Q1(a) #10 subspace: ① $\underline{0} \in$ set
② $\forall u, w \in$ set, and $\forall a, b \in F$
$au + bw \in$ set

(a) (b_1, b_2, b_3) $b_2 = b_3$
Suppose a vector $u = (a_1, a_2, a_3)$ $a_2 = a_3$
and a vector $w = (b_1, b_2, b_3)$ $b_2 = b_3$
$\therefore$ $u = (a_1, a_2, a_2)$ $w = (b_1, b_2, b_2)$
Suppose $\forall x, y \in F$
$\therefore$ $xu + yw = (xa_1 + yb_1, xa_2 + yb_2, xa_2 + yb_2)$
$\therefore$ $xa_3 + yb_2 = xa_2 + yb_2$
$\therefore$ $xu + yw$ is also belong to this plane.
If $a_1, a_2, a_3 = 0$ $u = \underline{0}$
$\therefore$ the plane of vectors (b_1, b_2, b_3) with $b_1 = b_2$ is a subspaces

(b) It is not a subspace, due to $\underline{0}$ is not one of the vector in set

(c) It is not a subspace, due to if u, w are vector satisfie $u_1 u_2 u_3 = 0$ and $w_1 w_2 w_3 = 0$, but $cv + dw$ will not always satisfie this rule.
$u = (1, 0, 1)$ $c = 2$ $1 \times 0 \times 1 = 0$
$w = (1, 1, 0)$ $d = 3$ $1 \times 1 \times 0 = 0$
$cu + dw = (2, 0, 2) + (3, 3, 0)$
$= (5, 3, 2)$
$5 \times 3 \times 2 = 30 \neq 0$

(d) It is a subspace
Suppose $\forall c, d \in F$
If $c = d = 0$
$cv + dw = 0 \times (1, 4, 0) + 0 \times (2, 2, 2)$
$= \underline{0}$
$\therefore \underline{0} \in$ set.

In addition $cv+dw$ is the equation of linear combination of v and w.

(e) It is a subspace

Simularily suppose. $\forall a, b \in F$ u, w satisfie

$u_1+u_2+u_3=0$ $w_1+w_2+w_3=0$

$\forall$ $u_1=u_2=u_3=0$

$\therefore u_1+u_2+u_3=0$

$\therefore \underline{0}$ belongs to this set of vectors.

$aw+bw$

$=(au_1+bw_1, au_2+bw_2, au_3+bw_3)$

$au_1+bw_1+au_2+bw_2+au_3+bw_3$

$=a(u_1+u_2+u_3)+b(w_1+w_2+w_3)$

$=a\times 0+b\times 0$

$=0$

$\therefore au+bw$ belongs to this set of vectors

(f) Suppose $u=(-2, 0, 2)$

$w=(1, 1, 1)$

$a=-1$

$b=1$

$au+bw=(2,0,-2)+(1,1,1)$

$=(3,1,-1)$

$3>1>-1$

$\therefore au+bw$ does not belong to this set

$\therefore$ It is not a subspace

#20 (a) $\begin{bmatrix} 1, & 4, & 2 \\ 2, & 8, & 4 \\ -1, & -4, & -2 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$

$\because R_2 = 2R_1$

$R_3 = -R_1$

$\therefore b_2 = 2b_1$

$b_3 = -b_1$

(b) $\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$ $\because$ $R_3 = -R_1$

$\therefore b_3 = -b_1$

(b)
P3.3

2 8 0
2 11 5

#18 $A = \begin{bmatrix} 1, & 4, & 0 \\ 2, & 11, & 5 \\ -1, & 2, & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1, & 4, & 0 \\ 0, & 3, & 5 \\ 0, & 6, & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1, & 4, & 0 \\ 0, & 3, & 5 \\ 0, & 0, & 0 \end{bmatrix}$

$\therefore$ the rank of A is 2.

$A^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & q \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & q-1 \end{bmatrix}$

$q-1 \neq 1$
$q \neq 2$

$\therefore$ the rank of A is 3, only if $q \neq 2$
if $q = 2$, then the rank will be 2.

#19 $AA^T = \begin{bmatrix} 1, & 1, & 5 \\ 1, & 0, & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 1+1+25 & 1+0+5 \\ 1+5 & 1+1 \end{bmatrix} = \begin{bmatrix} 27 & 6 \\ 6 & 2 \end{bmatrix}$

The rank is 2. for AA^T

$A^TA = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 5 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+1 & 1+0 & 5+1 \\ 1+0 & 1+0 & 5+0 \\ 5+1 & 5+0 & 25+1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 6 \\ 1 & 1 & 5 \\ 6 & 5 & 26 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2 & 1 & 6 \\ 0 & 1 & 4 \\ 0 & 2 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 6 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

the rank is 2 for A^TA

$$\begin{array}{cc} & \begin{array}{ccc} 4 & 6 & 10 \end{array} \\ AA^T = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = & \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix} \end{array} \Rightarrow \begin{bmatrix} 4 & 2 & 2 \\ 0 & 2 & 4 \\ 0 & 4 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 2 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

the rank is 2 for AA^T

$$A^TA = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 3 & 5 \end{bmatrix}$$

the rank is 2 for A^TA

(C) P3.4

#11 (a) $a = (1, 1, -1)$

$b = (-1, -1, 1)$

$a = -b$

$\therefore \forall x, y, z \in F$

$\because a = -b$

$\therefore xa + yb = xa - ya = (x-y)a$

$\therefore xa + yb = za$

$\therefore$ It a line in R^3

(b) A Plane in R^3

(c) All of R^3

(d) All of R^3

#26

(a) $\begin{bmatrix} 1, & 0, & 0 \\ 0, & 0, & 0 \\ 0, & 0, & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$A \begin{bmatrix} a & e & h \\ b & f & i \\ c & g & g \end{bmatrix} = \overset{A^T}{\begin{bmatrix} a & b & c \\ e & f & g \\ h & i & g \end{bmatrix}}$$

$$-A = \begin{bmatrix} -a & & \\ & -f & \\ & & -g \end{bmatrix}$$

(c) $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

Q2. Suppose $x, y \in \mathbb{R}$.

$$xu + yv = 0$$

$$[ax, bx] + [cy, dy] = 0$$

$$[ax + cy, bx + dy] = 0$$

$$\begin{cases} ax + cy = 0 & ① \\ bx + dy = 0 & ② \end{cases}$$

① $ax + cy = 0$

$$ax = -cy$$

$$x = -\frac{cy}{a}$$

take x to equation ②

② $b \cdot \left(-\frac{cy}{a}\right) + dy = 0$

$$-\frac{cyb}{a} + dy = 0$$

$$dy = \frac{cyb}{a}$$

$$ad = cb$$

$$ad - cb = 0$$

Q3

$$\begin{pmatrix}3\\2\\1\\-4\\1\end{pmatrix}x_1+\begin{pmatrix}2\\3\\0\\-1\\-1\end{pmatrix}x_2+\begin{pmatrix}1\\-6\\3\\-8\\7\end{pmatrix}x_3=0$$

$$\begin{bmatrix}3&2&1\\2&3&-6\\1&0&3\\-4&-1&-8\\1&-1&7\end{bmatrix}\times\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$$

$$\begin{matrix}2&3&-6\\-4&-1&-8\\1&-1&7\\2&-2&14\end{matrix}$$

$$3x_1+2x_2+x_3=0 \quad ①$$
$$2x_1+3x_2-6x_3=0 \quad ②$$
$$x_1-x_2+7x_3=0 \quad ③$$

$$\begin{array}{l}R_1\\R_2\\R_3\end{array}\begin{bmatrix}3&2&1\\2&3&-6\\1&-1&7\end{bmatrix}$$

$-8-2$

$3 \quad -3 \quad 21$

$$\begin{array}{l}\\R_2=3R_2-2R_1\\R_3=3R_3-R_1\end{array}\begin{bmatrix}3&2&1\\0&5&-20\\0&-5&20\end{bmatrix}$$

$$\begin{array}{l}\\ \\R_3=R_2+R_3\end{array}\begin{bmatrix}3&2&1\\0&5&-20\\0&0&0\end{bmatrix}$$

$$5x_2-20x_3=0$$
$$5x_2=20x_3$$
$$x_2=4x_3$$
$$x_1=-3x_3$$

take to operation ④

$$-4(-3x_3)-4x_3-8x_3=0$$
$$12x_3-4x_3-8x_3=0$$
$$0=0$$

take to operation ②

$$-3x_3+3x_3=0$$
$$0=0$$

$\therefore$ x_3 can be any value

$\therefore$ The set of these three vector is linear dependent

Q4 $\because \{u_1, \cdots, u_r, w_1, \cdots, w_s\}$ is a linearly independent subset of V

Suppose $\forall a_i \in F$

$\therefore a_1u_1 + \cdots + a_ru_r + a_{r+1}w_1 + \cdots + a_{r+s}w_s = 0$ if and only if $a_i = 0$

$\therefore \{u_1, \cdots, u_r\}$ and $\{w_1, \cdots w_s\}$ are also a linearly independent subset of V

Suppose. $u = x_1u_1 + x_2u_2 + \cdots + x_ru_r$ $\quad$ $u=0$ if and only if $x_i = 0$

$w = y_1w_1 + y_2w_2 + \cdots + y_sw_s$ $\quad$ $w=0$ if and only if $y_i = 0$

Suppose $u \cap w \neq \{0\}$.

Suppose $v = u \cap w$

$\therefore$ $v = x_1u_1 + x_2u_2 + \cdots + x_ru_r = y_1w_1 + y_2w_2 + \cdots + y_sw_s \neq 0$

$\therefore$ However, v can be zero, if only if $x_i = 0$ and $y_i = 0$

$\therefore$ CONTRADICTION

$\therefore \operatorname{span}(\{u_1, \cdots u_r\}) \cap \operatorname{span}(\{w_1, \ldots, w_s\}) = \{0\}$.

Q5

$$A^T = \begin{array}{c} \begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & 1 & 1 & 3 \\ 3 & 3 & 2 & 8 \\ 1 & 0 & 1 & 2 \end{bmatrix} \end{array} \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}$$

$$A^T x \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \\ 5 \end{bmatrix} \quad \left[\begin{array}{cccc|c} 2 & 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 3 & -2 \\ 0 & 3 & 4 & 7 & -8 \\ 0 & -1 & 2 & 1 & 6 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 2 & 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 3 & -2 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 4 & 4 & 4 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|c} 2 & 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 3 & -2 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$2x_3 + 2x_4 = 2$

$x_3 + x_4 = 1$

Suppose $x_4 = 1$

$x_3 = 0$

$x_2 + 2x_3 + 3x_4 = -2$

$x_2 + 3 = -2$

$x_2 = -5$

$2x_1 - 5 + 3 = 4$

$2x_1 - 2 = 4$

$2x_1 = 6$

$x_1 = 3$

$$\therefore \begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & 1 & 1 & 3 \\ 3 & 3 & 2 & 8 \\ 1 & 0 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 3 \\ -5 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \\ 5 \end{bmatrix}$$

$\therefore$ B belongs to row space of A

$$A^T x \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad \left[\begin{array}{cccc|c} 2 & 1 & 0 & 3 & 1 \\ 1 & 1 & 1 & 3 & 2 \\ 3 & 3 & 2 & 8 & 3 \\ 1 & 0 & 1 & 2 & 4 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 2 & 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 3 & 3 \\ 0 & 3 & 4 & 7 & 3 \\ 0 & -1 & 2 & 1 & 7 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|c} 2 & 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 3 & 3 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & 4 & 4 & 10 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 2 & 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 3 & 3 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right]$$ it is not solveable

Thus C does not belong to row space of A