FNCE 937 Homework 1
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1. Show $V_t^e = q_t k_{t+1}$ for $t = 0$. The cum-dividend value of the firm is

$$
\begin{aligned}
V_0 = \max_{\{i_t, k_{t+1}\}_{t=0}^{\infty}} & E_0 \left[\sum_{t=0}^{\infty} M_{0,t} d_t \right] \\
\text{s.t. } & d_t = \pi(k_t) - i_t - \phi(i_t, k_t) \\
& i_t = k_{t+1} - (1-\delta) k_t
\end{aligned}
$$

Let q_t be the Lagrange multiplier on the capital accumulation equation or marginal value of a unit of undepreciated capital. The first order conditions for this problem are

$$
\begin{aligned}
\text{wrt } [i_t]: \ & 1 + \phi_i(i_t, k_t) = q_t \\
\text{wrt } [k_{t+1}]: \ & E_0 [M_{0,t} q_t] = E_0 [M_{0,t+1} (\pi_k(k_{t+1}) - \phi_k(i_{t+1}, k_{t+1}) + q_{t+1}(1-\delta))]
\end{aligned}
$$

Given that π and ϕ exhibit constant returns to scale, the firm's problem can be re-written as

$$
\begin{aligned}
V_0 &= \max_{\{i_t, k_{t+1}\}_{t=0}^{\infty}} E_0 \left[\sum_{t=0}^{\infty} M_{0,t} \Big(\pi(k_t) - i_t - \phi(i_t, k_t) - q_t \left(k_{t+1} - (1-\delta) k_t - i_t \right) \Big) \right] \\
&= \max_{\{i_t, k_{t+1}\}_{t=0}^{\infty}} E_0 \left[\sum_{t=0}^{\infty} M_{0,t} \Big(\pi_k(k_t) k_t - i_t - \phi_k(i_t, k_t) k_t - \phi_i(i_t, k_t) i_t - q_t \left(k_{t+1} - (1-\delta) k_t - i_t \right) \Big) \right] \\
&= \max_{\{i_t, k_{t+1}\}_{t=0}^{\infty}} E_0 \left[\sum_{t=0}^{\infty} M_{0,t} \Big(\big(\pi_k(k_t) - \phi_k(i_t, k_t) + q_t(1-\delta) \big) k_t - \big(1 + \phi_i(i_t, k_t) - q_t \big) i_t - q_t k_{t+1} \Big) \right]
\end{aligned}
$$

Note that the term that multiplies i_t is 0 using the FOC wrt i_t.
Write out what remains in V_0 to get

$$
\begin{aligned}
V_0 = \max_{\{i_t, k_{t+1}\}_{t=0}^{\infty}} & E_0 \Big[\Big(\big(\pi_k(k_0) - \phi_k(i_0, k_0) + q_0(1-\delta) \big) k_1 - q_0 k_1 \Big) \\
& + M_{0,1} \Big(\big(\pi_k(k_1) - \phi_k(i_1, k_1) + q_1(1-\delta) \big) k_1 - q_1 k_2 \Big) + ... \\
& + M_{0,t} \Big(\big(\pi_k(k_t) - \phi_k(i_t, k_t) + q_t(1-\delta) \big) k_t - q_t k_{t+1} \Big) + ... \Big]
\end{aligned}
$$

Using the FOC wrt k_{t+1}, cancel terms to get

$$\begin{aligned}
V_0 &= \big(\pi_k(k_0) - \phi_k(i_0, k_0) + q_0(1-\delta)\big)k_1 + E_0 \lim_{t\to\infty} M_{0,t} q_t k_{t+1} \\
&= \big(\pi(k_0) - \phi(i_0, k_0) - i_0\big) + (\phi_i(i_0, k_0) + 1)i_0 + q_0(1-\delta)k_1 \\
&= d_0 + q_0 k_1
\end{aligned}$$

assuming no bubbles so that $E_0 \lim_{t\to\infty} M_{0,t} q_t k_{t+1} = 0$. The ex-dividend value of the firm is then $V_0^e = q_0 k_1$.

2.1a. Define the operator T on a space of bounded functions as

$$\begin{aligned}
(TV)(a,k) &= \max_{k',i}\{\pi - i - \phi + \beta E[V(a',k')|a,k]\} \\
&\text{s.t. } \pi = ak^{\theta_1} l^{\theta_2} - Wl \\
&\qquad i = k' - (1-\delta)k
\end{aligned}$$

If T satisfies monotonicity and discounting then by Blackwell's Theorem T is a contraction mapping.

Monotonicity: Suppose $V \le W$ (and subject to the same constraints above). Let $g_V(k)$ denote an optimal policy corresponding to V. Then for $k \in [0, \infty)$

$$\begin{aligned}
TV(a,k) &= \pi - i(g_V(k)) - \phi + \beta E[V(a', g_V(k))|a,k] \\
&\le \pi - i(g_V(k)) - \phi + \beta E[W(a', g_V(k))|a,k] \\
&\le \max_{k',i}\{\pi - i - \phi + \beta E[W(a',k')|a,k]\} \\
&= TW(a,k)
\end{aligned}$$

Discounting: Let c be some constant

$$\begin{aligned}
T(V+c)(a,k) &= \max_{k',i}\{\pi - i - \phi + \beta E[(W(a',k')) + c)|a,k]\} \\
&= \max_{k',i}\{\pi - i - \phi + \beta E[W(a',k')|a,k] + \beta c\} \\
&= TV(a,k) + \beta c
\end{aligned}$$

V needs to be a bounded function to apply Blackwell's theorem which can be achieved by setting a bound on k' and a.

2.1b. Given the following assumptions:

- $\beta \in (0,1)$

- If the capital and technology space k is a convex subset of R^L
- The correspondence $\Gamma : (k) \Rightarrow (k)$ is non-empty, compact-valued and continuous
- The correspondence Γ is convex
- The function $F = \pi - i - \phi$ is continuous and bounded
- The function F is strictly concave

Then the unique fixed point V^* is strictly concave and the optimal policy is a single-valued continuous function.

2.1c. With equity issuance costs, dividends can be expressed as $(1 + \chi\lambda)(\pi - i - \phi)$ where χ is an indicator function that is 1 when $d < 0$. Then

$$d = \begin{cases} \pi - i - \phi & \text{if } \pi - i - \phi \geq 0 \\ (1+\lambda)(\pi - i - \phi) & \text{if } \pi - i - \phi < 0 \end{cases}$$

The firm will only pay dividends when profits exceed optimal investment.

2.1d. First, solve for the labor decision which is a static problem:

$$\max_{l_t} \pi_t$$

Taking FOC gives

$$a_t k_t^{\theta_1} \theta_t l_t^{\theta_2 - 1} = W$$
$$l_t = \left(\frac{\theta_2 a_t k_t^{\theta_1}}{W} \right)^{\frac{1}{1-\theta_2}}$$

Re-write the problem as

$$\max_{k_{t+1}} E_0 \left[\sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \left(a_t k_t^{\theta_1} l_t^{\theta_2} - W l_t - k_{t+1} + (1-\delta) k_t \right) \right]$$

Taking FOC gives

$$E_0 \left[\frac{-1}{(1+r)^t} + \frac{1}{(1+r)^{t+1}} \left(a_{t+1} \theta_1 k_{t+1}^{\theta_1 - 1} l_{t+1}^{\theta_2} \right) + 1 - \delta \right] = 0$$
$$E_t \left[a_{t+1} \theta_1 k_{t+1}^{\theta_1 - 1} l_{t+1}^{\theta_2} \right] = r + \delta$$

Using the solution for l_t above:

$$E_t\left[a_{t+1}\theta_1 k_{t+1}^{\theta_1-1}\left(\frac{\theta_2 a_{t+1}k_{t+1}^{\theta_1}}{W}\right)^{\frac{\theta_2}{1-\theta_2}}\right] = r+\delta$$

$$E_t\left[a_{t+1}^{\frac{1}{1-\theta_2}}\theta_1 k_{t+1}^{\frac{\theta_1+\theta_2-1}{1-\theta_2}}\left(\frac{\theta_2}{W}\right)^{\frac{\theta_2}{1-\theta_2}}\right] = r+\delta$$

Note that k_{t+1} can be taken out of the expectation since it is a decision variable. Then

$$k_{t+1} = \left[\frac{\theta_1\left(\frac{\theta_2}{W}\right)^{\frac{\theta_2}{1-\theta_2}} E_t[a_{t+1}^{\frac{1}{1-\theta_2}}]}{r+\delta}\right]^{\frac{1-\theta_2}{1-\theta_1-\theta_2}}$$

Next, calculate $E_t[a_{t+1}^{\frac{1}{1-\theta_2}}]$ using the given process for a_t and properties of log normality,

$$\begin{aligned}
E_t[a_{t+1}^{\frac{1}{1-\theta_2}}] &= E_t[\exp\{\log(a_{t+1}^{\frac{1}{1-\theta 2}})\}] \\
&= E_t[\exp\{\frac{1}{1-\theta_2}(\log \overline{a}^{1-\rho}) \log a_t^{\rho} + \sigma\epsilon_t\}] \\
&= \overline{a}^{\frac{1-\rho}{1-\theta_2}} a_t^{\frac{\rho}{1-\theta_2}} \exp\{\frac{1}{2}\frac{\sigma^2}{(1-\theta_2)^2}\}
\end{aligned}$$

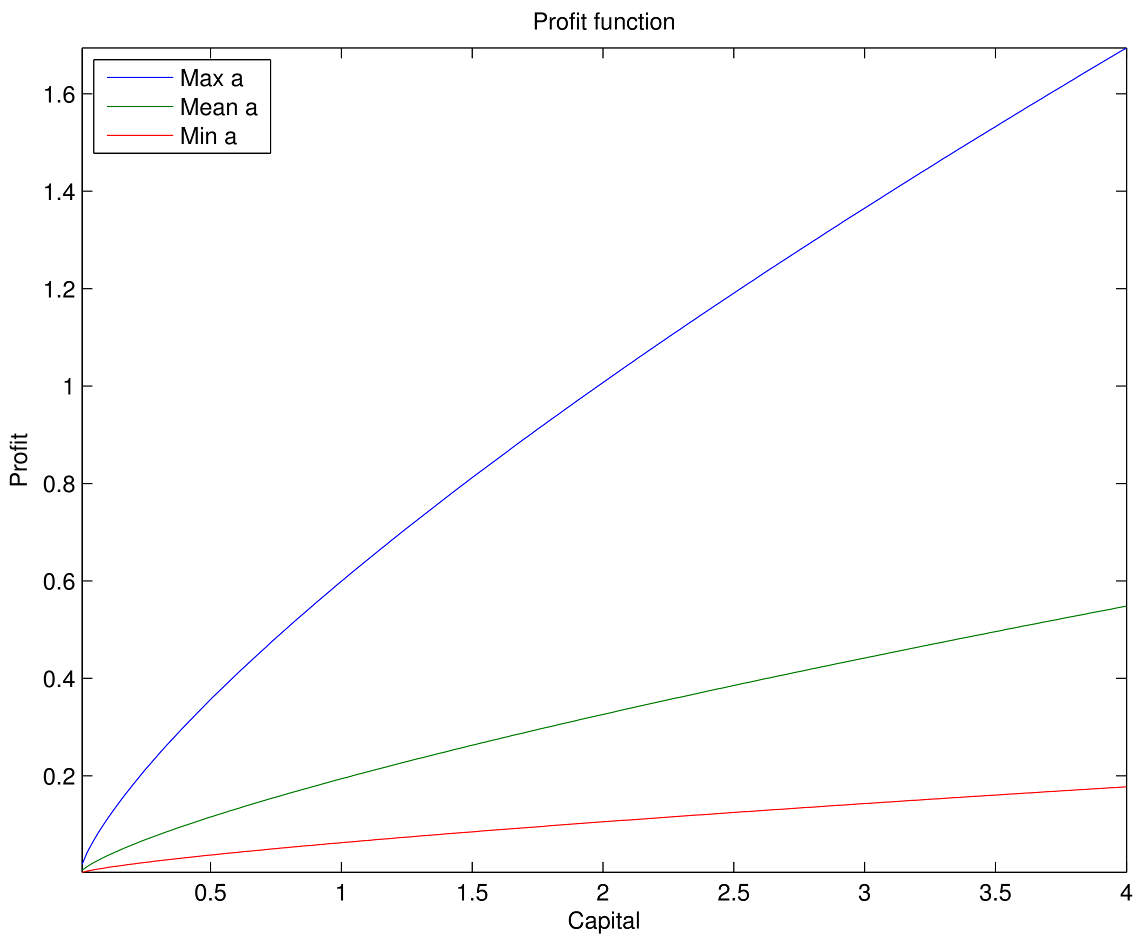
When $\overline{a} = a_t$,

$$E_t[a_{t+1}^{\frac{1}{1-\theta_2}}] = \overline{a}^{\frac{1}{1-\theta_2}} \exp\{\frac{1}{2}\frac{\sigma^2}{(1-\theta_2)^2}\} \tag{1}$$

Plug this in to the expression for optimal capital $k_{t+1} = V_t^e$ to get the value of the firm.
2.2a. Using calculations from the previous part, set $k_{t+1} = 1$ to calculate for $\overline{a}$

$$\overline{a} = \left(\frac{r+\delta}{\theta_1}\right)^{1-\theta_2}\left(\frac{\theta_2}{W}\right)^{-\theta_2} \exp\{\frac{-\sigma^2}{2(1-\theta_2)}\} \tag{2}$$

2.2b-d See code.



Profit function
Max a
Mean a
Min a
Profit
Capital
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0.5
1
1.5
2
2.5
3
3.5
4

Figure 1: 2.3a

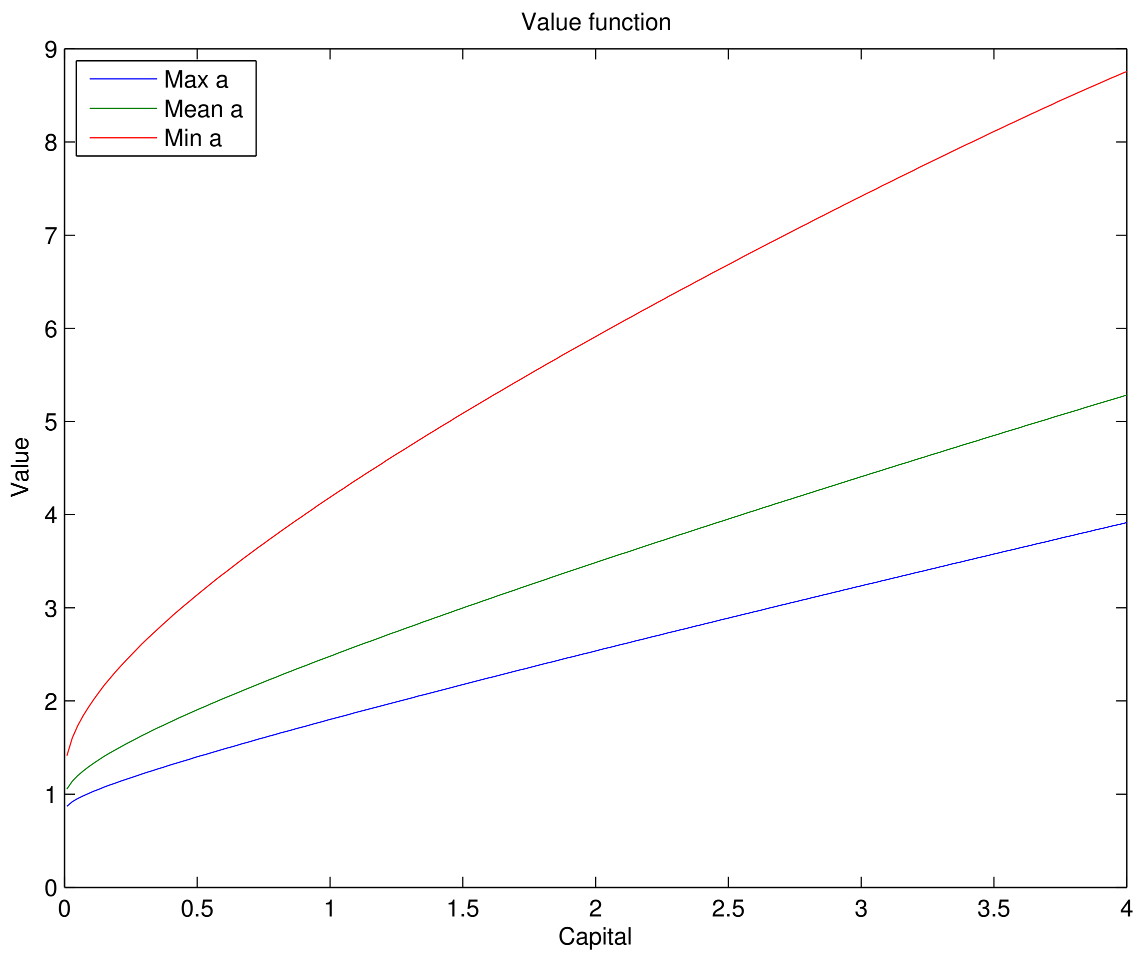


Figure 2: 2.3b value function

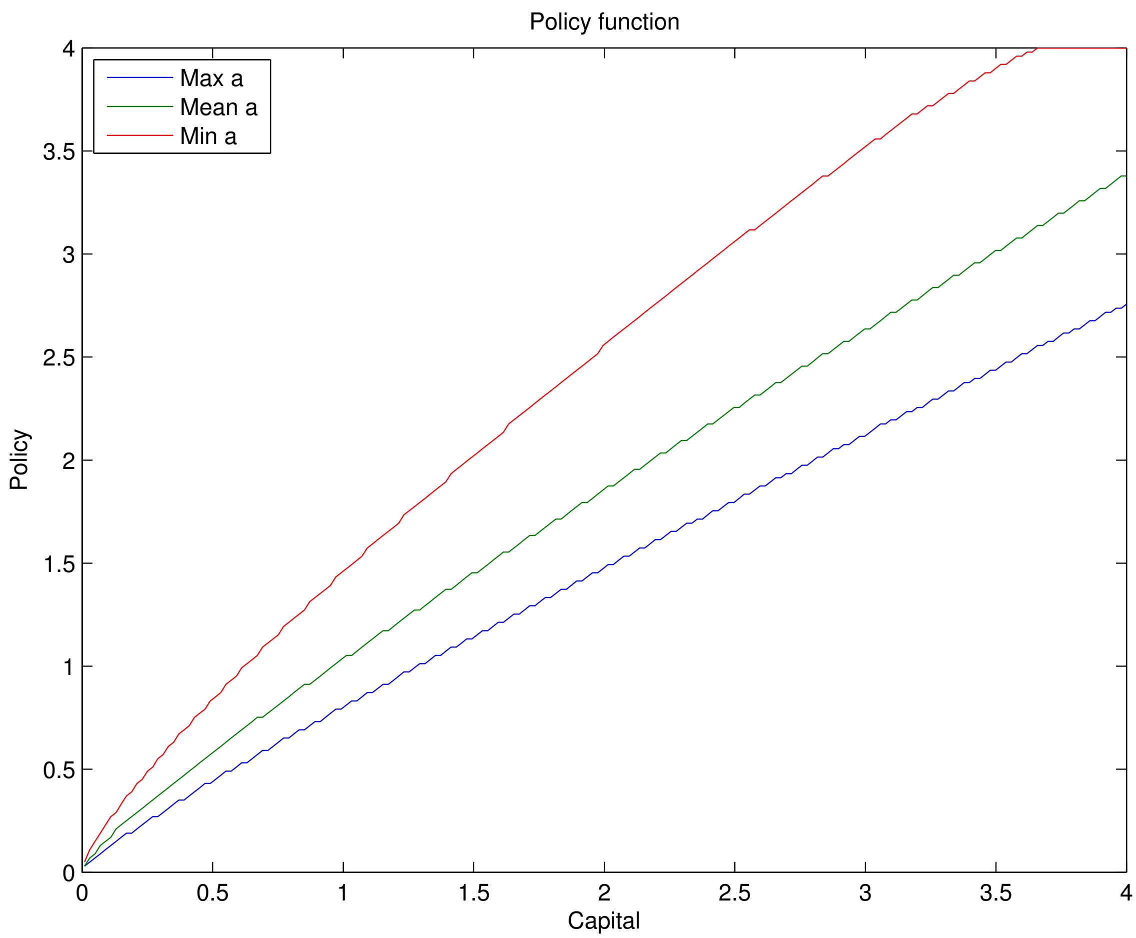


Figure 3: 2.3b policy function

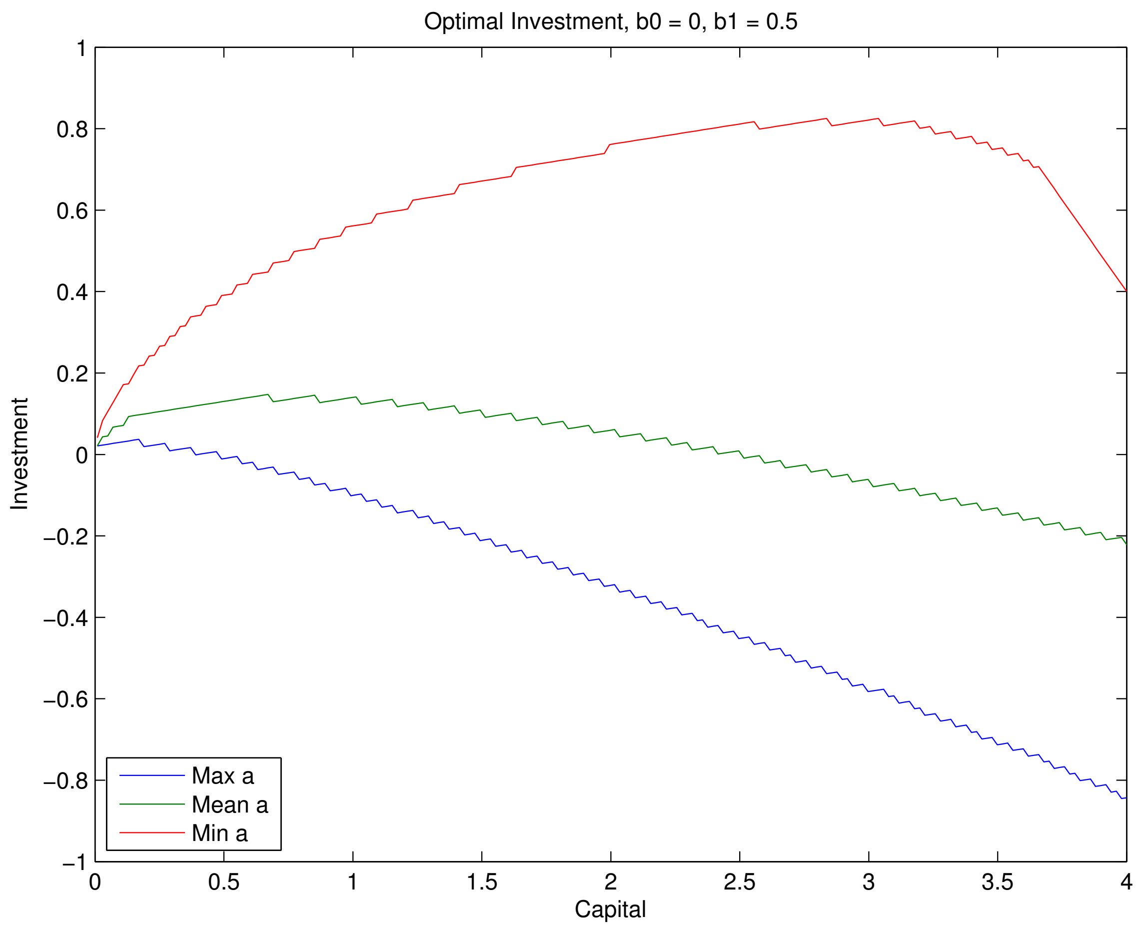


Figure 4: 2.3c optimal investment

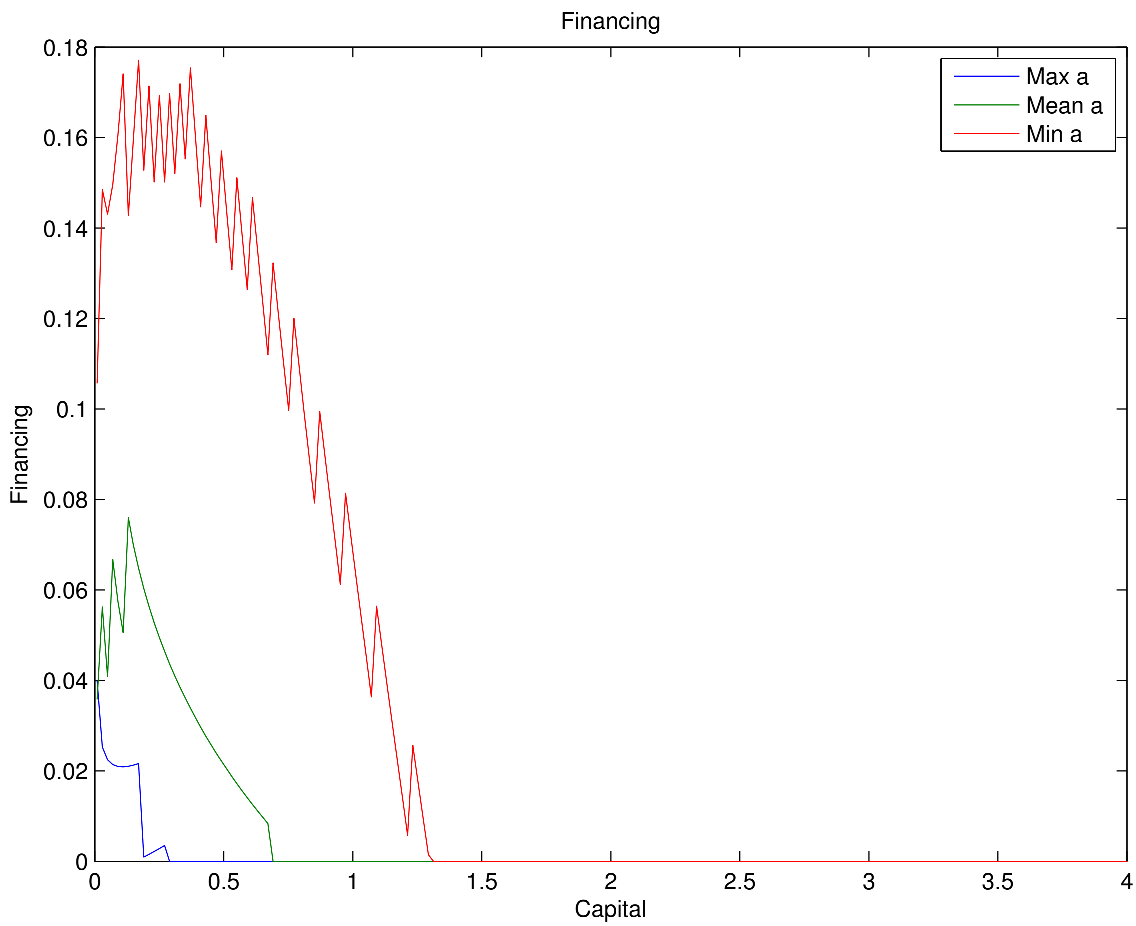


Figure 5: 2.3c financing

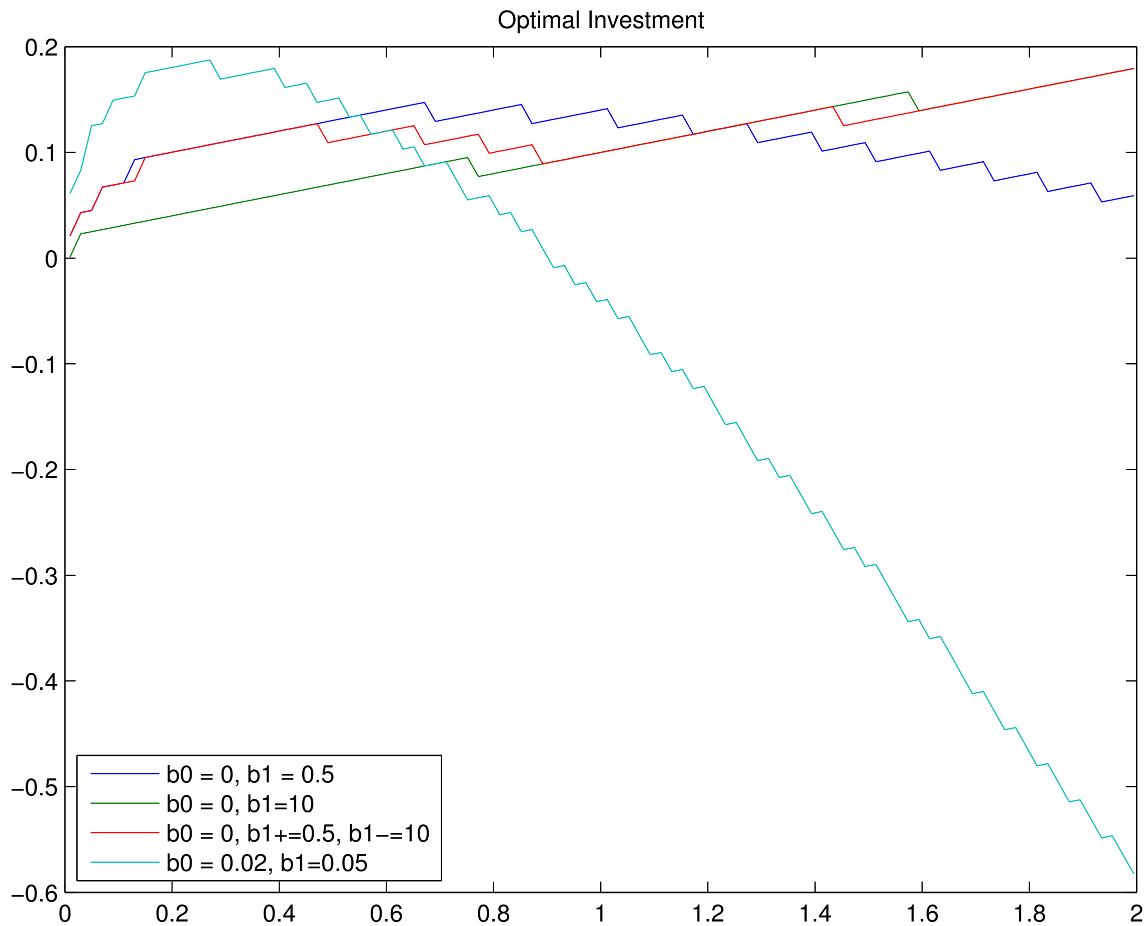

Optimal Investment
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−0.5
−0.6
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
b0 = 0, b1 = 0.5
b0 = 0, b1=10
b0 = 0, b1+=0.5, b1−=10
b0 = 0.02, b1=0.05

Figure 6: 2.3d

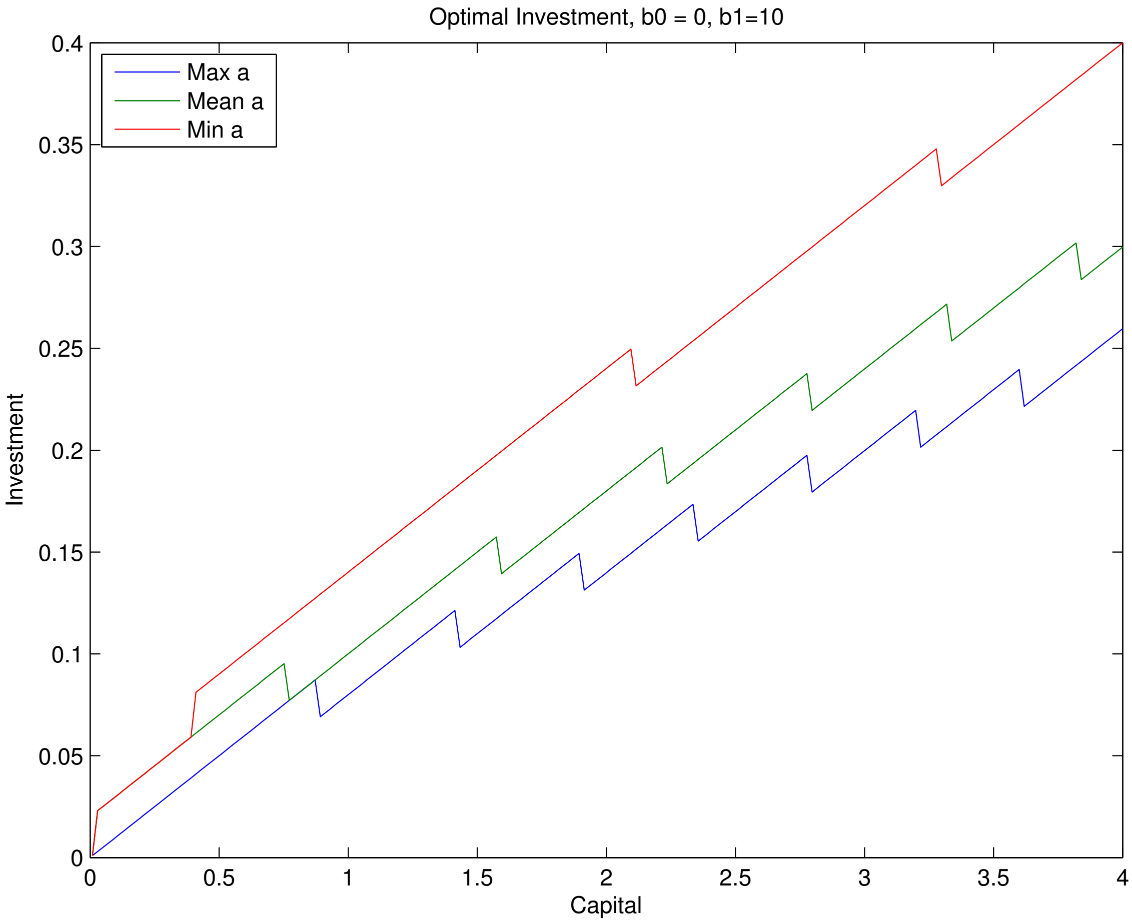


Figure 7: 2.3d-2

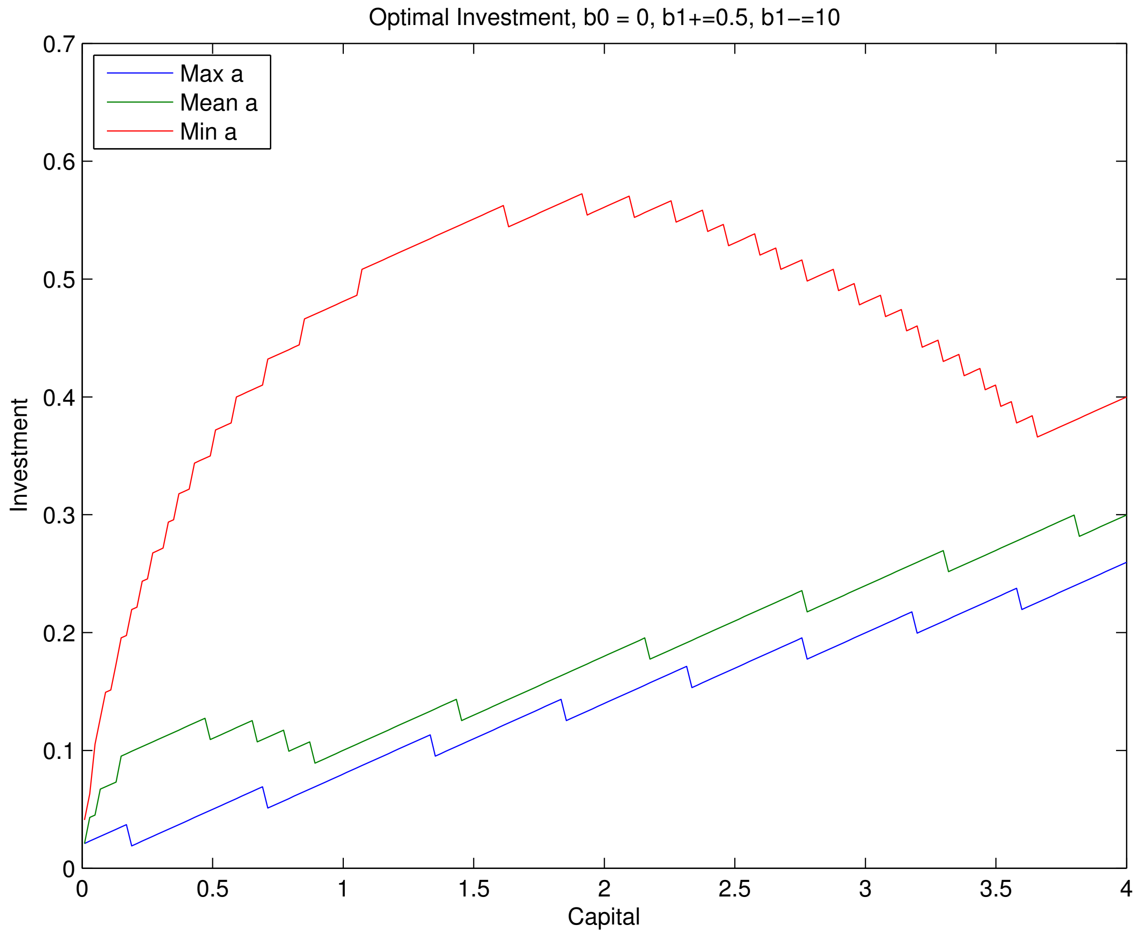


Figure 8: 2.3d-3

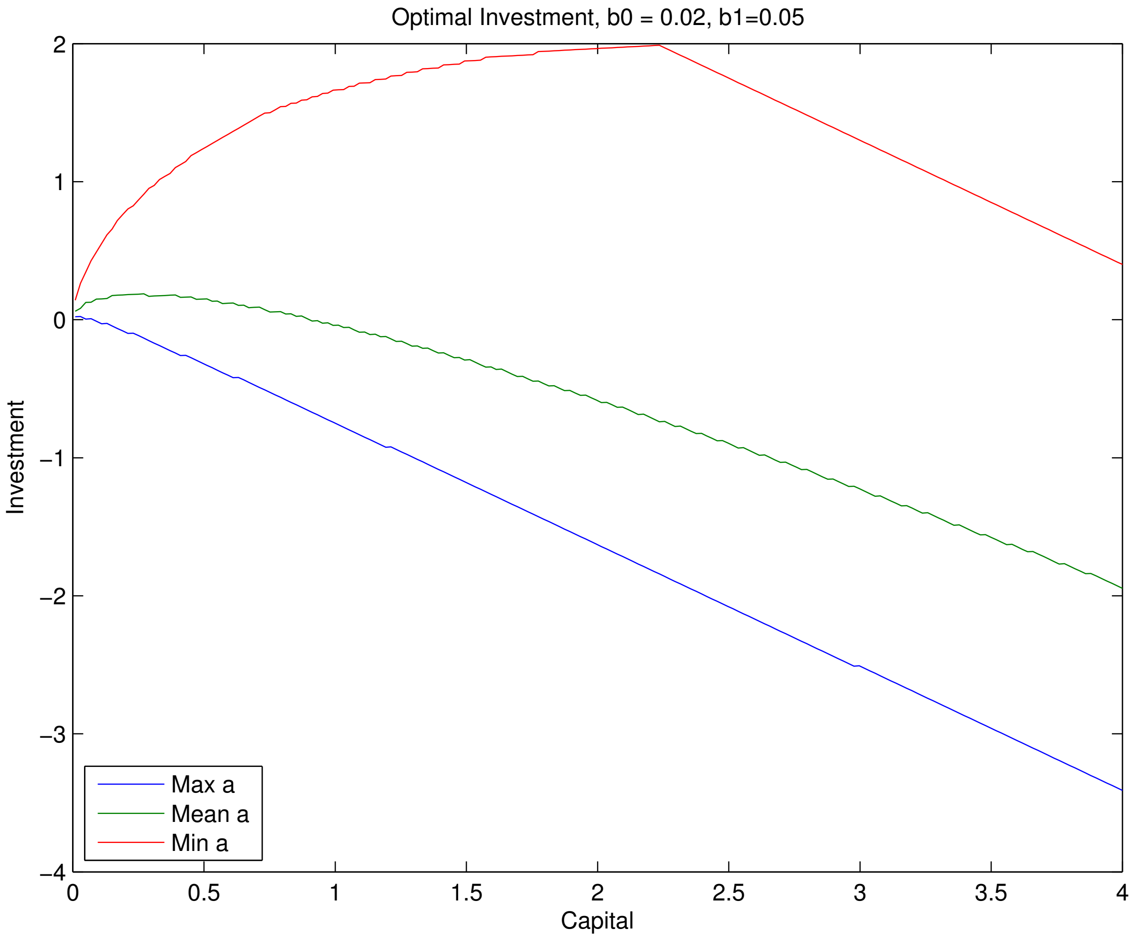


Figure 9: 2.3d-4