Q1

6.1 # 19

(a) the rank is 2

(b) $0 \times 1 \times 2 = 0$

(d) $1, \frac{1}{2}, \frac{1}{5}$

$(0.8-\lambda)(0.7-\lambda) - 0.3 \times 0.2$

$= 0.56 - 0.8\lambda - 0.7\lambda + \lambda^2 - 0.06$

$= 0.5 \cancel{-0.8\lambda} - \cancel{0.15\lambda} 1.5\lambda + \lambda^2$

$$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$-0.2x + 0.3y = 0$

$-2x + 3y = 0$

$3y = 2x$

$y = \frac{2}{3}x$

x	3
$\frac{2}{3}x$	2

$-\frac{1}{5}x + \frac{3}{10}y = 0$

$\frac{3}{10}y = \frac{1}{5}x$

$y = \frac{1}{5}x \cdot \frac{10}{3}$

1

$\frac{2}{3}$

24

trace of $A = 2+2+2 = 6$

$$\det(A) = \begin{vmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$\therefore \lambda_1 + \lambda_2 + \lambda_3 = 6$

$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 0$

$\because$ rank of A is 1

$\therefore 0, 0, 6$ is eignvalue of A

$$\begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix} - 0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix} = A - 0I \Rightarrow \begin{bmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$-1 - 1$

$2x + y + 2z = 0$

$2x = -y - 2z$

$x = -\frac{y}{2} - z$

$$x_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \qquad x_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} -4 & 1 & 2 \\ 4 & -4 & 4 \\ 2 & 1 & -4 \end{bmatrix} = A - 6I$$

$$\begin{bmatrix} -4 & 1 & 2 \\ 4 & -4 & 4 \\ 2 & 1 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} -4 & 1 & 2 \\ 0 & -3 & 6 \\ 0 & 3 & -6 \end{bmatrix} \Rightarrow \begin{array}{c} \begin{array}{ccc} 4 & 2 & -8 \end{array} \\ \begin{bmatrix} -4 & 1 & 2 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$-4x + y + 2z = 0$ $\qquad -4x = -y - 2z$

$-3y + 6z = 0$ $\qquad x = \dfrac{y + 2z}{4}$

$-3y = -6z$

$y = 2z$

Suppose $z = 1$ $\qquad y = 2$ $\qquad x = 1$

$$x_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

#27 rank of $A = 1$

trace of $A = 4$

$\det(A) = 0$

$\therefore$ eigenvalues of A is $0, 0, 0, 4$

$$\begin{array}{c} A \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array} \times \begin{array}{c} x \\ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{array} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} = 2x$$

$\therefore \lambda = 2.$

rank of $C = 2$

trace of $C = 4$

$\det(C) = 0$

$\therefore$ eigenvalues of C is $0, 0, 2, 2$

$\lambda_3 \cdot \lambda_4 = 4$

$\lambda_4 = 4/2 = 2$

29

$\because A$ is triangular matrix

$\therefore$ eigenvalues of A is its diagonal: $1, 4, 6$

rank of $B = 3$

trace$(B) = 2 = \lambda_1 + \lambda_2 + \lambda_3$

$\det(B) = -6 = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$.

$$\begin{array}{ccc} -\lambda + \frac{3}{\lambda} & 0 & -\frac{3}{\lambda} \end{array}$$

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 3 & 0 & -\lambda \end{vmatrix} \Rightarrow \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & -\lambda+\frac{3}{\lambda} \end{vmatrix} = 0$$

$$-\lambda \cdot (2-\lambda) \cdot \left(-\lambda + \frac{3}{\lambda}\right) = 0$$

$$(\lambda^2 - 3)(2-\lambda) = 0$$

$\lambda = 2$

$\lambda^2 = 3$

$\lambda = \pm\sqrt{3}$

$\therefore \lambda_1 = 2 \quad \lambda_2 = \sqrt{3} \quad \lambda_3 = -\sqrt{3}$

rank of $C = 1$

trace $C = 6$

$\det(C) = 0$

$\therefore$ eigenvalues of $C = 0, 0, 6$

Q_2 #4

$$\begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = A \cdot \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$$

$$G_{k+2} = 0.5\, G_{k+1} + 0.5\, G_k.$$

$$G_{k+1} = 1\, G_{k+1} + 0\, G_k$$

$$\therefore A = \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} \frac{1}{2}-\lambda & 0.5 \\ 1 & -\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\left(\tfrac{1}{2}-\lambda\right)(-\lambda) - 0.5 = 0$$

$$-\tfrac{1}{2}\lambda + \lambda^2 - \tfrac{1}{2} = 0$$

$$\lambda^2 - \tfrac{1}{2}\lambda - \tfrac{1}{2} = 0$$

$$\lambda^2 - \tfrac{1}{2}\lambda + \left(\tfrac{1}{4}\right)^2 = \tfrac{1}{2} + \left(\tfrac{1}{4}\right)^2$$

$$\left(\lambda - \tfrac{1}{4}\right)^2 = \tfrac{1}{2} + \tfrac{1}{16} = \tfrac{9}{16}$$

$$\lambda - \tfrac{1}{4} = \pm\tfrac{3}{4}$$

$$\lambda = \pm\tfrac{3}{4} + \tfrac{1}{4}$$

$$\lambda_1 = 1$$

$$\lambda_2 = -\tfrac{1}{2}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

for $\lambda_1 = 1$

$$A - I = \begin{bmatrix} -0.5 & 0.5 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -0.5 & 0.5 \\ 1 & -1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$-0.5x + 0.5y = 0$$

$$0.5y = 0.5x$$

$$y = x$$

$$\begin{matrix} 0.5 & 0.5 \\ 1 & 0 \end{matrix} \qquad \begin{matrix} 0.5 & 0 \\ 0 & 0.5 \end{matrix}$$

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for $\lambda_2 = -\frac{1}{2}$

$$A - \tfrac{1}{2}I = \begin{bmatrix} 1 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$x + \frac{y}{2} = 0$$

$$x = -\frac{y}{2}$$

$$x_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

#9 (b)

$$X = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \quad X^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

$$A^n = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \times \begin{bmatrix} (1)^n & 0 \\ 0 & (-\frac{1}{2})^n \end{bmatrix} \times \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

$$\therefore n \to \infty \quad A^n \Rightarrow X^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

(c) Suppose $k \to \infty$

$$\begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \approx \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

when $k \to \infty$

$G_{k+2} \Rightarrow \frac{2}{3}$

#19 $B = \begin{bmatrix} 5 & 1 \\ 0 & 4 \end{bmatrix}$

$\lambda_1 = 5 \quad \lambda_2 = 4$

for $\lambda_1 = 5$

$$\begin{bmatrix} 5 & 1 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$y = 0$

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

for $\lambda_2 = 4$

$$\begin{bmatrix} 5 & 1 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$x + y = 0$

$x = -y$

$$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \qquad X^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$B^k = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \times \begin{bmatrix} 5^k & 0 \\ 0 & 4^k \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5^k & 4^k \\ 0 & -4^k \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5^k & 5^k-4^k \\ 0 & 4^k \end{bmatrix}$$

#27

$$A-I \Rightarrow \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = 0 \qquad A-9I \Rightarrow \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$4x+4y=0 \qquad -4x+4y=0$$

$$x=-y$$

$$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \quad X^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Suppose $a^2 = A$

$$a = X\Lambda^{\frac{1}{2}}X^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \times \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Suppose $b^2 = B$

Then $b = X\Lambda^{\frac{1}{2}}X^{-1}$

However, $\Lambda^{\frac{1}{2}}$ can not solve $(-1)^{\frac{1}{2}}$ with real answer
Thus B has no real matrix square root.

Q3

rank of A is 3 . Thus one λ of A is zero

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 4 & 2 & 0 \\ 0 & -1-\lambda & 0 & 0 \\ 0 & 1 & -\lambda & 1 \\ 0 & 1 & 0 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 4 & 2 & 0 \\ 0 & -1-\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = (2-\lambda)(-1-\lambda)(-\lambda)(1-\lambda) = 0$$

$\lambda_1 = 0 \quad \lambda_2 = 2$
$\lambda_3 = -1 \quad \lambda_4 = 1$

for $\lambda_1 = 0 \qquad A - 0I = 0$

$$\begin{bmatrix} 2 & 4 & 2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 4 & 2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \\ k \end{bmatrix} = 0$$

$2x + 4y + 2z = 0$
$-y = 0 \Rightarrow y = 0$
$1y + k = 0 \qquad k = 0$

$2x + 2z = 0$
$2x = -2z$

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

for $\lambda_2 = 2 \quad A - 2I = 0$

$$\begin{bmatrix} 0 & 4 & 2 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \\ k \end{bmatrix} = 0$$

$$\begin{aligned} 4y + 2z &= 0 \\ -3y &= 0 \Rightarrow y = 0 \\ 1y - 2z + k &= 0 \qquad z = 0 \\ 1y - k &= 0 \qquad k = 0 \end{aligned}$$

$$X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

for $\lambda_3 = -1 \quad A + I = 0$

$$\begin{bmatrix} 3 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \\ k \end{bmatrix} = 0$$

$$\begin{aligned} 3x + 4y + 2z &= 0 \\ 1y + z + k &= 0 \\ y + 2k &= 0 \Rightarrow y = -2k. \end{aligned}$$

$$X_3 = \begin{bmatrix} -2 \\ 2 \\ -1 \\ -1 \end{bmatrix}$$

for $\lambda_4 = 1 \quad A - I = 0$

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \\ k \end{bmatrix} = 0$$

$$\begin{aligned} x + 4y + 2z &= 0 \\ -2y &= 0 \\ y - z + k &= 0 \\ y &= 0 \end{aligned}$$

$$X_4 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad X^{-1} = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 2 & 1 & 1 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 1 \end{bmatrix}$$

$$B = X \Lambda^{\frac{1}{3}} X^{-1} = \begin{bmatrix} \sqrt[3]{2} & 2\sqrt[3]{2} & \sqrt[3]{2} & \sqrt[3]{2} - 2 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Q4 $\because$ λ is an eigenvalue of A with eigenvector x

$\therefore$ $Ax = \lambda x$

$\because$ $(A - cI) \times x$

$= Ax - cIx$

$= \lambda x - cx$

$= (\lambda - c)x$

$\therefore$ $(A - cI)x = (\lambda - c)x$

$\therefore$ $\lambda - c$ is eigenvalue of $A - cI$ matrix

Q5 $\because$ $T^{-1}AT = \Lambda$ of A Suppose A is a $n \times n$ square matrix

$\therefore$ Λ is a square matrix that the eigenvalues of A is on the diagonal of Λ.

$\because$ the trace of A is sum of diagonal values, but also the sum of eigenvalues of A

$\therefore$ $\text{Trace}(A) = \sum_{i=1}^{n} \lambda_i = \text{Trace}(\Lambda) = T^{-1}AT$

$\therefore$ $\text{Trace}(A) = T^{-1}AT$