Q1 Problem Set 1.2

#5

$|v| = \sqrt{1^2 + 3^2} = \sqrt{10}$

$|w| = \sqrt{2^2+1^2+2^2} = \sqrt{9} = 3$

$u_1 = \frac{(1,3)}{\sqrt{10}}$

$= \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$

$u_2 = \frac{(2,1,2)}{3}$

$= \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$

Suppe. $U_1 = (x, y)$

$(x,y) \cdot v = 0$

$(x,y) \cdot (1,3) = 0$

$x + 3y = 0$

$3y = \cancel{x}$

$y = -\frac{x}{3}$

$\therefore$ the vector $(3,-1)$ or $(-3,1)$ is $\perp$ with v

$\therefore$ U_1 can be $(3,-1)/\sqrt{10}$ or $(-3,1)/\sqrt{10}$.

Suppose $V_2 = (x, y, z)$

$\because V_2 \perp w$

$\therefore (x,y,z) \cdot (2,1,2) = 0$

$2x + y + 2z = 0$

The vector $(1,-4,1)$ is $\perp$ with w

$\therefore$ U_2 can be $(1,-4,1)/\sqrt{18}$

However, due to w is a 3-dimission vector

The unit vector which is perpendicular to w

is a circle with radius 1

#6 (a) $\because w \perp v$

$\therefore (w_1, w_2) \cdot (2, -1) = 0$

$2w_1 - w_2 = 0$

$2w_1 = w_2$

$\therefore$ Every vector w which fits $2w_1 = w_2$ is perpendicular to v. And w is a straight line is 2-dimension space.

(b) All vectors perpendicular to $v = (1,1,1)$ lie on a plane in 3 dimensions. And suppose $u = (x, y, z) \perp v$, u should fit $x+y+z=0$.

(c) Suppose $u = (x, y, z)$ $u \perp (1,1,1)$ and $(1, 2, 3)$

R_1 $x+y+z=0$ $R_2 - R_1 =$ $y+2z=0$

R_2 $x+2y+3z=0$ $2z=-y$

$\therefore$ The vectors perpendicular to both $(1,1,1)$ and $(1,2,3)$ lie on a line in 3-dimension space. and the line should fit $2z=-y$

#13 Suppose $v = (x_v, y_v, z_v)$ $w = (x_w, y_w, z_w)$

$\because$ $v \perp (1,0,1)$ $w \perp (1,0,1)$ $v \perp w$

$\therefore$ $x_v + z_v = 0$ $x_w + z_w = 0$ $x_v \cdot x_w + y_v \cdot y_w + z_v \cdot z_w = 0$

$\therefore$ $x_v = -z_v$ $x_w = -z_w$

v can be $(-1, 0, 1)$, w can be $(0, 4, 0)$

#16 $\|v\| = \sqrt{1^2 + 1^2 + \cdots + 1^2} = \sqrt{9} = 3.$

$u = \frac{v}{\|v\|} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \cdots, \frac{1}{3})$ or $\frac{v}{3}$

The vector $(-2, 2, 0, 0, 0, 0, 0, 0, 0)$ is perpendicular to v

$w = (-2, 2, 0, 0, 0, 0, 0, 0, 0)/\sqrt{8}$

$= (-1, 1, 0, 0, 0, 0, 0, 0, 0)/\sqrt{2}$

Q2. $u = (u_1, u_2, u_3, \cdots u_n)$ $\quad$ $v = (v_1, v_2, v_3, \cdots, v_n)$

$w = (w_1, w_2, w_3, \ldots w_n)$

(a) $(u+v)\cdot w$

$= (u_1+v_1, u_2+v_2, u_3+v_3, \cdots u_n+v_n)\cdot(w_1, w_2, w_3, \ldots, w_n)$

$= \sum_{k=1}^{n} w_k (u_k + v_k)$

$u\cdot w + v\cdot w$

$= \sum_{k=1}^{n} w_k \cdot u_k + \sum_{k=1}^{n} w_k \cdot v_k$

$= \sum_{k=1}^{n} w_k (u_k + v_k)$

$\therefore (u+v)\cdot w = u\cdot w + v\cdot w$

(b) $\because \|u\| \geq 0$

$\therefore \sqrt{v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2} \geq 0$

$\therefore v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2 \geq 0$

$\therefore \sum_{k=1}^{n} v_k^2 \geq 0$

$\because \|u\| = 0$

$\therefore \sum_{k=1}^{n} v_k^2 = 0$

$\because$ All square number is greater than or equal to 'zero'; Thus, if and only if $v_1^2 = v_2^2 = v_3^2 = \cdots = v_n^2 = 0$. $\sum_{k=1}^{n} v_k^2$ can be equal to zero.

(c) $k(u+v)$

$= k\cdot(u_1+v_1, u_2+v_2, u_3+v_3, \cdots, u_n+v_n)$

$= (k(u_1+v_1), k(u_2+v_2), k(u_3+v_3), \cdots, k(u_n+v_n))$

$ku + kv$

$= (ku_1, ku_2, ku_3, \cdots, ku_n) + (kv_1, kv_2, kv_3, \cdots + kv_n)$

$= (ku_1 + kv_1, ku_2 + kv_2, \cdots, ku_n + kv_n))$

$= (k(u_1+v_1), k(u_2+v_2), \cdots, k(u_n+v_n))$

same.

$\therefore k(u+v) = ku + kv$

Q3 Suppose. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & & & \\ \vdots & & & & & \\ a_{m1} & & & & & a_{mn} \end{bmatrix}$ $m\times n$

$\times$

$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1m} \\ \vdots & \vdots & \vdots & & \\ b_{n1} & \cdots & & & b_{nm} \end{bmatrix}$ $n\times m \rightarrow m\times m$

$$AB = \begin{bmatrix} \sum_{i=1}^{i=n} a_{1i}b_{i1}, & \sum_{i=1}^{i=n} a_{1i}b_{i2}, & \cdots\cdots, & \sum_{i=1}^{i=n} a_{1i}b_{im} \\ \sum_{i=1}^{i=n} a_{2i}b_{i1}, & \sum_{i=1}^{i=n} a_{2i}b_{i2}, & \cdots, & \sum_{i=1}^{i=n} a_{2i}b_{im} \\ \vdots & \vdots & & \\ \sum_{i=1}^{i=n} a_{mi}b_{i1}, & \sum_{i=1}^{i=n} a_{mi}b_{i2}, & \cdots, & \sum_{i=1}^{i=n} a_{mi}b_{im} \end{bmatrix}$$

(a) if A has a zero row suppose. $x \in [1, m]$. x row of A is zero

$\therefore a_{x1}, a_{x2}, a_{x3}, a_{x4}, a_{x5}, \cdots, a_{xn} = 0$

$\because \sum_{i=1}^{i=n} a_{xi}b_{i1} = 0, \sum_{i=1}^{i=n} a_{xi}b_{i2} = 0, \cdots, \sum_{i=1}^{i=n} a_{xi}b_{im} = 0.$

And above is one row of AB. Thus if one row of A is zero, Then one row of AB should be zero

(b) if B has a zero column. suppose $y \in [1, n]$ y column of B is zero.

$\therefore b_{1y} = 0$ $\qquad \therefore \sum_{i=0}^{i=n} a_{1i}b_{iy} = 0$

$b_{2y} = 0$ $\qquad \sum_{i=0}^{i=n} a_{2i}b_{iy} = 0$

$\vdots$ $\qquad \vdots$

$b_{ny} = 0$ $\qquad \sum_{i=0}^{i=n} a_{mi}b_{iy} = 0$

$\therefore$ this is one column of AB.

$\therefore$ if one column of B is zero, one column of AB should be zero.

Q4 From the class, we have already prove ① $(AB)^{-1} = B^{-1}A^{-1}$
② $(AB)^T = B^T A^T$

(a) based on ② $(AB)^T = B^T A^T$

$$(A_1 A_2 \cdots A_{n-1} A_n)^T = A_n^T (A_1 A_2 \cdots A_{n-1})^T$$
$$= A_n^T \cdot A_{n-1}^T (A_1 A_2 \cdots A_{n-2})^T$$

Similar, repeat this process

We can get

$$= A_n^T A_{n-1}^T \cdots A_2^T A_1^T$$

(b) based on ① $(AB)^{-1} = B^{-1}A^{-1}$

$$(A_1 A_2 \cdots A_{n-1} A_n)^{-1} = A_n^{-1} (A_1 A_2 \cdots A_{n-1})^{-1}$$
$$= A_n^{-1} A_{n-1}^{-1} (A_1 A_2 \cdots A_{n-2})^{-1}$$

Repeat this process.

We can get

$$= A_n^{-1} A_{n-1}^{-1} A_{n-2}^{-1} \cdots A_2^{-1} A_1^{-1}$$

Q5 $\because$ A and B are unitary

$\therefore$ $A^* A = A A^* = I_n$ $\quad A^{-1} = A^* = A^H$

$B^* B = B B^* = I_n$ $\quad B^{-1} = B^* = A^H$

$\therefore$ $A \cdot A^H = A^H \cdot A = I_n$ $\quad A \cdot A^{-1} = A^{-1} \cdot A = I_n$

$\therefore$ A^H and A^{-1} are both unitary

$$(A^H B^{-1}) \cdot (A^H B^{-1})^H = (A^H B^{-1}) \cdot (B^{-1H} A^{HH})$$
$$= (A^H B^{-1}) \cdot (B A)$$
$$= A^H (B^{-1} B) A$$
$$= A^H I A$$
$$= A^H A$$
$$= I$$

$\therefore$ $A^H B^{-1}$ is also unitary

Q6

$$(A + A^H)^H = A^H + (A^H)^H = A^H + A = A + A^H$$

$\therefore$ $A + A^H$ is Hermitian.

$$(A - A^H)^H = A^H - (A^H)^H = A^H - A = -(A - A^H)$$

$\therefore$ $(A - A^H)^H$ is skew-Hermitian..

Suppose there is a square matrix D. and $A = 2D$

$B = (D + D^H)$

$C = (D - D^H)$

from above we know than $D + D^H$ is hermitian

$D - D^H$ is skew-hermitian.

$$B + C = D + D^H + D - D^H.$$
$$= 2D$$

we assume D is a square matrix

based on assumption $2D$ is also a square matrix

Thus $A = 2D = B + C$

$A = B + C$

$\because$ $2D$ is a square matrix

$\therefore$ A is also a square matrix.