{dist}}_P(u, v) = \mathsf{dist}_P(u, v) + \gamma(\bar{\tau}(\emptyset) - \min_{e \in P} \tau(e))$, where $\mathsf{dist}_P(u, v)$ is the path length of P, and $\gamma > 0$. For a tree T, by $\hat{\mathsf{dist}}_T(u, v)$ we mean $\hat{\mathsf{dist}}_P(u, v)$ where P is the path connecting u and v in T.*

The difference $\bar{\tau}(\emptyset) - \min_{e \in P} \tau(e))$ measures how much the minimum edge trussness of path P falls short of the maximum edge trussness of the graph G and γ controls the extent to which small edge trussness is penalized. The larger γ is, the more important edge trussness is in distance calculations. For instance, in the above example, $\bar{\tau}(\emptyset) = 4$ and for $\gamma = 3$, the truss distance of (q_2, q_3) in T_1 is $\hat{\mathsf{dist}}_{T_1}(q_2, q_3) = \mathsf{dist}_{T_1}(q_2, q_3) + 3 \cdot (4 - 2) = 3+6 = 8$, since the minimum edge trussness of T_1 is $\tau(q_1, t) = 2$. On the other hand, $\hat{\mathsf{dist}}_{T_2}(q_1, q_3) = \mathsf{dist}_{T_2}(q_1, q_3) + 3 \cdot (4 - 4) = 3 + 0 = 3$. Obviously, the Steiner tree T_2 has a smaller truss distances than T_1. It can be verified that its overall weight is smaller than that of T_1.

Find G_0 by expanding Steiner tree to graph. After obtaining the Steiner tree T for the query nodes, we locally expand the tree to a small graph G_t as follows. We firstly obtain the minimum trussness of edges in T as $k_t = \min_{e \in T} \tau(e)$. Then, we start from the nodes in T, and expand the tree to a graph in a BFS manner via edges of trussness no less than k_t, and iteratively insert these edges into G_t until the node size exceeds a threshold η, i.e., $|V(G_t)| \leq \eta$, were η is empirically tuned. Since G_t is a local expansion of T, the trussness of G_t will be at most k_t, i.e., $\tau(G_t) \leq k_t$. For ensuring the dense cohesive structure of identified communities, we apply a truss decompostion algorithm on G_t. Then, we extract the maximal connected k-truss subgraph H_t containing Q by removing all edges of trussness less than k_t from G_t, where $k \leq k_t$ is the maximum possible trussness.

Table 2: Network statistics (K $= 10^3$ and M $= 10^6$)

Network	$\|V_G\|$	$\|E_G\|$	d_{max}	$\bar{\tau}(\emptyset)$
Facebook	4K	88K	1,045	97
Amazon	335K	926K	549	7
DBLP	317K	1M	342	114
Youtube	1.1M	3 M	28,754	19
LiveJournal	4M	35M	14,815	352
Orkut	3.1M	117M	33,313	78

Reduce the diameter of G_0. We take the graph H_t with the maximum trussness k as input, and apply a variant of BulkDelete algorithm on H_t for returning the identified community. We implement a variant of BulkDelete algorithm, which is different from original BulkDelete w.r.t. the removed vertex set L. We reajust the furthest nodes to be removed, as $L' = \{u^* | \mathsf{dist}_{G_l}(u^*, Q) \geq d, u^* \in G_l\}$. This adjustment makes the algorithm not as efficent as BulkDelete in asymptotic running time complexity, but we still find it efficient in practice. On the other hand, in practice, this strategy can achieve a smaller graph diameter than BulkDelete, which provides a 2-approximation of the optimal. Moreover, in our implementation, in each iteration, we carefully remove only a subset of nodes in L', which have the largest a total of distances from all query nodes. As a result, more nodes with the largest query distance are removed from the community in the end. The reason is as follows. Suppose the largest query distance we found as d, in the real world, the number of nodes having query distance d may be large, due to the small-world property.

6. EXPERIMENTS

We conduct experimental studies using 6 real-world networks available from the Stanford Network Analysis Project[3], where all networks are treated as undirected. The network statistics are shown in Table 2. All networks except for Facebook contain 5,000 top-quality ground-truth communities.

To evaluate the efficiency and effectiveness of improved strategies, we test and compare three algorithms proposed in this paper, namely, Basic, BD, and LCTC. Here, Basic is the basic greedy approach Basic in Algorithm 1, which removes single furthermost node at each iteration. BD is the BulkDelete approach in Algorithm 4, which removes multiple furthermost nodes at each iteration. LCTC is the local exploration approach in Algorithm 5. For LCTC, we set the parameters $\eta = 1,000$ and $\gamma = 3$. For efficiency, we report runtime in seconds. We treat the runtime of a query as infinite if its runtime exceeds 1 hour. To evaluate the effectiveness of eliminating "free riders", we compare our methods with Truss (Algorithm 2), which finds the connected k-truss graph containing query nodes with the largest k only. We report the percentage of nodes that are kept in the resulting community by $\frac{|V(R)|}{|V(G_0)|}$, where R is the closest truss community found and G_0 is computed by Truss. The less percentage the more "free riders" being removed.

In addition, to evaluate the quality of closest truss community found, we compare our methods with two state-of-the-art community search approaches: the minimum degree-based community search (MDC) [23], which globally finds the dense subgraph containing all query nodes with the highest minimum degree under the distance and size constraints, and the query biased densest community search (QDC) [28], which shifts the detected community to the neighborhood of the query by integrating the edge density and nodes proximity to the query nodes. We use F1-score to measure the alignment between a discovered community C and a ground-truth community $\hat{C}$. Here, F1-score is defined as $F1(C, \hat{C}) = \frac{2 \cdot prec(C,\hat{C}) \cdot recall(C,\hat{C})}{prec(C,\hat{C}) + recall(C,\hat{C})}$ where $prec(C, \hat{C}) = \frac{|C \cap \hat{C}|}{|C|}$ is the precision and $recall(C, \hat{C}) = \frac{|C \cap \hat{C}|}{|\hat{C}|}$ is the recall.

[3] snap.stanford.edu

Table 3: Parameters of Query Nodes

Parameter	Range	Default
query size $\|Q\|$	1, 2, 4, 8, 16	3
degree rank Q_d	20%, 40%, 60%, 80%, 100%	80%
inter-distance l	1, 2, 3, 4, 5	2

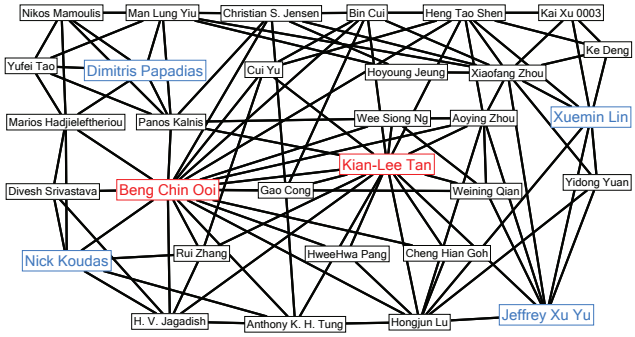


Figure 4: Closest 4-truss community containing {Jeffrey Xu Yu, Xuemin Lin, Dimitris Papadias, Nick Koudas}

We randomly generate sets of query nodes. Table 3 shows the three parameters, query size $|Q|$, degree rank Q_d, and inter-distance l, used for generating query nodes with their ranges and default values. Here, $|Q|$ is the number of query nodes, which is set to 3 by default. Q_d is the degree rank of query nodes. We sort all vertices in descending order of their degrees in a network. A node is said to be with degree rank of X%, if it has top highest X% degree in the network. The default value of Q_d is 80%, which means that a query node has degree higher than the degree of 20% nodes in the whole network. The inter-distance l is the inter-distance between all query nodes. The default $l = 2$ indicates that all query nodes are within distance of 2 to each other in the network.

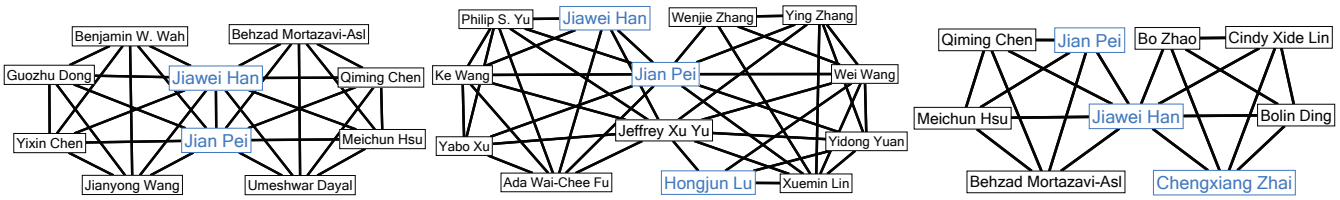
All algorithms are implemented in C++, and all the experiments are conducted on a Linux Server with Intel Xeon CUP X5570 (2.93 GHz) and 50GB main memory.

Exp-1: A Case Study on DBLP: We construct a collaboration network from the raw DBLP data set[4] for a case study. A vertex represents an author, and an edge between two authors indicates they have co-authored no less than 3 times. This DBLP graph contains 234,879 vertices and 541,814 edges.

First, we query closest truss community containing {"Jeffrey Xu Yu", "Xuemin Lin", "Dimitris Papadias", "Nick Koudas"}. As shown in Figure 4, the community is a 4-truss connected graph of 29 vertices and 92 edges, where two co-authors of each edge have at least two common co-authors. The diameter of this community is 3, and the distance between any two authors is no greater than 3. For example, "Dimitris Papadias" can reach "Xuemin Lin" via a path of length 3 as "Dimitris Papadias" $\rightarrow$ "Man Lung Yiu" $\rightarrow$ "Xiaofang Zhou" $\rightarrow$ "Xuemin Lin". All are densely connected in this community. Note that [28] uses the same query to detect two disjoint local communities which do not include all query authors in one community.

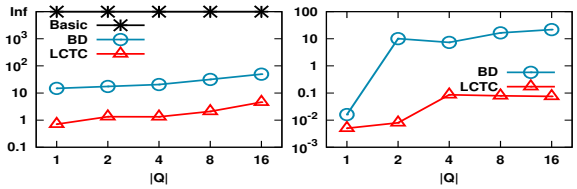
Second, we show that our closest truss community model can detect overlapping communities by similar but different queries. We test it using three queries: $Q_1 =$ {"Jiawei Han", "Jian Pei"}, $Q_2 =$ {"Jiawei Han", "Jian Pei", "Hongjun Lu"}, and $Q_3 =$ {"Jiawei Han", "Jian Pei", "Chengxiang Zhai"}. Q_2 and Q_3 add a different author into Q_1. The closest truss community for Q_1 is shown in Figure 5(a), which is a 6-truss of diameter 2. The closest truss community for Q_2 shown in Figure 5(b) is a 5-truss of diameter 2. The two communities for Q_1 and Q_2 are different. The closest truss community for Q_3 is in Figure 5(c), which is a 5-

[4] http://dblp.uni-trier.de/xml/



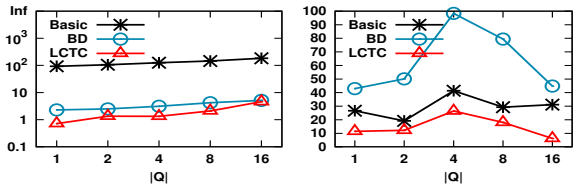
(a) $Q_1 = \{$Jiawei Han, Jian Pei$\}$ (b) $Q_2 = \{$Jiawei Han, Jian Pei, Hongjun Lu$\}$ (c) $Q_3 = \{$Jiawei Han, Jian Pei, C. X. Zhai$\}$

Figure 5: Overalpping communities detected using similar but different queries by our model



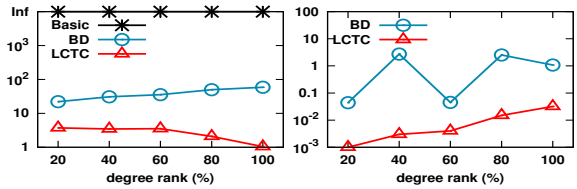
(a) Query Time (b) The percentage

Figure 6: DBLP: varying query size $|Q|$



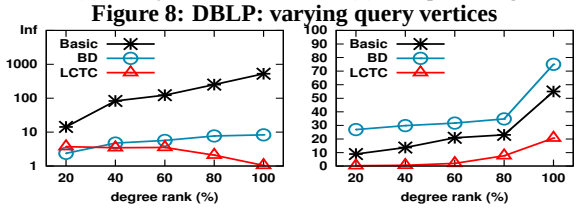
(a) Query Time (b) The percentage

Figure 7: Facebook: varying query size $|Q|$



(a) Query Time (b) The percentage

Figure 8: DBLP: varying query vertices



(a) Query Time (b) The percentage

Figure 9: Facebook: varying query vertices

truss of diameter 2. The left part of authors are close to "Jiawei Han" and "Jian Pei" as shown in Figure 5(a), and the right part is another group led by "Jiawei Han" and "Chengxiang Zhai" at UIUC.

Exp-2 Different Queries: We test our approaches using different queries on DBLP and Facebook (Table 2).

First, we vary the query size $|Q|$. We test 5 different $|Q|$ in $\{1, 2, 4, 8, 16\}$. For each $|Q|$, we randomly select 100 sets of $|Q|$ query nodes, and we report the average runtime and the average percentage of avoiding FRE.[5] The results for DBLP and Facebook are shown in Figure 6 and Figure 7, respectively.

For DBLP, as shown in Figure 6, LCTC outperforms BD in terms of both efficiency and the percentage of avoiding FRE. Basic cannot find communities in 1 hour limit. For Facebook, as shown in Figure 7, LCTC performs the best in both efficiency and the percentage of avoiding FRE. BD achieves the competitive efficiency with LCTC. This is because Facebook contains only 4K vertices, and the global method BD is effective on such a small network. Basic takes 100 seconds to find the results, given such a small network. As noticed, for the percentage of avoiding FRE, BD performs the worst, and has limited power to avoid FRE for a small network like Facebook.

Second, we vary the degree of query nodes. For a graph to be tested, we sort the vertices in descending order of their degrees, and partition them into 5 equal-sized buckets. For each bucket, we randomly select 100 different query sets of size 3, and we report the average runtime and the average percentage of avoiding FRE. The results for DBLP and Facebook are shown in Figure 8 and Figure 9, respectively. The performance in terms of runtime and the percentage of avoiding FRE are similar to the results by varying the query sizes. LCTC outperforms the others.

Third, we vary the inter-distance l within query nodes from 1 to 5. For each l value, we randomly select 100 sets of 3 query nodes, in which the inter-distance of query nodes is to be l. We report the average runtime and the average percentage of avoiding FRE. The results for DBLP and Facebook are shown in Figure 10 and Figure 11, respectively. The performance in terms of runtime and the percentage of avoiding FRE are similar to the results observed. All methods increase the percentage while the inter-distance l increases. This is because the diameter of community increases, and therefore the less number of nodes can be removed from graph. LCTC outperforms the others.

Table 4: Index size and index construction time

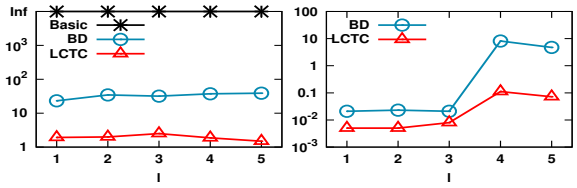
Network	Graph Size (M)	Index Size (M)	Index Time (s)
Facebook	0.9	1.3	7.4
Amazon	12	19	6.7
DBLP	13	20	14
Youtube	37	59	76
LiveJounarl	478	666	2,142
Orkut	1,640	2,190	21,012

We report the simple k-truss index in terms of index size (Megabytes) and index construction time (seconds) in Table 4. The size of the k-truss index is 1.6 times of the original graph size, which confirms that the simple k-truss indexing scheme has $O(m)$ space complexity and is very compact. The index construction is very efficient.

Exp-3 The Quality: To evaluate the effectiveness of different community models, we compare LCTC with three other methods MDC, QDC and Truss using the 5 networks, DBLP, Amazon, Youtube, LiveJournal, and Orkut, with ground-truth communities [32]. We randomly select query nodes that appear in a unique ground-truth community, and select 1,000 sets of such query nodes with the size randomly ranging from 1 to 16. We evaluate the accuracy by the F1-score of the detected community, and report the averaged F1-score over all query cases.

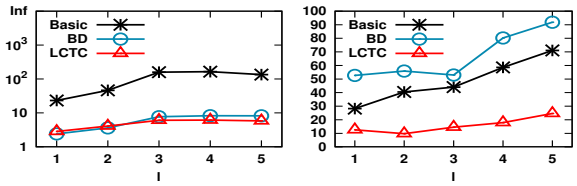
Figure 12(a) shows the F1-score. Our method achieves highest F1-score on most networks. QDC has the second best performance, which outperforms LCTC on Youtube network. MDC does not perform well due to the fixed distance and size constraints. Figure 12(b) shows that LCTC runs much faster than MDC and QDC, and is close to Truss. Figure 12(c) shows the size of communities detected by LCTC and Truss, in terms of the number of vertices and

[5]Notice that Basic, BD, LCTC are not optimal algorithms.



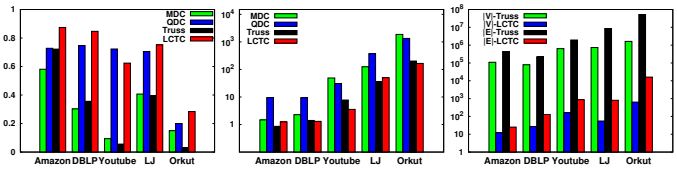
(a) Query Time (b) The percentage

Figure 10: DBLP: varying inner distance l



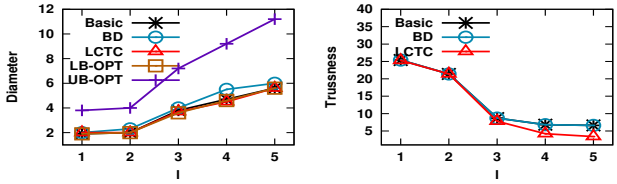
(a) Query Time (b) The percentage

Figure 11: Facebook: varying inner distance l



(a) F_1 score (b) Query time (c) Reduction

Figure 12: Quality evaluation on networks with ground-truth communities



(a) Diameter (b) Trussness

Figure 13: Varying the inner distance l on Facebook

edges. As we can see, the number of nodes ($|V|$-) and the number of edges ($|E|$-) in communities detected by LCTC are much less than those by Truss on all networks. It confirms the power of eliminating irrelevant nodes from discovered communities by our LCTC.

Diameter and Trussness Approximation: We evaluate the diameter approximation of detected communities by our methods on Facebook network. Here, we take the lower bound of the optimal diameter as the smallest query distance $\mathsf{dist}_R(R, Q)$, where R is the community detected by method Basic. We show the curve of $2\mathsf{dist}_R(R, Q)$, which serves as the upper bound of smallest diameter by Lemma 2. The averaged diameters of communities detected by different methods are reported in Figure 13(a), where we vary the inter-distance l. The diameters of detected communities obtained by all our methods are very close to the lower bound of optimal one. Figure 13(b) shows the maximum trussness of detected communities by our methods. Basic and BD globally search the k-truss containing query nodes on the entire graph, and the detected communities have the maximum trussness k. Since LCTC locally searches on a small graph, the trussness of detected subgraph found may not be the maximum. However, the trussness of communities detected by LCTC are very close to Basic and BD. LCTC balances the efficiency and effectiveness well.

Exp-4 Varying LCTC parameters: In this experiment, we test the performance of LCTC by varying parameters η and γ. We used the same query nodes that are selected in Exp-3 on DBLP network. The similar results can be also observed on other 4 networks in this paper. $\eta = 1000$ and $\gamma = 3$ is the default setting for LCTC. For the parameter η, we firstly vary it from 100 to 2000. The results of F1-score, the number of community vertices $|V|$ and the running time. are reported in Figure 14. As we can see, the number of community vertices increases when η increases from 100 to 500, and then keeps stable for larger η. It shows that the default setting $\eta = 1000$ is large enough. Moreover, LCTC achieves the stable performance of F1-score and running time by varying η. We also test the parameter γ, and report the results on Figure 15. The number of community vertices increases with the increased γ. Because LCTC with a larger γ can detected the community of a larger trussness, and the number of vertices to be removed is reduced. On the other hand, the F1-score increases with increasing γ at first, but it drop slightly when γ further increases. The running time of LCTC keeps table.

Exp-5 Synthetic Datasets: In this experiment, we generated synthetic graph datasets to test how close the communities detected by our methods are close to the optimal one. The graph contains 10000 vertices and 100000 edges. We generated 100 optimal communities of different size ranging from 10 to 50. Each community has the largest trussness value ranging between 5 and 20. We also add the noise nodes into graphs. The noise nodes do not belong to any communities. We randomly select 100 sets of 3 query nodes from optimal communities. We evaluate the averaged accuracy of detected communities compared with the optimal one by F1-score criterion in Figure 16 (a). LCTC achieve the highest F1-score, which shows the good approximation performance to the optimal communities. Basic performs nearly well as LCTC, and BD is worst. We also reported the averaged vertex size of detected communities in Figure 16 (b). As we can see, the size of communities detected by our methods Basic and LCTC both are very close to the optimal one.

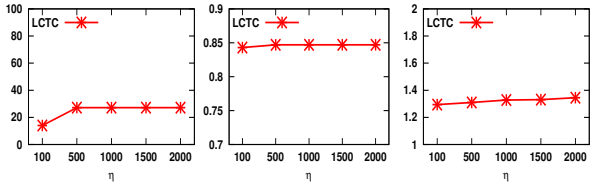
7. RELATED WORK AND DISCUSSION

The most related work to our study falls under community search, community detection, and dense subgraph mining.

7.1 Community Search

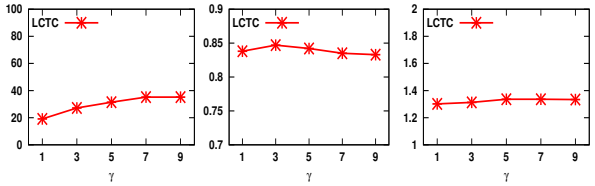
Recently, several community search models have been studied, including k-truss [13], quasi-clique [9], k-core [23, 10], influential community [16] and query biased densest subgraph [28]. Here, we compare these models with our proposed closest truss community model w.r.t. three aspects: (i) consideration of query nodes, (ii) cohesive structure, and (iii) quality approximation.

Query nodes. Cui et al. [9] have recently studied the problem of online search of overlapping communities for a query node by designing a new α-adjacency γ-quasi-k-clique model. Huang et al. [13] propose a k-truss community model based on triangle adjacency, to find all overlapping communities of a query node. They ignore the diameter of the resulting community. Cui et al. [10] find a k-core community for a query node using local search. In addition, influential community model [16] finds top-r communities with the highest influence scores over the entire graph; no query nodes are considered. Extending any of above models from one (or zero) query node to multiple query nodes raises new challenges. First, for the models with one query node and a parameter k, the search algorithm can easily start from this node to find qualified subgraphs. For multiple nodes, it is non-trivial for the search algorithm to determine the start point and search directions, which can quickly connect all query nodes. Second, for a given parameter k, the connected dense subgraph containing all query nodes may not exist. Thus, it requires the search algorithm to automatically determine the proper k for different query nodes.



(a) $|V|-$LCTC (b) F1-Score (c) Query Time

Figure 14: DBLP: varying parameter η of LCTC



(a) $|V|-$LCTC (b) F1-Score (c) Query Time

Figure 15: DBLP: varying parameter γ of LCTC

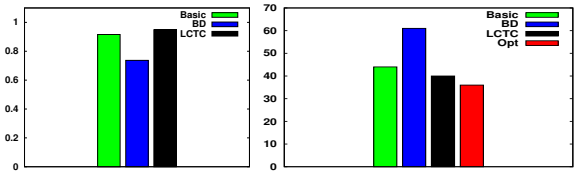
Cohesive structure. [23] and [28] support community search of multiple query nodes similarly to us, thus they are most related to our work. Sozio et al. [23] proposed a k-core based community model, called Cocktail Party model, with distance and size constraints. Conceptually, k-truss is a more cohesive definition than k-core, as k-truss is based on triangles whereas k-core simply considers node degree [25]. Our proposed closest truss community model is based on connected k-truss, and the Cocktail Party model is based on connected k-core. Most recently, Wu et al. [28] studied the query biased densest connected subgraph (QDC) problem for avoiding subgraphs irrelevant to query nodes in the community found. While QDC [28] is also defined based on a connected graph containing Q similarly to CTC, it optimizes a fundamentally different function called query biased edge density, which is calculated as the overall edge weight averaged over the weight of nodes in a community.

Quality approximation. Both problems proposed in [23] and [28] are NP-hard to compute, and do not admit approximations without further assumptions. [28] gives an approximation solution of QDC by relaxing the problem. Unfortunately, as the authors show themselves [28], this could fail in real applications, for two reasons. First, the algorithm may find a solution consisting of several connected components with query nodes split between them! Second, the approximation factor can be large, which can deteriorate further with a larger number of query nodes.

In contrast, we provide an efficient 2-approximation algorithm for finding the closest truss community containing any set of query nodes. We provide a heuristic algorithm based on local exploration which significantly improves the efficiency and show that on several real networks, it delivers a high-quality solution.

7.2 Community Detection

The goal of community detection is to identify all communities in the entire network. A typical method for finding communities is to optimize the modularity measure [19]. Generally, community detection falls into two major categories: non-overlapping [20, 22, 34] and overlapping community detection [21, 1, 31, 33]. All these methods consider static communities, where the networks are partitioned a priori. Query nodes are not considered since their focus is not community search. [15] surveys several community detection methods and evaluates their performance using rigorous tests. [30] proposes an online distributed algorithm for community detection in dynamic networks using label propagation. As such, these works on community detection are significantly different from our goal of query driven community search.



(a) F1-Score (b) Community Size

Figure 16: Quality evaluation on synthetic datasets

7.3 Dense Subgraph Mining

There is a very large body of work on mining dense subgraph patterns, including clique [3, 5, 26, 29], quasi-clique [24], k-core [2, 4], k-truss [7, 25, 35], dense neighborhood graph [27], to name a few.

Clique and quasi-clique enumeration methods include the classical algorithm [3], the external-memory H^*-graph algorithm [5], redundancy-aware clique enumeration [26], maximum clique computation using MapReduce [29], and optimal quasi-clique mining [24]. Various studies have been done on core decomposition and truss decomposition in different settings, including in-memory algorithms [2, 7, 35], external-memory algorithms [4, 25], and MapReduce [8]. [13, 35] designed an incremental algorithm for updating a k-truss with edge insertions/deletions. Wang et al. [27] studied a dense neighborhood graph based on common neighbors. None of these works considers query nodes, which as we have discussed earlier, raise major computational challenges.

8. CONCLUSION

In this paper, we study the closest truss community search problem over a graph, given a set of query nodes, that is, find a densely connected community, in which nodes are close to each other. Based on the dense subgraph definition of a k-truss, we formualte the CTC as a connected k-truss subgraph containing the query nodes with the largest k, and has the minimum diameter among such subgraphs. We showed the problem is NP-hard and is NP-hard to approximate within a factor better than 2. We also matched this lower bound by developing a greedy algorithmic framework that provides a 2-approximation to the optimal solution. To support the efficient search of a CTC, we make use of a truss index and develop efficient methods of truss idenfication and maintenance. Futhermore, we improve the efficiency of greedy framework further using the bulk deletion optimization and local exploration strategies. Extensive experimental results on large real-world networks with ground-truth communities demonstrate the effectivenss and efficiency of our proposed community search model and solutions.

It would be interesting to extend our search model and algorithms to directed graphs. Given the recent surge of interest in probabilisic graphs, an exciting question is how k-truss generalizes to probabilistic graphs. The challenge is to develop extensions that are widely useful and tractable. Last but not the least, it would be interesting to extend the notions and techniques to networks with interactions between nodes.

9. REFERENCES

[1] Y.-Y. Ahn et al. Link communities reveal multiscale complexity in networks. *Nature*, 466(7307):761–764, 2010.

[2] V. Batagelj and M. Zaversnik. An o (m) algorithm for cores decomposition of networks. *arXiv preprint cs/0310049*, 2003.

[3] C. Bron and J. Kerbosch. Finding all cliques of an undirected graph (algorithm 457). *Commun. ACM*, 16(9):575–576, 1973.

[4] J. Cheng et al. Efficient core decomposition in massive networks. *ICDE*, pp. 51–62, 2011.
[5] J. Cheng et al. Finding maximal cliques in massive networks by h*-graph. *SIGMOD*, pp. 447–458, 2010.
[6] N. Chiba and T. Nishizeki. Arboricity and subgraph listing algorithms. *SIAM J. Comput.*, 14(1):210–223, 1985.
[7] J. Cohen. Trusses: Cohesive subgraphs for social network analysis. Tech. rep., National Security Agency, 2008.
[8] J. Cohen. Graph twiddling in a mapreduce world. *Computing in Science and Eng.*, 11(4):29–41, 2009.
[9] W. Cui et al. Online search of overlapping communities. *SIGMOD*, pp. 277–288, 2013.
[10] W. Cui et al. Local search of communities in large graphs. *SIGMOD*, pp. 991–1002, 2014.
[11] J. Edachery et al. Graph clustering using distance-k cliques. *Int. Symp. on Graph Drawing*, pp. 98–106, 1999.
[12] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, 1979.
[13] X. Huang et al. Querying k-truss community in large and dynamic graphs. *SIGMOD*, pp. 1311–1322, 2014.
[14] L. Kou et al. A fast algorithm for steiner trees. *Acta informatica*, 15(2):141–145, 1981.
[15] A. Lancichinetti and S. Fortunato. Community detection algorithms: a comparative analysis. *Physical review E*, 80(5):056117, 2009.
[16] R.-H. Li et al. Influential community search in large networks. *PVLDB*, 8(5), 2015.
[17] J. J. McAuley and J. Leskovec. Learning to discover social circles in ego networks. *NIPS*, volume 272, pp. 548–556, 2012.
[18] K. Mehlhorn. A faster approximation algorithm for the steiner problem in graphs. *Information Processing Letters*, 27(3):125–128, 1988.
[19] M. E. Newman. Fast algorithm for detecting community structure in networks. *Physical review E*, 69(6):066133, 2004.
[20] M. E. Newman and M. Girvan. Finding and evaluating community structure in networks. *Physical review E*, 69(2):026113, 2004.
[21] G. Palla et al. Uncovering the overlapping community structure of complex networks in nature and society. *Nature*, 435(7043):814–818, 2005.
[22] M. Rosvall and C. T. Bergstrom. Maps of random walks on complex networks reveal community structure. *PNAS*, 105(4):1118–1123, 2008.
[23] M. Sozio and A. Gionis. The community-search problem and how to plan a successful cocktail party. *KDD*, pp. 939–948, 2010.
[24] C. E. Tsourakakis et al. Denser than the densest subgraph: Extracting optimal quasi-cliques with quality guarantees. *KDD*, pp. 104–112, 2013.
[25] J. Wang and J. Cheng. Truss decomposition in massive networks. *PVLDB*, 5(9):812–823, 2012.
[26] J. Wang et al. Redundancy-aware maximal cliques. *KDD*, pp. 122–130, 2013.
[27] N. Wang et al. On triangulation-based dense neighborhood graphs discovery. *PVLDB*, 4(2):58–68, 2010.
[28] Y. Wu et al. Robust local community detection: On free rider effect and its elimination. *PVLDB*, 8(7), 2015.
[29] J. Xiang et al. Scalable maximum clique computation using mapreduce. *ICDE*, pp. 74–85, 2013.
[30] J. Xie et al. Labelrankt: Incremental community detection in dynamic networks via label propagation. *Workshop on Dynamic Networks Management and Mining*, pp. 25–32, 2013.
[31] J. Xie et al. Overlapping community detection in networks: The state-of-the-art and comparative study. *ACM Comput. Surv.*, 45(4):43, 2013.
[32] J. Yang and J. Leskovec. Defining and evaluating network communities based on ground-truth. *ICDM*, pp. 745–754, 2012.
[33] J. Yang and J. Leskovec. Overlapping community detection at scale: a nonnegative matrix factorization approach. *WSDM*, pp. 587–596, 2013.
[34] Z. Yang et al. Clustering by nonnegative matrix factorization using graph random walk. *NIPS*, pp. 1079–1087, 2012.
[35] Y. Zhang and S. Parthasarathy. Extracting analyzing and visualizing triangle k-core motifs within networks. *ICDE*, pp. 1049–1060, 2012.