$R_2: R_2 - bR_1$

#3

① $1 > 0$

$9 - b^2 > 0$

$-b^2 > -9$

$b^2 < 9$

$-3 < b < 3$

$$\begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & b \\ 0 & 9-b^2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 9-b^2 \end{bmatrix} \times \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

② $2 > 0$

$2c - 16 > 0$

$2c > 16$

$c > 8$

$$\begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 4 \\ 0 & c-8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 0 \\ 0 & c-8 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$c \cdot \frac{b}{c}$ $\quad b \cdot \frac{b}{c} \Rightarrow \frac{b}{b}$ $\quad \frac{b^2}{c} \over c$

③ $c > 0$

$c^2 - b^2 > 0$

$c^2 > b^2$

$\because c > 0$

$\therefore c > b$

$$\begin{bmatrix} c & b \\ b & c \end{bmatrix} \Rightarrow \begin{bmatrix} c & b \\ 0 & \frac{c^2-b^2}{c} \end{bmatrix}$$

$c - \frac{b^2}{c}$

$\frac{c^2-b^2}{c}$

$$\begin{bmatrix} 1 & 0 \\ \frac{b}{c} & 1 \end{bmatrix} \times \begin{bmatrix} c & 0 \\ 0 & \frac{c^2-b^2}{c} \end{bmatrix} \times \begin{bmatrix} 1 & \frac{b}{c} \\ 0 & 1 \end{bmatrix}$$

7

$$4(x_1-x_2+2x_3)^2 = 4(x_1-x_2+2x_3)(x_1-x_2+2x_3)$$
$$= 4(\cancel{x_1^2}\ \cancel{-x_1x_2} + \cancel{2x_1x_3}\ \cancel{-x_1x_2} + \cancel{x_2^2} - 2x_2x_3$$
$$+\cancel{2x_1x_3} - 2x_2x_3 + 4x_3^2)$$
$$= 4(x_1^2 - 2x_1x_2 + x_2^2 + 4x_1x_3 - 4x_2x_3 + 4x_3^2)$$
$$= \underline{4x_1^2} - 8x_1x_2 + \underline{4x_2^2} + 16x_1x_3 - 16x_2x_3 + \underline{16x_3^2}$$

$$S = \begin{bmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & -4 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

pivots are 4
rank is 1

$$|S| = \begin{vmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16 \end{vmatrix} = \begin{vmatrix} 4 & -4 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$\det(S) = 0$

$\text{trace}(S) = \lambda_1 + \lambda_2 + \lambda_3 = 24$ only 1 is not 0

$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 0$

$\therefore \lambda_1 = 24 \quad \lambda_2 = \lambda_3 = 0$

22

$$S = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \quad \det(S-\lambda I) = \lambda^2 - 10\lambda + 9 = 0 \quad \lambda_1 = 1 \quad \lambda_2 = 9 \quad x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \times \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A^T \times A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} = S$$

$$S = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix} \quad \det(S-\lambda I) = \lambda^2 - 20\lambda + 64 = 0 \quad \lambda_1 = 4 \quad \lambda_2 = 16 \quad x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \times \frac{1}{\sqrt{2}} \times \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \times \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \times \frac{1}{\sqrt{2}} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$A^T \times A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix} = S$$

2 4

#26

$$S = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$C^T \times C = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \times \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 8 \end{bmatrix}$$

$$S: \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & \sqrt{5} \end{bmatrix}$$

$$C^T \times C = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & \sqrt{5} \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & \sqrt{5} \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 7 \end{bmatrix}$$

Q_2 Suppose $x_1 = [1, 0, 0, \dots, 0]^T$

$x_2 = [0, 1, 0, \dots, 0]^T$

$x_n = [0, 0, 0, \dots, 1]^T$

$\because$ positive definte

$\therefore$ $x_i^T A x > 0$ $\qquad i \in [0, n]$

Suppose $a_{ii} < 0$

$$x_i^T A x = [a_{i1}, a_{i2}, a_{i3}, a_{i4}, \dots a_{in}] \times \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

$1 \times n \quad n \times n$

$1 \times n$

$= a_{ii} < 0$

$\therefore$ It is not positive definte

$\therefore$ CONTRADICTION

$\therefore$ the diagonal elements of a positive definte matrix must be positive.

Q3

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \times \begin{bmatrix} 2 & -17 & 7 \\ -17 & -4 & -1 \\ 7 & -1 & -14 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x-17y+7z & -17x-4y-z & 7x-y-14z \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= 2x^2-17xy+7xz \quad -17xy-4y^2-zy+7xz-zy-14z^2$$

$$= 2x^2-4y^2-14z^2-34xy-2zy+14xz$$

$$\begin{vmatrix} 2-\lambda & -17 & 7 \\ -17 & -4-\lambda & -1 \\ 7 & -1 & -14-\lambda \end{vmatrix} =$$

$$\begin{vmatrix} 2 & -17 \\ -17 & -4 \end{vmatrix} = 2(-4)-17^2 = -297 < 0$$

$\therefore$ is not positive definite

from Q2. We also know that due to $-4, -14 < 0$ and they are diagonal elements

Q4 $\begin{vmatrix} 4 & -6 \\ -6 & -13 \end{vmatrix} = -88 < 0$

$\because$ not positive definite.

$\therefore$ No Cholesky decomposition

$4 \cdot \frac{2i}{4} = 2i \qquad -2i \cdot \frac{2i}{4} = 1 \qquad -2 \cdot \frac{2i}{4} = -i$

$R_2 : R_2 - \frac{2i}{4} R_1$

$R_3 : R_3 + \frac{1}{2} R_1$

$R_3 : R_3 - \frac{6i}{9} R_2$

$$\begin{bmatrix} 4 & -2i & -2 \\ 2i & 10 & -7i \\ -2 & 7i & 21 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & -2i & -2 \\ 0 & 9 & -6i \\ 0 & 6i & 20 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & -2i & -2 \\ 0 & 9 & -6i \\ 0 & 0 & 16 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & & \\ & 3 & \\ & & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ \frac{2i}{4} & 1 & 0 \\ -\frac{1}{2} & \frac{6i}{9} & 1 \end{bmatrix}^H = \begin{bmatrix} 2 & -i & -1 \\ 0 & 3 & -2i \\ 0 & 0 & 4 \end{bmatrix}$$

$$C^H \times C = \begin{bmatrix} 2 & 0 & 0 \\ i & 3 & 0 \\ -1 & 2i & 4 \end{bmatrix} \times \begin{bmatrix} 2 & -i & -1 \\ 0 & 3 & -2i \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -2i & -2 \\ 2i & 10 & -7i \\ -2 & 7i & 21 \end{bmatrix}$$

Q5 $\det(A-\lambda I) = (\lambda-3)(\lambda-4)\cdot(\lambda-4)(\lambda-5) = 0$

$\lambda_1 = 3 \qquad X_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 4 \qquad X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = X_3$

same eigenvalues

$\lambda_4 = 5 \qquad X_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 3 & & & \\ & 4 & & \\ & & 4 & \\ & & & 5 \end{bmatrix} \times \begin{bmatrix} -2 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}^{-1}$$

not indepent.

This can not inverse

$\therefore$ Can not find Jordan decomposition