BEPP 931 - Solution to Problem Set 8

Projection Method and Finite Difference Method

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Question 1 - Quality Ladder - Monopoly

We first solve for the per period profits $\pi(\omega)$ which are independent of investments x. Figure 1 plots the profits against the quality levels ω. Profits increase as quality improves but then reach a plateau when $\omega \geq 6$.

In figure 2 we plot the value function $V(\omega)$ and the policy function $x(\omega)$ against quality levels ω. The value function is monotonically increasing in quality but levels off for higher quality levels. Optimal investments $x(\omega)$ sharply increase when quality is low mirroring the sharp increase in profits when quality is still at a low level. When $\omega \geq 4$, optimal investments sharply decline and approach zero as we reach the highest quality levels.

We perform both value and policy function iteration. Our initial guess for the value function is $V^0(\omega) = \frac{\pi(\omega)}{1-\beta}$ corresponding to discounted profits when remaining at state ω forever. The value function iteration requires 127 iterations to converge. Our policy function iteration converges in only 7 iterations using Howard's improvement algorithm and we obtain the same value and policy function.

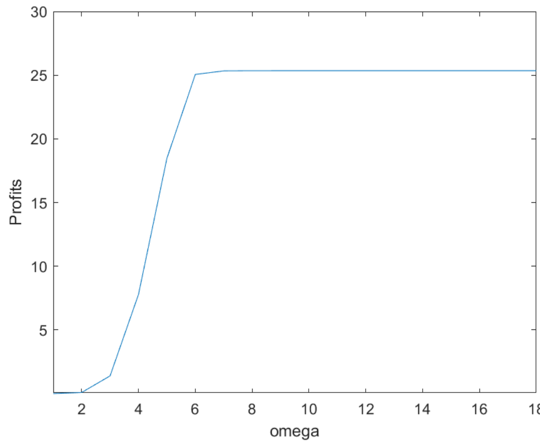

Figure 1. Quality Ladder - Monopoly Profits

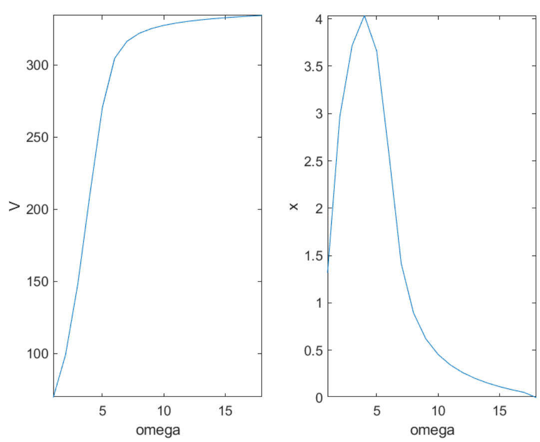

Figure 2. Quality Ladder - Monopoly Value and Policy Function

Question 3 - Quality Ladder - Duopoly

We now turn to the Quality Ladder model in a duopoly setting. Similar to the monopoly case, we can decouple the pricing decision from the optimal investments. That is, optimal price strategies ($p_1(\omega)$,$p_2(\omega)$) and profits ($\pi_1(\omega)$,$\pi_2(\omega)$) are functions of qualities of both players $\omega = (\omega_1, \omega_2)$ only and do not depend on investments.

We solve for (one of) the Markov perfect equilibria following Pakes & McGuire (1994) algorithm. Figure 3 displays the value function, policy function and profits for player 1. Profits for player 1 are low for low qualities ω_1 irrespective of player 2's quality ω_2. Profits are high when ω_1 is not in the lowest range and ω_2 is low. Profits sharply decline as ω_2 increases to medium levels. The value function for player 1 shows a similar pattern. Optimal investments for player 1 show a similar profile to the monopoly case keeping ω_2 fixed. When both ω_1 and ω_2 are low, the increase in x for player 1 is particularly steep.

We compare Gauss-Jacobi and Gauss-Seidel acceleration methods and combine it with extrapolation. In our Gauss-Jacobi algorithm, we first traverse the whole state space ω to then update $V(\omega)$. In contrast, with Gauss-Seidel we update those elements of $V(\omega)$ as soon as we have computed them and use the updated matrix for the next state. As expected, Gauss-Seidel is faster (157 vs 138 iterations without extrapolation). Adding extrapolation with $\lambda = 1.8$, more than halves the number of iterations required. We experimented with different initial guesses for the optimal investment matrix $x(\omega)$ (e.g. zero matrix, identity matrix and generating random numbers), but after all,the algorithm converges to the same equilibrium.

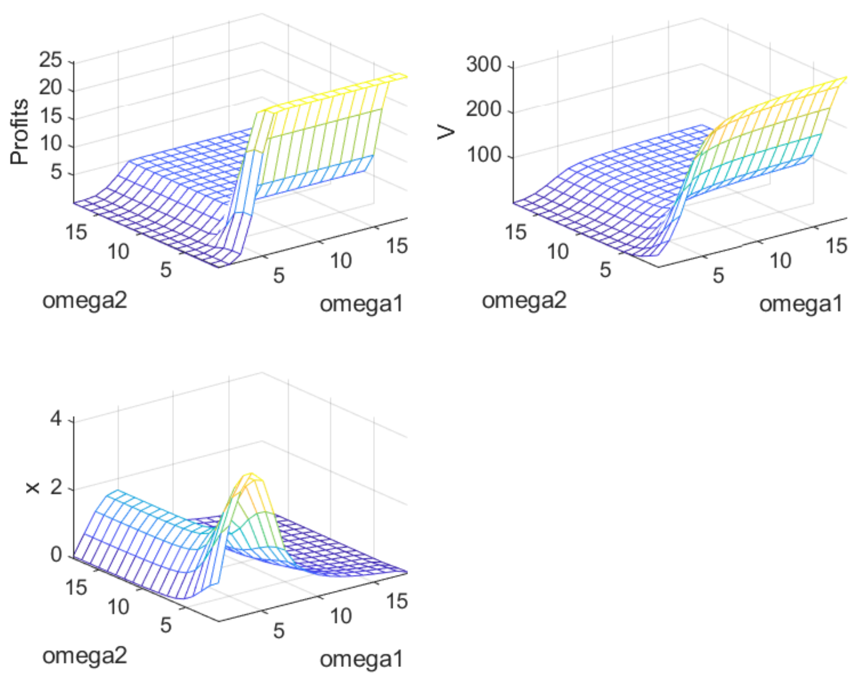

Figure 3. Quality Ladder - Duopoly

Question 2 - Learning By Doing - Monopoly

Learning-by-doing monopoly. The exercise below is inspired by the learning-by-doing model of Besanko, Doraszelski, Kryukov & Satterthwaite (2010).

The monopolist is characterized by its production experience $\omega \in \{1, \dots, L\}$. The marginal cost of production,

$c(\omega)$, depends on this stock of know-how. In particular, the monopolist faces a learning curve given by

$$c(\omega) = \begin{cases} \kappa\omega^{\eta} & \text{if} \quad 1 \leq \omega < l \\ \kappa l^{\eta} & \text{if} \quad l \leq \omega \leq L \end{cases}$$

where $\eta = \frac{\ln \rho}{\ln 2}$ for a learning curve with a slope of ρ percent, κ is the marginal cost with minimal know-how (normalized to be $\omega = 1$), and $l < L$ represents the stock of know-how at which the firm reaches the bottom of its learning curve.

The model is cast in discrete time and has an infinite horizon to avoid end effects.

Product market

The industry draws its customers from a large pool of potential buyers. In each period, one buyer enters the market and makes, at most, one purchase.

The utility consumer m derives from purchasing the product is $v - p + \epsilon_m$, where v represents the quality of the product and ϵ_m represents the buyer's idiosyncratic preference for the product. There is a no-purchase alternative with utility normalized to zero. I assume that the idiosyncratic shock ϵ_m is independently and identically logistically distributed across consumers. Moreover, the monopolist does not observe the buyer's idiosyncratic preference. The probability that the monopolist makes the sale therefore is

$$\Pr(q = 1) = D(p) = \frac{\exp(v - p)}{1 + \exp(v - p)}$$

.

Pricing dynamics

The monopolist's state ω represents its know-how in the present period. Its know-how in the subsequent period, ω', depends on whether or not it makes a sale and on whether or not its stock of know-how depreciates.

The probability that the stock of know-how depreciates is $\Pr(f = 1) = \Delta(\omega) = 1 - (1 - \delta)^{\omega}$, where $\delta \in [0, 1]$. This specification is conceptually similar to the deterministic "capital-stock" models of depreciation employed in the empirical work on organizational forgetting (Argote, Beckman & Epple 1990, Argote & Epple 1990, Benkard 2004), where the depreciation of know-how increases as the firm's stock of know-how increases.

Law of Motion

The law of motion[1] for the monopolist's stock of know-how is

$$\omega' = \omega + q - f$$

where $q \in \{0,1\}$ indicates whether the monopolist makes a sale and $f \in \{0,1\}$ represents organizational forgetting. The monopolist's stock of know-how therefore changes according the transition function

$$\Pr(\omega'|\omega, q) = \begin{cases} 1 - \Delta(\omega) & \text{if} \quad \omega' = \omega + q \\ \Delta(\omega) & \text{if} \quad \omega' = \omega + q - 1 \end{cases}$$

At the upper and lower boundaries of the state space, I take the transition function to be $\Pr(L|L, q = 1) = 1$ and $\Pr(1|1, q = 0) = 1$, respectively.

Bellman Equation

Let $V(\omega)$ denote the expected net present value of future cash flows to the monopolist if the current state is ω. The Bellman equation is

$$V(\omega) = \max_p D(p)(p - c(\omega)) + \beta \left[D(p)W_1(\omega) + (1 - D(p))W_0(\omega)\right]$$

where $\beta \in [0, 1)$ is the discount factor. The Bellman equation adds the current cash flow to the appropriately discounted expected future cash flow, where

$$W_1(\omega) = \sum_{\omega'=1}^{L} V(\omega') \Pr(\omega'|\omega, q = 1)$$
$$W_0(\omega) = \sum_{\omega'=1}^{L} V(\omega') \Pr(\omega'|\omega, q = 0)$$

are the expectation of the value function conditional on winning and losing the sale, respectively. Note that the monopolist is allowed to price below marginal cost or even set a negative price[2].

Optimality Conditions

The monopolist's pricing strategy is given by

$$p(\omega) = \arg\max_p D(p)(p - c(\omega)) + \beta \left[D(p)W_1(\omega) + (1 - D(p))W_0(\omega)\right]$$

.

[1]The probability of transition from the status quo to next periods state, determined by the state and the choice variables.
[2]This lack of limitation opens up to research areas of financing structure.

Let $h(p) = D(p)(p - c(\omega)) + \beta\left[D(p)W_1(\omega) + (1 - D(p))W_0(\omega)\right]$ denote the maximand on the RHS of the Bellman equation. Differentiating with respect to p and making use of the assumed functional forms yields

$$\frac{\partial h}{\partial p} = D(p)\left[1 - (p - c(\omega)) - \beta W_1(\omega) + h(p)\right]$$

.

Differentiating this again and combining terms gives

$$\frac{\partial^2 h}{\partial p^2} = -\frac{\partial h}{\partial p}(1 - 2D(p)) - D(p)$$

.

Thus, $\frac{\partial h}{\partial p} = 0 \Rightarrow \frac{\partial^2 h}{\partial p^2} = -D(p) < 0$, i.e., the objective function is strictly quasi-concave and the price choice $p(\omega)$ is therefore unique. It is found by numerically solving $\frac{\partial h}{\partial p} = 0$ or, equivalently,

$$0 = 1 - (1 - D(p))\left[(p - c(\omega)) + \beta\left(W_1(\omega) - W_0(\omega)\right)\right]$$

.

Parameterization

The number of know-how levels is $L = 30$, the slope of the learning curve is $\rho = 0.85$, the marginal cost of production with minimal know-how is $\kappa = 10$, the learning curve flattens out at $l = 15$ units of know-how, the quality of the good is $v = 10$, the depreciation probability is $\delta = 0.03$, and the discount factor is $\beta = \frac{1}{1.05}$, which corresponds to a yearly interest rate of 5%

Learning-by-Doing & Organizational Forgetting

Figure 4 presents the two main ingredients of the model above - learning by doing and organizational forgetting.

On the left axis is depicted the depreciation probability of the level of the state variable - know-how, which is an increasing and concave function in accordance with the Jost's second law of forgetting[3].

The red line shows the decreasing marginal cost as a function of know-how. Learning by doing, the company decreases its marginal cost of production as it moves down its learning curve until the bottom is hit.

Hence we can see that learning-by-doing and organizational forgetting are two opposite forces in deciding the firm's profit.

[3]I.e., given two associations of equal strength, the older will decrease less with the passage of time.

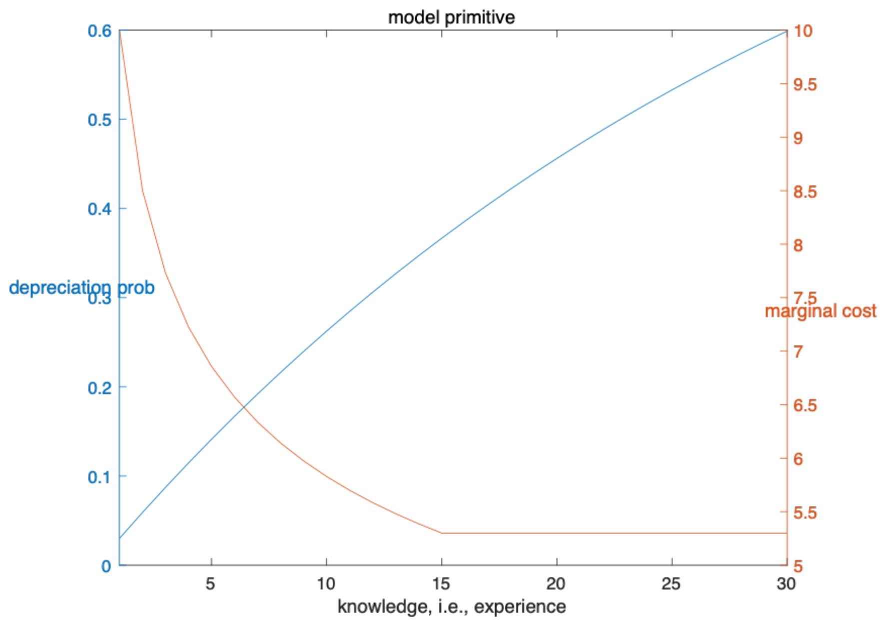

Figure 4. Setup - Learning by Doing

Solving the Bellman Equation by VFI & PFI

Figure 5 shows the solution to the Bellman Equation using both value function iteration and policy function iteration.

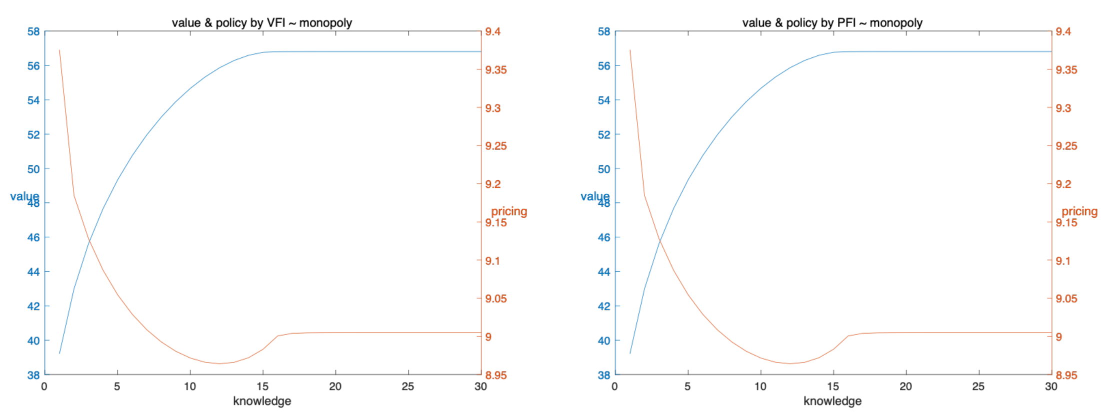

Figure 5. Monopoly

Using the same initial guess and tolerance level, the VFI and the PFI arrives at the same solution, and the result is intuitive - the value function increases with the knowledge stock, to the point around the learning bottom when the marginal cost stops decreasing and the effect of depreciation is all the way mild. The policy function is also seen to shift in patterns after the learning bottom.

Under the parameterization of $\delta = 0.03$, the effect of organizational forgetting is all the way mild, thus couldn't have conspicuous impact on the policy nor the value. To discover and make seen the depreciation effect, I let $\delta = 0.1$ and the solution comes out as in Figure 6. The change in both the value and the policy functions after the learning bottom is more legible.

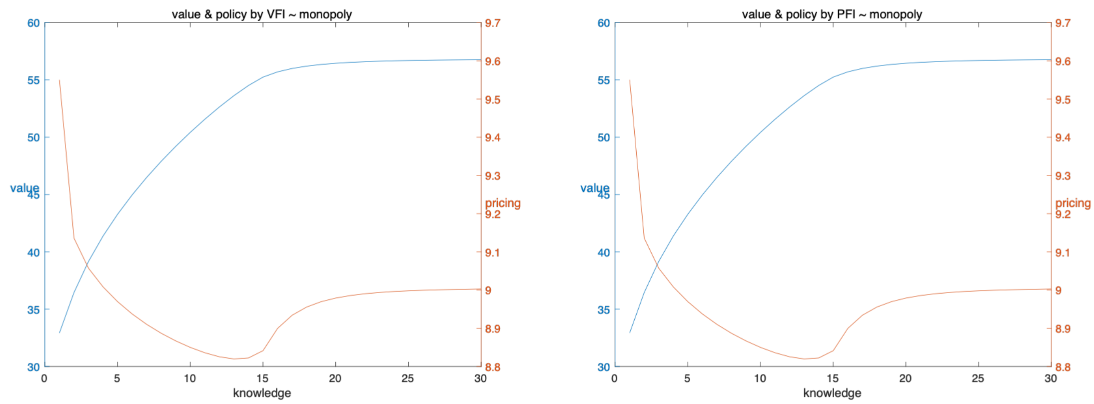

Figure 6. Monopoly - Radical depreciation

Question 4 - Learning By Doing - Duopoly

Learning-by-doing duopoly. This exercise introduces competition into the learning-by-doing model from Exercise $10-2$. since many of the ingredients of the model remain the same, I focus on pointing out the differences between the dynamic game and the single-agent dynamic programming problem from a previous exercise.

There are two firms. Firm n is characterized by its production experience $\omega_n \in \{1, \ldots, L\}$ The state space is thus $\Omega \in \{1, \ldots, L\}^2$. I refer to $\omega = (\omega_1, \omega_2)$ as the state of the industry and to ω_n as the state of firm n.

Product market

The probability that firm n makes the sale is

$$D_n(p_1, p_2) = \frac{\exp(v - p_n)}{1 + \sum_{k=1}^{2} \exp(v - p_k)}$$

.

Pricing dynamics

The Bellman equation for firm n is

$$V_n(\omega) = \max_{p_n} D_n\left(p_n, p_{-n}(\omega)\right)\left(p_n - c\left(\omega_n\right)\right) + \beta \sum_{k=0}^{2} D_k\left(p_n, p_{-n}(\omega)\right) W_{nk}(\omega)$$

,

where $p_{-n}(\omega)$ is the rival's pricing strategy, and

$$W_{n0}(\omega) = \sum_{\omega_1'=1}^{L} \sum_{\omega_2'=1}^{L} V_n\left(\omega'\right) \Pr\left(\omega_1' | \omega_1, q_1 = 0\right) \Pr\left(\omega_2' | \omega_2, q_2 = 0\right)$$
$$W_{n1}(\omega) = \sum_{\omega_1'=1}^{L} \sum_{\omega_2'=1}^{L} V_n\left(\omega'\right) \Pr\left(\omega_1' | \omega_1, q_1 = 1\right) \Pr\left(\omega_2' | \omega_2, q_2 = 0\right)$$
$$W_{n2}(\omega) = \sum_{\omega_1'=1}^{L} \sum_{\omega_2'=1}^{L} V_n\left(\omega'\right) \Pr\left(\omega_1' | \omega_1, q_1 = 0\right) \Pr\left(\omega_2' | \omega_2, q_2 = 1\right)$$

is the expectation of the value function of firm n conditional on the buyer purchasing good $k \in \{0, 1, 2\}$ (good 0 is the outside good).

Optimality Conditions

The pricing strategy of firm n is given by

$$p_n(\omega) = \arg\max_{p_n} D_n\left(p_n, p_{-n}(\omega)\right)\left(p_n - c\left(\omega_n\right)\right) + \beta \sum_{k=0}^{2} D_k\left(p_n, p_{-n}(\omega)\right) W_{nk}(\omega)$$

.

Let $h_n\left(p_n\right) = D_n\left(p_n, p_{-n}(\omega)\right)\left(p_n - c\left(\omega_n\right)\right) + \beta \sum_{k=0}^{2} D_k\left(p_n, p_{-n}(\omega)\right) W_{nk}(\omega)$ denote the maximand on the RHS of the Bellman equation. $h_n\left(p_n\right)$ is strictly quasi-concave and the price choice $p_n(\omega)$ is therefore unique. It is found by numerically solving $\frac{\partial h_n}{\partial p_n} = 0$ or, equivalently,

$$0 = 1 - \left(1 - D_n\left(p_n, p_{-n}(\omega)\right)\right)\left(p_n - c\left(\omega_n\right)\right) - \beta W_{nn}(\omega) + \beta \sum_{k=0}^{2} D_k\left(p_n, p_{-n}(\omega)\right) W_{nk}(\omega)$$

.

Equilibrium

The primitives are symmetric. I therefore restrict attention to symmetric Markov perfect equilibria (MPE). Such a MPE is characterized by a value function $V(\omega)$ and a policy function $p(\omega)$ such that, if $V(\omega)$ is firm 1 's value function, then firm 2 's value function is given by $V_2(\omega_1, \omega_2) = V(\omega_2, \omega_1)$. Similarly, if $p(\omega)$ is firm 1's policy function, then firm 2 's policy function is given by $p_2(\omega_1, \omega_2) = p(\omega_2, \omega_1)$. Existence of a symmetric MPE in pure strategies follows from the arguments in Doraszelski & Satterthwaite (2010) provided that prices are bounded.

Algorithm

To compute the MPE, Pakes & McGuire (1994) suggest an algorithm that essentially adapts value function iteration to dynamic games. The algorithm proceeds as follows:

1. Make initial guesses for the value and policy functions (or, more precisely, $L \times L$ matrices), $\mathbf{V}^0$ and $\mathbf{p}^0$, choose a stopping criterion $\epsilon > 0$, and initialize the iteration counter to $l = 1$.

2. 2. For all states $\omega \in \Omega$ compute

$$p^{l+1}(\omega) = \arg\max_{p_1} D_1\left(p_1, p_2^l(\omega_2, \omega_1)\right)(p_1 - c(\omega_1)) + \beta \sum_{k=0}^{2} D_k\left(p_1, p_2^l(\omega_2, \omega_1)\right) W_k^l(\omega)$$

.

and

$$V^{l+1}(\omega) = D_1\left(p_1^{l+1}(\omega), p_2^l(\omega_2, \omega_1)\right)\left(p_1^{l+1}(\omega) - c(\omega_1)\right) + \beta \sum_{k=0}^{2} D_k\left(p_1^{l+1}(\omega), p_2^l(\omega_2, \omega_1)\right) W_n^l(\omega)$$

,

where

$$W_0^l(\omega) = \sum_{\omega_1'=1}^{L} \sum_{\omega_2'=1}^{L} V^l(\omega') \Pr(\omega_1' | \omega_1, q_1 = 0) \Pr(\omega_2' | \omega_2, q_2 = 0)$$
$$W_1^l(\omega) = \sum_{\omega_1'=1}^{L} \sum_{\omega_2'=1}^{L} V^l(\omega') \Pr(\omega_1' | \omega_1, q_1 = 1) \Pr(\omega_2' | \omega_2, q_2 = 0)$$
$$W_2^l(\omega) = \sum_{\omega_1'=1}^{L} \sum_{\omega_2'=1}^{L} V^l(\omega') \Pr(\omega_1' | \omega_1, q_1 = 0) \Pr(\omega_2' | \omega_2, q_2 = 1)$$

3. If

$$\max_{\omega \in \Omega} \left| \frac{V^{l+1}(\omega) - V^l(\omega)}{1 + |V^{l+1}(\omega)|} \right| < \epsilon \quad \wedge \quad \max_{\omega \in \Omega} \left| \frac{p^{l+1}(\omega) - p^l(\omega)}{1 + |p^{l+1}(\omega)|} \right| < \epsilon$$

then stop; else increment the iteration counter l by one and go to step 2.

Unlike value function iteration for single-agent dynamic programming problems, there is no guarantee that the above algorithm converges. If it fails to converge, a trick that often works is to go through an additional dampening step before returning to step 2. This dampening step assigns

$$\mathbf{V}^{l+1} \leftarrow \lambda \mathbf{V}^{l+1} + (1 - \lambda) \mathbf{V}^l$$
$$\mathbf{p}^{l+1} \leftarrow \lambda \mathbf{p}^{l+1} + (1 - \lambda) \mathbf{p}^l$$

for some $\lambda \in (0, 1)$

The algorithm converges without dampening.

Figure 7 shows the value and the policy functions. The pricing policy is quite flat, except when both firms have little know-how and wants to get down the learning curve. This is also when depreciation/forgetting probability is quite low, so it's worth fighting a price war.

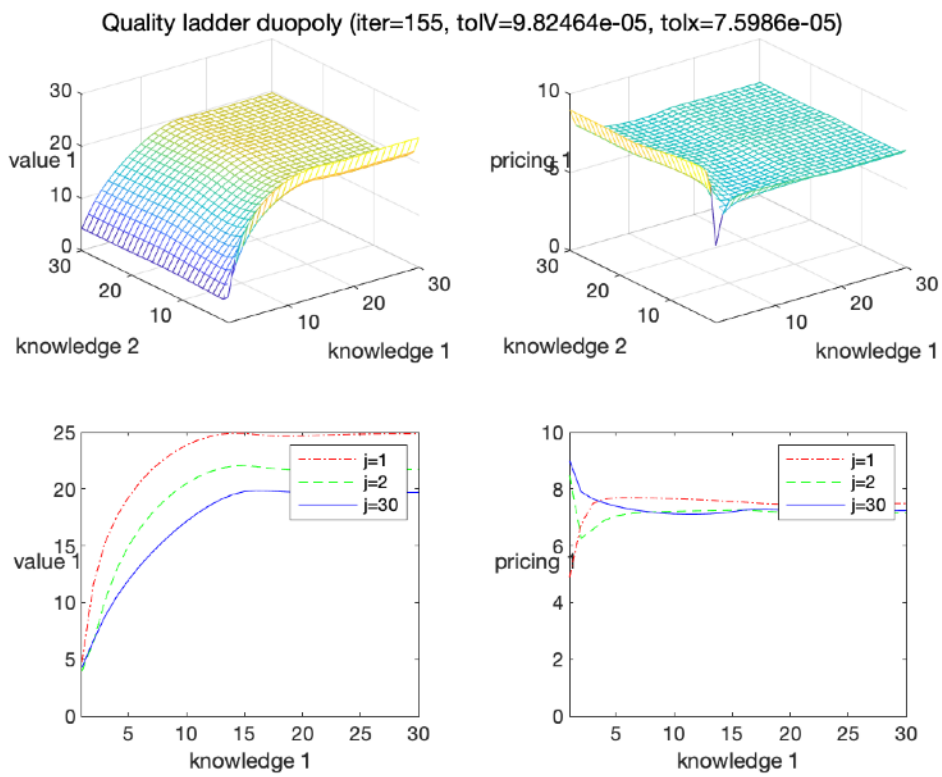

Figure 7. Duopoly - VFI

Figure 8 shows the transient distribution of the two firms, starting from . We can see that the industry starts out slightly asymmetric, but overtime the industry goes more symmetric. This is due to the well at the initial state.

This is perhaps one of the many equilibria, more of which could be found by Homotopy method (David Besanko, Ulrich Doraszelski, Yaroslav Kryukov, and Mark Satterthwaite, 2010).

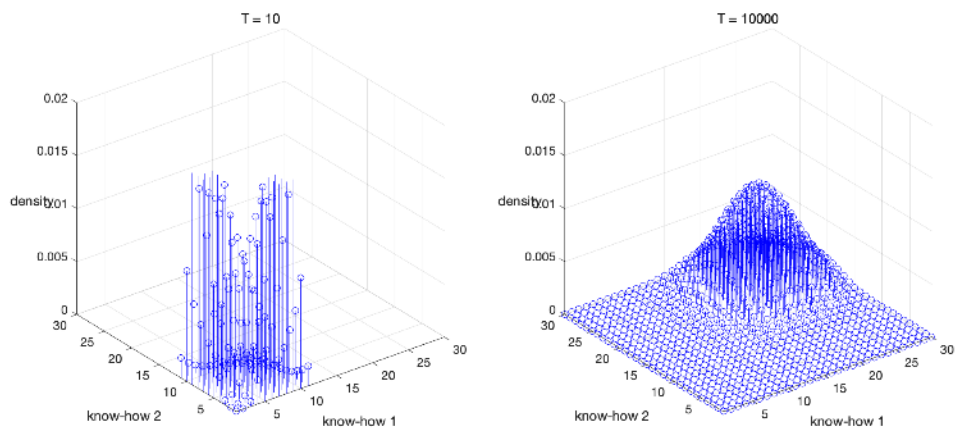

Figure 8. Duopoly - Transient Distribution given initial state (1,1)