Q1 (a) #8

$$|P| = \begin{vmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{vmatrix} = -\begin{vmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{vmatrix} = \begin{vmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{vmatrix} = i^3 \neq 0$$

Thus P is not invertible.

$$P^H = (P^*)^T = \begin{bmatrix} 0 & -i & 0 \\ 0 & 0 & -i \\ -i & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & -i \\ -i & 0 & 0 \\ 0 & -i & 0 \end{bmatrix} \neq P$$

Thus P is not hermitian.

$$PP^H = \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & -i \\ -i & 0 & 0 \\ 0 & -i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^HP = \begin{bmatrix} 0 & 0 & -i \\ -i & 0 & 0 \\ 0 & -i & 0 \end{bmatrix} \times \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$-i^2 = -i \cdot -i$
$= i^2 = -1$

$\underbrace{-1 \times -1 \times -1 \times \cdots \times -i}_{16}$
$1 \times -i$

Thus P is unitary

$$P^2 = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \qquad P^3 = \begin{bmatrix} -i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{bmatrix}$$

$$P^{100} = P^{99} \cdot P = (P^3)^{33} \cdot P$$
$$= (-i)^{33} \cdot P$$
$$= -iP$$

$|\lambda| = 1 \qquad \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = i^3 = -i$

$\det(A - \lambda I) = -\lambda^3 - i \qquad \lambda_1 = i \quad \lambda_2 = \frac{\sqrt{3} - i}{2} \quad \lambda_3 = \frac{\sqrt{3} - i}{2}$

#9

$$\lambda_1 = i \quad A-\lambda_1 I = \begin{bmatrix} -i & i & 0 \\ 0 & -i & i \\ i & 0 & -i \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$-xi+yi=0$

$-yi+zi=0 \quad zi=yi \quad z=y$

$xi-zi=0 \quad xi=zi \quad x=z$

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = \frac{-\sqrt{3}-i}{2} \quad \begin{bmatrix} \frac{\sqrt{3}+i}{2} & i & 0 \\ 0 & \frac{\sqrt{3}+i}{2} & i \\ i & 0 & \frac{\sqrt{3}+i}{2} \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\frac{\sqrt{3}+i}{2}x+yi=0 \quad \frac{\sqrt{3}+i}{2}x=-yi$

$\frac{\sqrt{3}+i}{2}y+zi=0 \quad \frac{\sqrt{3}+i}{2}y=-zi$

$xi+\frac{\sqrt{3}+i}{2}z=0 \quad xi=-\frac{\sqrt{3}+i}{2}z$

$$x_2 = \begin{bmatrix} \frac{i\sqrt{3}-1}{2} \\ \frac{-i\sqrt{3}-1}{2} \\ 1 \end{bmatrix}$$

$$\lambda_3 = \frac{\sqrt{3}-i}{2} \quad \begin{bmatrix} \frac{-\sqrt{3}+i}{2} & i & 0 \\ 0 & \frac{-\sqrt{3}+i}{2} & i \\ i & 0 & \frac{-\sqrt{3}+i}{2} \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\frac{-\sqrt{3}+i}{2}x+yi=0 \quad \frac{-\sqrt{3}+i}{2}x=-yi$

$\frac{-\sqrt{3}+i}{2}y+zi=0 \quad \frac{-\sqrt{3}+i}{2}y=-zi$

$xi+\frac{-\sqrt{3}+i}{2}z=0 \quad \frac{-\sqrt{3}+i}{2}z=-xi$

$$x_3 = \begin{bmatrix} \frac{-i\sqrt{3}-1}{2} \\ \frac{i\sqrt{3}-1}{2} \\ 1 \end{bmatrix}$$

$\frac{-\sqrt{3}+i}{2}\left(\frac{-\sqrt{3}+i}{2}i\right) = -\left(\frac{\sqrt{3}-i}{2}\,\frac{-\sqrt{3}+i}{2}\right)i$

$\frac{-i\sqrt{3}-1}{2} \quad \frac{-3+\sqrt{3}i+\sqrt{3}i+1}{4}$

$\frac{i\sqrt{3}+1}{2}\times i \quad \frac{2\sqrt{3}i-2}{4}$

$\frac{-\sqrt{3}+i}{2}$

$$Q = \begin{bmatrix} 1 & \frac{i\sqrt{3}-1}{2} & \frac{-i\sqrt{3}-1}{2} \\ 1 & \frac{-i\sqrt{3}-1}{2} & \frac{i\sqrt{3}-1}{2} \\ 1 & 1 & 1 \end{bmatrix}$$

The eigenvectors of unitary matrix are orthogonal

$$\left(\frac{i\sqrt{3}-1}{2}\right)^2+\left(\frac{-i\sqrt{3}-1}{2}\right)^2+1$$

$$=\frac{(i\sqrt{3})^2-2i\sqrt{3}+1}{4}+\frac{(-i\sqrt{3})^2+2i\sqrt{3}+1}{4}+1$$

$$=\frac{-3-\cancel{2i\sqrt{3}}+1\ \cancel{+}\ -3\cancel{+2i\sqrt{3}}+1}{4}\ \cancel{0}+1$$

$$= \cancel{-6+2}\ \frac{-4}{4}+1 = -1+1=0$$

11

$\because$ Q and U are unitary matrix

$\therefore QQ^H = Q^H Q = I \qquad Q^{-1} = Q^H$

$\because Q^{-1} = Q^H$

$\therefore (Q^{-1})^H = (Q^H)^H \Rightarrow (Q^{-1})^H = Q$

$\because QQ^H = I$

$\therefore (Q^{-1})^H \cdot Q^{-1} = Q^{-1}(Q^{-1})^H = I$

$\therefore Q^{-1}$ is unitary matrix

$\because$ U is unitary matrix $\qquad QU \cdot (QU)^H = I$?

$\therefore UU^H = U^H U = I \qquad U^{-1} = U^H$

$QU \cdot (QU)^H$

$= QU \cdot U^H Q^H$

$= QIQ^H$

$= QQ^H = I$

$\therefore QU$ is also unitary matrix.

#16

$\det(U - \lambda I) = (\cos\theta - \lambda)^2 + \sin^2\theta$

$= \lambda^2 - 2\cos\theta\,\lambda + 1 = 0$

$\lambda_1 = \cos\theta + \sqrt{\cos^2\theta - 1} = \cos\theta + \sqrt{\cancel{\cos^2\theta} - \cancel{\cos^2\theta} - \sin^2\theta} = \cos\theta + i\sin\theta$

$\lambda_2 = \cos\theta - \sqrt{\cos^2\theta - 1} = \cos\theta - \sqrt{\cancel{\cos^2\theta} - \cancel{\cos^2\theta} - \sin^2\theta} = \cos\theta - i\sin\theta$

$$U - \lambda_1 I = \begin{bmatrix} \cancel{\cos\theta} - \cancel{\cos\theta} - i\sin\theta & -\sin\theta \\ \sin\theta & \cancel{\cos\theta} - \cancel{\cos\theta} - i\sin\theta \end{bmatrix} \sim \begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix}$$

$-i\cancel{\sin\theta} \cdot x = \cancel{\sin\theta} \cdot y$

$\begin{matrix} 1 & 1 \\ -i & i \end{matrix}$

$$-ix = y \qquad x_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix} \qquad \bar{x}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$U - \lambda_2 I = \begin{bmatrix} \cancel{\cos\theta} - \cancel{\cos\theta} + i\sin\theta & -\sin\theta \\ \sin\theta & \cancel{\cos\theta} - \cancel{\cos\theta} + i\sin\theta \end{bmatrix} \sim \begin{bmatrix} i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{bmatrix}$$

$$\cancel{i\sin\theta}\, x = -\cancel{\sin\theta}\, y$$

$$x_2 = \begin{bmatrix} 1 \\ i \end{bmatrix} \qquad \bar{x}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \times \begin{bmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{bmatrix} \times \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$$

$$|\lambda| = 1$$

$$\begin{matrix} 1 & 1 \\ -i & i \end{matrix} \qquad \begin{matrix} 1 & 1 \\ i & -i \end{matrix}$$

$$\begin{matrix} 1 & i \\ 1 & -i \end{matrix}$$

Q2 (a) #4

$$(A^TCA)^T = A^TC^TA^{TT} = A^TC^TA$$

$$(A^TCA)^T = A^TCA \leftarrow\text{—— } C \text{ is symmetric } \therefore C = C^T$$

$\therefore A^TCA$ is symmetric

$A : 6\times3$

$A^T : 3\times6$

$A^TCA : 3\times6 \times \overset{6}{?} \times \overset{6}{?} \times 6\times3 \Rightarrow 3\times3$

$\therefore C : 6\times6$

#5

$\frac{2}{2-\lambda}$ $\frac{4}{2-\lambda}$ $\frac{4}{2-\lambda}$

$$\det(S-\lambda I) = \begin{vmatrix} 2-\lambda & 2 & 2 \\ 2 & -\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} = -\lambda^3+2\lambda^2+8\lambda = -\lambda(\lambda+2)(\lambda-4) = 0$$

$\lambda_1 = 0$

$\lambda_2 = -2$

$\lambda_3 = 4$

$$\lambda_1 = 0 \quad \begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$2x+2y+2z = 0$

$2x = 0 \quad \therefore x = 0$

$2y = -2z$

$y = -z$

$$x_1 = \pm\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad \bar{x}_1 = \pm\begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\lambda_2 = -2 \quad \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$4x+2y+2z = 0$

$2x+2y = 0 \quad 2x = -2y \quad x = -y$

$2x+2z = 0 \quad 2x = -2z \quad x = -z$

$$x_2 = \pm\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \qquad \bar{x}_2 = \pm\begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\lambda_3 = 4 \quad \begin{bmatrix} -2 & 2 & 2 \\ 2 & -4 & 0 \\ 2 & 0 & -4 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$-2x+2y+2z = 0$

$2x-4y = 0 \quad 2x = 4y \quad x = 2y$

$2x-4z = 0 \quad 2x = 4z \quad x = 2z$

$$x_3 = \pm\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \qquad \bar{x}_3 = \pm\begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

#7 $\det(S-\lambda I) = \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & -2 \\ 2 & -2 & -\lambda \end{vmatrix} = -\lambda^3+9\lambda = -\lambda(\lambda-3)(\lambda+3) = 0$ $\quad \lambda_1=0 \quad \lambda_2=3 \quad \lambda_3=-3$

$\lambda_1=0 \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\quad \begin{matrix} x+2z=0 \\ -y-2z=0 \\ 2x-2y=0 \end{matrix} \quad \begin{matrix} x=-2z \\ -y=2z \\ 2x=2y \end{matrix} \quad \begin{matrix} x=-2z \\ y=-2z \\ x=y \end{matrix} \quad x_1=\pm\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$

$\lambda_2=3 \quad \begin{bmatrix} -2 & 0 & 2 \\ 0 & -4 & -2 \\ 2 & -2 & -3 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\quad \begin{matrix} -2x+2z=0 \\ -4y-2z=0 \\ 2x-2y-3z=0 \end{matrix} \quad \begin{matrix} -2x=-2z \\ -4y=2z \\ \end{matrix} \quad \begin{matrix} x=z \\ -2y=z \\ \end{matrix} \quad x_2=\pm\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

$-1+3$

$\lambda_3=-3 \quad \begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & -2 \\ 2 & -2 & 3 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\quad \begin{matrix} 4x+2z=0 \\ 2y-2z=0 \\ 2x-2y+3z=0 \end{matrix} \quad \begin{matrix} 4x=-2z \\ y=z \\ \end{matrix} \quad -2x=z \quad x_3=\pm\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

$Q = \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$ with $\pm$ column vector can satisfie Q

#13

$\det(S-\lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2-6\lambda+8 = (\lambda-2)(\lambda-4)=0$ $\quad \lambda_1=2 \quad \lambda_2=4$

$\lambda_1=2 \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} \quad x+y=0 \quad x=-y \quad x_1=\pm\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2=4 \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} \quad -x+y=0 \quad x=y \quad x_2=\pm\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$S = 2\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + 4\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\det(B-\lambda I) = \begin{vmatrix} 9-\lambda & 12 \\ 12 & 16-\lambda \end{vmatrix} = \lambda^2 - 25\lambda = 0 \qquad \lambda_1 = 0, \quad \lambda_2 = 25$$

$$\lambda_1 = 0 \quad \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} \quad \begin{matrix} 9x+12y=0 \\ 12x+16y=0 \end{matrix} \quad \begin{matrix} 9x=-12y \\ 12x=-16y \end{matrix} \quad \begin{matrix} 3x=-4y \\ 3x=-4y \end{matrix} \quad x_1 = \pm\begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

$$\lambda_2 = 25 \quad \begin{bmatrix} -16 & 12 \\ 12 & -9 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} \quad \begin{matrix} -16x+12y=0 \\ 12x-9y=0 \end{matrix} \quad \begin{matrix} -16x=-12y \\ 12x=9y \end{matrix} \quad \begin{matrix} 4x=3y \\ 4x=3y \end{matrix} \quad x_2 = \pm\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ -3 \end{bmatrix} \times \begin{bmatrix} 4 & -3 \end{bmatrix} = \begin{bmatrix} 16 & -12 \\ -12 & 9 \end{bmatrix} \qquad \begin{bmatrix} 3 \\ 4 \end{bmatrix} \times \begin{bmatrix} 3 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$$

$$B = 0\begin{bmatrix} \frac{16}{25} & \frac{-12}{25} \\ \frac{-12}{25} & \frac{9}{25} \end{bmatrix} + 25\begin{bmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{bmatrix}$$

Q3 (a) $\because A$ is a unitary matrix

$\therefore AA^H = A^H A = I$ $\quad A^H = A^{-1}$

$\det(A) = \det(A^T) = \det(A^*)$

$AA^H = I$

$\det(AA^H) = \det(I) = 1$

$\det(A) \times \det(A^H) = 1$

$\det(A) \times (\det(A))^* = 1$

$|\det A|^2 = 1$

$|\det A| = 1$

(b Suppose A has 2 eigenvalues λ_1, λ_2 and $\lambda_1 \neq \lambda_2$

$A^H A = AA^H = I$

$Ax_1 = \lambda_1 x_1$

$Ax_2 = \lambda_2 x_2$

$A^H A x_1 = A^H \lambda_1 x_1$

$x_1 = A^H \lambda_1 x_1$

$x_2 = A^H \lambda_2 x_2$

$A^H = A^T$

$AA^H = I$

$x_2^H A x_1 = x_2^H \lambda_1 x_1 = \lambda_1 x_2^H x_1$

$x_2^H x_1 = 0$

$x_2^H A x_1 = ((x_2^H A)^H)^H x_1$

$= (A^H x_2)^H x_1$

$= (A^{-1} x_2)^H x_1$

$= (\lambda_2^* x_2)^H x_1$

$= x_2^H \lambda_2 x_1$

$= \lambda_2 x_2^H x_1$

$\lambda_1 x_2^H x_1 = \lambda_2 x_2^H x_1$

$\because \lambda_1 \neq \lambda_2$

$\therefore x_2^H \cdot x_1 = 0$

Q4 $\because U$ is unitary

$\therefore U^H U = U U^H = I$

$A = U^H B U$

$A^H = (U^H B U)^H = U^H B^H U^{HH} = U^H B^H U$

$AA^H = U^H B U U^H B^H U = U^H B I B^H U = U^H B B^H U$

$A^H A = U^H B^H U U^H B U = U^H B^H B U$

if B is normal

$AA^H = U^H B B^H U = U^H B^H B U = A^H A$

$\therefore A$ is normal

if A is normal

$A^H A = A^H A$

$U^H B^H B U = U^H B B^H U$

$B^H B = B \cdot B^H$

$\therefore B$ is normal

$\rightarrow \therefore B$ is normal if and only if A is normal

Q5 Suppose $B = A^H A$ $\quad \lambda$ is eigenvalues of B

$\because B^H = A^H A = B$

$\therefore B$ is hermitian matrix

$\because \lambda$ is eigenvalues

$\therefore Bx = \lambda x$ $\qquad \lambda =$

$(Bx)^H = (\lambda x)^H$

$x^H B^H = x^H \lambda^H$

$x^H B = \lambda x^H$ —— $B^H = B$

$x^H B x = \lambda x^H x$ $\qquad \|x\|^2 = x^H x$

$B = A^H A = \|A\|^2$ which means B is a number and $B > 0$

$\therefore x^H B x = B x^H x = \lambda x^H x$

$\therefore \lambda > 0$

Q6 $Ax = \lambda x$ $A^H x = \lambda^* x$

$x^H A x = x^H \lambda x = \lambda x^H x$

$(x^H A x)^H = x^H A^H x^{HH} = x^H A^H x$

$(\lambda x^H x)^H = \lambda^* x^H x$

$x^H A^H x = \lambda^* x^H x$

$x^H A^H x = x^H \lambda^* x$

$A^H x = \lambda^* x$

$\therefore$ x is an eigenvector of A^H corresponding to the conjugate λ^*

Q7

x_1^2	2
x_2^2	4
x_3^2	4
$x_1 \cdot x_2$	2
$x_2 \cdot x_3$	2
$x_1 \cdot x_3$	-2

$$Q = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 4 & 1 \\ -1 & 1 & 4 \end{bmatrix}$$

$\det(Q - \lambda I) = -\lambda^3 + 10\lambda^2 - 29\lambda + 20$

$\lambda_1 = 1$ $\lambda_2 = 4$ $\lambda_3 = 5$

$$\lambda_1 = 1 \quad \begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x + y - z = 0$ $y + z = 0$

$y = -z$

$-1-1$

$$x_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

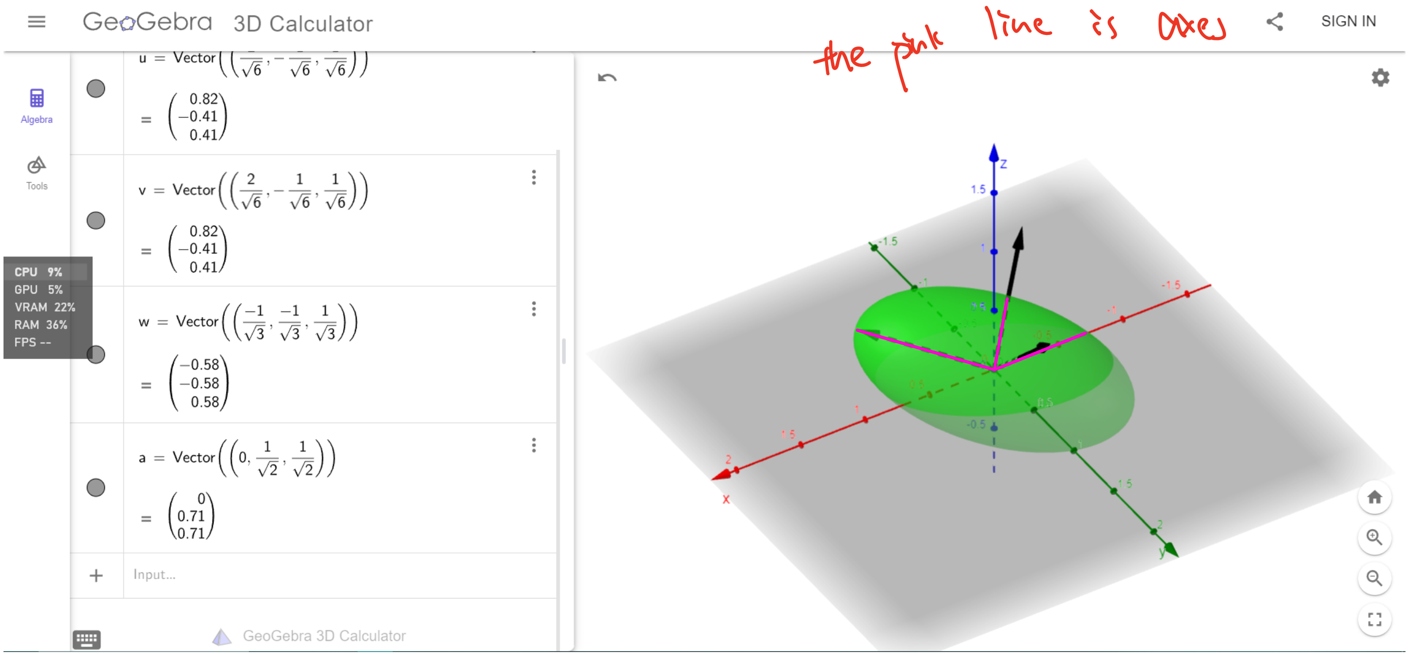
$$\lambda_2 = 4 \quad x_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \qquad \lambda_3 = 5 \quad x_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$4+1+1$

principal axes are $[2/\sqrt{6}, -1/\sqrt{6}, 1/\sqrt{6}]$, with length 1

$[-1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}]$ with length $1/2$

and $[0, 1/\sqrt{2}, 1/\sqrt{2}]$, with length $1/\sqrt{5}$



$$Q=\begin{bmatrix} 5 & 0 & -\sqrt{2} \\ 0 & 2 & 0 \\ -\sqrt{2} & 0 & 4 \end{bmatrix}$$

$$\det(Q-\lambda I) = -\lambda^3+11\lambda^2-36\lambda+36 = -(\lambda-2)(\lambda^2-9\lambda+18)$$

$$\lambda_1 = 2 \quad \lambda_2 = 3 \quad \lambda_3 = 6$$

for $\lambda_1 = 2$

$$Q-2I = \begin{bmatrix} 3 & 0 & -\sqrt{2} \\ 0 & 0 & 0 \\ -\sqrt{2} & 0 & 2 \end{bmatrix} \qquad x_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

for $\lambda_2 = 3$

$$Q-3I = \begin{bmatrix} 2 & 0 & -\sqrt{2} \\ 0 & -1 & 0 \\ -\sqrt{2} & 0 & 1 \end{bmatrix} \qquad x_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ 1 \end{bmatrix}$$

$$\left(\frac{\sqrt{2}}{2}\right)^2+1$$

$$\frac{2}{4}+1 = \sqrt{\frac{6}{4}} \qquad \frac{\sqrt{6}}{2}$$

$$\frac{\sqrt{2}}{2} \cdot \frac{2}{\sqrt{6}} \qquad \frac{1}{\sqrt{3}} \qquad \frac{2}{\sqrt{6}}$$

for $\lambda_3 = 6$

$$Q-6I = \begin{bmatrix} -1 & 0 & -\sqrt{2} \\ 0 & -4 & 0 \\ -\sqrt{2} & 0 & -2 \end{bmatrix} \qquad x_3 = \begin{bmatrix} -\sqrt{2} \\ 0 \\ 1 \end{bmatrix}$$

$$(\sqrt{2})^2+1 = 2+1 = 3$$

$$\frac{\sqrt{2}}{\sqrt{3}}$$

principal axes are $[0, 1, 0]$ with length $1/\sqrt{2}$

$\left[\frac{1}{\sqrt{3}}, 0, \frac{2}{\sqrt{6}}\right]$ with length $1/\sqrt{3}$

$\left[\frac{-\sqrt{2}}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}\right]$ with length $1/\sqrt{6}$

here is the unit vector
the pink line is axes