Q1

(a) #3

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 4 & 6 & 1 & 0 & 1 & 0 \\ -2 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & -4 & 1 & 0 \\ 0 & 4 & 0 & 2 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & -4 & 1 & 0 \\ 0 & 0 & 2 & -10 & 2 & -1 \end{array}\right]$$

$$\cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$R_3 = R_3 - 2R_2$, $R_3 = R_3 + 2R_1$, $R_2 = R_2 - 4R_1$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix}}_{E_{32}} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}}_{E_{31}} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}$$

(a) #25

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 2 \\ 3 & 5 & 7 & 6 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & -3 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

Thus . this system has no solution.

- Change "6" to "3"

$$\begin{array}{c} \begin{array}{cccc} 3 & 6 & 9 & 3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 2 \\ 3 & 5 & 7 & 3 \end{array}\right] \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This system will have infinitely solutions

(b) #25

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & -1 & 2 & 0 \\ 0 & 1 & 3 & -1 & 0 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & -1 & 2 & 0 \\ 0 & 0 & 8 & -2 & -2 & 6 \end{array}\right] \Rightarrow \begin{array}{c} \begin{array}{cccccc} 8 & 4 & 4 & 4 & 0 & 0 \end{array} \\ \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 4 & 0 & -1 & 3 & -1 \\ 0 & 0 & 4 & -1 & -1 & 3 \end{array}\right] \end{array}$$

$$\left[\begin{array}{ccc|ccc} 8 & 0 & 0 & 6 & -2 & -2 \\ 0 & 4 & 0 & -1 & 3 & -1 \\ 0 & 0 & 4 & -1 & -1 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ 0 & 3 & -3 & 1 & 2 & 0 \\ 0 & -3 & 3 & 1 & 0 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ 0 & 3 & -3 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 & 2 & 2 \end{array}\right] \varnothing$$

B is not invertible

(c) #5

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 4 & 2 & 0 & 1 & 0 \\ 6 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 4 & 2 & 0 & 1 & 0 \\ 0 & 0 & 5 & -3 & 0 & 1 \end{array}\right]$$

$$E \Rightarrow \begin{bmatrix} 1 & & \\ 0 & 1 & \\ -3 & 0 & 1 \end{bmatrix} \qquad EA = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix} = U$$

3 0 0 3 0 0

$$E \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -3 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right]$$

$$E^{-1} = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 3 & 0 & 1 \end{bmatrix} = L$$

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}$$

#13.

$$\begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ & b-a & b-a & b-a \\ & & c-b & c-b \\ & & & d-c \end{bmatrix}$$

1. $d-c \neq 0$

   $d \neq c$
2. $c \neq b$
3. $b \neq a$
4. $a \neq 0$

Q2

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots \\ a_{21} & a_{22} & \cdots & \cdots \\ a_{31} & & \ddots & \\ \vdots & & & \ddots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$\forall a_{ij} = 0$ when $i < j$

$a_{ii} \neq 0$. here is matrix A.

$$\begin{bmatrix} \times & 0 & 0 & 0 \\ \times & \times & 0 & 0 \\ \times & \times & \times & 0 \\ \times & \times & \times & \times \end{bmatrix} \quad \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}$$

$A \cdot A^{-1} = I$

Suppose A^{-1} is not lower triangular matrix. $B = A^{-1}$

$\forall b_{ij} \neq 0$ when $i < j$

$$(A \cdot B)_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}$$

$\because (A \cdot B)_{ij} = I_{ij}$

$\therefore$ when $i = j$ $\quad (A \cdot B)_{ij} = 1$

when $i \neq j$ $\quad (A \cdot B)_{ij} = 0$

$\because \forall b_{ij} \neq 0$ when $i < j$

$\forall a_{ij} = 0$ when $i < j$

$\therefore (A \cdot B)_{ij} \neq 0$ when $i < j$

$\therefore$ CONTRADICTION

$\therefore A^{-1}$ is a lower triangular matrix.

Q3 Suppose $[6,10,2] = x\cdot[1,3,2] + y[2,8,-1] + z[-1,9,2]$

$$1x + 2y - z = 6$$
$$3x + 8y + 9z = 10$$
$$2x - y + 2z = 2$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} 1 & 2 & -1 \\ 3 & 8 & 9 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 2 \end{bmatrix}$$

$R_2 = R_2 - 3R_1$

$R_3 = R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 12 \\ 0 & -5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ -10 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 0 & 1 & 6 & -4 \\ 0 & 0 & 1 & -\frac{15}{17} \end{array}\right]$$

$$x = \frac{43}{17} \quad y = \frac{22}{17} \quad z = -\frac{15}{17}$$

$$[6,10,2] = \frac{43}{17}\cdot[1,3,2] + \frac{22}{17}[2,8,-1] - \frac{15}{17}[-1,9,2]$$

Q4 Proof ① solution set $\in R^{n\times 1}$

② solution set is a subspace.

$Ax = b$

$$\begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ \vdots & \ddots & \\ a_{m1} & & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \quad x = a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdots a_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

② ⓐ $\underline{0}$ is one of x

$A\underline{0} = \underline{0}$

ⓑ Show that $\forall u, w \in S$ and $\forall a, b \in R$ $au + bw \in S$

u and w is solution for $Ax = b$

$au + bw$ may not a solution for $Ax = b$ due to if A may not be consistent. Then it won't have a solution in this case.

Thus the solution set is not a subspace of $R^{n\times 1}$

Q5 Suppose $\{U_1, U_2, U_3 \cdots U_n\}$ are the subspaces of V, and W is the intersection of subset of U_i

proof ① intersection of any number of subspace $\in V$

② intersection of any number of subspace is still a subspace

for proof ① $\because$ the U_i is the subspace of V

$\therefore$ $U_i \in V$

$\because$ the intersection of any number of subspace is still in the subspace of V

$\therefore$ $W \in V$

for proof ② ⓐ $\because$ the $\underline{0} \in U_i$

$\therefore$ $\underline{0} \in W$, due to any subspace has zero vector.

ⓑ Suppose $\forall x, y \in W$ $\forall a, b \in$ Field.

$\because$ x belongs to intersection

$\therefore$ ax also belongs to intersection

$\because$ y belongs to intersection

$\therefore$ by also belongs to intersection.

∵ ax, by belongs to intersection.

∴ $ax+by$ can also be intersection of all the subspace.

These intersection of any number of subspace is still a subspace

Sum up, the intersection of any number of subspace of a vector space V is a subspace of V

Q6 ① Show that $L(S) \subset V$

let $u \in L(S) \Rightarrow \exists c_i\text{'s} \in F : u = \sum_{k=1}^{n} c_k \alpha_k$

$\Rightarrow \exists c_i\text{'s} \in F : u = \sum_{k=1}^{n} c_k \alpha_k$

$\because \alpha_k$'s are in V

$\therefore u = \sum_{k=1}^{n} c_k \alpha_k \in V$

$\Rightarrow L(S) \subset V$

② $L(S)$ is a subspace

ⓐ $\underline{0} = 0\alpha_1 + 0\alpha_2 + \cdots + 0\alpha_n \in L(S)$

> 0

ⓑ Suppose $\forall u, w \in L(S)$ and $\forall a, b \in F$

$au + bw \in L(S)$

$u = \sum_{i=1}^{n} c_i \alpha_i$

$w = \sum_{i=1}^{n} d_i \alpha_i$

$\therefore au + bw = a \sum_{i=1}^{n} c_i \alpha_i + b \sum_{i=1}^{n} d_i \alpha_i$

$= \sum_{i=1}^{n} (ac_i)\alpha_i + \sum_{i=1}^{n} (bd_i)\alpha_i$

$$= \sum_{i=1}^{n} (ac_i + bd_i)\alpha_i \in L(S)$$

$\therefore L(S) \subset V$, $\underline{0} \in L(S)$. and $au + bw \in L(S)$

$\forall u, w \in L(S)$

$\forall a, b \in F$