Heterogeneous Firms, Equilibrium and Aggregation

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Models with Multiple Firms

Many theory papers study only individual firm behavior.
But often we want to understand specific features of the entire cross section of firms:

- Differences in stock returns
- Patterns in mergers, acquisitions or IPOs
- Differences in financing patterns
- The role of competition on innovation, etc..

Sometimes the cross sectional distribution of firms plays an important role in determining the behavior of the aggregate variables:

- Default by a few firms (or individuals) can feed back to the rest of the economy and affect aggregate values
- Generally the composition of firm investment, or demand for financing can affect aggregate savings and or asset prices.

Main Sources of Firm Level Data

- Most of the US firm level data about financial variables comes from CRSP and Compustat
- Additional data comes from recent panels like SDC and Sageworks
- Data about employment, production, productivity comes from the US Census Bureau which includes data on all firms.
- Most detailed about manufacturing but also newer data on services and retailers
- Much of the data is at the establishment or plant level as well
- General international data can be obtained from Compustat Global and now Amadeus and Osiris

Early Evidence on Size, Age and Growth - Evans (1988)

TABLE 1

Growth Rate, Exit Rate, and Sample Size

Number of Employees (1976)	Age (1976) 0–6	7–20	21–45	46–95	96+	Mean Rates and Total Survivors
1–19:						
Growth rate	8.88	4.16	2.78	5.99	2.87	5.70
Exit rate	39.59	22.11	20.09	17.41	.0	29.42
Number of survivors	3,079	3,018	1,761	185	7	8,050
20–49:						
Growth rate	2.72	.60	.15	1.50	7.42	.89
Exit rate	32.26	14.42	10.80	11.42	16.66	16.99
Number of survivors	466	1,032	925	155	5	2,583
50–99:						
Growth rate	−1.47	−.73	−.96	−.20	7.18	−.86
Exit rate	30.75	13.10	10.22	13.98	15.38	15.20
Number of survivors	385	1,001	1,071	203	11	2,671
100–249:						
Growth rate	−3.10	−2.55	−2.61	−2.41	−4.74	−2.64
Exit rate	25.32	12.16	10.94	6.54	10.52	12.71
Number of survivors	286	715	993	357	17	2,368
250–499:						
Growth rate	−5.54	−4.67	−5.17	−4.01	−2.84	−4.77
Exit rate	31.91	12.75	7.20	6.90	5.55	11.33
Number of survivors	96	260	451	256	17	1,080
500–999:						
Growth rate	−4.87	−7.70	−5.59	−4.38	−6.65	−5.55
Exit rate	21.21	9.85	6.43	4.58	5.00	7.51
Number of survivors	26	64	160	125	19	394
1,000 +:						
Growth rate	−11.83	−13.77	−9.20	−5.84	−3.33	−7.47
Exit rate	.0	12.82	8.92	2.09	8.00	5.59
Number of survivors	5	34	51	140	23	253
Mean growth rate	6.09	1.38	−.42	−1.37	−1.98	1.75
Mean exit rate	37.08	17.85	13.57	9.25	8.33	21.95
Surviving firms	4,343	6,124	5,412	1,421	99	17,399

Note.—Growth rate is the average annual logarithmic growth rate of employment for firms in the age-size category between 1976 and 1982 times 100. Exit rate is the percentage of firms in the age-size category that were in the sample in 1976 but not in 1982. Total number of survivors is the number of firms that were in the sample in both 1976 and 1982.

Recent Evidence on Age and Survival - Luttmer 2007

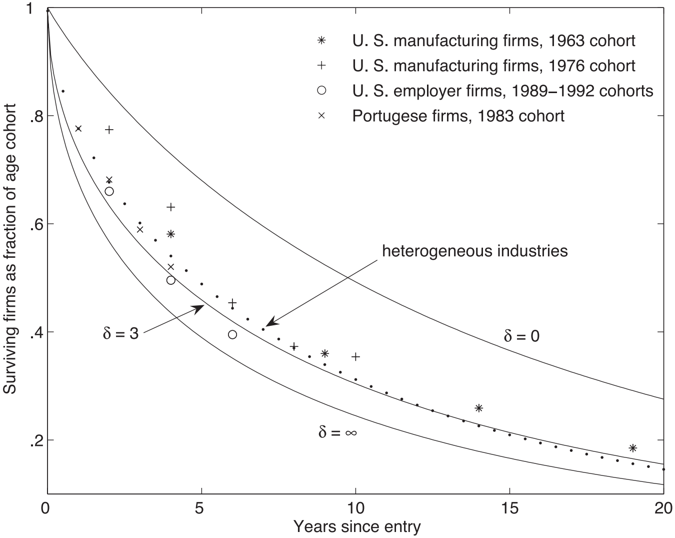


FIGURE IV
Survival Rates

Firm Size and Growth

Consider firms with DRS that rents capital every period to maximize per period profits:

- A Jorgensonian model without adjustment costs

$$k(a;R) = (\gamma a/(r+\delta))^{1/(1-\gamma)}$$

If $a_t = a_0 e^{gt}$, the growth rate of firm size over time is:

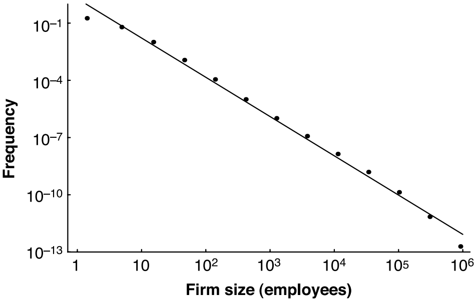
$$\frac{d \ln k(a;R)}{dt} = \frac{1}{1-\gamma}\frac{d \ln(\gamma a_t/R)}{dt} = 1/(1-\gamma)\frac{d \ln a_t}{dt} = \frac{g}{1-\gamma}$$

The firm's growth rate is independent of size - Gibrat's Law (Lucas (1978))

- We can think of this as a long run result
- Easy fix: add adjustment costs

Modern Evidence on Firm Size Distribution - Axtell (2001)

Power (Zipf's) law - Pareto distribution for firm employment size, l



$$
\begin{aligned}
\ln P(l > L) &= \eta(\ln \underline{l} - \ln L) \\
P(l > L) &= \left[\frac{\underline{l}}{L}\right]^{\eta}, \quad L > \underline{l}
\end{aligned}
$$

Empirically $\eta = 1.06$ - the **Pareto** tail parameter

The Theoretical Cross Section of Firms without Adjustment Costs

Given distribution of productivity $\Gamma_a(a) = Prob(a > A)$, the model implies an **endogenous** distribution of firm size $\Gamma_k(k) = Prob(k > K)$.

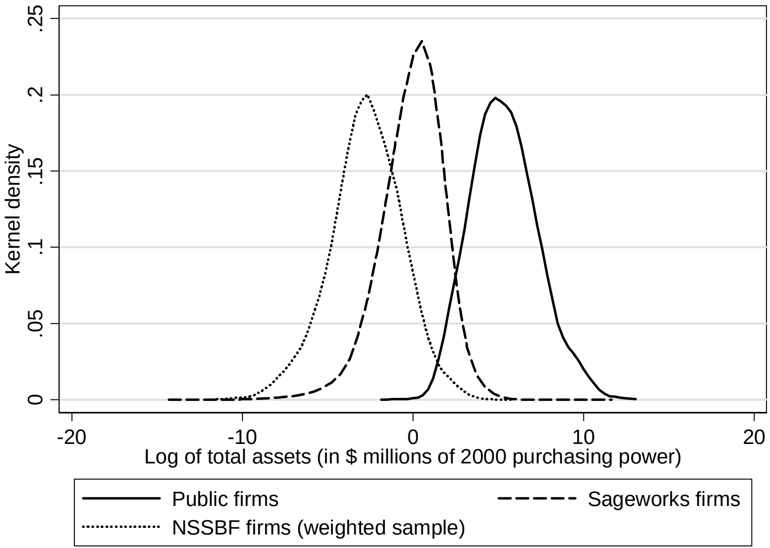
- Alternatively, we can use data about $\Gamma_k(k)$ to estimate/infer the underlying $\Gamma_a(a)$.

$$P(k > K) = P(a > (AR/\gamma)^{1-\gamma})$$

Useful assumptions to match the data

- Assume a Pareto distribution for productivity a
- Add fixed costs to generate a minimum scale of size $\underline{k}$
- To get a more precise quantitative match requires extra features

Public and Private Firms - Asker et al 2013



Public and Private Firms - Asker et al 2013

Fiscal year	No. of firms per year	No. of unique firms entering the sample	No. of unique firms exiting the sample	No. of firms per year	No. of unique firms entering the sample	No. of unique firms exiting the sample
	Full sample					
	Public			Private		
2002	3,352	3,352	0	2,535	2,535	0
2003	3,426	112	239	6,069	3,589	470
2004	3,293	110	269	13,147	7,663	1,251
2005	3,219	185	271	21,611	9,840	4,029
2006	3,134	167	368	26,267	8,577	7,515
2007	2,779	0	2,779	18,939	0	18,939
Total	19,203	3,926	3,926	88,568	32,204	32,204

Public and Private Firms - Asker et al 2013

	Full sample (F)	
	Public	*Private*
Total real assets ($m)	1,364.4	7.1
	2,958.1	*190.2*
Gross investment	0.045	0.076
	0.154	*0.261*
Net investment	0.022	0.033
	0.123	*0.205*
Sales growth	0.183	0.177
	0.674	*0.652*
ROA	0.065	0.075
	0.286	*1.069*
Cash holdings	0.225	0.152
	0.239	*0.202*
Book leverage	0.199	0.311
	0.230	*0.455*

Models of Cross Section of Firms

Equilibrium investment

- Lucas and Prescott (1971)

Dynamic Models of Industry Equilibrium - no investment

- Jovanovic (1982) - Bayesian learning plus entry and exit
- Hopenhayn (1992) - Competitive industry equilibrium entry and exit
- Ericson and Pakes (1995,1998) - Adds non-competitive behavior and suitable for empirical estimation

Quantitative Dynamic Models (calibrated to match data)

- **Gomes (2001)**, Adds investment and financing frictions
- Luttmer (2007) - Balanced growth

Pure Theory Papers: Problem of a Single Firm

Consider the classical dynamic investment problem

$$v(a,k) = \max_{k',i} \left[d(a,k,i) + \mathrm{E}_a M(a,a')v(a',k') \right]$$

We use k' and a' to denote the next period values of k and a and write

$$\begin{aligned} d(a,k,i) &= \pi(a,k) - I - \Phi(i,k) \\ k' &= i + (1-\delta)k \\ \mathrm{E}_a &= \mathrm{E}[\cdot|a] \end{aligned}$$

$M(a,a')$ is the discount factor, which may be stochastic and correlated with the firm's cash flows.

Quantitative Theory: Cross Section Distribution of Firms

Each individual firm can be described by its current individual state (a, k).

- We want to construct a cross-sectional of firms with a **distribution** defined over this state space.
- Formally μ is a measure such that

$$\forall (a, k) \in \mathbf{K} \times \mathbf{A}$$

  $\mu(a, k)$ denotes the mass of firms in the state (a, k).
- Looking at just the optimal policy function is equivalent to assuming that the cross-section distribution, μ, is uniform - all grid points are equally likely to occur.

Comparing Methods

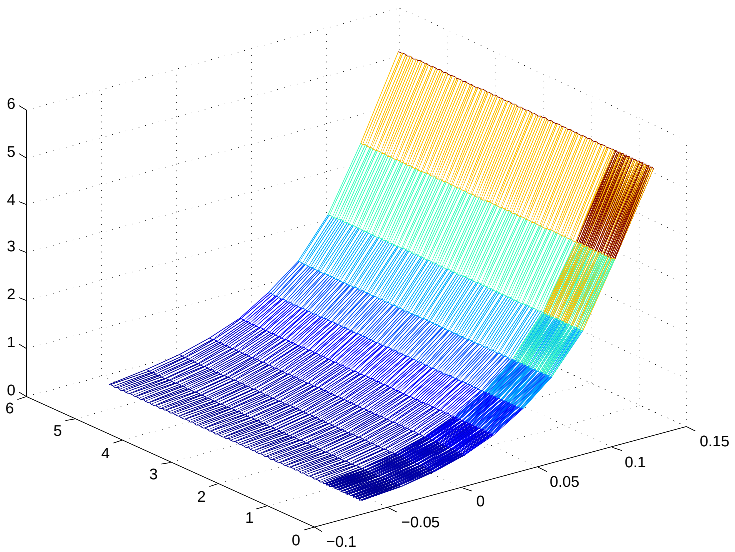
The cross sectional distribution is a measure $\mu(a,k)$ over the **same** state space as the policy or value functions $i(a,k)$ and $v(a,k)$

- But unlike the policy function it gives different **weights** to the different points in the space
- The weights depend on the simulated values of the policy functions

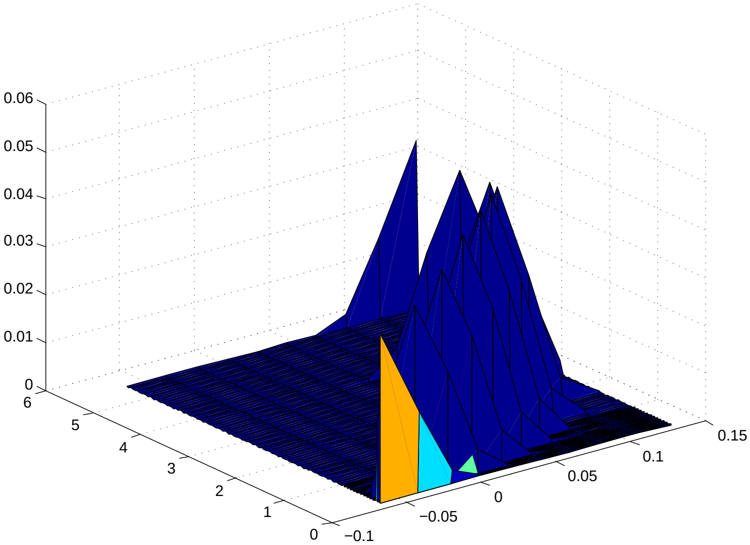
The differences can be quite substantial when making inferences about the core properties of the model.

- If the policy functions are very on-linear, or if discontinuities or corner solutions are important

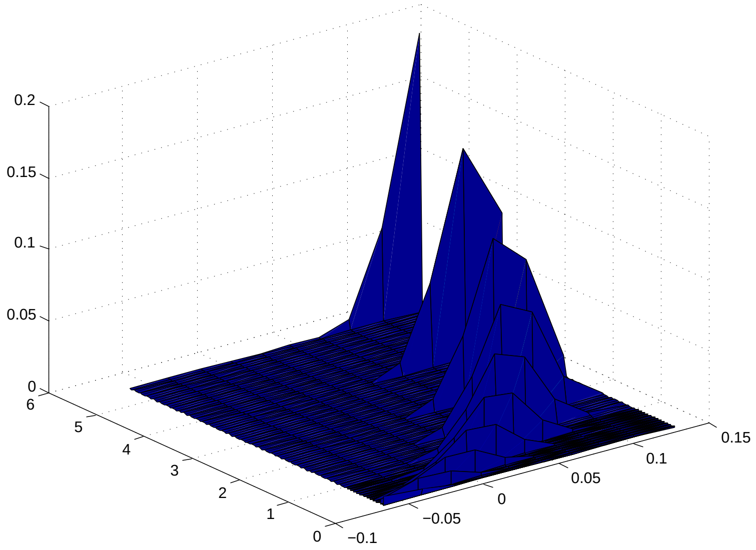
Optimal Investment Policy, $k'(a,k)$



Simulated Cross Sectional Distribution, $\mu(a, k)$



Cross Section Distribution of Firm Investment, $k'(a, k)$



Distribution of Firms: Law of Motion

To compute the distribution of firms we need to look at both at the transitions for the exogenous and the endogenous state variables

- In some cases we can first approximate the policy function with a continuous function to speed up the simulation.
- In general however the policy function will not be continuous or monotone and this method is not always useful

Law of Motion for the Exogenous State

For the exogenous state variable the law of motion is either

- An underlying continuous variable Markov process if we are using a continuous policy function
- Or, more generally, the approximate discrete Markov Chain summarized by the transition matrix P

If the latter the probability of moving between a current value a_j and future value a_i in the set $\mathrm{A} \in \mathbf{A}$ tomorrow is given by:

$$\int_{\mathrm{A}} \mathrm{P}(da' | a_t = a_j)$$

In the case of a Markov chain we get simply:

$$\sum_{a_i \in A} p_{i,j}$$

A 5-state Markov Chain

$$
\mathrm{P} = \begin{bmatrix} 0.7376 & 0.1947 & 0.0113 & 0.0001 & 0.0000 \\ 0.2473 & 0.5555 & 0.2221 & 0.0169 & 0.0002 \\ 0.0150 & 0.2328 & 0.5333 & 0.2328 & 0.0150 \\ 0.0002 & 0.0169 & 0.2221 & 0.5555 & 0.2473 \\ 0.0000 & 0.0001 & 0.0113 & 0.1947 & 0.7376 \end{bmatrix}
$$

In this case $p_{3,1}$ is probability of moving from state 1 today to state 3 tomorrow and

$$
\sum_j p_{i,j} = 1
$$

Law of Motion for the Endogenous State

The law of motion for the endogenous random variable k_t is generated by the optimal policy function $\mathbf{k}'(a, k)$

- The probability of reaching **any** possible set $\mathrm{K} \in \mathbf{K}$, given that the current state is $a_t = a$ and $k_t = k$, is equal to:

$$\mathrm{Q}(\mathrm{K}|(a, k)) = Prob(\mathrm{K}|(a, k)) = \begin{cases} 1 & \text{if } \mathbf{k}'(a, k) \in \mathrm{K}, \\ 0 & \text{if } \mathbf{k}'(a, k) \notin \mathrm{K}. \end{cases}$$

In other words

- a state $k \in \mathrm{K}$ next period can be reached from the current position (a, k) if and only if it is the optimal decision of the firm.
- There is no **uncertainty** about this transition

An Endogenous Transition Matrix - $n_k \times n_k n_a$

$$\mathrm{Q}(\mathrm{K}|(a,k)) = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

In this case $q_{3,1}$ is probability of moving from state $(a,k) = 1$ today to $k = 3$ tomorrow. Again:

$$\sum_j q_{i,j} = 1$$

but now there is no **uncertainty** about this transition

Law of Motion for the Cross-Sectional Distribution

Combining these two laws of motion we obtain the transition for the entire cross sectional distribution of firms

$$\mathrm{T}(\Theta|(a,k)) = \int_{\mathrm{A}} \mathrm{Q}(\mathrm{K}|(a,k))\mathrm{P}(da'|a),$$

for any feasible set $\Theta = (\mathrm{A}, \mathrm{K})$

- This captures all the conditional transition probabilities in the state space $\mathbf{A} \times \mathbf{K}$,

The law of motion for the distribution $\mu(a,k)$ is then

$$\mu'(\Theta) = \mathrm{T}(\Theta|(a,k))\mu(a,k)$$

Law of Motion for the Cross-Sectional Distribution: Extract of an Example

$$\mathrm{T}(\Theta|(a,k)) = \begin{bmatrix} 0.8099 & 0 & 0 & 0.8099 & 0 & 0 & 0.8099 & 0 & 0 \\ 0.1874 & 0 & 0 & 0.1874 & 0 & 0 & 0.1874 & 0 & 0 \\ 0.0027 & 0 & 0 & 0.0027 & 0 & 0 & 0.0027 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Stationary Distribution of Firms

An invariant distribution exists when

$$\mu' = \mu = \mu^*$$

- This invariant measure summarizes the distribution of firms in the long run or stationary environment and is usually the basis for comparing the model to the data
- Establishing existence is harder when policy functions are not continuous but it can be done (I offer an example in Gomes (2001))
- Uniqueness is even harder to establish and stability is not well understood at all

Computing the Stationary Distribution of Firms

The typical way to compute the stationary distribution is:

- Initialize the distribution of firms at an arbitrary $\mu_0(a, k)$
- Simulate this policy using the transition $\mathrm{T}(\cdot)$ for a very large number of periods (around 10000)
- Drop the first few thousand periods and use the rest as a proxy for the stationary distribution $\mu^*(a, k)$

This procedure is fine with continuous policies and transitions but it is both difficult and unnecessary with discrete a state space.

Computing the Stationary Distribution of Firms

Instead we can directly (and quickly) compute the stationary distribution as:

$$\mu^* = \lim_{j \to \infty} \mathrm{T}^j(:, i) \qquad \forall i$$

- It is important to note that $\mathrm{T}(\Theta|(a, k))$ is a very sparse matrix - i.e. full of zeros
- Computationally using sparse commands saves memory and makes the calculations faster

Entry and Exit

So far we have looked at models without entry and exit of firms.

- However the data is often made of severely unbalanced panels
- Ignoring entry and exit could lead to significant selection issues in our inference
- Moreover some economic phenomena require us to explicitly address entry and/or exit of firms
- Examples include studies on default, mergers and acquisitions, IPOs, innovation, etc.

Models with entry and exit can also be easier to solve!

Modeling Entry

The easiest way to model entry is to assume that there is an initial setup or entry cost, $e \geq 0$ to starting a new firm

- The model then implies that the following conditions must hold in equilibrium

$$\begin{aligned} v(a,k) &= e \qquad \text{if entry is positive} \\ v(a,k) &< e \qquad \text{if entry is 0} \end{aligned}$$

Properties of New Entrants

Many times it is convenient to assume that the entry decision is made **before** the shock a is observed, so that the entry condition becomes:

$$\begin{aligned} \mathrm{E}v(a,k) &= e \qquad \text{if entry is positive} \\ \mathrm{E}v(a,k) &< e \qquad \text{if entry is 0} \end{aligned}$$

- This introduces a simple but very powerful (and plausible) learning feature into the model for the very young firms (Jovanovic (1982))
- This also implies that the new entrants will be very diverse instead of self-selecting ex-ante
- As a result our model will produce very high failing rates among new young firms which is a common feature in almost all datasets

Properties of New Entrants

New entrants will generally be assumed to have 0 capital upon entry

- This means that young firms are very small - as we see in the data
- Combined with adjustment costs it may also mean that young firms stay small for many years
- Provided there are rising marginal costs to investment, so that large investment is very costly
- If capital also takes one period to become productive, young firms will actually not produce in their first period
- This is just a minor wrinkle and not that unrealistic

Cross Sectional Distribution of Firms with Entry

With entry, the law of motion for the $\mu(a, k)$ becomes

$$\mu'(a', k') = \mathrm{T}((a', k')|(a, k))\mu(a, k) + \gamma(a', 0)E$$

- The transition function $\mathrm{T}(\cdot)$ is computed exactly as before
- $\gamma(a', k' = 0)$ is the **probability** of a new entrant arriving next period with productivity shock a' and having capital $k' = 0$
- To save notation we will write $\gamma(a') = \gamma(a', k' = 0)$ henceforth
- E is the **number** or mass of new entrants
- This is the big new unknown in the problem

Distribution of New Entrants

A few comments about the probability distribution $\gamma(a')$

- Entrants generally are assumed to have no past history. This is convenient and generally will be consistent with the rest of the model
- This implies that entrants usually draw their shocks, a' from an i.i.d distribution.
- Sometimes we just assume that this is the long run, or invariant, distribution implied by the transition $\mathrm{P}(a'|a)$

Invariant Distribution of Firms with Entry

The law of motion for the cross sectional distribution can be written in matrix form as

$$\mu' = \mathrm{T}\mu + \gamma E$$

- This makes it apparent that the stationary distribution, if it exists, can now be constructed as

  $$\mu^* = (Ident - \mathrm{T})^{-1}\gamma E$$

  where *Ident* is an identity matrix
- However, as written, $(Ident - \mathrm{T})$ is a singular matrix!
- Intuitively entry produces more and more firms every time, so how can we possibly have a stationary equilibrium?
- The answer is we need exit too!

Modeling Exit

In reality firms often close when (equity) value turns negative

- Our model so far has no explicit debt so equity value equals the value function $v(a, k)$
- If our model forces firms to pay a fixed cost of production in every period it is quite possible that profits are negative
- This is very likely for small firms and those with very low productivities
- As an example remember that under Cobb-Douglas the profit function with fixed costs is

$$\pi(a, k) = aK^{\gamma} - f$$

Modeling Exit

To model exit we rewrite the problem of the firm as follows

$$v(a,k) = \max_{k'} \left[d(\cdot) + \mathrm{E} \max \underbrace{\{0, Mv(a',k')\}}|a \right]$$

- The term $\max\{0, Mv(a',k')\}$ captures the exit choice
- Exit occurs after a' is known
- Instead we can also assume that exit tomorrow is decided based only on today's information and write instead $\max\{0, \mathrm{E}Mv(a',k')|a\}$
- The first option smoothes a non-concavity in the firm problem but it is much harder to compute when prices are varying.

The Exit Decision

The optimal exit decision is given by

$$\mathbf{x}(a,k) = \begin{cases} 1 & a > \mathbf{a}^*(k) \qquad \text{(stay)} \\ 0 & a \leq \mathbf{a}^*(k) \qquad \text{(exit)} \end{cases}$$

where the optimal **exit threshold $\mathbf{a}^*(k)$** is defined implicitly by the relationship

$$\mathrm{E}[v(a',k')|a] = 0$$

- Intuitively firms choose to stay only if their future prospects (measured by the expected continuation value) are good enough.
- Assuming shocks are positively correlated, this is only true if the current level of productivity is above some threshold $a^*(k)$

The Exit Decision: Discussion

In most problems the value function $v(a, k)$ is increasing in both k and a

- It follows that exit is more likely for small and less productive firms
- The value of closing the firm could be different from 0.
- Here it might make more sense to have it equal to $(1-\delta)k'$ perhaps net of any adjustment costs
- Generally it seems reasonable to assume it equals the resale value of the assets of the firms

Cross Sectional Distribution with Entry and Exit

- Exit means that firms in some states will not survive into the next period
- Specifically the law of motion for the cross sectional distribution becomes

$$\mu' = \mathrm{T} \cdot \mathrm{X}\mu + \gamma E$$

- X is a matrix that assigns a value of 1 to the survival states and 0 to exit states, according to the optimal exit rule $\mathbf{x}(a, k)$
- The operator $\cdot$ means we are using the cartesian product (i.e. element by element) of the two matrices

Cross Sectional Distribution with Entry and Exit

With exit we can finally compute the invariant measure

$$\mu^*(a,k) = (Ident - \mathrm{T}\cdot\mathrm{X})^{-1}\gamma E$$

- This depends on the optimal investment and exit policies, $k'(\cdot)$ and $x(\cdot)$, as well as the exogenous shocks a

Importantly μ^* does not generally add to 1. Rather it scales linearly with E

- To express this linearity clear we write:

$$\mu^*(a,k) = N\mu_1^*(a,k)$$

- Where $\mu_1^*(a,k)$ is the stationary measure when the total measure of firms in the economy is $N = 1$.

Application - Investment and Cash Flow

Key question

- financing frictions important for corporate investment (e.g. Fazzari et al (1988), Kaplan and Zingales (1997), Erickson and Whited (2000))

Classical evidence in favor

- Profits/cash flow, π, is strongly correlated with investment
- Cash flow is very significant in investment regressions

Is the success of cash flow augmented investment regressions due to the fact that firms face financial constraints?

Investment and Cash Flow - Gomes (2001)

Equilibrium model of investment behavior with tractable financial market imperfections

- Good framework for many quantitative corporate finance/investment models
- The paper has a full general equilibrium model which we ignore here

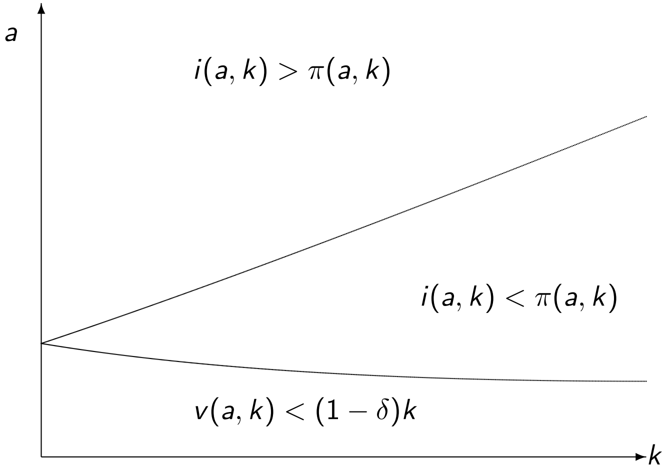
Problem of the firm

$$
\begin{aligned}
v(a,k) &= \max_{k'} \Big\{ \pi(a,k) - i(k',k) - \lambda(a,k,k') \\
&+ \beta \max\{(1-\delta)k', \int v(a',k') d\mathrm{P}(a'|a)\} \Big\}
\end{aligned}
$$

Costs of raising external finance (issuing equity)

$$
\lambda(a,k,k') = \begin{cases} \lambda_0 + \lambda_1[i(k',k) - \pi(z,k)], & \text{if } i(\cdot) > \pi(\cdot) \\ 0, & \text{else} \end{cases}
$$

Optimal Firm Behavior over the State Space



Calibration

Production Function and Output Elasticities

- $\alpha_k = 0.3, \alpha_l = 0.65$, - Census data estimates at firm level

Firm Level Productivity Shocks

- $\rho = 0.985$ and $\sigma = 0.04$ - estimates of cross sectional distribution of employment
- Today its common to instead use data on distribution of profits - more readily available

Financing Costs

- $\lambda_0 = 0.08, \lambda_1 = 0.028$, estimated directly from data on costs of issuing equity
- Recent estimates come up with similar numbers (e.g. Hennessy and Whited (2007))

Finally $\delta = 0.1$ and fixed costs, f, set to match entry/exit rates

Implications for Investment Regressions

After computing the stationary competitive equilibrium we can estimate theoretical investment regressions to replicate those in empirical studies

$$(i/k)_t = \alpha_0 + \alpha_1 Q_t + \alpha_2 (\pi/k)_t$$

Note: $Q = v(a,k)/k$.

Key benefit of having the detailed model

- Financing constraints are perfectly observable
- We know that many firms are constrained and/or using external funds

Investment Regressions in the Model

Regression Results

	Baseline	Augmented
Q	1.4	1.13
π/k		1.34
R^2	0.44	0.44

Key Finding:

- Cash flow, π, does not add overall *explanatory power* to investment regressions
- Even though we know model firms face (or may face) explicit financing constraint

Conclusions and Implications

Empirical success of cash flow augmented investment regressions **cannot be** informative about whether face financing constraints or not

- The paper illustrates this case with a calibrated version
- Strong investment cash flow correlations are more likely due to the existence of measurement error in Q

Most generally

- Financing costs, $\lambda(\cdot)$, are **by definition** always incorporated in the value of the firm $v(\cdot)$ and thus are already reflected in any (proper measured) value of Q
- The impact of financing constraints should be captured there and not in any separate variables

Other Applications: Corporate Restructuring

So far we focused on generic investment decisions

- What leads firms to expand or contract?

The same ideas can be used to explain why firms buy capital from each other

- The reallocation of used capital
- Mergers and acquisitions - Jovanovic and Rousseau (2002)
- Conglomerates - Gomes and Livdan (2004)

These ideas are also useful to think about models where distressed firms are forced to liquidate assets.

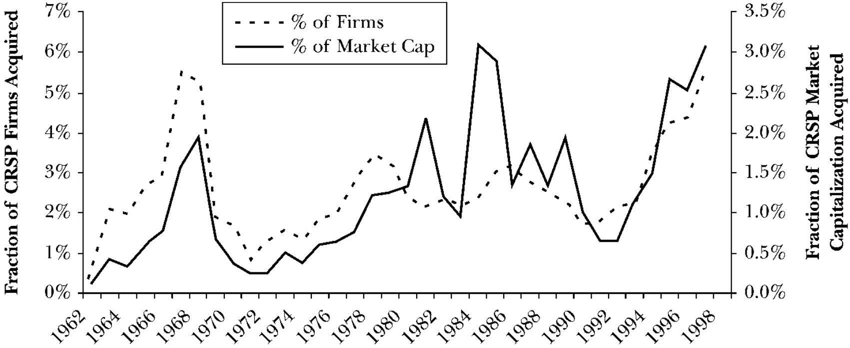
Some Mergers Facts - Andrade et al (2001)

Table 1

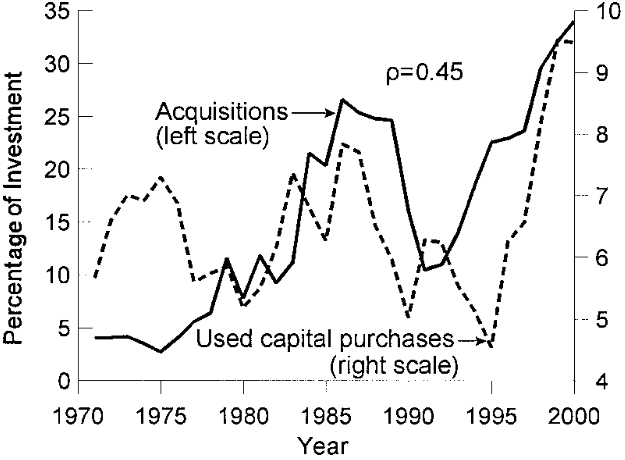
Characteristics and Descriptive Statistics by Decade, 1973–1998

	1973–1979	*1980–1989*	*1990–1998*	*1973–1998*
N	789	1,427	2,040	4,256
All Cash	38.3%	45.3%	27.4%	35.4%
All Stock	37.0%	32.9%	57.8%	45.6%
Any Stock	45.1%	45.6%	70.9%	57.6%
Hostile Bid at Any Point	8.4%	14.3%	4.0%	8.3%
Hostile Bid Successful	4.1%	7.1%	2.6%	4.4%
Bidders/Deal	1.1	1.2	1.0	1.1
Bids/Deal	1.6	1.6	1.2	1.4
Own Industry	29.9%	40.1%	47.8%	42.1%
Premium (Median)	47.2%	37.7%	34.5%	37.9%
Acquirer Leverage > Target Leverage	68.3%	61.6%	61.8%	62.9%
Acquirer Q > Target Q	68.4%	61.3%	68.3%	66.0%
Relative Size (Median)	10.0%	13.3%	11.2%	11.7%

Mergers Waves - Andrade et al (2001)



Mergers Waves and Acquisition of Used Capital - Jovanovic-Rousseau (2002)



Explaining Merger Activity: Jovanovic-Rousseau (2002)

Key idea:

- Investment is positively related to Q
- High Q firms should be buying capital while low Q firms should be selling it
- What if we use this to explain why high Q firms buy low Q firms directly?
- After all purchases of used capital and mergers move in very similar ways

Basic Environment

Many firms that maximize present value of future cash flows

- Production is linear ak

There are several ways to acquire capital

- Unbundled or standalone capital, i
  1. New capital purchased form households costs 1
- Bundled capital, acquired through a merger
  1. Old or used capital purchased form other firms costs $q < 1$

Capital accumulation can be done using two types of investments

$$k_{t+1} = (1 - \delta)k_t + i_t + m_t$$

Adjustment Costs

Let $\phi(ik, mk)k$ denote the adjustment costs to investment

- Where $ik = i/k$ and $mk = m/k$
- This is assumed to be smooth, convex and homogenous of degree one
- Later it is allowed for the adjustment costs to include a fixed cost of a merger (i.e. when $mk > 0$)

Firm Optimization

Recursive problem:

$$\begin{aligned} v(k,a) &= \max_{i,m} \{ak - i - qm - \phi(ik, mk)k + \\ &\quad \beta \mathrm{E}_a \max\{qk', v(k', a')\}\} \end{aligned}$$

Use CRS to rewrite as

$$\begin{aligned} Q(a) &= \max_{ik,mk} \{a - ik - qmk - \phi(ik, mk) + \\ &\quad (1 - \delta + ik + mk)\beta \mathrm{E}_a \max\{q, Q(a')\}\} \end{aligned}$$

Optimal Investment - Smooth $\phi(\cdot)$

Define

$$Q^\star(a) = \beta \mathrm{E}_a \max\{q, Q(a')\}$$

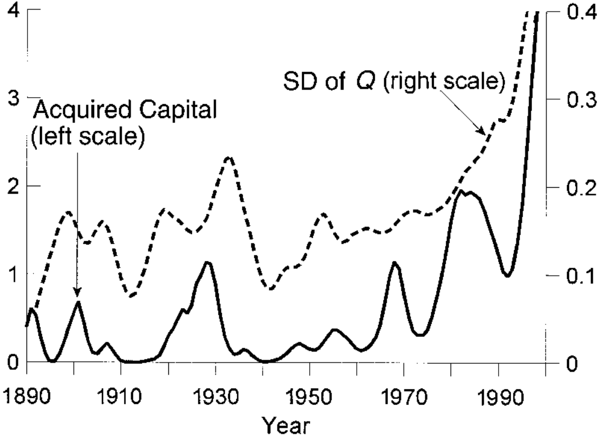
Optimal investment decisions obey the standard Q-theory FOCs

$$\phi_i(ik, mk) = Q^\star(a) - 1$$
$$\phi_m(ik, mk) = Q^\star(a) - q$$

Comments

- Merger investment is positively related to $Q^\star(a)$ and thus Q
- $Q^\star(a)$ is increasing in a - with positive autocorrelation
- Productive firms are buyers and less productive ones are sellers
- We should expect a lot of mergers when there is a large dispersion in a and thus $Q(a)$

Mergers and Dispersion in Q - Jovanovic-Rousseau (2002)



Optimal Decisions with Fixed Merger Cost

Suppose now there is a fixed cost $fk > 0$ whenever $mk > 0$.

- This ensures we will never see "small" size mergers

The optimal amount of total investment spending, $x = ik + mk$, solves

$$x + \phi(x, 0) = f + \min_{mk}\{(x - mk) + qmk + \phi(x - mk, mk)\}$$

Comments

- Initially, the firm buys unbundled capital,
- However, since $q < 1$ eventually a firm will choose to buy capital through a merger
- If x is large enough we will get that $mk > ik$.

Evidence about Conglomerates - Lang and Stulz (1994)

AVERAGE AND MEDIAN OF TOBIN'S q FOR GIVEN VALUES OF THE DIVERSIFICATION MEASURES FOR 1984

A. NUMBER OF SEGMENTS

	1	2	3	4	≥5
Tobin's q	1.53 (1.01) {580}	.91 (.71) {215}	.91 (.74) {272}	.77 (.63) {198}	.66 (.58) {184}

B. HERFINDAHL INDEX CONSTRUCTED FROM SALES

	$H = 1$	$.8 < H < 1$	$.6 < H < .8$	$.4 < H < .6$	$.0 < H < .4$
Tobin's q	1.53 (1.01) {580}	.85 (.69) {76}	.91 (.76) {160}	.86 (.66) {299}	.74 (.64) {305}

Are diversified firms destroying value?

- If so why would anyone own stock in diversified firms?

Explaining the Conglomerate Discount: Gomes and Livdan (2004)

Key idea:

- Diversification discount arises endogenously in a model of where firms maximize shareholder value

Model Environment

- Decreasing returns technology in each sector s

$$y^s = a^s k^{\alpha_k} l^{\alpha_l}, \qquad 0 < \alpha_k + \alpha_l < 1,$$

- Firm and sector specific shocks that are not perfectly correlated across sectors
- Firms can operate in sector, $s = 1$, $s = 2$, or in both, $s = 3$ (diversified firms)

Fixed Costs and Synergies

Conglomerates are efficient

- Production in any sector requires payment of fixed costs, f
- But joint operations in both sectors lower total fixed costs (overhead) to $(2 - \lambda)f$
- These savings are not available to shareholders artificially replicating a conglomerate by polling together investments in similar standalone firms

Firm Profits and Investment

Firm focused in sector s

$$\pi(s,k,a) = \max_{l} \{a^s k^{\alpha_k} l^{\alpha_l} - Wl - f\}$$

Conglomerate, operating in both sectors simultaneously:

$$\begin{aligned}\pi(3,k,a) &= \max_{l,0\leq\theta\leq 1} \left\{a^1 \theta^{\alpha_k+\alpha_l} k^{\alpha_k} l^{\alpha_l} + a^2 (1-\theta)^{\alpha_k+\alpha_l} k^{\alpha_k} l^{\alpha_l}\right. \\ &\qquad \left. - Wl - (2-\lambda) f\right\}\end{aligned}$$

where $a = (a^1, a^2)$

Investment

$$k' = (1-\delta)k + i,$$

Dynamic Optimization

Two Stages

- Optimal investment, $k' = \mathbf{k}(s, k, a)$, **conditional** on sectoral choice, s'

$$p(s', k, a) \equiv \max_{k'} \left\{ \pi(s', k, a) + (1 - \delta)k - k' + \beta \mathrm{E}_a v(s', k', a') \right\}$$

- Optimal sectoral choice, $s' = \mathbf{s}(s, k, a)$

$$v(s, k, a) = \max_{s'} \quad p(s', k, a)$$

$$s' \in \begin{cases} \{s, 3\}, & s = 1, 2 \\ \{1, 2, 3\}, & s = 3 \end{cases}$$

Optimal Diversification Decision

For a firm focused in sector s at the beginning of the period

- Diversification threshold, $\hat{k}(s,a)$ is

$$p(3,k,a) = p(s,k,a)$$

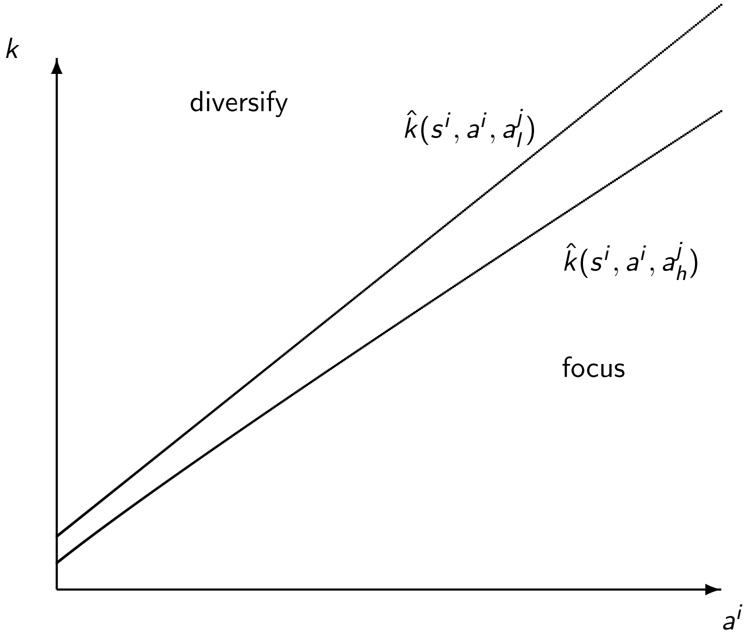
Diversification is always optimal if

- Fixed costs are 0
- Synergies are large ($\lambda \geq 1$)

For a previously specialized firm the threshold, $\hat{k}(s,a)$, is:

- Increasing in shock to profits in its current sector
- Decreasing in shock to profits in the other sector

Diversification Threshold



Generating a Diversification Discount: Quantitative Model

Specification

$$Q_{it} = b_0 + b_1 DIV_{it} + b_2 \ln(k_{it}) + \epsilon_{it}$$

$$DIV = \begin{cases} 1, & \text{if } s' = 3 \\ 0, & \text{else} \end{cases}$$

	All Firms		Q < 5	
Variable	**Data**	**Model**	**Data**	**Model**
DIV	−0.34	−0.20	−0.29	−0.07
(t-stat)	(−3.77)	(−5.39)	(−4.53)	(−3.71)
log(k)	−0.12	−0.70	−0.13	−0.31
(t-stat)	(−3.48)	(−5.26)	(−5.22)	(−5.29)

Why is there a Diversification Discount?

Because of self-selection

- Firms who optimally choose to become conglomerates are those who have low marginal productivities of capital in their initial sectors
- It is better to expand into new sectors

Conglomerate discount is more likely when synergies are small

- λ is low
- f is not too high
- Empirical implication: diversification into closely related sectors is less likely to lead to a discount

Equilibrium Models

After computing the stationary distribution of firms we have two basic options:

1. Use this distribution directly to compare the model's predictions with the data:
   - common and sometimes enough to address some problems
   - it avoids looking at (and computing) equilibrium prices
2. Use the distribution to construct economy-wide aggregates and impose consistency with other equilibrium conditions:
   - this requires clearing markets and finding equilibrium prices
   - it is only necessary if we care about the aggregate variables too

Adding Prices

To compute equilibrium models we will want to separate the effect of prices on firm decisions.

- There are at least two additional prices that can be introduced in the original problem of the firm (plus the SDF):
  1. a price of the output P
  2. a wage rate W
- So far we have subsumed in the exogenous state, a - implicitly assuming exogeneity
- But henceforth let $a = (z, P, W)$ to recognize explicitly the dependence of the value function and policy rules P and W

Problem of the Firm with Prices

The problem of the firm should then be formally written as

$$v(z,k;P,W) = \max_{k'} \left[d(z,k,k';P,W) + \mathrm{E}\left[Mv(z',k';P',W') | z,P,W \right] \right]$$

- Both P and W are de facto state variables now.
- The policy function is also dependent on prices, hence $k'(z,k;P,W)$
- It follows that the cross-sectional distribution of firms is also defined over the same state space, so $\mu(z,k;P,W)$

Aggregation

After we compute the cross-sectional distribution of firms $\mu(z, k; P, W)$ we can easily construct various aggregate quantities:

- Aggregate production or output

$$Y(\mu, P, W) = \int [zF(k, l(z, k, P, W))] \, d\mu(z, k; P, W),$$

- Aggregate investment

$$I(\mu, P, W) = \int \left[k'(z, k, P, W) - (1 - \delta)k\right] d\mu(z, k; P, W),$$

- Aggregate labor demand

$$L(\mu, P, W) = \int l(z, k, P, W) d\mu(z, k, P, W),$$

- And similarly for aggregate profits and market value, $\Pi(\mu, P, W)$, and $V(\mu, P, W)$, among others.

Equilibrium Prices and Quantities

Equilibrium prices require equalizing supply and demand:

- Equilibrium in the goods market

$$Y(\mu; P, W) = \int [zF(k, l(\cdot)) - f] \, d\mu(z, k, P, W) = Y^D(P, W)$$

- Equilibrium in the labor market

$$L(\mu; P, W) = \int l(\cdot) d\mu(z, k, P, W) = L^S(P, W)$$

- $Y^D(P, W)$ and $L^S(P, W)$ are demand for goods and supply of labor
  - Exogenous in industry/partial equilibrium models

Solving an Equilibrium Model

To compute the competitive equilibrium we then use the following iterative procedure:

- Start with an initial guess for prices P^0, W^0
- Compute the value function and policy rule for an individual firm $v(z, k; P^0, W^0)$ and $k'(z, k; P^0, W^0)$ using the tools discussed earlier in the course
- Compute the stationary cross-sectional distribution of firms $\mu^*(z, k; P^0, W^0)$ as discussed above
- Compute aggregate quantities $Y(\mu^*; P^0, W^0)$ and $L(\mu^*; P^0, W^0)$ and check whether the equilibrium conditions are satisfied
- If so stop, otherwise revise the price guesses and start again

Obviously this can take a while!

Faster Computation: Equilibrium Restrictions

Optimality restricts the nature of the exogenous functions $Y^D(P, W)$ and $L^S(P, W)$.

- For example, with Cobb-Douglas firm labor demand is:

$$\alpha_l zF(k, l(\cdot))/l \;=\; W \implies l = \alpha_l zF(k, l(\cdot))/W$$

- It follows that:

$$\begin{aligned} L(\mu; P, W) &= \frac{\alpha_l \int zF(k, l(\cdot))d\mu(z, k, P, W)}{W} \\ &= \frac{\alpha_l [Y(\mu; P, W) - Nf]}{W} \end{aligned}$$

Together with equilibrium this means we must have

$$WL^S(P, W) = \alpha_l [Y^D(P, W) - Nf]$$

This ties the equilibrium levels of P, W and N.

- Henceforth, use $W(P, N)$ and drop dependence on W.

Faster Computation: Exploiting Constant Returns

Next, the nature of the cross-sectional distribution, $\mu(\cdot)$, implies that in stationary equilibrium the aggregate supply of output equals:

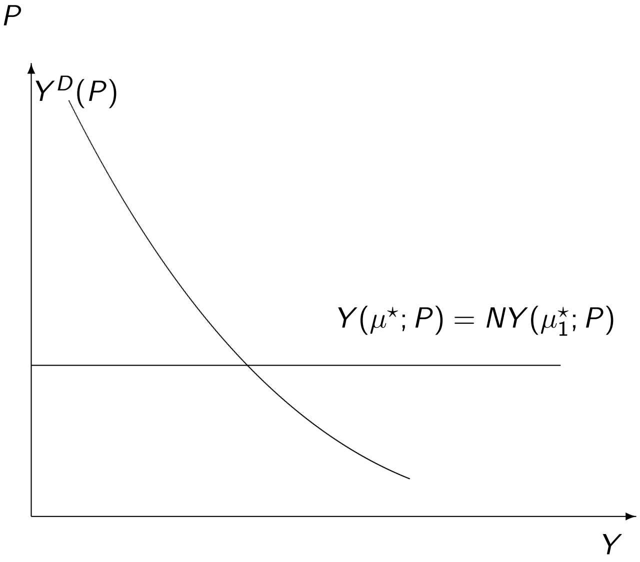
$$\begin{aligned} Y(\mu^\star; P) &= \int [zF(k, l(\cdot)) - f] \, d\mu^\star(z, k, P) = \\ &= N \cdot \int [zF(k, l(\cdot)) - f] \, d\mu_1^\star(z, k, P) = \\ &= N \cdot Y(\mu^\star; P) \end{aligned}$$

Key insight

- Aggregate production becomes **constant returns to scale**, in the mass of firms, N.
- Obviously the same applies to labor demand:

$$L(\mu^\star; P) = N \cdot L(\mu^\star; P)$$

Aggregate Supply with Free Entry



Computing the Equilibrium with Entry and Exit

The competitive equilibrium is given by

$$Y(\mu^*; P) = NY(\mu_1^\star; P) = Y^D(P)$$

- Computationally we can then solve for **either** N (easy) or P (hard)
- Because of CRS we cannot pin down the two numbers separately
- A minor wrinkle: we need to ensure that entry is positive in equilibrium
- To do this we return to the problem of the firm and make sure that the entry costs e is low enough so that

$$\mathrm{E}v(z, k; P) \quad = \quad e$$

  holds at the equilibrium price level, P

Transition Dynamics

Suppose there is a one off shock (e.g. every firm's productivity increases permanently).

- New stationary equilibrium, $\mu^{\star\star}, E^{\star\star}, P^{\star\star}$ can be computed as above.

How do we transition to the new steady-state?

$$\begin{aligned}
\mu_1(z,k;P_1) &= T\cdot X\mu^\star(z,k,P^\star)+\gamma E_1 \\
\mu_n(z,k;P_n) &= T\cdot X\mu_{n-1}(z,k,P_{n-1})+\gamma E_n \\
&= (T\cdot X)^n\mu^\star(z,k,P^\star)+\gamma\sum_{j=0}^{n-1}(T\cdot X)^j E_{n-j}
\end{aligned}$$

Imposing convergence after n steps:

$$\mu^{\star\star}(z,k,P^{\star\star})=\mu_n(z,k;P_n)$$

Transition Dynamics

The terminal condition is a system of $nk \times na$ linear equations in n unknowns.

- We know every element of the matrix $\mu^{\star\star}$
- We need to find out the levels of entry in every period, E^j, for $j = 1, 2, ..n$

After computing the level entry in every period we can obtain the sequence of distributions $\mu_j(z, k; P_j)$ for $j = 1, 2, ..n$

- And the associated sequence of aggregate supply of goods, $Y(\mu_j; P_j)$

It is then immediate to compute the equilibrium level of prices in every period:

$$Y(\mu_j; P_j) = Y^D(P_j)$$

Models with Aggregate Shocks

Idiosyncratic (i.e. firm specific) shocks will average out and will not affect aggregate quantities like $Y(\mu^*; P, W)$.

- In this case there is an equilibrium with a time invariant distribution of firms μ^*, and time invariant prices that clear markets:

$$Y(\mu^*; P, W) = Y^D(P)$$
$$L(\mu^*; P, W) = L^S(W)$$

Nondiversifiable shocks will move aggregate quantities and prices. They are important to study business cycles and asset prices

- Equilibrium models become very hard to solve when there are shocks that move aggregate quantities

Models with Aggregate Shocks

To simplify notation, let's again assume we eliminated W from the problem, and write:

$$z = z_i \times Z, \qquad \int z_i = 1$$

where Z is aggregate and z_i is specific to each firm i

- Since aggregate shocks move **all** firms' output systematically, there is now time variation in equilibrium prices

Firm's problem:

$$v(\cdot; P) = \max_{k'} \left[d(\cdot; P) + \mathrm{E} \left[Mv(\cdot; P') | z, P \right] \right]$$

Why this is much more difficult to solve:

- Just like when we wanted to compute the transition dynamics across stationary equilibria, we now have that $P' \neq P$
- This means we need to know **future** prices to solve each firms' problem
- The problem is that now this is not deterministic sequence but a **stochastic process**

Forecasting Equilibrium Prices

The key issue is that each firm must forecast P' in all states of the world to determine its optimal behavior today.

- Since $v(z', k')$ is conditional on P'
- This means optimal policies and thus the cross-sectional distribution $\mu(\cdot; P)$ become time-varying

Equilibrium implies that

$$Y(\mu; P) = Y^D(P) \implies P = P(\cdot; \mu)$$

- The **entire** cross sectional distribution matters for prices and thus becomes a **state variable** for the Bellman equation.

Forecasting Prices

Although future prices depend on future state variables, it can only be forecasted with **current** variables.

- A popular solution is to **assume** a simple forecasting rule for prices, $P'(\mu(z, k; P))$, e.g.:

$$P'(Z, K) = \alpha_0 + \alpha_1 \ln Z + \alpha_2 \ln K$$

where

$$K = \int k d\mu(z, k)$$

- This tries to summarize the entire measure $\mu(\cdot)$ with its first moments (Krusell and Smith (1998)).
- Of course we can also add higher order moments or quantiles.
- Generally, we want a good forecasting rule for future prices (Kogan and Mitra (2013)).

Solving Models with Aggregate Shocks

A possible algorithm to solve a model with aggregate shocks is:

- Start with a guess for the forecasting rule for prices, $P'^{0}(Z, \mu)$
- Given this rule compute the implied competitive equilibrium prices in all periods, $\{P_t\}_{t=0}^{\infty}$
- Use this time series to estimate a new forecasting rule $P'^{1}(Z, \mu)$, and compare with the initial one
- If similar stop, otherwise use the new rule and compute the equilibrium again.

A Model with Aggregate Shocks: Thomas (2001)

Question

- Does “lumpy” firm level investment matter for business cycles?

Earlier work - Caballero and Engel (1999)

- With aggregate shocks lumpy firm-level investment generates large movements in total investment demand
- If many firms concentrate near a positive/negative investment boundary, then aggregate investment becomes very responsive to shocks

Key finding here

- In general equilibrium, households' preference for smooth consumption generates large movements in prices which dampen impact of shocks on investment.
- Distribution of firms is not important for aggregate fluctuations!

Key Assumptions: Technology and Costs

Fixed **stochastic labor** cost of adjusting the capital stock, ϵw,

- i.i.d. across establishments with distribution $G(\epsilon)$

Decreasing returns production function for firm of type j:

$$y_j = Z k_j^{\alpha_k} l_j^{\alpha_l}$$

- Here j is effectively the number of periods since capital was last adjusted.
- Can also be interpreted as a **vintage** capital model

Comments

- No entry or exit of firms/plants
- No idiosyncratic productivity shocks

Optimal Investment

Conditional on investment all firms will choose the same capital stock, $k^\star$

- All investment costs are fixed and there are no differences in expected firm productivity
- Every period there is a cutoff level of costs $\bar{\epsilon}(j; Z, \mu)$ such that investment takes place for all firms with $\epsilon \leq \bar{\epsilon}(j; Z, \mu)$
- The mass of firms investing is then $G(\bar{\epsilon}(j; Z, \mu))$

If there is no investment then $k_{j,t+1} = (1 - \delta)k_{jt}$

Firm Distribution

The cross-sectional distribution of firms, $\mu(k)$, is just the measure of firms at each state j, μ_j, which evolves as

$$\mu'_j = \begin{cases} \mu_{j-1}[1 - G(\bar{\epsilon}(j; Z, \mu))], & j > 0 \\ \sum_j^{\infty} \mu_j G(\bar{\epsilon}(j; Z, \mu)), & j = 0 \end{cases}$$

- There is no dependence on idiosyncratic shocks

Extremely tractable but with one main implementation issue:

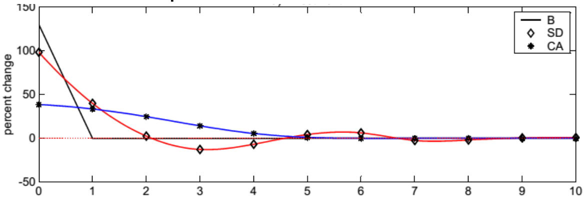
- We need to impose a finite upper bound on the time it takes for any firm to eventually adjust its capital $J < \infty$

Stationary distribution, $\mu(j)$:

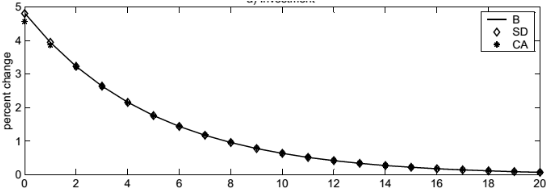
j	0	1	2	3	4	5
$G(\cdot)$	0.06	0.2	0.38	0.58	0.78	1
$\mu(\cdot)$	0.29	0.28	0.22	0.14	0.05	0

The Response of Investment to Aggregate Shocks

Response with Fixed Prices



Response in General Equilibrium



Thoughts on Solving Models with Aggregate Shocks

Models with endogenous entry and exit can be easier to compute

- As we have seen before we can solve the market clearing condition for either P or N
- What happens if we fix $P_t = 1$ (say) and solve for endogenous mass, N_t, **in every period**?
- Because of aggregate shocks the level of entry changes over time but prices will not
- However the problem of the firm only depends on prices P and P' and not on the level of entry
- As a result we do not need to
    1. Iterate on the problem of the firm as prices change
    2. Guess and verify the law of motion for prices

A Simple Model with Aggregate Shocks

Suppose the aggregate shock takes the form of a stochastic shift, B_t, in the aggregate demand for goods so that the equilibrium condition is

$$Y(\mu_t; P) = \int [zF(k, l(z,k)) - f] \, d\mu_t(z,k) = B_t Y^D(P)$$

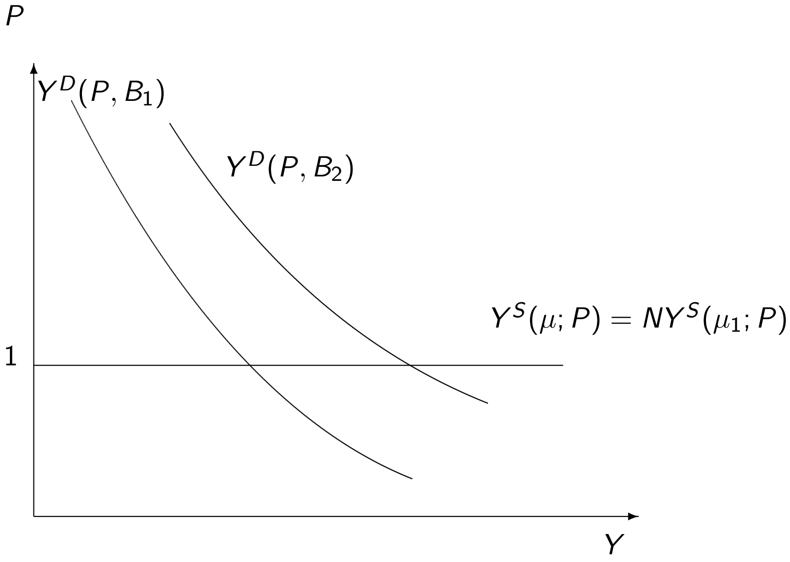
Suppose that $P = 1$ for all values of B_t then

- The problem of each firm is unchanged
- The investment and exit decisions remain time invariant

We can write the aggregate supply as:

$$\begin{aligned} Y(\mu_t; P=1) &= N_t \int [zF(k, l(z,k)) - f] \, d\mu_1(z,k; P=1) \\ &= N_t Y(\mu_1; P=1) \end{aligned}$$

Equilibrium with Demand Shocks



Equilibrium with Aggregate Shocks

Market clearing implies that the equilibrium number of firms obeys

$$N_t = B_t Y^D(P = 1)/Y(\mu_1; P = 1)$$

Next integrate the law of motion for the cross-sectional distribution to get

$$\begin{aligned} \int N' d\mu_1' &= \int T \cdot X N d\mu_1 + \int E d\gamma \\ N' &= (1 - \delta_N(P))N + E \end{aligned}$$

where

- $\delta_N(P)$ captures the endogenous "depreciation" in the stock of firms generated by the exit decision
- Note that exit is endogenous but time invariant and similarly $\mu_1 = \mu_1'$

Cross Section Distributions in Continuous Time

It is generally much easier to compute the transition dynamics in continuous time models.

- All we need is to apply the Finite Difference method to the appropriate **Kolmogorov** equations.

Simple example: let z be a scalar diffusion:

$$dz = \mu(z)dt + \sigma(z)dW, \qquad z(0) = z_0$$

What is the **density** of z at time t, $f(z,t)$?

- Similarly, what is the stationary distribution:

$$f(z) = \lim_{t \to \infty} f(z,t)$$

Transition Dynamics in Continuous Time

The **Kolmogorov forward equation** describes the dynamics of the pdf of **any** continuous time diffusion process:

$$\frac{\partial f(z,t)}{\partial t} = -\frac{\partial}{\partial z}[\mu(z)f(z,t)] + \frac{1}{2}\frac{\partial^2}{\partial z^2}[\sigma(z)^2 f(z,t)]$$

- This can be used to compute the transition dynamics of a model for a given **initial** distribution $f(z,0)$.
- There is also a corresponding Kolmogorov **backwards** equation that can be used to compute the distribution starting for a terminal condition. This is sometimes useful too.

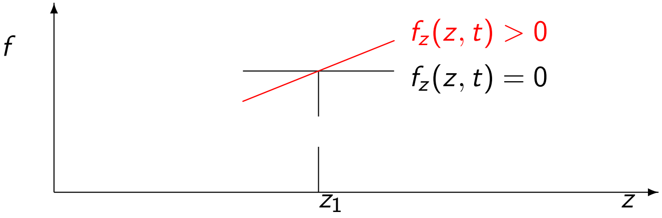
It follows that the stationary distribution obeys a (simpler) ODE

$$0 = -\frac{\partial}{\partial z}[\mu(z)f(z)] + \frac{1}{2}\frac{\partial^2}{\partial z^2}[\sigma(z)^2 f(z)]$$

Kolmogorov Forward Equation: Intuition

How does the mass of firms in point z_1 at period t changes over a small interval dt:

- If $\mu(z_1) > 0$ the drift keeps moving mass over time.
- Lower values of z are pushed towards z_1, while the mass initially at z_1 is pushed to higher values.
- The net impact depends on the slope of the distribution $f(z_1, t)$ with respect to z.
- If $f_z(z_1, t) = 0$, the distribution is locally uniform, the net effect of the drift cancels out.



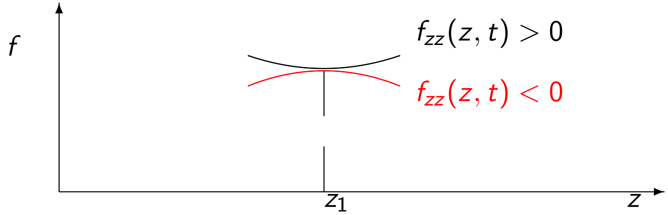
Kolmogorov Forward Equation: Intuition

Wiener process: no drift, unit variance:

$$\frac{\partial f(z,t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial z^2} f(z,t)$$

The impact of variance depends on the local convexity of $f(z,t)$:

- A locally convex function has more mass near z_1.
- It is more likely to observe more transitions to z_1 over the next instant, dt.



Application: Cross Sectional Distribution of Firms

The firm's problem in continuous time obeys the HJB equation:

$$rv(z,k) = d(z, i(z,k), k) + \frac{\partial v}{\partial k}[i(z,k) - \delta k] + \frac{\partial v}{\partial z}\mu(z) + \frac{\partial^2 v}{\partial z^2}\frac{\sigma(z)^2}{2}$$

- For simplicity ignore the role of prices and equilibrium.
- We use $i(z,k)$ to denote the optimal investment policy.

The (now two dimensional) Kolmogorov forward equation for the cross sectional **p.d.f.**, $f(z,k,t)$, is:

$$\begin{aligned} \frac{\partial f(z,k,t)}{\partial t} = & - \frac{\partial}{\partial k}[\dot{k} f(z,k,t)] - \frac{\partial}{\partial z}[\mu(z) f(z,k,t)] \\ & + \frac{1}{2}\frac{\partial^2}{\partial z^2}[\sigma(z)^2 f(z,k,t)] \end{aligned}$$

where $\dot{k} = i(z,k) - \delta k$

- This can be discretized using FD and solved forwards starting from any initial p.d.f. $f(z,k,0)$.