Finance 937
Firms and Investment

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Dynamic Quantitative Models of Firm Behavior

Our focus is on developing **quantitative** implications

- The more easily quantifiable ingredients of firm level decisions such as technology and costs
- Allow us to **quantify** and take to the data explicitly
- Pay less attention to conflicts of interest, distorted incentives, and property right issues which can be pretty vague

In addition, *technology and costs* are the true underlying source of agency issues and incomplete contracts

Basics of Accounting: Recovering Financial Data on Firms

Three main sources

- Balance sheet contains data on stock variables: assets and debt.
- Income statement contains flow data on accrued (but not necessarily realized) revenues, costs and profits.
- Statement of cash flows: contains flow data on all cash items.

Typical Balance Sheet

Firm i	
Cash and Equivalent	Accounts Payable
Accounts Receivable	Short Term Debt (b_{it}^{s})
Inventories	Long Term Debt, (b_{it}^{l})
Plant Prop. & Equip. (PP&E)	
Intangible Assets	Book Equity
Total Assets	Liabilities + Equity

Notes: PP&E is the “capital stock“ but at *historical* not market cost ($P_0^k k_{it}$). Book equity is also recorded at *historical* values. Debt is recorded at *face* value.

Typical Income Statement

Firm i	
Sales Revenues, $(P_{it}.y_{it})$	Cost of Goods Sold (COGS)
(Interest Income)	Sales, General and Admin. Cost (SG&A)
	Interest Costs $(r^b b_{it})$
	Estimated Depreciation Costs $(\delta P_0^k k_{it})$
	Taxes
Total Revenues	Total Costs

Operating Profits or *Earnings before interest, taxes and depreciation* (EBITDA)

$$
\begin{aligned}
\pi_{it} &= P_{it}.y_{it} - COGS - SGA \\
&= P_{it}.y_{it} - Wl - f
\end{aligned}
$$

Typical Statement of Cash Flow

Firm i

Operating Profits (π_{it})	Taxes
	Net Investment (i_{it})
Equity Issuance (n_{it})	Interest Payments ($r^b b_{it}$)
	Debt Repayments (b_{it})
Debt Issuance ($b_{i,t+1}$)	Dividend and Share Repurchases ($\tilde{d}_{it}$)
Cash Inflows	Cash Outflows

Firm Objectives

Key assumption in most models:

- the firm maximizes the present discounted value of dividends to its current shareholders.

The typical objective function for our firm is:

$$v_0 = \mathrm{E}_0 \sum_{t=0}^{\infty} M_{0,t} d_t$$

where

- $M_{0,t}$ is the relevant discount rate for the firm owners/shareholders between periods 0 and t.
- d_t is the **net distribution** to shareholders at time t.

Distributions are dividends *plus* share repurchases *minus* new shares issued.

Investors and Firm Value

The objective for the firm can be derived from the classical Euler equation for optimal portfolio allocation by shareholders:

$$1 = \mathrm{E}_0 \left[M_{0,1} R^s_{0,1} \right] \qquad \forall t > 0$$

and the definition of one period stock returns:

$$R^s_{0,1} = \frac{v^e_1 + d_1}{v^e_0}$$

Combine to obtain a recursive expression for the *ex-dividend* firm value:

$$v^e_0 = \mathrm{E}_0 \left[M_{0,1}(v^e_1 + d_t) \right]$$

Solving forward and imposing a TVC we obtain:

$$v^e_0 = \mathrm{E}_0 \sum_{t=1}^{\infty} M_{0,t} d_t$$

Most firm problems use cum-dividend value $v_0 = v^e_0 + d_0$.

Discounting: The specification of M

Partial equilibrium models (early on):

- Cross-sectional Asset Pricing (AP) papers
  - Specify an arbitrary exogenous process for the discount factor $M_{0,t}$.
- Corporate Finance (CF) and Industry/Growth/Trade (IO) Models:
  - Further assume no aggregate risk so that $M_{0,t} = M = R^{-t}$ is constant over time.
  - Now shocks to firm value are not correlated to discount factors and thus ex-ante firm returns are generally deterministic

General equilibrium:

- Explicitly link $M_{0,t}$ to the optimal investment choices of shareholders (households) in the economy
- Computationally hard to solve models with multiple firms
- Difficult to generate simultaneously realistic movements in $M_{0,t}$ and firm level decisions.

Firm Choices and Constraints

Very broadly, a typical firm maximizes ex-ante value by choosing optimal values for several possible policies:

- investment in productive capacity (i),
- production (y),
- hiring (l),
- payout policy (d),
- debt (b) and equity issuance (n) policies.

We start by focusing on models where only a few of these choices are available and then add others.

Constraints

In the most models optimal firm decisions are taken subject to three types of constraints:

- Technology,
- Capital accumulation,
- Sources and uses of funds

Technology and the Production Function

Typically firms derive revenues from the sale of a single good, y.

- Production of this good is done by using inputs, such as capital (k) and labor (l).
- Production function:

$$y = z.F(k, l)$$

  where z captures firm (total factor) productivity, managerial ability (Lucas (1978)), or even "organizational capital"
- Productivity may change over time, mostly exogenously
- Assume that $F : \mathcal{R}_+^2 \rightarrow \mathcal{R}_+$ is increasing, concave, and twice continuously differentiable. In addition:
  - $F(0,0) = F(0,l) = F(k,0) = 0$
  - $\lim_{k \rightarrow 0} F_k(k,l) = \infty$ and $\lim_{k \rightarrow \infty} F_k(k,l) = 0$ (the Inada conditions).

Revenues and Profits

Operating profits are given by:

$$\begin{aligned}\pi &= \max_{l}\{P.y - W.l - f\}\\ &= \max_{l}\{P.z.F(k,l) - W.l - f\}\end{aligned}$$

where

- P is the sale price of the good y,
- W is the wage rate, and
- f may be a fixed (overhead) cost of production.

Defining overhead costs as $f = W\bar{l}$, the problem of the firm could be rewritten as:

$$\pi = \max_{\tilde{l}}\left\{P.z.F(k, \tilde{l} - \bar{l}) - W.\tilde{l}\right\}$$

Revenues and Profits: Comments

At the firm level, technology and demand shocks are isomorphic: it does not matter whether uncertainty comes from P or z.

- Without any loss of generality, we generally set $P = 1$ and focus only on the properties of technology z.
- But remember this means we cannot identify pure technology shocks at the firm level without data on relative prices, P

Often we just optimize out any static variables such as labor.

- Assuming perfect competition in labor markets we obtain:

$$W = z.F_l(k, l)$$

Replacing above we obtain the **profit** function $\pi(k, z)$.

- This is generally a key model primitive
- These are **operational profits** (EBITDA)- before subtracting interest payments, taxes and depreciation expenses.

Returns to Scale

Generally $F(.)$ is of the Cobb-Douglas form

$$y = zk^{\alpha_k} l^{\alpha_l}$$

where $\alpha_k + \alpha_l \leq 1$ and the equality holds only in the case of constant returns to scale.

- The profit function becomes:

  $$\pi = ak^{\gamma} - f$$

  where

  $$\begin{aligned} a &= \alpha_l^{1/(1-\alpha_l)}(\alpha_l^{-\alpha_l} - 1)\left[z\left(\frac{W}{P}\right)^{-\alpha_l}\right]^{1/(1-\alpha_l)} \\ \gamma &= \frac{\alpha_k}{1-\alpha_l} \leq 1 \end{aligned}$$

- With constant returns to scale $f = 0$, since by definition production must be linearly homogenous in the inputs.

Constant Returns to Scale

Constant returns to scale (CRS) implies that the exact scale of the firm is indeterminate since per-period profits are linear in the scale of production

- Since firms can be of any size and produce any amount either
  - All firms have identical productivity and the exact number and size of firms in the economy is indeterminate
  - Only the highest productivity firm is producing at any point in time

Either way with CRS technologies we simply use one single firm to represent the whole economy.

- This more useful for macro models when we are not interested in explaining cross-section facts.

Non Constant Returns to Scale

To study differences across firms we *generally* need the profit function to exhibit non-constant returns to scale.

- Now the scale of production is well determinate since producing more, or less, than the optimal scale lowers profits.

Decreasing returns to scale (DRS) in operating profits, π, can follow from directly assuming DRS technology or from from combining CRS production with monopoly power:

$$\pi = P(y).y - Wl$$

With non constant returns to scale we can have an economy with:

- a fixed exogenous number of firms (perhaps just one) or,
- an endogenous number determined by free entry and exit.

Capital Accumulation and Investment

Dynamic models of firms further add accumulation of physical assets over time.

- The standard capital accumulation equation is:

$$\begin{aligned} i_t &= k_{t+1} - (1-\delta)k_t \\ &= (k_{t+1} - k_t) + \delta k_t \end{aligned}$$

  where δ denotes the rate of depreciation of the old capital stock and the subscript t denotes the period in which investment takes place.

- i_t is gross investment spending, including both net additions to the existing capital stock $(k_{t+1} - k_t)$, and replacement of depreciated or obsolete capacity (δk_t).
- Time to build: investments made this period will only affect capital (and profits) in the next period.

Sources and Uses of Funds

This is crucial in mapping the model to the data:

- Both an accounting and a cash flow (or budget) constraint.

A very detailed example (but ignoring taxes) is:

$$\widetilde{d}_t + i_t + (R^b - 1)b_t \quad = \quad \pi_t + (b_{t+1} - b_t) + n_t$$

Mapping to the data

- $\widetilde{d}_t$ is **gross** total distributions to shareholders in period t
  - dividends plus repurchases
- i_t is gross (total) capital expenditures including depreciation
- n_t is new equity issues in period t
- b_t is the stock of debt outstanding at the beginning of period time t
- $R^b = 1 + r^b$ is the **gross** interest rate on outstanding loans

Operating Profits and Corporate Earnings

Operating earnings, π_t equals

- Sales minus cost of goods old minus general expenditures (EBITDA)

Reported (before tax) corporate earnings, $\tilde{\pi}_t$, equal:

$$\tilde{\pi}_t = \pi_t - (R^b - 1)b_t - \delta \times k_t$$

- With taxes the budget constraint becomes:

$$\widetilde{d}_t + (k_{t+1} - k_t) = (1 - \tau)\tilde{\pi}_t + (b_{t+1} - b_t) + n_t$$

Equity Value with New Share Issuance

When new shares are issued the return for **existing** shareholders obeys the Euler equation:

$$\mathrm{E}_t M_{t,t+1} R^S_{t,t+1} = \mathrm{E}_t \left[M_{t,t+1} \frac{\widetilde{d}_{t+1} + \zeta_{t+1} v^e_{t+1}}{v^e_t} \right] = 1$$

where

- v^e_t is the ex-dividend value of equity:
- ζ_{t+1} is a shareholder dilution factor that obeys:

$$\zeta_{t+1} = 1 - \frac{n_{t+1}}{v^e_{t+1}}$$

- n_{t+1} is the value of new share issues.

Equity Value with New Share Issuance

Hence the value of the firm for equity holders obeys:

$$v_t^e = \mathrm{E}_t \left[M_{t,t+1} \left(\widetilde{d}_{t+1} + \zeta_{t+1} v_{t+1}^e \right) \right]$$

Define the corresponding *cum-dividend* value as:

$$v_t = v_t^e + (\widetilde{d}_t - n_t)$$

and the corresponding recursion for *cum-dividend* values remains as before:

$$v_t = (\widetilde{d}_t - n_t) + \mathrm{E}_t M_{t,t+1} v_{t+1}$$

Dividends and Equity Issues

Without equity market frictions we cannot separate $\widetilde{d}_t$ and n_t.

- Equity issues are just like **negative** dividends.

In this case we can just work with *net distributions*:

$$d_t = \widetilde{d}_t - n_t$$

and allowing this quantity to be negative.

- In general we say that a firm issues equity when $d_t < 0$ and pays positive dividends otherwise.

Without equity issuance costs, the budget constraint of the firm simplifies to:

$$d_t + (k_{t+1} - k_t) = (1 - \tau)\tilde{\pi}_t + (b_{t+1} - b_t)$$

Financial Frictions: Costly External Debt

Dynamic Program of the Firm

$$v_0 = \max_{\{d_t, k_{t+1}, b_{t+1}\}_{t=0}^{\infty}} \mathrm{E}_0 \left[\sum_{t=0}^{\infty} M_{0,t} d_t \right]$$

$$\text{s.t.} \qquad d_t + k_{t+1} - k_t = (1-\tau)\tilde{\pi}_t + (b_{t+1} - R^b b_t)$$

The first order condition for new debt issues implies:

$$1 - \mathrm{E} M R^b \leq 0$$

- Assuming R^b is constant (independent of b):

If issuing debt is expensive (for exogenous reasons) we have that $\mathrm{E} M R^b > 1$

- Optimal debt policy is $b_{t+1} = 0$

Financial Frictions: Optimal Debt Choices

To get interior solutions for b_t (explored later) we must have:

- Assume $R^b = R^b(b)$ - perhaps as a result of default risk.
- The first order condition for debt becomes:

$$1 = \mathrm{E}(R^b(b_{t+1}) + R'^b(b_{t+1})b_{t+1})$$

- which will pin down the optimal amount of b_{t+1}

Importantly:

- The optimal level b_{t+1} will impact the value function and the optimal investment choices
- Financing matters.

Firm Value under Modigliani-Miller

We will for now assume that $\mathrm{E} M R^b = 1$

- The ex-ante cost of (or **required market return** on) debt and equity are identical

Then the exact financing choice is **indeterminate**

- It will not affect the value of the firm - Modigliani-Miller (1958)

The problem of the firm can be simplified to:

$$
\begin{aligned}
v_0 = & \max_{\{d_t, k_{t+1}\}_{t=0}^{\infty}} \mathrm{E}_0 \left[\sum_{t=0}^{\infty} M_{0,t} d_t \right] \\
\text{s.t.} \quad & d_t + i_t = \pi_t
\end{aligned}
$$

where d_t now denotes net distributions to **both** equity and debt:

$$
d_t = \widetilde{d}_t - n_t + R^b b_t - b_{t+1}
$$

Firm Problem under Modigliani Miller

The **total** value of the firm can be written compactly as:

$$v_0 = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \mathrm{E}_0 \left[\sum_{t=0}^{\infty} M_{0,t} \left(\pi(k_t, a_t) + (1-\delta)k_t - k_{t+1} \right) \right]$$

- Since debt and equity values are arbitrary we can simply assume that $b_t = 0$ for all t.
- Total (or **enterprise**) value equals the value of a firm with 0 debt (unlevered firm)

Firm Value with Constant Returns to Scale

The first order condition for optimal capital accumulation is:

$$\mathrm{E}_0 M_{0,t} = \mathrm{E}_0 M_{0,t+1} \left[\pi_k(k_{t+1}, a_{t+1}) + (1-\delta) \right]$$

Under CRS we can rewrite the value of the firm as:

$$\begin{aligned}
v_0 \quad & = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \mathrm{E}_0 \left[\pi_k(k_0, a_0) + (1-\delta) \right] k_0 - k_1 \\
& + \quad M_{0,1} \left[\pi_k(k_1, a_1) + (1-\delta) \right] k_1 - M_{0,1} k_2 + \ldots \\
& + \quad M_{0,t} \left[\pi_k(k_t, a_t) + (1-\delta) \right] k_t - M_{0,t} k_{t+1} + \\
& + \quad M_{0,t+1} \left[\pi_k(k_{t+1}, a_{t+1}) + (1-\delta) \right] k_{t+1} - M_{0,t+1} k_{t+2} + \\
& + \quad \ldots
\end{aligned}$$

The FOC then implies that all intermediate terms cancel out.

Firm Value with Constant Returns to Scale

The value function the becomes

$$\begin{aligned} v_0 &= \pi(k_0, a_0) + (1-\delta)k_0 - \mathrm{E}_0 \lim_{t\to\infty} M_{0,t+1} k_{t+1} \\ &= d_0 + k_1 \end{aligned}$$

Assuming that the value of the firm is driven only by fundamentals and there are no bubbles:

$$\mathrm{E}_0 \lim_{t\to\infty} M_{0,t+1} k_{t+1} = 0$$

Moreover since:

$$d_0 = \pi(k_0, a_0) + (1-\delta)k_0 - k_1$$

Ex-dividend firm value is equal to the capital stock:

$$v_0^e = v_0 - d_0 = k_1$$

Value with Decreasing Returns to Scale

Suppose instead that:

$$\pi_k(k, a)k = \gamma\pi(k, a)$$

with $\gamma < 1$.
Then firm value is given by:

$$v_0^e = k_{t+1} + \mathrm{E}_0 \sum_{s=0}^{\infty} (1-\gamma) M_{0,t+s} \pi(k_{t+s}, a_{t+s})$$

the extra term

$$\mathrm{E}_0 \sum_{s=0}^{\infty} (1-\gamma) M_{0,t+s} \pi(k_{t+s}, a_{t+s}) > 0$$

captures the present value of any **future rents** or economic profits.

Optimal Investment under Modigliani Miller

The first order condition for optimal capital accumulation, k_{t+1}, is:

$$\mathrm{E}_0 M_{0,t} = \mathrm{E}_0 \left[M_{0,t+1} [\pi_k(k_{t+1}, a_{t+1}) + (1-\delta)] \right]$$

We can use the definition of the stochastic discount factor to construct the **one period ahead discount factor**:

$$M_{t,t+1} = \frac{M_{0,t+1}}{M_{0,t}}$$

Since all variables are known at time t we can use the law of iterated expectations to drop E_0 and rewrite the FOC as a function of contemporaneous information only as

$$1 = \mathrm{E}_t \left[M_{t,t+1} [\pi_k(k_{t+1}, a_{t+1}) + (1-\delta)] \right]$$

Optimal Investment

With decreasing returns to scale optimal investment is defined implicitly by the first order condition above since $\pi_k(k_{t+1}, a_{t+1})$ is a (declining) function of k_{t+1}.

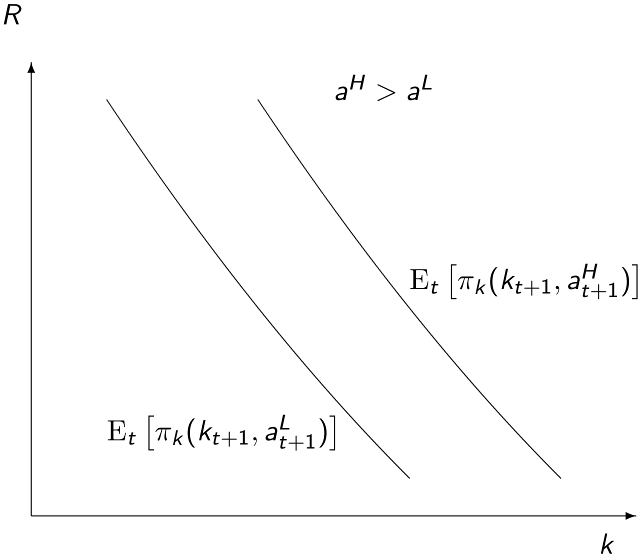
- For example using a constant SDF $M_{t,t+1} = 1/R^s = 1/(1+r^s)$ we get:

$$\mathrm{E}_t\left[\pi_k(k_{t+1}, a_{t+1})\right] = r^s + \delta$$

- Restricting to Cobb Douglas technology we get

$$k_{t+1} = \left(\frac{\gamma E_t\left[a_{t+1}\right]}{r^s + \delta}\right)^{\frac{1}{1-\gamma}}$$

Optimal Demand for Capital



Optimal Investment

Optimal firm size k_{t+1} is:

- increasing in **expected** productivity a_{t+1}
- declining in expected returns or (without aggregate risk) interest rates

Given firm size k_{t+1} we construct investment from capital accumulation as

$$i_t = k_{t+1} - (1 - \delta)k_t$$

Thus, in addition to expected productivity and discount rates investment is also

- negatively related to current size, i.e., large firms invest less than small firms

This is basically the **Jorgenson user cost** model of investment

Two Problems with Instantaneous Capital Adjustment

The optimal policy above is a good benchmark but it has two basic problems:

- It implies very fast (instantaneous in continuous time) adjustment of capital stock to shocks.
- It does not work with constant returns to scale, since $\pi_k(\cdot)$ is independent of k_{t+1}, so optimal firm size k_{t+1} is arbitrary.

Adjustment Costs

A common solution to both problems is to add adjustment costs to the accumulation of capital over time.

- Micro foundation: installing new capital disrupts production

Specifically, we assume that:

- Each dollar of added productive capacity requires $1 + \Phi(i_t)$ dollars of additional investment

With adjustment costs the uses and sources of funds equation becomes:

$$d_t + i_t + \Phi(i_t) = \pi_t$$

Adjustment Costs in the Capital Accumulation Equation

An alternative approach to introduce adjustment costs is to *instead* modify the capital accumulation equation to

$$k_{t+1} = (1 - \delta)k_t + G(i_t)$$

where $G(i_t) \leq i_t$

- Now we assume that spending one dollar in capital expenditures (and thus reducing distributions) will lead to less than one dollar in accumulated capital for next period.
- These two formulations of adjustment costs are formally equivalent.

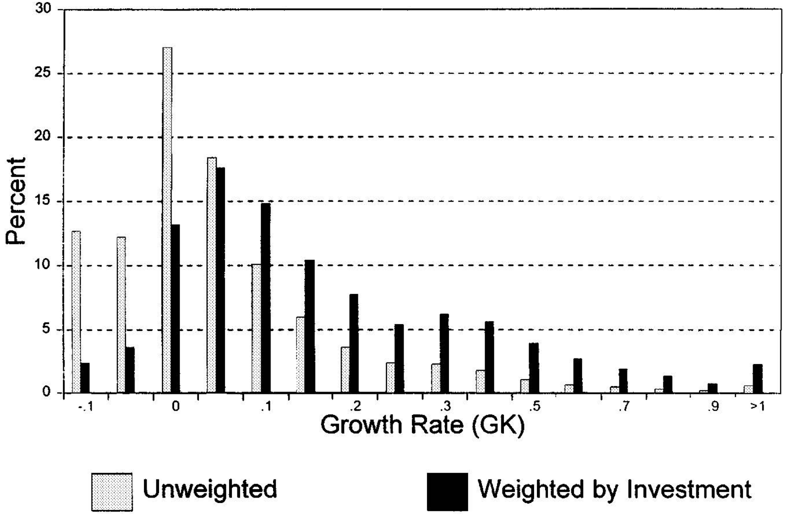
Sources of Adjustment Costs: Fixed Costs

Investment may entail disruptions that are independent of the actual amount of investment.

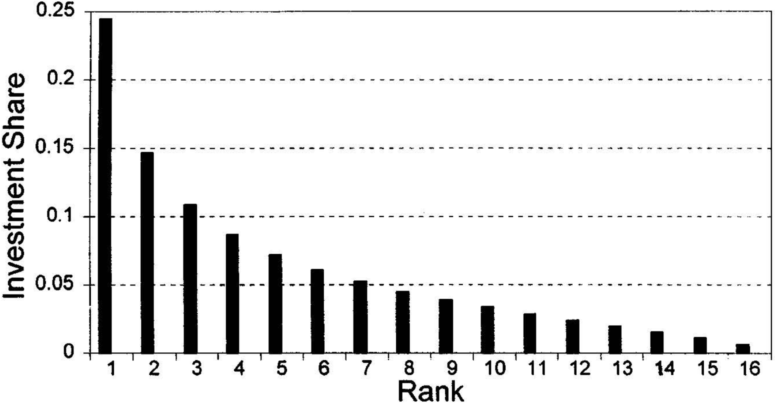
- This allow generates periods where it is optimal not to invest at all.
- Observations with zero investment are common in establishment and (smaller) firm level data. They are much less common at industry and aggregate levels.
- The larger the fixed cost the less frequent investment (and disinvestment) will be.
- Fixed costs also imply that investment, when it occurs, will be lumpy: It is never optimal to just invest a small amount

These costs are less relevant when working with aggregate/macro data.

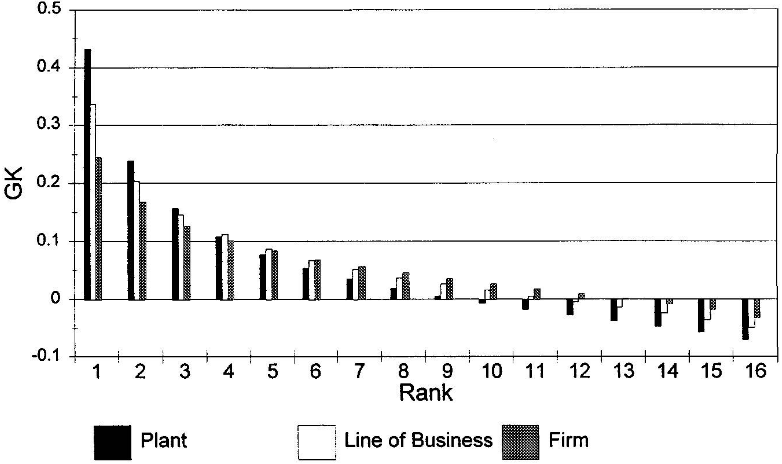
Evidence of Investment Inaction - Doms & Dunne (1998)



Evidence of Lumpy Investment - Doms & Dunne (1998)



Aggregation and Investment - Doms & Dunne (1998)



Sources of Adjustment Costs: Costly Reversibility

Suppose firms face different prices when buying and selling capital

- Let $p_I^+ = 1$ be the price of new capital purchases ($i_t^+ > 0$) and $p_I^- < 1$ be the price of capital sales ($i_t^- > 0$).
- If $p_I^- = 0$ investment is completely **irreversible**.

The budget constraint for the firm is then:

$$\begin{aligned} d_t + i_t^+ p_I^+ - i_t^- p_I^- &= \pi_t \\ d_t + p_i^+[i_t^+ - i_t^-] + i_t^-[p_I^+ - p_I^-] &= \pi_t \\ d_t + 1.i_t + i_t^-[1 - p_I^-] &= \pi_t \end{aligned}$$

where $i_t = i_t^+ - i_t^-$ is total investment.

Irreversibility - Ramey & Shapiro (2001)

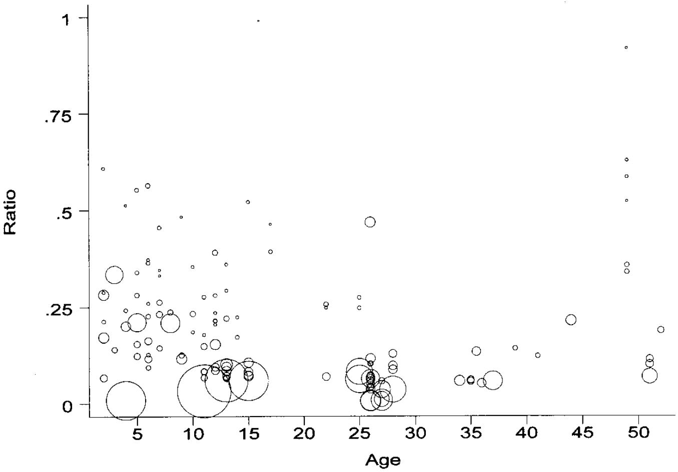


FIG. 2.—Ratio of sales price to reflated initial cost. The size of the circle is proportional to the reflated initial cost (not adjusted for depreciation).

A General Adjustment Costs Function

A very general and flexible adjustment function:

$$\Phi(i,k) = \begin{cases} \phi_0 + \phi_1^+ \frac{|i/k-\omega|^{1+\eta}}{1+\eta} k, & \text{if } i > \omega k \\ 0, & \text{if } i = \omega k \\ \phi_0 + \phi_1^- \frac{|i/k-\omega|^{1+\eta}}{1+\eta} k, & \text{if } i < \omega k \end{cases}$$

Two special cases of our adjustment cost function $\Phi(\cdot)$ with

- with irreversibility $\phi_1^+ = 0$ and $\phi_1^- = 1 - p_I^- > 0$ and $\eta = 0$.
- with fixed costs $\phi > 0$

ωk is the "normal" or "natural" amount of investment.

Natural Rate of Investment

What is the value of ω?

- A logical choice is to choose $\omega = 0$ so that it is costly to deviate from 0 investment.
- But it is also common to use $\omega = \delta$, implying that mere replacement of depreciated capital does not incur adjustment costs.

In practice, the choice is often guided by mathematical convenience.

The Marginal Adjustment Cost

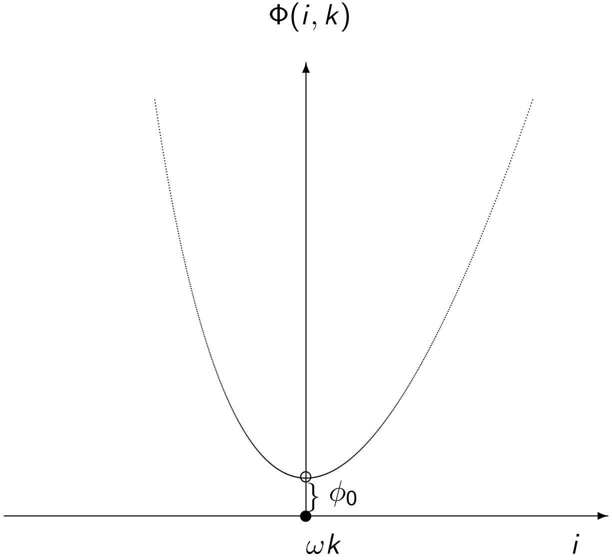
If $\eta > 0$, we have that $\Phi_{ii}(\cdot) > 0$, and the marginal cost of investment will be convex

- The cost of making adjustments to the capital stock will increase in investment

This implies that the firm will never invest "too much" and thus the capital stock will always respond slowly to shocks.

- The value of η helps regulate the volatility of investment expenditures in the model

Investment Adjustment Costs



Additional Comments on Adjustment Costs

If the firm/economy grows over time fixed adjustment costs become progressively less important.

- In this case we should scale the fixed cost by the amount of capital and use, $\phi_0 \times k$

It is common to assume the adjustment cost function is homogeneous of degree one in capital and investment.

- This is useful in CRS models but also seems fairly plausible

In this case we can write:

$$\Phi(i, k) = \phi(i/k)k$$

Financing Frictions as Adjustment Costs - Gomes et al (2006)

Equity issuance costs

- Assume they reduce the claim of current shareholders by a fraction $\Lambda_t \geq 1$ of the new equity issues, n_t
- Perhaps just because of issuance costs (underwriting fees)
- It might also be plausible to assume $\partial\Lambda(k, n, a)/\partial n \geq 0$

Defaultable debt, $b' = b_{t+1}$

- $R^b(k, b, a)$ is the cost of debt and obeys

$$\mathrm{E}[MR^b(k', b', a')] \geq 1$$

- Again, also plausible to have $\partial R^b(k, b, a)/\partial b \geq 0$

The Shareholder's Problem with Financing Frictions

The Bellman equation for **existing equity holders** is now:

$$
\begin{aligned}
e(k, b, a) \quad &= \quad \max_{d, b', k', n} \left\{ d - \Lambda n + \mathrm{E}\left[M' e(k', b', a') \right] \right\} \\
\text{s.t.} \qquad & \qquad c = \pi - i \\
& \qquad d = c + b' - R^b b + n \\
& \qquad k' = k(1 - \delta) + i \\
& \qquad d \geq 0,
\end{aligned}
$$

Optimal financing choices (a "pecking" order):

$$
\begin{cases}
b' > 0 \ \text{or} \ n > 0 & \Longrightarrow \qquad d = 0 \\
d > 0 & \Longrightarrow \quad b' \leq 0 \ \text{and} \ n = 0
\end{cases}
$$

Example 1: The case with no debt $b = b' = 0$

Resource constraint becomes:

$$d \quad = \quad c + n$$

Thus

$$n \quad = \quad i - \pi > 0 \Longrightarrow d = 0$$

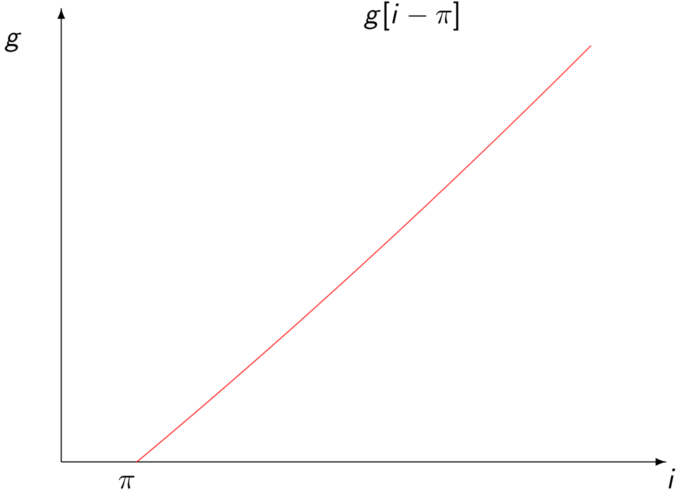
Bellman equation:

$$\begin{aligned} e(k,a) &= \max_{k',n}\{d - n - [\Lambda - 1]\, n + \mathrm{E}\left[M' e(k',a')\right]\} \\ &= \max_{k'}\{c - g + \mathrm{E}\left[M' e(k',a')\right]\} \end{aligned}$$

Thus

$$\begin{aligned} v(k,a) &\quad = \quad e(k,a) \\ g(k',k,a) &\quad = \quad \underbrace{\left[\Lambda\left(k',k,a\right) - 1\right]}_{\text{Financing Premium}} \times \underbrace{n(k',k,a)}_{\text{External Financing}} \end{aligned}$$

Fincial Frictions and Capital Adjustment Costs



Example 2: The case with no equity issues $n = 0$

Resource constraint is:

$$d = c + b' - R^b b$$

Bellman equation

$$\begin{aligned} e(k,b,a) &= \max_{k',b'} \{c + b' - R^b b + \mathrm{E}\left[M' e(k',b',a')\right]\} \\ e(k,b,a) + R^b b &= \max_{k',b'} \left\{c + b' + \mathrm{E}\left[M'(e(\cdot) + R^b b' - R^b b')\right]\right\} \\ v(k,a) &= \max_{k'} \left\{c - g + \mathrm{E}\left[M' v(k',a')\right]\right\} \end{aligned}$$

Thus

$$\begin{aligned} v(k,a) &= e(k,a) + R^b b \\ g(k',k,a) &= \underbrace{\mathrm{E}\left[M' R^b(k',k,a) - 1\right]}_{\text{Premium}} \times \underbrace{b'(k',k,a)}_{\text{Financing}} \end{aligned}$$

Financing Frictions as Adjustment Costs - Gomes et al (2006)

The general problem with financing frictions can often be rewritten as a frictionless problem with added adjustment costs

- We can ignore the financial details and instead just add a **financing cost** function to the problem:

$$v(k,a) = \max_{k'} \left\{ \pi - i - g + \mathrm{E}\left[M' v(k',a') \right] \right\}$$

where

$$g(k',k,a) = \mu(a,k) m(k',k,a)$$

Here $\mu(\cdot)$ is the marginal cost of external finance and

$$m \equiv R^b b + i - \pi$$

Now the amount of debt, b, is no longer a choice variable.

Problem of the Firm with Adjustment Costs

The problem of the firm becomes

$$\begin{aligned} v(a_0, k_0) &= \max_{\{k_{t+1}, i_t\}_{t=0}^{\infty}} \mathrm{E}_0 \left[\sum_{t=0}^{\infty} M_{0,t} \left(\pi(k_t, a_t) - i_t - \Phi(i_t, k_t) \right) \right] \\ \text{s.t.} \quad i_t &= k_{t+1} - (1-\delta) k_t \end{aligned}$$

Now

- We often optimize for both investment (control) and next period capital (state) separately.
- This problem can become complicated if $\Phi(\cdot)$ is not differentiable (or even continuous).

Given this, some of characterization below is slightly **informal** and meant as an approximation to the correct continuous time results.

Marginal q

When adjustment costs are smooth we solve this problem in two stages.

- First determine the optimal amount of investment conditional on the shadow value of capital
- Then determine the shadow value of capital

To this end define q_t as the multiplier on the capital accumulation equation in period t

$$q_t = \frac{\partial v_t}{\partial k_{t+1}}$$

- This is called **marginal** q and equals the marginal value of one unit of (newly) installed capital to the firm

Optimal Investment with Smooth Adjustment Costs

Optimal investment can be constructed from the static (sub)problem

$$\Psi(a_t, k_t) = \max_{i_t} \left[(q_t - 1)i_t - \Phi(i_t, k_t)\right]$$

- This determines the optimal amount of current period investment, given the current state variables
- Naturally $q_t = q(a_t, k_t)$ so $\Psi(\cdot)$ will be a function of the current state variables
- The first order condition for optimal investment is

$$q_t - 1 = \Phi_i(i_t, k_t)$$

- Without adjustment costs, $\Phi_i(i_t, k_t) = 0$.
- In that case $q_t = 1$ and this condition does not pin down investment

Average and Marginal Q

Suppose that technology and adjustment costs are CRS and markets are competitive so that $\pi = ak$

- Suppose also that $\Phi(\cdot)$ is continuously differentiable everywhere
- Then firm value can be show to equal:

$$v_t^e = q_t k_{t+1}$$

- Firm value is equal to the capital stock times the marginal value of a unit of capital

It follows that $q_t = v_t^e / k_{t+1} =$ "Average Q"

Q Theory of Investment - Tobin (1969), Hayashi (1982)

When $\Phi(i_t, k_t)$ is homogeneous of degree 1, its derivatives are a function of i/k only

- In this case we can write the optimal first condition to derive a closed form solution for the investment rate

$$\frac{i_t}{k_t} = \Phi_i^{-1}(q_t - 1)$$

- q is a sufficient statistic for investment
- Investment is above "normal" when the marginal value of capital is above 1, and below "normal" otherwise

Drivers of Q: Discount Rates, and Cash Flows/Profits

To further characterize (and measure) Q we can also use the Euler equation for the problem

$$q_t = E_t M_{t,t+1} \left[\pi_k(k_{t+1}, a_{t+1}) - \Phi_k(i_{t+1}, k_{t+1}) + (1-\delta) q_{t+1}\right]$$

Solve forward to get

$$q_t = E_t \sum_{s=1}^{\infty} M_{t,t+s} (1-\delta)^s \left[\pi_k(k_{t+s}, a_{t+s}) - \Phi_k(i_{t+s}, k_{t+s})\right]$$

Thus q (and investment) depends on **all** expected future discount rates, $M_{t,t+s}$, and cash flows, $\pi_k(k_{t+s}, a_{t+s}) - \Phi_k(i_{t+s}, k_{t+s})$.

- Abel and Blanchard (1986) construct marginal q directly by estimating a VAR for profits and interest rates to evaluate the sum

Application: Empirical Investment Equations

Suppose adjustment costs are smooth and symmetric, so that:

- $\phi_1^+ = \phi_1^- = \phi_1$
- $\phi_0 = 0$

In this case we can write the investment rate as

$$\frac{i_t}{k_t} = \omega + \frac{q_t - 1}{\phi_1}$$

- This has become a very popular empirical implementation of Q-theory
- $q = Q$ can be directly constructed from data of market value of firm and the value of its assets
- We can estimate the optimal investment equation **without solving** the full of the problem of the firm

Application: The Value of Intangibles - Hall (2001)

Motivation

- Investment and stock of intangible capital is hardly observable in the data.
- Securities value data can be used to infer the total stock of capital, and, thus, the stock of intangibles
- https://www.federalreserve.gov/releases/z1/

Solution

- Estimate value of intangibles with securities market data
- Assuming CRS and perfect competition

Compute (q_t, k_{t+1}) jointly from

$$v_t^e = q_t k_{t+1}$$
$$\phi_1 \frac{i_t}{k_t} = q_t - 1$$

- ϕ_1 is interpreted as the doubling time (in quarters) for the capital stock, facing a change in q_t from 1 to 2

Application: Alternative Adjustment Costs - (Hall 2001)

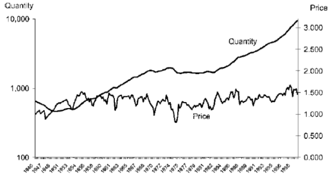


FIGURE 8. QUANTITY AND PRICE OF CAPITAL, WITH DOUBLING TIME OF EIGHT QUARTERS

FIGURE 9. QUANTITY AND PRICE OF CAPITAL, WITH DOUBLING TIME OF 32 QUARTERS

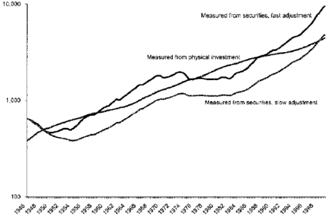


FIGURE 10. COMPARISON OF ESTIMATES OF THE QUANTITY OF CAPITAL

Cash Flow Effects - Fazzari et al 1988

Table 4. Effects of Q and Cash Flow on Investment, Various Periods, 1970–84[a]

Independent variable and summary statistic	*Class 1*	*Class 2*	*Class 3*
		1970–75	
Q_{it}	−0.0010	0.0072	0.0014
	(0.0004)	(0.0017)	(0.0004)
$(CF/K)_{it}$	0.670	0.349	0.254
	(0.044)	(0.075)	(0.022)
$\bar{R}^2$	0.55	0.19	0.13

Notes:

- $CF/K = \pi/k$.
- This strong correlation is often called the "cash flow effect" on investment

Investment, Size and Dividends- Fazzari et al (1988)

Table 2. Summary Statistics: Sample of Manufacturing Firms, 1970–84

Statistic	*Category of firm*		
	Class 1[a]	*Class 2*[b]	*Class 3*[c]
Number of firms	49	39	334
Average retention ratio	0.94	0.83	0.58
Percent of years with positive dividends	33	83	98
Average real sales growth (percent per year)	13.7	8.7	4.6
Average investment-capital ratio	0.26	0.18	0.12
Average cash flow–capital ratio	0.30	0.26	0.21
Average correlations of cash flow with investment (deviations from trend)[d]	0.92	0.82	0.20
Average of firm standard deviations of investment-capital ratios	0.17	0.09	0.06
Average of firm standard deviations of cash flow–capital ratios	0.20	0.09	0.06
Capital stock (millions of 1982 dollars)			
Average capital stock, 1970	100.6	289.7	1,270.0
Median capital stock, 1970	27.1	54.2	401.6

Investment with General Adjustment Costs

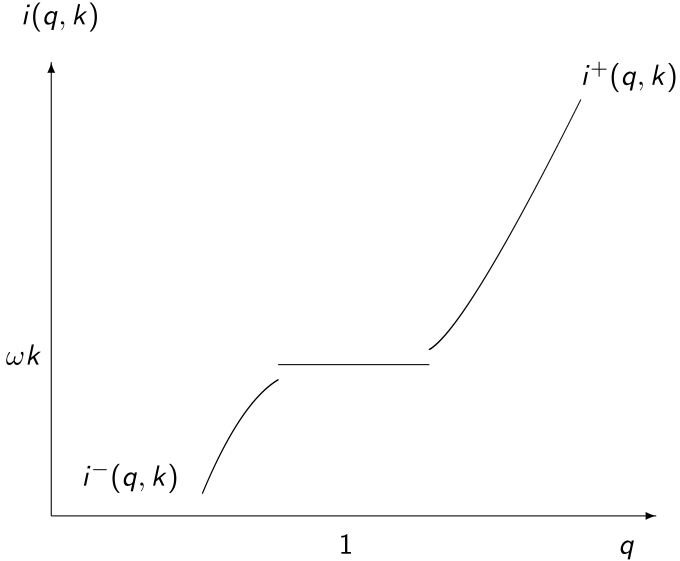
Away from the inaction region, optimal investment obeys the optimality conditions:

$$(i/k)_t^- = \omega + \left[\frac{q_t - 1}{\phi_1^-}\right]^{1/\eta} < 0 < \omega + \left[\frac{q_t - 1}{\phi_1^+}\right]^{1/\eta} = (i/k)_t^+$$

Note that, this implies the optimal investment policy is discontinuous in the capital stock around ω:

$$i_t^\star - \omega k_t = k \begin{cases} \left[(q_t - 1)/\phi_1^-\right]^{1/\eta}, & \text{if} \quad q_t = 1 + \Phi_i(0^-) \\ 0, & \text{if } \Phi_i(0^-) < q_t - 1 < \Phi_i(0^+) \\ \left[(q_t - 1)/\phi_1^+\right]^{1/\eta}, & \text{if} \quad q_t = 1 + \Phi_i(0^+) \end{cases}$$

Optimal Investment



Timing of Investment

The region where $i_t = \omega k_t$ is the **inaction** region.

- This region is increasing in the gap between ϕ_1^+ and ϕ_1^-
- There is more inaction when investment is more costly to reverse

To determine which region is relevant we need to compare the **value** of investing and disinvesting with that of inaction.

Timing of Investment

To determine the optimal timing of investment we need to compare the **value** of investing and/or disinvesting with that of inaction.

- Value of the firm

$$v(a, k) = \max[v_1(a, k), v_0(a, k)]$$

- Value of inactive firm

$$v_0(a, k) = \pi(a, k) - \delta k + \mathrm{E}[Mv(a', k)]$$

Note that for an inactive firm

$$\Phi(\delta k, k) = 0$$

Timing of Investment

- The value of a firm that (dis)invests optimally

$$v_1(a,k) = \pi(a,k) - i^\star - \Phi(i^\star,k) + \mathrm{E}[Mv(a',(1-\delta)k + i^\star)]$$

  where $i^\star$ is the value of the optimal (dis)investment policy.

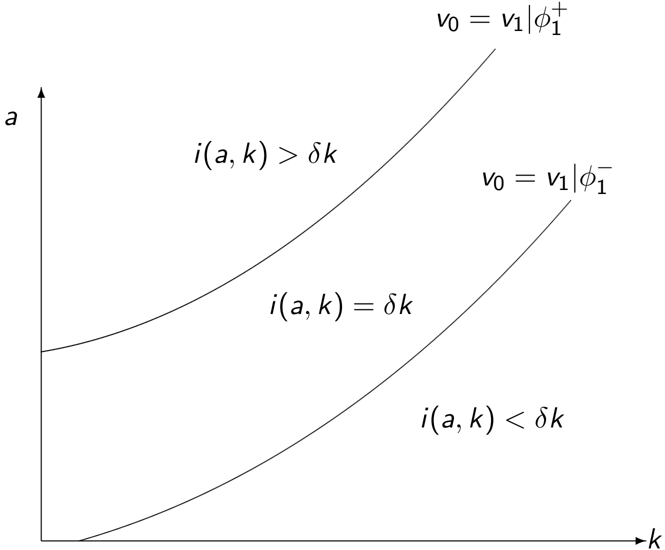
At the boundary of the inaction region the firm is exactly indifferent between the two choices so:

$$\begin{aligned} &v_1(a,k) = v_0(a,k) \\ &i^\star - \delta k + \Phi(i^\star,k) = \mathrm{E}M\left[v(a',(1-\delta)k + i^\star) - v(a',k)\right] \end{aligned}$$

Some comments:

- If $\phi_1^-(0) \neq \phi_1^+(0)$ we will have two solutions: one for high and one for low investment
- While $v(\cdot)$ is continuous in k the optimal policies are not: investment will jump discretely around ω
- These jumps will get larger if we also have $\phi_0 > 0$

Optimal Investment over the State Space



Application: Estimating Investment Behavior - Gala, Gomes & Liu (2019)

Optimal investment policy

$$q(k,z) \equiv v_k(k,z) = \Phi_I(i,k) \rightarrow \frac{i}{k} = \widetilde{G}(k, q(a,k))$$

where $a \equiv (z, P, W, \Omega)$. The key problem is that $q(k,z)$ is unknown

- Except under very extreme homogeneity assumptions

However the optimal investment policy is *by definition* a function of the state variables (SVs).

- It can **always** be approximated arbitrarily closely by a polynomial function of its SVs

$$\frac{i}{k} \simeq \sum_{i_k=0}^{n_k} \sum_{i_z=0}^{n_z} c_{i_k,i_z} \times k^{i_k} \times z^{i_z}$$

Measurement

Identifying/measuring the state variables

- k: *directly* observable $\rightarrow$ Net PPE
- Regarding the exogenous state variable, a
  - If $a = z \equiv$ TFP/Demand shocks, then it can be inferred directly from sales

$$y = zF(k, n) \implies a = y/F(k, n)$$

  - E.g. if $F(k, n) = k^{\alpha_k} n^{\alpha_n}$, productivity is a log-linear function of sales, labor and capital
  - If *firm-specific* wage shocks matter $\rightarrow W \in a$ now a is convolution of z and W and is inferable from profits/cash flows:

$$\pi = y - Wn = ak^{\gamma}$$

- To capture the impact of any aggregate SV, $\Omega \in a$, we can just add time effects.

Empirical State-Variable Approximations

No labor market shocks:

$$\frac{i_{jt}}{k_{jt}} \simeq \sum_{i_k=0}^{n_k} \sum_{i_y=0}^{n_y} \sum_{i_n=0}^{n_n} b_{i_k,i_y,i_n} \times k_{jt}^{i_k} \times y_{jt}^{i_y} \times n_{jt}^{i_n} + \delta_j + \eta_t + \epsilon_{jt}$$

where $\delta_j \equiv$ firm fixed effect ($\neq$ depreciation rates), $\eta_t \equiv$ time fixed effect.

Labor/wage shocks:

$$\frac{i_{jt}}{k_{jt}} \simeq \sum_{i_k=0}^{n_k} \sum_{i_\pi=0}^{n_\pi} b_{i_k,i_\pi} \times k_{jt}^{i_k} \times \pi_{jt}^{i_\pi} + \delta_j + \eta_t + \epsilon_{jt}$$

Empirical Investment Policy Functions

	(1)	(2)	(3)	(4)
Q	0.036	0.013		
	(0.003)	(0.003)		
CF	0.000	0.000		
	(0.000)	(0.000)		
$\ln k$		-0.147	-0.149	-0.177
		(0.006)	(0.006)	(0.008)
$\ln \frac{y}{k}$		0.200	0.201	0.067
		(0.007)	(0.007)	(0.008)
$(\ln k)^2$				0.017
				(0.001)
$(\ln \frac{y}{k})^2$				0.045
				(0.003)
$\overline{R}^2$	0.206	0.356	0.353	0.388

Investment in Continuous Time - Abel and Eberly (1994)

The discrete time version of this general problem can only be solved numerically.

- However continuous time we can often get some analytical solutions

Suppose

- The exogenous state, a, follows a continuous time diffusion:

$$da_t = \mu(a_t)dt + \sigma(a_t)dW, \qquad a(0) = a_0$$

  where dW is a standard Wiener process.

- Capital evolves according to

$$dk = (i_t - \delta k_t)dt \qquad k(0) = k_0$$

Firm Problem in Continuous Time

The value of the firm in sequence form

$$v(a_t, k_t) = \max_{i_{t+s},\nu_{t+s}} \int_0^{\infty} \pi(a_{t+s}, k_{t+s}) - \nu_{t+s}[i + \Phi(i_{t+s}, k_{t+s})]e^{-rs} ds$$

where

- $\nu_{t+s} \in \{0, 1\}$ is an indicator variable denoting the decision to undertake positive or negative investment
- r is the instantaneous discount/interest rate

The value of the firm in recursive form - the Hamilton Jacobi Bellman (HJB) equation

- Just a definition of the expected rate of return on the firm

$$rv(a, k)dt = \max_{i,\nu} \{\pi(a, k) - \nu[i + \Phi(i, k)] + \mathrm{E}dv(a, k)\}$$

Firm Value and Ito's Lemma

Calculate the expected capital gain using **Ito's Lemma**

- First order differential - adjusting for the fact that $(da)^2$ also includes a first order term (order dt)

$$dv(a,k) = v_k(a,k)dk + v_a(a,k)da + \frac{1}{2}v_{aa}(a,k)(da)^2$$

Rewrite firm's problem as

$$rv(\cdot) = \max_{i,\nu}\left\{\pi(a,k) - \nu\Phi(i,k) + \frac{\partial v}{\partial k}(i-\delta k) + \frac{\partial v}{\partial a}\mu(a) + \frac{\partial^2 v}{\partial a^2}\frac{\sigma(a)^2}{2}\right\}$$

Example: Geometric Brownian motion

Suppose that

- the stochastic process for a is a geometric Brownian motion (GBM) so that $\mu(a) = \mu a$ and $\sigma(a) = \sigma a$
- profits are $\pi = ak^{\theta}$

The value function in the region of zero investment follows a second order **differential** equation

$$rv(a,k) = \pi(a;k) + \Psi(a,k) + \mu a \frac{\partial v}{\partial a} + \frac{\sigma^2 a^2}{2} \frac{\partial^2 v}{\partial a^2}$$

where, as before:

$$\Psi(a,k) = \max_i \left[\underbrace{\frac{\partial v}{\partial k}}_{q} (i - \delta k) - \Phi(i,k) \right]$$

Example: The Value Function with GBM

Assume $\delta = 0$ so capital does not move without investment.
Then, in the region of no investment $v(\cdot)$ will have the form:

$$v(a,k) = B_0 + B_1 a^{\gamma_1} + B_2 a^{\gamma_2}$$

Verify and compute the values of B_0, B_1, B_2, plus γ_1, γ_2

- Derivatives

$$\frac{\partial v}{\partial a} = \frac{\partial B_0}{\partial a} + \gamma_1 B_1 a^{\gamma_1 - 1} + \gamma_2 B_2 a^{\gamma_2 - 1}$$
$$\frac{\partial^2 v}{\partial a^2} = \frac{\partial^2 B_0}{\partial a^2} + \gamma_1(\gamma_1 - 1) B_1 a^{\gamma_1 - 2} + \gamma_2(\gamma_2 - 1) B_2 a^{\gamma_2 - 2}$$

- Replace to get

$$r\left[B_0 + B_1 a^{\gamma_1} + B_2 a^{\gamma_2}\right] = \pi(a;k) + \mu \left[\frac{\partial B_0}{\partial a} a + \gamma_1 B_1 a^{\gamma_1} + \gamma_2 B_2 a$$
$$+ \left[\frac{\partial^2 B_0}{\partial a^2} a^2 + \gamma_1(\gamma_1 - 1) B_1 a^{\gamma_1} + \gamma_2(\gamma_2 - 1) B_2 a^{\gamma_2}\right] \frac{\sigma^2}{2}$$

Example: The Value Function with GBM

Match terms

$$
\begin{aligned}
r &= \mu\gamma_1 + \frac{\sigma^2}{2}\gamma_1(\gamma_1 - 1) && \text{terms in } B_1 a^{\gamma_1} \\
r &= \mu\gamma_2 + \frac{\sigma^2}{2}\gamma_2(\gamma_2 - 1) && \text{terms in } B_2 a^{\gamma_2} \\
rB_0 &= ak^\theta + \mu\frac{\partial B_0}{\partial a}a + \left[\frac{\partial^2 B_0}{\partial a^2}a^2\right]\frac{\sigma^2}{2} && \text{other terms}
\end{aligned}
$$

The first two equations are a quadratic equation for γ_1, γ_2

- Without loss of generality let $\gamma_1 > 1$ and $\gamma_2 < 0$
- Since $\gamma_2 < 0$ then $B_2 = 0$ - value function increases with a

The last equation implies B_0 is linear in a and obeys

$$
B_0(a, k) = \frac{ak^\theta}{r - \mu}
$$

Example: The Value Function and Option Value

In the region of zero investment the value function simplifies to

$$v(a,k) \quad = \quad \frac{ak^{\theta}}{r-\mu} + B_1 a^{\gamma_1}$$

Intuition

- The first term is the expected present discount value of future profits - Gordon growth formula with average growth μ and discount rate r.
- This is also called the **value of assets in place**
- This is the value of a firm that has no investment opportunities
- The second term is the **value of the growth options** of the firm. It is convex in a and arises only because the firm has an option to invest

Example: Value Matching and Smooth Pasting

Optimal investment decision: **endogenous** investment threshold at which the firm increases its capital

- Also provides boundary conditions for the value function and determines the value of the constant B_1

Endogenous boundaries for a firm with a single investment opportunity

$$v(a^\star, k+i) - i - \Phi(i) = v(a^\star, k) \qquad \text{Value Matching}$$

$$\frac{\partial v(a^\star, k+i)}{\partial a} = \frac{\partial v(a^\star, k)}{\partial a} \qquad \text{Smooth Pasting}$$

Replacing with the solution above implies

$$\frac{a^\star (k+i)^\theta}{r-\mu} + B_1 (a^\star)^{\gamma_1} - i - \Phi(i) = \frac{a^\star k^\theta}{r-\mu}$$

$$\frac{(k+i)^\theta}{r-\mu} + \gamma_1 B_1 (a^\star)^{\gamma_1 - 1} = \frac{k^\theta}{r-\mu}$$

Solve for B_1 and $a^\star$

Application - Effects of Uncertainty - Bloom (2007)

Motivation

- Uncertainty greatly increased during the Great Recession
- Can this generate a large downturn in investment

Develop structural framework to analyze the impact of **uncertainty shocks**

- Time-varying second moment

Key Findings

- Macro uncertainty shock produces a rapid drop and rebound

Application - Effects of Uncertainty - Bloom (2007)

Key Model details

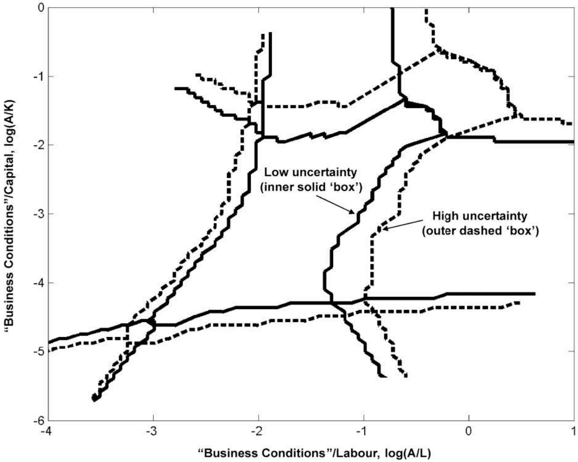
- Non convex costs in capital and labor
- Generates option value to investment and hiring
- Higher variance increases the value to waiting

Uncertainty shocks to technology

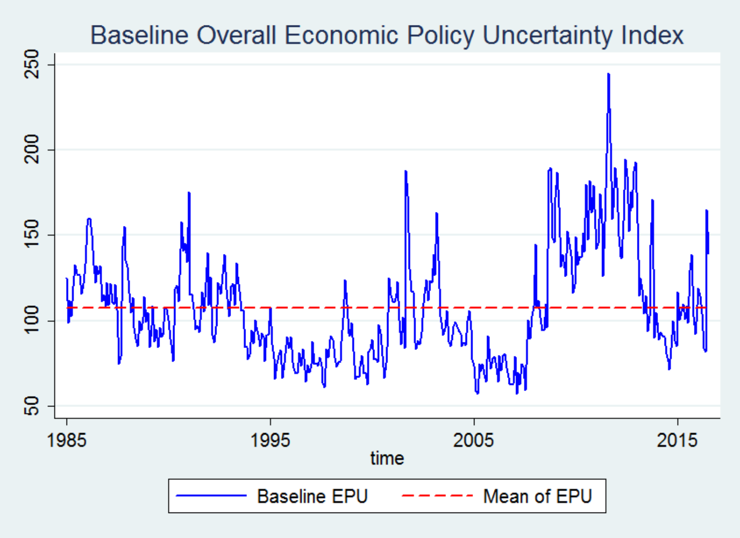
$$z_{t+1} = z_t(1 + \sigma_t \epsilon_t)$$

where σ_t, ϵ_t are independent random variables

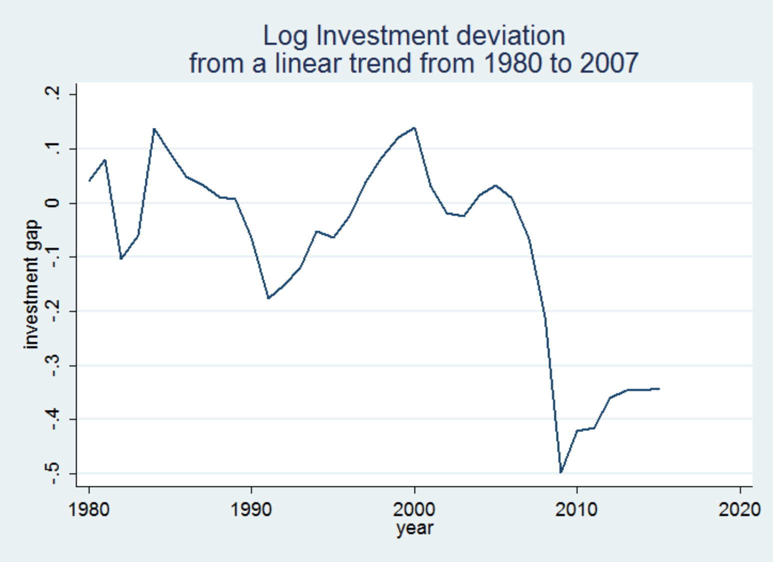
Application: Effects of Uncertainty - Bloom (2007)



Measuring Economic Policy Uncertainty - Baker et al (2016)



Application - Decline in Corporate Investment - Gala et al (2018)



Application - Decline in Corporate Investment - Gala et al (2018)

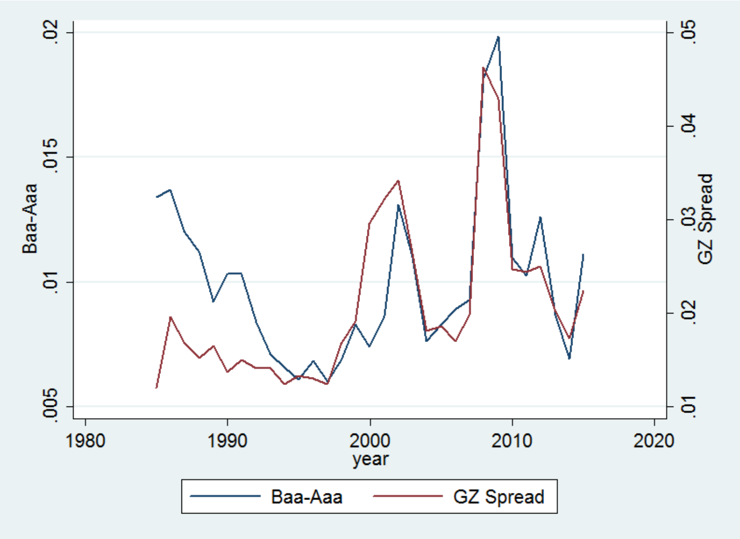
Can we explain this with a combination of

- Decline in aggregate (common) productivity (TFP), z_t
- Increase in uncertainty, σ_t
- Increase in financial frictions, μ_t

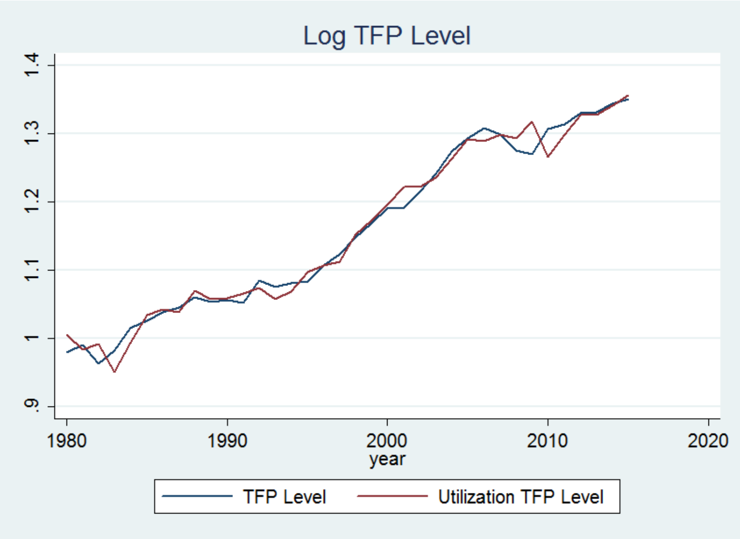
Degree of financial frictions is calibrated to match variation in bond yield (credit) spreads

- Glichrist and Zakrajsek (2012) propose an alternative index that is quite close

Quantifying Financial Frictions



Estimating TFP - San Francisco Fed



Quantifying the Decline in Investment