Journal of
Intelligent Material
Systems and Structures

Journal of Intelligent Material Systems and Structures
1–4

DOI: 10.1177/1045389X251336619
journals.sagepub.com/home/jim
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Exploring non-Hermitian wave dynamics in odd elastic metamaterials

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Abstract
Odd elastic metamaterials, enabled by nonreciprocal couplings implemented through active elements, break conventional symmetry constraints in density and elasticity tensors, giving rise to unconventional wave dynamics. These materials exhibit unique properties, including non-Hermiticity, 1D unidirectional wave amplification, 2D directional wave amplification, and topological phenomena such as skin modes. By redefining classical elasticity, odd elastic metamaterials open new avenues for wave manipulation, energy harvesting, and vibration control. This research emphasizes these materials' underlying principles and physical implementations, laying the groundwork for future advancements in non-Hermitian physics and elastic mechanics.

Keywords
Odd elasticity, odd density, non-Hermitian, wave dynamics

1. Introduction

Mechanical metamaterials, created by embedding local resonators into a background medium, exhibit frequency-dependent negative mass densities and elastic moduli (Liu et al., 2000). These properties enable novel phenomena such as negative refraction (Zhu et al., 2014), wave cloaking (Milton et al., 2006), and superlensing (Zhang and Liu, 2008). However, conventional metamaterials remain passive and obey symmetry constraints in the density and elasticity tensors. In contrast, active elements such as nonreciprocal couplings, enable the breaking of the major symmetry in the density and elasticity tensors such as $\rho_{ij} \neq \rho_{ji}$ and $C_{ijkl} \neq C_{klij}$ (Fruchart et al., 2023). Therefore, materials that display anti-symmetric density and elasticity tensors are termed as having odd elasticity and odd density (Fruchart et al., 2023). These asymmetries are essential for achieving non-Hermitian wave dynamics, which fundamentally rely on either odd density or odd elasticity. To demonstrate these mechanisms, we present a metabeam exhibiting odd elasticity and a plate model realizing odd density. These active metamaterials systematically introduce non-Hermitian effects in solid mechanics, leading to wave amplification (Chen et al., 2021), nonreciprocal responses (Wang et al., 2023; Wu et al., 2023), nontrivial topology (Wu et al., 2024), and skin effects (Scheibner et al., 2020; Wang et al., 2024).

2. Principles of odd density and odd elasticity

The elasticity refers to a material's ability to respond to deformation under load, or vice versa, whereas the density represents a material's mass per unit volume, influencing its inertial response to motion. In the linear regime, these behaviors adhere to the following generalized constitutive relations:

$$\sigma_{ij} = C_{ijkl}\partial_k u_l \tag{1}$$

$$p_i = \rho_{ij}\dot{u}_j \tag{2}$$

where the indices $i, j, k, l = 1, 2, 3$; p_i denotes the ith component of the momentum density vector; ρ_{ij} represents the density tensor; u_j is the jth component of the displacement vector; σ_{ij} refers to the stress tensor; and C_{ijkl} is the elasticity tensor. For passive media, the

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Data Availability Statement included at the end of the article.

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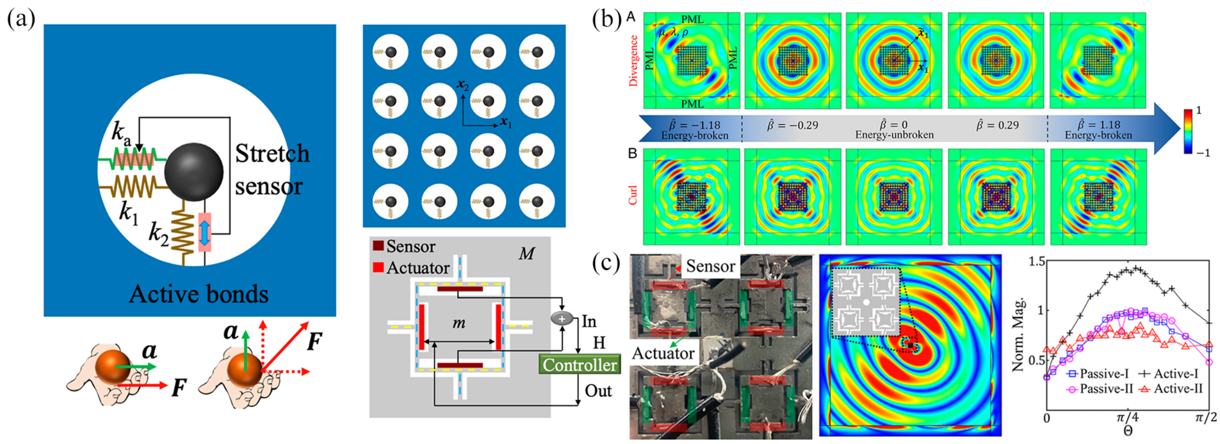

Figure 1. Physical realization and wave phenomena associated with odd density and elasticity: (a) unit cell (left panel), lattice structure (top right panel), and microstructure design (bottom right panel) of odd density media, (b) numerical demonstration of energy phase transition and directional wave amplification, and (c) directional wave amplification in 2D odd density media. Reprinted with permission from "Active metamaterials for realizing odd mass density," Wu Q., et al., Proceedings of the National Academy of Sciences*, 2023, © 2023 National Academy of Sciences, USA. The publication is available at https://doi.org/10.1073/pnas.2209829120.

density tensor, which can be expressed as the second derivatives of the kinetic energy, is symmetric ($\rho_{ij} = \rho_{ji}$) due to the symmetry of mixed partial derivatives. Similarly, the elasticity tensor exhibits major symmetry ($C_{ijkl} = C_{klij}$) because it can be expressed as the second derivatives of a thermodynamic potential function. To break these symmetry conditions, nonreciprocal couplings can be introduced into materials through active control. Nonreciprocal couplings produce nonzero work during cyclic deformation, leading to the absence of a corresponding kinetic energy associated with the density and a thermodynamic potential for elasticity. This results in the breakdown of the thermodynamics-based elasticity paradigm and,consequently, breaking of the symmetry of the elastic constant tensor ($\rho_{ij} \neq \rho_{ji}, C_{ijkl} \neq C_{klij}$; Fruchart et al., 2023). By utilizing the equations of motion

$$\dot{p}_i = \partial_j \sigma_{ij}, \tag{3}$$

and with the aid of equations (1) and (2), we derive

$$\rho_{ij}\ddot{u}_j = C_{ijkl}\partial_k \partial_j u_l. \tag{4}$$

The operator in this equation is not Hermitian, transitioning the system into the realm of non-Hermitian physics.

3. Physical realization of odd density

In metamaterials, local resonances are often introduced into the background medium, leading to an effective medium with a frequency-dependent diagonal density tensor. In Figure 1(a), nonreciprocal couplings are introduced into a mass-in-mass unit, resulting in an asymmetric, frequency-dependent density tensor (Wu et al., 2023)

$$\rho_{ij} = \begin{bmatrix} \rho_{11}(\omega) & \rho_{12}(\omega) \\ 0 & \rho_{22}(\omega) \end{bmatrix}. \tag{5}$$

The nonreciprocal coupling is achieved using a piezoelectric sensor-actuator system, as shown in the bottom right panel of Figure 1(a). The top and bottom piezoelectric patches sense vertical deformation and apply voltages to the left and right piezoelectric patches, generating a horizontal load. As a result, vertical loads induce horizontal motion. However, the inverse process does not occur; that is, horizontal loads cannot generate vertical motion. This asymmetry gives rise to the medium exhibiting odd density behavior. The odd density constant introduces non-Hermiticity into the system; however, non-Hermiticity alone is not sufficient to break energy conservation, as demonstrated in PT-unbroken phases (El-Ganainy et al., 2018). Our system satisfies generalized PT symmetry, resulting in real eigenvalues (energy conserved) when the magnitude of normalized odd density constant $\hat{\beta}$ is small, and complex eigenvalues (energy not conserved) when magnitude of $\hat{\beta}$ is large, as shown in Figure 1(b). Meanwhile, the phase in which energy is not conserved depends on the propagating angle of the waves. As shown in Figure 1(b), for a large negative value of $\hat{\beta} = -1.18$, the $-45°$ wave is amplified, whereas for a large positive value of $\hat{\beta} = 1.18$, the 45° wave is amplified. In Figure 1(c), we experimentally validate wave amplification at 45° for a large positive odd density constant using a 2-by-2 unit cell odd density material, as shown in the left panel. The displacement field, presented in the middle panel of Figure 1(c), aligns well with the numerical

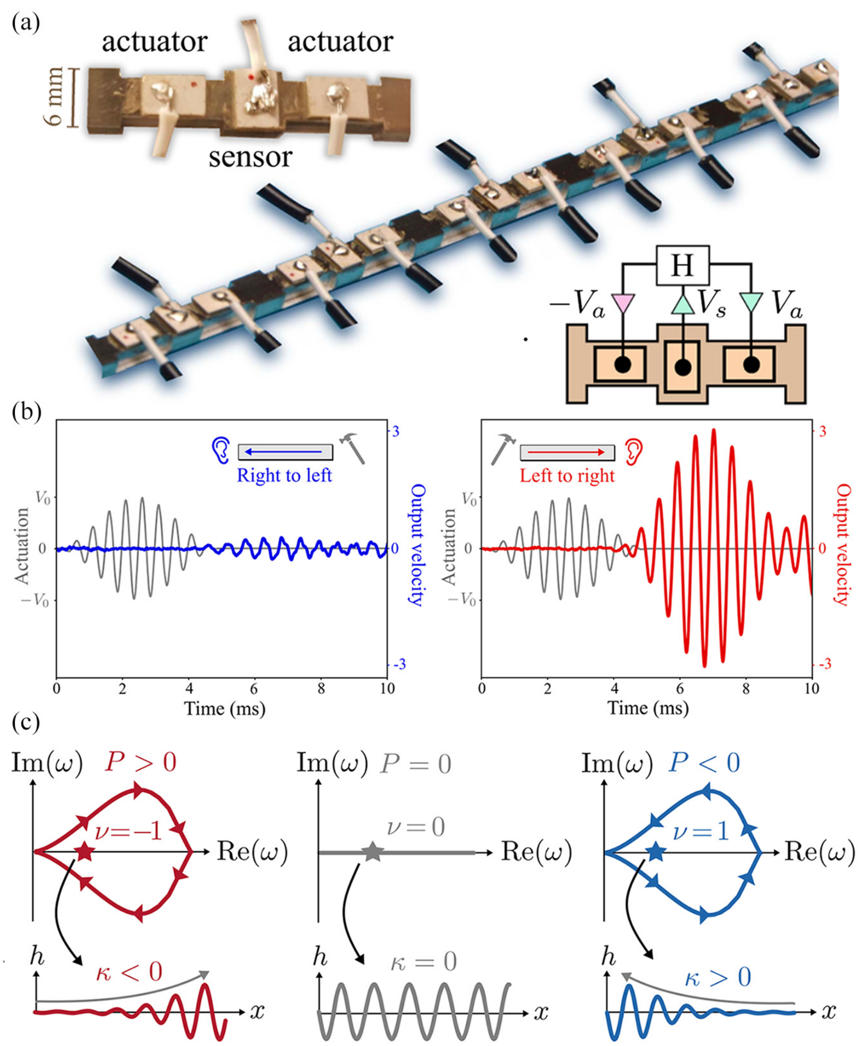

Figure 2. (a) Realization of an odd metabeam using a piezoelectric sensor-actuator system, (b) nonreciprocal wave propagation and unidirectional amplification, and (c) bulk topology of the frequency spectrum under periodic boundary conditions and the corresponding eigenmodes under open boundary conditions. Reprinted from "Realization of active metamaterials with odd micropolar elasticity," Chen Y., et al., Nature Communications*, 2021, under the terms of the Creative Commons Attribution License (CC BY). The publication is available at https://doi.org/10.1038/s41467-021-26034-z.

simulation shown in the rightmost panel of Figure 1(b). The corresponding normalized amplitude at different angles, shown in the right panel of Figure 1(c), demonstrates the amplification in the 45° direction.

4. Physical realization of odd elasticity

For the Timoshenko beam model, the generalized constitutive relation can be expressed as (Chen et al., 2021)

$$\begin{bmatrix} \sigma_{zx} \\ M \end{bmatrix} = \begin{bmatrix} \mu & \alpha+\beta \\ \alpha-\beta & B \end{bmatrix} \begin{bmatrix} s \\ b \end{bmatrix} \tag{6}$$

where M represents the bending moment, σ_{zx} denotes the shear stress, μ is the shear stiffness, B is the bending stiffness, α represents the symmetric cross-coupling, β denotes the antisymmetric cross-coupling, s is the shear strain, and b is the bending curvature. In the conventional case, the cross-coupling is zero, so the constitutive matrix is diagonal and symmetric. We now place a piezoelectric sensor at the center of the unit cell of the beam and add two actuators on either side of the unit cell, as shown in Figure 2(a). The sensor detects the bending deformation, while the shear load is applied to the beam by generating antisymmetric voltages across the two actuators. This sensing-actuating process couples the bending and shear deformations; however, it is nonreciprocal because shear strain cannot induce bending moments. As a result, the relations $\alpha+\beta=P$ and $\alpha-\beta=0$ are obtained, where P represents the odd coupling coefficient. The odd coupling coefficient introduces non-Hermiticity into the equations of motion, leading to complex asymmetric dispersion curves that indicate unidirectional amplification and attenuation. As shown in Figure 2(b), the signal detected on the left is attenuated under rightward excitation, while the signal detected on the right is amplified under leftward excitation. Thus, odd elasticity breaks the reciprocity theorem and violates energy conservation due to the exchange of energy between the odd beam and the circuits. The complex dispersion relation in the complex frequency domain forms a closed loop, naturally exhibiting topological properties characterized by the winding number, as shown in Figure 2(c). Remarkably, the nonzero winding numbers indicate the presence of skin modes in the beam under open boundary conditions, as predicted by the celebrated bulk-boundary correspondence (Okuma et al., 2020). The bulk-boundary correspondence between the winding number and skin modes reveals that the frequencies of skin modes are always enclosed by the complex dispersion curves, and the localization direction is determined by the sign of the winding numbers. As illustrated in Figure 2(c), the skin mode is localized to the right when the winding number is negative, localized to the left when the winding number is positive, and becomes extended when the winding number is zero.

5. Conclusion

In this research, we have explored odd elastic metamaterials, which break traditional symmetry constraints in density and elasticity tensors through the integration of active elements and nonreciprocal couplings. These materials exhibit unconventional wave dynamics, including non-Hermiticity, 1D unidirectional wave amplification, 2D directional wave amplification, and topological effects such as skin modes. These findings challenge conventional mechanics and open new opportunities for wave manipulation, energy harvesting, and vibration control, while providing a foundation for further advancements in non-Hermitian physics in elastic systems.

Acknowledgement

We would like to thank Dr. Yangyang Chen, Dr. Qian Wu, Dr. Xiaopeng Li, Dr. Xianchen Xu, and Mr. Honghua Qian for their contributions and participation in related works.

Declaration of conflicting interests

The authors declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding

The authors disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: This work is supported by the Air Force Office of Scientific Research under Grant No. AF 9550-18-1-0342 and AF 9550-20-1-0279 with Program Manager Dr. Byung-Lip (Les) Lee.

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Data availability statement

[AQ: 2] No data was used for the research described in the article.

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