BEPP 931 - Solution to Problem Set 4

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Question 1

We solve for the competitive general equilibrium by setting up a system of 11 (non-)linear equations. With 3 agents and 2 goods, we have to solve 6 FOCs, 3 budget constraints, 1 market clearing conditions (the second market will clear as well by Walras' Law) and the simplex condition for 11 unknowns (x^i_j, p_j, λ^i for $i = 1, 2, 3$ and $j = 1, 2$). When we use fsolve, we obtain:

$$\begin{aligned}
&p_1 = 0.814, p_2 = 0.186 \\
&\lambda^1 = 1.9553, \lambda^2 = 0.3229, \lambda^3 = 2.0809 \\
&x^1_1 = 0.6554, x^1_2 = 2.5078 \\
&x^2_1 = 1.9504, x^2_2 = 3.2170 \\
&x^3_1 = 1.3942, x^3_2 = 3.2752
\end{aligned}$$

We also tried to code up our own Newton algorithm by computing the Jacobian of the system of equations. The Jacobian is sparse since many derivatives are zero. Unfortunately, we run into troubles when solving for the step size: the Jacobian seems to be close to singular. Matlab's fsolve routine uses as default algorithm "trust-region-dogleg" which is a variant of the Powell hybrid method. The step size is a combination of the Newton step and and the Cauchy step that points to the direct of steepest descent.

As an alternative approach we use a fixed-point iteration to solve for the equilibrium prices. The fixed point iteration is executed on the specific function g:

$$\begin{aligned}
p^{k+1} &= g(p^k) \text{ and} \\
g_j(p) &= \frac{p_j + max(0, E_j(p))}{1 + \sum_{j=1}^{2} max(0, E_j(p))}
\end{aligned}$$

Within the fixed-point iteration we solve consumer's optimization program where we obtain their demands at given prices. This allows us to compute the excess demand and form $g(p)$. Since errors can propagate in hierarchical routines, we choose the default error bound of 10^{-6} for the demands and a looser bound of 10^{-4} for the fixed point. We also tried different bounds and different maximum iterations but we do not get close to the solution. In fact, the fixed point iteration does not converge and we obtain complex numbers for the prices.

Question 2

The asset pricing model is very similar to the CGE problem. We set up a system of non-linear equations. In particular, we have 6 FOCs for the 3 agents and 2 risky assets, and 2 market-clearing conditions for the 2 risky assets. By Walras Law the market for the safe asset clears as well. Each of the 6 FOCs is an expectation over the 4 equally probable states. We feed this system of equations into Matlab's fsolve routine and obtain the following asset prices and allocations:

$$\begin{aligned}
&p_1 = 1, p_2 = 1.0126, p_3 = 0.7864\\
&x_1^1 = 1.6180, x_2^1 = 0.7501, x_3^1 = 0.7915\\
&x_1^2 = 2.6867, x_2^2 = 0.2412, x_3^2 = 0.8161\\
&x_1^3 = 1.6953, x_2^3 = 0.0087, x_3^3 = 0.3923
\end{aligned}$$

Question 3

Goodwill Advertising

By design, the probability that a randomly chosen consumer purchases from firm 1 is

$$D_1\left(p_1, p_2; v_1, v_2\right) = \frac{\exp\left(v_1 - p_1\right)}{1 + \exp\left(v_1 - p_1\right) + \exp\left(v_2 - p_2\right)}.$$

The profit-maximization problem of firm 1 is thus given by

$$\max_{p_1 \geq 0} D_1\left(p_1, p_2; v_1, v_2\right) p_1.$$

The first-order condition (FOC) for an interior solution is

$$0 = 1 - \frac{1 + \exp\left(v_2 - p_2\right)}{1 + \exp\left(v_1 - p_1\right) + \exp\left(v_2 - p_2\right)} p_1.$$

The derivations for firm 2 are analogous.

There is no need to check for optimality apart from solving FOCs, since given goodwill levels v_1 and v_2, it can be shown that there exists a unique Nash equilibrium $\left(p_1^*\left(v_1, v_2\right), p_2^*\left(v_1, v_2\right)\right)$ of the product market game (Caplin Nalebuff 1991).

Now we perform the computations for $(v_1, v_2) \in \{0, 0.5, 1, \ldots, 10\}^2$ and plot the results.

From Figure 1 and Figure 2, it can be seen that the higher level of one's own goodwill, the lower level of the other's goodwill, the higher price strategy one adopts and ultimately the more profits one earns.

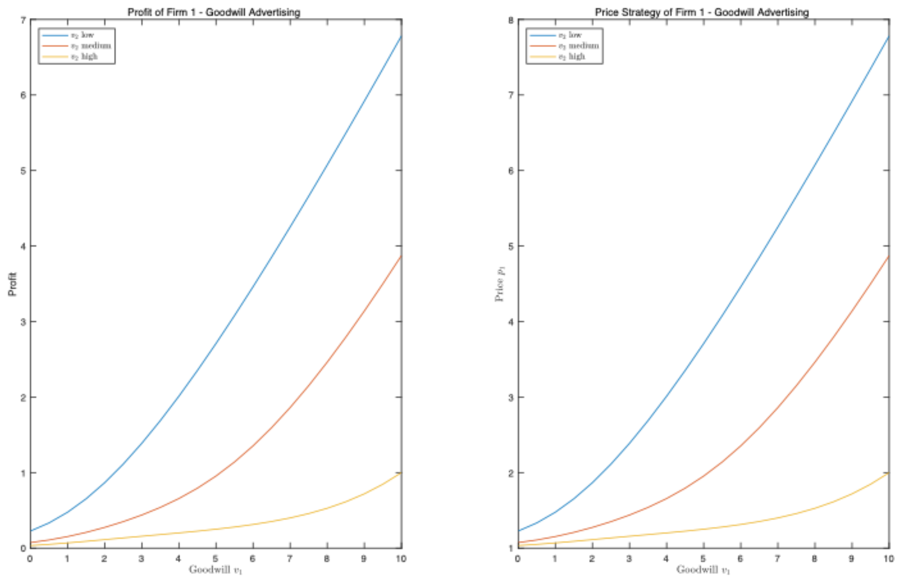

Figure 1. Solution Results for Goodwill Advertising - 2D

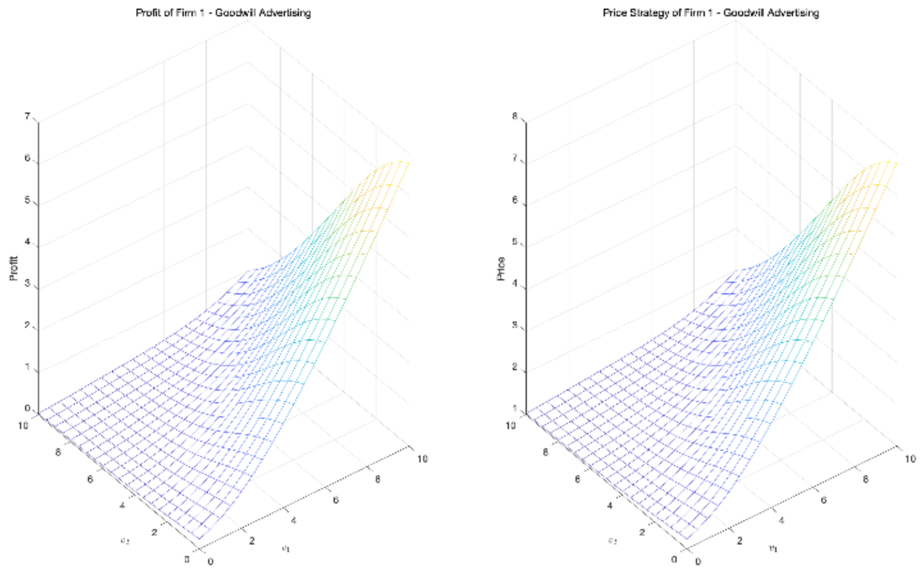

Figure 2. Solution Results for Goodwill Advertising - 3D

Awareness Advertising

Compute the Nash Equilibrium

By design, the probability that a randomly chosen consumer purchases from firm 1 is

$$D_1\left(p_1, p_2; s_1, s_2, v\right) = s_1\left(1-s_2\right)\frac{\exp\left(v-p_1\right)}{1+\exp\left(v-p_1\right)} + s_1 s_2 \frac{\exp\left(v-p_1\right)}{1+\exp\left(v-p_1\right)+\exp\left(v-p_2\right)}.$$

The profit-maximization problem of firm 1 is thus given by

$$\max_{p_1 \geq 0} D_1\left(p_1, p_2; s_1, s_2; v\right) p_1.$$

The first-order condition (FOC) for an interior solution is

$$\begin{aligned} 0 = &\left(1-s_2\right)\frac{\exp\left(v-p_1\right)}{1+\exp\left(v-p_1\right)}\left(1-\frac{1}{1+\exp\left(v-p_1\right)}p_1\right) \\ &+ s_2 \frac{\exp\left(v-p_1\right)}{1+\exp\left(v-p_1\right)+\exp\left(v-p_2\right)}\left(1-\frac{1+\exp\left(v-p_2\right)}{1+\exp\left(v-p_1\right)+\exp\left(v-p_2\right)}p_1\right) \end{aligned}.$$

The derivations for firm 2 are analogous.

Now we compute and plot the Nash equilibrium $\left(p_1^*\left(s_1, s_2, v\right), p_2^*\left(s_1, s_2, v\right)\right)$ and firms' profits $\left(\pi_1^*\left(s_1, s_2, v\right), \pi_2^*\left(s_1, s_2, v\right)\right)$ for $\left(s_1, s_2\right) \in \{0, 0.05, 0.1, \ldots, 1\}^2$ and $v \in \{4, 8, 9, 10\}$.

From Figure 3 and Figure 4, we no longer see pure monotonicity anymore as in the goodwill advertising case. This is reasonable since a firm faces a choice between setting a low price in order to be competitive and exploiting its captive segment by setting a high price. Roughly speaking, there is less room for profit when both firms' shares are high than when one's own high and the other's low, in which case firms both have to compete by setting prices low.

Optimality Check

Notice we have to check for global optimality since in contrast to goodwill advertising there may not be a Nash equilibrium in pure strategies. The reason is that a firm faces a choice between setting a low price in order to be competitive and exploiting its captive segment by setting a high price. This may give rise to a discontinuity in the firm's best reply function and ultimately lead to nonexistence.

I check for both global optimality by the definition of Nash Equilibrium and local optimality by calculating analytically FOCs and SOCs at the computed optimal points. All the solutions survive the check globally, but a few of them (around 10 in 21*21*4 cases) have SOCs calculated to be around positive 0.002, which though doesn't fail them in the local optimality check, since most of them are border points.

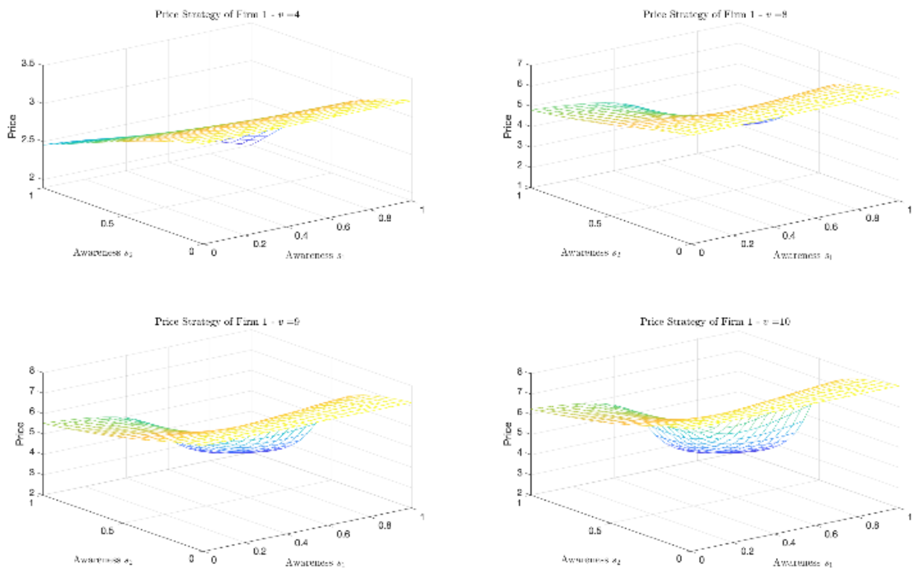

Figure 3. Price Strategy for Awareness Advertising

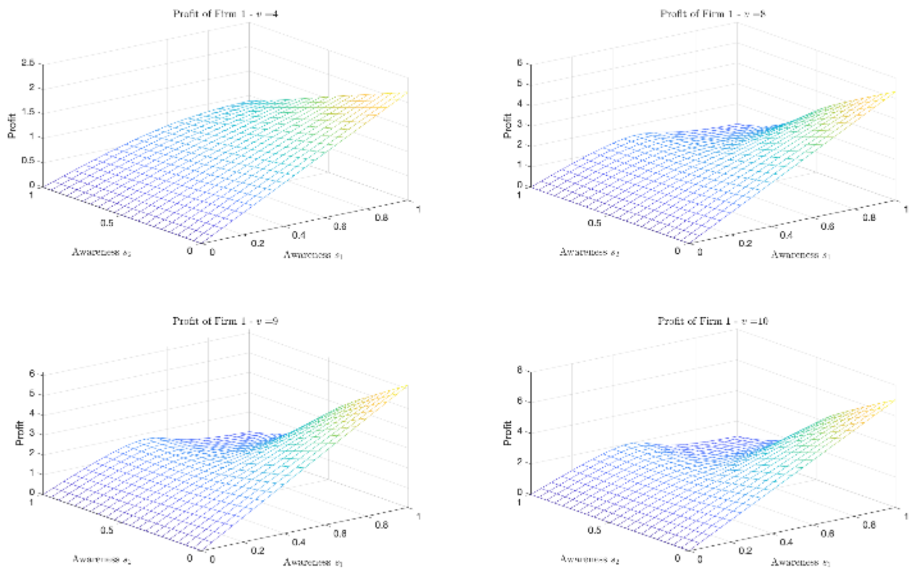

Figure 4. Profit for Awareness Advertising

Question 4

Exercise 6.1.

The degrees 1, 2, and 3 Taylor expansions near $x_0 = 1$, along with the log-linear approximation, the log-quadratic approximation, and the log-cubic approximation of f near $x_0 = 1$ are computed for the function:

$$f(x) = (x^{1/2} + 1)^{2/3} \qquad \textbf{Equation } 1.$$

To obtain the Taylor series, the symbolic math toolbox from MATLAB was used.

Similarly, the $(1, 1)$ Padé approximation based at $x_0 = 1$ was computed using the "pade" command from MATLAB.

To evaluate and compare the quality of the approximations, we use the following figure for $x \in [0.2, 3]$:

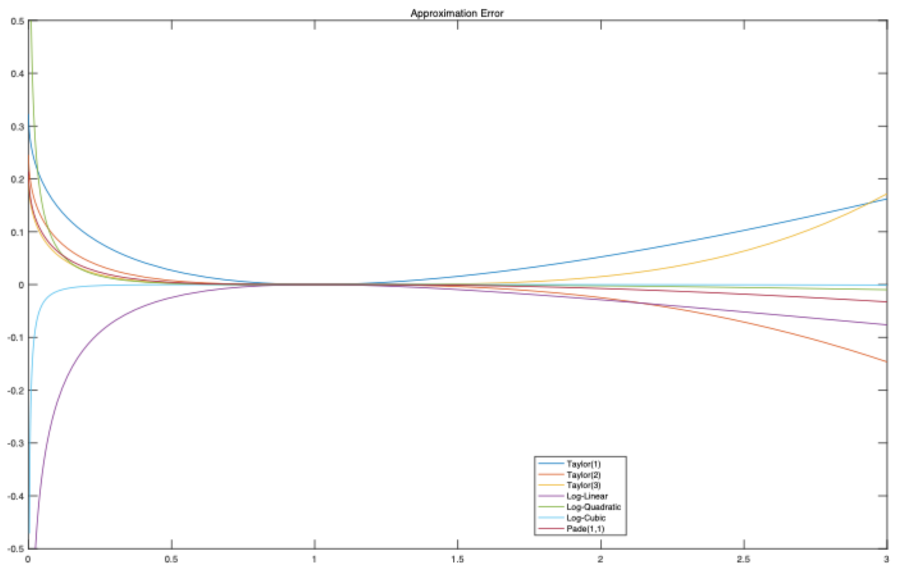

Figure 5. Approximation Errors

As can be observed from the figure above, the log-cubic approximation seems to do the better job in approximating the function on this particular interval. The next closest approximation seems to be the log-quadratic, whose errors fall fairly close to those of the log-cubic approximation, especially for the middle of the range.

The degree 1 Taylor approximation seems to perform the worst when compared to the others. The degree 3 Taylor approximation also performs significantly worse than the others in the upper half of the range.

Question 5

Exercise 6.7

We now compute the linear minimax approximation on $[0, 1]$ for the following function:

$$f(x) = e^x \qquad \textbf{Equation } 2.$$

The Legendre and Chebyshev approximations are also computed for this function, where for the Legendre approximation we use the "legendrefit" function in MATLAB. The following figure plots the errors of these three approximations on the interval $[0, 1]$:

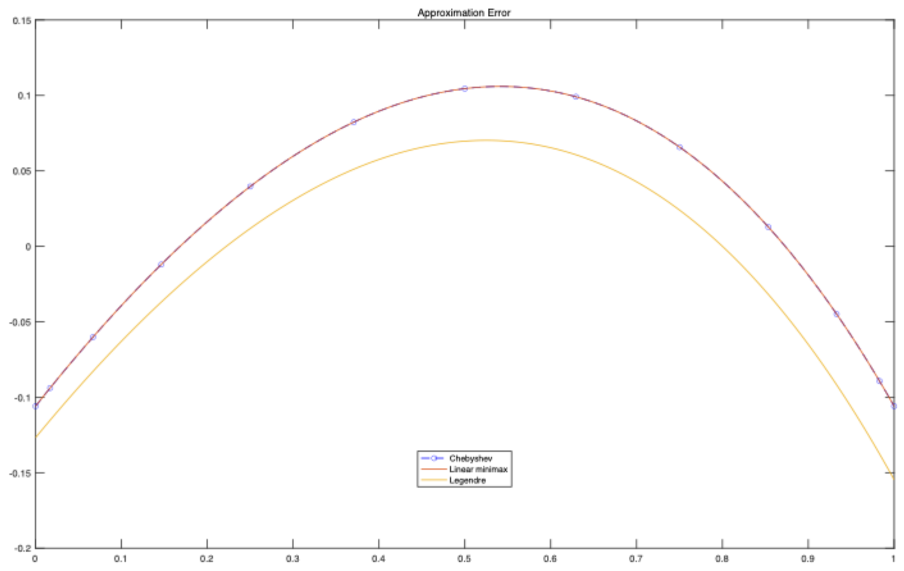

Figure 6. Approximation Errors

As can be observed from above, the Legendre approximation seems to perform significantly better than the Chebyshev and Linear Minimax approximations. The Chebyshev and Linear Minimax approximations seem to perform equally well for this function on this particular interval.

Question 6

We compute coefficients for the Lagrange and Hermite interpolating polynomials $p(x)$ to be applied on the values of $f(x) = x^{\frac{1}{3}}$ and its derivative $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$. The Lagrange interpolant uses only values of $f(x)$ whereas Hermite uses in addition derivative information to fit $p(x)$. We interpolate using 5 points in the set $X = \{\frac{i}{10} | i = 1, 2, 3, 4, 5\}$. The Lagrange polynomial is hence of degree 4 and the Hermite polynomial of degree 9. The figure below plots the approximation error of the Lagrange and Hermite interpolation. Unsurprisingly, the Hermite interpolation performs relatively better since it uses more information than Lagrange.

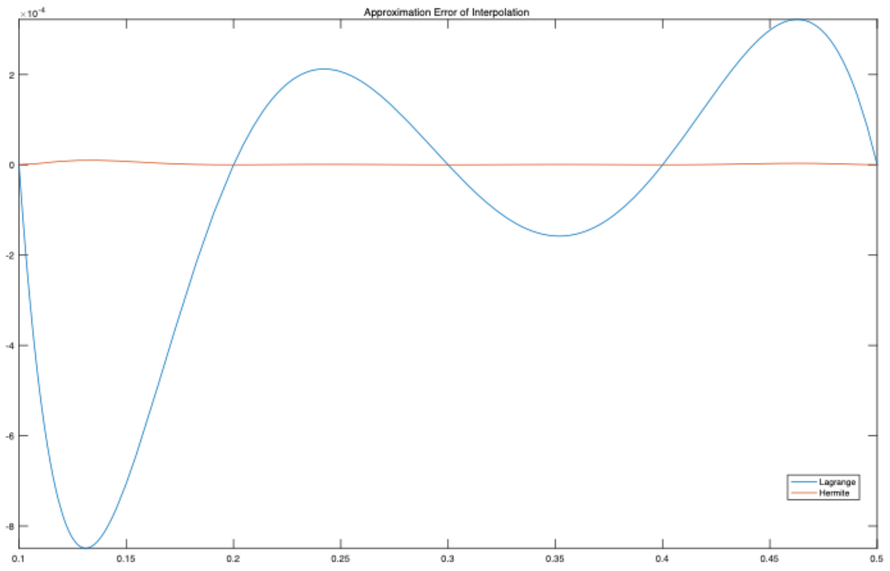

Figure 7. Errors of Approximations - Lagrange VS Hermite

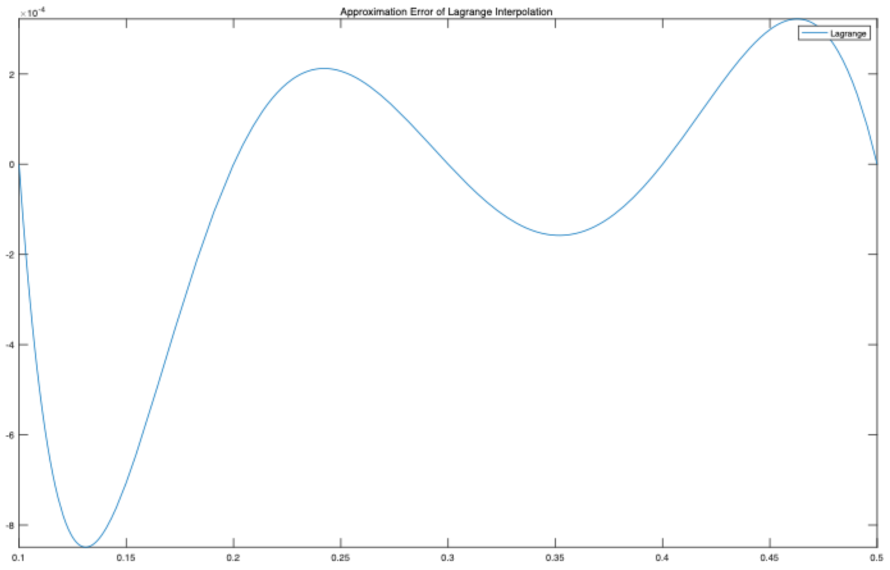

Figure 8. Lagrange Approximation Errors

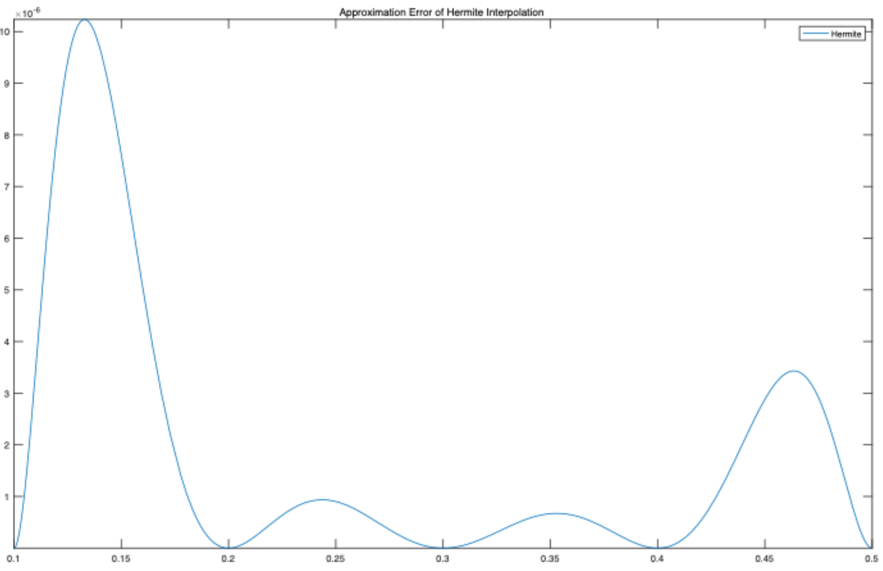

Figure 9. Hermite Approximation Errors

Question 7

Below we approximate three functions with each of its characteristics. The first function e^{2x+2} is an infinitely smooth function with no singularity. The second function $(x+1)^{1/4}$ is a smooth one with singularity at $x = -1$. The third one is a kinky one, $\min[\max[-1, 4(x-0.2)], 1]$.

We see from the approximation figures that for the first and last ones, Chebyshev performs the best, while for the second one, spline works the best. This is because Chebyshev performs well expect when handling singularities, while spline is better at dealing with singularities than Chebyshev.

In short, the Chebyshev interpolation approximation had the smallest maximum error and most nearly satisfied the equioscillation property, as predicted by our theorems, except when handling singularity.

Below I replicate Figures 6.7, 6.8, and 6.9. I examine each of the functions on the interval [-1, 1]. To be fair, we use for each function methods which have five free parameters.

For Spline Interpolation, I use Matlab's spline to implement the cubic spline interpolation.

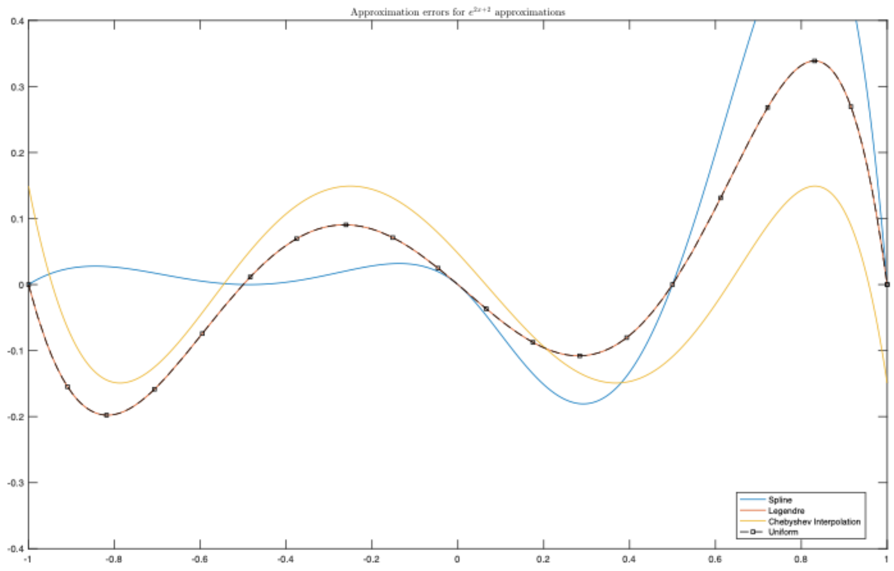

Figure 10. Function Interpolations 6-7

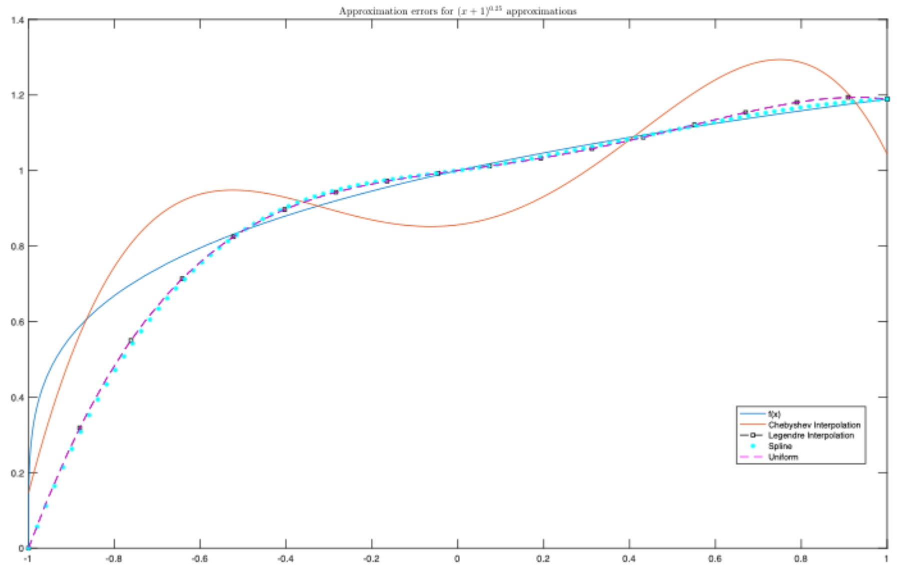

Figure 11. Function Interpolations 6-8

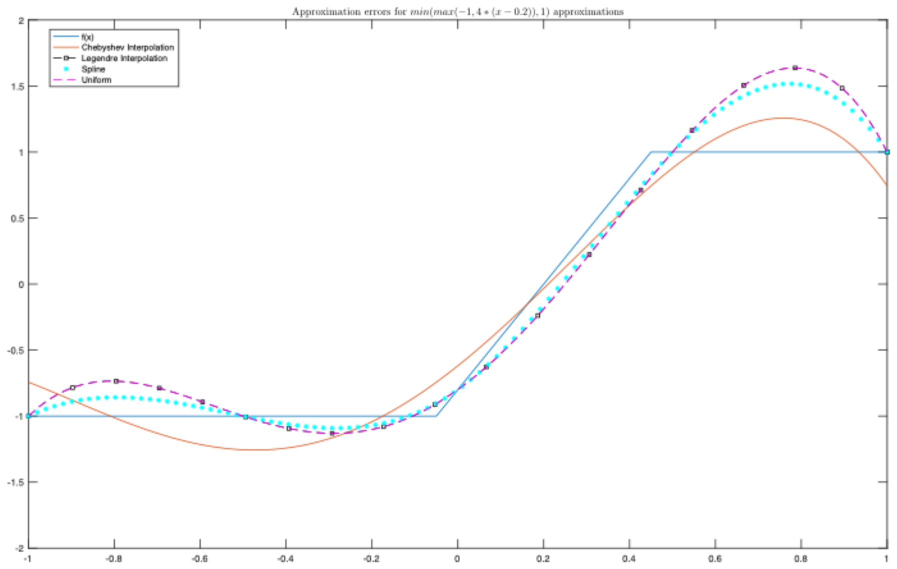

Figure 12. Function Interpolations 6-9