Q1

(a) 5.1 #14

$$\begin{bmatrix}1&2&3&0\\2&6&6&1\\-1&0&0&3\\0&2&0&7\end{bmatrix} \Rightarrow \begin{array}{c}\begin{matrix}2&4&6&0\end{matrix}\\\begin{bmatrix}1&2&3&0\\0&2&0&1\\0&2&3&3\\0&2&0&7\end{bmatrix}\end{array} \Rightarrow \begin{bmatrix}1&2&3&0\\0&2&0&1\\0&0&3&2\\0&0&0&6\end{bmatrix}$$

$$\det\begin{bmatrix}1&2&3&0\\2&6&6&1\\-1&0&0&3\\0&2&0&7\end{bmatrix} = \det\begin{bmatrix}1&2&3&0\\0&2&0&1\\0&0&3&2\\0&0&0&6\end{bmatrix} = 1\times2\times3\times6 = 36$$

$$\begin{array}{c}\begin{matrix}1&-\frac{1}{2}&0&0\end{matrix}\\\begin{bmatrix}2&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&2\end{bmatrix}\end{array} \Rightarrow \begin{array}{c}\begin{matrix}0&1&-\frac{2}{3}&0\end{matrix}\\\begin{bmatrix}2&-1&0&0\\0&\frac{3}{2}&-1&0\\0&-1&2&-1\\0&0&-1&2\end{bmatrix}\end{array} \Rightarrow \begin{array}{c}-\frac{3}{4}\\\begin{bmatrix}2&-1&0&0\\0&\frac{3}{2}&-1&0\\0&0&\frac{4}{3}&-1\\0&0&-1&2\end{bmatrix}\end{array} \Rightarrow \begin{array}{c}2-\frac{3}{4}=\frac{5}{4}\\\begin{bmatrix}2&-1&0&0\\0&\frac{3}{2}&-1&0\\0&0&\frac{4}{3}&-1\\0&0&0&\frac{5}{4}\end{bmatrix}\end{array}$$

$$\det\begin{bmatrix}2&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&2\end{bmatrix} = \det\begin{bmatrix}2&-1&0&0\\0&\frac{3}{2}&-1&0\\0&0&\frac{4}{3}&-1\\0&0&0&\frac{5}{4}\end{bmatrix} = 2\times\frac{3}{2}\times\frac{4}{3}\times\frac{5}{4} = 5$$

5.1 #16

$$A = \begin{bmatrix}1\\2\\3\end{bmatrix}\times\begin{bmatrix}1&-4&5\end{bmatrix} = \begin{array}{c}\begin{matrix}-2&8&10\end{matrix}\\\begin{bmatrix}1&-4&5\\2&-8&10\\3&-12&15\end{bmatrix}\end{array} \Rightarrow \begin{bmatrix}1&-4&5\\0&0&0\\0&0&0\end{bmatrix}$$

$\det(A)=0$ due to A is singular and U has zero row

$$\det(A) = \begin{vmatrix}0&1&3\\-1&0&4\\-3&-4&0\end{vmatrix} = -\begin{vmatrix}-1&0&4\\0&1&3\\-3&-4&0\end{vmatrix} = -\begin{array}{c}\begin{matrix}3&0&-12\end{matrix}\\\begin{vmatrix}-1&0&4\\0&1&3\\0&-4&-12\end{vmatrix}\end{array} = -\begin{array}{c}\begin{matrix}0&4&12\end{matrix}\\\begin{vmatrix}-1&0&4\\0&1&3\\0&0&0\end{vmatrix}\end{array}.$$

$$\det(A) = 0$$

$(b-a)\cdot\frac{\cancel{c-a}\ a-c}{b-a}$ $\quad (b^2+a^2)\cdot(c+\frac{\cancel{c-a}\ a-c}{b-a})$

5.1 #18

$$\det\begin{bmatrix}1&a&a^2\\1&b&b^2\\1&c&c^2\end{bmatrix} = \det\begin{array}{c}\begin{matrix}-1&-a&a^2\end{matrix}\\\begin{bmatrix}1&a&a^2\\0&b-a&b^2+a^2\\0&c-a&c^2+a^2\end{bmatrix}\end{array} = \det\begin{bmatrix}b-a&b^2+a^2\\c-a&c^2+a^2\end{bmatrix} = \det\begin{bmatrix}b-a&(b-a)(b+a)\\c-a&(c-a)(c+a)\end{bmatrix} = (b-a)(c-a)\times\det\begin{bmatrix}1&b+a\\1&c+a\end{bmatrix}$$

$$= (b-a)(c-a)\times\det\begin{array}{c}\begin{matrix}-1&-b-a\end{matrix}\\\begin{bmatrix}1&b+a\\0&c-b\end{bmatrix}\end{array} = (b-a)(c-a)(c-b)$$

5.1 #24 $\det(L)=1$

$\det(U)=3\times 2\times -1=-6$

$\det(A)=\det(L)\times\det(U)=-6$

$\det(U^{-1}L^{-1})=\det(U^{-1})\times\det(L^{-1})=\frac{1}{\det(U)}\times\frac{1}{\det(L)}=-\frac{1}{6}\times 1=-\frac{1}{6}$

$\det(U^{-1}L^{-1}A)=\det(U^{-1})\times\det(L^{-1})\times\det(A)=-\frac{1}{6}\times(-6)=1$

5.2 #2

$$A=\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}\Rightarrow\begin{array}{c}-1\ \ -1\ \ 0\\ \begin{bmatrix}1&1&0\\0&-1&1\\0&1&1\end{bmatrix}\end{array}\Rightarrow\begin{bmatrix}1&1&0\\0&-1&1\\0&0&2\end{bmatrix}$$

$\det A=-2$ independent.

$$B=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\Rightarrow\begin{array}{c}-7\ \ -14\ \ -21\\ \begin{bmatrix}1&2&3\\0&-3&-6\\0&-6&-12\end{bmatrix}\end{array}\Rightarrow\begin{array}{c}+6\ \ 12\\ \begin{bmatrix}1&2&3\\0&-3&-6\\0&0&0\end{bmatrix}\end{array}$$

$\det B=0$ dependent.

$$C=\begin{bmatrix}A&0\\0&A\end{bmatrix}$$

$\det C=A^2=4$ independent

$\det D=0$ due to it has dependent row in submatrix, dependent

5.2 #3

$\det A=x\times 0\times x=0$

cofactors of row 1 is $0,0,0$

$C_{11}=(-1)^{1+1}M_{11}=1\times\begin{vmatrix}0&x\\0&x\end{vmatrix}=0$

$C_{12}=(-1)^{3}M_{12}=-1\times\begin{vmatrix}0&x\\0&x\end{vmatrix}=0$

$C_{13}=(-1)^{4}M_{13}=1\times\begin{vmatrix}0&0\\0&0\end{vmatrix}=0$

$$A=\begin{bmatrix}x&x&x\\0&0&x\\0&0&x\end{bmatrix}\Rightarrow\begin{bmatrix}1&1&1\\0&0&1\\0&0&0\end{bmatrix}$$

rank of $A\le 2$

term 1 $=a_{11}(a_{22}a_{33}-a_{23}a_{32})=x(0\times x-x\times 0)=0$

term 2 $=a_{12}(a_{21}a_{33}-a_{23}a_{31})=x(0\times x-x\times 0)=0$

term 3 $=a_{13}(a_{21}a_{32}-a_{22}a_{31})=x(0-0)=0$

term 4 $=a_{21}(a_{12}a_{33}-a_{13}a_{32})=0(x^2-0)=0$

term 5 $=a_{32}(a_{11}a_{33}-a_{13}a_{31})=0(x^2-0)=0$

term 6 $=a_{23}(a_{11}a_{32}-a_{12}a_{31})=x(x\times 0-x\times 0)=0$

5.3 #1 $|A|=\begin{vmatrix}2&5\\1&4\end{vmatrix}=8-5=3$

$|B_1|=\begin{vmatrix}1&5\\2&4\end{vmatrix}=4-10=-6$

$|B_2|=\begin{vmatrix}2&1\\1&2\end{vmatrix}=4-1=3$

$x_1=-6/3=-2$

$x_2=3/3=1$

$$|A| = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 2\times(-1)^2\times(4-1) + 1\times(-1)^3\times(2-0) + 0 = 2\times3-2 = 4$$

$$|B_1| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 1\times(-1)^2\times(4-1) = 3$$

$$|B_2| = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 2\times(-1)^2\times0 + 1\times(-1)^3(2) + 0 = -2$$

$$|B_3| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1\times(-1)^4\times(1) = 1$$

$x_1 = \frac{3}{4}$

$x_2 = -\frac{1}{2}$

$x_3 = \frac{1}{4}$

#6

$$A = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 7 & 1 \end{vmatrix} \quad C = \begin{bmatrix} 3 & 0 & 0 \\ -2 & +1 & -7 \\ +0 & -0 & +3 \end{bmatrix} \quad C^T = \begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -7 & 3 \end{bmatrix}$$

$$\det A = 1\times(-1)^2\times(3) + 2\times(-1)^3\times0 + 0 = 3$$

$$A^{-1} = \begin{bmatrix} 1 & -\frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & -\frac{7}{3} & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} +3 & +2 & +1 \\ +2 & +4 & +2 \\ +1 & +2 & +3 \end{bmatrix} \quad C^T = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\det(A) = 2\times(-1)^2\times(4-1) + (-1)\times(-1)^3\times(-2) = 2\times3-2 = 4$$

$$A^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

Q_3 Suppose $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{22} & \vdots \\ & \ddots & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$

$$\det(kA) = \begin{vmatrix} ka_{11} & \cdots & ka_{1n} \\ ka_{21} & \cdots & ka_{2n} \\ \vdots & & \vdots \\ ka_{n1} & \cdots & ka_{nn} \end{vmatrix} = k \times \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ ka_{21} & \cdots & ka_{2n} \\ \vdots & \ddots & \vdots \\ ka_{n1} & \cdots & ka_{nn} \end{vmatrix} \cdots\cdots k^n \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = k^n \det(A)$$

Repeat the process. do row operation

$R_i \Rightarrow R_i / k$

Q_4

$A \cdot A^T = I = A^T A$

$\det(A \cdot A^T) = \det(I)$

$\det(A) \cdot \det(A^T) = \det(I)$

$\because \det(A) = \det(A^T)$

$\therefore [\det(A)]^2 = \det(I)$

$\det(A) = \sqrt{1}$

$\det(A) = \pm 1$

Q_5 $\begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \end{bmatrix}$

Suppose A is a upper triangular matrix.

$a_{ij} = 0$ when $i > j$

$C_{ij} = (-1)^{(i+j)} \times \det(M_{ij})$

when $i < j$

$\det(M_{ij}) = 0$, due to M_{ij} is a matrix without row i and column j

M_{ij} will must be upper triangular and have at least one zero on its diagonal.

$\therefore C_{ij} = 0$ when $i < j$

$\therefore$ cofactors of A is an lower triangular matrix

$\because Adj(A) = C^T$

$\therefore Adj(A)$ is an upper triangular

Similar, if A is an lower triangular.

We can still get cofactor matrix of A is an upper triangular matrix

Then $Adj(A)$ is an lower triangular matrix again

$\therefore$ Adjoint of an triangular matrix is still a triangular matrix

$M_{21} = \begin{matrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{matrix}$

$M_{11} = \begin{matrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{matrix}$

$M_{12} = \begin{matrix} 0 & x & x \\ 0 & x & x \\ 0 & 0 & x \end{matrix}$

Q6

a $A^{-1} = C^T \cdot \frac{1}{\det A}$

Suppose $a_{ii} = 0$ $i \in n$, A is invertible

$\det(A) = 0$

$\therefore C^T / \det(A)$ will not have an answer

$\therefore A^{-1}$ will not get an answer

$\therefore A$ is not invertible.

$\therefore$ CONJVATION

$\therefore$ when $a_{ii} \neq 0$ A is invertible.

$\therefore A$ is invertible when $a_{ii} \neq 0$

$\therefore A$ is invertible if only if each diagonal element $a_{ii} \neq 0$

b $A^{-1} = C^T \cdot \frac{1}{\det A}$

$A^{-1}_{ij} = C_{ji} \cdot \frac{1}{\det A}$

$\because i = j$

$\therefore A^{-1}_{ii} = C_{ii} \cdot \frac{1}{\det A}$

$\because A$ is triangular

$\therefore \det A = \prod_{i=1}^{n} a_{ii}$

$\therefore A^{-1}_{jj} = C_{jj} \cdot \frac{1}{\prod_{i=1}^{n} a_{ii}}$

$C_{jj} = (-1)^{j+j} \det M_{jj}$

$= \det M_{jj}$

$= \prod_{i=1}^{n} a_{ii} / a_{jj}$

$\therefore A^{-1}_{jj} = \cancel{\prod_{i=1}^{n} a_{ii}} / a_{jj} \times 1 / \cancel{\prod_{i=1}^{n} a_{ii}}$

$\therefore A^{-1}_{jj} = \frac{1}{a_{jj}}$

Q7

$$\begin{vmatrix} 2 & 2 & 5 & 3 \\ 2 & 3 & 3 & 2 \\ 3 & 3 & 7 & 4 \\ 3 & 2 & 9 & 7 \end{vmatrix} = \begin{array}{c} \begin{matrix} -6 & -6 & -15 & -9 \\ 6 & 4 & 18 & 14 \end{matrix} \\ \begin{vmatrix} 2 & 2 & 5 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & -2 & 3 & 5 \end{vmatrix} \end{array} = \begin{array}{c} \begin{matrix} 0 & 2 & -4 & -2 \end{matrix} \\ \begin{vmatrix} 2 & 2 & 5 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 3 \end{vmatrix} \end{array} = \begin{vmatrix} 2 & 2 & 5 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 4 \end{vmatrix}$$

$$= 2 \times 1 \times -1 \times 4$$

$$= -8$$

volume $= 8$